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On cardinality constrained cycle and path polytopes

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Volker Kaibel and Rüdiger Stephan

Abstract

Given a directed graph $D = (N, A)$ and a sequence of positive integers $1 \leq c_1 < c_2 < \dots < c_m \leq |N|$, we consider those path and cycle polytopes that are defined as the convex hulls of simple paths and cycles of D of cardinality c_p for some $p \in \{1, \dots, m\}$, respectively. We present integer characterizations of these polytopes by facet defining linear inequalities for which the separation problem can be solved in polynomial time. These inequalities can simply be transformed into inequalities that characterize the integer points of the undirected counterparts of cardinality constrained path and cycle polytopes. Beyond we investigate some further inequalities, in particular inequalities that are specific to odd/even paths and cycles.

1 Introduction

Let $D = (N, A)$ be a directed graph on n nodes that has neither loops nor parallel arcs, and let $c = (c_1, \dots, c_m)$ be a nonempty sequence of integers such that $1 \leq c_1 < c_2 < \dots < c_m \leq n$ holds. Such a sequence is called a *cardinality sequence*. For two different nodes $s, t \in N$, the *cardinality constrained (s, t) -path polytope*, denoted by $P_{s,t\text{-path}}^c(D)$, is the convex hull of the incidence vectors of simple directed (s, t) -paths P such that $|P| = c_p$ holds for some $p \in \{1, \dots, m\}$. The *cardinality constrained cycle polytope* $P_C^c(D)$, similar defined, is the convex hull of the incidence vectors of simple directed cycles C with $|C| = c_p$ for some p . Note, since D does not have loops, we may assume $c_1 \geq 2$ when we investigate cycle polytopes. The undirected counterparts of these polytopes are defined similarly. We denote them by $P_{s,t\text{-path}}^c(G)$ and $P_C^c(G)$, where G is an undirected graph. The associated polytopes without cardinality restrictions we denote by $P_{s,t\text{-path}}(D)$, $P_{s,t\text{-path}}(G)$, $P_C(D)$, and $P_C(G)$.

Cycle and path polytopes, with and without cardinality restrictions, defined on graphs or digraphs, are already well studied. For a literature survey on these polytopes see Table 1.

Those publications that treat cardinality restrictions, discuss only the cases $\leq k$ or $= k$, while we address the general case. In particular, we assume $m \geq 2$. The main contribution of this paper will be the presentation of IP-models (or IP-formulations) for cardinality constrained path and cycle polytopes whose inequalities generally define facets with respect to complete graphs and digraphs. Moreover, the associated separation problem can be solved in polynomial time.

The basic idea of this paper can be presented best for cycle polytopes. Given a finite set B and a cardinality sequence $b = (b_1, \dots, b_m)$, the set $\text{CHS}^b(B) := \{F \subseteq B : |F| = b_p \text{ for some } p\}$ is called a *cardinality homogenous set system*.

Table 1: Literature survey on path and cycle polyhedra

Schrijver [22], chapter 13:	dominant of $P_{s,t\text{-path}}(D)$
Stephan [20]:	$P_{s,t\text{-path}}^{(k)}(D)$
Dahl, Gouveia [7]:	$P_{s,t\text{-path}}^{\leq k}(D) := P_{s,t\text{-path}}^{(1,\dots,k)}(D)$
Dahl, Realfsen [8]:	$P_{s,t\text{-path}}^{\leq k}(D)$, D acyclic
Nguyen [19]:	dominant of $P_{s,t\text{-path}}^{\leq k}(G)$
Balas, Oosten [1]:	directed cycle polytope $P_C(D)$
Balas, Stephan [2]:	dominant of $P_C(D)$
Coullard, Pulleyblank [6], Bauer [3]:	undirected cycle polytope $P_C(G)$
Hartmann, Özlük [14]:	$P_C^{(k)}(D)$
Maurras, Nguyen [16, 17]:	$P_C^{(k)}(G)$
Bauer, Savelsbergh, Linderoth [4]:	$P_C^{\leq k}(G)$

Clearly, $P_C^c(D) = \text{conv}\{\chi^C \in \mathbb{R}^A \mid C \text{ simple cycle}, C \in CHS^c(A)\}$, where $CHS^c(A)$ is the cardinality homogeneous set system defined on the arc set A of D . According to Balas and Oosten [1], the integer points of the cycle polytope $P_C(D)$ can be characterized by the system

$$\begin{aligned}
x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= 0 && \text{for all } i \in N, \\
x(\delta^{\text{out}}(i)) &\leq 1 && \text{for all } i \in N, \\
-x((S : N \setminus S)) + x(\delta^{\text{out}}(i)) + x(\delta^{\text{out}}(j)) &\leq 1 && \text{for all } S \subset N, \\
&&& 2 \leq |S| \leq n-2, \\
&&& i \in S, j \in N \setminus S, \\
x(A) &\geq 2, \\
x_{ij} &\in \{0, 1\} && \text{for all } (i, j) \in A.
\end{aligned} \tag{1}$$

Here, $\delta^{\text{out}}(i)$ and $\delta^{\text{in}}(i)$ denote the set of arcs leaving and entering node i , respectively; for an arc set $F \subseteq A$ we set $x(F) := \sum_{(i,j) \in F} x_{ij}$; for any subsets S, T of N , $(S : T)$ denotes $\{(i, j) \in A \mid i \in S, j \in T\}$. Moreover, for any $S \subseteq N$, we denote by $A(S)$ the subset of arcs whose both endnodes are in S .

Grötschel [12] presented a complete linear description of a cardinality homogeneous set system. For $CHS^c(A)$, the model reads:

$$\begin{aligned}
0 &\leq x_{ij} \leq 1 && \text{for all } (i, j) \in A, \\
c_1 &\leq x(A) \leq c_m, \\
(c_{p+1} - |F|) x(F) - (|F| - c_p) x(A \setminus F) &\leq c_p(c_{p+1} - |F|) \\
&\text{for all } F \subseteq A \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \in \{1, \dots, m-1\}.
\end{aligned} \tag{2}$$

The *cardinality bounds* $c_1 \leq x(A) \leq c_m$ exclude all subsets of A whose cardinalities are out of the bounds c_1 and c_m , while the latter class of inequalities of model (2), which are called *cardinality forcing inequalities*, cut off all arc sets $F \subseteq A$ of forbidden cardinality between the bounds, since for each such F , the cardinality forcing inequality associated with F is violated by χ^F :

$$(c_{p+1} - |F|)\chi^F(F) - (|F| - c_p)\chi^F(A \setminus F) = |F|(c_{p+1} - |F|) > c_p(c_{p+1} - |F|).$$

However, for any $H \in CHS^c(A)$ the inequality associated with F is valid. If $|H| \leq c_p$, then $(c_{p+1} - |F|)\chi^H(F) - (|F| - c_p)\chi^H(A \setminus F) \leq (c_{p+1} - |F|)x(H \cap F) \leq c_p(c_{p+1} - |F|)$, and equality holds if $|H| = c_p$ and $H \subseteq F$. If $|H| \geq c_{p+1}$, then $(c_{p+1} - |F|)\chi^H(F) - (|F| - c_p)\chi^H(A \setminus F) \leq |F|(c_{p+1} - |F|) - (c_{p+1} - |F|)(|F| - c_p) = c_p(c_{p+1} - |F|)$, and equality holds if $|H| = c_{p+1}$ and $H \cap F = F$.

Combining both models results obviously in an integer characterization for the cardinality constrained cycle polytope $P_C^c(D)$. However, the cardinality forcing inequalities in this form are quite weak, that is, they define very low dimensional faces of $P_C^c(D)$. The key for obtaining stronger cardinality forcing inequalities for $P_C^c(D)$ is to count the nodes of a cycle rather than its arcs. The trivial, but crucial observation here is that, for the incidence vector $x \in \{0, 1\}^A$ of a cycle in D and for every node $i \in V$, we have $x(\delta^{\text{out}}(i)) = 1$ if the cycle contains node i , and $x(\delta^{\text{out}}(i)) = 0$ if it does not. Thus, for every $W \subseteq N$ with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$, the cardinality-forcing inequality

$$(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) - (|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \leq c_p(c_{p+1} - |W|),$$

is valid for $P_C^c(D)$, cuts off all cycles C , with $c_p < |C| < c_{p+1}$, that visit $\min\{|C|, |W|\}$ nodes of W , and is satisfied with equality by all cycles of cardinality c_p or c_{p+1} that visit $\min\{|C|, |W|\}$ nodes of W . Using these inequalities yields the following integer characterization for $P_C^c(D)$:

$$\begin{aligned} x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= 0 && \text{for all } i \in N, \\ x(\delta^{\text{out}}(i)) &\leq 1 && \text{for all } i \in N, \\ -x((S : N \setminus S)) + x(\delta^{\text{out}}(i)) + x(\delta^{\text{out}}(j)) &\leq 1 && \text{for all } S \subset N, \\ &&& 2 \leq |S| \leq n-2, \\ &&& i \in S, j \in N \setminus S, \end{aligned}$$

$$\begin{aligned} x(A) &\geq c_1, \\ x(A) &\leq c_m, \end{aligned}$$

$$\begin{aligned} (c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) \\ - (|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \\ - c_p(c_{p+1} - |W|) &\leq 0 && \forall W \subseteq N : \exists p \\ &&& \text{with } c_p < |W| < c_{p+1}, \end{aligned}$$

$$x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in A. \quad (3)$$

However, in the polyhedral analysis of cardinality constrained path and cycle polytopes we will focus on the directed cardinality constrained path polytope for a simple reason: valid inequalities for $P_{s,t\text{-path}}^c(D)$ can easily be transformed into valid inequalities for the other polytopes. In particular, from the IP-model for $P_{s,t\text{-path}}^c(D)$ that we present in section 3 we derive IP-models for the remaining polytopes \mathcal{P} , as illustrated in Figure 1, such that a transformed inequality is facet defining for \mathcal{P} when the original inequality is facet defining for $P_{s,t\text{-path}}^c(D)$. In addition, the subpolytopes $P_{s,t\text{-path}}^{(c_p)}(D)$ of $P_{s,t\text{-path}}^c(D)$ were studied in [20]. Theorem 2.3 in Section 2 and Table 1 in [20] imply that they are of codimension 1 whenever $4 \leq c_p \leq n-1$, provided that we have an appropriate digraph D . Thus, any facet defining inequality $\alpha x \leq \alpha_0$ for $P_{s,t\text{-path}}^{(c_p)}(D)$ which

is also valid for $P_{s,t\text{-path}}^c(D)$ can easily be shown to be facet defining also for $P_{s,t\text{-path}}^c(D)$ if $\alpha y = \alpha_0$ holds for some $y \in P_{s,t\text{-path}}^c(D) \setminus P_{s,t\text{-path}}^{(c_p)}(D)$. So, in the present paper many facet proofs must not be given from the scratch, but can be traced back to results in [20].

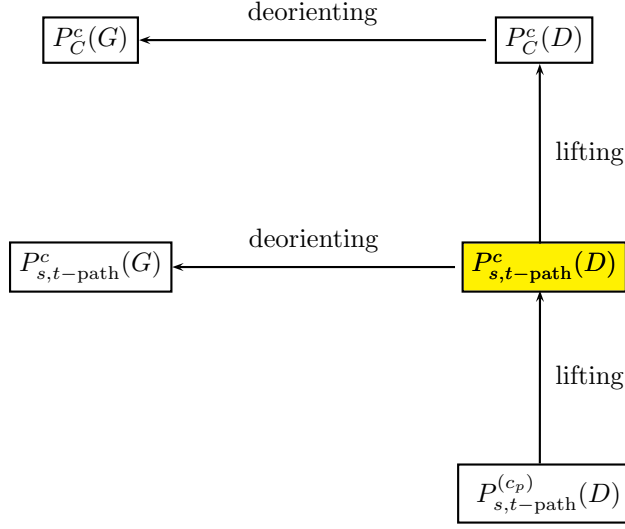


Figure 1. $P_{s,t\text{-path}}^c(D)$ and related polytopes.

In the following we investigate the cardinality constrained path polytope $P_{0,n\text{-path}}^c(D)$ defined on a digraph $D = (N, A)$ with node set $N = \{0, \dots, n\}$. In particular, $s = 0, t = n$. Since $(0, n)$ -paths do not use arcs entering 0 or leaving n , we may assume that $\delta^{\text{in}}(0) = \delta^{\text{out}}(n) = \emptyset$. Next, suppose that A contains the arc $(0, n)$ and the cardinality sequence c starts with $c_1 = 1$. Then the equation

$$\dim P_{0,n\text{-path}}^{(c_1, c_2, \dots, c_m)}(D) = \dim P_{0,n\text{-path}}^{(c_2, \dots, c_m)}(D) + 1$$

obviously holds. Moreover, an inequality $\alpha x \leq \alpha_0$ defines a facet of $P_{0,n\text{-path}}^{(c_2, \dots, c_m)}(D)$ if and only if the inequality $\alpha x + \alpha_0 x_{0n} \leq \alpha_0$ defines a facet of $P_{0,n\text{-path}}^{(1, c_2, \dots, c_m)}(D)$. Thus, the consideration of cardinality sequences starting with 1 does not give any new insights into the facial structure of cardinality constrained path polytopes. So we may assume that A does not contain the arc $(0, n)$. So, for our purposes it suffices to suppose that the arc set A of D is given by

$$A = \{(0, i), (i, n) : i = 1, \dots, n-1\} \cup \{(i, j) : 1 \leq i, j \leq n-1, i \neq j\}. \quad (4)$$

Therefore, by default, we will deal with the directed graph $\tilde{D}_n = (\tilde{N}_n, \tilde{A}_n)$, where $\tilde{N}_n = \{0, 1, \dots, n\}$ and $\tilde{A}_n = A$ is (4).

