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Multilevel Methods for Elliptic Problems on Domains not Resolved by the Coarse Grid

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Abstract. Elliptic boundary value problems are frequently posed on complicated domains which cannot be covered by a simple coarse initial grid as it is needed for multigrid like iterative methods. In the present article, this problem is resolved for selfadjoint second order problems and Dirichlet boundary conditions. The idea is to construct appropriate subspace decompositions of the corresponding finite element spaces by way of an embedding of the domain under consideration into a simpler domain like a square or a cube. Then the general theory of subspace correction methods can be applied.

Keywords: multilevel methods, complicated domains, subspace correction methods, subspace decompositions

MS (MOS) subject classification: 65N55, 65F10, 65N30

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Chapter 1

Introduction

By definition, a multilevel method for finite element equations is based on a sequence of refined triangulations. One starts with a coarse initial mesh crudely reflecting the properties of the boundary value problem under consideration. For the usual mathematical test problems like the Laplace equation on the unit square or on an L–shaped domain, this initial triangulation consists of only very few elements. Real–life problems, on the other hand, are often posed on very complicated regions, which can only be described by hundreds or thousands of finite elements.

A possibility to construct fast solvers for the resulting linear systems is to disregard any refinement history of the underlying grids and to decompose these grids a posteriori. This leads to some kind of algebraic multigrid methods. A recent approach is described in the paper of Bank and Xu in these proceedings.

In the present article, we follow the opposite direction of approximating complicated geometries in course of the refinement process. We restrict our attention to second order selfadjoint elliptic boundary value problems and Dirichlet boundary conditions. For this special class of problem we are able to construct nearly optimal iterative methods, which do not depend on the regularity of the boundary. For plane domains, even unphysical boundary conditions at a single point (which have no continuous counterpart) are allowed.

The idea is to construct appropriate subspace decompositions of the corresponding finite element spaces by way of an embedding of the domain under consideration into a simpler domain. Then the general theory of additive and multiplicative subspace correction methods can directly be applied. For a survey of this machinery, see the review articles of Xu [10] and of Yserentant [14], for example.

Our construction has originally been motivated by the numerical solution of obstacle problems; see [6], [5], [7]. In this application, the domain, on which linear elliptic problems have to be solved, is the subdomain where the current approximate solution does not touch the obstacle. This subdomain is unknown in advance of the computation and for this reason usually has no exact representation on coarser grids.

Chapter 2

The Discrete Elliptic Problem

Let $\Omega \subseteq \mathrm{IR}^2$ be a simple polygonal domain, e.g., a square. Let $\mathcal{T}_0$ be a coarse initial triangulation of Ω, which is refined several times, giving a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$. Despite the fact that all techniques and estimates in this paper can easily be generalized to the case of nonuniformly refined grids, we assume for ease of presentation that the triangles are uniformly refined. The triangles in $\mathcal{T}_{k+1}$ are generated from the triangles in $\mathcal{T}_k$ by subdividing these triangles in the usual fashion into four congruent subtriangles.

With the triangulations $\mathcal{T}_k$ we associate finite element spaces $\mathcal{S}_k$, consisting of the continuous functions, which are linear on the triangles in $\mathcal{T}_k$. The functions $u \in \mathcal{S}_k$ are given by their values at the nodes $x_i^{(k)}$, which are the vertices of the triangles in $\mathcal{T}_k$. With every node $x_i^{(k)}$ we associate a basis function $\psi_i^{(k)}$ of $\mathcal{S}_k$, taking the value 1 at this node and the value 0 at all other nodes.

