



RÜDIGER STEPHAN¹

On the cardinality constrained matroid polytope

¹Email: stephan@zib.de

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Abstract

In [8] it was shown how one can combine integer characterizations for cycle and path polytopes and Grötschel's cardinality forcing inequalities [6] to give facet defining integer representations for the cardinality restricted versions of these polytopes. Motivated by this work, we apply the same approach on the matroid polytope. It is well known that the so-called rank inequalities together with the nonnegativity constraints provide a complete linear description of the matroid polytope (see Edmonds [3]). By essentially adding the cardinality forcing inequalities, we obtain a complete linear description of the cardinality constrained matroid polytope which is the convex hull of the incidence vectors of those independent sets that have a feasible cardinality. Moreover, we show how the separation problem for the cardinality forcing inequalities can be reduced to that for the rank inequalities. We give also necessary and sufficient conditions for a cardinality forcing inequality to be facet defining.

1 Introduction

Let E be a finite set and \mathcal{I} a subset of the power set of E . The pair (E, \mathcal{I}) is called an *independence system* if (i) $\emptyset \in \mathcal{I}$ and (ii) whenever $I \in \mathcal{I}$ then $J \in \mathcal{I}$ for all $J \subset I$. If $I \subseteq E$ is in \mathcal{I} , then I is called an *independent set*, otherwise it is called a *dependent set*. Dependent sets $\{e\}$ with $e \in E$ are called *loops*. For any set $F \subseteq E$, $B \subseteq F$ is called a *basis* of F if $B \in \mathcal{I}$ and $B \cup \{e\}$ is dependent for all $e \in F \setminus B$. The *rank* of F is defined by $r_{\mathcal{I}}(F) := \max\{|B| : B \text{ basis of } F\}$. The set of all bases B of E is called a *basis system*. There are many different ways to characterize when an independence system is a matroid. For our purposes the following definition will be most comfortable. (E, \mathcal{I}) is called a *matroid*, and then it will be denoted by $M = (E, \mathcal{I})$, if

$$(iii) \quad I, J \in \mathcal{I}, |I| < |J| \Rightarrow \exists K \subseteq J \setminus I : |I \cup K| = |J|, K \cup I \in \mathcal{I}.$$

Equivalent to (iii) is the requirement that for each $F \subseteq E$ all its bases have the same cardinality. Throughout the paper we deal only with loopless matroids. The results of the paper can be easily brought forward to matroids containing loops.

Let $M = (E, \mathcal{I})$ be a matroid. A set $F \subseteq E$ is said to be *closed* if $r_{\mathcal{I}}(F) < r_{\mathcal{I}}(F \cup \{e\})$ for all $e \in E \setminus F$ and *inseparable* if there are no $F_1 \neq \emptyset \neq F_2$ with $F_1 \dot{\cup} F_2 = F$ such that $r_{\mathcal{I}}(F_1) + r_{\mathcal{I}}(F_2) \leq r_{\mathcal{I}}(F)$.

Given any independence system (E, \mathcal{I}) and any weights $w_e \in \mathbb{R}$ on the elements $e \in E$, the combinatorial optimization problem $\max w(I), I \in \mathcal{I}$, where $w(I) := \sum_{e \in I} w_e$, is called the *maximum weight independent set problem*. The convex hull of the incidence vectors of the feasible solutions $I \in \mathcal{I}$ is called the

independent set polytope and will be denoted by $P_{\mathcal{I}}(E)$. If (E, \mathcal{I}) is a matroid, then $P_{\mathcal{I}}(E)$ is also called the *matroid polytope*.

It is well-known, the maximum weight independent set problem on a matroid can be solved to optimality with the greedy algorithm. Moreover, the matroid polytope $P_{\mathcal{I}}(E)$ is determined by the rank inequalities and the nonnegativity constraints (see Edmonds [3]), i.e., $P_{\mathcal{I}}(E)$ is the set of all points $x \in \mathbb{R}^E$ satisfying

$$\begin{aligned} \sum_{e \in F} x_e &\leq r_{\mathcal{I}}(F) && \text{for all } \emptyset \neq F \subseteq E, \\ x_e &\geq 0 && \text{for all } e \in E. \end{aligned} \quad (1)$$

The rank inequality associated with F is facet defining for $P_{\mathcal{I}}(E)$ if and only if F is closed and inseparable (see Edmonds [3]).

Let $c = (c_1, \dots, c_m)$ be a finite sequence of integers with $0 \leq c_1 < c_2 < \dots < c_m$. Then, the *cardinality constrained independent set polytope* $P_{\mathcal{I}}^c(E)$ is defined to be the convex hull of the incidence vectors of the independent sets $I \in \mathcal{I}$ with $|I| = c_p$ for some $p \in \{1, \dots, m\}$, that is, $P_{\mathcal{I}}^c(E) = \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I}, |I| = c_p \text{ for some } p \in \{1, \dots, m\}\}$. If (E, \mathcal{I}) is a matroid, then $P_{\mathcal{I}}^c(E)$ is called the *cardinality constrained matroid polytope*. In the next section we will see that, if (E, \mathcal{I}) is a matroid, then the associated combinatorial optimization problem $\max w^T x$, $x \in P_{\mathcal{I}}^c(E)$ can be solved in polynomial time. Since c is linked to a cardinality constrained optimization problem, it is called a *cardinality sequence*.

Grötschel [6] gave a polyhedral analysis of the underlying basic problem of cardinality restrictions that enables us to provide a complete linear description of $P_{\mathcal{I}}^c(E)$. Given a finite set B and a cardinality sequence $c = (c_1, \dots, c_m)$, the set $\text{CHS}^c(B) := \{F \subseteq B : |F| = c_p \text{ for some } p\}$ is called a *cardinality homogenous set system*. The polytope associated with $\text{CHS}^c(B)$, namely the convex hull of the incidence vectors of elements of $\text{CHS}^c(B)$, is completely described by the *trivial inequalities* $0 \leq z_e \leq 1$, $e \in B$, the *cardinality bounds* $c_1 \leq \sum_{e \in B} z_e \leq c_m$, and the *cardinality forcing inequalities*

$$\begin{aligned} (c_{p+1} - |F|) \sum_{e \in F} z_e - (|F| - c_p) \sum_{e \in B \setminus F} z_e &\leq c_p(c_{p+1} - |F|) \\ \text{for all } F \subseteq B \text{ with } c_p < |F| < c_{p+1} &\text{ for some } p \in \{1, \dots, m-1\}, \end{aligned} \quad (2)$$

see Grötschel [6]. In [8] the cardinality forcing inequalities were successfully integrated in integer characterizations for cycle and path polytopes to provide characterizations of the integer points of cardinality constrained cycle and path polytopes by facet defining inequalities. Of course, with respect to $M = (E, \mathcal{I})$, $P_{\mathcal{I}}^c(E) = \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I} \cap \text{CHS}^c(E)\}$. By default, we assume that $c_m \leq r_{\mathcal{I}}(E)$. Our main result is that the system

$$\begin{aligned} \text{CF}_F(x) &:= (c_{p+1} - r_{\mathcal{I}}(F))x(F) - (r_{\mathcal{I}}(F) - c_p)x(E \setminus F) \leq c_p(c_{p+1} - r_{\mathcal{I}}(F)) \\ &\text{for all } F \subseteq E \text{ with } c_p < r_{\mathcal{I}}(F) < c_{p+1} \text{ for some } p \in \{0, \dots, m-1\}, \end{aligned} \quad (3)$$

$$x(E) \geq c_1, \quad (4)$$

$$x(E) \leq c_m, \quad (5)$$

$$x(F) \leq r_{\mathcal{I}}(F) \quad \text{for all } \emptyset \neq F \subseteq E, \quad (6)$$

$$x_e \geq 0 \quad \text{for all } e \in E \quad (7)$$

completely describes $P_{\mathcal{I}}^c(E)$. Here, for any $I \subseteq E$ we set $x(I) := \sum_{e \in I} x_e$. Of course, each $x \in P_{\mathcal{I}}^c(E)$ satisfies $c_1 \leq x(E) \leq c_m$. The cardinality forcing

inequality $\text{CF}_F(x) \leq c_p(c_{p+1} - r_{\mathcal{I}}(F))$ associated with F , where $c_p < r_{\mathcal{I}}(F) < c_{p+1}$, is valid as can be seen as follows. The incidence vector of any $I \in \mathcal{I}$ of cardinality at most c_p satisfies the inequality, since $r_{\mathcal{I}}(I \cap F) \leq c_p$:

$$\begin{aligned} (c_{p+1} - r_{\mathcal{I}}(F))\chi^I(F) - (r_{\mathcal{I}}(F) - c_p)\chi^I(E \setminus F) &\leq (c_{p+1} - r_{\mathcal{I}}(F))\chi^I(F) \\ &\leq (c_{p+1} - r_{\mathcal{I}}(F))c_p. \end{aligned}$$

