

# A Polyhedral Approach to Multicommodity Survivable Network Design

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## Abstract

The design of cost-efficient networks satisfying certain survivability constraints is of major concern to the telecommunications industry. In this paper we study a problem of extending the capacity of a network by discrete steps as cheaply as possible, such that the given traffic demand can be accommodated even when a single edge or node in the network fails. We derive valid and non-redundant inequalities for the polyhedron of capacity design variables, by exploiting its relationship to connectivity network design and knapsack-like subproblems. Computational work using this model and the additional inequalities is in progress.

## 1 Introduction

It is of major concern to telecommunication operators to design networks with a suitable degree of survivability towards component failures, cable cuts etc. Motivated by this, there has been a large amount of recent research on the design of networks satisfying specified connectivity constraints. Most of this research concentrates on **uncapacitated networks**, i.e., where each link can support all the traffic at once. However, for many important present and future telecommunication networks, capacities play a fundamental role.

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In this paper we study an integer programming model for the following integrated planning problem: decide which links to install in the network and which capacities to install on these links such that the network allows routing of point-to-point traffic even under single node or edge failures. Thus the model addresses MULTicommodity SURvivable Network design, for short: MULTISUN.

More specifically, we have given a set  $V$  of nodes and traffic demands between certain pairs of these nodes. Each demand represents a certain amount of point-to-point traffic to be routed in the network between origin and destination nodes. In addition, a set of edges joining pairs of nodes in  $V$  are given; these represent direct physical links (e.g., a fiber cable or a radio relay system). For each edge one wants to decide which capacity to install, chosen among a discrete set of alternatives, each with an associated building cost. The number of such alternatives depends on the application and the desired level of detail. We are interested in capacity extensions such that all demands can be routed simultaneously in the resulting network. Such a routing is called a multicommodity flow. Especially, we may require that the network allows a multicommodity flow also in certain failure situations, e.g., when a single edge or node fails. In our model, we allow traffic demands to be split up and routed on several paths, i.e., we consider continuous flows. The discreteness of the model lies in deciding the design/capacity extension. The optimization problem in MULTISUN is to find such a feasible network extension of minimum total building cost.

The purpose of this paper is to present and analyze an integer linear programming model for the MULTISUN problem using a polyhedral approach. We study properties of polytopes that are naturally associated with the model. Specifically, we present classes of non-redundant inequalities that strengthen the original formulation and may be (in fact, are) used in a cutting plane algorithm for solving real-world planning problems. In deriving these inequalities, we exploit relations to the knapsack problem and also the design of (uncapacitated) networks with connectivity constraints.

A large amount of work has been done by Minoux and others on the related model with a **continuous** cost function, see [15] and the references given there. Minoux [15] introduced a general survivability framework for multicommodity flow network design, based on telecommunication studies. Gavish et al. [8] considered an even more general model arising in fiber optic networks involving selection of different cable types and a discrete cost

function. They developed bounding procedures using Lagrangian relaxation. Balakrishnan and Graves [2] considered the problem of designing a (directed) network supporting multicommodity flow without survivability constraints and with a continuous piecewise linear cost function on each arc. A special case of the MULTISUN problem is the well known fixed charge network flow problem (assuming that only design costs are present), see e.g., [16]. For work on the design of uncapacitated networks satisfying connectivity constraints, see the work of Grötschel, Monma, and Stoer [9, 11, 19], and for directed networks, see Dahl [6].

This paper is organized as follows. The integer linear programming formulation of the MULTISUN problem is given in Section 2. Two models are introduced, one with and one without survivability requirements, and associated 0/1-polytopes are defined. We discuss basic properties (dimension and trivial facets) of these polytopes in Section 3. The remaining part of the paper discusses stronger formulations of the problem. In Section 4 classes of facet defining inequalities are derived from underlying knapsack structures of the original model. We then, in Section 5, exploit the mentioned relation of MULTISUN to connectivity design problems, and derive facet defining inequalities, so-called partition and lifted two-cover inequalities, from this. Some suggestions for further work are given in the concluding section.

We use fairly standard notation from graph theory and polyhedral theory, see [4, 18], but a few notions need to be explained.  $\mathbf{R}^E$  denotes the set of real vectors indexed by  $E$  (where  $E$  is a finite set). Let  $G = (V, E)$  be an undirected graph without loops and multiple edges. If  $w$  is a node in  $G$ , we let  $G - w$  denote the graph obtained from  $G$  by removing  $w$  and its incident edges from  $G$ . Similarly,  $G - e$  is the graph obtained by removing the edge  $e$ . The **cut**  $\delta_G(W)$  induced by a subset  $W$  of  $V$  is the set of edges with one end node in  $W$  and the other outside  $W$ ;  $W$  and  $V \setminus W$  are called **shores** of the cut. By  $G[W] = (W, E(W))$  we denote the graph induced by node set  $W$ . For two nodes  $u$  and  $v$ , a  $[u, v]$ -path  $P$  is a sequence of nodes and edges  $(v_0, e_1, v_1, e_2, \dots, v_{l-1}, e_l, v_l)$ , where each edge  $e_i$  is incident to the nodes  $v_{i-1}$  and  $v_i$  ( $i = 1, \dots, l$ ), where  $v_0 = u$  and  $v_l = v$ , and where no node or edge appears more than once in  $P$ . A graph  $G$  is said to be **2-edge** (or **2-node**) **connected** with respect to some given node set  $R$ , if between any two nodes  $u, v \in R$  there exist at least two edge- (or node-)disjoint  $[u, v]$ -paths. Similarly, we say that  $G$  is **connected** with respect to  $R$ , if  $G$  contains a  $[u, v]$ -path for each pair of nodes  $u, v \in R$ . If  $G$  is moreover an edge-minimal

connected graph w.r.t.  $R$ , then  $G$  is a **Steiner tree** with terminal set  $R$ .

A **network**  $\mathcal{N} = (G, c)$  is a graph  $G$  with weights (capacities or demands)  $c_e \geq 0$  associated with the edges  $e$ . Finally, given a **supply** network  $(G, c)$  and a **demand** network  $(H, d)$ , where  $G$  and  $H$  have the same node set, a **multicommodity flow** (w.r.t.  $(H, d)$ ) is defined as a collection of  $[u, v]$ -paths  $P_{uv}^i$  of  $G$  together with numbers  $\lambda_{uv}^i \geq 0$  (for all  $uv \in E(H)$ ,  $i = 1, \dots$ ), such that  $\sum_i \lambda_{uv}^i = d_{uv}$ , for each  $uv \in E(H)$ . The network  $(G, c)$  or the capacity vector  $c$  is said to **allow** a multicommodity flow w.r.t.  $(H, d)$ , if, moreover, for each edge  $e \in E(G)$  the sum of  $\lambda_{uv}^i$  over all paths containing  $e$  is at most its capacity  $c_e$ .

Given a vector  $x \in \mathbf{R}^I$  indexed by some set  $I$  and given a subset  $S$  of  $I$ , we write  $x(S)$  instead of  $\sum_{i \in S} x_i$ . By  $\chi^S \in \mathbf{R}^I$  we denote the incidence vector of  $S$ .

## 2 Mathematical Model

We present a mathematical model for the MULTISUN problem. In fact, we will study two models; Model 1, which does not take survivability into account, and Model 2, which does.

### 2.1 Model 1

The existing lines (for example transmission links), plus the lines that can be established, are given as a **supply** graph  $G = (V, E)$  with node set  $V$  and edge set  $E$ . The traffic demands are given as a **demand** graph  $H = (V, D)$  whose edges represent the different demands, and the amount of traffic  $d_{uv} > 0$  for each demand edge  $uv \in D$ .

