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Fair Ticket Prices in Public Transport

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Abstract

Ticket pricing in public transport usually takes a welfare or mnemonics maximization point of view. These approaches do not consider fairness in the sense that users of a shared infrastructure should pay for the costs that they generate. We propose an ansatz to determine fair ticket prices that combines concepts from cooperative game theory and integer programming. An application to pricing railway tickets for the intercity network of the Netherlands demonstrates that, in this sense, prices that are much fairer than standard ones can be computed in this way.

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1 Introduction

Public transport ticket prices are well studied in the economic literature on elasticities and welfare optimization as well as in the mathematical optimization literature on certain network design problems, see, e.g., the literature survey in [3]. To the best of our knowledge, however, there is no work on the fairness of ticket prices. The point is that typical pricing schemes are not related to infrastructure operation costs and, in this sense, favor some users, which do not fully pay for the costs they incur. For example, in this paper's (academic) example of the Dutch IC railway network, with the current distance tariff, the passengers in the central Randstad region of the country pay over 25% more than the costs they incur, and these excess payments subsidize operations elsewhere. We therefore investigate the construction of ticket prices that reflect operation costs better.

The ticket pricing can be seen as a cost allocation problem. This problem is widespread. Whenever it is necessary or desirable to divide a common cost between several users or items, a cost allocation method is needed. In the literature there are some examples of cost allocation applications using cooperative game theory, e.g, aircraft landing fees [7], water resource planning (or Tennessee Valley Authority) [8, 10], water resource development [12], distribution cost of gas and oil transportation [5], investment in electric power [6] and telephone billing rates [1].

In this paper we focus on a concept of game theory, namely the f-nucleolus. The nucleolus was originally suggested by Schmeidler [9] as a solution which minimizes the maximum discontent among all coalitions of the players in a

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cooperative game. Several modifications of the nucleolus concept have been suggested, such as the weak nucleolus (or per capita nucleolus) and the proportional nucleolus [12]. These three types are special cases of the so called f-nucleolus.

Our approach models ticket pricing as a cooperative cost allocation game to minimize overpayments. The f-nucleolus of this game can be computed by solving a sequence of linear programs, each of which has a number of constraints that is exponential in the number of players, and can be solved using a cutting plane approach, whose associated separation problem is a NP-hard combinatorial optimization problem.

The article is structured as follows. Section 2 recalls some concepts from cooperative game theory. A model that treats ticket pricing as a cost allocation game is presented in Section 3. The final Section 4 is devoted to the IC example.

2 Game Theoretical Setting

The *cost allocation game* deals with price determination and can be defined as follows, see [11] for a survey/an introduction. Given is a set of players $N = \{1, 2, \dots, n\}$, a cost function $c : 2^N \backslash \{\emptyset\} \to \mathbb{R}^+$, a polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\, x_i \geq 0 \;\forall i \in N\}$, which gives conditions on the prices x that the players are asked to pay, and a weight function $f : 2^N \backslash \{\emptyset\} \to \mathbb{R}^+$. For each vector $x \in P$, $x = (x_1, x_2, \dots, x_n) \in P$, and each *coalition* $S \subset N$, we define the *f-excess* of S at x as

$$e_f(S, x) := \frac{c(S) - x(S)}{f(S)}.$$

Here, $x(S) := \sum_{i \in S} x_i$, and we assume that the following set

$$\mathcal{X}(\Gamma) := \{x \in P \mid x(N) = c(N)\}$$

is non-empty. The f-excess represents the gain (or loss, if it is negative) of coalition S, if its members accept to pay $x(S)$ instead of operating some service themselves at cost $c(S)$; we will assume in this article that the weight function f has the form $f = \alpha + \beta|\cdot| + \gamma c$ with $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma > 0$. The excess measures price acceptability: the smaller $e_f(S, x)$, the less favorable is price x for coalition S, and for $e_f(S, x) < 0$, i.e., in case of a loss, x will be seen as unfair by the members of S. The cost allocation game $\Gamma = (N, c, P)$ is to determine a price $x \in \mathcal{X}(\Gamma)$ which minimizes the loss (or maximizes the gain) over all coalitions. Let us recall some related definitions from game theory.

