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Classification of integrable quadratic Hamiltonians on e(3) [3]

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Classification of integrable quadratic Hamiltonians on e(3) [§]

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1 Introduction.

In this work we consider quadratic Hamiltonians of the form

$$H = (M, AM) + (M, B\gamma) + (\gamma, C\gamma) + (P, M) + (Q, \gamma). \tag{1}$$

where $M = (M_1, M_2, M_3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ and A, C are constant symmetric matrices, B is a general constant matrix and P, Q are constant vectors. Without loss of generality we choose

$$A = \mathrm{diag}(a_1, a_2, a_3). \tag{2}$$

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The equations of motion are given by

$$\frac{dM_i}{dt} = \{H, M_i\}, \quad \frac{d\gamma_i}{dt} = \{H, \gamma_i\}$$

where Poisson brackets are defined by

$$\{M_i, M_j\} = \varepsilon_{ijk}\, M_k, \qquad \{M_i, \gamma_j\} = \varepsilon_{ijk}\, \gamma_k, \qquad \{\gamma_i, \gamma_j\} = 0 \tag{3}$$

with ε_{ijk} being the totally anti-symmetric tensor. These linear Poisson brackets are related to the Lie algebra $e(3)$.

Because they admit the two Casimir functions

$$J_1 = (\gamma, \gamma), \quad J_2 = (\gamma, M) \tag{4}$$

we need only one additional first integral for Liouville integrability [1]. In our paper we restrict ourselves to polynomial first integrals. The relevance of the combination of quadratic Hamiltonians (1) with Poisson brackets (3) arises from the Euler-Poinsot model describing motion of a rigid body around a fixed point under gravity and from the Kirchhoff model describing the motion of a rigid body in ideal fluid [2, 3, 4].

In section 2 we give a list of known integrable quadratic Hamiltonians with additional polynomial integrals of motion. Results of our search are given in section 3.

It is known from literature that for pairwise non-equal a_1, a_2, a_3 in (2) no extra integrable cases besides those mentioned in section 2 exist. In addition all Hamiltonians (1) with linear or quadratic first integral are known too. Hamiltonians with cubic first integral are discussed in [7]. We therefore study first integrals I of fourth degree and consider the case $a_1 = a_2 \neq a_3$.

The motivation for a systematic investigation of this ansatz arose from the recent finding of V. Sokolov of a new integrable Hamiltonian of this form with a fourth degree first integral. Our investigation is in so far more general as we consider quadratic Hamiltonians *with linear* terms. Our complete classification of this problem resulted in new integrable Hamiltonians with complex coefficients. The simplest example of such a kind is

$$H = M_1^2 + M_2^2 + 2M_3^2 + iM_1 + M_2$$

which commutes with some fourth degree polynomial under $so(3)$-brackets.

We also were inspired by the Goryachev-Chaplygin Hamiltonian [5, 6] which admits a fourth degree integral exclusively on the Casimir level $J_2 = 0$. Unfortunately our complete classification did not result in new integrable cases.

In section 4 we follow [17, 18] and consider the quantum counterpart

$$[M_i, M_j] = \varepsilon_{ijk} M_k, \qquad [M_i, \gamma_j] = \varepsilon_{ijk} \gamma_k, \qquad [\gamma_i, \gamma_j] = 0 \tag{5}$$

of the Poisson bracket (3). Here M_i, γ_j are elements of an associative algebra with commutator relations (5). The Hamiltonian is a (non-commutative) polynomial of second degree and first integrals are polynomials which commute with the Hamiltonian. Also here we classify quadratic Hamiltonians with linear terms having a fourth degree first integral and obeying the restriction $a_1 = a_2 \neq a_3$. We find four cases, among them the quantum analogue of the above mentioned integrable Hamiltonian of V. Sokolov.

In all three investigations an ansatz for the Hamiltonian and an ansatz for a fourth degree polynomial were made, the Poisson bracket or commutator were computed and set to zero, resulting in large non-linear over-determined algebraic systems. Although they are linear in the coefficients of H and linear in the coefficients of the first integral, the algebraic systems are still very large (see table 1) and challenging to solve.

