Secondary Hopf bifurcation caused by steady-state steady-state mode interaction

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Abstract: In two-parameter systems with symmetry two steady state bifurcation points of different symmetry types coalesce generically within one point. Under certain group theoretic conditions involving the action of the symmetry group on the kernels, we show that secondary Hopf bifurcation is borne by the mode interaction. We explain this phenomenon by using linear representation theory. For motivation an example with D_3-symmetry is investigated where the main properties causing the Hopf bifurcation are summarized.

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Contents

1. Introduction

We are interested in dynamical systems

$$\dot{x} = f(x, \lambda), \quad x \in \mathbb{R}^n, \tag{1}$$

depending on two–parameters $\lambda = (\lambda_1, \lambda_2)$. Furthermore we assume f to be equivariant with respect to a finite group G, i.e. $f(\delta(t)x, \lambda) = \delta(t)f(x, \lambda), \forall\, t \in G$, where δ is a faithful linear representation of G. Often the symmetry is the result of natural geometric arrangements. While x is a state variable, the parameters represent for example Reynolds number, load, aspect ratio, etc.
Steady state – steady state mode interaction we will consider here is defined by the coalescence of two steady state bifurcation points of *different* symmetry types. The symmetry type is defined by the action of the group G on the kernel.
In two–parameter systems mode interactions generically occur. Singularity theory and normal form theory have been used to study such phenomena. The result of this is summarized in Golubitsky, Stewart, Schaeffer [8].
In contrast to the above mentioned approaches, in [5] a mode interaction analysis concerning secondary steady state bifurcation is presented which does not depend on a special group (see also [14]).
The aim of this paper is to extend the approach in [5] to cases where steady-state steady-state mode interaction gives rise to secondary Hopf bifurcation (see Fig. 1).
There is no relation between our results and those in Golubitsky, Schaeffer [7] concerning tertiary Hopf bifurcation resulting from a $\mathbb{Z}_2 \times \mathbb{Z}_2$–mode interaction.

We start (Sec.2) with the case of a three–dimensional model system showing all essential characteristics. Here the dihedral group $G := D_3$ is acting like $\delta = \vartheta^2 \oplus \vartheta^3$ with the non-trivial one-dimensional irreducible representation ϑ^2 and the two-dimensional irreducible representation ϑ^3 of D_3.
This model system has been derived by computing the basic invariants and equivariants of this D_3 action with REDUCE giving the general form of an equivariant function.
Secondary Hopf bifurcation points with isotropy C_3 exist in this example as the result of steady state – steady state mode interaction of the two different symmetry types ϑ^2 and ϑ^3 being involved. The main condition for secondary Hopf bifurcation turns out to be that ϑ^3 being considered now as a representation of C_3 looses the property to

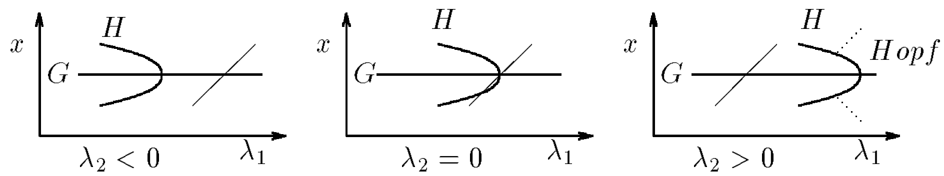

Figure 1: The three pictures show bifurcation diagrams for 3 different values of a second parameter λ_2. The subgroup H is a bifurcation subgroup of G of type ϑ. Interaction of symmetry breaking bifurcation points of type ϑ and η leads to symmetry breaking Hopf points of type ϱ, if $\varrho = \eta{\downarrow}H$ is of complex type.

be *absolutely* irreducible — it is of *complex type*. This is the motivation for the theory presented in the following sections.

Discussing this example essential group theoretic notations are introduced. In Sec. 3 innerconnectivity of linear representations is used to describe relations between symmetry breaking bifurcation points and Hopf points of different types. This description of the fundamental concept viewing non-standard group theory is a little more general than needed in Sec. 4 where we state our main results.

2. Example: D_3

Let $G := D_3 = \{1, r, r^2, s, sr, sr^2\}$ be a dihedral group where rotation r and reflection s generate the group.
Assume that $f : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ is equivariant with respect to D_3 acting as the representation $\delta : D_3 \to Gl(\mathbb{R}^3)$, generated by

$$\delta(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad \delta(s) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

δ is reducible and can be decomposed as $\delta = \vartheta^2 \oplus \vartheta^3$ with the non-trivial one-dimensional irreducible representation ϑ^2 and the two-dimensional irreducible representation ϑ^3 of D_3.

We use a REDUCE-program described in [6] to derive the general form of f. The first result of the program is that the ring of invariant polynomials $\mathbb{R}[x]_{\delta(D_3)}$ is generated by four invariants $\sigma_1(x), ..., \sigma_4(x)$ given in Table 1. The module of equivariant polynomial mappings $\mathbb{R}[x]^3_{\delta(D_3)}$ is generated over the ring $\mathbb{R}[x]_{\delta(D_3)}$ by some basic equivariants $b_1, \ldots, b_6$ (see Table 1). For the computation of fundamental invariants see also [12]. By Thm. 4.3 p. 46 and Thm. 5.3 p. 51 in [8] f has the form

$$f(x, \lambda) = \sum_{i=1}^{6} A^i(\sigma(x), \lambda) \cdot b_i(x), \quad \sigma(x) = (\sigma_1(x), ..., \sigma_4(x)). \tag{2}$$

Of course $f(0, \lambda) \equiv 0$. The mode interaction at $\lambda = (0, 0)$ is characterized by $f_x(0) = 0$. f may be the result of a center manifold reduction or a Lyapunov Schmidt reduction.