Definition 2.1. *Let*

$$\mathcal{C}_{\varepsilon,f}(\Gamma) := \{x \in \mathcal{X}(\Gamma) \mid e_f(S, x) \geq \varepsilon,\ \forall \emptyset \neq S \subsetneq N\}.$$

$\mathcal{C}_{\varepsilon,f}(\Gamma)$ is called the (ε, f)-core *of* Γ*. In particular, $\mathcal{C}_{0,f}(\Gamma)$ is the* core *of* Γ*. The* f-least core *of the game Γ, denoted $\mathcal{LC}_f(\Gamma)$, is the intersection of all nonempty (ε, f)-core. Equivalently, let $\varepsilon_f(\Gamma)$ be the largest ε such that $\mathcal{C}_{\varepsilon,f}(\Gamma) \neq \emptyset$, i.e.,*

$$\varepsilon_f(\Gamma) = \max_{x \in \mathcal{X}(\Gamma)} \min_{\emptyset \neq S \subsetneq N} e_f(S, x),$$

then $\mathcal{LC}_f(\Gamma) = \mathcal{C}_{\varepsilon_f(\Gamma), f}$. In other words, the f-least core is the set of all vectors in $\mathcal{X}(\Gamma)$ that maximize the minimum f-excess of proper subsets of N.

Lemma 2.1. *If $\mathcal{X}(\Gamma)$ is non-empty then the f-least core of Γ is non-empty.*

Proof. We consider the following LP

$$\begin{aligned}
&\max_{(x,\varepsilon)} \ \varepsilon \\
s.t. \ & x(S) + \varepsilon f(S) \leq c(S), \ \forall S \in 2^N \backslash \{\emptyset, N\} \\
& x \in \mathcal{X}(\Gamma).
\end{aligned} \tag{1}$$

The polyhedron defined by the constraints of (1) is non-empty because with each $x \in \mathcal{X}(\Gamma)$, since $f(S) > 0$ for all non-empty set S, we can choose ε sufficiently small such that

$$x(S) + \varepsilon f(S) \leq c(S), \ \forall S \in 2^N \backslash \{\emptyset, N\}.$$

On the other hand, the objective value of (1) is bounded for all feasible solutions:

$$\varepsilon \leq \frac{c(S)}{f(S)}, \ \forall S \in 2^N \backslash \{\emptyset, N\}.$$

Therefore, the LP (1) has an optimal solution. Let ε^* be the optimal value, then the f-least core of Γ is

$$\{x \in \mathcal{X}(\Gamma) \,|\, (x, \varepsilon^*) \text{ is a feasible solution of (1) }\},$$

which is non-empty. □

The f-least core contains, in general, more than one point. However, uniqueness can be enforced by imposing a lexicographic order as follows. For each $x \in \mathcal{X}(\Gamma)$, let $\theta_f(x)$ be the vector in $\mathbb{R}^{2^n-2}$ whose components are the f-excesses $e_f(S, x)$ of proper subsets S of N, arranged in nondecreasing order, i.e.,

$$\theta_f^i(x) \leq \theta_f^j(x), \ \ \forall 1 \leq i < j \leq 2^n - 2.$$

For $x, y \in \mathcal{X}(\Gamma)$, $\theta_f(x)$ is *lexicographically greater* than $\theta_f(y)$, denoted $\theta_f(x) \succ \theta_f(y)$, if there exists an index i_0 such that

$$\theta_f^i(x) = \theta_f^i(y) \ \forall i < i_0 \ \text{ and } \ \theta_f^{i_0}(x) > \theta_f^{i_0}(y).$$

We say x is more acceptable than y.

