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Abstract

Asymptotic analyses of the three dimensional compressible flow equations coupled with transport equations for the mixing ratios of water vapour, cloud water and rain water are described. We obtain reduced systems of equations for two particular regimes of length and time scales: Models for the long time evolution of deep convective columns and for the short time evolution of shallow convective layers.

The asymptotic deep convective column model is anelastic, yet the vertical motion is "pressure free", i.e., it evolves freely in interaction with buoyancy while the *horizontal divergence* adjusts to fullfil the anelastic constraint. The perturbation pressure guaranteeing compliance with the horizontal divergence constraint obeys a Poisson-type equation. Surprisingly, the *vertical velocity* plays an important role in the horizontal dynamics through the *Coriolis term*. The vertical acceleration in a saturated column is directly determined by the buoyancy induced by potential temperature differences relative to the background stratification. This potential temperature deviation is a conserved quantity.

Evaporation is the only important microphysical process in the undersaturated regime. The evaporation rate depends on the saturation deficit and the amount of rain water present and determines the (downward) vertical velocity and the distribution of water vapour.

To connect the deep convective column solutions to top and bottom boundary conditions, a different flow regime needs to be accounted for. Within shallow layers whose depth is comparable to the column diameters, adjustment to physical boundary conditions can take place. This is the second regime considered in this report. The shallow convective layer regime is shown to be asymptotically described by Boussinesq-type equations. These equations are closed by evolution equations which show that, in the saturated regime, the distributions of potential temperature and cloud water are determined by a condensation rate that is directly proportional to the vertical velocity. In the undersaturated regime, the potential temperature distribution is determined by the amount of rain present, since the water vapour in this case is shown to be a conserved quantity. In both regimes the distribution of rain water depends on the rain water flux.

Contents

Notation

Latin Symbols

a	radius of the Earth
A_{cr}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
A_{vs}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis; in the appendix abbreviated as A
B	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
c_p	specific heat capacity at constant pressure for dry air
C_{cr}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
$\mathcal{D}$	dimensionless number resulting from dimensional analysis
$\mathcal{F}$	constant exponent in dimensional analysis
e_∞	triple-point vapour pressure
g	acceleration of gravity
h	height
$\boldsymbol{k}$	vertical unit vector
K_{vc}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
l	length
L	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
L_{cond}	specific latent heat of condensation
p	pressure
P_{vs}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
r	mixing ratio
R	specific gas constant
R_{vs}	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis; in the appendix abbreviated as R
S	source term
S_θ	source term due to latent heat release
t	time coordinate
$\boldsymbol{v}$	velocity vector
v_t	terminal falling velocity of rain drops
V_r	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
w	vertical velocity
$\boldsymbol{x}$	horizontal coordinates with respect to h_{sc}
$\widetilde{\boldsymbol{x}}$	horizontal coordinates in dimensional form
z	vertical coordinate

Greek Symbols

α	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
β	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
γ	isentropic exponent
Γ	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
ε	asymptotic scaling parameter
θ	potential temperature

λ	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
μ	constant of $\mathcal{O}(1)$ resulting from asymptotic analysis
$\boldsymbol{\xi}$	horizontal coordinates with respect to εh_{sc}
ϱ	density
Ω	diurnal rotation frequency
$\boldsymbol{\Omega}$	Earth rotation vector
$\widehat{\boldsymbol{\Omega}}$	normalised Earth rotation vector

Dimensionless Numbers

Fr	Froude-Number
M	Mach-Number
Ro	Rossby-Number
Sr	Strouhal-Number

Mathematical Expressions and Operators

$\frac{D}{Dt}$	substantial derivative
$\exp[\cdot]$	exponential function
$\ln[\cdot]$	natural logarithm function (to the basis e)
$\max[\cdot]$	maximum function
$\mathcal{O}(\cdot)$	Landau-Symbol
$(\cdot)^{(i)}$	i: order of the asymptotic expansion
$(\cdot)_t$	differentiation with respect to t
$(\cdot)_z$	differentiation with respect to z
∇	Nabla-Operator
$\nabla_{\boldsymbol{x}}$	Nabla-Operator regarding $\boldsymbol{x}$
$\nabla_{\boldsymbol{\xi}}$	Nabla-Operator regarding $\boldsymbol{\xi}$
$\nabla^2 = \Delta$	Laplace-Operator

Indices

$_c$	cloud water
$_d$	dry air
$_{out}$	outside the deep convective column
$_r$	rain water
$_{ref}$	reference
$_{sat}$	saturated
$_{sc}$	scale
$_{SI}$	SI-unit of that quantity
$_{un}$	undersaturated
$_v$	water vapour
$_{vs}$	saturated water vapour
$(\cdot)_{\shortparallel}$	horizontal part of a vector
$(\cdot)_{\perp}$	vertical part of a vector

1 Introduction

Atmospheric moist convection plays an important role in many atmospheric systems. Moist convection may be divided into at least two coarse-grained types: deep convection and shallow convection. Deep convection is a thermodynamically driven process that transports momentum, heat, and moisture and usually results in strong precipitation which interacts with the atmosphere through evaporation. Emanuel, [2], explains that by the net release of latent heat integrated through the depth of the troposphere deep convection has strong effects on the dynamics and thermodynamics of the atmospheric circulation systems in which it is embedded. Shallow convection helps propagate moisture, heat and momentum from the boundary layer to the free atmosphere. The role of convective motions in the boundary layer in initiating and organizing deep convection has been studied among others by Crook, [1], who found that deep convection is sensitive to a number of thermodynamic parameters such as vertical decrease of moisture within the boundary layer.

Convection occurs on a range of temporal and spatial scales. Deep convection spans the whole depth of the troposphere whereas shallow convection has a vertical scale of 1 km. Vertical velocities in deep convective events can reach values of 10 m/s, and the width of the convective elements seem to be of the order of 1 km. Observations have revealed that the mean diameter does not vary much with height and the lifetime of individual cells typically is about 30 min. An excellent short review on deep precipitating convection may be found in [9].

This report is motivated by the need to elucidate the character of deep convection, in particular convective elements of $\mathcal{O}(1\text{ km})$ in the horizontal. While the integral effects of these elements are very interesting, here we address the dynamics of the individual elements themselves with the aim of gaining physical insight into the mechanisms involved. Such a detailed analysis will not only add to our knowledge on how convection works but also generate new ideas for the parameterization of convection in atmospheric models.

The main objective of this work is to carry out a scale analysis of a set of flow equations for the moist atmosphere involving the compressible flow equations in conjunction with a bulk microphysics closure scheme for the transport of water vapour, clouds, and rain water. A distinguished limit between the scaling parameters of the atmospheric motion and those of the phase change and coagulation processes is established, and the coupling and interaction of gas dynamics and humidity transport is analyzed through asymptotic analysis.

The method of deriving the asymptotic models is structured in three steps. The first step is to write the equations in nondimensional form by referring the dependent variables to characteristic reference values. Thus combinations of reference values build typical dimensionless numbers, e.g. Sr, M, Fr, Ro. These dimensionless numbers are then expressed in terms of a small parameter $\varepsilon \ll 1$. According to [7], the expansion parameter ε is identified as the cubic root of the universal acceleration ratio, $a\Omega^2/g$, where a is the Earth's radius, Ω the diurnal rotation frequency and g the acceleration of gravity. In practice, $\varepsilon \sim 1/7$. In Section 2 we derive the governing equations for both the gas dynamics and the humidity transport constituting the starting point of the asymptotic scale analysis. As second step, length and time coordinates are scaled with ε depending on the phenomena that are to be described by the model. In the third step asymptotic series expansions for all the dependent variables are introduced: Let $U = (p, \varrho, w, \boldsymbol{v}_{\shortparallel}, \theta, \dots)$, then $U = U(\boldsymbol{x}, z, t, ; \varepsilon) = \sum\limits_i \varepsilon^i U^{(i)}$ with, e.g., $U^{(i)} = U^{(i)}(\varepsilon^{-1}\boldsymbol{x}, \boldsymbol{x}, z, t) = U^{(i)}(\boldsymbol{\xi}, \boldsymbol{x}, z, t)$. As each equation has to be fulfilled in every order of ε, one single equation can be written as a series of equations arranged according to powers of ε. Each of the original equations has to be considered up to the order of ε that is necessary to obtain a closed system of equations for the desired quantity in leading order.

This work is related to that of Klein and Majda, [6], in which two deep convective multiple-scales regimes (vertical scale of 10 km) were investigated. These were the bulk micro / convective regimes with horizontal scales of 1 km and 10 km on a timescale of 2 min, and the convective / mesoscale regime with horizontal scales of 10 km and 100 km on a timescale of 20 min. In this report we deal with one deep convective and one shallow convective regime (vertical scale of 1 km), restriciting to a single scale in each spacial direction, and to a single time scale in both regimes. In Section 3 we consider the long time evolution (20 min) of deep convective columns with characteristic horizontal dimensions of 1 km. The model describing the life cycle of deep convective columns may be used to extend the multiscale model from Klein and Majda, [6], considering the horizontal bulk micro and convective scales on both the short initiation time scale of 2 min and the deep convective time scale of 20 min, resulting in a model with multiple length and time scales. In Section 4 the short time evolution (2 min) of shallow convective layers is addressed with isotropic bulk micro scalings of 1 km in the horizontal and vertical directions. The shallow convective case may contribute to the formulation of physically consistent boundary conditions for the deep convective system derived in Section 3. Finally, Section 5 summarizes the main results of the asymptotic analyses accomplished in this work.

Before we embark on the analysis, here we show in advance the asymptotic deep column model equations from Section 3.4 for saturated and undersaturated conditions.

In the saturated air regime the final system of equations contains the unknown variables $p^{(6)}$, $\theta^{(4)}$, $\boldsymbol{v}_{\shortparallel}^{(1)}$ and $w^{(0)}$. With these quantities the coupled mixing ratios of cloud and rain water can be determined. The equations read as follows:

Saturated Air

Mass Balance

$$\varrho^{(0)}\nabla_{\xi}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}+(\varrho^{(0)}w^{(0)})_z=0,$$

Horizontal Momentum Balance

$$\boldsymbol{v}_{\shortparallel t}^{(1)}+(\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\xi})\boldsymbol{v}_{\shortparallel}^{(1)}+w^{(0)}\boldsymbol{v}_{\shortparallel z}^{(1)}+(w^{(0)}\widehat{\boldsymbol{\Omega}}_{\shortparallel}\times\boldsymbol{k})+\frac{1}{\varrho^{(0)}}\nabla_{\xi}p^{(6)}=0,$$

Vertical Momentum Balance

$$w_t^{(0)}+\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\xi}w^{(0)}+w^{(0)}w_z^{(0)}=\theta^{(4)}-\theta_{\mathrm{out}}^{(4)},$$

Transport Equation for the Potential Temperature

$$\theta_t^{(4)}+\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\xi}\theta^{(4)}+w^{(0)}\theta_z^{(4)}=w^{(0)}(\theta_z^{(4)})_{\mathrm{out}},$$

Transport Equation for the Cloud Water Mixing Ratio

$$K_{vc}w^{(0)}r_{vs\,z}^{(0)}+C_{cr,1}r_c^{(0)}r_r^{(0)}=0,$$

Transport Equation for the Rain Water Mixing Ratio

$$r_{r\,t}^{(0)}+\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\xi}r_r^{(0)}+w^{(0)}r_{r\,z}^{(0)}-V_r\frac{1}{\varrho^{(0)}}({\varrho^{(0)}}^{\frac{1}{2}}r_r^{(0)})_z-C_{cr,2}r_c^{(0)}r_r^{(0)}=0.$$

The reduced system for the undersaturated air regime consists of the following equations with $p^{(6)}$, $r_v^{(0)}$, $r_r^{(0)}$, $\boldsymbol{v}_{\shortparallel}^{(1)}$ and $w^{(0)}$ as unknown variables:

Under-Saturated Air

Mass Balance

$$\varrho^{(0)}\nabla_{\xi}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}+(\varrho^{(0)}w^{(0)})_z=0,$$

Horizontal Momentum Balance

$$\boldsymbol{v}_{\shortparallel t}^{(1)} + (\boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(1)} + w^{(0)}\boldsymbol{v}_{\shortparallel z}^{(1)} + (w^{(0)}\widehat{\boldsymbol{\Omega}}_{\shortparallel} \times \boldsymbol{k}) + \frac{1}{\varrho^{(0)}}\nabla_{\boldsymbol{\xi}} p^{(6)} = 0,$$

Transport Equation for the Potential Temperature

$$w^{(0)}\theta_z^{(2)} = -\Gamma L E_r^{(0)},$$

Transport Equation for the Water Vapour Mixing Ratio

$$r_{vt}^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} r_v^{(0)} + w^{(0)} r_{vz}^{(0)} = E_r^{(0)},$$

Transport Equation for the Rain Water Mixing Ratio

$$r_{rt}^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} r_r^{(0)} + w^{(0)} r_{rz}^{(0)} - V\frac{1}{\varrho^{(0)}}\left((\varrho^{(0)})^{\frac{1}{2}} r_r^{(0)}\right)_z = -E_r^{(0)}.$$

2 Governing Equations

This section provides the basic set of equations to be used in this work. These are the three dimensional compressible flow equations including gravity and rotation of the earth and a bulk microphysics parameterization consisting of transport equations for the mixing ratios of water vapour, cloud water and rain water.

