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A Perturbation Result for Dynamical Contact Problems [1]

[1]Supported by the DFG Research Center Matheon, "Mathematics for key technologies: Modelling, simulation, and optimization of real-world processes", Berlin

ZIB-Report 08-27 (July 2008)

A Perturbation Result for Dynamical Contact Problems[†]

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July 10, 2008

Abstract

This paper is intended to be a first step towards the continuous dependence of dynamical contact problems on the initial data as well as the uniqueness of a solution. Moreover, it provides the basis for a proof of the convergence of popular time integration schemes as the Newmark method.

We study a frictionless dynamical contact problem between both linearly elastic and viscoelastic bodies which is formulated via the Signorini contact conditions. For viscoelastic materials fulfilling the Kelvin-Voigt constitutive law, we find a characterization of the class of problems which satisfy a perturbation result in a non-trivial mix of norms in function space. This characterization is given in the form of a stability condition on the contact stresses at the contact boundaries.

Furthermore, we present perturbation results for two well-established approximations of the classical Signorini condition: The Signorini condition formulated in velocities and the model of normal compliance, both satisfying even a sharper version of our stability condition.

AMS MSC 2000: 35L85, 74H55, 74M15

Keywords: Dynamical contact problems, stability, (visco-)elasticity, Signorini condition, Newmark method

1 Introduction

One of the most popular time discretization schemes for dynamical contact problems is the Newmark method. Unfortunately, this scheme may lead to artificial numerical oscillations at dynamical contact boundaries and an undesirable energy blow-up during time integration may occur [5, 20]. In [17], Kane et al. suggested a variant of Newmark's method which is energy dissipative at contact. Unfortunately, this scheme is still unable to circumvent the undesirable oscillations at contact boundaries, for which reason Deuflhard et al. suggested a contact-stabilized Newmark method [5, 20]. Up to now, the question of convergence of Newmark schemes in the

[†]Supported by the DFG Research Center MATHEON, "Mathematics for key technologies: Modelling, simulation, and optimization of real-world processes", Berlin

presence of contact has completely been avoided in both the engineering and the mathematical literature – a difficult problem due to the non-smoothness at contact boundaries. Our aim is to apply the established proof technique for discretizations of evolution problems by Hairer et al. (also known as "Lady Windermere's Fan", cf. [11]) to the contact-stabilized Newmark method. For this purpose, the necessary first step is to find a norm in which we can expect a perturbation result even in the presence of contact.

The classical approach modeling contact phenomena between elastic bodies employs Signorini's contact conditions which are based on the non-penetrability of mass and lead to nonsmooth and nonlinear variational inequalities. The first existence and uniqueness results for evolution problems in elasticity were obtained by Duvaut and Lions [6]. They studied the special case of prescribed normal stresses where the contact surface is known in advance. For the case of linear elasticity in conjunction with Signorini's contact conditions, to date, existence has only been provided in some simple geometric settings and for one-dimensional problems. A general theory for multi-dimensional dynamical contact problems is still missing.

Basically, the serious mathematical difficulties with the well-posedness of purely elastic problems result from the irregularity of the velocities at contact. However, the assumption of viscous material behavior allows the derivation of existence results. In [13] and [15], Jarušek proved the existence of a weak solution for the dynamical frictionless Signorini problem between a viscoelastic body with singular memory and a rigid foundation. More recently, the existence of weak solutions in viscoelasticity with Kelvin-Voigt constitutive law has been studied by Cocou [3] and Kuttler and Shillor [23]. Migòrski and Ochal [27] established the existence for a class of unilateral viscoelastic contact problems modeled by dynamical hemivariational inequalities. In 2008, Ahn and Stewart [1] proved an existence result for a frictionless dynamical contact problem between a linearly viscoelastic material of Kelvin-Voigt type and a rigid obstacle. A survey of existence and uniqueness results is given in the monograph [9].

Unfortunately, there are still fundamental and unresolved mathematical difficulties even in the analysis of viscoelastic contact. These are caused by the Signorini conditions on the unknown displacement field itself. Therefore, Jarušek and Eck investigated the solvability of dynamical contact problems with unilateral contact constraints on the velocity field (cf. [14, 16]). This approach yields the monotonicity of the corresponding multivalued operator.

Martins and Oden [26] proposed the normal compliance condition, likewise leading to a problem with a much simpler mathematical structure. Their model assumes that the normal stresses on the contact surface depend only on the normal displacement field which results in a relaxation of the non-penetration of mass. They presented existence and uniqueness results for linearly elastic and viscoelastic materials, but unfortunately their proof of uniqueness exhibits a fundamental error in the estimation of norms. Their model of normal compliance was used in various papers, see, e.g., [4, 19, 21, 22] and the monograph [18]. One of its main advantages is the higher regularity of the solutions in time [24]. However, for the medical applications that we have in mind (such as the movement of the knee joint, see [20]), a mutual interpenetration of the bodies is unacceptable and normal compliance models are ruled out.

Most of the papers cited above concern existence and uniqueness results for dynamical contact problems. However, to the best of our knowledge, there are still no mathematical results concerning the continuous dependence on the initial data. The lack of well-posedness results mainly originates from the hyperbolic structure of the problem which leads to shocks at the contact interfaces. The Signorini conditions in displacements seem to avoid a general regularity of such problems.

The paper is organized as follows. In Section 2, we will consider the frictionless dynamical contact problem between two linearly elastic bodies based on Signorini's conditions. We will give a short description of the underlying physical and mathematical model. Then, we will point out the essential mathematical difficulties in the derivation of a perturbation result for such materials. To this end, in Section 3, we will introduce the Kelvin-Voigt model for viscoelastic materials. We will find a characterization of a class of problems for which it is possible to prove a perturbation result in a non-trivial choice of mixed norms in function space. In Section 4, we will end with two famous approximations of the Signorini condition in linear viscoelasticity, namely the Signorini conditions in velocities and the normal compliance model. For these, we will give perturbation results yielding even the uniqueness of the approximated solutions.

2 The Signorini condition in linear elasticity

The first two sections of this paper deal with a perturbation result for dynamical contact problems with Signorini conditions in displacements. To the best of our knowledge, there exist no results concerning continuous dependence on the initial data in the mathematical literature, neither in the purely elastic nor in the viscoelastic case.

In Section 2.1, we will give a short description of the classical contact problem in linear elasticity which is formulated via the Signorini conditions. Then, in Section 2.2, we will analyze the fundamental problems with a characterization of linearly elastic contact problems which are satisfying a perturbation result.

2.1 Theoretical Background

We use the same model of dynamical contact between two linearly elastic bodies as in [5] which is based on Signorini's contact conditions. For the convenience of the reader, we will briefly present the notation of the paper and the formulation of the underlying mathematical model.

Notation. All domains treated here are understood to be bounded subsets in $\mathbb{R}^d$ with $d = 2, 3$ and indices i, j, l, m run from 1 to d throughout the paper. Let the two bodies be identified with the domains Ω^K, $K \in \{S, M\}$ where S and M stand for slave and master body, respectively. Let the solution be decomposed according to $\mathbf{u} = (\mathbf{u}^S, \mathbf{u}^M)$. Each of the boundaries $\partial\Omega^K$ with associated outward directed normal $\boldsymbol{\nu}^K$ shall be Lipschitz and is decomposed into three disjoint parts: Γ_D^K, the Dirichlet boundary, Γ_N^K, the Neumann boundary, and Γ_C^K, the possible contact boundary, see Figure 1. We assume a non-vanishing Dirichlet boundary, i.e. $meas(\Gamma_D) > 0$. The actual contact boundary is not known in advance, but is assumed to be contained in a compact strict subset of Γ_C^K. Set $\Omega = \Omega^S \cup \Omega^M$ and $\Gamma_* = \Gamma_*^S \cup \Gamma_*^M$ for $* \in \{D, N, C\}$.

