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Consistency Results for the Contact-Stabilized Newmark Method[1]

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Consistency Results for the Contact-Stabilized Newmark Method[†]

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Abstract

The paper considers the time integration of frictionless dynamical contact problems between viscoelastic bodies in the frame of the Signorini condition. Among the numerical integrators, interest focuses on the contact-stabilized Newmark method recently suggested by Deuflhard et al., which is compared to the classical Newmark method and an improved energy dissipative version due to Kane et al. In the absence of contact, any such variant is equivalent to the Störmer-Verlet scheme, which is well-known to have consistency order 2. In the presence of contact, however, the classical approach to discretization errors would not show consistency at all because of the discontinuity at the contact. Surprisingly, the question of consistency in the constrained situation has not been solved yet. The present paper fills this gap by means of a novel proof technique using specific norms based on earlier perturbation results due to the authors. The corresponding estimation of the local discretization error requires the bounded total variation of the solution. The results have consequences for the construction of an adaptive timestep control, which will be worked out subsequently in a forthcoming paper.

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1 Introduction

Dynamical contact problems play an important role in structural mechanics as well as in biomechanics. One of the main difficulties in the numerical treatment of such problems is the physically meaningful non-penetration condition, usually formulated via the classical Signorini contact condition, which is leading to non-smoothness at contact interfaces between bodies. In view of this difficulty, the non-penetration condition is quite often relaxed, both in analytical models and in numerical integration schemes. However, for the medical applications that we are treating (see [12]), any violation of this condition would be unacceptable.

As for the numerical integration of dynamical contact problems, various discretization schemes are known, see the monograph [13] for a survey. Among them the classical Newmark method is the most popular one. Unfortunately, its unsatisfactory handling of the non-penetration condition may lead to artificial oscillations at contact boundaries and even give rise to an undesirable energy blow-up [3, 12]. Therefore, Kane, Repetto, Ortiz, and Marsden [8] suggested an improved energy dissipative variant, which, however, is still unable to avoid the artificial oscillations at contact boundaries. For this reason, Deuflhard, Krause, and Ertel [3] (see also [12]) suggested a contact–stabilized Newmark method, which avoids unphysical oscillations and is still energy dissipative at contact. In view of challenging real life problems (e.g., the motion of a human knee), an adaptive timestep control would be desirable for the contact-stabilized Newmark scheme. Such a device requires a realistic estimation of the local discretization error (cf., e.g., [2]) based on consistency of the scheme and the corresponding consistency order, which is the topic of the present paper.

In the absence of contact, the classical Newmark scheme is equivalent to the Störmer-Verlet scheme and well–known to be second order consistent (see, e.g., the textbook [6]). In the presence of contact, however, the classical discretization error analysis would not supply any consistency at all because of the discontinuity at the contact. Up to now, to the best of our knowledge, consistency results for Newmark schemes in the presence of contact have not been given, neither in the engineering nor in the mathematical literature. In fact, the problem turned out to be really hard. As a preparatory step, we recently studied the stability of dynamical contact problems under perturbation of the initial data [10]. That study gave us the idea about a non-trivial mix of norms in function space which we will exploit here.

The paper is organized as follows. In Section 2, we fix notation and write down the dynamical Signorini contact problem together with the three variants of Newmark schemes to be treated below. Section 3 contains our main consistency result in a norm that may be interpreted as a sum of the physical kinetic and potential energies (including the viscoelastic part). This result requires a regularity assumption quite different from the classical approach. Finally, in Section 4, we give consistency results in some norm of the local error in displacements only. Numerical consequences of our results for the construction of an adaptive timestep control are postponed to a forthcoming paper.

2 Notation and Background

In order to fix notation, we write down the classical contact problem formulation via Signorini's contact conditions. Next, we present the three types of Newmark schemes for linearly viscoelastic materials, that we want to compare in the sequel.

2.1 Problem formulation

In this paper, we study dynamical contact problems between two linear viscoelastic bodies fulfilling the Kelvin-Voigt constitutive law. The model is based on Signorini's contact conditions and has been described at length in [10] for the viscoelastic case, or in [3], respectively, for the purely elastic case. For the convenience of the reader, we here merely collect the notation used therein.

Notation. Let the two bodies be identified with the union of two domains which are understood to be bounded subsets in $\mathbb{R}^d$ with $d = 2, 3$. Each of the boundaries are assumed to be Lipschitz and decomposed into three disjoint parts: Γ_D, the Dirichlet boundary, Γ_N, the Neumann boundary, and Γ_C, the possible contact boundary. The actual contact boundary is not known in advance, but is assumed to be contained in a compact strict subset of Γ_C. Tensor and vector quantities are written in bold characters, e.g., $\mathbf{v}$. Time derivatives are indicated by dots ($\dot{}$). Dirichlet boundary conditions give rise to subspaces

$$\mathbf{H}^1_D := \left\{ \mathbf{v} \,\middle|\, \mathbf{v} \in \mathbf{H}^1,\ \mathbf{v}|_{\Gamma_D} = \ 0 \right\} .$$

For given Banach space $\mathbf{X}$ and time interval $0 < T < \infty$, let $C([0,T],\mathbf{X})$ be the continuous functions $\mathbf{v} : [0,T] \to \mathbf{X}$. The space $\mathbf{L}^2(0,T;\mathbf{X})$ consists of all measurable functions $\mathbf{v} : (0,T) \to \mathbf{X}$ for which

$$\|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{X})} := \int_0^T \|\mathbf{v}(t)\|^2_{\mathbf{X}}\ \mathrm{d}t < \infty$$

holds. We identify $\mathbf{L}^2$ with its dual space and obtain the evolution triple

$$\mathbf{H}^1 \subset \mathbf{L}^2 \subset \left(\mathbf{H}^1\right)^*$$

where we denote by $\left(\mathbf{H}^1\right)^*$ the dual space to $\mathbf{H}^1$. With reference to this evolution triple, the Sobolev space $\mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ means the set of all functions $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1)$ that have generalized derivatives $\dot{\mathbf{v}} \in \mathbf{L}^2(0,T;\left(\mathbf{H}^1\right)^*)$, see, e.g., [16].

Variational problem formulation. At the contact interface Γ_C, the two bodies may come into contact but must not penetrate each other. We use the model of linearized non-penetration as it has been presented in [3]. The convex set of all admissible displacements is denoted by $\mathcal{K}$.

The materials under consideration are assumed to be linearly viscoelastic, i.e., the stresses satisfy the Kelvin-Voigt constitutive relation (cf. [10]). Both elasticity and viscoelasticity tensors are assumed to be sufficiently smooth, symmetric, and uniformly positive definite. If the viscoelasticity tensor vanishes, the constitutive law reduces to Hooke's law as used in [3] for purely linear elastic materials.