2.1 Equations of Mass, Momentum, Potential Temperature and Equation of State

The dimensionless equations for mass, momentum and potential temperature and the appropriate equation of state have been derived in [6] and are:

Mass Balance

$$\varrho_t + \nabla_{\boldsymbol{x}} \cdot (\varrho \boldsymbol{v}_{\shortparallel}) + (\varrho w)_z = 0, \tag{1}$$

Momentum Balance (horizontal and vertical)

$$(\varrho \boldsymbol{v}_{\shortparallel})_t + \nabla_{\boldsymbol{x}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} \circ \boldsymbol{v}_{\shortparallel}) + (\varrho \boldsymbol{v}_{\shortparallel} w)_z + \varepsilon (\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\shortparallel} + \varepsilon^{-4} \nabla_{\boldsymbol{x}} p = 0, \tag{2}$$

$$(\varrho w)_t + \nabla_{\boldsymbol{x}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} w) + (\varrho w w)_z + \varepsilon (\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\perp} + \varepsilon^{-4} p_z = -\varepsilon^{-4} \varrho, \tag{3}$$

Transport Equation for the Potential Temperature

$$\theta_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{x}} \theta + w \theta_z = S_\theta, \tag{4}$$

Equation of State

$$\theta \varrho = p^{1 - \Gamma \varepsilon}. \tag{5}$$

Here the equations have been made dimensionless by typical values of the atmospheric dynamics. The horizontal velocity component $\boldsymbol{v}_{\shortparallel}$ is scaled with the characteristic flow velocity speed in the atmosphere $u_{\mathrm{ref}} = 10$ m s^{-1}; w is the vertical velocity component also scaled with u_{ref}. The independent variables length $\boldsymbol{x}$ and time t are made dimensionless with $\ell_{\mathrm{ref}} = 10^4$ m and $t_{\mathrm{ref}} = \frac{\ell_{\mathrm{ref}}}{u_{\mathrm{ref}}} = 10^3$ s. The characteristic length ℓ_{ref} is equivalent to the pressure scale height, i.e., vertical distance with significant pressure drop. The density ϱ and the pressure p have been made dimensionless with $\varrho_{\mathrm{ref}} = 1.25$ kg m^{-3} and the typical value for the surface atmospheric pressure $p_{\mathrm{ref}} = 10^5$ kg m^{-1}s^{-2}, respectively. The characteristic value for the potential temperature θ scaling is chosen from the equation of state relation, i.e., $\theta_{\mathrm{ref}} = \frac{p_{\mathrm{ref}}}{R_d \varrho_{\mathrm{ref}}} = 273.16$ K, where $R_d = 287$ J kg^{-1} K^{-1} is the specific gas constant of dry air. The small parameter ε results from a distinguished limit relating the dimensionless parameters resulting from nondimensionalization, namely the Froude number Fr, Mach number M and Rossby number Ro_ℓ, as follows:

$$Fr = \frac{u_{\text{ref}}}{\sqrt{g\ell_{\text{ref}}}} \sim \varepsilon^2, \quad M = \frac{u_{\text{ref}}}{c_{\text{ref}}} \sim \varepsilon^2 \quad \text{and} \quad Ro_l = \frac{\Omega l_{\text{ref}}}{u_{\text{ref}}} \sim \varepsilon^{-1}. \tag{6}$$

Here $c_{\text{ref}} = \sqrt{p_{\text{ref}}/\varrho_{\text{ref}}} = \sqrt{g\ell_{\text{ref}}}$ is of the order of the speed of sound and is related to the speed of the gravity wave in a barotropic atmosphere, and $\Omega = 10^{-4}\ \text{s}^{-1}$ is the earth's rotation frequency. It is worth noting that from dimensional analysis consideration, one can consider ε in terms of properties of the rotating earth through the dimensionless parameter

$$\varepsilon = \left(\frac{a\Omega^2}{g}\right)^{\frac{1}{3}} \sim \frac{1}{7}, \tag{7}$$

where $a = 6 \times 10^6$ m is the earth's radius, $g = 10\ \text{m s}^{-2}$ acceleration due to gravity and rotation frequency Ω. The product $a\Omega$ can be considered as the absolute acceleration of points on the earth surface induced by its rotation. Thus ε can physically be interpreted as a ratio of two accelerations.
Equation (5) is obtained from the perfect gas equation of state, which when nondimensionalized leads to $p = (\varrho\theta)^\gamma$. The ratio γ of the thermodynamic specific heats c_p and c_v, at constant pressure and constant volume, respectively, $\left(\gamma = \frac{c_p}{c_v} = 1.4\right)$ for air, is useful in compressible flow studies. A Newtonian limit is introduced by setting

$$\frac{\gamma - 1}{\gamma} \sim \Gamma\varepsilon, \tag{8}$$

with $\Gamma = \mathcal{O}(1)$ as $\varepsilon \to 0$. See [6] for further discussion.

Finally, the source and sink term, S_θ, appearing in (4) includes latent heat release effects, moisture effects, and also addition and removal of heat by radiative effects. In this report we will neither take into account diffusive and turbulent transport mechanisms nor radiation effects.

2.2 Cloud Microphysics Equations

The cloud microphysics is introduced by considering the transport equations for mixing ratios of water vapour r_v, cloud water r_c, and rain water r_r. Here we consider the bulk paramaterization equations as given by Grabowski and Smolarkiewicz, [4], and neglect diffusive and turbulent transport mechanisms. The equations can be written compactly as follows:

$$\frac{Dr_v}{Dt} = -C_d + E_r, \tag{9}$$

$$\frac{Dr_c}{Dt} = C_d - A_r - C_r, \tag{10}$$

$$\frac{Dr_r}{Dt} = \frac{1}{\varrho}\frac{\partial}{\partial z}(\varrho v_t r_r) + A_r + C_r - E_r, \tag{11}$$

where $\frac{D}{Dt} = \partial_t + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{x}} + w\partial_z$ is the material derivative. The term $\frac{1}{\varrho}\frac{\partial}{\partial z}(\varrho v_t r_r)$ represents the redistribution of rain water as it falls with the terminal velocity v_t. We have assumed that the cloud water and water vapour have negligible terminal velocities. The microphysical interactions among water vapour, cloud water and rain water are represented through the rate of condensation of water vapour into cloud water (C_d), rate of evaporation of rain in the undersaturated environment (E_r) and coagulation processes; autoconversion of cloud water into rain (A_r) and accretion– collection of cloud water by rain (C_r). Thus, the model also assumes that rain only develops through autoconversion and accretion. Further, the model does not include evaporation of cloud water since we have assumed that the cloud water evaporates instantaneously if the air is undersaturated. The parameterizations of these interactions in their general (dimensionless) form are:

$$A_r = \mathcal{D}_1 \max\left[0, (r_c - 1)\right], \tag{12}$$

$$C_r = \mathcal{D}_2\, r_c\, r_r^{\mathcal{F}_1}, \tag{13}$$

$$E_r = \frac{p}{\varrho}\frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{\mathcal{F}_2}\left[\mathcal{D}_3 + \mathcal{D}_4\,(\varrho\, r_r)^{\mathcal{F}_3}\right]}{\mathcal{D}_5 p\, r_{vs} + \mathcal{D}_6}, \tag{14}$$

$$v_t = \mathcal{D}_7\, \varrho^{\mathcal{F}_4}\, r_r^{\mathcal{F}_5}. \tag{15}$$

Here $\mathcal{D}_1$ to $\mathcal{D}_7$ are dimensionless numbers composed by combinations of reference values and constants, $\mathcal{F}_1$ to $\mathcal{F}_5$ are constant exponents. These numbers and exponents are given in appendix A. The saturation mixing ratio of water vapour r_{vs} is described in 2.3.

The parameterizations for A_r and C_r do not depend on how autoconversion and accretion are achieved, these processes simply take place at a rate proportional to the mixing ratio of cloud water. The parameterization for evaporation indicates that evaporation cannot take place if the environment is saturated i.e. the rate of evaporation is proportional to $(r_{vs} - r_v)$, the saturation deficit. There is no explicit parameterization for the rate of condensation, however, it is assumed that condensation takes place when air is saturated and that the amount of condensed water vapour forms cloud water only.

We consider two distinct regimes in this report:

- Regime I: The water vapour is saturated in the presence of cloud water, i.e., $r_c > 0 \Rightarrow r_v = r_{vs}$. In this regime, the presence of cloud water creates the condition necessary for condensation, autoconversion and accretion. Evaporation cannot take place as the environment is saturated. Thus $C_d, A_r, C_r \geq 0$; $E_r = 0$.

- Regime II: The cloud water evaporates instantaneously in undersaturated conditions, i.e., $r_v < r_{vs} \Rightarrow r_c = 0$. In this regime, the environment is undersaturated hence evaporation takes place, but since there is no cloud water, condensation, autoconversion and accretion cannot take place. Thus $E_r \geq 0$; $C_d, A_r, C_r = 0$.

Finally, the nondimensionalization of the moisture transport equations requires reference values for the mixing ratios of the moist variables. According to [3] the water vapour mixing ratio in the atmosphere is usually smaller than $40\frac{\mathrm{g}}{\mathrm{kg}}$. A mass of 10 g water vapour per kg dry air can be chosen as typical value for the mixing ratios of water vapour and saturated water vapour, the mixing ratio of rain water is assumed to be of the same order of magnitude: $r_{v,\mathrm{ref}} = r_{vs,\mathrm{ref}} = r_{r,\mathrm{ref}} = 10^{-2}\frac{\mathrm{kg}}{\mathrm{kg}}$. Further, according to [4], the autoconversion threshold is typically between 10^{-4} and $10^{-3}\frac{\mathrm{kg}}{\mathrm{kg}}$. The arithmetic mean is chosen as reference value for the mixing ratio of cloud water: $r_{c,\mathrm{ref}} = 55 \,\mathrm{x}\, 10^{-5}\frac{\mathrm{kg}}{\mathrm{kg}}$.
After nondimensionalization with these reference values together with the substitution of (12) to (15), the parameterizations for autoconversion, accretion, evaporation and the terminal falling velocity of rain drops, the evolution equations, (9) – (11), can be written as

$$\begin{aligned} r_{vt} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{x}} r_v \quad &+ \quad w r_{vz} + \delta_s C_d \\ &- \quad \frac{p}{\varrho} \frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\,[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}]}{B_3 p\, r_{vs} + B_4} = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} r_{ct} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{x}} r_c + w r_{cz} \quad &+ \quad \delta_s A_{cr,1} \max\left[0, (r_c - 1)\right] \\ &+ \quad \frac{1}{\varepsilon}\left(\delta_s C_{cr,1} r_c r_r^{(1-\alpha\varepsilon)} - K_{vc}\delta_s C_d\right) = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} r_{rt} \quad &+ \quad \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{x}} r_r + w r_{rz} - V_r \frac{1}{\varrho}\frac{\partial}{\partial z}\left(\varrho^{(\frac{1}{2}+\beta\varepsilon)}\, r_r^{(1+\beta\varepsilon)}\right) - \varepsilon\delta_s A_{cr,2} \max\left[0, (r_c - 1)\right] \\ &- \quad \delta_s C_{cr,2} r_c r_r^{(1-\alpha\varepsilon)} + \frac{p}{\varrho}\frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\,[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}]}{B_3 p\, r_{vs} + B_4} = 0. \end{aligned} \tag{18}$$

Here we have introduced δ_s to distinguish between saturated and undersaturated regimes i.e. $\delta_s = 1$ for saturated regime and $\delta_s = 0$ in undersaturated conditions. As $\frac{r_{vs,ref}}{t_{ref}}$ is chosen to be the reference value for C_d, the factor $(K_{vc}\varepsilon^{-1})$ multiplying C_d in (17) is the ratio $\frac{r_{vs,ref}}{r_{c,ref}}$. Also as a result of different reference values for r_c and r_r, we have different order one terms $A_{cr,1}$, $A_{cr,2}$, $C_{cr,1}$ and $C_{cr,2}$ in equations (17) and (18).

