



Critical Edges in Perfect Graphs

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Introduction

The present work is devoted to the investigation of extremal graphs with respect to a rich and well-studied graph property: perfectness. We focus our study on perfect graphs that lose their perfection by simply deleting one edge, a so-called critical edge. We first briefly survey why so much attention has been paid to perfect graphs. Then we explain why studying critical edges is of interest, and outline some questions that arise in this context.

Various challenging problems in algorithmic graph theory involve at least one of the following four graph parameters:

$\alpha(G)$: *stability number*, the maximum size of a stable set of G ,
 $\theta(G)$: *clique covering number*, the least number of cliques that cover all nodes of G ,
 $\omega(G)$: *clique number*, the maximum size of a clique of G , and
 $\chi(G)$: *chromatic number*, the least number of stable sets needed to cover all nodes of G .

Since the intersection of a stable set and a clique can be at most one node, the four invariants are obviously related by

$$\alpha(G) \leq \theta(G) \tag{1}$$

$$\omega(G) \leq \chi(G) \tag{2}$$

for an arbitrary graph G . But requiring equality for these two min-max relations (1) and (2) does not give much information about the structure of the graph: every graph G augmented with a sufficiently large stable set and clique satisfies (1) and (2), respectively, with equality. However, requiring equality in (1) and (2) not only for G but

$$\alpha(G') = \theta(G') \tag{3}$$

$$\omega(G') = \chi(G') \tag{4}$$

for all of its induced subgraphs G' yields a class of graphs with very interesting structural properties. By a suggestion of Berge [3], graphs satisfying (3) and (4) have been termed *perfect* and all other graphs *imperfect*. Since complementation transforms stable sets into cliques and colorings into clique coverings, the inequalities (1), (2) and equalities (3), (4) are dual to one another by $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \theta(\overline{G})$, where \overline{G} denotes the complementary graph of G . It is clear by this duality that a graph fulfills (3) if and only if its complement satisfies (4), and vice versa. A much stronger relation was conjectured by Berge [3] and proven by Lovász [42], namely that (3) and (4) are in fact equivalent (for finite graphs). This fundamental result is nowadays known as *Perfect Graph Theorem*.

Moreover, from an algorithmic point of view perfect graphs behave nicely with respect to the graph parameters α , θ , ω , and χ . While these four invariants are difficult to evaluate for arbitrary graphs, polynomial time algorithms solving the corresponding optimization problems even in their weighted versions have been designed by Grötschel, Lovász, and Schrijver [21] for perfect graphs. In particular, determining $\alpha(G)$ corresponds to maximizing the linear function $\mathbb{1}^T x$ over the *stable set polytope* $\text{STAB}(G)$, a 0/1-polytope defined as the convex hull of the incidence vectors of all stable sets in G . In order to calculate $\mathbb{1}^T x$ over $\text{STAB}(G)$, an explicit description of $\text{STAB}(G)$ as the solution set of a system of linear inequalities is required but, unfortunately, unknown for most graphs (see [22] for a detailed account). Since a stable set and a clique have at most one node in common, for every clique Q of G , the so-called clique constraint $x(Q) \leq 1$ is satisfied by every incidence vector of a stable set in G . Hence, all clique constraints are valid for $\text{STAB}(G)$ and define, together with the nonnegativity constraints $x(v) \geq 0 \ \forall v \in V(G)$, a polytope $\text{QSTAB}(G)$ which obviously contains the stable set polytope of G , i.e.,

$$\text{STAB}(G) \subseteq \text{QSTAB}(G) \tag{5}$$

holds. The nonnegativity and maximal clique constraints define facets of $\text{STAB}(G)$ for every graph G and, thus, they are necessarily contained in any description of $\text{STAB}(G)$ by means of linear inequalities. Perfect graphs are polyhedrally characterized that these two classes are even sufficient (see [9]): a graph G is perfect if and only if $\text{STAB}(G)$ and $\text{QSTAB}(G)$ coincide, i.e., if and only if (5) is satisfied with equality.

For all those important properties and polynomial time algorithms for otherwise hard combinatorial optimization problems, the recognition problem for perfect graphs is still open. In particular, the characterization of perfect graphs via the famous *Strong Perfect Graph Conjecture* of Berge [3] has not been verified so far. This conjecture states that a graph is perfect if and only if it does not contain any subgraph isomorphic to a chordless odd cycle of length at least five, termed *odd hole*, or to the complement of such a cycle, an *odd antihole*. It is just a simple observation that odd holes and odd antiholes are imperfect, but they are even more, namely *minimally imperfect graphs*: imperfect graphs with the property that removing any of its nodes yields a perfect graph. Using this term introduced by Padberg [50], the Strong Perfect Graph Conjecture reads that odd holes and odd antiholes are the only minimally imperfect graphs. In order to give a characterization of minimally imperfect graphs (and so to verify or falsify the Strong Perfect Graph Conjecture), many fascinating structural properties of such graphs have been discovered. So, e.g., every minimally imperfect graph $G = (V, E)$ admits an extraordinary symmetry with respect to its maximum cliques and stable sets:

$$G \text{ has precisely } |V| \text{ maximum cliques (stable sets),} \tag{6}$$

$$\text{every node is contained in exactly } \omega(G) \text{ such cliques } (\alpha(G) \text{ such stable sets).} \tag{7}$$

Furthermore, node pairs with certain neighborhood structures must not occur in minimally imperfect graphs, e.g.,

comparable pairs: two nodes $x, y \in V$ with $N(x) - y \subseteq N(y)$,
twins: two nodes $x, y \in V$ with $N(x) - y = N(y) - x$, and
antitwins: two nodes $x, y \in V$ with $N(x) - y = V - N(y) - \{x, y\}$,

to mention just a few of them. (Chapter 1 provides all graph theoretical notions and presents further results on perfect and minimally imperfect graphs.)

The starting point of this thesis was the following consideration, my supervisor Professor Grötschel focused my attention on in spring 1996. Imagine you are creating a graph on a certain node set. You start with a complete graph and consecutively delete one edge until you get a stable set, or you take a stable set and consecutively add one edge until the complete graph is reached. So you create a sequence of graphs starting and ending up with a perfect graph. But, if you choose the edges to be deleted or added randomly, most graphs of your sequence will be imperfect. In order to avoid imperfect graphs in our sequence, we face the following problems in this context.

Problem 0.1 *If you have obtained a perfect graph by deleting or adding a certain edge set, is there a rule to choose the next edge in order to keep perfectness?*

Problem 0.2 *For a perfect graph, is there an order of the edges to be deleted (added) so that we get a sequence of perfect graphs ending up with a stable set (a complete graph)?*

We have called an edge e of a perfect graph G *critical* if $G - e$ is imperfect. The most simple example of a critical edge is a single chord of a cycle of odd length ≥ 5 which forms a triangle with two edges of the cycle, called a *short chord*. So one rule in the sense of Problem 0.1 is “never omit a single short chord of an odd cycle”. But could we obtain, by removing a certain edge set from a complete graph, a perfect graph every of its edges is such a chord? This graph would be a “dead end graph” for our procedure of consecutively removing edges, so we ask:

Problem 0.3 *Are there perfect graphs such that you cannot delete or add any edge without losing perfectness?*

Problem 0.4 *If such “dead end graphs” exist, do they admit a structure similarly interesting as that of minimally imperfect graphs?*

Problem 0.5 *In which classes of perfect graphs do such graphs appear?*

We have termed perfect graphs that admit only critical edges *critically perfect*. The attempt to construct a first example, e.g., a perfect graph all of its edges are single short chords of an odd cycle, has yielded the graph depicted in Figure 3.17 on page 63. Further examples followed soon (Figure 3.1 on page 30 shows the three smallest critically perfect graphs, Figure 3.2 on page 31 further examples). Hougardy has discovered: all the first examples are line graphs of bipartite graphs. It has turned out in the sequel that this was not a pure chance: line graphs of bipartite graphs and their complements play a central role in the context of critically perfect graphs.

Problem 0.1 and Problem 0.2 are investigated in Chapter 2. As already mentioned, we say that an edge e of a perfect graph G is critical if $G - e$ is imperfect. Furthermore, we call e *anticritical* with respect to the complementary graph \overline{G} : every critical edge e of G is a non-edge of \overline{G} the addition of which yields the imperfect graph $\overline{G} + e$ due to the Perfect Graph Theorem. In particular, whenever we delete and add a critical and anticritical edge e of a perfect graph G , we create in $G - e$ and $G + e$, respectively, *minimally* imperfect subgraphs. In order to attack Problem 0.1 the Sections 2.1 and 2.2 study those subgraphs $G' \subseteq G$ which yield the minimally imperfect subgraphs $G' - e$ and $G' + e$ of $G - e$ and $G + e$, respectively. Many properties of those subgraphs rely on properties of minimally imperfect graphs. So we obtain, e.g., that the endnodes of a critical or anticritical edge must neither form a comparable pair, nor twins, nor antitwins. Moreover, the endnodes of every critical (anticritical) edge are contained in a clique (stable set) of size three and in an even hole (antihole).

Section 2.3 is dedicated to Problem 0.1 with respect to graphs that belong to a certain subclass of perfect graphs. (The large abundance of classes of perfect graphs has led us to confine our studies, besides some "classical" ones, only to classes which seem to be quite large.) In other words, we investigate in which classes perfect graphs may occur that admit critical or anticritical edges at all. One answer gives a new characterization of *Meyniel graphs*. They have been introduced in [45] as graphs all odd cycles of length ≥ 5 of which admit at least two chords. So critical edges in form of single short chords of odd cycles do obviously not occur in Meyniel graphs. Actually, removing any edge from a Meyniel graph keeps perfectness by [29] and even more holds: *a graph is Meyniel if and only if it does not admit any critical edge*. Two further main results of this section establish characterizations of critical and anticritical edges in line graphs. An edge of a line graph is critical iff it is a single short chord of an odd cycle, non-edges between the endnodes of even paths of length ≥ 4 are precisely the anticritical edges.

Section 2.4 deals with Problem 0.2. We present several classes of perfect graphs for which the answer is in the affirmative and we give specific rules how to delete or add the edges. It turns out that it does not suffice to identify non-critical or non-anticritical edges: we can, e.g., certainly remove an arbitrary edge from a Meyniel graph keeping perfectness but, at present, we do not know anything about critical edges of the resulting graph. So what we have to do is to look for edges the deletion or addition of which preserves membership within the corresponding *subclass* of perfect graphs. By this way, we provide the studied ordering of edges to be deleted or added for, e.g., every bipartite, triangulated, and weakly triangulated graph.

The subject of Chapter 3 is the investigation of "dead end graphs" for the procedure of consecutively deleting or adding edges, i.e., we deal with Problem 0.3, Problem 0.4, and Problem 0.5. We have already mentioned some examples of critically perfect graphs, hence Problem 0.3 is solved. All non-edges of the complement of a critically perfect graph are anticritical, we call the complements of critically perfect graphs *anticritically perfect*. Thus the complements of all graphs shown in Figure 3.1, Figure 3.2, and Figure 3.17 are anticritically perfect. Moreover, the first graph depicted in Figure 3.1 and all graphs from

Figure 3.2 and Figure 3.17 (i.e., our first example!) are even both, critically *and* anticritically perfect, but not the second and the third graph in Figure 3.1. Hence the classes of critically and anticritically perfect graphs intersect but do not coincide. Note further that every graph which is critically *and* anticritically perfect is a singleton in the graph of all perfect graphs. I.e., you cannot reach another perfect graph from such a graph by the deletion or the addition of one edge.

Section 3.1 is devoted to properties of critically and anticritically perfect graphs in order to tackle Problem 0.4. But looking for structures similarly interesting as that of minimally imperfect graphs has, unfortunately, not revealed such strong structural results. Critically and anticritically perfect graphs do not admit any symmetry with respect to their maximum cliques and stable sets (in contrast to the properties (6) and (7) of minimally imperfect graphs). The third graph depicted in Figure 3.1, e.g., has three nodes which do not belong to any of its maximum stable sets. However, the results obtained in Section 2.1 and Section 2.2 reveal that critically and anticritically perfect graphs share some forbidden substructures with minimally imperfect graphs, e.g., they do not contain comparable pairs, twins, or antitwins.

The next two sections provide the two known key tools to construct critically and anticritically perfect graphs. Section 3.2 studies several graph operations known to preserve perfectness and the “behavior” of critical and anticritical edges under applying those operations. Clearly, a graph operation preserving critical or anticritical perfectness enables us to generate, from two critically or anticritically perfect graphs, new graphs belonging to these two classes. E.g., if we construct a graph G by identifying two perfect graphs G_1 and G_2 in a clique $Q \subseteq G_1, G_2$ then G is a perfect graph that “inherits” all critical edges from G_1 and G_2 . Thus, the class of critically perfect graphs is closed under applying clique identification. But we have established an even stronger result: an edge e in Q is critical in G even if it is neither critical in G_1 nor in G_2 , but belongs to a triangle in G_1 and an even hole in G_2 (then e is a single short chord of an odd cycle in G). This observation provides the opportunity to *create a critically perfect graph from two perfect but non-critical graphs*. The graph shown in Figure 3.9 on page 50 admits precisely one non-critical edge. Taking two copies of this graph and identifying them in the endnodes of their only non-critical edge yields a critically perfect graph which does not contain any critically perfect proper subgraph.

Section 3.3 gives characterizations of critically and anticritically perfect line graphs, based on the two main results of Section 2.3. Here we prove that every critically and anticritically perfect line graph is the line graph of a bipartite graph with a certain structure, called *H-graph* and *A-graph*, respectively. (In the line graph of a bipartite H-graph, every edge is a single short chord of an odd cycle, where every two non-adjacent nodes are the endnodes of a chordless even path of length ≥ 4 in the line graph of a bipartite A-graph.) Moreover, the line graph of every 3-connected bipartite graph is both, critically and anticritically perfect. The importance of the results obtained in these two sections is that *the only known critically or anticritically perfect graphs are line graphs or graphs created by using line graphs and the operations considered in Section 3.2*.

Section 3.4 deals with Problem 0.5. First we study whether all critically perfect graphs may belong to a certain subclass of perfect graphs. For that, we first observe that *every perfect graph F may occur as subgraph of a critically perfect graph*: cover all non-critical edges of F by cliques Q_1, \dots, Q_k , choose critically perfect graphs G_1, \dots, G_k with $Q_i \subseteq G_i$ for $1 \leq i \leq k$, and identify F with all graphs G_i in Q_i each. The arising graph contains F as subgraph and is critically perfect. Consequently, *critically and anticritically perfect graphs cannot be characterized by forbidden subgraphs* (the only forbidden subgraphs are minimally imperfect). Therefore, the classes of critically or anticritically perfect graphs are not contained in any F -free class of perfect graphs for some perfect graph F (the uniquely colorable perfect graphs build the only subclass of perfect graphs known to the author which is not characterized by forbidden subgraphs). Conversely, every proper subgraph $F' \subset F$ is F -free and perfect but, if it has less than nine nodes, not critically perfect (cf. the smallest critically perfect graphs in Figure 3.1). Hence, *no inclusion relations exist between the classes of critically or anticritically perfect graphs and the most other classes of perfect graphs*. So we study the intersections of the corresponding classes, using results from Section 2.3 and Section 2.4. Clearly, the classes of Meyniel, bipartite, triangulated, and weakly triangulated graphs do not contain any critically perfect graph.

Chapter 4 is dedicated to the investigation of a further aspect: how imperfect is a graph that arises by deleting a critical edge e of a perfect graph G . In order to decide whether $G - e$ is still “almost perfect” or already “very imperfect”, we have to introduce a measure for the imperfectness of a graph. For that, we make use of the polyhedral characterization of perfect graphs [9]: a graph is perfect iff nonnegativity and clique constraints suffice to describe its stable set polytope. An imperfect graph obtained from a perfect graph G by deleting a critical edge e does not possess this property anymore. The stable set polytope associated with $G - e$ admits facets of other types in addition.

In Section 4.1, we consider a hierarchy of inequalities valid for $\text{STAB}(G)$, starting with the clique constraints

$$x(Q, \mathbb{1}) \leq 1$$

where $\mathbb{1}$ stands for a node-weighting of the clique $Q \subseteq G$. By $\alpha(Q) = 1$ for every clique Q , all clique constraints are special *rank constraints*

$$x(G', \mathbb{1}) \leq \alpha(G')$$

associated with subgraphs $G' \subseteq G$ which are obviously valid for $\text{STAB}(G)$. Whenever G' has $|G'|$ stable sets of size $\alpha(G')$ so that the incidence vectors are affinely independent, the rank constraint $x(G', \mathbb{1}) \leq \alpha(G')$ is facet-defining for $\text{STAB}(G')$ (this is true for, e.g., every minimally imperfect graph G' by Padberg [50]). In the case $G' \subset G$, the rank constraint associated with G' does not induce a facet of $\text{STAB}(G)$ in general. But we may strengthen $x(G', \mathbb{1}) \leq \alpha(G')$ to a facet

$$x(G, a) \leq \alpha(G')$$

of $\text{STAB}(G)$, called *weak rank constraint*, by determining appropriate integer coefficients a_i for every node $v_i \in G - G'$ via sequential lifting [49]. (Note $a_i = 1$ holds for $v_i \in G'$

and, in particular, every rank constraint $x(G', \mathbb{1}) \leq \alpha(G')$ is a weak rank constraint $x(G, a) \leq \alpha(G')$ with $a_i = 1 \ \forall v_i \in G'$ and $a_i = 0 \ \forall v_i \in G - G'$.)

Following a suggestion in [22], we consider two polytopes associated with the two inequality classes above: we define $\text{RSTAB}(G)$ and $\text{WSTAB}(G)$ as the polytope described by rank constraints and weak rank constraints, respectively, together with the nonnegativity constraints. The two polytopes are obviously relaxations of $\text{STAB}(G)$ refining (5) as follows:

$$\text{STAB}(G) \subseteq \text{WSTAB}(G) \subseteq \text{RSTAB}(G) \subseteq \text{QSTAB}(G).$$

We call a graph G *rank-perfect* if $\text{STAB}(G)$ and $\text{RSTAB}(G)$ coincide and *weakly rank-perfect* if $\text{STAB}(G) = \text{WSTAB}(G)$. Clearly, every perfect graph is rank-perfect since, in this case, even $\text{STAB}(G)$ and $\text{QSTAB}(G)$ coincide by [9], and every rank-perfect graph is weakly rank-perfect in particular. So the classes of all rank-perfect and all weakly rank-perfect graphs constitute super-classes of perfect graphs and yield the studied *measure for the imperfectness of a graph*.

I.e., in order to decide how imperfect a graph is that we obtain by deleting a critical edge e from a perfect graph G means to *decide whether $G - e$ is rank-perfect or weakly rank-perfect*. For that, we have to study the facets of $\text{STAB}(G - e)$.

Problem 0.6 *Describe certain types of facet-defining inequalities of $\text{STAB}(G - e)$ or $\text{STAB}(G + e)$!*

Problem 0.7 *Which types of facets provide, together with the nonnegativity and clique constraints, a complete description of $\text{STAB}(G - e)$ or $\text{STAB}(G + e)$?*

In Section 4.2 and Section 4.3, we deal with Problem 0.6 and establish weak rank constraints of $\text{STAB}(G - e)$ where G is a perfect line graph and the complement of such a graph, respectively. The characterizations of critical and anticritical edges in line graphs from Section 2.3 provide the knowledge of *all* minimally imperfect graphs which occur in $G - e$. We give the weak rank constraints obtained by lifting rank constraints associated with one or the union of some minimally imperfect subgraphs of $G - e$. If G is a perfect line graph and e a critical edge, then odd holes are the only minimally imperfect subgraphs in $G - e$. One odd hole or the subgraph induced by some odd holes of $G - e$ yield the line graph of a hypomatchable graph which induces a rank constraint of $\text{STAB}(G - e)$ by Edmonds and Pulleyblank [17]. Each rank constraint associated with the line graph of a hypomatchable graph distinct from an odd hole of length five is lifted to a weak rank facet of $\text{STAB}(G - e)$, where lifting coefficients equal to one or zero suffice. An odd hole of length five in $G - e$ may induce two different weak rank facets of $\text{STAB}(G - e)$, where a lifting coefficient equal to two may be needed additionally. In the special case that G is the line graph of a bipartite graph we don't have to distinguish between these two cases, the corresponding facets of $\text{STAB}(G - e)$ are particular rank constraints. If G is the complement of a perfect line graph and e a critical edge, odd antiholes are the only minimally imperfect subgraphs in $G - e$ (due to the characterization of anticritical

edges in line graphs). Here we lift rank constraints associated with one odd antihole or the subgraph induced by two odd antiholes to weak rank facets of $\text{STAB}(G - e)$. The description of certain types of facet-defining inequalities for $\text{STAB}(G - e)$ partially solves Problem 0.6. The polyhedral consequences of the addition of anticritical edges have not been studied so far.

Throughout Sections 4.2 and 4.3, we have studied weak rank facets in stable set polytopes associated with certain perfect graphs G that become imperfect by removing a critical edge e . All examples of such imperfect graphs $G - e$ found by the author in this context have the property that their stable set polytopes only admit weak rank facets besides nonnegativity facets. This gives reason to conjecture that *$G - e$ is weakly rank-perfect if G is a perfect line graph or the complement of such a graph.* In particular, since the corresponding facets of $\text{STAB}(G - e)$ are rank constraints if G is the line graph of a *bipartite* graph, $G - e$ would be even rank-perfect in that case. In Section 4.4 we verify the above conjecture for graphs $G - e$ in the special case that G is the complement of the line graph of a bipartite graph: we ensure that nonnegativity, maximal clique, and the weak rank constraints studied in Section 4.3 suffice to describe $\text{STAB}(G - e)$. This gives at least a partial answer to Problem 0.7.

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Chapter 1

Basics

This chapter provides the reader with all elementary graph theoretical notions and notations used throughout this thesis. Moreover, we summarize some known results on perfect and minimally imperfect graphs which we will frequently make use of in later chapters.

1.1 Graph Theoretical Notions

A **graph** G consists of a pair of finite sets, the **node set** $V(G)$ and the **edge set** $E(G)$. The elements of $V(G)$ and $E(G)$ are the **nodes** and **edges** of G , respectively. $V(G)$ is non-empty and $|G| = |V(G)|$ is the **size** of G , often referred by n . $E(G)$ is a subset of the set of all unordered pairs $e = \{x, y\}$ of nodes $x, y \in V(G)$. The two nodes x and y are then called the **endnodes** of e and we briefly write $e = xy$. If the endnodes of an edge coincide, it is called a **loop**, two distinct edges with the same endnodes are **parallel**. Unless specified otherwise, we only deal with **simple** graphs that neither possess loops nor parallel edges. If there is no danger of confusion, we define a graph by $G = (V, E)$ and use V and E instead of $V(G)$ and $E(G)$, respectively.

Every edge is **incident** to its endnodes. Two edges e, e' are defined to be **incident** if they share an endnode, otherwise e and e' are called **independent**. A set of independent edges is said to be a **matching**. The endnodes of an edge are **adjacent** or **neighbors**. For a node set $V' \subseteq V(G)$, the **neighborhood** of a node $x \in V(G)$ with respect to V' is $N_{V'}(x) = \{y \in V' : xy \in E(G)\}$ and the **degree** of x with respect to V' is $d_{V'}(x) = |N_{V'}(x)|$. If $V' = V(G)$ holds, we write $N(x)$ and $d(x)$ instead of $N_{V(G)}(x)$ and $d_{V(G)}(x)$, respectively. Nodes x with $d(x) = 0$ are called **isolated** and **universal** if $d(x) = |V(G)|$ holds. Furthermore, $\delta(G) = \min\{d(x) : x \in V(G)\}$ denotes the **minimum degree** of G and $\Delta(G) = \max\{d(x) : x \in V(G)\}$ its **maximum degree**. Every graph G with $\delta(G) = k = \Delta(G)$ is **k -regular**.

Let $G = (V, E)$ be a graph, $V' \subseteq V$ and $E' \subseteq E$, then we call $G' = (V', E')$ a **partial subgraph** of G and write $G' \subseteq G$. $G - E'$ means the partial subgraph of G with node set V and edge set $E - E'$; we briefly write $G - e$ for $G - \{e\}$. $G + e$ stands for the graph $(V, E \cup \{e\})$. If $G' \subseteq G$ where $E(G')$ consists of all edges in E with both endnodes in $V' = V(G')$, then $G' = G[V']$ is the **subgraph of G induced by V'** . *When we are just*

talking of subgraphs or using $G' \subseteq G$, we will always mean *induced subgraphs* and refer to G' only using its node set V' and, conversely, identify V' with $G[V']$. The notation $G - V'$ means the subgraph of G induced by $V - V'$ and we write $G - v$ instead of $G - \{v\}$. For $V' \subset V$, we call $G' = G[V']$ a **proper** subgraph of G , denoted by $G' \subset G$.

A sequence of nodes v_0, v_1, \dots, v_k is a **path (cycle)** in G , if $v_i \neq v_j$ holds for $0 \leq i, j \leq k$ ($1 \leq i, j \leq k$ and $v_0 = v_k$) and $v_{i-1}v_i$ is an edge of G for $1 \leq i \leq k$. The number k is the **length** of this path (cycle) which is called **even** and **odd** if k is even and odd, respectively. Every edge v_iv_j in the subgraph of G induced by the nodes v_0, \dots, v_k with $j \neq i+1$ is a **chord** of the path (cycle), a chord v_iv_j with $j = i+2$ is called **short**. Chordless paths (cycles) are also called **induced**. Every path $P = v_0, v_1, \dots, v_k$ connecting the nodes v_0 and v_k is said to be a (v_0, v_k) -**path**. The nodes v_0, v_k are called the **endnodes** of P , v_1, \dots, v_{k-1} its **internal nodes**, and $v_0v_1, v_{k-1}v_k$ its **endedges**. Referring to a subpath of P , we use its endnodes and write $P[v_i, v_j] = v_i, \dots, v_j$, $P(v_i, v_j) = v_i, \dots, v_{j-1}$, $P(v_i, v_j] = v_{i+1}, \dots, v_j$, and $P(v_i, v_j) = v_{i+1}, \dots, v_{j-1}$. Two paths P and P' are **open-disjoint** if they admit common endnodes but do not share any internal node. A graph G is called **k -connected** if there are k open-disjoint (x, y) -paths in G for every pair x, y of different nodes in G . For $k = 1$ we just say that G is **connected**. If G is not connected we call it **disconnected**. Every non-empty, maximal connected subgraph of G is called a **component**. (Note that *minimal* and *maximal* always refer to set inclusion, but *minimum* and *maximum* always refer to size.) A subset $C \subset V(G)$ is called a **cutset** of G if the number of components in $G - C$ is strictly greater than in G , a cutset C with $|C| = 1$ is just a **cutnode**. A **block** is a subgraph of G which is either a single edge or a maximal 2-connected subgraph. Every block admitting only one cutnode is called an **endblock**.

A **stable set (clique)** in a graph $G = (V, E)$ is a set of nodes any two of which are non-adjacent (adjacent). If a stable set (clique) has size k it is called a **k -stable set (k -clique)**. The size of a maximum stable set (maximum clique) of G is its **stability number $\alpha(G)$ (clique number $\omega(G)$)**. A **coloring** and a **clique covering** of G is a partition of V into disjoint stable sets, called **color classes**, and cliques, respectively. We call a coloring (clique covering) using k stable sets (cliques) a **k -coloring (k -clique covering)**. The minimum number of stable sets (cliques) needed for such a partition of V is the **chromatic number $\chi(G)$ (clique covering number $\theta(G)$)**.

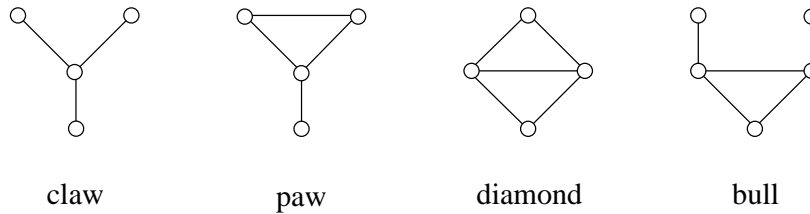


Figure 1.1

Graphs G of size n with $\omega(G) = n$ are called **complete** and usually denoted by K_n . P_k and C_k denotes a chordless path and cycle, respectively, on k nodes, C_3 is also called a

triangle and C_k with $k \geq 4$ a **hole**. Every graph G admitting a k -coloring is **k -partite**, we also say that G is **bipartite** for $k = 2$. If a graph with color classes V_1, \dots, V_k has the property that every node in V_i is adjacent to every node in V_j whenever $i \neq j$ holds, it is **complete k -partite**. $K_{a,b}$ stands for a complete bipartite graph with color classes A, B and $a = |A|, b = |B|$. $K_{1,b}$ is also called a **star** and the star $K_{1,3}$ a **claw**. The only node of degree b in a star $K_{1,b}$ is sometimes called its **center**. Adding an arbitrary edge to a claw yields a **paw**, deleting an arbitrary edge in a K_4 a $K_4 - e$ or **diamond** (see Figure 1.1). The graph on 5 nodes, called **bull**, is also shown in Figure 1.1.

For a graph $G = (V, E)$, the unique graph \overline{G} with the same node set V but $\{xy \in V \times V : x \neq y, xy \notin E\}$ as edge set is defined to be the **complementary graph** or just the **complement** or **co-graph** of G . Clearly, the edges of G become the non-edges of \overline{G} and vice versa. The operation to take the complement of a graph is said **complementation**. Let \mathcal{C} stand for a certain class of graphs, then we denote by $\text{co-}\mathcal{C}$ its **complementary class** or **co-class** for short, i.e., the class of all graphs with $G \in \text{co-}\mathcal{C}$ iff $\overline{G} \in \mathcal{C}$.

A further graph operator frequently used in the sequel is the line operator. Applying it to a graph G yields the **line graph** $L(G)$ of G by taking the edges of G as nodes of $L(G)$ and joining two nodes of $L(G)$ by an edge iff the corresponding edges of G are incident. Note the opportunity to uniquely reconstruct the underlying graph $G = L^{-1}(L(G))$ from its line graph $L(G)$ whenever $G \neq K_3, K_{1,3}$.

Two graphs G_1 and G_2 are **isomorphic** if there is a bijection $\phi : V(G_1) \rightarrow V(G_2)$ preserving adjacency, i.e., with $\phi(x)\phi(y) \in E(G_2)$ if and only if $xy \in E(G_1)$ holds. We then often write $G_1 = G_2$ for $G_1 \cong G_2$. If $F_i \subseteq G_i$ for $i = 1, 2$ holds and a graph isomorphism $\phi : F_1 \rightarrow F_2$ exists, then the **identification of G_1 and G_2 in F_1 and F_2** means to identify every node $x \in V(F_1)$ with $\phi(x) \in V(F_2)$ and every edge $xx' \in E(F_1)$ with $\phi(x)\phi(x') \in E(F_2)$. Every graph isomorphic to its complement is called **self-complementary**. If a graph G does not admit any subgraph $G' \subseteq G$ isomorphic to a certain graph F , we say that G is **F -free**. A graph not containing any cycle as partial subgraph is called a **tree** in particular.

An **embedding** of a graph G in a surface \mathcal{F} is a mapping ϕ that assigns to each node of G a point in \mathcal{F} and to each edge of G a simple curve in \mathcal{F} such that two curves meet in a point x iff the corresponding edges share an endnode v and $x = \phi(v)$ holds. The (topological) closures of the connected components of $\mathcal{F} - \phi(G)$ are the faces of $\phi(G)$. A graph embeddable in \mathcal{F} is called a **triangulation** of \mathcal{F} if all faces of $\phi(G)$ are triangles. Every graph is said to be **planar (toroidal)** if it admits an embedding in the plane (torus). To **subdivide** an edge xy in a graph G means to replace it by a $P_3 = xzy$ where z is a new node only adjacent to x and y . A **subdivision** $\mathcal{U}(G)$ of G is any graph derivable from G by recursively subdividing some of its edges. The **contraction** of two nodes x and y in a graph is the operation that replaces x and y by a single node z , i.e., that adds a new node z , joins it to all neighbors of x and y by an edge, and finally deletes x and y . Every embedding is clearly invariant under any subdivision and under contraction of adjacent nodes x and y if one of the nodes has degree two.

Let \wp be a graph parameter and $\wp(G) \neq \wp(G - e)$, then the edge e is called \wp -**critical** in G . We denote the set of all \wp -critical edges of G by $E_\wp(G)$. If the partial subgraph $G_\wp = (V, E_\wp)$ of $G = (V, E)$ is connected, we call G \wp -**connected** and \wp -**critical** if $E_\wp = E$. A \wp -critical edge of G is \wp -**anticritical** in \overline{G} .

1.2 Known Results on Perfect and Minimally Imperfect Graphs

In his classical paper [3], BERGE proposed to call a graph **perfect** if, for each of its subgraphs, the chromatic number equals the clique number, otherwise the graph is said to be **imperfect**. At this time, the perfectness of a few graph classes was already known. So it is obvious for bipartite graphs and is equivalent to a theorem of KÖNIG [40] for co-bipartite graphs. KÖNIG [39] and [40] yield the perfectness of line graphs of bipartite graphs and their complements, respectively. The same result for **triangulated graphs**, defined to be C_k -free for $k \geq 4$, was established by BERGE [3] and for co-triangulated graphs by HAJNAL and SURÁNYI [23]. The observation that, in all these cases, perfectness holds for the graph class as well as for its co-class has made BERGE to conjecture that a graph is perfect iff its complement is. This question was answered in the affirmative by LOVÁSZ in [42] and is known as *Perfect Graph Theorem*.

Another observation is that odd holes and their complements, termed odd **antiholes**, are imperfect and so each graph containing an odd hole or an odd antihole as subgraph. BERGE conjectured in [3] that a graph is perfect iff it contains neither odd holes nor odd antiholes as subgraphs, such graphs are nowadays said to be **Berge**. This still outstanding conjecture, famous as *Strong Perfect Graph Conjecture* (for short SPGC), has already been verified for several classes of Berge graphs. To mention just a few of them, the SPGC is known to be true for F -free Berge graphs where F is a claw (PARTHASARATHY and RAVINDRA [51]), a paw (OLARIU [48]), a P_4 (JUNG [38], SEINSCHKE [53]), a diamond (TUCKER [61]), a K_4 (TUCKER [59]), or a bull (CHVÁTAL and SBIHI [11]), and also for planar and toroidal Berge graphs due to TUCKER [58] and GRINSTEAD [20], respectively.

Since the occurrence of odd cycles with certain chords seems to be important with respect to perfectness, several graph classes have been studied in this context. E.g., **line-perfect graphs**, defined as graphs that do not admit any cycle of odd length ≥ 5 as partial subgraph, are perfect due to TROTTER [57] (note that this class consists of all graphs whose line graphs are perfect). A further important class of perfect graphs was introduced by MEYNIEL [45] to contain all graphs whose cycles of odd length ≥ 5 have at least two chords, nowadays known as **Meyniel graphs**.

PADBERG [50] introduced the notion of **minimally imperfect graphs**, namely of imperfect graphs with the property that all of whose proper subgraphs are perfect. Clearly, a graph is minimally imperfect iff its complement is, due to the Perfect Graph Theorem [42]. In these terms, the SPGC states that the odd holes and the odd antiholes are the only minimally imperfect graphs. Therefore, minimally imperfect Berge graphs are

called **monsters** since the existence of this third type of minimally imperfect graphs would contradict the SPGC. In order to characterize minimally imperfect graphs, many properties have been found, e.g., properties reflecting an extraordinary amount on symmetry of their maximum cliques and stable sets.

Theorem 1.1 (LOVÁSZ [42]) *Every minimally imperfect graph G has exactly $\alpha\omega + 1$ nodes and, for every node x of G , the graph $G - x$ can be partitioned into α cliques of size ω and into ω stable sets of size α , where $\alpha = \alpha(G)$ and $\omega = \omega(G)$ holds.*

Theorem 1.2 (PADBERG [50]) *Every minimally imperfect graph G on n nodes has precisely n maximum stable sets and precisely n maximum cliques. Each node of G is contained in precisely $\alpha(G)$ maximum stable sets and in precisely $\omega(G)$ maximum cliques. For every maximum clique Q (stable set S) there is a unique maximum stable set S (clique Q) with $Q \cap S = \emptyset$.*

BLAND, HUANG, and TROTTER suggested in [7] to call a graph **partitionable** if it satisfies the conditions of Theorem 1.1 for some integers α , ω and verified Theorem 1.2 for all partitionable graphs.

Various forbidden substructures of minimally imperfect graphs have been discovered, we will confine ourself to mention only these which we will frequently use in later chapters. Let us start with results concerned to special cutsets. An immediate consequence of the definition of minimally imperfect graphs is that they cannot contain a **clique-cutset**, i.e., any cutset with pairwise adjacent nodes. In particular, every minimally imperfect graph is at least 2-connected. CHVÁTAL generalized this relation to cutsets C containing one node that is adjacent to all remaining nodes in C , termed **star-cutsets**, by proving that no minimally imperfect graph contains a star-cutset (*Star-Cutset Lemma* [10]). Investigating cutsets with pairwise non-adjacent nodes, called **stable-cutsets**, TUCKER [60] pointed out that the odd holes are the only minimally imperfect graphs containing stable-cutsets.

Turning to some node pairs forbidden in minimally imperfect graphs, consider two distinct nodes x and y of a graph G . We say that x **dominates** y if $N(y) \subseteq N(x) \cup \{x\}$ holds and call then x, y a **comparable pair**. A comparable pair x, y is said to be **strict** if x, y are non-adjacent and **weak** otherwise. An easy consequence of Theorem 1.1 is that minimally imperfect graphs neither contain strict nor weak comparable pairs. Hence minimally imperfect graphs do also not admit nodes whose neighborhood induces a clique, called **simplicial nodes**. The same is true for **antisimplicial nodes**, defined to be simplicial nodes of the complement, by the Perfect Graph Theorem [42]. If all induced (x, y) -paths in G have even length (odd length, length k), x and y are called an **even pair** (**odd pair**, **k -pair**, respectively). An odd pair x, y is said to be **strict** if x, y are non-adjacent and **weak** otherwise. MEYNIEL proved in [46] that minimally imperfect graphs do not contain any even pair (*Even Pair Lemma*). A natural analogue, with "even" replaced by "odd", is also conjectured but only proven for 3-pairs by HOÀNG [31] so far. Note that this conjecture is equivalent for strict and weak odd pairs due to a result of HOUGARDY [32]: a graph G with a weak odd pair x, y is minimally imperfect iff $G - xy$ is. Furthermore, if every node in $G - \{x, y\}$ is adjacent either to both or to none of x and y (to either x

or y) then x, y are called **twins** (**antitwins**). Twins (antitwins) x, y are said to be **true** if x, y are adjacent and **false** otherwise. Minimally imperfect graphs neither admit twins due to the so-called *Replacement Lemma* [42], the key lemma in the proof of the Perfect Graph Theorem [42], nor antitwins due to OLARIU (*Antitwin Lemma* [47]).

The knowledge of forbidden substructures of minimally imperfect graphs has led to various definitions and characterizations of classes of perfect graphs. The concept of star-cutsets has been used by HAYWARD to prove the perfectness of **weakly triangulated graphs** and **murky graphs**, which are C_k - and \overline{C}_k -free for every $k \geq 5$ [25] and C_5 -, P_6 -, and \overline{P}_6 -free [26], respectively. CHVÁTAL [10] defined the **star-closure** \mathcal{C}^* of a graph class \mathcal{C} to contain a graph G if and only if $G \in \mathcal{C}$ or $G - x \in \mathcal{C}^* \forall x \in V(G)$ and G or \overline{G} has a star-cutset, and proved that \mathcal{C}^* is perfect if \mathcal{C} is. Let $TRIV$ stand for the class of all graphs with at most two nodes, then $TRIV^*$ is the class of weakly triangulated graphs [25]. Another interesting class of perfect graphs is BIP^* , where BIP denotes the class of bipartite graphs. BIP^* contains all Meyniel graphs and some further classes of perfect graphs (see [10]).

Applying the concept of even pairs, MEYNIEL [46] defined a graph to be in the class of **strict quasi parity graphs** (for short SQP) if each of its non-complete subgraphs has an even pair. Furthermore, the class of **quasi parity graphs** (QP for short) is defined in [46] that for each subgraph G' of a graph in QP should hold that G' or \overline{G}' owns an even pair. Meyniel established the perfectness of all graphs in SQP and QP as well as several inclusion relations, e.g., that all Meyniel graphs are in SQP and $SQP \subset QP$ holds. HAYWARD, HOÀNG, and MAFFRAY proved in [28] that weakly triangulated graphs are in SQP by characterizing them as graphs whose non-complete subgraphs contain a 2-pair. Another graph class concerned to even pairs was defined by BERTSCHI in [5]. He said a graph G to be **even contractile** if there is a sequence of graphs G_0, G_1, \dots, G_k such that $G = G_0$ holds, G_i is obtained via the contraction of an even pair in G_{i-1} for $1 \leq i \leq k$, and G_k is complete. An even contractile graph need not to be perfect but a graph each of its non-complete subgraphs is even contractile, called **perfectly contractile**, is in SQP . Note that Meyniel graphs are shown to be perfectly contractile in [5].

A stable set of a graph G has been termed **strong** if it has a non-empty intersection with all maximal cliques of G . Since no minimally imperfect graph contains a strong stable set by Theorem 1.2, **strongly perfect graphs**, introduced in [4] to be graphs all of whose subgraphs admit a strong stable set, are perfect. The notion of strong stable sets was generalized by HAMMER and MAFFRAY in [24]. They called a node set $V' \subseteq V$ of $G = (V, E)$ **absorbant** if every node in $V - V'$ has at least one neighbor in V' , and then a graph **absorbantly perfect** if every of its subgraphs owns a minimal absorbant set having a non-empty intersection with all its maximal cliques. Absorbantly perfect graphs are a superclass of strongly perfect graphs by definition. The perfectness of absorbantly perfect graphs was shown by characterizing them to possess either a strong stable set or a weak comparable pair (see [24]). Finally, PREISSMANN [52] suggested to call a graph G **locally perfect** if each of its subgraphs $G' \subseteq G$ admits a coloring which uses only $\omega(N_{G'}(x))$ colors in $N_{G'}(x) \forall x \in G'$ and proved the perfectness of these graphs.

Chapter 2

Critical Edges with Respect to Perfectness

We introduce the notions of critical and anticritical edges in perfect graphs. In order to tackle Problem 0.1 to find necessary conditions when an edge is not critical or anticritical, we provide several properties of graphs admitting critical or anticritical edges through the Sections 2.1 and 2.2. We use this knowledge in Section 2.3 to investigate the occurrence of such edges in several classes of perfect graphs. Section 2.4 is devoted to Problem 0.2: order the edges (non-edges) of a perfect graph such that deleting (adding) the edges in this order yields a sequence of perfect graphs.

2.1 Critical Edges

We define an edge e of a perfect graph G to be **critical** if $G - e$ is imperfect. In particular, for every critical edge e of a perfect graph G there is necessarily at least one subgraph $G_e \subseteq G$ such that $G_e - e$ is *minimally* imperfect. According to the three possible types of minimally imperfect graphs, we distinguish between three different types of critical edges. We say that an edge e of a perfect graph G is **H-critical** (**A-critical**, **M-critical**) if there is a subgraph $G_e \subseteq G$ such that $G_e - e$ is isomorphic to an odd hole (an odd antihole, a monster, respectively). To get familiarly with critical edges, let us make some considerations on the corresponding subgraphs G_e .

For every H-critical edge e , there is a subgraph G_e isomorphic to an odd cycle of length ≥ 5 which admits precisely one chord, namely e . Moreover, e has to be a *short* chord of this cycle, forming a triangle with two of its edges, since G_e must not contain an odd hole. Unless it is defined otherwise, we always refer to a subgraph G_e where $G_e - e$ is an odd hole by the graph having $\{v_1, \dots, v_{2k+1}\}$ with $k \geq 2$ as nodes, $v_i v_{i+1}$ for $1 \leq i \leq 2k+1$ (mod $2k+1$) and $e = v_1 v_{2k}$ as edges. The nodes v_1, v_{2k} and v_{2k+1} form a triangle, the only maximum clique of G_e , and v_1, \dots, v_{2k} induce an even hole.

If e is an A-critical edge there is a subgraph G_e isomorphic to a \overline{P}_{2k+1} with $k \geq 2$. Therefore, it is natural to use the convention that G_e has $\{v_1, \dots, v_{2k+1}\}$ as nodes and $v_i v_{i+1}$ as non-edges for $1 \leq i \leq 2k$, i.e., we have $e = v_1 v_{2k+1}$. Then all nodes with odd index

form the only maximum clique of G_e with size $k+1$. Since $\overline{P}_{2i+1} \subset \overline{P}_{2k+1}$ holds whenever $i < k$, the subgraph G_e contains other A-critical edges than e for $k > 2$. In the case $G_e = \overline{P}_5$, the edge e is A-critical as well as H-critical since \overline{P}_5 is precisely a cycle on 5 nodes with just one short chord.

Of course, such a description cannot be given for M-critical edges and, if the SPGC is true, there are no M-critical edges at all. Hence we are interested in general properties of G_e caused by properties of all the minimally imperfect graphs $G_e - e$.

Lemma 2.1 *Let G be a perfect graph, $e = xy$ a critical edge of G , and $G_e - e$ a minimally imperfect subgraph of $G - e$, then the following conditions are simultaneously satisfied.*

- (i) *The edge e is ω -critical but not χ -critical with respect to G_e ; x and y are contained in the unique maximum clique of G_e .*
- (ii) *The edge e is θ -critical but not α -critical with respect to G_e , deleting e destroys every minimum clique covering of G_e .*
- (iii) *There is a maximum stable set of $G_e - e$ containing x and y .*

Proof. Consider a critical edge $e = xy$ of the perfect graph G and a minimally imperfect subgraph $G_e - e \subseteq G - e$ corresponding to e . We have $\omega(G_e) = \chi(G_e)$ but $\omega(G_e - e) < \chi(G_e - e)$. Because the removal of an edge cannot increase the chromatic number, $\chi(G_e) = \chi(G_e - e)$ and, therefore, $\omega(G_e) > \omega(G_e - e)$ follows. Moreover, the deletion of e does not only destroy all maximum cliques in G_e , but there is only one such clique by [55] and (i) is shown. Due to the Perfect Graph Theorem [42], $\alpha(G_e) = \theta(G_e)$ but $\alpha(G_e - e) < \theta(G_e - e)$ holds. Since the stability number cannot be decreased by the removal of an edge, $\alpha(G_e) = \alpha(G_e - e)$ yields $\theta(G_e) < \theta(G_e - e)$. Thus, deleting e must destroy every minimum clique covering of G_e and we have obtained (ii). In order to prove (iii) consider the graph $\overline{G}_e + e$ and assume the edge $e = xy$ of $\overline{G}_e + e$ to be not contained in any maximum clique of $\overline{G}_e + e$. Then the graph \overline{G}_e is partitionable by TUCKER [59] in contradiction to \overline{G}_e perfect due to [42]. Hence, x and y lie in a maximum clique of $\overline{G}_e + e$ and, therefore, in a maximum stable set of $G_e - e$. \square

Remark. If $G_e - e$ is an odd hole or antihole, both endnodes of e appear in precisely one maximum stable set each. But it is unknown whether this is true for all critical edges, i.e., if this property applies to M-critical edges.

The deletion of e causes only local changes on the subgraphs G_e . Hence, we can certainly state that e is ω -critical, θ -critical and not χ -critical or α -critical with respect to G_e , but not provide a similar result with respect to G . That the occurrence of a critical edge can also have an effect on G itself is shown by the next two lemmas. The first one is concerned to forbidden substructures in G , the other one gives some further properties, which we will frequently make use of in the sequel.

Lemma 2.2 *Suppose $e = xy$ to be a critical edge of a perfect graph G . Then x and y are neither true twins, true antitwins, a weak comparable pair, nor a weak odd pair of G . Moreover, neither x nor y is a universal or a simplicial node in G .*

Proof. Consider a perfect graph G , a critical edge $e = xy$ of G , and a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. Then x and y do not form a weak comparable pair, true twins, and true antitwins of G , else they would be a strict comparable pair, false twins, and false antitwins in $G_e - e$ leading to a contradiction to a well-known fact, the Replacement Lemma [42], and the Antitwin Lemma [47], respectively. Now, neither x nor y can be a universal or a simplicial node, since x or y had to form either a weak comparable pair or true twins with one of its neighbors in G otherwise. Suppose finally x and y to be a weak odd pair, then they would be a strict odd pair in $G_e - e$. Due to a result of HOUGARDY [32], $G_e - e$ can be minimally imperfect only if G_e is, hence we have a contradiction to G_e perfect. \square

Lemma 2.3 *Let $e = xy$ be a critical edge of the perfect graph G and $G_e - e \subseteq G - e$ minimally imperfect.*

- (i) $N_{G_e - e}(x) \cap N_{G_e - e}(y) \neq \emptyset$ holds, hence the nodes x and y occur in a triangle of G .
- (ii) There is an odd induced (x, y) -path in $G_e - e$, thus an even hole through x, y in G .
- (iii) If e is M -critical, it is the middle edge of a diamond and belongs to a K_5 in $G_e \subseteq G$.