The external forces are represented by a linear functional f_{ext} on $\mathbf{H}^1_D$ which accounts for the volume forces and the tractions on the Neumann boundary. The internal forces can be written as a bilinear form a in $\mathbf{H}^1$ for the linearly elastic part, respectively b for the viscous part. Both bilinear forms are bounded in $\mathbf{H}^1$. The sum of internal elastic and external forces can be represented by

$$\langle \mathbf{F}(\mathbf{w}), \mathbf{v} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} = a(\mathbf{w}, \mathbf{v}) - f_{\text{ext}}(\mathbf{v}), \qquad \mathbf{v}, \mathbf{w} \in \mathbf{H}^1$$

and the viscoelastic forces can be written as

$$\langle \mathbf{G}(\mathbf{w}), \mathbf{v} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} = b(\mathbf{w}, \mathbf{v}), \qquad \mathbf{v}, \mathbf{w} \in \mathbf{H}^1 .$$

Via integration by parts and exploiting the boundary conditions, see [4] and [9], the contact problem in the weak formulation can be written as a hyperbolic variational inequality: For almost every $t \in [0, T]$ find $\mathbf{u} \in \mathcal{K}$ with $\mathbf{u}(\cdot, t) \in C([0, T], \mathbf{H}^1)$ and $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0, T; \mathbf{H}^1, \mathbf{L}^2)$ such that

$$\langle \ddot{\mathbf{u}}, \mathbf{v} - \mathbf{u} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} + \langle \mathbf{F}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} + \langle \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v} - \mathbf{u} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \geq 0, \ \forall \, \mathbf{v} \in \mathcal{K} \tag{1}$$

and

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \ \dot{\mathbf{u}}(x, 0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega . \tag{2}$$

Incorporating the constraints $\mathbf{v}(t) \in \mathcal{K}$ for almost all $t \in [0, T]$ by the characteristic functional $I_{\mathcal{K}}(\mathbf{v})$,

$$I_{\mathcal{K}}(\mathbf{v}) := \begin{cases} 0 & \text{if } \mathbf{v} \in \mathcal{K} \\ \infty & \text{else} \end{cases}, \qquad \mathbf{v} \in \mathbf{H}^1_D$$

the variational inequality (1) can equivalently be formulated as the variational inclusion

$$0 \in \ddot{\mathbf{u}} + \mathbf{F}(\mathbf{u}) + \mathbf{G}(\dot{\mathbf{u}}) + \partial I_{\mathcal{K}}(\mathbf{u}) \tag{3}$$

utilizing the subdifferential $\partial I_{\mathcal{K}}$ of $I_{\mathcal{K}}$ (see, e.g., [5]). For a given solution $\mathbf{u}$ of this variational inequality, we define for almost every $t \in [0, T]$ the contact forces $\mathbf{F}_{\text{con}}(\mathbf{u}) \in \left(\mathbf{H}^1\right)^*$ via

$$\langle \mathbf{F}_{\text{con}}(\mathbf{u}), \mathbf{v} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \quad = \quad \langle \ddot{\mathbf{u}} + \mathbf{F}(\mathbf{u}) + \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v} \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1}, \ \mathbf{v} \in \mathbf{H}^1 . \tag{4}$$

As shown, for instance, in [1], the unilateral contact problem between a viscoelastic body and a rigid foundation has at least one weak solution.

2.2 Newmark schemes

Here, we give details about the spatio-temporal discretization of the dynamical contact problem (1). It is based on the Rothe approach, which means a discretization first in time and then in space. In this setting, the temporal discretization can be entirely formulated in function space, which, apart from a less clumsy notation, makes it independent of any spatial discretization argument. The arising constrained minimization problems in space can be solved using adaptive monotone multigrid methods (see [11, 12]). In what follows, we consider three variants of the Newmark method. These three schemes have already been described in [3] for the purely elastic case, but – in view of perturbation results in [10] – we here give the generalization to the viscoelastic case. Special attention is paid to the predictor step, which will turn out to be crucial in the analysis of consistency errors given below. In order to fix notation, let the continuous time interval $[0,T]$ be subdivided by $N_\triangle + 1$ discrete time points

$$0 < t_1 < \cdots < t_{N_\triangle} = T$$

forming a mesh $\triangle = \{0, t_1, \ldots, T\}$ on $[0,T]$. In addition, let

$$\tau_n = t_{n+1} - t_n\,, \qquad n = 0, \ldots, N_\triangle$$

denote the (not necessarily equidistant) step size.

Classical Newmark scheme (N-CL). This scheme can be found, e.g., in the textbooks [13, 14]. For our contact problem (1) it reads

$$\begin{aligned} \mathbf{u}^{n+1}_{\text{pred}} &= \mathbf{u}^n + \tau\dot{\mathbf{u}}^n \\ 0 &\in \mathbf{u}^{n+1} - \mathbf{u}^{n+1}_{\text{pred}} + \tfrac{1}{2}\tau^2\big(\mathbf{F}^{2\beta}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}^{2\beta}(\dot{\mathbf{u}}^n, \dot{\mathbf{u}}^{n+1}) - \widetilde{\mathbf{F}}^{2\beta}_{\text{con}}(\mathbf{u}^n, \mathbf{u}^{n+1})\big) \\ \dot{\mathbf{u}}^{n+1} &= \dot{\mathbf{u}}^n - \tau\left(\mathbf{F}^{\gamma}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}^{\gamma}(\dot{\mathbf{u}}^n, \dot{\mathbf{u}}^{n+1}) - \mathbf{F}^{\gamma}_{\text{con}}(\mathbf{u}^n, \mathbf{u}^{n+1})\right) \end{aligned} \tag{5}$$

where, for ease of writing, we introduced the shorthand notations

$$\mathbf{F}^{\lambda}(\mathbf{u}^n, \mathbf{u}^{n+1}) := (1-\lambda)\mathbf{F}(\mathbf{u}^n) + \lambda\mathbf{F}(\mathbf{u}^{n+1})\,, \qquad \lambda \in [0,1] \tag{6}$$

and

$$\widetilde{\mathbf{F}}^{2\beta}_{\text{con}}(\mathbf{u}^n, \mathbf{u}^{n+1}) := (1-2\beta)\mathbf{F}_{\text{con}}(\mathbf{u}^n) - 2\beta\partial I_{\mathcal{K}}(\mathbf{u}^{n+1})\,. \tag{7}$$

We will concentrate on the most popular symmetric case $2\beta = \gamma = \frac{1}{2}$ which guarantees consistency order 2 in the absence of contact. In the presence of contact, however, the energy of the system cannot be guaranteed to remain bounded during time integration and an undesirable energy blow–up may occur (see [13]). In addition, depending on the parameter specification, the improper handling of the non–penetration condition may give rise to spurious numerical oscillations at contact boundaries.

