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G_2 generating technique for minimal $D = 5$ supergravity and black rings

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ZIB-Report 10-14 (July 2010)

DTP-MSU/07-19, LAPTH-1203/07

G_2 generating technique for minimal $D = 5$ supergravity and black rings

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(Dated: July 8, 2010)

A solution generating technique is developed for $D = 5$ minimal supergravity with two commuting Killing vectors based on the G_2 U-duality arising in the reduction of the theory to three dimensions. The target space of the corresponding 3-dimensional sigma-model is the coset $G_{2(2)}/(SL(2,R) \times SL(2,R))$. Its isometries constitute the set of solution generating symmetries. These include two electric and two magnetic Harrison transformations with the corresponding two pairs of gauge transformations, three $SL(2,R)$ S-duality transformations, and the three gravitational scale, gauge and Ehlers transformations (altogether 14). We construct a representation of the coset in terms of 7×7 matrices realizing the automorphisms of split octonions. Generating a new solution amounts to transforming the coset matrices by one-parametric subgroups of $G_{2(2)}$ and subsequently solving the dualization equations. Using this formalism we derive a new charged black ring solution with two independent parameters of rotation.

PACS numbers: 04.20.Jb, 04.50.+h, 04.65.+e

I. INTRODUCTION

The discovery of rotating black rings [1] (for a recent review see [2] and references therein) has attracted new interest in five-dimensional minimal supergravity [3, 4]. Within this theory supersymmetric charged black ring solutions were found [5, 6]. The bosonic sector of five-dimensional minimal supergravity is Einstein-Maxwell theory with a Chern-Simons term, the structure of the Lagrangian being similar to that of eleven-dimensional supergravity [7, 8]. While in pure Einstein-Maxwell theory in five and higher dimensions no charged black hole solution, generalizing the uncharged Myers-Perry black holes [9], is known, the Chern-Simons term endows five-dimensional Einstein-Maxwell theory with more hidden symmetries, implying the existence of exact charged rotating black hole solutions [10–12], Meanwhile the most general black ring solution which might possess mass, two angular momenta, electric charge and magnetic moment as independent parameters is still not found. Here we propose a new generating technique which can solve this problem. It is based on the duality symmetries of the three-dimensional reduction of the theory.

The hidden symmetries arising upon dimensional reduction of five-dimensional minimal supergravity to three dimensions were studied by Mizoguchi and Ohta [7], by Cremmer, Julia, Lu and Pope [13] using the technique of [14], and were more recently investigated both in the bosonic and fermionic sectors by Possel [15] (see also [16]). The corresponding classical U-duality group is the non-compact version of the lowest rank exceptional group G_2 [17]. In three dimensions one obtains the gravity-coupled sigma-model with the homogeneous target space $G_{2(2)}/SO(4)$ for

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the Lorentzian signature of the 3-space, or $G_{2(2)}/(SL(2,R)\times SL(2,R))$ in the Euclidean case. Some further aspects of these symmetries were discussed in [8], their infinite-dimensional extension upon reduction to two and one dimensions was also explored [18].

Here we investigate the $G_{2(2)}/(SL(2,R)\times SL(2,R))$ sigma model in the context of the solution generating technique which has proved to be extremely useful in various non-linear theories from pure gravity, Einstein-Maxwell theory [19–23] and dilatonic gravity [24–30] to more general supergravity models [31–33] and string theory [34]. Some partial use of hidden symmetries of this kind to generate new rotating rings recently became a rapidly developing industry. One direction was to use the $SL(2,R)$ subgroup of the U-duality group [35, 36]. Another line was related to the purely gravitational sector (without the Maxwell field) which leads to $SL(3,R)$ U-duality in three dimensions [20, 37, 38]. Further reduction to two dimensions gives rise to a Belinsky-Zakharov type integrable model which was extensively used to construct soliton solutions [38–49]. However, the full G_2 symmetry was never used for generating purposes for lack of a convenient representation of the coset $G_{2(2)}/(SL(2,R)\times SL(2,R))$ in terms of the target space variables. Although the 14-dimensional (adjoint) representation was given explicitly in [7], it is still too complicated for practical generating applications. Here we construct a suitable representation in terms of 7×7 matrices and give two examples of its application: a sigma-model construction of the electrically charged rotating black hole and the generation of a non-BPS doubly rotating charged black ring from the black ring with two angular momenta of [47].

Five-dimensional minimal supergravity contains a graviton, two $N=2$ symplectic-Majorana gravitini (equivalent to a single Dirac gravitino), and one $U(1)$ gauge field. The bosonic part of the Lagrangian is very similar to that of $D=11$ supergravity, being endowed with a Chern-Simons term [3, 4]:

$$S_5=\frac{1}{16\pi G_5}\left[\int d^5x\sqrt{-\hat{g}}\left(\hat{R}-\frac{1}{4}\hat{F}^2\right)-\frac{1}{3\sqrt{3}}\int \hat{F}\wedge\hat{F}\wedge\hat{A}\right], \tag{1}$$

where $\hat{F}=d\hat{A}$. This theory can be obtained as a suitably truncated Calabi-Yau compactification of $D=11$ supergravity [50].

Our purpose is to construct a generating technique for classical solutions with two commuting Killing symmetries. Dimensional reduction leads to a three-dimensional sigma-model possessing a $G_{2(2)}$ target space symmetry [7]. To explore it fully we need a convenient representation of the action of symmetries on the target space variables. We give here an alternative derivation of the three-dimensional sigma-model which has the advantage of being more explicit and easy to use for solution generation. The reduction is performed in Sect. 2 in two steps, first to four, then to three dimensions. In Sect. 3 we reveal the symmetries of the three-dimensional sigma-model using a direct (computer assisted) solution of the corresponding Killing equations[1]. The resulting symmetry transformations are identified in the usual terms of gauge, S-duality and Harrison-Ehlers sectors. Then we reformulate in Sect. 4 the problem in terms of a covariant (with respect to the two-Killing plane) reduction which is more suitable for constructing the matrix representation, and give the coset matrix representative as a symmetrical 7×7 matrix. In Sect. 5 we identify the charging transformation, and apply it to the construction of the doubly rotating charged black ring.

II. DIMENSIONAL REDUCTION

A. D=4

Assuming that the five-dimensional metric and the Maxwell field $\hat{A}$ do not depend on a space-like coordinate z, we arrive at the four-dimensional Einstein theory with two Maxwell fields, a dilaton and an axion. We parametrize the five-dimensional interval and the Maxwell one-form as

$$ds_5^2 = \mathrm{e}^{-2\phi}(dz+C_\mu dx^\mu)^2+\mathrm{e}^{\phi}ds_4^2, \tag{2}$$

$$\hat{A} = A_\mu dx^\mu+\sqrt{3}\kappa dz, \tag{3}$$

($\mu=1\ldots4$). The corresponding four-dimensional action reads

$$S_4=\frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\left[R-\frac{3}{2}(\partial\phi)^2-\frac{3}{2}\mathrm{e}^{2\phi}(\partial\kappa)^2-\frac{1}{4}\mathrm{e}^{-3\phi}G^2-\frac{1}{4}\mathrm{e}^{-\phi}\tilde{F}^2-\frac{1}{2}\kappa FF^*\right], \tag{4}$$

---

[1] A purely algebraic construction of the Killing vectors will be presented elsewhere [51].

where

$$G_4 = G_5/2\pi R_5, \quad G = dC, \quad F = dA, \quad \tilde{F} = F + \sqrt{3}C \wedge d\kappa, \tag{5}$$

and F^* is the four-dimensional Hodge dual of F.

The dilaton ϕ and the axion κ parametrize the coset $SL(2,R)/U(1)$. To reveal the $SL(2,R)$ S-duality symmetry in the sector of vector fields (A is inherited from 5D theory, C is the Kaluza-Klein vector) one has to reparametrize them using some dualization [8]. We will reveal S-duality later on the level of the further 3D reduction.

The field equations in terms of the four-dimensional variables read

$$\nabla^2\phi - \mathrm{e}^{2\phi}(\partial\kappa)^2 + \frac{1}{4}\mathrm{e}^{-3\phi}G^2 + \frac{1}{12}\mathrm{e}^{-\phi}\tilde{F}^2 \;=\; 0, \tag{6}$$

$$\nabla_\mu\left(\mathrm{e}^{2\phi}\nabla^\mu\kappa\right) - \frac{1}{3}\left[\sqrt{3}\,\nabla_\mu(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}C_\nu) + \frac{1}{2}F_{\mu\nu}F^{*\mu\nu}\right] \;=\; 0, \tag{7}$$

$$\nabla_\mu\left(\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu} + 2\kappa F^{*\mu\nu}\right) \;=\; 0, \tag{8}$$

$$\nabla_\mu\left(\mathrm{e}^{-3\phi}G^{\mu\nu}\right) + \sqrt{3}\,\mathrm{e}^{-\phi}\tilde{F}^{\mu\nu}\partial_\mu\kappa \;=\; 0, \tag{9}$$

and the Bianchi identities are

$$\nabla_\mu F^{*\mu\nu} = 0, \qquad \nabla_\mu G^{*\mu\nu} = 0. \tag{10}$$

It is convenient to introduce the modified Maxwell tensors

$$\mathcal{F}^{\mu\nu} \;=\; \mathrm{e}^{-\phi}\tilde{F}^{\mu\nu} + 2\kappa F^{*\mu\nu}, \tag{11}$$

$$\mathcal{G}^{\mu\nu} \;=\; \mathrm{e}^{-3\phi}G^{\mu\nu} + \sqrt{3}\,\left(\mathrm{e}^{-\phi}\kappa\tilde{F}^{\mu\nu} + \kappa^2F^{*\mu\nu}\right), \tag{12}$$

in terms of which the Maxwell equations have the divergence form

$$\nabla_\mu\mathcal{F}^{\mu\nu} = 0, \qquad \nabla_\mu\mathcal{G}^{\mu\nu} = 0. \tag{13}$$