2.3 Saturation Mixing Ratio

The saturation mixing ratio characterises the maximum amount of water vapour an air parcel can hold. The saturation mixing ratio r_{vs} is given in terms of the the saturated vapour pressure e_s by

$$r_{vs} = \frac{\epsilon e_s}{p - e_s}, \tag{19}$$

where $\epsilon = \frac{R_d}{R_v} = \frac{287.04}{461.50} = 0.622$ is the ratio of the gas constants of dry air and water vapour. Experimental results show that e_s depends only on temperature T and here we use the Clausius Clapeyron formula

$$\frac{de_s}{dT} = \frac{L_{\text{cond}}}{R_v T^2} e_s, \tag{20}$$

where L_{cond} is the latent heat of condensation which we will assume to be constant. By integrating this relation with respect to T we get

$$e_s = e_\infty \exp\left(\frac{L_{\text{cond}}}{R_v T_0} \frac{(T - T_0)}{T}\right), \tag{21}$$

where $T_0 = T_{\text{ref}} = 273.16\text{K}$ and $e_\infty = 611.2\text{kgm}^{-1}\text{s}^{-2}$, saturated vapour pressure at T_0.

Making the saturated pressure dimensionless with p_{ref} leads to $\frac{e_\infty}{p_{\text{ref}}} = \varepsilon^2 P_{vs}$ and $\epsilon \frac{e_\infty}{p_{ref}} = \varepsilon^3 R_{vs}$. By defining $A = \frac{L_{\text{cond}}}{R_v T_0} = 19.83 = \varepsilon^{-1} A_{vs}$ and using the reference value for r_{vs} from section 2.2 and the relation $\frac{(T-T_0)}{T} = 1 - T^{-1} = 1 - \theta^{-1} p^{\Gamma\varepsilon} = 1 - \frac{\varrho}{p}$, the water vapour saturation mixing ratio can be written as

$$r_{vs} = \left(\frac{R_{vs} \exp\left[\frac{1}{\varepsilon} A_{vs}\left(1 - \frac{\varrho}{p}\right)\right]}{p - \varepsilon^3 P_{vs} \exp\left[\frac{1}{\varepsilon} A_{vs}\left(1 - \frac{\varrho}{p}\right)\right]}\right). \tag{22}$$

3 Deep Convective Columns

This section investigates the evolution of deep convective columns with characteristic horizontal dimensions of 1 km on deep convective time scales or $\sim$ 20 min. To this end, with respect to the horizontal coordinates, a new length scale which is a factor of ε smaller than the scale height ($\varepsilon \ell_{\mathrm{ref}} \approx 1\,\mathrm{km}$), and the associated horizontal stretched coordinate $\boldsymbol{\xi} = \varepsilon^{-1}\boldsymbol{x}$ are introduced. Thus mass and momentum balances and the transport equation for potential temperature – (1) to (4) – as well as the moisture transport equations – (16) to (18) – are rewritten as:

<u>Mass Balance</u>

$$\varrho_t + \frac{1}{\varepsilon}\nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel}) + (\varrho w)_z = 0, \tag{23}$$

<u>Momentum Balance (horizontal and vertical)</u>

$$(\varrho \boldsymbol{v}_{\shortparallel})_t + \frac{1}{\varepsilon}\nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} \circ \boldsymbol{v}_{\shortparallel}) + (\varrho \boldsymbol{v}_{\shortparallel} w)_z + \varepsilon(\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^5}\nabla_{\boldsymbol{\xi}} p = 0, \tag{24}$$

$$(\varrho w)_t + \frac{1}{\varepsilon}\nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} w) + (\varrho w w)_z + \varepsilon(\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^4} p_z = -\frac{1}{\varepsilon^4}\varrho, \tag{25}$$

<u>Transport Equation for the Potential Temperature</u>

$$\theta_t + \frac{1}{\varepsilon}\boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}}\theta + w\theta_z = S_\theta, \tag{26}$$

<u>Water Vapour Transport Equation</u>

$$\begin{aligned} r_{vt} + \frac{1}{\varepsilon}\boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_v \quad &+ \quad w r_{vz} + \delta_s C_d \\ &- \quad \frac{p}{\varrho}\frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\left[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}\right]}{B_3 p r_{vs} + B_4} = 0, \end{aligned} \tag{27}$$

<u>Cloud Water Transport Equation</u>

$$\begin{aligned} r_{ct} + \frac{1}{\varepsilon}\boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_c + w r_{cz} \quad &+ \quad \delta_s A_{cr,1} \max\left[0, (r_c - 1)\right] \\ &+ \quad \frac{1}{\varepsilon}\left(\delta_s C_{cr,1} r_c r_r^{(1-\alpha\varepsilon)} - K_{vc}\delta_s C_d\right) = 0, \end{aligned} \tag{28}$$

<u>Rain Water Transport Equation</u>

$$\begin{aligned} r_{rt} \quad + \quad & \frac{1}{\varepsilon}\boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_r + w r_{rz} - V_r \frac{1}{\varrho}\frac{\partial}{\partial z}\left(\varrho^{(\frac{1}{2}+\beta\varepsilon)}\, r_r^{(1+\beta\varepsilon)}\right) - \varepsilon\delta_s A_{cr,2} \max\left[0, (r_c - 1)\right] \\ - \quad & \delta_s C_{cr,2} r_c r_r^{(1-\alpha\varepsilon)} + \frac{p}{\varrho}\frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\left[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}\right]}{B_3 p r_{vs} + B_4} = 0. \end{aligned} \tag{29}$$

Subsequently for all the dependent variables, asymptotic series expansions are introduced. If these variables are contained in $U = (p, \varrho, w, \boldsymbol{v}_{\shortparallel}, \theta, \dots)$, then the expansion

$$U(\boldsymbol{\xi}, z, t, ; \varepsilon) = \sum_i \varepsilon^i U^{(i)}(\boldsymbol{\xi}, z, t) \tag{30}$$

is assumed.
The system constituted by the flow and moisture equations described in the previous section is obtained under the assumptions $\theta^{(0)} = 1$, $\theta^{(1)} = 0$ (see [6] for further discussion) and $p_t^{(i)} = 0$ for $i = 0, \dots, 3$ with the aim of deriving a closed system of equations for the vertical velocity in leading order $w^{(0)}$.

3.1 Analysis of the Mass and Momentum Balance Equations

We shall begin by analysing the mass and momentum balance equations since they are independent of the regime under consideration. Substitute the expansion (30) into (23) - (25) and collect all terms multiplied by powers of ε and equate these to zero to get:

<u>Mass Balance</u>
In the mass balance equation terms of order ε^{-1} appear and thus we obtain as first condition

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(0)} = 0. \tag{31}$$

We shall show later in (3.3) that $\varrho^{(0)} = \varrho^{(0)}(z)$. Trivial solutions to this equation include $\boldsymbol{v}_{\shortparallel}^{(0)} = 0$ or $\boldsymbol{v}_{\shortparallel}^{(0)} = \boldsymbol{v}_{\shortparallel}^{(0)}(z)$. The solution $\boldsymbol{v}_{\shortparallel}^{(0)} = \boldsymbol{v}_{\shortparallel}^{(0)}(z)$ is appropriate when one wants to include influence of the vertical wind shear on deep convection. The importance of the vertical shear in maintaining the convection has been studied by [8]. However in the subsequent analysis in the present paper we will assume that $\boldsymbol{v}_{\shortparallel}^{(0)} = 0$ in a first step.
Next, we consider the mass equation at leading order and again make use of $\varrho^{(0)} = \varrho^{(0)}(z)$ to obtain the equation

$$\varrho^{(0)} \nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)} + (\varrho^{(0)} w^{(0)})_z = 0. \tag{32}$$

This form of continuity equation constitutes the anelastic approximation. It retains leading order density variation with height and provides a non-homogeneous divergence constraint for the horizontal flow. Note that in this equation the *first order* horizontal, but the *leading order* vertical velocities appear. This is in accordance with the anisotropic spatial scales in this regime.

Horizontal Momentum Balance

$$\nabla_{\boldsymbol{\xi}} p^{(i)} = 0; \quad i = 0, \ldots, 5. \tag{33}$$

$$\boldsymbol{v}_{\shortparallel t}^{(1)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_{\shortparallel}^{(1)} + w^{(0)} \boldsymbol{v}_{\shortparallel z}^{(1)} + (w^{(0)} \widehat{\boldsymbol{\Omega}}_{\shortparallel} \times \boldsymbol{k}) + \frac{1}{\varrho^{(0)}} \nabla_{\boldsymbol{\xi}} p^{(6)} = 0. \tag{34}$$

The pressure $p^{(6)}$ in (34) is the Lagrangian multiplier corresponding to equation (32), i.e., it has to adjust in such a way that the divergence constraint from the continuity equation is fulfilled. The setting here is a bit different than in classical incompressible or anelastic flows, because here the vertical velocity is developing freely independent of the pressure, while only the horizontal divergence is controlled by a pressure field.
A notable feature of the horizontal momentum balance (34) is the Coriolis term due to vertical motion $(w^{(0)} \widehat{\boldsymbol{\Omega}}_{\shortparallel} \times \boldsymbol{k})$. It is the product of the vertical velocity in leading order and the horizontal Coriolis parameter, the horizontal component of Earth's rotation vector. This horizontal Coriolis parameter is usually neglected in meteorological applications. However, this part of the Coriolis force could account for the onset of large scale rotation within an otherwise uncorrelated agglomeration of unstable updrafts. The value of this term is the product of the leading order vertical velocity and the cosine of the degree of latitude. This means that it is zero at the poles, while being maximal near the equator.

Vertical Momentum Balance

$$p_z^{(i)} = -\varrho^{(i)}; \quad i = 0, \ldots, 3. \tag{35}$$

$$\varrho^{(0)} w_t^{(0)} + \varrho^{(0)} \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} w^{(0)} + \varrho^{(0)} w^{(0)} w_z^{(0)} + p_z^{(4)} + \varrho^{(4)} = 0. \tag{36}$$

Equation (36) is non-hydrostatic and shows that $w^{(0)}$ may change locally by advection, by background pressure $p^{(4)}$ and buoyancy. The pressure in the vertical and horizontal momentum equations appears in different orders, $p^{(4)}$ and $p^{(6)}$ respectively.
Since $\nabla_{\boldsymbol{\xi}} p^{(4)} = 0$, the pressure $p^{(4)}$ is imposed on the air column by the background flow as in boundary layer like models. This means that $p^{(4)}$ at any point is equal to hydrostatic environment pressure at the same level i.e. $\frac{\partial p^{(4)}}{\partial z} = -\varrho_{\text{out}}^{(4)}$. Thus by making use of the expansion of the equation of state (43), we can write (36) as

$$w_t^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} w^{(0)} + w^{(0)} w_z^{(0)} = \theta^{(4)} - \theta_{\text{out}}^{(4)}. \tag{37}$$

This equation implies that the vertical motion production is mostly affected by the buoyancy term $\theta^{(4)} - \theta_{\text{out}}^{(4)}$. Convection is enhanced if $\theta^{(4)}$ remains higher than $\theta_{\text{out}}^{(4)}$ at all heights.
The pressure $p^{(5)}$ in the pressure expansion has the same behaviour as $p^{(4)}$ and does not play any role in the dynamics described here.

Combining (34) and (37), one gets an expression for the perturbation pressure $p^{(6)}$ which obeys a Poisson type equation

$$\nabla_{\boldsymbol{\xi}}^2 p^{(6)} = -\nabla \cdot (\varrho^{(0)} \boldsymbol{v}^{(1)} \cdot \nabla \boldsymbol{v}^{(1)}) - \varrho^{(0)} \nabla_{\boldsymbol{\xi}} \cdot (w^{(0)} \widehat{\boldsymbol{\Omega}}_{\shortparallel} \times \boldsymbol{k}) - \frac{\partial}{\partial z} (\theta^{(4)}_{\text{out}} - \theta^{(4)}) \varrho^{(0)}, \quad (38)$$

where

$$\boldsymbol{v}^{(1)} \equiv \boldsymbol{v}^{(1)}_{\shortparallel} + w^{(0)} \boldsymbol{k}.$$

On the left hand side there is the horizontal Laplacian of $p^{(6)}$. The first term on the right hand side is related to the gradient of velocity rotation and shear terms. The second term acts to redistribute Coriolis force induced by the motion, and the third term is due to the vertical gradient of temperature deviation attributable to the difference in temperature within and outside the column. It is important to note that the vertical velocity plays a significant role in the horizontal pressure redistribution through the Coriolis term.
Equations of state and the saturation mixing ratio expansions are also valid for both regimes under consideration, and the expansions are:

<u>Equations of State</u>

$$\varrho^{(0)} - p^{(0)} = 0, \tag{39}$$

$$\varrho^{(1)} + \Gamma \varrho^{(0)} \ln \varrho^{(0)} - p^{(1)} = 0, \tag{40}$$

$$\varrho^{(2)} + \theta^{(2)} \varrho^{(0)} - \frac{1}{2} \Gamma^2 \varrho^{(0)} \left(\ln \varrho^{(0)} \right)^2 + \Gamma p^{(1)} \ln \varrho^{(0)} + \Gamma p^{(1)} - p^{(2)} = 0, \tag{41}$$

$$\begin{aligned} &\varrho^{(3)} + \theta^{(2)} \varrho^{(1)} + \theta^{(3)} \varrho^{(0)} + \frac{1}{6} \Gamma^3 \varrho^{(0)} \left(\ln \varrho^{(0)} \right)^3 - \frac{1}{2} \Gamma^2 p^{(1)} \left(\ln \varrho^{(0)} \right)^2 \\ &-\Gamma^2 p^{(1)} \ln \varrho^{(0)} + \Gamma p^{(2)} \ln \varrho^{(0)} + \Gamma p^{(2)} - p^{(3)} + \frac{1}{2} \Gamma \frac{{p^{(1)}}^2}{\varrho^{(0)}} = 0, \end{aligned} \tag{42}$$

$$\begin{aligned} &\varrho^{(4)} + \theta^{(2)} \varrho^{(2)} + \theta^{(3)} \varrho^{(1)} + \theta^{(4)} \varrho^{(0)} \\ &-\frac{1}{24} \Gamma^4 \varrho^{(0)} \left(\ln \varrho^{(0)} \right)^4 + \frac{1}{6} \Gamma^3 p^{(1)} \left(\ln \varrho^{(0)} \right)^3 + \frac{1}{2} \Gamma^3 p^{(1)} \left(\ln \varrho^{(0)} \right)^2 \\ &-\frac{1}{2} \Gamma^2 p^{(2)} \left(\ln \varrho^{(0)} \right)^2 - \Gamma^2 p^{(2)} \ln \varrho^{(0)} + \Gamma p^{(3)} \ln \varrho^{(0)} + \Gamma p^{(3)} - p^{(4)} \\ &-\frac{1}{2} \Gamma^2 \frac{{p^{(1)}}^2}{\varrho^{(0)}} \ln \varrho^{(0)} - \frac{1}{2} \Gamma^2 \frac{{p^{(1)}}^2}{\varrho^{(0)}} + \Gamma \frac{p^{(1)} p^{(2)}}{\varrho^{(0)}} - \frac{1}{6} \Gamma \frac{{p^{(1)}}^3}{{\varrho^{(0)}}^2} = 0. \end{aligned} \tag{43}$$

Water Vapour Saturation Mixing Ratio

$$r_{vs}^{(0)} = R_{vs} {p^{(0)}}^{A\Gamma-1}, \tag{44}$$

$$\begin{aligned} r_{vs}^{(1)} &= R_{vs} {p^{(0)}}^{A\Gamma-1}\left[A\theta^{(2)} - \frac{1}{2}A\Gamma^2\left(\ln p^{(0)}\right)^2\right] \\ &\quad + R_{vs}{p^{(0)}}^{A\Gamma-2}\left[p^{(1)}\Big(A\Gamma - 1\Big)\right], \end{aligned} \tag{45}$$

$$\begin{aligned} r_{vs}^{(2)} &= R_{vs}{p^{(0)}}^{A\Gamma-1}\Bigg[A\theta^{(3)} + \frac{1}{2}A^2{\theta^{(2)}}^2 - A\Gamma\theta^{(2)}\ln p^{(0)} \\ &\qquad -\frac{1}{2}A^2\Gamma^2\theta^{(2)}\left(\ln p^{(0)}\right)^2 + \frac{1}{6}A\Gamma^3\left(\ln p^{(0)}\right)^3 \\ &\qquad +\frac{1}{8}A^2\Gamma^4\left(\ln p^{(0)}\right)^4\Bigg] \\ &\quad + R_{vs}{p^{(0)}}^{A\Gamma-2}\Bigg[p^{(1)}\Bigg(A^2\Gamma\theta^{(2)} - A\theta^{(2)} - A\Gamma^2\ln p^{(0)} \\ &\qquad + \frac{1}{2}A\Gamma^2\left(\ln p^{(0)}\right)^2 - \frac{1}{2}A^2\Gamma^3\left(\ln p^{(0)}\right)^2\Bigg) \\ &\qquad + p^{(2)}\Big(A\Gamma - 1\Big)\Bigg] \\ &\quad + R_{vs}{p^{(0)}}^{A\Gamma-3}\left[{p^{(1)}}^2\left(1 - \frac{3}{2}A\Gamma + \frac{1}{2}A^2\Gamma^2\right)\right]. \end{aligned} \tag{46}$$

All the terms appearing in equations (39) to (46) are pure functions of z except the two terms in equation of state (43) involving $\varrho^{(4)}$ and $\theta^{(4)}$ which are functions of $\boldsymbol{\xi}$ and t in addition to z.

3.2 Analysis of the Moist Thermodynamics Equations

From the assumption $p_t^{(i)} = 0$ for $i = 0, 1, 2, 3$ and the hydrostatic equations (35) it follows that $\varrho_t^{(i)} = 0$ for $i = 0, 1, 2, 3$. This implies from the equations of state (41), (42) and from the saturation mixing ratio equations (44) - (46) that

$$\theta_t^{(i)} = 0, \quad \text{and} \quad r_{vs\,t}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3. \tag{47}$$

Also taking the horizontal gradient of (35) and making use of (33) results in $\nabla_{\boldsymbol{\xi}}\varrho^{(i)} = 0$ for $i = 0, 1, 2, 3$ which again implies from the equations of state (41), (42) and from the saturation mixing ratio equations (44) - (46) that

$$\nabla_{\boldsymbol{\xi}}\theta^{(i)} = 0 \quad \text{and} \quad \nabla_{\boldsymbol{\xi}} r_{vs}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3. \tag{48}$$

Transport Equation for the Potential Temperature

By making use of (47) and (48), the expansions of the potential temperature are:

$$w^{(0)}\theta_z^{(2)} = S_\theta^{(2)}, \tag{49}$$

$$w^{(0)}\theta_z^{(3)} + w^{(1)}\theta_z^{(2)} = S_\theta^{(3)}, \tag{50}$$

$$\theta_t^{(4)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}}\theta^{(4)} + w^{(0)}\theta_z^{(4)} + w^{(1)}\theta_z^{(3)} + w^{(2)}\theta_z^{(2)} = S_\theta^{(4)}. \tag{51}$$

Source Term

The crucial mechanism for the development of a deep convective system is the release of latent heat by an ascending air parcel. The source term S_θ in equation (26) represents this mechanism in the following way:

$$S_\theta = \varepsilon^2 \Gamma \frac{\theta}{p} \varrho L (C_d - E_r).$$

In the saturated air regime $r_v = r_{vs}$, then (27) implies that $C_d = -\frac{D}{Dt} r_{vs}$. In the undersaturated air regime $(C_d - E_r) = -E_r$. Thus the source term can be written as

$$S_\theta = -\varepsilon^2 \Gamma \frac{\theta}{p} \varrho L \left(\delta_s \frac{D}{Dt} r_{vs} + \frac{p}{\varrho} \frac{(r_{vs} - r_v)(\varrho\, r_r)^{\left(\frac{1}{2} + \lambda \varepsilon^2\right)} \left[B_1 + B_2 (\varrho\, r_r)^{\mu\varepsilon}\right]}{B_3 p r_{vs} + B_4} \right), \tag{52}$$

where $\frac{D}{Dt} r_{vs} = r_{vst} + \frac{1}{\varepsilon} \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_{vs} + w r_{vsz}$, and E_r has been replaced by the corresponding parameterization.

3.2.1 Saturated Air Regime

The perturbation expansion of (52) for the saturated regime and making use of (47) and (48) lead to

$$S_\theta^{(2)} = -\Gamma L w^{(0)} {r_{vs}}_z^{(0)}, \tag{53}$$

$$S_\theta^{(3)} = -\Gamma L \left[w^{(0)} \left({r_{vs}}_z^{(1)} - \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(0)} \right) + w^{(1)} {r_{vs}}_z^{(0)} \right], \tag{54}$$

$$\begin{aligned} S_\theta^{(4)} = -\Gamma L \Bigg[& w^{(0)} {r_{vs}}_z^{(2)} - w^{(0)} \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(1)} + w^{(0)} \left(\frac{1}{2} \Gamma^2 \left(\ln \varrho^{(0)}\right)^2 - \right. \\ & \left. \Gamma \frac{p^{(1)}}{p^{(0)}} \right) {r_{vs}}_z^{(0)} - w^{(1)} \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(0)} + w^{(1)} {r_{vs}}_z^{(1)} + w^{(2)} {r_{vs}}_z^{(0)} \Bigg]. \end{aligned} \tag{55}$$

By substituting these in (49)– (51) together with the analytical solutions for $p^{(0)}$, $\varrho^{(0)}$ and $p^{(1)}$, $\varrho^{(1)}$ given by (67)–(70) and assuming that $w^{(0)} \neq 0$, the expansions for the potential temperature reduce to

$$\theta_z^{(2)} = -\Gamma L {r_{vs}}_z^{(0)}, \tag{56}$$

$$\theta_z^{(3)} = -\Gamma L \left({r_{vs}}_z^{(1)} - \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(0)} \right), \tag{57}$$

$$\begin{aligned} \theta_t^{(4)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} \theta^{(4)} + w^{(0)} \theta_z^{(4)} &= -\Gamma L w^{(0)} \Bigg({r_{vs}}_z^{(2)} - \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(1)} \\ &+ \left(\frac{1}{2} \Gamma^2 \left(\ln \varrho^{(0)} \right)^2 - \Gamma \frac{p^{(1)}}{p^{(0)}} \right) {r_{vs}}_z^{(0)} \Bigg). \end{aligned} \tag{58}$$

Let us assume a weak temperature gradient approximation outside the column. For a derivation see [5]. In the weak temperature gradient approximation, the horizontal temperature fluctuations are weak making the dominant balance between vertical advection of potential temperature and the total heating [7], then

$$w^{(0)} (\theta_z^{(4)})_{\text{out}} = -\Gamma L w^{(0)} \left({r_{vs}}_z^{(2)} - \Gamma \ln \varrho^{(0)} {r_{vs}}_z^{(1)} + \left(\frac{1}{2} \Gamma^2 \left(\ln \varrho^{(0)} \right)^2 - \Gamma \frac{p^{(1)}}{p^{(0)}} \right) {r_{vs}}_z^{(0)} \right) \tag{59}$$

since $\nabla_{\boldsymbol{\xi}} {r_{vs}}^{(i)} = 0$ for $i = 0, 1, 2$ and thus

$$\theta_t^{(4)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} \theta^{(4)} + w^{(0)} \theta_z^{(4)} = w^{(0)} (\theta_z^{(4)})_{\text{out}}. \tag{60}$$

This equation implies that the deviation of the potential temperature from the background, $\theta^{(4)} - \theta_{\text{out}}^{(4)}$, is a conserved quantity.

Finally, the description of the Saturated Air Regime is completed by the leading order transport equations of the cloud water and rain water mixing ratios:

Transport Equation for the Cloud Water Mixing Ratio

$$K_{vc} w^{(0)} {r_{vs}}_z^{(0)} + C_{cr,1} r_c^{(0)} r_r^{(0)} = 0, \tag{61}$$

Transport Equation for the Rain Water Mixing Ratio

$${r_r}_t^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} r_r^{(0)} + w^{(0)} {r_r}_z^{(0)} - V_r \frac{1}{\varrho^{(0)}} (\varrho^{(0)\,\frac{1}{2}} r_r^{(0)})_z - C_{cr,2} r_c^{(0)} r_r^{(0)} = 0. \tag{62}$$

3.2.2 Undersaturated Air Regime

Transport Equation for the Potential Temperature

The expansion of the source term (52) (due to evaporation) in the undersaturated regime ($\delta_s = 0$) leads to

$$S_\theta^{(2)} = \frac{\Gamma L (r_v^{(0)} - r_{vs}^{(0)}) (\varrho^{(0)} r_r^{(0)})^{\frac{1}{2}} (B_1 + B_2)}{B_3 p^{(0)} r_{vs}^{(0)} + B_4}, \tag{63}$$

and the following potential temperature relation from (51) will hold

$$w^{(0)}\theta_z^{(2)} = \frac{\Gamma L(r_v^{(0)} - r_{vs}^{(0)})(\varrho^{(0)} r_r^{(0)})^{\frac{1}{2}}(B_1 + B_2)}{B_3 p^{(0)} r_{vs}^{(0)} + B_4}. \tag{64}$$

Thus the vertical velocity can be diagnosed from this weak temperature gradient equation. It depends on the leading order evaporation rate which in turn directly depends on the leading order saturation deficit and the amount of rain water. The evaporation rate determines the consumption of latent heat in the undersaturated environment.