Proof. Suppose $e = xy$ to be a critical edge of a perfect graph G and $G_e - e$ to be a minimally imperfect subgraph of $G - e$ corresponding to e . Since x and y must not form an even pair in $G_e - e$ according to the Even Pair Lemma [46] and $G_e - e$ is connected, an odd induced (x, y) -path exists in $G_e - e$ and yields (ii). We know from Lemma 2.1.(i) that x and y are contained in the only maximum clique of G_e . Hence $\omega(G_e - e) = 1$ would be the consequence of $N_{G_e - e}(x) \cap N_{G_e - e}(y) = \emptyset$ and we get assertion (i). Similarly, $\omega(G_e - e) < 4$ follows if e is not contained in a K_5 and, therefore, $G_e - e$ cannot be a monster in this case due to a result of TUCKER [59]. In order to complete the proof of assertion (iii), consider a node $z \in N_{G_e - e}(x) \cap N_{G_e - e}(y)$. The nodes x, z , and y induce a P_3 in $G_e - e$, hence a hole C is running through them in $G_e - e$ by HOÁNG [30]. If $G_e - e$ is supposed to be a monster, C is even. $G[V(C) - z]$ is a chordless cycle of odd length in G_e , hence $|C| = 4$ follows. \square

2.2 Anticritical Edges

We define a critical edge e of the perfect graph \overline{G} to be **anticritical** with respect to G , i.e., the addition of an anticritical edge $e \notin E(G)$ to G yields an imperfect graph. Clearly, there is a subgraph $G_e \subseteq G$ for every anticritical edge e of a perfect graph G such that $G_e + e$ is minimally imperfect. According to the three types of critical edges, we say that e is an **H-anticritical** (**A-anticritical**, **M-anticritical**) edge of G if e is H-critical (A-critical, M-critical, respectively) in \overline{G} . Looking at this close relation between critical and anticritical edges, it is not surprising that most of the results concerned to critical edges can be easily translated to properties with respect to anticritical edges.

Investigating the subgraphs G_e corresponding to an anticritical edge e , let us make use of the conventions already given for \overline{G}_e in the previous section. If $G_e + e$ is an odd antihole,

the endnodes v_1 and v_{2k} of e form together with v_{2k+1} the only maximum stable set of G_e , and v_1, \dots, v_{2k} induce an even antihole. For every A-anticritical edge e , there is a subgraph G_e isomorphic to P_{2k+1} with $k \geq 2$. Again, other A-anticritical edges occur in G_e for $k > 2$, while e is A-anticritical as well as H-anticritical for $k = 2$. In analogy to M-critical edges, no detailed description can be given for M-anticritical edges. Hence we are interested in properties of G_e caused by properties of the minimally imperfect graph $G_e + e \subseteq G + e$ again.

Lemma 2.4 *Let G be a perfect graph, $e = xy \notin E(G)$ an anticritical edge, and $G_e + e$ a minimally imperfect subgraph of $G + e$ corresponding to e .*

- (i) *The edge e is χ -anticritical but not ω -anticritical with respect to G_e ; x and y admit the same color in every minimum coloring of G_e .*
- (ii) *The edge e is α -anticritical but not θ -anticritical with respect to G_e ; x and y are contained in the unique maximum stable set of G_e .*
- (iii) *x and y appear in a maximum clique of $G_e + e$.*

Proof. All assertions of this lemma are immediate consequences of Lemma 2.1 by the duality of $\chi(G)$, $\omega(G)$ and $\theta(\overline{G})$, $\alpha(\overline{G})$ and the Perfect Graph Theorem [42]. \square

Remark. The addition of e also causes only local changes on the subgraph G_e , so we cannot provide results similar to (i) and (ii) with respect to G . Furthermore, if $G_e + e$ is an odd hole or antihole, the endnodes of e appear in precisely one maximum clique of G_e each. But, again, it is unknown whether this property applies to M-anticritical edges.

The occurrence of anticritical edges leads, in analogy to critical edges, to the following forbidden substructures in G , which are immediate consequences of some properties of the minimally imperfect graphs $G_e + e$.

Lemma 2.5 *Let $e = xy$ be a critical edge of the perfect graph \overline{G} , then x and y are neither false twins, false antitwins, a strict comparable pair, a strict odd pair, nor a 2-pair of G . Moreover, neither x nor y is an isolated or an antisimplicial node in G , i.e., a simplicial node in \overline{G} .*

Proof. Consider a perfect graph G , an anticritical edge $e = xy \notin E(G)$, and a subgraph $G_e \subseteq G$ such that $G_e + e \subseteq G - e$ is minimally imperfect. Then Lemma 2.2 implies that x and y does not form a strict comparable pair, false twins, and false antitwins of G and neither x nor y can be an isolated or an antisimplicial node. If x and y are supposed to be a strict odd pair, they would be a weak odd pair in $G_e + e$, a contradiction to HOUGARDY [32]. Now assume x and y to form a 2-pair. In the case $N_{G_e}(x) - N_{G_e}(y) = \emptyset$, the node y would dominate x in $G_e + e$, hence there is a node $x' \in N_{G_e}(x) - N_{G_e}(y)$. Since x and y are a 2-pair in G_e and x' and y are non-adjacent, all (x', y) -paths in $G_e + e$ must contain x or a node in $N_{G_e}(x) \cap N_{G_e}(y)$ in contradiction to the Star-Cutset Lemma [10]. \square

We can establish a result concerned to 2-pairs and anticritical edges, while nothing similar is known for critical edges. Clearly, the reason is that, in the context of anticritical edges, pairs of non-adjacent nodes are of interest. So we also obtain an additional property besides these related to Lemma 2.3.

Lemma 2.6 *Suppose $e = xy$ to be a critical edge of the perfect graph \overline{G} .*

- (i) *The nodes x and y belong to a stable set of size 3 in G .*
- (ii) *G admits an even antihole containing x and y .*
- (iii) *If e is M -anticritical, then x and y are the nodes of degree 0 of a $\overline{K_4 - e}$, and are contained in a stable set of size 5.*
- (iv) *The edge e is only A -anticritical, if x and y form an even pair.*

Proof. Consider a perfect graph G and an anticritical edge $e = xy \notin E(G)$. Then the assertions (i), (ii), and (iii) follow directly from the corresponding statements of Lemma 2.3. Hence suppose x and y to form an even pair. Since x and y must neither be a 2-pair in G nor in G_e by Lemma 2.5, there is an induced (x, y) -path P in G_e of even length ≥ 4 . Then $P + e$ is an odd hole and $P = G_e$ follows, hence e is A -anticritical. \square

2.3 Which Graph Classes admit Critical Edges?

This section is devoted to the investigation of Problem 0.1. We focus on subclasses of perfect graphs and try to find out whether membership in such a subclass makes it easier to characterize critical or anticritical edges.

The first question is, of course: In which perfect graphs do critical or anticritical edges occur? More precisely, are there classes of perfect graphs so that no graph in this class has a critical or anticritical edge? For some graph classes, this follows immediately using some of the properties concerned with critical edges that are established in the previous sections. By Lemma 2.3.(i), e.g., every critical edge is contained in a triangle, so *bipartite graphs do not possess any critical edge*. That *the same is true for triangulated graphs* is an immediate consequence of Lemma 2.3.(ii), stating that every critical edge has to occur in an even hole. Both properties together yield the existence of an odd cycle of length at least 5 in the union of the triangle and the even hole, so *line-perfect graphs cannot admit critical edges*, too. The next theorem provides the *characterization of perfect graphs without any critical edge*. It also yields a new characterization of Meyniel graphs. The proof of this theorem relies on the perfectness of a superclass of Meyniel graphs, namely **slim graphs** introduced by HERTZ [29]. If $G = (V, E)$ is Meyniel and $V' \subseteq V$, the slim graph $G(V')$ is obtained by deleting every edge of G with both endnodes in V' .

Theorem 2.7 (HOUGARDY, WAGLER [37]) *A perfect graph does not admit any critical edge iff it is Meyniel.*

Proof. (*Only if*) Let G be a perfect graph without any critical edge. Assume G to be not a Meyniel graph then G admits an odd cycle C with length ≥ 5 which has at most one chord. Since G is perfect, C cannot be chordless, hence C possess precisely one chord e . If e were not a short chord of C , an odd hole would be contained in G . Thus e has to be a short chord of C and so it is an H-critical edge of G in contradiction to the precondition. \diamond

(*If*) Suppose $G = (V, E)$ to be a Meyniel graph. Then, for every subset $V' \subseteq V$, the graph $G(V')$ generated from G by deleting every edge of G with both endnodes in V' is a slim graph, still perfect by HERTZ [29]. Thus, if we choose an arbitrary pair of adjacent nodes in a Meyniel graph and delete the edge connecting them, we get a perfect graph again, i.e., a Meyniel graph cannot admit any critical edge. \square

Remark. It is obvious that H-critical edges do not occur in Meyniel graphs by definition. Furthermore, in Section 2.1 it is observed that, for every A-critical edge e , there is a subgraph G_e isomorphic to \overline{P}_{2k+1} with $k \geq 2$. Since then $\overline{P}_5 \subseteq \overline{P}_{2k+1}$ holds, we immediately see that A-critical edges cannot be contained in Meyniel graphs. But no proof independent from HERTZ's result in [29] is known so far showing that Meyniel graphs do not possess any M-critical edge.

Making use of this new characterization of Meyniel graphs, a perfect graph does not admit any anticritical edge if and only if it is co-Meyniel. Hence *a graph is free of critical as well as anticritical edges iff it is contained in the intersection of the class of Meyniel graphs and its complementary class*. The next theorem proves that all complete k -partite graphs are contained in the intersection of these two graph classes.

Theorem 2.8 *No complete k -partite graph contains a critical or anticritical edge.*

Proof. Suppose G to be a complete k -partite graph with color classes S_1, \dots, S_k . Assume $e = xy$ to be a critical edge of G and G_e a subgraph with $G_e - e$ minimally imperfect. Then G_e is complete k' -partite for some $k' \leq k$ and $x \in S_i, y \in S_j$ holds with $i \neq j$ and $i, j \leq k'$. Since x and y occur in the intersection of all maximum cliques of G_e by Lemma 2.1.(i), we conclude $S_i = \{x\}$ and $S_j = \{y\}$. Thus x and y are true twins of G_e in contradiction to Lemma 2.2 and G does not admit any critical edge. Now, consider two non-adjacent nodes $x, y \in S_i$ for one $i \in \{1, \dots, k\}$. Due to $N(x) = N(y) = G - S_i$, the nodes x and y are false twins. Hence xy cannot be an anticritical edge of G by Lemma 2.5 and the assertion of the lemma is true. \square

Theorem 2.7 implies that critical edges occur in all perfect graphs G not contained in the class of Meyniel graphs. Our aim is to characterize the critical edges e in these perfect graphs G , i.e., to find out which minimally imperfect subgraphs $G_e - e$ may occur in $G - e$. Our main tools of investigation are properties of the graphs $G_e \subseteq G$ discussed throughout Section 2.1 and knowledge about forbidden subgraphs in the classes of perfect graphs under consideration.

Lemma 2.9 *Let G be a perfect graph and e a critical edge of G .*

- (i) *Is G a P_5, K_5 -free Berge graph, then $G - e$ has C_5 and \overline{C}_7 as the only minimally imperfect subgraphs.*
- (ii) *If G is diamond-free Berge, K_4 -free Berge, or planar Berge, then $G - e$ neither contains a \overline{C}_{2k+1} with $k \geq 3$ nor a monster.*
- (iii) *Is G toroidal Berge, $G - e$ has odd holes and \overline{C}_7 as the only minimally imperfect subgraphs.*
- (iv) *For every locally perfect graph G , the graph $G - e$ cannot contain a \overline{C}_{2k+1} with $k \geq 3$.*
- (v) *Is G weakly triangulated or bull-free Berge, no C_{2k+1} with $k \geq 3$ appears in $G - e$.*
- (vi) *If G is murky, $G - e$ has C_5 and monsters as the only minimally imperfect subgraphs.*

Proof. Let G be perfect, e a critical edge of G , and $G_e - e \subseteq G - e$ minimally imperfect.

Every P_5, K_5 -free Berge graph G is perfect due to MAFFRAY and PREISSMANN [44]. We observe that G obviously neither admit any H-critical edge with $|G_e| > 5$ nor any A-critical edge with $|G_e| > 7$. M-critical edges do not occur by Lemma 2.3.(iii) and (i) is true.

Lemma 2.3.(iii) also shows that neither diamond-free Berge graphs nor K_4 -free Berge graphs admit M-critical edges. Since a diamond is contained in \overline{P}_{2k+1} for $k > 2$, we obtain $G_e = \overline{P}_5$ for every odd antihole $G_e - e$ corresponding to an A-critical edge e and we infer that diamond-free Berge graphs only have H-critical edges. That $\omega(\overline{P}_{2k+1}) = k + 1$ holds, completes the assertion for K_4 -free Berge graphs. Now, let G be planar Berge, then G cannot possess any $\mathcal{U}(K_5)$, a subdivision of the K_5 , as subgraph by the characterization of planar graphs of KURATOVSKY [41]. Thus G has no M-critical edge by Lemma 2.3.(iii) again. Furthermore, $K_5 \subseteq \overline{P}_{2k+1}$ for $k > 3$ and $\mathcal{U}(K_5) \subseteq \overline{P}_{2k+1}$ for $k = 3$ implies that all A-critical edges e in planar Berge graphs have to yield a C_5 in $G - e$, i.e., they are H-critical. This shows (ii).

Let G be toroidal Berge, we prove that $G - e$ does not admit any minimally imperfect subgraph $G_e - e$ with $\omega(G_e - e) \geq 4$. First note that $G - e$ is still toroidal. Since every minimally imperfect toroidal graph either has clique number < 4 or is 6-regular and triangulates the torus, due to GRINSTEAD [20], we have to show that every such triangulation is maximal with respect to the addition of edges. Following ALTSHULER [1], every 6-regular triangulation of the torus can be represented as the graph $T(r, s, t)$ shown in Figure 2.1. The nodes $v_{i,s+1}$ have to be identified with $v_{i,1}$ for $1 \leq i \leq r$, but the nodes $v_{r+1,j}$ with $v_{1,j-t}$, where the second subscript is taken modulo s , and t is fixed. Then the neighborhood of every node $v_{i,j} \in T(r, s, t)$ contains a cycle C of length 6 as partial subgraph, encircling $v_{i,j}$ as is emphasized by bold lines in Figure 2.1. Obviously, $v_{i,j}$ cannot be linked to a node in $T(r, s, t) - (C \cup \{v_{i,j}\})$ without crossing one edge of C . Thus, there is no toroidal graph $G_e \subseteq G$ such that the deletion of e results in a minimally imperfect toroidal graph $G_e - e$ which is 6-regular and triangulates the torus. By [20] we get, therefore, $\omega(G_e - e) < 4$ for all minimally imperfect subgraphs $G_e - e \subseteq G - e$ if G

is toroidal Berge. A result of TUCKER [59] states that every monster must contain a K_4 . Thus minimally imperfect graphs with clique number < 4 are the odd holes and the \overline{C}_7 which completes the proof of assertion (iii).

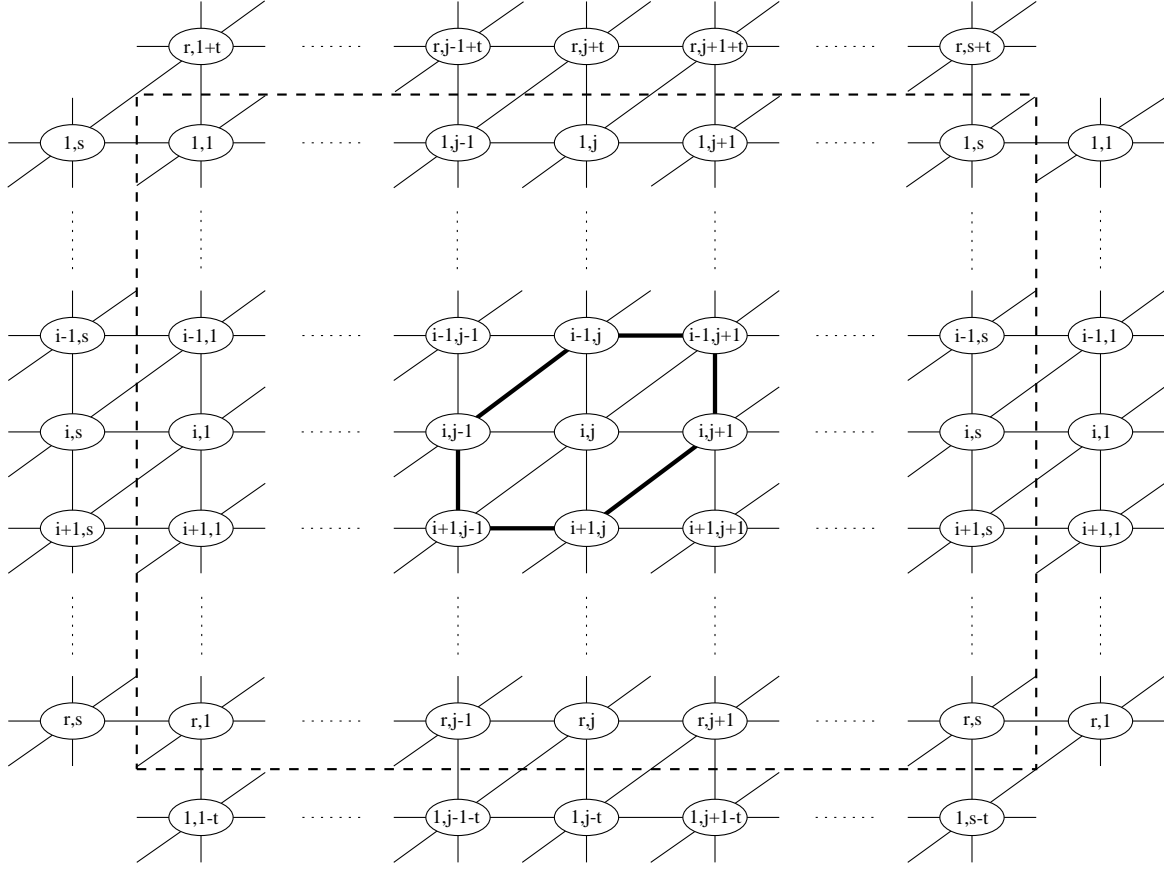


Figure 2.1: $T(r, s, t)$ representing every 6-regular triangulation of the torus.

In order to show (iv), note that the \overline{P}_7 is minimally non-locally perfect. $\overline{P}_7 \subseteq \overline{P}_{2k+1}$ for $k \geq 3$ implies that locally perfect graphs cannot possess any A-critical edge e with $|G_e| \geq 7$.

If G is weakly triangulated, we can easily exclude H-critical edges e with $|G_e| \geq 7$, since $C_k \not\subseteq G$ holds for $k \geq 5$ by definition. Furthermore, if e is H-critical and $|G_e| \geq 7$, then the nodes $v_1, v_2, v_{2k-1}, v_{2k}, v_{2k+1}$ induce a bull in G_e , thus we can exclude H-critical edges in bull-free Berge graphs, too, and (v) is true.

Finally, murky graphs do not contain any C_5 , P_6 , and \overline{P}_6 . Thus, they do not possess H-critical and A-critical edges e with $|G_e| \geq 7$, since the nodes $v_{2k+1}, v_1, \dots, v_5$ and v_1, \dots, v_6 induce a P_6 and \overline{P}_6 , respectively, for $k \geq 3$. \square

Remark. The absence of M-critical edges in planar and toroidal Berge graphs can also be directly inferred from the results of TUCKER [58] and GRINSTEAD [20] that establish

the perfectness of planar and toroidal Berge graphs, respectively. Moreover, it is not necessary to check separately which kinds of anticritical edges appear in certain classes of perfect graphs. If all graphs G belonging to a graph class \mathcal{C} must not admit any H-critical, A-critical, and M-critical edge, then all graphs \overline{G} in $\text{co-}\mathcal{C}$ do not possess any H-anticritical, A-anticritical, and M-anticritical edge, respectively.

The previous lemma provides all results concerning critical edges in certain subclasses of perfect graphs that we can infer from the present knowledge of their forbidden subgraphs. There is only one further subclass of perfect graphs for which we are able to characterize even both critical *and* anticritical edges: the class of all perfect line graphs. That characterization bases on the special structure of line graphs and of the opportunity to uniquely reconstruct the underlying graph from its line graph [66] (provided the line graph is different from K_3).

It is well-known that the line graph G of a graph F is perfect iff F is line-perfect, i.e., iff F does not contain any odd cycle of length at least 5 as partial subgraph. Note that every *partial* subgraph F' of F corresponds to an *induced* subgraph $L(F')$ of $L(F) = G$. Furthermore, every node $v_i \in V(G)$ corresponds to an edge $e_i \in E(F)$, we refer to this by using the corresponding index i for $1 \leq i \leq n = |V(G)| = |E(F)|$.

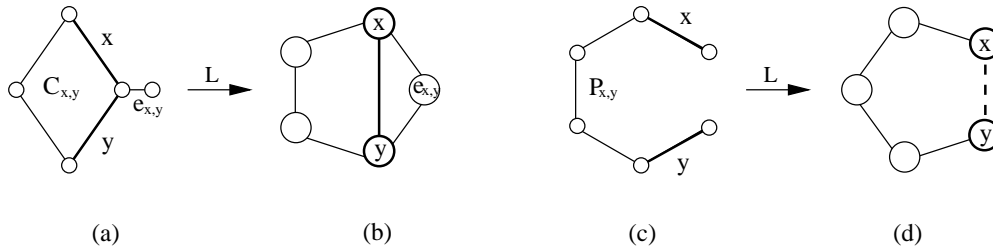


Figure 2.2

In order to characterize critical and anticritical edges in $L(F)$, we define two structures in the underlying line-perfect graphs F . We say that two incident edges x and y form an **H-pair** in F if there is an edge $e_{x,y}$ incident to the common node of x and y and if there is a (not necessarily induced) even cycle $C_{x,y}$ containing x and y but only one endnode of $e_{x,y}$ (see Figure 2.2(a)). Then $L(C_{x,y})$ is an even hole and the node in $L(F)$ corresponding to $e_{x,y}$ has precisely two neighbors on $L(C_{x,y})$, namely x and y (see Figure 2.2(b)).

Two non-incident edges x and y are called an **A-pair** if they are the endedges of a (not necessarily induced) odd path $P_{x,y}$ with length at least five (see Figure 2.2(c)). Then $L(P_{x,y})$ is an even, chordless path of length at least four with endnodes x and y or, in other words, $L(P_{x,y})$ is an odd hole where the edge between x and y is missing (see Figure 2.2(d)). It is straightforward that the deletion and addition of the edge xy in $L(C_{x,y} \cup e_{x,y})$ and $L(P_{x,y})$, respectively, yields a minimally imperfect graph. The next two theorems even establish that xy is a critical and anticritical edge in $L(F)$ only if x, y form an H-pair and A-pair, respectively, in F .

Theorem 2.10 *Let G be the line graph of a line-perfect graph F . An edge $e = xy$ of G is critical iff x and y form an H-pair in F .*

Proof. (If) Let x and y form an H-pair in the line-perfect graph F , then there is an edge $e_{x,y}$ incident to both and an even cycle $C_{x,y}$ containing x and y but only one endnode of $e_{x,y}$. In $G = L(F)$, the edge $e = xy$ belongs to the even hole $L(C_{x,y})$ and the node of G corresponding to $e_{x,y}$ has only x and y as neighbors on $L(C_{x,y})$, i.e., e is H-critical in G . \diamond

(Only if) Suppose G to be the line graph of a line-perfect graph F and e to be a critical edge of G . In order to show that the edges corresponding to the endnodes of e form an H-pair in F , we prove the following three claims according to the three types of critical edges.

Claim 1: *For all H-critical edges $e = xy$ of G , the edges x and y form an H-pair in F .*

Claim 2: *If e is A-critical and $G_e - e$ a corresponding odd antihole, $|G_e| = 5$ must hold.*

Claim 3: *G cannot admit any M-critical edge.*

Then the assertion of the theorem is clear for H-critical edges by Claim 1. For A-critical edges, it follows by Claim 2 and Claim 1 (since A-critical edges e with $|G_e| = 5$ are H-critical, too), and is irrelevant for M-critical edges by Claim 3.

Proof of Claim 1. Let $e = xy$ and $G_e - e$ be an odd hole corresponding to e which is isomorphic to C_{2k+1} with $k \geq 2$. Then, by convention, $x = v_1$ and $y = v_{2k}$ holds and the nodes $\{v_1, \dots, v_{2k}\}$ induce an even hole C in G with $N_C(v_{2k+1}) = \{x, y\}$. In F , the nodes v_1, \dots, v_{2k} correspond to the edges of an even cycle $L^{-1}(C)$ containing the incident edges e_1 and e_{2k} , i.e., we have $L^{-1}(C) = C_{x,y}$. By $N_C(v_{2k+1}) = \{x, y\}$, the edge e_{2k+1} is incident to e_1 and e_{2k} in F , but does neither form a triangle with them, nor has both endnodes on $C_{x,y}$, hence $e_{2k+1} = e_{x,y}$ and x, y form an H-pair in F .

Proof of Claim 2. Let $e = xy$ be an A-critical edge of G and consider $G_e = \overline{P}_{2k+1}$ with $k \geq 2$. Then, by convention, G_e has the nodes v_1, \dots, v_{2k+1} where $x = v_1$ and $y = v_{2k+1}$ holds, and $v_i v_{i+1}$ are the non-edges of G_e for $1 \leq i \leq 2k$. For $k \geq 3$, the nodes $v_1, v_2, v_3, v_{2k}, v_{2k+1}$ induce a subgraph of G which is forbidden according to BEINEKE's characterization of line graphs [2]. Hence, we conclude $k = 2$ and, therefore, $|G_e| = 5$ follows.

Proof of Claim 3. Suppose $e = xy$ to be an M-critical edge of G and $G_e - e$ to be a corresponding monster. Due to [2], G_e is $K_{1,3}$ -free but $G_e - e$ is not since the SPGC is true for $K_{1,3}$ -free graphs by a result of PARTHASARATHY and RAVINDRA [51]. So let us consider a $K_{1,3} = \{w, x, y, z\}$ with center z in $G_e - e$. The nodes x, z, w and y, z, w induce P_3 's which have to be contained in two holes C_x and C_y , respectively (see HOÀNG [30]). Let $x = x_0, \dots, x_k, w, z$ and $y = y_0, \dots, y_l, w, z$ be the nodes of C_x and C_y , respectively. Assume $yx_1 \in E(G_e)$ and consider the partial subgraph $L^{-1}(C_x \cup \{y\})$ of F . Then $L^{-1}(C_x)$ is an even cycle (otherwise C_x would be an odd hole in G_e) and the edge y has both endnodes

on $L^{-1}(C_x)$, forming a triangle with x and x_1 (note xy is not a critical edge of G if y is parallel to x in F). Hence $(L^{-1}(C_x) - \{x, x_1\}) \cup \{y\}$ is an odd cycle of length ≥ 5 in F (since $yw \notin E(G_e)$), and $(C_x - \{x, x_1\}) \cup \{y\}$ is an odd hole in G_e , a contradiction. Therefore, we have $yx_1 \notin E(G_e)$ and, analogously, $xy_1 \notin E(G_e)$. Choose i, j with $i+j \geq 2$ smallest possible and $x_i y_j \in E(G_e)$ (note at least $x_k y_l \in E(G_e)$ else x_k, y_l, z and w would induce a $K_{1,3}$ in G_e) and let $C = x \dots x_i y_j \dots y$. Then C is a chordless cycle different from a triangle (by $yx_1, xy_1 \notin E(G_e)$), i.e., C has to be an even hole. The definition of C_x and C_y implies $N_C(z) = \{x, y\}$, thus $(C \cup \{z\}) - e$ is an odd hole in $G_e - e$, a contradiction to $G_e - e$ Berge by assumption. \square

The above theorem implies that *every perfect line graph only admits H-critical edges*. The next theorem shows that *perfect line graphs only have A-anticritical edges* and, therefore, complements of perfect line graphs only possess A-critical edges (see also [63]).

Theorem 2.11 *Let G be the line graph of a line-perfect graph F . An edge $xy \notin E(G)$ is anticritical iff x and y form an A-pair in F .*

Proof. (*If*) Let x and y form an A-pair in the line-perfect graph F , then x and y are the endedges of an odd path $P_{x,y}$ of length at least 5. Applying the line operator to $P_{x,y}$, we get an even induced path $L(P_{x,y})$ of length at least 4 with endnodes x and y . The addition of the edge xy to $L(P_{x,y})$ yields an odd hole in $L(F)$, i.e., the edge $e = xy$ is A-anticritical. \diamond

(*Only if*) Let G be the line graph of a line-perfect graph F and $e \notin E(G)$ an anticritical edge. In order to show that the edges corresponding to the endnodes of e form an A-pair in F , we prove the following three claims according to the three types of anticritical edges.

Claim 1: *For all A-anticritical edges $e = xy$ of G , the edges x and y form an A-pair in F .*

Claim 2: *If e is H-anticritical and $G_e + e \subseteq G + e$ an odd antihole, $|G_e| = 5$ must hold.*

Claim 3: *G cannot admit any M-anticritical edge.*

The assertion of the theorem is clear for A-anticritical edges by Claim 1, follows for H-anticritical edges by Claims 2 and 1, and is irrelevant for M-anticritical edges by Claim 3.

Proof of Claim 1. Consider an A-anticritical edge $e = xy$ and a corresponding odd hole $G_e + e$ isomorphic to C_{2k+1} with $k \geq 2$. Then we have $G_e = P_{2k+1}$ where x and y are the endnodes of this path. Thus $L^{-1}(P_{2k+1})$ must be a path of length $2k+1$ with endedges x and y , i.e., $L^{-1}(P_{2k+1}) = P_{x,y}$ and the edges x, y form an A-pair in F .

Proof of Claim 2. Let $e = xy$ be an H-anticritical edge of G and $G_e + e \subseteq G + e$ an odd antihole \overline{C}_{2k+1} with $k \geq 2$ corresponding to e . Then, by convention, G_e has the nodes v_1, \dots, v_{2k+1} where $x = v_1$ and $y = v_{2k}$ holds, while xy and $v_i v_{i+1}$ for $1 \leq i \leq 2k+1 \pmod{2k+1}$ are the non-edges of G_e . For $k \geq 3$, the nodes $v_1, v_3, v_{2k}, v_{2k+1}$ induce a $K_{1,3}$ with center v_3 in G , a contradiction to [2]. Hence $k = 2$ and, therefore, $|G_e| = 5$ must hold.

Proof of Claim 3. Suppose $e = xy$ to be an M-anticritical edge of G and $G_e + e \subseteq G + e$ a corresponding monster. G_e is as line graph $K_{1,3}$ -free [2], but $G_e + e$ is not, since the SPGC is true for $K_{1,3}$ -free graphs by [51]. Consider a $K_{1,3} = \{w, x, z; y\}$ with center y in $G_e + e$. The nodes w, y, z and x, y, z induce P_3 's in $G_e + e$ contained in two even holes C_w and C_x , respectively (cf. [30]). Hence we have the hole C_w and the path $P_x = C_x - e$ in G_e ; let $w = w_0, \dots, w_k, z, y$ be the nodes of C_w and $x = x_0, \dots, x_l, z, y$ be the nodes of P_x . Consider the smallest index i such that $N_{C_w}(x_i) \neq \emptyset$ and choose j minimal with $x_i w_j \in E(G_e)$ (note that at least $w_k x_l \in E(G_e)$). Then we have $N_{C_w}(x_{i'}) = \emptyset \forall i' < i$ and $N_{C_w}(x_i) = \{w_j, w_{j+1}\}$ (since G_e is a line graph). Therefore, $P = y w w_1 \dots w_j x_i \dots x$ and $Q = y z w_k \dots w_{j+1} x_i \dots x$ are induced paths where either P or Q has even length, since either $y w w_1 \dots w_j$ or $y z w_k \dots w_{j+1}$ is even (note that $C_w - w_j w_{j+1}$ has an odd number of edges). Due to $xw \notin E(G_e)$ and $xz \notin E(G_e)$, neither P nor Q has length 2. Hence one of them has even length ≥ 4 and $G_e + e$ contains one of the odd holes $P + e$ and $Q + e$ in contradiction to $G_e + e$ Berge by assumption. \square

Remark. Two operators similar to the line operator are the Gallai operator Γ and the wing operator W , where $\Gamma(F)$ and $W(F)$ are defined to take the edges of F as nodes and to join two nodes by an edge if the corresponding edges of F are the endedges of a P_3 and P_4 , respectively. We can certainly determine for all graphs F whether $\Gamma(F)$ and $W(F)$, respectively, contains critical edges. But conversely, we cannot describe all the possible structures in the underlying graph F similar to H-pairs and A-pairs. The reason is that we are able to uniquely reconstruct F from $L(F)$ if $F \neq K_3, K_{1,3}$ by WHITNEY [66], but not from $\Gamma(F)$ or $W(F)$. Here different graphs F have the same image in $\Gamma(F)$ or $W(F)$.

All results with respect to the occurrence of the three kinds of critical edges investigated in this section are summarized in Appendix A.2.

2.4 Perfect Edge Orders

After considering the occurrence of critical edges in several classes of perfect graphs, we turn to Problem 0.2: we are interested whether it is possible, for certain perfect graphs, to successively delete or add edges keeping perfectness until a stable set or a complete graph is reached. For that, we use knowledge from the previous sections.

Let $G = (V, E)$ be a perfect graph. We call a numbering of its edge set $E = \{e_1, \dots, e_m\}$ a **perfect edge order** if, for $G = G_0$, all graphs $G_i := G_{i-1} - e_i$ are perfect for $1 \leq i \leq m$. Clearly, e_i has to be a non-critical edge of G_{i-1} for $1 \leq i \leq m$, and G_m is a stable set. Analogously, we say that a perfect graph G admits a **co-perfect edge order** iff its complement \overline{G} has a perfect edge order. Here we simply use the numbering of the edges of \overline{G} for the non-edges of G and get finally a complete graph.

Note that it does not always suffice to identify non-critical or non-anticritical edges in several kinds of perfect graphs. E.g., we can certainly delete an arbitrary edge of a Meyniel graph keeping perfectness by [29], but we may obtain a slim graph that is not Meyniel.

Hence, we cannot provide perfect edge orders of Meyniel graphs, although the graphs in this class are even characterized that they do not contain any critical edge due to Theorem 2.7. So we mainly have to look for edges such that their deletion or addition preserves the membership to the corresponding *subclass* of perfect graphs.

Starting with the simplest example of graphs having this property, we first mention that *bipartite graphs obviously admit perfect edge orders* and, therefore, co-bipartite graphs co-perfect edge orders. The same is true for line-perfect graphs and every other class of perfect graphs defined by some forbidden *partial* subgraphs. In the case of line-perfect graphs we have even more: that are precisely those perfect graphs such that *every* of its edge orders is perfect.

Theorem 2.12 *A graph is line-perfect iff all edge orders are perfect.*

Proof. (*If*) A line-perfect graph G does not contain any odd cycle of length ≥ 5 as partial subgraph. Obviously, $G - e$ does also not contain any odd cycle of length ≥ 5 $\forall e \in E(G)$ and, therefore, is still line-perfect. Thus every ordering of $E(G)$ is perfect. \diamond

(*Only if*) Assume G to be perfect but not line-perfect. Then G admits a cycle C of odd length at least 5 as partial subgraph. C cannot be chordless and every edge order of G that deletes all chords of C before an edge of C is not perfect. \square

Next, consider a triangulated graph $G = (V, E)$. According to a well-known characterization of triangulated graphs, G admits a simplicial node x , hence no edge e incident to x is critical by Lemma 2.2. The graph $G - e$ is not only still perfect, but even still triangulated, since x is also simplicial in $G - e$ and $G[V - x]$ remains unchanged. Thus G admits a perfect edge order $E = \{e_1, \dots, e_m\}$ where e_i is incident to a simplicial node of G_{i-1} for $1 \leq i \leq m$. Analogously, we can add an edge incident to an antisimplicial node of every complement of a triangulated graph keeping the graph co-triangulated. This yields a co-perfect edge order of triangulated graphs.

It is natural to ask for perfect edge orders in weakly triangulated graphs, defined as a common generalization of triangulated and co-triangulated graphs. Let G be non-complete and weakly triangulated. Then a 2-pair x, y occurs in G due to a characterization of weakly triangulated graphs given by HAYWARD, HOÀNG, and MAFFRAY in [28]. The graph $G + xy$ is not only perfect by Lemma 2.5 but still weakly triangulated by a result of SPINRAD and SRITHARAN [56] where a *co-perfect edge order for weakly triangulated graphs* is presented.

That the class of weakly triangulated graphs is closed under complementation yields particularly a *perfect edge order for every weakly triangulated graph* as studied by HAYWARD in [27]. Clearly, that provides also *perfect and co-perfect edge orders for triangulated graphs* and enables us in addition to establish that *bipartite graphs G admit co-perfect edge orders*: either there are non-adjacent nodes a, b in different color classes of G and $G + ab$ is still bipartite, or G is as complete bipartite graph weakly triangulated.

Nothing is known so far to the author about perfect or co-perfect edge orders in other subclasses of perfect graphs. In contrast to results concerning the occurrence of critical edges, where only local properties are used, we here need structural knowledge about the subclass of perfect graphs under consideration. Only if the deletion or the addition of an edge lets this special structure unchanged, we can ensure the existence of such perfect orders for all graphs in this class.

All results with respect to the occurrence of perfect and co-perfect edge orders investigated in this section are summarized in Appendix A.2.

Chapter 3

Critical and Anticritical Perfectness

This chapter is dedicated to the investigation of perfect graphs where we cannot delete or add one edge without losing perfectness. Problem 0.3 asks for the existence of such graphs: the answer is yes. Section 3.1 introduces perfect graphs such that the deletion and addition of any edge yields an imperfect graph as critically and anticritically perfect graphs, respectively. We study their properties in order to tackle Problem 0.4 whether they admit a structure similarly interesting as that of minimally imperfect graphs. The answer seems to be no: in contrast to minimally imperfect graphs, e.g., critically and anticritically perfect graphs have no symmetry with respect to their maximum cliques and stable sets. However, we establish some common forbidden substructures.

In view of Problem 0.5, in which classes of perfect graphs critically and anticritically perfect graphs appear, we first provide a large range of example graphs. In Section 3.2, graph operations known to preserve perfectness are studied to figure out whether the classes of critically and anticritically perfect graphs are closed under applying them. Section 3.3 characterizes critically and anticritically perfect line graphs, using the characterization of critical and anticritical edges of line graphs in Section 2.3. It is natural to ask for a constructive characterization of the classes of all critically or anticritically perfect graphs: every critically or anticritically perfect graph is either a basic graph (e.g., a line graph or a co-line graph) or can be decomposed into smaller critically or anticritically perfect graphs using the operations investigated in Section 3.2. The attempt to give such a characterization failed: there are critically perfect graphs constructed by taking non-critical graphs as a starting point. However, no critically and anticritically perfect graphs are known so far created without using one of the techniques presented in Section 3.2 and 3.3.

Moreover, we establish in Section 3.4 that critically and anticritically perfect graphs do not admit any perfect forbidden subgraph. Consequently, neither critically nor anticritically perfect graphs can be characterized by forbidden subgraphs, and the classes of critically and anticritically perfect graphs are incomparable to all classes of perfect graphs having a perfect forbidden subgraph. Hence, investigating Problem 0.5, in which classes of perfect graphs critically and anticritically perfect graphs occur, mainly means to consider the intersections of the corresponding classes. (Note that we study, besides some "classical" classes of perfect graphs, only those which seem to be quite large.) This is done in Section 3.4 using results obtained in the Sections 2.3 and 2.4.

3.1 Definitions and Properties

We define a graph to be **critically perfect** if it is perfect and has only critical edges but, for convenience, no isolated node. Let $C\text{-PERF}$ stand for the class of all critically perfect graphs. Analogously, we call G **anticritically perfect** if it is a perfect graph where the addition of an arbitrary edge $e \notin E(G)$ yields an imperfect graph, and no node of G is universal. The class of anticritically perfect graphs is denoted by $A\text{-PERF}$ in the sequel. Clearly, a graph G belongs to $A\text{-PERF}$ iff $\overline{G} \in C\text{-PERF}$. As we will see later in this chapter, critically and anticritically perfect graphs are not rare, taking the “strange” requirements to be satisfied into account. A computer search of HOUGARDY [33] provides a description of small critically perfect graphs.

Lemma 3.1 (HOUGARDY [33]) *There are no critically perfect graphs with fewer than 9 nodes. Precisely 3 and 10 critically perfect graphs on 9 and 10 nodes, respectively, exist.*

Clearly, Lemma 3.1 remains true if “critically perfect” is replaced by “anticritically perfect”. Figure 3.1 shows the three critically perfect graphs on nine nodes. The first graph is self-complementary and, therefore, also anticritical. The other two graphs are not anticritical, but their complements are. Every of the critically perfect graphs with ten nodes contains one of the three smallest examples as subgraph; how to add the corresponding nodes is described in Section 3.3. These ten critically perfect graphs are not anticritical (but, of course, their complements are).

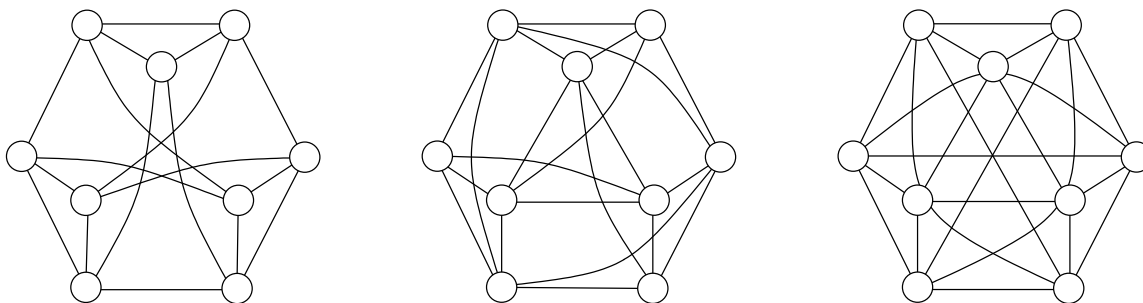


Figure 3.1

Especially, we have obtained $A\text{-PERF} \cap C\text{-PERF} \neq \emptyset$ but $A\text{-PERF} \neq C\text{-PERF}$. Even the strong requirement that the deletion as well as the addition of an arbitrary edge yields an imperfect graph is satisfied by many graphs. Figure 3.2 shows only a few graphs belonging to $A\text{-PERF} \cap C\text{-PERF}$. Note that the graphs in the first and third row are the complements of the graphs contained in the second and fourth row, respectively; moreover, all graphs in the first column are isomorphic to the first graph shown in Figure 3.1. Many further examples of critically and anticritically perfect graphs are presented later in this chapter.

Turning to Problem 0.4 whether critically and anticritically perfect graphs admit a structure similarly interesting as that of minimally imperfect graphs, we study properties of

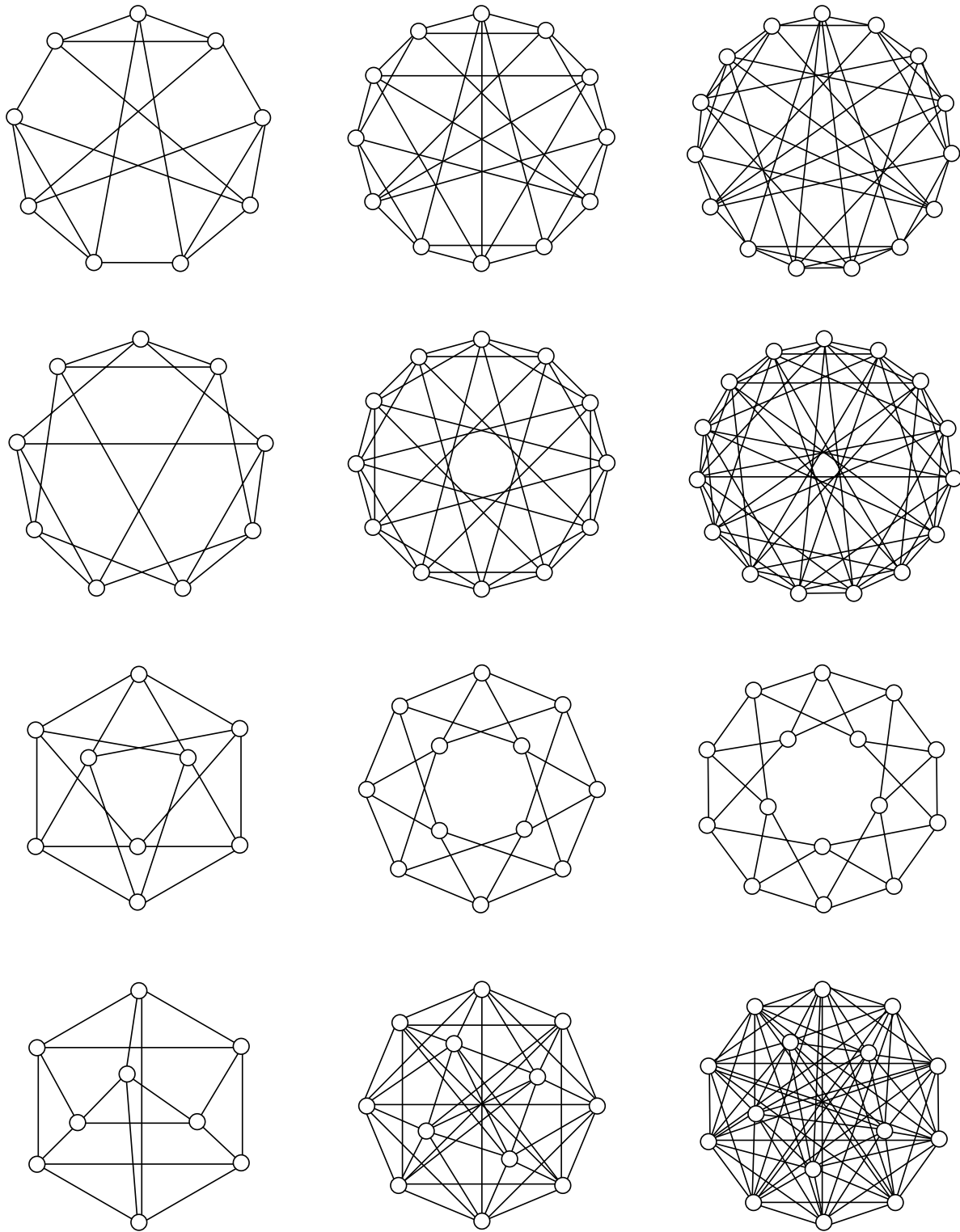


Figure 3.2: Examples for graphs which are critically and anticritically perfect.

a graph caused by its critical and anticritical perfectness. We first observe that, in contrast to Theorem 1.2, critically and anticritically perfect graphs have no symmetry with respect to their maximum cliques and stable sets: while every node of the first graph in Figure 3.1 occurs in the same number of maximum cliques and stable sets, three nodes of the third graph in Figure 3.1 are not contained in any maximum stable set. However, critically and anticritically perfect graphs share some forbidden structures with minimally imperfect graphs which are immediate consequences from the results given in Section 2.1 and Section 2.2. Furthermore, the connectivity, the minimum as well as the maximum degree are considered.

Lemma 3.2 *Let G be a critically perfect graph, then the following conditions are satisfied.*

- (i) *G has no true twins, true antitwins, a weak comparable pair, or a weak odd pair.*
- (ii) *G contains no simplicial node, i.e., every node of G belongs to at least two maximal cliques. There is no node v in G such that $N(v) - v'$ is a clique for one $v' \in N(v)$.*
- (iii) *All edges of G occur in a triangle and in an even hole.*
- (iv) *Every block of G is critically perfect.*
- (v) *G has minimum degree $\delta(G) \geq 4$ and maximum degree $\Delta(G) \leq n - 3$.*

Proof. Consider a graph $G \in C\text{-}PERF$ and let denote $n = |V(G)|$. Then (i) follows by Lemma 2.2 and G does not admit any simplicial node. Thus every node of G must belong to two maximal cliques of G . Condition (iii) is a consequence of Lemma 2.3. To prove the remaining assertion of (ii), suppose that $N(v) - v'$ is a clique for a node v of G and $v' \in N(v)$. Then every common neighbor of v and v' , existing by (iii), would dominate v in contradiction to (i).