B. D=3

Further reduction to $D = 3$ is performed with respect to time, assuming the standard parametrization of the stationary four-metric

$$ds_4^2 = -f(dt - \omega_i dx^i)^2 + f^{-1}h_{ij}dx^i dx^j. \tag{14}$$

The spatial part of the Bianchi identities (10) is solved introducing the electric potentials $C_0 = \bar{v}_1$, $A_0 = \bar{v}_2$, so that

$$G_{i0} = \partial_i\bar{v}_1, \qquad F_{i0} = \partial_i\bar{v}_2. \tag{15}$$

Similarly, the spatial components of the Maxwell equations (13) are solved by introducing magnetic potentials $\bar{u}_1$, $\bar{u}_2$:

$$\mathcal{G}^{ij} = \frac{f}{\sqrt{h}}\,\epsilon^{ijk}\partial_k\bar{u}_1, \qquad \mathcal{F}^{ij} = \frac{f}{\sqrt{h}}\,\epsilon^{ijk}\partial_k\bar{u}_2. \tag{16}$$

The time components of the corresponding equations then give the second order equations for these potentials. Straightforwardly we can find (with the convention $\epsilon^{ijk} = -\epsilon^{0ijk}$)

$$G^{ij} \;=\; \frac{f}{\sqrt{h}}\mathrm{e}^{3\phi}\,\epsilon^{ijk}(w_1)_k, \qquad (w_1)_k := \partial_k\bar{u}_1 - \sqrt{3}\;\kappa(\partial_k\bar{u}_2 - \kappa\partial_k\bar{v}_2), \tag{17}$$

$$\tilde{F}^{ij} \;=\; \frac{f}{\sqrt{h}}\mathrm{e}^{\phi}\,\epsilon^{ijk}(w_2)_k, \qquad (w_2)_k := \partial_k\bar{u}_2 - 2\kappa\partial_k\bar{v}_2, \tag{18}$$

$$\tilde{F}_{i0} \;=\; (z_2)_i, \qquad (z_2)_i := \partial_i\bar{v}_2 - \sqrt{3}\bar{v}_1\partial_i\kappa. \tag{19}$$

The remaining components of the Maxwell tensors are obtained using the following relations valid for any second rank four-dimensional antisymmetric tensor $W_{\mu\nu}$ with the assumed form of the metric (14):

$$W^{i0} = W^{ij}\omega_j - h^{ij}W_{j0}, \qquad W_{ij} = f^{-2}h_{ik}h_{jl}W^{kl} + 2W_{0[i}\omega_{j]}. \tag{20}$$

Using this we find:

$$
\begin{aligned}
G^{i0} &= \frac{f}{\sqrt{h}} \mathrm{e}^{3\phi}\, \epsilon^{ijk} \omega_j (w_1)_k - \partial^i \bar{v}_1, &(21)\\
G_{ij} &= f^{-1}\sqrt{h}\mathrm{e}^{3\phi}\, \epsilon_{ijk}(w_1)^k + 2\omega_{[i}\partial_{j]}\bar{v}_1, &(22)\\
\tilde{F}^{i0} &= \frac{f}{\sqrt{h}} \mathrm{e}^{\phi}\, \epsilon^{ijk} \omega_j (w_2)_k - (z_2)^i, &(23)\\
\tilde{F}_{ij} &= f^{-1}\sqrt{h}\mathrm{e}^{\phi}\, \epsilon_{ijk}(w_2)^k + 2\omega_{[i}(z_2)_{j]}, &(24)
\end{aligned}
$$

and for the squared quantities:

$$
G^2 = -2(\partial\bar{v}_1)^2 + 2\mathrm{e}^{6\phi}(w_1)^2, \qquad \tilde{F}^2 = -2(z_2)^2 + 2\mathrm{e}^{2\phi}(w_2)^2, \tag{25}
$$

where $(w_1)^2 = (w_1)_i (w_1)^i$.

Now we turn to the Einstein equations:

$$
R_{\mu\nu} - \frac{3}{2}\left(\partial_\mu\phi\partial_\nu\phi + \mathrm{e}^{2\phi}\partial_\mu\kappa\partial_\nu\kappa\right) - \frac{1}{2}\mathrm{e}^{-3\phi}\left(G_{\mu\alpha}G_\nu{}^\alpha - \frac{1}{4}G^2 g_{\mu\nu}\right) - \frac{1}{2}\mathrm{e}^{-\phi}\left(\tilde{F}_{\mu\alpha}\tilde{F}_\nu{}^\alpha - \frac{1}{4}\tilde{F}^2 g_{\mu\nu}\right) = 0. \tag{26}
$$

The Ricci tensor decomposes as follows

$$
\begin{aligned}
R_{00} &= \frac{1}{2}\left(f\nabla^2 f - (\partial f)^2 + \tau^2\right), &(27)\\
R^i{}_0 &= -\frac{f}{2\sqrt{h}}\, \epsilon^{ijk}\partial_j\tau_k, &(28)\\
R^{ij} &= f^2\mathcal{R}^{ij} - \frac{1}{2}\left[\partial^i f\partial^j f + \tau^i\tau^j\right] + h^{ij}R_{00}, &(29)
\end{aligned}
$$

where

$$
\tau^i = -\frac{f^2}{\sqrt{h}}\, \epsilon^{ijk}\partial_j\omega_k. \tag{30}
$$

The $0i$-part of (26) can be solved introducing the twist potential χ via

$$
\tau_i = \partial_i\chi + \frac{1}{2}\left\{\bar{v}_1\partial_i\bar{u}_1 - \bar{u}_1\partial_i\bar{v}_1 + \bar{v}_2\partial_i\bar{u}_2 - \bar{u}_2\partial_i\bar{v}_2 + \sqrt{3}\,\left[\kappa^2\bar{v}_1\partial_i\bar{v}_2 - \bar{v}_2\partial_i(\kappa^2\bar{v}_1)\right] - \sqrt{3}\,\left[\kappa\bar{v}_1\partial_i\bar{u}_2 - \bar{u}_2\partial_i(\kappa\bar{v}_1)\right]\right\}. \tag{31}
$$

Using this relation, we can rewrite the 00-component of the Einstein equations as

$$
R_{00} = \frac{1}{4}f\left\{\mathrm{e}^{-3\phi}\left[(\partial\bar{v}_1)^2 + \mathrm{e}^{6\phi}(w_1)^2\right] + \mathrm{e}^{-\phi}\left[(z_2)^2 + \mathrm{e}^{2\phi}(w_2)^2\right]\right\}, \tag{32}
$$

and present the space-space part as

$$
\begin{aligned}
R^{ij} &= \frac{3}{2}f^2 h^{ia}h^{jb}\left(\partial_a\phi\partial_b\phi + \mathrm{e}^{2\phi}\partial_a\kappa\partial_b\kappa\right)\\
&- \frac{1}{2}\mathrm{e}^{-3\phi}f\left[\partial^i\bar{v}_1\partial^j\bar{v}_1 + \mathrm{e}^{6\phi}(w_1)^i(w_1)^j\right] + \frac{1}{4}\mathrm{e}^{-3\phi}fh^{ij}\left[(\partial\bar{v}_1)^2 + \mathrm{e}^{6\phi}(w_1)^2\right]\\
&- \frac{1}{2}\mathrm{e}^{-\phi}f\left[(z_2)^i(z_2)^j + \mathrm{e}^{2\phi}(w_2)^i(w_2)^j\right] + \frac{1}{4}\mathrm{e}^{-\phi}fh_{ij}\left[(z_2)^2 + \mathrm{e}^{2\phi}(w_2)^2\right]. &(33)
\end{aligned}
$$

From the Eq. (28) we then find that the Ricci tensor built on the three-dimensional metric h_{ij} will satisfy the following three-dimensional Einstein equation

$$
\begin{aligned}
\mathcal{R}_{ij} &= \frac{1}{2f^2}\left(\partial_i f\partial_j f + \tau_i\tau_j\right) + \frac{3}{2}\left(\partial_i\phi\partial_j\phi + \mathrm{e}^{2\phi}\partial_i\kappa\partial_j\kappa\right)\\
&- \frac{1}{2f}\left[\mathrm{e}^{-3\phi}\partial_i\bar{v}_1\partial_j\bar{v}_1 + \mathrm{e}^{3\phi}(w_1)_i(w_1)_j + \mathrm{e}^{-\phi}(z_2)_i(z_2)_j + \mathrm{e}^{\phi}(w_2)_i(w_2)_j\right]. &(34)
\end{aligned}
$$

These equations derive from the action

$$S_3 = \int \left(\mathcal{R} - G_{AB} \frac{\partial \Phi^A}{\partial x^i} \frac{\partial \Phi^B}{\partial x^j} h^{ij} \right) \sqrt{h} d^3 x \tag{35}$$

describing the three-dimensional gravity coupled sigma model with eight scalar fields $\Phi^A = \{f, \chi, \phi, \kappa, \bar{v}_1, \bar{v}_2, \bar{u}_1, \bar{u}_2\}$ and the target space metric G_{AB} which will be given shortly. It can be checked that the field equations for these quantities are equivalent to the equations resulting from variation of this action over the Φ^A.