The remainder of the paper is organized as follows: In Section 2, we examine the relationship between directed path and cycle polytopes. In Section 3, we consider the inequalities of the IP-model for the directed cardinality constrained

path polytope $P_{0,n\text{-path}}^c(\tilde{D}_n)$ and give necessary and sufficient conditions for them to be facet defining. Moreover, we present some further classes of inequalities that also cut off forbidden cardinalities. Finally, in Section 4, we transform facet defining inequalities for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ into facet defining inequalities for the other polytopes.

2 The relationship between directed path and cycle polytopes

This section generalizes the results in [20], Section 2. Denote by \mathcal{P} the set of simple $(0, n)$ -paths P in $\tilde{D}_n = (\tilde{N}_n, \tilde{A}_n)$. Let D' be the digraph that arises by removing node 0 from \tilde{D}_n and identifying $\delta^{\text{out}}(0)$ with $\delta^{\text{out}}(n)$. Then, D' is a complete digraph on node set $\{1, \dots, n\}$ and \mathcal{P} becomes the set \mathcal{C}^n of simple cycles that visit node n . The convex hull of the incidence vectors of cycles $C \in \mathcal{C}^n$ in turn is the restriction of the cycle polytope defined on D' to the hyperplane $x(\delta^{\text{out}}(n)) = 1$. Balas and Oosten [1] showed that the *degree constraint*

$$x(\delta^{\text{out}}(i)) \leq 1$$

induces a facet of the cycle polytope defined on a complete digraph. Hence, the path polytope $P_{0,n\text{-path}}(\tilde{D}_n)$ is isomorphic to a facet of the cycle polytope $P_C(D')$. From the next theorem we conclude that this relation holds also for cardinality constrained path and cycle polytopes. We start with some preliminary statements from linear algebra.

Lemma 2.1. *Let $k \neq \ell$ be natural numbers, let $x^1, x^2, \dots, x^r \in \mathbb{R}^p$ be vectors satisfying the equation $1^T x^i = k$, and let $y \in \mathbb{R}^p$ be a vector satisfying the equation $1^T y = \ell$. Then the following holds:*

- (i) *y is not in the affine hull of the set $\{x^1, \dots, x^r\}$.*
- (ii) *The points x^1, \dots, x^r are affinely independent if and only if they are linearly independent.* \square

According to the terminology of Balas and Oosten [1], for any digraph $D = (N, A)$ on n nodes we call the polytope

$$P_{CL}^c(D) := \{(x, y) \in P_C^c(D) \times \mathbb{R}^n : y_i = 1 - x(\delta^{\text{out}}(i)), i = 1, \dots, n\}$$

the *cardinality constrained cycle-and-loops polytope*. Its integer points are the incidence vectors of spanning unions of a simple cycle and loops.

Lemma 2.2. *The points $x^1, \dots, x^p \in P_C^c(D)$ are affinely independent if and only if the corresponding points $(x^1, y^1), \dots, (x^p, y^p) \in P_{CL}^c(D)$ are affinely independent.*

Proof. The map $f : P_{CL}^c(D) \rightarrow P_C^c(D)$, $(x, y) \mapsto x$ is an affine isomorphism. \square

Theorem 2.3. *Let $D_n = (N, A)$ be the complete digraph on $n \geq 3$ nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 2$. Then the following holds:*

- (i) *The dimension of $P_C^c(D_n)$ is $(n - 1)^2$.*

(ii) For any node $i \in N$, the degree inequality $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$.

Proof. (i) Balas and Oosten [1] proved that $\dim P_C(D_n) = (n-1)^2$, while Theorem 1 of Hartmann and Özlük [14] says that

$$\dim P_C^{(k)}(D_n) = \begin{cases} |A|/2 - 1, & \text{if } k = 2, \\ n^2 - 2n, & \text{if } 2 < k < n \text{ and } n \geq 5, \\ n^2 - 3n + 1, & \text{if } k = n \text{ and } n \geq 3, \end{cases} \quad (5)$$

and $\dim P_C^{(3)}(D_4) = 6$. Since $P_C^c(D_n) \subseteq P_C(D_n)$, it follows immediately that $\dim P_C^c(D_n) \leq (n-1)^2$. When $n = 3$, $m \geq 2$ implies $P_C^c(D_n) = P_C(D_n)$, and thus $\dim P_C^{(2,3)}(D_3) = 4$. When $n = 4$, the statement can be verified using a computer program, for instance, with `polymake` [11]. For $n \geq 5$ the claim follows from (5) and Lemma 2.1 (i) unless $c = (2, n)$: it exists some cardinality c_p , with $2 < c_p < n$, and thus there are $n^2 - 2n + 1$ affinely independent vectors $x^r \in P_C^{(c_p)}(D_n) \subset P_C^c(D_n)$. Moreover, since $m \geq 2$, there is a vector $y \in P_C^c(D_n)$ of another cardinality which is affinely independent from the points x^r . Hence, $P_C^c(D_n)$ contains $n^2 - 2n + 2$ affinely independent points proving $\dim P_C^c(D_n) = (n-1)^2$.

The case $c = (2, n)$ requires a higher amount of technical detail, because the dimensions of both polytopes $P_C^2(D_n)$ and $P_C^n(D_n)$ are less than $n^2 - 2n$. Setting $d_n := \dim P_C^{(n)}(D_n)$, we see that there are $d_n + 1 = n^2 - 3n + 2$ linearly independent points $x^r \in P_C^{(2,n)}(D_n) \cap P_C^{(n)}(D_n)$ satisfying $1^T x^r = n$. Clearly, the points $(x^r, y^r) \in P_{CL}^{(2,n)}$ are also linearly independent. Next, consider the point (x^{23}, y^{23}) , where x^{23} is the incidence vector of the 2-cycle $\{(2, 3), (3, 2)\}$, and $n - 1$ further points (x^{1i}, y^{1i}) , where x^{1i} is the incidence vector of the 2-cycle $\{(1, i), (i, 1)\}$. The incidence matrix Z whose rows are the vectors (x^r, y^r) , $r = 1, 2, \dots, d_n + 1$, (x^{23}, y^{23}) , and (x^{1i}, y^{1i}) , $i = 2, 3, \dots, n$, is of the form

$$Z = \begin{pmatrix} X & \mathbf{0} \\ Y & L \end{pmatrix},$$

where

$$L = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 1 & \dots & 1 \\ \hline \mathbf{0} & & E - I & & & \end{array} \right).$$

E is the $(n-1) \times (n-1)$ matrix of all ones and I the $(n-1) \times (n-1)$ identity matrix. $E - I$ is nonsingular, and thus L is of rank n . X is of rank $d_n + 1$, and hence $\text{rank}(Z) = d_n + 1 + n = n^2 - 2n + 2$. Together with Lemma 2.2, this yields the desired result.

(ii) When $n \leq 4$, the statement can be verified using a computer program. When $n \geq 5$ and $4 \leq c_p < n$ for some index $p \in \{1, \dots, m\}$, the claim can be showed along the lines of the proof to part (i) using Theorem 11 of Hartmann and Özlük [14] saying that the degree constraint defines a facet of $P_C^{(c_p)}(D_n)$.

It remains to show that the claim is true for $c \in \{(2, 3), (2, n), (3, n), (2, 3, n)\}$, $n \geq 5$. W.l.o.g. consider the inequality $x(\delta^{\text{out}}(1)) \leq 1$. When $c = (2, 3)$, consider all 2- and 3-cycles whose incidence vectors satisfy $x(\delta^{\text{out}}(1)) = 1$. This are exactly $n^2 - 2n + 1$ cycles, namely the 2-cycles $\{(1, j), (j, 1)\}$, $j = 2, \dots, n$, and the 3-cycles $\{(1, j), (j, k), (k, 1)\}$ for all arcs (j, k) that are not incident

with node 1. Their incidence vectors are affinely independent, and hence, the degree constraint is facet defining for $P_C^{(2,3)}(D_n)$. This implies also that it induces a facet of $P_C^{(2,3,n)}(D_n)$. Turning to the case $c = (2, n)$, note that the degree constraint is satisfied with equality by all Hamiltonian cycles. Hence, we have $d_n + 1$ linearly independent Hamiltonian cycles and again, the 2-cycles $\{(1, i), (i, 1)\}$, which are linearly independent of them. Finally, let $c = (3, n)$. Beside $d_n + 1$ Hamiltonian cycles, consider the 3-cycles $(1, 3), (3, 4), (4, 1)$ and $\{(1, 2), (2, j), (j, 1)\}$, $j = 3, \dots, n$. Then the $n^2 - 2n + 1$ corresponding points in $P_{CL}^c(D_n)$ build a nonsingular matrix. Hence, by Lemma 2.2, it follows the desired result. \square

Given a cardinality sequence $c = (c_1, \dots, c_m)$ with $m \geq 2$ and $c_1 \geq 2$, Theorem 2.3 implies that $\dim P_{0,n-\text{path}}^c(\tilde{D}_n) = n^2 - 2n$. From Theorem 2.3 another important fact can be derived. Facet defining inequalities for $P_{0,n-\text{path}}^c(\tilde{D}_n)$ can easily be lifted to facet defining inequalities for $P_C^c(D_n)$. For sequential lifting, see Nemhauser and Wolsey [18].

Theorem 2.4. *Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Let $\alpha x \leq \alpha_0$ be a facet defining inequality for $P_{0,n-\text{path}}^c(\tilde{D}_n)$ and γ the maximum of $\alpha(C)$ over all cycles C in \tilde{D}_n with $|C| = c_p$ for some p . Setting $\alpha_{ni} := \alpha_{0i}$ for $i = 1, \dots, n - 1$, the inequality*

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \alpha_{ij} x_{ij} + (\gamma - \alpha_0) x(\delta^{\text{out}}(n)) \leq \gamma \quad (6)$$

defines a facet of $P_C^c(D_n)$. \square

No similar relationship seems to hold between undirected cycle and path polytopes.

3 Facets of $P_{0,n-\text{path}}^c(\tilde{D}_n)$

Let $D = (N, A)$ be a digraph on node set $N = \{0, \dots, n\}$. The integer points of $P_{0,n-\text{path}}^c(D)$ are characterized by the following system:

$$\begin{aligned} x(\delta^{\text{out}}(i)) - x(\delta^{\text{in}}(i)) &= \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \in N \setminus \{0, n\}, \\ -1 & \text{if } i = n, \end{cases} \\ x(\delta^{\text{out}}(i)) &\leq 1 && \text{for all } i \in N \setminus \{0, n\}, \\ x((S : N \setminus S)) - x(\delta^{\text{in}}(j)) &\geq 0 && \forall S \subset N : 0, n \in S, j \in N \setminus S, \\ x(A) &\geq c_1, \\ x(A) &\leq c_m, \end{aligned}$$

$$\begin{aligned} &\frac{(c_{p+1} - |W| + 1) \sum_{i \in W} x(\delta^{\text{out}}(i))}{-(|W| - 1 - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i))} - c_p(c_{p+1} - |W| + 1) \leq 0 \\ &\quad \forall W \subseteq N : 0, n \in W, \exists p \text{ with } c_p < |W| - 1 < c_{p+1}, \end{aligned}$$

$$x_{ij} \in \{0, 1\} \quad \text{for all } (i, j) \in A. \quad (7)$$

Here, the cardinality forcing inequalities arise in another form, since the number of nodes that are visited by a simple path is one more than the number of arcs in difference to a simple cycle. The first three and the integrality constraints ensure that x is the incidence vector of a simple $(0, n)$ -path P (cf. [20]). The cardinality bounds and the cardinality forcing inequalities guarantee that $|P| = c_p$ for some p .

Dahl and Gouveia [7] gave a complete linear description of $P_{0,n\text{-path}}^{(1,2,3)}(D')$, where $D' = D \cup \{(0, n)\}$. So, we have also one for $P_{0,n\text{-path}}^{(2,3)}(D)$. Consequently, from now on we exclude the case $c = (2, 3)$ with respect to directed path polytopes. More precisely, in the sequel we consider only the set of cardinality sequences $\text{CS} := \{c = (c_1, \dots, c_m) : m \geq 2, 2 \leq c_1 < \dots < c_m \leq n, c \neq (2, 3)\}$.

Due to the flow conservation constraints, two different inequalities that are valid for $P_{0,n\text{-path}}^c(D)$ may define the same face. The next theorem, which is an adaption of a result of Hartmann and Özlük [14], says how those inequalities can be identified.

Theorem 3.1. *Let $\alpha x \geq \alpha_0$ be a valid inequality for $P_{0,n\text{-path}}^c(D)$ and let T be a spanning tree of D . Then for any specified set of coefficients β_{ij} for the arcs $(i, j) \in T$, there is an equivalent inequality $\alpha' x \geq \alpha_0$ for $P_{0,n\text{-path}}^c(D)$ such that $\alpha'_{ij} = \beta_{ij}$ for $(i, j) \in T$. \square*

Similarly to the cycle-and-loops polytope, the *cardinality constrained $(0, n)$ -path-and-loops polytope* is defined to be the polytope

$$P_{PL}^c(D) := \{(x, y) \in P_{0,n\text{-path}}^c(D) \times \mathbb{R}^{n-1} : y_i = 1 - x(\delta^{\text{out}}(i)), i = 1, \dots, n-1\}.$$

Given a valid inequality $cx \leq c_0$, a $(0, n)$ -path P is said to be *tight* if $c(P) = c_0$.

3.1 Facets related to cardinality restrictions

The cardinality bounds $x(\tilde{A}_n) \geq c_1$ and $x(\tilde{A}_n) \leq c_m$ define facets of the cardinality constrained path polytope $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if $4 \leq c_i \leq n-1$ for $i = 1, m$ (see Table 1 of [20]).

Next, we turn to the cardinality forcing inequalities. Due to the easier notation, we analyze them for the polytope $P^* := \{x \in P_C^c(D_n) | x(\delta^{\text{out}}(1)) = 1\}$ which is isomorphic to $P_{0,n\text{-path}}^c(\tilde{D}_n)$.