Since we are interested in Hopf points with isotropy C_3 we concentrate on special representatives of the contact equivalence class. The fixed point space of C_3 is given by

$$Fix(C_3) = \{(x_1, 0, 0), x_1 \in \mathbb{R}\}.$$

For $i = 4, 5, 6$ there is $b_i(x_1, 0, 0) = 0$ and $d_x b_i(x_1, 0, 0) = 0$. We restrict to linear A^1, A^2, A^3 and set $A^i = 0, i = 4, 5, 6$, because b_4, b_5, b_6 will not influence our calculations. Then the equivariant function f with $f(0) = 0$ and $f_x(0) = 0$ may be assumed to have the form

$$f(x, \lambda) = \left(\sum_{i=1}^{4} a_i \sigma_i(x) + \lambda_1 \right) \cdot b_1(x) + \left(\sum_{i=1}^{4} c_i \sigma_i(x) + c_5 \lambda_1 + \lambda_2 \right) \cdot b_2(x) + b_3(x).$$

$\sigma_1(x) = x_2^2 + x_3^2, \qquad \sigma_2(x) = x_1^2,$ $\sigma_3(x) = -x_2^3 + 3x_2x_3^2, \quad \sigma_4(x) = -3x_1x_2^2x_3 + x_1x_3^3 .$
$b_1(x) = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}, \quad b_2(x) = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}, \quad b_3(x) = \begin{pmatrix} 0 \\ x_1x_3 \\ -x_1x_2 \end{pmatrix},$ $b_4(x) = \begin{pmatrix} 0 \\ -x_2^2 + x_3^2 \\ 2x_2x_3 \end{pmatrix}, \quad b_5(x) = \begin{pmatrix} -3x_2^2x_3 + x_3^3 \\ 0 \\ 0 \end{pmatrix}, \quad b_6(x) = \begin{pmatrix} 0 \\ -2x_1x_2x_3 \\ -x_1x_2^2 + x_1x_3^2 \end{pmatrix}.$

Table 1: Invariants and basic equivariants of D_3 acting as $\vartheta^2 \oplus \vartheta^3$.

The parameters have been chosen in such a way that they unfold the linear part.
In the sense of singularity theory [8] one would like to show that all other equivariant functions satisfying $f(0) = 0$ and $f_x(0) = 0$ are given by a nonlinear change of coordinates $S(x,\lambda)f(X(x,\lambda),\Lambda(\lambda))$ preserving the equivariance (G-contact equivalence).

Now we discuss the bifurcation points. The linearization along the trivial solution is given by

$$f_x(0,\lambda) = \left(\begin{array}{c|cc} (\lambda_2 + c_5\lambda_1) & 0 & 0 \\ \hline 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{array}\right) \tag{3}$$

having obviously the eigenvalue $(\lambda_2 + c_5\lambda_1)$ and the double eigenvalue λ_1.
Hence for each fixed $\lambda_2 \neq 0$, there is a bifurcation point of type $\vartheta^3_{D_3}$ at $x = (0,0,0), \lambda_1 = 0$, while at $x = (0,0,0), \lambda_1 = \frac{-\lambda_2}{c_5}$ there is a bifurcation point of type $\vartheta^2_{D_3}$ (assuming $c_5 \neq 0$). By the equivariant branching lemma [16], there are branches with isotropy C_3 and $\mathbb{Z}_2$, respectively, bifurcating from these bifurcation points. We say that C_3 is a *bifurcation subgroup* for $\vartheta^2_{D_3}$ ([1], [3]) (and analogously that $\mathbb{Z}_2$ is a bifurcation subgroup for $\vartheta^3_{D_3}$).

$(x,\lambda) = (0,0)$ is an exceptional point with multiple eigenvalue zero and reducible kernel. Since $\dim(Fix(C_3) \cap \mathbb{R}^3) = 1$, the generalization of the equivariant branching lemma (Thm. 3.5 [8] p. 83) applies yielding a branch of equilibria with isotropy C_3 of $f(x,\lambda_1,0) = 0$. Solutions within $Fix(C_3)$ are given by $x_2 = x_3 = 0$ and

$$f(x_1,0,0,\lambda) = (c_2x_1^2 + c_5\lambda_1 + \lambda_2)\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = 0. \tag{4}$$

Hence the branch with isotropy C_3 bifurcating at the pitchfork $x = 0, \lambda_1 = \frac{-\lambda_2}{c_5}$ is given by $x_1 = \pm\sqrt{-\frac{c_5\lambda_1+\lambda_2}{c_2}}$, with $-\frac{(c_5\lambda_1+\lambda_2)}{c_2} > 0$ (assuming $c_5 \neq 0$ and $c_2 \neq 0$). The solutions are *conjugate* ($\delta(s)x^+ = x^-$) to each other.

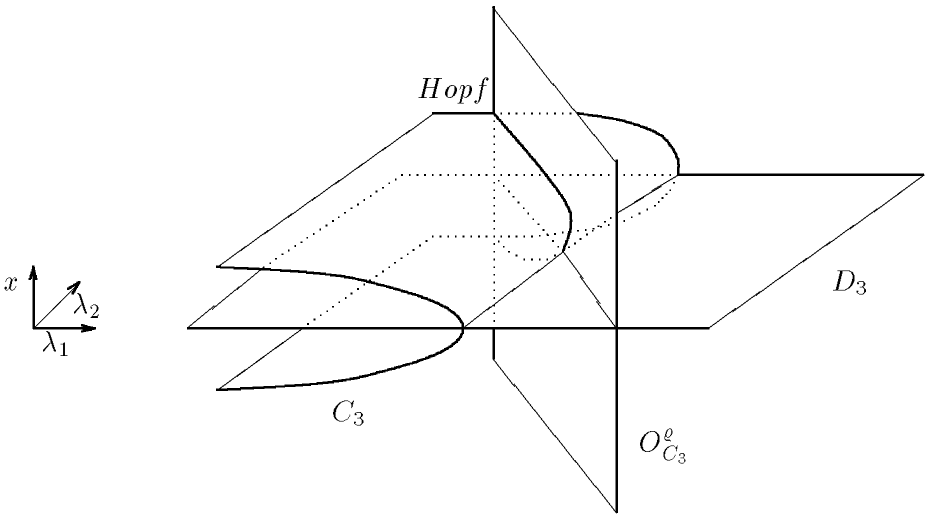

Figure 2: Manifolds of D_3– and C_3–invariant equilibria and points with eigenvalues with zero real part. Intersection of steady-state steady-state solution leads to symmetry breaking Hopf points.