For each edge  $e \in E$  one has to choose a capacity  $y_e$  from among a small set of discrete capacity extensions with associated costs, such that the so constructed network satisfies all traffic demands and is of minimum total cost. Below we describe in detail how the cost function and the multicommodity flow constraints are modeled.

The discrete capacities for each edge  $e \in E$  are given by numbers  $0 < M_e^1 < M_e^2 < \dots < M_e^{T_e}$  (where  $T_e \geq 1$ ), denoting, for instance, sizes of cables and/or of terminal equipment. The capacity  $M_e^{T_e}$  is supposed to be at least as large as the sum of all demands. This can always be achieved with

a sufficiently high cost, if necessary. Define  $M_e^0$  to be 0. Let  $m_e^t$  denote the extension steps  $M_e^t - M_e^{t-1}$ , for  $t = 1, \dots, T_e$ . The cost of extending the capacity from  $M_e^{t-1}$  to  $M_e^t$  for  $t = 1, \dots, T_e$  is given by  $c_e^t \geq 0$ . So the cost of installing capacity  $M_e^t$  on edge  $e$  is  $\sum_{\tau=1}^t c_e^\tau m_e^\tau$ .

One can view the capacity/cost function on each edge  $e$  as a step function with step lengths  $m_e^t$  and step heights  $c_e^t$ . In [7] we introduced a more general model with a piecewise linear cost function, where sloped steps are allowed. But in this paper we only consider step functions.

The cost function is modeled with binary **design variables**  $x_e^t$  (sometimes written  $x(e, t)$ ) for all edges  $e \in E$ , indicating the incremental capacity installations. The values of  $x_e^t$  are required to decrease with increasing  $t$ . We define  $I := \{(e, t) \mid t = 1, \dots, T_e, e \in E\}$  as the index set of all the design variables  $x_e^t$ , and let  $x \in \mathbf{R}^I$  be the vector of all these variables. For a design vector  $x \in \mathbf{R}^I$  the corresponding cost is  $c^T x$  and the **associated capacity vector**  $y$  is given by  $y_e = \sum_{t=1}^{T_e} m_e^t x_e^t$ .

Let  $\bar{y}$  be the capacity vector associated with some design vector  $x$ . The network  $(G, \bar{y})$  is supposed to allow a multicommodity flow carrying all traffic. The feasibility of a capacity vector  $\bar{y}$  can be expressed in terms of linear inequalities as follows. For some given nonnegative vector  $\mu \in \mathbf{R}^E$  and demand edge  $f \in D$  let  $\pi_f^\mu$  denote the shortest path length in  $G$  between the two end nodes of  $f$  with respect to edge lengths  $\mu_e$ . It can be shown (see [12]) that  $\bar{y}$  is feasible if and only if

$$\sum_{e \in E} \mu_e \bar{y}_e \geq \sum_{f \in D} \pi_f^\mu d_f \quad \text{for all } \mu \geq 0. \quad (1)$$

The necessity of these inequalities stems from the fact that the cheapest way to route a multicommodity flow, if no capacity constraints but edge costs  $\mu_e$  are given, is to route each flow individually on its shortest path w.r. to  $\mu$ . Thus a lower bound for  $\mu^T \bar{y}$  is the sum of the shortest-path-lengths multiplied with the demand value. The sufficiency of the system (1) comes from linear programming duality. This characterization of feasible capacities is known as the ‘‘Japanese theorem’’, first stated in [12, 17]. The inequalities (1) are called **metric inequalities**. This name is motivated by the fact that, for complete input and demand graphs, any vector  $\mu \in \mathbf{R}^E$  defining a non-redundant inequality in (1) induces a (pseudo-)metric on  $G$ , that is, it is nonnegative, symmetric and satisfies the triangle inequality  $\mu_{uv} + \mu_{vw} \geq \mu_{uw}$  for any three nodes  $u, v, w$ . (Here,  $\mu_{uu}$  is supposed to be 0).

In the inequality system (1) we can restrict ourselves to the inequalities defined by vectors  $(\mu, \pi)$  in the set  $\Pi$  of extreme rays of the cone

$$\{\mu \in \mathbf{R}^E, \pi \in \mathbf{R}^D \mid \mu \geq 0, \pi_f = \pi_f^\mu \text{ for all } f \in D\}.$$

Non-redundant metric inequalities were investigated in [1, 13].

A special type of metric inequality is the so-called **cut inequality** defined by a node set  $W \neq \emptyset, W \neq V$

$$\bar{y}(\delta_G(W)) \geq d(\delta_H(W)). \quad (2)$$

This can be seen to be a metric inequality when  $\mu$  is set as the incidence vector of  $\delta_G(W)$ , and when we assume that  $G[W]$  and  $G[V \setminus W]$  are connected. Cut inequalities express the fact that the total demand crossing a cut should not exceed its capacity.

Model 1 with cost function and multicommodity flow constraints can now be stated by inserting  $\bar{y}_e := \sum_{t=1}^{T_e} m_e^t x_e^t$  into the equation system (1):

### Model 1

$$\min c^T x$$

subject to

$$\begin{aligned} \text{(i)} \quad & 1 \geq x_e^1 \geq x_e^2 \geq \dots \geq x_e^{T_e} \geq 0 && \text{for all } e \in E \\ \text{(ii)} \quad & \sum_{e \in E} \mu_e \sum_{t=1}^{T_e} m_e^t x_e^t \geq \sum_{f \in D} \pi_f^\mu d_f && \text{for all } (\mu, \pi) \in \Pi \\ \text{(iii)} \quad & x_e^t \text{ integer} && \text{for all } (e, t) \in I. \end{aligned} \quad (3)$$

The constraints (i) are called **ordering constraints**, and the constraints (ii) are called **metric inequalities**, as inequalities (1). If  $S \subseteq I$  is such that its incidence vector  $\chi^S$  satisfies all the constraints in Model 1, we say that  $\chi^S$  and  $S$  are **feasible**.

This concludes the discussion of the validity of Model 1 as valid integer linear programming formulation of the MULTISUN problem when no survivability is taken into account.

## 2.2 Model 2

The second model includes survivability constraints, which means that if one of the lines or nodes of the network fails all the traffic should still be accommodated in the remaining operating network. More precisely, we require that the installed capacity vector  $\bar{y}$  satisfies the following constraints:

- (i) for each  $e \in E$  the capacity vector  $\bar{y}$  restricted to  $E \setminus \{e\}$  allows a multicommodity flow w.r.t.  $(H, d)$ ,
- (ii) for each  $v \in V$  the capacity vector  $\bar{y}$  restricted to  $E \setminus \delta_G(v)$  allows a multicommodity flow w.r.t.  $(H - v, d)$ .

The constraints (i), called **edge failure constraints**, assure that the network  $(G, \bar{y})$  has sufficient reserve capacity to protect against any single edge failure. Similarly, the **node failure constraints** (ii) protect the network against any single node failure. Note that if a node fails, then all the demands originating in this node are deleted from the demand graph. In some applications one may have more complex changes of the demand graph in case of node failures (see [7]), but this is not considered here.