In order to show (iv), assume in contrary that G admits a critical edge e such that $G_e \cap B_i \neq \emptyset$ holds for $i = 1, 2$ where $G_e - e \subseteq G - e$ is minimally imperfect and B_1, B_2 are blocks of G . Then G_e would be at most 1-connected in contradiction to $G_e - e$ at least 2-connected. Hence for all critical edges e of G , $G_e \subseteq B_i$ must hold for one block B_i , i.e., B_i itself is critically perfect.

Now, let us come to the minimum degree $\delta(G) \geq 4$. Edges incident to nodes with degree < 3 are not critical according to (iii). Assume that there is $x \in V(G)$ with $N(x) = \{x_1, x_2, x_3\}$. Condition (iii) implies that the edge xx_1 belongs to a triangle. Without loss of generality, let x_1x_2 be an edge, then $N(x) - \{x_3\}$ would be a clique in contradiction to (ii). Finally, turn to the bound for the maximum degree $\Delta(G) \leq n - 3$. A node of degree $n - 1$ would dominate all other nodes in contradiction to (i). Consider $u \in V(G)$ with $d(u) = n - 2$ and $w \notin N(u)$. Since $vw \notin E$, $v \in N(u)$ would imply that u dominates v , we have $N(u) = N(w)$. Let e be an edge incident to u and $G_e - e$ a corresponding minimally imperfect subgraph. Then $w \notin G_e - e$ holds (otherwise w would dominate v in $G_e - e$), and u is contained in exactly one stable set of size two in $G_e - e$, again a contradiction to [50], and condition (v) is satisfied. \square

Lemma 3.3 *Every anticritically perfect graph G fulfills the following conditions.*

- (i) G does not admit false twins, false antitwins, a strict comparable pair, a strict odd pair, or a 2-pair.
- (ii) G has no antisimplicial node, i.e., every node of G belongs to two maximal stable sets. There is no node v in G so that $N_{\overline{G}}(v) - v'$ is a stable set for one $v' \in N_{\overline{G}}(v)$.
- (iii) Every two non-adjacent nodes occur in a stable set of size 3 and in an even antihole.
- (iv) G is 2-connected.
- (v) G has minimum degree $\delta(G) \geq 2$ and maximum degree $\Delta(G) \leq n - 5$.

Proof. Consider a graph $G \in A\text{-}PERF$. Then (i), (ii), (iii), and (v) follows by Lemma 2.5 and 3.2. We only have to show the 2-connectivity of G . If G is disconnected, no edge e linking two nodes in different components of G can be anticritical, since every subgraph of $G + e$ containing both endnodes of e is at most 1-connected. Now assume G to be 1-connected and $x \in B_1, y \in B_2$ to be neighbors of the cutnode c belonging to two blocks B_1 and B_2 of G . Then every subgraph of $G + xy$ containing x and y has one of the sets $\{x, y\}$ or $\{c, x, y\}$ as clique-cutset or one of the nodes x or y as simplicial node. Thus $G + xy$ is perfect and xy not an anticritical edge of G . \square

Remark. The two previous lemmas provide some necessary conditions for a graph to belong to $C\text{-}PERF$ or $A\text{-}PERF$. In particular, that every two adjacent and non-adjacent nodes must admit a common neighbor and non-neighbor in critically and anticritically perfect graphs, respectively, is easy to check. The same is true for the occurrence of twins, comparable pairs, and for the bounds of the minimum and maximum degree.

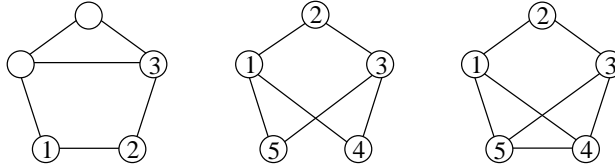
Let us investigate further parameters of graphs in $C\text{-}PERF$ and $A\text{-}PERF$. The following lemma gives bounds of the stability number in critically perfect graphs (hence also for the clique covering number by [42]). Clearly, this implies bounds for the clique number and the chromatic number of graphs in $A\text{-}PERF$.

Lemma 3.4 *Every critically perfect graph G fulfills $3 \leq \alpha(G) \leq |V(G)| - 6$.*

Proof. Consider a critically perfect graph $G = (V, E)$ and let $|V| = n$. In order to prove $3 \leq \alpha(G)$, assume in contrary $\omega(\overline{G}) \leq 2$ for the anticritically perfect graph \overline{G} . Then \overline{G} must be bipartite, let A and B denote its color classes. If there are non-adjacent nodes $a \in A, b \in B$ the graph $G + ab$ is still bipartite. Consequently, \overline{G} is a complete bipartite graph. But now, every two nodes in the same color class form a 2-pair (see also the co-perfect edge order for bipartite graphs given in Section 2.4). Hence, we have either a contradiction to Lemma 2.5 or to Lemma 3.1 and infer $3 \leq \alpha(G)$.

In order to show $\alpha(G) \leq n - 6$, consider a maximum stable set S of G . By $\delta(G) \geq 4$ follows $|S| \leq n - 4$ immediately. If $|S| = n - 4$, we have $N(w) = V - S = \{v_1, v_2, v_3, v_4\} \forall w \in S$. Due to Lemma 3.2.(iii), $w \in S$ and v_1 have to admit a common neighbor in $V - S$, say

v_2 . Since v_1, v_2 must neither form true twins nor a comparable pair by Lemma 3.2.(i), we obtain $v_1v_3, v_2v_4 \in E$ but $v_1v_4, v_2v_3 \notin E$. Furthermore, $v_3v_4 \in E$ follows, since otherwise v_1 and v_2 would dominate v_3 and v_4 , respectively. Hence, $G[V - S] \cong C_4$ holds and it is straightforward to check by Lemma 3.2.(iii) that none of the edges in $G[V - S]$ is critical. Now assume $|S| = n - 5$ and let $V - S = \{v_1, \dots, v_5\}$, $S = \{w_6, \dots, w_n\}$. Consider a node $v_i \in V - S$. We obtain $d_{V-S}(v_i) < 4$, because v_i must not dominate its neighbors in S by Lemma 3.2.(i) (note: v_i has at least one neighbor in S since S is maximal in particular). Moreover, $d_{V-S}(v_i) > 1$ holds, otherwise there is at least one edge incident to v_i which is not contained in a triangle or the only neighbor of v_i in $V - S$ dominates v_i in contradiction to Lemma 3.2. Thus $G[V - S]$ is isomorphic to one of the following three graphs.



Let $N_{V-S}(v_2) = \{v_1, v_3\}$ and $N_{V-S}(v_3) = \{v_2, v_4, v_5\}$ using a suitable numbering of the nodes in $V - S$. The edge v_2v_3 is not contained in a triangle in $G[V - S]$, hence v_2v_3 is H-critical due to Lemma 2.1.(i), v_2 and v_3 possess a common neighbor $w_6 \in S$, and we have to look for an even hole C with $N_C(w_6) = \{v_2, v_3\}$. If $N_C(v_2) = \{v_1, v_3\}$, then $N(w_6) = \{v_2, v_3, v_4, v_5\}$ follows, hence v_3 would dominate w_6 in contradiction to Lemma 3.2.(i). Thus, we get $N_C(v_2) = \{v_3, w_7\}$ and $N(w_7) = \{v_1, v_2, v_4, v_5\}$. Then C has the nodes v_2, w_7, v_4, v_3 where $v_4 \in N(v_1)$ holds (for $v_4 \notin N(v_1)$, the nodes v_1, w_7, v_4, v_3, w_6 induce a C_5 in the first graph shown above). Now consider the edge v_3w_6 . It is H-critical by Lemma 2.1.(i) since v_3w_6 is not contained in any K_4 . The only neighbor of w_6 not adjacent to v_3 is the node v_1 , and Lemma 3.2.(iii) yields an even hole through v_3w_6 that contains v_1 . Thus, there is a common neighbor of v_3 and w_6 not linked to v_1 by an edge. Hence $v_1v_5 \notin E$ follows and $G[V - S]$ is the first graph shown above. But the nodes $v_1, v_3, w_7, w_6, v_4, v_2, v_5$ induce a \overline{C}_7 in G . This final contradiction provides $\alpha(G) < n - 5$. \square

The next lemma is concerned to bounds of the clique number of critically perfect graphs. Clearly, these results give also bounds for the chromatic number and, furthermore, can easily be translated for the stability number and the clique covering number of anticritically perfect graphs.

Lemma 3.5 *Critically perfect graphs G satisfy $3 \leq \omega(G) \leq |V(G)| - 5$.*

Proof. Consider a critically perfect graph $G = (V, E)$ and let $|V| = n$. The lower bound for the clique number is an immediate consequence of Lemma 3.2.(iii). Turning to the upper bound for the clique number, let Q denote a maximum clique of G . $\Delta(G) \leq n - 3$ implies $|Q| \leq n - 2$ (see Lemma 3.2.(v)). For $|Q| = n - 2$, all nodes of Q are simplicial in contradiction to Lemma 3.2.(ii). In the case $|Q| = n - 3$, every node $w \in Q$ is either simplicial or has a neighbor $v \in N(w) - Q$ such that $N(w) - v$ is a clique, again a contradiction to Lemma 3.2.(ii). Hence assume $|Q| = n - 4$, let $V - Q = \{v_1, v_2, v_3, v_4\}$, and

consider a node $w \in Q$. Due to Lemma 3.2.(ii), $d_{V-Q}(w) > 1$ follows and, together with $\Delta(G) \leq n - 3$, we obtain $d_{V-Q}(w) = 2 \forall w \in Q$. Furthermore, two nodes $w, w' \in Q$ with $N_{V-Q}(w) = N_{V-Q}(w')$ are true twins of G in contradiction to Lemma 3.2.(i). Denote $w_{ij} \in Q$ for $N_{V-Q}(w_{ij}) = \{v_i, v_j\}$, then $|Q| \leq 6$ follows by $|V - Q| = 4$ particularly. Every edge $w_{ij}w_{ik}$ has to be critical, hence it is contained in an even hole C by Lemma 3.2.(iii). Obviously, $C \cap Q = \{w_{ij}, w_{ik}\}$ holds. Since both nodes w_{ij} and w_{ik} admit only one common non-neighbor $C = w_{ij}v_jv_kw_{ik}$ follows. Especially, we get $v_jv_k \in E$ from every edge $w_{ij}w_{ik} \in E(Q)$. Hence, $|Q| \geq 5$ due to Lemma 3.1 yields $G[V - Q] \cong K_4$ and G is, therefore, the complement of a bipartite graph. Obviously, $G - v_iw_{ij}$ is still co-bipartite $\forall v_i \in V - Q$ and $\forall w_{ij} \in Q$. The contradiction to G critically perfect finally implies $\omega(G) < n - 4$. \square

Note that the bounds for all the graph parameters presented throughout this section are best possible as is shown by the three graphs in Figure 3.1. All results concerned to the forbidden substructures and graph parameters of graphs in *C-PERF* or *A-PERF* are summarized in Appendix A.2.

3.2 Operations Preserving Critical Perfectness

A graph operation transforms graphs G_1, \dots, G_i into a new graph G . If an operation transfers a common property \mathcal{P} of G_1, \dots, G_i to G , this operation preserves the property \mathcal{P} and the class of all graphs admitting \mathcal{P} is closed under this operation. In order to find graph operations the classes *C-PERF* and *A-PERF* of critically and anticritically perfect graphs, respectively, are closed under applying them, we check known perfectness preserving operations. Since we know *A-PERF* \neq *C-PERF* from the previous section, complementation neither preserves critical nor anticritical perfectness. So let us concentrate on those perfectness preserving operations that take two graphs G_1 and G_2 as arguments. To describe a basic relation between operations preserving critical and anticritical perfectness, let us define the operation $\bar{*}$ with

$$G_1 \bar{*} G_2 = \overline{\overline{G_1} * \overline{G_2}}$$

to be the **antiooperation** of the graph operation $*$ (note $\overline{\overline{G_1} \bar{*} \overline{G_2}} = \overline{\overline{G_1} * \overline{G_2}}$ and $\bar{\bar{*}} = *$). We observe the following relation between a graph operation and its antiooperation.

Proposition 3.6 *A graph operation preserves critical perfectness iff its antiooperation preserves anticritical perfectness.*

Proof. Consider two critically perfect graphs G_1, G_2 . If the graph operation $*$ preserves critical perfectness $\overline{G_1}, \overline{G_2}, \overline{G_1 * G_2} \in A\text{-PERF}$ follows. Since the definition of $\bar{*}$ implies $\overline{G_1 * G_2} = \overline{\overline{G_1} \bar{*} \overline{G_2}}$, so $\bar{*}$ preserves anticritical perfectness. Conversely, let *A-PERF* be closed under applying $\bar{*}$. Then $\overline{G_1} \bar{*} \overline{G_2}$ is anticritically perfect, hence $\overline{G_1 * G_2} \in A\text{-PERF}$. Thus $G_1 * G_2$ is critically perfect, i.e., $*$ preserves critical perfectness. \square

As first example, consider the **disjoint union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The arising graph $G = (V, E)$ with $V = V_1 \cup V_2$, $E = E_1 \cup E_2$ is obviously critically perfect iff G_1 and G_2 are. The complementary graph \overline{G} has the edge set

$$\{xy : x, y \in V_1, xy \notin E_1\} \cup \{xy : x, y \in V_2, xy \notin E_2\} \cup \{xy : x \in V_1, y \in V_2\}.$$

The antioperation of the disjoint union is the **complete join** preserving anticritical perfectness by Proposition 3.6. To figure out whether it also preserves critical perfectness can be done by checking whether $A\text{-}PERF$ is closed under applying the disjoint union. Since anticritically perfect graphs have to be 2-connected following Lemma 3.3.(iv), the disjoint union neither preserves anticritical perfectness, nor the complete join critical perfectness by Proposition 3.6.

The next graph operation under consideration is the **substitution**, introduced in [42]. Let v be a node of a graph $G_1 = (V_1, E_1)$ (see Figure 3.3(a)), then substituting v by another graph $G_2 = (V_2, E_2)$ means to delete v and to join every node of $N_{G_1}(v)$ to every node of G_2 . The arising graph G has node set $V'_1 \cup V_2$ with $V'_1 = V_1 - \{v\}$ and edge set

$$\{xy : x, y \in V'_1, xy \in E_1\} \cup \{xy : x, y \in V_2, xy \in E_2\} \cup \{xy : x \in N_{G_1}(v), y \in V_2\}$$

(see Figure 3.3(b)). Investigating \overline{G} , we point out that

$$\{xy : x, y \in V'_1, xy \notin E_1\} \cup \{xy : x, y \in V_2, xy \notin E_2\} \cup \{xy : x \in N_{\overline{G}_1}(v), y \in V_2\}$$

is the edge set of \overline{G} . Hence, \overline{G} is generated by substituting v in \overline{G}_1 by \overline{G}_2 , i.e., *the substitution equals its antioperation* (the substitution is the only graph operation known to the author with this property). Turning to the investigation of critical edges in G , first note that $G[V_2] \cong G_2$ and $G[V'_1 \cup \{u\}] \cong G_1$ holds $\forall u \in V_2$. Every subgraph G_e with $G_e - e$ minimally imperfect is, consequently, contained in G for every critical edge e of G_i for one $i \in \{1, 2\}$. The following lemma states that a graph G obtained by substitution cannot admit any critical edge not inherited from one of the original graphs.

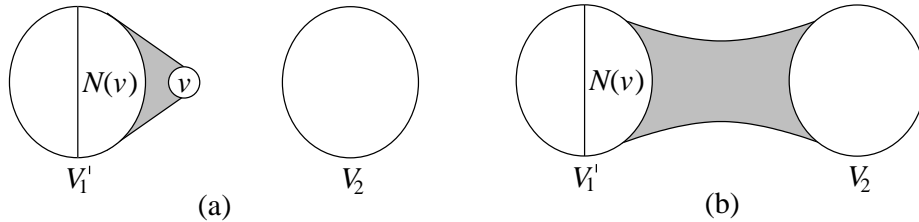


Figure 3.3

Lemma 3.7 *Let G be perfect and the graph generated from $G_1 = (V_1, E_1)$ by substituting a node $v \in V_1$ by $G_2 = (V_2, E_2)$. If $e \in E(G)$ is a critical edge and $G_e \subseteq G$ a subgraph such that $G_e - e$ is minimally imperfect, then either $G_e \subseteq G[V'_1 \cup \{u\}]$ holds for one $u \in V_2$ or $G_e \subseteq G[V_2]$ with $V'_1 = V_1 - \{v_1\}$.*

Proof. Consider the perfect graph G generated from G_1 by substituting a node $v \in V(G_1)$ by another graph G_2 , and let $V'_1 = V(G_1) - \{v\}$, $V_2 = V(G_2)$. If e is intended to be a critical edge of G , there is a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. In order to show the assertion of this lemma, assume in contrary $V(G_e) \cap V'_1 \neq \emptyset$ and $u, u' \in (V(G_e) \cap V_2)$. Then $V(G_e) \cap N_{G_1}(v) \neq \emptyset$ holds, since otherwise $\overline{G_e - e}$ cannot be 2-connected (note: $\overline{G_e - e}$ is even disconnected if both endnodes of e are contained in $N_{G_1}(v)$ or none of them) and, therefore, neither $\overline{G_e - e}$ nor $G_e - e$ is minimally imperfect. But now, $G_e - e$ contains a star-cutset C in contradiction to CHVÁTAL's Star-Cutset Lemma [10]: if u is not an endnode of e , then C consists of $\{u\} \cup (N_{G_1}(v) \cap V(G_e))$, otherwise we have $C = \{u'\} \cup (N_{G_1}(v) \cap V(G_e))$. \square

With the help of the above lemma, the following theorem yields a characterization when a graph obtained by substitution is critically perfect.

Theorem 3.8 *Let G_1 and G_2 be disjoint graphs and $v \in V(G_1)$. The graph obtained from G_1 by substituting v by G_2 is critically perfect iff one of the following conditions is satisfied.*

- (i) G_1 is critically perfect, G_2 is perfect and admits only critical edges.
- (ii) $G_1 - v$ and G_2 are critically perfect, and $N_{G_1}(v) = \emptyset$ holds.

Proof. Consider $G_1 = (V_1, E_1)$, the graph G generated from G_1 by substituting one of its nodes v by another graph $G_2 = (V_2, E_2)$, and let $V'_1 = V_1 - \{v\}$.

(If) G is perfect due to LOVÁSZ's Replacement Lemma [42], since G_1 and G_2 are perfect in both cases. Thus, we have to show that G has only critical edges but no isolated node, if (i) or (ii) is satisfied. Let condition (i) be fulfilled, then G does not admit any isolated node, since G_1 does not have one. The existence of subgraphs $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect can be inherited from G_1 and G_2 for every edge $e = xy$ of G : if $x \in V'_1$, then $G_e \subseteq G[V'_1 \cup \{u\}]$ holds either for an arbitrary node $u \in V_2$ (if $y \in V'_1$) or for $u = y$ otherwise, while $G_e \subseteq G[V_2]$ follows from $x, y \in V_2$. The case that G_1 and G_2 satisfy (ii) is trivial since G is the disjoint union of the critically perfect graphs $G_1 - v$ and G_2 . \diamond

(Only if) We show that G cannot be critically perfect if both conditions (i) and (ii) fail. First, if one of the graphs G_1 and G_2 is imperfect, G is imperfect since $G_i \subseteq G$ holds for $i = 1, 2$. Hence, assume G_1 and G_2 to be perfect. In the case that v is an isolated node of G_1 , condition (i) is violated. The assertion follows since G is the disjoint union of $G_1 - v$ and G_2 . If G_1 does not admit any isolated node, condition (ii) fails. Since (i) is also supposed to be violated, there must be a non-critical edge $e = xy$ in G_1 or G_2 . If e is intended to be a critical edge of G , we have to look for a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect and $G_e \not\subseteq G_i$ holds for $i = 1, 2$. But we can conclude by Lemma 3.7 that there is no such subgraph G_e . Hence, one of the conditions (i) and (ii) is necessary for G to be critically perfect. \square

Consequently, *substitution preserves critical perfectness*. Moreover, *it also preserves anticritical perfectness* by Proposition 3.6, since the substitution is its own antioperation.

Remark. Theorem 3.8 is stronger than the result that $C\text{-PERF}$ is closed under applying substitution, since condition (i) in Theorem 3.8 is a relaxation of $G_1, G_2 \in C\text{-PERF}$. To satisfy condition (i), the graph G_2 can even be a stable set. In other words, condition (i) of Theorem 3.8 states particularly that the **multiplication**, introduced in [42], preserves critical perfectness. The antioperation of the multiplication is the special case that v in G_1 is substituted by a clique G_2 . Hence, both conditions (i) and (ii) are violated, so the multiplication cannot preserve anticritical perfectness.

Now, we turn to the **composition** introduced by BIXBY in [6]. Consider two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and two nodes $v_1 \in V_1, v_2 \in V_2$ (see Figure 3.4(a)). Then composing G_1 and G_2 in v_1 and v_2 means to remove v_1, v_2 and to add an edge between every node of $N_{G_1}(v_1)$ and of $N_{G_2}(v_2)$. The arising graph $G = (V, E)$ has node set $V = V'_1 \cup V'_2$ with $V'_i = V_i - \{v_i\}$ for $i = 1, 2$ (see Figure 3.4(b)) and edge set

$$E = \{xy : x, y \in V'_i, xy \in E_i, i = 1, 2\} \cup \{xy : x \in N_{G_1}(v_1), y \in N_{G_2}(v_2)\}.$$

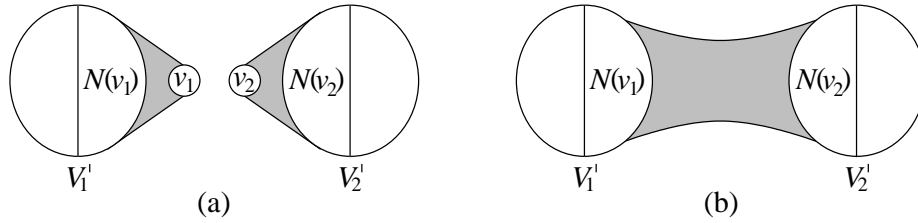


Figure 3.4

Considering \overline{G} , we immediately see that the composition cannot equal its antioperation. Thus, we have to ask separately whether composition and anticomposition preserve critical perfectness. Investigating critical edges in a graph G generated by composition, first note that $G[V'_1 \cup \{u_2\}] \cong G_1$ holds $\forall u_2 \in N_{G_2}(v_2)$ and $G[V'_2 \cup \{u_1\}] \cong G_2$ $\forall u_1 \in N_{G_1}(v_1)$. Every subgraph G_e with $G_e - e$ minimally imperfect is, in analogy to the substitution, contained in G for every critical edge e of G_i for one $i \in \{1, 2\}$. The following lemma is concerned to this relation of critical edges in G and G_i for $i = 1, 2$.

Lemma 3.9 *Let G be a perfect graph generated from disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by composition with respect to $v_i \in V_i$ and let $V'_i = V_i - \{v_i\}$ for $i \in \{1, 2\}$. If $e \in E(G)$ is a critical edge and $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect, then $G_e \subseteq G[V'_1 \cup \{u_2\}]$ holds for one $u_2 \in N_{G_2}(v_2)$ or $G_e \subseteq G[V'_2 \cup \{u_1\}]$ for one $u_1 \in N_{G_1}(v_1)$.*

Remark. If $e \in E(G)$ is a critical edge joining $N_{G_1}(v_1)$ and $N_{G_2}(v_2)$, there may be graphs $G_e \subseteq G[V'_1 \cup \{u_2\}]$ for $u_2 \in N_{G_2}(v_2)$ and $G'_e \subseteq G[V'_2 \cup \{u_1\}]$ for $u_1 \in N_{G_1}(v_1)$, while either $G_e \subseteq G[V'_1 \cup \{u_2\}]$ or $G_e \subseteq G[V'_2 \cup \{u_1\}]$ must hold for all other critical edges e of G . Thus, G cannot admit any critical edge not inherited from the original graphs G_1 and G_2 .

Proof. Suppose G to be the graph composed from perfect graphs G_1 and G_2 in two nodes $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ and denote $V'_i = V(G_i) - \{v_i\}$ for $i = 1, 2$. Let $e = xy$ be a critical edge of G and $G_e \subseteq G$ a subgraph such that $G_e - e$ is minimally imperfect. In order to show the assertion of this lemma, assume in contrary $G_e \not\subseteq G_i$ for $i = 1, 2$. In the case $G_e \cap N_{\overline{G_i}}(v_i) = \emptyset$ for $i = 1$ and $i = 2$, $\overline{G_e}$ is disconnected by $G_e \not\subseteq G_i$ for $i = 1, 2$. Hence $\overline{G_e - e}$ cannot be 2-connected and, therefore, $G_e - e$ not minimally imperfect. Without loss of generality, let $w_1 \in G_e \cap N_{\overline{G_1}}(v_1)$. Since G_e is disconnected by $G_e \not\subseteq G_1$ if $G_e \cap N_{G_2}(v_2) = \emptyset$, there is a node $u_2 \in G_e \cap N_{G_2}(v_2)$. Furthermore, $G[V'_1 \cup \{u_2\}] \cong G_1$ but $G_e \not\subseteq G_1$ provides the existence of a node $w_2 \in G_e \cap V'_2$ with $w_2 \neq u_2$. Now, $C = \{u_2\} \cup (G_e \cap N_{G_1}(v_1))$ is a star-cutset of G_e separating w_1 from w_2 . If $G_e \cap N_{G_1}(v_1) \subseteq N_{G_e - e}(u_2)$, then C is also a star-cutset of $G_e - e$ separating w_1 and w_2 in contradiction to CHVÁTAL's Star-Cutset Lemma [10]. Thus, we conclude $u_2 = x$ and $y \in N_{G_1}(v_1)$. In the case $G_e \cap N_{G_2}(v_2) = \{u_2\}$, C is still a star-cutset of $G_e - e$, hence there is a node $u'_2 \in G_e \cap N_{G_2}(v_2)$ with $u'_2 \neq u_2$. But now, $\{u'_2\} \cup (G_e \cap N_{G_1}(v_1))$ is a star-cutset of $G_e - e$ separating w_1 from u_2 , yielding the final contradiction to [10]. Therefore, $G_e \subseteq G_1$ or $G_e \subseteq G_2$ must hold for every critical edge e of G . \square

With help of the above lemma, the following theorem implies a characterization when a graph obtained by composition is critically perfect.

Theorem 3.10 *Let G_1 and G_2 be disjoint graphs and $v_i \in V(G_i)$ for $i = 1, 2$. The graph G obtained from G_1 and G_2 by composition in v_1 and v_2 is critically perfect iff one of the following conditions is satisfied.*

- (i) G_1 and G_2 are perfect graphs without isolated nodes, every edge xy of G with $x, y \in V(G_i) - \{v_i\}$ is a critical edge of G_i for $i \in \{1, 2\}$, while for every edge xy of G with $x \in N_{G_1}(v_1)$, $y \in N_{G_2}(v_2)$ holds that xv_1 is a critical edge of G_1 or v_2y of G_2 .
- (ii) $G_i - v_i$ is critically perfect for $i = 1, 2$ and $N_{G_1}(v_1) = \emptyset$ or $N_{G_2}(v_2) = \emptyset$.

Proof. Consider $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, the graph G generated from G_1 and G_2 by composing them in $v_1 \in V_1$, $v_2 \in V_2$, and let $V'_i = V_i - \{v_i\}$ for $i = 1, 2$.

(If) G is perfect due to BIXBY [6], since G_1 and G_2 are perfect graphs in both cases. Thus, we have to show that G has only critical edges but no isolated node, if (i) or (ii) is satisfied. In the case that condition (i) is fulfilled, G does not admit any isolated node, since neither G_1 nor G_2 has one. In order to show that every edge $e = xy$ of G is critical, we must find a subgraph $G_e \subseteq G$ with $G_e - e$ minimally imperfect for every $e \in E(G)$. If $x, y \in V'_1$, then $G_e \subseteq G[V'_1 \cup N_{G_2}(v_2)]$, since e is a critical edge of G_1 and $G[V'_1 \cup \{u_2\}] \cong G_1$ holds $\forall u_2 \in N_{G_2}(v_2)$. Analogously, e is critical for $x, y \in V'_2$. Now, suppose $x \in V'_1$ and $y \in V'_2$, then $x \in N_{G_1}(v_1)$ and $y \in N_{G_2}(v_2)$ holds particularly. If xv_1 is a critical edge of G_1 , then $G_e \subseteq G[V'_1 \cup \{y\}]$ follows, since every node in $N_{G_2}(v_2)$ replaces v_1 of G_1 in G . A symmetric argument yields $G_e \subseteq G[V'_2 \cup \{x\}]$ in the case that v_2y is a critical edge of G_2 . On the other hand, if G_1 and G_2 satisfy condition (ii), G is critically perfect since it is the disjoint union of the two critically perfect graphs $G_1 - v_1$ and $G_2 - v_2$. \diamond

(Only if) We show that G cannot be critically perfect if both conditions (i) and (ii) are failed. First, if one of the graphs G_1 and G_2 is imperfect, it is obvious that G is imperfect since $G_i \subseteq G$ holds for $i = 1, 2$. Hence, assume G_1 and G_2 to be perfect. If v_1 is an isolated node of G_1 or v_2 is an isolated node of G_2 , condition (i) is violated. G is the disjoint union of $G_1 - v_1$ and $G_2 - v_2$, hence G cannot be critically perfect if one of the graphs $G_1 - v_1$ and $G_2 - v_2$ is not. If neither G_1 nor G_2 admit any isolated node, condition (ii) fails. Supposing that (i) is also violated, there is either a non-critical edge $e = xy$ with $x, y \in V'_i$ in G_i for one $i \in \{1, 2\}$, or there are two non-critical edges $e_1 = xv_1$ and $e_2 = v_2y$ in G_1 and G_2 , respectively. In both cases, the edge xy of G cannot be critical since otherwise, a subgraph $G_{xy} \subseteq G$ with $G_{xy} - xy$ minimally imperfect had to exist with $G_{xy} \not\subseteq G_i$ for $i = 1, 2$ in contradiction to Lemma 3.9. Hence, one of the conditions (i) and (ii) is necessary for G to be critically perfect. \square

As a consequence, since both conditions (i) and (ii) in Theorem 3.10 are relaxations of $G_1, G_2 \in C\text{-}PERF$, we can conclude that *the composition preserves critical perfectness*. In order to figure out whether the composition also preserves anticritical perfectness, we first establish for graphs G generated by composition which minimally imperfect subgraphs occur in $G + xy$ for non-adjacent nodes x, y of G . Then we state, with help of this lemma, a theorem characterizing when a graph obtained by composition is anticritically perfect.

Lemma 3.11 *Let G be a perfect graph obtained by composition of disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $v_i \in V_i$, $V'_i = V_i - \{v_i\}$ for $i \in \{1, 2\}$, $e = xy \notin E(G)$ an anticritical edge of G , and $G_e \subseteq G$ a subgraph such that $G_e + e$ is minimally imperfect. Then G_e is an even chordless (x, y) -path of length > 2 in G for $x \in N_{\overline{G}_1}(v_1)$ and $y \in N_{\overline{G}_2}(v_2)$. Otherwise, $G_e \subseteq G[V'_1 \cup \{u_2\}]$ holds for one $u_2 \in N_{G_2}(v_2)$ or $G_e \subseteq G[V'_2 \cup \{u_1\}]$ for one $u_1 \in N_{G_1}(v_1)$.*

Proof. Assume G to be a perfect graph arising from disjoint graphs G_1 and G_2 by composition in $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ and write $V'_i = V(G_i) - \{v_i\}$ for $i = 1, 2$. Suppose $e = xy \notin E(G)$ to be an anticritical edge and $G_e \subseteq G$ with $G_e + e$ minimally imperfect. In order to prove the assertion of this lemma, let $G_e \cap V'_i \neq \emptyset$ for $i = 1, 2$. If $G_e \cap N_{G_i}(v_i) = \emptyset$ for one $i \in \{1, 2\}$, then G_e is disconnected and $G_e + e$ can never be 2-connected, hence there is a node $u_i \in G_e \cap N_{G_i}(v_i)$ for $i = 1, 2$. In the case $G_e \cap V'_2 = \{u_2\}$, $G_e \subseteq G_1$ holds by $G_1 \cong G[V'_1 \cup \{u_2\}]$ and, analogously, $G_e \subseteq G_2$ for $G_e \cap V'_1 = \{u_1\}$. So we conclude $|G_e \cap V'_i| \geq 2$ for $i = 1, 2$ and let $u'_2 \in G_e \cap (V'_2 - \{u_2\})$ particularly. Furthermore, \overline{G}_e is disconnected if $G_e \cap N_{\overline{G}_i}(v_i) = \emptyset$ for $i = 1, 2$ holds, hence let $G_e \cap N_{\overline{G}_1}(v_1) \neq \emptyset$ without loss of generality. Now, $\{u_2\} \cup (G_e \cap N_{G_1}(v_1))$ is a star-cutset of G_e separating u'_2 from every node in $G_e \cap N_{\overline{G}_1}(v_1)$. Since $G_e + e$ must not admit any star-cutset due to [10], $x \in N_{\overline{G}_1}(v_1)$ and $y \in V'_2 - \{u_2\}$ follows. Especially, we even have $y \in N_{\overline{G}_2}(v_2)$, otherwise $\{y\} \cup (G_e \cap N_{G_1}(v_1))$ is a star-cutset of $G_e + e$ separating x and u_2 . We are done if there is an induced (x, y) -path P of even length in G_e , since P must have even length ≥ 4 , hence $P + e$ is an odd hole contained in $G_e + e$ and $P = G_e$ follows. Otherwise, x and y would be a strict odd pair in G_e and Lemma 2.5 yields the final contradiction. \square

Theorem 3.12 *Let the composition of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ in $v_i \in V_i$ for $i = 1, 2$ generate the graph G , then G is anticritically perfect iff one of the following conditions is satisfied.*

- (i) G_i is perfect but $G_i + xy$ imperfect $\forall xy \notin E_i$ for $i = 1, 2$. One of the nodes v_i does not occur in an even pair of G_i , $N_{G_i}(v_i) = V_i - \{v_i\}$ holds for at most one $i \in \{1, 2\}$ and, if v_1 is universal in G_1 , there is no universal node in G_2 , and vice versa.
- (ii) $G_i - v_i$ is anticritically perfect and v_i is a universal node of G_i for $i = 1, 2$.

Remark. The properties required in (i) and (ii) necessarily cause the connectivity of G_1 and G_2 . If $N_{G_1}(v_1) = V_1 - \{v_1\}$ holds for precisely one $i \in \{1, 2\}$, condition (i) means that G_2 has to be anticritically perfect, and vice versa.

Proof. Consider disjoint graphs G_1, G_2 , the graph $G = (V, E)$ generated from G_1 and G_2 by composing them in $v_i \in V(G_i)$, and let $V'_i = V(G_i) - \{v_i\}$ for $i = 1, 2$.

(If) G is perfect by [6], since G_1 and G_2 are perfect if one of the above conditions is fulfilled. Thus, we have to show that $G + xy$ is imperfect for every of its non-edges $xy \notin E$ but no universal node exists in G , if (i) or (ii) is satisfied. In the case that condition (ii) is true, G simply corresponds to the complete join of $G_1 - v_1$ and $G_2 - v_2$. Hence G is anticritical since $G_1 - v_1$ and $G_2 - v_2$ are. Turning to the other case, let G_1 and G_2 fulfill condition (i) and $N_{\overline{G}_1}(v_1) \neq \emptyset$. If v_2 is universal in G_2 , then G_1 is anticritical and \overline{G} obtained by substituting v_1 in \overline{G}_1 by $\overline{G_2} - v_2$. Thus, Theorem 3.8.(i) is satisfied by \overline{G}_1 and $\overline{G_2} - v_2$, hence \overline{G} critically perfect and G anticritically perfect. So we can assume $N_{\overline{G}_2}(v_2) \neq \emptyset$ and get $\Delta(G) < |V| - 1$, since no node in $N_{\overline{G}_1}(v_1)$ is linked by an edge to any node in $N_{\overline{G}_2}(v_2)$. In order to show that every edge $e = xy \notin E$ is anticritical, we must find a subgraph $G_e \subseteq G$ such that $G_e + e$ is minimally imperfect $\forall e \notin E$. We simply can inherit G_e from G_1 or G_2 in the cases that $x, y \in V'_i$ for one $i \in \{1, 2\}$, $x \in N_{\overline{G}_1}(v_1)$ and $y \in N_{G_2}(v_2)$, or $x \in N_{G_1}(v_1)$ and $y \in N_{\overline{G}_2}(v_2)$. Thus, we only have to find G_e where $x \in N_{\overline{G}_1}(v_1)$ and $y \in N_{\overline{G}_2}(v_2)$.

According to Lemma 3.11, we look for an even, chordless (x, y) -path in G . Assume in contrary that all chordless (x, y) -paths in G have odd length. We investigate chordless (x, u_1) -paths P_{u_1} with $P_{u_1} \cap N_{G_1}(v_1) = \{u_1\}$ and chordless (u_2, y) -paths P_{u_2} with $P_{u_2} \cap N_{G_2}(v_2) = \{u_2\}$. Then every chordless (x, y) -path P is of the form $P_{u_1} \cup \{u_1 u_2\} \cup P_{u_2}$ and, since every P is supposed to have odd length, all paths P_{u_1} and P_{u_2} must have the same parity. If all these paths were even then x, v_1 and y, v_2 would be strict odd pairs of G_1 and G_2 in contradiction to Lemma 2.5 since xv_1 and yv_2 are anticritical edges of G_1 and G_2 , respectively. Otherwise, all these paths are odd, hence x, v_1 and y, v_2 even pairs of G_1 and G_2 , respectively, in contradiction to (i). (Note that every chordless (x, v_1) -path and (y, v_2) -path in G_i has the form $P_{u_i} \cup \{u_i v_i\}$ with $u_i \in N_{G_i}(v_i)$ for $i = 1$ and $i = 2$, respectively.) Therefore, at least one even, chordless (x, y) -path exists in G and has length ≥ 4 , because x and y do not admit any common neighbor in G . Thus $G + xy$ is also imperfect for every two non-adjacent nodes x, y of G if condition (i) is true. \diamond

(Only if) We show that G cannot be anticritically perfect if both conditions (i) and (ii) are failed. Assume G_1 and G_2 to be perfect, since G is clearly imperfect if one of the graphs G_1 and G_2 is. Condition (i) is violated if $N_{G_i}(v_i) = V_i - \{v_i\}$ holds for $i = 1, 2$. Then the arising graph G equals the complete join of $G_1 - v_1$ and $G_2 - v_2$, hence G cannot

be anticritical if $G_1 - v_1$ or $G_2 - v_2$ is not. Now assume $N_{\overline{G}_1}(v_1) \neq \emptyset$, i.e., let (ii) fail. If v_2 is universal in G_2 , then \overline{G} is the result of substituting v_1 in \overline{G}_1 by $\overline{G}_2 - v_2$. According to Theorem 3.8, G can never be anticritical if G_1 is not or G_2 admits non-adjacent nodes x, y such that $G_2 + xy$ is still perfect. We finally have to investigate the case $N_{\overline{G}_1}(v_1), N_{\overline{G}_2}(v_2) \neq \emptyset$. Since no node in $N_{\overline{G}_1}(v_1)$ is linked to any node in $N_{\overline{G}_2}(v_2)$ by an edge, $\Delta(G) < |V| - 1$ holds and we must find non-adjacent nodes $x, y \in V$ such that $G + xy$ is still perfect. If a non-anticritical edge occurs in G_1 or G_2 , this assertion follows from Lemma 3.11. Hence, suppose x, v_1 and y, v_2 to be even pairs of G_1 and G_2 , respectively. For every chordless (x, y) -path P in G follows $P \cap N_{G_i}(v_i) = \{u_i\}$ for $i = 1, 2$. $P[x, u_1]$ and $P[u_2, y]$ are chordless paths of odd length, since x, v_1 and y, v_2 are even pairs. Thus, every chordless (x, y) -path $P = P[x, u_1] \cup \{u_1 u_2\} \cup P[u_2, y]$ has odd length, i.e., x and y form a strict odd pair in G and xy cannot be an anticritical edge of G due to Lemma 2.5 or Lemma 3.11. This proves $G \notin A\text{-PERF}$ if G_1 and G_2 fail both conditions (i) and (ii). \square

Remark. The above characterization when a graph obtained by composition is anticritically perfect does *not provide that A-PERF is closed under composition*. Theorem 3.12.(i) is not a relaxation of $G_1, G_2 \in A\text{-PERF}$, since one of the nodes v_1, v_2 must not appear in an even pair to insure that the graph obtained by composition with respect to v_1 and v_2 is anticritical. This difficulty is caused by the fact that, in contrast to substitution and composition, the anticomposition creates edges which are originally neither contained in G_1 nor in G_2 . Figure 3.5 shows two anticritically perfect graphs G_1 and G_2 (complements of the third graph in Figure 3.1), which contain even pairs x, v_1 and y, v_2 , respectively (the neighbors of v_1 and v_2 are emphasized by grey filled nodes). Indeed, the graph obtained by composing them with respect to v_1 and v_2 admits the strict odd pair x, y and, therefore, does not belong to $A\text{-PERF}$ by Lemma 2.5.

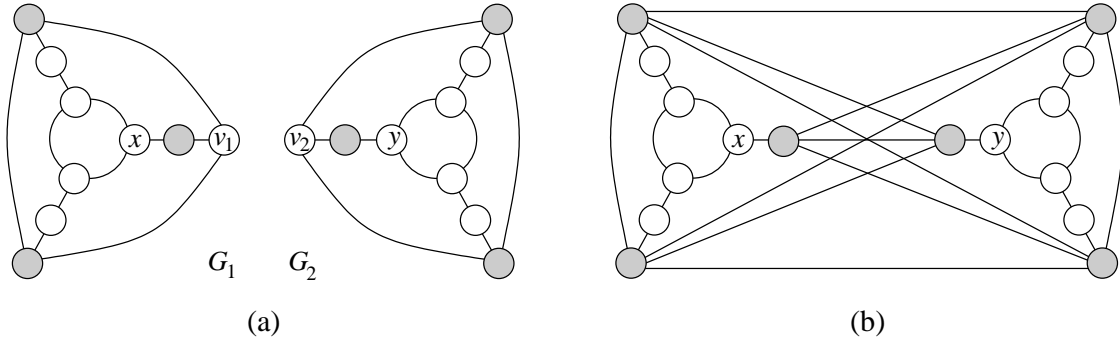


Figure 3.5

It is noticeable that in the previous theorems, compositions reduces to substitution or even to disjoint union in some special cases. So it seems natural to wonder whether there is a graph operation generalizing all of them. CORNUÉJOLS and CUNNINGHAM introduced such a general operation Φ_{jk} in [13]. Consider two disjoint graphs G_1 and G_2 , cliques $Q_i \cup Q'_i \subseteq G_i$ of size $k + j$ with $|Q_i| = k$, $V(Q'_i) = \{v_1^i, \dots, v_j^i\}$, and denote $V(G_i) - V(Q_i \cup Q'_i)$ by V_i' for $i = 1, 2$. Let every node $v \in V_i'$ be adjacent to at most one node in Q'_i and, if $vv_l^i \in E(G_i)$ for one $l \in \{1, \dots, j\}$, let v be adjacent to all nodes in Q_i

for $i = 1, 2$ (see Figure 3.6(a)). Then the graph $G_1 \Phi_{jk} G_2$ is obtained by identifying the cliques Q_1 and Q_2 , joining every neighbor of v_i^1 in V_1' to every neighbor of v_i^2 in V_2' , and deleting Q_1' and Q_2' (see Figure 3.6(b)).

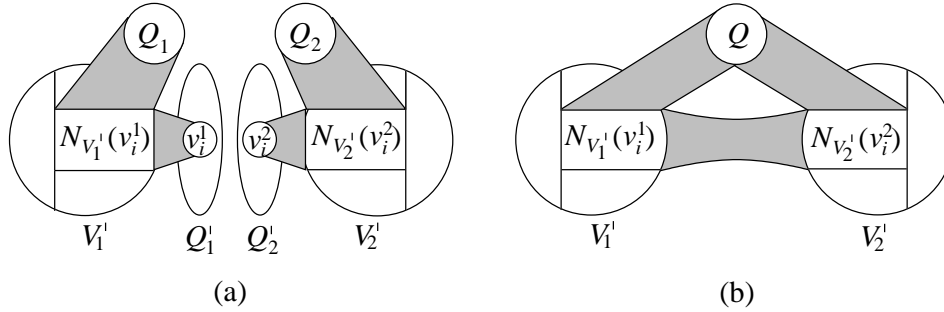


Figure 3.6

Obviously, Φ_{00} is the disjoint union and Φ_{10} the composition. Φ_{1k} is a generalization of the composition known as **amalgam**. BURLET and FONLUPT [8] introduced the amalgam and established that it preserves not only perfectness but even the property to be a Meyniel graph. CORNUÉJOLS and CUNNINGHAM [13] proved that the **2-amalgam** Φ_{2k} is perfectness preserving, while this property seems not to be true for Φ_{jk} with $j > 2$. Actually, an example that Φ_{30} can create an imperfect graph from two perfect graphs is given in [13]. On the other hand, it is well-known that the **clique-identification** Φ_{0k} preserves perfectness for every $k \geq 1$.

Of course, we are interested when *C-PERF* or *A-PERF* is closed under applying one of the perfectness preserving Φ_{jk} -operations. The first step is to figure out if at all a critically or anticritically perfect graph fulfills the conditions that enable applying Φ_{jk} to it. First, let us look at the situation in G_1 and G_2 described in the definition of Φ_{jk} with respect to critical perfectness. $Q_i \cup Q_i'$ is a clique where every neighbor in V_i' of a node in Q_i' has to be adjacent to every node in Q_i , i.e., every node $v \in Q_i$ dominates every node $v_i^i \in Q_i'$. Thus vv_i^i cannot be a critical edge due to Lemma 2.2, so Φ_{jk} can be applied to graphs in *C-PERF* only for $j = 0$ or $k = 0$. Hence, the amalgam Φ_{1k} is applicable to critically perfect graphs only in its special case Φ_{10} , the already investigated composition. Similarly, a 2-amalgamation Φ_{2k} of critically perfect graphs is possible if at all for $k = 0$. But then the nodes $v_1^i, v_2^i \in Q_i'$ would not possess any common neighbor and the edge connecting v_1^i and v_2^i could never be critical by Lemma 2.3.(i). Thus, you cannot apply any 2-amalgam to a graph in *C-PERF*. But Φ_{jk} -operations with $j = 0$, i.e., identifications in k -cliques, are possible for critically perfect graphs.

Consider the graph $G = G_1 \Phi_{0k} G_2$, created from G_1 and G_2 by identification in k -cliques Q_i , let $V_i' = V_i - Q_i$ for $i \in \{1, 2\}$, and denote $Q_i \subseteq G$ just by Q . In order to check when G is in *C-PERF*, we first consider the critical edges in G . Recall for that the convention concerned to these graphs G_e which correspond to an H-critical edge e and yield an odd hole $G_e - e$. Here $G_e - e$ is isomorphic to C_{2k+1} with node set $\{v_1, \dots, v_{2k+1}\}$, where $e = v_1 v_{2k}$ holds and v_1, \dots, v_{2k} induce an even hole, but v_1, v_{2k}, v_{2k+1} a triangle.

Lemma 3.13 *Let $G = G_1 \Phi_{0k} G_2$ be a perfect graph, $e = xy$ a critical edge of G , and $G_e \subseteq G$ a subgraph such that $G_e - e$ is minimally imperfect. Then $G_e \subseteq G[V'_i \cup Q]$ holds for $i \in \{1, 2\}$ or, in the case $x, y \in Q$, G_e has $\{v_1, \dots, v_{2k+1}\}$ as node set, $G_e \cap Q = \{x, y\}$, $x = v_1$ and $y = v_{2k}$ holds; $G_1[v_1, \dots, v_{2k}]$ is an even hole where v_1, v_{2k}, v_{2k+1} induce a triangle in G_2 or vice versa.*

Remark. If e is a critical edge of G which has one endnode in V'_i , then $G_e \subseteq G_i$ follows, since $G_e - e$ must not contain a clique-cutset. In the case $x, y \in Q$, all three possibilities described in the lemma may occur simultaneously.