Let Ω' be an arbitrarily complicated, nasty subdomain of Ω, possibly without any regularity property. Consider the nested subspaces

$$\mathcal{S}_k' = \operatorname{span}\{\, \psi_i^{(k)} \mid \psi_i^{(k)}(x) = 0 \ \text{ for } \ x \notin \Omega' \} \tag{2.1}$$

of the spaces $\mathcal{S}_k$ and in particular the space $\mathcal{S}' = \mathcal{S}_j'$. The supports of the functions in the subspaces $\mathcal{S}_k'$ exhaust Ω' from interior. Our aim is to construct and to analyze fast iterative solution procedures for the discrete boundary value problem to find the function $u \in \mathcal{S}'$ satisfying

$$a(u, v) = f^*(v), \quad v \in \mathcal{S}'. \tag{2.2}$$

Here f^* is a given linear functional on $\mathcal{S}'$. $a(u, v)$ denotes a symmetric coercive bilinear form on $\mathcal{S}'$ with the property that

$$\delta |u|_1^2 \leq a(u, u) \leq M |u|_1^2 \tag{2.3}$$

holds for all $u \in \mathcal{S}'$. δ and M are positive constants and $|u|_1 = |u|_{1;\Omega'}$ denotes the usual seminorm on $H^1(\Omega')$ given by

$$|u|_1^2 = \sum_{i=1}^{2} \int_{\Omega'} |D_i u(x)|^2 \mathrm{d}x. \tag{2.4}$$

We assume that $|\cdot|_1$ is a norm on $\mathcal{S}'$.

Chapter 3

Subspace Correction Methods

Let $\langle u, v\rangle$ be an arbitrary inner product on $\mathcal{S}'$. Define the operator $A : \mathcal{S}' \to \mathcal{S}'$ by the relation

$$\langle Au, v\rangle = a(u, v), \quad v \in \mathcal{S}' , \tag{3.1}$$

and determine the right hand side $f \in \mathcal{S}'$ by

$$\langle f, v\rangle = f^*(v), \quad v \in \mathcal{S}'. \tag{3.2}$$

Then the discrete boundary value problem (2.2) is equivalent to the operator equation

$$Au = f. \tag{3.3}$$

To solve (3.3) iteratively, one specifies subspaces

$$\mathcal{W}_k \subseteq \mathcal{S}'_k, \quad k = 0, 1, \ldots, j\,, \tag{3.4}$$

where the case $\mathcal{W}_k = \mathcal{S}'_k$ is explicitly included. Introducing the a–orthogonal projections $P_k : \mathcal{S}' \to \mathcal{W}_k$ by

$$a(P_k u, w_k) = a(u, w_k), \quad w_k \in \mathcal{W}_k, \tag{3.5}$$

the basic building blocks of the iterative methods considered here, are the *subspace corrections*

$$\tilde{u} \;\leftarrow\; \tilde{u} + P_k(u - \tilde{u}) \tag{3.6}$$

with respect to the spaces $\mathcal{W}_k$. After the subspace correction (3.6), the error $u - \tilde{u}$ between the exact solution u and the new approximation $\tilde{u}$ is a–orthogonal to the space $\mathcal{W}_k$. Utilizing the projections $Q_k : \mathcal{S}' \to \mathcal{W}_k$, given by

$$\langle Q_k u, w_k\rangle = \langle u, w_k\rangle, \quad w_k \in \mathcal{W}_k, \tag{3.7}$$

and the operators $A_k : \mathcal{W}_k \to \mathcal{W}_k$, defined analogously to $A : \mathcal{S}' \to \mathcal{S}'$, the correction step (3.6) can be written as

$$\tilde{u} \;\leftarrow\; \tilde{u} + A_k^{-1} Q_k(f - A\tilde{u}). \tag{3.8}$$

For large subspaces $\mathcal{W}_k$, as considered here, the correction steps (3.8) are too expensive. Therefore, one replaces these correction steps by the approximate correction steps

$$\tilde{u} \;\leftarrow\; \tilde{u} + B_k^{-1} Q_k(f - A\tilde{u}) \tag{3.9}$$

with symmetric and positive definite operators $B_k : \mathcal{W}_k \to \mathcal{W}_k$. These operators should have the property that the correction term

$$d_k = B_k^{-1} Q_k (f - A\tilde{u}) \tag{3.10}$$

can easily be computed as the solution of the linear system

$$\langle B_k d_k, w_k \rangle = \langle f - A\tilde{u}, w_k \rangle, \quad w_k \in \mathcal{W}_k. \tag{3.11}$$

It should be noted that the evaluation of

$$\langle f - A\tilde{u}, w_k \rangle = f^*(w_k) - a(\tilde{u}, w_k) \tag{3.12}$$

neither requires an explicit knowledge of the abstract operator A nor of the right hand side f.

