Facets for the Multiple Knapsack Problem

C.E. Ferreira* A. Martin R. Weismantel

Konrad-Zuse-Zentrum für Informationstechnik Berlin, Germany

In this paper we consider the multiple knapsack problem which is defined as follows: given a set N of items with weights f_i, $i \in N$, a set M of knapsacks with capacities F_k, $k \in M$, and a profit function $c_{ik}, i \in N, k \in M$; find an assignment of a subset of the set of items to the set of knapsacks that yields maximum profit (or minimum cost). With every instance of this problem we associate a polyhedron whose vertices are in one to one correspondence to the feasible solutions of the instance. This polytope is the subject of our investigations. In particular, we present several new classes of inequalities and work out necessary and sufficient conditions under which the corresponding inequality defines a facet. Some of these conditions involve only properties of certain knapsack constraints, and hence, apply to the generalized assignment polytope as well. The results presented here serve as the theoretical basis for solving practical problems. The algorithmic side of our study, i.e., separation algorithms, implementation details and computational experience with a branch and cut algorithm are discussed in the companion paper [**?**].

* On leave from University of São Paulo, Brazil.

1 Introduction and Notation

In this paper we investigate the multiple knapsack polytope, i.e., the polytope associated with the multiple knapsack problem. The *multiple knapsack problem* is defined as follows: Given a set N of *items* with weights $f_i > 0$, $i \in N$, a set M of *knapsacks* with capacities $F_k > 0$, $k \in M$, and a profit function $c_{ik}, i \in N, k \in M$. Find an assignment of a subset of the set of items to the set of knapsacks that yields maximum profit (or minimum cost). Without loss of generality we assume throughout the paper that $N = \{1, \dots, n\}$, $n \geq 1$ and $M = \{1, \dots, m\}$, $m \geq 1$.

The multiple knapsack problem is known to be $\mathcal{NP}$-complete, since, in case $m = 1$, it coincides with the 0/1 single knapsack problem, which is $\mathcal{NP}$-complete (cf. [?]). From an algorithmical point of view the multiple knapsack problem was extensively studied (see, for example, the excellent survey of Martello and Toth in [?]). Unfortunately, the strategies that provide an exact solution for the problem have a prohibitive running time for instances of large scale. Our objective is to develop a cutting plane algorithm for the multiple knapsack problem and study its performance, when applied to real world large scale instances.

The instances we have in mind arise in two different applications, namely in the layout of electronic circuits and in the design of processors for main frame computers.

One major problem arising in the layout of electronic circuits is the placement problem. Roughly speaking, this problem consists of assigning a given set of so-called cells to locations on a given rectangle, usually called a master. Physically, every cell represents a logic function. For our purposes, a cell can be viewed as a rectangle whose size is known in advance. A common approach to attack this problem is via decomposition techniques. Working in this scheme, a first step is to assign the cells to subareas of the master, subdivide every subarea and continue until every subarea contains at most one cell. In knapsack terminology, the cells correspond to the items. The weight of an item is represented by the area of the corresponding cell, whereas a knapsack corresponds to a subarea of the master and its capacity is given by the size of the subarea. The task is to assign cells to subareas such that a certain cost function is minimized. Thus, algorithms for the solution of the multiple knapsack problem are used as subroutines for the solution of the placement problem in chip design.

Similarly, in the design of main frame computers, a given set of "components" has to be assigned to a given number of "modules". The capacity of a module as well as the weight of a component is defined by the corresponding area. The coefficients c_{ik}, $i \in N$, $k \in M$, of the profit function reflect the production cost of installing component i on module k.

In reality both applications are much more complicated than indicated here. For instance, in the main frame computer design application, subsets of the components must be connected by wires and, hence, space has to be reserved for doing this. Moreover, each module has an additional capacity limiting the number of wires that may leave the module. For modelling the placement problem of electronic circuits, the wireability of the nets, the length of the resulting wires, and the resulting communication cost must be considered. We refrain from explaining all these technical details here and refer the reader to [?], [?], [?] and [?] where it is shown how the multiple knapsack problem comes up and is used in the solution process.

This paper is devoted to the theoretical underpinning of our cutting plane approach to the multiple knapsack problem and is organized as follows. In section 2 the multiple knapsack, the single knapsack and the generalized assignment polytope are defined. Here, we give a brief review of the state of the art concerning these polytopes and present the basic concepts that are of importance for a study of the multiple knapsack polytope. In sections 3 and 4 our polyhedral investigations are presented. In section 3 we deal with the so-called heterogeneous two-cover inequality and the multiple cover inequality. In particular, necessary and sufficient conditions such that the corresponding inequalities are valid (facet-defining) will be given. A procedure for extending facet-defining inequalities will enable us to generalize several existing classes of inequalities. This is done in section 4. The discussions will end with some conclusions. Our experiences with a cutting plane algorithm and computational results are presented in the follow-up paper [**?**].

We introduce some notation that will be used throughout the paper. Let $e_i \in \mathbb{R}^n$ be the unit vector with a value of one in the i-th component and zero otherwise. We denote by e_{ik} the unit vector in the vector space $\mathbb{R}^{nm}$.

For $x \in \mathbb{R}^n$ and $I \subseteq N$ we set $x(I) := \sum_{i \in I} x_i$. For $I \subseteq N$, we denote by χ^I the *incidence vector* of I in $\mathbb{R}^n$, i.e., $\chi^I_i = 1$, if $i \in I$,and $\chi^I_i = 0$, otherwise.