Proof. Consider a perfect graph G arising by identification of two disjoint graphs G_1 and G_2 in a clique Q and let V'_i denote $V(G_i) - Q$ for $i = 1, 2$. If $e = xy$ is supposed to be a critical edge of G , there must be a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. In order to prove the assertion of this lemma, we can confine ourself to considering the case $G_e \cap V'_i \neq \emptyset$ for $i = 1, 2$. G_e cannot be disconnected, hence $G_e \cap Q \neq \emptyset$ holds and $G_e \cap Q$ is a clique-cutset of G_e . Since $G_e - e$ must not possess any clique-cutset, we have $x, y \in Q$. Now, $(G_e - e) \cap Q$ is a star-cutset of $G_e - e$ if it contains more than two nodes, thus $(G_e - e) \cap Q = \{x, y\}$ follows due to [10]. Therefore, $G_e - e$ has the stable-cutset $\{x, y\}$ and is, consequently, an odd hole by TUCKER [60]. Taking into account that no node in V'_1 is linked by an edge to any node in V'_2 , the assertion follows immediately. \square

With help of this lemma, we can easily characterize when a graph obtained by clique-identification is critically perfect.

Theorem 3.14 *Let G be the graph created by identifying two disjoint graphs G_1 and G_2 in a clique Q , then G is critically perfect iff one of the following conditions is satisfied.*

- (i) $|Q| \geq 2$ holds, G_1 and G_2 are perfect graphs without isolated nodes. For $i = 1, 2$, every edge of G_i with one endnode in $V(G_i) - Q$ is critical in G_i , while $e = xy$ with $x, y \in Q$ is critical in G_1 , in G_2 , or x and y are contained in an even hole in G_1 and have a common neighbor in $V(G_2) - Q$ or vice versa.
- (ii) $|Q| < 2$ holds, G_1 and G_2 are critically perfect.

Proof. Consider disjoint graphs G_1, G_2 , the graph $G = (V, E)$ generated from G_1 and G_2 by identification in a clique Q , and let $V'_i = V(G_i) - Q$ for $i = 1, 2$.

(If) Since G_1 and G_2 are perfect graphs without isolated nodes in both cases, G is perfect with minimum degree $\delta(G) > 0$. We have to check whether all edges $e \in E$ are critical, i.e., whether there is $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. If $|Q| < 2$ and G_1, G_2 are critically perfect, this is obviously true. So let us consider an edge $e = xy$ in the case that condition (i) is satisfied. For $x \in V'_i$, the edge e is critical in G_i , hence $G_e \subseteq G[V'_i \cup Q]$ for $i = 1, 2$. Otherwise, e has both endnodes in Q and $G_e \subseteq G_i$ holds for one $i \in \{1, 2\}$ or e is H-critical in G by (i). \diamond

(Only if) We show that G cannot be critically perfect if both conditions (i) and (ii) are failed. G is imperfect and contains isolated nodes, if one of the graphs G_1 and G_2 is

imperfect and has minimum degree $\delta(G_i) = 0$, respectively, since $G[V'_i \cup Q] \cong G_i$ holds for $i = 1, 2$. Hence, let G_1 and G_2 be perfect graphs without isolated nodes. If $|Q| < 2$ holds, G is obviously not critically perfect if one of the graphs G_1 and G_2 is not. So let $|Q| \geq 2$ but condition (i) fail. Either, there is a non-critical edge $e = xy$ in G_i with $x \in V'_i$ for one $i \in \{1, 2\}$. Then e is not critical in G , since none of the subgraphs G_e described in Lemma 3.13 can occur by $G_e \not\subseteq G_i$ and $x \notin Q$. Otherwise, there is an edge $e = xy$ with both endnodes in Q , which is neither critical in G_1 , nor in G_2 , nor x and y are contained in an even hole in one of the graphs G_1 and G_2 and have a common neighbor in the other graph. Thus, e cannot be a critical edge of G due to Lemma 3.13 again and we have shown that G is not critically perfect if both conditions (i) and (ii) are violated. \square

As a consequence, since both conditions (i) and (ii) in Theorem 3.14 are relaxations of $G_1, G_2 \in C\text{-}PERF$, we have obtained that *the clique-identification Φ_{0k} and its antioperation preserve critical and anticritical perfectness, respectively.*

Now, let us turn to the question whether $A\text{-}PERF$ is closed under one of the operations Φ_{jk} . In contrary to the case of critically perfect graphs, no restrictions are known for applying Φ_{jk} to anticritically perfect graphs. We have already shown that $A\text{-}PERF$ is not closed under the disjoint union Φ_{00} and the composition Φ_{10} . So it seems natural to expect that the same is true for Φ_{1k} with $k > 0$. Indeed, for every $k > 0$ we can easily construct anticritically perfect graphs G'_1 and G'_2 such that $G'_1 \Phi_{1k} G'_2 \notin A\text{-}PERF$ holds. Consider two anticritically perfect graphs G_1 and G_2 with even pairs x, v_1 and y, v_2 , respectively. Duplicating v_1 and v_2 by k -cliques Q_1 and Q_2 with $k > 0$, respectively, i.e., substituting v_i in G_i by a $(k + 1)$ -clique $Q_i \cup \{v_i\}$ for $i = 1$ and $i = 2$, yields $G'_1, G'_2 \in A\text{-}PERF$ by Theorem 3.8 and Proposition 3.6. The amalgam Φ_{1k} can be applied to G'_1 and G'_2 with respect to v_1, Q_1 and v_2, Q_2 . But in $G'_1 \Phi_{1k} G'_2$, the nodes x and y form a strict odd pair, hence xy cannot be an anticritical edge of $G'_1 \Phi_{1k} G'_2$ by Lemma 2.5 and, therefore, $G'_1 \Phi_{1k} G'_2 \notin A\text{-}PERF$ follows.

Figure 3.7 shows two anticritically perfect graphs G'_1 and G'_2 , which are constructed from the graphs G_1 and G_2 in Figure 3.5 by duplicating v_1 and v_2 by the nodes q_1 and q_2 , respectively. Applying the amalgam Φ_{11} to G'_1 and G'_2 with respect to v_1, q_1 and v_2, q_2 yields the graph in Figure 3.7. $G'_1 \Phi_{11} G'_2$ admits the strict odd pair x, y and, therefore, does not belong to $A\text{-}PERF$ by Lemma 2.5.

We can neither prove that $A\text{-}PERF$ is closed under applying Φ_{2k} nor present a counterexample to this statement. So let us close our investigations concerned to Φ_{jk} -operations with the clique-identification Φ_{0k} . In order to check when a graph G , created from G_1 and G_2 by identification in a k -clique Q , is in $A\text{-}PERF$, we first consider the anticritical edges in G .

Lemma 3.15 *Let $G = G_1 \Phi_{0k} G_2$ be a perfect graph, Q its clique-cutset, $V'_i = V(G_i) - Q$ for $i \in \{1, 2\}$, $e = xy \notin E(G)$ an anticritical edge, and $G_e \subseteq G$ a subgraph such that $G_e + e$ is minimally imperfect. Then either $G_e \subseteq G[V'_i \cup Q]$ holds for $i \in \{1, 2\}$ or, if $x \in V'_1$ and $y \in V'_2$, G_e is an induced (x, y) -path of even length ≥ 4 in G .*

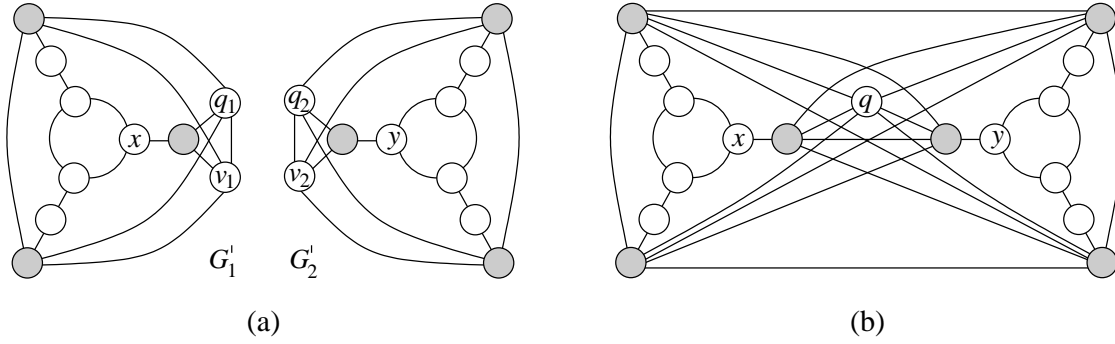


Figure 3.7

Proof. Consider a perfect graph G arising by identification of two disjoint graphs G_1 and G_2 in a clique Q and let V_i' denote $V(G_i) - Q$ for $i = 1, 2$. Since $e = xy$ is supposed to be an anticritical edge of G , there must be a subgraph $G_e \subseteq G$ such that $G_e + e$ is minimally imperfect. In order to prove the assertion of this lemma, consider the case $G_e \cap V_i' \neq \emptyset$ for $i = 1, 2$. G_e cannot be disconnected, hence $G_e \cap Q \neq \emptyset$ holds. For $x \in V_i'$ and $y \in V(G_i)$ for one i , we have $G_e \cap Q$ as clique-cutset of $G_e + e$. Hence assume $x \in V_1'$ and $y \in V_2'$ and denote $Q_x = N_{G_e}(x) \cap Q$, $Q_y = N_{G_e}(y) \cap Q$. First, suppose there is a node $q \in Q_x$. Then $G_e \cap V_1' = \{x\}$ follows, since otherwise $\{x\} \cup (G_e \cap Q)$ would be a star-cutset with center q of $G_e + e$ in contradiction to [10]. But now, x is contained in at most two maximal cliques $\{x\} \cup Q_x$ and $\{x, y\} \cup (Q_x \cap Q_y)$ of $G_e + e$. Hence x appears in at most two maximum cliques of $G_e + e$ and $\omega(G_e + e) = 2$ follows by PADBERG [50] (see Theorem 1.2). Finally, suppose $Q_x = \emptyset$, then x and y cannot admit any common neighbor in $G_e + e$. If there is an even, induced (x, y) -path P in G_e , it must have length ≥ 4 . Then $P + e$ is an odd hole contained in and, therefore, isomorphic to $G_e + e$. Otherwise, if x and y are connected in G_e only by odd induced paths, xy cannot be an anticritical edge due to Lemma 2.5 in contradiction to the assumption. Thus $G_e + e$ is an odd hole in both cases. \square

Now, for every $k > 0$ we can easily construct anticritically perfect graphs G_1' and G_2' such that $G_1' \Phi_{0k} G_2' \notin A\text{-PERF}$ holds. Consider two anticritically perfect graphs G_1 and G_2 . Substituting two nodes v_1 and v_2 in G_1 and G_2 by $(k + 1)$ -cliques $Q_1 \cup \{x\}$ and $Q_2 \cup \{y\}$, respectively, yields $G_1', G_2' \in A\text{-PERF}$ by Theorem 3.8 and Proposition 3.6. But in $G_1' \Phi_{0k} G_2'$, the graph obtained by identifying G_i in Q_i , the nodes x and y form a 2-pair, i.e., there is no induced path of even length ≥ 4 connecting $x \in V(G_1) - Q_1$ and $y \in V(G_2) - Q_2$. Hence xy cannot be an anticritical edge of $G_1' \Phi_{0k} G_2'$ by Lemma 3.15 and $G_1' \Phi_{0k} G_2' \notin A\text{-PERF}$ follows. Note that the identification of two anticritically perfect graphs in cliques which are non-maximal with respect to both of them never yields a graph in $A\text{-PERF}$ due to the above Lemma 3.15. Especially, Φ_{01} does never preserve anticritical perfectness.

Let us finally investigate the **stable set-identification** Φ_s . Since the identification of two graphs in one node is a clique-identification as well as a stable set identification, we already know that the stable set-identification does not preserve anticritical perfectness. Actually, the identification in stable sets of size ≥ 2 may create odd holes, hence it is

even not perfectness preserving in general, but CORNEIL and FONLUPT proved in [12] a sufficient condition for that. Consider two disjoint perfect graphs G_1, G_2 and stable sets $S \subseteq G_i$ for $i = 1, 2$. The graphs G_1 and G_2 satisfy the **strong chain condition** on S , introduced in [12], if G_i is connected, for every pair of nodes $v, v' \in S$, there is an induced path $P \subseteq G_i$ with $P \cap S = \{v, v'\}$, and all induced (v, v') -paths in G_i have the same parity for $i = 1, 2$. (Note that all $v, v' \in S$ either form strict odd pairs or even pairs by this definition.) Then the graph $G = G_1 \Phi_S G_2$ generated from G_1 and G_2 by identification in S is perfect due to [12]. Although we know that *A-PERF* is not closed under stable set-identification, let us start with a lemma concerned to anticritical edges in a perfect graph generated by this operation.

Lemma 3.16 *Let G_1 and G_2 be disjoint perfect graphs, $S \subseteq G_i$ a stable set for $i = 1, 2$ satisfying the strong chain condition, and denote $V'_i = V(G_i) - S$. Consider $G = G_1 \Phi_S G_2$, an anticritical edge $e = xy \notin E(G)$, and $G_e \subseteq G$ a subgraph such that $G_e + e$ is minimally imperfect. Then $G_e \subseteq G[V'_i \cup S]$ holds for one $i \in \{1, 2\}$ or e is A-anticritical.*

Proof. Consider the graph G obtained by identifying G_1 and G_2 in a stable set S and denote $V'_i = V(G_i) - S$ for $i = 1, 2$. Since G_1 and G_2 are supposed to fulfill the strong chain condition on S , G is perfect. If $e = xy \notin E(G)$ is intended to be an anticritical edge, there is a subgraph $G_e \subseteq G$ such that $G_e + e$ is minimally imperfect. We can confine ourself to considering subgraphs G_e with $G_e \cap V'_i \neq \emptyset$ for $i = 1, 2$ again and show that $G_e + e$ must be an odd hole in every case.

If $x \in V'_i$ for one $i \in \{1, 2\}$ and $y \in V_i$, then $G_e \cap S$ is a stable-cutset of $G_e + e$ separating $G_e \cap V'_1$ from $G_e \cap V'_2$, hence $G_e + e$ is an odd hole due to TUCKER [60]. For $x \in V'_1$ and $y \in V'_2$, consider a maximum clique of $G_e + e$ containing x and y (such a clique exists by Lemma 2.4.(iii)). If at all x and y possess common neighbors, they belong to S and $\omega(G_e + e) \leq 3$ follows. $G_e + e \not\cong \overline{C}_7$ and TUCKER [59] imply that $G_e + e$ is an odd hole.

So let us finally turn to the case $x, y \in S$. The nodes x and y would form a strict odd pair of G in contradiction to Lemma 2.5, if all induced (x, y) -paths were odd. Thus, the strong chain condition yields that all induced paths of G connecting two nodes of S must have even length. In particular, if G_e contains an induced (x, y) -path of even length ≥ 4 , $G_e + e$ must be isomorphic to an odd hole. Hence suppose all induced (x, y) -paths to have length 2. Furthermore, there is $z \in G_e \cap (S - \{x, y\})$, because x and y would be a clique-cutset of $G_e + e$ otherwise. Now x, z and y, z are even pairs of G_e . Since $G_e + e$ must not possess any even pair due to [46], there is an induced (x, z) -path P in G_e which neither admits y nor a node adjacent to y and, analogously, an induced (y, z) -path P' in G_e which neither contains x nor one of its neighbors. If there were an edge connecting P and P' in G_e , consider the first node $v \in P$ (seen from x) linked to a node in P' and let w be the first node in $N_{P'}(v)$ (seen from y). Then $P[x, v] \cup \{vw\} \cup P'[w, y]$ is an induced (x, y) -path in G_e and we would yield a contradiction either since every induced (x, y) -path in G_e has length 2 or due to $N_{G_e}(y) \cap P = \emptyset$ and $N_{G_e}(x) \cap P' = \emptyset$. Hence, no edge between the induced paths P and P' exists in G_e . Consequently, $P \cup \{xy\} \cup P'$ is an odd hole in $G_e + e$ and is, therefore, isomorphic to $G_e + e$. I.e., e is an A-anticritical edge. \square

According to the above lemma, a graph $G = G_1 \Phi_S G_2$ can be anticritically perfect only if all of its anticritical edges xy with $x \in V'_i$, $y \in V(G_i)$ for one $i \in \{1, 2\}$ are inherited from G_i and x, y are connected by an induced path of even length ≥ 4 if $x \in V'_1$ and $y \in V'_2$ holds. For $x, y \in S$ and $|S| \geq 3$, xy is not required to be an anticritical edge of G_i for one $i \in \{1, 2\}$, since there is an induced (x, y) -path of even length ≥ 4 in G running through a node $z \in S - \{x, y\}$, compound from an even (x, z) -path in G_1 and an even (y, z) -path in G_2 existing due to the strong chain condition. (Note that $x, y \in S$ must not form a strict odd pair if G is intended to be anticritically perfect due to Lemma 2.5.)

But this fact does not provide the opportunity to create an anticritically perfect graph G by substituting nodes v_1 and v_2 in two anticritically perfect graphs G_1 and G_2 , respectively, by stable sets S of the same size ≥ 3 and identifying the arising graphs in these stable sets. Then every edge added between two nodes in S is certainly anticritical, but no edge xy with $x \in N_{G_1}(v_1)$ and $y \in N_{G_2}(v_2)$. More generally, there must not be nodes $x \in G_1$ and $y \in G_2$ that are adjacent to all nodes in S if $G = G_1 \Phi_S G_2$ is intended to be anticritically perfect. In this case, x and y were a 2-pair in G , hence xy not an anticritical edge of G due to Lemma 2.5 or 3.16. Consider, e.g., the two anticritically perfect graphs G_1 and G_2 shown in Figure 3.5. The nodes x, v_1 and y, v_2 form even pairs in G_1 and G_2 , respectively, and the strong chain condition is satisfied for $\{x, v_1\}$ and $\{y, v_2\}$, respectively. But identifying G_1 and G_2 in these even pairs does not yield an anticritically perfect graph G , since there are two nodes in G_1 and G_2 which are adjacent to $\{x, v_1\}$ and $\{y, v_2\}$, respectively. Indeed, to insure that xy is an anticritical edge of G in the case $x \in V'_1$ and $y \in V'_2$ is the difficulty.

Now, let us turn to the relation of stable set-identification and critically perfect graphs. We again provide a lemma concerned to critical edges in a graph generated by Φ_S .

Lemma 3.17 *Let $G = G_1 \Phi_S G_2$ arise from disjoint perfect graphs G_1 and G_2 , $S \subseteq G_i$ be a stable set for $i = 1, 2$ satisfying the strong chain condition, $e = xy$ a critical edge of G , and $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. Then $G_e \subseteq G[V'_i \cup S]$ holds for one $i \in \{1, 2\}$ and $V'_i = V(G_i) - S$ or e is H -critical.*

Proof. Consider the graph G obtained by identifying G_1 and G_2 in a stable set S and denote $V'_i = V(G_i) - S$ for $i = 1, 2$. G is perfect if G_1 and G_2 fulfill the strong chain condition on S . If $e = xy \in E(G)$ is intended to be a critical edge, there is a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. In order to prove the assertion of this lemma, it suffices to consider the case $G_e \cap V'_i \neq \emptyset$ for $i = 1, 2$ since $G[V'_i \cup S] \cong G_i$ holds. $G_e \cap S$ is non-empty (otherwise G_e is disconnected). Hence $G_e \cap S$ is a stable-cutset of $G_e - e$ separating $G_e \cap V'_1$ from $G_e \cap V'_2$ and $G_e - e$ is an odd hole by TUCKER [60]. \square

We see that an edge of $G = G_1 \Phi_S G_2$ can be critical even it is not critical in G_1 or G_2 . But the effort to characterize all the cases creating critical edges in G seems to be hopeless. Too many possibilities for such subgraphs $G_e \subseteq G$ with $G_e \cap V'_i \neq \emptyset$ for $i = 1, 2$ may occur. Figure 3.8 lists only the cases for $G_e - e = C_7$ (here the grey ellipses mark $G_e \cap S$ and the bold lines the critical edges e), while subgraphs $G_e - e$ isomorphic to odd holes

C_{2k+1} appear in a much larger abundance for $k \geq 4$. But we can, of course, insure that identifying two critically perfect graphs in a stable set yields a critically perfect graph if it is perfect at all. This is immediately seen since we can inherit the critical edges of $G_1 \Phi_S G_2$ from G_1 and G_2 (recall that every subgraph-identification does not create any edge which is neither contained in G_1 nor in G_2).

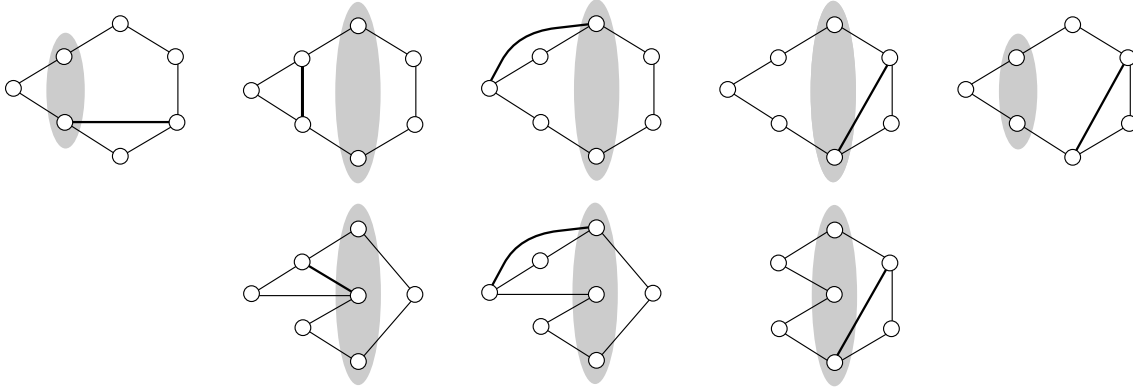


Figure 3.8

Finally, two observations are worth noting. On the one hand, Lemma 3.7, 3.9, and 3.13 do not only enable us to characterize when a graph G obtained by the corresponding graph operations substitution, composition, and clique-identification is critically perfect. Moreover, Lemma 3.7, 3.9, 3.13, and 3.17 provide that all critical edges of G which are not inherited from the underlying graphs G_1 and G_2 are H-critical. To obtain analogous results for anticritical edges is more difficult, because the corresponding antioperations create edges which are originally neither contained in G_1 nor in G_2 . But Lemma 3.11, 3.15, and 3.16 state similar facts. Hence, we can even insure that, except for amalgam and 2-amalgam, *none of the investigated graph operations creates any M-critical or M-anticritical edge*. In other words, if the graphs G_1 and G_2 are free of M-critical or M-anticritical edges, so is the arising graph $G_1 * G_2$ if $*$ stands for the substitution, composition, clique-, or stable set-identification. Looking at the Strong Perfect Graph Conjecture [3], this result is welcome.

On the other hand, Theorem 3.8, 3.10, 3.14, and Lemma 3.17 provide the opportunity to *create* critically perfect graphs from graphs G_1 and G_2 that do not necessarily belong to $C\text{-}PERF$. Most notably, the clique-identification enables us to construct critically perfect graphs from graphs $G_1, G_2 \notin C\text{-}PERF$. Consider the graph shown in Figure 3.9 (it is the line graph of the bipartite graph shown in Figure 3.10 where the edges ac and ad are added). The graph admits precisely one non-critical edge, emphasized by a bold line in the picture. Taking two copies of this graph and identifying them in the endnodes of their only non-critical edge, i.e., in the grey nodes, yields a critically perfect graph due to Theorem 3.14. The resulting graph is even *minimally* critical perfect, since it does not contain any critically perfect graph as proper subgraph.

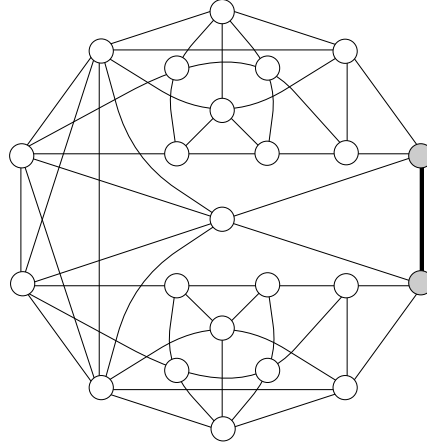


Figure 3.9

All results when $C\text{-PERF}$ or $A\text{-PERF}$ is closed under applying one of the operations investigated in this section are summarized in Appendix A.2.

3.3 Critically and Anticritically Perfect Line Graphs

We start with the investigation of critically perfect line graphs. For this, recall the structure of H-pairs in the underlying graphs F introduced on page 23 in Section 2.3, which guarantees that edges of $L(F)$ are critical. Two incident edges x and y are defined to be an H-pair, if they are contained in a $K_{1,3}$ with edges $x, y, e_{x,y}$ and in an even cycle $C_{x,y}$ that admits only one endnode of the edge $e_{x,y}$ (see Figure 2.2(a)). By Theorem 2.10, an edge xy of $L(F)$ is critical iff x and y form an H-pair in F . Hence, if $L(F)$ is intended to be critically perfect, *every* pair of incident edges in F must form an H-pair. We call graphs with this property **H-graphs** if they are connected and admit more than one edge. Recall that every disconnected critically perfect graph only admits critically perfect components (since it does not admit any isolated node by definition), i.e., it can be created from connected critically perfect graphs by taking their disjoint union. Hence, it suffices to consider connected graphs. The next theorem yields a characterization of critically perfect line graphs.

Theorem 3.18 *Let G be the line graph of F . G is critically perfect if and only if F is a bipartite H-graph.*

Proof. (*If*) Let F be a bipartite H-graph, then its line graph G is perfect due to KÖNIG [39]. Every edge of G is critical by Theorem 2.10, and G is connected since F is, thus G is critically perfect. \diamond

(*Only if*) Let G be critically perfect and the line graph of F . We know from the characterization of critical edges in G by Theorem 2.10 that every two incident edges of F

form an H-pair, consequently, F is an H-graph. Furthermore, F is line-perfect since its line graph G is perfect. Thus, in order to prove that F is bipartite, we only have to show that the occurrence of a triangle in F violates its line-perfectness. Suppose F to admit a triangle with nodes a, b, c . Since F is an H-graph, ab and ac form an H-pair, i.e., there is an even cycle $C_{ab,ac}$, running through them, and an edge $e_{ab,ac}$ with $e_{ab,ac} \cap C_{ab,ac} = \{a\}$. We obtain $|C_{ab,ac}| = 4$, since $C_{ab,ac} - a$ together with the edge bc would form an odd cycle of length ≥ 5 , otherwise. Let a, b, c, d be the nodes of $C_{ab,ac}$ and $e_{ab,ac} = ae$ with $e \notin \{b, c, d\}$. Since ab, ae form an H-pair, too, they are contained in an even cycle $C_{ab,ae}$ and we get an odd cycle $C \subseteq F$ of length ≥ 5 in any case: First, assuming $c \notin C_{ab,ae}$, we yield C from $C_{ab,ae}$ replacing the edge ab by ac and cb . Supposing $c \in C_{ab,ae}$ in contrary, then we have, as part of $C_{ab,ae}$, a path $P = a, e, \dots, c$ with $b \notin P$. If P has odd length, then C consists of P and the edges ab and bc . In the case that P has even length > 2 , we obtain C from P and the edge ac between its endnodes. Otherwise, $P = a, e, c$ and $C = a, e, c, d, b$ holds. Since the existence of an odd cycle in F contradicts the perfectness of G , we conclude that F is bipartite. \square

Finding examples of critically perfect line graphs means looking for bipartite H-graphs according to the above theorem. In order to get familiar with the notion of H-graphs, we investigate some properties and examples. Let $F = (A \cup B, E)$ be a bipartite H-graph with color classes A, B and, without loss of generality, let $|A| \leq |B|$ hold in the sequel. Directly from the definition of H-graphs, $\delta(F) \geq 3$ follows and, consequently, $3 \leq |A|$ and $3|B| \leq |E|$. Furthermore, since F is connected and a cycle runs through every two of its incident edges, an H-graph must be 2-connected. Figure 3.10 shows a 3-regular, 2-connected, bipartite graph. Since the node b appears on every cycle containing the other two neighbors of a , it is not an H-graph. Thus, minimum degree ≥ 3 and 2-connectivity are necessary, but not sufficient for a bipartite graph to be an H-graph. A sufficient, but not necessary condition is provided by the next lemma.

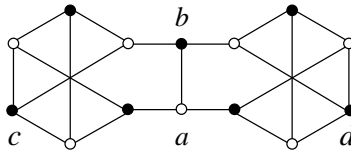


Figure 3.10

Lemma 3.19 *Every 3-connected, bipartite graph is an H-graph.*

Proof. Consider a 3-connected, bipartite graph $F = (A \cup B, E)$ and two arbitrary incident edges ab_1, ab_2 of F with $a \in A$ and $b_1, b_2 \in B$. We show that ab_1 and ab_2 form an H-pair in F . Since F is 3-connected, $\delta(F) \geq 3$ implies that there is a node $b_3 \in N(a) - \{b_1, b_2\}$ and $F - \{a, b_3\}$ is still connected. In particular, b_1 and b_2 are linked by a path P in $F - \{a, b_3\}$. Hence, we obtain $C_{ab_1,ab_2} = P \cup \{b_2a, ab_1\}$ and $e_{ab_1,ab_2} = ab_3$, i.e., ab_1 and ab_2 form an H-pair in F . \square

For example, every complete bipartite graph $K_{a,b}$ with $3 \leq a, b$ is 3-connected, hence an H-graph by the previous lemma. On the other hand, every bipartite H-graph $F = (A \cup B, E)$ with $3 = |A|$ has to be a complete bipartite graph, since every node in B must be adjacent to all three nodes in A by $3 \leq \delta(F)$. Thus, we have obtained the following characterization of bipartite H-graphs with only three nodes in one color class.

Lemma 3.20 *Let $F = (A \cup B, E)$ be a bipartite graph, $|A| = 3$, and $|B| = b \geq 3$. Then F is an H-graph iff it is isomorphic to $K_{3,b}$.*

Indeed, $K_{3,3}$ is the smallest bipartite H-graph and the only one with $|B| = 3$. Its line graph is the first of the three smallest critically perfect graphs on nine nodes shown in Figure 3.1, the first column of Figure 3.2 contains four different representations of this graph, too. Furthermore, the second smallest critically perfect line graph must admit 12 nodes, since $|E| \geq 3|B| = 12$ follows for every bipartite H-graph with $|B| = 4$, i.e., there cannot be any critically perfect line graph on 10 or 11 nodes. The line graph of $K_{3,4}$ admits 12 nodes and is shown as the middle graph in the first row of Figure 3.2.

In view of Lemma 3.20, we are mainly interested in non-complete, bipartite H-graphs with $|A| \geq 4$ in the sequel. In order to find the smallest examples of such graphs, let us consider graphs with $|B| = 4, 5$. Every bipartite graph F with $|A| = |B| = 4$ and $\delta(F) \geq 3$ must satisfy $3|B| = 12 \leq |E| \leq 16 = 4|B|$. The only such graph with least number of edges is the 3-regular graph H_1 in Figure 3.11. H_1 is an H-graph since it is 3-connected, its line graph is the middle graph in the third row of Figure 3.2. For $|A| = |B| = 4$ and each number of edges $13 \leq |E| \leq 16$, there is precisely one bipartite graph, arising from H_1 by adding a respective number of edges. These graphs are H-graphs in each case, because they are 3-connected since H_1 is.

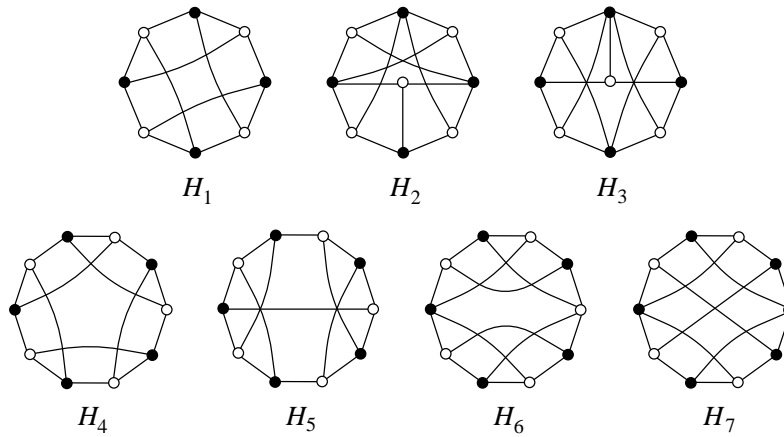


Figure 3.11: Examples of bipartite H-graphs.

Every bipartite graph with $|B| = 5$ and minimum degree 3 must admit at least 15 edges. The line graph of $K_{3,5}$ admits 15 nodes and is the last graph shown in the first row of Figure 3.2. For $|A| = 4$, the graphs H_2 and H_3 in Figure 3.11 are the only graphs with

minimal number of edges. H_2 and H_3 are H-graphs since they are 3-connected. Again, all bipartite H-graphs with $|A| = 4$, $|B| = 5$, and $|E| > 15$ arise from H_2 and H_3 by successively adding of edges, until $K_{4,5}$ is reached. For $|A| = 5$, there are two 3-regular, 3-connected graphs H_4, H_5 shown in Figure 3.11. The line graph of H_4 is presented as last graph in the third row of Figure 3.2. With $|E| = 16$, we find two further H-graphs H_6 and H_7 which do not contain any H-graph as proper partial subgraph. All other bipartite H-graphs with $|A| = |B| = 5$ arise from H_4, \dots, H_7 by successively adding edges. Note that H_6 is not 3-connected, but we show in the next lemma that the addition of an edge keeping the graph bipartite preserves the property of being an H-graph in general.

Lemma 3.21 *Let $F = (A \cup B, E)$ be a bipartite H-graph and $a \in A, b \in B$ non-adjacent, then $F + ab$ is an H-graph.*

Proof. Consider a non-complete, bipartite H-graph $F = (A \cup B, E)$ and two nodes $a \in A, b \in B$ with $ab \notin E$. We show that ab, ab' is an H-pair in $F + ab$ for an arbitrary node $b' \in N_F(a)$ (a symmetric argument applied to all neighbors of b in F yields then the assertion of the lemma). If there is an a -free (b, b') -path P in F and a node $b'' \in N_F(a) - b'$ with $b'' \notin P$, we are ready with $C_{ab,ab'} = P \cup \{b'a, ab\}$ and $e_{ab,ab'} = ab''$. Otherwise, all nodes of $N_F(a) - b'$ belong to every a -free (b, b') -path in F , a contradiction to F 2-connected because F cannot admit two open-disjoint (b, b') -paths in that case: one of the paths contains the whole set $N_F(a) - b'$ (note $|N_F(a) - b'| \geq 2$ by $\delta(F) \geq 3$), but the other path contains a and, therefore, also a neighbor of a in $N_F(a) - b'$. \square

Thus, in order to find all bipartite H-graphs, it suffices to know these H-graphs F such that $F - e$ is not an H-graph $\forall e \in E(F)$. Let us call H-graphs with this property *critical*. H-graphs that do not possess another H-graph as proper partial subgraph must be critical. Hence, the graphs H_1, \dots, H_7 are examples of critical H-graphs. The first graph shown in Figure 3.12 is a further graph with this property. Even all complete bipartite graphs $K_{3,b}$ with $b \geq 3$ are critical, since $\delta(K_{3,b} - e) = 2$ holds $\forall e \in E(K_{3,b})$, although we have $K_{3,3} \subset K_{3,b}$ for $b > 3$.

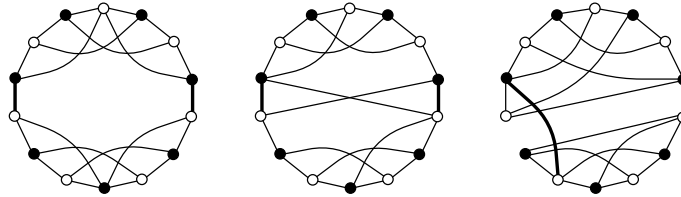


Figure 3.12

It is immediately seen that every H-graph, all edges of which are incident to a node of degree 3, is critical. Unfortunately, this condition is only sufficient, it does not provide a characterization of critical H-graphs. For example, consider the 3-regular graph F shown in Figure 3.10 again. Add an arbitrary edge $a'b'$ to F , keeping the graph bipartite, where a' and b' appear in different components of $F - \{a, b\}$. It is easy to check that the resulting graph F' is an H-graph, one graph constructed in this way is the second graph

shown in Figure 3.12. Furthermore, F' is critical since $\delta(F' - e) = 2$ holds $\forall e \neq a'b'$, and $F' - a'b' = F$ is known to be not an H-graph. Hence, F' is a critical H-graph and has two adjacent nodes of degree > 3 , namely a' and b' . The third graph in Figure 3.12 is a critical H-graph as well: it contains even two edges e, e' with endnodes of degree > 3 (emphasized by bold lines in the picture). But removing e or e' destroys the 2-connectivity, and deleting an arbitrary edge distinct from e and e' yields a node of degree two. Therefore, non-critical H-graphs cannot be characterized as H-graphs admitting an edge, for which both of its endnodes have degree > 3 .

Now, let us turn to anticritically perfect line graphs. Analogously to the previous investigations of line graphs in *C-PERF*, we make use of the structure of A-pairs in the underlying graphs F , introduced on page 23 in Section 2.3. Recall that two non-incident edges x and y form an A-pair if they are the endedges of an odd path $P_{x,y}$ with length at least five (see Figure 2.2(c)). Theorem 2.11 shows that xy is an anticritical edge of $L(F)$ iff x, y form an A-pair in F . Hence, in order to describe anticritically perfect line graphs, we are interested in these graphs F where *every* pair of non-incident edges is an A-pair. We call graphs F with this property **A-graphs** if they are connected and admit non-incident edges. Note that this definition excludes, besides the occurrence of isolated nodes in F , only the cases $F = K_3$ and $F = K_{1,k}$ for some $k \geq 1$. The next theorem yields a characterization of A-graphs whose line graphs are anticritically perfect.

Theorem 3.22 *The line graph of a graph F is anticritically perfect if and only if F is a bipartite A-graph.*

Proof. (*If*) Let F be a bipartite A-graph, then its line graph G is perfect by KÖNIG [39]. Every edge $e \notin E(G)$ is anticritical by Theorem 2.11 and $\Delta(G) < |V(G)| - 1$ holds by the definition of A-graphs, thus G is anticritically perfect. \diamond

(*Only if*) Let G be anticritically perfect and the line graph of F . We know from the characterization of anticritical edges in G by Theorem 2.11 that every two non-incident edges of F form an A-pair, consequently, F is an A-graph. Furthermore, F is line-perfect since its line graph G is perfect. Thus, in order to prove that F is bipartite, we only have to show that the occurrence of a triangle in F violates its line-perfectness. Suppose F to contain a triangle with nodes a, b, c . There must be a further node d , adjacent to a node of the triangle by the definition of A-graphs, let c be adjacent to d . Then ab and cd are non-incident edges of F , i.e., they have to be the endedges of an odd path $P_{ab,cd}$ of length at least 5. We get an odd cycle $C \subseteq F$ of length ≥ 5 in any case: Without loss of generality, let a be one endnode of $P_{ab,cd}$. If $P_{ab,cd} = a, b, \dots, c, d$ holds, F admits the cycle $C = P_{ab,cd}[a, c] \cup \{ac\}$. Otherwise, i.e., in the case $P_{ab,cd} = a, b, \dots, d, c$, we obtain $C = P_{ab,cd}[b, c] \cup \{bc\}$. The existence of an odd cycle in F yields an odd hole in G contradicting its perfectness, hence F has to be a bipartite graph. \square

According to the above theorem, complements of line graphs are obviously critically perfect if and only if the underlying graph is a bipartite A-graph. Hence we are only interested in analyzing bipartite A-graphs $F = (A \cup B, E)$, analogously to the case of H-graphs.

Without loss of generality, let $|A| \leq |B|$ hold again. By the definition of A-graphs, F contains a pair of non-incident edges and an odd path of length ≥ 5 . Thus, we obtain $3 \leq |A|$ again. The smallest example of an A-graph is the graph $A_1 = K_{3,3}$ shown in Figure 3.13. Recall that $K_{3,3}$ is also the smallest H-graph and $L(K_{3,3}) = \overline{L(K_{3,3})}$ holds.

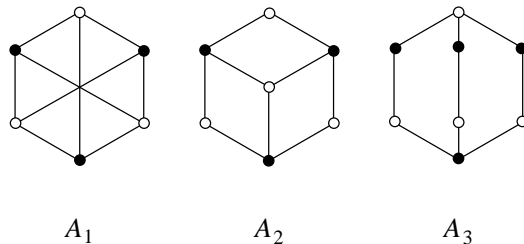


Figure 3.13

Many H-graphs are A-graphs as well, e.g., all complete bipartite graphs with $|A| \geq 3$ and the graphs in Figure 3.11. But the graphs in Figure 3.12 show the existence of H-graphs which are not A-graphs. The non-incident edges, emphasized by bold lines in the picture, do not form an A-pair. Actually, A-graphs occur in a larger abundance than H-graphs since a graph has to satisfy weaker requirements to be an A-graph than to be an H-graph. A-graphs have not necessarily minimum degree 3, see the A-graphs A_2 and A_3 in Figure 3.13. Note that $\overline{L(A_1)}$, $\overline{L(A_2)}$, and $\overline{L(A_3)}$ shown in Figure 3.1 are the only critically perfect graphs on nine nodes. Figure 3.14 contains five A-graphs with 10 edges, arising from A_1 , A_2 , or A_3 by the addition of one edge and, in three cases, the addition of one node.

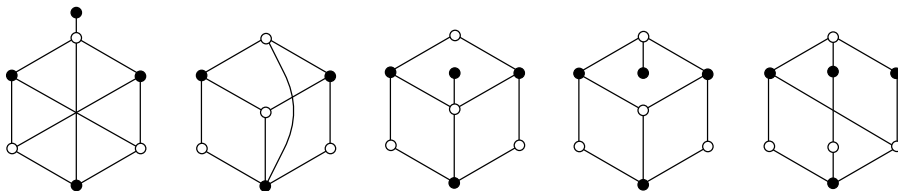


Figure 3.14

These five graphs are the only simple, bipartite A-graphs with nine edges. Note that the graph, generated from A_3 by joining its nodes of degree 3 by an edge, is not an A-graph. Hence, $F + ab$ is not necessarily an A-graph for *arbitrary* non-adjacent nodes $a \in A, b \in B$ of F if F is an A-graph. But it seems to be true that there is at least one non-edge in every non-complete bipartite A-graph such that its addition yields a bipartite A-graph again. Hence, it is also interesting to know *critical* A-graphs F with the property that $F - e$ is not an A-graph anymore $\forall e \in E(F)$. Obviously, the graphs A_1, A_2 , and A_3 are critical A-graphs, but the graphs shown in Figure 3.14 are not. Further examples of critical A-graphs are presented in Figure 3.15; note that they are critical with respect to deletion of an arbitrary edge although $A_2 \subset A_5$ and $A_3 \subset A_4, A_8, A_9$ holds.

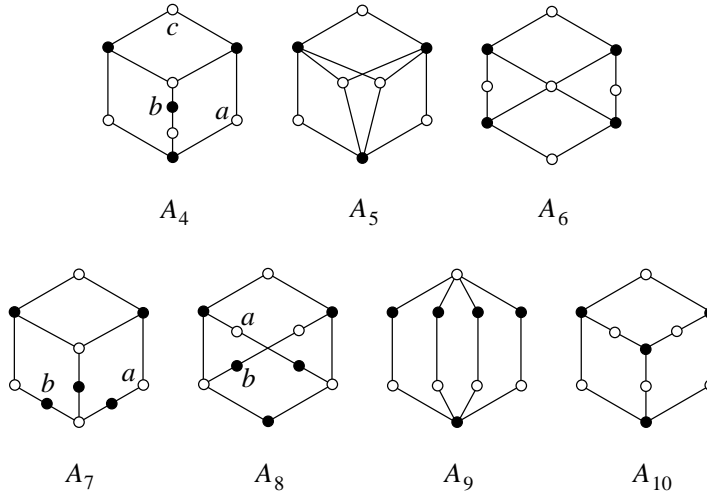


Figure 3.15

But if, in contrast to the case of H-graphs, the addition of an arbitrary non-edge to an A-graph F does not necessarily yield an A-graph, a similar result is true for $F + ab$ if a and b are *adjacent*. Then $F + ab$ is a bipartite multigraph, the parallel edges in $F + ab$ correspond to true twins in $L(F + ab)$. Thus, $L(F + ab)$ arises from $L(F)$ by substituting its node ab by a K_2 , and the next lemma is a consequence of Theorem 3.8.(i).

Lemma 3.23 *Is F a bipartite A-graph and $ab \in E(F)$, then $F + ab$ is an A-graph.*

There are five A-graphs, generated from A_1 , A_2 , and A_3 by duplicating one edge. The complements of their line graphs are the remaining five critically perfect graphs on 10 nodes. Thus, all critically perfect graphs on at most 10 nodes are complements of line graphs of bipartite A-graphs. Note that the addition of parallel edges to an H-graph F does not yield an H-graph again, since the corresponding nodes of $L(F)$ are true twins, hence the edge connecting them cannot be critical by Lemma 2.2. Since the line graphs of bipartite multigraphs can be obtained by substitution as described above, we are only interested in simple, bipartite A-graphs in the sequel.

The three A-graphs with minimum degree 1 in Figure 3.14 are examples that A-graphs do not necessarily possess a further property which is required for H-graphs, namely the 2-connectivity. The following lemma yields a characterization of all A-graphs which are not 2-connected.

Lemma 3.24 *Let F be a connected, bipartite graph, different from a star $K_{1,k}$. F is an A-graph iff it satisfies the following three conditions.*

- (1) *Every block of F is an A-graph or a single edge.*
- (2) *At most one cutnode appears in every color class.*
- (3) *If $a \in F$ is a cutnode and $b \in N(a)$, then there is no $a_1 \in N(b) - a$ such that $\{a, a_1\}$ separates b from $N(a) - b$.*

Remark. A consequence of condition (3) is that no neighbor of any cutnode in F can have degree 2. Furthermore, no block of F can consist of the edge connecting two cutnodes $a \in A$, $b \in B$ of F since even $\{a\}$ would separate b from $N(a) - b$ in this case. The non-incident edges b_1a, ba_1 with $b_1 \in N(a) - b$ and $a_1 \in N(b) - a$ could never be an A-pair by (3). Thus, a block of F can be a single edge only if it is an endblock, and every cutnode has degree ≥ 3 .

Proof. Suppose $F = (A \cup B, E)$ to be a connected, bipartite graph admitting non-incident edges; let $a, a_i \in A$, and $b, b_i \in B$.

(If) In order to show that F is an A-graph if it fulfills the conditions (1), (2), and (3), we prove the following two claims.

Claim 1. *Let B_1, B_2 are two blocks of F with $B_1 \cap B_2 = \{a\}$. Then every two non-incident edges $e_1 \in B_1$ and $e_2 \in B_2$ form an A-pair.*

Consider two blocks B_1, B_2 of F with $B_1 \cap B_2 = \{a\}$ and two non-incident edges $a_1b_1 \in B_1$, $a_2b_2 \in B_2$. If both B_1 and B_2 were single edges, they would not admit any pair of non-incident edges. Without loss of generality, let B_2 be different from a single edge and assume $a \neq a_2$. In B_1 , there is an odd path $P_1 = b_1, a_1, \dots, a$ of length ≥ 1 (note: $a \neq a_1$ does not hold necessarily). We have to find an even path $P_2 = a, \dots, b_2, a_2$ of length ≥ 4 in B_2 . The existence of an odd, a_2 -free (a, b_2) -path of length ≥ 3 follows for $ab_2 \notin E$ by the 2-connectivity of B_2 , and for $ab_2 \in E$ by condition (3). Thus, P_2 exists and we obtain $P_{a_1b_1, a_2b_2} = P_1 \cup P_2$.

Claim 2. *Let B_1, B_2 are two blocks of F with $B_1 \cap B_2 = \emptyset$. Then every two edges $e_1 \in B_1$ and $e_2 \in B_2$ form an A-pair.*

If F admits two blocks B_1, B_2 with $B_1 \cap B_2 = \emptyset$, then two cutnodes a, b appear in F . By condition (2), they are contained in different color classes, i.e., $a \in A$ and $b \in B$ hold. Let $a \in B_1$ and $b \in B_2$. Moreover, there is a further block B_3 of F with $a, b \in B_3$. Now consider two edges $a_1b_1 \in B_1$, $a_2b_2 \in B_2$ and find the corresponding path $P_{a_1b_1, a_2b_2}$. Since $B_3 \neq \{ab\}$ holds as consequence of condition (3), there is an odd (a, b) -path $P_3 \subset B_3$ of length ≥ 3 . Furthermore, there are two odd paths $P_1 = b_1, a_1, \dots, a \subseteq B_1$ and $P_2 = b, \dots, b_2, a_2 \subseteq B_2$ of length ≥ 1 , and we obtain $P_{a_1b_1, a_2b_2} = P_1 \cup P_3 \cup P_2$.