Along this C_3–branch the Jacobian is given by

$$f_x(x_1, 0, 0, \lambda) = \left(\begin{array}{c|cc} 3c_2x_1^2 + \lambda_2 + c_5\lambda_1 & 0 & 0 \\ \hline 0 & a_2x_1^2 + \lambda_1 & x_1 \\ 0 & -x_1 & a_2x_1^2 + \lambda_1 \end{array} \right) \tag{5}$$

with eigenvalues $3c_2x_1^2 + \lambda_2 + c_5\lambda_1$ and $a_2x_1^2 + \lambda_1 \pm ix_1$. So for $a_2x_1^2 + \lambda_1 = 0$, there is a pair of complex conjugate eigenvalues on the imaginary axis. This gives the manifold of codimension one in Fig. 2 denoted by $O^{\varrho}_{C_3}$. It intersects the solution manifold with isotropy C_3 in one curve.
This *Hopf curve* is given by

$$x_1 = \pm\sqrt{-\frac{\lambda_2}{c_2 - a_2c_5}}, \quad x_2 = x_3 = 0, \quad \lambda_1 = \frac{\lambda_2 a_2}{c_2 - a_2c_5},$$

where $c_2 - a_2c_5 \neq 0$ and $-\frac{\lambda_2}{c_2 - a_2c_5} > 0$. The Hopf curve consists of Hopf points (except $(x, \lambda) = (0, 0)$) with respect to λ_1 for fixed $\lambda_2 \neq 0$. Indeed the real part of the eigenvalue $\mu = a_2x_1^2 + \lambda_1 \pm ix_1$ along the C_3-branch satisfies $\frac{d\Re(\mu)}{d\lambda_1} = -\frac{c_5a_2}{c_2} + 1$ which is generically nonzero. The other eigenvalue is nonzero too. This eigenvalue crossing conditions assure the bifurcation of periodic orbits, see [10].

Observing the basic equivariants ones recognizes that the equivariant b_3 (because of its derivative) is responsible for the existence of Hopf points. On the other hand the existence of the Hopf points is no accident of calculation, but an example of a more general principle as we will show.

We **summarize** the important aspects of this example which appear as conditions in the general Theorem 4.12 that steady-state steady-state mode interaction leads to bifurcation of periodic orbits.

- The kernel U of the linearization at $(x, \lambda_1, \lambda_2) = (0,0,0)$ is not irreducible. Two non-isomorphic irreducible representations $\vartheta = \vartheta^2$ and $\eta = \vartheta^3$ are acting on the kernel $U = \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^2 = U_\vartheta \oplus U_\eta$.

- The bifurcation subgroup C_3 for ϑ satisfies the condition of the equivariant branching lemma (Thm. 3.5 [8] p.83):

$$\dim(Fix(C_3) \cap U_\vartheta) = \dim(Fix(C_3) \cap (U_\vartheta + U_\eta)) = 1.$$

- The bifurcating C_3-branch is the result of a pitchfork bifurcation because the *normalizer* $N_{D_3}(C_3) = \{t \in D_3 | \; tC_3t^{-1} = C_3\}$ satisfies $N_{D_3}(C_3)/C_3 \cong \mathbb{Z}_2$ (see [1]).

- The linearizations in (3) and (5) have block diagonal structure because $f_x(0,0)$ in (3) commutes with D_3 acting as $\delta = \vartheta \oplus \eta$ and the linearization in (5) commutes with C_3 acting as the restriction of δ to C_3. For a theory of matrices having the *symmetry* of a group see SERRE [11] and FÄSSLER, STIEFEL [2]. The blockdiagonal structures of (3) and (5) are related to each other in the same way how δ is related to its restriction $\delta{\downarrow}C_3$. This kind of relation and its consequences for bifurcation points and Hopf points including their numeric are discussed in Sec. 3.

- The lower block in (5) commutes with the two-dimensional irreducible representation $\varrho := \eta \downarrow C_3$ of C_3. This block has generically pairs of complex conjugate eigenvalues. This is a typical property of irreducible representations of *complex type*. Indeed, ϱ is of complex type in contrast to η which is *absolutely irreducible*, see [8], [13].

- The eigenvalues depend C^1 on the variables x and the parameters λ.

- The situation of the example is generic within 2 parameters and thus the manifolds in Fig. 1 intersect each other transversally. This may be expressed by a non-vanishing determinant of derivatives.

3. Perturbation of eigenvalues of commuting matrices

We are interested in special steady state solutions of dynamical systems (1). We assume f to be equivariant with respect to a faithful orthogonal representation δ of a finite group G. An important consequence which we will exploit is that the linearization commutes with the isotropy group of the point (x, λ), where it is evaluated:

$$f_x(x,\lambda)\delta(t) = \delta(t) f_x(x,\lambda) \quad \forall\, t \in G_x. \tag{6}$$

Since the numerical detection and computation of bifurcation points and Hopf points are based on the eigenvalues of the Jacobian $f_x(x, \lambda)$ as well as the theoretical considerations in Sec. 4., matrices with the property (6) will be investigated in this section.

Consider a path $(x(s), \lambda(s))$ having the isotropy G for $s = s_0$, but for $s \neq s_0$ having the isotropy of a proper subgroup H. Then the Jacobians commute with G *and–or* H. Especially, in a neighborhood of s_0 the relation between both commuting properties is relevant.

Let D be a representation of a group H (which will be assumed in the sequel to be a subgroup of G) on a finite dimensional real vector space V. Let

$$C(D) := \{A \in Hom_{\mathbb{R}}(V) : D(h)A = AD(h) \text{ for all } h \in H\}$$

consist of all endomorphism A of V which commute with D.

Using a symmetry adapted basis of V, each $A \in C(D)$ has blockdiagonal form where each block is associated with irreducible representations of H appearing in D (see [2]). More precisely: Let η be an irreducible representation of H of dimension n_η. Then the diagonal block associated with η contains m_η equal subblocks A_η with $m_\eta = n_\eta$ for an absolutely irreducible representation and $m_\eta = n_\eta/2$ for an irreducible representation of complex type (we do not consider the quaternion case here). Now, eigenvalues μ of $A \in C(D)$ can be obviously associated with irreducible representations η if $\mu \in \sigma(A_\eta)$.