We used the computer algebra program Crack which was originally designed to solve over-determined PDE-systems. But its interactive capabilities allowed to solve large algebraic systems at first interactively and in doing so to learn how to take advantage of the bi-linearity. Due to an ongoing effort to generalize gathered experience and to incorporate it into the program it is now able to solve large bi-linear systems, for example the first and third system in table 1 automatically and the second system with only few manual interactions. An overview of essential features of the program and other applications requiring the solution of bi-linear systems is given in [7].

type of problem	$e(3)$	$e(3), J_2 = 0$	$e(3)$ quant.
# of unknowns (H,I,total)	17,200,217	17,176,193	17,200,217
# of equations	451	396	451
total # of terms	5469	5243	9681
average # of terms/equ.	12.1	13.2	21.5
time to solve	18h 53min	$\approx$ 15h	11h 43min
details in section	3.1	3.2	4

Table 1. An overview of the solved algebraic systems

Times are measured on a 1.7GHz Pentium 4 running a 120 MByte REDUCE session under Linux.

2 Known integrable Hamiltonian on $e(3)$

In the following we list all known integrable Hamiltonian of type (1) on $e(3)$. First let us consider all cases where the matrices $A = \{a_{ij}\}$, $B = \{b_{ij}\}$ and $C = \{c_{ij}\}$ are diagonal, i.e. the Hamiltonian takes the form:

$$\begin{aligned} H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + 2b_1 M_1\gamma_1 + 2b_2 M_2\gamma_2 + 2b_3 M_3\gamma_3 + \\ c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2 + p_1 M_1 + p_2 M_2 + p_3 M_3 + q_1\gamma_1 + q_2\gamma_2 + q_3\gamma_3. \end{aligned} \tag{6}$$

Kirchhoff's problem of the motion of a rigid body in ideal fluid. In this case the Hamiltonian does not contain linear terms, i.e. we have $p_i = q_i = 0$. For this problem there are three known classical integrable cases of Kirchhoff [8], Clebsch [9] and Steklov-Lyapunov [10, 11].

The Kirchhoff case is defined by the identities

$$a_1 = a_2, \qquad b_1 = b_2, \qquad c_1 = c_2.$$

The additional integral I is linear: $I = M_3$.

For the Clebsch case the coefficients a_i are arbitrary and the remaining parameters satisfy the following conditions:

$$b_1 = b_2 = b_3,$$

$$\frac{c_1 - c_2}{a_3} + \frac{c_3 - c_1}{a_2} + \frac{c_2 - c_3}{a_1} = 0.$$

If not all a_i are equal then the Hamiltonian can be represented in the form

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + a_2 a_3 \gamma_1^2 + a_3 a_1 \gamma_2^2 + a_1 a_2 \gamma_3^2.$$

The additional quadratic integral admitted by H is

$$I = M_1^2 + M_2^2 + M_3^2 + (a_2 + a_3)\gamma_1^2 + (a_3 + a_1)\gamma_2^2 + (a_1 + a_2)\gamma_3^2.$$

If $a_1 = a_2 = a_3$ then we get the Neumann Hamiltonian

$$H = M_1^2 + M_2^2 + M_3^2 + c_1\gamma_1^2 + c_2\gamma_2^2 + c_3\gamma_3^2.$$

The additional integral in this case coincides with H.

For the Steklov-Lyapunov case the coefficients a_i are arbitrary and the remaining parameters satisfy the following conditions:

$$\frac{b_1 - b_2}{a_3} + \frac{b_3 - b_1}{a_2} + \frac{b_2 - b_3}{a_1} = 0,$$

$$c_1 - \frac{(b_2-b_3)^2}{a_1} = c_2 - \frac{(b_3-b_1)^2}{a_2} = c_3 - \frac{(b_1-b_2)^2}{a_3}.$$