If the single subspace correction steps (3.9) are repeated in a cyclic order, one gets a *multiplicative subspace correction method*. These methods generalize the classical Gauss–Seidel iteration, where the subspaces are one–dimensional and are spanned by basis functions. The corresponding *additive subspace correction method*

$$\tilde{u} \leftarrow \tilde{u} + \sum_{k=0}^{j} B_k^{-1} Q_k (f - A\tilde{u}) \tag{3.13}$$

is a Jacobi–type iteration. It is usually applied in form of a preconditioner for the conjugate gradient method.

A classical multigrid method for the solution of (2.2) and (3.3), respectively, would correspond to the choice $\mathcal{W}_k = \mathcal{S}_k'$ and to a simple symmetric Gauss–Seidel or Jacobi–iteration B_k^{-1}.

The general framework outlined here arose from the abstract formulation of domain decomposition methods. A breakthrough in the analysis of these methods were the papers [2] and [3] of Bramble, Pasciak, Wang and Xu, in which the first satisfying convergence proof for the multiplicative case has been given. For detailed references and a thorough discussion, we refer to [10], [14], or to Oswald's paper in these proceedings.

Chapter 4

Subspace Decompositions

The basic step in the convergence analysis of the subspace correction methods is to find a subspace decomposition

$$\mathcal{S}' = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_j \tag{4.1}$$

of the space $\mathcal{S}' = \mathcal{S}_j'$ into subspaces

$$\mathcal{V}_k \subseteq \mathcal{W}_k, \quad k = 0, 1, \ldots, j\,, \tag{4.2}$$

such that

$$\sum_{k=0}^{j} \langle B_k v_k, v_k \rangle \;\le\; K_1 \, \| \sum_{k=0}^{j} v_k \|^2 \tag{4.3}$$

holds for all $v_k \in \mathcal{V}_k$. The norm on the right hand side of this equation is the energy norm

$$\|u\| = a(u,u)^{1/2} \tag{4.4}$$

associated with the boundary value problem under consideration. The constant K_1 (possibly still depending on the number j of refinement levels) describes the *stability of the decomposition* (4.1).

If the B_k are taken as usual (Jacobi method, symmetrized Gauss–Seidel iterations, etc.) and if the level 0 correction is exactly computed, the estimate (4.3) is equivalent to the estimate

$$|v_0|_1^2 + \sum_{k=1}^{j} 4^k \|v_k\|_0^2 \;\le\; \widetilde{K}_1 \, | \sum_{k=0}^{j} v_k |_1^2 \tag{4.5}$$

for the functions $v_k \in \mathcal{V}_k$, or follows at least from this estimate. The (semi–) norm on the right–hand side of (4.5) is given by (2.4), and the inner product

$$(u,v) \;=\; \sum_{T \in \mathcal{T}_0} \frac{1}{\mathrm{area}(T)} \int_T uv \, \mathrm{d}x \tag{4.6}$$

induces the norm $\|v\|_0 = (v,v)^{1/2}$. The task of the weights $1/\mathrm{area}(T)$ is to make the estimates independent of the size of the triangles in the initial triangulation. In the three–dimensional case, these factors have to be replaced by other factors behaving like $1/\mathrm{diam}(T)^2$. The factors $4^k = (2^k)^2$ arise from the fact that the diameters of the triangles shrink by the factor 2 from one refinement level to the next. For a detailed exposition of the relation between (4.3) and (4.5), see [14].