It is convenient to introduce index sets representing the **operating states** of the network. Let  $S^E$  ( $S^V$ ) have one element for each  $e \in E$  ( $v \in V$ ) corresponding to the failure of  $e$  ( $v$ , resp.), and let  $S^0$  represent the case without any node or edge failure. In the following, we are interested in the following subsets of operating states, namely  $S := S^0$ ,  $S := S^0 \cup S^E$ , and  $S := S^0 \cup S^E \cup S^V$ . Furthermore, for an operating state  $s \in S^0 \cup S^E \cup S^V$  we define sets  $E(s)$  and  $D(s)$  of operating supply and demand edges:

- For  $s \in S^0$ , let  $E(s) := E$  and  $D(s) := D$ .
- For  $s \in S^E$  representing the failure of edge  $e$ , define  $E(s) := E \setminus \{e\}$ , and  $D(s) := D$ .
- For  $s \in S^V$  representing the failure of edge  $v$ , define  $E(s) := E \setminus \delta_G(v)$ , and  $D(s) := D \setminus \delta_H(v)$ .

For any operating state  $s$ , the shortest-path values  $\pi_f^\mu$  are defined with respect to the graph of operating supply edges  $(V, E(s))$ , and the set  $\Pi(s)$  is defined analogous to  $\Pi$  in Model 1.

The integer linear programming formulation of the MULTISUN problem with survivability constraints now becomes:

**Model 2**

$$\min c^T x$$

subject to

$$\begin{aligned} \text{(i)} \quad & 1 \geq x_e^1 \geq x_e^2 \geq \dots \geq x_e^{T_e} \geq 0 && \text{for all } e \in E \\ \text{(ii)} \quad & \sum_{e \in E(s)} \mu_e \sum_{t=1}^{T_e} m_e^t x_e^t \geq \sum_{f \in D(s)} \pi_f^\mu d_f && \text{for all } \pi \in \Pi(s), s \in S \\ \text{(iii)} \quad & x_e^t \text{ integer} && \text{for all } (e, t) \in I. \end{aligned} \tag{4}$$

We remark that this model could be used for more general failure situations as well, for example the simultaneous failure of two or more nodes. This would be reflected in other choices of  $E(s)$  and  $D(s)$ . For  $S := S^0$ , Model 2 is equivalent with Model 1.

### 2.3 Polytopes Associated with the Models

We introduce the polytopes associated with the MULTISUN models.

$$\text{MSUN}_S(G, m, H, d) := \text{conv} \{ x \in \mathbf{R}^J \mid x \text{ satisfies (4)(i)–(iii)} \}$$

If no misunderstanding is possible, we drop the parameter list and write  $\text{MSUN}_S$ . When we consider only edge failures  $S := S^0 \cup S^E$ , we also write  $\text{MSUN}_E$ ; when  $S := S^0 \cup S^E \cup S^V$ , we write  $\text{MSUN}_{E \cup V}$ , and for  $S := S^0$ , we denote the polytope by  $\text{MSUN}_\emptyset$ .

Since the vertices of the polytope  $\text{MSUN}_S$  are exactly the feasible  $x$  with integer coefficients, the MULTISUN problem can now also be written as

$$\min c^T x \quad \text{subject to } x \in \text{MSUN}_S.$$

To optimize over a polyhedron using a linear programming code, one needs to know all, or at least “sufficiently” many, of its defining inequalities. The inequalities in (4) are generally not enough to achieve good lower bounds. So, a main purpose of the subsequent study is to find more classes of valid and non-redundant inequalities for  $\text{MSUN}_S$ .



### 3 Basic Properties of MULTISUN Polyhedra

We begin the polyhedral investigations of MULTISUN polyhedra by a study of their dimension. Let a supply graph  $G = (V, E)$  and a demand graph  $H = (V, D)$  with demand vector  $d$  be given.

**Proposition 5** *Let  $MSUN_S$  be one of the polytopes  $MSUN_\emptyset$ ,  $MSUN_E$  or  $MSUN_{E \cup V}$ .*

*Then  $MSUN_S(G, m, H, d)$  is full-dimensional if and only if  $MSUN_S(G - e, m, H, d)$  is nonempty for all  $e \in E$ .*

**Proof.** Assume that  $MSUN_S(G - e, m, H, d)$  is empty for some  $e \in E$ . Then either  $MSUN_S(G, m, H, d)$  is empty, or each feasible  $x \in \mathbf{R}^I$  satisfies  $x_e^1 = 1$ , hence  $MSUN_S$  is not full-dimensional.

To prove the sufficiency, we assume that  $MSUN_S$  is not full-dimensional, i.e., there is a linear equation  $a^T x = \alpha$  with nonzero  $a$  satisfied by each point in  $MSUN_S$ . Let  $f \in E$ . By hypothesis, there exists a solution  $x \in MSUN_S$  with  $x_f^t = 0$  for  $t = 1, \dots, T_f$ . By monotonicity, the solutions  $x^k$  obtained from  $x$  by changing  $x_e^t$  to 1 for  $t = 1, \dots, k$  are feasible. Thus we have  $a^T x = a^T x^1 = a^T x^2 = \dots = \alpha$ , and by subtraction, we get  $a_e^t = 0$  for all  $t = 1, \dots, T_e$ . Since  $f$  was chosen arbitrarily in  $E$ , we get  $a = 0$ , a contradiction. Thus the equality system of  $MSUN_S$  is empty, and the polyhedron is full-dimensional.  $\square$

We hereafter assume that each MSUN-polytope under consideration is full-dimensional; this simplifies polyhedral arguments. In most applications the supply graph will be sufficiently connected anyway.

**Proposition 6** *Let  $MSUN_S$  be one of the polytopes  $MSUN_\emptyset$ ,  $MSUN_E$  or  $MSUN_{E \cup V}$ .*

- (i) *The inequalities  $1 \geq x_e^1 \geq x_e^2 \geq \dots \geq x_e^{T_e}$  define facets of  $MSUN_S$  for all  $e \in E$ .*
- (ii) *The inequality  $x(e, T_e) \geq 0$  defines a facet of  $MSUN_S$  if and only if for all  $f \neq e$  the polytope  $MSUN_S(G - f, m, H, d) \cap \{x \in \mathbf{R}^I \mid x(e, T_e) = 0\}$  is nonempty.*

**Proof.** (i): For each  $(f, k) \in I$ , define the vector  $x^{f,k} \in \{0, 1\}^I$  by setting  $x^{f,k}(e, t) := 1$  for all  $e \neq f$  and all  $t$ ,  $x^{f,k}(f, t) := 1$  if  $t < k$ , and  $x^{f,k}(f, t) := 0$  if  $t \geq k$ . Define furthermore  $x' \in \mathbf{R}^I$  to be the vector of all 1s. These  $|I| + 1$  points are feasible and affinely independent. Each inequality in (i) holds with equality for exactly  $|I|$  of the points. Thus the face induced by this inequality is a facet.

(ii): If  $\text{MSUN}_S(G - f, m, H, d) \cap \{x \mid x(e, T_e) = 0\}$  is empty for some  $f$ , then  $x(e, T_e) = 0$  implies  $x_f^1 = 1$ , therefore the non-negativity inequality cannot define a facet. If, on the other hand, the given condition holds, then the  $|I|$  vectors  $y^{f,k}$ , derived from  $x^{f,k}$  by setting the component  $x(e, T_e)$  to 0 for all  $(f, k) \in I$ , are feasible and affinely independent.  $\square$

The inequalities (4)(ii) do not define facets of  $\text{MSUN}_S$  except in very special cases. This indicates the need of stronger formulations than the LP relaxation given by (4). In the remaining part of the paper we therefore give different classes of inequalities that lead to improved LP formulations of the MULTISUN problem.