If F does not possess any cutnode, it is an A-graph by condition (1). In the case that F admits precisely one cutnode, the assertion follows by (1) and Claim 1. Otherwise, precisely two cutnodes appear in F by condition (2), and we deduce the assertion from condition (1), Claim 1, and Claim 2. \diamond

(Only if) In order to show that (1), (2), and (3) are necessary for F to be an A-graph, we distinguish three cases, where F is supposed to fail condition (i) in Case i.

Case 1. Assume, F admits a block B which is neither a single edge nor an A-graph. Then B contains two non-incident edges e_1 and e_2 , not forming an A-pair in B . There is no path of odd length ≥ 5 with e_1 and e_2 as endedges in B and, therefore, in F .

Case 2. Suppose F to contain two cutnodes $a_1, a_2 \in A$. Consider components F_1 of $F - a_1$ with $a_2 \notin F_1$, F_2 of $F - a_2$ with $a_1 \notin F_2$, and neighbors $b_i \in F_i$ of a_i for $i = 1, 2$. The edges b_1a_1 and a_2b_2 are non-incident, but not an A-pair since every path with b_1a_1 and a_2b_2 as endedges must possess the endnodes b_1 and b_2 , i.e., it must be even.

Case 3. Let there be a cutnode $a \in A$, components F_1, F_2 of $F - a$, $b_1 \in N_{F_1}(a)$, and $a_1 \in N_{F_1}(b_1) - a$ such that $\{a, a_1\}$ separates b_1 from $N(a) - b_1$. Consider a neighbor $b_2 \in F_2$ of a . Then the non-incident edges b_2a, b_1a_1 do not form an A-pair: every path P with endedges b_2a, b_1a_1 must start in b_2 , thus every such path of odd length ends in a_1 . But, by assumption, the only path $P = b_2ab_1a_1$ with this property has length 3. \square

Thus, we have seen that two properties required for H-graphs, namely 2-connectivity and minimum degree 3, are not needed for A-graphs. However, H-graphs and A-graphs share some properties. The 3-connectivity is a common sufficient condition for a graph to be an H-graph as well as an A-graph.

Lemma 3.25 *Every 3-connected, bipartite graph is an A-graph.*

Proof. Consider a 3-connected, bipartite graph $F = (A \cup B, E)$ and two non-incident edges a_1b_1 and a_2b_2 of F with $a_i \in A$, $b_i \in B$. We show that a_1b_1 and a_2b_2 form an A-pair in F . Since F is 3-connected, there are open-disjoint, odd (a_1, b_2) -paths P_1, \dots, P_k with $k \geq 3$. At most two paths among P_1, \dots, P_k can contain a_2 or b_1 ; without loss of generality, let P_3, \dots, P_k be a_2, b_1 -free. If there is a path P_i with $3 \leq i \leq k$ of length ≥ 3 , then $b_1P_i a_2$ is the studied path $P_{a_1b_1, a_2, b_2}$. Otherwise, the only a_2, b_1 -free (a_1, b_2) -path is the edge a_1b_2 , we obtain $k = 3$ and let $b_1 \in P_1$, $a_2 \in P_2$. By the 3-connectivity of F again, there must be an a_1, b_2 -free (a_2, b_1) -path Q of odd length ≥ 3 and we get $P_{a_1b_1, a_2, b_2} = b_2Qa_1$. Otherwise, if the edge a_2b_1 were the only a_1, b_2 -free (a_2, b_1) -path, $\{a_1, a_2\}$ would be a cutset of size two, separating the nodes of $P_1(b_1, b_2)$ from $P_2(a_1, a_2)$ in contradiction to F 3-connected. \square

Consequences. We summarize the results established in this section. Due to Theorem 3.18 and Theorem 3.22, respectively, we know that a graph $L(F)$ is critically and anticritically perfect if and only if F is a bipartite H-graph and A-graph. The reverse results are true for complements of line graphs, here $\overline{L(F)}$ belongs to C-PERF and A-PERF iff F is a bipartite A-graph and H-graph, resp. (a characterization of critically perfect complements of line graphs independent of Theorem 3.22 can be found in [63]). Moreover, $L(F)$ is contained in $A\text{-PERF} \cap C\text{-PERF}$ iff F is an H-graph as well as an A-graph. This is satisfied at least for all 3-connected bipartite graphs due to Lemma 3.19 and Lemma 3.25. Especially, every line graph of a 3-connected bipartite graph is a singleton in the graph of perfect graphs, i.e., you cannot reach another perfect graph from such a graph by adding or deleting one edge.

Finally it is worth noting that all graphs in $C\text{-PERF}$ and $A\text{-PERF}$ known to the author so far are related to line graphs of bipartite H- and A-graphs in some way. Most of the known critically perfect graphs are line graphs of bipartite H-graphs, complements of line graphs of bipartite A-graphs, or can be generated by applying the operations investigated in the previous section to them. Even the graph presented in Figure 3.9 is the line graph of a bipartite graph which is almost an H-graph: only the two edges corresponding to the grey nodes do not form an H-pair. Particularly, it is unknown to the author whether there is another way to construct a non-critically perfect graph, admitting no critically perfect subgraph, with the property that all of its non-critical edges belong to one clique (this property is obviously needed to construct a critically perfect graph from graphs not belonging to $C\text{-PERF}$ by clique-identification).

3.4 Which Graph Classes Contain Critically Perfect Graphs?

This section deals with Problem 0.5 in which subclasses of perfect graphs do critically and anticritically perfect graphs appear. We first show that both classes $C\text{-PERF}$ and $A\text{-PERF}$ are incomparable to all classes of perfect graphs whose members admit forbidden subgraphs different from minimally imperfect graphs.

Theorem 3.26 *Let \mathcal{C} be a class of perfect graphs. If there is a perfect graph F such that no graph in \mathcal{C} can admit F as (induced) subgraph, then neither $C\text{-PERF}$ nor $A\text{-PERF}$ contains or is contained in the class \mathcal{C} .*

Proof. Consider a class \mathcal{C} of perfect graphs and a perfect graph F which is a minimal forbidden subgraph for all graphs contained in \mathcal{C} . First, we construct a critically perfect graph G with $F \subset G$. For this cover all non-critical edges and all isolated nodes of F by cliques Q_1, \dots, Q_k , choose graphs $G_1, \dots, G_k \in C\text{-PERF}$ with $Q_i \subseteq G_i$ for $1 \leq i \leq k$, and identify F with all graphs G_i in Q_i each. The arising graph G contains F as subgraph and is critically perfect by Theorem 3.14 (in the case that isolated nodes occur in F , recall that the disjoint union preserves critical perfectness). Hence $C\text{-PERF} \not\subseteq \mathcal{C}$ follows. Conversely, consider a proper subgraph $F' \subset F$ with $|V(F')| < 9$ to insure $F' \notin C\text{-PERF}$ due to Lemma 3.1. The graph F' belongs to \mathcal{C} since F is minimal forbidden for \mathcal{C} , hence $C\text{-PERF}$ cannot contain this class. Applying the same construction to \overline{F} yields that there is no inclusion relation between $C\text{-PERF}$ and the class containing the complements of all graphs in \mathcal{C} . Thus we can finally conclude that $A\text{-PERF}$ and \mathcal{C} are incomparable. \square

The proof of Theorem 3.26 yields particularly that the only forbidden subgraphs of critically and anticritically perfect graphs are minimally imperfect. Since the whole class of perfect graphs consists of all graphs without any minimally imperfect subgraph, we cannot characterize critically and anticritically perfect graphs by giving a complete list of forbidden subgraphs.

Corollary 3.27 *$C\text{-PERF}$ and $A\text{-PERF}$ cannot be characterized by forbidden subgraphs.*

For most of the classes \mathcal{C} of perfect graphs there is a perfect graph F which is a minimal forbidden subgraph for all graphs in \mathcal{C} , hence $C\text{-PERF}$ and $A\text{-PERF}$ are incomparable to all these graph classes \mathcal{C} . But there are some classes of perfect graphs not defined by forbidden subgraphs, e.g., the class of uniquely colorable perfect graphs (such graphs admit a unique minimum coloring).

In order to answer Problem 0.5 we study, therefore, the intersection of $C\text{-PERF}$ or $A\text{-PERF}$ and other subclasses of perfect graphs. Actually, there are some graph classes with an empty intersection with $C\text{-PERF}$ or $A\text{-PERF}$. Clearly, the class of Meyniel graphs does not contain any critically perfect graph by Theorem 2.7. All graph classes contained in the intersection of all Meyniel and all co-Meyniel graphs must even have an empty intersection with $C\text{-PERF}$ and $A\text{-PERF}$. All graph classes whose members are perfect and admit perfect edge orders and co-perfect edge orders have obviously an empty intersection with $C\text{-PERF}$ and $A\text{-PERF}$, respectively, too (see Section 2.4). The following theorem establishes this result for a further class of perfect graphs. **Clique separable graphs** have been defined by GAVRIL [19] as these graphs whose connected subgraphs without any clique-cutset are all complete k -partite or the complete join of a connected bipartite graph and a clique.

Theorem 3.28 *Clique separable graphs are neither critically nor anticritically perfect.*

Proof. Assume G to be clique separable. Every graph can be decomposed into connected subgraphs G_0, \dots, G_l not admitting any clique-cutset, and can be reconstructed from these graphs G_0, \dots, G_l by successively clique identification as described in [65]: Let $G^0 = G_0$ and G^i arise from G^{i-1} and G_i by identification in a clique Q_i such that $G^l = G$ holds, G^i is connected, and $G_i \neq Q_i$ for $1 \leq i \leq l$. Then G_0, \dots, G_l are complete k -partite or the complete join of a connected bipartite graph and a clique since G is clique separable.

We first show that G cannot be critically perfect. Due to $G_i \neq Q_i$, every graph G_i particularly contains an edge $e = xy$ with $y \notin Q_i$ (note: G_i is connected). If e is intended to be a critical edge of G and G_e to be a subgraph with $G_e - e$ minimally imperfect, then $G_e \subseteq G_i$ follows from Lemma 3.13, i.e., e is a critical edge of G_i . Since complete k -partite graphs do not possess any critical edge due to Theorem 2.8, we conclude that G_i is the complete join of a bipartite graph and a clique Q . Every node of Q dominates all remaining nodes in G_i , hence no edge incident to a node in Q is critical by Lemma 2.2. But for $Q = \emptyset$, the graph G_i is bipartite and without any critical edge making use of Lemma 2.3. Hence G admits at least one non-critical edge.

Now, we prove the assertion that G is not anticritically perfect. First suppose G_i to contain non-adjacent nodes x, y and $G_{xy} + xy$ to be a minimally imperfect subgraph of $G + xy$. Then $G_{xy} \subseteq G_i$ holds by Lemma 3.15, hence xy is an anticritical edge of G_i . Because complete k -partite graphs do not admit any anticritical edge due to Theorem 2.8, G_i must be the complete join of a bipartite graph F and a clique. If there are non-adjacent nodes x and y in different color classes of F , then $F + xy$ is still bipartite and $G_i + xy$ still perfect. Otherwise, F is a complete bipartite graph, thus every two of its non-adjacent

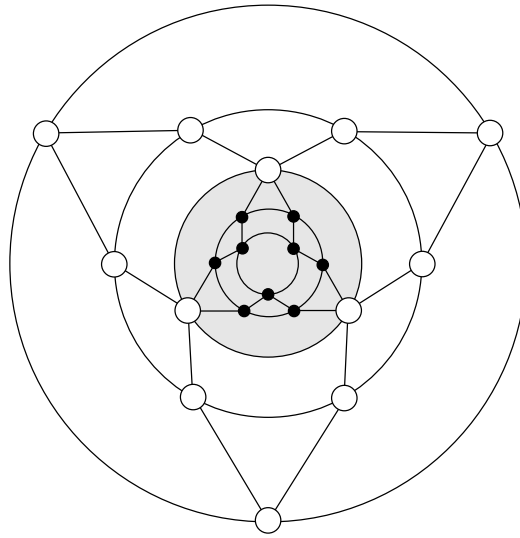


Figure 3.16(a)

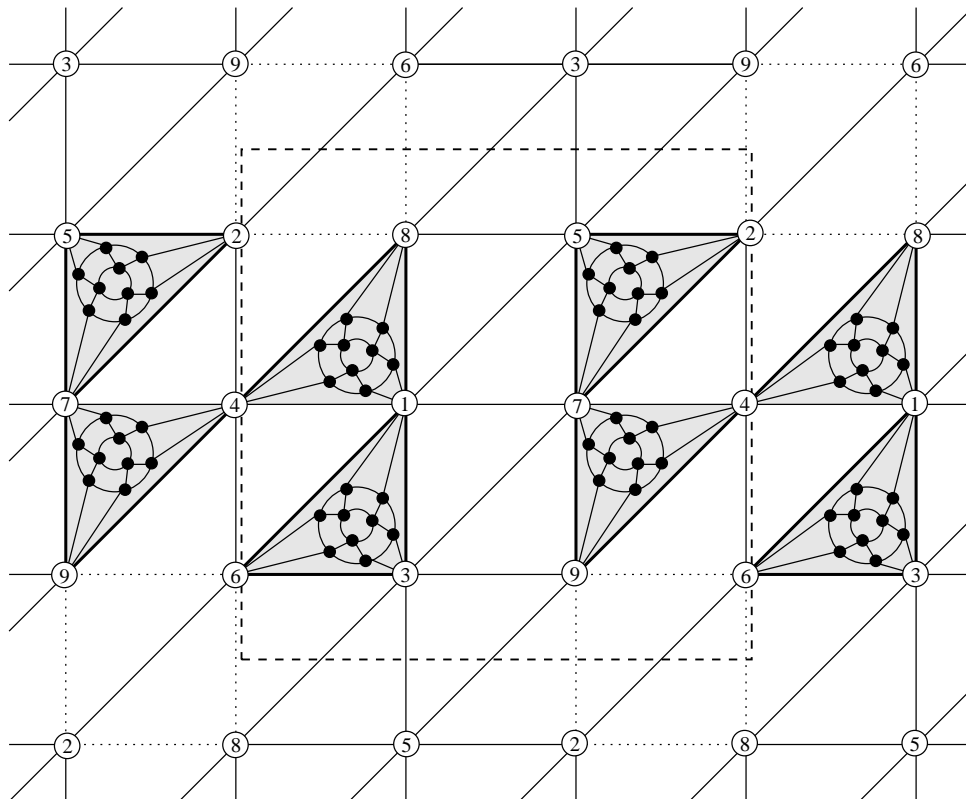


Figure 3.16(b)

nodes x and y are false twins and $G_i + xy$ is perfect following Lemma 2.5. Hence G cannot be anticritically perfect if one of the graphs G_i is non-complete and we finally assume G_i to be a clique for $1 \leq i \leq l$. Consider a node $x \in G_l - Q_l$ and note $G^{l-1} \neq Q_l$, otherwise G would be a clique (because G_l is) and not anticritically perfect by definition. Then there is a node $y \in G^{l-1} - Q_l$ with $Q_l \cap N_G(y) \neq \emptyset$ such that $Q_l \cap N_G(y)$ is a clique-cutset separating x and y in G (note: every $z \in Q_l - N_G(y)$ belongs to another G_i than y , i.e., all (y, z) -paths of G run through $Q_l \cap N_G(y)$ and, therefore, z is not contained in any induced (x, y) -path of G by $Q_l \cap N_G(y) \subseteq N_G(x)$). Hence x and y form a 2-pair and $G + xy$ cannot be imperfect by Lemma 2.5. \square

So we were able to prove that *C-PERF* has an empty intersection with the classes of Meyniel graphs, weakly triangulated graphs, and clique separable graphs. No similar result has been established so far for *BIP**, the class of bull-free Berge graphs, slim graphs, strict quasi-parity graphs, and strongly perfect graphs but there are no critically perfect graphs known to the author belonging to one of these classes. Note that absorbantly perfect graphs are characterized in [24] to possess either a strong stable set (that meets all maximal cliques of the graph) or a weak comparable pair (a node that dominates one of its neighbors). Hence the class of absorbantly perfect graphs cannot contain any critically perfect graph that is not strongly perfect due to Lemma 2.2.

Next, the classes of weakly triangulated and clique separable graphs also have an empty intersection with *A-PERF*, and no anticritically perfect graphs are known belonging to *BIP**, the class of bull-free Berge graphs, K_5 -, P_5 -free Berge graphs, and slim graphs. (For an overview, see also the corresponding charts and tables in Appendix A.1 and A.2, respectively.)

In order to give some examples of critically or anticritically perfect graphs contained in certain classes of perfect graphs, recall on the one hand the results in Section 2.3 concerning the occurrence of different kinds of critical edges and on the other hand the significance of line graphs mentioned in the previous section.

First, we have observed in Lemma 2.9(ii) that diamond-free, K_4 -free, or planar Berge graphs only admit H-critical edges. So we expect by Theorem 2.10 and Theorem 2.11 a relation of line graphs and critically or anticritically perfect graphs in these three classes. Indeed, all critically or anticritically perfect line graphs are diamond-free by Theorem 3.18 and Theorem 3.22. A line graph $L(F)$ is obviously K_4 -free, critically, and anticritically perfect iff F is a bipartite H-graph and A-graph, respectively, with $\Delta(F) = 3$. Moreover, the line graphs of H_1 (cf. Figure 3.11) and of A_2, A_3, A_4, A_7, A_8 , and A_{10} (see Figure 3.13 and Figure 3.15) admit an embedding in the plane. But not all critically or anticritically perfect graphs contained in these three graph classes are line graphs. Figure 3.16(a) shows a diamond-free, K_4 -free, and planar graph. It is obtained by identifying one copy of $L(H_1)$ in its internal triangle (grey filled in the picture) with the external triangle of a further copy of $L(H_1)$. This graph belongs to $C-PERF \cap A-PERF$ but is not a line graph.

All mentioned planar graphs and in addition $L(A_1)$, $\overline{L(A_2)}$, $L(A_6)$, and $L(A_9)$ are toroidal as well as locally perfect. A further graph embeddable in the torus and locally perfect is presented in Figure 3.17; it was the first example of a critically perfect graph found by the author. With respect to Lemma 2.9(iii), the question arises whether there is a toroidal graph in $C\text{-}PERF$ that admits an A-critical edge the deletion of which yields a $\overline{C_7}$. The graph G in Figure 3.16(b) answers this question in the affirmative. G is generated from a perfect graph on nine nodes by covering its non-critical edges (emphasized by bold lines in the picture) by the grey filled triangles and identifying four copies of $L(H_1)$ in these cliques. The arising graph G is critically perfect due to Theorem 3.14 and the nodes v_1, \dots, v_7 induce a $\overline{P_7}$ in G . This graph and $L(A_5)$ are toroidal but not locally perfect, while $\overline{L(A_3)}$ is locally perfect but not toroidal.

The remaining classes of perfect graphs mentioned in Section 2.3 are only known to admit A-critical edges. But hoping for a similar relation of co-line graphs and critically or anticritically perfect graphs in these classes is unmet. The class of weakly triangulated graphs cannot admit any critically or anticritically perfect graph (see the perfect edge order and co-perfect edge order described for weakly triangulated graphs in Section 2.4). It is easy to check that there are at least no bull-free complements of *line* graphs in $C\text{-}PERF$ or $A\text{-}PERF$. Furthermore, $\overline{L(A_1)}$ is the only co-line graph in the class of murky graphs, $\overline{L(A_3)}$ and $\overline{L(A_{10})}$ the only P_5 - and K_5 -free complements of line graphs.

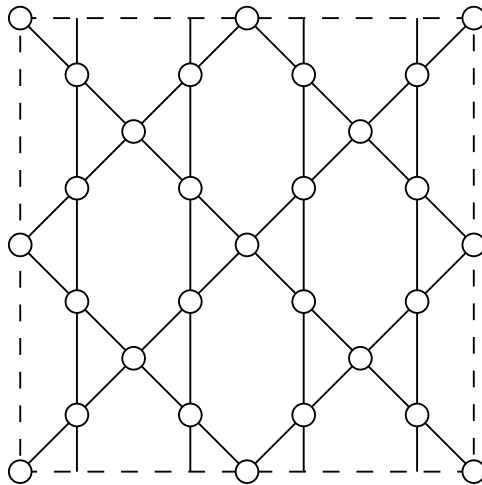


Figure 3.17

Turning to other examples of critically or anticritically perfect graphs contained in certain classes of perfect graphs, note that $L(A_3)$ and $L(A_9)$ are strongly perfect (thus also absorbantly perfect) and perfectly contractile. These two graphs together with $L(A_4)$, $L(A_7)$, and $L(A_8)$ belong to the class SQP . Consequently, they are also quasi-parity and the complements of them are examples of critically perfect graphs in the class QP .

A complete list of all these graph classes considered here together with the information whether or not we know examples of critically or anticritically perfect graphs belonging to them is provided in Appendix A.1 and A.2.

Finally, we mention that certain graphs in $A\text{-PERF}$ and $C\text{-PERF}$ are minimal counterexamples for classes of perfect graphs concerned to even pairs and for preperfect graphs (see the definition below). Even pairs play an important role in conjunction with perfectness and not only the definition of many classes of perfect graphs relies on MEYNIEL's Even Pair Lemma [46]. HOUGARDY established in [34]: proving that minimally imperfect graphs are *minimally* even pair-free implies the SPGC. Concerning the description of all minimally even pair-free graphs, he conjectured in [32] the following relation.

Conjecture 3.29 (HOUGARDY [32]) *Every minimally even pair-free graph is either an odd hole, an antihole of length at least seven, or the line graph of a bipartite graph.*

By a computer search [33], HOUGARDY proved this conjecture for all graphs on less than 17 nodes. Checking his list, we have found anticritically perfect graphs among these minimally even pair-free graphs. E.g., the line graphs of A_2 , $A_4 + ab$, A_6 , $A_7 + ab$, $A_8 + ab$, A_{10} (see Figure 3.15 for the edges to be inserted), and of the bipartite A-graphs depicted in Figure 3.18 range among these example graphs. Furthermore, note that the complements of their line graphs are also even pair-free. Hence all these graphs, their complements, and in addition $L(A_4 + bc)$ and $\overline{L(A_4 + bc)}$ (see Figure 3.15) are even minimally non-quasi parity graphs.

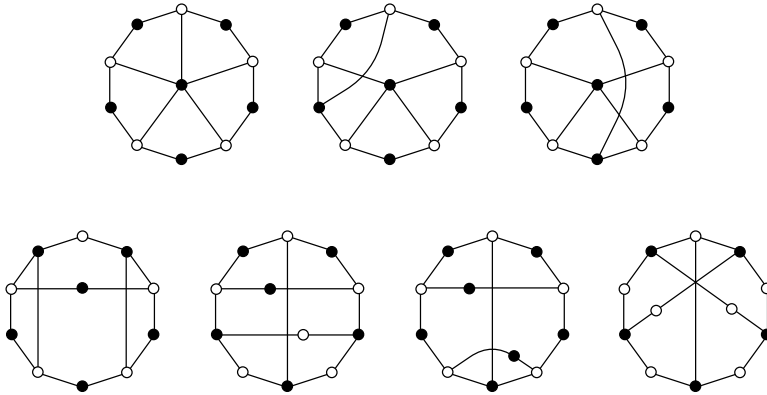


Figure 3.18

The list of HOUGARDY [33] only contains line graphs $L(F)$ such that F is either

- (i) the $K_{2,3}$,
- (ii) a bipartite A-graph, or
- (iii) a graph obtained from another bipartite graph with minimally even pair-free line graph by a simple operation (replace one edge by a C_4 and an edge with precisely one endnode on the C_4).

The operation in (iii) creates graphs, the complements of their line graphs admit an even pair, so the minimally even pair-free line graphs known to be in $A\text{-}PERF$ coincide with the minimally non-quasi parity graphs mentioned above. Unfortunately, a general relation between minimally even pair-free line graphs and anticritically perfect graphs has not been established so far.

Another subclass of perfect graphs having minimal counterexamples in $A\text{-}PERF$ and $C\text{-}PERF$ is the class of preperfect graphs, introduced by HAMMER and MAFFRAY [24]. They defined a node x of a graph G to **predominate** a node y of G if one of the following situations occurs:

- (i) $V(G) = \{x\} = \{y\}$,
- (ii) $x \neq y$, $xy \in E(G)$, every maximum clique containing y also contains x ,
- (iii) $x \neq y$, $xy \notin E(G)$, every maximum stable set containing x also contains y .

HAMMER and MAFFRAY proved in [24] that no minimally imperfect graph can own a node predominating another one. Hence **preperfect graphs**, defined to be graphs each of whose subgraphs has a predominating node, are perfect. The class of preperfect graphs seems to be quite large (see the inclusion relations of several classes of perfect graphs shown in Appendix A.1).

HAMMER and MAFFRAY presented in [24] an infinite sequence of perfect but minimally non-preperfect graphs. A different sequence of such graphs was found (but not published) by HOUGARDY, MAFFRAY, and SEBÖ [36]. All these graphs are line graphs of special bipartite graphs, including the graphs shown in the two rows below in Figure 3.2. Motivated by this observation, minimally non-preperfect graphs with small maximum degree have been investigated in [62] and characterized for $\Delta \leq 4$ as follows.

Theorem 3.30 (TUZA and WAGLER [62]) *A graph of maximum degree 4 is minimally non-preperfect iff it is either an odd hole, the \overline{C}_7 , or the line graph of a 3-regular, 3-connected bipartite graph.*

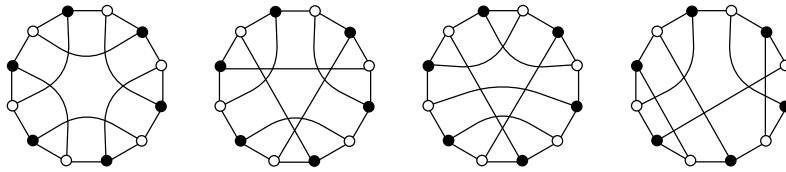


Figure 3.19

We already know from Lemma 3.19 and Lemma 3.25 that every 3-connected bipartite graph is an H-graph as well as an A-graph. So the perfect graphs among the examples of minimally non-preperfect graphs described in Theorem 3.30 belong to $A\text{-}PERF \cap C\text{-}PERF$, e.g., the line graphs of $K_{3,3}$, H_1 , H_4 , H_5 (see Figure 3.11), and the line graphs

of the 3-connected, bipartite graphs shown in Figure 3.19. It is worth noting that all the perfect, minimally non-preperfect graphs known so far admit a maximum degree equal to four, so all of them belong to $A\text{-}PERF \cap C\text{-}PERF$.

Note finally that there are also critically and anticritically perfect graphs known to belong to the class of preperfect graphs. E.g., the line graphs of A_2, \dots, A_{10} are preperfect, hence their complements, too.

Chapter 4

Some Polyhedral Consequences

This chapter investigates how imperfect a graph is that arises by deleting a critical edge of a perfect graph. Section 4.1 introduces a measure for the imperfectness of a graph using the polyhedral characterization of perfect graphs [9]: the polytope $\text{QSTAB}(G)$ given by nonnegativity and clique constraints coincides with the stable set polytope $\text{STAB}(G)$ iff the associated graph is perfect. We examine several classes of inequalities valid for the stable set polytope of an imperfect graph, in particular rank constraints and weak rank constraints. Following a suggestion in [22], we consider the polytopes $\text{RSTAB}(G)$ and $\text{WSTAB}(G)$ given by rank constraints and weak rank constraints, respectively, together with the nonnegativity constraints. That clique constraints are special rank constraints and weak rank constraints are obtained from certain rank constraints by PADBERG's lifting procedure [49] implies $\text{STAB}(G) \subseteq \text{WSTAB}(G) \subseteq \text{RSTAB}(G) \subseteq \text{QSTAB}(G)$. We introduce the graph classes where $\text{RSTAB}(G)$ and $\text{WSTAB}(G)$ coincide with $\text{STAB}(G)$, and term them rank-perfect and weakly rank-perfect graphs, respectively. Thus the classes of all rank-perfect and all weakly rank-perfect graphs constitute superclasses of perfect graphs and yield the studied measure for the imperfectness of a graph.

Deciding how imperfect a graph is that arises by deleting a critical edge e of a perfect graph G means, therefore, to figure out whether $G - e$ is rank-perfect or weakly rank-perfect. For that, we have to study the facets of $\text{STAB}(G - e)$. In Section 4.2 and 4.3, we deal with Problem 0.6 and investigate $\text{STAB}(G - e)$ where G is a perfect line graph and the complement of such a graph, respectively. The characterizations of critical and anticritical edges in line graphs from Section 2.3 provide the knowledge of *all* minimally imperfect graphs which occur in $G - e$. We give the weak rank constraints obtained by lifting rank constraints associated with one or the union of some minimally imperfect subgraphs of $G - e$.

In view of Problem 0.7, we conjecture that the facets of $\text{STAB}(G - e)$ described in Section 4.2 are the only new facets resulting from the deletion of an edge in a perfect line graph G . Then, all these graphs $G - e$ would be weakly rank-perfect and, graphs arising from line graphs of bipartite graphs, even rank-perfect. We are able to show in Section 4.4 that all graphs generated from the complement of the line graph of a bipartite graph by the deletion of an arbitrary edge are contained in the class of weakly rank-perfect graphs. This gives at least a partial answer to Problem 0.7.

4.1 Rank-Perfect and Weakly Rank-Perfect Graphs

Let $G = (V, E)$ be a graph and $n = |V|$. For every subset $V' \subseteq V$, let $\chi^{V'}$ denote its **incidence vector**, i.e., the vector defined by

$$\chi_i^{V'} := \begin{cases} 1 & \text{if } v_i \in V' \\ 0 & \text{otherwise.} \end{cases}$$

The **stable set polytope**

$$\text{STAB}(G) := \text{conv}\{\chi^S \in \mathbb{R}^n : S \subseteq V \text{ is a stable set}\}$$

is defined as the convex hull of the incidence vectors χ^S of all stable sets S of the graph G . A linear inequality $a^T x \leq b$ is said to be **valid** for $\text{STAB}(G)$ if it holds for all $x \in \text{STAB}(G)$, i.e., if $\text{STAB}(G) \subseteq \{x \in \mathbb{R}^n : a^T x \leq b\}$. Since $\text{STAB}(G)$ is a full-dimensional polytope there is a unique minimal set of valid inequalities for $\text{STAB}(G)$ (up to positive scalar multiplication) that defines $\text{STAB}(G)$. A valid inequality belongs to this minimal set, called the set of **facets** of $\text{STAB}(G)$, iff it is satisfied as an equality by n affinely independent incidence vectors of stable sets of G . Among these facets are, of course, the "trivial" facets demanding nonnegativity:

$$x_i \geq 0 \quad \forall v_i \in V. \quad (4.0)$$

All remaining "nontrivial" facets are of the form

$$\sum_{v_i \in G} a_i x_i \leq b$$

where a_1, \dots, a_n are nonnegative integers and b is a positive integer (see [54]). We often write $x(G, a) \leq b$ for such an inequality, where we interpret the vector $a = (a_1, \dots, a_n)$ to be a node weighting of G associating the weight a_i to the node v_i for $1 \leq i \leq n$ and denote the **weighted graph** by (G, a) . Since χ^\emptyset cannot satisfy any inequality with $b > 0$ with equality, there actually have to be n stable sets of G with *linearly* independent incidence vectors satisfying the corresponding *nontrivial* facet with equality. Let $V' = \{v_i \in V(G) : a_i > 0\}$, then we say that the facet $a^T x \leq b$ is associated to V' and $G[V']$, or that V' and $G[V']$ are facet-inducing. Since the determination of the facets of $\text{STAB}(G)$ is very difficult in general, one often tries to find classes of valid inequalities for $\text{STAB}(G)$ and to investigate when such inequalities yield facets of $\text{STAB}(G)$. One natural way to obtain a class of valid nontrivial inequalities is to consider **rank constraints** of the form

$$\sum_{v_i \in G'} x_i \leq \alpha(G') \quad (4.1)$$

for $G' \subseteq G$ (note $\alpha(G')$ is said to be the rank of $V(G')$). For convenience, we often write (4.1) as $x(G', \mathbb{1}) \leq \alpha(G')$ with $\mathbb{1} = (1, \dots, 1)$ or $x(G') \leq \alpha(G')$. Clearly, a rank constraint is satisfied by the incidence vector of every stable set of G , so it is valid for $\text{STAB}(G)$. In particular, whenever G' has $\alpha(G')$ -stable sets $S_1, \dots, S_{|G'|}$ the incidence vectors of which

are linearly independent, $x(G') \leq \alpha(G')$ is facet-inducing for $\text{STAB}(G)$ at least if $G' = G$. PADBERG [49] showed that special rank constraints, the **clique constraints**,

$$\sum_{v_i \in Q} x_i \leq 1 \quad (4.1a)$$

are facet-inducing for $\text{STAB}(G)$ iff Q is a maximal clique of G . Further classes of rank constraints are the **odd hole constraints**

$$\sum_{v_i \in C_{2k+1}} x_i \leq k \quad (4.1b)$$

where $C_{2k+1} \subseteq G$ and the **odd antihole constraints**

$$\sum_{v_i \in \overline{C}_{2k+1}} x_i \leq 2 \quad (4.1c)$$

where $\overline{C}_{2k+1} \subseteq G$. The inequalities (4.1b) and (4.1c) are facets of $\text{STAB}(G)$ at least for $G = C_{2k+1}$ and $G = \overline{C}_{2k+1}$, respectively. PADBERG showed in [50] that $x(G) \leq \alpha(G)$ is facet-inducing for $\text{STAB}(G)$ for every minimally imperfect graph G . BLAND, HUANG, and TROTTER [7] generalized this result to all partitionable graphs. Furthermore, CHVÁTAL established in [9] this property for α -connected graphs (the partial subgraph on all nodes with only the α -critical edges is connected), EDMONDS and PULLEYBLANK [17] via matching theory for line graphs of 2-connected hypomatchable graphs. A graph H is called **hypomatchable** if $\forall v \in V(H)$, the subgraph $H - v$ admits a matching meeting all of its nodes. The matchings of H correspond to the stable sets of its line graph $L(H)$, hence

$$\sum_{v_i \in L(H)} x_i \leq \frac{|V(H)| - 1}{2} \quad (4.1d)$$

defines a rank constraint and is facet-inducing for $\text{STAB}(L(H))$ iff H is 2-connected [17]. (Note that the odd hole constraints (4.1b) are special cases of the rank constraints of type (4.1d)). But in all these cases, a rank constraint $x(G') \leq \alpha(G')$ does not need to provide a facet of $\text{STAB}(G)$ if $G' \subset G$. However, (4.1) can be strengthened to a facet

$$\sum_{v_i \in G'} x_i + \sum_{v_i \in G - G'} a_i x_i \leq \alpha(G') \quad (4.2)$$

of $\text{STAB}(G)$ by determining appropriate integer coefficients a_i for $v_i \in G - G'$ using sequential lifting (see [49]). Let us call every facet of the form (4.2) a **weak rank constraint** of $\text{STAB}(G)$ iff it is obtained by lifting the rank constraint $x(G') \leq \alpha(G')$ and G' has $|G'|$ stable sets of size $\alpha(G')$ the incidence vectors of which are affinely independent. Thus, every lifting of a constraint (4.1b) or (4.1c) is a weak rank constraint by [50], every lifting of a constraint (4.1d) by [17]. Note that the facet obtained by lifting may depend on the order in which the nodes are lifted. Hence a rank constraint $x(G') \leq \alpha(G')$ associated with a subgraph $G' \subset G$ that is facet-inducing for $\text{STAB}(G')$ may give rise to *several* weak rank facets of $\text{STAB}(G)$, one for each valid lifting. In the next two sections,

we describe weak rank facets of $\text{STAB}(G - e)$ basing on rank constraints (4.1b) or (4.1d) and (4.1c) where G and \overline{G} , respectively, is a perfect line graph. In the latter case we are even able to prove that these special weak rank constraints yield the only facets besides (4.0) and (4.1a) (see Theorem 4.11 in Section 4.4).

As an example of a weak rank facet, consider the 5-wheel G shown in Figure 4.1(a). The facet $\sum_{1 \leq i \leq 5} x_i + 2x_6 \leq 2$ is a weak rank facet of $\text{STAB}(G)$ since $x(G - v_6) \leq 2$ is a rank facet of $\text{STAB}(G - v_6)$. However, $\text{STAB}(G)$ may admit nontrivial facets that are not weak rank constraints. Consider, e.g., the graph G shown in Figure 4.1(b). The inequality $\sum_{1 \leq i \leq 6} x_i + 2x_7 \leq 3$ is a facet of $\text{STAB}(G)$, but not a weak rank constraint since no proper subgraph $G' \subset G$ induces a rank facet $x(G') \leq 3$ of $\text{STAB}(G')$ that could be lifted to the above facet. Note that $x(G - v_7) \leq 3$ is a rank constraint, but not a facet of $\text{STAB}(G - v_7)$.

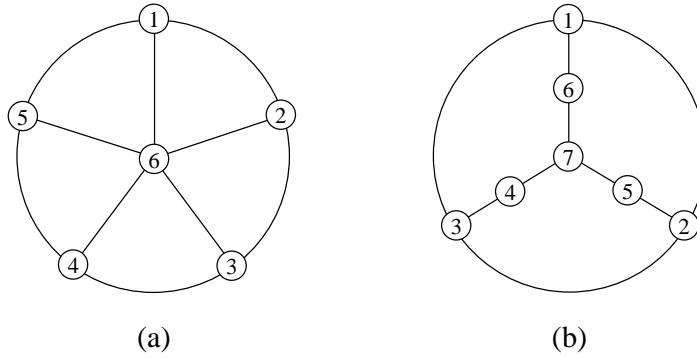


Figure 4.1

Let us consider the vector a of coefficients of an arbitrary nontrivial facet $a^T x \leq b$ of $\text{STAB}(G)$ to be a node weighting of G . Define the **weighted stability number** of (G, a) by $\alpha(G, a) = \max\{\sum_{v_i \in S} a_i : S \subseteq G \text{ is a stable set}\}$, then $b = \alpha(G, a)$ holds and we get

$$x(G, a) \leq \alpha(G, a). \quad (4.3)$$

We obtain a hierarchy of inequalities by stepwise refinement of this general nontrivial facet. As a first special case, we may rewrite the weak rank constraints (4.2) in the form

$$x(G, a) \leq \alpha(G', \mathbb{1})$$

where $G' \subseteq G$ with $V(G') \subseteq \{v_i \in V(G) : a_i = 1\}$ holds. Further, the rank constraints (4.1) can be expressed as $a^T x \leq \alpha(G', \mathbb{1})$ where $G' \subseteq G$ and $a = \chi^{V(G')}$ holds, i.e., as

$$x(G', \mathbb{1}) \leq \alpha(G', \mathbb{1}).$$

The previous inequality can finally be restricted to a clique constraint (4.1a) by

$$x(G', \mathbb{1}) \leq 1$$

where $G' \subseteq G$ is a clique.

In [22], the following concept is studied. Given an inequality class (\mathcal{L}) valid for $\text{STAB}(G)$, define the polyhedron $\mathcal{L}\text{STAB}(G)$ described only by nonnegativity constraints and inequalities in (\mathcal{L}) . Then investigate \mathcal{L} -perfect graphs, i.e., all graphs with $\mathcal{L}\text{STAB}(G) = \text{STAB}(G)$. Following this suggestion, we get by the hierarchy of the inequality classes above a chain of polytopes associated with a graph G and define classes of graphs for which these polytopes coincide with $\text{STAB}(G)$. By definition, the stable set polytope of every graph is entirely described by all "trivial" facets (4.0) and all "nontrivial" facets (4.3). With the above inequality classes, we analogously define three further polytopes associated with G : $\text{WSTAB}(G)$ as the polytope described by facets of type (4.0) and (4.2), $\text{RSTAB}(G)$ defined by the inequality classes (4.0) and (4.1), and $\text{QSTAB}(G)$ to be the polytope with (4.0) and (4.1a) as facet system. Then, by the relations of the inequality classes (4.3), (4.2), (4.1), and (4.1a) mentioned above, we immediately see the following inclusion relations of these polytopes:

$$\text{QSTAB}(G) \supseteq \text{RSTAB}(G) \supseteq \text{WSTAB}(G) \supseteq \text{STAB}(G).$$

Each of the polytopes $\text{QSTAB}(G)$, $\text{RSTAB}(G)$, and $\text{WSTAB}(G)$ is a relaxation of $\text{STAB}(G)$ and, very often, these inclusions are proper. In a few important cases, however, equality holds. Most notably, perfect graphs are characterized in [9] to be precisely the graphs G with

$$\text{QSTAB}(G) = \text{STAB}(G).$$

Let us call all graphs G with

$$\text{RSTAB}(G) = \text{STAB}(G)$$

rank-perfect. The smallest not rank-perfect graph is the 5-wheel shown in Figure 4.1(a). But, obviously, every perfect graph is rank-perfect, and there are further interesting classes of rank-perfect graphs. SHEPHERD [55] proposed to call a graph G **near-perfect** if its stable set polytope has, besides facets of type (4.0) and (4.1a), only the rank facet $x(G) \leq \alpha(G)$ associated with G itself. He strengthened the result of PADBERG [50], that every minimally imperfect graph is near-perfect, to the following characterization: the minimally imperfect graphs are precisely these imperfect graphs G where both G and \overline{G} are near-perfect. Further examples of rank-perfect graphs are **t-perfect** graphs, by CHVÁTAL [9] defined to be the graphs whose stable set polytopes admit only rank facets induced by edges and chordless odd cycles. (Note that "t" stands for "trou", the French word for hole, and that every C_{2k+1} with $k \geq 1$ is here considered to be a hole.) By definition [22], a generalization of t-perfect graphs is the class of **h-perfect** graphs (from hole-perfect) where (4.0), (4.1a), and (4.1b) suffice to describe the associated stable set polytopes. Although h-perfect graphs do not seem to occur in such abundance as perfect graphs, there are some interesting classes of such graphs known (see [22]). Line graphs are another class of rank-perfect graphs since every nontrivial facet of their stable set polytope different from (4.1a) is a rank facet of type (4.1d) induced by the line graph of an induced 2-connected hypomatchable graph by EDMONDS and PULLEYBLANK [17].

As a natural generalization of rank-perfect graphs, we analogously define graphs G with

$$\text{WSTAB}(G) = \text{STAB}(G)$$

to be **weakly rank-perfect**. Note that the graph shown in Figure 4.1(b) is the smallest graph with $\text{WSTAB}(G) \supset \text{STAB}(G)$. We will prove in Section 4.4 that the class $\text{co-}L(\text{BIP}) - e$, obtained from complements of line graphs of bipartite graphs by the deletion of an arbitrary edge, is one subclass of weakly rank-perfect graphs. A similar result is conjectured for the class $L(\text{LINE}) - e$, obtained from line graphs of line-perfect graphs by the deletion of an arbitrary edge.

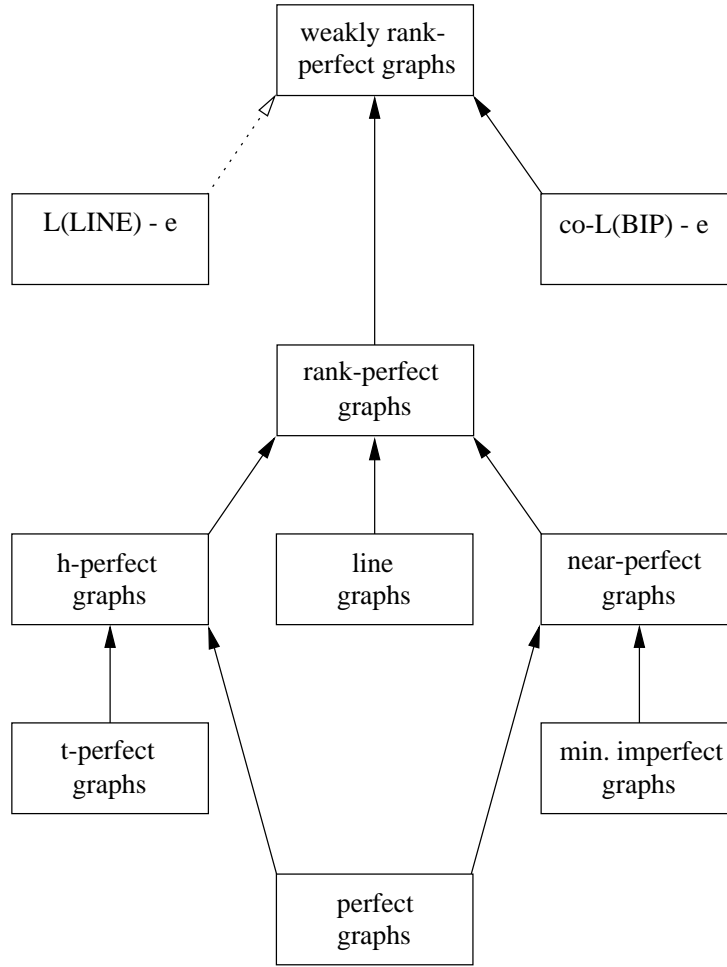


Figure 4.2: Classes of weakly rank-perfect graphs.

The classes of all rank-perfect and all weakly rank-perfect graphs constitute superclasses of perfect graphs and yield, thereby, a measure for the imperfectness of a graph. All inclusion relations of the graph classes mentioned in this section are summarized in Figure 4.2. All these graph classes are defined in terms of polyhedral theory. Even a polyhedral characterization of perfect [9] and minimally imperfect graphs [55] is known whereas their graph theoretical characterization concerned with BERGE's Strong Perfect Graph Conjecture [3] still remains open.

The stability number $\alpha(G)$ may be calculated as the maximum value of the linear function $\mathbb{1}^T x$ over $\text{STAB}(G)$. Since $\text{RSTAB}(G)$ and $\text{WSTAB}(G)$ approximate $\text{STAB}(G)$ "locally" in the neighborhood of the maximum stable sets of G , the knowledge of the structure of $\text{RSTAB}(G)$ and $\text{WSTAB}(G)$ would immediately imply a min-max characterization of $\alpha(G)$. Actually, in [22] it was shown that the stable set problem for perfect, t-perfect, and h-perfect graphs can be solved in polynomial time. For line graphs, this property follows from matching theory [16]. But it is unknown whether the polynomial solubility of the stable set problem extends to the class of rank-perfect or weakly rank-perfect graphs.

4.2 Facets of $\text{STAB}(G - e)$ with G a Perfect Line Graph

Let e stand for a critical edge in the line graph G of a simple line-perfect graph F . The main objective of this section is to study those facets of $\text{STAB}(G - e)$ that are neither nonnegativity nor maximal clique constraints. The graphs $G - e$ are imperfect but, in general, no line graphs anymore, so the result of EDMONDS and PULLEYBLANK [17] does not give a description of $\text{STAB}(G - e)$. However, it turns out that the "new" facets of $\text{STAB}(G - e)$ are related to hypomatchable graphs as considered in [17]: the line graphs of some hypomatchable graphs H appear in $G - e$, and weak rank facets of $\text{STAB}(G - e)$ base on rank constraints of type (4.1d) associated with $L(H)$. We will see that those graphs $L(H) \subseteq G - e$ are odd holes or the union of some odd holes of $G - e$. (Recall that every minimally imperfect subgraph of $G - e$ is an odd hole, since the line graph G can only admit H-critical edges e by Theorem 2.10, cf. Section 2.3.) Further, the studied hypomatchable graphs H with $L(H) \subseteq G - e$ do not occur in the line-perfect graph F , so we have to identify those subgraphs $F_H \subseteq F$ with $L(F_H) - e = L(H)$.

First, we provide some general properties of hypomatchable graphs. Recall that a graph H is called hypomatchable if it does not admit a perfect matching but $H - v$ does for all $v \in V(H)$, where a matching is **perfect** if it meets all nodes of the graph. LOVÁSZ [43] proved the following characterization of hypomatchable graphs which we will frequently use in the sequel. A graph H is hypomatchable if and only if there is a sequence $H_0, H_1, \dots, H_k = H$ of graphs such that H_0 is an odd cycle and for $1 \leq i \leq k$, H_i is obtained from H_{i-1} by adding an odd path that joins two (not necessarily distinct) nodes of H_{i-1} and has all internal nodes outside H_{i-1} . The odd path in H_i having its endnodes in H_{i-1} but all internal nodes outside H_{i-1} is called the **ear** E_i , and the sequence $H_0, H_1, \dots, H_k = H$ of graphs an **ear decomposition** of H . Every ear decomposition $H_0, H_1, \dots, H_k = H$ defines a sequence E_0, E_1, \dots, E_k of corresponding ears with $E_0 = H_0$. Hence, we may also characterize H by this ear sequence E_0, E_1, \dots, E_k and use both sequences to describe the same ear decomposition.