Modified Newmark scheme (N-M). In [8], Kane, Repetto, Ortiz, and Marsden suggested a modification of the above classical scheme, replacing the term $\widetilde{\mathbf{F}}^{2\beta}_{\rm con}(\mathbf{u}^n, \mathbf{u}^{n+1})$ by $\partial I_{\mathcal{K}}(\mathbf{u}^{n+1})$, so that, for our contact problem (1), we arrive at

$$\begin{aligned} \mathbf{u}^{n+1}_{\rm pred} &= \mathbf{u}^n + \tau\dot{\mathbf{u}}^n \ , \\ 0 &\in \mathbf{u}^{n+1} - \mathbf{u}^{n+1}_{\rm pred} + \tfrac{1}{2}\tau^2\big(\mathbf{F}^{2\beta}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}^{2\beta}(\dot{\mathbf{u}}^n, \dot{\mathbf{u}}^{n+1}) + \partial I_{\mathcal{K}}(\mathbf{u}^{n+1})\big) \\ \dot{\mathbf{u}}^{n+1} &= \dot{\mathbf{u}}^n - \tau\big(\mathbf{F}^{\gamma}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}^{\gamma}(\dot{\mathbf{u}}^n, \dot{\mathbf{u}}^{n+1}) - \mathbf{F}_{\rm con}(\mathbf{u}^{n+1})\big) \, . \end{aligned} \tag{8}$$

In the symmetric case $2\beta = \gamma = \frac{1}{2}$, the fully implicit treatment of the non-penetration constraints leads to energy dissipativity of the scheme (as has been shown in [3]). Unfortunately, this modification is still not able to get rid of the artificial oscillations at contact interfaces.

Contact–stabilized Newmark scheme (N-CS). In [3], Deuflhard, Krause, and Ertel suggested a contact-stabilized version of (N-M) that avoids the occurrence of undesirable oscillations and is still energy dissipative in the presence of contact. This desirable feature is achieved by introducing an additional discrete $\mathbf{L}^2$-projection at contact interfaces per each time step, which can easily be incorporated into (N-M) by exchanging the predictor step. This is therefore the scheme in the main focus of the present paper.

Assume that the spatial quantities corresponding to $\mathbf{u}^n$ are obtained via finite elements $\mathbf{S}_h$ with $h > 0$ a spatial mesh size parameter. Note that now $\mathcal{K} \subset \mathbf{S}_h$ has to be understood as a discrete approximation of the set of admissible displacements. For details concerning this part of the discretization, we refer the reader to [9, 11]. Then

$$\begin{aligned} 0 &\in \mathbf{u}^{n+1}_{\rm pred} - (\mathbf{u}^n + \tau_n\dot{\mathbf{u}}^n) + \partial I_{\mathcal{K}}(\mathbf{u}^{n+1}_{\rm pred}) \\ 0 &\in \mathbf{u}^{n+1} - \mathbf{u}^{n+1}_{\rm pred} + \tfrac{1}{2}\tau_n^2\big(\mathbf{F}^{1/2}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}(\tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}) + \partial I_{\mathcal{K}}(\mathbf{u}^{n+1})\big) \\ \dot{\mathbf{u}}^{n+1} &= \dot{\mathbf{u}}^n - \tau_n\big(\mathbf{F}^{1/2}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}(\tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}) - \mathbf{F}_{\rm con}(\mathbf{u}^{n+1})\big) \end{aligned} \tag{9}$$

where the contact forces $\mathbf{F}_{\rm con}(\mathbf{u}^{n+1})$ are defined via

$$\begin{aligned} &\langle \mathbf{F}_{\rm con}(\mathbf{u}^{n+1}), \mathbf{v}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\ &= \langle \mathbf{u}^{n+1} - \mathbf{u}^{n+1}_{\rm pred} + \tfrac{1}{2}\tau_n^2\big(\mathbf{F}^{1/2}(\mathbf{u}^n, \mathbf{u}^{n+1}) + \mathbf{G}(\tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n})\big), \mathbf{v}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \ , \qquad \mathbf{v} \in \mathbf{H}^1 \, . \end{aligned} \tag{10}$$

The variational inclusion defining the modified predictor in (N-CS) requires the evaluation of the normal trace of $\mathbf{u}$. Although the trace of a finite element function in $\mathbf{S}_h$ is always well-defined, this is not the case for an arbitrary $\mathbf{L}^2$-function. This corresponds to the fact that the values of a function $\mathbf{u} \in \mathbf{S}_h$ at the boundary also serve as values of the finite element function within the small strip given by the elements at the boundary. By discretization in space the boundary gets assigned to a mass, although for the continuous case the boundary has measure zero. It is this double role of the degrees of freedom on the boundary as interior and boundary values, which, on one hand, causes artificial oscillations at the contact interface and, on the other hand, allows for removing the oscillations by means of the discrete $\mathbf{L}^2$-projection. Both schemes (N-CL) and (N-M) translate this discrete boundary mass into forces at the contact boundary, which thus can be understood to be the main causes for the artificial oscillations at the interfaces. In (N-CS), by means of the contact–stabilization, this non-physical part of the boundary forces is removed.

Before we dive into the technicalities of the consistency proofs, we want to mention a trivial, but important observation. In the spatial limit $h \to 0$, the discrete $\mathbf{L}^2$-projection introduced in (N-CS) vanishes, since the boundaries have measure zero in $\mathbf{L}^2$. This leads to the following proposition:

Proposition 2.1. *The predictors (N-CL), (N-M) and (N-CS) coincide in the $\mathbf{L}^2$-sense, i.e. the relation*

$$(\mathbf{u}^{n+1}_{pred}, \mathbf{v})_{\mathbf{L}^2} = (\mathbf{u}^n + \tau_n \dot{\mathbf{u}}^n, \mathbf{v})_{\mathbf{L}^2} \qquad \textit{for all } \mathbf{v} \in \mathbf{L}^2 \tag{11}$$

holds for all three variants.

For the proofs to follow in the subsequent Sections 3 and 4, we introduce a convenient

Notation. Let

$$\epsilon(t_n, \mathbf{u}, \tau_n) := \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \tag{12}$$

denote the *consistency error in the position variables*, and

$$\epsilon(t_n, \dot{\mathbf{u}}, \tau_n) := \dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^{n+1} \tag{13}$$

the *consistency error in the velocities*, respectively. We will also need

$$\bar{\epsilon}(s, \dot{\mathbf{u}}, \tau_n) := \dot{\mathbf{u}}(s) - \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau_n}, \quad s \in [t_n, t_{n+1}]. \tag{14}$$

In order to analyze and estimate these errors, the following norm will play the central role:

$$\begin{aligned} E(t_n, \tau_n) := {} & \tfrac{1}{2}\|\epsilon(t_n, \dot{\mathbf{u}}, \tau_n)\|^2_{\mathbf{L}^2} + \tfrac{1}{2} a(\epsilon(t_n, \mathbf{u}, \tau_n), \epsilon(t_n, \mathbf{u}, \tau_n)) \\ & + \int_{t_n}^{t_{n+1}} b\left(\bar{\epsilon}(s, \dot{\mathbf{u}}, \tau_n), \bar{\epsilon}(s, \dot{\mathbf{u}}, \tau_n)\right) \, ds \, . \end{aligned} \tag{15}$$

Obviously, this norm may be interpreted as a sum of the kinetic energy, measured in $\mathbf{L}^2$, and the potential energy, measured in the usual energy norm in $\mathbf{H}^1$, including the viscoelastic part. We will therefore call it the *physical energy norm* throughout the paper.