Definition 2.2. *The* f-nucleolus *of a cost allocation game Γ, denoted by $\mathcal{N}_f(\Gamma)$, is the set of vectors in $\mathcal{X}(\Gamma)$ that maximize θ_f with respect to the lexicographic ordering.*

For $S \subset N$, let χ_S denote the incidence vector of S, i.e., χ_S^i is 1 if $i \in S$ and 0 else. For a set Σ of sets $S \subset N$, we denote

$$\chi_\Sigma := \{\chi_S \,|\, S \in \Sigma\}.$$

The following algorithm computes points in the f-least core, and terminates with the f-nucleolus.

Algorithm 1. *Compute the f-nucleolus of $\Gamma = (N, c, P)$.*

1. *Set $k := 0$, $\mathcal{A}_1 := \{\chi_N\}$, and $P_1 := \mathcal{X}(\Gamma)$.*

2. *Set $k := k+1$, and solve the linear program*

$$\begin{aligned} &\max_{(x,\varepsilon)} \ \varepsilon \\ s.t.\ & x(S) + \varepsilon f(S) \le c(S),\ \forall S \in \mathcal{S}_k \\ & x \in P_k, \end{aligned} \tag{2}$$

where

$$\mathcal{S}_k := 2^N \backslash \{S \subset N \mid \chi_S \in \text{span}\mathcal{A}_k\}.$$

If the problem is infeasible then stop, $\mathcal{N}_f(\Gamma) = \emptyset$. Otherwise, let (x^k, ε^k) and (λ^k, μ^k) be primal and dual optimal solutions.

3. *Define*

$$\begin{aligned} \Pi_{k+1} &:= \{S \in \mathcal{S}_k \mid \lambda_S^k > 0\}, \\ \mathcal{B}_{k+1} \subset \Pi_{k+1} &: \chi_{\mathcal{B}_{k+1}} \text{ is a basis of } \text{span}\chi_{\Pi_{k+1}} \\ P_{k+1} &:= \{x \in P_k \mid x(S) = c(S) - \varepsilon^k f(S),\ \forall S \in \mathcal{B}_{k+1}\}, \\ \mathcal{A}_{k+1} &:= \mathcal{A}_k \cup \chi_{\mathcal{B}_{k+1}}. \end{aligned}$$

4. *If $|\mathcal{A}_{k+1}| < n$ then goto 2, else $\{x^k\}$ is the f-nucleolus of Γ.*

Theorem 2.1. *If $\mathcal{X}(\Gamma)$ is non-empty then the f-nucleolus is non-empty and contains a unique point. Algorithm 1 gives a point in the f-least core of Γ after each step and terminates after at most $n-1$ steps.*

Proof. By induction, using an argument which is similar to the one in the proof of Lemma 2.1, we can easily prove that the LP (2) has an optimal solution for every $k \ge 1$. The fact that $x_k \in \mathcal{LC}_f(\Gamma)$ for every $k \ge 1$ is trivial. We now prove by induction that

$$\mathcal{N}_f(\Gamma) \subset P_k \neq \emptyset \tag{3}$$

holds for every $k \ge 1$ and that the number of steps is bounded by $n-1$. P_k is non-empty because x^{k-1} belongs to P_k for every $k \ge 2$ and $P_1 = \mathcal{X}(\Gamma) \neq \emptyset$. If $\mathcal{N}_f(\Gamma)$ is empty then (3) is true. We consider the case that $\mathcal{N}_f(\Gamma)$ is non-empty and prove (3) by induction. With $k = 1$, (3) holds since

$$\mathcal{N}_f(\Gamma) \subset \mathcal{X}(\Gamma) = P_1.$$

Assume that (3) holds for $k = \bar{k}$, i.e., $\mathcal{N}_f(\Gamma) \subset P_{\bar{k}}$. Clearly, for each vector $\tilde{x}$ in the f-nucleolus, $(\tilde{x}, \varepsilon^{\bar{k}})$ is an optimal solution of (2) with $k = \bar{k}$. If it is not the case, then $(\tilde{x}, \varepsilon^{\bar{k}})$ is infeasible. But since $\tilde{x} \in \mathcal{N}_f(\Gamma) \subset P_{\bar{k}}$, we have

$$\min_{S \in \mathcal{S}_{\bar{k}}} e_f(S, \tilde{x}) < \varepsilon^{\bar{k}}.$$