## 4 Improved Formulations Based on Knapsack Substructures

### 4.1 Band Inequalities

Several ideas may lead to improved, i.e., stronger, formulations of the MULTISUN problem. We describe a class of valid inequalities for  $\text{MSUN}_\emptyset$  called **band inequalities**. They are derived as facet-defining inequalities for a relaxation of  $\text{MSUN}_\emptyset$ , the so-called ICOV-polytope.

Let  $\sum_{e \in E} \sum_{t=1}^{T_e} g_e^t x_e^t \geq b$  be a metric inequality (3)(ii), where  $g_e^t$  is defined as  $\mu_e m_e^t$ . We will assume that  $g_e^{T_e} \geq b$  for all  $e \in E$ . This can be done without loss of generality, because we assume that the highest capacity of each edge is “large”. Let  $F$  be the support of a given metric inequality, that is, the set of edges with positive  $g_e^1$ , and consider the following polytope

$$\begin{aligned} \text{ICOV}(g, b) := \text{conv}\{ & (x_e^t : t = 1, \dots, T_e, e \in F) \mid \\ & \sum_{e \in F} \sum_{t=1}^{T_e} g_e^t x_e^t \geq b, \\ & 1 \geq x_e^1 \geq x_e^2 \geq \dots \geq x_e^{T_e} \geq 0 \text{ for all } e \in F, \\ & x \text{ integral} \}. \end{aligned} \tag{7}$$

The polytope  $\text{ICOV}(g, b)$  can be viewed as a knapsack polytope with additional ordering constraints. Facial properties of knapsack polytopes have been studied in recent years, see e.g., [5, 3]. Knapsack polytopes with the additional constraints  $\sum_t x_e^t \leq 1$  (for all  $e \in F$ ), and  $x \geq 0$ , have been studied in [14, 20] under the names “multiple-choice knapsack problem” or “knapsack problem with generalized upper bounds” (which is actually a larger class of problems). The polytope  $\text{ICOV}(g, b)$  can be linearly transformed into a knapsack polytope with generalized upper bounds, so all results pertaining to that polytope apply also to the  $\text{ICOV}$ -polytope.

Any inequality that is valid for  $\text{ICOV}(g, b)$  is clearly also valid for  $\text{MSUN}_\emptyset$ , if the missing coefficients (for  $(e, t)$  with  $e \notin F$ ) are filled up with zeros.

To describe the band inequalities we introduce some notation. Define the index set  $I(F) := \{(e, t) \in I \mid t = 1, \dots, T_e, e \in F\}$  for each  $F \subseteq E$ . (For simplicity, we write  $I(e)$  in stead of  $I(\{e\})$ ). A **band**  $B$  of  $F$  is a subset of  $I(F)$  containing exactly one element  $(e, t_e^B)$  in each  $I(e)$ ,  $e \in F$ . Let  $B^< = \{(e, t) \in I(F) \mid t < t_e^B\}$ , and define similarly  $B^>$ . A band  $B$  is called **valid** if  $g(B^<) < b$ . We say that a band  $B'$  of  $F$  is **above** a band  $B$  of  $F$  if  $t_e^B \leq t_e^{B'}$  for all  $e \in F$ , and  $t_e^B < t_e^{B'}$  for at least one  $e \in F$ .

Whenever  $B$  is a valid band, the inequality

$$x(B) := \sum_{(e,t) \in B} x_e^t \geq 1, \tag{8}$$

is valid for  $\text{ICOV}(g, b)$ . It is called a **band inequality**. The band inequalities are, after transformation, equivalent to the GUB cover inequalities (in [20]) for the knapsack problem with generalized upper bounds, but their non-redundancy is not proved there.

Figure 1 illustrates a band inequality with  $F = \{e_1, \dots, e_4\}$  and  $b = 4$ . Each column depicts a different edge. The  $(e, t)$  are represented by a box of width 1 and height  $g_e^t$ . For each  $e$ , the boxes  $(e, 1), (e, 2), \dots, (e, T_e)$  are stacked on top of each other, with  $(e, 1)$  being lowest, and  $(e, T_e)$  being highest. In our example,  $g_e^1 = 1$  for each  $e$ , and  $b = 4$ . The 1-coefficients of the band inequality are depicted inside the boxes. The value  $g(B^<)$  equals the area below the 1's.

We next characterize when a band inequality defines a facet of  $\text{ICOV}(g, b)$ , which is of some interest for the strength of the similar  $\text{MSUN}$  inequality. First, however, we remark that  $\text{ICOV}(g, b)$  is full-dimensional. This follows

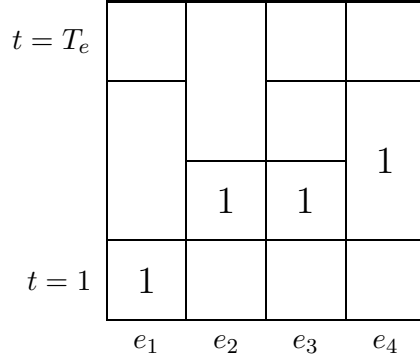


Figure 1: Band inequality

from the assumption  $g_e^{T_e} \geq b$  and  $|F| \geq 2$ , using the same construction as in the proof of Proposition 5.

**Proposition 9** *Let  $B$  be a band in  $F$ , where  $|F| \geq 2$ . Then the band inequality  $x(B) \geq 1$  defines a facet of  $\text{ICOV}(g, b)$  if and only if there is no valid band above  $B$ .*

**Proof.** Assume that  $B'$  is a valid band above  $B$ . It follows from the ordering constraints that  $x(B) \geq x(B')$  for each  $x \in \text{ICOV}(g, b)$ , so  $x(B) \geq 1$  is implied by other valid inequalities and therefore redundant.

To prove the converse, assume that there is no valid band above  $B$ . This implies that, for all  $e \in F$  and all  $t \geq t_e^B$ , the incidence vectors of

$$B^< \cup \{(e, 1), (e, 2), \dots, (e, t)\}$$

lie in the face defined by the present band inequality. Moreover, by the assumption that  $g_e^{T_e} \geq b$ , we have that, for all  $e, f \in F$ , and all  $t < t_e^B$ , the incidence vectors of

$$I(f) \cup \{(e, 1), (e, 2), \dots, (e, t)\}$$

lie in the face. These are sufficiently many affinely independent vectors to prove (by standard polyhedral methods) that the band inequality defines a facet of  $\text{ICOV}(g, b)$ .  $\square$

A special case are the band inequalities derived from cut inequalities (2). They are a subclass of the partition inequalities for Model 1 studied in Subsection 5.1, many of which define facets also for  $MSUN_\emptyset$ .

The separation problem for band inequalities for  $ICOV(g, b)$  or  $MSUN_\emptyset$  is easily seen to be equivalent to the NP-complete multiple-choice knapsack problem.

Besides the band inequalities there are other non-redundant inequalities for  $ICOV(g, b)$ . Some properties of their coefficient structure are stated in the next proposition.