In terms of the slightly rearranged potentials

$$v_1 := \bar{v}_1, \quad u_1 := \bar{u}_1 - \kappa^3 \bar{v}_1, \quad v_2 := \bar{v}_2 - \sqrt{3}\,\kappa \bar{v}_1, \quad u_2 := \bar{u}_2 - \sqrt{3}\,\kappa^2 \bar{v}_1, \tag{36}$$

the equation for the twist potential χ will simplify to

$$\tau_i = \partial_i \chi + \frac{1}{2}(v_1 \partial_i u_1 - u_1 \partial_i v_1 + v_2 \partial_i u_2 - u_2 \partial_i v_2). \tag{37}$$

The other variables will read

$$\begin{aligned} (w_1)_i &:= \partial_i u_1 + \kappa^3 \partial_i v_1 - \sqrt{3}\,\kappa(\partial_i u_2 - \kappa \partial_i v_2), \\ (w_2)_i &:= \left(\partial_i u_2 - 2\kappa \partial_i v_2 - \sqrt{3}\,\kappa^2 \partial_i v_1 \right), \\ (z_2)_i &:= \left(\partial_i v_2 + \sqrt{3}\,\kappa \partial_i v_1 \right). \end{aligned} \tag{38}$$

Finally, denoting $\xi = \ln f$ and shifting the twist potential

$$\chi \to \chi + \frac{1}{2}(v_1 u_1 + v_2 u_2), \tag{39}$$

we can present the metric of the space of potentials $\Phi^A = (\xi,\ \phi,\ \kappa,\ \chi,\ v_1,\ u_1,\ v_2,\ u_2)$ as

$$\begin{aligned} dl^2 &= G_{AB} d\Phi^A d\Phi^B = \frac{1}{2} \Bigg\{ d\xi^2 + \mathrm{e}^{-2\xi}(d\chi + v_1 du_1 + v_2 du_2)^2 + 3\left(d\phi^2 + \mathrm{e}^{2\phi} d\kappa^2 \right) \\ &- \mathrm{e}^{-\xi} \Bigg[\mathrm{e}^{-3\phi} dv_1^2 + \mathrm{e}^{3\phi} \left[du_1 + \kappa^3 dv_1 - \sqrt{3}\kappa(du_2 - \kappa dv_2) \right]^2 \\ &+ \mathrm{e}^{-\phi} \left(dv_2 + \sqrt{3}\kappa dv_1 \right)^2 + \mathrm{e}^{\phi} \left(du_2 - 2\kappa dv_2 - \sqrt{3}\kappa^2 dv_1 \right)^2 \Bigg] \Bigg\}. \end{aligned} \tag{40}$$

The target space is an eight-dimensional space with signature $(++++----)$ similar to that of four-dimensional dilaton-axion gravity with two Maxwell fields [29]. In the latter case it was identified as the $SU(2,2)/S(U(2)\times U(2))$ coset space possessing a 15-dimensional isometry group and a 7-dimensional isotropy subgroup. In our case the symmetry group is $G_{2(2)}$ [7, 13]. In what follows we present an independent computer-assisted way to reveal the geometric structure of the space (40).

III. ISOMETRIES OF THE TARGET SPACE

We used MAPLE to calculate the Riemann tensor of the target space and to check that all its covariant derivatives vanish, indicating that it is a symmetric space. The set of Killing vectors was identified using the CRACK code on REDUCE [52] to solve the Killing equations for the metric (40)

$$X_{A;B} + X_{B;A} = 0. \tag{41}$$

The code gives 14 Killing vectors X_M (we use $M,\, N\, \ldots = 1,\ldots,14$ for numbering the Killing vectors and $A,\, B\, \ldots = 1,\ldots,8$ to denote the target space coordinates) from which the first 9 are relatively simple:

$$
\begin{aligned}
X_1 &= \partial_\chi, \\
X_2 &= \partial_{u_1}, \\
X_3 &= \partial_{u_2}, \\
X_4 &= -u_1\partial_\chi + \partial_{v_1}, \\
X_5 &= -u_2\partial_\chi + \partial_{v_2}, \\
X_6 &= \partial_\xi + \chi\partial_\chi + \partial_\phi - \kappa\partial_\kappa + 2v_1\partial_{v_1} - u_1\partial_{u_1} + v_2\partial_{v_2}, \\
X_7 &= 3\partial_\xi + 3\chi\partial_\chi + \partial_\phi - \kappa\partial_\kappa + 3v_1\partial_{v_1} + 2v_2\partial_{v_2} + u_2\partial_{u_2}, \\
X_8 &= v_2^2\partial_\chi - \partial_\kappa - \sqrt{3}u_2\partial_{u_1} + \sqrt{3}v_1\partial_{v_2} - 2v_2\partial_{u_2}, \\
X_9 &= {u_2}^2\partial_\chi + \kappa^2\partial_\kappa - e^{-2\,\phi}\partial_\kappa - 2\,\kappa\partial_\phi - \sqrt{3}u_1\partial_{u_2} + \sqrt{3}v_2\partial_{v_1} - 2\,u_2\partial_{v_2}.
\end{aligned}
\tag{42}
$$

The remaining five are complicated in terms of the chosen coordinates, we will present them in an alternative form later on. The first eight Killing vectors span a Borel subalgebra, they are used below to construct a matrix representation of the coset.

Already the number 14 of generators and the signature of the target space indicate that one deals with the real non-compact form $G_{2(2)}$ of the exceptional group G_2. The same code provides the following list of commutation relations:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0	0	0	0	X_1	$3X_1$	0	0	$3X_5$
X_2		0	0	$-X_1$	0	$-X_2$	0	0	$-\sqrt{3}X_3$	$\sqrt{3}X_8$
X_3			0	0	$-X_1$	0	X_3	$-\sqrt{3}X_2$	$-2X_5$	$3X_6-2X_7$
X_4				0	0	$2X_4$	$3X_4$	$\sqrt{3}X_5$	0	0
X_5					0	X_5	$2X_5$	$-2X_3$	$\sqrt{3}X_4$	$2X_9$
X_6						0	0	X_8	$-X_9$	0
X_7							0	X_8	$-X_9$	X_{10}
X_8								0	$3X_6-X_7$	$-2X_{12}$
X_9									0	$-3X_{13}$
X_{10}										0

	X_{11}	X_{12}	X_{13}	X_{14}
X_1	X_2	$-3X_3$	$\sqrt{3}X_4$	$3X_6-3X_7$
X_2	0	0	$2\sqrt{3}X_6-\sqrt{3}X_7$	$-3X_{11}$
X_3	0	$2X_8$	X_9	X_{12}
X_4	X_6	$-\sqrt{3}X_9$	0	$-\sqrt{3}X_{13}$
X_5	$\frac{\sqrt{3}}{3}X_8$	$-X_7$	0	$-X_{10}$
X_6	$2X_{11}$	X_{12}	$-X_{13}$	X_{14}
X_7	$3X_{11}$	$2X_{12}$	0	$3X_{14}$
X_8	0	$-3\sqrt{3}X_{11}$	$-X_{10}$	0
X_9	$-\frac{\sqrt{3}}{3}X_{12}$	$-2X_{10}$	0	0
X_{10}	0	$3X_{14}$	0	0
X_{11}	0	0	$\frac{\sqrt{3}}{3}X_{14}$	0
X_{12}		0	0	0
X_{13}			0	0
X_{14}				0

The Killing metric constructed with the structure constants gives the following set of non-zero scalar products $(X_M, X_N) \equiv \eta_{MN}$:

$$
\begin{aligned}
&(X_1, X_{14}) = \sqrt{3}(X_2, X_{13}) = (X_3, X_{10}) = -3(X_4, X_{11}) = (X_5, X_{12}) = \\
&\quad = \tfrac{3}{2}(X_6, X_6) = (X_6, X_7) = \tfrac{1}{2}(X_7, X_7) = (X_8, X_9) = 24.
\end{aligned}
\tag{43}
$$

The Cartan subalgebra of G_2 is spanned by X_6, X_7, the basis orthogonal with respect to the metric (43) can be chosen as

$$H_1 = X_7 - \frac{3}{2}X_6, \quad H_2 = \frac{\sqrt{3}}{2}X_6. \tag{44}$$

The remaining generators can be put in the Cartan-Weyl form

$$[H_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}}, \tag{45}$$
$$[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}] = \alpha_i H_i, \tag{46}$$
$$[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}] = N_{\boldsymbol{\alpha}\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \tag{47}$$

where $i = 1, 2$ and the roots $\boldsymbol{\alpha}_{\pm a}$, $a = 1, \ldots, 6$ are (with H_1 corresponding to the abscissa and H_2 to the ordinate)

$$\begin{aligned}
\boldsymbol{\alpha}_{\pm 1} &= \pm(1, 0), \\
\boldsymbol{\alpha}_{\pm 2} &= \pm(-3/2, \sqrt{3}/2), \\
\boldsymbol{\alpha}_{\pm 3} &= \pm(-1/2, \sqrt{3}/2), \\
\boldsymbol{\alpha}_{\pm 4} &= \pm(1/2, \sqrt{3}/2), \\
\boldsymbol{\alpha}_{\pm 5} &= \pm(3/2, \sqrt{3}/2), \\
\boldsymbol{\alpha}_{\pm 6} &= \pm(0, \sqrt{3}).
\end{aligned} \tag{48}$$

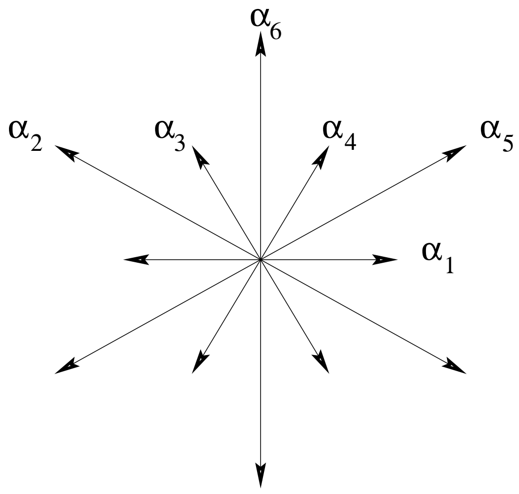

FIG. 1: The root diagram for g_2.