The incidence vector of any $I \in \mathcal{I}$ of cardinality at least c_{p+1} satisfies also the inequality, since $r_{\mathcal{I}}(I \cap F) \leq r_{\mathcal{I}}(F)$ and thus $r_{\mathcal{I}}(I \cap (E \setminus F)) \geq c_{p+1} - r_{\mathcal{I}}(F)$:

$$\begin{aligned} &(c_{p+1} - r_{\mathcal{I}}(F))\chi^I(F) - (r_{\mathcal{I}}(F) - c_p)\chi^I(E \setminus F) \\ &\leq (c_{p+1} - r_{\mathcal{I}}(F))r_{\mathcal{I}}(F) - (r_{\mathcal{I}}(F) - c_p)\chi^I(E \setminus F) \\ &\leq (c_{p+1} - r_{\mathcal{I}}(F))r_{\mathcal{I}}(F) - (r_{\mathcal{I}}(F) - c_p)(c_{p+1} - r_{\mathcal{I}}(F)) \\ &= c_p(c_{p+1} - r_{\mathcal{I}}(F)). \end{aligned}$$

However, it is not hard to see that some incidence vectors of independent sets I with $c_p < |I| < c_{p+1}$ violate the inequality.

When $M = (E, \mathcal{I})$ is the trivial matroid, i.e., all $F \subseteq E$ are independent sets, then $\mathcal{I} \cap \text{CHS}^c(E) = \text{CHS}^c(E)$. Thus, cardinality constrained matroids are a generalization of cardinality homogenous set systems.

The paper is organized as follows. In Section 2 we prove that the system (3)-(7) provides a complete linear description of the cardinality constrained matroid polytope. Next, we will give sufficient conditions for the cardinality forcing inequalities to be facet defining. Finally, we show that the separation problem for the cardinality forcing inequalities can be reduced to that for the rank inequalities. This results in a polynomial time separation routine based on Cunningham's separation algorithm for the rank inequalities. In Section 3 we briefly discuss some consequences for cardinality constrained combinatorial optimization problems and in particular for the intersection of two cardinality constrained matroid polytopes.

2 Polyhedral analysis of $P_{\mathcal{I}}^c(E)$

Let $M = (E, \mathcal{I})$ be a matroid. As already mentioned, $P_{\mathcal{I}}(E)$ is determined by (1). For any natural number k , the independence system $M' := (E, \mathcal{I}')$ defined by $\mathcal{I}' := \{I \in \mathcal{I} : |I| \leq k\}$ is again a matroid and is called the k -truncation of M . Therefore, the matroid polytope $P_{\mathcal{I}'}^c(E)$ associated with the k -truncation of M is defined by system (1), where the rank inequalities are indexed with \mathcal{I}' instead of \mathcal{I} . Following an argument of Gamble and Pulleyblank [5], the only set of the k -truncation which maybe closed and inseparable with respect to the truncation, but not with respect to the original matroid M is E itself, and the rank inequality associated with E is the cardinality bound $x(E) \leq k$. Hence, in context of the original matroid M , $P_{\mathcal{I}'}^c(E)$ is described by

$$\begin{aligned} x(F) &\leq r_{\mathcal{I}}(F) \quad \text{for all } \emptyset \neq F \subseteq E, \\ x(E) &\leq k, \\ x_e &\geq 0 \quad \text{for all } e \in E. \end{aligned} \tag{8}$$

Of course, the connection to cardinality constraints is obvious, since $P_{\mathcal{I}'}^c(E) = P_{\mathcal{I}}^{(0, \dots, k)}(E)$. The basis system of M' is the set of all bases B of E (with respect

to M') and in case of $r_{\mathcal{I}}(E) \geq r_{\mathcal{I}'}(E)$ the bases are all of cardinality k . Assuming $r_{\mathcal{I}}(E) \geq r_{\mathcal{I}'}(E)$, the associated polytope

$$\text{conv}\{\chi^B \in \mathbb{R}^E : B \text{ basis of } E \text{ with respect to } M'\}$$

is determined by

$$\begin{aligned} x(F) &\leq r_{\mathcal{I}}(F) && \text{for all } \emptyset \neq F \subseteq E, \\ x(E) &= k, \\ x_e &\geq 0 && \text{for all } e \in E. \end{aligned} \tag{9}$$

On a basis system of a matroid one can optimize in polynomial time by application of the greedy algorithm. Thus, for each member c_p of a cardinality sequence $c = (c_1, \dots, c_m)$ an optimal solution I^p of the linear optimization problem $\max w(I)$, $I \in \mathcal{I}$, $|I| = c_p$ can be found in polynomial time. The best of the solutions I^p , $p = 1, \dots, m$ with respect to the linear objective w is then the optimal solution of $\max w(I)$, $I \in \mathcal{I} \cap \text{CHS}^c(E)$. Since $0 \leq c_1 < \dots < c_m \leq r_{\mathcal{I}}(E) \leq |E|$ and thus $m \leq |E|$, it can be found by at most $|E| + 1$ applications of the greedy algorithm.

These preliminary remarks are sufficient to present our main theorem. In the sequel, we denote the rank function by r instead of r_I . Given a valid inequality $ax \leq a_0$ with $a \in \mathbb{R}^E$, $F \subseteq E$ is said to be *tight* if $a\chi^F = a_0$. A valid inequality $ax \leq a_0$ is *dominated* by another valid inequality $bx \leq b_0$, if $\{x \in P_{\mathcal{I}}^c(E) : ax = a_0\} \subseteq \{x \in P_{\mathcal{I}}^c(E) : bx = b_0\}$. It is said to be *strictly dominated* by $bx \leq b_0$, if $\{x \in P_{\mathcal{I}}^c(E) : ax = a_0\} \subsetneq \{x \in P_{\mathcal{I}}^c(E) : bx = b_0\}$.

2.1 A complete linear description

Theorem 2.1. *The cardinality constrained matroid polytope $P_{\mathcal{I}}^c(E)$ is completely described by system (3)-(7).*

Proof. Since all inequalities of system (3)-(7) are valid, $P_{\mathcal{I}}^c(M)$ is contained in the polyhedron defined by (3)-(7). To show the converse, we consider any valid inequality $bx \leq b_0$ for $P_{\mathcal{I}}^c(M)$ and associate with the inequality the following subsets of E :

$$\begin{aligned} P &:= \{e \in E : b_e > 0\}, \\ Z &:= \{e \in E : b_e = 0\}, \\ N &:= \{e \in E : b_e < 0\}. \end{aligned}$$

We will show by case by case enumeration that the inequality $bx \leq b_0$ is dominated by some inequality of the system (3)-(7). By definition, $E = P \dot{\cup} Z \dot{\cup} N$, and hence, if $P = Z = N = \emptyset$, then $E = \emptyset$, and it nothing to show. By a scaling argument we may assume that either $b_0 = 1$, $b_0 = 0$, or $b_0 = -1$.

(1) $b_0 = -1$.

(1.1) $c_1 = 0$. Then $0 \in P_{\mathcal{I}}^c(E)$, and hence $0 = b \cdot 0 \leq -1$, a contradiction.

(1.2) $c_1 > 0$.