Theorem 3.2. *Let $D_n = (N, A)$ be the complete digraph on $n \geq 4$ nodes and W a subset of N with $1 \in W$ and $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-forcing inequality*

$$(c_{p+1} - |W|) \sum_{i \in W} x(\delta^{\text{out}}(i)) - (|W| - c_p) \sum_{i \in N \setminus W} x(\delta^{\text{out}}(i)) \leq c_p(c_{p+1} - |W|) \quad (8)$$

defines a facet of P^ if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n - 1$.*

Proof. Assuming that $|W| + 1 = c_{p+1} < n$, we see that (8) is dominated by nonnegativity constraints $x_{ij} \geq 0$ for $(i, j) \in N \setminus W$. When $c_{p+1} = n$ and $n - |W| \geq 2$, (8) is dominated by another inequality of the same form for some $W' \supset W$ with $|W'| = n - 1$. Therefore, if inequalities (8) are not facet defining,

then they are dominated by other inequalities of the IP-model that are facet defining for P^* .

Suppose that $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$. By choice, $|W| \geq 3$ and $|N \setminus W| \geq 3$. Moreover, assume that the equation $bx = b_0$ is satisfied by all points that satisfy (8) at equality. Setting $\iota := c_{p+1} - |W|$, we will show that

$$\begin{aligned} b_{1i} &= \iota & \forall i \in N \setminus \{1\} \\ b_{i1} &= \iota & \forall i \in W \setminus \{1\}, \\ b_{ij} &= \kappa & \forall i \in W \setminus \{1\}, j \in N \setminus \{1\}, \\ b_{ij} &= \lambda & \forall i \in N \setminus W, j \in N \setminus \{1\}, \\ b_{i1} &= \mu & \forall i \in N \setminus W \end{aligned} \tag{9}$$

for some $\kappa \neq 0, \lambda, \mu$. Then, considering a tight cycle of length c_p and two tight cycles of length c_{p+1} , one using an arc in $(N \setminus W : \{1\})$, the other not, yields the equation system

$$\begin{aligned} b_0 &= 2\iota + (c_p - 2)\kappa \\ b_0 &= \iota + (|W| - 1)\kappa + (c_{p+1} - |W| - 1)\lambda + \mu \\ b_0 &= 2\iota + (|W| - 2)\kappa + (c_{p+1} - |W|)\lambda \end{aligned}$$

which solves to

$$\begin{aligned} b_0 &= 2\iota + (c_p - 2)\kappa \\ \mu &= \iota + \left(\frac{|W| - c_p}{|W| - c_{p+1}} - 1\right)\kappa \\ \lambda &= \frac{|W| - c_p}{|W| - c_{p+1}}\kappa. \end{aligned}$$

Thus, $bx = b_0$ is the equation

$$\begin{aligned} \iota x(\delta^{\text{out}}(1)) + \iota x(\delta^{\text{in}}(1)) + \left(\frac{|W| - c_p}{|W| - c_{p+1}} - 1\right)\kappa \sum_{i \in N \setminus W} x_{i1} \\ + \kappa \sum_{i \in W \setminus \{1\}} x(\delta_1^{\text{out}}(i)) + \frac{|W| - c_p}{|W| - c_{p+1}}\kappa \sum_{i \in N \setminus W} x(\delta_1^{\text{out}}(i)) &= 2\iota + (c_p - 2)\kappa, \end{aligned}$$

where $\delta_1^{\text{out}}(i) := \delta^{\text{out}}(i) \setminus \{(i, 1)\}$. Adding $\kappa - \iota$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$ and multiplying the resulting equation with $-\frac{|W| - c_{p+1}}{\kappa}$, we see that $bx = b_0$ is equivalent to (8).

To show (9), we may assume without loss of generality that $2 \in W$ and $b_{1i} = c_{p+1} - |W|$, $i \in N \setminus \{1\}$, and $b_{21} = c_{p+1} - |W|$, by Theorem 3.1. Next, let \mathcal{R} be the set of subsets of N of cardinality c_{p+1} that contain W , i.e.,

$$\mathcal{R} := \{R \subset N \mid |R| = c_{p+1}, R \supset W\}.$$

For any $R \in \mathcal{R}$, the c_{p+1} -cycles on R are tight tours on R . Theorem 23 of Grötschel and Padberg [13] implies that there are $\tilde{\alpha}_i^R, \tilde{\beta}_i^R$ for $i \in R$ such that $b_{ij} = \tilde{\alpha}_i^R + \tilde{\beta}_j^R$ for all $(i, j) \in A(R)$. Setting

$$\begin{aligned} \alpha_i^R &:= \tilde{\alpha}_i^R - \tilde{\alpha}_1^R & (i \in R), \\ \beta_i^R &:= \tilde{\beta}_i^R - \tilde{\alpha}_1^R & (i \in R), \end{aligned} \tag{10}$$

yields $\alpha_i^R + \beta_j^R = b_{ij}$ for all $(i, j) \in A(R)$. Since $\alpha_1^R = 0$ and $b_{1i} = \iota$, it follows that $\beta_i^R = \iota$ for all $i \in R \setminus \{1\}$. In a similar manner one can show for any $S \in \mathcal{R}$ the existence of α_i^S, β_i^S for $i \in S$ with $\alpha_1^S = 0$, $\beta_j^S = \iota$ for $j \in S \setminus \{1\}$, and $\alpha_i^S + \beta_j^S = b_{ij}$ for all $(i, j) \in A(S)$. This implies immediately that $\alpha_i^R = \alpha_i^S$ and

$\beta_i^R = \beta_i^S$ for all $i \in R \cap S$. Thus, there are α_i, β_i for all $i \in N$ such that $\alpha_1 = 0$, $\beta_i = \iota$ for $i \in N \setminus \{1\}$, and $b_{ij} = \alpha_i + \beta_j$ for all $(i, j) \in A$.

Next, consider a tight c_p -cycle that contains the arcs $(1, k), (k, j)$ but does not visit node ℓ for some $j, k, \ell \in W$. Replacing node k by node ℓ yields another tight c_p -cycle, and therefore $b_{1k} + b_{kj} = b_{1\ell} + b_{\ell j}$, which implies that $\alpha_k = \alpha_\ell$ for all $k, \ell \in W \setminus \{1\}$. Thus, there is κ such that $b_{ij} = \kappa$ for all $i \in W \setminus \{1\}$, $j \in N \setminus \{1\}$. Moreover, it follows immediately that $b_{i1} = \iota$ for all $i \in W \setminus \{1\}$. One can show analogously that $\alpha_i = \alpha_j$ for all $i, j \in N \setminus W$. This implies the existence of λ, μ with $b_{ij} = \lambda$ for all $i \in N \setminus W$, $j \in N \setminus \{1\}$ and $b_{i1} = \mu$ for all $i \in N \setminus W$.

Finally, when $|W| + 1 = c_{p+1} = n$, we show that there are $n^2 - 2n$ affinely independent points $x \in P^*$ satisfying (8) at equality. Without loss of generality, let $W = \{1, \dots, n-1\}$. Because each tour is tight with respect to (8), it exist $n^2 - 3n + 2$ linearly independent points $(x^r, y^r) \in Q := \{(x, y) \in P_{CL}^c(D_n) \mid x(\delta^{\text{out}}(1) = 1)\}$ with $y^r = 0$. Furthermore, consider the incidence vectors of the $n-2$ cycles $(1, 2, \dots, c_p)$, $(1, 3, 4, \dots, c_p + 1), \dots, (1, n-2, n-1, 2, 3, \dots, c_p - 2)$, $(1, n-1, 2, 3, \dots, c_p - 1)$. The corresponding points in Q are linearly independent and they are also linearly independent of the points (x^r, y^r) . Hence, (8) is also facet defining if $|W| + 1 = c_{p+1} = n$. \square

Theorem 3.3. *Let $D_n = (N, A)$ be the complete digraph on n nodes, and let $1 \in W \subset N$ with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-subgraph inequality*

$$2x(A(W)) - (|W| - c_p - 1)[x((W : N \setminus W)) + x((N \setminus W : W))] \leq 2c_p \quad (11)$$

is valid for P^ and induces a facet of P^* if and only if $p+1 < m$ or $c_{p+1} = n = |W| + 1$.*

Proof. A cycle of length less or equal to c_p uses at most c_p arcs of $A(W)$ and thus its incidence vector satisfies (11). A cycle C of length greater or equal to c_{p+1} uses at most $|W| - 1$ arcs in $A(W)$ and if C indeed visits any node in W , then it uses at least 2 arcs in $(W : N \setminus W) \cup (N \setminus W : W)$ and hence,

$$\begin{aligned} 2\chi^C(A(W)) - (|W| - c_p - 1)[\chi^C((W : N \setminus W)) + \chi^C((N \setminus W : W))] \\ \leq 2(|W| - 1) - 2(|W| - c_p - 1) = 2c_p. \end{aligned}$$

In particular, all cycles of feasible length that visit node 1 satisfy (11).

To prove that (11) is facet defining, assume that $p+1 = m$ and $c_m < n$. When $c_{p+1} - c_p = 2$ holds, then (11) does not induce a facet of P^* for the same reason as the corresponding cardinality forcing inequality does not induce a facet of P^* . Indeed, both inequalities define the same face. When $c_{p+1} - c_p > 2$, then it is easy to see that the face induced by (11) is a proper subset of the face defined by the cardinality forcing inequality (8), and thus, it is not facet defining. The same argumentation holds when $p+1 = m$, $c_m = n$, and $n - |W| > 1$.

To show that (11) defines a facet, when the conditions are satisfied, we suppose that the equation $bx = b_0$ is satisfied by every $x \in P^*$ that satisfies (11) at equality. Using Theorem 3.1 we may assume that $b_{w1} = 2$ for some $w \in W$, $b_{i1} = 2$ for all $i \in W$, and $b_{iw} = -(|W| - c_p - 1)$ for all $i \in N \setminus W$.

Let $q, r \in N \setminus W$ be two nodes that are equal if $c_{p+1} = |W| + 1$ and otherwise different. Then, all (q, r) -paths of length $|W| + 1$ whose internal nodes are all

the nodes of W satisfies the equation $bx = b_0$. (Note, in case $c_{p+1} = |W| + 1$, the paths are Hamiltonian cycles.) Thus, it exist α_q , β_r , and α_j , β_j for $j \in W$ with

$$\begin{aligned} b_{qj} &= \alpha_q + \beta_j & (j \in W) \\ b_{ir} &= \alpha_i + \beta_r & (i \in W) \\ b_{ij} &= \alpha_i + \beta_j & ((i, j) \in A(W)). \end{aligned}$$

Without loss of generality we may assume that $\beta_w = 0$. Since $b_{1j} = 2$, it follows that $\alpha_1 = 2$, $\beta_j = 0$ for all $j \in W \setminus \{1\}$, and $\alpha_q = |W| - c_p - 1$. When $c_p = 2$, then the cycles $\{(1, j), (j, 1)\}$ for $j \in W \setminus \{1\}$. When $c_p \geq 3$, then consider a tight c_p -cycle that starts with $(1, i)$, (i, j) and skips node k for some $i, j, k \in W \setminus \{1\}$. Replacing the arcs $(1, i)$, (i, j) by $(1, k)$, (k, j) yields another tight c_p -cycle, and thus the equation $b_{1i} + b_{ij} = b_{1k} + b_{kj}$. In either case, it follows that $b_{j1} = 2$ for $j \in W \setminus \{1\}$ and there is λ such that $b_{ij} = \lambda$ for all $(i, j) \in A(W \setminus \{1\})$. Summarizing our intermediate results and adding further, easy obtainable observations, we see that

$$\begin{aligned} b_{1i} &= 2 & (i \in W \setminus \{1\}) \\ b_{i1} &= 2 & (i \in W \setminus \{1\}) \\ b_{ij} &= \lambda & ((i, j) \in A(W \setminus \{1\})) \\ b_{qi} &= -(|W| - c_p - 1) & (i \in W \setminus \{1\}) \\ b_{q1} &= -(|W| - c_p - 1) + 2 - \lambda \\ b_{ir} &= -(|W| - c_p - 1)(\lambda - 1) & (i \in W \setminus \{1\}) \\ b_{1r} &= -(|W| - c_p - 1)(\lambda - 1) + 2 - \lambda \\ b_0 &= 4 + (c_p - 2)\lambda \end{aligned} \tag{12}$$

holds.

So, when $c_{p+1} = n$, we have $q = r$ and $N \setminus W = \{q\}$, and thus, $bx = b_0$ is the equation

$$\begin{aligned} 2x(\delta^{\text{out}}(1)) - \lambda x_{1q} + 2x(\delta^{\text{in}}(1)) - \lambda x_{q1} + \lambda x(A(W \setminus \{1\})) \\ - (|W| - c_p - 1)x(\delta^{\text{out}}(q)) - (|W| - c_p - 1)(\lambda - 1)x(\delta^{\text{in}}(q)) = 4 + (c_p - 2)\lambda. \end{aligned}$$

Adding $(1 - \frac{\lambda}{2})(|W| - c_p - 1)$ times the equation $x(\delta^{\text{out}}(q)) - x(\delta^{\text{in}}(q)) = 0$ and $(\lambda - 2)$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$, we see that $bx = b_0$ is equivalent to (11), and hence (11) is facet defining.