For the analysis of the additive version one needs the second essential condition that

$$\| \sum_{k=0}^{j} w_k \|^2 \;\leq\; K_2 \sum_{k=0}^{j} \langle B_k w_k, w_k \rangle \tag{4.7}$$

holds for all $w_k \in \mathcal{W}_k$, or equivalently, as above, the estimate

$$\Big| \sum_{k=0}^{j} w_k \Big|_1^2 \;\leq\; \widetilde{K}_2 \Big\{ |w_0|_1^2 + \sum_{k=1}^{j} 4^k \|w_k\|_0^2 \Big\}. \tag{4.8}$$

As (4.8) is known to hold for all functions $w_k \in \mathcal{S}_k$ (the spaces associated with the basic domain Ω) with a constant $\widetilde{K}_2$ not depending on j, nothing has to be shown here. For a proof of (4.8), see [1] or [14]. Similarly, the Cauchy–Schwarz inequality, needed for the analysis of the multiplicative procedure (see [10] or [14], for example), is a direct consequence of the corresponding property for the full spaces $\mathcal{S}_k$.

The speed of convergence of the optimally scaled additive method (3.13), or of its conjugate gradient–accelerated version, can be estimated in terms of the constants K_1 and K_2 in (4.3) and (4.7). Similar results hold for the multiplicative version. For a detailed exposition, we refer again to the survey articles [10] and [14].

Chapter 5

A Subspace Decomposition by Interpolation Operators

The remaining task is to construct a decomposition (4.1) of the discrete solution space $\mathcal{S}'$ with the property (4.5). In this section, we consider decompositions generated by interpolation–like operators $I'_k : \mathcal{S}' \to \mathcal{S}'_k$ given by

$$(I'_k u)(x_i^{(k)}) = \begin{cases} u(x_i^{(k)}) & \psi_i^{(k)} \in \mathcal{S}'_k \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.1}$$

Recall that such splittings are related to the hierarchical basis method [11], [13]. Because of the decomposition

$$u = I'_0 u + \sum_{k=1}^{j} (I'_k u - I'_{k-1} u) \tag{5.2}$$

of the functions $u \in \mathcal{S}'$, the space $\mathcal{S}'$ is the direct sum of $\mathcal{S}'_0$ and of the subspaces

$$\mathcal{V}_k = \{ I'_k u - I'_{k-1} u \,|\, u \in \mathcal{S}' \} \tag{5.3}$$

of the spaces $\mathcal{S}'_k$.

The analytic foundation for the proof of the stability of this decomposition is the following

Lemma 5.1 *There exists a constant c depending only on the shape regularity of the triangles $T \in \mathcal{T}_k$ such that*

$$|u(x) - u(y)| \leq c\sqrt{j-k+1}\,|u|_{1;T} \tag{5.4}$$

holds for all functions $u \in \mathcal{S}_j$ and all points $x, y \in T$.

This estimate can be proved along the lines given in [11]. The fact, that we are dealing with two space dimensions, enters the proof of this lemma. For three space dimensions, the estimate (5.4) is wrong.

With Lemma 5.1, we can estimate the norm of the modified interpolation operators (5.1). This is, in a certain sense, the key result of this section.

Lemma 5.2 *There exists a constant c depending only on the shape regularity of the triangles in $\mathcal{T}_0$ such that*

$$|I'_k u|_1 \leq c\sqrt{j-k+1}\,|u|_1 \tag{5.5}$$

holds for all functions u in the subspace $\mathcal{S}' = \mathcal{S}'_j$ of $\mathcal{S}_j$.

Proof. We estimate $|I'_k u|^2_{1;T}$ for the triangles $T \in \mathcal{T}_k$. Two cases have to be distinguished. The first case is that T is an "interior" triangle of Ω', i.e., that the basis functions $\psi_i^{(k)}$ associated with all three vertices of T belong to $\mathcal{S}'_k$. In this case, the restriction of $I'_k u$ to T is simply the linear interpolant of u at the vertices of T. Therefore, the estimate

$$|I'_k u|_{1;T} \leq c_1 \sqrt{j-k+1}\, |u|_{1;T} \tag{5.6}$$

follows from Lemma 5.1 by a simple scaling argument.