**Proposition 10**

- (i) *All non-redundant inequalities for  $ICOV(g, b)$  that are not equivalent to any of the ordering constraints  $1 \geq x_e^1 \geq x_e^2 \geq \dots \geq x_e^{T_e}$  have nonnegative coefficients. The same holds for non-redundant inequalities of  $MSUN_\emptyset$ .*
- (ii) *For all non-redundant inequalities of  $ICOV(g, b)$  that are not equivalent to any ordering constraint, the sum of coefficients over  $I(e)$  is equal to its right-hand side, for all  $e \in F$ .*

**Proof.** (i): If the inequality has a negative coefficient for some  $(e, t)$ , and if  $t > 1$ , then any vector satisfying this inequality with equality must also satisfy  $x_e^{t-1} = x_e^t$ . Otherwise we could construct a contradiction to the validity of the inequality by increasing  $x_e^t$ . Since the inequality is non-redundant, it is equivalent to the ordering constraint  $x_e^{t-1} \geq x_e^t$ . By a similar argument, if the coefficient of some  $(e, 1)$  is negative then the inequality is equivalent to  $x_e^1 \leq 1$ .

(ii): Since the incidence vector of  $I(e)$  is feasible for all  $e \in F$ , and the given inequality is valid, the sum of coefficients for each fixed edge is not less than the right-hand side. Suppose that for some  $f \in F$  it is strictly larger than the right-hand side. Since the face  $P$  defined by the given inequality is not contained in the face defined by  $x(f, T_f) \leq 0$  there exists a set  $Z \subseteq I$  whose incidence vector is in  $P$  and satisfies  $x(f, T_f) > 0$ , which implies that  $I(f)$  is contained in  $Z$ . By (i), the inequality has nonnegative coefficients, so the incidence vector of  $Z$  cannot satisfy the inequality with equality. This contradicts our choice of  $Z$ . Therefore the sum of coefficients over  $I(e)$  is equal to the right-hand side of the inequality, for all  $e \in E$ .  $\square$

The band inequalities (8) are exactly those non-redundant inequalities with integral coefficients whose coefficients in  $I(e)$  sum up to 1 for each edge  $e \in F$ .

The band inequalities can be generalized to  **$k$ -band inequalities** whose integer coefficients sum up to  $k$  for each fixed edge, but we do not discuss this generalization further in this paper.

## 4.2 Strengthened Band Inequalities (Model 2)

Unfortunately, the band inequalities for Model 1 are not facet-defining for  $\text{MSUN}_E$  (with edge survivability constraints). But it is possible, as can be seen in the next lemma, to raise the right-hand side of a band inequality for  $\text{MSUN}_E$  to 2 to achieve a valid inequality (under some further assumptions).

**Lemma 11** *Let  $g^T x \geq b > 0$  be a metric inequality for  $\text{MSUN}_\emptyset(G, m, H, d)$ , let  $F$  be the set of edges with positive  $g_e^1$ , and let  $B$  be a band of  $F$ . If  $g(B^< \setminus I(e)) < b$  for all  $e \in F$  then*

$$x(B) \geq 2 \tag{12}$$

*is a valid inequality for  $\text{MSUN}_E$ .*

**Proof.** First note that  $g^T x \geq b$  is also a valid inequality for  $\text{MSUN}_\emptyset(G - e, m, H, d)$ , for all  $e \in F$ . Since  $g(B^< \setminus I(e)) < g(B^<) < b$ , the inequality  $x(B) \geq 1$  is valid for  $\text{MSUN}_\emptyset(G - e, m, H, d)$ , for all  $e \in F$ . Suppose there is a vertex  $\chi^Z$  of  $\text{MSUN}_E(G, m, H, d)$  satisfying  $\chi^Z(B) = 1$ , say  $B \cap Z = \{f\}$ . But then  $\sum_{e \neq f} \sum_{t=1}^{T_e} g_e^t \chi^Z(e, t) \leq g(B^< \setminus I(f)) < b$ , which contradicts the survivability condition when edge  $f$  is deleted from the supply graph. So  $\chi^Z(B) \geq 2$  for all vertices  $\chi^Z$  of  $\text{MSUN}_E$ .  $\square$

We call inequality (12) **strengthened band inequality**.

When the given band was derived from a cut inequality, then the strengthened band inequality can be shown to define a facet of  $\text{MSUN}_E$  in many cases. This is done in Subsection 5.2.

## 5 Improved Formulations Based on Relations to Connectivity Network Design

In this section we exploit the relation of the MULTISUN problem to the design of networks satisfying connectivity requirements. This leads to several classes of facet defining inequalities for the MSUN-polytopes.

First, we give a simple, but useful lemma. It describes the main relations between the two classes of problems and will be used in later facet proofs. To simplify the presentation, we assume throughout this section that the set of demand edges and its adjacent nodes induce a connected graph with node set  $R$ , the **terminal set**. We assume that the supply graph  $G$  is connected as well. Let the polytope CON in  $\mathbf{R}^E$  be the convex hull of incidence vectors of subgraphs of  $G$  that are connected with respect to  $R$ . Similarly, let 2ECON (2NCON) be the polytope associated with 2-edge connected (2-node connected) subgraphs with respect to  $R$ . For  $F \subseteq E$  and a band  $B$  of  $F$ , let  $I(F)$ ,  $B^<$  and  $B^>$  be defined as in Section 4.

**Lemma 13** *Let  $F \subseteq E$ . Then the following statements hold.*

- (i) *If  $(V, F)$  is connected w.r.t.  $R$ , then  $I(F)$  is feasible for  $MSUN_\emptyset$ .*
- (ii) *If  $(V, F)$  is 2-edge connected (2-node connected) w.r.t.  $R$ , then  $I(F)$  is feasible for  $MSUN_E$  ( $MSUN_{E \cup V}$ ).*
- (iii) *If  $a^T y \geq \alpha$  is a valid inequality for CON (or 2ECON, 2NCON, resp.), then  $\sum_e a_e x_e^1 \geq \alpha$  is a valid inequality for  $MSUN_\emptyset$  ( $MSUN_E$ ,  $MSUN_{E \cup V}$  resp.)*

**Proof.** Straightforward, using the fact that we can install a sufficiently “large” capacity on each edge.  $\square$

Among the nonredundant inequalities listed in [9, 11] for CON (2ECON) are partition inequalities, cut, and lifted two-cover inequalities.

**Partition inequalities** for CON are defined as

$$x(\cup_{i=1}^p \delta(V_i)) \geq p - 1 \tag{14}$$

for all partitions  $\{V_1, \dots, V_p\}$  of  $V$  (here  $p \geq 2$ ) where each  $V_i$  contains some node of  $R$ . Such partitions are called **proper partitions**. Associated with  $\mathcal{P}$  we have the graph  $\hat{G} = (\hat{V}, \hat{E})$  obtained by shrinking each node set  $V_i$  into one node.

**Remark 15** By [9], partition inequalities define facets of CON if and only if (i)  $G[V_i]$  is connected for all  $i$ , (ii)  $G[V_i]$  is 2-edge connected w.r.t.  $R$  for all  $i$ , and (iii)  $\hat{G}$  is 2-node connected.

**Cut inequalities** for 2ECON are defined as

$$x(\delta(W)) \geq 2 \tag{16}$$

for all subsets  $W$  of  $V$  such that both  $W$  and  $V \setminus W$  contain nodes of  $R$ .

**Remark 17** By [11, 19], the cut inequality defines a facet of 2ECON if  $G[W]$  and  $G[V \setminus W]$  are 2-edge connected and  $G$  is 3-edge connected. (Actually, somewhat weaker conditions are already sufficient and at the same time necessary).