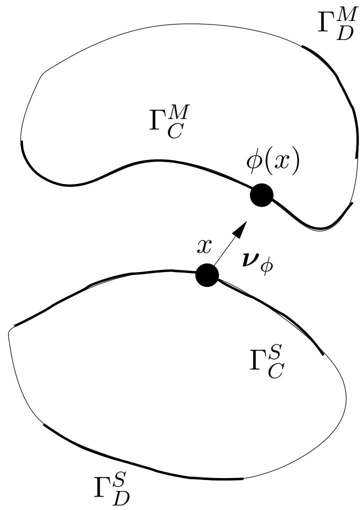

Figure 1: Two body contact problem and decomposition of the boundary

Tensor and vector quantities are written in bold characters, e.g., $\boldsymbol{\sigma}$ and $\mathbf{v}$ with components σ_{ij} and v_i, respectively. The partial derivative with respect to the spatial variable x_j is indicated by a subindex j, e.g., $v_{,j}$. The dot ($\dot{}$) denotes the derivative with respect to the time $t > 0$. We write the Euclidean vector norm in $\mathbb{R}^d$ as $|\cdot|$. Let $\mathbf{L}^2(\Omega^K) = (L^2(\Omega^K))^d$ and $\mathbf{L}^2 = \mathbf{L}^2(\Omega^S) \times \mathbf{L}^2(\Omega^M)$. The Sobolev space of functions with weak derivative in $\mathbf{L}^2(\Omega^K)$ is denoted by $\mathbf{H}^1(\Omega^K) = (H^1(\Omega^K))^d$ and $\mathbf{H}^{-1}(\Omega^K)$ is the corresponding dual space. For Dirichlet boundary conditions we define the subspaces

$$\mathbf{H}_D^1(\Omega^K) = \left\{ \mathbf{v} \,\middle|\, \mathbf{v} \in \mathbf{H}^1(\Omega^K),\, \mathbf{v}|_{\Gamma_D^K} = \mathbf{0} \right\} ,$$

and $\mathbf{H}_D^1 = \mathbf{H}_D^1(\Omega^S) \times \mathbf{H}_D^1(\Omega^M)$. Let $\mathbf{H}_D^1$ be equipped with the norm

$$\|\mathbf{v}\|^2_{\mathbf{H}_D^1} = \|\mathbf{v}^S\|^2_{\mathbf{H}^1(\Omega^S)} + \|\mathbf{v}^M\|^2_{\mathbf{H}^1(\Omega^M)} .$$

Scalar products are written in the form $(\cdot,\cdot)_{\mathbf{L}^2(\Omega^K)}$ and $(\cdot,\cdot)_{\mathbf{H}^1(\Omega^K)}$ with induced norms $\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega^K)} = (\mathbf{v},\mathbf{v})_{\mathbf{L}^2(\Omega^K)}$, $\|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega^K)} = (\mathbf{v},\mathbf{v})_{\mathbf{H}^1(\Omega^K)}$. For a Banach space $\mathbf{X}$ and $0 < T < \infty$ we denote by $C([0,T],\mathbf{X})$ all continuous functions $\mathbf{v} : [0,T] \to \mathbf{X}$.

The space $\mathbf{L}^2(0,T;\mathbf{X})$ consists of all measurable functions $\mathbf{v} : (0,T) \to \mathbf{X}$ for which

$$\|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{X})} = \int_0^T \|\mathbf{v}(t)\|^2_{\mathbf{X}} \, \mathrm{d}t < \infty$$

holds. We identify $\mathbf{L}^2$ with its dual space and obtain the evolution triple

$$\mathbf{H}^1 \subset \mathbf{L}^2 \subset \mathbf{H}^{-1}$$

with dense, continuous and compact embeddings (cf., e.g., [31]). With reference to this evolution triple, the Sobolev space $\mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ means the set of all functions $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1)$ that have generalized derivatives $\dot{\mathbf{v}} \in \mathbf{L}^2(0,T;\mathbf{H}^{-1})$.

Non-penetration condition. At the contact interface Γ_C, the two bodies may come into contact but must not penetrate each other. We assume a bijective mapping $\phi : \Gamma_C^S \longrightarrow \Gamma_C^M$ between the two possible contact surfaces to be given and, following [8], we define linearized non-penetration with respect to ϕ by

$$[\mathbf{u} \cdot \boldsymbol{\nu}]_\phi(x,t) = \mathbf{u}^S(x,t) \cdot \boldsymbol{\nu}_\phi(x) - \mathbf{u}^M(\phi(x),t) \cdot \boldsymbol{\nu}_\phi(x) \leq g(x)\,, \quad x \in \Gamma_C^S\,.$$

This condition is given with respect to the initial gap

$$\Gamma_C^S \ni x \mapsto g(x) = |x - \phi(x)| \in \mathbb{R}$$

between the two bodies in the reference configuration and we have set

$$\boldsymbol{\nu}_\phi = \begin{cases} \dfrac{\phi(x) - x}{|\phi(x) - x|}\,, & \text{if} \quad x \neq \phi(x)\,, \\ \boldsymbol{\nu}^S(x) = -\boldsymbol{\nu}^M(x)\,, & \text{if} \quad x = \phi(x)\,. \end{cases}$$

Variational problem formulation. For the weak formulation of the contact problem, we denote the convex set of all admissible displacements by

$$\mathcal{K} = \{\mathbf{v} \in \mathbf{H}^1_D \,|\, [\mathbf{v} \cdot \boldsymbol{\nu}]_\phi \leq g\}\,. \tag{1}$$

For the data we assume $\mathbf{f}(\cdot,t) \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\pi}(\cdot,t) \in \mathbf{H}^{-1/2}(\Gamma_N)$ for all $t \in [0,T]$. On $\mathbf{H}^1_D$, the linear functional f_{ext}, which accounts for the volume forces and the tractions on the Neumann boundary, is given by

$$f_{\mathrm{ext}}(\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbf{L}^2} + \langle \boldsymbol{\pi}, \mathbf{v} \rangle_{\mathbf{H}^{-1/2}(\Gamma_N) \times \mathbf{H}^{1/2}(\Gamma_N)}\,.$$

The internal forces can be written as a bilinear form

$$a(\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{E}\,\boldsymbol{\varepsilon}(\mathbf{v}^K) : \boldsymbol{\varepsilon}(\mathbf{w}^K) \, \mathrm{d}x\,, \qquad \mathbf{v},\mathbf{w} \in \mathbf{H}^1$$

with the elasticity tensor $\mathbf{E}$ and the linearized second-order strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)\,, \qquad \mathbf{v} \in \mathbf{H}^1\,.$$

Then, the sum of internal and external forces can be represented by

$$\langle \mathbf{F}(\mathbf{w}), \mathbf{v} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} = a(\mathbf{w}, \mathbf{v}) - f_{\mathrm{ext}}(\mathbf{v}) , \qquad \mathbf{v}, \mathbf{w} \in \mathbf{H}^1 .$$

Via integration by parts and exploiting the boundary conditions, see [7] and [18], we can write the contact problem in the weak formulation as a hyperbolic variational inequality: For almost every $t \in [0,T]$ find $\mathbf{u}(\cdot,t) \in \mathcal{K}$ with $\ddot{\mathbf{u}}(\cdot,t) \in \mathbf{H}^{-1}$ such that

$$\langle \ddot{\mathbf{u}}, \mathbf{v}-\mathbf{u} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{F}(\mathbf{u}), \mathbf{v}-\mathbf{u} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \geq 0 , \qquad \forall\, \mathbf{v} \in \mathcal{K} \tag{2}$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega . \tag{3}$$

Incorporating the constraints $\mathbf{v}(t) \in \mathcal{K}$ for almost all $t \in [0,T]$ by the characteristic functional $I_{\mathcal{K}}(\mathbf{v})$,

$$I_{\mathcal{K}}(\mathbf{v}) = \begin{cases} 0 & \text{if } \mathbf{v} \in \mathcal{K} \\ \infty & \text{else} \end{cases} , \qquad \mathbf{v} \in \mathbf{H}^1_D$$

the variational inequality (2) can equivalently be formulated as the variational inclusion

$$0 \in \ddot{\mathbf{u}} + \mathbf{F}(\mathbf{u}) + \partial I_{\mathcal{K}}(\mathbf{u}) \tag{4}$$

utilizing the subdifferential $\partial I_{\mathcal{K}}$ of $I_{\mathcal{K}}$ (see, e.g., [10]). For a given solution $\mathbf{u}$ of this variational inequality, we define for almost every $t \in [0,T]$ the contact forces $\mathbf{F}_{\mathrm{con}}(\mathbf{u}) \in \mathbf{H}^{-1}$ via

$$\langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \quad = \quad \langle \ddot{\mathbf{u}} + \mathbf{F}(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} , \qquad \mathbf{v} \in \mathbf{H}^1 . \tag{5}$$

2.2 A perturbation result

In the following we analyze the continuous dependence on the initial data for the linearly elastic contact problem presented above. We start with a formal derivation of a relation which describes the dynamical behavior of an initial perturbation in the energy norm of the system.