Let $(\bar{x}, \varepsilon^{\bar{k}})$ be an optimal solution of (2) with $k = \bar{k}$, then

$$\min_{S \in \mathcal{S}_{\bar{k}}} e_f(S, \tilde{x}) < \varepsilon^{\bar{k}} = \min_{S \in \mathcal{S}_{\bar{k}}} e_f(S, \bar{x}).$$

On the other hand, since $\tilde{x}, \bar{x} \in P_{\bar{k}}$, there holds

$$e_f(S, \tilde{x}) = e_f(S, \bar{x}),\ \forall S \in \{T \in \mathcal{S}_1 \mid \chi_T \in \text{span}\mathcal{A}_{\bar{k}}\} = \mathcal{S}_1 \backslash \mathcal{S}_{\bar{k}}.$$

Therefore, we have

$$\{e_f(S,\bar{x}) \mid S \in \mathcal{S}_1:\ e_f(S,\bar{x}) < \varepsilon^{\bar{k}}\} \subsetneq \{e_f(S,\tilde{x}) \mid S \in \mathcal{S}_1:\ e_f(S,\tilde{x}) < \varepsilon^{\bar{k}}\}.$$

From this it follows that

$$\theta_f(\bar{x}) \succ \theta_f(\tilde{x}),$$

which contradicts the assumption that $\tilde{x}$ belongs to the f-nucleolus. Now let $\tilde{x}$ be a vector in the f-nucleolus $\mathcal{N}_f(\Gamma)$. Since $(\tilde{x}, \varepsilon^{\bar{k}})$ is an optimal solution of (2) with $k = \bar{k}$, due to the complementary slackness theorem, we have

$$\tilde{x}(S) = c(S) - \varepsilon^{\bar{k}} f(S),\ \forall S \in \mathcal{B}_{\bar{k}+1}.$$

Hence, since $\mathcal{N}_f(\Gamma) \subset P_{\bar{k}}$, and from the definition of $P_{\bar{k}+1}$, there holds

$$\tilde{x} \in P_{\bar{k}+1}.$$

It means that (3) holds for $k = \bar{k} + 1$.

We now consider the k-th. step of Algorithm 1 with $|\mathcal{A}_k| < n$. Trivially, the set $\mathcal{S}_k$ is non-empty. Let (λ^k, μ^k) be a dual optimal solution of (2). Then we have

$$\sum_{S \in \mathcal{S}_k} f(S)\lambda_S^k \geq 1 > 0.$$

Therefore, since $f(S) > 0$ for all $S \in \mathcal{S}_k$, the set Π_{k+1} is non-empty. Hence, $\mathcal{B}_{k+1}$ is non-empty and

$$|\mathcal{A}_{k+1}| - |\mathcal{A}_k| \geq 1.$$

On the other hand, we have $\mathcal{A}_1 = \{\chi_N\}$. Therefore

$$k \leq |\mathcal{A}_k| \leq n,\ \forall 1 \leq k \leq n.$$

So there exists $1 \leq k \leq n$ such that

$$|\mathcal{A}_k| = n. \tag{4}$$

Let $k^* \leq n$ be the smallest number k satisfying (4). The algorithm stops after $k^* - 1$ steps. Clearly, $\mathcal{A}_{k^*}$ is independent. Therefore, P_{k^*} contains exactly a point, since P_{k^*} is non-empty due to (3). Moreover, due to (3), if the f-nucleolus is non-empty then it contains a unique point, namely, the point in P_{k^*}.