Let η be an absolutely irreducible representation of H. Then an eigenvalue μ of $A \in C(D)$ is called η-simple, if the corresponding block A_η of the symmetry adapted block diagonal form of $A \in C(D)$ has the simple eigenvalue μ. It is clear that when μ_0 is a η-simple eigenvalue of $A_0 \in C(D)$ then there is a neighborhood U_0 of A_0 in $C(D)$ and a C^∞–function $\mu : U_0 \to \mathbb{C}$ such that $\mu(A_0) = \mu_0$ and $\mu(A)$ is an η-simple eigenvalue of A.

The case of an irreducible representation ϱ of H of *complex type* needs more attention. Let A_ϱ be the ϱ-block of $A \in C(D)$ in the symmetry adapted block diagonal form (Th. 2.3 in [13]) which has the form

$$A_\varrho = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R}^{c,c}.$$

The eigenvalues of A_ϱ appear in complex conjugate pairs $(\mu, \bar{\mu})$. The case $\mu = \bar{\mu} \in \mathbb{R}$ is not excluded, it occurs generically in a one–parameter family of matrices in $C(D)$, leads to an eigenspace of dimension two or larger and is considered in the Perturbation Lemma 3.3 below.

We call the pair $(\mu, \bar{\mu})$ of eigenvalues of $A \in C(D)$ ϱ–simple if and only if the real eigenspace of A_ϱ associated with the pair of eigenvalues has dimension two and the eigenvalues are semisimple. (That means that either there is a complex conjugate pair of algebraically simple, nonreal eigenvalues of A_ϱ or that there is an algebraically and geometrically double real eigenvalue of A_ϱ).

Now we state

Lemma 3.1 *Let $(\mu_0, \bar{\mu}_0)$ be a ϱ-simple eigenvalue pair of $A_0 \in C(D)$.*
Then there is a neighborhood U_0 of A_0 in $C(D)$ and real C^∞–functions $r_\varrho, s_\varrho : U_0 \to \mathbb{R}$ such that

$$r_\varrho(A_0) + is_\varrho(A_0) = \mu_0$$

and $r_\varrho(A) \pm is_\varrho(A)$ is a ϱ–simple pair of eigenvalues of A for each $A \in U_0$.

Let H be a subgroup of G and δ a representation of G on $\mathbb{R}^n$. Then by

$$[\delta\downarrow H] : H \to Gl(\mathbb{R}^n), \quad [\delta\downarrow H](t) := \delta(t), \quad \forall\, t \in H$$

a *restricted representation* is defined.
Of course $C(\delta) \subset C(D)$ with $D := \delta\downarrow H$. Let η be an absolutely irreducible representation of G. We intend to look at the deformation of η-simple eigenvalues of $A_0 \in C(\delta)$ within $C(D)$.

The key for understanding this deformation is the canonical decomposition of an irreducible representation η of G considered as restricted representation $\eta\downarrow H$.

Definition 3.2 *([4]) Let η be an irreducible representation of G and let δ^i denote the irreducible representations of H. Then the multiplicities $m_{\delta^i}(\eta\downarrow H)$ in the decomposition*

$$\eta\downarrow H = \sum_{i=1}^{h} m_{\delta^i}(\eta\downarrow H)\delta^i$$

are called innerconnectivity multiplicities.

For example the two-dimensional irreducible representation $\eta = \vartheta^3$ of $G = D_3$ in Sec. 2. has the canonical decomposition

$$\vartheta^3\downarrow C_3 = 0 \cdot \delta^1 \oplus 1 \cdot \delta^2,$$

where δ^1, δ^2 are the one-dimensional irreducible representation and the two-dimensional irreducible representation of $H = C_3$. Thus $m_{\delta^1}(\vartheta^3\downarrow C_3) = 0$ and $m_{\delta^2}(\vartheta^3\downarrow C_3) = 1$.

Now assume that η and ϱ are absolutely irreducible representations of G and H, respectively. Let $m_\varrho(\eta\downarrow H) = 1$. Let $\mu_0 \in \mathbb{C}$ be an η–simple eigenvalue of A_0, which means that μ_0 is an algebraically simple eigenvalue of the η–block A_η. Assume that μ_0 is not an eigenvalue of type θ for all irreducible representations θ with $m_\varrho(\theta\downarrow H) \geq 1$. Then there is a neighborhood U_0 of A_0 in $C(\delta\downarrow H)$ and a function $\mu : U_0 \to \mathbb{C}$ such that $\mu(A)$ is a ϱ-simple eigenvalue of A.

For an irreducible representation ϱ of H of complex type the situation is the same as long as $\mu_0 \in \mathbb{C}$ is nonreal. For real μ_0 we apply Lemma 3.1:

Lemma 3.3 (Perturbation–Lemma) *Let η be an absolutely irreducible representation of G and ϱ an irreducible representation of H of complex type. Let $m_\varrho(\eta\downarrow H) = 1$. Let $\mu_0 \in \mathbb{R}$ be an η–simple eigenvalue of $A_0 \in C(\delta)$ such that $\mu_0 \notin \sigma(A_\theta)$ for all irreducible representations θ with $m_\varrho(\theta\downarrow H) \geq 1$.*
Then there is a neighborhood U_0 of A_0 in $C(\delta\downarrow H)$ and real C^∞–functions $r_\varrho, s_\varrho : U_0 \to \mathbb{R}$ such that

$$r_\varrho(A_0) = \mu_0,\ s_\varrho(A_0) = 0$$

and $r_\varrho(A) \pm i s_\varrho(A)$ is a ϱ–simple pair of eigenvalues of A for each $A \in U_0$.

A simple example is again $G = D_3, H = C_3, \eta{\downarrow}H = \varrho$ as in Sec. 2.

Now we apply the theory above to the dynamical system (1) which is equivariant with respect to $\delta : G \to Gl(\mathbb{R}^n)$. In [14], [15] the concept of *test functions* is used to detect and compute bifurcation points and Hopf points numerically. These test functions are monitoring the Jacobian $A := f_x(x, \lambda)$ and vanishing if A has a zero-eigenvalue or a pair of purely imaginary complex conjugate eigenvalues, respectively. This concept has turned out to be useful also for the proof of mode interaction results ([5], [14]) and will be used here too. Numerical aspects like efficiency does not play any role now. Hence it appears most convenient to use eigenvalues for the definition of test functions t.