**Lifted two-cover inequalities** for 2ECON are defined by pairwise disjoint nonempty node sets  $W_1, \dots, W_p$ , each containing some node of  $R$ , and edges  $e_1, \dots, e_k$  ( $3 \leq k \leq p$ ) with  $e_i \in \delta(W_i) \cap \delta(W)$  for  $i = 1, \dots, k$ . Here  $W$  denotes the union of all  $W_i$ , and  $k$  is assumed to be odd. The lifted two-cover inequalities have the form

$$x((\cup_{i=1}^p \delta(W_i)) \setminus \{e_1, \dots, e_k\}) \geq p - \lfloor k/2 \rfloor. \tag{18}$$

**Remark 19** In [19] it was shown that the lifted two-cover inequality defines a facet of 2ECON if  $G$  is complete, and the graphs induced by  $V \setminus W$  and by  $W_i$  for  $i = 1, \dots, p$ , are three-edge connected.

By Lemma 13 (iii) any valid inequality  $a^T y \geq \alpha$  for CON (2ECON, 2NCON) can be transformed to a valid inequality for the respective MSUN-polyhedron. When  $a$  is a 0/1-vector, the set  $B := \{(e, 1) \in I \mid a_e > 0\}$  can be interpreted as a band of the support of  $a^T y \geq \alpha$ . In the next subsections we will see how the inequality  $\sum_e a_e x_e^1 \geq \alpha$  can be made facet-inducing for the respective MSUN-polyhedron by finding a band above  $B$  such that the inequality stays valid. Certainly this technique can be applied as well to other valid inequalities for CON (or 2ECON, 2NCON) not listed here. Especially we observe that it can be applied to the so-called node-partition inequalities (in [11]) valid for 2NCON, but not 2ECON. They are a generalization of partition inequalities (14).



## 5.1 Partition Inequalities

Consider a proper partition  $\mathcal{P}$  of  $V$  into node sets  $V_i$ ,  $i = 1, \dots, p$ , called **shores**. Let  $\hat{G} = (\hat{V}, \hat{E})$  be defined as above, and similarly, let  $\hat{H} = (\hat{V}, \hat{D})$  be obtained from  $H$  by shrinking each node set  $V_i$  into one node. Note that the edge set  $\hat{E}$ , called a **multicut**, consists of those edges in  $E$  with end nodes belonging to different shores.

**Definition 20** A band  $B$  in  $\hat{E}$  is called a **valid  $\mathcal{P}$ -band** if for each nonempty  $W \subset \hat{V}$  with connected shores (i.e.,  $\hat{G}[W]$  and  $\hat{G}[\hat{V} \setminus W]$  are connected) we have that

$$m(\{(e, t) \in B^< \mid e \in \delta_{\hat{G}}(W)\}) < d(\delta_{\hat{H}}(W)). \quad (21)$$

A valid  $\mathcal{P}$ -band  $B$  is called **maximal** if no valid  $\mathcal{P}$ -band above  $B$  exists.

**Proposition 22** Let  $\mathcal{P}$  be a proper partition and assume that  $B$  is a valid  $\mathcal{P}$ -band. Then the **partition inequality**

$$x(B) \geq p - 1 \quad (23)$$

is valid for  $MSUN_{\emptyset}$ .

**Proof.** Let  $u$  and  $v$  be two distinct nodes in  $\hat{V}$ , and let  $W'$  be a node subset of  $\hat{V}$  such that  $u \in W'$  and  $v \notin W'$ . We observe that if, say,  $\hat{G}[W']$  is not connected, then we can find  $W \subseteq W'$  or  $W \supseteq W'$  such that the cut  $\delta_{\hat{G}}(W)$  is a subset of  $\delta_{\hat{G}}(W')$ , has connected shores and separates  $u$  and  $v$  (recall that  $\hat{G}$  is connected). Since  $B$  is a valid  $\mathcal{P}$ -band, each feasible  $Z \subseteq I$  satisfies  $(e, t_e^B) \in Z$  for at least one edge  $e \in \delta_{\hat{G}}(W) \subseteq \delta_{\hat{G}}(W')$  (otherwise, the capacity across this cut would be too small). Thus, by Menger's theorem, the subgraph of  $\hat{G}$  consisting of those edges  $e$  with  $(e, t_e^B) \in Z$  contains a  $[u, v]$ -path, and since  $u$  and  $v$  were arbitrary, this graph contains a spanning tree of  $p - 1$  edges. This proves that the partition inequality holds for  $\chi^Z$ , and, by convexity, the inequality is valid for  $MSUN_{\emptyset}$ .  $\square$

We next discuss those partition inequalities that define a facet of  $MSUN_{\emptyset}$ .

**Theorem 24** Let  $\mathcal{P}$  be a proper partition and assume that  $B$  is a valid  $\mathcal{P}$ -band. Then the partition inequality (23) defines a facet of  $MSUN_{\emptyset}$  if the following conditions hold:

(i) The partition inequality (14) defined by partition  $\mathcal{P}$  induces a facet of CON (see Remark 15), and

(ii)  $B$  is maximal.

Moreover, condition (ii) is also necessary for (23) to define a facet of  $\text{MSUN}_\emptyset$ .

**Proof.** Necessity of (ii): If there is a valid  $\mathcal{P}$ -band  $B'$  above  $B$ , then the partition inequality for  $B$  can be obtained by summing the (valid) partition inequality for  $B'$  and certain (at least one) ordering constraints, which contradicts the facet assumption. Thus we have proved the necessity of the conditions (i) and (ii).

To prove sufficiency, assume that both conditions hold, and let  $a^T x \geq \alpha$  be the inequality (23) (with  $\alpha = p - 1$ ). Let  $F(a, \alpha) = \{x \in \text{MSUN}_\emptyset \mid a^T x = \alpha\}$  be the face of  $\text{MSUN}_\emptyset$  induced by this inequality. We may assume that  $F(a, \alpha) \subseteq F(b, \beta)$ , where  $F(b, \beta)$  is the facet of  $\text{MSUN}_\emptyset$  induced by some inequality  $b^T x \geq \beta$ . We want to prove that  $(a, \alpha)$  is a scalar multiple of  $(b, \beta)$ .

First, consider an edge  $e \notin \hat{E}$ , say,  $e \in E[V_1]$ . There is a Steiner tree  $F \in E \setminus \{e\}$  such that  $\chi^F$  satisfies (14) with equality (since this inequality defines a facet of CON). Thus we see from Lemma 13 (i) that  $\chi^{I(F)} \in F(a, \alpha)$ . Define  $I(F)_{(t)} := I(F) \cup \{(e, 1), \dots, (e, t)\}$  for  $t = 1, \dots, T_e$ , and observe that the incidence vector of  $I(F)_{(t)}$  also lies in  $F(a, \alpha)$ . Thus both the incidence vectors of  $I(F)$  and  $I(F)_{(t)}$  lie in  $F(b, \beta)$ , which leads to  $b_e^t = 0$  for  $t = 1, \dots, T_e$ , as claimed. By similar arguments, one can show that  $b_e^t = 0$  for each  $(e, t) \in B^<$ .