In the following, we prove that the point in P_{k^*} belongs to the f-nucleolus. Denote this point by x^*. For each $1 \leq j < k^*$, since the optimal solution (x^j, ε^j) of (2) with $k = j$ is also a feasible solution of (2) with $k = j + 1$, we have

$$\varepsilon^j \leq \varepsilon^{j+1}. \tag{5}$$

As already mentioned, with $|\mathcal{A}_j| < n$ the set $\mathcal{B}_{j+1}$ is non-empty. Let T be a subset of N that belongs to $\mathcal{B}_{j+1}$. Clearly, $T \in \mathcal{S}_j \backslash \mathcal{S}_{j+1}$, and hence there holds

$$\min_{S \in \mathcal{S}_j \backslash \mathcal{S}_{j+1}} e_f(S,x) \leq e_f(T,x) = \varepsilon^j,\ \forall x \in P_{j+1}. \tag{6}$$

On the other hand, since $x^j \in P_{j+1}$, we have

$$e_f(S,x) = e_f(S,x^j),\ \forall S \in \mathcal{S}_1 \backslash \mathcal{S}_{j+1},\ x \in P_{j+1}. \tag{7}$$

As (x^j, ε^j) is a feasible solution of (2) with $k = j$, there holds

$$e_f(S, x^j) \geq \varepsilon^j,\ \forall S \in \mathcal{S}_j. \tag{8}$$

Combining (7), $\mathcal{S}_j \subset \mathcal{S}_1$ and (8) yields

$$e_f(S, x) \geq \varepsilon^j,\ \forall S \in \mathcal{S}_j \backslash \mathcal{S}_{j+1},\ x \in P_{j+1}. \tag{9}$$

From (6) and (9) follows

$$\min_{S \in \mathcal{S}_j \backslash \mathcal{S}_{j+1}} e_f(S, x) = \varepsilon^j,\ \forall x \in P_{j+1}.$$

Especially, with $x = x^*$, we have

$$\min_{S \in \mathcal{S}_j \backslash \mathcal{S}_{j+1}} e_f(S, x^*) = \varepsilon^j. \tag{10}$$

Since $\mathcal{S}_{k^*} = \emptyset$, there holds

$$\mathcal{S}_k = \cup_{j=k}^{k^*-1} (\mathcal{S}_j \backslash \mathcal{S}_{j+1}),\ \forall 1 \leq k < k^*. \tag{11}$$

Combining (5), (10) and (11) yields

$$\min_{S \in \mathcal{S}_k} e_f(S, x^*) = \varepsilon^k,\ \forall 1 \leq k < k^*. \tag{12}$$

Now let y be an arbitrary vector in $\mathcal{X}(\Gamma) \backslash \{x^*\}$. As $P_1 = \mathcal{X}(\Gamma)$ and $P_{k^*} = \{x^*\}$, and since

$$\mathcal{X}(\Gamma) \backslash \{x^*\} = \cup_{j=1}^{k^*-1} (P_j \backslash P_{j+1}),$$

there exists $1 \leq l < k^*$ satisfying

$$y \in P_l \backslash P_{l+1},$$

i.e., there exists a set $U \in \mathcal{B}_{l+1} \subset \Pi_{l+1}$ such that

$$x(U) + \varepsilon^l f(U) \neq c(U). \tag{13}$$

Recalling

$$\Pi_{l+1} = \{S \in \mathcal{S}_l \,|\, \lambda_S^l > 0\},$$

where (λ^l, μ^l) is a dual optimal solution of (2) with $k = l$. Clearly, it holds

$$\min_{S \in \mathcal{S}_l} e_f(S, y) < \varepsilon^l. \tag{14}$$

Otherwise, since $y \in P_l$ and ε^l is the optimal value of (2) with $k = l$, (y, ε^l) is an optimal solution of (2) with $k = l$, which contradicts (13) due to the complementary slackness theorem. On the other hand, since $x^*, y \in P_l$, we have

$$e_f(S, x^*) = e_f(S, y),\ \forall S \in \mathcal{S}_1 \backslash \mathcal{S}_l. \tag{15}$$

From (12), (14) and (15) follows

$$\{e_f(S, x^*) \,|\, S \in \mathcal{S}_1 :\ e_f(S, x^*) < \varepsilon^l\} \subsetneq \{e_f(S, y) \,|\, S \in \mathcal{S}_1 :\ e_f(S, y) < \varepsilon^l\}.$$