- (1.2.1) $P = Z = \emptyset, N \neq \emptyset$. Assume that there is some tight $I \in \mathcal{I}$ with $|I| = c_p, p \geq 2$. Then, for any $J \subset I$ with $|J| = c_1$ holds: $\chi^J \in P_{\mathcal{I}}^c(E)$ and $b\chi^J > b\chi^I = -1$, a contradiction. Therefore, if any $I \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|I| = c_1$. Thus, $bx \leq -1$ is dominated by the cardinality bound $x(E) \geq c_1$.
- (1.2.2) $P \cup Z \neq \emptyset, N = \emptyset$. Then, $by \geq 0$ for all $y \in P_{\mathcal{I}}^c(E)$, a contradiction.
- (1.2.3) $P \cup Z \neq \emptyset, N \neq \emptyset$. If $c_1 \leq r(P \cup Z)$, then there is some independent set $I \subseteq P \cup Z$ of cardinality c_1 , and hence, $b\chi^I \geq 0$, a contradiction. Thus, $c_1 > r(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight independent set J of cardinality c_p with $p \geq 2$. If $J \subseteq N$, then the incidence vector of any $K \subset J$ with $|K| = c_1$ violates $bx \leq -1$. Hence, $J \cap (P \cup Z) \neq \emptyset$. On the other hand, $J \cap N \neq \emptyset$ due to $c_p > c_1 > r(P \cup Z)$. However, by removing any $(c_p - c_1)$ elements in $N \cap J$, we obtain some independent set K of cardinality c_1 whose incidence vector violates the inequality $bx \leq -1$, a contradiction. Therefore, if any $T \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|T| = c_1$. Thus, $bx \leq -1$ is dominated by the bound $x(E) \geq c_1$.
- (2) $b_0 = 0$.
- (2.1) $P \cup Z \neq \emptyset, N = \emptyset$. Then, either $bx \leq 0$ is not valid or $b = 0$.
- (2.2) $P = \emptyset, Z \cup N \neq \emptyset$. Then, $bx \leq 0$ is dominated by the nonnegativity constraints $x_e \geq 0$ for $e \in N$ or $b = 0$.
- (2.3) $P \neq \emptyset, N \neq \emptyset$.
- (2.3.1) $c_1 > 0$. If $c_1 \leq r(P \cup Z)$, then there is some independent set $I \subseteq P \cup Z$ with $I \cap P \neq \emptyset$ of cardinality c_1 , and hence, $b\chi^I > 0$, a contradiction. Thus, $c_1 > r(P \cup Z)$. Assume, for the sake of contradiction, that there is some tight independent set J of cardinality c_p with $p \geq 2$. Since $c_p > c_1 > r(P \cup Z)$ and J is tight, $J \cap (P \cup Z) \neq \emptyset \neq J \cap N$. From here, the proof for this case can be finished as the proof for the case (1.2.3) with $b_0 = 0$ instead of $b_0 = -1$ in order to show that $bx \leq 0$ is dominated by the cardinality bound $x(E) \geq c_1$.
- (2.3.2) $c_1 = 0$. As in case (2.3.1), it follows immediately that $c_2 > r(P \cup Z)$, and if $I \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|I| = c_1 = 0$, that is, $I = \emptyset$, or $|I| = c_2$. Moreover, if $I \in \mathcal{I}$ with $|I| = c_2$ is tight, then follows $|I \cap (P \cup Z)| = r(P \cup Z)$. Hence, $bx \leq b_0$ is dominated by the cardinality forcing inequality $\text{CF}_F(x) \leq 0$ with $F = P \cup Z$.
- (3) $b_0 = 1$.
- (3.1) $P = \emptyset, Z \cup N \neq \emptyset$. Then, $b \leq 0$, and hence $bx \leq 1$ is dominated by any nonnegativity constraint $x_e \geq 0, e \in E$.
- (3.2) $P \cup Z \neq \emptyset, N = \emptyset$. Assume that there is some $I \in \mathcal{I}, I \notin \text{CHS}^c(E)$ with $|I| < c_m$ that violates $bx \leq 1$. Then, of course, all independent sets $J \supset I$ violate $bx \leq 1$, in particular, those J with $|J| = c_m$, a

contradiction. Hence, $bx \leq 1$ is not only a valid inequality for $P_{\mathcal{I}}^c(E)$ but also for $P_{\mathcal{I}}^{(0,1,\dots,c_m)}(E)$, that is, $bx \leq 1$ is dominated by some inequality of the system (8) with $k = c_m$.

(3.3) $P \neq \emptyset, N \neq \emptyset$. Let $p \in \{1, \dots, m\}$ be minimal such that there is a tight independent set I^* of cardinality c_p . Of course, $c_p > 0$, because otherwise I^* could not be tight. If $p = m$, then $bx \leq 1$ is dominated by the cardinality bound $x(E) \leq c_m$, because then all tight $J \in \mathcal{I} \cap \text{CHS}^c(E)$ have to be of cardinality $c_p = c_m$. So, let $0 < c_p < c_m$. We distinguish 2 subcases.

(3.3.1) $c_p \geq r(P \cup Z)$. Suppose, for the sake of contradiction, that there is some tight independent set I of cardinality c_p such that $|I \cap (P \cup Z)| < r(P \cup Z)$. Then, $I \cap (P \cup Z)$ could be completed to a basis B of $P \cup Z$, and since $|B| \leq |I|$, there is some $K \subseteq I \setminus B$ such that $I' := B \cup K \in \mathcal{I}$ and $|I'| = |I|$. K is maybe the empty set. Anyway, by construction, I' is of cardinality c_p and violates the inequality $bx \leq 1$. Thus, $|I \cap (P \cup Z)| = r(P \cup Z)$. For the same reason, any tight $J \in \mathcal{I} \cap \text{CHS}^c(E)$ satisfies $|J \cap (P \cup Z)| = r(P \cup Z)$, and since p is minimal, $|J| \geq c_p$. Now, with similar arguments as in case (1.2.3) one can show that if $T \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|T| = c_p$. Thus, $c_p = c_1 > 0$ and $bx \leq 1$ is dominated by the cardinality bound $x(E) \geq c_1$.

(3.3.2) $c_p < r(P \cup Z)$. Following the argumentation line in (3.3.1), we see that $I \subseteq P \cup Z$ and $|I \cap P|$ has to be maximal for any tight independent set I of cardinality c_p . Assume that $c_{p+1} \leq r(P \cup Z)$. Then, from any tight independent set I with $|I| = c_p$ we can construct a tight independent set J with $|J| = c_{p+1}$ by adding some elements $e \in Z$. However, it is not hard to see that there is no tight $K \in \mathcal{I} \cap \text{CHS}^c(E)$ that contains some $e \in N$. Thus, when $c_{p+1} \leq r(P \cup Z)$, $bx \leq 1$ is dominated by the nonnegativity constraints $y_e \geq 0, e \in N$. Therefore, $c_{p+1} > r(P \cup Z)$. The following is now immediate: If $I \in \mathcal{I} \cap \text{CHS}^c(E)$ is tight, then $|I| = c_p$ or $|I| = c_{p+1}$; if $|I| = c_p$, then $I \subset P \cup Z$, and if $|I| = c_{p+1}$, then $|I \cap (P \cup Z)| = r(P \cup Z)$ and $c_{p+1} > r(P \cup Z)$. Thus, $bx \leq 1$ is dominated by the cardinality forcing inequality $\text{CF}_{P \cup Z}(x) \leq c_p(c_{p+1} - r(P \cup Z))$.