The leading order transport equations of the water vapour and rain water mixing ratios round off the picture of the Undersaturated Air Regime:

Transport Equation for the Water Vapour Mixing Ratio

$$r_{v\,t}^{(0)} + \boldsymbol{v}_{_\parallel}^{(1)} \cdot \nabla_{\xi} r_v^{(0)} + w^{(0)} r_{v\,z}^{(0)} = \frac{(r_{vs}^{(0)} - r_v^{(0)})(\varrho^{(0)} r_r^{(0)})^{\frac{1}{2}}(B_1 + B_2)}{(B_3 p^{(0)} r_{vs}^{(0)} + B_4)}, \tag{65}$$

Transport Equation for the Rain Water Mixing Ratio

$$\begin{aligned} r_{r\,t}^{(0)} + \boldsymbol{v}_{_\parallel}^{(1)} \cdot \nabla_{\xi} r_r^{(0)} + w^{(0)} r_{r\,z}^{(0)} \;\; = \;\; & \frac{(r_v^{(0)} - r_{vs}^{(0)})(\varrho^{(0)} r_r^{(0)})^{\frac{1}{2}}(B_1 + B_2)}{(B_3 p^{(0)} r_{vs}^{(0)} + B_4)} \\ & + V \frac{1}{\varrho^{(0)}} \left((\varrho^{(0)})^{\frac{1}{2}} r_r^{(0)} \right)_z . \end{aligned} \tag{66}$$

In the undersaturated Air Regime the potential temperature is only expanded up to $\theta^{(2)}$ (see Section 3.4.2 for the explanation).

3.3 Analytical Solutions and Closures

As far as possible, analytical solutions for the variables in lower orders are derived to end up with a system with as many equations as (higher order) unknowns.

We can determine the leading order $p^{(0)}$, $\varrho^{(0)}$ and first order $p^{(1)}$, $\varrho^{(1)}$ from the leading and first order momentum equations as follows. Combining equations (33, $i = 0$), (35, $i = 0$) and (39) yields an ordinary differential equation and thus the following explicit formulas for $p^{(0)}$ and $\varrho^{(0)}$:

$$p^{(0)}(z) = \exp(-z), \tag{67}$$

$$\varrho^{(0)}(z) = \exp(-z). \tag{68}$$

Also solving the ordinary differential equation formed by equations (33, $i=1$), (35, $i=1$) and (40) provides explicit expressions for $p^{(1)}$ and $\varrho^{(1)}$:

$$p^{(1)}(z) = \Gamma\left(-\frac{1}{2}z^2\right)\exp(-z), \tag{69}$$

$$\varrho^{(1)}(z) = \Gamma\left(z-\frac{1}{2}z^2\right)\exp(-z). \tag{70}$$

Equations (35, $i=2$), (41) and (56) provide $p^{(2)}$, $\varrho^{(2)}$ and $\theta^{(2)}$:

$$\theta^{(2)}(z) = \underbrace{\theta^{(2)}(0)}_{C_1} - \Gamma LR\Big[\exp\big(-(A\Gamma-1)z\big)-1\Big], \tag{71}$$

$$\begin{aligned}\varrho^{(2)}(z) &= \Gamma^2\left(\left(-\frac{C_1}{\Gamma^2}-\frac{ALR}{A\Gamma-1}\right)+\left(\frac{C_1}{\Gamma^2}+\frac{LR}{\Gamma}\right)z+z^2-\frac{5}{6}z^3+\frac{1}{8}z^4\right)\exp(-z)\\ &\quad +\Gamma^2\frac{ALR}{A\Gamma-1}\exp(-A\Gamma z),\end{aligned} \tag{72}$$

$$\begin{aligned}p^{(2)}(z) &= \Gamma^2\left(\left(-\frac{ALR}{A\Gamma-1}+\frac{LR}{\Gamma}\right)+\left(\frac{C_1}{\Gamma^2}+\frac{LR}{\Gamma}\right)z-\frac{1}{3}z^3+\frac{1}{8}z^4\right)\exp(-z)\\ &\quad +\Gamma^2\left(\frac{ALR}{A\Gamma-1}-\frac{LR}{\Gamma}\right)\exp(-A\Gamma z).\end{aligned} \tag{73}$$

Equations (35, $i=3$), (42) and (57) yield $p^{(3)}$, $\varrho^{(3)}$ and $\theta^{(3)}$:

$$\begin{aligned}\theta^{(3)}(z) &= \underbrace{\theta^{(3)}(0)}_{C_2}+\Big[a_0+a_1z+a_2z^2\Big]\exp\big(-(A\Gamma-1)z\big)\\ &\quad +b_0\exp\big(-2(A\Gamma-1)z\big)+c_0,\end{aligned} \tag{74}$$

$$\begin{aligned}\varrho^{(3)}(z) &= \Big[d_0+d_1z+d_2z^2+d_3z^3+d_4z^4+d_5z^5+d_6z^6\Big]\exp(-z)\\ &\quad +\Big[e_0+e_1z+e_2z^2\Big]\exp(-A\Gamma z)+f_0\exp\big(-(2A\Gamma-1)z\big),\end{aligned} \tag{75}$$

$$\begin{aligned}p^{(3)}(z) &= \Big[g_0+g_1z+g_2z^2+g_3z^3+g_4z^4+g_5z^5+g_6z^6\Big]\exp(-z)\\ &\quad +\Big[h_0+h_1z+h_2z^2\Big]\exp(-A\Gamma z)+i_0\exp\big(-(2A\Gamma-1)z\big).\end{aligned} \tag{76}$$

The coefficients a_0 to i_0 of the polynomials in z are listed in appendix B.

3.4 Summary of the Asymptotic Deep Column Model Equations

3.4.1 Closed System of Equations of the Saturated Air Regime

The final system is constituted by equations (32), (34), (37) and (60) containing the variables $p^{(6)}$, $\theta^{(4)}$, $\boldsymbol{v}_{\shortparallel}^{(1)}$ and $w^{(0)}$.

To complete the picture of the Saturated Air Regime, the leading order transport equations of the cloud water and rain water mixing ratios are needed which are given by equations (61) and (62). These are 'stand-alone-equations' that can be solved by using the results of the closed system of equations mentioned right above.

Below all the equations describing the Saturated Air Regime are shown once again:

Mass Balance

$$\varrho^{(0)}\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(1)} + (\varrho^{(0)}w^{(0)})_z = 0,$$

Horizontal Momentum Balance

$$\boldsymbol{v}_{\shortparallel t}^{(1)} + (\boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(1)} + w^{(0)}\boldsymbol{v}_{\shortparallel z}^{(1)} + (w^{(0)}\widehat{\boldsymbol{\Omega}}_{\shortparallel} \times \boldsymbol{k}) + \frac{1}{\varrho^{(0)}}\nabla_{\boldsymbol{\xi}} p^{(6)} = 0,$$

Vertical Momentum Balance

$$w_t^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} w^{(0)} + w^{(0)}w_z^{(0)} = \theta^{(4)} - \theta_{\text{out}}^{(4)},$$

Transport Equation for the Potential Temperature

$$\theta_t^{(4)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}}\theta^{(4)} + w^{(0)}\theta_z^{(4)} = w^{(0)}(\theta_z^{(4)})_{\text{out}},$$

where $(\theta_z^{(4)})_{\text{out}}$ is given by (59),

Transport Equation for the Cloud Water Mixing Ratio

$$K_{vc}w^{(0)}r_{vs\,z}^{(0)} + C_{cr,1}r_c^{(0)}r_r^{(0)} = 0,$$

Transport Equation for the Rain Water Mixing Ratio

$$r_{r\,t}^{(0)} + \boldsymbol{v}_{\shortparallel}^{(1)} \cdot \nabla_{\boldsymbol{\xi}} r_r^{(0)} + w^{(0)}r_{r\,z}^{(0)} - V_r\frac{1}{\varrho^{(0)}}(\varrho^{(0)\,\frac{1}{2}}r_r^{(0)})_z - C_{cr,2}r_c^{(0)}r_r^{(0)} = 0.$$

From the potential temperature equation, the vertical stability of the deep convective column can be influenced by the vertical gradient of the fourth order potential temperature $\theta_z^{(4)}$ outside the column. We define $f(z)$ as

$$f(z) = (\theta_z^{(4)})_{\text{out}}. \tag{77}$$

We make use of the analytical expressions for p, ϱ and θ and the constants of $\mathcal{O}(1)$ appearing in $f(z)$ known from the dimensional analysis: $A = 2.83$, $R = 0.38$, $L = 1.75$, $\Gamma = 2$; with $C_1 = \theta^{(2)}(z=0)$ and $C_2 = \theta^{(3)}(z=0)$ left as parameters, $f(z)$ can be written as

$$\begin{aligned} f(z) = {} & 395.12 \exp(-13.98z) \\ & + \left[(-334.73 - 264.07C_1) - 486.58z + 962.98z^2\right] \exp(-9.32z) \\ & + \Big[(27.42 + 118.31C_1 + 24.82{C_1}^2 + 17.54C_2) + (235.05 + 176.73C_1)z \\ & \quad + (-76.04 - 181.01C_1)z^2 - 543.27z^3 + 330.04z^4)\Big] \exp(-4.66z). \end{aligned} \tag{78}$$

To get an idea of this function's shape Figure 1 shows the variation of $(\theta_z^{(4)})_{\text{out}}$ with height where $\theta^{(3)}(z=0)$ is set to be 0, $\theta^{(2)}(z=0)$ is left as free parameter in equation (78) and is varied between -2.0 and 2.0.

3.4.2 Closed System of Equations for the Undersaturated Air Regime

As p and ϱ are independent of the horizontal scale in the orders appearing in equation (41), it follows that $\theta^{(2)}$ does not change in horizontal layers either, and it is imposed by the background state. Assuming the background state to be moist adiabatic according to equation (71) of the saturated case means that equations (35, $i = 2$) and (41) again give the analytical solutions for $p^{(2)}$ and $\varrho^{(2)}$ (equations (72) and (73)).

In the Undersaturated Air Regime it is not necessary to expand the potential temperature up to $\theta^{(4)}$ because already with the transport equation for $\theta^{(2)}$ a closed system of equations to determine $w^{(0)}$ is given.

Thus the final system in the Undersaturated Air Regime is constituted by equations (32), (34), (64), (65) and (66) containing the variables $p^{(6)}$, $r_v^{(0)}$, $r_r^{(0)}$, $\boldsymbol{v}_{\shortparallel}^{(1)}$ and $w^{(0)}$. This means that unlike in the Saturated Air Regime in this case the moisture transport equations are necessary to actually close the system of equations. Below all the equations characterising the Undersaturated Air Regime are shown once again:

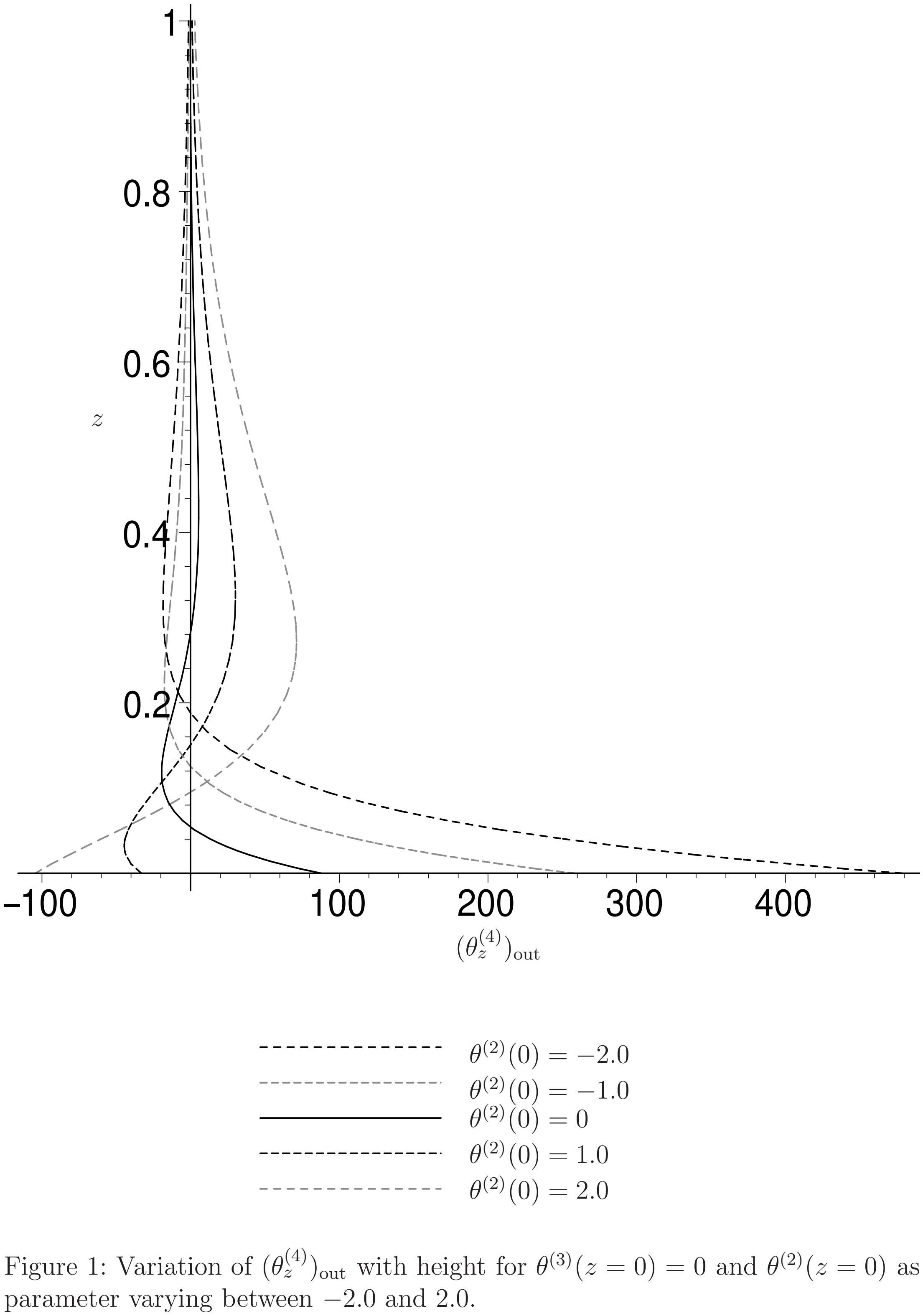

Figure 1: Variation of $(\theta_z^{(4)})_{\mathrm{out}}$ with height for $\theta^{(3)}(z=0) = 0$ and $\theta^{(2)}(z=0)$ as parameter varying between -2.0 and 2.0.