Let $\mathbf{u}$ and $\tilde{\mathbf{u}}$ be two solutions of the linearly elastic contact problem (2) with initial values $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ and $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ in Ω. We assume $\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}$ to be contained in $\mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ which is of course not satisfied in general. Nevertheless, we formally integrate (5) from 0 to t with $t \in [0,T]$ and we use $\mathbf{v} = \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}$ as a trial function. This yields

$$\begin{aligned} &\int_0^t \langle \ddot{\mathbf{u}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1}\ \mathrm{d}s + \int_0^t a(\mathbf{u}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}})\ \mathrm{d}s \\ &= \int_0^t f_{\mathrm{ext}}(\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}})\ \mathrm{d}s + \int_0^t \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}), \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1}\ \mathrm{d}s . \end{aligned}$$

Performing the same formal calculation with (5) for $\tilde{\mathbf{u}}$ instead of $\mathbf{u}$, we can subtract the resulting equation for $\tilde{\mathbf{u}}$ from the equation for $\mathbf{u}$ above. By the linearity of the functional f_{ext} we find

$$\begin{aligned}
&\int_0^t \langle \ddot{\mathbf{u}} - \ddot{\tilde{\mathbf{u}}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \, \mathrm{d}s + \int_0^t a(\mathbf{u} - \tilde{\mathbf{u}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}) \, \mathrm{d}s \\
&= \int_0^t \langle \mathbf{F}_{\text{con}}(\mathbf{u}) - \mathbf{F}_{\text{con}}(\tilde{\mathbf{u}}), \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \, \mathrm{d}s \,.
\end{aligned}$$

Under the regularity assumption $\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ on the velocities, we can reformulate the left-hand side of this expression by applying integration by parts in time (see, e.g., Prop. 23.23 in [31]). This calculation gives rise to the following result.

Lemma 2.1. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (2) *with initial values* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *Assume that* $\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$. *Then, for all* $t \in [0,T]$,

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\
&= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + 2 \int_0^t \langle \mathbf{F}_{\text{con}}(\mathbf{u}(s)) - \mathbf{F}_{\text{con}}(\tilde{\mathbf{u}}(s)), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s) \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \, ds \,.
\end{aligned} \tag{6}$$

In the absence of contact, the two left-hand terms and the first two right-hand terms in (6) show the continuous dependence of the solution from the initial values. In the presence of contact, this structure is disturbed by the additional integral term on the right-hand side.

In order to estimate this term, we need information on the time derivatives of the displacements at the contact boundaries. Unfortunately, a purely elastic formulation is in general not able to provide such information. This fact originates from the underlying assumption that the dependence of internal stresses on velocities can be neglected. Therefore, there is no way to define a class of elastic contact problems just by demanding a kind of stability on the contact stresses. A perturbation result for linearly elastic contact problems would necessarily require estimates on the velocities at the contact boundaries.

Actually, this problem is linked to the additional regularity assumption on the solutions which is needed for the derivation of Lemma 2.1. Due to the hyperbolic structure of purely elastic contact problems, the Signorini solutions are not as smooth as the initial data allow. Therefore, in the following section, we will turn to materials with viscoelastic behavior.

3 The Signorini condition in viscoelasticity

In this section we will characterize a class of viscoelastic contact problems which are stable with respect to perturbations of the initial data.

In Section 3.1, we will present the weak formulation of the dynamical contact problem for viscoelastic materials of Kelvin-Voigt type with Signorini's contact conditions. After that, in Section 3.2, we will discuss the so-called persistency condition for linearly elastic problems in comparison with the viscoelastic case. It will play a major role in the interpretation of our stability condition for viscoelastic contact problems. Finally, in the subsequent Section 3.3, we will present a characterization of a class of viscoelastic contact problems which satisfy continuous dependence on the initial data. Under this condition we will prove a perturbation result in a special mix of norms in function space.

3.1 Theoretical Background

In the following, we assume the materials under consideration to be linearly viscoelastic, i.e., the stresses $\boldsymbol{\sigma}$ satisfy the Kelvin-Voigt constitutive relation

$$\boldsymbol{\sigma}\left(\mathbf{u}^K, \dot{\mathbf{u}}^K\right) = \mathbf{E}^K \,\boldsymbol{\varepsilon}(\mathbf{u}^K) + \mathbf{V}^K \,\boldsymbol{\varepsilon}(\dot{\mathbf{u}}^K)\,, \qquad \mathbf{u}^K, \dot{\mathbf{u}}^K \in \mathbf{H}^1$$

where $\mathbf{E}^K$ and $\mathbf{V}^K$ are the fourth-order elasticity and viscosity tensors, respectively (see, e.g., [12]). Both tensors are assumed to be sufficiently smooth (with $E^K_{ijml}, V^K_{ijml} \in L^\infty(\Omega^K)$), symmetric, and uniformly positive definite, i.e., there are constants $E_0, V_0 > 0$ such that

$$(\boldsymbol{E}\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq E_0|\boldsymbol{\zeta}|^2, \quad (\boldsymbol{V}\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq V_0|\boldsymbol{\zeta}|^2 \tag{7}$$

for all symmetric second-order tensors $\boldsymbol{\zeta} = (\zeta_{ij})$ where $|\boldsymbol{\zeta}| = \left(\sum_{i,j} \zeta_{ij}^2\right)^{1/2}$. When $\mathbf{V^K} = 0$, this constitutive law reduces to Hooke's law as used for linearly elastic materials.

Variational problem formulation. For the weak formulation, the convex set of all admissible displacements is given by

$$\mathcal{K} \;=\; \{\mathbf{v} : (0,T) \to \mathbf{H}^1_D \,|\, \dot{\mathbf{v}} \in \mathbf{H}^1,\; [\mathbf{v}\cdot\boldsymbol{\nu}]_\phi \leq g\}\,. \tag{8}$$

The viscous part of the internal forces can be written as the bilinear form

$$b(\mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{V}\,\boldsymbol{\varepsilon}(\mathbf{v}^K) \colon \boldsymbol{\varepsilon}(\mathbf{w}^K)\;\mathrm{d}x\,, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{H}^1$$

which can be represented by

$$\langle \mathbf{G}(\mathbf{w}), \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} = b(\mathbf{w}, \mathbf{v})\;, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{H}^1\;.$$

As in the linearly elastic case, we write the viscoelastic contact problem in the weak formulation as a hyperbolic variational inequality: For almost every $t \in [0,T]$ find $\mathbf{u}(\cdot,t) \in \mathcal{K}$ with $\mathbf{u}(\cdot,t) \in C([0,T],\mathbf{H}^1)$ and $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ such that

$$\langle \ddot{\mathbf{u}}, \mathbf{v}-\mathbf{u}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{F}(\mathbf{u}), \mathbf{v}-\mathbf{u}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v}-\mathbf{u}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \geq 0\,, \qquad \forall\, \mathbf{v} \in \mathcal{K} \tag{9}$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega\,. \tag{10}$$

We define the contact forces $\mathbf{F}_{\text{con}}(\mathbf{u}) \in \mathbf{H}^{-1}$ for almost every $t \in [0,T]$ as

$$\langle \mathbf{F}_{\text{con}}(\mathbf{u}), \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \quad = \quad \langle \ddot{\mathbf{u}} + \mathbf{F}(\mathbf{u}) + \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1}\,, \qquad \mathbf{v} \in \mathbf{H}^1\,. \tag{11}$$

As shown, for instance, in [1] and [23] the unilateral contact problem between a viscoelastic body and a rigid foundation has at least one weak solution.