□

2.2 Facets

We first study the facial structure of a single cardinality constrained matroid polytope $P_{\mathcal{I}}^{(k)}(E)$. All points of $P_{\mathcal{I}}^{(k)}(E)$ satisfy the equation $x(E) = k$, and hence, any inequality $x(F) \leq r(F)$ is equivalent to the inequality $x(E \setminus F) \geq k - r(F)$. Motivated by this observation, we introduce the following definitions. For any $F \subseteq E$, the number $r^k(F) := k - r(E \setminus F)$ is called the k -rank of F . Due to the submodularity of r we have $r^k(F_1) + r^k(F_2) \leq r^k(F)$ for all F_1, F_2 with $F = F_1 \dot{\cup} F_2$, and F is said to be k -separable if equality holds for some $F_1 \neq \emptyset \neq F_2$, otherwise k -inseparable. Due to the equation $x(E) = k$, $\dim P_{\mathcal{I}}^{(k)}(E) \leq |E| - 1$, and in fact, in the most cases we have equality. However,

if $\dim P_{\mathcal{I}}^{(k)}(E) < |E| - 1$, then at least one rank inequality $x(F) \leq r(F)$ with $\emptyset \neq F \subsetneq E$ is an implicit equation. As is easily seen, this implies that an inequality $x(F') \leq r(F')$ (or $x(F') \geq r^k(F')$) does not necessarily induce a facet of $P_{\mathcal{I}}^{(k)}(E)$, although F is inseparable (k -inseparable). To avoid the challenges involved, we only characterize the polytopes $P_{\mathcal{I}}^{(k)}(E)$ of dimension $|E| - 1$.

Lemma 2.2. *Let $M = (E, \mathcal{I})$ be a matroid and for any $k \in \mathbb{N}$, $0 < k < r(E)$, $M_k = (E, \mathcal{I}_k)$ the k -truncation of M with rank function r_k . Then, E is inseparable with respect to r_k .*

Proof. Let $E = F_1 \dot{\cup} F_2$ with $F_1 \neq \emptyset \neq F_2$ be any partition of E . We have to show that $r_k(F_1) + r_k(F_2) > r_k(E)$. By definition, $r_k(E) = k$. First, let $r(F_i) \leq k$ for $i = 1, 2$. Then, $r_k(F_i) = r(F_i)$ and consequently, $r_k(F_1) + r_k(F_2) = r(F_1) + r(F_2) \geq r(E) > k$ due to the submodularity of r . Next, let w.l.o.g. $r(F_1) > k$. Then, $r_k(F_1) = k$ and, since $F_2 \neq \emptyset$, $r_k(F_2) > 0$. Thus, $r_k(F_1) + r_k(F_2) = k + r_k(F_2) > k$. \square

Lemma 2.3. *Let $M = (E, \mathcal{I})$ be a matroid, $M_k = (E, \mathcal{I}_k)$ its k -truncation with rank function r_k , $\emptyset \neq F \subseteq E$, and $\bar{F} = E \setminus F$ be closed with $r(\bar{F}) < k < r(E)$. Then, F is k -inseparable with respect to r_k .*

Proof. $r(\bar{F}) < k$ implies $r_k(\bar{F}) = r(\bar{F})$, and since beyond it \bar{F} is closed with respect to r , it is also closed with respect to r_k . Let $F = F_1 \dot{\cup} F_2$ be a proper partition of F . We have to show that $r_k^k(F_1) + r_k^k(F_2) < r_k^k(F)$. First, suppose that $I \in \mathcal{I}$ with $|I| = k$ and $|I \cap \bar{F}| = r_k(\bar{F})$ implies $I \cap F_1 = \emptyset$ or $I \cap F_2 = \emptyset$. Since \bar{F} is closed with respect to r_k , it follows that $r_k^k(F_1) = r_k^k(F_2) = 0$, while $r_k^k(F) = k - r_k(\bar{F}) > 0$. So assume that there is some independent set I' of cardinality k such that $|I' \cap \bar{F}| = r_k(\bar{F})$ and $I' \cap F_i \neq \emptyset$ for $i = 1, 2$. Since $k < r(E)$, there is some element e such that $I := I' \cup \{e\}$ is independent with respect to r . Set $I_1 := I \setminus \{f_1\}$ and $I_2 := I \setminus \{f_2\}$ for $f_1 \in I \cap F_1, f_2 \in I \cap F_2$. Then, $r_k^k(F_1) \leq |I_1 \cap F_1|$ and $r_k^k(F_2) \leq |I_2 \cap F_2|$. Hence, $r_k^k(F_1) + r_k^k(F_2) \leq |I_1 \cap F_1| + |I_2 \cap F_2| < |I_1 \cap F_1| + |I_1 \cap F_2| = |I_1 \cap F| = r^k(F)$. \square

Lemma 2.4. *Let $M = (E, \mathcal{I})$ be a matroid, $\emptyset \neq F \subseteq E$, and A the matrix whose rows are the incidence vectors of $I \in \mathcal{I}$ with $|I| = k$ that satisfy the inequality $x(F) \geq r^k(F)$ at equality. Moreover, denote by A_F the submatrix of A restricted to F . Then, $\text{rank}(A_F) = |F|$ if and only if $r^k(F) \geq 1$, $\bar{F} := E \setminus F$ is closed, and (i) F is k -inseparable or (ii) $k < r(E)$.*

Proof. Necessity. The inequality $x(F) \geq r^k(F)$ is valid for $P_{\mathcal{I}}^{(k)}(E)$. As is easily seen, if $r^k(F) \leq 0$, then $\text{rank}(A_F) < |F|$. Next, assume that \bar{F} is not closed. Then, there is some $e \in F$ such that $r(\bar{F} \cup \{e\}) = r(\bar{F})$ which is equivalent to $r^k(F) = r^k(F \setminus \{e\})$. Thus, $x(F) \geq r^k(F)$ is the sum of the inequalities $x(F \setminus \{e\}) \geq r^k(F \setminus \{e\})$ and $x_e \geq 0$. This implies $\chi_e^I = 0$ for all incidence vectors of independent sets I with $|I| = k$ satisfying $x(F) \geq r^k(F)$ at equality. Again, it follows $\text{rank}(A_F) < |F|$. Finally, suppose that neither $k < r(E)$ nor F is k -inseparable. Then, $k = r(E)$ and F is $r(E)$ -separable. Thus, the inequality $x(F) \geq r^{r(E)}(F)$ is the sum of the valid inequalities $x(F_1) \geq r^{r(E)}(F_1)$ and $x(F_2) \geq r^{r(E)}(F_2)$ for some $F_1 \neq \emptyset \neq F_2$ with $F = F_1 \dot{\cup} F_2$. Setting $\lambda := r^{r(E)}(F_2)\chi_{F_1}^{F_1} - r^{r(E)}(F_1)\chi_{F_2}^{F_2}$, we see that for any $|F| \times |F|$ submatrix \tilde{A}_F of A_F we have $\tilde{A}_F \lambda = 0$, that is, the columns of \tilde{A}_F are linearly dependent which implies $\text{rank}(A_F) < |F| - 1$.

Sufficiency. First, let $k = r(E)$. Suppose $\text{rank}(A_F) < |F|$. Then, $A_F \lambda = 0$ for some $\lambda \in \mathbb{R}^F$, $\lambda \neq 0$. Since \bar{F} is closed and $r^k(F) \geq 1$ (that is, $r(\bar{F}) < k$), for each $e \in F$ there is an independent set I with $|I| = k$ that contains e and whose incidence vector satisfies $x(F) \geq r^k(F)$ at equality. Thus, A_F does not contain a zero-column. Moreover, $A_F \geq 0$, and hence, $F_1 := \{e \in F : \lambda_e > 0\}$ and $F_2 := \{e \in F : \lambda_e \leq 0\}$ defines a proper partition of F . Let $J \subseteq \bar{F}$ with $|J| = r(\bar{F})$ be an independent set. For $i = 1, 2$, let $B_i \subseteq F$ be an independent set such that $J \cup B_i$ is a basis of E and $J \cup (B_i \cap F_i)$ is a basis of $\bar{F} \cup F_i$. Set $S_i := B_i \cap F_i$ and $T_i := B_i \setminus S_i$ ($i = 1, 2$). By construction, $T_1 \subseteq F_2$ and $T_2 \subseteq F_1$. By matroid axiom (iii), to $J \cup S_1$ there is some $U_1 \subseteq J \cup B_2$ such that $K := J \cup S_1 \cup U_1$ is a basis of F . Clearly, $U_1 \subseteq (B_2 \cap F_2) = S_2$. Since the incidence vectors of $J \cup B_1$ and K are rows of A , it follows immediately $\lambda(T_1) = \lambda(U_1)$. With an analogous construction one can show that there is some $U_2 \subseteq S_1$ such that $\lambda(U_2) = \lambda(T_2)$. It follows, $\lambda(T_2) = -\lambda(S_2) \geq -\lambda(U_1) = -\lambda(T_1) = \lambda(S_1) \geq \lambda(U_2) = \lambda(T_2)$. Thus, between all terms we have equality implying $\lambda(S_1) = \lambda(U_2)$. Moreover, since $U_2 \subseteq S_1$ and $\lambda_e > 0$ for all $e \in S_1$, it follows $S_1 = U_2$. Hence, $K = J \cup S_1 \cup S_2$. This, in turn, implies that F is k -separable, a contradiction.

