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Cardinality Constrained Combinatorial Optimization: Complexity and Polyhedra

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Abstract

Given a combinatorial optimization problem and a subset N of natural numbers, we obtain a cardinality constrained version of this problem by permitting only those feasible solutions whose cardinalities are elements of N . In this paper we briefly touch on questions that addresses common grounds and differences of the complexity of a combinatorial optimization problem and its cardinality constrained version. Afterwards we focus on polytopes associated with cardinality constrained combinatorial optimization problems. Given an integer programming formulation for a combinatorial optimization problem, by essentially adding Grötschel's cardinality forcing inequalities [11], we obtain an integer programming formulation for its cardinality restricted version. Since the cardinality forcing inequalities in their original form are mostly not facet defining for the associated polyhedra, we discuss possibilities to strengthen them. In [13] a variation of the cardinality forcing inequalities were successfully integrated in the system of linear inequalities for the matroid polytope to provide a complete linear description of the cardinality constrained matroid polytope. We identify this polytope as a master polytope for our class of problems, since many combinatorial optimization problems can be formulated over the intersection of matroids.

1 Introduction, Basics, and Complexity

Given a combinatorial optimization problem and a subset N of natural numbers, we obtain a cardinality constrained version of this problem by permitting only those feasible solutions whose cardinalities are elements of N . Well-known examples of cardinality constrained combinatorial optimization problems are the traveling salesman problem and the minimum odd cycle problem. Both problems are for itself combinatorial optimization problems, but in the line of sight of the minimum cycle problem, they are cardinality restricted versions of the latter problem. More formally, let E be a finite set, \mathcal{I} a subset of the power set 2^E of E , and $w : E \rightarrow \mathbb{R}$, $e \mapsto w(e)$ a weight function. For any $F \subseteq E$ and any $y \in \mathbb{R}^E$, we set $y(F) := \sum_{e \in F} y_e$. The mathematical program

$$\max\{w(F) : F \in \mathcal{I}\}$$

is called a *combinatorial optimization problem* (COP). We also refer to it as the triple $\Pi = (E, \mathcal{I}, w)$. Elements of \mathcal{I} are called *feasible solutions*. This definition entails that no element of E is counted more than one time in a feasible solution. By setting cardinality constraints on the set of feasible solutions,

the COP becomes a *cardinality constrained combinatorial optimization problem* (CCCOP). Formally, for any finite set $N \in \mathbb{N}$, the cardinality constrained version Π_N of Π is the mathematical program

$$\max\{w(F) : F \in \mathcal{I}, |F| \in N\},$$

also denoted by $\Pi_N = (E, \mathcal{I}, w, N)$. Π_N is, considered for itself, again a COP. If $N = \{k\}$ for some k , we speak of a k -COP and write Π_k instead of $\Pi_{\{k\}}$. An overview on k -COP's is given by Bruglieri et al. [4].

As usual we apprehend a combinatorial optimization problem also as the collection of all its problem instances. But we do not distinguish between the problem Π^* and its instances $\Pi = (E, \mathcal{I}, w)$. Moreover, if we say that an algorithm A solves the COP $\Pi = (E, \mathcal{I}, w)$ in polynomial time, then we mean, strictly speaking, that A solves all problem instances $\Pi \in \Pi^*$ in polynomial time.

If k is fixed, then any k -COP Π_k can be solved in polynomial time by enumeration on all $\binom{n}{k}$ subsets I of E of cardinality k . Consequently, if N is fixed, then Π_N can be solved in polynomial time by enumeration.

Several polynomial time solvable COP's of the form

$$\min\{w(F) | F \in \mathcal{I}\}$$

with a nonnegative weight function $w : E \rightarrow \mathbb{R}_+$ become NP-hard if one imposes cardinality constraints (cf. Bruglieri et al. [4]). For example, one can find in polynomial time a shortest simple cycle, that is, a simple cycle of minimum weight, if the weight function is nonnegative, but the TSP, which arises by taking k as the number of nodes, is NP-hard. Or, in order to give another example, the min-cut problem can be solved in polynomial time for a nonnegative weight function, but most of the cardinality constrained cut problems are NP-hard (e.g. the equicut problem).

We believe, however, that less the cardinality restriction itself is responsible for the NP-hardness of some k -COP, but rather the fact that the original problem is also NP-hard as soon as one admits negative weights, or, in order to formulate it more tentatively, both ingredients "arbitrary weights" and "cardinality restriction" for a COP seem to be equivalent in many cases with respect to the complexity of a COP. One argument for this hypothesis is that the restriction to nonnegative weights is irrelevant for k -COP's: For any $M \in \mathbb{R}$ and any two feasible solutions F_1, F_2 of a k -COP Π_k we have $w(F_1) \leq w(F_2)$ if and only if $w'(F_1) \leq w'(F_2)$, where $w'_e := w_e + M$. By taking M large enough the weights w'_e are nonnegative, and for appropriate $M \geq 0$ the weights w'_e are nonnegative. That means, Π_k is invariant under shifting of the weights by a constant.

We would like to enrich our hypothesis by two examples. The above mentioned polynomial-time solvable shortest cycle problem becomes NP-hard if arbitrary weights are admitted or the set of feasible solutions is limited to Hamiltonian cycles, see Garey and Johnson [9]. Denoting by CYCLE the shortest cycle problem, the latter fact implies that CYCLE_k is NP-hard for arbitrary k . For instance, let $G = (V, E)$ with $n := |V|$ be an instance of the TSP. Adding to G a set V' of n isolated nodes, we obtain a graph $G' = (V \cup V', E)$ of order $m := 2n$. Every Hamiltonian cycle in G is obviously a cycle of cardinality $\lfloor m/2 \rfloor$ in G' , and vice versa. Thus, the TSP can be polynomially reduced to the problem of

finding in a graph on n nodes a shortest cycle of cardinality $\lfloor n/2 \rfloor$. This implies that $\text{CYCLE}_{\lfloor n/2 \rfloor}$ is NP-hard. Even more, if CYCLE_k is NP-hard for arbitrary k , then it is very unlikely to find a polynomial time algorithm that solves the general cardinality constrained shortest cycle problem CYCLE_N for arbitrary N .

Another example is the optimization problem over an independence system of a matroid. We will give a short introduction to independence systems and matroids, because they will play an important role in the next sections. Recall that $\mathcal{I} \subseteq 2^E$ is called an *independence system* if (i) $\emptyset \in \mathcal{I}$ and (ii) $I \in \mathcal{I}$ implies $J \in \mathcal{I}$ for each $J \subseteq I$. The subsets of E belonging to \mathcal{I} are called *independent*, otherwise *dependent*. The minimal dependent sets are called *circuits* of \mathcal{I} and the set \mathcal{C} of all circuits is called the *circuit system* of \mathcal{I} . For any $F \subseteq E$, $B \subseteq F$ is called a *basis* of F if $B \in \mathcal{I}$ and $B \cup \{e\} \notin \mathcal{I}$ for all $e \in F \setminus B$. The *rank* and the *lower rank* of any set $F \subseteq E$ is defined by $r(F) := \max\{|B| : B \text{ basis of } F\}$ and $r_u(F) := \min\{|B| : B \text{ basis of } F\}$, respectively. The independence system \mathcal{I} is called a *matroid* if (iii) for each $F \subseteq E$ its bases have the same cardinality, and consequently $r_u(F) = r(F)$ for all $F \subseteq E$. In order to indicate that \mathcal{I} is a matroid, we will write $M = (E, \mathcal{I})$.

Let IND be the optimization problem over an independence system of a matroid given by an independence oracle. This problem can be solved in polynomial time with the greedy algorithm for arbitrary weights. In accordance with our hypothesis, also the cardinality constrained version of this problem IND_N can be solved in polynomial time for any N , see [13].

To our knowledge there is no polynomial time solvable combinatorial optimization problem discussed in the literature for which the cardinality constrained version is NP-hard (supposed that arbitrary weights are admitted). However, classes of polynomial time solvable combinatorial optimization problems that are completely irrelevant but whose cardinality constrained versions are NP-hard, can be constructed quite easily. For instance, let $G = (V, E, w)$ be a weighted graph on $n = |V|$ nodes and consider the *embedded traveling salesman problem* (ETSP) defined as follows:

$$\min\{w(T) : T \subseteq E, \text{ if } |T| = n, \text{ then } T \text{ is a tour}\}.$$

It can be obviously solved in polynomial time. To this end, let $T^* := \{e \in E : w_e < 0\}$. If $|T^*| \neq n$ or $|T^*| = n$ and T^* is a Hamiltonian cycle, then T^* is optimal. Otherwise, that is, in the case $|T^*| = n$ and T^* is not a Hamiltonian cycle, let $e^- \in T^*$, $e^+ \in E \setminus T^*$ (if $E \setminus T^* \neq \emptyset$) be edges of maximal and minimal weight, respectively. By construction, $w(e^-) < 0$ and $w(e^+) \geq 0$. Now it follows immediately that $w(F) \geq \min\{w(T^* \setminus \{e^-\}), w(T^* \cup \{e^+\})\}$ for all $F \subseteq E$ with $|F| \neq n$. Moreover, for any Hamiltonian cycle T we have $w(T) \geq w(T^* \setminus \{e^-\})$. Hence, $T^* \setminus \{e^-\}$ or $T^* \cup \{e^+\}$ is the optimal solution. So, the ETSP can indeed be solved in polynomial time. However, the cardinality constrained version ETSP_n of ETSP is the TSP which is known to be NP-hard. Of course, such a construction – namely the embedding of a NP-hard combinatorial optimization problem into a trivial setting – can be done not only for the TSP but also for other NP-hard COP's as the linear ordering problem.