The simple roots are $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}_2$, the remaining ones can be obtained as follows:

$$\boldsymbol{\alpha}_{\pm 3} = \pm(\boldsymbol{\alpha} + \boldsymbol{\beta}), \ \boldsymbol{\alpha}_{\pm 4} = \pm(2\boldsymbol{\alpha} + \boldsymbol{\beta}), \ \boldsymbol{\alpha}_{\pm 5} = \pm(3\boldsymbol{\alpha} + \boldsymbol{\beta}), \ \boldsymbol{\alpha}_{\pm 6} = \pm(3\boldsymbol{\alpha} + 2\boldsymbol{\beta}). \tag{49}$$

The $E_{\boldsymbol{\alpha}}$ can be identified in terms of the X_M as follows

$$\begin{aligned}
E_1 &= \sqrt{\frac{1}{6}}X_{10}, & E_{-1} &= \sqrt{\frac{3}{2}}X_3, \\
E_2 &= \sqrt{\frac{3}{2}}X_2, & E_{-2} &= \sqrt{\frac{1}{2}}X_{13}, \\
E_3 &= \sqrt{\frac{1}{2}}X_8, & E_{-3} &= \sqrt{\frac{1}{2}}X_9, \\
E_4 &= \sqrt{\frac{1}{6}}X_{12}, & E_{-4} &= \sqrt{\frac{3}{2}}X_5, \\
E_5 &= \sqrt{\frac{1}{6}}X_{14}, & E_{-5} &= \sqrt{\frac{3}{2}}X_1, \\
E_6 &= -\sqrt{\frac{3}{2}}X_{11}, & E_{-6} &= \sqrt{\frac{3}{2}}X_4.
\end{aligned} \tag{50}$$

The following structure constants are non-zero up to the standard symmetries:

$$-N_{12} = N_{14} = N_{25} = N_{34} = \sqrt{\frac{3}{2}}, \quad N_{13} = \sqrt{2}. \tag{51}$$

The symmetry of the root diagram under reflection implies the existence of a target space coordinate transformation $\Phi^A \to \tilde{\Phi}^A = (\tilde{\xi}, \tilde{\phi}, \tilde{\kappa}, \tilde{\chi}, \tilde{v}_1, \tilde{u}_1, \tilde{v}_2, \tilde{u}_2)$ such that

$$E_{-\boldsymbol{\alpha}}(\Phi) = E_{\boldsymbol{\alpha}}(\tilde{\Phi}). \tag{52}$$

In terms of the tilded variables the remaining 5 Killing vectors can be written as (non-normalized)

$$\begin{aligned} X_{10} &= \partial_{\tilde{u}_2}, \\ X_{11} &= -\tilde{u}_1 \partial_{\tilde{\chi}} + \partial_{\tilde{v}_1}, \\ X_{12} &= -\tilde{u}_2 \partial_{\tilde{\chi}} + \partial_{\tilde{v}_2}, \\ X_{13} &= \partial_{\tilde{u}_1}, \\ X_{14} &= \partial_{\tilde{\chi}}. \end{aligned} \tag{53}$$

The target space isometries generated by the above Killing vectors can be classified as gauge, scale, S-duality and Harrison-Ehlers transformations. The first, generated by X_1, is the gravitational gauge transformation consisting in the shift of the twist potential:

$$\chi \to \chi + \lambda_1; \ (\phi, \ \xi, \ v_1, \ v_2, \ u_1, \ u_2, \ \kappa \text{ invariant}), \tag{54}$$

where λ_1 is the parameter. The next two, generated by X_2, X_3, are magnetic gauge transformations shifting the magnetic potentials u_1 (corresponding to the Kaluza-Klein vector field) and u_2 (corresponding to the 5D Maxwell field):

$$\begin{aligned} &u_1 \to u_1 + \lambda_2; \ (\phi, \ \xi, \ \chi, \ v_1, \ v_2, \ u_2, \ \kappa \text{ invariant}), \\ &u_2 \to u_2 + \lambda_3; \ (\phi, \ \xi, \ \chi, \ v_1, \ v_2, \ u_1, \ \kappa \text{ invariant}). \end{aligned} \tag{55}$$

The Killing vectors X_4, X_5 generate two electric gauge transformations:

$$\begin{aligned} &v_1 \to v_1 + \lambda_4, \ \chi \to \chi - u_1 \lambda_4; \ (\phi, \ \xi, \ v_2, \ u_1, \ u_2, \ \kappa \text{ invariant}), \\ &v_2 \to v_2 + \lambda_5, \ \chi \to \chi - u_2 \lambda_5; \ (\phi, \ \xi, \ v_1, \ u_1, \ u_2, \ \kappa \text{ invariant}). \end{aligned} \tag{56}$$

From the sector X_6, X_7, one can separate the scale transformation generated by $X_6 - X_7$

$$\begin{aligned} \kappa &\to \kappa, \quad u_1 \to e^{\lambda_6} u_1, \quad v_1 \to e^{\lambda_6} v_1, \quad \chi \to e^{2\lambda_6} \chi, \\ \phi &\to \phi, \quad u_2 \to e^{\lambda_6} u_2, \quad v_2 \to e^{\lambda_6} v_2, \quad e^{\xi} \to e^{2\lambda_6} e^{\xi}. \end{aligned} \tag{57}$$

The second independent element of the Cartan subalgebra

$$H = \frac{1}{2}\left(\sqrt{3} H_2 - H_1\right) = \frac{1}{2}\left(3X_6 - X_7\right), \tag{58}$$

together with

$$L_- = X_9, \quad L_+ = 4\sqrt{3}\, X_8, \tag{59}$$

form the $SL(2,R)$ S-duality algebra

$$[H, L_\pm] = \mp L_\pm, \quad [L_-, L_+] = H. \tag{60}$$

The corresponding finite transformations consist of the scaling

$$\begin{aligned} \chi &\to \chi, \quad \kappa \to e^{-2\lambda_7} \kappa, \quad u_1 \to e^{-3\lambda_7} u_1 \quad u_2 \to e^{-\lambda_7} u_2 \\ \xi &\to \xi, \quad e^{\phi} \to e^{2\lambda_7} e^{\phi} \quad v_1 \to e^{3\lambda_7} v_1, \quad v_2 \to e^{\lambda_7} v_2, \end{aligned} \tag{61}$$

the axidilaton shift

$$\begin{aligned}
\kappa &\to \kappa - \lambda_8, \\
\phi &\to \phi, \\
u_1 &\to -\lambda_8\sqrt{3}u_2 + {\lambda_8}^3 v_1 + {\lambda_8}^2\sqrt{3}v_2 + u_1, \\
u_2 &\to u_2 - \sqrt{3}{\lambda_8}^2 v_1 - 2\,\lambda_8\, v_2, \\
v_1 &\to v_1, \\
v_2 &\to \lambda_8\sqrt{3}v_1 + v_2, \\
\xi &\to \xi, \\
\chi &\to {v_1}^2{\lambda_8}^3 + \sqrt{3}v_2\, v_1\, {\lambda_8}^2 + {v_2}^2\lambda_8 + \chi,
\end{aligned} \tag{62}$$

and the shift of the inverted axidilaton

$$\begin{aligned}
\kappa &\to \frac{\kappa\,(1-\lambda_9\,\kappa) - \lambda_9\, e^{-2\,\phi}}{(1-\lambda_9\,\kappa)^2 + {\lambda_9}^2 e^{-2\,\phi}}, \\
e^{\phi} &\to \lambda_9^2\, e^{-\phi} + e^{\phi}(\lambda_9\,\kappa - 1)^2, \\
u_1 &\to u_1, \\
u_2 &\to -\lambda_9\, u_1\sqrt{3} + u_2, \\
v_1 &\to \lambda_9\sqrt{3}v_2 + v_1 - {\lambda_9}^2 u_2\sqrt{3} + {\lambda_9}^3 u_1, \\
v_2 &\to -2\, u_2\,\lambda_9 + v_2 + {\lambda_9}^2 u_1\sqrt{3}, \\
\xi &\to \xi, \\
\chi &\to {\lambda_9}^3{u_1}^2 - {\lambda_9}^2 u_2\sqrt{3}u_1 + \lambda_9\, {u_2}^2 + \chi.
\end{aligned} \tag{63}$$

The remaining generators $X_{10}, X_{11}, X_{12}, X_{13}, X_{14}$, form the Ehlers-Harrison sector. In terms of the tilded quantities (53) the corresponding finite transformations are found straightforwardly. They consist of two magnetic Harrison transformations (X_{13}, X_{10}):

$$\begin{aligned}
&\tilde{u}_1 \to \tilde{u}_1 + \tilde{\lambda}_{13};\ (\tilde{\phi},\ \tilde{\xi},\ \tilde{\chi},\ \tilde{v}_1,\ \tilde{v}_2,\ \tilde{u}_2,\ \tilde{\kappa}\ \text{invariant}), \\
&\tilde{u}_2 \to \tilde{u}_2 + \lambda_{10};\ (\tilde{\phi},\ \tilde{\xi},\ \tilde{\chi},\ \tilde{v}_1,\ \tilde{v}_2,\ \tilde{u}_1,\ \tilde{\kappa}\ \text{invariant}),
\end{aligned} \tag{64}$$

two electric Harrison transformations (X_{11}, X_{12}):

$$\begin{aligned}
&\tilde{v}_1 \to \tilde{v}_1 + \lambda_{11},\ \tilde{\chi} \to \tilde{\chi} - u_1\lambda_{11};\ (\tilde{\phi},\ \tilde{\xi},\ \tilde{v}_2,\ \tilde{u}_1,\ \tilde{u}_2,\ \tilde{\kappa}\ \text{invariant}), \\
&\tilde{v}_2 \to \tilde{v}_2 + \lambda_{12},\ \tilde{\chi} \to \tilde{\chi} - u_2\lambda_{12};\ (\tilde{\phi},\ \tilde{\xi},\ \tilde{v}_1,\ \tilde{u}_1,\ \tilde{u}_2,\ \tilde{\kappa}\ \text{invariant}),
\end{aligned} \tag{65}$$

and an Ehlers transformation (X_{14}):

$$\tilde{\chi} \to \tilde{\chi} + \lambda_{14};\ (\tilde{\phi},\ \tilde{\xi},\ \tilde{v}_1,\ \tilde{v}_2,\ \tilde{u}_1,\ \tilde{u}_2,\ \tilde{\kappa}\ \text{invariant}). \tag{66}$$

The above classification of the target space isometry transformations is standard in dealing with four-dimensional theories, where they have a particularly simple sense when applied to stationary axisymmetric asymptotically flat configurations: Harrison transformations are interpreted as charging transformations, while the Ehlers transformation is interpreted as generating a NUT parameter. In the five-dimensional setting this is not so. Since one of the two four-dimensional vector fields is Kaluza-Klein, the associated Harrison transformations are no longer charging, one of them (X_{11}) being a five-dimensional gauge transformation. On the other hand, the Ehlers transformation now generates a five-dimensional rotation [53] (note that generation of rotation by sigma-model dualities is not possible at all in four-dimensional theories).

To construct a new solution one has first to identify the target space coordinates in terms of the seed solution, then to apply the above transformations to find new target space variables, and finally to perform inverse dualization to get the metric and the vector potential. However, to apply the most interesting Harrison-Ehlers transformations we need to know an explicit transformation from the initial to the tilded potentials. To find it in a concise form is still an open problem.