Otherwise, that is, if $p+1 < m$, (12) holds for each pair of nodes $q, r \in N \setminus W$. Moreover, letting $k \neq l \in W \setminus \{1\}$, it can be seen that every (k, l) -path P of length $c_{p+1} - |W| + 1$ or $c_m - |W| + 1$ whose internal nodes are in $N \setminus W$ satisfies the equation $bx = -\lambda(|W| - c_p - 1)$. Thus, there are π_k , π_l , and $\{\pi_j | j \in N \setminus W\}$ such that

$$\begin{aligned} b_{kj} &= \pi_k - \pi_j & (j \in N \setminus W) \\ b_{jl} &= \pi_j - \pi_l & (j \in N \setminus W) \\ b_{ij} &= \pi_i - \pi_j & ((i, j) \in A(N \setminus W)). \end{aligned}$$

Since $b_{kj} = -(|W| - c_p - 1)(\lambda - 1)$ for $j \in N \setminus W$, it follows that $\pi_i = \pi_j$ for all $i, j \in N \setminus W$ which implies that $b_{ij} = 0$. Hence, $bx = b_0$ is the equation

$$\begin{aligned} 2x(\delta^{\text{out}}(1)) + 2x(\delta^{\text{in}}(1)) - \lambda \sum_{i \in N \setminus W} (x_{1i} + x_{i1}) \\ + \lambda x(A(W \setminus \{1\})) - (|W| - c_p - 1)x((N \setminus W) : W) \\ - (|W| - c_p - 1)(\lambda - 1)x((W : N \setminus W)) = 4 + (c_p - 2)\lambda. \end{aligned}$$

Adding $(1 - \frac{\lambda}{2})(|W| - c_p - 1)$ times the equation

$$x((N \setminus W : W)) - x((W : N \setminus W)) = 0$$

and $(\lambda - 2)$ times the equations $x(\delta^{\text{out}}(1)) = 1$ and $x(\delta^{\text{in}}(1)) = 1$, we see that $bx = b_0$ is equivalent to (11), and hence (11) is facet defining. \square

3.2 Facets unrelated to cardinality restrictions

Theorem 3.4. *Let $c \in CS$ and $n \geq 4$. The nonnegativity constraint*

$$x_{ij} \geq 0 \tag{13}$$

defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if $c \neq (2, n)$ or $c = (2, n)$, $n \geq 5$, and (i, j) is an inner arc.

Proof. By Theorem 3.1 of [20], (13) defines a facet of $P_{0,n\text{-path}}^{(k)}(\tilde{D}_n)$ whenever $4 \leq k \leq n - 1$. Hence, Lemma 2.1 implies that (13) is facet defining for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if $n \geq 5$ and there is an index p with $4 \leq c_p \leq n - 1$.

It remains to examine the cases $c = (2, n)$, $(3, n)$, and $(2, 3, n)$. When $n = 4$, the assertion can be verified using a computer program. For $n \geq 5$, inequalities (13) are facet defining for the asymmetric traveling salesman polytope $P_C^{(n)}(D_n)$ (see Grötschel and Padberg [13]). Thus, there are $n^2 - 3n + 1$ linearly independent points x^r in $P_{0,n\text{-path}}^c(\tilde{D}_n)$ satisfying $1^T x^r = n$ and $x_{ij}^r = 0$. When $c = (2, n)$ and the arc (i, j) is incident with a terminal node, there are only $n^2 - 2n - 1$ affinely independent points in $P_{0,n\text{-path}}^c(\tilde{D}_n)$ satisfying (13) with equality, since we have only $n - 2$ paths from 0 to n of length 2 that do not use (i, j) . When $c = (2, n)$ and (i, j) is an inner arc or $c = (3, n)$, (13) can be shown to induce a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ applying the same techniques as in the proof to Theorem 2.3. Clearly then, (13) induces also a facet of $P_{s,t\text{-path}}^{(2,3,n)}(D)$. \square

Theorem 3.5. *Let $c \in CS$, $n \geq 4$, and i be an internal node of \tilde{D}_n . The degree constraint*

$$x(\delta^{\text{out}}(i)) \leq 1 \tag{14}$$

induces a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ unless $c = (2, n)$.

Proof. When $n \geq 5$ and $4 \leq c_p \leq n - 1$ for some index p , (14) can be shown to induce a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ using Lemma 2.1 of this paper and Theorem 3.2 of [20], saying that (14) induces a facet of $P_{0,n\text{-path}}^{(c_p)}(\tilde{D}_n)$. So, let $c \in \{(2, n), (3, n), (2, 3, n)\}$. All Hamiltonian $(0, n)$ -paths satisfy $x(\delta^{\text{out}}(i)) = 1$, and $n^2 - 3n + 2$ of them are linearly independent. So, when $c = (2, n)$, we have only one further affinely independent $(0, n)$ -path, namely the path $\{(0, i), (i, n)\}$. Thus, in this case (14) is not facet defining. Against it, when $c = (3, n)$, we consider the paths $\{(0, i), (i, j), (j, n)\}$ for all internal nodes $j \neq i$. These $n - 2$ paths are linearly independent, and they are also linearly independent of the $n^2 - 3n + 2$ Hamiltonian paths which can be proved with the same approach as in the proof to Theorem 2.3. Consequently, (14) induces a facet of $P_{0,n\text{-path}}^{(3,n)}(\tilde{D}_n)$ and $P_{0,n\text{-path}}^{(2,3,n)}(\tilde{D}_n)$. \square

Lemma 3.6. *Let $4 \leq p \leq n - 1$ and $\tilde{N}_n = \{0\} \dot{\cup} Q \dot{\cup} \{v\} \dot{\cup} R \dot{\cup} \{n\}$ be a partition of \tilde{N}_n with Q and R nonempty and $|Q| \leq p - 2$. Moreover, let $bx = b_0$ be an equation, with $b_{0i} = 0$ for $i \in Q \cup \{v\}$ and $b_{0j} = 1$ for $j \in R$, that is satisfied by each $(0, n)$ -path of length p that satisfies the equation $x((Q \cup \{0\} : R)) - x((R : \{v\})) = 0$. If $|Q| \geq 5$ or $|R| \geq 5$, then there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that*

$$\begin{aligned}
b_{ij} &= \alpha & \forall (i, j) \in \tilde{A}_n(Q), \\
b_{ij} &= \alpha & \forall (i, j) \in \tilde{A}_n(R), \\
b_{ij} &= 1 + \alpha & \forall (i, j) \in (Q : R), \\
b_{ij} &= \beta & \forall (i, j) \in (R : Q), \\
b_{in} &= \gamma & \forall i \in Q, \\
b_{iv} &= \delta & \forall i \in Q, \\
b_{vi} &= \epsilon & \forall i \in Q, \\
b_{in} &= \zeta & \forall i \in R, \\
b_{iv} &= \eta & \forall i \in R, \\
b_{vi} &= \theta & \forall i \in R.
\end{aligned} \tag{15}$$

Proof. First, we assume that $|Q| \geq 5$ which implies $p \geq 7$. Let P_1 be any $(0, v)$ -path of length $|Q|$ that visits all nodes in Q but one, say l . Complete P_1 to a $(0, n)$ -path of cardinality p that satisfies the equation $x((Q \cup \{0\} : R)) - x((R : \{v\})) = 0$, and hence also $bx = b_0$. Substituting any two arcs $(i, k), (k, j) \in \tilde{A}_n(Q) \cap P$ by the arcs $(i, l), (l, j)$ yields a new $(0, n)$ -path that satisfies also the equation $bx = b_0$, and hence, $b_{ik} + b_{kj} = b_{il} + b_{lj}$. Obviously, we can derive such an equation for all distinct nodes $i, j, k, l \in Q$. By Lemma 2 of Hartmann and Özlük [14], it exist α, π_0, π_v , and $\{\pi_j | j \in Q\}$ with

$$\begin{aligned}
b_{0i} &= \alpha + \pi_0 - \pi_i & \forall i \in R, \\
b_{iv} &= \alpha + \pi_i - \pi_v & \forall i \in R, \\
b_{ij} &= \alpha + \pi_i - \pi_j & \forall (i, j) \in \tilde{A}_n(Q).
\end{aligned}$$

Without loss of generality, we may assume that $\pi_z = 0$ for some $z \in Q$, and since $b_{0i} = 0$ for $i \in Q$, we have even $\pi_i = 0$ for all $i \in Q$. Consequently, $b_{ij} = \alpha$ for all $(i, j) \in \tilde{A}_n(Q)$ and $b_{iv} = \alpha - \pi_v =: \delta$ for all $i \in Q$.

Next, consider a $(0, n)$ -path P of length p with $b(P) = b_0$ starting with the arc $(0, j)$ for some $j \in R$ and ending with the arcs $(k, l), (l, m), (m, n)$ for any $k, l, m \in Q$ that skips some node $i \in Q$. Substituting the arcs $(0, j), (k, l), (l, m)$ by the arcs $(0, i), (i, j), (k, m)$, yields a path \tilde{P} for which $b(\tilde{P}) = b_0$ holds. Thus, $b_{0j} + b_{kl} + b_{lm} = b_{0i} + b_{ij} + b_{lm}$, which implies that $b_{ij} = 1 + \alpha$ for all $(i, j) \in (Q : R)$.

Next, consider a $(0, n)$ -path P of length p with $b(P) = b_0$ that ends with the arcs $(i, j), (j, n)$ and skips node k for some $i, j, k \in Q$. Then the path $\tilde{P} := (P \setminus \{(i, j), (j, n)\}) \cup \{(i, k), (k, n)\}$ satisfies $b(\tilde{P}) = b_0$, and thus, $b_{ij} + b_{jn} = b_{ik} + b_{kn}$ which implies $b_{jn} = b_{kn}$, since $b_{ij} = b_{ik} = \alpha$. Consequently, $b_{in} = \gamma$ for some γ and all $i \in Q$. With a similar construction one can show that there is ϵ such that $b_{vi} = \epsilon$ for $i \in Q$. When $|R| = 1$, then we have no more to show. Otherwise, let $(i, j) \in R$ and P be a $(0, n)$ -path starting with $(0, i), (i, j)$, skipping a node $k \in Q$, and satisfying $b(P) = b_0$. Substituting node i by k , yields the equation $b_{0i} + b_{ij} = b_{0k} + b_{kj}$. Since $b_{0i} = 1$, $b_{0k} = 0$, and $b_{kj} = 1 + \alpha$, it follows $b_{ij} = \alpha$. It can be shown with similar arguments that the remaining equations of system (15) also holds.

Second, let $|R| \geq 5$. When $p \geq 5$, the proof can be performed similarly by showing firstly that the equation $b_{ik} + b_{kj} = b_{il} + b_{lj}$ holds for any distinct nodes $i, j, k, l \in R$, and applying Lemma 2 of Hartmann and Özlük [14]. When $p = 4$, the method fails, because it does not exist a $(0, n)$ -path of length 4 that satisfies the equation $bx = b_0$ and visits 3 nodes in R . So, let $p = 4$ which implies $|Q| \leq 2$. Consider the paths $\{(0, i), (i, v), (v, k), (k, n)\}$ and $\{(0, j), (j, v), (v, k), (k, n)\}$ for some $i, j \in R$ and $k \in Q$. Since both paths satisfy the equation $bx = b_0$, it follows that $b_{iv} = b_{jv}$. Hence, $b_{iv} = \eta$ for some η , for $i \in R$. This in turn implies the existence of α such that $b_{ij} = \alpha$ holds for all $(i, j) \in \tilde{A}_n(R)$, by considering the paths $\{(0, i), (i, j), (j, v), (v, n)\}$. Analogous constructions show successively that also the remaining equations of system (15) hold. \square

Theorem 3.7. *Let $c = (c_1, \dots, c_m) \in CS$, $n \geq 4$, $S \subset \tilde{N}_n$, $0, n \in S$, and $v \in \tilde{N}_n \setminus S$. The one-sided min-cut inequality*

$$x((S : \tilde{N}_n \setminus S)) - x(\delta^{in}(v)) \geq 0 \quad (16)$$

induces a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if $|\tilde{N}_n \setminus S| \geq 2$, $|S| \geq c_1 + 1$, and $c \neq (2, n)$.

Proof. Necessity. When $\tilde{N}_n \setminus S = \{v\}$, (16) becomes the trivial inequality $0x \geq 0$, and thus it is not facet defining. When $|S| \leq c_1$, all feasible $(0, n)$ -paths P satisfy $|P \cap (S : \tilde{N}_n \setminus S)| \geq 1$, and hence, (16) can be obtained by summing up the inequality $x((S : \tilde{N}_n \setminus S)) \geq 1$ and the degree constraint $-x(\delta^{in}(v)) \geq -1$. Suppose that $c = (2, n)$, $|\tilde{N}_n \setminus S| \geq 2$, and $|S| \geq 3$. With respect to the polytope $P_{0,n\text{-path}}^{(n)}(\tilde{N}_n)$, (16) is equivalent to the inequality $x(\tilde{A}_n(S)) \leq |S| - 1$. The latter inequality is an equivalent of a subtour elimination constraint defined on D_n which is known to be facet defining for the ATSP. Thus, (16) induces a facet of $P_{0,n\text{-path}}^{(n)}(\tilde{D}_n)$ and consequently, there are $n^2 - 3n + 1$ linear independent incidence vectors of Hamiltonian $(0, n)$ -paths satisfying (16) at equality. From the $(0, n)$ -paths of length 2 are only the paths $\{(0, i), (i, n)\}$, $i \in (S \setminus \{0, n\}) \cup \{v\}$ tight which are at most $n - 2$. Consequently, there are not enough affinely independent points satisfying (16) at equality, and hence, (16) does not induce a facet of $P_{0,n\text{-path}}^{(2,n)}(\tilde{D}_n)$.

Sufficiency. By Theorem 3.4 of [20], (16) induces a facet of $P_{0,n\text{-path}}^{(k)}(\tilde{D}_n)$ for $4 \leq k \leq n - 2$ if and only if $|S| \geq k + 1$ and $|\tilde{N}_n \setminus S| \geq 2$. Hence, when $|S| \geq c_i + 1$ for some index $i \in \{1, \dots, m\}$ with $c_i \geq 4$ and $|\tilde{D}_n \setminus S| \geq 2$, inequality (16) is facet defining for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ by applying Lemma 2.1. In particular, this finishes the proof when $i = 1$. Note that in case $i = m$, $c_i \geq 4$ and $|S| \geq c_i + 1$ imply $4 \leq c_m \leq n - 2$, since $|S| \leq n - 1$.

It remains to consider the cases $c_1 = 2$ and $c_1 = 3$. Set $Q := S \setminus \{0, n\}$ and $R := \tilde{N}_n \setminus (S \cup \{v\})$. Suppose that $R \neq \emptyset$, $|Q| \geq c_1 - 1$, and $c \neq (2, n)$.