That means

$$\theta_f(x^*) \succ \theta_f(y).$$

Therefore, x^* belongs to the f-nucleolus $\mathcal{N}_f(\Gamma)$. $\square$

3 Ticket Pricing as a Cooperative Game

To apply the framework of Section 2 to the ticket pricing problem, we define a suitable cost allocation game $\Gamma = (N, c, P)$. Consider a railway network as a graph $G = (V, E)$, and let $N \subseteq V \times V$ be a set of origin-destination (OD) pairs, between which passengers want to travel, i.e., we consider each (set of passengers of an) OD-pair as a player. We next define the cost $c(S)$ of a coalition $S \subseteq N$ as the minimum operation cost of a network of railway lines in G that service S. Using the classical line planning model of [4], $c(S)$ can be computed by solving the integer program

$$
\begin{aligned}
c(S) := \min_{(\xi,\rho)} \quad & \sum_{(r,f)\in\mathcal{R}\times\mathcal{F}} (c^1_{r,f}\xi_{r,f} + c^2_{r,f}\rho_{r,f}) \\
s.t. \quad & \sum_{r\in\mathcal{R}, r\ni e} \sum_{f\in\mathcal{F}} c_{cap} f(m\xi_{r,f} + \rho_{r,f}) \geq \sum_{i\in S} P^i_e, \ \forall e \in E \\
& \sum_{r\in\mathcal{R}, r\ni e} \sum_{f\in\mathcal{F}} f\xi_{r,f} \geq F^i_e, \ \forall (i,e) \in S \times E \\
& \rho_{r,f} - (M - m)\xi_{r,f} \leq 0, \ \forall (r,f) \in \mathcal{R}\times\mathcal{F} \\
& \sum_{f\in\mathcal{F}} \xi_{r,f} \leq 1, \ \forall r \in \mathcal{R} \\
& \xi \in \{0,1\}^{|\mathcal{R}\times\mathcal{F}|}, \ \rho \in \mathbb{Z}^{|\mathcal{R}\times\mathcal{F}|}_{\geq 0}.
\end{aligned}
$$

The model assumes that the P^i passengers of each OD-pair i travel on a unique shortest path $\mathcal{P}^i$ (with respect to some distance in space or time) through the network, such that demands P^i_e on capacities of edges e arise, and, likewise, demands F^i_e on frequencies of edges. These demands can be covered by a set $\mathcal{R}$ of possible routes (or lines) in G, which can be operated at a (finite) set of possible frequencies $\mathcal{F}$, and with a minimal and maximal number of wagons m and M in each train. c_{cap} is the capacity of a wagon, $c^1_{r,f}$ and $c^2_{r,f}$, $(r,f) \in \mathcal{R}\times\mathcal{F}$, are cost coefficients for the operation of route r at frequency f. The variable $\xi_{r,f}$ equals 1 if route r is operated at frequency f, and 0 otherwise, while variable $\rho_{r,f}$ denotes the number of wagons in addition to m on route r with frequency f. The constraints guarantee sufficient capacity and frequency on each edge, link the two types of route variables, and ensure that each route is operated at a single frequency.

Finally, we define the polyhedron P, which gives conditions on the prices x that the players are asked to pay, as follows. Let (u_{j-1}, u_j), $j = 1, \ldots, l$, be OD-pairs such that u_j, $j = 0, \ldots, l$, belong to the travel path $\mathcal{P}^{st}$ associated with some OD-pair (s,t), $u_0 = s$, and $u_l = t$, and let (u,v) be an arbitrary OD-pair such that u and v also lie on the travel path $\mathcal{P}^{st}$ from s to t. We then stipulate that the prices x_i/P_i, which individual passengers of OD-pair i have to pay, must satisfy the monotonicity properties

$$0 \leq \frac{x_{uv}}{P_{uv}} \leq \frac{x_{st}}{P_{st}} \leq \sum_{j=1}^{l} \frac{x_{u_{j-1}u_j}}{P_{u_{j-1}u_j}}.$$