It remains to show that the statement is true if $k < r(E)$. Let $M_k = (E, \mathcal{I}_k)$ be the k -truncation of M with rank function r_k . By hypothesis, all conditions of Lemma 2.3 hold. Hence, F is k -inseparable with respect to r_k . Thus, all conditions of the theorem hold for r_k instead of r and hence, $\text{rank}(A_F) = |F|$. \square

Theorem 2.5. *Let $M = (E, \mathcal{I})$ be a matroid and $k \in \mathbb{N}$, $0 < k \leq r(E)$.*

- (a) $P_{\mathcal{I}}^{(k)}(E)$ has dimension $|E| - 1$ if and only if E is inseparable or $k < r(E)$.
- (b) Let $\dim P_{\mathcal{I}}^{(k)}(E) = |E| - 1$ and $\emptyset \neq F \subsetneq E$. The inequality $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^{(k)}(E)$ if and only if F is closed and inseparable, $r(F) < k$, and (i) $\bar{F} := E \setminus F$ is k -inseparable or (ii) $k < r(E)$.

Proof. (a) First, let $k = r(E)$. For any $\emptyset \neq F \subseteq E$, the rank inequality $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}(E)$ if and only if F is closed and inseparable. Consequently, the polytope $P_{\mathcal{I}}^{(r(E))}(E)$, which is a face of $P_{\mathcal{I}}(E)$, has dimension $|E| - 1$ if and only if E is inseparable. Next, let $0 < k < r(E)$. By Lemma 2.2, E is inseparable with respect to the rank function r_k of the k -truncation $M_k = (E, \mathcal{I}_k)$. Consequently, $x(E) \leq r_k(E) = k$ defines a facet of $P_{\mathcal{I}_k}(E)$ and hence, $\dim P_{\mathcal{I}}^{(k)}(E) = |E| - 1$.

(b) Clearly, $x(F) \leq r(F)$ does not induce a facet of $P_{\mathcal{I}}^{(k)}(E)$ if F is separable or not closed, since $\dim P_{\mathcal{I}}^{(k)}(E) = |E| - 1$, and hence, any inequality that is not facet defining for $P_{\mathcal{I}}(E)$ is also not facet defining for $P_{\mathcal{I}}^{(k)}(E)$. Next, if $r(F) \geq k$, then holds obviously $x(F) \leq x(E) = k \leq r(F)$, that is, either F is not closed, $x(F) \leq r(F)$ is an implicit equation, or the face induced by $x(F) \leq r(F)$ is the emptyset. Finally, assume that F is closed but neither (i) nor (ii) holds. Then, $k = r(E)$ and \bar{F} is k -separable. Thus, there are nonempty subsets \bar{F}_1, \bar{F}_2 of \bar{F} with $\bar{F} = \bar{F}_1 \dot{\cup} \bar{F}_2$ such that $r^k(\bar{F}) = r^k(\bar{F}_1) + r^k(\bar{F}_2)$. Now, the inequality $x(\bar{F}) \geq r^k(\bar{F})$, which is equivalent to $x(F) \leq r(F)$, is the sum of the valid inequalities $x(\bar{F}_i) \geq r^k(\bar{F}_i)$, $i = 1, 2$, both not being implicit equations.

To show the converse, let F satisfy all conditions mentioned in Theorem 2.5 (b). The restriction of $M = (E, \mathcal{I})$ to F is again a matroid. Denote it by $M' = (F, \mathcal{I}')$ and its rank function by r' . F remains inseparable with respect to r' . Thus, the restriction of $x(F) \leq r(F)$ to F , denoted by $x_F(F) \leq r(F) = r'(F)$, induces a facet of $P_{\mathcal{I}'}(F)$. A set of affinely independent vectors whose sum of components is equal to some ℓ , is also linearly independent. Thus, there are $|F|$ linearly independent vectors $\chi^{I'_j}$ of independent sets $I'_j \in \mathcal{I}'$ of cardinality $r'(F)$ ($j = 1, \dots, |F|$). The sets I'_j are also independent sets with respect to \mathcal{I} . Due to the matroid axiom (iii), $\bar{P} := I'_1$ can be completed to an independent set I_1 of cardinality k . Since $P \subseteq F$ and $|P| = r(F)$, $Q := I_1 \setminus P \subseteq \bar{F}$. Now, $I'_j, I_1 \in \mathcal{I}$, $I'_j \subseteq F$, and $r(F) = |I'_j| < |I_1| = k$. Hence, $I_j := I'_j \cup Q \in \mathcal{I}$ for all j . Consequently, we have $|F|$ linearly independent vectors $\chi^{I_j} \in P_{\mathcal{I}}^{(k)}(E)$ satisfying $x(F) \leq r(F)$ at equality.

Next, let A be the matrix whose rows are the incidence vectors of tight independent sets and $A_{\bar{F}}$ its restriction to \bar{F} . By Lemma 2.4, $A_{\bar{F}}$ contains a $|\bar{F}| \times |\bar{F}|$ submatrix B of full rank. By construction, each row B_i of B is an incidence vector of an independent set $J'_i \subseteq \bar{F}$ with $|J'_i| = r^k(\bar{F})$. W.l.o.g. we may assume that $B_1 = \chi^Q$, that is, $Q = J'_1$. By a similar argument as above, the independent sets $J_i := J'_i \cup P$ are tight and its incidence vectors are linearly independent.

Altogether we have $|F|$ linearly independent vectors χ^{I_j} with $I_j \cap \bar{F} = Q$ and $|\bar{F}|$ linearly independent vectors χ^{J_i} with $J_i \cap F = P$, where $J_1 = I_1$. As is easily seen, this yields a system of $|F| + |\bar{F}| - 1 = |E| - 1$ linearly independent vectors satisfying $x(F) \leq r(F)$ at equality. \square

Theorem 2.6. $P_{\mathcal{I}}^c(E)$ is fulldimensional unless $c = (0, r(E))$ and E is separable.

Proof. Clearly, $\dim P_{\mathcal{I}}^c(E) \geq \dim P_{\mathcal{I}}^{(c_p)}(E) + 1$ for all p , since the equation $x(E) = c_p$ is satisfied by all $y \in P_{\mathcal{I}}^{(c_p)}(E)$ but violated by at least one vector $z \in P_{\mathcal{I}}^c(E)$.

If $0 < c_p < r(E)$ for some p , then, by Theorem 2.5, $\dim P_{\mathcal{I}}^{(c_p)}(E) = |E| - 1$, and consequently $\dim P_{\mathcal{I}}^c(E) = |E|$. If there is no such p , then $c = (0, r(E))$. Again by Theorem 2.5, $\dim P_{\mathcal{I}}^{(r(E))}(E) = |E| - 1$ if and only if E is inseparable. Since $\dim P_{\mathcal{I}}^{(0, r(E))}(E) = \dim P_{\mathcal{I}}^{(r(E))}(E) + 1$, it follows the claim. \square

Theorem 2.7. For any $\emptyset \neq F \subseteq E$, the rank inequality $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^c(E)$ if and only if one of the following conditions holds.