First, let $c = (3, n)$ and $2 \leq |Q| \leq n - 3$. We have already mentioned that inequality (16) induces a facet of $P_{0,n\text{-path}}^{(n)}(\tilde{D}_n)$, which implies that there are $n^2 - 3n + 1$ linearly independent incidence vectors x^r of Hamiltonian paths satisfying (16) at equality. In addition, consider the paths $(0, i), (i, j), (j, n)$ for some $(i, j) \in \tilde{A}_n(Q)$, and $(0, k), (k, v), (v, n)$ for each $k \in Q \cup R$. All these $n - 1$ paths are tight and with linearly independent incidence vectors, and considering the corresponding points in $P_{PL}^c(\tilde{D}_n)$ one can see that they are also linearly

independent of the points x^r . Consequently, we have found $n^2 - 2n$ affinely independent points satisfying (16) with equality.

Next, let $c_1 = 3$, $c \neq (3, n)$, and $2 \leq |Q| \leq c_2 - 2$. Since $c \neq (3, n)$, it follows that $4 \leq c_2 \leq n - 1$. When $n \leq 10$, the assertion can be verified using a computer program. So, let $n \geq 11$, which implies that Q or R contains at least five nodes. Furthermore, let $bx = b_0$ be an equation that is satisfied by all $(0, n)$ -paths of feasible length satisfying (16) at equality. By Theorem 3.1, we may assume that $b_{0i} = 0$ for $i \in Q \cup \{v\}$, $b_{vn} = 0$, and $b_{0j} = 1$ for $j \in R$. By Lemma 3.6, there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that equation system (15) holds. Since, $b_{vn} = 0$, a path $(0, i), (i, v), (v, n)$ with $i \in Q \cup R$ yields $\delta = b_0$ and $\eta = b_0 - 1$. Next, consider a tight $(0, n)$ -path of length c_2 that visits firstly all nodes in Q , then $q \geq 0$ nodes in R , and finally node v . In either case, such a path yields the equation $(c_2 - 3)\alpha = 0$, and hence, $\alpha = 0$. Moreover, from the paths $(0, i), (i, j), (j, n)$ and $(0, v), (v, k), (k, n)$ for some $i, j, k \in Q$ we deduce that $\gamma = b_0$ and $\epsilon = 0$. Again, consider a tight $(0, n)$ -path of length c_2 that visits firstly some nodes in R , then v , and finally some nodes in Q . Such a path yields the equation $2b_0 = b_0$, and thus, $b_0 = 0$. Now it is easy to see that $\beta = \zeta$ and $\zeta + \theta = 0$. Summarizing the results, we conclude that $bx = b_0$ is the equation

$$x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) + \theta[x(\delta^{\text{out}}(v)) - x((\tilde{N}_n \setminus S : S))] = 0.$$

Since $x(\delta^{\text{out}}(v)) = x(\delta^{\text{in}}(v))$ and $x((S : \tilde{N}_n \setminus S)) = x((\tilde{N}_n \setminus S : S))$ for all $x \in P_{0,n\text{-path}}^c(\tilde{D}_n)$, $bx = b_0$ and

$$x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0.$$

are equivalent which proves that (16) defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ for all c with $c_1 = 3$ and Q with $2 \leq |Q| \leq c_2 - 2$.

Finally, let $c_1 = 2$ and $c \neq (2, n)$. When $c = (2, 3, n)$ and $2 \leq |Q| \leq n - 3$, the assertion follows immediately, because (16) defines already a facet of $P_{0,n\text{-path}}^{(3,n)}(\tilde{D}_n)$. When $c = (2, 3, n)$ and $|Q| = 1$, say $Q = \{q\}$, then we have beside $n^2 - 3n + 1$ linearly independent tight Hamiltonian paths, $n - 2$ paths $(0, i), (i, v), (v, n)$ for $i \in Q \cup R$, and the path $(0, q), (q, n)$. These $n^2 - 2n$ paths can be shown to be linearly independent proving the assertion. Finally, let $4 \leq c_2 \leq n - 1$ and $1 \leq |Q| \leq c_2 - 2$. When $n \leq 10$, the claim can be shown using a computer program. When $n \geq 11$, again we assume that an equation $bx = b_0$ is satisfied by all points $y \in P_{0,n\text{-path}}^c(\tilde{D}_n)$ that satisfies (16) at equality. Moreover, we suppose that $b_{0i} = 0$ for $i \in Q \cup \{v\}$, $b_{vn} = 0$, $b_{0i} = 1$ for $i \in R$, and there are $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$, and θ such that (15) holds. This time, it follows immediately $b_0 = 0$ and $\gamma = 0$ by considering the paths $(0, i), (i, n)$ for $i \in Q \cup \{v\}$. We distinguish three cases:

$|R| = 1$: Since $|R| = 1$ implies $|Q| = n - 3$ and $c_2 = n - 1$, the $(0, n)$ -paths of length c_2 whose internal node are in $Q \cup \{v\}$ yield the equations $(c_2 - 3)\alpha + \delta$, $(c_2 - 3)\alpha + \epsilon = 0$, and $(c_2 - 4)\alpha + \delta + \epsilon = 0$. This in turn gives $\alpha = \delta = \epsilon = 0$. Now it can be easily derived that $\eta = -1$, $\beta = \zeta$, and $\zeta + \theta = 0$. Hence, the equation $bx = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

$|R| \geq 2$ and $p = 4$: The paths

$$\begin{aligned} &\{(0, q), (q, i), (i, v), (v, n)\}, \\ &\{(0, i), (i, v), (v, q), (q, n)\}, \\ &\{(0, v), (v, i), (i, j), (j, n)\}, \\ &\{(0, v), (v, i), (i, q), (q, n)\}, \\ &\{(0, i), (i, v), (v, j), (j, n)\}, \\ &\{(0, q), (q, v), (v, j), (j, n)\}, \quad q \in Q, i, j \in R \end{aligned}$$

yield an equation system that solves to $\alpha = \delta = \epsilon = 0$, $\eta = -1$, $\beta = \zeta$, and $\zeta + \theta = 0$. Thus, also in this case we see that $bx = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

$|R| \geq 2$ and $p \geq 5$: For any subsets N_1, N_2, \dots, N_p of \tilde{N}_n we say that a path P goes along N_1, N_2, \dots, N_k (denoted by $N_1 \rightarrow N_2 \rightarrow \dots \rightarrow N_k$) when P can be decomposed in subsequences P_1, P_2, \dots, P_{k-1} such that $P_i \subseteq N_i \cup (N_i : N_{i+1})$ and $P_i \cap (N_i : N_{i+1}) \neq \emptyset$ for $i = 1, \dots, k-1$. From $(0, n)$ -paths of length c_2 going along $0 \rightarrow Q \rightarrow R \rightarrow v \rightarrow n$, $0 \rightarrow Q \rightarrow v \rightarrow R \rightarrow n$, $0 \rightarrow R \rightarrow v \rightarrow Q \rightarrow n$, $0 \rightarrow v \rightarrow R \rightarrow Q \rightarrow n$, and $0 \rightarrow R \rightarrow v \rightarrow R \rightarrow Q \rightarrow n$, we obtain an equation system which solves to $\alpha = \epsilon = 0$, $\eta = -1$, $\theta + \beta = 0$, and $\delta + \zeta + \theta = 0$. When $|Q| = 1$, then any $(0, n)$ -path of length c_2 that skips the node in Q and ends with $(v, r), (r, n)$, yields the equation $\zeta + \theta = 0$ which implies also $\delta = 0$ and $\beta = \zeta$. Otherwise, there is a $(0, n)$ -path of length c_2 along $0 \rightarrow Q \rightarrow v \rightarrow R \rightarrow Q \rightarrow n$ from which we obtain the equation $\delta + \theta + \beta = 0$. Hence, we obtain also $\delta = 0$, $\beta = \zeta$, and $\zeta + \theta = 0$. In either case, we see that $bx = b_0$ is equivalent to the equation $x((S : \tilde{N}_n \setminus S)) - x(\delta^{\text{in}}(v)) = 0$.

Thus, inequality (16) defines also a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ for all c and Q with $c_1 = 2$ and $1 \leq |Q| \leq c_2 - 2$, which finishes the proof. \square

We introduce a further class of inequalities whose undirected pendants we need later for the characterization of the integer points of $P_C^c(K_n)$.

Theorem 3.8. *Let $c \in CS$, $n \geq 4$, $S \subset \tilde{N}_n$, and $0, n \in S$. The min-cut inequality*

$$x((S : \tilde{N}_n \setminus S)) \geq 1 \tag{17}$$

is valid for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if $|S| \leq c_1$ and facet defining for it if and only if $3 \leq |S| \leq c_1$ and $|\tilde{N}_n \setminus S| \geq 2$.

Proof. When $c \neq (3, n)$, the theorem follows from Theorem 3.3 of [20], Lemma 2.1, and the fact that $m \geq 2$. When $c = (3, n)$, then $|S| = 3$ and $|\tilde{N}_n \setminus S| = n - 2$. W.l.o.g. let $S = \{0, 1, n\}$ and $\tilde{N}_n \setminus S = \{2, \dots, n-1\}$. Identifying nodes 0 and n we see that the min-cut inequality is equivalent to a subtour elimination constraint. Thus, there are $n^2 - 3n + 1$ linearly independent Hamiltonian $(0, n)$ -paths in \tilde{D}_n that satisfy (17) at equality. To see that (17) is facet defining, consider also the 3-paths $\{(0, 1), (1, i), (i, n)\}$ for $i = 2, \dots, n-1$ and $\{(0, 2), (2, 3), (3, n)\}$. These are $n-1$ affinely independent points, and they are also affinely independent of the $n^2 - 3n + 1$ Hamiltonian paths which is easily seen by considering the corresponding points in $P_{PL}^c(\tilde{D}_n)$. Thus, we have constructed $n^2 - 2n$ points in $P_{0,n\text{-path}}^c(\tilde{D}_n)$ satisfying (17) at equality. \square

3.3 Inequalities specific to odd or even paths

Theorem 3.9. *Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 2$, and c_p even for $1 \leq p \leq m$, and let $\tilde{N}_n = S \dot{\cup} T$ be a partition of \tilde{N}_n with $0 \in S$, $n \in T$. The odd path exclusion constraint*

$$x(\tilde{A}_n(S)) + x(\tilde{A}_n(T)) \geq 1 \quad (18)$$

is valid for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ and defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if (i) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2} + 1$, or (ii) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2}$.

Proof. Clearly, each $(0, n)$ -path of even length uses at least one arc in $\tilde{A}_n(S) \cup \tilde{A}_n(T)$. Thus, inequality (18) is valid.

When $|S|$ or $|T|$ is less than $\frac{c_2}{2}$, then there is no $(0, n)$ -path of length c_p , $p \geq 2$, that satisfies (19) at equality which implies that (19) cannot be facet defining for $P_{0,n\text{-path}}^c(\tilde{D}_n)$. Thus $|S|, |T| \geq \frac{c_2}{2}$ holds if (18) is facet defining. For $c_1 = 2$ we have to require even $|S|, |T| \geq \frac{c_2}{2} + 1$. For the sake of contradiction assume w.l.o.g. that $|S| = \frac{c_2}{2}$. Then follows $|T| \geq \frac{c_2}{2} + 1$. However, for an inner arc $(i, j) \in \tilde{A}_n(S)$ there is no tight $(0, n)$ -path of cardinality c_2 that uses (i, j) .

Next, let (i) or (ii) be true. The conditions imply that for $p = 1$ or $p = 2$ $c_p \geq 4$ and $|S|, |T| \geq \frac{c_p}{2} + 1$ holds. Restricted to the polytope $P_{0,n\text{-path}}^{(c_p)}(\tilde{D}_n)$ inequality (18) is equivalent to the max-cut inequality $x((S : T)) \leq \frac{c_p}{2}$ which were shown to be facet defining for $P_{0,n\text{-path}}^{(c_p)}(\tilde{D}_n)$ (see Theorem 3.5 of [20]). Thus there are $n^2 - 2n - 1$ linearly independent points in $P_{0,n\text{-path}}^c(\tilde{D}_n) \cap P_{0,n\text{-path}}^{(c_p)}(\tilde{D}_n)$ satisfying (18) at equality. Moreover, the conditions ensure that there is also a tight $(0, n)$ -path of cardinality c_q , where $q = 3 - p$. By Lemma 2.1 (i), the incidence vector of this path is affinely independent of the former points, and hence, (18) defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$. \square

Theorem 3.10. *Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $\tilde{N}_n = S \dot{\cup} T$ be a partition of \tilde{N}_n with $0, n \in S$. The even path exclusion constraint*

$$x(\tilde{A}_n(S)) + x(\tilde{A}_n(T)) \geq 1 \quad (19)$$

is valid for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ and defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$ if and only if (i) $c_1 = 3$, $|S| - 1 \geq \frac{c_2+1}{2}$, and $|T| \geq \frac{c_2-1}{2}$, or (ii) $c_1 \geq 5$ and $\min(|S| - 1, |T|) \geq \frac{c_2-1}{2}$.

Proof. Since the start and end node 0 and n are in the same partition, each $(0, n)$ -path of odd length uses at least one arc in $\tilde{A}_n(S) \cup \tilde{A}_n(T)$. Thus, inequality (19) is valid.

To show necessity assume that $|S| - 1$ or $|T|$ is less than $\frac{c_2-1}{2}$. Then there is no $(0, n)$ -path of length c_r , $r \geq 2$, that satisfies (19) at equality. Thus, (19) cannot be facet defining. Hence, we must have $|S| - 1, |T| \geq \frac{c_2-1}{2}$. In case $c_1 = 3$ we have to require even $|S| - 1 \geq \frac{c_2+1}{2}$, because otherwise there would be no tight $(0, n)$ -path using an inner arc $(i, j) \in \tilde{A}_n(S)$.

To show sufficiency we firstly suppose that (i) holds. Let $bx = b_0$ be an equation that is satisfied by all $(0, n)$ -paths of feasible length satisfying (19) at equality. By Theorem 3.1, we may assume that $b_{0i} = 0$ for $i \in T$, $b_{in} = 1$ for $j \in S \setminus \{0\}$, and $b_{uv} = 0$ for some particular arc $(u, v) \in (T : S \setminus \{0, n\})$.