The previous paragraph shows that, in general, we are not able to extrapolate from the polynomial time solvability of a COP to the polynomial time solvability of its cardinality constrained version. We can only give a much weaker result.

Let $\Pi = (E, \mathcal{I}, w)$ be a COP, let $n_{\max} = \max\{|I| : I \in \mathcal{I}\}$, and $n_{\min} = \min\{|I| : I \in \mathcal{I}\}$. The *maximum cardinality COP* Π_{\max} is the optimization problem

$$\max\{w(I) : I \in \mathcal{I}, |I| = n_{\max}\}.$$

Similarly, the *minimum cardinality COP* Π_{\min} is the optimization problem

$$\max\{w(I) : I \in \mathcal{I}, |I| = n_{\min}\}.$$

Theorem 1.1. *If a COP $\Pi = (E, \mathcal{I}, w)$ can be solved in polynomial time for all weightings $w : E \rightarrow \mathbb{R}$, the same holds for Π_{\max} and Π_{\min} . Hence, if Π_{\min} or Π_{\max} is NP-hard, then Π is too.*

Proof. The optimal solutions of Π_{\max} and Π_{\min} are invariant under shifting of the weights by a constant. For any instance $\Pi = (E, \mathcal{I}, w)$, set $M := |E| \cdot W + 1$, where $W = \max\{|w_e| : e \in E\}$. Then, an optimal solution X^1 of Π with weights $w_e^1 := w_e + M$ is of maximum cardinality and an optimal solution X^2 of Π with weights $w_e^2 := w_e - M$ is of minimum cardinality. In particular, X^1 and X^2 are optimal solutions for Π_{\max} and Π_{\min} , respectively. Since the transformations are polynomial, the claim follows. \square

Perhaps better results are obtainable if one excludes such artificial COP's as the ETSP. This is maybe done by adding requirements on the homogeneity of the feasible solutions. For instance, one could require that all feasible solutions of Π have some common property P independent of the cardinality of the solutions. The ETSP could be easily excluded by adding the constraint that each feasible solution has to be a cycle.

The remainder of this paper is organized as follows. In Section 2 we present a general framework for the polyhedral investigation of CCCOP's. The central subject of this investigation are Grötschel's cardinality forcing inequalities [11]. These inequalities together with the constraint $\min N \leq x(E) \leq \max N$ cut off solutions that are feasible for Π , but forbidden for Π_N . This results into an integer programming formulation for Π_N provided we have one for Π . Since the cardinality forcing inequalities in their original form are mostly not facet defining for the associated polyhedra, we discuss in Section 3 some possibilities to strengthen them. It turns out that often matroidal relaxations of a COP Π help to provide facet defining integer formulations of Π_N , if there are any. This is based on the consideration that on the one hand many combinatorial optimization problems can be described over the intersection of matroids, and on the other hand we have a complete linear and tractable description of the cardinality constrained matroid polytope, see [13]. In Section 4, we briefly touch the problem to derive a complete linear description of the polyhedra associated with Π_N provided we know a complete linear description of the polyhedron associated with Π .

2 Polyhedral Investigation of CCCOP's

The fundamental class of inequalities of this and the following section are Grötschel's cardinality forcing inequalities [11]. Given a finite set E , the sequence $c = (c_1, \dots, c_m)$ of integers with $0 \leq c_1 < c_2 < \dots < c_m \leq |E|$ is called a *cardinality sequence*, and the set $\text{CHS}^c(E) := \{F \subseteq E : |F| = c_p \text{ for some } p\}$

is called a *cardinality homogenous set system*. The polytope associated with $\text{CHS}^c(E)$, that is, the convex hull of the incidence vectors of $I \in \text{CHS}^c(E)$, is completely described by the *trivial inequalities* $0 \leq x_e \leq 1$, $e \in E$, the *cardinality bounds* $c_1 \leq x(E) \leq c_m$, and the *cardinality forcing inequalities*

$$(c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) \leq c_p(c_{p+1} - |F|) \quad (1)$$

for all $F \subseteq E$ with $c_p < |F| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$,

see Grötschel [11]. The cardinality bounds exclude all subsets of E whose cardinalities are out of the bounds c_1 and c_m , while the cardinality forcing inequalities do this for all subsets of E with forbidden cardinality between the two bounds. To see this, let $F \in E$ with $c_p < |F| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then the cardinality forcing inequality associated with F is violated by the incidence vector χ^F of F :

$$(c_{p+1} - |F|)\chi^F(F) - (|F| - c_p)\overbrace{\chi^F(E \setminus F)}^{=0} = |F|(c_{p+1} - |F|) > c_p(c_{p+1} - |F|).$$

However, every $I \in \text{CHS}^c(E)$ satisfies the inequality associated with F . If $|I| \leq c_p$, then

$$(c_{p+1} - |F|)\chi^I(F) - \overbrace{(|F| - c_p)\chi^I(E \setminus F)}^{>0} \leq \overbrace{(c_{p+1} - |F|)\chi^I(I \cap F)}^{\geq 0} \leq c_p(c_{p+1} - |F|),$$

and equality holds if $|I| = c_p$ and $I \subseteq F$. If $|I| \geq c_{p+1}$, then

$$(c_{p+1} - |F|)\chi^I(F) - (|F| - c_p)\chi^I(E \setminus F) \leq (c_{p+1} - |F|)|F| - (|F| - c_p)(c_{p+1} - |F|) = c_p(c_{p+1} - |F|),$$

and equality holds if $|I| = c_{p+1}$ and $I \cap F = F$.

Although the class of cardinality forcing inequalities consists of exponentially many members, Grötschel [11] showed that the associated separation problem is solvable in polynomial time by the greedy algorithm. Let $x^* \in \mathbb{R}^E$ be any nonnegative vector. Sort the components of x^* such that $x_{e_1}^* \geq x_{e_2}^* \geq \dots \geq x_{e_{|E|}}^*$. Then, for each integer q with $c_p < q < c_{p+1}$, x^* satisfies the cardinality forcing inequality associated with $F' := \{e_1, \dots, e_q\}$ if and only if x^* satisfies all cardinality forcing inequalities associated with sets $F \subseteq E$ of cardinality q , see Grötschel [11].

For any COP Π and any cardinality sequence $c = (c_1, \dots, c_m)$, we denote Π_N , where $N = \{c_1, \dots, c_m\}$ also by Π_c . To each integer (linear) programming formulation in binary variables for a COP we can add the cardinality bounds and cardinality forcing inequalities to derive integer programming formulations for the cardinality constrained versions of these optimization problems.

Theorem 2.1. *Let*

$$\begin{aligned} \max \quad & w^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x_e \in \{0, 1\} \quad \text{for all } e \in E \end{aligned} \quad (2)$$

be an integer programming formulation for a COP $\Pi = (E, \mathcal{I}, w)$ and $c = (c_1, \dots, c_m)$ a cardinality sequence. Then, system (2) together with the cardinality bounds $c_1 \leq x(E) \leq c_m$ and the cardinality forcing inequalities (1) provide an integer programming formulation for Π_c . \square

Theorem 2.1 remains true even if Π incorporates cardinality restrictions a priori as for perfect matchings, minimal spanning trees, or the TSP. Of course, in this case the approach is nonsense.

Adding the cardinality forcing inequalities (1), however, does not necessarily result in facet defining inequalities of the associated polytopes. For instance, consider cardinality constrained matroids. The linear program

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & x(F) \leq r(F) \quad \text{for all } \emptyset \neq F \subseteq E, \\ & x_e \geq 0 \quad \text{for all } e \in E \end{aligned} \quad (3)$$

is a well-known formulation for finding a maximum weight independent set in a matroid $M = (E, \mathcal{I})$, see Edmonds [7]. Recall that for any $F \subseteq E$, $r(F)$ denotes the *rank* of F , that is, the maximum size of an independent set $I \subseteq F$. Given a cardinality sequence $c = (c_1, \dots, c_m)$ with $0 \leq c_1 < \dots < c_m \leq |E|$, the cardinality restricted version of this problem can be formulated as follows:

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & x(F) \leq r(F) \quad \text{for all } \emptyset \neq F \subseteq E, \quad (4) \\ & (c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) \leq c_p(c_{p+1} - |F|) \quad \text{for all } F \subseteq E \quad (5) \\ & \quad \text{with } c_p < |F| < c_{p+1} \text{ for some } p, \\ & x(E) \geq c_1, \quad (6) \\ & x(E) \leq c_m, \quad (7) \\ & x_e \in \{0, 1\} \quad \text{for all } e \in E. \quad (8) \end{aligned}$$

Clearly, the integer points of the associated *cardinality constrained matroid polytope*

$$P_M^c(E) := \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I} \cap \text{CHS}^c(E)\}$$

are described by (4)-(8). However, the above IP-formulation for finding a maximum weight independent set $I \in \mathcal{I} \cap \text{CHS}^c(E)$ is quite weak, since in general none of the cardinality forcing inequalities is facet defining for the cardinality constrained matroid polytope.