An alternative way consists in constructing a matrix representation of the target space. First we have to reveal the nature of the isotropy group $\mathcal{H} \in G = G_{2(2)}$. This can be done as follows. One looks for a representative of the coset $N(\Phi) \in G/\mathcal{H}$ which transforms by the global left multiplication by $g \in G$ and the local right multiplication by $h(\Phi) \in \mathcal{H}$:

$$N \to gNh(\Phi). \tag{67}$$

Infinitesimally this reads

$$X_M N(\Phi) = x_M N(\Phi) + q^\alpha_M(\Phi) N(\Phi) y_\alpha, \tag{68}$$

where X_M, $M = 1, \dots, 14$ are Killing vectors acting as differential operators, x_M are the corresponding matrices of the $g_{2(2)}$ algebra, y_α, $\alpha = 1, \dots, 6$ are generators of the isotropy subalgebra $\mathcal{H}$ and $q^\alpha_M(\Phi)$ are "compensating" functions. Consider the 7×7 matrix representation of $g_{2(2)}$ as Z-matrix (6.2) in [54] with omitted "i" to get the non-compact real form of the g_2 algebra. It is easy to find matrices x_M corresponding to the differential operators X_M. Solving the Eqs.(68) for the Borel subalgebra $M = 1, \dots, 8$ with the assumption that for these values of M the functions $q^\alpha_M(\Phi) = 0$, one obtains the matrix $N(\Phi)$. Then applying (68) for $M = 9, \dots, 14$ we find $q^\alpha_M(\Phi)$ together with the corresponding y_α. This calculation, requiring the explicit form of the "complicated" generators X_M, was performed using MAPLE. The resulting isotropy subalgebra $\mathcal{H}$ turns out to be spanned by

$$\begin{aligned} &Y_1 = X_8 - X_9,\ Y_2 = X_{11} - X_4,\ Y_3 = 3X_5 + X_{12}, \\ &Y_4 = 3X_3 + X_{10},\ Y_5 = X_2 + \frac{1}{\sqrt{3}} X_{13},\ Y_6 = 3X_1 - X_{14}, \end{aligned} \tag{69}$$

which form $sl(2, R) \times sl(2, R)$.

Using the matrix $N(\Phi)$ one can then pass to the gauge-independent matrix

$$M = N \eta N^{-1},$$

where η is some constant matrix invariant under $\mathcal{H}$ transformations

$$h \eta h^{-1} = \eta, \quad h \in \mathcal{H},$$

such that the target space metric reads

$$dl^2 = \frac{1}{8} Tr \left(dM M^{-1} dM M^{-1} \right). \tag{70}$$

The matrix M by construction will be invariant under the local action of $\mathcal{H}$. Before giving its explicit form we would like to go over to a more concise notation using the 2-dimensional covariance of our construction with respect to the reduced dimensions t, z.

IV. 2D-COVARIANT REPRESENTATION

The covariant parametrisation for dimensional reduction of five-dimensional minimal supergravity with two commuting isometries is [7, 20]

$$ds_5^2 = \lambda_{ab}(dz^a + a^a_i dx^i)(dz^b + a^b_j dx^j) + \tau^{-1} h_{ij} dx^i dx^j, \tag{71}$$

$$\hat{A} = \sqrt{3} \psi_a dz^a + A_i dx^i \tag{72}$$

$(a, b = 0, 1,\ z^0 = t, z^1 = z\,; i, j = 1, 2, 3)$, where $\tau \equiv -\det \lambda$. This is related to our previous parametrisation arising from the two-step reduction (2), (14) by

$$\lambda = \begin{pmatrix} -e^{\phi+\xi} + e^{-2\phi} v_1^2 & e^{-2\phi} v_1 \\ e^{-2\phi} v_1 & e^{-2\phi} \end{pmatrix}, \quad a^0_i = -\omega_i, \quad a^1_i = C_i + v_1 \omega_i, \quad \psi = \begin{pmatrix} v_2/\sqrt{3} + \kappa v_1 \\ \kappa \end{pmatrix}. \tag{73}$$

Dualization of $F_{ij} \equiv \partial_i A_j - \partial_j A_i$ gives the scalar μ via

$$\frac{1}{\sqrt{3}} F^{ij} = a^{aj} \partial^i \psi_a - a^{ai} \partial^j \psi_a + \frac{1}{\tau \sqrt{h}} \epsilon^{ijk} \left(\partial_k \mu + \epsilon^{ab} \psi_a \partial_k \psi_b \right). \tag{74}$$

This is related to previous quantities by $\mu = (u_2 - \kappa v_2)/\sqrt{3}$. Dualization of $G^b_{ij} \equiv \partial_i a^b_j - \partial_j a^b_i$ gives the two-vector ω^a:

$$\tau\lambda_{ab}G^{bij} = \frac{1}{\sqrt{h}}\epsilon^{ijk}\left[\partial_k\omega_a - \psi_a\left(3\partial_k\mu + \epsilon^{bc}\psi_b\partial_k\psi_c\right)\right], \tag{75}$$

which is expressed as

$$\omega = \begin{pmatrix} -\chi - \kappa v_2\left(\kappa v_1 + 2v_2/\sqrt{3}\right)/\sqrt{3} \\ u_1 - \kappa^2 v_2/\sqrt{3} \end{pmatrix}. \tag{76}$$

In this notation the target space metric (40) reads:

$$dl^2 = \frac{1}{4}\mathrm{Tr}\left(\lambda^{-1}d\lambda\lambda^{-1}d\lambda\right) + \frac{1}{4}\tau^{-2}d\tau^2 + \frac{3}{2}d\psi^T\lambda^{-1}d\psi - \frac{1}{2}\tau^{-1}V^T\lambda^{-1}V - \frac{3}{2}\tau^{-1}\left(d\mu + \epsilon^{ab}\psi_a d\psi_b\right)^2, \tag{77}$$

(with $\epsilon^{01} = 1$), where the vector-valued one-form V_a is

$$V_a = d\omega_a - \psi_a\left(3d\mu + \epsilon^{bc}\psi_b d\psi_c\right). \tag{78}$$

The three coordinates μ and ω_a are cyclic.

The first eight ("simple") Killing vectors $X_1, \ldots, X_8$ together with X_{11} can be regrouped covariantly as follows. The mixed tensor

$$M_a{}^b = 2\lambda_{ac}\frac{\partial}{\partial\lambda_{cb}} + \omega_a\frac{\partial}{\partial\omega_b} + \delta^b_a\omega_c\frac{\partial}{\partial\omega_c} + \psi_a\frac{\partial}{\partial\psi_b} + \delta^b_a\mu\frac{\partial}{\partial\mu} \tag{79}$$

generates linear transformations in the (t, z) plane obeying the gl(2,R) subalgebra,

$$\left[M_a{}^b, M_c{}^d\right] = \delta^b_c M_a{}^d - \delta^d_a M_c{}^b. \tag{80}$$

Three mutually commuting operators are associated with the three cyclic "magnetic" coordinates:

$$N^a = \frac{\partial}{\partial\omega_a}, \qquad Q = \frac{\partial}{\partial\mu}. \tag{81}$$

Their commutation relations are

$$\left[M_a{}^b, N^c\right] = -(\delta^c_a N^b + \delta^b_a N^c), \tag{82}$$

$$\left[M_a{}^b, Q\right] = -\delta^b_a Q. \tag{83}$$

$$\left[N^a, N^b\right] = 0, \tag{84}$$

$$[Q, N^a] = 0. \tag{85}$$

A doublet operator associated with gauge transformations of the ψ_a reads:

$$R^a = \left[\frac{\partial}{\partial\psi_a} + 3\mu\frac{\partial}{\partial\omega_a} - \epsilon^{ab}\psi_b\left(\frac{\partial}{\partial\mu} + \psi_c\frac{\partial}{\partial\omega_c}\right)\right]. \tag{86}$$

The corresponding commutation relations are

$$\left[M_a{}^b, R^c\right] = -\delta^c_a R^b, \tag{87}$$

$$\left[N^a, R^b\right] = 0, \tag{88}$$

$$[Q, R^a] = 3N^a, \tag{89}$$

$$\left[R^a, R^b\right] = 2\epsilon^{ab}Q. \tag{90}$$

The correspondence with the previous operators is as follows:

$$\begin{gathered} X_1 = -N^0,\;\; X_2 = N^1,\;\; X_3 = \frac{1}{\sqrt{3}}Q,\;\; X_4 = M_1{}^0,\;\; X_5 = \frac{1}{\sqrt{3}}R^0, \\ X_6 = M_0{}^0 - M_1{}^1,\;\; X_7 = 2M_0{}^0 - M_1{}^1,\;\; X_8 = -R^1,\;\; X_{11} = -M_0{}^1. \end{gathered} \tag{91}$$

Two more vectors L_a : $X_{14} = 3L_0$, $X_{13} = -\sqrt{3}L_1$ complete the subalgebra $sl(3,R) \in g_{2(2)}$:

$$\begin{aligned} \left[M_a{}^b, L_c\right] &= (\delta^b_c L_a + \delta^b_a L_c)\,, &(92)\\ [N^a, L_b] &= M_b{}^a\,, &(93)\\ [L_a, L_b] &= 0\,. &(94) \end{aligned}$$

Their commutation with Q gives another doublet:

$$[Q, L_a] = P_a\,, \quad X_{12} = \sqrt{3}P_0, \quad X_9 = -P_1. \tag{95}$$

Finally the algebra $g_{2(2)}$ is closed with the singlet operator T, such that

$$[R^a, L_b] = \delta^a_b T\,, \tag{96}$$

which is identified as $X_{10} = -\sqrt{3}T$. The remaining commutation relations are

$$\begin{aligned} \left[M_a{}^b, P_c\right] &= \delta^b_c P_a\,,\\ \left[M_a{}^b, T\right] &= \delta^a_b T\,,\\ [N^a, P_b] &= \delta^a_b Q\,,\\ [N^a, T] &= R^a\,,\\ [Q, P_a] &= -2\epsilon_{ab} R^b\,,\\ [Q, T] &= M_c{}^c\,,\\ [R^a, P_b] &= -3M^a{}_b + \delta^a_b M_c{}^c\,,\\ [R^a, T] &= 2\epsilon^{ab} P_b\,,\\ [L_a, P_b] &= 0\,,\\ [L_a, T] &= 0\,,\\ [P_a, P_b] &= 2\epsilon_{ab} T\,,\\ [P_a, T] &= 3L_a\,. \end{aligned} \tag{97}$$