3 Consistency error in the physical energy norm

This section contains our main consistency results, in the presence of contact, for the classical, the modified, and the contact-stabilized Newmark method. We derive an estimate for the local discretization error in a special norm suggested by our previous perturbation results in [10].

We start with the analysis of the discretization errors for the Newmark schemes (N-M) and (N-CS), which we afterwards transfer to the classical scheme (N-CL).

Lemma 3.1. *Assume that $\ddot{\mathbf{u}}, \mathbf{F}_{\text{con}}(\mathbf{u}) \in (\mathbf{H}^1)^*$ for all $t \in [0,T]$. Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$\begin{aligned} & E(t_n,\tau_n) - \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\ &= -\int_{t_n}^{t_{n+1}} a\left(\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}, \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\right) ds \\ &\quad + \int_{t_n}^{t_{n+1}} \langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})), \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, ds \\ &\quad - \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right) \langle \ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1}), \bar{\epsilon}(t_n,\dot{\mathbf{u}},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, ds\,. \end{aligned} \tag{16}$$

Proof. A simple calculation shows that for (N-M) and (N-CS)

$$\dot{\mathbf{u}}^{n+1} = \dot{\mathbf{u}}^n + \frac{2}{\tau_n}\left(\mathbf{u}^{n+1} - \mathbf{u}^{n+1}_{\text{pred}}\right)\,.$$

By means of Proposition 2.1, we can write the local error of the velocities in $\mathbf{L}^2$-sense as

$$\tfrac{1}{2}\|\dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^{n+1}\|^2_{\mathbf{L}^2} = \tfrac{1}{2}\|\dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^n - \tfrac{2}{\tau_n}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right)\|^2_{\mathbf{L}^2}\,.$$

Since $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$, integration by parts (see, e.g., Prop. 23.23 in [16]) yields

$$\begin{aligned} & \tfrac{1}{2}\|\dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^{n+1}\|^2_{\mathbf{L}^2} \\ &= \tfrac{1}{2}\left\|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}^n - \tfrac{2(s-t_n)}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right)\right\|^2_{\mathbf{L}^2}\Bigg|_{s=t_n}^{s=t_{n+1}} \\ &= \int_{t_n}^{t_{n+1}} \Big\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \\ &\qquad\qquad \dot{\mathbf{u}}(s) - \dot{\mathbf{u}}^n - \tfrac{2(s-t_n)}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right)\Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds\,. \end{aligned}$$

By means of

$$\begin{aligned} & \dot{\mathbf{u}}(s) - \dot{\mathbf{u}}^n - \tfrac{2(s-t_n)}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right) \\ &= \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n} - \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right)\left(\dot{\mathbf{u}}^n - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\right)\,, \end{aligned}$$

we can split up the term on the right-hand side as

$$
\begin{aligned}
&\tfrac{1}{2}\|\dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^{n+1}\|^2_{\mathbf{L}^2} \\
&= \int_{t_n}^{t_{n+1}} \Big\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&\quad - \int_{t_n}^{t_{n+1}} \Big(1 - \tfrac{2(s-t_n)}{\tau_n}\Big)\Big\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \bar{\epsilon}(t_n, \dot{\mathbf{u}}, \tau_n)\Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds\,.
\end{aligned}
$$

Inserting the definitions of the continuous and discrete contact forces (4) and (10) into the first term, we find that (due to the linearity of the external forces)

$$
\begin{aligned}
&\tfrac{1}{2}\|\dot{\mathbf{u}}(t_{n+1}) - \dot{\mathbf{u}}^{n+1}\|^2_{\mathbf{L}^2} \\
&= -\int_{t_n}^{t_{n+1}} a\Big(\mathbf{u}(s) - \tfrac{\mathbf{u}^n+\mathbf{u}^{n+1}}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds \\
&\quad - \int_{t_n}^{t_{n+1}} b\Big(\dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds \\
&\quad + \int_{t_n}^{t_{n+1}} \Big\langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&\quad - \int_{t_n}^{t_{n+1}} \Big(1 - \tfrac{2(s-t_n)}{\tau_n}\Big)\Big\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \bar{\epsilon}(t_n, \dot{\mathbf{u}}, \tau_n)\Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds\,.
\end{aligned}
$$

The first term representing the purely elastic material behaviour can be reformulated by the fundamental theorem of calculus (cf. [16]) as

$$
\begin{aligned}
&-\int_{t_n}^{t_{n+1}} a\Big(\mathbf{u}(s) - \tfrac{\mathbf{u}^n+\mathbf{u}^{n+1}}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds \\
&= -\tfrac{1}{2}\int_{t_n}^{t_{n+1}} a\Big(\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds \\
&\quad - \int_{t_n}^{t_{n+1}} a\Big(\mathbf{u}(s) - \tfrac{\mathbf{u}^n+\mathbf{u}(t_{n+1})}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds \\
&= -\tfrac{1}{2}a\left(\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}, \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right) \\
&\quad - \int_{t_n}^{t_{n+1}} a\Big(\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n}\Big) ds\,.
\end{aligned}
$$

The third term containing the contact forces can be written as

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&= \int_{t_n}^{t_{n+1}} \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})), \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&\quad + \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} .
\end{aligned}$$

Summing up these expressions yields

$$\begin{aligned}
&\tfrac{1}{2}\|\epsilon(t_n,\dot{\mathbf{u}},\tau_n)\|_{\mathbf{L}^2}^2 + \tfrac{1}{2}a(\epsilon(t_n,\mathbf{u},\tau_n),\epsilon(t_n,\mathbf{u},\tau_n)) + \int_{t_n}^{t_{n+1}} b\left(\bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n),\bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\right) ds \\
&- \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\
&= -\int_{t_n}^{t_{n+1}} a\left(\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}, \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\right) ds \\
&\quad + \int_{t_n}^{t_{n+1}} \langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})), \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&\quad - \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right) \left\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \bar{\epsilon}(t_n,\dot{\mathbf{u}},\tau_n)\right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds .
\end{aligned}$$

Due to

$$\int_{t_n}^{t_{n+1}} \left(1 - \frac{2(s-t_n)}{\tau_n}\right) ds = 0 ,$$

we can replace the constant term $\frac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right)$ in the last line by an arbitrary functional in $\left(\mathbf{H}^1\right)^*$ that is constant in time. Choosing $\ddot{\mathbf{u}}(t_{n+1})$, we can reformulate the last term on the right-hand side as

$$\begin{aligned}
&- \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right) \left\langle \ddot{\mathbf{u}}(s) - \tfrac{2}{\tau_n^2}\left(\mathbf{u}^{n+1} - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n\right), \bar{\epsilon}(t_n,\dot{\mathbf{u}},\tau_n)\right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&= - \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right) \langle \ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1}), \bar{\epsilon}(t_n,\dot{\mathbf{u}},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds .
\end{aligned}$$

□

Note that the above proof works for both Newmark schemes due to the fact that they both use the Störmer-Verlet discretization in the absence of contact. Due to the admissibility of the continuous and discrete solutions $\mathbf{u}(t_{n+1})$ and $\mathbf{u}^{n+1}$, it holds (see [3])

$$\left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \leq 0 . \tag{17}$$

Hence, in a next step, we can omit this term on the left-hand side of equality (16). In order to derive consistency error estimates for both schemes, we will exploit the inequalities of Korn and Young.