Consider an edge  $e \in \hat{E}$  for which  $t_e^B < T_e$ . Since  $B$  is assumed maximal, the band obtained from  $B$  by replacing  $(e, t_e^B)$  by  $(e, t_e^B + 1)$  is **not** a valid  $\mathcal{P}$ -band. Thus there exists a nonempty  $W \subset \hat{V}$  with connected shores and  $e \in \delta_{\hat{G}}(W)$ , such that

$$m(\{(g, t) \in B^< \mid g \in \delta_{\hat{G}}(W)\}) + m(e, t_e^B) \geq d(\delta_{\hat{H}}(W)). \quad (25)$$

Now, choose a spanning tree  $T$  in  $\hat{G}$  that contains  $e$  but no other edge of  $\delta_{\hat{G}}(W)$ . This is possible since the shores of this cut both are connected. Define  $Z$  as the union of the four index sets  $I(E \setminus \hat{E})$ ,  $I(T \setminus \{e\})$ ,  $\{(g, t) \in B^< \mid g \in \delta_{\hat{G}}(W)\}$ , and  $\{(e, t_e^B)\}$ . We claim that  $Z$  is feasible for  $\text{MSUN}_\emptyset$ . To prove this, we describe a feasible multicommodity flow in the network

$\mathcal{N} = (G, c)$ , where  $c$  is the capacity vector associated with the design  $Z$ . By connectedness of each  $G[V_i]$  and  $\hat{G}[W]$  (and the “large” capacities there), we can clearly route within  $W$  (or  $V \setminus W$ ) all demands with both end nodes belonging to  $W$  (or  $V \setminus W$ ). For the same reason we can also route in  $\mathcal{N}$  all demands crossing the cut defined by  $W$  if the demand across the cut does not exceed the capacity of  $\mathcal{N}$  in this cut. This is indeed so, according to (25). Thus  $Z$  is feasible. Furthermore,  $\chi^Z$  satisfies  $a^T x \geq \alpha$  with equality. By monotonicity,  $Z' = Z \cup \{(e, t_e^B + 1)\}$  is also feasible and its incidence vector also lies in  $F(a, \alpha)$ , from which we obtain (as above) that  $b(e, t_e^B + 1) = 0$ . Continuing in this manner, we conclude that  $b_e^t = 0$  for  $t > t_e^B$  and  $e \in \hat{E}$ .

Thus, so far we have shown that  $b_e^t = 0$  for each  $(e, t) \in I \setminus B$ . Finally we prove that all  $b_e^t$  with  $(e, t) \in B$  must be equal. In fact, since (14) defines a facet of CON (whose affine hull does not contain the origin), there are  $|E|$  linearly independent incidence vectors of edge sets containing Steiner trees such that each of these vectors satisfies (14) with equality. Let  $A$  be a  $|\hat{E}| \times |\hat{E}|$ -dimensional matrix whose columns are these vectors; note that  $A$  is nonsingular. Also let  $b'$  ( $a'$ ) be the  $|\hat{E}|$ -dimensional vector obtained from  $b$  ( $a$ , resp.) by removing the components for  $(e, t) \in I \setminus B$ ; all shown above to be zero. Thus we have  $(b')^T A = \beta \mathbf{1}$  and  $(a')^T A = (p - 1) \mathbf{1}$ , which implies that  $(a, \alpha)$  is a scalar multiple of  $(b, \beta)$ . Therefore  $F(a, \alpha)$  is a facet.  $\square$

Unfortunately, it may not be easy to check algorithmically whether a band is a valid  $\mathcal{P}$ -band. However, for small  $p$  it can be done by simply checking all cuts explicitly. Furthermore, we remark that a band is valid provided that the following multicommodity flow problem is feasible. First, we define a capacity vector  $c$  by increasing with (a suitably small)  $\epsilon$  the total capacity below the band  $B$  on each edge, and let  $d$  be the demand as usual. The multicommodity flow problem of interest is then obtained by reversing the roles of  $c$  and  $d$  (that is, viewing  $c$  as demands and  $d$  as capacities).

## 5.2 Cut Inequalities

In the following sections we consider the polytope  $\text{MSUN}_E(G, H, m, d)$  associated with Model 2. Let  $R$  be the terminal set, and let  $y(\delta(W)) \geq 2$  be a cut inequality (16) for 2ECON, defined by node set  $W$ , such that both  $W$  and  $V \setminus W$  contain nodes of  $R$  (the terminal set). Let  $B$  be a band of  $\delta_G(W)$

with the property

$$m(B^< \setminus I(e)) < d(\delta_H(W)) \quad \text{for all } e \in \delta_G(W). \quad (26)$$

Then the **cut inequality** for  $\text{MSUN}_E$  defined by  $W$  and  $B$  is

$$x(B) \geq 2. \quad (27)$$

Since this inequality can also be seen as a strengthened band inequality derived from the metric inequality  $y(\delta_G(W)) \geq d(\delta_H(W))$ , it is valid for  $\text{MSUN}_E$  (by Lemma 11) and therefore also for  $\text{MSUN}_{E \cup V}$ .

The next theorem gives a sufficient condition for a cut inequality (27) to define a facet of  $\text{MSUN}_E$ .

**Theorem 28** *The cut inequality defined by  $W$  and  $B$  defines a facet of  $\text{MSUN}_E(G, H, m, d)$  if the following conditions are satisfied:*

- (i) *The cut inequality  $y(\delta(W)) \geq 2$  defines a facet for 2ECON (see Remark 17).*
- (ii) *For  $g \in \delta(W)$  minimizing  $m(B^< \cap I(g))$  and for all  $f \in \delta(W) \setminus \{g\}$ , the edge set  $E(W) \cup E(V \setminus W) \cup \{f, g\}$  defines a 2-edge connected graph with respect to  $R$ .*
- (iii) *There is no band above  $B$  satisfying condition (26).*

**Proof.** Let  $b^T x \geq \beta$  be a valid inequality defining a facet of  $\text{MSUN}_E$  that contains the face defined by  $x(B) \geq 2$ .

With condition (i) and Lemma 13 (ii) one can prove  $b_e^t = 0$  for all  $(e, t) \in I(E(W) \cup E(V \setminus W))$  and for all  $(e, t) \in B^<$ .

Let  $f$  be some edge in  $\delta_G(W)$  with  $t_f^B < T_f$  and let  $g$  minimize  $m(B^< \cap I(g))$ . If  $g = f$ , choose a  $g \neq f$  minimizing  $m((B^< \setminus I(f)) \cap I(g))$ . Since, by condition (iii), it is not possible to replace  $t_f^B$  by  $t_f^B + 1$  without violating (26), we have  $m(B^< \setminus I(g)) + m(f, t_f^B) \geq d(\delta_H(W))$ . Condition (ii) ensures that the incidence vector of

$$I(E(W) \cup E(V \setminus W)) \cup B^< \cup \{(f, t_f^B)\} \cup I(g)$$

is feasible and satisfies  $x(B) = 2$ . Especially, if edge  $g$  is failing, the cut  $\delta_G(W)$  has sufficiently high capacity to route all demands between  $W$  and

$V \setminus W$ . By increasing the capacity of edge  $f$ , we can prove  $b_f^t = 0$  for all  $t > t_f^B$ . This shows that  $b_e^t = 0$  for all  $(e, t) \in B^>$ .

With condition (i) one can now prove that all  $b_e^t$  with  $(e, t) \in B$  have the same value, as in the last part of the proof of Theorem 24. Therefore all coefficients of the band must be the same. This implies that  $b^T x \geq \beta$  is a positive scalar multiple of  $x(B) \geq 2$  as desired.  $\square$

### 5.3 Lifted Two-Cover Inequalities

The lifted two-cover inequalities (18) for the 2ECON polytope can be generalized in much the same way as the cut and partition inequalities.

Let  $W_1, \dots, W_p, e_1, \dots, e_k$  define a lifted two-cover inequality (18). We assume that  $\delta(W_i) \cap D \neq \emptyset$ , for  $i = 1, \dots, p$  (recall that  $D$  is the set of demand edges). Now define, for each  $i \leq k$ , a band  $B_i$  of  $\delta(W_i) \setminus \{e_i\}$  such that  $x(B_i) \geq 1$  is valid for  $\text{MSUN}_\emptyset(G - e_i, m, H, d)$ . Define also, for  $i = k+1, \dots, p$ , a band  $B_i$  of  $\delta(W_i)$ , such that  $x(B_i) \geq 2$  is valid for  $\text{MSUN}_E(G, m, H, d)$ .