If η is an absolutely irreducible representation of G, a test function $t : Fix(G) \times \mathbb{R}^2 \to \mathbb{R}$ corresponding to a symmetry breaking bifurcation point (x_0, λ_0) with isotropy G of type η is defined by $t(x, \lambda) = \mu(A_\eta)$, where μ denotes the eigenvalue of type η and $\mu(A_\eta)$ is assumed to be algebraically simple. Obviously, (x_0, λ_0) is an element of the set of niveau level zero

$$S_G^\eta := \{(x, \lambda) \in Fix(G) \times \mathbb{R}^2 |\, f_x(x, \lambda) \text{ has a zero-eigenvalue of type } \eta\}.$$

Analogously, a test function $t : Fix(G) \times \mathbb{R}^2 \to \mathbb{R}, t(x, \lambda) = \Re\mu(A_\eta)$ is defined for Hopf points of type η, where η is a representation of G which is absolutely irreducible or of complex type. $\mu(A_\eta)$ is assumed to be nonreal and algebraically simple.
The zero level set O_G^η consists of points $(x, \lambda) \in Fix(G) \times \mathbb{R}^2$ where the block A_η of the Jacobian $f_x(x, \lambda)$ has a purely imaginary complex conjugate pair of eigenvalues. Hopf points of type η are elements of O_G^η.

The sets S_H^ϱ and O_H^ϱ are defined analogously for irreducible representations ϱ of subgroups H.

Lemma 3.4 *Let H be a subgroup of G and η and ϱ absolutely irreducible representations of G and H, respectively. If $m_\varrho(\eta{\downarrow}H) = 1$ then*

$$S_G^\eta \subset S_H^\varrho \quad \textit{and} \quad O_G^\eta \subset O_H^\varrho.$$

This lemma reflects the fact that test functions $t : Fix(G) \times \mathbb{R}^2 \to \mathbb{R}$ may be extended to $t : Fix(H) \times \mathbb{R}^2 \to \mathbb{R}$. So the numerical computation of bifurcation points of type η is related to those of type ϱ.
It is this relation which is used in [14], [5] to show the existence of secondary bifurcation points caused by steady-state steady-state mode interaction.

In view of the Perturbation-Lemma 3.3 we are interested in cases when there is a relation between Hopf points with isotropy H of type ϱ and steady state bifurcation points with isotropy G of type η. To cover the limit case of real μ_0 in Lemma 3.3 we allow the Hopf test function $t(x, \lambda) = \Re\mu(A_\varrho)$, associated with the irreducible representation ϱ of complex type to be evaluated also at points (x, λ) with algebraically and geometrically double real eigenvalues $\mu(A_\varrho)$. In this sense we also slightly extend the notion of O_H^ϱ by allowing these eigenvalues to vanish for $(x, \lambda) \in O_H^\varrho$.

Lemma 3.5 *Let H be a subgroup of G, η an absolutely irreducible representation of G and ϱ an irreducible representation of H of complex type. Assume $m_\varrho(\eta{\downarrow}H) = 1$. Then*

$$S_G^\eta \subset O_H^\varrho.$$

In the motivating example we additionally have $N_G(H)/H \equiv \mathbb{Z}_2$. A solution with isotropy H comes with a conjugate solution $\delta(g)x \neq x$, $g \in N_G(H) \setminus H$. Both linearizations $f_x(x,\lambda)$ and $f_x(\delta(g)x,\lambda)$ commute with $\delta{\downarrow}H$ and are essentially the same:

$$f_x(\delta(g)x,\lambda) = \delta(g) f_x(x,\lambda)\delta(g^{-1}), \quad g \in N_G(H) \setminus H.$$

So for $g \in N_G(H) \setminus H$ we define an operator $N : C(\delta \downarrow H) \to C(\delta \downarrow H)$, $A \to \tilde{A} := \delta(g)A\delta(g^{-1})$. N is independent of g. It is clear that A and $\tilde{A}$ have the same spectrum of eigenvalues. But the corresponding irreducible representations may be different.

Definition 3.6 *([9]) For $g \in G - H$ a* conjugate *irreducible representation of $H' = gHg^{-1}$ of $\delta : H \to Aut(V)$ is defined by*

$$\delta^g : H' \to Aut(V) \quad \delta^g(gsg^{-1}) = \delta(s) \quad \forall\, s \in H.$$

For each representation ϱ of H and $g \in G \setminus H$ there is a *conjugate* representation ([9]) of $H' = gHg^{-1}$ defined by

$$\varrho^g : H' \to Aut(V) \quad \varrho^g(ghg^{-1}) := \varrho(h) \quad \forall\, h \in H.$$

Lemma 3.7 *Under the assumptions of Lemma 3.3 and $N_G(H)/H \equiv \mathbb{Z}_2$, the statements of this Lemma can be sharpened as follows: U_0 can be chosen in such a way that $N(U_0) = U_0$. If $\varrho^g = \varrho$ for $g \in N_G(H) \setminus H$, then*

$$r(N(A)) = r(A),\ s(N(A)) = -s(A).$$

4. Mode interaction and secondary Hopf bifurcation

Steady–state mode interaction we are interested in, is defined as follows:

Definition 4.8 *A point $(x_0,\lambda_0) \in Fix(G) \times \mathbb{R}^2$ is called ϑ–η–(steady–state) mode interaction point if $f(x_0,\lambda_0) = 0$ and the kernel U of $f_x^0 = f_x(x_0,\lambda_0)$ is the sum of two G–irreducible subspaces U_ϑ and U_η,*

$$U = U_\vartheta + U_\eta,$$

of nontrivial, different, absolutely irreducible types ϑ and η.

In the following we will refer to mode interaction conditions, part of them is given by

(IC_1) H is a bifurcation symmetry in ϑ and in $\vartheta \oplus \eta$, ($m_1(\eta{\downarrow}H) = 0$, $m_1(\vartheta{\downarrow}H) = 1$, m_1 denoting the multiplicity of the trivial representation).

Then $U \cap Fix(H)$ is spanned by one vector $v_H \neq 0$. There exists $z_H \in Fix(H)$ with $z_H \neq 0$ and $z_H^T f_x^0 = 0$ and there exist vectors $v_1 = (\hat{v}_1, 1, 0)^T, v_2 = (\hat{v}_2, 0, 1)^T$ with $\hat{v}_i \in Fix(G), f_y(x_0, \lambda_0)v_i = 0, i = 1, 2$, where $y = (x, \lambda)$.