Remark. Note that the initial conditions (10) are formulated in a meaningful way: The generalized derivative $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ is determined only up to changes on a set of measure zero on $[0,T]$. However, since the embedding $\mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2) \subset C(0,T;\mathbf{L}^2)$ is continuous (cf. [31], sec. 33.1), there exists a uniquely determined representative

$$\dot{\mathbf{u}} \in C([0,T],\mathbf{L}^2)\,.$$

The initial conditions (10) are to be understood in this sense.

Korn's inequality. On Ω, the inequality of Korn

$$c_k\,\|\mathbf{v}\|^2_{\mathbf{H}^1} \leq \|\mathbf{v}\|^2_{\mathbf{L}^2} + \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{\mathbf{L}^2}\,, \qquad \forall\, \mathbf{v} \in \boldsymbol{H}^1$$

holds where $c_k > 0$ is a constant depending only on Ω and Γ_D. Under our additional assumption that $\Gamma_D \subset \partial\Omega$ is connected with $meas(\Gamma_D) > 0$, the inequality reduces to

$$c_k\,\|\mathbf{v}\|^2_{\mathbf{H}^1} \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{\mathbf{L}^2}\,, \qquad \forall\, \mathbf{v} \in \boldsymbol{H}^1_D\,. \tag{12}$$

With the ellipticity condition (7), it follows that $\|\cdot\|_{\mathbf{H}^1}$ and $a(\cdot,\cdot)^{1/2}$ are equivalent norms on $\mathbf{H}^1$. A proof of Korn's inequality can, for instance, be found in [28] and [29].

Gronwall's inequality. Let δ, λ be two mappings from an interval $[0,T]$ into $[0,\infty)$. Assume δ is continuous, λ is integrable, $C \in [0,\infty)$ and

$$\delta(t) \leq C + \int_0^t \lambda(s)\delta(s)\ \mathrm{d}s\,, \qquad \forall\, t \in [0,T]\,.$$

Then

$$\delta(t) \leq C \exp\left(\int_0^t \lambda(s)\ \mathrm{d}s\right)\,, \qquad \forall\, t \in [0,T]\,.$$

A proof of this generalized version of the inequality of Gronwall can be found in [30].

3.2 The notorious persistency condition

In the case of linear elasticity, we expect the energy of the whole system to be constant in time. As far as we know, up to now, energy conservation can only be shown for solutions $\mathbf{u}$ with sufficiently smooth velocities $\dot{\mathbf{u}}$ and under an additional assumption at the contact boundary, known as *persistency condition* (see, e.g., [25]). Analyzing the proof of energy conservation in [25], we find that this condition can be weakened to

$$\langle \mathbf{F}_{\text{con}}(\mathbf{u}), \dot{\mathbf{u}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} = 0\,, \quad \text{a.e. } t \in [0,T]\,. \tag{13}$$

The solutions of the purely linearly elastic contact problem are in general not smooth enough to satisfy the persistency condition (13). By contrast, we are able to prove the validity of this condition for solutions of our viscoelastic problem. For this aim, we follow the presentation for unilateral contact problems in [1] and references therein.

Theorem 3.1. *Let* $\mathbf{u}(\cdot,t) \in \mathcal{K}$ *fulfill* (9) *with* $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$*. Then* $\mathbf{u}$ *satisfies the generalized persistency condition* (13)*.*

Proof. The definition of the contact forces and the continuity of the linearly elastic forces (see, e.g., Lemma 1.1 in [8]) lead to $\mathbf{F}_{\text{con}}(\mathbf{u}) \in \mathbf{L}^2(0,T;\mathbf{H}^{-1})$ since

$$\begin{aligned}
&\|\mathbf{F}_{\text{con}}(\mathbf{u})\|_{\mathbf{L}^2(0,T;\mathbf{H}^{-1})} \\
&\le c\left(\|\ddot{\mathbf{u}}\|_{\mathbf{L}^2(0,T;\mathbf{H}^{-1})} + \|\mathbf{F}(\mathbf{u})\|_{\mathbf{L}^2(0,T;\mathbf{H}^{-1})} + \|\mathbf{G}(\mathbf{u})\|_{\mathbf{L}^2(0,T;\mathbf{H}^{-1})}\right) \\
&\le c\left(\|\ddot{\mathbf{u}}\|_{\mathbf{L}^2(0,T;\mathbf{H}^{-1})} + \|\mathbf{u}\|_{\mathbf{L}^2(0,T;\mathbf{H}^{1})} + \|\dot{\mathbf{u}}\|_{\mathbf{L}^2(0,T;\mathbf{H}^{1})}\right) \\
&< \infty\,.
\end{aligned}$$

By the generalized main theorem of calculus (compare, e.g., prob. 23.5 in [31]), the time derivative $\dot{\mathbf{u}}(t)$ exists for almost every $t \in [0,T]$ even in the classical sense because the generalized derivative $\dot{\mathbf{u}}$ is contained in $\mathbf{L}^2(0,T;\mathbf{H}^1)$. We choose t such that $\dot{\mathbf{u}}(t)$ exists and condition (9) holds. Then,

$$\left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t)), \tfrac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} \right\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \quad = \quad \begin{cases} \ge 0 & \text{if } h > 0 \\ \le 0 & \text{if } h < 0 \end{cases}$$

and, for almost every $t \in [0,T]$, we find

$$\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t)), \dot{\mathbf{u}}(t) \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} = \lim_{h\to 0} \left\langle \mathbf{F}_{\text{con}}(\mathbf{u}(t)), \frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} \right\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} = 0\,.$$

□

The weakened persistency condition will play an important role in the interpretation of our stability condition for viscoelastic contact problems presented in the next section.

3.3 A perturbation result

In what follows, we analyze the continuous dependence of linearly viscoelastic contact problems with Signorini conditions on the initial data. Viscosity leads to higher regularity of the solutions which justifies the formal calculation for linearly elastic problems as performed in Section 2.2. Furthermore, linearly viscoelastic models are in general capable of providing information on the time derivatives of the solutions at the contact boundaries.

We start with a result for the linearly viscoelastic problem corresponding to Lemma 2.1 in the purely elastic case.

Lemma 3.2. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (9) *with initial values* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *Then, for all* $t \in [0,T]$,

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\
&+ 2 \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s)) \; ds \\
&= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + 2 \int_0^t \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \mathbf{F}_{\mathrm{con}}(\tilde{\mathbf{u}}(s)), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s) \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \; ds\,.
\end{aligned} \tag{14}$$

Proof. As in the linearly elastic case, we integrate the viscoelasticity equation (9) for $\mathbf{u}$ and $\tilde{\mathbf{u}}$ from 0 to t, test them with $\mathbf{v} = \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}$, subtract one from the other and obtain

$$\begin{aligned}
&\int_0^t \langle \ddot{\mathbf{u}} - \ddot{\tilde{\mathbf{u}}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \; \mathrm{d}s + \int_0^t a(\mathbf{u} - \tilde{\mathbf{u}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}) \; \mathrm{d}s + \int_0^t b(\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}, \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}) \; \mathrm{d}s \\
&= \int_0^t \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}) - \mathbf{F}_{\mathrm{con}}(\tilde{\mathbf{u}}), \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \; \mathrm{d}s\,.
\end{aligned}$$

Integration by parts (see, e.g., Prop. 23.23 in [31]) leads to the expression of the theorem. □

Even in the case of linear viscoelasticity, the dynamical contact problem with Signorini conditions may be ill-posed in the presence of contact. Hence, we give a characterization of problems for which the continuous dependence of the solutions on the initial values holds. The result presented in the lemma above shows that we have to impose such a stability condition on the time integral over the contact forces applied to the velocities.