- (i) $0 < r(F) < c_{m-1}$ and F is closed and inseparable.
- (ii) $0 < c_{m-1} = r(F) < c_m < r(E)$, and F is closed and inseparable.
- (iii) $0 < c_{m-1} = r(F) < c_m = r(E)$, F is closed and inseparable, \bar{F} is c_m -inseparable, and E is inseparable.
- (iv) $0 < c_{m-1} < c_m = r(F)$, $F = E$, and $c_m < r(E)$ or E inseparable.
- (v) $c_{m-1} = c_1 = 0$, $c_m = r(E)$, and $r(F) + r(E \setminus F) = r(E)$.

Proof. We prove the theorem by case by case enumeration.

(a) Let $0 < r(F) < c_{m-1}$. It is not hard to see that if F is separable or not closed, then $x(F) \leq r(F)$ does not define a facet of $P_{\mathcal{I}}^c(E)$. So, let F be closed and inseparable. By Theorem 2.5, $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^{(c_{m-1})}(E)$ and $\dim P_{\mathcal{I}}^{(c_{m-1})}(E) = |E| - 1$. Thus, it defines also a facet of $P_{\mathcal{I}}^c(E)$.

(b) Let $0 < c_{m-1} = r(F) < c_m < r(E)$. Clear by interchanging c_{m-1} and c_m in item (a).

(c) Let $0 < c_{m-1} = r(F) < c_m = r(E)$. The conditions mentioned in (iii) are equivalent to the postulation that $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^{(c_m)}(E)$ and $\dim P_{\mathcal{I}}^{(c_m)}(E) = |E| - 1$. If, indeed, the latter is true, then $x(F) \leq r(F)$ induces a facet also of $P_{\mathcal{I}}^c(E)$. To show the converse, suppose, for the sake of contradiction, that $x(F) \leq r(F)$ does not induce a facet of $P_{\mathcal{I}}^{(c_m)}(E)$ or $\dim P_{\mathcal{I}}^{(c_m)}(E) < |E| - 1$. Let $\mathcal{B} := \{\chi^{I_j} : I_j \in \mathcal{I}, |I_j| = c_m, j = 1, \dots, z, \}$ be an affine basis of the face of $P_{\mathcal{I}}^{(c_m)}(E)$ induced by $x(F) \leq r(F)$. By hypothesis, $z \leq |E| - 2$. Moreover, set $J := I_1 \cap F$ and $K := I_1 \setminus J$. Then, any incidence vector of an independent set $L \subseteq F$ with $|L| = c_{m-1}$ can be obtained as an affine combination of the set $\mathcal{B}' := \mathcal{B} \cup \{\chi^J\}$, which can be seen as follows: $L, I_1 \in \mathcal{I}$, and $|L| = r(F)$ implies $L \cup K \in \mathcal{I}$. Consequently, $\chi^L = \chi^{L \cup K} - \chi^K$. Now, $\chi^K = \chi^{I_1} - \chi^J$ and $\chi^{L \cup K} = \sum_{j=1}^z \lambda_j \chi^{I_j}$ with $\sum_{j=1}^z \lambda_j = 1$, since $L \cup K$ is tight. Thus, $\chi^L = \sum_{j=1}^z \lambda_j \chi^{I_j} - \chi^{I_1} + \chi^J$, that is, χ^L is in the affine hull of \mathcal{B}' . Since $|\mathcal{B}'| \leq |E| - 1$, $x(F) \leq r(F)$ is not facet defining for $P_{\mathcal{I}}^c(E)$, a contradiction.

(d) Let $0 < c_{m-1} < r(F) < c_m$. Since none of the independent sets I with $|I| = c_p$ is tight for $p = 1, \dots, m-1$, $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^c(E)$ if and only if it is an implicit equation for $P_{\mathcal{I}}^{(c_m)}(E)$ and $\dim P_{\mathcal{I}}^{(c_m)}(E) = |E| - 1$. However, $\dim P_{\mathcal{I}}^{(c_m)}(E) = |E| - 1$ implies $c_m < r(E)$ or E is inseparable. In either case, it follows that $x(F) \leq r(F)$ is an implicit equation for $P_{\mathcal{I}}^{(c_m)}(E)$ if and only if $F = E$. Thus, $r(F) = c_m$, a contradiction.

(e) Let $0 < c_{m-1} < c_m = r(F)$. Clearly, if $F \subset E$, then $x(F) \leq r(F)$ is strictly dominated by the cardinality bound $x(E) \leq c_m$. Consequently, $F = E$ and $x(F) \leq r(F)$ is an implicit equation for $P_{\mathcal{I}}^{(c_m)}(E)$. For the same reasons as in (d), $\dim P_{\mathcal{I}}^{(c_m)}(E) = |E| - 1$. Hence, $c_m < r(E)$ or E is inseparable.

(f) Let $c_{m-1} = c_1 = 0$. Again, $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^c(E)$ if and only if it is an implicit equation for $P_{\mathcal{I}}^{(c_m)}(E)$. This is the case if and only if $c_m = r(E)$ and $r(F) + r(E \setminus F) = r(E)$.

(g) Let $r(F) > c_m$. Then, $x(F) \leq x(E) \leq c_m < r(F)$, that is, the face induced by $x(F) \leq r(F)$ is the empty set. \square

Theorem 2.8. *Let $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then, the cardinality forcing inequality $CF_F(x) \leq c_p(c_{p+1} - r(F))$ defines a facet of $P_{\mathcal{I}}^c(E)$ if and only if*

(a) $c_p = c_1 = 0$ and the inequality $x(F) \leq r(F)$ defines a facet of $P_{\mathcal{I}}^{(c_{p+1})}(E)$,
or

(b) $c_p > 0$, F is closed and (i) $\bar{F} := E \setminus F$ is c_{p+1} -inseparable or (ii) $c_{p+1} < r(E)$.

Proof. For $P_{\mathcal{I}}^{(c_{p+1})}(E)$, the inequality $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ is equivalent to $x(F) \leq r(F)$, while for $P_{\mathcal{I}}^{(c_p)}(E)$, it is equivalent to $x(F) \leq c_p$. Hence, in case $c_p = c_1 = 0$, $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ induces a facet of $P_{\mathcal{I}}^c(E)$ if and only if it induces a facet of $P_{\mathcal{I}}^{(c_{p+1})}(E)$. When $\dim P_{\mathcal{I}}^{(c_{p+1})}(E) = |E| - 1$, this is the case if and only if F is closed and inseparable and (i) \bar{F} is c_{p+1} -inseparable or (ii) $c_{p+1} < r(E)$, see Theorem 2.5 (b).

In the following, let $c_p > 0$. Let A be the matrix whose rows are the incidence vectors of $I \in \mathcal{I}$ with $|I| = c_p$ or $|I| = c_{p+1}$ that satisfy the inequality $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ at equality. Denote by A_F and $A_{\bar{F}}$ the restriction of A to F and \bar{F} , respectively. By Theorem 2.6, $P_{\mathcal{I}}^c(E)$ is fulldimensional. Hence, $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ is facet defining if and only if the affine rank of A is equal to $|E|$.

If F is not closed, then there is some $e \in \bar{F}$ with $r(F \cup \{e\}) = r(F)$. Thus, $\text{CF}_{F'}(x) \leq c_p(c_{p+1} - r(F'))$ is a valid inequality for $P_{\mathcal{I}}^c(E)$, where $F' := F \cup \{e\}$, and $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ is the sum of this inequality and $-(c_{p+1} - c_p)x_e \leq 0$. Next, assume that neither (i) nor (ii) holds. Then, $c_{p+1} = r(E)$ and \bar{F} is $r(E)$ -separable. Thus, there is a proper partition $\bar{F} = \bar{F}_1 \dot{\cup} \bar{F}_2$ of \bar{F} with $r^{r(E)}(\bar{F}_1) + r^{r(E)}(\bar{F}_2) = r^{r(E)}(\bar{F})$. Since F is closed, it is not hard to see that $r^{r(E)}(\bar{F}_i) > 0$ which implies $c_p < r(F \cup \bar{F}_i) < r(E)$ for $i = 1, 2$, and hence, the inequalities $\text{CF}_{F \cup \bar{F}_1}(x) \leq c_p(c_{p+1} - r(F \cup \bar{F}_1))$ and $\text{CF}_{F \cup \bar{F}_2}(x) \leq c_p(c_{p+1} - r(F \cup \bar{F}_2))$ are valid. One can check again that then $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ is the sum of these both cardinality forcing inequalities.