The most convenient way to find a matrix representative of the coset consists in exponentiating the Borel subgroup. In our case this amounts to using the one-parametric subgroups corresponding to generators $X_1, \ldots, X_8$, or, in the two-covariant notation, N^a, Q, R^a and three independent components of $M_a{}^b$. The latter, together with the N^a, generate representatives of the vacuum coset $SL(3,R)/SL(2,R)$ [20] which is a submanifold of the full target space. These representatives are of the form

$$M_1 = \mathcal{V}_\omega^T M_0 \mathcal{V}_\omega, \quad \mathcal{V}_\omega = e^{\omega_a n^a}, \tag{98}$$

where M_0 is the 7×7 matrix

$$M_0 = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0\\ 0 & -\tau^{-1} & 0 & 0 & 0\\ 0 & 0 & \lambda^{-1} & 0 & 0\\ 0 & 0 & 0 & -\tau & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{99}$$

and λ and λ^{-1} are 2×2 blocks. Then the full $G_{2(2)}/(SL(2,R) \times SL(2,R))$ coset matrix can be constructed as

$$M = \mathcal{V}^T M_0 \mathcal{V}, \quad \mathcal{V} = \mathcal{V}_\psi \mathcal{V}_\mu \mathcal{V}_\omega, \tag{100}$$

where

$$\mathcal{V}_\mu = e^{\mu q}, \quad \mathcal{V}_\psi = \mathrm{e}^{\psi_a r^a}, \tag{101}$$

with ω_a, μ, ψ_a the target space coordinates and n^a, q, r^a the 7×7 matrices of the $g_{2(2)}$ algebra given in Appendix A. The computation gives the coset matrix in the symmetrical block form

$$M = \begin{pmatrix} A & B & \sqrt{2}U\\ B^T & C & \sqrt{2}V\\ \sqrt{2}U^T & \sqrt{2}V^T & S \end{pmatrix}, \tag{102}$$

with

$$
\begin{aligned}
A &= \begin{pmatrix} \big[(1-y)\lambda + (2+x)\psi\psi^T - \tau^{-1}\tilde\omega\tilde\omega^T \\ \quad +\mu(\psi\psi^T\lambda^{-1}J - J\lambda^{-1}\psi\psi^T)\big] & \tau^{-1}\tilde\omega \\ \tau^{-1}\tilde\omega^T & -\tau^{-1} \end{pmatrix}, \\
B &= \begin{pmatrix} (\psi\psi^T - \mu J)\lambda^{-1} - \tau^{-1}\tilde\omega\psi^T J & \big[(-(1+y)\lambda J - (2+x)\mu + \psi^T\lambda^{-1}\tilde\omega)\psi \\ & \quad +(z - \mu J\lambda^{-1})\tilde\omega\big] \\ \tau^{-1}\psi^T J & -z \end{pmatrix}, \\
C &= \begin{pmatrix} (1+x)\lambda^{-1} - \lambda^{-1}\psi\psi^T\lambda^{-1} & \lambda^{-1}\tilde\omega - J(z - \mu J\lambda^{-1})\psi \\ \tilde\omega^T\lambda^{-1} + \psi^T(z + \mu\lambda^{-1}J)J & \big[\tilde\omega^T\lambda^{-1}\tilde\omega - 2\mu\psi^T\lambda^{-1}\tilde\omega \\ & -\tau(1+x-2y-xy+z^2)\big] \end{pmatrix}, \\
U &= \begin{pmatrix} (1+x-\mu J\lambda^{-1})\psi - \mu\tau^{-1}\tilde\omega \\ \mu\tau^{-1} \end{pmatrix}, \\
V &= \begin{pmatrix} (\lambda^{-1} + \mu\tau^{-1}J)\psi \\ \psi^T\lambda^{-1}\tilde\omega - \mu(1+x-z) \end{pmatrix}, \\
S &= 1 + 2(x-y)\,,
\end{aligned}
\tag{103}
$$

where

$$\tilde\omega = \omega - \mu\psi\,, \quad x = \psi^T\lambda^{-1}\psi\,, \quad y = \tau^{-1}\mu^2\,, \quad z = y - \tau^{-1}\psi^T J\tilde\omega\,, \tag{104}$$

and J is the 2×2 matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The matrix M solves the equation

$$M^{-1}dM = \mathcal{J} = 16\eta^{MN}\bar{J}_M j_N, \tag{105}$$

where

$$\bar{J_M} = G_{AB}J^A_M d\Phi^B \tag{106}$$

are the one-forms dual to the Killing vectors $J_M = J^A_M\partial/\partial\Phi^A$. In the present case,

$$
\begin{aligned}
\mathcal{J} \;=\; & \left(\bar{M}^a_b - \frac{1}{3}\delta^a_b Tr\bar{M}\right) m_a{}^b + \bar{N}^a n^{aT} + \bar{L}_a \ell_a{}^T \\
& + \frac{1}{3}\left(\bar{R}^a r^{aT} + \bar{P}_a p_a{}^T + \bar{Q}q^T + \bar{T}t^T\right).
\end{aligned}
\tag{107}
$$

The action of $G_{2(2)}$ on the coset matrix is

$$M'(\Phi^A) = \mathcal{C}^T M(\Phi^A)\,\mathcal{C}, \quad \mathcal{C} \in G, \tag{108}$$

where $\mathcal{C}$ is the exponential of some Lie algebra generator. The strategy to generate a new solution consists in the following steps. First, one must identify the target space coordinates corresponding to the seed solution and form the matrix $M(\Phi^A)$ as function of these variables (in the 2-covariant form $\Phi^A = (\lambda_{ab}, \omega_a, \psi_a, \mu)$. This involves solving the duality equations (74), (75) for the magnetic-type potentials. Then one chooses a transformation $\mathcal{C}$ of the desired type and computes the transformed matrix $M'(\Phi^A)$ in terms of the same target space variables. The new target space variables $\Phi'^A = (\lambda'_{ab}, \omega'_a, \psi'_a, \mu')$ can then be found by solving the set of equations

$$M(\Phi'^A) = M'(\Phi^A). \tag{109}$$

Finally, the metric functions and the vector potential of the new solution must be calculated. This step also involves solving differential equations to dualize back the potentials.

We have tested our formalism by generating the charged rotating black hole solution of five-dimensional supergravity from the Myers-Perry black hole. This is achieved by applying a combination of an electric Harrison (X_{12}) transformation accompanied by the corresponding gauge transformation in order to preserve asymptotic flatness of the solution. The resulting solution can be transformed to the previously known one by some coordinate transformation. This is given in the Appendix B.

V. CHARGED BLACK RING WITH TWO PARAMETERS OF ROTATION

As an example of application of our formalism we derive here the electrically charged version of the rotating black ring with two parameters of rotation constructed by Pomeransky and Senkov using the inverse scattering technique [47]

$$\begin{aligned} ds^2 = & -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\phi^2 - 2\frac{J(x,y)}{H(y,x)}d\phi d\psi \\ & + \frac{F(y,x)}{H(y,x)}d\psi^2 + \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left[\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right], \end{aligned} \tag{110}$$

where the variables (t, x, y, ϕ, ψ) vary in the range $-\infty < t < +\infty, -1 \le x \le 1, -\infty < y < -1, 0 \le (\phi, \psi) < 2\pi,\ k,\ \nu,\ \lambda$ are parameters, and the rotation one-form is two-component $\Omega = \Omega_\psi d\psi + \Omega_\phi d\phi$. Explicitly,

$$\Omega = -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)}\left[(1-x^2)y\sqrt{\nu}d\psi + \frac{1+y}{1-\lambda+\nu}[1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y)]\,d\phi\right], \tag{111}$$

and the functions G, H, J, F are (we use here the original notation of [47], not to be confused with our fields λ, x, y):

$$\begin{aligned} G(x) &= (1-x^2)(1+\lambda x+\nu x^2), \\ H(x,y) &= 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2x\lambda(1-y^2\nu^2)+x^2y^2\nu(1-\lambda^2-\nu^2), \\ J(x,y) &= \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}\left[1+\lambda^2-\nu^2+2(x+y)\lambda\nu-xy\nu(1-\lambda^2-\nu^2)\right], \\ F(x,y) &= \frac{2k^2}{(x-y)^2(1-\nu)^2}\Big\{G(x)(1-y^2)\Big([(1-\nu)^2-\lambda^2](1+\nu)+y\lambda(1-\lambda^2+2\nu-3\nu^2)\Big)+G(y)\Big[2\lambda^2 \\ &+ x\lambda[(1-\nu)^2+\lambda^2]+x^2[(1-\nu)^2-\lambda^2](1+\nu)+x^3\lambda(1-\lambda^2-3\nu^2+2\nu^3)-x^4(1-\nu)\nu(-1+\lambda^2+\nu^2)\Big]\Big\}. \end{aligned} \tag{112}$$

Regularity of the black ring implies the inequalities $0 \le \nu < 1,\ 2\sqrt{\nu} \le \lambda < 1+\nu$. The mass and angular momenta can be read out from the asymptotic expansion of the metric:

$$M = \frac{3k^2\pi\lambda}{G_N(1-\lambda+\nu)}, \quad J_\psi = \frac{4k^3\pi\lambda\sqrt{\nu}\sqrt{(1+\nu)^2-\lambda^2}}{G_N(1-\nu)^2(1-\lambda+\nu)}, \quad J_\phi = \frac{2k^3\pi\lambda(1+\lambda-6\nu+\lambda\nu+\nu^2)\sqrt{(1+\nu)^2-\lambda^2}}{G_N(1-\nu)^2(1-\lambda+\nu)^2}, \tag{113}$$

where G_N is the Newton constant. This solution is free of conical and Dirac string singularities.