Stability condition. Measuring the perturbation in the displacements and the velocities in the canonical norm on $\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$, we demand for all $t \in [0,T]$

$$\begin{aligned} &\left| \int_0^t \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(t)) - \mathbf{F}_{\mathrm{con}}(\tilde{\mathbf{u}}(t)), \dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t) \rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1(\Omega)} \,\mathrm{d}t \right| \\ &\le \epsilon(t) \left(\|\kappa\,(\mathbf{u}-\tilde{\mathbf{u}})\|_{\mathbf{L}^2(0,t;\mathbf{H}^1(\Omega))} + \left\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\right\|_{\mathbf{L}^2(0,t;\mathbf{H}^1(\Omega))} \right) \left\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\right\|_{\mathbf{L}^2(0,t;\mathbf{H}^1(\Omega))} \end{aligned} \tag{15}$$

where $0 \le \kappa \in L^2(0,T)$ and $\epsilon(t) \ge 0$ is sufficiently small.

The precise meaning of the requirement "ϵ sufficiently small" will be given in the proof of the following perturbation theorem. We will see that, for the derivation of a perturbation result in the viscoelastic case, the validity of this demand is absolutely fundamental.

For the class of viscoelastic problems satisfying the stability condition above, the following perturbation result holds.

Theorem 3.3. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (9) *with initial conditions* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *For* $T > 0$, *assume the stability condition* (15) *with* (17). *Then, for all* $t \in [0,T]$,

$$\begin{aligned} &\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\ &+ 2\left(\alpha - \frac{\sup_{s\in[0,T]}\epsilon(s)}{V_0\,c_k}\right) \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\; ds \\ &\le \left(\|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \right) \cdot e^{\int_0^t \tilde{\kappa}^2(s)\; ds} \end{aligned} \tag{16}$$

with $\alpha \in [0,1)$ *and* $\tilde{\kappa}^2(s) = \frac{\sup_{t\in[0,T]}\epsilon^2(t)}{2\,E_0\,V_0\,c_k^2(1-\alpha)} \cdot \kappa^2(s)$ *for* $s \in (0,T)$.

Proof. Using Lemma 3.2 the stability condition (15) leads for all $t \in (0,T)$ to

$$\begin{aligned} &\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\ &+ 2\int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\;\mathrm{d}s \\ &\le \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\ &\quad + 2\left| \int_0^t \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}(s)) - \mathbf{F}_{\mathrm{con}}(\tilde{\mathbf{u}}(s)), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s) \rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1}\;\mathrm{d}s \right| \\ &\le \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\ &\quad + 2\,\epsilon(t)\,\|\kappa\,(\mathbf{u}-\tilde{\mathbf{u}})\|_{\mathbf{L}^2(0,t;\mathbf{H}^1)} \left\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\right\|_{\mathbf{L}^2(0,t;\mathbf{H}^1)} + 2\,\epsilon(t) \left\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\right\|^2_{\mathbf{L}^2(0,t;\mathbf{H}^1)} . \end{aligned}$$

For an arbitrary parameter $\alpha \in [0,1)$, we use Young's inequality in the form

$$2ab \leq \frac{1}{2\,V_0\,c_k\,(1-\alpha)}\,a^2 + 2\,V_0\,c_k\,(1-\alpha)\,b^2$$

to find

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\
&+ 2 \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ \mathrm{d}s \\
&\leq \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + \tfrac{\epsilon^2(t)}{2\,V_0\,c_k\,(1-\alpha)}\ \|\kappa\,(\mathbf{u} - \tilde{\mathbf{u}})\|^2_{\mathbf{L}^2(0,t;\mathbf{H}^1)} + 2(\epsilon(t) + V_0\,c_k\,(1-\alpha))\,\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\|^2_{\mathbf{L}^2(0,t;\mathbf{H}^1)}\,.
\end{aligned}$$

Remember our assumption $meas(\Gamma_D) > 0$. For almost every $t > 0$, the velocities $\dot{\mathbf{u}}(t)$ are contained in $\mathbf{H}^1(\Omega)$ with a trace on the boundary. Hence, it follows from $\mathbf{u}(t) = 0$ on Γ_D that even $\dot{\mathbf{u}}(t) = 0$ on Γ_D for almost every t. Then, Korn's inequality (12) yields

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\
&+ 2 \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ \mathrm{d}s \\
&\leq \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + \tfrac{\epsilon^2(t)}{2\,E_0\,V_0\,c_k^2\,(1-\alpha)} \int_0^t \kappa^2(s)\,a(\mathbf{u}(s) - \tilde{\mathbf{u}}(s), \mathbf{u}(s) - \tilde{\mathbf{u}}(s))\ \mathrm{d}s \\
&\quad + 2\left(\tfrac{\epsilon(t)}{V_0\,c_k} + (1-\alpha)\right) \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ \mathrm{d}s
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\
&+ 2\left(\alpha - \tfrac{\epsilon(t)}{V_0\,c_k}\right) \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ \mathrm{d}s \\
&\leq \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + \tfrac{\epsilon^2(t)}{2\,E_0\,V_0\,c_k^2\,(1-\alpha)} \int_0^t \kappa^2(s)\,a(\mathbf{u}(s) - \tilde{\mathbf{u}}(s), \mathbf{u}(s) - \tilde{\mathbf{u}}(s))\ \mathrm{d}s\,.
\end{aligned}$$

In order to ensure the non-negativity of the integral term on the left-hand side, we demand $\epsilon(t)$ to be such small that

$$\frac{\epsilon(t)}{V_0\,c_k} < 1\,, \qquad \forall\, t \in [0,T]\,. \tag{17}$$

Introducing the supremum of $\epsilon(t)$ over the whole time intervall $[0,T]$ yields

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t)-\dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2}+a(\mathbf{u}(t)-\tilde{\mathbf{u}}(t),\mathbf{u}(t)-\tilde{\mathbf{u}}(t))\\
&+2\left(\alpha-\tfrac{\sup\epsilon(t)}{V_0\,c_k}\right)\int_0^t b(\dot{\mathbf{u}}(s)-\dot{\tilde{\mathbf{u}}}(s),\dot{\mathbf{u}}(s)-\dot{\tilde{\mathbf{u}}}(s))\,\mathrm{d}s\\
&\le \|\dot{\mathbf{u}}_0-\dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2}+a(\mathbf{u}_0-\tilde{\mathbf{u}}_0,\mathbf{u}_0-\tilde{\mathbf{u}}_0)\\
&\quad+\int_0^t \tilde{\kappa}^2(s)\,a(\mathbf{u}(s)-\tilde{\mathbf{u}}(s),\mathbf{u}(s)-\tilde{\mathbf{u}}(s))\,\mathrm{d}s
\end{aligned}$$

with

$$\tilde{\kappa}^2(s)=\frac{\sup\epsilon^2(t)}{2\,E_0\,V_0\,c_k^2(1-\alpha)}\cdot\kappa^2(s)\in L^2(0,T)\,.$$

We note that $\mathbf{u}\in C([0,T],\mathbf{H}^1)$ and $\dot{\mathbf{u}}\in C([0,T],\mathbf{L}^2)$. Together with Lebesgue's theorem and $\dot{\mathbf{u}}\in\mathbf{L}^2(0,T;\mathbf{H}^1)$, this fact yields the continuity of the left-hand side w.r.t. $t\in[0,T]$. Theorem 3.3 is a direct consequence of Gronwall's lemma as it has been presented in Section 3.1. □

3.4 Interpretation of the stability condition

In order to interpretate and to motivate our stability condition (15), our first aim is to localize the contact stresses on a part of the possible contact boundaries. On the basis of such a localization, we will give a sufficient criterion for the validity of the condition.

Our considerations are motivated by the intuition that perturbations in the contact forces are effective only on a small part of the contact boundaries, namely where the original solution is in contact and the perturbed is not, or vice versa. Due to lack of regularity of the solutions, however, we resort to a heuristic argumentation rather than a rigorous proof.

Localization of the contact stresses. For simplification, we assume that the possible contact boundaries and the bijective mappings between the two possible contact boundaries coincide, i.e. $\Gamma_C=\tilde{\Gamma}_C$ and $\phi=\tilde{\phi}$.