Hypomatchable graphs have an odd number of nodes, are non-bipartite and connected, but not necessarily 2-connected. By definition, an ear may be an odd cycle attached to a single node of the previous graph. However, these degenerated ears can be avoided if H is 2-connected: CORNUÉJOLS and PULLEYBLANK proved in [14] that every 2-connected hypomatchable graph H admits an ear decomposition $H_0, H_1, \dots, H_k = H$ with H_i

2-connected for $0 \leq i \leq k$. The next lemma provides an even stronger assertion for 2-connected hypomatchable graphs on more than four nodes.

Lemma 4.1 *Let H be a 2-connected hypomatchable graph and $|V(H)| \geq 5$. Then there is an ear decomposition $H_0, H_1, \dots, H_k = H$ of H such that each H_i is 2-connected and H_0 is an odd cycle of length at least five.*

Proof. Since H is 2-connected, it admits an ear decomposition $H_0, H_1, \dots, H_k = H$ with H_i 2-connected for $0 \leq i \leq k$ by [14]. We are ready if H_0 is an odd cycle of length ≥ 5 , hence assume H_0 is a triangle. From $|V(H)| > 3$ follows that there is an ear $\neq H_0$ with at least three edges, let $i \in \{1, \dots, k\}$ be the smallest index such that E_i has length ≥ 3 . Then $V(H_{i-1}) = V(H_0)$ holds and E_i has two distinct nodes $v, v' \in V(H_0)$ as endnodes (since H_i is 2-connected). Hence $(H_0 - vv') \cup E_i, vv', E_1, \dots, E_{i-1}$ is an ear decomposition of H_i starting with an odd cycle of length ≥ 5 and defining only 2-connected graphs. The ears E_{i+1}, \dots, E_k complete this ear decomposition to the studied decomposition of H . \square

We observe the following: let $H_0, H_1, \dots, H_k = H$ be an ear decomposition of a hypomatchable graph H and E_j with $1 \leq j < k$ an ear of length one. So $V(H_j) = V(H_{j-1})$ holds and the endnodes of each ear E_i with $j < i \leq k$ are contained in $H_{i-1} - E_j$. Thus the ear sequence $E_0, \dots, E_{j-1}, E_{j+1}, \dots, E_k, E_j$ also characterizes H . In other words, we can always reorder the ears E_1, \dots, E_k such that the ear decomposition starts with all ears of length ≥ 3 and ends up with all ears of length one. In the sequel, we call an ear decomposition $H_0, H_1, \dots, H_k = H$ of a 2-connected hypomatchable graph H with $|V(H)| \geq 5$ **proper** if each H_i is 2-connected, H_0 is an odd cycle of length ≥ 5 , and there is an index $j \in \{1, \dots, k\}$ such that E_1, \dots, E_j have length ≥ 3 and E_{j+1}, \dots, E_k have length one.

Now let us study those hypomatchable graphs the line graphs of which may occur in $G - e$ where G is the line graph of a simple line-perfect graph F and e is a critical edge of G . First, we observe that we are only interested in hypomatchable graphs H which are 2-connected (otherwise the rank constraint associated with $L(H)$ is never facet-inducing by [17]) and have at least five nodes (otherwise $H = K_3$ holds and the rank constraint associated with $L(H)$ is a clique constraint). Thus, we only consider hypomatchable graphs H that possess a proper ear decomposition by Lemma 4.1 and contain an odd cycle of length ≥ 5 in particular. Consequently, $H \not\subseteq F$ and $L(H) \not\subseteq G$ holds. Let x and y denote the endnodes of the critical edge e of G , then $x, y \in E(H)$ and $x, y \in V(L(H))$ follows immediately. Let F_H stand for the subgraph of F with $L(F_H) - e = L(H)$. Then $E(H) = E(F_H)$ clearly holds by $V(L(H)) = V(L(F_H))$, i.e., we deduce $x, y \in E(F_H)$ in particular. By $E(L(H)) = E(L(F_H)) - xy$, the edges x and y are not incident in H but incident in F_H , and every pair of edges e, e' distinct from x, y is incident in H iff e, e' are incident in F_H . In addition to these simple observations, we point out five further technical properties of H and F_H which we will frequently make use of in the sequel.

Lemma 4.2 *Consider the line graph G of a simple line-perfect graph F and a critical edge $e = xy$ of G . Let H be a 2-connected hypomatchable graph with $|V(H)| \geq 5$ and F_H the subgraph of F with $L(F_H) - e = L(H)$. Let u_1, z_1 denote the endnodes of x and u_2, z_2 the endnodes of y in H and F_H and assume $z_1 = z_2$ in F_H . Then we get the following.*

- (i) H is simple.
- (ii) Every odd cycle of length at least five in H contains x and y .
- (iii) Every cycle $C \subseteq H$ running through x and y contains the edge $e_{x,y} = z_1 z_2$. The path $x = u_1 z_1$, $e_{x,y} = z_1 z_2$, $y = z_2 u_2$ in H corresponds to the edges $x = u_1 z_1$, $e_{x,y} = u_3 z_1$, $y = u_2 z_1$ in F_H (note: u_3 is a new node not contained in $V(H)$), all remaining edges of C form with x and y a cycle in F_H . Moreover, C has either length 4 or odd length ≥ 5 .
- (iv) If a cycle of length four runs through x and y in H , then all other cycles of H containing x and y have length 5.
- (v) Let $H_0, H_1, \dots, H_k = H$ be a proper ear decomposition of H . Then none of the ears E_1, \dots, E_k has z_1 or z_2 as an endnode (i.e., $d(z_1) = d(z_2) = 2$ holds in H).

Proof. In order to show assertion (iii), let $C \subseteq H$ be a cycle with $x, y \in E(C)$. Since x and y are non-incident in H , we may represent C as shown in Figure 4.3(a) with $i, j > 0$. In F_H , the edges $x = u_1 z_1$ and $y = z_2 u_2$ are incident and $z_1 = z_2$ holds by assumption. For $z_2 = z$ (see Figure 4.3(b)), e_1 and f_1 would become incident in F_H , a contradiction. So we deduce $z_2 = z'$ (see Figure 4.3(c)). Here e_i is incident to x in F_H only for $i = 1$, we have to insert a new node u_3 in F_H in order to keep F_H loop-less, and get $e_1 = e_{x,y}$. The edges y, f_1, \dots, f_j, x form a cycle in F_H and, therefore, j is either equal to 1 or is even (since F_H is line-perfect). Thus C has either length 4 or odd length ≥ 5 and (iii) is true.

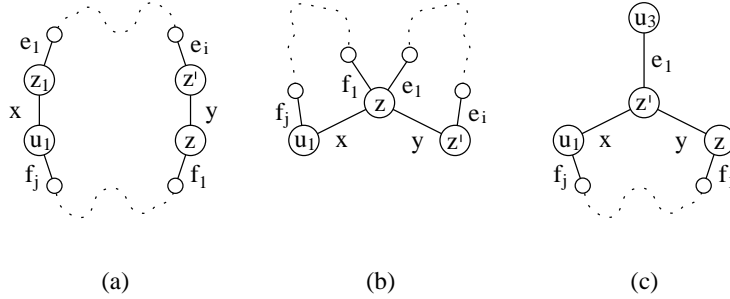


Figure 4.3

Now, let a cycle C of length 4 run through x and y in H . Then C has the edges $x = u_1 z_1$, $e_{x,y} = z_1 z_2$, $y = z_2 u_2$, and $f = u_2 u_1$ by (iii). Since F_H is simple, C is the only cycle of even length running through x and y in H by (iii). Suppose H admits a cycle with edges $x, e_{x,y}, y, f_1, \dots, f_j$ and $j \geq 4$ even. Then the odd cycle $f_1, \dots, f_j, u_1 u_2$ of length ≥ 5 appears in F_H in contradiction to F_H line-perfect. Thus every cycle through x and y in H distinct from C has length 5 and (iv) is true.

If there is an odd cycle $C \subseteq H$ with length ≥ 5 but $x \notin E(C)$ or $y \notin E(C)$, then $L(C) = L(C) - xy$ would imply that $L(F)$ contains the odd hole $L(C)$. The line-perfectness of F yields assertion (ii).

H is simple because F is and edges parallel to x or y in H are forbidden: H contains by Lemma 4.1 an odd cycle C of length ≥ 5 with $x, e_{x,y}, y \in E(C)$ by (ii),(iii) and let C' denote the even cycle with edges $E(C) - e_{x,y}$ in F_H . An edge x' in H parallel to x would have u_1, u_3 as endnodes in F_H (otherwise the incidence is not preserved) and $(C' - x) \cup \{x', e_{x,y}\}$ is an odd cycle of length ≥ 5 in F_H , a contradiction to F_H line-perfect. A symmetric argument applied to edges parallel to y yields assertion (i).

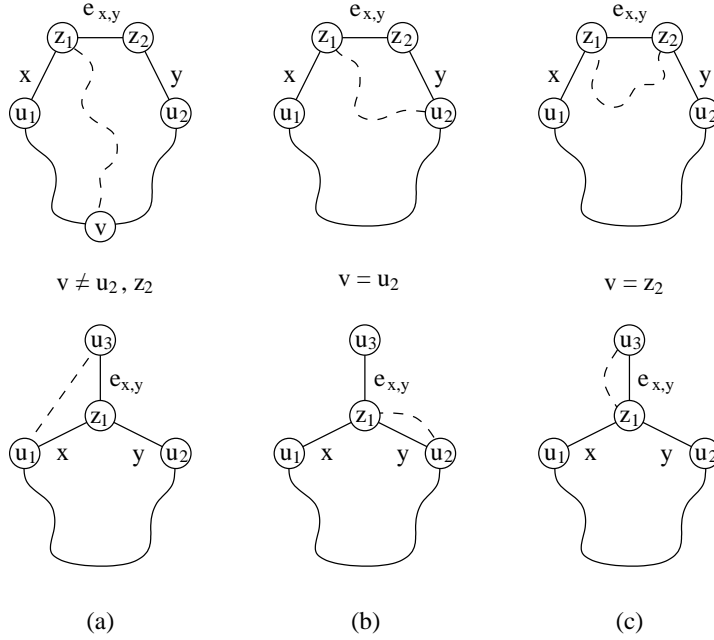


Figure 4.4

Finally, turn to the prove of assertion (v). According to Lemma 4.1 H admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$ such that each H_i is 2-connected and H_0 is an odd cycle of length at least 5. Let F_{H_i} denote the partial subgraph of F_H with $L(H_i) = L(F_{H_i}) - xy$ for $0 \leq i \leq k$. H_0 contains x and y by (ii) and also $e_{x,y} = z_1 z_2$ by (iii). Chose $i \in \{1, \dots, k\}$ minimal such that z_1 or z_2 is an endnode of E_i . Let, without loss of generality, z_1 and v be the endnodes of E_i and e_1 be the edge of E_i incident to z_1 . Then the only edges incident to z_1 in H_{i-1} are $e_{x,y}$ and x , the only edges incident to z_2 in H_{i-1} are $e_{x,y}$ and y . H_{i-1} is 2-connected, so two open-disjoint (z_1, v) -paths P_x and $P_{e_{x,y}}$ exist in H_{i-1} with $x \in E(P_x)$ and $e_{x,y} \in E(P_{e_{x,y}})$. (In Figure 4.4, the paths P_x and $P_{e_{x,y}}$ are depicted with solid lines, the ear E_i with dashed lines.) If $v \neq z_2, u_2$ holds, e_1 has to be incident to $e_{x,y}$ and x but not to y in F_{H_i} . Hence $e_1 = u_3 u_1$ follows (see Figure 4.4(a)) and implies $e_1 = z_1 u_1$ in H . Thus e_1 would be an edge parallel to x in contradiction to (i). For $v = u_2$ (see Figure 4.4(b)), E_i consists of the single edge e_1 , otherwise $E_i \cup P_{e_{x,y}}$ is an odd cycle of length ≥ 5 in H_i that does not contain x in contradiction to (ii). Thus $e_1 = z_1 u_2$ is an edge in F_{H_i} parallel to y and we get a contradiction to F_H simple. In the remaining case $v = z_2$ (see Figure 4.4(c)), $E_i \cup P_x$ is a cycle in H_i running through x and y . So E_i is the single edge $e_1 = z_1 z_2$ in H_i according to (iii). Hence e_1 becomes an edge of F_{H_i} parallel to $e_{x,y}$ and we have a contradiction to F_H simple again. \square

Remark. Every cycle $C \subseteq H$ of odd length ≥ 5 yields an odd hole in $L(H) \subseteq G - xy$. Since G is perfect and each minimally imperfect subgraph of $G - xy$ is an odd hole by Theorem 2.10, it is straightforward that every such cycle $C \subseteq H$ has to contain x and y . Moreover, an edge xy of G is critical if and only if it corresponds to an H-pair x and y in F according to Theorem 2.10 again. Recall the definition of an H-pair on page 23: two incident edges x and y form an H-pair iff there is a $K_{1,3}$ with edges $x, y, e_{x,y}$ and an even cycle $C_{x,y}$ that contains x and y but only one endnode of $e_{x,y}$ (see Figure 2.2(a)). Then $L(C_{x,y} \cup e_{x,y}) - xy$ is an odd hole of $G - xy$ (see Figure 2.2(b)) and it is not surprising that every cycle $C \subseteq H$ of odd length ≥ 5 corresponds to $e_{x,y}$ and such an even cycle $C_{x,y}$ in F_H .

To accumulate some findings of the above lemma, we introduce the following notions for later convenience.

Agreement. Let H stand for a simple 2-connected hypomatchable graph with $x, y, e_{x,y} \in E(H)$, $|V(H)| \geq 5$ and F_H for the simple line-perfect graph with $x, y, e_{x,y} \in E(F_H)$ and $L(H) = L(F_H) - xy$. Denote the endnodes of the edges $x, y, e_{x,y}$ as follows: $x = u_1z_1$, $y = z_2u_2$, $e_{x,y} = z_1z_2$ in H and $x = u_1z$, $y = u_2z$, $e_{x,y} = u_3z$ in F_H (all other edges have the same endnodes in H and F_H). Further, if $H_0, H_1, \dots, H_k = H$ is an ear decomposition of H , write F_{H_i} for the partial subgraph of F_H with $L(H_i) = L(F_{H_i}) - xy$ for $0 \leq i \leq k$.

The next two lemmas provide descriptions of ear decompositions of H in the two cases that H does not contain and contains, respectively, a cycle of length 4 through x and y .

Lemma 4.3 *If H has no cycle of length four running through x and y , then H admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$ with $x, y \in E(H_0)$ such that*

- (i) *for every ear E_i with $1 \leq i \leq j$ of length ≥ 3 , there is no even path between its endnodes in $F_{H_{i-1}} - z$,*
- (ii) *for every ear E_i with $j < i \leq k$ of length 1, there is no edge and no even path between its endnodes in $F_{H_{i-1}} - z$,*
- (iii) *H is triangle-free, and F_H is bipartite.*

Remark. Note that the edge u_1u_2 must not occur in H since H has no cycle of length four running through x and y . The assertions (i) and (ii) include that the endnodes of every ear E_1, \dots, E_k are distinct from z_1 and z_2 . In other words, F_H consists of $e_{x,y}$ and the union of some even cycles $C_{x,y}$. We say in the sequel that $F_H - u_3$ admits a **pseudo-ear decomposition** E'_0, E_1, \dots, E_k starting with one of those even cycles $C_{x,y}$ as E'_0 where the endnodes of every ear E_1, \dots, E_k are distinct from z .

Proof. Let H does not admit a cycle of length 4 through x and y . Then H does not have any cycle through x and y with even length by Lemma 4.2(iii). Since H is 2-connected with $|V(H)| \geq 5$ (according to the above agreement), it admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$ by Lemma 4.1. I.e., H_i is 2-connected for $0 \leq i \leq k$, H_0 is an odd cycle of length ≥ 5 , and there is an index $j \in \{1, \dots, k\}$ such that E_1, \dots, E_j have

length ≥ 3 and E_{j+1}, \dots, E_k have length one. In order to prove the assertions of this lemma we show the following two claims.

Claim 1. *H is triangle-free.*

H_0 is triangle-free. So if H contains a triangle, there is an index $i > 0$ with H_{i-1} triangle-free but H_i not. Let Δ denote a triangle in H_i , so $\Delta \cap E_i \neq \emptyset$ holds. E_i has two distinct endnodes v, v' and odd length, so $\Delta \cap E_i \neq \emptyset$ implies $E_i = vv'$. Let v'' denote the third node of Δ . In particular, we know that v, v' , and v'' are distinct from z_1, z_2 according to Lemma 4.2(v). Since H_i is 2-connected, there is a cycle running through the edges $e_{x,y} = z_1 z_2$ and E_i , let C denote the shortest such cycle. For $v'' \notin C$ let $C' = (C \cup \{v''\}) - vv'$. If $v'' \in C$, the choice of C implies that v'' is a neighbor of either v or v' in C , and so define either $C' = (C - \{v\}) \cup v'v''$ or $C' = (C - \{v'\}) \cup vv''$, respectively. In every case, the cycle C' also contains the edge $e_{x,y}$, so both cycles C and C' run through x and y by Lemma 4.2(v). That either C or C' has even length contradicts the condition on H and Lemma 4.2(iii). \diamond

Claim 2. *For every ear E_i , there is no even path between its endnodes in $F_{H_{i-1}} - z$.*

Lemma 4.2(v) implies that the endnodes v, v' of E_i are distinct from z_1, z_2 . If there is an even (v, v') -path P in $H_{i-1} - e_{x,y}$, then $P \cup E_i$ is an odd cycle in H_i and F_{H_i} . Since F_{H_i} is line-perfect, we deduce that $P \cup E_i$ is a triangle. The contradiction to Claim 1 yields that every (v, v') -path in $H_{i-1} - e_{x,y}$ and, therefore, in $F_{H_{i-1}} - z$ has odd length. \diamond

Since there is no ear with z_1 or z_2 as endnode by Lemma 4.2(v), every ear E_1, \dots, E_k is determined by Claim 2. The endnodes v, v' of an ear E_i with length one must not be adjacent in H_{i-1} since H is simple. Hence the assertions (i) and (ii) are true. Finally, H is triangle-free by Claim 1 and, therefore, $F_H - z$ is so. Since there is no cycle of length 4 containing x and y in H , F_H is triangle-free. This proves assertion (iii). \square

Lemma 4.4 *If H has a cycle of length four running through x and y , then H admits a proper ear decomposition $H_0, H_1 = H$ with $x, y \in E(H_0)$ such that*

- (i) H_0 is an odd cycle of length 5 and
- (ii) E_1 is the edge $u_1 u_2$.

Proof. Let a cycle of length 4 run through x and y in H and $H_0, H_1, \dots, H_k = H$ be a proper ear decomposition of H . H_0 has length 5 by Lemma 4.2(iv), this proves assertion (i). Moreover, since each H_i is 2-connected every ear $E_i \neq E_0$ has two distinct endnodes which are different from z_1 and z_2 according to Lemma 4.2(v). Assume there is an ear E_i with $i \in \{1, \dots, k\}$ minimal such that E_i has length ≥ 3 . Then $V(H_{i-1}) = V(H_0)$ holds. Hence, there is a cycle in H_i that contains all edges of E_i and $x, e_{x,y}, y$, since z_1 and z_2 are distinct from the endnodes of E_i . This cycle has length ≥ 6 and we get a contradiction to Lemma 4.2(iv). Thus every ear E_1, \dots, E_k of H must be just an edge. Parallel edges to edges of H_0 are forbidden since H is simple by Lemma 4.2(i), chords of

H_0 incident to z_1 and z_2 must not occur by Lemma 4.2(v). So the only possible ears connect u_1 and u_2 . We have at least one ear u_1u_2 since H contains a cycle of length 4 through x and y , and at most one ear u_1u_2 since H is simple and, therefore, assertion (ii) is true. \square

The result of EDMONDS and PULLEYBLANK [17] characterizes hypomatchable graphs H which are facet-inducing for matching polytopes. Hence, the line graphs $L(H)$ induce rank constraints (4.1d) of the stable set polytope associated with $G - xy$. The two previous lemmas describe the hypomatchable graphs H with $L(H) \subseteq G - xy$ and the corresponding subgraphs $F_H \subseteq F$. This enables us to show how to lift the rank constraint associated with $L(H) = L(F_H) - xy$ to a weak rank facet of $\text{STAB}(G - xy)$. We distinguish the two cases $|V(H)| > 5$ and $|V(H)| = 5$. The first theorem treats the case $|V(H)| > 5$ where H is given by Lemma 4.3. Again, we use the agreement for H and F_H from page 77.

Theorem 4.5 *Let G be the line graph of a simple line-perfect graph F and xy be a critical edge of G . Then every subgraph $F_H \subseteq F$ induces in $\text{STAB}(G - xy)$ the (weak) rank facet*

$$\sum_{v_i \in L(F_H) - xy} x_i + \sum_{v_i \in N_{x,y}} x_i \leq \alpha(L(F_H) - xy) \quad (4.2a)$$

with $N_{x,y} = \{v_i \in (G - xy) - L(F_H) : v_i \in N(x) \cap N(y)\}$ if

- (i) $x, y \in E(F_H)$ but $u_1u_2 \notin E(F_H)$ holds,
- (ii) $F_H - \{z, u_3\}$ is an induced subgraph of F , and
- (iii) there is a 2-connected, hypomatchable graph H with $|V(H)| > 5$, $u_1u_2 \notin E(H)$, and $L(H) = L(F_H) - xy$.

Proof. Let F be simple and line-perfect, $G = L(F)$, and $F_H \subseteq F$ fulfill the conditions (i), (ii), and (iii). In order to prove the assertion of the theorem, we essentially make use of the correspondence between stable sets in $L(F_H) - xy$ and matchings in H . First, we show that the inequality (4.2a) is valid for $\text{STAB}(G - xy)$ by the following claim.

Claim 1. *There is no maximum stable set S of $L(F_H) - xy$ such that $S \cup \{v_i\}$ is a stable set in $G - xy$ for any $v_i \in N_{x,y}$.*

Assume in contrary there is a maximum stable set S of $L(F_H) - xy$ and some $v_i \in N_{x,y}$ adjacent to none of the nodes in S . Let e_i be the edge of F corresponding to the node v_i of $G - xy$ and S correspond to a maximum matching M of H . By $v_i \in N_{x,y}$ but $u_1u_2 \notin E(F)$, the edge e_i is incident to the common endnode z of x , $e_{x,y}$, and y in F , hence M must neither contain x , $e_{x,y}$, nor y . So M does not meet the two nodes z_1 and z_2 of H by Lemma 4.2(v) in contradiction to M maximum. \diamond

According to the definition of $N_{x,y} \subseteq V(G - xy)$, the corresponding edges in $E(F)$ are all incident to z by $u_1u_2 \notin E(F)$. Hence $N_{x,y}$ induces a clique in $G - xy$ and Claim 1 yields that (4.2a) is valid for $\text{STAB}(G - xy)$ by $\alpha(G[L(F_H) \cup N_{x,y}] - xy) = \alpha(L(F_H) - xy)$. In order to find $|V(G)|$ many stable sets in $G - xy$ that satisfy (4.2a) with equality and have

affinely independent incidence vectors, we use the result of EDMONDS and PULLEYBLANK [17] as starting point. They showed that a hypomatchable subgraph is facet-inducing for the matching polytope of the whole graph iff it is 2-connected and induced. By the conditions (ii), (iii), Lemma 4.2(iii), and Lemma 4.2(v), F_H corresponds to an induced, 2-connected, hypomatchable graph H (recall $u_3 \notin V(H)$ by Lemma 4.2(iii), z_1 and z_2 have degree two in H by Lemma 4.2(v)). We explicitly give here a construction similarly to that done in [17], since we will need it for the proof of Claim 3.

Claim 2. *There is a set \mathcal{M} of $|E(H)|$ maximum matchings of H the incidence vectors of which are affinely independent.*

H satisfies the conditions of Lemma 4.3 and has a proper ear decomposition H_0, H_1, \dots, H_k as described there, especially, H_i is 2-connected for $0 \leq i \leq k$. By induction each H_i has $|E(H_i)|$ maximum matchings whose incidence vectors are affinely independent: The assertion is obviously satisfied by the odd cycle H_0 . Suppose the result is true for H_{i-1} and consider H_i . Let v and v' be the endnodes of the ear E_i (recall: $v \neq v'$ by H_i 2-connected). By induction there are $|E(H_{i-1})|$ independent maximum matchings of H_{i-1} . Each can be extended to a maximum matching of H_i by adding the 2nd, 4th, \dots edges of E_i . Then $|E(E_i)| - 1$ additional matchings can be obtained by considering each internal node w of E_i , if one constructs a matching of E_i which does not meet w and $w' \in \{v, v'\}$ such that the (w, w') -path in E_i has odd length. We extend this matching of E_i to a maximum matching of H_i by adding a matching of H_{i-1} which does not meet only the endnode of E_i distinct from w' . This is the only matching constructed so far not meeting w , so it is independent of all others. We get the last matching by starting with the 1st, 3rd, \dots edges of E_i and combining it with a perfect matching of $H_{i-1} - v$ from which the edge incident to v' has been removed (if the ear E_i has length one, this is the only new matching to construct). The last matching contains one edge of H_{i-1} less than the others, so it is independent of all others. By this construction, we obtain the studied set \mathcal{M} of $|E(H)|$ maximum matchings of H whose incidence vectors are affinely independent. \diamond

For later convenience, let \mathcal{M}_v denote the set of matchings belonging to \mathcal{M} which do not meet $v \in V(H)$, i.e., which are perfect matchings of $H - v$. Since matchings of H correspond to stable sets of $L(H) = L(F_H) - xy$, \mathcal{M} represents a set of $|E(H)| = |E(F_H)| = n_H$ many stable sets, say S_1, \dots, S_{n_H} of $G - xy$. In order to show that (4.2a) is a facet of $\text{STAB}(G - xy)$, we construct the remaining stable sets S_{n_H+1}, \dots, S_n of $G - xy$ with $n = |E(F)| = |V(G)|$ such that S_1, \dots, S_n satisfy (4.2a) with equality and have affinely independent incidence vectors. For that, we partition the edge set $E(F) - E(F_H)$ with respect to F_H as follows:

$$\begin{aligned} N_0 &:= \{e_i \in E(F) - E(F_H) : e_i = vv' \text{ with } v, v' \notin V(F_H)\} \\ N_1 &:= \{e_i \in E(F) - E(F_H) : e_i = vw \text{ with } v \notin V(F_H), w \in V(F_H) - z\} \\ N_2 &:= \{e_i \in E(F) - E(F_H) : e_i = u_3w \text{ with } w \in V(F_H) - z\} \\ N_{x,y} &:= \{e_i \in E(F) - E(F_H) : e_i = zw\} \end{aligned}$$

(note that edges in $E(F) - E(F_H)$ with both endnodes in $V(F_H)$ are incident to either

u_3 or z by (ii), hence those edges are contained in either N_2 or $N_{x,y}$). We have to find, for each $e_i \in N_0 \cup N_1 \cup N_2$, a matching $M \in \mathcal{M}$ such that $M \cup \{e_i\}$ corresponds to a stable set of $G - xy$. Clearly, we cannot find such a matching for any $e_i \in N_{x,y}$ by Claim 1. But $\forall e_i \in N_{x,y}$, we have to look for a matching $M \in \mathcal{M}$ such that $(M - \{e_{x,y}\}) \cup \{e_i\}$ corresponds to a stable set of $G - xy$. For that, we provide the following claim (note: it is easy to find some appropriate maximum matching M of H , but we have to look for a suitable one which is a member of \mathcal{M}).

Claim 3. *Let $w \in V(F_H) - \{z, u_3\}$. If w has the same color as z in F_H , then there is a matching $M \in \mathcal{M}_w$ with $x, y \in M$. Otherwise, there is $M \in \mathcal{M}_w$ with $e_{x,y} \in M$.*

F_H is bipartite by Lemma 4.3 and H admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$ as described there. We inductively construct a maximum matching M_j of that H_j for $j \in \{0, \dots, k\}$ minimal with $w \in V(H_j)$. The assertion is satisfied for $j = 0$ in both cases by Lemma 4.2(iii). Assume the assertion is true for $j = i - 1$ and consider the case $j = i$. Then w is an internal node of the ear E_i . Let v' be the endnode of E_i with the same color as w and let v be the other one. (Note: E_i connects nodes of different colors in F_H by Lemma 4.3(i).) Then by induction there is a perfect matching M_{i-1} of $H_{i-1} - v'$ that fulfills the assertion. We extend M_{i-1} to M_i by a maximum matching of E_i that contains the edge of E_i incident to v' and does not meet w and v . We yield the studied matching M of H by extending M_i if we choose the 2nd, 4th, \dots edges of every ear E_{i+1}, \dots, E_k : M contains either x, y or $e_{x,y}$, it does not meet w but all other nodes of H , and by comparing the constructions used here and in Claim 2, M belongs to \mathcal{M} . \diamond

Now, we are prepared to construct the stable sets S_{n_H+1}, \dots, S_n of $G - xy$ which we need, one for each $e_i \in E(F) - E(F_H)$:

$$S_i = \begin{cases} \{e_i\} \cup M & \text{if } e_i \in N_0 \text{ with } M \in \mathcal{M} \\ \{e_i\} \cup M & \text{if } e_i \in N_1 \text{ with } M \in \mathcal{M}_w \text{ for } w \neq u_3, \\ & \text{with } M \in \mathcal{M}_{z_1} \text{ for } w = u_3 \\ \{e_i\} \cup M & \text{if } e_i \in N_2 \text{ with } M \in \mathcal{M}_w \text{ and } x, y \in M \\ \{e_i\} \cup (M - e_{x,y}) & \text{if } e_i \in N_{x,y} \text{ with } M \in \mathcal{M} \text{ and } e_{x,y} \in M \text{ for } w \notin V(F_H), \\ & \text{with } M \in \mathcal{M}_w \text{ and } e_{x,y} \in M \text{ for } w \in V(F_H). \end{cases}$$

(For each $e_i = u_3w \in N_2$, the node w has the same color as z in F_H since F is line-perfect, hence Claim 3 yields the existence of the above matching in \mathcal{M}_w . For each $e_i = zw \in N_{x,y}$ with $w \in V(F_H)$, the node w is distinct from u_3 and not contained in the same color class of the bipartite graph F_H as z , otherwise we have a contradiction either to F line-perfect or to $|V(H)| > 5$. Again, Claim 3 yields the existence of the studied matching in \mathcal{M}_w .)

The incidence vectors $\chi^{S_{n_H+1}}, \dots, \chi^{S_n}$ are affinely independent, since $\chi_i^{S_i}$ is the only non-zero entry of $\chi_i^{S_1}, \dots, \chi_i^{S_n} \forall e_i \in E(F) - E(F_H)$ and $\lambda_i = 0$ follows for $n_H + 1 \leq i \leq n$ by

$$\sum_{1 \leq i \leq n} \lambda_i \chi^{S_i} = 0.$$

The vectors $\chi^{S_1}, \dots, \chi^{S_n}$ are affinely independent by Claim 2. Finally, the stable sets S_1, \dots, S_n are chosen such that $|S_i \cap G[L(F_H) \cup N_{x,y}]| = \alpha(L(F_H) - xy)$ holds for $1 \leq i \leq n$, i.e., $\chi^{S_1}, \dots, \chi^{S_n}$ satisfy (4.2a) with equality and so the constraint induces a facet of $\text{STAB}(G - xy)$. \square

Remark. At the beginning of this section, we have mentioned that the graphs $L(H) \subseteq G - xy$ are odd holes or the union of some odd holes. From Theorem 2.10 we know that odd holes are the only minimally imperfect subgraphs of $G - xy$ and every critical edge xy of G corresponds to an H-pair x, y in F . We have remarked to Lemma 4.3 that F_H consists of an edge $e_{x,y}$ and the union of some even cycles $C_{x,y}$. Recall that $L(C_{x,y} \cup e_{x,y}) - xy$ is an odd hole in $G - xy$ for every of those cycles $C_{x,y}$ (cf. Figure 2.2(a),(b) on page 23). So it is worth to study the relationship of the H-pair x, y in F and the subgraphs $L(F_H) - xy \subseteq G - xy$ the rank constraints (4.1d) are associated with. Let $\mathcal{N}_{x,y} = \{e \in E(F) : e = zw, w \neq u_1, u_2\}$ be the set of all edges incident to x and y in z (we still use the agreement for F_H on page 77). Furthermore, let $\mathcal{C}_{x,y}$ denote the set of all even cycles $C_{x,y} \subseteq F$ through x and y such that there is an edge $zw \in \mathcal{N}_{x,y}$ with $w \notin C_{x,y}$. Then $F_H - u_3$ corresponds to a subset $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ (cf. the remark to Lemma 4.3), and $zu_3 \cup N_{x,y}$ to $\mathcal{N}_{x,y}$. I.e., $F_H \cup N_{x,y} = \mathcal{C}_{x,y}^H \cup \mathcal{N}_{x,y}$ holds and constraint (4.2a) can be read as

$$\sum_{e_i \in \mathcal{C}_{x,y}^H} x_i + \sum_{e_i \in \mathcal{N}_{x,y}} x_i \leq \frac{1}{2} |V(\mathcal{C}_{x,y}^H)|.$$

We may combine $\mathcal{C}_{x,y}^H$ with every suitable $e_{x,y} \in \mathcal{N}_{x,y}$ to a graph F_H , and we always get the above facet as lifting. In order to find the facets (4.2a) in $\text{STAB}(G - xy)$ we look for the subsets $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ with a pseudo-ear decomposition such that there is an edge $zw \in \mathcal{N}_{x,y}$ with $w \notin \mathcal{C}_{x,y}^H$ (cf. the remark to Lemma 4.3). Hence, using the notions introduced above, we can specify Theorem 4.5 as follows: *Let G be the line graph of a simple line-perfect graph F and x, y an H-pair in F . Then every subset $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ with $|V(\mathcal{C}_{x,y}^H)| > 4$ induces in $\text{STAB}(G - xy)$ the rank facet*

$$\sum_{e_i \in \mathcal{C}_{x,y}^H} x_i + \sum_{e_i \in \mathcal{N}_{x,y}} x_i \leq \frac{1}{2} |V(\mathcal{C}_{x,y}^H)|$$

if $\mathcal{C}_{x,y}^H$ admits a pseudo-ear decomposition, there is an edge $zw \in \mathcal{N}_{x,y}$ with $w \notin \mathcal{C}_{x,y}^H$, and $\mathcal{C}_{x,y}^H - z$ is an induced subgraph of F . I.e., all graphs $L(F_H) - xy \in G - xy$ that base on the same $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ with $|V(\mathcal{C}_{x,y}^H)| > 4$ yield the same rank facet of $\text{STAB}(G - xy)$. In order to illustrate that, look at the line-perfect graph F and its line graph G shown in Figure 4.5.

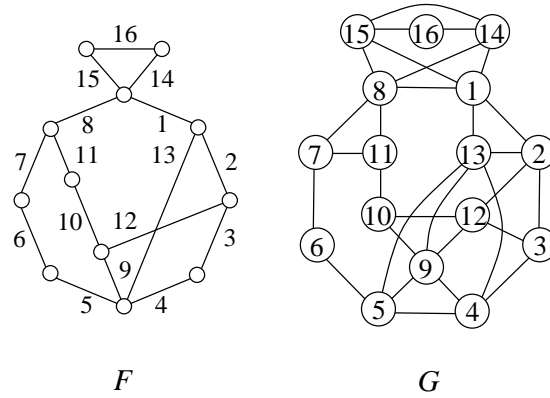


Figure 4.5

The edges $x = e_1$ and $y = e_8$ form an H-pair in F . We get $\mathcal{N}_{x,y} = \{e_{14}, e_{15}\}$ and find in F

$$\begin{aligned} C_{x,y}^{(1)} &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \\ C_{x,y}^{(2)} &= \{e_1, e_2, e_3, e_4, e_9, e_{10}, e_{11}, e_8\} \\ C_{x,y}^{(3)} &= \{e_1, e_2, e_{12}, e_9, e_5, e_6, e_7, e_8\} \\ C_{x,y}^{(4)} &= \{e_1, e_2, e_{12}, e_{10}, e_{11}, e_8\} \\ C_{x,y}^{(5)} &= \{e_1, e_{13}, e_4, e_3, e_{12}, e_{10}, e_{11}, e_8\} \\ C_{x,y}^{(6)} &= \{e_1, e_{13}, e_5, e_6, e_7, e_8\} \\ C_{x,y}^{(7)} &= \{e_1, e_{13}, e_9, e_{10}, e_{11}, e_8\} \end{aligned}$$

as cycles that run through x and y . Since the two edges in $\mathcal{N}_{x,y}$ have only one endnode on each of those cycles, $C_{x,y}^{(1)}, \dots, C_{x,y}^{(7)}$ build the set $\mathcal{C}_{x,y}$. Eight subsets of $\mathcal{C}_{x,y}$ are suitable: the cycles $C_{x,y}^{(4)}, C_{x,y}^{(6)}, C_{x,y}^{(7)}$ as cases (a),(b),(c) and the five subsets in Figure 4.6(d), \dots , (h).

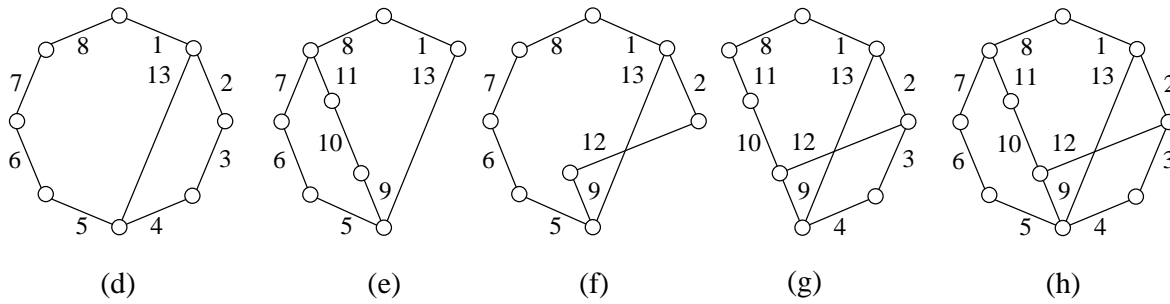


Figure 4.6

The next table lists, for the five non-trivial cases (d), \dots , (h), of which cycles those subsets consist and gives a pseudo-ear decomposition E'_0, E_1, \dots, E_k .

- (d) $\mathcal{C}_{x,y}^H = C_{x,y}^{(1)} \cup C_{x,y}^{(6)}, \quad E'_0 = C_{x,y}^{(1)}, E_1 = \{e_{13}\}$
(e) $\mathcal{C}_{x,y}^H = C_{x,y}^{(6)} \cup C_{x,y}^{(7)}, \quad E'_0 = C_{x,y}^{(6)}, E_1 = \{e_9, e_{10}, e_{11}\}$
(f) $\mathcal{C}_{x,y}^H = C_{x,y}^{(3)} \cup C_{x,y}^{(6)}, \quad E'_0 = C_{x,y}^{(3)}, E_1 = \{e_{13}\}$
(g) $\mathcal{C}_{x,y}^H = C_{x,y}^{(2)} \cup C_{x,y}^{(4)} \cup C_{x,y}^{(5)} \cup C_{x,y}^{(7)}, E'_0 = C_{x,y}^{(2)}, E_1 = \{e_{12}\}, \quad E_2 = \{e_{13}\}$
(h) $\mathcal{C}_{x,y}^H = \mathcal{C}_{x,y}, \quad E'_0 = C_{x,y}^{(1)}, E_1 = \{e_9, e_{10}, e_{11}\}, E_2 = \{e_{12}\}, E_3 = \{e_{13}\}$

Hence $\text{STAB}(G - xy)$ admits the following eight facets of type (4.2a).

- (a) $x_1 + x_2 \quad \quad \quad + x_8 \quad \quad + x_{10} + x_{11} + x_{12} \quad \quad + x_{14} + x_{15} \leq 3$
(b) $x_1 \quad \quad \quad + x_5 + x_6 + x_7 + x_8 \quad \quad \quad + x_{13} + x_{14} + x_{15} \leq 3$
(c) $x_1 \quad \quad \quad + x_8 + x_9 + x_{10} + x_{11} \quad \quad + x_{13} + x_{14} + x_{15} \leq 3$
(d) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \quad \quad \quad + x_{13} + x_{14} + x_{15} \leq 4$
(e) $x_1 \quad \quad \quad + x_5 + x_6 + x_7 + x_8 + x_9 \quad x_{10} + x_{11} \quad \quad + x_{13} + x_{14} + x_{15} \leq 4$
(f) $x_1 + x_2 \quad \quad \quad + x_5 + x_6 + x_7 + x_8 + x_9 + \quad \quad \quad + x_{12} + x_{13} + x_{14} + x_{15} \leq 4$
(g) $x_1 + x_2 + x_3 + x_4 \quad \quad \quad + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 4$
(h) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 5$

If we consider another H-pair $x = e_1, y = e_2$ in F , we get $\mathcal{N}_{x,y} = \{e_{13}\}$ and there are the following four even cycles in F containing x and y .

$$\begin{aligned} C_{x,y}^{(1)} &= \{e_1, e_8, e_7, e_6, e_5, e_4, e_3, e_2\} \\ C_{x,y}^{(2)} &= \{e_1, e_8, e_7, e_6, e_5, e_9, e_{12}, e_2\} \\ C_{x,y}^{(3)} &= \{e_1, e_8, e_{11}, e_{10}, e_9, e_4, e_3, e_2\} \\ C_{x,y}^{(4)} &= \{e_1, e_8, e_{11}, e_{10}, e_{12}, e_2\} \end{aligned}$$

Since the only edge in $\mathcal{N}_{x,y}$ is a chord of $C_{x,y}^{(1)}, C_{x,y}^{(2)}$, and $C_{x,y}^{(3)}$ (see Figure 4.5), these three cycles do not belong to $\mathcal{C}_{x,y}$ by definition. Hence $\mathcal{C}_{x,y} = C_{x,y}^{(4)}$ holds and, therefore, $C_{x,y}^{(4)}$ is the only subset of $\mathcal{C}_{x,y}$ which fulfills all the conditions above. Thus $\text{STAB}(G - xy)$ admits only one facet of type (4.2a) which is a zero-lifted odd hole constraint (4.1b) in particular.

$$x_1 + x_2 \quad \quad \quad + x_8 \quad \quad + x_{10} + x_{11} + x_{12} + x_{13} \leq 3$$

Let us consider another H-pair in F , namely $x = e_4$ and $y = e_5$. Here $\mathcal{N}_{x,y} = \{e_9, e_{13}\}$ holds and two even cycles run through x and y .

$$\begin{aligned} C_{x,y}^{(1)} &= \{e_4, e_3, e_2, e_1, e_8, e_7, e_6, e_5\} \\ C_{x,y}^{(2)} &= \{e_4, e_3, e_{12}, e_{10}, e_{11}, e_7, e_6, e_5\} \end{aligned}$$

Since $e_9 \in \mathcal{N}_{x,y}$ has only one endnode on $C_{x,y}^{(1)}$ and $e_{13} \in \mathcal{N}_{x,y}$ has only one endnode on $C_{x,y}^{(2)}$, the set $\mathcal{C}_{x,y}$ consists in those two cycles. We find three subsets of $\mathcal{C}_{x,y}$ which admit a pseudo-ear decomposition E'_0, E_1, \dots, E_k , namely $C_{x,y}^{(1)}$ as case (a), $C_{x,y}^{(2)}$ as case (b), and

$$(c) \quad \mathcal{C}_{x,y}^H = \mathcal{C}_{x,y} \quad E'_0 = C_{x,y}^{(1)} \quad E_1 = \{e_{11}, e_{10}, e_{12}\}.$$

There is no edge in $\mathcal{N}_{x,y}$ with only one endnode in the whole set $\mathcal{C}_{x,y}$, hence the latter case (c) does not fulfill the conditions above. $\text{STAB}(G - xy)$ admits, therefore, only two facets of type (4.2a).

$$\begin{aligned} (a) \quad & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 & + x_{13} & \leq 4 \\ (b) \quad & & x_3 + x_4 + x_5 + x_6 + x_7 & + x_9 + x_{10} + x_{11} + x_{12} + x_{13} & \leq 4 \end{aligned}$$

We have studied how to lift the rank constraints (4.1d) basing on a graph $L(H) \subseteq G - xy$ with $|V(H)| > 5$ to a weak rank facet (4.2a) of $\text{STAB}(G - xy)$. Now, let us turn to the second case where $|V(H)| = 5$ holds (we still use the agreement from page 77). If H does not admit any cycle of length 4 running through x and y , i.e., if $u_1u_2 \notin E(H)$, H admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$ as given by Lemma 4.3. We deduce $H_0 = C_5$ and $k = 0$, since H admits only five nodes and is simple. Otherwise, for $u_1u_2 \in E(H)$, the only possible ear decomposition $H_0, H_1 = H$ is given by Lemma 4.4. Then $H_0 = C_5$ and $E_1 = u_1u_2$ follows. In the latter case, a weak rank facet may appear in $\text{STAB}(G - xy)$ which does not base on $L(H)$ but only on $L(H_0)$, i.e., on the odd hole of length five itself. Thus we concentrate our attention on this $C_5 \subseteq G - xy$. To state and prove the following theorem, we make use of the convention to number the nodes of $C_5 \subseteq G - xy$ introduced in Section 2.1 for convenience: let v_1, \dots, v_5 be the nodes of C_5 and $v_i v_{i+1}$ for $1 \leq i \leq 5 \pmod{5}$ its edges, where $e = v_1v_4$ is the critical edge and $v_1v_4v_5$ the triangle in $C_5 + e$ (i.e., we have $x = e_1$, $y = e_4$, and $e_{x,y} = e_5$). Furthermore, we define in $(G - e) - C_5$ three node sets with respect to C_5 by

$$\begin{aligned} N_3 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_4, v_5\}\}, \\ N_4 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_2, v_3, v_4\}\}, \\ N_5 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_2, v_3, v_4, v_5\}\}. \end{aligned}$$

(Note that $N_3 \cup N_5$ corresponds to the set $N_{x,y}$ from Theorem 4.5, N_4 is either empty or contained in $L(F_H)$, and the sets N_4 and N_5 consist of at most one node.)

Theorem 4.6 *Let G be the line graph of a simple line-perfect graph F and $e = v_1v_4$ a critical edge of G . Every $C_5 \subseteq G - e$ yields in $\text{STAB}(G - e)$ the facet*

$$\sum_{v_i \in C_5} x_i + \sum_{v_i \in N_3 \cup N_4 \cup N_5} x_i \leq 2 \quad (4.2a)$$

except $N_4 = \emptyset$ but $N_5 \neq \emptyset$ and, except the case $N_4 \neq \emptyset$ but $N_5 = \emptyset$, in addition the facet

$$\sum_{v_i \in C_5} x_i + \sum_{v_i \in N_3} x_i + 2 \sum_{v_i \in N_5} x_i \leq 2. \quad (4.2b)$$

Remark. To illustrate the assertion of this theorem, look at the odd holes with nodes v_1, \dots, v_5 that arise in the line graphs in Figure 4.7 by the deletion of the critical edge v_1v_4 . Both facets (4.2a) and (4.2b) appear only in the case $N_4 \neq \emptyset$ and $N_5 \neq \emptyset$ (see Figure 4.7(a), here the facet of type (4.2b) does not base on the rank constraint (4.1d) associated with the $C_5 \cup N_4$, but only on the odd hole constraint (4.1b) associated with the C_5). For $N_4 = \emptyset$ but $N_5 \neq \emptyset$ (see Figure 4.7(b)), the C_5 only induces a facet of type

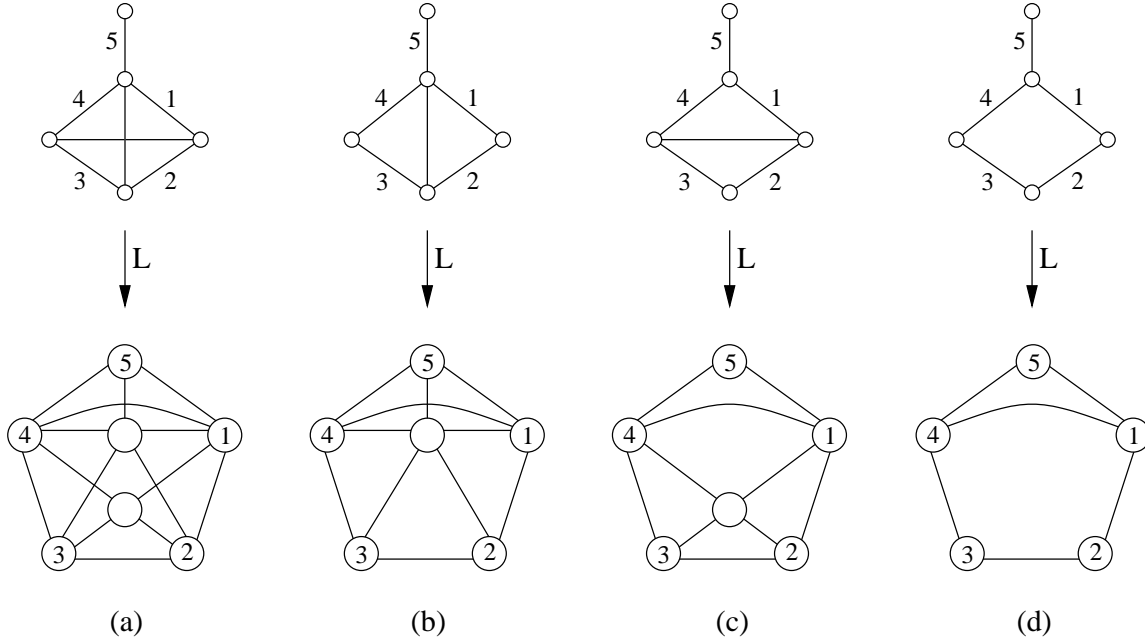


Figure 4.7

(4.2b). For $N_4 \neq \emptyset$ but $N_5 = \emptyset$ (see Figure 4.7(c)), the C_5 defines only a rank facet of type (4.2a). In the case $N_4 = N_5 = \emptyset$ shown in Figure 4.7(d), the facets (4.2a) and (4.2b) coincide.