We would like to endow this solution with an electric charge. This may be done by applying our formalism with the reduction along (t, ψ), (t, ϕ), or t and a linear combination of ψ and ϕ. We will here consider both choices (t, ψ) and (t, ϕ), and show that they lead to the same result. The target space potentials corresponding to dimensional reduction with respect to (t, ψ) are

$$\lambda_{00} = -\frac{H(y,x)}{H(x,y)}, \quad \lambda_{01} = -\frac{H(y,x)}{H(x,y)}\Omega_\psi, \quad \lambda_{11} = \frac{F(y,x)}{H(y,x)} - \frac{H(y,x)}{H(x,y)}\Omega_\psi^2, \quad \tau = \frac{F(y,x)}{H(x,y)}, \tag{114}$$

$$a^0_\phi = \Omega_\phi - a^1_\phi\Omega_\psi, \qquad a^1_\phi = -\frac{J(x,y)}{F(y,x)}, \tag{115}$$

and the reduced three-dimensional metric is

$$h_{ij}dx^i dx^j = \frac{2k^2}{(1-\nu)^2(x-y)^2}\left[F(y,x)\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right) - \frac{2k^2 G(x)G(y)}{(x-y)^2}d\phi^2\right]. \tag{116}$$

The "hatted" potentials $\hat{\lambda}_{ab}$, $\hat{\tau}$, $\hat{a}^a_\psi$ and three-dimensional metric $\hat{h}_{ij}$ corresponding to dimensional reduction with respect to (t, ϕ) are obtained from (114) by making the exchange $F(x, y) \leftrightarrow -F(y, x)$ and $\Omega_\phi \leftrightarrow \Omega_\psi$.

To generate the electric potential $\bar{v}_2 = \sqrt{3}\psi_0$ one must perform the Harrison transformation generated by $P_0 = X_{12}/\sqrt{3}$. It turns out, however, that this transformation alone does not preserve asymptotic flatness of the solution. To fix this, one must add the corresponding gauge transformation $R^0 = \sqrt{3}X_5$, so that the resulting charging transformation $\mathcal{C} = P_0 + R^0 = (X_{12} + 3X_5)/\sqrt{3}$ belongs to the $SL(2,R) \times SL(2,R)$ isotropy subgroup (see Eq. (69)). The corresponding one-parameter subgroup is given by the exponential

$$\mathcal{C} = \mathrm{e}^{\alpha(p^0 + r_0)} = \begin{pmatrix} c^2 & 0 & 0 & s^2 & 0 & 0 & \sqrt{2}sc \\ 0 & c & 0 & 0 & 0 & s & 0 \\ 0 & 0 & c & 0 & -s & 0 & 0 \\ s^2 & 0 & 0 & c^2 & 0 & 0 & \sqrt{2}sc \\ 0 & 0 & -s & 0 & c & 0 & 0 \\ 0 & s & 0 & 0 & 0 & c & 0 \\ \sqrt{2}sc & 0 & 0 & \sqrt{2}sc & 0 & 0 & c^2 + s^2 \end{pmatrix} = \mathcal{C}^T, \tag{117}$$

where α is a parameter, and $c \equiv \cosh\alpha$, $s \equiv \sinh\alpha$. Applying this transformation to the seed matrix M built from the potentials (114) leads to the transformed matrix

$$M' = \mathcal{C}^T M \mathcal{C}, \tag{118}$$

and extracting the transformed potentials we obtain

$$\begin{aligned}
\tau' &= D^{-1}\tau\,, \\
\lambda'_{00} &= D^{-2}\lambda_{00}\,, \\
\lambda'_{01} &= D^{-2}[c^3\lambda_{01} + s^3\lambda_{00}\omega_0]\,, \\
\omega'_0 &= D^{-2}[c^3(c^2 + s^2 + 2s^2\lambda_{00})\omega_0 - s^3(2c^2 + (c^2+s^2)\lambda_{00})\lambda_{01}]\,, \\
\omega'_1 &= \omega_1 + D^{-2}s^3[-c^3\lambda_{01}^2 + s(2c^2 - \lambda_{00})\lambda_{01}\omega_0 - c^3\omega_0^2]\,, \\
\psi'_0 &= D^{-1}sc(1 + \lambda_{00})\,, \\
\psi'_1 &= D^{-1}sc(c\lambda_{01} - s\omega_0)\,, \\
\mu' &= D^{-1}sc(c\omega_0 - s\lambda_{01})\,,
\end{aligned} \tag{119}$$

where

$$D = c^2 + s^2\lambda_{00} = 1 + s^2(1 + \lambda_{00}) \tag{120}$$

is common to the two reductions.

The seed potentials ω_a or $\hat{\omega}_a$ are obtained by dualizing the a^a_ϕ or $\hat{a}^a_\psi$ via Eq. (75). Inspection of the relations (119) shows that it is not necessary to compute ω_1, while the computation of ω_0 yields simply

$$\omega_0(x,y) = -\Omega_\phi(y,x)\,, \qquad \hat{\omega}_0(x,y) = \Omega_\psi(y,x)\,. \tag{121}$$

To write down the charged solution, there remains to dualize back the ω'_a and μ' to the a'^a_ϕ and A'_ϕ. It is easy to show (without explicit dualization) from the above relations, that $a'^1_\phi = a^1_\phi$, while the transformed gravimagnetic and magnetic fields are given by

$$G'^{0i\phi} = c^3 G^{0i\phi} + s^3\left[-\omega_1 G^{1i\phi} + \frac{\lambda_{00}^2}{\tau\sqrt{h}}\epsilon^{ij}\partial_j\Omega_\psi\right], \tag{122}$$

$$F'^{i\phi} = a'^{0\phi}\partial^i\psi'_0 + a'^{1\phi}\partial^i\psi'_1 - \frac{\sqrt{3}sc}{D\tau\sqrt{h}}\epsilon^{ij}(c\partial_j\omega_0 + s\lambda_{00}^2\partial_j\Omega_\psi)\,, \tag{123}$$

leading to

$$\begin{aligned}
a'^0_\phi(x,y) &= c^3 a^0_\phi(x,y) + s^3 a^0_\psi(y,x)\,, \\
A'_\phi(x,y) &= -cH(y,x)\Omega_\phi(x,y) - sH(x,y)\Omega_\psi(y,x)\,,
\end{aligned} \tag{124}$$

for dimensional reduction relative to (t,ψ), and

$$\begin{aligned}
\hat{a}'^0_\psi(x,y) &= c^3\hat{a}^0_\psi(x,y) - s^3\hat{a}^0_\phi(y,x)\,, \\
\hat{A}'_\psi(x,y) &= -cH(y,x)\Omega_\psi(x,y) + sH(x,y)\Omega_\phi(y,x)\,,
\end{aligned} \tag{125}$$

for dimensional reduction relative to (t,ϕ). Both dimensional reductions lead to the same final charged black ring metric

$$\begin{aligned} ds'^2 &= -D^{-2}\frac{H(y,x)}{H(x,y)}(dt+\Omega')^2 + D\Big[-\frac{F(x,y)}{H(y,x)}d\phi^2 - 2\frac{J(x,y)}{H(y,x)}d\phi d\psi \\ &\quad + \frac{F(y,x)}{H(y,x)}d\psi^2 + \frac{2k^2H(x,y)}{(1-\nu)^2(x-y)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right)\Big], \end{aligned} \tag{126}$$

$$\Omega' = (c^3\Omega_\psi(x,y) - s^3\Omega_\phi(y,x))d\psi + (c^3\Omega_\phi(x,y) + s^3\Omega_\psi(y,x))d\phi\,, \tag{127}$$

$$\begin{aligned} A' &= \frac{\sqrt{3}sc}{DH(x,y)}\Big[2\lambda(1-\nu)(x-y)(1-\nu xy)\,dt + \\ &+\Big(sH(x,y)\Omega_\phi(y,x) - cH(y,x)\Omega_\psi(x,y)\Big)d\psi \\ &-\Big(sH(x,y)\Omega_\psi(y,x) + cH(y,x)\Omega_\phi(x,y)\Big)d\phi\Big], \end{aligned} \tag{128}$$

with

$$D = 1 + s^2\frac{2\lambda(1-\nu)(x-y)(1-\nu xy)}{H(x,y)}\,. \tag{129}$$

This is to be compared with the charged black ring given in [55], Sect. 4. A difference is that the authors of [55] start with a seed having an extra parameter (dipole charge), which can be fine tuned so that Dirac-Misner strings are absent. Such string singularities arise if the orbits of ∂_ψ [2] do not close off at $x=1$. In the present case it is clear that both $\Omega'_\psi(1,y)$ and $A'_\psi(1,y)$ are proportional to $\Omega_\phi(y,1)$, which does not vanish, so that string singularities are unavoidable. Specifically,

$$\Omega'_\psi(1,y) = -s^3\frac{4k\lambda}{\sqrt{(1+\nu)^2-\lambda^2}}\,. \tag{130}$$

To eliminate this singularity one can apply a further transformation depending on an extra parameter which can be adjusted to get rid of the Dirac string [56].

VI. CONCLUSION

In this paper we have presented a new solution-generating technique for $D=5$ minimal supergravity based on the hidden symmetry G_2. This opens the possibility of finding new families of solutions possessing two commuting Killing symmetries. In this case the bosonic equations of motion reduce to those of a three-dimensional gravity coupled sigma-model on a symmetric space. Here we have elaborated in detail the case of one time-like and one space-like Killing vectors, leading to the $G_{2(2)}/(SL(2,R)\times SL(2,R))$ target space. The case of two space-like symmetries can be dealt with along the same lines, the target space being instead $G_{2(2)}/SO(4)$. In four-dimensional theories these two cases are usually interpreted as corresponding to stationary axisymmetric and plane-wave space-times. In five dimensions one has more freedom to choose a pair of two commuting symmetries, so one can use this approach in wider classes of space-times. Moreover, for a similar reason the generating symmetries can be more efficient. For instance, in the four-dimensional case one cannot generate the Kerr metric from the Schwarzschild metric using only sigma-model symmetries. To achieve this, one must proceed to higher level transformations [57] belonging to the Geroch group associated with the infinite extension of the sigma-model symmetries for configurations depending only on two variables. In five dimensions, rotating black holes can be generated from static black holes by using only the sigma-model symmetries [53].

An interesting application can be expected for black rings in $D=5$ minimal supergravity. The most general (pure black ring) solutions obtained so far are three-parametric: the charged black ring with one rotation parameter [55]

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[2] The ψ and ϕ of [55] should be exchanged to conform to the notations of [47].

and the doubly rotating uncharged black ring [47]. It was conjectured that the generic solution of this kind should contain five independent parameters: a mass, an electric charge, a dipole charge, and two angular momenta. To obtain such a solution free of conical and Dirac singularities (if it exists) one should incorporate primarily seven free parameters, with two extra parameters to be fixed by imposing the regularity conditions. One can show that out of the fourteen $G_{2(2)}$ transformations six preserve the asymptotic behaviour of the black ring. From these six three are gauge, while the other three generate a charge, a dipole moment and an angular momentum. So from the three-parameter unbalanced neutral black ring as given in [2] it is possible in principle to generate a six-parameter solution, which should lead to a four-parameter non-singular black ring [56]. Here we have given only a four-parameter solution (doubly rotating charged black ring) which is plagued by the Dirac string singularity.