In mode interaction points, there exist bifurcating branches breaking the symmetry of the group G, if the conditions of the general equivariant branching lemma ([8] Thm. 3.5, p. 83) hold. We reformulate this lemma for two parameters in our language.

Lemma 4.9 *Let $f : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ be a G-equivariant C^∞ vectorfield with respect to $\delta : G \to Gl(\mathbb{R}^n)$. Assume that (x_0, λ_0) is a ϑ–η–mode interaction point satisfying (IC_1) and that the transversality condition*

$$z_H^T f_{xy}^0 [v_H, \beta_1 v_1 + \beta_2 v_2] \neq 0, \tag{7}$$

holds for $\beta \in \mathbb{R}^2, \beta \neq 0$, where $f_{xy}^0 = f_{xy}(x_0, \lambda_0)$. Then there exist a two-dimensional manifold

$$\mathcal{C}^G = \{(x_G(\lambda), \lambda) : \|\lambda - \lambda_0\| < \varepsilon\} \subset X^G \times \mathbb{R}^2, \quad \varepsilon > 0 \tag{8}$$

of G–invariant equilibria and a two–dimensional manifold

$$\mathcal{C}^H = \{(x_H(s_1, s_2), \lambda(s_1, s_2)) : |s_1| < \varepsilon, |s_2| < \varepsilon\} \subset X^H \times \mathbb{R}^2, \quad \varepsilon > 0 \tag{9}$$

of H–invariant equilibria of $\dot{x} = f(x, \lambda)$ with $(x_H(0), \lambda(0)) = (x_0, \lambda_0)$. Secondly, there exist a branch $\mathcal{C}_\vartheta$ of bifurcation points of type ϑ.

Proof: As in [5] and [14] we start with a Lyapunov-Schmidt reduction within $Fix(H)$. Since f is G-equivariant and x_0 is G-invariant, the linearisation f_x^0 has the symmetry of δ ($\delta(t) f_x^0 = f_x^0 \delta(t), \ \forall t \in G$) and the vectors $f_{\lambda_i}(x_0, \lambda_0), i = 1, 2$ are G-invariant. Thus the restriction of f_x^0 to $Fix(H)$ maps into $Fix(H)$ again. Since $\dim(U \cap Fix(H)) = 1$, the $corange(f_x^0) \cap Fix(H)$ is spanned by a vector $z_H \in Fix(H), z_H \neq 0, z_H \notin Fix(G)$. Since $f_{\lambda_1}(x_0, \lambda_0)$ and $f_{\lambda_2}(x_0, \lambda_0)$ are G-invariant and f_x^0 is regular on $Fix(G)$, the vectors $v_1 = (\hat{v}_1, 1, 0)^T, v_2 = (\hat{v}_2, 0, 1)^T$ exist with $\hat{v}_i \in Fix(G), f_y(x_0, \lambda_0)v_i = 0, i = 1, 2$. A Lyapunov-Schmidt reduction within $Fix(H)$ results into a reduced scalar bifurcation function

$$g(u) = z_H^T f(y_0 + u_1 v_1 + u_2 v_2 + u_3 v_3 + w^*(u)),$$

where $y_0 = (x_0, \lambda_0)^T$ and $v_3 = (v_H, 0, 0)^T$. There is a local one-to-one correspondence between solutions $(x, \lambda) \in Fix(H) \times \mathbb{R}^2$ of $f(x, \lambda) = 0$ and solutions $u \in \mathbb{R}^3$ of $g(u) = 0$. Points $u = (u_1, u_2, u_3)$ correspond to G-invariant points for $u_3 = 0$ and to points with isotropy H for $u_3 \neq 0$.
Since f_x^0 is regular on $Fix(G)$, there is locally a two-dimensional manifold $\mathcal{C}^G$ of equilibria $x_G \in Fix(G)$ (which correspond to $u = (u_1, u_2, 0)^T$).
$g(u_1, u_2, 0) \equiv 0$ implies the existence of a function $h(u)$ with $g(u) = u_3 h(u)$. In terms of g and h the transversality condition (7) reads with $\beta = (\beta_1, \beta_2) \neq 0$

$$0 \neq z_H^T f_{xy}^0 [v_H, \beta_1 v_1 + \beta_2 v_2] = g_{u_3 u_1}(0)\beta_1 + g_{u_3 u_2}(0)\beta_2 = h_{u_1}(0)\beta_1 + h_{u_2}(0)\beta_2. \tag{10}$$

Essentially this is the bifurcation (eigenvalue crossing) condition $(D_\lambda \nu)\beta \neq 0$, where ν denotes the real eigenvalue corresponding to $Fix(H)$ getting zero at (x_0, λ_0). Now the

Implicit Function Theorem applied to $h(u) = 0$ yields a two-dimensional manifold of solutions $h(u(s_1, s_2)) \equiv 0$ which corresponds to a manifold of solutions

$$\mathcal{C}^H := \{(x_H(s_1, s_2), \lambda(s_1, s_2)) : |s_1| < \varepsilon, |s_2| < \varepsilon\} \subset Fix(H) \times \mathbb{R}^2, \quad \varepsilon > 0.$$

The condition (10) allows another application of the Implicit Function Theorem applied to $h(u_1, u_2, 0) = 0$: the result is a curve $(u_1(s), u_2(s), 0)$ satisfying $h(u_1(s), u_2(s), 0) \equiv 0$ which corresponds to a branch

$$\mathcal{C}_\vartheta = \{(x_G(s), \lambda(s)) : |s| < \varepsilon\} \subset Fix(G) \times \mathbb{R}^2, \quad \varepsilon > 0,$$

with $f(x_G(s), \lambda(s)) \equiv 0$ such that the kernel of $f_x(x_G(s), \lambda(s))$ contains an irreducible space of type ϑ and $(x_G(0), \lambda(0)) = (x_0, \lambda_0)$.
$\mathcal{C}_\vartheta$ is the intersection of $\mathcal{C}^H$ with $Fix(G)$. For some fixed s with $(x_G(s), \lambda(s)) \neq (x_0, \lambda_0)$ the point $(x_G(s), \lambda(s))$ is a symmetry breaking bifurcation point of type ϑ of

$$F(x, \nu) = 0, \quad F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad F(x, \nu) := f(x, \lambda(s) + \nu \cdot \zeta),$$

where $\zeta \neq 0$ is a suitable chosen vector. ∎

In [5] we have been interested in bifurcation points on $\mathcal{C}^H$ of type ϱ being an absolutely irreducible representation of H. In contrast to that we are interested here in ϱ being of *complex type*. More precisely we are interested in Hopf points of type ϱ on the H–manifold $\mathcal{C}^H$ emerging from the mode interaction point. For this purpose we define