Lemma 3.2. *Assume that* $\ddot{\mathbf{u}}, \mathbf{F}_{\text{con}}(\mathbf{u}) \in (\mathbf{H}^1)^*$ *for all* $t \in [0,T]$. *Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$
\begin{aligned}
E(t_n,\tau_n)^{1/2} \leq C \Bigg[&\left(\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n)\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} + \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \tfrac{\dot{\mathbf{u}}(t_n)+\dot{\mathbf{u}}(t_{n+1})}{2}\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|^2_{(\mathbf{H}^1)^*} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1}))\|^2_{(\mathbf{H}^1)^*} \, ds \right)^{1/2} \Bigg] .
\end{aligned} \tag{18}
$$

Proof. Omitting the term (17) on the left-hand side of (16) by positivity and splitting up the last term on the right-hand side, we find

$$
\begin{aligned}
E(t_n,\tau_n) \leq &- \int_{t_n}^{t_{n+1}} a\left(\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}, \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n) \right) ds \\
&+ \int_{t_n}^{t_{n+1}} \langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})), \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n) \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \, ds \\
&- \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n} \right) \langle \ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})), \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n) \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \, ds \\
&+ \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n} \right) \langle \ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})), \dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n) \rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \, ds \, .
\end{aligned}
$$

After using the continuity of the bilinear forms a and b in $\mathbf{H}^1$, we can apply the inequality of Young to estimate

$$
\begin{aligned}
E(t_n,\tau_n) \leq \Bigg[&\left(\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1}))\|^2_{(\mathbf{H}^1)^*} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|^2_{(\mathbf{H}^1)^*} \, ds \right)^{1/2} \Bigg] \cdot \left(\int_{t_n}^{t_{n+1}} \|\bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|^2_{(\mathbf{H}^1)^*} \, ds \right)^{1/2} \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n)\|^2_{\mathbf{H}^1} \, ds \right)^{1/2} .
\end{aligned} \tag{19}
$$

Due to the inequality of Korn and Cauchy-Schwarz, a simple calculation shows that

$$\left(\int_{t_n}^{t_{n+1}} \|\bar\epsilon(s,\dot{\mathbf{u}},\tau_n)\|^2_{\mathbf{H}^1}\,ds\right)^{1/2}$$

$$\leq C\left(\int_{t_n}^{t_{n+1}} b\left(\bar\epsilon(s,\dot{\mathbf{u}},\tau_n),\bar\epsilon(s,\dot{\mathbf{u}},\tau_n)\right)\,ds + \int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \tfrac{\dot{\mathbf{u}}^n+\dot{\mathbf{u}}^{n+1}}{2}\|^2_{\mathbf{L}^2}\,ds\right)^{1/2}$$

$$\leq C\left(\int_{t_n}^{t_{n+1}} b\left(\bar\epsilon(s,\dot{\mathbf{u}},\tau_n),\bar\epsilon(s,\dot{\mathbf{u}},\tau_n)\right)\,ds + \tfrac{\tau_n}{4}\|\epsilon(t_n,\dot{\mathbf{u}},\tau_n)\|^2_{\mathbf{L}^2}\right)^{1/2}$$

$$+\,C\left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \tfrac{\dot{\mathbf{u}}(t_n)+\dot{\mathbf{u}}(t_{n+1})}{2}\|^2_{\mathbf{L}^2}\,ds\right)^{1/2}.$$

Hence, (19) yields an inequality of the type

$$E(t_n,\tau_n) \leq 2a\cdot E(t_n,\tau_n)^{1/2} + b^2$$

where $a, b \geq 0$. Writing

$$\left(E(t_n,\tau_n)^{1/2} - a\right)^2 \leq b^2\,,$$

we obtain the inequality

$$E(t_n,\tau_n)^{1/2} \leq a + b\,.$$

Finally, b^2 consists of several summands, some of the form $\alpha\cdot\beta$. Application of the inequality $\alpha\cdot\beta \leq \frac{1}{2}(\alpha^2+\beta^2)$, yields the result of the lemma. □

Classical consistency theory. For the sake of comparison, let us have a look into the classical theory. There we would assume that the continuous solution is k-times continuously differentiable with respect to time t. In order to obtain the highest possible consistency order, we would make the assumption that the accelerations, respectively the contact forces, satisfy

$$\dot{\mathbf{u}} \in C^1([0,T],\mathbf{H}^1)\,,\quad \ddot{\mathbf{u}} \in C^1([0,T],(\mathbf{H}^1)^*)$$

and

$$\mathbf{F}_{\text{con}}(\mathbf{u}) \in C^1([0,T],(\mathbf{H}^1)^*)\,.$$

Inserting these regularity assumptions into the right-hand side of (18) would yield an error estimate of the kind

$$E(t_n,\tau_n)^{1/2} = \mathcal{O}(\tau_n^{3/2})\,, \tag{20}$$

i.e. a consistency order 3/2. Upon applying the standard techniques (like "Lady Windermere's Fan", [7]), we would lose one order of τ for convergence. Unfortunately, in the presence of contact, we cannot even assume continuity of velocities and accelerations. If we assume only boundedness of these quantities, we come to an estimate of the form

$$E(t_n, \tau_n)^{1/2} = \mathcal{O}(\tau_n^{1/2}), \tag{21}$$

i.e. to a consistency order 1/2. This, again upon applying standard techniques, would not give rise to any convergence! That is why a detailed discretization error analysis for the Newmark schemes has been missing up to now. Obviously, a more advanced concept is required to treat this complex situation.

Bounded variation. Let $(X; \|\cdot\|_X)$ be a Banach space. The (total) variation of a function $f : [a,b] \to X$ is defined as

$$\mathrm{Var}(f, [a,b], X) := \sup \left\{ \sum_{j=1}^{n} \|f(t_j) - f(t_{j-1})\|_X : a = t_0 < t_1 < \ldots < t_n = b \right\},$$

i.e. as the supremum of the above differences taken over all partitions of $[a,b]$ into finitely many subintervals. As usual, $\mathrm{BV}([a,b], X)$ denotes the set of all functions from $[a,b]$ into X that have bounded variation, i.e. for which the property $\mathrm{Var}(f, [a,b], X) < \infty$ holds. Let $f : [a,b] \to X$ be a function of bounded variation. Note that $\mathrm{Var}(f, [a,b], X)$ is only a seminorm on the linear space $\mathrm{BV}(a,b; X)$, while the norm

$$\|f\|_{\mathrm{BV}([a,b];X)} := \|f(a)\|_X + \mathrm{Var}(f, [a,b], X)$$

makes $\mathrm{BV}([a,b]; X)$ complete. Moreover, the intriguing property

$$\mathrm{Var}(f, [a,c], X) + \mathrm{Var}(f, [c,b], X) = \mathrm{Var}(f, [a,b], X) \quad \text{for } a < c < b. \tag{22}$$

holds for every function of bounded variation. Observe that the left- and right-hand limes exists at every $t \in [a,b]$. However, a function with bounded variation need not to be continuous. It can be shown that it is only continuous except at countably many points of $[a,b]$ (compare, e.g., [15, 19.21] for these notations and results).