To give another example, in [12] it was shown that the inequalities

$$\begin{aligned} (c_{p+1} - |F|)x(F) - (|F| - c_p)x(A \setminus F) &\leq c_p(c_{p+1} - |F|) \\ \text{for all } F \subseteq A \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, \dots, m-1\}, \end{aligned}$$

induce very low dimensional faces of the *cardinality constrained cycle polytope*,

$$P_C^c(D) := \text{conv}\{\chi^C \in \mathbb{R}^A : C \text{ simple directed cycle, } C \in \text{CHS}^c(A)\},$$

defined on a directed graph $D = (V, A)$.

3 Three Recommendations

In this section we will give three recommendations how to derive stronger inequalities than inequalities (1) to cut off solutions of forbidden cardinality.

3.1 Rank Induced Cardinality Forcing Inequalities

The reason, why the cardinality forcing inequalities in their natural form are quite weak, is quickly found when analyzing cardinality constrained matroids. By Edmonds [7], a rank inequality $x(F) \leq r(F)$ is facet defining for the matroid polytope if and only if F is closed and inseparable. Recall that any $F \subseteq E$ is said to be *closed* if $r(F \cup \{e\}) > r(F)$ for all $e \in E \setminus F$. It is called *inseparable* if $r(F_1) + r(F_2) > r(F)$ for all nonempty partitions $F = F_1 \dot{\cup} F_2$ of F .

When we renounce of these properties and, in addition, substitute the right hand side of the inequality by $|F|$, then we obtain a valid inequality, and that is already all. So, the first and most important reason, why the rank inequality associated with a closed and inseparable subset F of E is facet defining, arises from the fact that the bound $r(F)$ is tighter than $|F|$ (unless $r(F) = |F|$). The second reason is connected to the exposed position of F among subsets F' of E with the same rank as F .

The first observation ($r(F)$ instead of $|F|$) can be immediately incorporated into cardinality forcing inequalities:

$$(c_{p+1} - r(F))x(F) - (r(F) - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r(F))$$

for all $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ for some $p \in \{0, \dots, m-1\}$.

From the second observation (F closed and inseparable) we can easily adapt the closeness, since if F and $F' := F \cup \{e\}$ for some $e \in E \setminus F$ have the same rank k , where $c_p < k < c_{p+1}$, then the rank induced cardinality forcing inequality associated with F is the sum of the rank induced cardinality forcing inequality associated with F' and the inequality $-(c_{p+1} - c_p)x_e \leq 0$, which is a multiple of the nonnegativity constraint $-x_e \leq 0$. In contrast, the separability seems not to fit into the framework of cardinality constrained matroids.

Theorem 3.1 ([13]). *Let $F \subseteq E$ such that $0 < c_p < r(F) < c_{p+1} < r(E)$ for some $p \in \{1, \dots, m-1\}$. Then, the cardinality forcing inequality $CF_F(x) \leq c_p(c_{p+1} - r(F))$ defines a facet of $P_M^c(E)$ if and only if F is closed.* \square

The statement is obviously independent of the fact whether F is separable or not.

The observations in the previous paragraphs yield the first of three recommendations made in this section to find valid inequalities that are specific to cardinality restrictions. Let $c = (c_1, \dots, c_m)$ be a cardinality sequence, $\Pi = (E, \mathcal{I}, w)$ a COP, and Π_c its cardinality constrained version. Moreover, denote by $P_{\mathcal{I}}(E)$ and $P_{\mathcal{I}}^c(E)$ the polytope associated with Π and Π_c , respectively. In analogy to matroid theory, we define a rank function r by $r(F) := \max\{|I \cap F| : I \in \mathcal{I}\}$ for all $F \subseteq E$. Moreover, any subset F of E is called *closed* if $r(F \cup \{e\}) > r(F)$ for all $e \in E \setminus F$.

Recommendation 1. *Instead of investigating the original cardinality forcing inequalities (1), analyze the rank induced cardinality forcing inequalities*

$$(c_{p+1} - r(F))x(F) - (r(F) - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r(F)), \quad (9)$$

$F \subseteq E$ closed with $c_p < r(F) < c_{p+1}$ for some p .

Evidently, inequalities (9) are valid. Moreover, they are stronger than inequalities (1). To see this, let F and G be subsets of E such that $F \subseteq G$ and

$c_p < |F| = r(G) < c_{p+1}$ for some p . Then, the cardinality forcing inequality associated with F is the sum of the rank induced cardinality forcing inequality associated with G and the inequalities $-(c_{p+1} - c_p)x_e \leq 0$ for $e \in G \setminus F$. Moreover, it is not hard to see that the closeness of F is a necessary condition for inequality (9) to be facet defining. Namely, if F is not closed, that is, there exists $e \in E \setminus F$ such that $r(F \cup \{e\}) = r(F)$, then (9) is the sum of the inequalities $\text{CF}_{F \cup \{e\}}(x) \leq c_p(c_{p+1} - r(F \cup \{e\}))$ and $-(c_{p+1} - c_p)x_e \leq 0$. Indeed, our computational results with the convex hull codes **polymake** [10] and **PORTA** [5] as well as our theoretical results confirm that the cardinality forcing inequalities in the latter form frequently appear in the linear descriptions of many polyhedra associated with CCCOP's. Unfortunately, they are still not necessarily facet defining, in general not separable in polynomial time unless $P = NP$, and sometimes hard to identify.

3.2 Matroidal Relaxations

In case Π is the maximum independent set problem over a matroid, Π_c can be solved via the linear program (3) extended by the inequalities (9) and the cardinality bounds $c_1 \leq x(E) \leq c_m$. Moreover, inequalities (9) are separable in polynomial time, see [13].

The question, how we can benefit from the nice polyhedral structure of cardinality constrained matroids, leads to the second recommendation.

Recommendation 2. *Find a “good” combinatorial relaxation (or matroidal relaxation) $\Pi' = (E, \mathcal{J}, w)$ of the COP of consideration $\Pi = (E, \mathcal{I}, w)$, or even better, directly of $\Pi_c = (E, \mathcal{I}, w, c)$.*

Here, a COP $\Pi' = (E, \mathcal{J}, w)$ is called a *combinatorial relaxation (matroidal relaxation)* of $\Pi = (E, \mathcal{I}, w)$ if $\mathcal{J} \supseteq \mathcal{I}$ (and \mathcal{J} is a matroid). Of course, $\mathcal{J} \supseteq \mathcal{I}$ or $\mathcal{J} \supseteq (\mathcal{I} \cap \text{CHS}^c(E))$ implies that $(\mathcal{J} \cap \text{CHS}^c(E)) \supseteq (\mathcal{I} \cap \text{CHS}^c(E))$. Hence, valid inequalities for $P_{\mathcal{J}}^c(E)$ are also valid for $P_{\mathcal{I}}^c(E)$.

The hope behind Recommendation 2 is that “good” combinatorial relaxations yield strong inequalities for $P_{\mathcal{I}}^c(E)$. In particular, we are interested in rank induced cardinality forcing inequalities for $P_{\mathcal{J}}^c(E)$ that are facet defining for $P_{\mathcal{I}}^c(E)$. In the best case “good” means that, firstly, $P_{\mathcal{J}}^c(E)$ has a tractable facial description, and secondly, the facet defining inequalities for $P_{\mathcal{J}}^c(E)$ are also facet defining for $P_{\mathcal{I}}^c(E)$. If, for instance, \mathcal{J} is a matroid, then $P_{\mathcal{J}}^c(E)$ has a tractable facial structure, but this alone says nothing about the tightness of its facet defining inequalities for $P_{\mathcal{I}}^c(E)$.

It is quite obvious that the tightness of a combinatorial (matroidal) relaxation influences the strength of the associated inequalities with respect to $P_{\mathcal{I}}^c(E)$. For instance, an independence system \mathcal{I} defined on some ground set E is the intersection of finitely many matroids defined on the same set E : The circuit system \mathcal{C} associated to \mathcal{I} has only finitely many members. Each circuit $C \in \mathcal{C}$ can be used to define a matroid \mathcal{I}_C by setting $\mathcal{I}_C := \{I \subseteq E : C \not\subseteq I\}$. Then, $\mathcal{I} = \bigcap_{C \in \mathcal{C}} \mathcal{I}_C$. This is, however, usually not an efficient way to describe \mathcal{I} , since in general \mathcal{I} is the intersection of less matroids. From a polyhedral point of view a small description of \mathcal{I} by matroids should usually lead to strong inequalities for $P_{\mathcal{I}}^c(E)$.