Mass Balance

$$\varrho^{(0)}\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{v}_{\shortparallel}^{(1)}+(\varrho^{(0)}w^{(0)})_z=0,$$

Horizontal Momentum Balance

$$\boldsymbol{v}_{\shortparallel t}^{(1)}+(\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\boldsymbol{\xi}})\boldsymbol{v}_{\shortparallel}^{(1)}+w^{(0)}\boldsymbol{v}_{\shortparallel z}^{(1)}+(w^{(0)}\widehat{\boldsymbol{\Omega}}_{\shortparallel}\times\boldsymbol{k})+\frac{1}{\varrho^{(0)}}\nabla_{\boldsymbol{\xi}}p^{(6)}=0,$$

Transport Equation for the Potential Temperature

$$w^{(0)}\theta_z^{(2)}=-\Gamma L E_r^{(0)},$$

Transport Equation for the Water Vapour Mixing Ratio

$$r_{vt}^{(0)}+\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\boldsymbol{\xi}}r_v^{(0)}+w^{(0)}r_{vz}^{(0)}=E_r^{(0)},$$

Transport Equation for the Rain Water Mixing Ratio

$$r_{rt}^{(0)}+\boldsymbol{v}_{\shortparallel}^{(1)}\cdot\nabla_{\boldsymbol{\xi}}r_r^{(0)}+w^{(0)}r_{rz}^{(0)}-V\frac{1}{\varrho^{(0)}}\left((\varrho^{(0)})^{\frac{1}{2}}r_r^{(0)}\right)_z=-E_r^{(0)}.$$

The evaporation term in the above equations reads as follows:

$$E_r^{(0)}=\frac{(r_{vs}^{(0)}-r_v^{(0)})(\varrho^{(0)}r_r^{(0)})^{\frac{1}{2}}(B_1+B_2)}{(B_3p^{(0)}r_{vs}^{(0)}+B_4)}.$$

4 Shallow Convective Layers

From the analysis in the previous section, there is a great deal of interest in determining vertical fluxes of energy and momentum in the lower troposphere. This can be done by considering the temperature, moisture and vertical velocity at the boundary layer. Thus we consider a shallow layer of fluid near the surface of depth of 1 km (i.e $\varepsilon h_{\rm sc}$) and a horizontal length scale of the same order of magnitude. Hence our new co-ordinate system is built by $\boldsymbol{\xi} = \varepsilon^{-1}\boldsymbol{X}_{\shortparallel}$ and $\eta = \varepsilon^{-1}z$. We consider the time scale associated with the horizontal advection i.e. $\tau = \varepsilon^{-1}t$. Substituting these scalings into the governing equations (1)–(4), (16)–(18) yields

<u>Mass Balance</u>

$$\varrho_\tau + \nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel}) + (\varrho w)_\eta = 0, \tag{79}$$

<u>Horizontal Momentum Balance</u>

$$\varepsilon^4(\varrho \boldsymbol{v}_{\shortparallel})_\tau + \varepsilon^4 \nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} \circ \boldsymbol{v}_{\shortparallel}) + \varepsilon^4 (\varrho \boldsymbol{v}_{\shortparallel} w)_\eta + \varepsilon^6 (\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\shortparallel} + \nabla_{\boldsymbol{\xi}} p = 0, \tag{80}$$

<u>Vertical Momentum Balance</u>

$$\varepsilon^4(\varrho w)_\tau + \varepsilon^4 \nabla_{\boldsymbol{\xi}} \cdot (\varrho \boldsymbol{v}_{\shortparallel} w) + \varepsilon^4 (\varrho w w)_\eta + \varepsilon^6 (\widehat{\boldsymbol{\Omega}} \times \varrho \boldsymbol{v})_{\perp} + p_\eta = -\varepsilon \varrho, \tag{81}$$

<u>Potential Temperature Equation</u>

$$\theta_\tau + \boldsymbol{v}_{\shortparallel} . \nabla_{\boldsymbol{\xi}} \theta + w\theta_\eta = \varepsilon S_\theta, \tag{82}$$

<u>Water Vapour Equation</u>

$$\begin{aligned} r_{v\tau} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_v \quad &+ \quad w r_{v\eta} + \varepsilon \Bigg(\delta_s C_d \\ &- \quad \frac{p}{\varrho} \frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\,[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}]}{B_3 p r_{vs} + B_4} \Bigg) = 0, \end{aligned} \tag{83}$$

<u>Cloud Water Equation</u>

$$\begin{aligned} r_{c\tau} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_c + w r_{c\eta} - K_{vc}\delta_s C_d \quad &+ \quad \varepsilon \delta_s A_{cr,1} \max{[0, (r_c - 1)]} \\ &+ \quad \delta_s C_{cr,1} r_c r_r^{(1-\alpha\varepsilon)} = 0, \end{aligned} \tag{84}$$

<u>Rain Water Equation</u>

$$\begin{aligned} r_{r\tau} \quad + \quad & \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\boldsymbol{\xi}} r_r + w r_{r\eta} - V_r \frac{1}{\varrho} \frac{\partial}{\partial \eta} (\varrho^{(\frac{1}{2}+\beta\varepsilon)}\, r_r^{(1+\beta\varepsilon)}) - \varepsilon^2 \delta_s A_{cr,2} \max{[0, (r_c - 1)]} \\ - \quad & \varepsilon \delta_s C_{cr,2} r_c r_r^{(1-\alpha\varepsilon)} + \varepsilon \frac{p}{\varrho} \frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\,[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}]}{B_3 p r_{vs} + B_4} = 0, \end{aligned} \tag{85}$$

with $\delta_s = 1$ for saturated regime and $\delta_s = 0$ for undersaturated conditions.

Here again, the parameter ε provides the basis of the expansion for any dependent variable $U \in \{\boldsymbol{v}_{\shortparallel}, w, \varrho, p, \theta, r_v, r_r, r_c, r_{vs}\}$. We assume that the solution can be represented asymptotically by the expression

$$U(\boldsymbol{\xi}, \eta, \tau, ; \varepsilon) = \sum_i \varepsilon^i U^{(i)}(\boldsymbol{\xi}, \eta, \tau). \tag{86}$$

4.1 Momentum Equations

The leading order momentum equations of these expansions

$$\nabla_{\boldsymbol{\xi}} p^{(0)} = 0, \quad \text{and} \quad p^{(0)}_{\eta} = 0. \tag{87}$$

These implies that $p^{(0)} = p^{(0)}(\tau)$ and without loss of generality let us assume that $p^{(0)}$ is independent of time τ and equal to the surface pressure, i.e.

$$p^{(0)} = 1, \tag{88}$$

and therefore, from equation of state (39), we have

$$\varrho^{(0)} = p^{(0)} = 1. \tag{89}$$

To simplify the equations of state we anticipate the result $\theta^{(2)} = 0$ and take the potential temperature distribution within the layer as

$$\theta = 1 + \varepsilon^3 \theta^{(3)}(\boldsymbol{\xi}, \eta, \tau) + \mathcal{O}\left(\varepsilon^4\right). \tag{90}$$

Thus the equations of state (40)–(42) reduce to

$$\varrho^{(1)} = p^{(1)}, \tag{91}$$

$$\varrho^{(2)} = p^{(2)} - \Gamma p^{(1)}, \tag{92}$$

$$\varrho^{(3)} = p^{(3)} - \theta^{(3)} - \Gamma p^{(2)} + \frac{1}{2}\Gamma {p^{(1)}}^2. \tag{93}$$

We can further determine the solution for the next orders of pressure $p^{(i)}$ for $i = 1, 2, 3$ from order $\mathcal{O}(\varepsilon^i)$ momentum equations, namely

$$\nabla_{\boldsymbol{\xi}} p^{(i)} = 0 \quad \text{and} \quad p^{(i)}_{\eta} = -\varrho^{(i-1)}, \tag{94}$$

and hence density $\varrho^{(i)}$ for $i = 1, 2$ from (91) and (92). Based on the same reasoning as above and assuming that $p^{(i)}$'s vanish at the surface, we find that

$$p^{(1)} = \varrho^{(1)} = -\eta, \quad p^{(2)} = \frac{\eta^2}{2}, \quad \varrho^{(2)} = \frac{\eta^2}{2} + \Gamma\eta, \quad p^{(3)} = \frac{\eta^3}{6} - \Gamma\frac{\eta^2}{2} + \eta. \tag{95}$$

The $\mathcal{O}(\varepsilon^4)$ momentum equations provides us with the evolution equations for velocity components $\boldsymbol{v}_{_{\shortparallel}}^{(0)}$ and $w^{(0)}$

$$\boldsymbol{v}_{_{\shortparallel}\tau}^{(0)} + \boldsymbol{v}_{_{\shortparallel}}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_{_{\shortparallel}}^{(0)} + w^{(0)} \boldsymbol{v}_{_{\shortparallel}\eta}^{(0)} + \nabla_{\boldsymbol{\xi}} p^{(4)} = 0, \tag{96}$$

$$w_{\tau}^{(0)} + \boldsymbol{v}_{_{\shortparallel}}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} w^{(0)} + w^{(0)} w_{\eta}^{(0)} + p_{\eta}^{(4)} + \varrho^{(3)} = 0. \tag{97}$$

This result contains the basic physical processes associated with the development of downdrafts and updrafts in a shallow convective layer. The term $p_\eta^{(4)}$ represents the effect of the pressure gradient perturbation on the vertical motion and the term $\varrho^{(3)}$ represents the effect of buoyancy on vertical motion. This term is given by (93).

The perturbation pressure $p^{(4)}$ is due to the advection nonlinearity and the moist processes, and it redistributes momentum so that the divergence condition for velocity is satisfied. Please note that the coriolis acceleration is not significant at this leading order. One can write the equation describing $p^{(4)}$ as

$$\nabla^2 p^{(4)} = -\nabla \cdot (\boldsymbol{v}.\nabla \boldsymbol{v}) + \frac{\partial \theta^{(3)}}{\partial \eta} + \varrho^{(2)}. \tag{98}$$

4.2 Continuity Equation

With $\varrho^{(0)} = 1$, the leading order mass balance equation is

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{_{\shortparallel}}^{(0)} + w_{\eta}^{(0)} = 0. \tag{99}$$

The leading order velocity field has zero divergence.

4.3 Moist Thermodynamics Equations

In absence of radiation, diffusive and turbulent mechanisms, we assume that the only heat source is the release of latent heat due to condensation / evaporation. Recall that from (52) the source term S_θ in the potential temperature equation can be written in terms of the latent heat as

$$S_\theta = -\varepsilon^2 \Gamma \frac{\theta}{p} \varrho L \left(\delta_s \frac{Dr_{vs}}{D\tau} + \frac{p}{\varrho} \frac{(r_{vs} - r_v)\,(\varrho\, r_r)^{(\frac{1}{2}+\lambda\varepsilon^2)}\,[B_1 + B_2\,(\varrho\, r_r)^{\mu\varepsilon}]}{B_3 p r_{vs} + B_4} \right), \tag{100}$$

where

$$\frac{Dr_{vs}}{D\tau} = \frac{1}{\varepsilon} \left(\frac{\partial r_{vs}}{\partial \tau} + \boldsymbol{v}_{_{\shortparallel}}.\nabla_{\boldsymbol{\xi}} r_{vs} + w \frac{\partial r_{vs}}{\partial \eta} \right). \tag{101}$$

Since $p^{(0)} = 1$, the leading order and the next order of the water vapour saturation mixing ratio r_{vs} are:

$$r_{vs}^{(0)} = R_{vs}, \tag{102}$$

$$r_{vs}^{(1)} = R_{vs}(\Gamma A_{vs} - 1) p^{(1)}. \tag{103}$$

Thus

$$S_\theta^{(0)} = S_\theta^{(1)} = 0, \quad S_\theta^{(2)} = -\Gamma L \left(\delta_s w^{(0)} \frac{\partial}{\partial \eta} r_{vs}^{(1)} + \frac{(r_{vs}^{(0)} - r_v^{(0)})(r_r^{(0)})^{1/2}[B_1 + B_2]}{(r_{vs}^{(0)} B_3 + B_4)} \right). \tag{104}$$

But we know that

$$\nabla_{\boldsymbol{\xi}} \theta^{(2)} = 0 \qquad \text{and} \qquad \theta_t^{(2)} = 0 \tag{105}$$

from (94), thus

$$w^{(0)} \theta_\eta^{(2)} = S_\theta^{(1)} = 0. \tag{106}$$

Therefore $\theta_\eta^{(2)} = 0$, and without loss of generality let $\theta^{(2)} = 0$ as anticipated in equation (90).