If not all the a_i are equal then the Hamiltonian can be represented in the form

$$H = a_1M_1^2 + a_2M_2^2 + a_3M_3^2 + 2a_2a_3M_1\gamma_1 + 2a_3a_1M_2\gamma_2 + 2a_1a_2M_3\gamma_3$$
$$+a_1(a_2-a_3)^2\gamma_1^2 + a_2(a_3-a_1)^2\gamma_2^2 + a_3(a_1-a_2)^2\gamma_3^2.$$

The additional integral is quadratic:

$$I = M_1^2 + M_2^2 + M_3^2 + 2(a_2+a_3)M_1\gamma_1 + 2(a_3+a_1)M_2\gamma_2 + 2(a_1+a_2)M_3\gamma_3$$
$$+(a_2-a_3)^2\gamma_1^2 + (a_3-a_1)^2\gamma_2^2 + (a_1-a_2)^2\gamma_3^2.$$

If $a_1 = a_2 = a_3$ we have to interchange the Hamiltonian H and the integral I just as in the Clebsch case.

Recently in the paper [12] by Sokolov a Hamiltonian with non-diagonal matrix B having an integral of fourth degree was presented. One of the possible forms of this Hamiltonian is:

$$H = M_1^2 + M_2^2 + 2\,M_3^2 + 2\,(a_1\gamma_1 + a_2\gamma_2)\,M_3 - (a_1^2 + a_2^2)\,\gamma_3^2. \tag{7}$$

The integral can be represented as a product $I = k_1\,k_2$, where $k_1 = M_3$ and

$$k_2 = (M_1^2 + M_2^2 + M_2^2)\,M_3 + 2\,(a_1M_1 + a_2M_2)\,(M_1\gamma_1 + M_2\gamma_2)$$
$$+2\,(a_1\gamma_1 + a_2\gamma_2)\,M_3^2 + (a_1\gamma_1 + a_2\gamma_2)^2\,M_3$$
$$-(a_1^2 + a_2^2)\,(2M_1\gamma_1 + 2M_2\gamma_2 + M_3\gamma_3)\,\gamma_3.$$

This case appears to be similar in its properties to the Kowalewski case given below.

The problem of motion of a rigid body around a fixed point. Hamiltonians describing such situations have the form (6), where $b_i = c_i = p_i = 0$. The following integrable cases are known.

The Lagrange case:

$$H = M_1^2 + M_2^2 + a_3M_3^2 + q_3\gamma_3 \tag{8}$$

where a_3, q_3 are arbitrary. The additional integral I is linear: $I = M_3$.

The Euler case:

$$H = a_1M_1^2 + a_2M_2^2 + a_3M_3^2 \tag{9}$$

with an additional quadratic integral:

$$I = M_1^2 + M_2^2 + M_3^2.$$

The Kowalewski case:

$$H = M_1^2 + M_2^2 + 2M_3^2 + q_1\gamma_1 + q_2\gamma_2 \tag{10}$$

with arbitrary parameters q_1, q_2. The additional integral of fourth degree can be written as $I = G_1^2 + G_2^2$, where

$$G_1 = M_1^2 - M_2^2 - q_1\gamma_1 + q_2\gamma_2, \qquad G_2 = 2M_1M_2 - q_2\gamma_1 - q_1\gamma_2.$$

We note, that if $q_2 \neq iq_1$, then the transformation $M \to T\,M$, $\Gamma \to T\,\Gamma$, where

$$T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

can be used to make q_2 to zero.

Generalizations. Terms linear in the moments M_i that occur in the Hamiltonian could be interpreted as an action of hydrostatic forces ([4]). We have no comments on the physical meaning of other mixed Hamiltonians (1) (i.e. Hamiltonians having both $B \neq 0$ or $C \neq 0$, and $Q \neq 0$).