If there is a basis function $\psi_i^{(k)}$ associated with a vertex $x_i^{(k)}$ of T, which does not belong to $\mathcal{S}'_k$, the situation is slightly more complicated. In this case, there exists at least one point $\bar{x} \notin \Omega'$ in $T_1 = T$ or in another triangle $T_1 \in \mathcal{T}_k$ with vertex $x_i^{(k)}$. The functions in $\mathcal{S}'$ vanish at $\bar{x}$. Therefore, for every $x \in T$, one gets

$$\begin{aligned} |u(x)| &\leq |u(x) - u(x_i^{(k)})| + |u(x_i^{(k)}) - u(\bar{x})| \\ &\leq c\sqrt{j-k+1}\, \{\, |u|_{1;T} + |u|_{1;T_1} \}\,. \end{aligned}$$

As above, this yields the estimate

$$|I'_k u|_{1;T} \leq c_2 \sqrt{j-k+1}\, |u|_{1;T\cup T_1}. \tag{5.7}$$

As each triangle in $\mathcal{T}_k$ intersects only a limited number of other triangles in $\mathcal{T}_k$, the proposition follows from (5.6) and (5.7). ■

The functions v_k in the space $\mathcal{V}_k$ satisfy the estimate

$$4^k \|v_k\|_0^2 \leq c\, |v_k|_1^2 \tag{5.8}$$

with a constant c depending again only on the shape regularity of the triangles under consideration. This estimate relies on the observation that every node $x_i^{(k)}$ has a neighbor $x_l^{(k)}$ of first or second degree at which the functions in $\mathcal{V}_k$ vanish. The scaling factor 4^k depends only on the number k of refinement levels, because the L_2–like norm induced by the inner product (4.6) is scaled by the areas of the triangles in the initial triangulation.

As an immediate consequence of Lemma 5.2 and of (5.8) (compare [11], [12]), we can state

Theorem 5.3 *There exists a constant C depending only on the shape regularity of the triangles in $\mathcal{T}_0$ (and not on the domain Ω'!) such that*

$$|I'_0 u|_1^2 + \sum_{k=1}^{j} 4^k \|I'_k u - I'_{k-1} u\|_0^2 \leq C j^2 |u|_1^2 \tag{5.9}$$

holds for all functions $u \in \mathcal{S}'_j$.

Hence, the decomposition of $\mathcal{S}'$ into $\mathcal{S}'_0$ and the subspaces (5.3) is stable in the sense of (4.3) with a constant

$$K_1 \sim j^2 \tag{5.10}$$

growing only logarithmically with $1/h \sim 2^j$. Therefore, for any choice

$$\mathcal{V}_k \subseteq \mathcal{W}_k \subseteq \mathcal{S}'_k\,, \tag{5.11}$$

one gets a nearly optimal multilevel method. The number of iteration steps, needed to reduce the error by a given factor, increases at most logarithmically, when the gridsize tends to zero. Classical multigrid methods correspond to the choice $\mathcal{W}_k = \mathcal{S}'_k$, whereas the other extreme $\mathcal{W}_k = \mathcal{V}_k$ leads hierarchical basis type iterative methods. This considerably generalizes related results in [6].

Note that absolutely no regularity assumption concerning the boundary of the domain $\Omega' \subseteq \Omega$ entered. Even unphysical boundary conditions at a single point, which have no continuous counterpart, are allowed. On the other hand, the construction works only for two space dimensions.