From the tight path $(0, u), (u, v), (v, n)$ we deduce that $b_0 = 1$. Furthermore, for $i \in T, j \in S$ the paths $(0, i), (i, j), (j, n)$ are tight, $b_{0i} = 0$, and $b_{jn} = 1$. Thus, $b_{ij} = 0$ for all $i \in T, j \in S \setminus \{0, n\}$. Next, we show that $b_{ij} = 0$ for all $i \in S \setminus \{0, n\}, j \in T$. Let $a_1 = (i_1, j_1), \dots, a_p = (i_p, j_p) \in (S \setminus \{0, n\} : T)$ be any non-adjacent arcs, where $p := \frac{c_2-1}{2}$. Associate with each $q \in \{1, \dots, p\}$ a tight $(0, n)$ -path P^q of length c_2 that uses all arcs a_1, \dots, a_p with the exception of a_q and whose associated coefficients b_{ij} of the remaining arcs are known. For instance, if $q = 1$ then

$$(0, j_1), (j_1, i_2), (i_2, j_2), \dots, (j_{p-1}, i_p), (i_p, j_p), (j_p, i_1), (i_1, n)$$

is such a path. These paths yield an equality system of the form

$$(E - I)b = 0,$$

where E is the $p \times p$ matrix of all ones, I is the $p \times p$ identity matrix, and b is the vector of variables $b_k = b_{a_k}$ for $1 \leq k \leq p$. Since $E - I$ is a nonsingular matrix, we have that $b_{a_k} = 0$ for $1 \leq k \leq p$. Since the arcs a_j were arbitrarily chosen, it follows that $b_{ij} = 0$ for all $i \in S \setminus \{0, n\}, j \in T$. Next, considering the paths $(0, i), (i, j), (j, n)$ for $i \in S \setminus \{0, n\}, j \in T$ we see that $b_{jn} = \gamma$ and $b_{0i} = 1 - \gamma$ for some γ . Further, from the paths $(0, i), (i, j), (j, n)$ with $(i, j) \in \tilde{A}_n(T)$ we derive that $b_{ij} = 1 - \gamma$ for all $(i, j) \in \tilde{A}_n(T)$. Finally, from any tight path using an arc $(i, j) \in \tilde{A}_n(S \setminus \{0, n\})$, we derive that $b_{ij} = 1 - \gamma$ for those arcs (i, j) . Thus, $bx = b_0$ is the equation

$$(1 - \gamma)x(\tilde{A}_n(S)) + (1 - \gamma)x(\tilde{A}_n(T)) + \gamma x(\delta^{\text{in}}(n)) = 1.$$

Subtracting γ times the equation $x(\delta^{\text{in}}(n)) = 1$ and dividing by $1 - \gamma$, we see that $bx = b_0$ is equivalent to (19). Consequently, (19) defines a facet of $P_{0,n\text{-path}}^c(\tilde{D}_n)$.

Finally, suppose that (ii) holds. Note, inequality (19) is equivalent to the *max-cut inequality* $x((S : T)) \leq \lfloor \frac{c_1}{2} \rfloor$ with respect to $P_{0,n\text{-path}}^{(c_1)}(\tilde{D}_n)$. From Theorem 3.5 of [20] follows that the latter inequality defines a facet of $P_{0,n\text{-path}}^{(c_1)}(\tilde{D}_n)$, hence also the former. Since there is also a tight $(0, n)$ -path of length c_2 , it follows the claim by Lemma 2.1 (i). \square

Theorem 3.11. *Let $D_n = (N, A)$ be the complete digraph on $n \geq 6$ nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 3$, $c_1 \geq 2$, $c_m \leq n$, and $c_{p+2} = c_{p+1} + 2 = c_p + 4$ for some $2 \leq p \leq m - 2$. Moreover, let $N = P \dot{\cup} Q \dot{\cup} \{r\}$ be a partition of N , where P contains node 1 and satisfies $|P| = c_p + 1 = c_{p+1} - 1$. Then the modified cardinality forcing inequality*

$$\sum_{v \in P} x(\delta^{\text{out}}(v)) - \sum_{v \in Q} x(\delta^{\text{out}}(v)) + x((Q : \{r\})) - x((P : \{r\})) \leq c_p \quad (20)$$

defines a facet of $P^ = \{x \in P_C^c(D_n) | x(\delta^{\text{out}}(1)) = 1\}$.*

Proof. The arcs that are incident with node r have coefficients zero. Let C be a cycle that visits node 1 and is of feasible length. If C does not visit node r , C satisfies clearly (20), since the restriction of (20) to the arc set $A(N \setminus \{r\})$ is an ordinary cardinality forcing inequality (8). When C visits node r and

uses at most c_p arcs whose corresponding coefficients are equal to one, then C satisfies also (20), since all those coefficients that are not equal to 1 are 0 or -1 . So, let C with $|C| \geq c_{p+1}$ visit node r and use as many arcs whose corresponding coefficients are equal to one as possible. That are exactly $|P|$ arcs which are contained in $A(P) \cup (P : Q)$. But then C must use at least one arc in $A(Q) \cup (Q : P)$ whose coefficient is -1 . Hence, also in this case C satisfies (20), which proves the validity of (20).

To show that (20) is facet defining, suppose that the equation $bx = b_0$ is satisfied by all points that satisfy (20) at equality. By Theorem 3.1, we may assume that $b_{1r} = b_{r1} = 0$ and $b_{1i} = 1$ for $i \in N \setminus \{1, r\}$. By considering the c_{p+1} -cycles with respect to $P \cup \{j\}$ for $j \in N \setminus P$, one can show along the lines of the proof of Theorem 3.2 that there are $\alpha_k, \beta_k, k \in N$, with $b_{ij} = \alpha_i + \beta_j$ for all $(i, j) \in A$, $\alpha_1 = 0, \beta_r = 0$, and $\beta_j = 1$. In particular, when $c_p = 2$, the tight 2-cycles $\{(1, i), (i, 1)\}, i \in P$ yield $\alpha_k = \alpha_\ell$ for $k, \ell \in P \setminus \{1\}$. Otherwise one can show as in the proof of Theorem 3.2 that $\alpha_k = \alpha_\ell$ for all $k, \ell \in P \setminus \{1\}$. Thus, there is κ such that $\alpha_i = \kappa$ for $i \in P, i \neq 1$. This in turn implies that there is λ with $\alpha_j = \lambda$ for $j \in Q$ by considering tight c_{p+1} -cycles. Then, the equation $b_{r1} = 0$, a tight cycle of length c_p , and two tight cycles of length c_{p+1} , one visiting node r , the other a node $j \in Q$, yield the equation system

$$\begin{aligned} b_{r1} &= 0 \\ b_0 &= (c_p - 1)(\kappa + 1) + \beta_1 \\ b_0 &= c_p(\kappa + 1) \\ b_0 &= c_p(\kappa + 1) + \lambda + \beta_1 + 1 \end{aligned}$$

which solves to

$$\begin{aligned} b_0 &= c_p(\kappa + 1) \\ \lambda &= -\kappa - 2 \\ \beta_1 &= \kappa + 1 \\ \alpha_r &= -\kappa - 1. \end{aligned}$$

Next, consider for $i \in P \setminus \{1\}, j, k \in Q$ a c_{p+2} -cycle C that starts in node 1, then visits all nodes in $P \setminus \{1, i\}$, followed by the nodes j, r, i, k , and finally returns to 1. Since C is tight, we can derive the equation

$$1 + (c_p - 1)(\kappa + 1) + b_{jr} + (\alpha_r + 1) + (\kappa + 1) + (\lambda + \beta_1) = b_0$$

which solves to $b_{jr} = \kappa$. By considering further tight c_{p+2} -cycles one can deduce that $b_{ri} = -\kappa$ for $i \in Q$ and $b_{jk} = -\kappa - 1$ for $(j, k) \in A(Q)$. Thus, $bx = b_0$ is the equation

$$\begin{aligned} x(\delta^{\text{out}}(1) \setminus \{(1, r)\}) - x((Q : \{1\})) + (2\kappa + 1)x((P \setminus \{1\} : \{1\})) \\ + (\kappa + 1) \sum_{i \in P \setminus \{1\}} x(\delta^{\text{out}}(i) \setminus \{(i, 1), (i, r)\}) \\ - (\kappa + 1) \sum_{i \in Q} x(\delta^{\text{out}}(i) \setminus \{(i, 1), (i, r)\}) \\ - \kappa x(\delta^{\text{out}}(r) \setminus \{(r, 1)\}) + \kappa x(\delta^{\text{in}}(r) \setminus \{(1, r)\}) = c_p(\kappa + 1). \end{aligned}$$

Adding κ times the equations $x(\delta^{\text{out}}(1)) - x(\delta^{\text{in}}(1)) = 0$ and $x(\delta^{\text{out}}(r)) - x(\delta^{\text{in}}(r)) = 0$, we see that $bx = b_0$ is equivalent to (20), and hence, (20) defines a facet. \square

3.4 Separation

All inequalities of the IP-model (7) as well as the min-cut inequalities (17) and the modified cardinality forcing inequalities (20) can be separated in polynomial

time. For the one-sided min-cut inequalities (16), separation consists in finding a minimum $\{0, n\}$ - l -cut in \tilde{D}_n for each node $l \in \tilde{N}_n \setminus \{0, n\}$. The cardinality forcing inequalities can be separated with a greedy algorithm. To this end, let $x^* \in \mathbb{R}_+^{\tilde{A}_n}$ be a fractional point. Set $y_i^* := x^*(\delta^{\text{out}}(i))$ for $i = 0, \dots, n-1$, and apply the greedy separation algorithm 8.27 of Grötschel [12] on input data y^*, \tilde{N}_n , and c . To separate the modified cardinality forcing inequalities this algorithm can be applied $n-1$ times as subroutine, namely: for each internal node r of \tilde{N}_n , apply it on the subgraph induced by $\tilde{N}_n \setminus \{r\}$.

Next, the separation problem for the odd (even) path exclusion constraints is equivalent to the maximum cut problem which is known to be NP-hard. Turning to the cardinality-subgraph inequalities (11), it seems to be very unlikely that there is a polynomial time algorithm that solves the separation problem for this class of inequalities. Assume that there is given an instance $(D' = (N', A'), c = (c_1, \dots, c_m), x^*)$ of the separation problem, where $x^* \in A'$ is a fractional point satisfying $x^*(\delta^{\text{out}}(1)) = 1$. (We consider the separation problem for P^* .) In the special case of $m = 2$ and $c_m = c_2 - c_1 = 2$ the separation problem for the inequalities (11) and x^* reduces to find a subset W^* of N' of cardinality $k := c_1 + 1$ such that $1 \in W^*$ and $x^*(A'(W^*)) > 2c_p$. This problem can be tackled on the underlying graph $G' = (N', E')$ with edge weights $w_e := x_{ij}^* + x_{ji}^*$ for $e = [i, j] \in E'$, where x_{ij}^* is set to zero if the arc (i, j) is not in A' . The associated optimization problem $\max w(E'(W)), W \subseteq N', 1 \in W, |W| = k$, is a variant of the weighted version of the densest k -subgraph problem which is known to be NP-hard (see Feige and Seltser [9]).

4 Facets of the other polytopes

In this section, we derive facet defining inequalities for related polytopes mentioned in the introduction from facet defining inequalities for the cardinality constrained path polytope $P_{0,n-\text{path}}^c(\tilde{D}_n)$.

4.1 Facets of the directed cardinality constrained cycle polytope

Corollary 4.1. *Let $D_n = (N, A)$ be the complete digraph on $n \geq 3$ nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Then the following statements hold:*

- (a) *The nonnegativity constraint $x_{ij} \geq 0$ defines a facet of $P_C^c(D_n)$.*
- (b) *The degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$ for every $i \in N$.*
- (c) *Let S be a subset of N with $2 \leq |S| \leq n-2$, let $v \in S$ and $w \in N \setminus S$. The multiple cycle exclusion constraint*

$$x(\delta^{\text{out}}(v)) + x(\delta^{\text{out}}(w)) - x((S : N \setminus S)) \leq 1 \quad (21)$$

induces a facet of $P_C^c(D_n)$ if and only if $|S|, |N \setminus S| \geq c_1$ and $c \notin \{(2, 3), (2, n)\}$.

- (d) *For any $S \subset N$ with $|S|, |N \setminus S| \leq c_1 - 1$, the min-cut inequality*

$$x((S : N \setminus S)) \geq 1 \quad (22)$$

is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if and only if $|S|, |N \setminus S| \geq 2$.

(e) Let S be a subset of N and $j \in N \setminus S$. The one-sided min-cut inequality

$$x((S : N \setminus S)) - x(\delta^{\text{out}}(j)) \geq 0 \quad (23)$$

defines a facet of $P_C^c(D_n)$ if and only if $|S| \geq c_1$ and $2 \leq |N \setminus S| \leq c_1 - 1$.

(f) The cardinality bound $x(A) \geq c_1$ defines a facet of $P_C^c(D_n)$ if and only if $c_1 = 3$ and $n \geq 5$ or $4 \leq c_1 \leq n - 1$. Analogously, $x(A) \leq c_m$ defines a facet of $P_C^c(D_n)$ if and only if $c_m = 3$ and $n \geq 5$ or $4 \leq c_m \leq n - 1$.

(g) Let W be a subset of N with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-forcing inequality (8) defines a facet of $P_C^c(D_n)$ if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n - 1$.

(h) Let W be a subset of N such that $c_p < |W| < c_{p+1}$ holds for some $p \in \{1, \dots, m-1\}$. The cardinality-subgraph inequality (11) is valid for $P_C^c(D_n)$ and induces a facet of $P_C^c(D_n)$ if and only if $p+1 < m$ or $c_{p+1} = n = |W| + 1$.