In Recommendation 2 we suggest to find directly combinatorial (matroidal)

relaxations of Π_c instead of Π . Of course, the combinatorial relaxations of Π and Π_c are usually the same, but not necessarily. If $\mathcal{J} \supseteq \mathcal{I}$, then $\mathcal{J} \supseteq (\mathcal{I} \cap \text{CHS}^c(E))$. However, $\mathcal{K} \supseteq (\mathcal{I} \cap \text{CHS}^c(E))$ for some $\mathcal{K} \subseteq 2^E$ does not necessarily imply $\mathcal{K} \supseteq \mathcal{I}$. This fact also affects the facial structures of the polytopes associated with Π and Π_c . Consider again an artificial COP, for instance, the *embedded directed odd cycle problem* (EDOCP)

$$\min\{w(C) : C \subseteq E, \text{ if } |C| \geq 3 \text{ is odd, then } C \text{ is a simple directed cycle}\}$$

defined on a digraph $D = (V, A)$. The associated polytope, namely the *embedded directed odd cycle polytope* $P_{\text{EDOC}}(A)$, which is the convex hull of all incidence vectors of the feasible solutions of the EDOCP, is fulldimensional, since $\mathbf{0} \in P_{\text{EDOC}}(A)$ and $u_a \in P_{\text{EDOC}}(A)$ for all $a \in A$. Here, $\mathbf{0}$ denotes the zero vector, and u_a denotes the a th unit vector. Moreover, a trivial inequality $x_a \leq 1$ defines a facet of $P_{\text{EDOC}}(A)$, since the vectors u_a and $u_a + u_b$ for all $b \in A \setminus \{a\}$ belong to $P_{\text{EDOC}}(A)$, are linearly independent, and satisfy the inequality at equality. The trivial inequalities $x_a \leq 1$ for $a \in A$ can be interpreted as rank inequalities for the trivial matroid $\mathcal{I} = 2^A$. As is easily seen, the singletons $\{a\}$, where $a \in A$, are the closed and inseparable sets with respect to \mathcal{I} . Now, restricting the feasible solutions of the EDOC to odd cardinalities ≥ 3 , we obtain the so called *directed odd cycle problem* (DOCP). Of course, an inequality $x_{ij} \leq 1$, with $(i, j) = a$, is valid for the polytope associated with the DOCP, the so called *directed odd cycle polytope*

$$P_{\text{DOC}}(A) := \text{conv}\{\chi^C \in \mathbb{R}^A : C \text{ is a simple directed cycle with } |C| \text{ odd}\},$$

but now the inequality is the consequence of the valid inequalities $x(\delta^{\text{out}}(i)) \leq 1$ and $-x_{ik} \leq 0$ for all $k \in \delta^{\text{out}} \setminus \{j\}$, and hence, definitely not facet defining if $|\delta^{\text{out}}(i)| \geq 2$. Conversely, $x(\delta^{\text{out}}(i)) \leq 1$ is not valid for $P_{\text{EDOC}}(A)$ unless $|\delta^{\text{out}}(i)| = 1$.

We close the discussion with two examples for a favorable application of Recommendation 2.

Example A: Cardinality Constrained Matchings.

A *matching* of a graph $G = (V, E)$ is a set of mutually disjoint edges. Given any edge weights $w_e \in \mathbb{R}$, to find a maximum weight matching in G is one of the hardest combinatorial optimization problems solvable in polynomial time.

The problem of finding a maximum weight matching of cardinality $k \leq \lfloor |V|/2 \rfloor$ can be easily transformed into the perfect matching problem. Add $\ell := |V| - 2k$ new nodes u_1, \dots, u_ℓ and join each of them with every node $v \in V$ by a (zero-weight) edge. Denote the resulting graph by $G' = (V', E')$. Then, the restriction of any perfect matching M in G' to G is a matching of cardinality k , since the node set $\{u_1, \dots, u_\ell\}$ is a stable set. Consequently, for any cardinality sequence c , the associated cardinality constrained matching problem can be solved in polynomial time.

Let \mathcal{M} be the collection of all matchings of G . The *matching polytope* $P_{\text{MATCH}}(E)$ of $G = (V, E)$ is the convex hull of the incidence vectors of all matchings $M \in \mathcal{M}$. By Edmonds [6], the matching polytope is determined by

the following inequalities:

$$x_e \geq 0 \quad \text{for all } e \in E, \quad (10)$$

$$x(\delta(v)) \leq 1 \quad \text{for all } v \in V, \quad (11)$$

$$x(E(W)) \leq \lfloor \frac{1}{2}|W| \rfloor \quad \text{for all } W \subseteq V, |W| \text{ odd}. \quad (12)$$

The set of all matchings \mathcal{M} of G form an independence system. Inequalities (11) are the facet defining rank inequalities associated with the independence system \mathcal{M} of G . A natural translation of inequalities (11) to cardinality constraints are the inequalities

$$(2c_{p+1} - |W|) \sum_{v \in W} x(\delta(v)) - (|W| - 2c_p) \sum_{v \in V \setminus W} x(\delta(v)) \leq 2c_p(2c_{p+1} - |W|) \\ \text{for all } W \subseteq V \text{ with } 2c_p < |W| < 2c_{p+1}, p = 1, \dots, m, \quad (13)$$

which are easily seen to be valid for the *cardinality constrained matching polytope* $P_{\text{MATCH}}^c(E) := \text{conv}\{\chi^M \in \mathbb{R}^E : M \in \mathcal{M} \cap \text{CHS}^c(E)\}$. It is not hard to see that inequalities (13) do not belong to rank induced cardinality forcing inequalities in the strict sense, since their coefficients have three different values: $4c_{p+1} - 2|W|$, $4c_p - 2|W|$, and $2c_p + 2c_{p+1} - 2|W|$.

Turning to matroidal relaxations, Fekete, Firla, and Spille [8] discuss the problem to describe \mathcal{M} as intersection of a minimum number of matroids. Although it is hard to determine the involved matroids, there is at least one matroid that can be derived by studying inequalities (12). Let $F \subseteq E$ and $G' = (W, F)$ the subgraph of G induced by F . Moreover, let $H_i = (W_i, F_i)$, $i = 1, \dots, k$, be the connected components of G' , and define $r(F) := \sum_{i=1}^k \lfloor \frac{1}{2}|W_i| \rfloor$. Then, it is not hard to see that r is the rank function of a matroid, say $M^{1/2} = (E, \mathcal{I})$. This matroid is closely related to the graphical matroid. The inseparable and closed sets with respect to $M^{1/2}$ are just the edge sets $E(W)$ with $|W|$ odd. Consequently, inequalities (12) are rank inequalities, and hence, a cardinality constrained counterpart of inequalities (12) can be derived for $P_{\text{MATCH}}^c(E)$. The next theorem indicates that those rank induced cardinality forcing inequalities are part of a facial description of $P_{\text{MATCH}}^c(E)$.

For the next proofs we introduce an useful definition. Let $ax \leq \alpha$ be a valid inequality for the polyhedron $P_{\mathcal{I}}(E)$ associated with a COP $\Pi = (E, \mathcal{I}, w)$. The feasible solution $I \in \mathcal{I}$ is said to be *tight* if $a\chi^I = \alpha$.

Theorem 3.2. *Let $K_n = (V, E)$ be the complete graph on n nodes, $c_1 \geq 1$ and $c_m \leq \lfloor \frac{n}{2} \rfloor - 1$. Let $W_1 \dot{\cup} W_2 \dot{\cup} \dots \dot{\cup} W_k$ be a disjoint union of odd subsets of V such that $c_p < r := \sum_{i=1}^k \frac{|W_i| - 1}{2} < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Setting $F := \sum_{i=1}^k E(W_i)$, the inequality*

$$(c_{p+1} - r)x(F) - (r - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r) \quad (14)$$

defines a facet of $P_{\text{MATCH}}^c(E)$.

Proof. By definition, $r = r(F)$ and F is closed with respect to $M^{1/2}$. Since $\mathcal{M} \subseteq \mathcal{I}$, it is not hard to see that inequality (14) is valid. To show that (14) defines a facet of $P_{\text{MATCH}}^c(E)$, we assume that there is an equation $bx = b_0$ that is satisfied by all points in $P_{\text{MATCH}}^c(E)$ which satisfy (14) at equality. If $c_p = 1$,

then it is easy to see that $b_e = b_0 = \frac{b_0}{c_p}$ for all $e \in F$. Next, let $c_p \geq 2$ which implies $r \geq 3$. We will show that for any two edges $e, f \in F$, $b_e = b_f$ holds. If e and f are non-adjacent, then there is a matching M of cardinality $c_p + 1$ with $e, f \in F$ due to $r_u(F) = r \geq 3$. The matchings $M_e := M \setminus \{e\}$ and $M_f := M \setminus \{f\}$ are tight. Hence, $b_0 = b\chi^{M_e} = b\chi^{M_f}$ which implies immediately $b_e = b_f$. If e and f are adjacent, then there is some edge $g \in F$ which is adjacent neither to e nor to f . By the former argumentation, $b_e = b_g$ and $b_f = b_g$, and thus, $b_e = b_f$. Any tight matching $M \subseteq F$ yields now $b_e = \frac{b_0}{c_p}$ for all $e \in F$.