Assumption 3.3. Throughout the remaining paper, we want to restrict our considerations to dynamical contact problems where the assumptions

$$\dot{\mathbf{u}} \in \mathrm{BV}([0,T], \mathbf{H}^1), \quad \ddot{\mathbf{u}} \in \mathrm{BV}([0,T], (\mathbf{H}^1)^*) \tag{23}$$

hold.

Since the displacements $\mathbf{u}$ are absolutely continuous in $\mathbf{H}^1$, they are especially of bounded variation, i.e. $\mathbf{u} \in \mathrm{BV}([0,T], \mathbf{H}^1)$. By the definition of the contact forces (4), Assumption 3.3 leads to $\mathbf{F}_{\mathrm{con}}(\mathbf{u}) \in \mathrm{BV}(0,T; (\mathbf{H}^1)^*)$. In particular, the velocities $\dot{\mathbf{u}}$, the accelerations $\ddot{\mathbf{u}}$ and the contact forces $\mathbf{F}_{\mathrm{con}}(\mathbf{u})$ have to be defined in $\mathbf{H}^1$, respectively in $(\mathbf{H}^1)^*$, for every time $t \in [0,T]$. We remark that the assumption of bounded variation excludes the case of highly oscillatory functions in time.

In all of the subsequent theorems, the term

$$R(\mathbf{u},[t_n,t_{n+1}]) := \mathrm{Var}(\mathbf{u},[t_n,t_{n+1}],\mathbf{H}^1) + \mathrm{Var}(\dot{\mathbf{u}},[t_n,t_{n+1}],\mathbf{H}^1) \\ + \mathrm{Var}(\ddot{\mathbf{u}},[t_n,t_{n+1}],(\mathbf{H}^1)^*) \tag{24}$$

will arise.

Theorem 3.4. *Let Assumption* 3.3 *hold. Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$E(t_n,\tau_n)^{1/2} = \mathcal{O}(\tau_n^{1/2}) \cdot R(\mathbf{u},[t_n,t_{n+1}])\,. \tag{25}$$

Proof. Under the regularity assumptions on the solution, we find that

$$\begin{aligned} E(t_n,\tau_n)^{1/2} = \mathcal{O}(\tau_n^{1/2}) \Big(& \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}\|_{\mathbf{H}^1} \\ & + \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n)\|_{\mathbf{H}^1} + \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1})\|_{\mathbf{H}^1} \\ & + \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|_{(\mathbf{H}^1)^*} \\ & + \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1}))\|_{(\mathbf{H}^1)^*}\Big) \\ = \mathcal{O}(\tau_n^{1/2}) \big(& \mathrm{Var}(\mathbf{u},[t_n,t_{n+1}],\mathbf{H}^1) + \mathrm{Var}(\dot{\mathbf{u}},[t_n,t_{n+1}],\mathbf{H}^1) \\ & + \mathrm{Var}(\ddot{\mathbf{u}},[t_n,t_{n+1}],(\mathbf{H}^1)^*) + \mathrm{Var}(\mathbf{F}_{\mathrm{con}}(\mathbf{u}),[t_n,t_{n+1}],(\mathbf{H}^1)^*)\big)\,. \end{aligned}$$

□

We want to transfer our consistency result for the modified and the contact–stabilized Newmark scheme to the classical Newmark scheme. Since the predictor steps of all three schemes coincide in $\mathbf{L}^2$-sense (see Proposition 2.1), we have to modify the above proofs only marginally.

Theorem 3.5. *Let Assumption* 3.3 *hold. Then, for initial values on the exact solution, the local error of (N-CL) satisfies*

$$E(t_n,\tau_n)^{1/2} = \mathcal{O}(\tau_n^{1/2}) \cdot R(\mathbf{u},[t_n,t_{n+1}])\,. \tag{26}$$

Proof. Performing the same calculations as in the proof of Lemma 3.1, we just have to replace $\mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1})$ in expression (19) by $\frac{1}{2}\big(\mathbf{F}_{\mathrm{con}}(\mathbf{u}^n) + \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1})\big)$. Then, we can write

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \Big\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \tfrac{\mathbf{F}_{\mathrm{con}}(\mathbf{u}^n)+\mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1})}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n} \Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\ = & \int_{t_n}^{t_{n+1}} \Big\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \tfrac{\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_n))+\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1}))}{2}, \dot{\mathbf{u}}(s) - \tfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau_n} \Big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\ & + \tfrac{1}{2}\big\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\big\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \end{aligned}$$

where the last term on the right-hand side is again non-positive. In analogy to Lemma 3.1, we find that

$$
\begin{aligned}
&E(t_n,\tau_n) - \tfrac{1}{2}\left\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\
&= -\int_{t_n}^{t_{n+1}} a\left(\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}, \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\right) ds \\
&\quad + \int_{t_n}^{t_{n+1}} \left\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \tfrac{\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_n))+\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1}))}{2}, \bar{\epsilon}(s,\dot{\mathbf{u}},\tau_n)\right\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} ds \\
&\quad - \int_{t_n}^{t_{n+1}} \left(1 - \tfrac{2(s-t_n)}{\tau_n}\right) \langle \ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1}), \bar{\epsilon}(t_n,\dot{\mathbf{u}},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, ds\,.
\end{aligned}
$$

Applying again the inequalities of Korn and Young, we find the corresponding result to Lemma 3.2, namely

$$
\begin{aligned}
E(t_n,\tau_n)^{1/2} \le C &\left[\left(\int_{t_n}^{t_{n+1}} \|\mathbf{u}(s) - \tfrac{\mathbf{u}(t_n)+\mathbf{u}(t_{n+1})}{2}\|^2_{\mathbf{H}^1}\, ds \right)^{1/2} \right.\\
&+ \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n)\|^2_{\mathbf{H}^1}\, ds \right)^{1/2} + \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \tfrac{\dot{\mathbf{u}}(t_n)+\dot{\mathbf{u}}(t_{n+1})}{2}\|^2_{\mathbf{H}^1}\, ds \right)^{1/2} \\
&+ \left(\int_{t_n}^{t_{n+1}} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|^2_{(\mathbf{H}^1)^*}\, ds \right)^{1/2} \\
&\left.+ \left(\int_{t_n}^{t_{n+1}} \|\mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \tfrac{\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_n))+\mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1}))}{2}\|^2_{(\mathbf{H}^1)^*}\, ds \right)^{1/2} \right].
\end{aligned}
$$

Introducing the total variations as in the proof of Theorem 3.4 yields the estimate of the theorem. □

At first glance, it looks as if the results above were no progress beyond the mere boundedness condition of the classical consistency theory, see (21). In fact, the total variations on the right-hand side of our estimate do not contribute to any additional order in τ_n. However, due to the telescoping property (22), the terms $R(\mathbf{u},[t_n,t_{n+1}])$ on the right-hand side sum up to total variations over the whole time interval $[0,T]$. Hence, we expect to be able to show global convergence without loosing the order τ as in the classical theory. A precise proof of this expectation would require a discrete analog of our continuous perturbation result [10] in the physical energy norm. It is therefore postponed to a forthcoming paper dedicated to the issue of *convergence* of the contact–stabilized Newmark method.