Definition 4.10 *$(x, \lambda) \in Fix(H) \times \mathbb{R}^2$ is called a potential Hopf point of $f : \mathbb{R}^n \times R^2 \to \mathbb{R}^n$ of type ϱ, if $f(x, \lambda) = 0$ and $f_x(x, \lambda)$ has a ϱ–simple pair $(\mu, \bar{\mu})$ of eigenvalues on the imaginary axis ($\Re(\mu) = 0$), see Sec.3.*

Observe that a ϑ–η–mode interaction point $(x_0, \lambda_0, \alpha_0)$ and even any G–invariant bifurcation point of type η is a special (degenerate) potential Hopf point of type ϱ if $m_\varrho(\eta{\downarrow}H) = 1$ holds: the eigenvalue pair in the preceding Def.4.10 is $(0, 0)$.

In this definition some essential conditions are missing which are necessary to show the bifurcation of periodic orbits. But the conditions used in numerical computations are captured.

Some group theoretic conditions are necessary to show the existence of secondary Hopf bifurcation points.

Definition 4.11 *([5]) Let ϑ and η be irreducible representations of G, let H be a bifurcation symmetry of ϑ and let ϱ be a nontrivial irreducible representation of H. We say that the* **interaction condition IC***$(\vartheta, \eta, H, \varrho)$ holds iff*

(IC_1) $m_1(\vartheta{\downarrow}H) = 1$ *and* $m_1(\eta{\downarrow}H) = 0$
(equivalently H is a bifurcation symmetry in $U_\vartheta + U_\eta$ and in U_ϑ),

(IC_2) $m_\varrho(\eta{\downarrow}H) = 1$ *and* $m_\varrho(\vartheta{\downarrow}H) = 0$.

Theorem 4.12 *Let $f : I\!R^n \times I\!R^2 \to I\!R^n$ be a G-equivariant C^∞ vectorfield with respect to $\delta : G \to Gl(I\!R^n)$. Assume that (x_0, λ_0) is a ϑ–η–mode interaction point and that $IC(\vartheta, \eta, H, \varrho)$ holds with ϱ being an irreducible representation of H of* complex type.
Let z_H, v_H, v_1, v_2 be as in Lemma 4.9. Assume that the transversality condition

$$det \begin{pmatrix} z_H^T f_{xy}^0[v_H, v_1] & z_H^T f_{xy}^0[v_H, v_2] \\ \\ \frac{\partial \mu}{\partial \lambda_1}(x_0, \lambda_0) & \frac{\partial \mu}{\partial \lambda_2}(x_0, \lambda_0) \end{pmatrix} \neq 0 \qquad (11)$$

holds, where $\mu(x, \lambda)$ denotes the real eigenvalue of $f_x(x, \lambda)$ for $x \in Fix(G)$ of symmetry type η getting zero at (x_0, λ_0).
Then there exists a branch of secondary potential Hopf points with isotropy group H of type ϱ of $\dot{x} = f(x, \lambda)$.

Proof: We follow the lines of the proof of Lemma 4.9 using the notations therein. Since the transversality condition (11) implies (7), the manifolds $\mathcal{C}^G, \mathcal{C}^H, \mathcal{C}_\vartheta$ exist due to Lemma 4.9.
Secondly the transversality condition implies that a directional derivative satisfies

$$(D_\lambda \mu(x_0, \lambda_0))\gamma \neq 0,$$

for some $\gamma = (\gamma_1, \gamma_2) \neq 0$. Now the Implicit Function Theorem, applied to $\mu(x, \lambda) = 0, x \in Fix(G)$ yields a branch

$$\mathcal{C}_\eta = \{(x_G(s), \lambda(s)) | \, |s| < \varepsilon\} \subset Fix(G) \times \mathrm{I\!R}^2, \varepsilon > 0$$

of G-invariant solutions through (x_0, λ_0), such that the kernel of $f_x(x_G(s), \lambda(s))$ contains an irreducible space of type η.

By the Perturbation-Lemma 3.3, the function $\mu(x, \lambda)$ can be extended to a C^∞– (test) function $t : Fix(H) \times \mathrm{I\!R}^2 \to \mathrm{I\!R}, t(x, \lambda) := \Re\mu(x, \lambda)$, where $\mu, \bar{\mu}$ is a pair of conjugate complex eigenvalues of $f_x(x, \lambda)$.
Recall the Lyapunov-Schmidt reduction $g(u) = u_3 h(u)$. In these coordinates the test function t is written as $\tau : \mathrm{I\!R}^3 \to \mathrm{I\!R}, \tau(u) = t(y_0 + u_1 v_1 + u_2 v_2 + u_3 v_3)$, where $y_0 = (x_0, \lambda_0)$ and v_1, v_2, v_3 are defined as in the proof of Lemma 4.9.
Observe that solutions of $h(u) = 0, \tau(u) = 0$ correspond to potential Hopf points of type ϱ. If

$$\det \begin{pmatrix} h_{u_1}(0) & h_{u_2}(0) \\ \tau_{u_1}(0) & \tau_{u_2}(0) \end{pmatrix} \neq 0,$$

being exactly the transversality condition (11), the Implicit Function Theorem yields the desired branch of potential Hopf points of type ϱ

$$\mathcal{C}_\varrho = \{(x_H(s), \lambda(s)) \in O_H^\varrho \cap \mathcal{C}^H | \ |s| < \varepsilon\} \subset Fix(H) \times \mathrm{I\!R}^2, \ \varepsilon > 0.$$

Remark:

i.) The transversality condition means that $\mathcal{C}_\vartheta$ and $\mathcal{C}_\eta$ intersect transversally in (x_0, λ_0). Secondly, it assures that $\mathcal{C}^H$ and O_H^ϱ intersect transversally, where O_H^ϱ is defined in Sec. 3.

ii.) The condition (IC_2) implies $S_G^\eta \subset O_H^\varrho$ see Lemma 3.5 and thus $\mathcal{C}_\eta \subset O_H^\varrho$. But for generic f condition (IC_2) implies $\mathcal{C}_\vartheta \not\subset O_H^\varrho$.