Next, consider the coefficients $b_e, e \in E \setminus F$. If $c_{p+1} = r + 1$, then one can easily deduce that $b_e = -b_0 \frac{r-c_p}{c_p} = -b_0 \frac{r-c_p}{c_p(c_{p+1}-r)}$. So, let $c_{p+1} > r + 1$ and $e^* \in E \setminus F$ be any edge. No matter whether e^* is incident with no, one, or two nodes of $W_1 \dot{\cup} W_2 \dot{\cup} \dots \dot{\cup} W_k$, there is some matching $M^* \subseteq F$ with $|M^*| = r$ such that $M^* \cup \{e^*\}$ is also a matching. Moreover, $M^* \cup \{e^*\}$ can be completed to a matching M' with $|M'| = c_{p+1} + 1$ even if $c_{p+1} = c_m$, since $c_m \leq \lfloor \frac{n}{2} \rfloor - 1$. The matchings $M'_f := M' \setminus \{f\}, f \in M' \setminus M^*$ are tight with respect to (14) which implies $b\chi^{M'_f} = b_0$ for all $f \in M' \setminus M^*$. Hence, it is not hard to see that $b_f = -b_0 \frac{r-c_p}{c_p(c_{p+1}-r(F))}$ for all $f \in M' \setminus M^*$. In particular, $b_{e^*} = -b_0 \frac{r-c_p}{c_p(c_{p+1}-r(F))}$, and since e^* were arbitrarily chosen, it follows that $b_e = -b_0 \frac{r-c_p}{c_p(c_{p+1}-r(F))}$ for all $e \in E \setminus F$. Thus, $b\chi = b_0$ is a multiple of (14), which finishes the proof. \square

Example B: Cardinality Constrained Paths.

Let $D = (V, A)$ be a directed graph and $c = (c_1, \dots, c_m)$ a cardinality sequence. For any $v \in V$, we denote by $\delta^{\text{in}}(v)$ and $\delta^{\text{out}}(v)$ the set of arcs entering and leaving node v , respectively. For any two disjoint nodes $s, t \in V$, let $\mathcal{P}_{s,t}(D)$ be the collection of simple directed (s, t) -paths of D . Then,

$$P_{s,t\text{-path}}^c(D) := \text{conv}\{\chi^P \in \mathbb{R}^A : P \in \mathcal{P}_{s,t}(D) \cap \text{CHS}^c(A)\}$$

is called the *cardinality constrained path polytope*, and according to [12], the integer points of $P_{s,t\text{-path}}^c(D)$ can be described by the system

$$\begin{aligned} x_{uv} &\in \{0, 1\} && \text{for all } (u, v) \in A. \\ x(\delta^{\text{out}}(s)) &= x(\delta^{\text{in}}(t)) = 1, && (15) \\ x(\delta^{\text{in}}(s)) &= x(\delta^{\text{out}}(t)) = 0, && (16) \\ x(\delta^{\text{out}}(v)) - x(\delta^{\text{in}}(v)) &= 0 && \text{for all } v \in V \setminus \{s, t\}, \quad (17) \\ x(\delta^{\text{out}}(v)) &\leq 1 && \text{for all } v \in V \setminus \{s, t\}, \quad (18) \\ x(\delta^{\text{in}}(v)) - x((S : V \setminus S)) &\leq 0 && \forall S \subset V : s, t \in S, v \in V \setminus S, \quad (19) \\ x(A) &\geq c_1, \\ x(A) &\leq c_m, \end{aligned}$$

$$(c_{p+1} - |W|) \sum_{v \in W} x(\delta^{\text{out}}(v)) - (|W| - c_p) \sum_{v \in V \setminus W} x(\delta^{\text{out}}(v)) \leq c_p(c_{p+1} - |W|)$$

for all $W \subseteq V : s \in W, t \in V \setminus W$ with $c_p < |W| < c_{p+1}$ for some p . (20)

It is worthwhile to have a closer look at the cardinality constrained path polytope from a matroidal point of view, disclosing that the cardinality forcing

inequalities (20) are originated from matroids. The collection of all simple (s, t) -paths is contained in the intersection of the same three matroids that are used to formulate the asymmetric traveling salesman problem (ATSP) by matroids. The three matroids are the two partition matroids $M^{\text{out}} = (A, \mathcal{I}^{\text{out}})$, $M^{\text{in}} = (A, \mathcal{I}^{\text{in}})$ whose independence systems are defined by

$$\begin{aligned}\mathcal{I}^{\text{out}} &:= \{B \subseteq A : |B \cap \delta^{\text{out}}(v)| \leq 1 \text{ for all } v \in V\}, \\ \mathcal{I}^{\text{in}} &:= \{B \subseteq A : |B \cap \delta^{\text{in}}(v)| \leq 1 \text{ for all } v \in V\},\end{aligned}$$

respectively, and the graphic matroid $M^F = (A, \mathcal{I}^F)$, where \mathcal{I}^F denotes the collection of all forests of D . Consequently, the rank and cardinality inequalities associated with these matroids are valid inequalities for $P_{s,t\text{-path}}^c(D)$.

The facet defining rank inequalities for $P_{M^{\text{out}}}^c(A)$ are exactly the inequalities $x(\delta^{\text{out}}(v)) \leq 1$ for $v \in V$. Thus, inequalities (18) are originated from the partition matroid M^{out} . The facet defining cardinality forcing inequalities for $P_{M^{\text{out}}}^c(A)$ are of the form

$$(c_{p+1} - |U|) \sum_{v \in U} x(\delta^{\text{out}}(v)) - (|U| - c_p) \sum_{v \in V \setminus U} x(\delta^{\text{out}}(v)) \leq c_p(c_{p+1} - |U|), \quad (21)$$

where $U \subseteq V$ with $c_p < |U| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. If $s \in U$ and $t \in V \setminus U$, then, with respect to $P_{s,t\text{-path}}^c(D)$, inequality (21) is equivalent to

$$(c_{p+1} - |U|) \sum_{v \in U \cup \{t\}} x(\delta^{\text{out}}(v)) - (|U| - c_p) \sum_{v \in V \setminus (U \cup \{t\})} x(\delta^{\text{out}}(v)) \leq c_p(c_{p+1} - |U|),$$

due to $x(\delta^{\text{out}}(t)) = 0$. Setting $W := U \cup \{t\}$, we see that this inequality is an inequality among (20). Thus, inequalities (20) are originated from the cardinality constrained version of M^{out} . By Theorem 3.2 of [12] these inequalities define facets of $P_{s,t\text{-path}}^c(D)$.

Due to the flow conservation constraints (17), the inequalities that can be derived from facet defining rank and cardinality forcing inequalities for $P_{M^{\text{in}}}^c(A)$ are equivalent to any of the inequalities (18) and (20).

The facet defining rank inequalities for $P_{M^F}^c(A)$ are of the form $x(A(U)) \leq |U| - 1$ for $\emptyset \neq U \subseteq V$. Due to the equations (15), (16), the face induced by the rank inequality associated with some U is contained in the face induced by the rank inequality associated with $U' := U \setminus \{s, t\}$ (with respect to $P_{s,t\text{-path}}^c(D)$). However, the inequality $x(A(U')) \leq |U'| - 1$ is still not facet defining for $P_{s,t\text{-path}}^c(D)$. To this end, consider an inequality among (19) with $S := V \setminus U'$ and some $u \in U'$:

$$x(\delta^{\text{in}}(u)) - x((S : V \setminus S)) \leq 0.$$

Adding the inequalities $x(\delta^{\text{in}}(v)) \leq 1$ for $v \in U' \setminus \{u\}$, we obtain $x(A(U')) \leq |U'| - 1$. Also, none of the cardinality forcing inequalities

$$(c_{p+1} - r_F(W))x(W) - (r_F(W) - c_p)x(A \setminus W) \leq c_p(c_{p+1} - r_F(W))$$

for closed sets $W \subseteq A$ with respect to the graphic matroid is facet defining for $P_{s,t\text{-path}}^c(D)$ regardless in which partition $W, A \setminus W$ are s and t .

3.3 Iterated Inequality-Strengthening

A favorite method in order to obtain insights about the facial structure of a polytope is to compute the H-representation of a polytope, given by its V-representation, with convex-hull codes such as PORTA [5] or `polymake` [10]. However, since the used routines have exponential running time (indeed, a polynomial algorithm for the convex hull problem is unknown), this approach works only for small problem instances. Here might general lifting procedures come into play, because we are perhaps not able to compute the H-representation of a polytope, but able to determine quickly the affine space associated with the face induced by a valid inequality. For example, PORTA returns a set of linearly independent equations that are satisfied by all points in the face F_a of a polyhedron P induced by a valid inequality $ax \leq \alpha$. In other words, the set of equations determines the affine hull of F_a . However, P intersects usually both half spaces induced by such an equation $bx = \beta$, which means that neither $bx \leq \beta$ nor $bx \geq \beta$ is a valid inequality for P . In this case the following procedure can be applied.

Procedure 3.3. Inequality-Strengthening.

Input: A 0-1-polytope $P \in \mathbb{R}^d$ given by its vertex set \mathcal{V} , a valid inequality $ax \leq \alpha$, and an equation $bx = \beta$ that is satisfied by all points $v \in \mathcal{V}$ that satisfy $ax \leq \alpha$ at equality.