$\Gamma = (N, c, P)$ defines a cost allocation game to determine cost-covering prices for using the railway network G, in which coalitions S consider the option to bail out of the common system and set up their own, private one. Computing prices in the f-least core or the f-nucleolus of Γ requires to solve several linear programs of type (2). This can be done using cutting planes. We start with a (small) subset $\emptyset \neq \Sigma \subset \mathcal{S}_k$ and consider the LP obtained from (2) by deleting the constraints corresponding to the coalitions $S \in \mathcal{S}_k \backslash \Sigma$, i.e.,

$$\begin{aligned} & \max_{(x,\varepsilon)} \ \varepsilon \\ s.t. \ & x(S) + \varepsilon f(S) \leq c(S), \ \forall S \in \Sigma \\ & x \in P_k. \end{aligned} \tag{16}$$

Let (x^*, ε^*) be an optimal solution of this LP. The separation problem for (x^*, ε^*) is to find a coalition $T \in \mathcal{S}_k$ such that (x^*, ε^*) violates the constraint

$$x(T) + \varepsilon f(T) \leq c(T). \tag{17}$$

Recalling $f = \alpha + \beta|\cdot| + \gamma c$. If $\varepsilon^* \geq 0$, then, since $\alpha, \beta \geq 0$, there holds for each $S \in \Sigma$

$$x^*(S) + \varepsilon^* \gamma c(S) \leq x^*(S) + \varepsilon^* f(S). \tag{18}$$

On the other hand, since (x^*, ε^*) is a feasible solution of (16), we have

$$x^*(S) + \varepsilon^* f(S) \leq c(S). \tag{19}$$

From (18), (19), $x^*(S) \geq 0$ and $c(S) > 0$ follows

$$\varepsilon^* \gamma \leq 1. \tag{20}$$

Trivially, the inequality (20) holds for $\varepsilon^* < 0$, as well. Therefore, the separation problem can be formulated for our application as the integer program

$$\begin{aligned} \max_{(\xi,\rho,z)} & \sum_{i \in N} (x_i^* + \beta \varepsilon^*) z_i + (\gamma \varepsilon^* - 1) \sum_{(r,f) \in \mathcal{R} \times \mathcal{F}} (c^1_{r,f} \xi_{r,f} + c^2_{r,f} \rho_{r,f}) + \alpha \varepsilon^* \\ s.t. \quad & \sum_{r \in \mathcal{R}, r \ni e} \sum_{f \in \mathcal{F}} c_{cap} f (m \xi_{r,f} + \rho_{r,f}) - \sum_{i \in N} P^i_e z_i \geq 0, \ \forall e \in E \\ & \sum_{r \in \mathcal{R}, r \ni e} \sum_{f \in \mathcal{F}} f \xi_{r,f} - F^i_e z_i \geq 0, \ \forall (i,e) \in N \times E \\ & \rho_{r,f} - (M - m) \xi_{r,f} \leq 0, \ \forall (r,f) \in \mathcal{R} \times \mathcal{F} \\ & \sum_{f \in \mathcal{F}} \xi_{r,f} \leq 1, \ \forall r \in \mathcal{R} \\ & \xi \in \{0,1\}^{|\mathcal{R} \times \mathcal{F}|}, \ \rho \in \mathbb{Z}^{|\mathcal{R} \times \mathcal{F}|}_{\geq 0}, \ z \in \chi_{\mathcal{S}_k}. \end{aligned} \tag{21}$$

A violated constraint exists iff the optimum is larger than 0. If the optimal value is not positive, then (x^*, ε^*) is a feasible solution of (2). Otherwise, we can find a feasible solution $(\bar{\xi}, \bar{\rho}, \bar{z})$ of (21) with a positive objective function value. Define $T := \{i \in N \,|\, \bar{z}_i = 1\}$, then (x^*, ε^*) violates the constraint (17).