(i) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 2$, and c_p even for $1 \leq p \leq m$, and let $N = S \dot{\cup} T \dot{\cup} \{n\}$ be a partition of N . The odd cycle exclusion constraint

$$x(A(S)) + x(A(T)) + x((T : \{n\})) - x((\{n\} : T)) \geq 0 \quad (24)$$

is valid for $P_C^c(D_n)$ and defines a facet of $P_C^c(D_n)$ if and only if (α) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2}$, or (β) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2} - 1$.

(j) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \dot{\cup} T$ be a partition of N . The even cycle exclusion constraint

$$x(A(S)) + x(A(T)) \geq 1 \quad (25)$$

is valid for $P_C^c(D_n)$ and defines a facet of $P_C^c(D_n)$ if and only if $|S|, |T| \geq \frac{c_2-1}{2}$.

(k) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 3$, $c_1 \geq 2$, $c_m \leq n$, $n \geq 6$, and $c_{p+2} = c_{p+1} + 2 = c_p + 4$ for some $2 \leq p \leq m-2$. Moreover, let $N = P \dot{\cup} Q \dot{\cup} \{r\}$ be a partition of N , with $|P| = c_p + 1 = c_{p+1} - 1$. Then the modified cardinality forcing inequality (20) defines a facet of $P_C^c(D_n)$.

Proof. (a) When $n \leq 4$, the statement can be verified using a computer program. When $c = (2, 3)$ and $n \geq 5$, we apply Theorem 10 of Hartmann and Özlük which says that $x_{ij} \geq 0$ defines a facet of $P_C^{(p)}(D_n)$ whenever $p \geq 3$ and $n \geq p + 1$. Thus, there are $n^2 - 2n$ 3-cycles satisfying $x_{ij} \geq 0$ at equality. Together with Lemma 2.1 applied on these tight 3-cycles and any 2-cycle not using arc (i, j) , we get the desired result. The remainder statements of (a) follow by application of Theorem 3.4 and Theorem 2.4.

(b) First, when $c = (2, 3)$ one can show along the lines of the proof to Proposition 5 of Balas and Oosten [1] that $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$. Next, when $(2, 3) \neq c \neq (2, n)$, the degree constraint can be shown to induce a facet using theorems 3.5 and 2.4. Finally, when $c = (2, n)$, consider any $n^2 - 3n + 2$ affinely independent tours and the $n - 1$ additional 2-cycles $\{(i, j), (j, i)\}$ for $j \in N \setminus \{i\}$. Note, all tours satisfy the degree constraint at equation, and hence, these $n^2 - 2n + 1$ constructed points are tight. Moreover, using Lemma 2.2 they

can be easily shown to be facet defining for $P_C^c(D_n)$.

(c) Supposing that $c = (2, 3)$, the inequality (21) is dominated by the nonnegativity constraint $x_{ij} \geq 0$ for any arc $(i, j) \in (S : N \setminus S) \cup (N \setminus S : S)$ that is neither incident with v nor with w . Next, suppose that $c = (2, n)$. Inequality (21) is equivalent to the subtour elimination constraint $x(A(S)) \leq |S| - 1$ with respect to the ATSP $P_C^{(n)}(D_n)$. Thus, we have $n^2 - 3n + 1$ tours satisfying (21) at equality. But we have only $n - 1$ tight 2-cycles, and consequently, (21) does not induce a facet. Next, if $|S| \leq c_1 - 1$, then (21) is the sum of the valid inequalities $x(\delta^{\text{out}}(v)) - x((S : N \setminus S)) \leq 0$ and $x(\delta^{\text{out}}(w)) \leq 1$. Finally, if $|N \setminus S| \leq c_1 - 1$, then (21) is the sum of the inequalities $x(\delta^{\text{out}}(w)) - x((S : N \setminus S)) \leq 0$ and $x(\delta^{\text{out}}(v)) \leq 1$ (cf. Hartmann and Özlük [14, p. 162]).

Suppose that the conditions in (c) are satisfied. First, consider the inequality (21) on the polytope $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(1)) = 1\}$ which is isomorphic to the path polytope $P_{0,n-\text{path}}^c(\tilde{D}_n)$. Then, (21) is equivalent to the one-sided min-cut inequality (16) which defines a facet of Q by Theorem 3.7. Thus, also (21) defines a facet of Q . Now, by application of Theorem 2.4 on Q and (21) we obtain the desired result. (When $c_1 \geq 4$, then the statement can be proved also with Theorem 14 of Hartmann and Özlük [14].)

(d) Assuming $|S| = 1$ or $|N \setminus S| = 1$ implies that (22) is an implicit equation. So, let $|S|, |N \setminus S| \geq 2$ which implies that $c_1 \geq 3$. From Theorem 3.8 follows that (22) defines a facet of $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(i)) = 1\}$, and hence, by Theorem 2.4, it defines also a facet of $P_C^c(D_n)$.

(e) When $|N \setminus S| \geq c_1$, (23) is obviously not valid. When $|N \setminus S| = 1$, (23) is the flow constraint $x(\delta^{\text{in}}(j)) - x(\delta^{\text{out}}(j)) = 0$. When $|S| \leq c_1 - 1$ and $|N \setminus S| \leq c_1 - 1$, (23) is the sum of the valid inequalities $x((S : N \setminus S)) \geq 1$ and $-x(\delta^{\text{out}}(j)) \geq -1$.

Suppose that $|S| \geq c_1$ and $2 \leq |N \setminus S| \leq c_1 - 1$. Then in particular $c_1 \geq 3$ holds. For any node $i \in S$, (23) defines a facet of $Q := \{x \in P_C^c(D_n) : x(\delta^{\text{out}}(i)) = 1\}$, by Theorem 3.7. Applying Theorem 2.4 we see that therefore (23) defines also a facet of $P_C^c(D_n)$.

(f) Since $\dim\{x \in P_C^c(D_n) : x(A) = c_i\} = \dim P_C^{(c_i)}(D_n)$, the claim follows directly from Theorem 1 of Hartmann and Özlük [14].

(g)-(i) Necessity can be proved as in the corresponding part of the proof to Theorem 3.2 (3.3, 3.9) while sufficiency can be shown by applying Theorem 2.4 on Theorem 3.2 (3.3, 3.9).

(j) By Theorem 15 of Hartmann and Özlük, (25) defines a facet of $P_C^{(c_1)}(D_n)$. Moreover, the cardinality conditions for S and T ensure that there is a tight cycle of cardinality c_2 , and hence, by Lemma 2.1, (25) defines a facet of $P_C^c(D_n)$.

(k) Apply Theorem 2.4 on Theorem 3.11. \square

4.2 Facets of the undirected cardinality constrained cycle polytope

In this section, we consider the undirected cardinality constrained cycle polytope $P_C^c(K_n)$ defined on the complete graph $K_n = (N, E)$, where c is a cardinality sequence with $3 \leq c_1 < \dots < c_m \leq n$ and $m \geq 2$. It was shown in [15] and [17] that $\dim P_C^{(p)}(K_n) = |E| - 1$ for $3 \leq p \leq n - 1$ and $n \geq 5$. Thus, it is

easy to verify that $\dim P_C^c(K_n) = |E| = n(n-1)/2$ for all $n \geq 4$, since $m \geq 2$. Note, in case of $n = 4$, $P_C^c(K_n) = P_C(K_n)$, and by Theorem 2.3 of Bauer [3], $\dim P_C(K_4) = 6 = |E|$.

Facet defining inequalities for $P_C^c(K_n)$ can be derived directly from the inequalities mentioned in Corollary 4.1 (b)-(h), since these inequalities are equivalent to symmetric inequalities. A valid inequality $cx \leq \gamma$ for $P_C^c(D_n)$ is said to be *symmetric* if $c_{ij} = c_{ji}$ holds for all $i < j$. Due to the flow conservation constraints, it is equivalent to a symmetric inequality if and only if the system $t_i - t_j = c_{ij} - c_{ji}$ is consistent (see Hartmann and Özlük [14] and Boros et al [5]). One can show that the undirected counterpart $\sum_{1 \leq i < j \leq n} c_{ij} y_{ij}$ of a symmetric inequality $cx \leq \gamma$ is valid for $P_C^c(K_n)$. Moreover, it induces a facet of $P_C^c(K_n)$ if $cx \leq \gamma$ induces a facet of $P_C^c(D_n)$. This follows from an argument of Fischetti [10], originally stated for the ATSP and STSP, which is also mentioned in Hartmann and Özlük [14] in the context of directed and undirected p -cycle polytopes $P_C^{(p)}(D_n)$ and $P_C^{(p)}(K_n)$.

Corollary 4.2. *Let $K_n = (N, E)$ be the complete graph on $n \geq 3$ nodes and $c = (c_1, \dots, c_m)$ a cardinality sequence with $m \geq 2$ and $c_1 \geq 3$. Then holds:*

(a) *For any $e \in E$, the nonnegativity constraint $y_e \geq 0$ defines a facet of $P_C^c(K_n)$ if and only if $n \geq 5$.*

(b) *The degree constraint $y(\delta(i)) \leq 2$ defines a facet of $P_C^c(K_n)$ for every $i \in N$.*

(c) *Let S be a subset of N with $c_1 \leq |S| \leq n - c_1$, let $v \in S$ and $w \in N \setminus S$. Then, the two-sided min-cut inequality*

$$y(\delta(v)) + y(\delta(w)) - y((S : N \setminus S)) \leq 2 \quad (26)$$

induces a facet of $P_C^c(K_n)$.

(d) *For any $S \subset N$ with $|S|, |N \setminus S| \leq c_1 - 1$, the min-cut inequality*

$$y((S : N \setminus S)) \geq 2 \quad (27)$$

is valid for $P_C^c(K_n)$ and induces a facet of $P_C^c(K_n)$ if and only if $|S|, |N \setminus S| \geq 2$.

(e) *Let S be a subset of N and $j \in N \setminus S$. The one-sided min-cut inequality*

$$y((S : N \setminus S)) - y(\delta(j)) \geq 0 \quad (28)$$

defines a facet of $P_C^c(K_n)$ if and only if $|S| \geq c_1$ and $2 \leq |N \setminus S| \leq c_1 - 1$.

(f) *The cardinality bound $y(E) \geq c_1$ defines a facet of $P_C^c(K_n)$. The cardinality bound $y(E) \leq c_m$ defines a facet of $P_C^c(K_n)$ if and only if $c_m < n$.*

(g) *Let W be a subset of N with $c_p < |W| < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-forcing inequality*

$$(c_{p+1} - |W|) \sum_{i \in W} y(\delta(i)) - (|W| - c_p) \sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W|) \quad (29)$$

defines a facet of $P_C^c(K_n)$ if and only if $c_{p+1} - |W| \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n - 1$.

(h) *Let W be a subset of N such that $c_p < |W| < c_{p+1}$ holds for some $p \in \{1, \dots, m-1\}$. The cardinality-subgraph inequality*

$$2y(E(W)) - (|W| - c_p - 1)y((W : N \setminus W)) \leq 2c_p \quad (30)$$

is valid for $P_C^c(K_n)$ and induces a facet of $P_C^c(K_n)$ if and only if $p+1 < m$ or $c_{p+1} = n = |W| + 1$.

(i) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \dot{\cup} T$ be a partition of N . The even cycle exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \quad (31)$$

is valid for $P_C^c(K_n)$ and defines a facet of $P_C^c(K_n)$ if and only if $|S|, |T| \geq \frac{c_2-1}{2}$.

Proof. (a) When $n \leq 5$ the statement can be verified using a computer program. When $n \geq 6$, the claim follows from Proposition 2 of Kovalev, Maurras, and Vaxés [15], Proposition 2 of Maurras and Nguyen [17], and the fact that $m \geq 2$.

(b)-(i) All directed inequalities occurring in Corollary 4.1 (b)-(h) and (j) are equivalent to symmetric inequalities. For example, the degree constraint $x(\delta^{\text{out}}(i)) \leq 1$ is equivalent to $x(\delta^{\text{out}}(i)) + x(\delta^{\text{in}}(i)) \leq 2$. Via the identification $y(\delta(i)) \cong x(\delta^{\text{out}}(i)) + x(\delta^{\text{in}}(i))$ we see that $y(\delta(i)) \leq 2$ defines a facet of $P_C^c(K_n)$ if $x(\delta^{\text{out}}(i)) \leq 1$ defines a facet of $P_C^c(D_n)$.

Necessity can be shown with similar arguments as for the directed counterparts of these inequalities. \square

The inequalities mentioned in Corollary 4.2 (a)-(c), (e)-(g) together with the integrality constraints $y_e \in \{0, 1\}$ for $e \in E$ provide a characterization of the integer points of $P_C^c(K_n)$. In this context note that if $|N \setminus S| = 2$ the inequalities in (e) are equivalent to the well-known parity constraints

$$y(\delta(j) \setminus \{e\}) - y_e \geq 0 \quad (j \in N, e \in \delta(j))$$

mentioned for example in [3].

The odd cycle exclusion constraints as well as the modified cardinality forcing inequalities from Corollary 4.1 are not symmetric nor equivalent to symmetric inequalities. Hence, we did not derive counterparts of these inequalities for $P_C^c(K_n)$. Of course, given a valid inequality $cx \leq c_0$ for $P_C^c(D_n)$, one obtains a valid inequality $\tilde{c}y \leq 2c_0$ for $P_C^c(K_n)$ by setting $\tilde{c}_{ij} := c_{ij} + c_{ji}$ for $i < j$. However, it turns out that the counterparts of these two classes of inequalities are irrelevant for a linear description of $P_C^c(K_n)$.

4.3 Facets of the undirected cardinality constrained path polytope

The undirected cardinality constrained $(0, n)$ -path polytope $P_{0,n\text{-path}}^c(K_{n+1})$ is the symmetric counterpart of $P_{0,n\text{-path}}^c(\tilde{D}_n)$. Here, $K_{n+1} = (N, E)$ denotes the complete graph on node set $N = \{0, \dots, n\}$.