For $t>0$, we want to consider the part $\Gamma_C(t)$ of the possible contact boundaries where the solution of the dynamical viscoelastic contact problem is actually in contact. The natural definition of the *actual contact boundaries* is

$$\Gamma_C(t)=\{x\in\Gamma_C\,|\,[\mathbf{u}\cdot\boldsymbol{\nu}]_\phi=g\}\subset\Gamma_C\,,\qquad t\ge 0\,,$$

respectively

$$\tilde{\Gamma}_C(t)=\{x\in\Gamma_C\,|\,[\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi=g\}\subset\Gamma_C\,,\qquad t\ge 0$$

for the perturbed solution. Unfortunately, in the general case, the lack of regularity results for dynamical contact problems prohibits the introduction of actual contact boundaries in this way. This is due to the fact that the definition of the admissible set $\mathcal{K}$ yields only

$$[\mathbf{u}\cdot\boldsymbol{\nu}]_\phi \le g \qquad \text{for } \textit{almost every } x\in\Gamma_C\,,$$

i.e. up to boundary sets of measure zero. Hence, the definition above necessitates the additional assumption that the solutions $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are continuous on the possible contact boundaries Γ_C and $\tilde{\Gamma}_C$. This is satisfied, e.g., if $\mathbf{u}\in\mathbf{L}^2(0,T;\mathbf{H}^2(\Omega))$.

Now, we mean by the *critical part of the actual contact boundaries* the set where the solution is in contact and the perturbed solution is not, or vice versa. It is given by the symmetric difference

$$\begin{aligned}\Gamma_C^*(t) &= (\Gamma_C(t)\cup\tilde{\Gamma}_C(t))\setminus(\Gamma_C(t)\cap\tilde{\Gamma}_C(t))\\ &= \{x\in\Gamma_C(t)\cup\tilde{\Gamma}_C(t)\,|\,[\mathbf{u}\cdot\boldsymbol{\nu}]_\phi<g,\ [\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi=g \text{ or}\\ &\qquad\qquad [\mathbf{u}\cdot\boldsymbol{\nu}]_\phi=g,\ [\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi<g\}\,,\qquad t\ge 0\,.\end{aligned}$$

Finally, we introduce the space

$$\mathbf{V}_C = \{\mathbf{v}\in\mathbf{L}^2(0,T;\mathbf{H}_D^1)\,|\,\dot{\mathbf{v}}\in\mathbf{L}^2(0,T;\mathbf{H}^1),\ [\mathbf{v}\cdot\boldsymbol{\nu}]=0 \text{ on } \Gamma_C(t)\cap\tilde{\Gamma}_C(t)\}$$

containing functions which are zero on the intersection of the two actual contact boundaries.

Using these preliminary definitions we want to write for all $t\in[0,T]$ the difference of the contact forces as an operator on the critical contact boundary Γ_C^*, i.e.

$$\begin{aligned}&\textstyle\int_0^t\langle\mathbf{F}_{\mathrm{con}}(\mathbf{u})-\mathbf{F}_{\mathrm{con}}(\tilde{\mathbf{u}}),\dot{\mathbf{v}}\rangle_{\mathbf{H}^{-1}(\Omega)\times\mathbf{H}^1(\Omega)}\ \mathrm{d}s\\ &=\textstyle\int_0^t\langle(\hat{\boldsymbol{\sigma}}(\mathbf{u},\dot{\mathbf{u}})-\hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}},\dot{\tilde{\mathbf{u}}}))^*,\dot{\mathbf{v}}\rangle_{\mathbf{H}^{-1/2}(\Gamma_C^*(s))\times\mathbf{H}^{1/2}(\Gamma_C^*(s))}\ \mathrm{d}s\,,\qquad \mathbf{v}\in\mathbf{V}_C\end{aligned}\tag{18}$$

with a functional $(\hat{\boldsymbol{\sigma}}(\mathbf{u},\dot{\mathbf{u}})-\hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}},\dot{\tilde{\mathbf{u}}}))^*\in\mathbf{L}^2(0,T;\mathbf{H}^{-1/2}(\Gamma_C^*(t)))$.

We give a brief sketch how to validate the representation (18). The proof is based on a trace theorem which generalizes the definition of the normal stresses in the strong sense. Some formulations of such a theorem can be found, e.g., in [2] and [18]. We mention that it is possible to prove generalized versions of this theorem under less strict assumptions.

The first idea of the argumentation is the fact that the contact forces are supported by the union of the actual contact boundaries $\Gamma_C(t)\cup\tilde{\Gamma}_C(t)$. This yields a representation of the contact forces via a functional in $\mathbf{L}^2(0,T;\mathbf{H}^{-1/2}(\Gamma_C(t)\cup\tilde{\Gamma}_C(t))$. In a second idea, the reduction of the contact forces onto the subset $\Gamma_C^*(t)$ is motivated in analogy to the persistency condition in Section 3.2: If a function $\mathbf{v}\in\mathbf{V}_C$ satisfies $[\mathbf{v}(t)\cdot\boldsymbol{\nu}]=0$ on $\Gamma_C(t)\cap\tilde{\Gamma}_C(t)$, it follows that even $[\dot{\mathbf{v}}(t)\cdot\boldsymbol{\nu}]=0$ for almost every $x\in\Gamma_C(t)\cap\tilde{\Gamma}_C(t)$. Under the regularity assumption that $\operatorname{div}\boldsymbol{\sigma}(\mathbf{u},\dot{\mathbf{u}})\in\mathbf{L}^2(0,T;\mathbf{H}^{-1})$, the trace theorem mentioned above gives a representation of the contact forces via (18).

Note that the difference $\mathbf{u}-\tilde{\mathbf{u}}$ is contained in $\mathbf{V}_C$ by definition. Thus, we can reformulate Lemma 3.2 in the following way.

Lemma 3.4. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (9) *with initial values* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *For* $T > 0$, *assume that* (18) *is valid. Then,*

$$\begin{aligned} &\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\ &+ 2\int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\; ds \\ &= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\ &\quad + 2\int_0^t \langle (\hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}))^*,\ \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1/2}(\Gamma^*_C(s)) \times \mathbf{H}^{1/2}(\Gamma^*_C(s))}\; ds\,. \end{aligned} \tag{19}$$

Using this new representation of the integral term on the right-hand side, we find a variant of our stability condition (15) for the contact stresses. This variant is a sufficient criterion for the validity of the original stability condition.

Localized stability condition. For all $t \in [0,T]$, let

$$\begin{aligned} &\left\| (\hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}))^* \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1/2}(\Gamma^*_C(s)))} \\ &\le \epsilon(t) \left(\|\kappa\,(\mathbf{u} - \tilde{\mathbf{u}})\|_{\mathbf{L}^2(0,t;\mathbf{H}^1(\Omega))} + \left\| \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^1(\Omega))} \right) \end{aligned} \tag{20}$$

where $0 \le \kappa \in L^2(0,T)$ and $\epsilon(t) \ge 0$ is sufficiently small.

The quasistatic contact problem. In order to show that our stability condition (20) is reasonable, we want to discuss it for the special case of quasistatic contact problems. These problems follow from the dynamic viscoelastic contact problem (9) by setting

$$\ddot{\mathbf{u}} = 0$$

which yields the following problem formulation in the form of a variational inequality: For almost every $t \in [0,T]$ find $\mathbf{u}(\cdot,t) \in \mathcal{K}$ with $\mathbf{u}(\cdot,t) \in C([0,T],\mathbf{H}^1)$ and $\dot{\mathbf{u}} \in \mathbf{L}^2(0,T;\mathbf{H}^1)$ such that

$$\langle \mathbf{F}(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{H}^{-1} \times \mathbf{H}^1} + \langle \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v} \rangle_{\mathbf{H}^{-1} \times \mathbf{H}^1} \quad = \quad \langle \mathbf{F}_{\mathrm{con}}(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{H}^{-1} \times \mathbf{H}^1}\,, \qquad \forall\, \mathbf{v} \in \mathbf{H}^1$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega\,.$$