Output: A valid inequality $cx \leq \gamma$ such that the face induced by this inequality contains the face induced by $ax \leq \alpha$.

1. Set $\tilde{\mathcal{V}} = \{v \in \mathcal{V} : bv > \beta\}$.
2. If $\tilde{\mathcal{V}} = \emptyset$, return “ $ax \leq \alpha$ ”.
3. Set $\lambda_v := \frac{\alpha - av}{bv - \beta}$ for all $v \in \tilde{\mathcal{V}}$.
4. Set $\lambda^* := \min\{\lambda_v : v \in \tilde{\mathcal{V}}\}$.
5. Define a new inequality $cx \leq \gamma$ by $c := a + \lambda^* \cdot b$ and $\gamma := \alpha + \lambda^* \cdot \beta$. Return “ $cx \leq \gamma$ ”.

$\tilde{\mathcal{V}}$ is the set of all vertices of P that violate the inequality $bx \leq \beta$. Thus, $av < \alpha$ and $\lambda_v > 0$ for all $v \in \tilde{\mathcal{V}}$, which in turn implies $\lambda^* > 0$. Now, for every $v \in \mathcal{V} \setminus \tilde{\mathcal{V}}$ we have: $av \leq \alpha$, $bv \leq \beta$, and hence $cv \leq \gamma$. Moreover, for each $v \in \tilde{\mathcal{V}}$ we have: $(a + \lambda b) \cdot v \leq \alpha + \lambda\beta$ for $0 \leq \lambda \leq \lambda_v$. Hence, $cx \leq \gamma$ is satisfied by every $v \in \tilde{\mathcal{V}}$. Consequently, $cx \leq \gamma$ is a valid inequality for P . Furthermore, if $\tilde{\mathcal{V}} \neq \emptyset$, then by choice of λ^* , the face induced by $ax \leq \alpha$ is strictly contained in the face induced by $cx \leq \gamma$. So, applying Procedure 3.3 iteratively, results in a facet defining inequality for P . Of course, the running time of the procedure is linear in $|\mathcal{V}|$, but in general not polynomial in the dimension d of the space.

Based on these observations, we give the following recommendation.

Recommendation 3. *If the cardinality forcing inequalities are not facet defining for the polytope of consideration, then they are probably a good starting point to derive stronger cardinality specific inequalities. Algorithmically this can be done with Procedure 3.3.*

To give an application for Procedure 3.3, we consider cardinality constrained cuts. Let $G = (V, E)$ be a graph. For any $S \subseteq V$, we denote by $\delta(S)$ the set of edges connecting S and $V \setminus S$. A subset C of E is called a cut if $C = \delta(S)$ for some $S \subseteq V$. The sets S and $V \setminus S$ are the *shores* of C . The collection of all cuts of G is denoted by \mathcal{C} . The max cut problem $\max\{w(C) : C \in \mathcal{C}\}$, is NP-hard. In the following, we install cardinality restrictions acting on the shores of the cuts.

Let $C \subseteq E$ be a cut with shores S and T . Then, $|T| = |V| - |S|$, that is, the cardinality of S determines that of T and vice versa. Moreover, $\min\{|S|, |T|\} \leq \lfloor \frac{n}{2} \rfloor$. Consequently, it is sufficient to force only the cardinality of the smaller shore which can be done with cardinality sequences $c = (c_1, \dots, c_m)$ with $1 \leq c_m < \dots < c_1 \leq \lfloor \frac{n}{2} \rfloor$. These observations give reason to define the *node cardinality constrained cut polytope*

$$P_{\text{Cut}}^c(E) := \text{conv}\{\chi^{\delta(S)} \in \mathbb{R}^E : S \in \text{CHS}^c(V)\}.$$

In the sequel, let $P_{\text{Cut}}^c(E)$ be defined on the complete graph $K_n = (V, E)$ on n nodes. Since the cardinality constraints in form of the cardinality sequence c are not setted on E but on V , it seems to be hard to incorporate the cardinality forcing inequalities. However, requiring not only $S \in \text{CHS}^c(V)$ for a shore S of a cut, but also $s \notin S$ for a fixed node s , opens the doors to these inequalities. Denote by \mathcal{S} the collection of all subsets of V not containing s . Then, $\bar{P}_{\text{Cut}}^c(E) := \text{conv}\{\chi^{\delta(U)} \in \mathbb{R}^E : U \in \mathcal{S} \cap \text{CHS}^c(V)\}$ is a slight variation of $P_{\text{Cut}}^c(E)$. Both polytopes are connected as follows: Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $1 \leq c_m < \dots < c_1 \leq \lfloor \frac{n}{2} \rfloor$ and $\bar{c} = (c_1, \dots, c_m, n - c_m, \dots, n - c_1)$. Then, $P_{\text{Cut}}^c(E) = P_{\text{Cut}}^{\bar{c}}(E)$ and $\bar{P}_{\text{Cut}}^c(E) = \bar{P}_{\text{Cut}}^{\bar{c}}(E)$, but in general $P_{\text{Cut}}^c(E) \neq \bar{P}_{\text{Cut}}^c(E)$. Thus, $\bar{P}_{\text{Cut}}^c(E)$ generalizes $P_{\text{Cut}}^c(E)$. The difference in the facial structure, however, is small. For instance, if we restrict ourselves to $c = (k)$, then $\bar{P}_{\text{Cut}}^{(k)}(E)$ is a face of $P_{\text{Cut}}^{(k)}(E)$. If $k \leq \lfloor \frac{n}{2} \rfloor$, then it is induced by $x(\delta(s)) \geq k$, if $k > \lfloor \frac{n}{2} \rfloor$, then by $x(\delta(s)) \leq k$.

Since $\{s, v\} \in E$ for all $v \in V \setminus \{s\}$, we have $|\delta(U) \cap \delta(s)| = |U|$ for all $U \in \mathcal{S} \cap \text{CHS}^c(V)$. Thus, the cardinality forcing inequality

$$(c_{p+1} - |W|)x(\delta(s) \cap \delta(W)) - (|W| - c_p)x(\delta(s) \cap \delta(V \setminus W)) \leq c_p(c_{p+1} - |W|)$$

is valid for all $W \in \mathcal{S}$ with $c_p < |W| < c_{p+1}$ for some p . The inequalities are not facet defining, but by application of Procedure 3.3 and the right choice of equations generated by PORTA they can be strengthened. With this approach we identified a very simple class of $n \cdot (m - 1)$ inequalities

$$x(E) - (n - c_p - c_{p+1})x(\delta(s)) \leq c_p c_{p+1} \quad \forall s \in V, p \in \{1, \dots, m - 1\}, \quad (22)$$

which will be shown to be facet defining in Theorem 3.7.

Example. We consider the complete graph $K_{13} = (V, E)$, the cardinality sequence $c = (2, 6, 7, 11)$, and the cardinality forcing inequality

$$2 \sum_{v \in Y} x_{sv} - 2 \sum_{v \in Z} x_{sv} \leq 4,$$

where $s = 1$, $Y = \{2, 3, 4, 5\}$, and $Z = \{6, \dots, 13\}$. All points in $Q := P_{\text{Cut}}^c(E)$ satisfying the inequality at equality satisfy the equation

$$x(E(Z)) - 6x((Z : \{s\})) = 0.$$

The minimum value λ^* in Procedure 3.3 will be attained by 4. This results in the inequality

$$2 \sum_{v \in Y} x_{sv} - 26 \sum_{v \in Z} x_{sv} + 4x(E(Z)) \leq 4.$$

Going so on, we obtain

$$\begin{aligned} 8x(E(Y)) + x((Y : Z)) + 2x(E(Z)) &= 48, \\ \lambda^* &= \frac{2}{3}, \text{ and after scaling} \\ 6 \sum_{v \in Y} x_{sv} - 78 \sum_{v \in Z} x_{sv} + 16x(E(Y)) + 2x((Y : Z)) + 16x(E(Z)) &\leq 108. \end{aligned}$$

Finally,

$$\begin{aligned} -6 \sum_{v \in Y} x_{sv} + 6 \sum_{v \in Z} x_{sv} - x(E(Y)) + x((Y : Z)) - x(E(Z)) &= 0, \\ \lambda^* &= 7, \\ -36x(\delta(s)) + 9x(E(V \setminus \{s\})) &\leq 108. \end{aligned}$$

The last inequality is a multiple of an inequality of the form (22).

The goal of the remainder of this section is to show that inequalities (22) define facets of $P_{\text{Cut}}^c(E)$. An inequality among (22) is equivalent to $x(\delta(s)) \geq c_p$ with respect to $P_{\text{Cut}}^{(c_p)}(E)$, since $x(E) = c_p(n - c_p)$ for all $x \in P_{\text{Cut}}^{(c_p)}(E)$. An analogous observation holds for c_{p+1} . Thus, in order to show that the inequalities (22) are indeed facet defining, we first study the inequalities $x(\delta(s)) \geq k$ with respect to $P_{\text{Cut}}^{(k)}(E)$.