Theorem 4.3. *Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \dots, n\}$, $n \geq 3$, and let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$ and $c_1 \geq 2$. Then*

$$\dim P_{0,n\text{-path}}^c(K_{n+1}) = |E| - 3.$$

Proof. All points $y \in P_{0,n\text{-path}}^c(K_{n+1})$ satisfy the equations

$$y_{0n} = 0, \quad (32)$$

$$y(\delta(0)) = 1, \quad (33)$$

$$y(\delta(n)) = 1. \quad (34)$$

Thus, the dimension of $P_{0,n-\text{path}}^c(K_{n+1})$ is at most $|E| - 3$. When $4 \leq c_i < n$ for some $i \in \{1, \dots, m\}$, then the statement is implied by Theorem 4.7 of [20], saying that $\dim P_{0,n-\text{path}}^{(c_i)}(K_{n+1}) = |E| - 4$, and the fact that $m \geq 2$. So, assume that $c = (2, 3)$, $c = (2, n)$, $c = (3, n)$, or $c = (2, 3, n)$.

When $c = (2, 3)$, assume that the equation $by = b_0$ is satisfied by $[0, n]$ -paths of length 2 and 3. Then the equations

$$\begin{array}{ccccccc} y_{0i} & + & y_{ij} & + & y_{jn} & = & 0, \\ y_{0j} & + & y_{ij} & + & y_{in} & = & 0, \\ y_{0i} & & & + & y_{in} & = & 0, \\ y_{0j} & & & + & y_{in} & = & 0 \end{array}$$

imply $b_{ij} = 0$, $b_{0i} = b_{0j}$, and $b_{in} = b_{jn}$ for $1 \leq i, j \leq n - 1$. Thus, $bx = b_0$ is a linear combination of the equations (32)-(34).

When $c = (2, n)$, $c = (3, n)$, or $c = (2, 3, n)$, the statement can be verified using Theorem 4.7 of [20] which says that $\dim P_{0,n-\text{path}}^{(n)}(K_{n+1}) = |E| - n - 2$. Thus, there are $|E| - n - 1$ linearly independent Hamiltonian paths. Via the approach on the path-and-loops polytope defined on an undirected graph we see that indeed $\dim P_{0,n-\text{path}}^c(K_{n+1}) = |E| - 3$. \square

In the sequel we confine ourselves to the set CS of cardinality sequences $c = (c_1, \dots, c_m)$ with $m \geq 2$, $c_1 \geq 2$, and $c \neq (2, 3)$. Next, we show that the nonnegativity constraints $y_e \geq 0$ define facets of $P_{0,n-\text{path}}^c(K_{n+1})$. Since this requires for those path polytopes with $c \in \{(3, n), (2, 3, n)\}$ a higher amount of technical detail we start with a preparing Lemma and a following definition.

Lemma 4.4. *Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \dots, n\}$, $n \geq 4$. For any $e \in E$, $e \neq [0, n]$, the nonnegativity constraint $y_e \geq 0$ defines a facet of $P_{0,n-\text{path}}^{(n)}(K_{n+1})$ if and only if e is incident to 0 or n and $n \geq 4$ or e is not incident to both 0 and n and $n \geq 5$.*

Proof. Let $n \geq 5$ and $e = [i, j]$ be an edge whose endnodes are not incident with 0 and n , say w.l.o.g. $e = [n - 2, n - 1]$. Consider the Hamiltonian $[n - 2, n - 1]$ -path polytope $P_{n-2, n-1-\text{path}}^{(n-2)}(G)$ defined on the complete subgraph $G = (N', E')$ induced by the node set $N' := \{1, \dots, n - 1\}$. Its dimension is $|E'| - n$ by Theorem 4.7 of [20]. Thus, there are $|E'| - n + 1$ linearly independent Hamiltonian $[n - 1, n - 2]$ -paths in G that can be extended to Hamiltonian $[0, n]$ -paths in K_{n+1} by adding to each path the edges $[0, n - 2]$ and $[n - 1, n]$. Their incidence vectors, say X^r , $r = 1, \dots, |E'| - n + 1$, are linearly independent and satisfy $x_{n-1, n-2}^r = 0$ by construction. But they satisfy also $x_{0, n-1}^r = x_{n-2, n}^r = 0$ and $x_{0, k}^r = x_{k, n}^r = 0$ for $k = 1, \dots, n - 3$. Next, denote by $Y^{[0, k], [l, n]}$ the incidence vector of a Hamiltonian $[0, n]$ -path that does not use edge $[n - 2, n - 1]$, starts with $[0, k]$, and ends with $[l, n]$. Then, it is easy to see that the points $Y^{[0, n-1], [1, n]}$, $P^{[0, 1], [n-2, n]}$, and $P^{[0, i], [n-1, n]}$, $P^{[0, n-2], [i, n]}$ for $i = 1, \dots, n - 3$ are linearly independent, and they are also linearly independent of the points X^r . Since it is a total of $(|E'| - n + 1) + (2n - 4) = |E| - n - 2$ points, the inequality $y_{n-2, n-1} \geq 0$ defines a facet of $P_{0,n-\text{path}}^{(n)}(K_{n+1})$. When $n \leq 4$, then it follows immediately that $y_{n-2, n-1} \geq 0$ is not facet defining.

When the edge e is incident to 0 or n , then a simpler construction starting with Hamiltonian $[i, j]$ -paths of the complete subgraph $G = (N', E')$ induced by the node set $N' := (N \setminus \{0, n\}) \cup \{i, j\}$ yields the desired result. \square

The undirected cardinality constrained $[0, n]$ -path-and-loops polytope is the polytope

$$P_{PL}^c(K_{n+1}) := \left\{ (y, z) \in P_{0,n\text{-path}}^c(K_{n+1}) \times \mathbb{R}^{n-1} \mid \begin{array}{l} z_i = 2 - y(\delta(i)), \\ i = 1, \dots, n-1 \end{array} \right\}.$$

Some points $y^1, \dots, y^p \in P_{0,n\text{-path}}^c(K_{n+1})$ are affinely independent if and only if the corresponding points $(y^1, z^1), \dots, (y^p, z^p) \in P_{PL}^c(K_{n+1})$ are affinely independent.

Theorem 4.5. *Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \dots, n\}$ with $n \geq 4$, and let $c = (c_1, \dots, c_m) \in CS$ be a cardinality sequence. For any $e \in E, e \neq [0, n]$, the nonnegativity constraint $y_e \geq 0$ defines a facet of $P_{0,n\text{-path}}^c(K_{n+1})$ if and only if $c \neq (2, n)$ or $c = (2, n)$ and e is an internal edge.*

Proof. When $4 \leq c_i < n$ for some $i \in \{1, \dots, m\}$, then the claim follows from Theorem 4.9 of [20] and the fact that $m \geq 2$. Otherwise, $c = (2, n)$, $c = (3, n)$, or $c = (2, 3, n)$ for $n \geq 4$. By Lemma 4.4, $y_e \geq 0$ defines a facet of $P_{0,n\text{-path}}^{(n)}(K_{n+1})$. Thus, there are $|E| - n - 2$ linearly independent points $y^r \in P_{0,n\text{-path}}^{(n)}(K_{n+1})$ satisfying $y_e^r = 0$ and $1^T y^r = n$. Considering the corresponding points in the path-and-loops polytope $P_{PL}^{(3,n)}(K_{n+1})$, we see that they can easily be completed to a set of $|E| - 3$ affinely independent points in $P_{0,n\text{-path}}^{(3,n)}(K_{n+1})$ satisfying $y_e = 0$. Thus, $y_e \geq 0$ defines a facet. Thus, it induces also a facet of $P_{0,n\text{-path}}^{(2,3,n)}(K_{n+1})$. Finally, when $c = (2, n)$, we have $n-1$ 2-paths satisfying $y_e = 0$ if e is an internal edge, and otherwise only $n-2$. In the former case, these $n-1$ 2-paths are affinely independent of the points y^r (consider the corresponding points in the path-and-loops polytope $P_{PL}^{(2,n)}(K_{n+1})$), and hence, $y_e \geq 0$ is facet defining. In the latter case, we have only a total of $|E| - 4$ affinely independent points satisfying $y_e = 0$, and thus, $y_e \geq 0$ does not induce a facet of $P_{0,n\text{-path}}^{(2,n)}(K_{n+1})$. \square

Finally, we use the concept of symmetric inequalities to transform facet defining inequalities for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ into those for $P_{0,n\text{-path}}^c(K_{n+1})$. A valid inequality $cx \leq c_0$ for the directed path polytope $P_{0,n\text{-path}}^c(\tilde{D}_n)$ is said to be *pseudo-symmetric* if $c_{ij} = c_{ji}$ for all $1 \leq i < j \leq n-1$. It is equivalent to a pseudo-symmetric inequality if and only if the system $t_i - t_j = c_{ij} - c_{ji}$ for $1 \leq i < j \leq n-1$ is consistent. In [20] it was shown that the undirected counterpart $\bar{c}y \leq c_0$ of a pseudo-symmetric inequality $cx \leq c_0$ (obtained by setting $\bar{c}_{0i} = c_{0i}, \bar{c}_{in} = c_{in}$ for all internal nodes i and $\bar{c}_{ij} = c_{ij} = c_{ji}$ for all $1 \leq i < j \leq n-1$) is facet defining for $P_{0,n\text{-path}}^{(p)}(K_{n+1})$ if $cx \leq c_0$ is facet defining for $P_{0,n\text{-path}}^{(p)}(\tilde{D}_n)$. The same holds for $P_{0,n\text{-path}}^c(\tilde{D}_n)$ and $P_{0,n\text{-path}}^c(K_{n+1})$.

Corollary 4.6. *Let $K_{n+1} = (N, E)$ be the complete graph on node set $N = \{0, \dots, n\}$ with $n \geq 4$, and let $c = (c_1, \dots, c_m) \in CS$ be a cardinality sequence. Then we have:*

- (a) *The degree constraint $y(\delta(i)) \leq 2$ defines a facet of $P_{0,n\text{-path}}^c(K_{n+1})$ for every node $i \in N \setminus \{0, n\}$ unless $c = (2, n)$.*
- (b) *Let S be a subset of N with $0, n \in S$ and $|S| \leq c_1$. Then, the min-cut inequality*

$$y((S : N \setminus S)) \geq 2 \tag{35}$$

induces a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $|S| \geq 3$ and $|V \setminus S| \geq 2$.

(c) Let $S \subset N$ with $0, n \in S$, $j \in N \setminus S$, and $|S| \geq c_1 + 1$. Then, the one-sided min-cut inequality

$$y((S : N \setminus S)) - y(\delta(j)) \geq 0 \quad (36)$$

is valid for $P_{0,n-path}^c(K_{n+1})$ and induces a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $|N \setminus S| \geq 2$.

(d) The cardinality bound $y(E) \geq c_1$ defines a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $c_1 \geq 4$. The cardinality bound $y(E) \leq c_m$ defines a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $c_m < n$.

(e) Let W be a subset of N with $0, n \in W$ and $c_p < |W| - 1 < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-forcing inequality

$$(c_{p+1} - |W| + 1) \sum_{i \in W} y(\delta(i)) - (|W| - c_p - 1) \sum_{i \in N \setminus W} y(\delta(i)) \leq 2c_p(c_{p+1} - |W| + 1) \quad (37)$$

defines a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $c_{p+1} - |W| + 1 \geq 2$ and $c_{p+1} < n$ or $c_{p+1} = n$ and $|W| = n$.

(f) Let W be a subset of N such that $0, n \in W$ and $c_p < |W| - 1 < c_{p+1}$ for some $p \in \{1, \dots, m-1\}$. The cardinality-subgraph inequality

$$2y(E(W)) - (|W| - c_p - 2)y((W : N \setminus W)) \leq 2c_p \quad (38)$$

is valid for $P_{0,n-path}^c(K_{n+1})$ and induces a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if $p+1 < m$ or $c_{p+1} = n = |W|$.

(g) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 2$, and c_p even for $1 \leq p \leq m$, and let $N = S \dot{\cup} T$ be a partition of N with $0 \in S$, $n \in T$. The odd path exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \quad (39)$$

is valid for $P_{0,n-path}^c(K_{n+1})$ and defines a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if (i) $c_1 = 2$ and $|S|, |T| \geq \frac{c_2}{2} + 1$, or (ii) $c_1 \geq 4$ and $|S|, |T| \geq \frac{c_2}{2}$.

(h) Let $c = (c_1, \dots, c_m)$ be a cardinality sequence with $m \geq 2$, $c_1 \geq 3$, and c_p odd for $1 \leq p \leq m$, and let $N = S \dot{\cup} T$ be a partition of N with $0, n \in S$. The even path exclusion constraint

$$y(E(S)) + y(E(T)) \geq 1 \quad (40)$$

is valid for $P_{0,n-path}^c(K_{n+1})$ and defines a facet of $P_{0,n-path}^c(K_{n+1})$ if and only if (α) $c_1 = 3$, $|S| - 1 \geq \frac{c_2+1}{2}$, and $|T| \geq \frac{c_2-1}{2}$, or (β) $c_1 \geq 5$ and $\min(|S| - 1, |T|) \geq \frac{c_2-1}{2}$. \square

As already mentioned, the modified cardinality forcing inequalities (20) are not equivalent to pseudo-symmetric inequalities.

5 Concluding remarks

Restricting the set of feasible solutions of a combinatorial optimization problem to those that satisfy some specified cardinality constraints always can be done by adding the corresponding cardinality forcing inequalities inherited from the polytope associated with the respective cardinality homogeneous set system. However, as we have demonstrated at the example of paths and cycles, one may end with rather weak formulations unless this is done carefully: Imposing the restrictions on the number of vertices leads to formulations with facet defining inequalities, while the straight-forward approach using the arcs does not result in strong inequalities.

It would be interesting to see whether this is similar for cardinality restricted versions of other optimization problems. Moreover, we believe that there should be other interesting situations where knowledge on a master polyhedron (like the cardinality homogeneous set systems polyhedron) and on a polyhedron associated with particular combinatorial structures (like paths and cycles) can be brought into fruitful interplay.

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