Proof. Let G be the line graph of a simple line-perfect graph F , $e = v_1v_4$ a critical edge of G , and $n = |E(F)| = |V(G)|$. Consider the corresponding odd hole $C_5 \subseteq G - e$ with nodes v_1, \dots, v_5 . We partition the nodes v_i in $(G - e) - C_5$ with respect to their neighborhood on C_5 . Looking at the underlying graph F , we know by Theorem 2.10 that $C_5 + e$ is the image of an even cycle $C \subseteq F$ with edges e_1, \dots, e_4 and of e_5 that is incident to e_1 and e_4 but has only one endnode on C . Listing all possibilities for edges e_i with $6 \leq i \leq n$ in F to be incident to the edges e_1, \dots, e_5 yields the studied partition of the nodes in $(G - e) - C_5$:

$$\begin{aligned}
 N_0 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \emptyset\} \\
 N_1 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_5\}\} \\
 N_2 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_j, v_{j+1}\} \text{ with } 1 \leq j \leq 3 \text{ or} \\
 &\quad \{v_j, v_{j+1}, v_5\} \text{ with } j = 2\} \\
 N_3 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_4, v_5\}\}, \\
 N_4 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_2, v_3, v_4\}\}, \\
 N_5 &= \{v_i \in (G - e) - C_5 : N_{C_5}(v_i) = \{v_1, v_2, v_3, v_4, v_5\}\}.
 \end{aligned}$$

(Since F is line-perfect, it cannot admit any odd cycle of length ≥ 5 as partial subgraph. Hence F cannot contain an edge e_i incident to e_2, e_3, e_5 and an edge e_j incident to e_1, e_2, e_3, e_4 , otherwise the odd cycle e_1, e_j, e_3, e_i, e_5 appears in F . For an edge e_i incident to e_j, e_{j+1}, e_5 with $1 \leq j \leq 3$ follows $j = 2$, otherwise either e_i, e_2, e_3, e_4, e_5 or e_1, e_2, e_3, e_i, e_5 is an odd cycle in F .)

Define $A_i = \{v_i, v_{i+2}\}$ (the indices are considered modulo 5), then A_1, \dots, A_5 are the maximum stable sets of C_5 . In order to show that (4.2a) and (4.2b) are facet-inducing for $\text{STAB}(G - e)$ we prove the following two claims for both of them.

Claim 1. *The constraint is valid for $\text{STAB}(G - e)$.*

Claim 2. *In $G - e$ there are stable sets S_1, \dots, S_n whose incidence vectors are affinely independent and satisfy the constraint with equality.*

Proof of Claim 1 for (4.2a). Checking the validity of (4.2a) for $\text{STAB}(G - e)$, it suffices to show $\alpha(C_5 \cup \{v_i, v_{i'}\}) = 2$ for $v_i, v_{i'} \in N_3 \cup N_4 \cup N_5$ with $v_i v_{i'} \notin E(G - e)$. The nodes in $N_3 \cup N_5$ correspond to edges of F that are incident to the common endnode of e_1 and e_4 , hence $N_3 \cup N_5$ induces a clique in $G - e$. So let $v_i \in N_4$ and $v_{i'} \in N_3 \cup N_5$ (recall that N_4 contains at most one node since F is simple). The only non-neighbor of v_i on C_5 is v_5 , which is a neighbor of $v_{i'}$, so $C_5 \cup \{v_i, v_{i'}\}$ has only stable sets of size two, and (4.2a) is valid for $\text{STAB}(G - e)$. \diamond

Proof of Claim 2 for (4.2a). In order to exhibit that the inequality (4.2a) induces a facet of $\text{STAB}(G - e)$, we define the following stable sets S_1, \dots, S_n of $G - e$.

$$S_i = \begin{cases} A_i & \text{if } v_i \in C_5 \\ A_1 \cup \{v_i\} & \text{if } v_i \in N_0 \cup N_1 \\ A_{j+2} \cup \{v_i\} & \text{if } v_i \in N_2 \\ \{v_2, v_i\} & \text{if } v_i \in N_3 \\ \{v_5, v_i\} & \text{if } v_i \in N_4 \\ N_4 \cup \{v_i\} & \text{if } v_i \in N_5 \end{cases}$$

(Recall, $j \in \{1, \dots, 5\}$ is determined by $N_{C_5}(v_i)$ for each $v_i \in N_2$. Furthermore, by the assumption $N_5 = \emptyset$ if $N_4 = \emptyset$, we do not need the stable set $S_n = N_4 \cup N_5$ in the case $N_4 = \emptyset$.) The incidence vectors $\chi^{S_1}, \dots, \chi^{S_n}$ are affinely independent: Due to

$$\sum_{1 \leq i \leq n} \lambda_i \chi^{S_i} = 0,$$

$\lambda_n = 0$ is immediately seen. Then χ^{S_i} is the only nonzero entry of $\chi^{S_1}, \dots, \chi^{S_{n-1}}$ for each $v_i \in N_0 \cup N_1 \cup N_2 \cup N_3 \cup N_4$ and $\lambda_i = 0$ follows $\forall v_i \in (G - e) - C_5$. The vectors $\chi^{S_1}, \dots, \chi^{S_5}$ are obviously affinely independent. Furthermore, the stable sets S_1, \dots, S_n are chosen such that $|S_i \cap (C_5 \cup N_3 \cup N_4 \cup N_5)| = 2$ holds for $1 \leq i \leq n$. I.e., (4.2a) is satisfied with equality by $\chi^{S_1}, \dots, \chi^{S_n}$ and is, consequently, facet-inducing for $\text{STAB}(G - e)$. \diamond

Now, let us turn to constraint (4.2b). Corresponding to the coefficients in this constraint, define a weighting on the nodes of $G - e$ as follows

$$a(v_i) = \begin{cases} 1 & \text{if } v_i \in C_5 \\ 0 & \text{if } v_i \in N_0 \cup N_1 \cup N_2 \\ 1 & \text{if } v_i \in N_3 \\ 0 & \text{if } v_i \in N_4 \\ 2 & \text{if } v_i \in N_5 \end{cases}$$

Proof of Claim 1 for (4.2b). We have clearly $\alpha(C_5, a) = 2$. The only nodes of $G - e$ with $a(v_i) > 0$, not adjacent to any node in N_3 , are the adjacent nodes v_2 and v_3 . The node $v_n \in N_5$ (if it exists at all) is adjacent to all nodes $v_i \in G - v_n$ with $a(v_i) > 0$. We have $a(S) \leq 2$ for all stable sets $S \subseteq G - e$ and (4.2b) is valid for $\text{STAB}(G - e)$. \diamond

Proof of Claim 2 for (4.2b). Recall that the maximum stable sets of the C_5 are denoted by A_1, \dots, A_5 and define the following stable sets of $G - e$.

$$S_i = \begin{cases} A_i & \text{if } v_i \in C_5 \\ A_1 \cup \{v_i\} & \text{if } v_i \in N_0 \cup N_1 \\ A_{j+2} \cup \{v_i\} & \text{if } v_i \in N_2 \\ \{v_2, v_i\} & \text{if } v_i \in N_3 \\ N_5 \cup \{v_i\} & \text{if } v_i \in N_4 \\ \{v_i\} & \text{if } v_i \in N_5 \end{cases}$$

(Again, $j \in \{1, \dots, 5\}$ is determined by $N_{C_5}(v_i)$ for each $v_i \in N_2$. Furthermore, according to the assumption that $N_5 \neq \emptyset$ holds if $N_4 \neq \emptyset$, we are able to build the stable set $S_i = N_5 \cup \{v_i\}$ with $v_i \in N_4$.) The incidence vectors $\chi^{S_1}, \dots, \chi^{S_n}$ are affinely independent:

$$\sum_{1 \leq i \leq n} \lambda_i \chi^{S_i} = 0$$

immediately implies $\lambda_n = 0$, then $\chi_i^{S_i}$ is the only nonzero entry of $\chi_i^{S_1}, \dots, \chi_i^{S_{n-1}}$ for $5 < i < n$ and, therefore, $\lambda_i = 0$ follows for $5 < i < n$. The vectors $\chi^{S_1}, \dots, \chi^{S_5}$ are obviously affinely independent again. Finally, the stable sets S_1, \dots, S_n are chosen such that $a(S_i) = 2$ holds for $1 \leq i \leq n$. Thus $\chi^{S_1}, \dots, \chi^{S_n}$ satisfy (4.2b) with equality, i.e., the constraint is facet-inducing for $\text{STAB}(G - e)$. \square

Remark. Similar to Theorem 4.5, let us reinterpret Theorem 4.6 by looking at the H-pair $x = e_1, y = e_4$ in F that gives rise to the $C_5 \subseteq G - xy$. According to Theorem 2.10, every C_5 in $G - xy$ corresponds in F to an edge $e_{x,y} = e_5$ and a cycle $C_{x,y} = e_1 e_2 e_3 e_4$. In the special case considered here, we have to split the set $\mathcal{N}_{x,y} = \{e \in E(F) : e \text{ forms a } K_{1,3} \text{ with } x \text{ and } y\}$ (see the remark to Theorem 4.5) with respect to the studied cycle $C_{x,y}$ as follows:

$$\begin{aligned} N_5(C_{x,y}) &= \{e \in \mathcal{N}_{x,y} : e \text{ has both endnodes in } C_{x,y}\} \\ N_3(C_{x,y}) &= \mathcal{N}_{x,y} - N_5(C_{x,y}) \end{aligned}$$

(note that N_4 does not depend on the special choice of $C_{x,y}$). Then $e_{x,y} \in N_3(C_{x,y})$, $C_5 \cup N_3 = C_{x,y} \cup N_3(C_{x,y})$ and $N_5 = N_5(C_{x,y})$ holds. Moreover, the set $C_{x,y} \cup N_3(C_{x,y})$

corresponds to the union of $|N_3(C_{x,y})|$ many C_5 's in $G - xy$, hence all C_5 's in $G - xy$ basing on the same $C_{x,y}$ in F yield the same facets. Thus, using the notions introduced above Theorem 4.6 may be restated as follows. *Let G be the line graph of a simple line-perfect graph F and x, y be an H -pair in F . Then every $C_{x,y}$ with $|C_{x,y}| = 4$ and $N_3(C_{x,y}) \neq \emptyset$ induces, if $N_5(C_{x,y}) = \emptyset$ for $N_4 = \emptyset$, in $\text{STAB}(G - xy)$ the rank facet*

$$\sum_{e_i \in C_{x,y}} x_i + \sum_{e_i \in N_{x,y}} x_i + \sum_{e_i \in N_4} x_i \leq 2$$

and, if $N_5(C_{x,y}) \neq \emptyset$ for $N_4 \neq \emptyset$, in addition the facet

$$\sum_{e_i \in C_{x,y}} x_i + \sum_{e_i \in N_3(C_{x,y})} x_i + 2 \sum_{v_i \in N_5(C_{x,y})} x_i \leq 2.$$

For illustration, look at the line-perfect graph F and its line graph G shown in Figure 4.8.

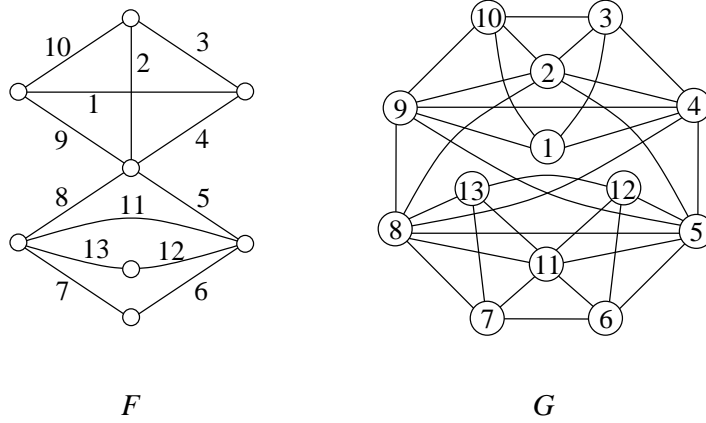


Figure 4.8

The edges $x = e_4$ and $y = e_9$ form an H -pair in F . There is only one even cycle in F through x and y , namely

$$C_{x,y} = \{e_4, e_3, e_{10}, e_9\}, \quad N_3(C_{x,y}) = \{e_5, e_8\}, \quad N_4 = \{e_1\}, \quad N_5(C_{x,y}) = \{e_2\}.$$

By $|N_3(C_{x,y})| = 2$ two odd holes $\{v_4, v_3, v_{10}, v_9, v_5\}$ and $\{v_4, v_3, v_{10}, v_9, v_8\}$ appear in $G - xy$ basing on $C_{x,y}$ and yield common liftings of type (4.2a) and (4.2b).

$$\begin{array}{lll} x_1 + x_2 + x_3 + x_4 + x_5 & + x_8 + x_9 + x_{10} & \leq 2 \\ 2x_2 + x_3 + x_4 + x_5 & + x_8 + x_9 + x_{10} & \leq 2 \end{array}$$

Next, choose the H -pair $x = e_5$ and $y = e_6$ in F . Then we have in F only one even cycle that contains x and y , namely

$$C_{x,y} = \{e_5, e_8, e_7, e_6\}, \quad N_3(C_{x,y}) = \{e_{12}\}, \quad N_4 = \emptyset, \quad N_5(C_{x,y}) = \{e_{11}\}.$$

Hence $G - xy$ contains only one odd hole $\{v_5, v_8, v_7, v_6, v_{12}\}$ by $|N_3(C_{x,y})| = 1$. The rank constraint associated with this odd hole is lifted to only one facet

$$x_5 + x_6 + x_7 + x_8 + 2x_{11} + x_{12} \leq 2$$

of type (4.2b) in $\text{STAB}(G - xy)$, since $N_4 = \emptyset$ but $N_5(C_{x,y}) \neq \emptyset$ holds.

Now, consider the H-pair $x = e_5$ and $y = e_8$ in F . Two cycles $C_{x,y}$ appear in F , and we have to consider the following two cases:

- (a) $C_{x,y}^{(1)} = \{e_5, e_6, e_7, e_8\}$, $N_3(C_{x,y}^{(1)}) = \{e_2, e_4, e_9\}$, $N_4 = \{e_{11}\}$, $N_5(C_{x,y}^{(1)}) = \emptyset$,
- (b) $C_{x,y}^{(2)} = \{e_5, e_{12}, e_{13}, e_8\}$, $N_3(C_{x,y}^{(2)}) = \{e_2, e_4, e_9\}$, $N_4 = \{e_{11}\}$, $N_5(C_{x,y}^{(2)}) = \emptyset$.

For $i \in \{1, 2\}$, $|N_3(C_{x,y}^{(i)})| = 3$ implies that each $C_{x,y}^{(i)}$ causes three different odd holes of length five in $G - xy$. All three odd holes basing on the same cycle $C_{x,y}^{(i)}$ induce one facet of type (4.2a) in $\text{STAB}(G - xy)$ for $i = 1, 2$ (note that $N_5(C_{x,y}^{(1)}) = N_5(C_{x,y}^{(2)}) = \emptyset$ but $N_4 \neq \emptyset$ holds). So we obtain the following two rank facets in $\text{STAB}(G - xy)$.

$$\begin{array}{llll} (a) & x_2 & + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 & + x_{11} & \leq 2 \\ (b) & x_2 & + x_4 + x_5 & + x_8 + x_9 & + x_{11} + x_{12} + x_{13} & \leq 2 \end{array}$$

Turning to the case with $N_4 = N_5(C_{x,y}) = \emptyset$, look at the line-perfect graph F and its line graph G shown in Figure 4.5 again. The edges $x = e_2$ and $y = e_3$ form an H-pair. We find three cycles $C_{x,y}$ that run through x and y .

$$\begin{aligned} C_{x,y}^{(1)} &= \{e_2, e_1, e_8, e_7, e_6, e_5, e_4, e_3\} \\ C_{x,y}^{(2)} &= \{e_2, e_1, e_8, e_{11}, e_{10}, e_9, e_4, e_3\} \\ C_{x,y}^{(3)} &= \{e_2, e_{13}, e_4, e_3\} \end{aligned}$$

The only edge e_{12} incident to x and y has both endnodes on $C_{x,y}^{(2)}$, hence $C_{x,y}^{(2)}$ does not belong to $\mathcal{C}_{x,y}$ by definition and we get $\mathcal{C}_{x,y} = \{C_{x,y}^{(1)}, C_{x,y}^{(3)}\}$. We have to apply Theorem 4.6 to handle the odd hole basing on $C_{x,y}^{(3)}$ (see case (a)) and we have to look for subsets $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ with $|\mathcal{C}_{x,y}^H| > 4$ that satisfy the conditions of Theorem 4.5 (there is only one such subset, see case (b)).

$$\begin{array}{lll} (a) & C_{x,y}^{(3)} = \{e_2, e_3, e_4, e_{13}\} & N_3(C_{x,y}^{(3)}) = \{e_{12}\} \quad N_4 = N_5(C_{x,y}^{(3)}) = \emptyset \\ (b) & \mathcal{C}_{x,y}^H = \mathcal{C}_{x,y} & H'_0 = C_{x,y}^{(1)} \quad E_1 = \{e_{13}\} \end{array}$$

So $C_{x,y}^{(3)}$ induces by Theorem 4.6 a zero-lifted odd hole constraint ($N_4 = N_5(C_{x,y}^{(3)}) = \emptyset$ implies that the two facets (4.2a) and (4.2b) coincide in that case). The whole set $\mathcal{C}_{x,y}$ induces a facet of type (4.2a) by Theorem 4.5, because e_{12} is incident to x and y but has only one endnode in $\mathcal{C}_{x,y}^H$. Thus $\text{STAB}(G - xy)$ admits the following two rank facets of type (4.2a).

$$\begin{array}{llll} (a) & x_2 + x_3 + x_4 & + x_{12} + x_{13} & \leq 2 \\ (b) & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 & + x_{12} + x_{13} & \leq 4 \end{array}$$

Finally, the following observations are worth noting in order to figure out in which situation we have to apply Theorem 4.5 and when Theorem 4.6. Let $x = u_1z$ and $y = zu_2$ be an H-pair in the simple line-perfect graph F and G be the line graph of F .

First, let F contain a cycle $C_{x,y} = u_1zu_2w$ of length four through x and y . For $N_5(C_{x,y}) \neq \emptyset$ (i.e., $C_{x,y}$ has the chord zw), $C_{x,y}$ is the *only* even cycle through x and y in F , otherwise F would possess one of the odd cycles of length at least five emphasized by bold lines in Figure 4.9. Thus, if there is a cycle $C_{x,y} = u_1zu_2w$ in F and $zw \in E(F)$, then $\mathcal{C}_{x,y} = C_{x,y}$ holds (since x and y form an H-pair in F) and $\text{STAB}(G - xy)$ has precisely one facet of type (4.2b) and, if $u_1u_2 \in E(F)$, in addition precisely one facet of type (4.2a) according to Theorem 4.6.

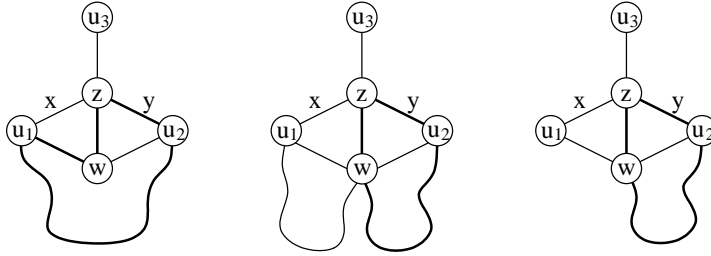


Figure 4.9

For $N_4 \neq \emptyset$ (i.e., F contains the edge u_1u_2), $\mathcal{C}_{x,y}$ does not contain any cycle of length > 4 according to Lemma 4.2(iv). Thus, $\text{STAB}(G - xy)$ has $|\mathcal{C}_{x,y}|$ facets of type (4.2a) and, for each $C_{x,y} \in \mathcal{C}_{x,y}$ with $N_5(C_{x,y}) \neq \emptyset$, one facet of type (4.2b). Hence, for $N_5(C_{x,y}) \neq \emptyset$ for one cycle $C_{x,y} \in \mathcal{C}_{x,y}$ or $N_4 \neq \emptyset$, all weak rank facets of $\text{STAB}(G - xy)$ are described only by the latter Theorem 4.6.

Moreover, $\mathcal{C}_{x,y}$ admits cycles $C_{x,y}$ of length 4 *as well as* cycles of length > 4 only if $N_4 = \emptyset$ and $N_5(C_{x,y}) = \emptyset$ holds for all $C_{x,y} \in \mathcal{C}_{x,y}$ of length 4. Then every odd hole of length five in $G - xy$ induces in $\text{STAB}(G - xy)$ only one rank facet since the two facets described in Theorem 4.6 coincide in that case. Hence, for $N_4 = \emptyset$ and $N_5(C_{x,y}) = \emptyset$ for all $C_{x,y} \in \mathcal{C}_{x,y}$ of length 4, $\text{STAB}(G - xy)$ has precisely one facet of type (4.2a) for every subset $\mathcal{C}_{x,y}^H \in \mathcal{C}_{x,y}$ such that $\mathcal{C}_{x,y}^H - z$ is an induced subgraph of F , there is an edge incident to x and y with only one endnode in $\mathcal{C}_{x,y}^H$, and $\mathcal{C}_{x,y}^H$ admits a pseudo-ear decomposition. We have the situation $N_4 = \emptyset$ and $N_5(C_{x,y}) = \emptyset$ for all $C_{x,y} \in \mathcal{C}_{x,y}$ of length 4 at least if F is a bipartite graph, hence we get the following corollary (using the agreement from page 77 and all the notions introduced above).

Corollary 4.7 *Let G be the line graph of a simple bipartite graph F and $x = u_1z, y = zu_2$ be an H-pair in F . Then every subset $\mathcal{C}_{x,y}^H \subseteq \mathcal{C}_{x,y}$ induces in $\text{STAB}(G - xy)$ the rank facet*

$$\sum_{e_i \in \mathcal{C}_{x,y}^H} x_i + \sum_{e_i \in \mathcal{N}_{x,y}} x_i \leq \frac{1}{2} |V(\mathcal{C}_{x,y}^H)|$$

if $\mathcal{C}_{x,y}^H$ admits a pseudo-ear decomposition, there is an edge $zw \in \mathcal{N}_{x,y}$ with $w \notin \mathcal{C}_{x,y}^H$, and $\mathcal{C}_{x,y}^H - z$ is an induced subgraph of F (where $\mathcal{N}_{x,y} = \{e \in E(F) : e = zw, w \neq u_1, u_2\}$ and $\mathcal{C}_{x,y}$ stands for the set of all even cycles $C_{x,y}$ through x and y such that there is an edge $zw \in \mathcal{N}_{x,y}$ with $w \notin C_{x,y}$).

A proof of Corollary 4.7 independent of Theorem 4.5 and 4.6 can be found in [64].

4.3 Facets of $\text{STAB}(G - e)$ with \overline{G} a Perfect Line Graph

Through this section, let G_F denote the complement of the line graph of a line-perfect graph F and the node $v_i \in V(G_F)$ correspond to the edge $e_i \in E(F)$ for $1 \leq i \leq n = |V(G_F)| = |E(F)|$. Again, our main objective is to study facets of $\text{STAB}(G_F - e)$ that are neither nonnegativity nor maximal clique constraints. The graph $G_F - e$ has odd antiholes as only minimally imperfect subgraphs: Theorem 2.11 characterizes anticritical edges in perfect line graphs. Consequently, we obtain a characterization of critical edges in complements of line graphs. By Theorem 2.11 we know, therefore, that G_F only admits A-critical edges e , i.e., every minimally imperfect subgraph $G_e - e \subseteq G_F - e$ is an odd antihole. The following theorem shows how to lift the rank constraint (4.1c) associated with an odd antihole of $G_F - e$ to a weak rank facet of $\text{STAB}(G_F - e)$.

Theorem 4.8 *Let $G_F = \overline{L(F)}$, F be a line-perfect graph, and $e = xy$ a critical edge of G_F . Then every minimally imperfect subgraph $G_e - e \subseteq G_F - e$ induces the facet*

$$\sum_{v_i \in G_{F'}} x_i + \sum_{v_i \in G_{M'}} 2x_i \leq 2 \quad (4.2c)$$

in $\text{STAB}(G_F - e)$ for every pair of partial subgraphs $F', M' \subseteq F$ such that

- (i) F' is connected and has maximum degree $\Delta(F') = 2$,
- (ii) $G_e \subseteq G_{F'}$ and no edge of F incident to the edges x and y is contained in F' ,
- (iii) M' is a matching in $F - V(F')$, and
- (iv) every edge in $F - E(F' \cup M')$ is incident to x and y , to two incident edges of F' , or to one edge of M' .

Remark. A-critical edges $e = xy$ in G_F correspond to A-pairs x, y in F by Theorem 2.11. Hence recall from page 23 that, for every A-pair x and y , there is a path $P_{x,y} \subseteq F$ of odd length ≥ 5 with x and y as endedges. The edges of $P_{x,y}$ correspond to the nodes of the odd antihole $G_e - e \subseteq G_F - e$ (cf. Figure 2.2(c),(d) on page 23). In order to find the weak rank facets (4.2c) of $\text{STAB}(G_F - e)$ associated with $G_e - e$, we first have to find the partial subgraph $F' \subseteq F$ containing $P_{x,y}$ by condition (ii). Since F is line-perfect, F' is either a chordless path, a triangle, or an even hole by condition (i), hence a path of length ≥ 5 or a hole of even length ≥ 8 by (ii) (if F' is an even hole, $G_{F'}$ is the union of two odd

antiholes in $G_F - e$). Then we have to find a matching M' in the remaining part of F (see condition (iii)) such that condition (iv) is satisfied. Since there may be several such paths $P_{x,y}$ in F , each of them corresponding to the nodes of an odd antihole in $G_F - e$, we have to apply this procedure to all these paths $P_{x,y}$ in order to get the whole set of facets (4.2c) of $\text{STAB}(G_F - e)$.

Before we prove Theorem 4.8, let us illustrate how the partial subgraphs F' , $M' \subseteq F$ look like. For that, consider the line-perfect graph F shown in Figure 4.10 and the complement of its line graph G_F .

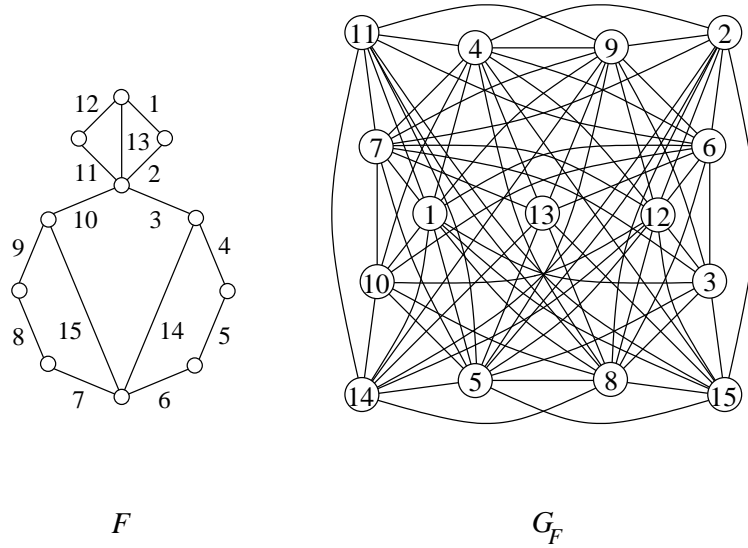


Figure 4.10

The edges $x = e_1$ and $y = e_4$ form an A-pair in F . Hence $e = v_1v_4$ is a critical edge of G_F . If we delete e from G_F , the following facets different from nonnegativity and clique constraints occur in $\text{STAB}(G_F - e)$. All lifting coefficients are included.

$$\begin{array}{llllllllllllllll}
 x_1 + x_2 & & + x_4 + & x_5 + & x_6 + & x_7 + & x_8 + & x_9 + & x_{10} & & + 1x_{12} & & & & & & \leq 2 & (1) \\
 x_1 + x_2 & & + x_4 + & x_5 + & x_6 & & + 2x_8 & & + x_{10} & & + 1x_{12} & & & + x_{15} & \leq 2 & (2) \\
 x_1 & & + x_3 + x_4 & & + 2x_6 & & & + 2x_9 & & + x_{11} + & x_{12} & & & & & & \leq 2 & (3a) \\
 x_1 & & + x_3 + x_4 & & + 2x_6 & & & + 2x_8 & & + x_{11} + & x_{12} & & & & & & \leq 2 & (3b) \\
 x_1 & & + x_3 + x_4 + 1x_5 & & & & + 2x_8 & & & + x_{11} + & x_{12} & & & + 2x_{15} & \leq 2 & (3c) \\
 x_1 & & + x_3 + x_4 + 1x_5 & & & + 2x_7 & & + 2x_9 & & + x_{11} + & x_{12} & & & & & & \leq 2 & (3d) \\
 x_1 & & + x_3 + x_4 + 1x_5 + 1x_6 & & & + 2x_8 & & & & + x_{11} + & x_{12} & & & + 1x_{15} & \leq 2 & (3e) \\
 x_1 & & + x_3 + x_4 + 1x_5 + 1x_6 + 1x_7 & & & + 2x_9 & & & & + x_{11} + & x_{12} & & & & & & \leq 2 & (3f) \\
 x_1 & & + x_3 + x_4 + 1x_5 + 1x_6 & & & + 1x_8 + 1x_9 & & & & + x_{11} + & x_{12} & & & + 1x_{15} & \leq 2 & (3g) \\
 x_1 & & + x_3 + x_4 + 1x_5 + 1x_6 + 1x_7 + 1x_8 + 1x_9 & & & & & & & + x_{11} + & x_{12} & & & & & & \leq 2 & (3h) \\
 x_1 & & & + x_4 + 1x_5 & & & + x_7 + & x_8 + & x_9 + & x_{10} + & x_{11} + & x_{12} & & + x_{14} & & & \leq 2 & (4) \\
 x_1 & & & + x_4 + 1x_5 & & & & + 2x_8 & & + x_{10} + & x_{11} + & x_{12} & & + x_{14} + & x_{15} & \leq 2 & (5) \\
 x_1 & & & + x_4 + & x_5 + & x_6 + & x_7 + & x_8 + & x_9 + & x_{10} & & & + x_{13} & & & & \leq 2 & (6) \\
 x_1 & & & + x_4 + & x_5 + & x_6 & & + 2x_8 & & + x_{10} & & & + x_{13} & & + x_{15} & \leq 2 & (7)
 \end{array}$$

The numbering of these constraints is concerned to the odd antihole $G_e^{(1)} - e, \dots, G_e^{(7)} - e$ that appear in $G_F - e$. $G_e^{(6)} - e$ yields a zero-lifted odd antihole constraint (4.1c), the rank constraints associated with $G_e^{(1)} - e, G_e^{(2)} - e, G_e^{(4)} - e, G_e^{(5)} - e, G_e^{(7)} - e$ admit a unique lifting, while we obtain eight different weak rank constraints (4.2c) from $G_e^{(3)} - e$. The odd antihole result from the seven different paths $P_{x,y}^{(1)}, \dots, P_{x,y}^{(7)}$ of odd length ≥ 5 with endedges $x = e_1$ and $y = e_4$ in F :

$$\begin{aligned} P_{x,y}^{(1)} &= e_1 e_2 e_{10} e_9 e_8 e_7 e_6 e_5 e_4, \\ P_{x,y}^{(2)} &= e_1 e_2 e_{10} e_{15} e_6 e_5 e_4, \\ P_{x,y}^{(3)} &= e_1 e_{12} e_{11} e_3 e_4, \\ P_{x,y}^{(4)} &= e_1 e_{12} e_{11} e_{10} e_9 e_8 e_7 e_{14} e_4, \\ P_{x,y}^{(5)} &= e_1 e_{12} e_{11} e_{10} e_{15} e_{14} e_4, \\ P_{x,y}^{(6)} &= e_1 e_{13} e_{10} e_9 e_8 e_7 e_6 e_5 e_4, \\ P_{x,y}^{(7)} &= e_1 e_{13} e_{10} e_{15} e_6 e_5 e_4. \end{aligned}$$

The partial subgraphs $F'_{(i)}$ and $M'_{(i)}$ of F , satisfying the conditions of Theorem 4.8 and corresponding to the facets of $\text{STAB}(G_F - xy)$ above, are

$F'_{(1)} = \{e_{12}\} \cup P_{x,y}^{(1)},$	$M'_{(1)} = \emptyset$	(see(1)),
$F'_{(2)} = \{e_{12}\} \cup P_{x,y}^{(2)},$	$M'_{(2)} = \{e_8\}$	(see(2)),
$F'_{(3a)} = P_{x,y}^{(3)},$	$M'_{(3a)} = \{e_6, e_9\}$	(see(3a)),
$F'_{(3b)} = P_{x,y}^{(3)}$	$M'_{(3b)} = \{e_6, e_8\}$	(see(3b)),
$F'_{(3c)} = P_{x,y}^{(3)} \cup \{e_5\},$	$M'_{(3c)} = \{e_8, e_{15}\}$	(see(3c)),
$F'_{(3d)} = P_{x,y}^{(3)} \cup \{e_5\},$	$M'_{(3d)} = \{e_7, e_9\}$	(see(3d)),
$F'_{(3e)} = P_{x,y}^{(3)} \cup \{e_5, e_6, e_{15}\},$	$M'_{(3e)} = \{e_8\}$	(see(3e)),
$F'_{(3f)} = P_{x,y}^{(3)} \cup \{e_5, e_6, e_7\},$	$M'_{(3f)} = \{e_9\}$	(see(3f)),
$F'_{(3g)} = P_{x,y}^{(3)} \cup \{e_5, e_6, e_{15}, e_9, e_8\},$	$M'_{(3g)} = \emptyset$	(see(3g)),
$F'_{(3h)} = P_{x,y}^{(3)} \cup \{e_5, e_6, e_7, e_8, e_9\},$	$M'_{(3h)} = \emptyset$	(see(3h)),
$F'_{(4)} = P_{x,y}^{(4)} \cup \{e_5\},$	$M'_{(4)} = \emptyset$	(see(4)),
$F'_{(5)} = P_{x,y}^{(5)} \cup \{e_5\},$	$M'_{(5)} = \{e_8\}$	(see(5)),
$F'_{(6)} = P_{x,y}^{(6)},$	$M'_{(6)} = \emptyset$	(see(6)),
$F'_{(7)} = P_{x,y}^{(7)},$	$M'_{(7)} = \{e_8\}$	(see(7)).

In order to see in which abundance the different liftings of a rank facet associated with only one odd antihole may occur, let us consider the A-pair $x = e_4, y = e_8$ in F and the $C_5 = v_4 v_5 v_6 v_7 v_8$ in $G_F - xy$. The following weak rank constraints appear in $\text{STAB}(G_F - xy)$:

$$\begin{array}{llllll}
2x_1 & +x_4+x_5+x_6+x_7+x_8 & +2x_{10} & \leq 2 & (a) \\
& +x_4+x_5+x_6+x_7+x_8 & +2x_{10} & +2x_{12} & \leq 2 & (b) \\
2x_1 & +x_4+x_5+x_6+x_7+x_8+1x_9 & +2x_{11} & \leq 2 & (c) \\
+2x_2 & +x_4+x_5+x_6+x_7+x_8+1x_9 & & +2x_{12} & \leq 2 & (d) \\
& +x_4+x_5+x_6+x_7+x_8+1x_9 & & +2x_{13} & \leq 2 & (e) \\
2x_1 & +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & & \leq 2 & (f) \\
& +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & +2x_{12} & \leq 2 & (g) \\
2x_1 & +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10}+1x_{11} & & \leq 2 & (h) \\
2x_1 & +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9 & +1x_{11} & \leq 2 & (i) \\
+1x_2 & +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & +2x_{12} & \leq 2 & (j) \\
+1x_2+1x_3 & +x_4+x_5+x_6+x_7+x_8+1x_9 & +2x_{12} & \leq 2 & (k) \\
1x_1 & +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & +1x_{13} & \leq 2 & (l) \\
1x_1 & +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10}+1x_{11}+1x_{12} & & \leq 2 & (m) \\
1x_1+1x_2 & +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & +1x_{12} & \leq 2 & (n) \\
& +x_4+x_5+x_6+x_7+x_8+1x_9+1x_{10} & +1x_{12}+1x_{13} & \leq 2 & (o) \\
1x_1 & +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9 & +1x_{13} & \leq 2 & (p) \\
1x_1 & +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9 & +1x_{11}+1x_{12} & \leq 2 & (q) \\
1x_1+1x_2 & +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9 & +1x_{12} & \leq 2 & (r) \\
& +1x_3+x_4+x_5+x_6+x_7+x_8+1x_9 & +1x_{12}+1x_{13} & \leq 2 & (s)
\end{array}$$

All these facets come from the odd antihole of $G_F - xy$ corresponding to the path $P_{x,y} = e_4, e_5, e_6, e_7, e_8 \subseteq F$. The partial subgraphs $F'_{(i)}$ and $M'_{(i)}$ of F , satisfying the conditions of Theorem 4.8 and corresponding to the facets of $\text{STAB}(G_F - xy)$ above, are

$$\begin{array}{llll}
F'_{(a)} = & P_{x,y}, & M'_{(a)} = & \{e_1, e_{10}\} \quad (\text{see(a)}), \\
F'_{(b)} = & P_{x,y}, & M'_{(b)} = & \{e_{10}, e_{12}\} \quad (\text{see(b)}), \\
F'_{(c)} = & P_{x,y} \cup \{e_9\}, & M'_{(c)} = & \{e_1, e_{11}\} \quad (\text{see(c)}), \\
F'_{(d)} = & P_{x,y} \cup \{e_9\}, & M'_{(d)} = & \{e_2, e_{12}\} \quad (\text{see(d)}), \\
F'_{(e)} = & P_{x,y} \cup \{e_9\}, & M'_{(e)} = & \{e_{13}\} \quad (\text{see(e)}), \\
F'_{(f)} = & P_{x,y} \cup \{e_9, e_{10}, e_3\}, & M'_{(f)} = & \{e_1\} \quad (\text{see(f)}), \\
F'_{(g)} = & P_{x,y} \cup \{e_9, e_{10}, e_3\}, & M'_{(g)} = & \{e_{12}\} \quad (\text{see(g)}), \\
F'_{(h)} = & P_{x,y} \cup \{e_9, e_{10}, e_{11}\}, & M'_{(h)} = & \{e_1\} \quad (\text{see(h)}), \\
F'_{(i)} = & \{e_{11}, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(i)} = & \{e_1\} \quad (\text{see(i)}), \\
F'_{(j)} = & P_{x,y} \cup \{e_9, e_{10}, e_2\}, & M'_{(j)} = & \{e_{12}\} \quad (\text{see(j)}), \\
F'_{(k)} = & \{e_2, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(k)} = & \{e_{12}\} \quad (\text{see(k)}), \\
F'_{(l)} = & P_{x,y} \cup \{e_9, e_{10}, e_{13}, e_1\}, & M'_{(l)} = & \emptyset \quad (\text{see(l)}), \\
F'_{(m)} = & P_{x,y} \cup \{e_9, e_{10}, e_{11}, e_{12}, e_1\}, & M'_{(m)} = & \emptyset \quad (\text{see(m)}), \\
F'_{(n)} = & P_{x,y} \cup \{e_9, e_{10}, e_2, e_1, e_{12}\}, & M'_{(n)} = & \emptyset \quad (\text{see(n)}), \\
F'_{(o)} = & P_{x,y} \cup \{e_9, e_{10}, e_{13}, e_{12}\}, & M'_{(o)} = & \emptyset \quad (\text{see(o)}), \\
F'_{(p)} = & \{e_1, e_{13}, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(p)} = & \emptyset \quad (\text{see(p)}), \\
F'_{(q)} = & \{e_1, e_{12}, e_{11}, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(q)} = & \emptyset \quad (\text{see(q)}), \\
F'_{(r)} = & \{e_{12}, e_1, e_2, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(r)} = & \emptyset \quad (\text{see(r)}), \\
F'_{(s)} = & \{e_{12}, e_{13}, e_3\} \cup P_{x,y} \cup \{e_9\}, & M'_{(s)} = & \emptyset \quad (\text{see(s)}).
\end{array}$$

In most cases, the graph $F'_{(\cdot)} \subseteq F$ is an odd or an even path. $F'_{(\cdot)}$ is sometimes identical with $P_{x,y}$ (see (a),(b)), extends $P_{x,y}$ on one end (see (c),(d),(e)), or on both ends (see (p),(q),(r),(s)). In a few cases, $F'_{(\cdot)}$ is an even cycle as in (f) and (g). Note that two different paths $P_{x,y}$ and $P'_{x,y}$ complement one another to this cycle. In that case, the facet of type (4.2c) of $\text{STAB}(G_F - xy)$ bases on a subgraph of $G_F - xy$ which is the union of two odd antiholes.

Proof of Theorem 4.8. Let G_F be the complement of the line graph of a line-perfect graph F and e a critical edge of G_F . Consider two partial subgraphs $F', M' \subseteq F$ that satisfy the conditions (i),..., (iv) of the theorem. Let $G_{F'} = \overline{L(F')}$ and $G_{M'} = \overline{L(M')}$ denote the corresponding subgraphs of G_F and let the node v_i of G_F correspond to the edge e_i of F for $1 \leq i \leq n = |E(F)| = |V(G_F)|$. According to Theorem 2.11, e is an A-critical edge of G_F , i.e., every minimally imperfect subgraph $G_e - e \subseteq G_F - e$ is an odd antihole. We make use of the convention from Section 2.1 to number the nodes of $G_e - e = \overline{C}_{2k+1}$, $k \geq 2$ by v_1, \dots, v_{2k+1} with $v_i v_{i+1} \notin E(\overline{C}_{2k+1})$ for $1 \leq i \leq 2k+1 \pmod{2k+1}$ and $e = v_1 v_{2k+1}$. Then condition (ii) implies $E(F') = \{e_1, \dots, e_{k'}\}$ with $2k+1 \leq k'$ and $E(M') = \{e_{m'}, \dots, e_n\}$ with $k' < m'$ follows from (iii). By (i), F' is a chordless path or a chordless cycle, let the edges e_i and e_{i+1} be incident for $1 \leq i \leq k'-1$. Furthermore, let $\{e_{k'+1}, \dots, e_{l'}\}$ be the edges of F that are incident to e_1 and e_{2k+1} or to the common node of e_j and e_{j+1} for some $j \in \{1, \dots, k'-1\}$. Denote by $\{e_l, \dots, e_{m'-1}\}$ the edges of F that are incident to some e_j for $j \in \{m', \dots, n\}$ (note $l \leq l' + 1$ holds by condition (iv)).

In order to show that the inequality (4.2c)

$$\sum_{v_i \in G_{F'}} x_i + \sum_{v_i \in G_{M'}} 2x_i \leq 2$$

induces a facet of $\text{STAB}(G_F - e)$ we prove the following two claims.

Claim 1. (4.2c) is valid for $\text{STAB}(G_F - e)$.

Claim 2. In $G_F - e$ there are stable sets S_1, \dots, S_n whose incidence vectors are affinely independent and satisfy (4.2c) with equality.

Corresponding to the coefficients in (4.2c), define a weighting on the nodes of $G_F - e$ as follows

$$a(v_i) = \begin{cases} 1 & \text{if } v_i \in G_{F'} \\ 2 & \text{if } v_i \in G_{M'} \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Claim 1. In order to check the validity of (4.2c) for $\text{STAB}(G_F - e)$ it suffices to show $a(S) \leq 2$ for all stable sets $S \subseteq G_{F' \cup M'} - e$. The conditions (i) and (ii) imply $\alpha(G_{F'} - e) = 2$, therefore $a(S) \leq 2$ for all stable sets $S \subseteq G_{F'} - e$. Due to (iii), $G_{M'}$ is a clique, thus $\alpha(G_{M'}) = 1$ holds and $a(S) = 2$ for all (non-empty) stable sets $S \subseteq G_{M'}$.

Furthermore, every node in $G_{F'}$ is adjacent to every node in $G_{M'}$ by (iii), i.e., there is no stable set with nodes from $G_{F'}$ and from $G_{M'}$. Thus (4.2c) is valid for $\text{STAB}(G_F - e)$. \diamond

Proof of Claim 2. By (i) and (ii), $A_0 = \{v_1, v_{2k+1}\}$ and $A_i = \{v_i, v_{i+1}\}$ for $1 \leq i \leq k' - 1$ are maximum stable sets of $G_{F'} - e$. Define the following stable sets of $G_F - e$

$$S_i = \begin{cases} A_{i-1} & \text{if } 1 \leq i \leq k' \\ A_j \cup \{v_i\} & \text{if } k' + 1 \leq i \leq l' \\ \{v_j, v_i\} & \text{if } l' + 1 \leq i \leq m' - 1 \\ \{v_i\} & \text{if } m' \leq i \leq n \end{cases}$$

(For v_i with $i \in \{k' + 1, \dots, l'\}$ and $i \in \{l' + 1, \dots, m' - 1\}$, choose the smallest $j \leq k' - 1$ and $j \geq m'$ such that $\{v_j, v_{j+1}\} \subseteq N(v_i)$ and $\{v_j\} \subseteq N(v_i)$, respectively, holds.) The incidence vectors $\chi^{S_1}, \dots, \chi^{S_n}$ of these stable sets are affinely independent: $\chi_i^{S_i}$ is the only non-zero entry of $\chi_i^{S_1}, \dots, \chi_i^{S_n}$ for $i \in \{k' + 1, \dots, m' - 1\}$. By

$$\sum_{1 \leq i \leq n} \lambda_i \chi^{S_i} = 0,$$

$\lambda_i = 0$ holds therefore for $i \in \{k' + 1, \dots, m' - 1\}$ and $\lambda_i = 0$ follows for $i \geq m'$, too. Furthermore, from $\lambda_i + \lambda_{i+1} = 0$ for $2k + 2 \leq i \leq k' - 1$ and $\lambda_{k'} = 0$ we know $\lambda_i = 0 \ \forall i \in \{2k + 2, \dots, k'\}$. Finally, $\chi^{S_1}, \dots, \chi^{S_{2k+1}}$ are affinely independent by PADBERG [50]. The stable sets S_1, \dots, S_n are chosen such that $a(S_i) = 2$ holds for $1 \leq i \leq n$, thus $\chi^{S_1}, \dots, \chi^{S_n}$ satisfy (4.2c) with equality and the constraint is facet-inducing for $\text{STAB}(G_F - e)$. \square

In the special case that G_F is the complement of the line graph of a bipartite graph F , we obtain precisely the same facets as described in the theorem above. A proof of this fact independent of Theorem 4.8 can be found in [64].

4.4 A Class of Weakly Rank-Perfect Graphs

In the last two sections, we have studied facets of $\text{STAB}(G - e)$ where G is a perfect line graph or the complement of such a graph. The stable set polytopes of all these graphs $G - e$ that appeared during this investigations admit only weak rank facets besides nonnegativity facets. In view of Problem 0.7 this gives reason to state the following conjectures.