To show the converse, let $M^F = (F, \mathcal{I}^F)$ with $\mathcal{I}^F := \{I \cap F : I \in \mathcal{I}\}$ be the restriction of M to F and $M_{c_p}^F = (F, \mathcal{I}_{c_p}^F)$ the c_p -truncation of M^F . Since $0 < c_p < r(F)$, Lemma 2.2 implies that F is inseparable with respect to the rank function of $M_{c_p}^F$. Consequently, the restriction of $x(F) \leq c_p$ to F defines a facet of $P_{\mathcal{I}^F}(F)$. Hence, A contains a $|F| \times |E|$ submatrix B such that B_F is nonsingular and $B_{\bar{F}} = 0$. Next, since F is closed, $r^{c_{p+1}}(\bar{F}) \geq 1$, and (i) \bar{F} is c_{p+1} -inseparable or (ii) $c_{p+1} < r(E)$, Lemma 2.4 implies that A contains a $|\bar{F}| \times |E|$ submatrix C such that $C_{\bar{F}}$ is nonsingular. Thus,

$$D := \begin{pmatrix} B_F & 0 \\ C_F & C_{\bar{F}} \end{pmatrix}$$

is a nonsingular $|E| \times |E|$ submatrix of A (or a row permutation of A). \square

2.3 Separation problem

Given any $P_{\mathcal{I}}^c(E)$ and any $x^* \in \mathbb{R}^E$, the separation problem consists of finding an inequality among (3)-(7) violated by x^* if there is any. This problem should be solvable efficiently, due to the equivalence of optimization and separation (see Grötschel, Lovász, and Schrijver [7]). By default, we may assume that x^* satisfies the cardinality bounds (4), (5) and the nonnegativity constraints (7). A violated rank inequality among (6) (if there is any) can be found by a polynomial time algorithm proposed by Cunningham [1]. So, we are actually interested only in finding an efficient algorithm that solves the separation problem for the class of cardinality forcing inequalities (3). We shall see, however, that the separation problem for the cardinality forcing inequalities can be transformed to that for the rank inequalities.

The separation problem for the class of cardinality forcing inequalities consists of checking whether or not

$$(c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) \leq c_p(c_{p+1} - r(F))$$

for all $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ for some $p \in \{0, \dots, m-1\}$.

For any $F \subseteq E$,

$$\begin{aligned} & (c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) \leq c_p(c_{p+1} - r(F)) \\ \Leftrightarrow & (c_{p+1} - c_p)x^*(F) - (r(F) - c_p)x^*(E) \leq c_p(c_{p+1} - r(F)) \\ \Leftrightarrow & x^*(F) \leq \frac{c_p(c_{p+1} - r(F)) + (r(F) - c_p)x^*(E)}{(c_{p+1} - c_p)} =: \gamma_F. \end{aligned}$$

Moreover, for any $k \in \{1, \dots, r(E)\}$, the right hand sides of the inequalities $x^*(F) \leq \gamma_F$ for $F \subseteq E$ with $r(F) = k$ are equal and differ only by a constant to the right hand sides of the corresponding rank inequalities $x(F) \leq r(F) = k$. Thus, both the separation problem for the rank inequalities and cardinality forcing inequalities could be solved by finding, for each $k \in \{1, \dots, |E|\}$, a set $F^* \subseteq E$ of rank k that maximizes $x^*(F)$. If $x^*(F^*) > k$, then the inequality $x(F^*) \leq r(F^*)$ is violated by x^* . If, in addition, $c_p < k < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$ and $x^*(F^*) > \gamma_{F^*}$, then x^* violates the cardinality forcing inequality associated with F^* .

This natural approach, however, seems not to result in an efficient separation routine. In order to mark the difficulties, we investigate the above approach for the class of rank inequalities, when $M = (E, \mathcal{I})$ is the graphic matroid defined on some graph $G = (V, E)$. It is well-known, the closed and inseparable rank inequalities for the graphic matroid are of the form $x(E(W)) \leq |W| - 1$ for $\emptyset \neq W \subseteq V$. If we would tackle the separation problem for this class of inequalities by finding, for each $k \in \{1, \dots, |W|\}$ separately, a set W_k^* that maximizes $x^*(E(W))$ such that $|W| = k$, then we would run into trouble, since for each k , such a problem is the weighted version of the densest k -subgraph problem which is known to be NP-hard (see Feige and Seltser [4]).

The last line of argument indicates that it is probably not a good idea to split the separation problem for the cardinality forcing inequalities (3) into separation problems for the subclasses $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ with $r(F) = k$, $k \in \{c_1 + 1, \dots, c_m - 1\} \setminus \{c_2, c_3, \dots, c_{m-1}\}$. It would be rather better to approach it as “non-cardinality constrained” problem. And this is exactly that what Cunningham did for the rank inequalities.

In the sequel, we firstly remind of some important facts regarding Cunningham’s algorithm for the separation of the rank inequalities. Afterwards, we show how the separation problem for the cardinality forcing inequalities can be reduced to that for the rank inequalities.

The theoretical background of Cunningham’s separation routine is the following min-max relation.

Theorem 2.9 (Edmonds [2]). *For any $x^* \in \mathbb{R}_+^E$, $\max\{y(E) : y \in P_M(E), y \leq x^*\} = \min\{r(F) + x^*(E \setminus F) : F \subseteq E\}$. \square*

Indeed, for any $y \in P_M(E)$ with $y \leq x^*$, $y(E) = y(F) + y(E \setminus F) \leq r(F) + x^*(E \setminus F)$, and equality will be attained if only if $y(F) = r(F)$ and $y(E \setminus F) = x^*(E \setminus F)$. Theorem 2.9 guarantees that any F minimizing $r(F) +$

$x^*(E \setminus F)$ maximizes $x^*(F) - r(F)$. For any matroid $M = (E, \mathcal{I})$ given by an independence testing oracle and any $x^* \in \mathbb{R}_+^E$, Cunningham's algorithm finds a $y \in P_M(E)$ with $y \leq x^*$ maximizing $y(E)$, a decomposition of y as convex combination of incidence vectors of independent sets, and a set $F^* \subseteq E$ with $r(F^*) + x^*(E \setminus F^*) = y(E)$ in strongly polynomial time. The vector y will be constructed by path augmentations along shortest paths in an auxiliary digraph.

Next, we return to the separation problem for the cardinality forcing inequalities (3). In the sequel, we suppose that x^* satisfies the rank inequalities (6).

Lemma 2.10. *Let $x^* \in \mathbb{R}_+^E$ satisfying all rank inequalities (6). If a cardinality forcing inequality $CF_F(x) \leq c_p(c_{p+1} - r(F))$ with $c_p < r(F) < c_{p+1}$ is violated by x^* , then $c_p < x^*(E) < c_{p+1}$.*

Proof. First, let $x^*(E) \leq c_p$. Then $x^*(F) \leq c_p$, and hence,

$$\begin{aligned} & (c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) \\ & \leq (c_{p+1} - r(F))c_p - (r(F) - c_p)x^*(E \setminus F) \\ & \leq c_p(c_{p+1} - r(F)). \end{aligned}$$

Next, let $x^*(E) \geq c_{p+1}$. By hypothesis, x^* satisfies all rank inequalities (6), in particular, $x(F) \leq r(F)$. Thus,

$$\begin{aligned} & (c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) \\ & = (c_{p+1} - c_p)x^*(F) - (r(F) - c_p)x^*(E \setminus F) \\ & \leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)x^*(E) \\ & \leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)c_{p+1} \\ & = c_p(c_{p+1} - r(F)). \end{aligned}$$