We assume the characterization (18) of the contact forces in the form of stresses at the contact boundaries to be valid. Then, the definition of the contact forces and the continuity of the linearly elastic forces (see, e.g., Lemma 1.1 in [8]) directly lead to

$$\begin{aligned}
&\left\| \hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}) \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1/2}(\Gamma_C \cup \tilde{\Gamma}_C))} \\
&\leq \epsilon(t) \left(\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\tilde{\mathbf{u}})\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1})} + \left\| \mathbf{G}(\dot{\mathbf{u}}) - \mathbf{G}(\dot{\tilde{\mathbf{u}}}) \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1})} \right) \\
&\leq \epsilon(t) \left(\|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1})} + \left\| \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1})} \right)
\end{aligned}$$

for almost every $t \in [0, T]$. Naturally, $\epsilon(t)$ is equal to zero if the solution $\mathbf{u}(t)$ as well as the perturbation $\tilde{\mathbf{u}}(t)$ have a vanishing actual contact boundary in the time intervall $[0, t]$. Under the assumption that we have for all $t \in (0, T)$ an estimate of the form

$$\begin{aligned}
&\left\| \left(\hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}) \right)^* \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1/2}(\Gamma_C^*(s)))} \\
&\leq \epsilon'(t) \left\| \hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}) \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1/2}(\Gamma_C \cup \tilde{\Gamma}_C))}
\end{aligned}$$

with $\epsilon'(t)$ sufficiently small, we find the stability condition (15) to be satisfied for quasistatic contact problems:

$$\begin{aligned}
&\left\| \left(\hat{\boldsymbol{\sigma}}(\mathbf{u}, \dot{\mathbf{u}}) - \hat{\boldsymbol{\sigma}}(\tilde{\mathbf{u}}, \dot{\tilde{\mathbf{u}}}) \right)^* \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{-1/2}(\Gamma_C^*(s)))} \\
&\leq \epsilon'(t)\, \epsilon(t) \left(\|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{L}^2(0,t;\mathbf{H}^{1})} + \left\| \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \right\|_{\mathbf{L}^2(0,t;\mathbf{H}^{1})} \right)
\end{aligned}$$

In the quasistatic case, our localized stability assumption is equivalent to the requirement that "$\epsilon'(t)\, \epsilon(t)$ is sufficiently small" for all $t \in [0, T]$. This corresponds to our intuition that, averaged over every complete time intervall $[0, t]$, the critical part of the actual contact boundaries is only a small part of the possible contact boundaries.

Our characterization seems to be reasonable at least in the case of a small spatial variation in the velocities. Thus, if the dynamical contact problem shows a behavior similar to the one of the quasistatic problem, it might satisfy continuous dependence on the initial data.

4 Viscoelastic Approximations of the Signorini condition

In the last section we will consider two well-established approximations of the viscoelastic contact problem with the classical Signorini condition in displacements: First we will proof a perturbation result for the contact problem where the Signorini contact constraints are formulated on the velocity field, as it is used by Jarušek and

Eck (compare, e.g., [9]). Then, we will analyze the widely-used normal compliance model of contact introduced by Martins and Oden in [26]. This model is based on a penalization of the exact Signorini condition in displacements. It will further motivate our stability condition as formulated in the previous section, since the approximating solution satisfies the assumption even in a sharper version.

4.1 The Signorini condition in velocities

Jarušek and Eck formulate the Signorini contact condition on the field of velocities instead of displacements, i.e. they replace the convex set (8) by

$$\mathcal{K} = \{\mathbf{v} : (0,T) \to \mathbf{H}^1_D \,|\, \dot{\mathbf{v}} \in \mathbf{H}^1,\ [\dot{\mathbf{v}} \cdot \boldsymbol{\nu}]_\phi \le 0\}\,. \tag{21}$$

This modified contact condition leads to a much simpler mathematical structure since it yields a monotonicity property of the contact problem. Unfortunately, employing the Signorini contact conditions in velocities describes the physical behavior correctly only in a short period of time. Once the two bodies have lost contact, they will never regain it. Thus, the bodies can only come into contact if they touch each other already at initial time. Actually, the model of Jarušek and Eck is only applicable to the process of loosing contact.

The weak formulation of the corresponding problem can be written as the following variational inequality: For almost every $t \in [0,T]$ find $\mathbf{u} \in C([0,T], \mathbf{H}^1)$ with $\dot{\mathbf{u}}(\cdot,t) \in \mathcal{K}$ and $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}([0,T], \mathbf{H}^1, \mathbf{L}^2)$ such that

$$\begin{aligned}&\langle \ddot{\mathbf{u}}, \mathbf{v} - \dot{\mathbf{u}}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{F}(\mathbf{u}), \mathbf{v} - \dot{\mathbf{u}}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v} - \dot{\mathbf{u}}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \\ &= \langle \mathbf{F}_{\mathrm{con}}(\dot{\mathbf{u}}), \mathbf{v} - \dot{\mathbf{u}}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} \ge 0\,, \qquad \forall\, \mathbf{v} \in \mathcal{K}\end{aligned} \tag{22}$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega\,. \tag{23}$$

Using the monotonicity property of the contact operator $\mathbf{F}_{\mathrm{con}}(\dot{\mathbf{u}})$, we find the following perturbation result:

Theorem 4.1. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (22) *with initial conditions* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *Assume that* $\phi = \tilde{\phi}$. *Then, for all* $t \in [0,T]$,

$$\begin{aligned}&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t) - \tilde{\mathbf{u}}(t), \mathbf{u}(t) - \tilde{\mathbf{u}}(t)) \\ &+ 2\int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ ds \\ &\le \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0)\,.\end{aligned} \tag{24}$$

Proof. We make exactly the same calculations as in the proof of Lemma 3.2 for the original Signorini problem in viscoelasticity and find

$$\begin{aligned}
&\|\dot{\mathbf{u}}(T) - \dot{\tilde{\mathbf{u}}}(T)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(T) - \tilde{\mathbf{u}}(T), \mathbf{u}(T) - \tilde{\mathbf{u}}(T)) \\
&+ 2 \int_0^T b(\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t), \dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)) \, \mathrm{d}t \\
&= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0) \\
&\quad + 2 \int_0^T \langle \mathbf{F}_{\text{con}}(\dot{\mathbf{u}}) - \mathbf{F}_{\text{con}}(\dot{\tilde{\mathbf{u}}}), \dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}} \rangle_{\mathbf{H}^{-1} \times \mathbf{H}^1} \, \mathrm{d}t \,.
\end{aligned}$$

The admissibility of the velocities $\dot{\mathbf{u}}$ and $\dot{\tilde{\mathbf{u}}}$ yields together with the variational inequality (22) the estimate of the theorem. $\square$

We remark that this perturbation result leads to the unique solvability of the dynamical contact problem with the Signorini condition in velocities.

Corollary 4.2. *There exists at most one solution of the dynamical contact problem* (22) *based on the Signorini condition in velocities.*

Proof. Assume that $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are two solutions of (22) with the same initial conditions. Then, Theorem 4.1 yields that

$$\|\mathbf{u}(t) - \tilde{\mathbf{u}}(t)\|_{\mathbf{H}^1} = 0 \,, \quad \|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|_{\mathbf{L}^2} = 0 \;\text{ for all } t \geq 0$$

and

$$\|\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}}\|_{\mathbf{L}^2(0,t;\mathbf{H}^1)} = 0 \;\text{ for a.e. } t \geq 0 \,.$$

This gives us the uniqueness of a solution $\mathbf{u} \in C([0,T], \mathbf{H}^1)$ with generalized derivative $\dot{\mathbf{u}} \in \mathbf{L}^2(0,T;\mathbf{H}^1)$. $\square$

4.2 The Normal Compliance Problem

We dedicate the last section of this paper to the continuous dependence of viscoelastic normal compliance problems on the initial data.