Chapter 6

The Decomposition of the Solution Space by L_2–like Projections

Assuming a certain regularity of Ω', one can also utilize the L_2–like decomposition

$$u = Q_0'u + \sum_{k=1}^{j}(Q_k'u - Q_{k-1}'u) \tag{6.1}$$

of the functions $u \in \mathcal{S}_j'$, where the $Q_k' : \mathcal{S}' \to \mathcal{S}_k'$ are the orthogonal projections with respect to the inner product (4.6). Recall that this decomposition played a crucial role in [4]. It turns out that the decomposition (6.1) of $\mathcal{S}_j'$ into $\mathcal{S}_0'$ and the subspaces

$$\mathcal{V}_k = \{Q_k'u - Q_{k-1}'u) \,|\, u \in \mathcal{S}'\} \tag{6.2}$$

is stable, if $\mathrm{IR}^2 \backslash \Omega'$ is "rich enough". Different from the construction in the last section, this approach works also for three space dimensions.

For simplicity, let $\mathcal{S}_k'$ be a subspace of $H_0^1(\Omega) \subseteq H^1(\mathrm{IR}^2)$. We call $T \in \mathcal{T}_k$ a boundary triangle of Ω', if at least one basis function $\psi_i^{(k)}$, associated with a vertex $x_i^{(k)}$ of T, is not contained in $\mathcal{S}_k'$. We make the following regularity assumption on Ω': For every boundary triangle, there exists a circle B such that every triangle in $\mathcal{T}_k$ intersecting T is completely overlapped by B and such that the area of B can be estimated as

$$\operatorname{meas} B \leq c_1 \operatorname{meas} B\backslash\Omega' \,, \tag{6.3}$$

and the diameter of B as

$$\operatorname{diam} B \leq c_2 \operatorname{diam} T. \tag{6.4}$$

The property (6.3) excludes that the complement of Ω' consists of single points or lines. This was allowed in the last section. Nevertheless the condition is extremely weak. It covers domains which some people would call "fractal". Oswald [9] discusses a related condition for the solution of our problem.

For a given boundary triangle $T \in \mathcal{T}_k$ and for the associated circle B, one can define the operator $\Pi : L_2(B) \to L_2(B)$ by

$$(\Pi u)(x) = \frac{1}{\operatorname{meas} B\backslash\Omega'} \int_{B\backslash\Omega'} u(y)\,\mathrm{d}y \,. \tag{6.5}$$

It maps the functions in $L_2(B)$ to constants. The square of the L_2–norm of this operator is

$$\frac{\operatorname{meas} B}{\operatorname{meas} B\backslash\Omega'} \;\le\; c_1\,. \tag{6.6}$$

As Π reproduces constant functions, (6.6) yields

$$\|u-\Pi u\|_{L_2(B)} \;\le\; (1+\sqrt{c_1})\,\inf_{\alpha\in\mathbb{R}} \|u-\alpha\|_{L_2(B)}. \tag{6.7}$$

With help of the Poincaré–inequality for the circle B and with (6.4), one obtains the error estimate

$$\|u-\Pi u\|_{0;B} \;\le\; c\,2^{-k}|u|_{1;B}, \quad u\in H^1(B), \tag{6.8}$$

with respect to the weighted L_2–norm induced by the inner product (4.6) on the left hand side. Therefore, the functions $u\in H^1(\mathbb{R}^2)$ vanishing on $B\backslash\Omega'$ and in particular the functions in $u\in\mathcal{S}'$ satisfy

$$\|u\|_{0;B} \;\le\; c\,2^{-k}|u|_{1;B}\,. \tag{6.9}$$

The constant c in this estimate depends only on the constants c_1 and c_2 in (6.3) and (6.4), respectively and on the shape regularity of $\mathcal{T}_0$.
Next, we introduce the L_2–bounded quasi–interpolants $M_k' : \mathcal{S}'\to\mathcal{S}_k'$ by

$$M_k' u \;=\; \sum_{\psi_i^{(k)}\in\mathcal{S}_k'} \frac{(u,\psi_i^{(k)})}{(1,\psi_i^{(k)})}\,\psi_i^{(k)}. \tag{6.10}$$