The $\mathcal{O}(\varepsilon^3)$ leading order potential temperature equation is

$$\theta_\tau^{(3)} + \boldsymbol{v}_{\shortparallel}^{(0)} . \nabla_{\boldsymbol{\xi}} \theta^{(3)} + w^{(0)} \theta_\eta^{(3)} = S_\theta^{(2)}. \tag{107}$$

4.3.1 Moist Variables Equations: Saturated Regime

From (104) we find that (107) becomes

$$\frac{\partial \theta^{(3)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)} . \nabla_{\boldsymbol{\xi}} \theta^{(3)} + w^{(0)} \frac{\partial \theta^{(3)}}{\partial \eta} = \Gamma L (\Gamma A_{vs} - 1) w^{(0)}. \tag{108}$$

Here we see that the latent heat release due to condensation is proportional to the vertical velocity $w^{(0)}$. We see that $w^{(0)} > 0$ creates a positive $\theta^{(3)}$ and $w^{(0)} < 0$ creates a negative $\theta^{(3)}$. The physical process captured is latent heat release due to condensation in upward moving air parcels and evaporative cooling in downward moving ones.

The evolution equations for cloud water and rain water are given by

$$\frac{\partial r_c^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)} . \nabla_{\boldsymbol{\xi}} {r_c}^{(0)} + w^{(0)} \frac{\partial {r_c}^{(0)}}{\partial \eta} = -K_{vc}(\Gamma A_{vs} - 1) w^{(0)} - C_{cr,1} r_c^{(0)} r_r^{(0)} \tag{109}$$

and

$$\frac{\partial r_r^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)} . \nabla_{\boldsymbol{\xi}} {r_r}^{(0)} + w^{(0)} \frac{\partial {r_r}^{(0)}}{\partial \eta} = V_r \frac{\partial r_r^{(0)}}{\partial \eta}, \tag{110}$$

respectively.
At this order, the microphysical processes that are important in a saturated air regime are condensation and accretion.

4.3.2 Moist Variables Equations: Undersaturated Regime

The thermodynamic equation for undersaturated regime is given by

$$\frac{\partial \theta^{(3)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}} \theta^{(3)} + w^{(0)} \frac{\partial \theta^{(3)}}{\partial \eta} = -\Gamma L \frac{(r_{vs}^{(0)} - r_v^{(0)})(r_r^{(0)})^{1/2}[B_1 + B_2]}{(r_{vs}^{(0)} B_3 + B_4)} \tag{111}$$

from (104) and (107). The equation for water vapour is

$$\frac{\partial r_v^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}} {r_v}^{(0)} + w^{(0)} \frac{\partial {r_v}^{(0)}}{\partial \eta} = 0 \tag{112}$$

and for rain water is

$$\frac{\partial r_r^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}} {r_r}^{(0)} + w^{(0)} \frac{\partial {r_r}^{(0)}}{\partial \eta} = V_r \frac{\partial r_r^{(0)}}{\partial \eta}. \tag{113}$$

At leading order, evaporation of precipitation into undersaturated layer is a heat sink and not a source of water vapour, thus reducing the tendency of the shallow layer to saturate. If r_v is initially constant then (112) implies that it will remain constant at all time and thus constant $r_{vs} - r_v$. This implies that the distribution of $\theta^{(3)}$ depends mainly on the amount of rain water present. (Of course, these statements pertain to the short 2 min time scale analyzed in this section only, and not to the deep convective time scales of about 20 min considered earlier.)

4.4 Summary of the Shallow Convective Equations

Below is a summary of the set of simplifed asymptotic equations describing the 1 km by 1 km shallow convective elements described above.

$$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{v}_{\shortparallel}^{(0)} + w_{\eta}^{(0)} = 0,$$

$$\boldsymbol{v}_{\shortparallel\tau}^{(0)} + \boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v}_{\shortparallel}^{(0)} + w^{(0)} \boldsymbol{v}_{\shortparallel\eta}^{(0)} + \nabla_{\boldsymbol{\xi}} p^{(4)} = 0,$$

$$w_{\tau}^{(0)} + \boldsymbol{v}_{\shortparallel}^{(0)} \cdot \nabla_{\boldsymbol{\xi}} w^{(0)} + w^{(0)} w_{\eta}^{(0)} + p_{\eta}^{(4)} + \varrho^{(3)} = 0,$$

$$\varrho^{(3)} = p^{(3)} - \theta^{(3)} - \Gamma p^{(2)} + \frac{1}{2} \Gamma {p^{(1)}}^2,$$

with

$$p^{(1)} = \varrho^{(1)} = \eta, \quad p^{(2)} = \frac{\eta^2}{2} \quad \text{and} \quad p^{(3)} = \frac{\eta^3}{6} - \Gamma \frac{\eta^2}{2} + \eta.$$

Saturated Regime

$$\frac{\partial \theta^{(3)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}\theta^{(3)} + w^{(0)}\frac{\partial \theta^{(3)}}{\partial \eta} = \Gamma L(\Gamma A_{vs} - 1)w^{(0)}$$

$$\frac{\partial r_c^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}{r_c}^{(0)} + w^{(0)}\frac{\partial {r_c}^{(0)}}{\partial \eta} = -K_{vc}\Gamma L(\Gamma A_{vs} - 1)w^{(0)} - C_{cr,1}r_c^{(0)}r_r^{(0)},$$

$$\frac{\partial r_r^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}{r_r}^{(0)} + w^{(0)}\frac{\partial {r_r}^{(0)}}{\partial \eta} = V_r\frac{\partial r_r^{(0)}}{\partial \eta}.$$

Undersaturated Regime

$$\frac{\partial \theta^{(3)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}\theta^{(3)} + w^{(0)}\frac{\partial \theta^{(3)}}{\partial \eta} = -\Gamma L\frac{(r_{vs}^{(0)} - r_v^{(0)})(r_r^{(0)})^{1/2}[B_1 + B_2]}{(r_{vs}^{(0)}B_3 + B_4)},$$

$$\frac{\partial r_v^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}{r_v}^{(0)} + w^{(0)}\frac{\partial {r_v}^{(0)}}{\partial \eta} = 0,$$

$$\frac{\partial r_r^{(0)}}{\partial \tau} + \boldsymbol{v}_{\shortparallel}^{(0)}.\nabla_{\boldsymbol{\xi}}{r_r}^{(0)} + w^{(0)}\frac{\partial {r_r}^{(0)}}{\partial \eta} = V_r\frac{\partial r_r^{(0)}}{\partial \eta}.$$

A steady state solution of the shallow layer system can be used to provide the lower boundary conditions for the deep convective column system, derived in Section 3, using the ideas of matched asymptotics.

5 Conclusions

Asymptotic analyses of two convective systems on the horizontal bulk micro scale of 1 km for the moist atmosphere are presented: The first one is characterized by the deep convective vertical scale of 10 km and the corresponding convective time scale of 20 min, the second one consists of the shallow convective vertical scale of 1 km and the corresponding short time scale of 2 min. The shallow convective system may serve as physically consistent boundary condition for the deep convective system. The models derived in this report show some special features, namely:

Deep Convective Columns

- The continuity equation provides an anelastic divergence constraint for the horizontal flow. The perturbation pressure representing the corresponding Lagrange multiplicator obeys a two dimensional Poisson equation.
- The horizontal momentum balance contains the product of the vertical velocity in leading order and the horizontal Coriolis parameter which is usually neglected in meteorological applications.
- In the saturated column the vertical velocity is directly determined by the potential temperature deviation between inside the column and outside which is a conserved quantity. In the undersaturated regime the vertical velocity depends on the leading order evaporation rate which in turn is constituted by the saturation deficit and the amount of rain water present.

Shallow Convective Layer

- The flow is described by Boussinesq-type equations.
- At leading order, the important microphysical processes are condensation and evaporation, but there are no coagulation processes.
- The latent heat release due to condensation is proportional to the vertical velocity.
- Downdrafts that penetrate the surface do not occur unless precipitation is occurring.
- The temperature distribution depends mainly on the amount of rain water present in the understaurated regime.

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Bibliography

[1] N. Andrew Crook. Sensitivity of Moist Convection Forced by Boundary Layer Processes to Low-Level Thermodynamic Fields. *Monthly Weather Review*, 124(8):1767–1785, August 1996.

[2] Kerry A. Emanuel. *Atmospheric Convection*. Oxford University Press, New York, Oxford, 1994.

[3] Dieter Etling. *Theoretische Meteorologie: Eine Einführung*. Springer-Verlag, Berlin, Heidelberg, New York, Barcelona, Hongkong, London, Mailand, Paris, Tokio, second edition, 2002.

[4] Wojciech W. Grabowski and Piotr K. Smolarkiewicz. Two-Time-Level Semi-Lagrangian Modeling of Precipitating Clouds. *Monthly Weather Review*, 124(3):487–497, March 1996.

[5] Rupert Klein. Asymptotic Analyses for Atmospheric Flows and the Construction of Asymptotically Adaptive Numerical Methods. *Z. angew. Mathem. Mech.*, 80(11-12):765–785, 2000.

[6] Rupert Klein and Andrew J. Majda. Systematic Multiscale Models for Deep Convection on Mesoscales. *Theoretical and Computational Fluid Dynamics*, 20:525–551, 2006.

[7] Andrew J. Majda and Rupert Klein. Systematic Multiscale Models for the Tropics. *Journal of the Atmospheric Sciences*, 60:393–408, 2003.

[8] Richard Rotunno, Joseph B. Klemp, and Morris L. Weisman. A Theory for Strong, Long-lived Squall-lines. *Journal of the Atmospheric Sciences*, 45:463–485, 1988.

[9] Bjorn Stevens. Atmospheric Moist Convection. *Annual Review of Earth and Planetary Sciences*, 33:605–643, 2005.

A Parameters in the Microphysics Parameterizations

$$\mathcal{D}_1 = 10^{-3}\frac{t_{ref}}{t_{SI}} r_{c,ref}$$

$$\mathcal{D}_2 = 2.2\frac{t_{ref}}{t_{SI}} r_{c,ref} r_{r,ref}^{\mathcal{F}_1}$$

$$\mathcal{D}_3 = 4.26 \text{ x } 10^{-4} \left(\frac{\varrho_{ref}}{\varrho_{SI}}\right)^{(\mathcal{F}_2-1)} r_{r,ref}^{\mathcal{F}_2} \frac{t_{ref}}{t_{SI}}$$

$$\mathcal{D}_4 = 8.08 \text{ x } 10^{-3} \left(\frac{\varrho_{ref}}{\varrho_{SI}}\right)^{(\mathcal{F}_2+\mathcal{F}_3-1)} r_{r,ref}^{(\mathcal{F}_2+\mathcal{F}_3)} \frac{t_{ref}}{t_{SI}}$$

$$\mathcal{D}_5 = 5.4$$

$$\mathcal{D}_6 = 2.55 \text{ x } 10^{3} \frac{p_{SI}}{p_{ref} r_{vs,ref}}$$

$$\mathcal{D}_7 = 14.34\frac{t_{ref} l_{SI}}{t_{SI} l_{ref}} \left(\frac{\varrho_{ref}}{\varrho_{SI}}\right)^{(\mathcal{F}_4+0.5)} r_{r,ref}^{\mathcal{F}_5}$$

$$\mathcal{D}_{10} = \frac{e_\infty}{p_{ref}}$$

$$\mathcal{F}_1 = 0.875$$

$$\mathcal{F}_2 = 0.525$$

$$\mathcal{F}_3 = 0.2046$$

$$\mathcal{F}_4 = -0.3654$$

$$\mathcal{F}_5 = 0.1346$$

The symbols of the above equations have the following meaning:

- A variable with the index SI stands for the SI-unit of that quantity.
- R_d: specific gas constant of dry air
- R_v: specific gas constant of water vapour
- e_∞: triple-point vapour pressure
- L_{cond}: specific latent heat of condensation
- c_p: specific heat capacity at constant pressure for dry air