To simplify the next proofs we recall some facts from Linear Algebra. Denote the kernel and the image of a matrix $A \in \mathbb{R}^{m \times n}$ by $\ker(A)$ and $\text{im}(A)$, respectively. Denote by A_i the i th column of A . Let $v_1, \dots, v_k \in \ker(A)$ and $v_{k+1}, \dots, v_r \in \mathbb{R}^n$ be any vectors. In order to show that these vectors are linearly independent, it is sufficient to do this for the vectors v_1, \dots, v_k and Av_{k+1}, \dots, Av_r separately. Moreover, we need the following lemma.

Lemma 3.4. *Let $n \in \mathbb{N}$, $n \geq 2$, and $\alpha, \beta \in \mathbb{R}$. The $n \times n$ matrix A defined by*

$$a_{ij} = \begin{cases} \alpha & \text{if } i = j, \\ \beta & \text{otherwise} \end{cases}$$

has full rank if and only if $\alpha \neq \beta$ and $\alpha + (n-1)\beta \neq 0$.

Proof. Clearly, if $\alpha = \beta$, then $\text{rank}(A) < n$. If $\alpha + (n-1)\beta = 0$, then $\sum_{i=1}^n A_i = 0$, which implies that $\text{rank}(A) < n$. To show the converse, define a new column $A_{n+1} := \frac{\beta}{\alpha + (n-1)\beta} \sum_{i=1}^n A_i$. All entries of A_{n+1} are equal to β . Thus, the matrix B defined by $B_i := A_i - A_{n+1}$ for $i = 1, \dots, n$, and $B_{n+1} = A_{n+1}$ has entries $\alpha - \beta$ on the diagonal of the first n entries. Since $\alpha \neq \beta$, it follows that $\text{rank}(B) = n$. We conclude that $\text{rank}(A) = \text{rank}(A, A_{n+1}) = \text{rank}(B) = n$. \square

Denoting by $\mathbf{1}_n$ the $n \times n$ matrix of all ones and by I_n the $n \times n$ identity matrix, the matrix defined in Lemma 3.4 is equal to $\beta \mathbf{1}_n + (\alpha - \beta)I_n$.

Theorem 3.5. *Let $K_n = (V, E)$ be the complete graph on n nodes and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ be an integer. Then,*

$$\dim P_{\text{Cut}}^{(k)}(E) = \begin{cases} n-1, & \text{if } k=1, \\ |E| - n, & \text{if } n \text{ is even and } k=n/2, \\ |E| - 1, & \text{otherwise.} \end{cases} \quad (23)$$

Proof. In case $k = 1$ the statement is clear. When n is odd and $k = \lfloor \frac{n}{2} \rfloor$, $P_{\text{Cut}}^{(k)}(E)$ is the face of $P_{\text{Cut}}(E)$ induced by the inequality $x(E) \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$, which by Theorem 2.1 of Barahona and Mahjoub [3] defines a facet of $P_{\text{Cut}}(E)$. Since $P_{\text{Cut}}(E)$ is fulldimensional, it follows $\dim P_{\text{Cut}}^{(k)}(E) = |E| - 1$. Next, let n be even and $k = n/2$. Then, for any $s \in V$, $|V^s|$ is odd and $\ell := k - 1 = \lfloor \frac{|V^s|}{2} \rfloor$. Hence, $\dim P_{\text{Cut}}^{(\ell)}(E^s) = |E^s| - 1 = (|E| - n + 1) - 1 = |E| - n$. Since any cut $\delta^s(W)$ with $|W| = \ell$ in K_n^s can be augmented to a cut in K_n by adding node s to W , it follows immediately that $\dim P_{\text{Cut}}^{(k)}(E) \geq |E| - n$. In order to show equality, we remark that any cut $\delta(W)$ of K_n with $|W| = \frac{n}{2}$ satisfies the n linearly independent equations $x(\delta(v)) = \frac{n}{2}$, $v \in V$.

Finally, let $2 \leq k < \lfloor \frac{n}{2} \rfloor$. Since all points in $P_{\text{Cut}}^{(k)}(E)$ satisfy the equation

$$x(E) = k(n - k), \quad (24)$$

it follows that $\dim P_{\text{Cut}}^{(k)}(E) \leq |E| - 1$. To show equality, let $bx = \beta$ be an equation that is satisfied by all $x \in P_{\text{Cut}}^{(k)}(E)$. Our goal is to show that $bx = \beta$ is a multiple of (24).

Let $V = S \dot{\cup} T \dot{\cup} U \dot{\cup} \{v\} \dot{\cup} \{w\}$ be a partition of V such that $|S| = |T| = k - 1$ and $|U| = n - 2k$. Similar as in the proof of Lemma 2.5 of Barahona, Grötschel, and Mahjoub [2] one can show that $b(s : U) = b(t : U)$ by considering the cuts

$$\begin{aligned} C_1 &:= (S \cup \{s\} \cup U : T \cup \{t\}), & C_2 &:= (S \cup \{s\} : T \cup \{t\} \cup U) \\ C_3 &:= (S \cup \{t\} \cup U : T \cup \{s\}), & C_4 &:= (S \cup \{t\} : T \cup \{s\} \cup U). \end{aligned}$$

Here, for any $y \in V$ and $Z \subseteq V \setminus \{y\}$, $b(y : Z)$ denotes the sum $\sum_{v \in Z} b_{yv}$. Now let $W \subseteq V \setminus \{s, t\}$ be any node set of cardinality $n - 2k + 1$. Since U was arbitrarily chosen, we have $b(s : W \setminus \{v\}) = b(t : W \setminus \{v\})$ for each $v \in W$. Defining $z_v := b_{sv} - b_{tv}$ for $v \in W$, we can write this set of equations as equation system $(\mathbf{1}_{|W|} - I_{|W|})z = 0$. By Lemma 3.4, $\mathbf{1}_{|W|} - I_{|W|}$ is a nonsingular matrix, and thus $z = 0$ is the only solution implying $b_{sv} = b_{tv}$ for all $v \in W$. Since s, t , and W were arbitrarily chosen, we can conclude that $b_e = \sigma$ for all $e \in E$ for some $\sigma \in \mathbb{R}$, and hence $\beta = \sigma k(n - k)$. Consequently, $bx = \beta$ is a multiple of (24). \square

Theorem 3.6. *Let $K_n = (V, E)$ be the complete graph on n nodes and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ integer. Then for any $s \in V$, the inequality*

$$x(\delta(s)) \geq k \quad (25)$$

is valid for $P_{\text{Cut}}^{(k)}(E)$. It defines a facet of $P_{\text{Cut}}^{(k)}(E)$ if and only if $k = 1$, otherwise a face of dimension $|E| - n$.

Proof. Let F be the face induced by (25). In case $k = 1$, the cuts $\delta(w)$ for $w \in V \setminus \{s\}$ are tight with respect to (25). Since their incidence vectors are linearly independent, we conclude that F has dimension $n - 2$, that is, (25) defines a facet. When n is even and $k = \frac{n}{2}$, then (25) is satisfied with equality by all points $x \in P_{\text{Cut}}^{(k)}(E)$. Consequently, $F = P_{\text{Cut}}^{(k)}(E)$ which by Theorem 3.5 implies $\dim F = |E| - n$.

Next, let $2 \leq k \leq \frac{n}{2} - 1$. The incidence vector of a feasible tight cut satisfies equation (24) and the $n - 1$ equations

$$x_{sv} - \frac{x(\delta(v)) - k}{n - 2k} = 0, \quad v \in V^s. \quad (26)$$

Since these equations are linearly independent, it follows that $\dim F \leq |E| - n$. The inequality $\dim F \geq |E| - n$ follows from the fact that $\dim P_{\text{Cut}}^{(k)}(E^s) = |E^s| - 1 = |E| - n$ and any cut $\delta^s(W)$ with $|W| = k$ in K_n^s corresponds to a tight cut $\delta(W)$ in K_n .

Finally, let n be odd and $k = \lfloor \frac{n}{2} \rfloor$. Clearly, $\dim F \leq |E| - n$, since the incidence vector of a tight cut satisfies the equations (24) and (26). To show equality, consider the cut polytope $P_{\text{Cut}}^{(k)}(E^s)$ which can be obtained by projecting F to \mathbb{R}^{E^s} . Its dimension is $|E^s| - |V^s|$. Consequently, there are $r := |E^s| - |V^s| + 1$ linearly independent incidence vectors of cuts, say $\delta^s(W_1), \dots, \delta^s(W_r)$ of K_n^s with $|W_i| = k$. Since the shores of a cut $\delta^s(W_i)$ have the same cardinality, we may assume w.l.o.g. that for some $t \in V^s$, $t \in W_i$ for $i = 1, \dots, r$. Of course, the cuts $C_i := \delta(W_i \cup \{s\})$, $i = 1, \dots, r$ are tight with respect to (25), and their incidence vectors are linearly independent, too. In addition, beside (24) and (26), these vectors satisfy the $n - 2$ equations

$$x_{sv} - x_{tv} + x_{st} = 0 \quad \text{for all } v \in \tilde{V}, \quad (27)$$