Moreover, we assume that these bands are “consistent” in the sense that if an edge  $e$  lies in two bands  $B_i$  and  $B_j$ , then  $t_e^{B_i} = t_e^{B_j}$ . Denote the union of all  $B_i$  by  $B$ . Then the **lifted two-cover inequality** defined by all  $W_i, e_i$  and  $B$  is

$$x(B) \geq p - \lfloor k/2 \rfloor. \quad (29)$$

**Lemma 30** *The lifted two-cover inequality (29) is valid for  $\text{MSUN}_E$ .*

**Proof.** Note that  $x(B_i) \geq 1$  for  $i = 1, \dots, k$ , is also valid for  $\text{MSUN}_E$ . Let  $W$  be the union of all  $W_i$ . Add the valid inequalities

$$\begin{aligned} x(B_i) &\geq 1 && \text{for } i = 1, \dots, k, \\ x(B_i) &\geq 2 && \text{for } i = k+1, \dots, p, \\ x(e, t_e^B) &\geq 0 && \text{for all } e \in \delta(W) \setminus \{e_1, \dots, e_k\}, \end{aligned}$$

divide the result by 2, and round up the coefficients of the right-hand side. The resulting valid inequality is the lifted two-cover inequality (29).  $\square$

From the validity proof it can be seen that the lifted two-cover inequality is valid, but redundant, if  $k$  is even. Note that, for odd  $k$ , an integer vector  $x \in \text{MSUN}_E$  satisfies (29) with equality if and only if it satisfies with equality all except one of the inequalities used in the validity proof. The next theorem gives a sufficient facet condition for the lifted two-cover inequalities.

**Theorem 31** *The lifted two-cover inequality (29) defines a facet of  $\text{MSUN}_E$  if*

- (i) *the lifted two-cover inequality (18) defined by the  $W_i$  and  $e_i$  induces a facet of  $2\text{ECON}$  (see Remark 19);*
- (ii) *for any  $i = k + 1, \dots, p$  and for any two edges  $f, g \in \delta(W_i)$ , there exists a vertex  $y$  of  $2\text{ECON}$  with  $y_f = y_g = 1$ , satisfying (18) and  $y(\delta(W_i)) \geq 2$  with equality; and*
- (iii) *the band  $B$  is maximal in the sense that it is not possible to increase any  $t_e^B$  without violating the conditions of the definition.*

**Proof.** Let  $\{x \in \text{MSUN}_E \mid x(B) = p - \lfloor k/2 \rfloor\} \subseteq \{x \in \text{MSUN}_E \mid b^T x = \beta\}$ , where the last set is a facet of  $\text{MSUN}_E$ . Since (18) defines a facet of  $2\text{ECON}$ , it is possible to prove  $b_e^t = 0$  for all  $e \in E$  and  $t < t_e^B$  (or  $t \leq T_e$  if  $t_e^B$  does not exist).

To prove  $b_f^t = 0$  for all  $(f, t) \in B^>$ , pick some edge  $f \in \delta(W_i)$ . By the maximality of  $B$  it is not possible to increase  $t_f^B$  by 1, so there exists  $W_j$  adjacent to  $f$  such that  $x(B_j) \geq 1$  (or 2) is not valid any more. Suppose  $j = p > k$ . Then  $B_p$  must violate (26), that is, there must exist an edge  $g \neq f$  such that  $m(B_p^< \setminus I(g)) + m(f, t_f^B) \geq d(\delta_H(W_p)) > 0$ . Now, construct a feasible solution  $x$  with  $b^T x = \beta$  by equipping edge  $f$  with exactly capacity  $m(f, t_f^B)$ , edge  $g$  with highest capacity, and all other edges in  $\delta(W_p)$  with capacity  $m(e, t_e^B - 1)$ . (So far, exactly two nonzero-coefficients of the lifted two-cover inequality have been “used”.) To complete the present solution with high-capacity edges, we use condition (ii) for  $i = p$ . Furthermore all edges with  $b(I(e)) = 0$  can be equipped with high capacity. The so constructed vector  $x$  is feasible for  $\text{MSUN}_E$ , because when edge  $g$  fails, only the demands originating in  $W_p$  need to be routed on low-capacity edges, and the capacity of  $\delta(W_p)$  is high enough to make this possible. Moreover,  $G[W_p]$  is connected, otherwise (18) would not define a facet of  $2\text{ECON}$ . Since any further capacity extension of edge  $f$  also leads to a feasible solution with  $b^T x = \beta$  we have  $b_f^t = 0$  for all  $t > t_f^B$ . Similar constructions can be done for all other edges  $f$  in  $\cup \delta(W_i) \setminus \{e_1, \dots, e_k\}$ . So we have proved that all coefficients of value 0 in the lifted two-cover inequality have also value 0 in  $b^T x \geq \beta$ . To prove that both inequalities are in fact equivalent, we can use the

$|E|$  affinely independent integer vectors in 2ECON multiplied with the high capacity (and proceed as in the last part of the proof of Theorem 24).  $\square$

Concerning the separation problems for the inequalities listed here, it was shown in [10] that the problem of determining a violated partition inequality (14) for CON and a violated lifted two-cover inequality (18) for 2ECON, given some nonnegative vector  $y \in \mathbf{R}^E$ , is NP-complete. Therefore the separation problems for inequalities (23) and (29) are NP-complete, too. The NP-complete separation problem of band inequalities (8) for  $\text{ICOV}(g, b)$  can be reduced to separation of cut inequalities (27) for  $\text{MSUN}_E(G, m, H, d)$  with suitably defined  $(G, m, H, d)$ , hence the latter problem is also NP-complete.

Despite these negative results, it is possible to design heuristics to determine violated inequalities for MSUN-polyhedra. Since partition, cut, and lifted two-cover inequalities played a major role in solving connectivity problems to optimality, see [10], the corresponding inequalities will probably be useful in finding good lower bounds for MULTISUN-problems.

## 6 Conclusions and Future Work

We have studied an integer linear programming model for the multicommodity survivable network design problem (MULTISUN): find a minimum cost capacitated network that allows certain multicommodity flows under single edge and node failures. By exploiting connections to both knapsack-like problems and (uncapacitated) connectivity design problems, we found several classes of valid inequalities (band-, partition and lifted two-cover inequalities) for MULTISUN polytopes. Furthermore we discussed facet conditions for these inequalities.

It is of main interest to direct future work towards the development of a cutting plane algorithm for solving MULTISUN problems. This work can be based on polyhedral results in this paper. In particular, work needs to be done concerning separation for the different facet defining inequalities introduced here. Some encouraging preliminary results along these lines will be reported in a forthcoming paper. Another interesting area is to develop heuristics for the MULTISUN problem accompanied with worst-case analysis. (For connectivity design problems, empirical good heuristic methods have been found).

The models presented here can be extended or varied in various ways. For instance, one may include flow (routing) costs in the objective function. Also, from an algorithmic point of view, it may be useful to exploit possible further structure on the cost function in certain applications. We would also like to study additional constraints on the routing, like regional constraints and diversification (saying that, for each demand, only a certain fraction of the flow may go through each node or edge, see [7]). These modifications are interesting from an applied point of view and are worthwhile to be investigated theoretically.

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