The normal compliance model of contact introduced by Martins and Oden is based on a penalty approximation of the exact Signorini conditions which leads to a higher regularity of the solutions. Introducing the penalty parameter $\epsilon > 0$, the contact forces $\mathbf{F}_{\text{con}}$ of the problem with Signorini conditions in displacements are replaced by the normal compliance operator $\boldsymbol{P}$ given by

$$\boldsymbol{P}^S\left(\mathbf{u}^S\right)(x^s) = -\frac{1}{\epsilon}\left([\mathbf{u}^S \cdot \boldsymbol{\nu}]_\phi - g\right)_+ \cdot \boldsymbol{\nu}_\phi \,, \qquad \text{on } \Gamma_C^S \times [0,T] \tag{25}$$

and

$$\boldsymbol{P}^M\left(\mathbf{u}^M\right)\left(x^M\right) = -\boldsymbol{P}^S\left(\mathbf{u}^S\right)\left(\phi^{-1}\left(x^M\right)\right) \,, \qquad \text{on } \Gamma_C^M \times [0,T] \tag{26}$$

where $(\cdot)_+ = \max(0,\cdot)$ denotes the positive part of a function. Defining

$$\mathcal{K} \,=\, \{\mathbf{v} : (0,T) \to \mathbf{H}^1_D \,|\, \dot{\mathbf{v}} \in \mathbf{H}^1\} \tag{27}$$

we can write the normal compliance problem in the weak formulation as a partial differential equality: For almost every $t \in [0,T]$ find $\mathbf{u}(\cdot,t) \in C([0,T],\mathbf{H}^1)$ with $\dot{\mathbf{u}} \in \mathbf{W}^{1,2}(0,T;\mathbf{H}^1,\mathbf{L}^2)$ such that

$$\begin{aligned}&\langle \ddot{\mathbf{u}}, \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{F}(\mathbf{u}), \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1} + \langle \mathbf{G}(\dot{\mathbf{u}}), \mathbf{v}\rangle_{\mathbf{H}^{-1}\times\mathbf{H}^1}\\ &= \langle \boldsymbol{P}(\mathbf{u}), \mathbf{v}\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}\,, \qquad \forall\, \mathbf{v} \in \mathcal{K}\end{aligned} \tag{28}$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),\ \dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x) \text{ in } \Omega\,. \tag{29}$$

For this penalty approach we are able to prove a perturbation result in the same mixed norm as used in the previous section for the original Signorini contact condition in displacements.

Theorem 4.3. *Let* $\mathbf{u}$ *and* $\tilde{\mathbf{u}}$ *be two solutions of* (28) *with initial conditions* $\mathbf{u}(x,0) = \mathbf{u}_0(x)$, $\dot{\mathbf{u}}(x,0) = \dot{\mathbf{u}}_0(x)$ *and* $\tilde{\mathbf{u}}(x,0) = \tilde{\mathbf{u}}_0(x)$, $\dot{\tilde{\mathbf{u}}}(x,0) = \dot{\tilde{\mathbf{u}}}_0(x)$ *in* Ω. *Then, for all* $t \in [0,T]$,

$$\begin{aligned}&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t)-\tilde{\mathbf{u}}(t), \mathbf{u}(t)-\tilde{\mathbf{u}}(t))\\ &+ 2\alpha \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ ds\\ &\leq \Big(\|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0)\Big)\cdot e^{\frac{2\,c_s^2}{\epsilon^2\,E_0\,V_0\,c_k^2\,(1-\alpha)}\cdot t}\end{aligned} \tag{30}$$

with $\alpha \in [0,1)$.

Proof. Performing the same calculations as in the proof of Lemma 3.2, we find the corresponding result with $\mathbf{F}_{\mathrm{con}}(\mathbf{u})$ replaced by $\boldsymbol{P}(\mathbf{u})$:

$$\begin{aligned}&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t)-\tilde{\mathbf{u}}(t), \mathbf{u}(t)-\tilde{\mathbf{u}}(t))\\ &+ 2 \int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\ \mathrm{d}s\\ &= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0)\\ &\quad + 2 \int_0^t \langle \boldsymbol{P}(\mathbf{u}(s)) - \boldsymbol{P}(\tilde{\mathbf{u}}(s)), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s)\rangle_{\mathbf{H}^{-1/2}(\Gamma)\times\mathbf{H}^{1/2}(\Gamma)}\ \mathrm{d}s\\ &= \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0 - \tilde{\mathbf{u}}_0, \mathbf{u}_0 - \tilde{\mathbf{u}}_0)\\ &\quad - \tfrac{2}{\epsilon} \int_0^t \int_{\Gamma_C^S} \Big[([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ - ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+\Big]\cdot[(\dot{\mathbf{u}} - \dot{\tilde{\mathbf{u}}})\cdot\boldsymbol{\nu}]_\phi\ \mathrm{d}a^S\ \mathrm{d}s\end{aligned}$$

Since the normal compliance model does not satisfy the persistency condition, we need an estimate for the contact forces on the whole contact boundaries. We arrive at the following case distinction:
If $([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ \cdot ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+ > 0$, then

$$|\,([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ - ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+\,| = |[(\mathbf{u}-\tilde{\mathbf{u}})\cdot\boldsymbol{\nu}]_\phi|$$

and if $([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ \cdot ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+ = 0$, we estimate

$$|\,([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ - ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+\,| \leq |[(\mathbf{u}-\tilde{\mathbf{u}})\cdot\boldsymbol{\nu}]_\phi|\,.$$

Thus, we conclude

$$\|\,([\mathbf{u}\cdot\boldsymbol{\nu}]_\phi - g)_+ - ([\tilde{\mathbf{u}}\cdot\boldsymbol{\nu}]_\phi - g)_+\,\|^2_{\mathbf{H}^{1/2}(\Gamma_C^S)} \leq 2\,\|\mathbf{u}-\tilde{\mathbf{u}}\|^2_{\mathbf{H}^{1/2}(\Gamma_C^S)}$$

which corresponds to the localized stability condition for the problem with Signorini's contact condition. We use Young's inequality in the form

$$2ab \leq \frac{c_s}{2\,V_0\,c_k\,(1-\alpha)}\,a^2 + \frac{2\,V_0\,c_k\,(1-\alpha)}{c_s}\,b^2$$

where $\alpha \in [0,1)$ and c_s denotes the Sobolev embedding constant. This leads to

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t)-\tilde{\mathbf{u}}(t), \mathbf{u}(t)-\tilde{\mathbf{u}}(t)) \\
&+ 2\int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\;\mathrm{d}s \\
&\leq \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0-\tilde{\mathbf{u}}_0, \mathbf{u}_0-\tilde{\mathbf{u}}_0) \\
&\quad + \tfrac{1}{\epsilon^2}\tfrac{2\,c_s}{V_0\,c_k\,(1-\alpha)}\int_0^t \|\mathbf{u}-\tilde{\mathbf{u}}\|^2_{\mathbf{H}^{1/2}(\Gamma_C)}\;\mathrm{d}s + \tfrac{2\,V_0\,c_k\,(1-\alpha)}{c_s}\int_0^t \|\dot{\mathbf{u}}-\dot{\tilde{\mathbf{u}}}\|^2_{\mathbf{H}^{1/2}(\Gamma_C)}\;\mathrm{d}s\,.
\end{aligned}$$

Korn's inequality (12) yields

$$\begin{aligned}
&\|\dot{\mathbf{u}}(t) - \dot{\tilde{\mathbf{u}}}(t)\|^2_{\mathbf{L}^2} + a(\mathbf{u}(t)-\tilde{\mathbf{u}}(t), \mathbf{u}(t)-\tilde{\mathbf{u}}(t)) \\
&+ 2\alpha\int_0^t b(\dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s), \dot{\mathbf{u}}(s) - \dot{\tilde{\mathbf{u}}}(s))\;\mathrm{d}s \\
&\leq \|\dot{\mathbf{u}}_0 - \dot{\tilde{\mathbf{u}}}_0\|^2_{\mathbf{L}^2} + a(\mathbf{u}_0-\tilde{\mathbf{u}}_0, \mathbf{u}_0-\tilde{\mathbf{u}}_0) + \tilde{\kappa}^2_\epsilon \int_0^t a(\mathbf{u}-\tilde{\mathbf{u}}, \mathbf{u}-\tilde{\mathbf{u}})\;\mathrm{d}s
\end{aligned}$$

where

$$\tilde{\kappa}^2_\epsilon = \frac{1}{\epsilon^2}\cdot\frac{2\,c_s^2}{E_0\,V_0\,c_k^2\,(1-\alpha)}\,.$$

Now, the result follows from Gronwall's lemma as presented in Section 3.1. $\square$

Note that our perturbation result yields the uniqueness of the normal compliance problem for two viscoelastic bodies of Kelvin-Voigt type.

Corollary 4.4. *There exists at most one solution of the normal compliance problem* (28)*.*

Proof. Compare the proof of Corollary 4.2. □

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