For example, an obvious hybrid of Lagrange's and Kirchhoff's Hamiltonians (with additional hydrostatic member) is

$$H = M_1^2 + M_2^2 + s_1M_3^2 + s_2\gamma_3M_3 + s_3\gamma_3^2 + s_4M_3 + s_5\gamma_3, \tag{12}$$

3 Results

3.1 The classical case.

For the Hamiltonian (1) on $e(3)$ we consider the case of A being diagonal:

$$A = \mathrm{diag}(a_1, a_2, a_3),$$

where

$$a_1 = a_2 \neq a_3, \qquad a_i \neq 0, \qquad i = 1, 2, 3.$$

Without loss of generality matrix B can be considered as low-triangular, i.e.

$$b_{12} = b_{13} = b_{23} = 0.$$

Note, that the addition of Casimir functions (4) to the Hamiltonian does not influence the equation of motion. By subtracting an appropriate linear combination of J_1 and J_2 we can make $b_{11}=c_{11}=0$ and get

$$\begin{aligned} H &= M_1^2+M_2^2+a_3M_3^2+ \\ & b_{21}\gamma_1M_2+b_{31}\gamma_1M_3+b_{32}\gamma_2M_3+b_{22}\gamma_2M_2+b_{33}\gamma_3M_3+ \\ & c_{12}\gamma_1\gamma_2+c_{13}\gamma_1\gamma_3+c_{22}\gamma_2^2+c_{23}\gamma_2\gamma_3+c_{33}\gamma_3^2+ \\ & p_1M_1+p_2M_2+p_3M_3+q_1\gamma_1+q_2\gamma_2+q_3\gamma_3. \end{aligned} \tag{13}$$

Lemma. For a Hamiltonian of type (13), a canonical transformation can be used to obtain $b_{21}=0$.

Thus we will consider only the following Hamiltonian:

$$\begin{aligned} H &= M_1^2+M_2^2+a_3M_3^2+b_{31}\gamma_1M_3+b_{32}\gamma_2M_3+b_{22}\gamma_2M_2+b_{33}\gamma_3M_3+ \\ & c_{12}\gamma_1\gamma_2+c_{13}\gamma_1\gamma_3+c_{22}\gamma_2^2+c_{23}\gamma_2\gamma_3+c_{33}\gamma_3^2+ \\ & p_1M_1+p_2M_2+p_3M_3+q_1\gamma_1+q_2\gamma_2+q_3\gamma_3. \end{aligned} \tag{14}$$

Theorem 1. *The Hamiltonian of the above form commutes with a polynomial integral of* 4^{th} *degree iff it coincides with one of the following*:

- (12), *where* s_i *- are arbitrary, or*
- $$\begin{aligned} & H=M_1^2+M_2^2+M_3^2+2s_1\gamma_3M_3-s_1^2\gamma_3^2+s_2\gamma_1+s_3\gamma_2+s_4\gamma_3+ \\ & \lambda\Big(2s_1M_3^2+s_2M_1+s_3M_2+s_4M_3+s_1(s_2\gamma_1+s_3\gamma_2)\Big), \end{aligned} \tag{15}$$
- $$\begin{aligned} & H=M_1^2+M_2^2+2M_3^2+s_1(i\gamma_1+\gamma_2)M_3+ \\ & s_2(-iM_1+M_2)+s_3M_3+s_4(i\gamma_1+\gamma_2)-s_1s_2\gamma_3, \end{aligned} \tag{16}$$
- $$\begin{aligned} & H=M_1^2+M_2^2+2M_3^2+2(s_1\gamma_1+s_2\gamma_2)\,M_3-(s_1^2+s_2^2)\gamma_3^2+ \\ & s_3\,(M_3+s_1\gamma_1+s_2\gamma_2)+s_4\gamma_1+s_5\gamma_2, \end{aligned} \tag{17}$$

  where s_i *are parameters constrained only by the condition*:

  $$s_2s_5+s_1s_4=0.$$

Remarks

1. The case (15) was founded by Rubanovskiyi [13].
2. Under the conditions $s_1=s_2=s_3=0$ the formula (17) describes the Kowalewski case. If $s_3=s_4=s_5=0$ we have the Sokolov case for which the general formula (17) was presented in [14] and the Lax pair was found by Sokolov and A.V. Tsiganov [15].
3. The Hamiltonian (16) is probably new.

3.2 Classical case with additional condition.

We now consider the case of the area integral J_2 being equal to zero. In this case the following additional integrable Hamiltonians are known.