□

Lemma 2.11. *Let $x^* \in \mathbb{R}_+^E$ satisfying all rank inequalities (6), and let $c_p < x^*(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Then for any $F \subseteq E$ holds: If $(c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) > c_p(c_{p+1} - r(F))$, then $c_p < r(F) < c_{p+1}$.*

Proof. Let $F \subseteq E$, and assume that $r(F) \leq c_p$. Then,

$$\begin{aligned} & (c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) - c_p(c_{p+1} - r(F)) \\ & = (c_{p+1} - c_p)x^*(F) - (r(F) - c_p)x^*(E) - c_p(c_{p+1} - r(F)) \\ & \leq (c_{p+1} - c_p)r(F) - (r(F) - c_p)x^*(E) - c_p(c_{p+1} - r(F)) \\ & = \underbrace{(c_{p+1} - x^*(E))}_{>0} \underbrace{(r(F) - c_p)}_{\leq 0} \leq 0. \end{aligned}$$

Next, if $r(F) \geq c_{p+1}$, then

$$\begin{aligned} & (c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) - c_p(c_{p+1} - r(F)) \\ & = (c_{p+1} - c_p)x^*(F) - (r(F) - c_p)x^*(E) - c_p(c_{p+1} - r(F)) \\ & \leq (c_{p+1} - c_p)x^*(E) - (r(F) - c_p)x^*(E) - c_p(c_{p+1} - r(F)) \\ & = \underbrace{(c_{p+1} - r(F))}_{\leq 0} \underbrace{(x^*(E) - c_p)}_{>0} \leq 0. \end{aligned}$$

Thus, $(c_{p+1} - r(F))x^*(F) - (r(F) - c_p)x^*(E \setminus F) > c_p(c_{p+1} - r(F))$ at most if $c_p < r(F) < c_{p+1}$. □

Theorem 2.12. *For any $P_{\mathcal{I}}^c(E)$ and any $x^* \in \mathbb{R}_+^E$ satisfying all rank inequalities (6), the separation problem for x^* and the cardinality forcing inequalities (3) can be solved in strongly polynomial time.*

Proof. By Lemmas 2.10 and 2.11 we know that x^* violates a cardinality forcing inequality at most if $c_p < x^*(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. Thus, if $x^*(E) = c_q$ for some $q \in \{1, \dots, m\}$, then $x^* \in P_{\mathcal{I}}^c(E)$.

Suppose that $c_p < x^*(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. We would like to find some $F' \subseteq E$ such that

$$(c_{p+1} - r(F'))x^*(F') - (r(F') - c_p)x^*(E \setminus F') - c_p(c_{p+1} - r(F')) > 0$$

if there is any. Lemma 2.11 says that $c_p < r(F') < c_{p+1}$, and thus, the inequality $\text{CF}_{F'}(x) \leq c_p(c_{p+1} - r(F'))$ is indeed a cardinality forcing inequality among (3) violated by x^* . If there is no such F' , then for all $F \subseteq E$ with $c_p < r(F) < c_{p+1}$ the associated cardinality forcing inequality with F is satisfied by x^* , and by Lemma 2.10, all other cardinality forcing inequalities among (3) are also satisfied by x^* .

To find such a subset F' of E , set $\delta := \frac{x^*(E) - c_p}{c_{p+1} - c_p}$. Since $c_p < x^*(E) < c_{p+1}$, $0 < \delta < 1$. Moreover, $\frac{c_{p+1} - x^*(E)}{c_{p+1} - c_p} = 1 - \delta$. For any $F \subseteq E$ it now follows:

$$\begin{aligned} (c_{p+1} - c_p)x^*(F) - (r(F) - c_p)x^*(E) - c_p(c_{p+1} - r(F)) &> 0 \\ \Leftrightarrow x^*(F) - \frac{r(F)x^*(E) + c_p x^*(E) - c_p c_{p+1} + c_p r(F)}{c_{p+1} - c_p} &> 0 \\ \Leftrightarrow x^*(F) - r(F) \frac{x^*(E) - c_p}{c_{p+1} - c_p} - c_p \frac{c_{p+1} - x^*(E)}{c_{p+1} - c_p} &> 0 \\ \Leftrightarrow x^*(F) - r(F) \delta &> c_p(1 - \delta) \\ \Leftrightarrow \frac{x^*(F)}{\delta} - r(F) &> c_p \frac{(1 - \delta)}{\delta}. \end{aligned}$$

Setting $x' := \frac{1}{\delta}x^*$, we see that the last inequality is equivalent to $x'(F) - r(F) > c_p \frac{(1 - \delta)}{\delta}$. Thus, we can apply Cunningham's algorithm to find some $F \subseteq E$ that maximizes $x'(F) - r(F)$. If $x'(F) - r(F) > c_p \frac{(1 - \delta)}{\delta}$, then $c_p < r(F) < c_{p+1}$ and the cardinality forcing inequality associated with F is violated by x^* . \square

Consequently, we suggest a separation routine that works as follows. First, compute with Cunningham's algorithm a subset F of E maximizing $x^*(F) - r(F)$. If $x^*(F) - r(F) > 0$, then the associated rank inequality $x(F) \leq r(F)$ is violated by x^* . However, in the special case that $c_p < r(F) < c_{p+1}$ and $c_p < x^*(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$, also the cardinality forcing inequality associated with F is violated by x^* , and it is not hard to see that then the former inequality is dominated by the latter one. If $x^*(F) - r(F) \leq 0$, then x^* satisfies all rank inequalities (6), and if, in addition, $x^*(E) = c_p$ for some p , then we know that $x^* \in P_{\mathcal{I}}^c(E)$. Otherwise, i.e., if $c_p < x^*(E) < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$, then we check whether or not there is a violated cardinality forcing inequality among (3) by applying Cunningham's algorithm on $M = (E, \mathcal{I})$ and $x' = \frac{1}{\delta}x^*$.

3 Concluding remarks

The cardinality constrained matroid polytope turns out to be a useful object to enhance the theory of polyhedra associated with cardinality constrained combi-

natorial optimization problems. Imposing cardinality constraints on a combinatorial optimization problem does not necessarily turn it into a harder problem: The cardinality constrained version of the maximum weight independent set problem in a matroid is manageable on the algorithmic as well as on the polyhedral side without any difficulties. Facets related to cardinality restrictions (cardinality forcing inequalities) are linked to well-known attributes of matroid theory (closed subsets of E). The analysis of the separation problem for the cardinality forcing inequalities discloses that it is sometimes better not to split a cardinality constrained problem into simpler cardinality constrained problems but to transform it into one or more non-cardinality restricted problems.

It stands to reason to investigate the intersection of two matroids with regard to cardinality restrictions. It is well-known, if an independence system \mathcal{I} defined on some ground set E can be described as the intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, then the optimization problem $\max w(I)$, $I \in \mathcal{I}$ can be solved in polynomial time, for instance with Lawler's weighted matroid intersection algorithm [9]. This algorithm solves also the cardinality constrained version $\max w(I)$, $I \in \mathcal{I} \cap \text{CHS}^c(E)$, since for each cardinality $p \leq r(E)$ it generates an independent set I of cardinality p which is optimal among all independent sets J of cardinality p . Thus, from an algorithmic point of view the problem is well studied. However, there is an open question regarding the associated polytope. It is well known, $P_{\mathcal{I}}(E) = P_{\mathcal{I}_1}(E) \cap P_{\mathcal{I}_2}(E)$, that is, the non-cardinality constrained independent set polytope $P_{\mathcal{I}}(E)$ is determined by the nonnegativity constraints $x_e \geq 0$, $e \in E$, and the rank inequalities $x(F) \leq r_j(F)$, $\emptyset \neq F \subseteq E$, $j = 1, 2$, where r_j is the rank function with respect to \mathcal{I}_j . We do not know, however, whether or not $P_{\mathcal{I}}^c(E) = P_{\mathcal{I}_1}^c(E) \cap P_{\mathcal{I}_2}^c(E)$ holds. So far, we have not found any counterexample contradicting the hypothesis that equality holds.

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E-mail address: `stephan@math.tu-berlin.de`