Conjecture 4.9 *Let G be the line graph of a line-perfect graph F and e an edge of G . Then a description of $\text{STAB}(G - e)$ is given by the nonnegativity constraints (4.0), the maximal clique constraints (4.1a), and the weak rank constraints of type (4.2a) and (4.2b).*

Conjecture 4.10 *Let G_F be the complement of the line graph of a line-perfect graph F and e an edge of G_F . Then $\text{STAB}(G_F - e)$ is entirely described by the nonnegativity constraints (4.0), the maximal clique constraints (4.1a), and the weak rank constraints of type (4.2c).*

Thus we conjecture that $G - e$ is weakly rank-perfect if G is a perfect line graph or the complement of a perfect line graph. By Corollary 4.7, the first conjecture can be strengthened that $G - e$ is even rank-perfect if G is the line graph of a bipartite graph. For complements of line graphs we are able to prove the second conjecture in the special case that "line-perfect" is replaced by "bipartite".

Theorem 4.11 *Let G_F be the complement of the line graph of a bipartite graph F and e an edge of G_F . A description of $\text{STAB}(G_F - e)$ is given by the nonnegativity constraints (4.0), the maximal clique constraints (4.1a), and the lifted odd antihole constraints (4.2c).*

Proof. Consider a bipartite graph F , the complement of its line graph G_F , and an edge e of G_F . By KÖNIG's Matching-Theorem [40], G_F is perfect. If e is not a critical edge of G_F , then $G_F - e$ is perfect by definition, i.e., the constraints (4.0) and (4.1a) suffice to describe $\text{STAB}(G_F - e)$. Thus let e be a critical edge and

$$\sum_{v_i \in G_F} a_i x_i \leq b$$

be a facet of $\text{STAB}(G_F - e)$ different from (4.0) and (4.1a). Then we have to show that it is a lifted odd antihole constraint of $\text{STAB}(G_F - e)$ as described in Theorem 4.8, i.e., that it is a facet of type (4.2c). Consider a to be a node weighting of G_F , then $b = \alpha(G_F, a)$ holds and there are $|G_F|$ maximum weighted stable sets of (G_F, a) whose incidence vectors are linearly independent. Let B denote the non-singular matrix with those vectors as rows. Consider the subgraph G_{F^*} of G_F induced by $\{v_i \in G_F : a_i > 0\}$ and let B^* be the matrix with all columns of B that correspond to nodes in G_{F^*} (i.e., to all nodes with non-zero weight). The columns of B^* are linearly independent since B has full column rank and, therefore, B^* has $|G_{F^*}|$ linearly independent rows. Furthermore, $\alpha(G_F, a) = \alpha(G_{F^*}, a^*)$ holds where a^* is a restriction of a to G_{F^*} . Hence, the facet $ax \leq b$ of $\text{STAB}(G_F - e)$ can be read as

$$\sum_{v_i \in G_{F^*}} a_i x_i \leq b \quad (*)$$

(the nodes v_i of G_F with $a_i = 0$ do not play an important role in the study of the facet). In other words, it suffices to exhibit that $(*)$ is a facet of type (4.2c) of $\text{STAB}(G_{F^*} - e)$. Let F^* be the partial subgraph of F with $V(F^*) = \{w \in V(F) : w \text{ incident to an } e_i \in E(F) \text{ with } a_i > 0\}$ and $E(F^*) = \{e_i \in E(F) : a_i > 0\}$ (i.e., $G_{F^*} = \overline{L(F^*)}$) and let $n^* = |G_{F^*}| = |E(F^*)|$. Since $(*)$ is particularly a facet of $\text{STAB}(G_{F^*} - e)$, there have to be maximum weighted stable sets S_1, \dots, S_{n^*} in $(G_{F^*} - e, a)$ with $\alpha(S_i, a) = b$, whose incidence vectors $\chi^{S_1}, \dots, \chi^{S_{n^*}}$ are linearly independent. If we count, in the sequel, the numbers of maximal stable sets, maximum weighted stable sets, or maximum weighted stable sets with linearly independent incidence vectors in any subgraph G'_F of G_F , let us use the following abbreviations for convenience:

$$\begin{aligned} \text{MSS}(G'_F) &: \text{number of maximal stable sets in } G'_F \\ \text{MWSS}(G'_F, a) &: \text{number of maximum weighted stable sets in } (G'_F, a) \\ \text{MWSS}_{\text{ind}}(G'_F, a) &: \text{number of maximum weighted stable sets in } (G'_F, a) \\ &\quad \text{whose incidence vectors are linearly independent} \end{aligned}$$

(we still use a as the node weighting restricted to $G'_F \subseteq G_F$). Then $\text{MSS}(G'_F) \geq \text{MWSS}(G'_F, a) \geq \text{MWSS}_{\text{ind}}(G'_F, a)$ holds and $\text{MWSS}_{\text{ind}}(G_{F^*} - e, a) \geq n^*$, since $(*)$ is a facet of $\text{STAB}(G_{F^*} - e)$ by assumption. Note that stable sets $S \subseteq G_{F^*}$ correspond to edge sets $E_S \subseteq E(F^*)$ with the property that there is a node in $V(F^*)$ all edges in E_S are incident to. Consequently, if F_i and F_j are different components of F^* and S is a stable set in $G_{F_i \cup F_j}$, we either have $S \subseteq G_{F_i}$ or $S \subseteq G_{F_j}$. Hence, in order to determine $\text{MSS}(G_{F_i \cup F_j})$, $\text{MWSS}(G_{F_i \cup F_j}, a)$, or $\text{MWSS}_{\text{ind}}(G_{F_i \cup F_j}, a)$, we often count the corresponding numbers with respect to G_{F_i} and G_{F_j} separately. Furthermore, we write v_i for the node of G_F corresponding to $e_i \in E(F)$ and use

$$\begin{aligned} S_w &= \{v_i \in V(G_{F^*}) : e_i \in E(F^*) \text{ is incident to } w \in V(F^*)\} \\ V_2(F) &= \{w \in V(F) : d_F(w) \geq 2\} \end{aligned}$$

to denote a stable set in G_{F^*} corresponding to the set of all edges in F^* incident to $w \in V(F^*)$ and to denote the set of all nodes in F with degree at least 2, respectively. Finally, let $e = xy$, then we have $x, y \in V(G_{F^*})$ and $x, y \in E(F^*)$, since $(*)$ is neither of type (4.0) nor of type (4.1a). Furthermore, by Theorem 2.11 follows that x and y form an A-pair in F^* and, therefore, are non-incident edges of F^* .

The proof that $(*)$ is a facet of type (4.2c) of $\text{STAB}(G_F - e)$ is organized in the following claims which show that F^* satisfies the conditions (i), (ii), and (iii) of Theorem 4.8.

Claim 1. $|V_2(F_i)| = \text{MSS}(G_{F_i}) \geq \text{MWSS}(G_{F_i}, a)$ is satisfied by every component F_i of F^* with $|E(F_i)| > 1$.

Claim 2. Every component F_i of F^* with $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$ is either isomorphic to a single edge or to an even hole.

Claim 3. The edges x and y are contained in the same component F' of F^* . Furthermore, $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |G_{F'}|$ and $\text{MWSS}_{\text{ind}}(G_{F^* - F'} - xy, a) \leq |G_{F^* - F'}|$ hold.

Claim 4. F' is either isomorphic to a chordless path or to an even hole of length ≥ 6 ; we obtain $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) = |G_{F'}|$.

Claim 5. In F' , every path between x and y has an odd number of edges ≥ 3 ; all edges of F' have weight $b/2$.

Claim 6. The subgraph $M' \subseteq F^*$ induced by $V(F^*) - V(F')$ is a matching; all edges of M' have weight b .

Proof of Claim 1. Let F_i be a component of F^* with $|E(F_i)| > 1$. Suppose the edge $e_0 = ww'$ of F_i is the only edge incident to $w \in V(F_i)$. Then the node $v_0 \in V(G_{F_i})$ is contained in the stable sets $S_w, S_{w'}$ of G_{F_i} . Since F_i is connected and we have $|E(F_i)| > 1$ by precondition, $d_{F_i}(w') \geq 2$ and, therefore, $S_w = \{v_0\} \subset S_{w'}$ must hold. Thus S_w cannot

be a maximal stable set of G_{F_i} for all $w \in V(F_i) - V_2(F_i)$ and, as a consequence,

$$|V_2(F_i)| = \text{MSS}(G_{F_i}).$$

$\text{MSS}(G_{F_i}) \geq \text{MWSS}(G_{F_i}, a)$ obviously holds since a is a nonnegative node weighting. \diamond

Proof of Claim 2. Suppose F_i to be a component of F^* with $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$. In the case $|E(F_i)| = 1$, F_i is isomorphic to a single edge. Now, assume $|E(F_i)| > 1$. By $\text{MWSS}(G_{F_i}, a) = |G_{F_i}|$ and Claim 1, we get $|V_2(F_i)| \geq \text{MWSS}(G_{F_i}, a) = |V(G_{F_i})| = |E(F_i)|$. On the other hand, $|E(F_i)| \geq |V_2(F_i)|$ obviously holds by $|E(F_i)| > 1$. Thus $|E(F_i)| = |V_2(F_i)|$ follows and F_i has to be a chordless cycle. Since F^* is bipartite, F_i is isomorphic to an even hole. \diamond

Proof of Claim 3. Let F_1, \dots, F_k be the components of F^* . In the case $|E(F_i)| = 1$, $\text{MWSS}(G_{F_i}, a) \leq |V(G_{F_i})| = 1$ obviously holds. If $|E(F_i)| > 1$, then $|V(G_{F_i})| = |E(F_i)| \geq |V_2(F_i)| \geq \text{MWSS}(G_{F_i}, a)$ follows on account of $|E(F_i)| \geq |V_2(F_i)|$ and Claim 1, i.e.,

$$|V(G_{F_i})| \geq \text{MWSS}(G_{F_i}, a)$$

holds. Now, assume $x \in E(F_j)$ and $y \in E(F_l)$ for $j \neq l$. Then $\{x, y\}$ is the only stable set of $G_{F^*} - xy$ containing x and y , thus we have $\text{MWSS}(G_{F^*}, a) \geq \text{MWSS}_{\text{ind}}(G_{F^*}, a) \geq n^* - 1$.

$$\sum_{1 \leq i \leq k} \text{MWSS}(G_{F_i}, a) \geq \sum_{1 \leq i \leq k} \text{MWSS}_{\text{ind}}(G_{F_i}, a) \geq \sum_{1 \leq i \leq k} |E(F_i)| - 1 = \sum_{1 \leq i \leq k} |V(G_{F_i})| - 1$$

holds particularly and, by $|V(G_{F_i})| \geq \text{MWSS}(G_{F_i}, a)$ for $1 \leq i \leq k$, we have equality for at least $k - 1$ components. Without loss of generality, let $|V(G_{F_i})| = \text{MWSS}(G_{F_i}, a)$ for F_1, \dots, F_{k-1} . Then F_1, \dots, F_{k-1} are either isomorphic to single edges or to even holes by Claim 2. $\text{MWSS}_{\text{ind}}(\overline{C}_{2k}, a) \leq 2k - 1$ implies F_1, \dots, F_{k-1} have to be single edges. Without loss of generality, let $j \in \{1, \dots, k - 1\}$ (i.e., let $F_j = \{x\}$). Then we certainly have

$$\sum_{1 \leq i \leq k-1} \text{MWSS}(G_{F_i}, a) = k - 1$$

but one of these maximum weighted stable sets of G_{F^*} is $\{x\}$ which cannot be a maximum weighted stable set of $G_{F^*} - xy$. Therefore it must also hold $|V(G_{F_k})| = \text{MWSS}(G_{F_k}, a)$. In the case of $|E(F_k)| = 1$, the maximum weighted stable sets of G_{F^*} corresponding to F_j and F_l are $\{x\}$ and $\{y\}$, respectively. Thus $\{x\}$ and $\{y\}$ are contained in precisely one maximal stable set of $G_{F^*} - xy$, namely $\{x, y\}$. Then $\text{MWSS}(G_{F^*} - xy, a) \leq |V(G_{F^*})| - 1$ would contradict the precondition that $(*)$ is a facet of $\text{STAB}(G_{F^*} - xy)$. Hence we have $|E(F_k)| > 1$. But now, F_k has to be isomorphic to an even hole by Claim 2 and $\text{MWSS}_{\text{ind}}(\overline{C}_{2k}, a) < 2k$ leads to a contradiction. Thus x and y have to appear in the same component of F^* , let us denote this component by F' . We have

$$\text{MWSS}(G_{F^* - F'} - xy, a) \leq |V(G_{F^* - F'})| \text{ and } \text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |V(G_{F'})|$$

since $(*)$ is a facet of $\text{STAB}(G_{F^*} - xy)$. \diamond

Proof of Claim 4. Consider the component $F' \subseteq F^*$ containing x and y . Decompose F' into a set $\mathcal{C} = \{C_1, \dots, C_k\}$ of node disjoint holes and a set $\mathcal{T} = \{T_1, \dots, T_{k'}\}$ of node disjoint trees such that the subgraph of F' induced by $V(F') - V(\mathcal{C})$ admits precisely $T_1, \dots, T_{k'}$ as its components. Denote by

$$\begin{aligned} l^{\mathcal{C}} &: \text{ the number of edges in } E(F') - E(\mathcal{C}) \text{ with both endnodes in } V(\mathcal{C}), \\ l_i^{\mathcal{C}} &: \text{ the number of edges linking the tree } T_i \text{ with } \mathcal{C}. \end{aligned}$$

Let $T_i^{\mathcal{C}}$ stand for the union of the tree T_i and the $l_i^{\mathcal{C}}$ edges connecting T_i with \mathcal{C} and

$$\begin{aligned} l_i &: \text{ the number of nodes of degree 1 in } T_i \text{ which are adjacent to some node in } \mathcal{C}, \\ l'_i &: \text{ the number of nodes of degree 1 in } T_i \text{ which are not linked to any node in } \mathcal{C}. \end{aligned}$$

In particular, $l_i^{\mathcal{C}} \geq l_i$ holds for $1 \leq i \leq k'$ by definition and $l_i > 0$ since F' is connected. Moreover, we observe

$$|E(F')| = \sum_{1 \leq j \leq k} |E(C_j)| + l^{\mathcal{C}} + \sum_{1 \leq i \leq k'} (|E(T_i)| + l_i^{\mathcal{C}}). \quad (\text{i})$$

We know from Claim 3 that $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \geq |G_{F'}| = |E(F')|$ holds. In order to estimate $\text{MWSS}_{\text{ind}}(G_{F'}, a)$ with respect to $|E(F')|$, first recall that every maximal stable set of $G_{F'}$ is a stable set S_w with $w \in V_2(F')$. Because the sets \mathcal{C} and \mathcal{T} define a node partition of F' , let us count the number of maximal stable sets of $G_{F'}$ corresponding to nodes in \mathcal{C} and \mathcal{T} separately. Consider $C_j \in \mathcal{C}$ and let C'_j stand for C_j together with all edges incident to C_j (i.e., together with all edges $ww' \in E(F')$ with $w \in C_j$, $w' \in (\mathcal{C} - C_j) \cup \mathcal{T}$). Then $V_2(C'_j) = V_2(C_j)$ implies $\text{MSS}(G_{C'_j}) = |V(C_j)|$ by Claim 1. Let M be the matrix with the incidence vectors of the $|V(C_j)|$ maximal stable sets of $G_{C'_j}$ as rows. Choose the columns of M such that the first $|E(C_j)|$ columns correspond to the edges of C_j . Then the remaining columns of M , corresponding to the edges incident to C_j , do not play any role for determining the rang of M . The maximal stable set matrix of the even antihole G_{C_j} (and so M) has rang $|E(C_j)| - 1$, therefore

$$\text{MWSS}_{\text{ind}}(G_{\mathcal{C}}, a) \leq \sum_{1 \leq j \leq k} (|E(C_j)| - 1) - l^{\mathcal{C}} = |E(F[\mathcal{C}])| - k - l^{\mathcal{C}} \quad (\text{ii})$$

follows. Particularly, the edges connecting some hole in \mathcal{C} and some tree in \mathcal{T} do not belong to $E(F[\mathcal{C}])$. Turn now to the remaining part of F' consisting of $T_1^{\mathcal{C}}, \dots, T_{k'}^{\mathcal{C}}$. First note that $T_i^{\mathcal{C}}$ is not necessarily a tree for $1 \leq i \leq k'$ since two edges may link T_i with the same node $w \in \mathcal{C}$. But the maximal stable set S_w of $G_{F'}$ is already included in $\text{MWSS}_{\text{ind}}(G_{\mathcal{C}}, a)$ (see (ii)), so only maximal stable sets $S_{w'}$ of $G_{F'}$ with $w' \in \mathcal{T}$ are left to count. Hence, in order to estimate $\text{MWSS}_{\text{ind}}(G_{T_i^{\mathcal{C}}}, a)$ with respect to $|E(T_i^{\mathcal{C}})|$, we may $T_i^{\mathcal{C}}$ consider to be a tree for $1 \leq i \leq k'$. Suppose $T_i^{\mathcal{C}}$ yields $|E(T_i)| + l_i^{\mathcal{C}}$ maximum weighted stable sets in $G_{F'}$, we get

$$|V_2(T_i^{\mathcal{C}})| \geq \text{MWSS}(G_{T_i^{\mathcal{C}}}, a) = |E(T_i)| + l_i^{\mathcal{C}} \quad (\text{iii})$$

by Claim 1. The assumption for $T_i^{\mathcal{C}}$ implies $|V_2(T_i^{\mathcal{C}})| = |V(T_i)| - l'_i$, so we infer

$$|V(T_i)| - l'_i = |V_2(T_i^{\mathcal{C}})| \geq |E(T_i)| + l_i^{\mathcal{C}} = |V(T_i)| - 1 + l_i^{\mathcal{C}} \geq |V(T_i)| - 1 + l_i$$

by inequality (iii), $|E(T_i)| = |V(T_i)| - 1$ (recall that T_i is a tree), and $l_i^C \geq l_i$. Hence, we obtain $l_i + l'_i \leq 1$. That F' is connected yields $l_i > 0$, so we conclude $l_i = 1$ and $l'_i = 0$. Thus T_i consists of precisely one node. But now, $|V_2(T_i^C)| = 0$ holds in contradiction to (iii) and, therefore, T_i^C cannot yield $|E(T_i)| + l_i^C$ maximum weighted stable sets in $G_{F'}$. We obtain

$$\text{MWSS}_{\text{ind}}(G_{T_i^C}, a) \leq \text{MWSS}(G_{T_i^C}, a) \leq |E(T_i)| + l_i^C - 1 \quad (\text{iv})$$

for $1 \leq i \leq k'$. So we combine (ii) and (iv) to

$$\text{MWSS}_{\text{ind}}(G_{F'}, a) \leq |G_{F'}| - (k + l^C + k'). \quad (\text{v})$$

Since F' is bipartite, there are at most two edges z, z' of F' which are incident to both x and y . Thus only if there is a C_4 with edges $\{x, z, y, z'\}$ in F' , there can be two stable sets in $G_{F'} - xy$ containing x and y . Therefore

$$\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) - \text{MWSS}_{\text{ind}}(G_{F'}, a) \leq 2$$

must hold and we get $k + l^C + k' \leq 2$ by Claim 3 and (v). Provided $k + l^C + k' = 2$, F' would be disconnected for $k = 2$ or $k' = 2$. Hence assume $k, k' = 1$ and refrain from using indices for the only hole $C \in \mathcal{C}$ and the only tree $T \in \mathcal{T}$. By $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) - \text{MWSS}_{\text{ind}}(G_{F'}, a) = 2$, F' contains a C_4 with edges x, z, y, z' . Without loss of generality, let $E(C) = \{x, z, y, z'\}$. Since $\{x, z\}$, $\{z, y\}$, $\{y, z'\}$, and $\{z', x\}$ are not maximal stable sets in $G_{F'} - xy$, at least three nodes of C are linked to T (then the edges incident to these three nodes may yield maximum weighted stable sets) and $l^C \geq 3$ follows. Estimating $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a)$, we get on one hand

$$\begin{aligned} |E(F')| &\leq \text{MWSS}_{\text{ind}}(G_{F'} - xy, a) = \text{MWSS}_{\text{ind}}(G_{F'}, a) + 2 \\ &\leq (|E(C)| - 1) + |V_2(T^C)| + 2 \\ &= |E(C)| - 1 + (|V(T)| - l') + 2 \\ &= |E(C)| + |E(T)| - l' + 2 \end{aligned}$$

by Claim 3, our assumption, (ii) and Claim 1, the definition of T^C , and T tree. On the other hand, $|E(C)| + l^C + |E(T)| \leq |E(F')|$ holds by (i), we deduce $l^C + l' \leq 2$, and get a contradiction to $l^C \geq 3$ and $l' \geq 0$. So $k + k' = 1$ follows by $x, y \in E(F')$. In the case of $k = 1$ and $k' = 0$, F' is isomorphic to an even hole. If F' were a C_4 with edges $\{x, z, y, z'\}$, $G_{F'} - xy$ would have at most the maximum weighted stable sets $\{x, z, y\}$ and $\{x, z', y\}$ in contradiction to Claim 3. Hence F' is an even hole of length ≥ 6 and $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \leq |G_{F'}|$ holds by (v). In the case of $k = 0$ and $k' = 1$, F' is isomorphic to a tree. So F' cannot contain a C_4 with edges $\{x, z, y, z'\}$ in particular and $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \leq \text{MWSS}_{\text{ind}}(G_{F'}, a) + 1$ follows. Let l' stand for the number of nodes with degree 1 in F' , then

$$|E(F')| + 1 - l' = |V(F')| - l' = |V_2(F')| \geq \text{MWSS}(G_{F'}, a) \geq |E(F')| - 1$$

holds since F' is a tree, by the definition of l' , Claim 1, and Claim 3. From $l' \leq 2$ follows $l' = 2$. I.e., F' is a chordless path and we obtain $\text{MWSS}_{\text{ind}}(G_{F'} - xy, a) \leq |V(G_{F'})|$ by

(iv). Therefore, $\text{MWSS}_{\text{lind}}(G_{F'} - xy, a) \leq |V(G_{F'})|$ is true in both cases, if F' is an even hole of length ≥ 6 or a chordless path, and the assertion of the claim follows by Claim 3. \diamond

Proof of Claim 5. Consider the component $F' \subseteq F^*$ containing x and y . By Claim 4, F' is either an even hole of length ≥ 6 or a chordless path. Let $E(F') = \{e_1, \dots, e_m\}$, where e_i and e_{i+1} are incident edges for $1 \leq i \leq m-1$ (and e_m and e_1 if F' is an even hole). Let S_1, \dots, S_{m-1} with $S_i = \{v_i, v_{i+1}\}$ be these maximal stable sets that yield $|V(G_{F'})| - 1$ maximum weighted stable sets with linearly independent incidence vectors in $G_{F'} - xy$. Furthermore, let S_{xy} denote this stable set of $G_{F'} - xy$ containing x and y (recall that there is only one such stable set by Claim 4). Then, the incidence vectors of S_1, \dots, S_{m-1} and S_{xy} have to be linearly independent. Without loss of generality, let $x = e_j$ with $1 \leq j < m-1$ and $y = e_l$ with $j+1 < l \leq m$ (actually, $y \neq e_m$ if $x = e_1$ and F' even hole). Assume $l = j + 2k + 1$ with $k > 0$, then $S_j, \dots, S_{l-1}, S_{xy}$ would be the maximal stable sets of an even antihole in $G_{F'} - xy$, i.e., their incidence vectors $\chi^{S_j}, \dots, \chi^{S_{l-1}}, \chi^{S_{xy}}$ could not be linearly independent. Thus we have $l = j + 2k$ with $k > 0$. For $k = 1$, $S_j = \{x, v_{j+1}\}$ and $S_{j+1} = \{v_{j+1}, y\}$ are certainly maximal stable sets in $G_{F'}$, but not in $G_{F'} - xy$. Thus Claim 1 implies $\text{MWSS}_{\text{lind}}(G_{F'} - xy, a) \leq |V(G_{F'})| - 1$ but Claim 4 lead us to a contradiction. Hence $k > 1$ yields that every path between x and y has an odd number of edges ≥ 3 and, therefore, $S_{xy} = \{x, y\}$ must hold.

Now turn to the edge weights in F' . Since $S_1, \dots, S_{m-1}, S_{xy}$ have to be maximum weighted stable sets in $G_{F'} - xy$, we obtain $a_i + a_{i+1} = b$ for $1 \leq i \leq m-1$ and $a_j + a_l = b$. Furthermore, $a_i = a_{i'}$ holds whenever i and i' have the same parity. In particular, we have $a_j = a_l = b/2$ by $a(S_{xy}) = b$ and $l = j + 2k$, so we finally obtain $a_i = b/2$ for $1 \leq i \leq m$. \diamond

Proof of Claim 6. Consider the subgraph $M' \subseteq F^*$ induced by $V(F^*) - V(F')$. Claim 3 and Claim 4 imply $\text{MWSS}_{\text{lind}}(G_{M'} - xy, a) = |V(G_{M'})|$. Since M' is defined to be the union of all components F_i of F^* neither containing x nor y , $\text{MWSS}_{\text{lind}}(G_{M'} - xy, a) = \text{MWSS}_{\text{lind}}(G_{M'}, a)$ follows. Thus we infer $\text{MWSS}_{\text{lind}}(G_{F_i}, a) = |V(G_{F_i})|$ for all $F_i \subseteq M'$. By Claim 2, every F_i has to be isomorphic to a single edge (note $\text{MWSS}_{\text{lind}}(G_{C_{2k}}, a) = 2k - 1$), i.e., M' is a matching in F^* . Therefore, every G_{F_i} consists of a single node v_i . Since G_{F_i} have to yield an maximum weighted stable set of G_{F^*} for all $F_i \subseteq M'$ by $\text{MWSS}_{\text{lind}}(G_{M'} - xy, a) = |V(G_{M'})|$, $\{v_i\}$ must be this maximum weighted stable set. We obtain that every node $v_i \in G_{M'}$ and every edge $e_i \in M'$, respectively, has the weight b . \diamond

Thus, F' satisfies Theorem 4.8.(i) by Claim 4 and Theorem 4.8.(ii) by Claim 5, where M' satisfies Theorem 4.8.(iii) by Claim 6. I.e., $(*)$ has the form

$$\sum_{v_i \in G_{F'}} \frac{b}{2} x_i + \sum_{v_i \in G_{M'}} b x_i \leq b$$

and is obviously a facet of type (4.2c) in $\text{STAB}(G_{F^*} - xy)$.

In order to establish this statement for $\text{STAB}(G_F - xy)$, recall that the inequality

$$\sum_{v_i \in G_F} a_i x_i \leq b$$

is supposed to be a facet of $\text{STAB}(G_F - xy)$. With scaling $\frac{b}{2} = 1$, we have shown $a_i = 1 \forall v_i \in G_{F'}$ and $a_i = 2 \forall v_i \in G_{M'}$. For all remaining nodes v_i with $a_i = 0$, Theorem 4.8.(iv) must also hold since the above inequality is a facet of $\text{STAB}(G_F - xy)$ by assumption. Thus every facet of $\text{STAB}(G_F - e)$ different from (4.0) and (4.1a) is indeed a facet of type (4.2c). \square

Since the facets of type (4.2c) are weak rank constraints, the following corollary is an immediate consequence of Theorem 4.11 and gives at least a partial answer to Problem 0.7.

Corollary 4.12 *All graphs, arising from complements of line graphs of bipartite graphs by deletion of an arbitrary edge, are weakly rank-perfect.*

As already mentioned in Section 3.2 and Section 3.3, all critically perfect graphs known so far are line graphs of bipartite graphs, their complementary graphs, or created by the disjoint union, the multiplication, the substitution, the composition, the clique-, or the stable set-identification of suitable graphs. If Conjecture 4.9 turns out to be true for line graphs of bipartite graphs, we are able to describe $\text{STAB}(G - e)$ whenever G is a critically perfect line graph or a critically perfect co-line graph (cf. Theorem 3.18 and Theorem 3.22). Then we would obtain the whole facet system of $\text{STAB}(G - e)$ for all critically perfect graphs G which are generated from line graphs or co-line graphs G_1, G_2 only using the disjoint union, the multiplication, the substitution, or the composition: since all critical edges of G must be inherited from one of the original graphs G_1 or G_2 in that case (cf. Lemma 3.7 and Lemma 3.9), we just have to use the results concerning the effect on the stable set polytopes under applying those operations due to CHVÁTAL [9] and CUNNINGHAM [15]. Moreover, if e is a critical edge of either G_1 or G_2 and G is created from G_1 and G_2 by clique- or stable set-identification, we would obtain the whole facet system of $\text{STAB}(G - e)$ by a result of FONLUPT and HADJAR [18]. E.g., if all minimally imperfect subgraphs of $G - e$ are contained in G_1 , the descriptions of $\text{STAB}(G_1 - e)$ and $\text{STAB}(G_2)$ would yield one for $\text{STAB}(G - e)$ by [18]. But Lemma 3.13 and Lemma 3.17 show that $G - e$ may admit minimally imperfect subgraphs $G_e - e$ with $G_e \not\subseteq G_i$ for $i \in \{1, 2\}$. In that case, the results provided in [18] does not suffice for finding all facets of $\text{STAB}(G - e)$. But looking for a complete description of $\text{STAB}(G - e)$ seems to be hopeless if we take into account in which abundance minimally imperfect subgraphs $G_e - e$ of $G - e$ with $G_e \not\subseteq G_i$ for $i \in \{1, 2\}$ may occur (see Section 3.2).

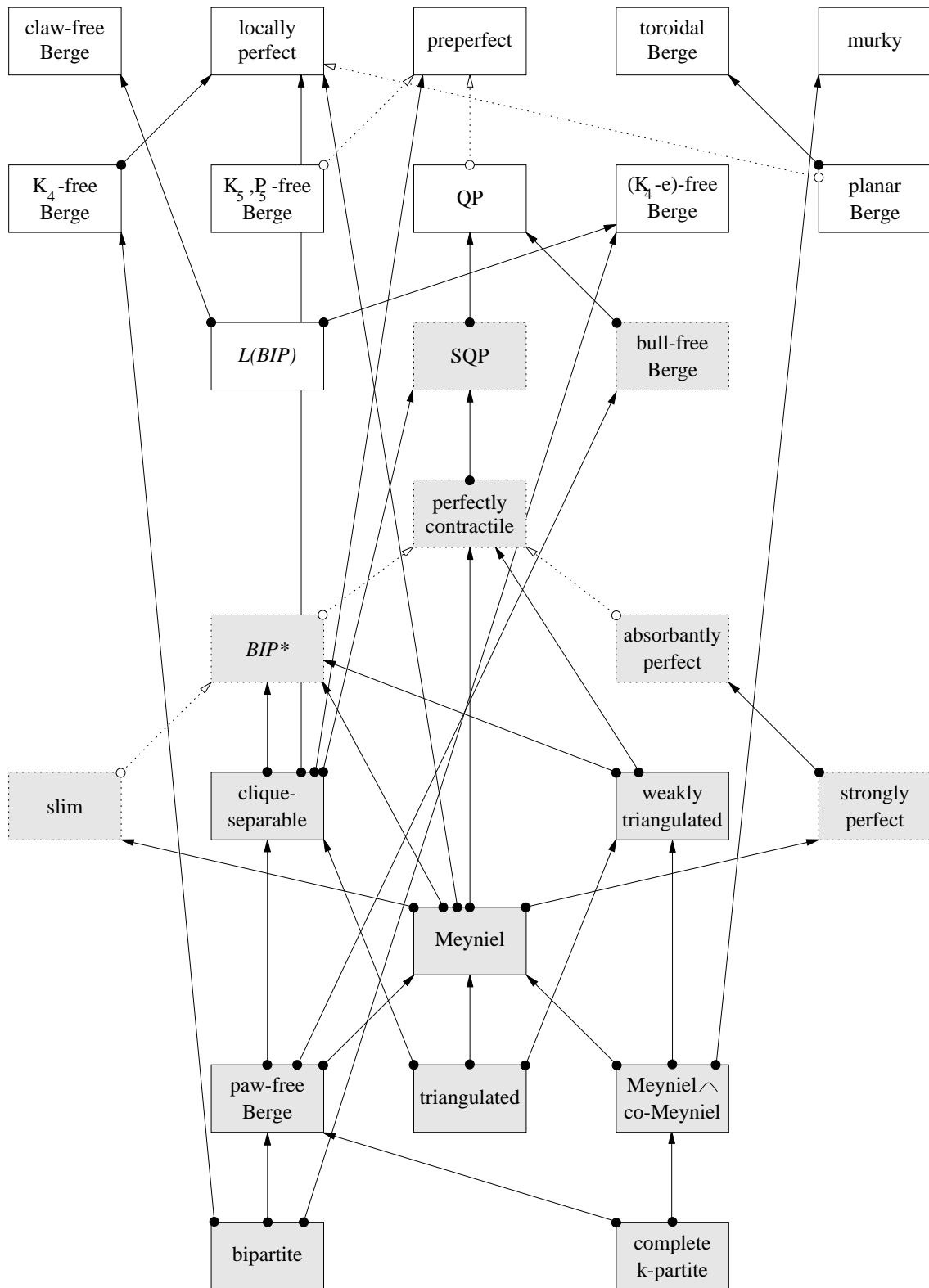
Finally, FONLUPT and HADJAR studied in [18] also which effect the addition of an edge to a perfect graph on the stable set polytope has. But [18] gives a description of $\text{STAB}(G + e)$ only for some special cases, where e is not an anticritical edge of G . So further efforts are needed to study the stable set polytopes associated with graphs obtained from anticritically perfect graphs by the addition of an edge.

Appendix A

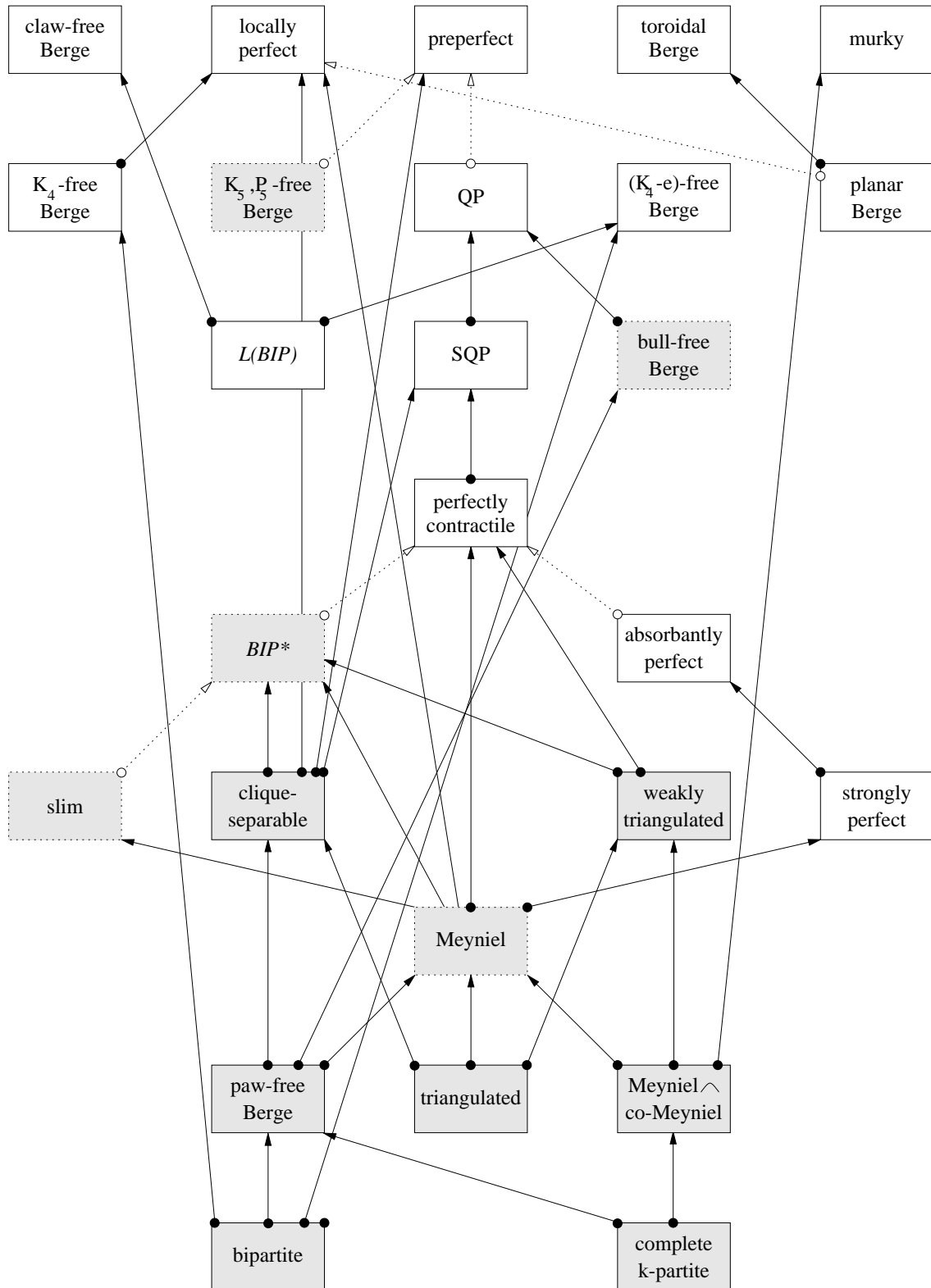
A.1 Charts of Inclusion Relations

Many classes of perfect graphs are mentioned throughout this thesis. The following two charts give an overview on their inclusion relations. Every class of perfect graphs occurring in this thesis, except for *C-PERF* and *A-PERF*, appears in the two charts (the denotation "graphs" is omitted in every name of a graph class). All inclusion relations not mentioned in the chapters above are quoted from HOUGARDY [35]. A solid arrow stands for a proven inclusion relation, a dotted arrow for a conjectured one.

As a consequence of Theorem 3.26 we know that *C-PERF* and *A-PERF* are incomparable with the most classes of perfect graphs, in particular, there are no inclusion relations to any of the classes \mathcal{C} in the charts. But these two charts illustrate the knowledge on the intersection of *C-PERF* and *A-PERF*, respectively, and every of these classes \mathcal{C} obtained in Section 3.4 as follows. If \mathcal{C} appears in the chart in a not filled rectangle, then there are graphs in $\mathcal{C} \cap C\text{-PERF}$ and $\mathcal{C} \cap A\text{-PERF}$, respectively (see Section 3.4 for examples). Otherwise, i.e., if the corresponding rectangle is filled with grey color, we do not know any graph in $\mathcal{C} \cap C\text{-PERF}$ and $\mathcal{C} \cap A\text{-PERF}$, respectively. Here a rectangle edged in solid lines stands for a proven empty intersection again, a rectangle edged in dotted lines if this fact has not been established so far.



Intersections of C - $PERF$ and classes of perfect graphs.

Intersections of A -PERF and classes of perfect graphs.

A.2 Tables Summarizing the Results

We provide the reader a brief overview on the results established through Chapters 2 and 3.

Occurrence of Critical Edges

The following table summarizes all results about which kinds of critical edges may occur in the graph classes investigated in Section 2.3.

	H-critical	A-critical	M-critical
bull-free Berge graphs	$k = 2$	$k \geq 2$?
co- $L(LINE)$	$k = 2$	$k \geq 2$	—
diamond-free Berge graphs	$k \geq 2$	$k = 2$	—
K_4 -free Berge graphs	$k \geq 2$	$k = 2$	—
$L(LINE)$	$k \geq 2$	$k = 2$	—
locally perfect graphs	$k \geq 2$	$k = 2$?
Meyniel graphs	—	—	—
murky graphs	$k = 2$	$k = 2$?
P_5 , K_5 -free Berge graphs	$k = 2$	$k \leq 3$	—
planar Berge graphs	$k \geq 2$	$k = 2$	—
toroidal Berge graphs	$k \geq 2$	$k \leq 3$	—
weakly triangulated graphs	$k = 2$	$k \geq 2$?

— the corresponding kind of critical edges must not occur

? it is unknown whether M-critical edges occur here

else the possible size of $k \geq 2$ for $G_e - e \cong C_{2k+1}, \overline{C}_{2k+1}$ is given

Perfect Edge Orders and Perfect Non-Edge Orders

We give an overview on graph classes which possess perfect edge orders (PEO) or co-perfect edge orders (CEO) due to Section 2.4.

	PEO	CEO
bipartite graphs	+	+
line-perfect graphs	+	?
triangulated graphs	+	+
weakly triangulated graphs	+	+

— the corresponding graph class admits such an order

? it is unknown whether the corresponding graph class admits such an order

Forbidden Substructures

We sum up all aspects of forbidden substructures for critical and anticritical edges in Section 2.1 and 2.2, as well as for critically and anticritically perfect graphs in Section 3.1.

	xy critical	$G \in C\text{-PERF}$	xy anticritical	$G \in A\text{-PERF}$
2-pair	/	+	−	−
weak odd pair	−	−	/	+
strict odd pair	/	+	−	−
true (anti)twin	−	−	/	+
false (anti)twin	/	+	−	−
weak comparable pair	−	−	/	+
strict comparable pair	/	+	−	−
universal node	−	−	/	/
isolated node	/	/	−	−
simplicial node	−	−	/	+
antisimplicial node	/	+	−	−

- + the corresponding substructure may occur
 − the corresponding substructure must not occur
 / x and y does not have this property by definition

Operations Preserving Critically and Anticritically Perfectness

This table summarizes the results obtained in Section 3.2, whether or not the considered graph classes are closed under the following perfection preserving operations.

	$C\text{-PERF}$	$A\text{-PERF}$	$C\text{-PERF} \cap A\text{-PERF}$
multiplication	+	−	−
substitution	+	+	+
disjoint union Φ_{00}	+	−	−
composition Φ_{10}	+	−	−
amalgam Φ_{1k} , $k > 0$	/	−	/
2-amalgam Φ_{2k}	/	?	/
clique identification Φ_{0k} , $k > 0$	+	−	−
stable set identification	+	−	−

- + the graph class is closed under applying this operation
 − the graph class is not closed under applying this operation
 ? it is unknown whether the class is closed under this operation
 / this operation cannot be applied to any graph in the class

Graph Parameters for Critically and Anticritically Perfect Graphs

The bounds for the graph parameters investigated in Section 3.1 are listed in the next table. Recall that all these bounds are tight by the graphs shown in Figure 3.1.

	$G \in C\text{-PERF}$	$G \in A\text{-PERF}$
minimum degree	$4 \leq \delta(G) \leq n - 3$	$2 \leq \delta(G) \leq n - 5$
maximum degree	$4 \leq \Delta(G) \leq n - 3$	$3 \leq \Delta(G) \leq n - 5$
clique number	$3 \leq \omega(G) \leq n - 5$	$3 \leq \omega(G) \leq n - 6$
stability number	$3 \leq \alpha(G) \leq n - 6$	$3 \leq \alpha(G) \leq n - 5$

Occurrence of Critically and Anticritically Perfect Graphs

This table provides the summarized information obtained in Section 3.4, which classes of perfect graphs are known to contain critically or anticritically perfect graphs.

	$C\text{-PERF}$	$A\text{-PERF}$	$C\text{-PERF} \cap A\text{-PERF}$
absorbantly perfect graphs	?	+	?
BIP^*	?	?	?
clique separable graphs	—	—	—
$\text{co-}L(LINE)$	+	+	+
diamond-free Berge graphs	+	+	+
K_4 -free Berge graphs	+	+	+
$L(LINE)$	+	+	+
locally perfect graphs	+	+	+
Meyniel graphs	—	?	—
murky graphs	+	+	+
P_5 , K_5 -free Berge graphs	+	?	?
perfectly contractile graphs	?	+	?
planar Berge graphs	+	+	?
preperfect graphs	+	+	+
quasi-parity graphs	+	+	+
strict quasi-parity graphs	?	+	?
strongly perfect graphs	?	+	?
toroidal Berge graphs	+	+	+
weakly triangulated graphs	—	—	—

- +
 -
 - ?
- the intersection of these graph classes is known to be non-empty
the intersection of the two graph classes is known to be empty
no graph contained in the intersection of these graph classes is known

Notation Index

$V(G)$	9	node set of the graph G
$E(G)$	9	edge set of the graph G
$ G $	9	equals $ V(G) $
$e = xy$	9	edge with endnodes x and y
$G = (V, E)$	9	graph with node set V and edge set E
$N_{V'}(x)$	9	set of neighbors of x in $V' \subseteq V(G)$
$d_{V'}(x)$	9	number of neighbors of x in $V' \subseteq V(G)$
$N(x)$	9	set of neighbors of x in $V(G)$
$d(x)$	9	number of neighbors of x in $V(G)$
$\delta(G)$	9	minimum degree of the graph G
$\Delta(G)$	9	maximum degree of the graph G
$G' \subseteq G$	9	G' is a subgraph of G
$G - E'$	9	graph with node set $V(G)$ and edge set $E(G) - E'$
$G - e$	9	graph with node set $V(G)$ and edge set $E(G) - \{e\}$
$G + e$	9	graph with node set $V(G)$ and edge set $E(G) \cup \{e\}$
$G[V']$	9	subgraph of G induced by the set $V' \subseteq V(G)$
$G - V'$	10	equals $G[V(G) - V']$
$G - v$	10	equals $G[V(G) - \{v\}]$
$G' \subset G$	10	G' is a proper subgraph of G
$P[v_i, v_j]$	10	path with nodes v_i, \dots, v_j
$P[v_i, v_j)$	10	path with nodes v_i, \dots, v_{j-1}
$P(v_i, v_j]$	10	path with nodes v_{i+1}, \dots, v_j
$P(v_i, v_j)$	10	path with nodes v_{i+1}, \dots, v_{j-1}
$\alpha(G)$	10	stability number of G
$\omega(G)$	10	clique number of G
$\chi(G)$	10	chromatic number of G
$\theta(G)$	10	clique covering number of G
K_n	10	complete graph on n nodes
P_k	10	chordless path on k nodes
C_k	10	chordless cycle on k nodes
$K_{a,b}$	11	complete bipartite graph with color classes of size a and b
$K_{1,b}$	11	star
$K_{1,3}$	11	claw
$K_4 - e$	11	diamond

\overline{G}	11	complementary graph of G
$L(G)$	11	line graph of G
$L^{-1}(G)$	11	reconstruction of the original graph from the line graph G
$G_1 \cong G_2$	11	G_1 is isomorphic to G_2
$G_1 = G_2$	11	equals $G_1 \cong G_2$
$\mathcal{U}(G)$	11	subdivision of G
SPGC	12	Strong Perfect Graph Conjecture
\mathcal{C}^*	14	star-closure of the graph class \mathcal{C}
<i>TRIV</i>	14	class of all graphs with at most two nodes
<i>BIP</i>	14	class of all bipartite graphs
<i>SQP</i>	14	class of all strict quasi parity graphs
<i>QP</i>	14	class of all quasi parity graphs
$T(r, s, t)$	21	6-regular triangulation of the torus
$e_{x,y}$	23	edge incident to the common node of an H-pair x, y
$C_{x,y}$	23	even cycle running through an H-pair x, y
$P_{x,y}$	23	odd path of length ≥ 5 with the A-pair x, y as endedges
<i>C-PERF</i>	30	class of all critically perfect graphs
<i>A-PERF</i>	30	class of all anticritically perfect graphs
$\overline{*}$	35	antioperation of the graph operation $*$
Φ_{jk}	42	graph operation introduced in [13]
Φ_{00}	43	disjoint union
Φ_{10}	43	composition
Φ_{1k}	43	amalgam
Φ_{2k}	43	2-amalgam
Φ_{0k}	43	clique-identification
Φ_S	46	stable set-identification
$F = (A \cup B, E)$	51	bipartite graph with color classes A and B
H_1, \dots, H_7	52	critical bipartite H-graphs
A_1, \dots, A_3	55	critical bipartite A-graphs
A_4, \dots, A_{10}	56	critical bipartite A-graphs
$\chi^{V'}$	68	incidence vector of $V' \subseteq V(G)$
$\text{STAB}(G)$	68	stable set polytope associated with the graph G
$x(G, a)$	68	equals $\sum_{v_i \in G} a_i x_i$ where a_i is the weight of v_i
(G, a)	68	graph G with a node weighting a
$\mathbb{1}$	68	vector with only 1-entries
$x(G)$	68	equals $x(G, \mathbb{1})$
$\alpha(G, a)$	70	weighted stability number
$\text{WSTAB}(G)$	71	polytope solely defined by (4.0) and (4.2)
$\text{RSTAB}(G)$	71	polytope solely defined by (4.0) and (4.1)
$\text{QSTAB}(G)$	71	polytope solely defined by (4.0) and (4.1a)
<i>LINE</i>	72	class of all line-perfect graphs

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