If one aims at a higher consistency order, a weaker norm will be needed. In the next section, we will present such a norm, which, however as it turns out, will be a discrete norm (depending on τ_n).

4 Consistency error in a discrete displacement norm

In addition to the previous results concerning the local error of Newmark schemes in the physical energy norm, we here want to give a consistency result in a discrete norm containing the displacements only.

Notation. We denote the *discrete displacement norm* by

$$\begin{aligned}\|\epsilon(t_n,\mathbf{u},\tau_n)\|^2_{\tau_n} := \|\epsilon(t_n,\mathbf{u},\tau_n)\|^2_{\mathbf{L}^2} + \tfrac{\tau_n^2}{4} a(\epsilon(t_n,\mathbf{u},\tau_n),\epsilon(t_n,\mathbf{u},\tau_n)) \\ + \tfrac{\tau_n}{2} b(\epsilon(t_n,\mathbf{u},\tau_n),\epsilon(t_n,\mathbf{u},\tau_n))\,.\end{aligned} \tag{27}$$

The proofs to follow are less complicated than the ones in Section 3, but based on similar principles. Again the total variation of the continuous solution will show up. We start with a result equivalent to the one of Lemma 3.1 which gives a representation of the error in the considered norm.

Lemma 4.1. *Assume that* $\dot{\mathbf{u}} \in \mathbf{H}^1$ *and* $\ddot{\mathbf{u}}, \mathbf{F}_{\text{con}}(\mathbf{u}) \in (\mathbf{H}^1)^*$ *for all* $t \in [0,T]$. *Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$\begin{aligned}&\|\epsilon(t_n,\mathbf{u},\tau_n)\|^2_{\tau_n} - \tfrac{\tau_n^2}{2}\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\ &= \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \langle \ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, dr \right) ds \\ &\quad + \tfrac{\tau_n^2}{4} a(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)) \\ &\quad + \tfrac{\tau_n}{2} \int_{t_n}^{t_{n+1}} b(\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n))\, ds\,.\end{aligned} \tag{28}$$

Proof. Due to the result of Proposition 2.1, we can write the consistency error of the positions in the $\mathbf{L}^2$-norm as

$$\begin{aligned}&\|\mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\|^2_{\mathbf{L}^2} \\ &= (\mathbf{u}(t_{n+1}) - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n, \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1})_{\mathbf{L}^2} + \tfrac{\tau_n^2}{2} a\left(\tfrac{\mathbf{u}^n + \mathbf{u}^{n+1}}{2}, \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right) \\ &\quad + \tfrac{\tau_n^2}{2} b\left(\tfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau_n}, \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right) - \tfrac{\tau_n^2}{2}\langle \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\ &= (\mathbf{u}(t_{n+1}) - \mathbf{u}^n - \tau_n\dot{\mathbf{u}}^n, \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1})_{\mathbf{L}^2} \\ &\quad - \tfrac{\tau_n^2}{2}\langle \ddot{\mathbf{u}}(t_{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\ &\quad + \tfrac{\tau_n^2}{2} a\left(\tfrac{\mathbf{u}^n + \mathbf{u}^{n+1}}{2} - \mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right) \\ &\quad + \tfrac{\tau_n^2}{2} b\left(\tfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau_n} - \dot{\mathbf{u}}(t_{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\right) \\ &\quad + \tfrac{\tau_n^2}{2}\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
&\|\epsilon(t_n,\mathbf{u},\tau_n)\|^2_{\mathbf{L}^2} + \tfrac{\tau_n^2}{4} a(\epsilon(t_n,\mathbf{u},\tau_n),\epsilon(t_n,\mathbf{u},\tau_n)) + \tfrac{\tau_n}{2} b(\epsilon(t_n,\mathbf{u},\tau_n),\epsilon(t_n,\mathbf{u},\tau_n)) \\
&-\tfrac{\tau_n^2}{2}\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\
&= (\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n) - \tau_n \dot{\mathbf{u}}(t_n), \epsilon(t_n,\mathbf{u},\tau_n))_{\mathbf{L}^2} \\
&\quad - \tfrac{\tau_n^2}{2}\langle \ddot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\
&\quad + \tfrac{\tau_n^2}{4} a(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)) \\
&\quad + \tfrac{\tau_n^2}{2} b\left(\tfrac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\tau_n} - \dot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n))\right) .
\end{aligned}
$$

Since we look for a solution $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$, we can use integration by parts (see, e.g., Prop. 23.23 in [16]) to write

$$
\begin{aligned}
&\|\epsilon(t_n,\mathbf{u},\tau_n)\|^2_{\tau_n} - \tfrac{\tau_n^2}{2}\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \\
&= \int_{t_n}^{t_{n+1}} (\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_n), \epsilon(t_n,\mathbf{u},\tau_n))_{\mathbf{L}^2}\, ds \\
&\quad - \int_{t_n}^{t_{n+1}} (s-t_n)\langle \ddot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, ds \\
&\quad + \tfrac{\tau_n^2}{4} a(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)) + \tfrac{\tau_n}{2}\int_{t_n}^{t_{n+1}} b(\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n))\, ds \\
&= \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \langle \ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1}\, dr \right) ds \\
&\quad + \tfrac{\tau_n^2}{4} a(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n)) + \tfrac{\tau_n}{2}\int_{t_n}^{t_{n+1}} b(\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1}), \epsilon(t_n,\mathbf{u},\tau_n))\, ds\,.
\end{aligned}
$$

□

We notice again that due to

$$
\frac{\tau_n^2}{2}\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\mathrm{con}}(\mathbf{u}^{n+1}), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^*\times\mathbf{H}^1} \le 0\,, \tag{29}
$$

we can omit this term on the left-hand side of (28). In order to find a suitable estimate for the right-hand side of this error representation, we use the same techniques as those in the proof of Lemma 3.2.