where $\tilde{V} := V^s \setminus \{t\}$. Since $r + (n - 2) = |E| - n + 1$, it suffices to construct $(n - 2)$ further tight cuts whose incidence vectors are linearly independent and linearly independent of the former points. To this end, let w.l.o.g. $V' = \{1, \dots, n - 2\}$ and $U = \{s, 1, \dots, n - k\}$. For each v with $1 \leq v \leq n - k$, $\tilde{C}_v := \delta(U \setminus \{v\})$ is a feasible tight cut. Moreover, for each $v \in \{n - k + 1, \dots, n - 2\}$ and any $u, \tilde{u} \in \{1, \dots, n - k\}$, the cut $\tilde{C}_v := \delta((U \cup \{v\}) \setminus \{u, \tilde{u}\})$ is tight. Let A be the matrix associated with the left hand side of the equations (27). Since $\chi^{C_i} \in \ker(A)$, it remains to show that the matrix $B := [A\chi^{\tilde{C}_1}, \dots, A\chi^{\tilde{C}_{n-2}}]$ has full rank. Indeed, B is of the form

$$B = \begin{bmatrix} 2I_{n-k} & \mathbf{0} \\ * & 2(\mathbf{1}_{k-2} - I_{k-2}) \end{bmatrix},$$

which implies immediately $\text{rank}(B) = n - 2$. \square

Theorem 3.7. *Let $K_n = (V, E)$ be the complete graph on n nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $1 \leq c_1 < \dots < c_m \leq \lfloor \frac{n}{2} \rfloor$. Then, $P_{\text{Cut}}^c(E)$ is fulldimensional. Moreover, the inequality*

$$x(E) - (n - c_p - c_{p+1})x(\delta(s)) \leq c_p c_{p+1} \quad (28)$$

defines a facet of $P_{\text{Cut}}^c(E)$ for all $s \in V$.

Proof. W.l.o.g let $V^s = \{1, \dots, n - 1\}$. Let F be the face of $P_{\text{Cut}}^c(E)$ induced by (28). We will show that $\dim F = |E| - 1$, which implies that $P_{\text{Cut}}^c(E)$ is fulldimensional due to the fact that not all feasible cuts are tight. Since $m \geq 2$, it follows that $c_{p+1} \geq 2$. Inequality (28) is equivalent to $x(\delta(s)) \geq c_{p+1}$ with respect to $P_{\text{Cut}}^{(c_{p+1})}(E)$. Hence, by Theorem 3.6, there are $q := |E| - n + 1$

linearly independent incidence vectors of tight cuts $C_i := \delta(W_i)$ with $s \in W_i$ and $|W| = n - c_{p+1}$. In the sequel, we distinguish three cases.

(1) Let $c_{p+1} < \lfloor \frac{n}{2} \rfloor$. Then, the vectors χ^{C_i} satisfy the $n - 1$ equations

$$x_{sv} + \frac{x(\delta(s)) - x(\delta^s(v))}{n - 2c_{p+1} - 1} = 0 \quad \forall v \in V^s. \quad (29)$$

Denote by A the matrix associated with the left hand side of (29). To construct $n - 1$ further points, consider the set $U = \{s, 1, \dots, r\}$ with $r = n - c_p - 2$. For each node $v \in \{r + 1, \dots, n - 1\}$, the cut $\tilde{C}_v := \delta(U \cup \{v\})$ is tight, and for each node $v \in \{1, \dots, r\}$ and any two disjoint nodes $t, u \in \{r + 1, \dots, n - 1\}$, the cut $\tilde{C}_v := \delta((U \cup \{t, u\}) \setminus \{v\})$ is tight. Since $\chi^{C_i} \in \ker(A)$ for $i = 1, \dots, q$, it is sufficient to prove that the matrix $B := [A\chi^{\tilde{C}_1}, \dots, A\chi^{\tilde{C}_{n-1}}]$ has full rank. It is not hard to see that B is of the form

$$B = \begin{bmatrix} \mathbf{0} & \sigma \cdot I_r \\ \sigma(\mathbf{1}_{\bar{r}} - I_{\bar{r}}) & * \end{bmatrix},$$

where $\bar{r} = n - 1 - r$ and $\sigma = 1 + \frac{1+2c_p-n}{n-2c_{p+1}-1}$. Clearly, I_r and $\mathbf{1}_{\bar{r}} - I_{\bar{r}}$ have full rank, and hence, also B .

(2) Let $c_{p+1} = \frac{n}{2}$. This time the vectors χ^{C_i} satisfy the equations

$$x(\delta(v)) - x(\delta(s)) = 0 \quad \forall v \in V^s. \quad (30)$$

Let A be the matrix associated with the left hand side of (30). Of course, $\chi^{C_i} \in \ker(A)$ for $i = 1, \dots, q$. Next, consider again the cuts \tilde{C}_v , $v = 1, \dots, n - 1$. The matrix $B := [A\chi^{\tilde{C}_1}, \dots, A\chi^{\tilde{C}_{n-1}}]$ is of the form

$$B = \begin{bmatrix} \mathbf{0} & (n - 2c_p)I_r \\ (n - 2c_p)(\mathbf{1}_{\bar{r}} - I_{\bar{r}}) & * \end{bmatrix}.$$

Since B has obviously full rank, (28) defines a facet if $c_{p+1} = \frac{n}{2}$.

(3) Let $c_{p+1} = \frac{n-1}{2}$. When $c_p > 1$, reverse the roles of c_p and c_{p+1} and apply (1). So, in the sequel we may assume that $c_p = 1$. The vectors χ^{C_i} satisfy the equations

$$x(\delta^s(v)) - x(\delta(s)) = 0 \quad \forall v \in V^s. \quad (31)$$

Denote the left hand side of system (31) by the matrix A . Since $\chi^{C_i} \in \ker(A)$, it remains to show that the images of the incidence vectors of the cuts $\delta(i)$, $i = 1, \dots, n - 1$ are linearly independent. Now, for each $i \in \{1, \dots, n - 1\}$,

$$x(\delta^s(j)) - x(\delta(s)) = \begin{cases} n - 3 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $A \cdot [\chi^{\delta(1)}, \dots, \chi^{\delta(n-1)}] = (n - 3)I_{n-1}$. \square

Since the argumentation for showing that inequalities (22) are facet defining uses the facial structure of the polytopes $P_{\text{Cut}}^{(k)}(E)$, very similar results can be obtained for $\bar{P}_{\text{Cut}}^{(k)}(E)$ where it does not matter whether $k \leq \lfloor \frac{n}{2} \rfloor$ or $k > \lfloor \frac{n}{2} \rfloor$.

4 Extensions

The paper shows that the incorporation of the combinatorial structure of a COP $\Pi = (E, \mathcal{I}, w)$ into cardinality forcing inequalities may result in strong inequalities that cut off feasible solutions $I \in \mathcal{I}$ of forbidden cardinality. In particular, well-known attributes of matroid theory (closeness) and matroidal relaxations might play an important role in this context.

It is natural to search for complete linear descriptions of polyhedra $P_{\mathcal{I}}^c(E)$ associated with CCCOP's at least for those problems for which a complete linear description of the polyhedron $P_{\mathcal{I}}(E)$ associated with the original COP is known. For instance, the matching polytope $P_{\text{MATCH}}(E)$ is determined by the inequalities (10)-(12). However, we do not know whether it is sufficient to add inequalities (13), (14), and the cardinality bound $c_1 \leq x(E) \leq c_m$ in order to obtain a complete linear description of $P_{\text{MATCH}}^c(E)$.

If we have a complete linear description of $P_{\mathcal{I}}^{(c_i)}(E)$ for $i = 1, \dots, m$, then an extended formulation for $P_{\mathcal{I}}^c(E)$ can be obtained via *disjunctive programming*, which is optimization over the union of polyhedra. Below we restate a well-known result of Balas [1].

Theorem 4.1. *Given r polyhedra $P^i = \{x \in \mathbb{R}^n : A^i x \geq b^i\} = \text{conv}(V^i) + \text{cone}(R^i)$, the following system:*

$$\begin{aligned} y &= \sum_{i=1}^r x^i \\ A^i x^i &\geq \lambda^i b^i, & i = 1, \dots, r \\ \sum_{i=1}^r \lambda^i &= 1 \\ \lambda_i &\geq 0, & i = 1, \dots, r \end{aligned} \tag{32}$$

provides an extended formulation for the polyhedron

$$P := \text{conv}\left(\bigcup_{i=1}^r V^i\right) + \text{cone}\left(\bigcup_{i=1}^r R^i\right).$$

□

In our context, $r = m$ and $P^i = P_{\mathcal{I}}^{(c_i)}(E)$ for $i = 1, \dots, m$. In addition, in many cases the linear descriptions of the polyhedra $P_{\mathcal{I}}^{(c_i)}(E)$ will only differ in the cardinality constraints $x(E) = c_i$, that is, there is some common constraint system $Ax \geq b$ such that $P_{\mathcal{I}}^{(c_i)}(E) = \{x \in \mathbb{R}^n : Ax \geq b, x(E) = c_i\}$ for $i = 1, \dots, m$. This in turn simplifies system (32).

We currently formulate CCCOP's as disjunctive programs and try to project down the associated extended formulations to the original spaces in order to derive complete linear descriptions, for instance, we do that for cardinality constrained matchings or the intersection of two cardinality constrained matroids – but so far, without any success even for $m = 2$.

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