The Goryachev-Chaplygin Hamiltonian [5, 6]

$$H = M_1^2 + M_2^2 + 4M_3^2 - s_1\gamma_1 \tag{18}$$

belongs to the class of Hamiltonians describing the motion of a rigid body around a fixed point. The additional integral is of third degree.

For the Kirchhoff problem is known the Hamiltonian by Chaplygin:

$$H = M_1^2 + M_2^2 + 2M_3^2 + c\gamma_1^2 - c\gamma_2^2. \tag{19}$$

Here the additional integral has fourth degree.

Theorem 2. *The Hamiltonian of the form* (14) *commutes with a polynomial integral of fourth degree with additional condition* $(M, \gamma) = 0$, *iff it coincides with one of the Hamiltonian from Theorem 1 or with the following*:

- $$H = M_1^2 + M_2^2 + 2M_3^2 + s_1(\gamma_1^2 - \gamma_2^2) + s_2\gamma_1\gamma_2 + s_3M_3 + s_4\gamma_1 + s_5\gamma_2, \tag{20}$$

  where s_i *- are arbitrary parameters.*

The Hamiltonian (20) is the generalization of Chaplygin's case [5, 6, 16].

4 Quantum case.

In the papers by Komarov, Sklyanin [17, 18, 19] and others quantum generalizations of classical Hamiltonians on $e(3)$ and $so(4)$ were considered. Similarly, instead of the Poisson bracket (3) we investigate the commutator relations (5) in an associative algebra with generators $M_1, M_2, M_3,\ \gamma_1, \gamma_2, \gamma_3$. We can consider M_i, γ_i to be operators. Due to commutator relations (5) any monomial always could be ordered such, that M_i come before any γ_i and indices increase (such a presentation is unique). Thus Hamiltonians have the same form (1) as in the classical case, but the multiplication is now noncommutative. In this case, any element I from the associative algebra that satisfies $[H, I] = 0$ is called an integral.

Theorem 3. *The Hamiltonian of the above form commutes with a polynomial integral of* 4^{th} *degree iff it coincides with one of the following*:

-
$$H = M_1^2 + M_2^2 + s_1 M_3^2 + s_2 \gamma_3 M_3 + s_3 \gamma_3^2 + s_4 M_3 + s_5 \gamma_3, \tag{21}$$
where s_i - are arbitrary parameters;

-
$$\begin{gathered} H = M_1^2 + M_2^2 + M_3^2 + 2 s_1 \gamma_3 M_3 - s_1^2 \gamma_3^2 + s_2 \gamma_1 + s_3 \gamma_2 + s_4 \gamma_3 + \\ \lambda \Big(2 s_1 M_3^2 + s_2 M_1 + s_3 M_2 + s_4 M_3 + s_1 (s_2 \gamma_1 + s_3 \gamma_2) \Big), \end{gathered} \tag{22}$$

-
$$\begin{gathered} H = M_1^2 + M_2^2 + 2M_3^2 + s_1 (i\gamma_1 + \gamma_2) M_3 + \\ s_2 (-i M_1 + M_2) + s_3 M_3 + s_4 (i\gamma_1 + \gamma_2) - s_1 s_2 \gamma_3, \end{gathered} \tag{23}$$

-
$$\begin{gathered} H = M_1^2 + M_2^2 + 2M_3^2 + 2(s_1 \gamma_1 + s_2 \gamma_2)\, M_3 - (s_1^2 + s_2^2) \gamma_3^2 + \\ s_3 \,(M_3 + s_1 \gamma_1 + s_2 \gamma_2) + s_4 \gamma_1 + s_5 \gamma_2, \end{gathered} \tag{24}$$
where s_i are parameters constrained only by the condition:
$$s_2 s_5 + s_1 s_4 = 0;$$

Notice that this list looks exactly as the list of subsection 3.1 although *a priori* this coincidence is not predictable. The most interesting here is Hamiltonian (24), which is a quantum generalization of the Sokolov Hamiltonian. If $s_1 = s_2 = s_3 = 0$ we have the quantum Kowalewski Hamiltonian [17, 20].

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