Then, utilizing (6.9) and the Poincaré–inequality, one can show the error estimate

$$\|u-M_k'u\|_0 \le \hat{c}\,2^{-k}|u|_1. \tag{6.11}$$

The proof relies essentially on the fact that the operators M_k' reproduce functions on a triangle $T\in\mathcal{T}_k$, which are constant in a neighborhood of T. Details on this technique can be found in [12].
The estimate (6.11) implies the error estimate

$$\|u-Q_k'u\|_0 \le \hat{c}\,2^{-k}|u|_1 \tag{6.12}$$

for the orthogonal projections $Q_k' : \mathcal{S}'\to\mathcal{S}_k'$. Using in addition that

$$|u-Q_0'u|_1^2 \;\le\; \widetilde{K}_1 \sum_{k=1}^{j} 4^k\|Q_k'u-Q_{k-1}'u\|_0^2 \tag{6.13}$$

holds for $u\in\mathcal{S}'$ (this is a consequence of (4.8)), one finally obtains

Theorem 6.1 *Provided that the subregion Ω' has the properties described at the beginning of this section, there exists a constant C such that*

$$|Q_0'u|_1^2 + \sum_{k=1}^{j} 4^k \|Q_k'u - Q_{k-1}'u\|_0^2 \leq Cj\,|u|_1^2 \tag{6.14}$$

holds for all functions $u \in \mathcal{S}'$.

We remark that the optimality of the decomposition (6.1) can be shown in a similar way; see [9]. Trying to keep the conditions on the boundary of Ω' as weak (and simple) as possible, we did not attempt to prove such a result here.

Based on Theorem 6.1, one obtains nearly optimal subspace correction methods for $\mathcal{W}_k = \mathcal{S}_k'$; again the number of iteration steps needed to reduce the error by a given factor grows only at most logarithmically, when the gridsize decreases.

Chapter 7

Numerical Experiences and Final Remarks

As a first illustrating example, we consider the unit square $\Omega = (0,1)\times(0,1)$ with the initial triangulation $\mathcal{T}_0$ depicted in Figure 7.1 and with final triangulations $\mathcal{T}_j$ obtained by a successive uniform refinement of $\mathcal{T}_0$ as described in Section 2. The bilinear form $a(u,v)$ of Section 2 is the Dirichlet–integral, leading to a boundary value problem for the Laplace equation. The right–hand side $f^*(v)$ is given by the integral of v over Ω.

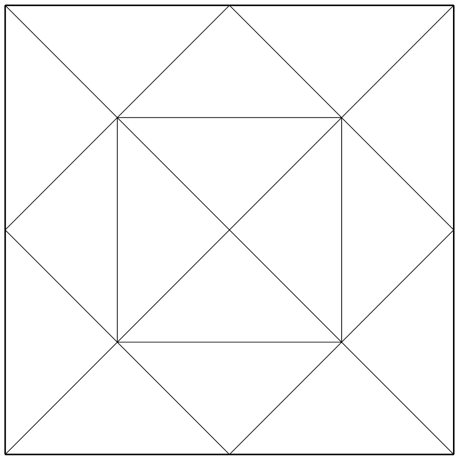

Figure 7.1: Initial triangulation

We compared the convergence rates of a classical multigrid method (the multiplicative subspace correction method corresponding to $\mathcal{W}_k = \mathcal{S}'_k$ and the symmetric Gauss–Seidel–method as approximate solver B_k^{-1}) for $\Omega'_1 = \Omega$ with the rates obtained for $\Omega'_2 = (0, 1-2^{-(j+2)}) \times (0,1)$ as the other extreme. The observed convergence rates are shown in Table 7.1.

	j=2	j=3	j=4	j=5	j=6	j=7
Ω'_1	0.28	0.30	0.31	0.32	0.33	0.33
Ω'_2	0.49	0.58	0.65	0.71	0.75	0.78

Table 7.1: Convergence rates for the first example

The convergence rates for the two domains differ considerably, a fact which can be easily explained. Of course, the unit square Ω_1' gives an optimal performance, whereas, for Ω_2', the supports of the functions in $\mathcal{S}_k'$ overlap only the rectangle $(0, 1-2^{-(k+2)}) \times (0,1)$ so that $\Omega_2' = (0, 1-2^{-(j+2)}) \times (0,1)$ is not exhausted very well. The convergence rates for Ω_2' are typical for the convergence rates that we observed for many other domains with critical boundaries.