4 Fair IC Ticket Prices

We now use our ansatz to compute ticket prices for the intercity network of the Netherlands, which is shown in Figure 1. Our data is a simplified version of that published in [4], namely, we consider all 23 cities, but reduce the number of OD-pairs to 85 by removing pairs with small demand. However, with $2^{85}-1$ possible coalitions, the problem is still very large. Solving LP (2) (with $\mathcal{S}_k = \mathcal{S}_1 = 2^N \setminus \{\emptyset, N\}$), and separating the coalition constraints by solving IPs (21), we determine a point x^* in the c-least core (i.e., $f = c$) and define *c-least core ticket prices* (lc-prices) for each OD-pair i as $p_i^* := x_i^*/P^i$.

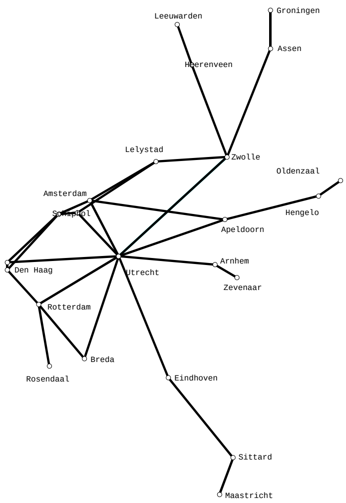

Figure 1: The intercity network of the Netherlands.

Figure 2 and Figure 3 compare these lc-prices p^* with the distance dependent prices $\overline{p}$ that have been used by the railway operator NS Reizigers for this network as reported in [2]. Figure 2 plots the relative c-profits $\frac{c(S)-x(S)}{c(S)}$ with respect to $x = x^*$ and $x = \overline{x} = \overline{p} \circ P$ ($\circ$ denotes the coordinate-wise product) of some 8000 coalitions computed in the course of the cutting plane algorithm, and sorted in non-decreasing order. Note that the core of this particular game is empty, and some coalitions have to pay more than their cost. The maximum c-loss of any coalition with respect to the lc-prices is a mere 1.1%. This hardly noticeable unfairness is in stark contrast with the 25.67% maximum c-loss with respect to the distance prices. In fact, there are 10 other coalitions with losses of more than 20%. Even worse, the coalition with the maximum loss is the main coalition of passengers traveling in the center of the country, i.e., in our model, a major coalition would earn a substantial benefit from shrinking the network.

How do the lc-prices look like? Figure 3 plots the distribution of the ratio between the lc-prices and the distance prices. It can be seen that lc-prices are lower, equal, or slightly higher for most passengers. However, some passengers, mainly in the periphery of the country, pay more to cover the costs that they produce. The increment factor is at most 3.78 except for two OD-pairs. The top of the list is the OD-pair Den Haag HS to Den Haag CS, which gets 14.4 times more expensive. The reason is that the travel path of this OD-pair consists of a single edge that is not used by any other travel route, i.e., the network is *too dense* at this point.

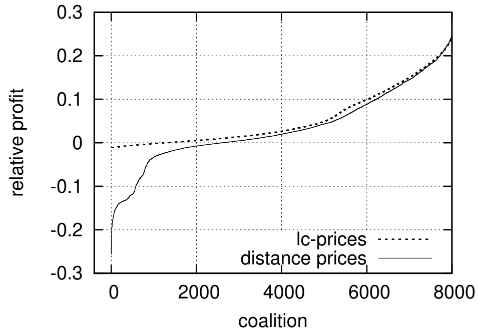

Figure 2: Relative profits of coalitions.

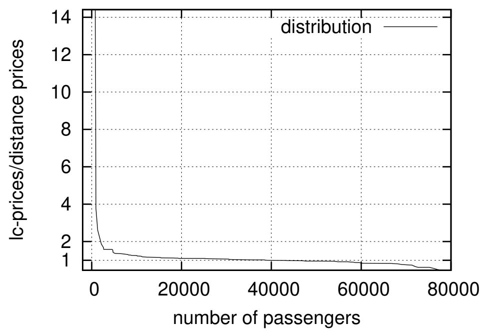

Figure 3: The distribution of the ratio $\frac{lc\text{-}prices}{distance\ prices}$.

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