Lemma 4.2. *Assume that* $\dot{\mathbf{u}} \in \mathbf{H}^1$ *and* $\ddot{\mathbf{u}}, \mathbf{F}_{\mathrm{con}}(\mathbf{u}) \in (\mathbf{H}^1)^*$ *for all* $t \in [0,T]$. *Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$\begin{aligned} \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} = \mathcal{O}(\tau_n^{-1/2}) \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \|\ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1})\|_{(\mathbf{H}^1)^*}\, dr \right) ds \\ + \mathcal{O}(\tau_n^{3/2}) \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1} \\ + \mathcal{O}(\tau_n^{1/2}) \int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1})\|_{\mathbf{H}^1}\, ds\,. \end{aligned} \tag{30}$$

Proof. Omitting 29 and applying the continuity of the bilinear forms a and b in $\mathbf{H}^1$ to (28), we find the estimate

$$\begin{aligned} \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n}^2 \le \left(\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \|\ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1})\|_{(\mathbf{H}^1)^*}\, dr \right) ds \right) \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1} \\ + \mathcal{O}(\tau_n^2) \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1} \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1} \\ + \mathcal{O}(\tau_n) \left(\int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1})\|_{\mathbf{H}^1}\, ds \right) \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1}\,. \end{aligned}$$

After inserting the inequality of Korn in the form

$$\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1}^2 \le C \left(\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{L}^2}^2 + b(\epsilon(t_n, \mathbf{u}, \tau_n), \epsilon(t_n, \mathbf{u}, \tau_n)) \right)\,,$$

we can divide the whole inequality by the square root of the left-hand side. This yields the result of the theorem. □

In the presence of contact, we make the same assumption on the solution of (1) as in Section 3. This leads again to an estimate for the consistency error which uses the total variation of the continuous solution.

Theorem 4.3. *Let Assumption* 3.3 *hold. Then, for initial values on the exact solution, the local errors of (N-M) and (N-CS) satisfy*

$$\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} = \mathcal{O}(\tau_n^{3/2}) \cdot R(\mathbf{u}, [t_n, t_{n+1}])\,. \tag{31}$$

Proof. Due to our regularity assumption, we can estimate expression (30) as follows:

$$\begin{aligned} \|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} &= \mathcal{O}(\tau_n^{3/2}) \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\ddot{\mathbf{u}}(s) - \ddot{\mathbf{u}}(t_{n+1})\|_{(\mathbf{H}^1)^*} \\ &\quad + \mathcal{O}(\tau_n^{3/2}) \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1} \\ &\quad + \mathcal{O}(\tau_n^{3/2}) \sup\nolimits_{s\in[t_n,t_{n+1}]} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1})\|_{\mathbf{H}^1} \\ &= \mathcal{O}(\tau_n^{3/2}) \left(\mathrm{Var}(\ddot{\mathbf{u}}, [t_n, t_{n+1}], (\mathbf{H}^1)^*) + \mathrm{Var}(\mathbf{u}, [t_n, t_{n+1}], \mathbf{H}^1) \right. \\ &\qquad \left. + \mathrm{Var}(\dot{\mathbf{u}}, [t_n, t_{n+1}], \mathbf{H}^1) \right)\,. \end{aligned}$$

□

We prove again a corresponding consistency result for the classical Newmark scheme, as done in Theorem 3.5 for the physical energy norm.

Theorem 4.4. *Let Assumption* 3.3 *hold. Then, for initial values on the exact solution, the local error of (N-CL) satisfies*

$$\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} = \mathcal{O}(\tau_n^{3/2}) \cdot R(\mathbf{u}, [t_n, t_{n+1}])\,. \tag{32}$$

Proof. Following the proof of Lemma 4.1 above, we have to replace $\mathbf{F}_{\text{con}}(\mathbf{u}^{n+1})$ by $\frac{1}{2}\left(\mathbf{F}_{\text{con}}(\mathbf{u}^n) + \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1})\right)$ which yields

$$\begin{aligned}
&\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} - \tfrac{\tau_n^2}{4}\langle \mathbf{F}_{\text{con}}(\mathbf{u}^{n+1}) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})), \mathbf{u}^{n+1} - \mathbf{u}(t_{n+1})\rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1} \\
&= -\int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \langle \ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1}), \epsilon(t_n, \mathbf{u}, \tau_n)\rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1}\, dr \right) ds \\
&\quad + \tfrac{\tau_n^2}{4} a(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \epsilon(t_n, \mathbf{u}, \tau_n)) + \tfrac{\tau_n}{2} \int_{t_n}^{t_{n+1}} b(\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1}), \epsilon(t_n, \mathbf{u}, \tau_n))\, ds \\
&\quad + \tfrac{\tau_n^2}{4}\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1})) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_n)), \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1}\rangle_{(\mathbf{H}^1)^* \times \mathbf{H}^1}\,.
\end{aligned}$$

Compared to Lemma 4.2 we have obtained an additional term including contact forces on the right-hand side. Following the proof of Lemma 4.2 we end up with

$$\begin{aligned}
\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\tau_n} &= \mathcal{O}(\tau_n^{-1/2}) \int_{t_n}^{t_{n+1}} \left(\int_{t_n}^{s} \|\ddot{\mathbf{u}}(r) - \ddot{\mathbf{u}}(t_{n+1})\|_{(\mathbf{H}^1)^*}\, dr \right) ds \\
&+ \mathcal{O}(\tau_n^{3/2}) \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_{\mathbf{H}^1} + \mathcal{O}(\tau_n^{1/2}) \int_{t_n}^{t_{n+1}} \|\dot{\mathbf{u}}(s) - \dot{\mathbf{u}}(t_{n+1})\|_{\mathbf{H}^1}\, ds \\
&+ \mathcal{O}(\tau_n^{3/2}) \|\mathbf{F}_{\text{con}}(\mathbf{u}(t_n)) - \mathbf{F}_{\text{con}}(\mathbf{u}(t_{n+1}))\|_{(\mathbf{H}^1)^*}\,.
\end{aligned}$$

Introducing the total variation as in the proof of Theorem 4.3 yields the result of the theorem. ☐

Finally, we want to improve our estimate for the displacements measured in the a-norm by means of the inequality of Korn.

Corollary 4.5. *Let Assumption* 3.3 *hold. Then, for initial values on the exact solution, the local errors of (N-CL), (N-M), and (N-CS) satisfy*

$$a(\epsilon(t_n, \mathbf{u}, \tau_n), \epsilon(t_n, \mathbf{u}, \tau_n))^{1/2} = \mathcal{O}(\tau_n) \cdot R(\mathbf{u}, [t_n, t_{n+1}])\,. \tag{33}$$

Proof. The inequality of Korn yields

$$\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1} \le C \left(\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{L}^2}^2 + b(\epsilon(t_n, \mathbf{u}, \tau_n), \epsilon(t_n, \mathbf{u}, \tau_n)) \right)^{1/2}\,.$$

Hence, we find, due to Theorem 4.3,

$$\|\epsilon(t_n, \mathbf{u}, \tau_n)\|_{\mathbf{H}^1} = \mathcal{O}(\tau_n) \cdot R(\mathbf{u}, [t_n, t_{n+1}])\,.$$

The continuity of the bilinear form a in $\mathbf{H}^1$ gives the result of the corollary. ☐

5 Conclusion

This paper has worked out a consistency theory for three types of Newmark discretizations — the classical, the modified, and the contact–stabilized one. Consistency error estimates have been given in a physical energy norm (including the viscoelastic part) and a discrete displacement norm. Both estimates require the solution together with its first and second derivative, in the presence of contact, to be in the function space BV. With these results, the basis is laid, on one hand, for a theoretical convergence theory and, on the other hand, for the derivation of an automatic step-size control of the contact–stabilized Newmark scheme.

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