A shortcomimg of Theorem 4.12 is that only the existence of potential Hopf points is shown. For a potential Hopf point to be a Hopf point with bifurcating periodic orbits it remains to show (see [10])

i.) $\Im(\mu) \neq 0$,

ii.) $(\Re(\mu))' \neq 0$,

iii.) there are no other eigenvalues on the imaginary axis.

Theorem 4.13 *Assume that the conditions of Theorem 4.12 hold. Let* $b := (v_H, 0, 0)^T + \gamma_1 v_1 + \gamma_2 v_2$ *be the direction of the branch of potential Hopf points given by* $z_H^T f_{xy}[v_H, b] = 0$, $\frac{d\Re\mu}{dy}(x_0, \lambda_0)b = 0$. *If*

$$(D\Im\,\mu)(x_0, \lambda_0)b \neq 0, \tag{12}$$

holds for the eigenvalue μ *of type* ϱ *and for all irreducible representations* θ *of* H *with* $m_\theta(\eta\!\downarrow\! H) \geq 1$ *and* $\kappa(x, \lambda)$ *the eigenvalue corresponding to* θ *with* $\kappa(x_0, \lambda_0) = 0$ *holds*

$$(D\Re\kappa)(x_0, \lambda_0)b \neq 0, \tag{13}$$

then a point $(x(s^*), \lambda(s^*)) \neq (x_0, \lambda_0)$ *on the branch of potential Hopf points is a Hopf point of* $\dot{x} = F(x, \nu), F : I\!R^n \times I\!R \to I\!R^n, \quad F(x, \nu) = f(x, \lambda(s^*) + \nu\zeta)$, *where* $\zeta \in I\!R^2$ *is a suitable chosen vector.*

Proof: The condition (12) implies that $(x(s^*), \lambda(s^*))$ has a pair of eigenvalues $\mu, \bar{\mu}$ with $\Re(\mu) = 0, \Im(\mu) \neq 0$ of type ϱ, if $(x(s^*), \lambda(s^*))$ is sufficiently close to (x_0, λ_0). Assume that $(0, \zeta) \in \mathbb{R}^{n+2}$ is chosen transversal to the direction of the Hopf branch in $(x(s^*), \lambda(s^*))$. Since there is a branch of equilibria with isotropy H through $(x(s^*), \lambda(s^*))$, it may be assumed that this branch is parametrized by $(x(\nu), \lambda(\nu))$ with eigenvalue $\omega(\nu)$ satisfying $\Re(\omega(0)) = 0$. It remains to show

$$(\Re\omega)'(0) \neq 0.$$

This derivative is given by the directional derivative

$$D(\Re\,\mu)(x(s^*), \lambda(s^*))(u, \zeta)^T,$$

for some vector u which is nonzero because of the transversality condition (11) and the transversal choice of ζ. Since (13) holds we may also assume that there are no other eigenvalues on the imaginary axis. ∎

We remark that condition (12) cannot be satisfied if f is a gradient system.

There are several examples for the group theoretic conditions in Theorem 4.12 and 4.13 to be fulfilled. They hold for $G = D_m = \{1, r, \ldots, r^{m-1}, s, sr, \ldots, sr^{m-1}\}$ with ϑ being the one-dimensional irreducible representation with $\vartheta(r^i) = 1, \vartheta(sr^i) = -1, i = 1, \ldots, m$ and η the faithful two-dimensional irreducible representation. Then $H := C_m = \{1, r, \ldots, r^{m-1}\}$ is the bifurcation subgroup of D_m of type ϑ. The irreducible representation ϱ of complex type is given by $\eta{\downarrow}C_m$. Obviously, the normalizer is given by $D_m = N_{D_m}(C_m)$ and $D_m/C_m \cong \mathbb{Z}_2$. Our model example in Sec.2 is of this type ($m=3$).

The symmetric group $G := S_4$ of permutations of four elements is a second example. The faithful, three-dimensional, irreducible representation ϑ has C_3 and C_4 as bifurcation subgroups. Our conditions are fulfilled, if η is the other three-dimensional representation or the two-dimensional irreducible representation and if ϱ is the irreducible representation of complex type of C_3 or C_4, respectively.

In all these examples we have $N_G(H)/H \equiv \mathbb{Z}_2$. We do not know any example where an *asymmetric* bifurcation symmetry H (for ϑ) possesses an irreducible representation of complex type.

If $N_G(H)/H \equiv \mathbb{Z}_2$, the parametrization of $\mathcal{C}^H$ can be chosen in such a way that

$$(x_H(-s_1,s_2), \lambda(-s_1,s_2)) = (\delta(g)x_H(s_1,s_2), \lambda(-s_1,s_2))$$

for some $g \in N_G(H) \setminus H$. If additionally $\varrho^g = \varrho$, Lemma 3.7 implies that the test function $t(x,\lambda)$ is invariant in the sense that $t(\delta(g)x,\lambda) = t(x,\lambda)$ for $x \in Fix(H)$. Then the potential Hopf points given by $\mathcal{C}_\varrho$ arise always in g–conjugate pairs which can be expressed by the parametrization property

$$x(-s) = \delta(g)x(s), \quad \lambda(-s) = \lambda(s).$$

Theorem 4.14 *Assume that the conditions of Theorem 4.12 hold. If $N_G(H)/H \equiv \mathbb{Z}_2$ and $\varrho^g = \varrho$ for some $g \in N_G(H) \setminus H$, then the curve of potential Hopf points of type ϱ consists of points which are conjugate to each other.*

Observe that the Hopf points appear in conjugate pairs in the example in Sec. 2.

Theorem 4.14 completes our explaination why certain steady-state steady-state mode interactions lead to secondary Hopf bifurcation — under special group theoretic conditions.

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