B Coefficients of the Analytical Solutions

$$
\begin{aligned}
a_0 &= \frac{1}{(1-A\Gamma)^3}\Big(- AC_1\Gamma LR - A\Gamma^2 L^2 R^2 + 3A^2 C_1 \Gamma^2 LR + 3A^2\Gamma^3 L^2 R^2 \\
&\qquad -3A^3 C_1 \Gamma^3 LR - 3A^3\Gamma^4 L^2 R^2 + A^4 C_1 \Gamma^4 LR + A^4 \Gamma^5 L^2 R^2 \\
&\qquad +\Gamma^2 LR - 2A\Gamma^3 LR + A^2\Gamma^4 LR\Big)
\end{aligned}
$$

$$
a_1 = \frac{1}{(1-A\Gamma)^3}\Big(A^3\Gamma^5 LR + 3A\Gamma^3 LR - 3A^2\Gamma^4 LR - \Gamma^2 LR\Big)
$$

$$
a_2 = \frac{1}{(1-A\Gamma)^3}\Big(- A^4\Gamma^6 LR + \frac{7}{2}A^3\Gamma^5 LR + \frac{5}{2}A\Gamma^3 LR - \frac{9}{2}A^2\Gamma^4 LR - \frac{1}{2}\Gamma^2 LR\Big)
$$

$$
b_0 = \frac{1}{(1-A\Gamma)}\Big(- A^2\Gamma^3 L^2 R^2 + A\Gamma^2 L^2 R^2\Big)
$$

$$
\begin{aligned}
c_0 &= \frac{1}{(1-A\Gamma)^3}\Big(AC_1\Gamma LR - 3A^2 C_1\Gamma^2 LR + 3A^3 C_1 \Gamma^3 LR - A^4 C_1 \Gamma^4 LR \\
&\qquad -\Gamma^2 LR + 2A\Gamma^3 LR - A^2\Gamma^4 LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
d_0 &= -C_2 \\
&\quad + \frac{1}{(1-A\Gamma)}\Big(- A^2\Gamma^3 L^2 R^2 + A\Gamma^2 L^2 R^2\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^2}\Big(\frac{1}{2}A^2\Gamma^3 L^2 R^2 - \frac{1}{2}A\Gamma^2 L^2 R^2\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^3}\Big(3A^2 C_1\Gamma^2 LR + 3A^2\Gamma^3 L^2 R^2 - AC_1\Gamma LR - A\Gamma^2 L^2 R^2 \\
&\qquad +A^4 C_1\Gamma^4 LR + A^4\Gamma^5 L^2 R^2 - 3A^3 C_1\Gamma^3 LR - 3A^3\Gamma^4 L^2 R^2 \\
&\qquad -2A^2\Gamma^4 LR + A\Gamma^3 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^4}\Big(- 2A^3\Gamma^5 LR - A\Gamma^3 LR + 3A^2\Gamma^4 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^6}\Big(AC_1\Gamma LR + A\Gamma^2 L^2 R^2 - 5A^2 C_1\Gamma^2 LR - 5A^2\Gamma^3 L^2 R^2 \\
&\qquad +10A^3 C_1\Gamma^3 LR + 10A^3\Gamma^4 L^2 R^2 - 10A^4 C_1\Gamma^4 LR \\
&\qquad -10A^4\Gamma^5 L^2 R^2 - A^6 C_1\Gamma^6 LR - A^6\Gamma^7 L^2 R^2 + 5A^5 C_1\Gamma^5 LR \\
&\qquad +5A^5\Gamma^6 L^2 R^2 - \Gamma^2 LR + A^3\Gamma^5 LR + 3A\Gamma^3 LR - 3A^2\Gamma^4 LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= C_2 - 2C_1\Gamma \\
&\quad + \frac{1}{(1-A\Gamma)}\Big(2A\Gamma^3LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^3}\Big(AC_1\Gamma LR - 3A^2C_1\Gamma^2LR - A^4C_1\Gamma^4LR + 3A^3C_1\Gamma^3LR \\
&\qquad\qquad + 2A\Gamma^3LR - A^2\Gamma^4LR - \Gamma^2LR\Big)
\end{aligned}
$$

$$
d_2 = \frac{5}{2}C_1\Gamma + 2\Gamma^2LR + \frac{1}{(1-A\Gamma)}\Big(-\frac{1}{2}A\Gamma^3LR\Big)
$$

$$
d_3 = -\frac{1}{2}C_1\Gamma - \frac{1}{2}\Gamma^2LR + \Gamma^3
$$

$$
d_4 = -\frac{13}{12}\Gamma^3
$$

$$
d_5 = \frac{7}{24}\Gamma^3
$$

$$
d_6 = -\frac{1}{48}\Gamma^3
$$

$$
\begin{aligned}
e_0 &= \frac{1}{(1-A\Gamma)^3}\Big(-3A^2C_1\Gamma^2LR - 3A^2\Gamma^3L^2R^2 + AC_1\Gamma LR + A\Gamma^2L^2R^2 \\
&\qquad - A^4C_1\Gamma^4LR - A^4\Gamma^5L^2R^2 + 3A^3C_1\Gamma^3LR + 3A^3\Gamma^4L^2R^2 \\
&\qquad + 2A^2\Gamma^4LR - A\Gamma^3LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^4}\Big(2A^3\Gamma^5LR + A\Gamma^3LR - 3A^2\Gamma^4LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^6}\Big(-AC_1\Gamma LR - A\Gamma^2L^2R^2 + 5A^2C_1\Gamma^2LR + 5A^2\Gamma^3L^2R^2 \\
&\qquad - 10A^3C_1\Gamma^3LR - 10A^3\Gamma^4L^2R^2 + 10A^4C_1\Gamma^4LR \\
&\qquad + 10A^4\Gamma^5L^2R^2 + A^6C_1\Gamma^6LR + A^6\Gamma^7L^2R^2 - 5A^5C_1\Gamma^5LR \\
&\qquad - 5A^5\Gamma^6L^2R^2 + \Gamma^2LR - A^3\Gamma^5LR - 3A\Gamma^3LR + 3A^2\Gamma^4LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
e_1 &= \frac{1}{(1-A\Gamma)^3}\Big(-3A^2\Gamma^4LR+2A^3\Gamma^5LR+A\Gamma^3LR\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^4}\Big(-3A\Gamma^3LR+10A^2\Gamma^4LR-11A^3\Gamma^5LR+4A^4\Gamma^6LR\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^6}\Big(-A\Gamma^3LR+4A^2\Gamma^4LR-A^5\Gamma^7LR-6A^3\Gamma^5LR\\
&\qquad+4A^4\Gamma^6LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
e_2 &= \frac{1}{(1-A\Gamma)^3}\Big(\frac{1}{2}\Gamma^2LR-2A\Gamma^3LR+A^4\Gamma^6LR+\frac{7}{2}A^2\Gamma^4LR-3A^3\Gamma^5LR\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^6}\Big(-\frac{1}{2}\Gamma^2LR+\frac{7}{2}A\Gamma^3LR-10A^2\Gamma^4LR+15A^3\Gamma^5LR\\
&\qquad-A^6\Gamma^8LR-\frac{25}{2}A^4\Gamma^6LR+\frac{11}{2}A^5\Gamma^7LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
f_0 &= \frac{1}{(1-A\Gamma)}\Big(A^2\Gamma^3L^2R^2-A\Gamma^2L^2R^2\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^2}\Big(-\frac{1}{2}A^2\Gamma^3L^2R^2+\frac{1}{2}A\Gamma^2L^2R^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
g_0 &= \Gamma^2LR\\
&\quad+\frac{1}{(1-A\Gamma)}\Big(A\Gamma^3LR-A^2\Gamma^3L^2R^2+A\Gamma^2L^2R^2\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^2}\Big(\frac{1}{2}A^2\Gamma^3L^2R^2-\frac{1}{2}A\Gamma^2L^2R^2\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^3}\Big(3A^2\Gamma^3L^2R^2-A\Gamma^2L^2R^2+A^4\Gamma^5L^2R^2-3A^3\Gamma^4L^2R^2\\
&\qquad-3A^2\Gamma^4LR+3A\Gamma^3LR-\Gamma^2LR\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^4}\Big(-2A^3\Gamma^5LR-A\Gamma^3LR+3A^2\Gamma^4LR\Big)\\
&\quad+\frac{1}{(1-A\Gamma)^6}\Big(AC_1\Gamma LR+A\Gamma^2L^2R^2-5A^2C_1\Gamma^2LR-5A^2\Gamma^3L^2R^2\\
&\qquad+10A^3C_1\Gamma^3LR+10A^3\Gamma^4L^2R^2-10A^4C_1\Gamma^4LR\\
&\qquad-10A^4\Gamma^5L^2R^2-A^6C_1\Gamma^6LR-A^6\Gamma^7L^2R^2+5A^5C_1\Gamma^5LR\\
&\qquad+5A^5\Gamma^6L^2R^2-\Gamma^2LR+A^3\Gamma^5LR+3A\Gamma^3LR-3A^2\Gamma^4LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
g_1 &= C_2 + \Gamma^2 LR \\
&\quad + \frac{1}{(1-A\Gamma)}\Big(A\Gamma^3 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^3}\Big(AC_1\Gamma LR - 3A^2C_1\Gamma^2 LR - A^4C_1\Gamma^4 LR + 3A^3C_1\Gamma^3 LR \\
&\qquad + 2A\Gamma^3 LR - A^2\Gamma^4 LR - \Gamma^2 LR\Big)
\end{aligned}
$$

$$
g_2 = C_1\Gamma + \frac{1}{2}\Gamma^2 LR + \frac{1}{(1-A\Gamma)}\Big(-\frac{1}{2}A\Gamma^3 LR\Big)
$$

$$
g_3 = -\frac{1}{2}C_1\Gamma - \frac{1}{2}\Gamma^2 LR
$$

$$
g_4 = -\frac{1}{4}\Gamma^3
$$

$$
g_5 = \frac{1}{6}\Gamma^3
$$

$$
g_6 = -\frac{1}{48}\Gamma^3
$$

$$
\begin{aligned}
h_0 &= -\Gamma^2 LR \\
&\quad + \frac{1}{(1-A\Gamma)}\Big(-A\Gamma^3 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^3}\Big(3A^2\Gamma^4 LR - 3A\Gamma^3 LR + \Gamma^2 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^4}\Big(2A^3\Gamma^5 LR + A\Gamma^3 LR - 3A^2\Gamma^4 LR\Big) \\
&\quad + \frac{1}{(1-A\Gamma)^6}\Big(-AC_1\Gamma LR - A\Gamma^2 L^2R^2 + 5A^2C_1\Gamma^2 LR + 5A^2\Gamma^3 L^2R^2 \\
&\qquad - 10A^3C_1\Gamma^3 LR - 10A^3\Gamma^4 L^2R^2 + 10A^4C_1\Gamma^4 LR \\
&\qquad + 10A^4\Gamma^5 L^2R^2 + A^6C_1\Gamma^6 LR + A^6\Gamma^7 L^2R^2 - 5A^5C_1\Gamma^5 LR \\
&\qquad - 5A^5\Gamma^6 L^2R^2 + \Gamma^2 LR - A^3\Gamma^5 LR - 3A\Gamma^3 LR + 3A^2\Gamma^4 LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
h_1 &= \frac{1}{(1-A\Gamma)}\Big(A\Gamma^3LR\Big) \\
&+\frac{1}{(1-A\Gamma)^3}\Big(-6A^2\Gamma^4LR+3A^3\Gamma^5LR+4A\Gamma^3LR-\Gamma^2LR\Big) \\
&+\frac{1}{(1-A\Gamma)^4}\Big(-11A^3\Gamma^5LR-3A\Gamma^3LR+10A^2\Gamma^4LR+4A^4\Gamma^6LR\Big) \\
&+\frac{1}{(1-A\Gamma)^6}\Big(-6A^3\Gamma^5LR-A\Gamma^3LR+4A^2\Gamma^4LR-A^5\Gamma^7LR \\
&\qquad +4A^4\Gamma^6LR\Big)
\end{aligned}
$$

$$
\begin{aligned}
h_2 &= \frac{1}{2}\Gamma^2LR \\
&+\frac{1}{(1-A\Gamma)^3}\Big(-A^2\Gamma^4LR+\frac{1}{2}A^3\Gamma^5LR+\frac{1}{2}A\Gamma^3LR\Big) \\
&+\frac{1}{(1-A\Gamma)^6}\Big(-\frac{1}{2}\Gamma^2LR+15A^3\Gamma^5LR+\frac{7}{2}A\Gamma^3LR-10A^2\Gamma^4LR \\
&\qquad +\frac{11}{2}A^5\Gamma^7LR-A^6\Gamma^8LR-\frac{25}{2}A^4\Gamma^6LR\Big)
\end{aligned}
$$

$$
i_0 = \frac{1}{(1-A\Gamma)^2}\Big(-\frac{1}{2}A^2\Gamma^3L^2R^2+\frac{1}{2}A\Gamma^2L^2R^2\Big)
$$