One possibility to improve the convergence rate is to use enlarged correction spaces $\mathcal{W}_k = \mathcal{S}_k^\vee$, as obtained from the extension of $\mathcal{S}_k'$ by truncated basis functions of $\mathcal{S}_k$ (c.f. [7],[8]). In this way, one reaches nearly the same convergence rates as for regular problems. For the domain Ω_2' above, the convergence rates are asymptotically equal to the rates for $\Omega_1' = \Omega$.

As a second example, we consider the Laplacian on a subdomain Ω' of the unit square Ω with "fractal" boundary. Starting with $\Omega_0' = \Omega$, the domain Ω' is approximated by a sequence of domains Ω_j', which are triangulated by subsets $\mathcal{T}_j' \subset \mathcal{T}_j$. The boundaries of Ω_4' and Ω_5' are shown in Figure 7.2. We solved the boundary value problem on these domains. The convergence rates of the multiplicative methods with $\mathcal{W}_k = \mathcal{S}_k'$ and with extended spaces $\mathcal{W}_k = \mathcal{S}_k^\vee$ are given in Table 7.2.

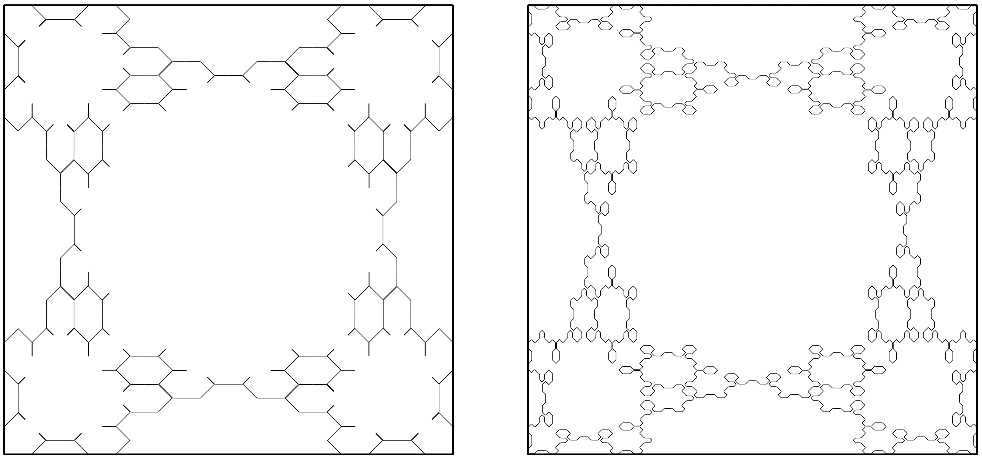

Figure 7.2: Approximate boundaries of the fractal domain

Difficulties in the analysis of the modified versions do not arise from the stability estimate (4.3), because one can use the same subspaces $\mathcal{V}_k \subseteq \mathcal{S}'_k \subseteq \mathcal{S}^\vee_k$ as before, but from the estimates (4.7), (4.8). The optimal estimate (4.8) does not transfer automatically from the full spaces $\mathcal{S}_k$ to the spaces $\mathcal{W}_k = \mathcal{S}^\vee_k$ as in the case of correction spaces $\mathcal{W}_k \subseteq \mathcal{S}_k$. However, a very crude argument, using only the triangle inequality, shows that (4.7) still holds with a constant $K_2 \sim j$. Hence, these problems can be easily remedied at the cost of an additional power of j.

	j=2	j=3	j=4	j=5	j=6	j=7
$\mathcal{S}_k'$	0.30	0.61	0.58	0.68	0.65	0.71
$\mathcal{S}_k^\vee$	0.28	0.30	0.32	0.32	0.33	0.34

Table 7.2: Convergence rates for the second example

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