An even farther reaching perspective consists in further reducing our sigma model to two dimensions and constructing a Belinski-Zakharov type integrable model or deriving Bäklund transformations. Formally this can be done in the same way as was recently used [38–49] in the purely vacuum sector, but the rank of matrices involved is increased from three to seven (though in the block form) which makes the problem technically more difficult.

Acknowledgments

The authors would like to thank Paul Sorba for numerous illuminating discussions and helpful advice in dealing with the g_2 algebra. D.G. is grateful to the LAPTH for hospitality in July 2007 while the paper was written. C-M.C. would like to thank the LAPTH for hospitality in August 2006 when this work was initiated. The work of C-M.C. was supported by the National Science Council of the R.O.C. under the grant NSC 96-2112-M-008-006-MY3, and in part by the National Center of Theoretical Sciences and Center for Mathematics and Theoretic Physics (NCU). T.W. acknowledges the use of SHARCNET computing facilities (www.sharcnet.ca).

Appendix A: Matrix representation

Starting with the real form of g_2 in the split octonion basis as given in [54], it is straightforward to find the desired representation for the generators used in the main text. Their generic block decomposition is

$$j = \begin{pmatrix} S & \tilde{V} & \sqrt{2}U \\ -\tilde{U} & -S^T & \sqrt{2}V \\ \sqrt{2}V^T & \sqrt{2}U^T & 0 \end{pmatrix} , \tag{A1}$$

where S is a 3×3 matrix, U and V are 3-component column matrices, and $\tilde{U}$, $\tilde{V}$ are the 3×3 dual matrices $\tilde{U}_{ij} = \epsilon_{ijk}U_k$. The matrices ${m_a}^b$, n^a and ℓ_a generating SL(3,R) are of type S, the corresponding 3×3 blocks being

$$S_{{m_0}^0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \quad S_{{m_0}^1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad S_{{m_1}^0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad S_{{m_1}^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$
$$S_{n^0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad S_{n^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} , \quad S_{\ell_0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad S_{\ell_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \tag{A2}$$

The matrices p_a and q are of type U, the corresponding 1×3 blocks being

$$U_{p_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad U_{p_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad U_q = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} . \tag{A3}$$

The matrices r^a and t are of type V, the corresponding 1×3 blocks being

$$V_{r^0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad V_{r^1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad V_t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{A4}$$

Note that the transposed matrices j_A^T are related to the original matrices j_A by

$$j_A^T = -Kj_AK , \tag{A5}$$

where the involution K has the block structure

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{A6}$$

The real matrix representing the coset $G_{2(2)}/(SL(2,R) \times SL(2,R))$ may be chosen to have the symmetrical block structure

$$M = \begin{pmatrix} A & B & \sqrt{2}U \\ B^T & C & \sqrt{2}V \\ \sqrt{2}U^T & \sqrt{2}V^T & S \end{pmatrix}, \tag{A7}$$

where A and C are symmetrical 3×3 matrices, B is a 3×3 matrix, U and V are 3-component column matrices, and S is a scalar such that the inverse matrix reads

$$M^{-1} = KMK = \begin{pmatrix} C & B^T & -\sqrt{2}V \\ B & A & -\sqrt{2}U \\ -\sqrt{2}V^T & -\sqrt{2}U^T & S \end{pmatrix}. \tag{A8}$$

Appendix B: Black holes

Here we illustrate the application of our technique to generate the charged rotating non-BPS black holes of five-dimensional supergravity, starting with the five-dimensional vacuum Kerr metric [9]:

$$ds^2 = -dt^2 + \frac{\rho^2}{4\Delta}dx^2 + \rho^2 d\theta^2 + (x+a^2)\sin^2\theta d\phi^2 + (x+b^2)\cos^2\theta d\psi^2 + \frac{r_0^2}{\rho^2}\left(dt + a\sin^2\theta d\phi + b\cos^2\theta d\psi\right)^2, \tag{B1}$$

where

$$\rho^2 = x + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Delta = (x+a^2)(x+b^2) - r_0^2 x. \tag{B2}$$

Choosing $z = \psi$ as the reduced direction, we find the following identification with our variables:

$$\begin{aligned} \lambda_{00} &= -1 + \frac{r_0^2}{\rho^2}, \quad \lambda_{01} = \frac{r_0^2}{\rho^2} b\cos^2\theta, \quad \lambda_{11} = (x+b^2)\cos^2\theta + \frac{r_0^2}{\rho^2}b^2\cos^4\theta, \quad \tau = \left(x + b^2 - r_0^2 + \frac{r_0^2}{\rho^2}a^2\cos^2\theta\right)\cos^2\theta, \\ a_\phi^0 &= -\tau^{-1}\frac{r_0^2}{\rho^2}(x+b^2)a\sin^2\theta\cos^2\theta, \qquad a_\phi^1 = \tau^{-1}\frac{r_0^2}{\rho^2}ab\sin^2\theta\cos^2\theta, \end{aligned} \tag{B3}$$

and the invariant three-metric

$$h_{ij}dx^i dx^j = \tau\left(\frac{\rho^2}{4\Delta}dx^2 + \rho^2 d\theta^2 + \frac{\Delta}{\tau}\sin^2\theta\cos^2\theta d\phi^2\right), \qquad \sqrt{h} = \frac{1}{2}\tau\rho^2\sin\theta\cos\theta. \tag{B4}$$

The dualization of the vector fields gives

$$\omega_0 = -\frac{r_0^2}{\rho^2}a\cos^2\theta, \qquad \omega_1 = -\frac{r_0^2}{\rho^2}ab\cos^4\theta. \tag{B5}$$

The action of our charge-generating transformation with parameter α ($c = \cosh\alpha$, $s = \sinh\alpha$) on this neutral seed leads to the transformed potentials according to (119). Performing the inverse dualization, we obtain the charged black hole solution

$$\begin{aligned} ds'^2 &= -D^{-2}\left(1 - \frac{r_0^2}{\rho^2}\right)(dt + \Omega')^2 + D\left[\frac{\rho^2 dx^2}{4\Delta} + \rho^2 d\theta^2 + \left(x + a^2 + \frac{r_0^2 a^2}{\rho^2 - r_0^2}\sin^2\theta\right)\sin^2\theta d\phi^2\right. \\ &\quad \left. + 2\frac{r_0^2 ab}{\rho^2 - r_0^2}\sin^2\theta\cos^2\theta d\phi d\psi + \left(x + b^2 + \frac{r_0^2 b^2}{\rho^2 - r_0^2}\cos^2\theta\right)\cos^2\theta d\psi^2\right], \end{aligned} \tag{B6}$$

$$\Omega' = -r_0^2\left[\left(\frac{c^3 a}{\rho^2 - r_0^2} + \frac{s^3 b}{\rho^2}\right)\sin^2\theta d\phi + \left(\frac{c^3 b}{\rho^2 - r_0^2} + \frac{s^3 a}{\rho^2}\right)\cos^2\theta d\psi\right], \tag{B7}$$

$$A' = \sqrt{3}scD^{-1}\frac{r_0^2}{\rho^2}\left[dt + (ca + sb)\sin^2\theta d\phi + (cb + sa)\cos^2\theta d\psi\right], \tag{B8}$$

with

$$D = 1 + s^2 \frac{r_0^2}{\rho^2}. \tag{B9}$$

The same solution is obtained if reduction is carried out with respect to the angular variable ϕ instead of ψ. Note that it is regular on the polar axis $\sin\theta = 0$.

This solution should be compared with the solution for the charged rotating black hole first given in [58] and rederived, in a different parametrization, in [10]. The comparison with [58] is straightforward. The solution presented in [10] is, for $g = 0$,

$$d\bar{s}^2 = -dt^2 - \frac{2\bar{q}}{\bar{\rho}^2}\bar{\nu}(dt - \bar{\omega}) + \frac{\bar{f}}{\bar{\rho}^4}(dt - \bar{\omega})^2 + \frac{\bar{\rho}^2 r^2}{\bar{\Delta}} dr^2 + \bar{\rho}^2 d\theta^2 + (r^2 + \bar{a}^2)\sin^2\theta d\phi^2 + (r^2 + \bar{b}^2)\cos^2\theta d\psi^2, \tag{B10}$$

$$\bar{A} = \frac{\sqrt{3}\bar{q}}{\bar{\rho}^2}(dt - \bar{\omega}), \tag{B11}$$

where

$$\bar{\nu} = \bar{b}\sin^2\theta d\phi + \bar{a}\cos^2\theta d\psi, \qquad \bar{\omega} = \bar{a}\sin^2\theta d\phi + \bar{b}\cos^2\theta d\psi, \qquad \bar{f} = 2\bar{m}\bar{\rho}^2 - \bar{q}^2, \tag{B12}$$

$$\bar{\Delta} = (r^2 + \bar{a}^2)(r^2 + \bar{b}^2) + \bar{q}^2 + 2\bar{a}\bar{b}\bar{q} - 2\bar{m}r^2, \qquad \bar{\rho}^2 = r^2 + \bar{a}^2\cos^2\theta + \bar{b}^2\sin^2\theta. \tag{B13}$$

The metrics ds'^2 and $d\bar{s}^2$ are related by the following coordinate and parameter transformation:

$$r^2 = x + s^2(r_0^2 - a^2 - b^2) - 2absc, \qquad 2\bar{m} = (1 + 2s^2)r_0^2, \qquad \bar{q} = -scr_0^2, \qquad \bar{a} = -ca - sb, \qquad \bar{b} = -cb - sa, \tag{B14}$$

implying

$$\bar{\rho}^2 = D\rho^2 = \rho^2 + s^2 r_0^2, \qquad \bar{\Delta} = \Delta. \tag{B15}$$

Comparing then the electromagnetic potentials, we find $\bar{A} = -A'$, so the two solutions are identical under charge conjugation (or a simultaneous sign change of t, ϕ and ψ).

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