Given an item $i \in N$, let us define the set $I_i := \{j \in N \mid f_j = f_i\}$. For a vector $x' \in \{0,1\}^{nm}$ and some knapsack $k \in M$, we define $B_k(x') := \{i \in N \mid x'_{ik} = 1\}$.

If $a^T x \leq \alpha$ is a valid inequality for some polytope P, we set $EQ(P, a^T x \leq \alpha) := \{x' \in P \mid a^T x' = \alpha\}$. In order to simplify notation, we frequently abbreviate $EQ(P, a^T x \leq \alpha)$ by $EQ(a^T x \leq \alpha)$, if there is no way of confusion.

We abbreviate an instance of the multiple knapsack problem by the quadruple (N, M, f, F) where N denotes the set of items, M the set of knapsacks, the i-th component of the vector $f \in R^N$ denotes the weight of item i in N and the k-th component of the vector $F \in R^M$ denotes the capacity of knapsack $k \in M$. Of course, by (N, M, f, F) an instance of the multiple knapsack problem is specified up to the cost function $c_{ik}, i \in N, k \in M$. We neglect this function here, because, for our polyhedral investigations, the cost function is of no interest. If it will be (especially in our compagnion paper) we speak of the *weighted multiple knapsack problem* and abbreviate an instance by the quintuple (N, M, f, F, c). A 0/1 vector $x \in \{0,1\}^{nm}$ satisfying $\sum_{k \in M} x_{ik} \leq 1$ for all $i \in N$ will be called an *assignment*. If in addition for every $k \in M$, the inequality $\sum_{i \in N} f_i x_{ik} \leq F_k$ holds, the assignment will be called *valid*.

Given a set V. We call nonempty subsets $V_1, \ldots, V_p$ of V a *partition* of V, if $\bigcup_{i=1}^{p} V_i = V$ and $V_i \cap V_j = \emptyset$ for all $i, j \in \{1, \ldots, p\}, i \neq j$.

Let be given some $k \in M$. A set $S \subseteq N$ is a *cover* with respect to knapsack k if $\sum_{i \in S} f_i > F_k$. The cover is *minimal* with respect to k, if $\sum_{i \in S \setminus \{s\}} f_i \leq F_k$ for all $s \in S$. Let $N' \subseteq N$ be some nonempty subset of the set of items and $z \in N \setminus N'$. The set $N' \cup \{z\}$ is called *(1,d)-configuration* with respect to knapsack k, if

1. $\sum_{j \in N'} f_j \leq F_k$;
2. $K \cup \{z\}$ is a minimal cover with respect to k, for all $K \subset N'$ with $|K| = d$.

For $N' \subseteq N$, we use the symbol $f(N')$ to denote the value $\sum_{i \in N'} f_i$. Accordingly, we use the notation $F(M') := \sum_{k \in M'} F_k$ for a subset $M' \subseteq M$.

Finally, for the exposition of the proofs that will be given in sections 3 and 4 we need some graphtheoretic notation. Let $G = (V, E)$ denote a graph with *node set* V and *edge set* E. A *path* in G from $u \in V$ to $v \in V$ is an edge set $\{e_1, \ldots, e_r\}$ such that $e_i = \{u_i, u_{i+1}\}$ $i = 1, \ldots r-1$, $u_1 = u$, $u_r = v$ and $u_i \neq u_j$ for all $i, j \in \{1, \ldots, r\}, i \neq j$. A graph is *connected*, if for every pair of distinct nodes u and v there exists a path from u to v in G. A *spanning tree* in G is an edge set T such that the graph (V, T) is connected and $|T| = |V| - 1$.

2 The Multiple Knapsack and Related Polytopes: a Brief Overview

Let be given an instance (N, M, f, F) of the multiple knapsack problem. It will turn out that we often refer to subinstances of the problem where certain items are not feasible for certain knapsacks. Thus, we define the polyhedron in a more general frame. Suppose $A_i \subseteq N$ and $B_i \subseteq M$ for $i = 1, \ldots, t$ are given, and let $T := \bigcup_{i=1}^{t} A_i \times B_i$. Define the *multiple knapsack polytope* by

$$MK(T, f, F) := \text{conv}\{x \in \mathbb{R}^T \mid \begin{array}{ll} \sum_{i:(i,k) \in T} f_i x_{ik} \leq F_k, & k \in \bigcup_{l=1}^{t} B_l, \\ \sum_{k:(i,k) \in T} x_{ik} \leq 1, & i \in \bigcup_{j=1}^{t} A_j, \\ x_{ik} \in \{0, 1\}, & (i,k) \in T\}. \end{array}$$

The first type of inequalities, $\sum_{i:(i,k) \in T} f_i x_{ik} \leq F_k$, are called *knapsack constraints*. Using this notation, the polytope corresponding to the multiple knapsack problem defined at the beginning coincides with $MK(N \times M, f, F)$, which we often abbreviate by MK. It is easy to see that MK is full dimensional if and only if $f_i \leq F_k$ for all $i \in N$ and $k \in M$. Similarly, the dimension of the polytope $MK(T, f, F)$ equals $|T| = \sum_{i=1}^{t} |A_i||B_i|$ if and only if $f_i \leq F_k$ for all $(i, k) \in T$. in the following we assume that $f_i \leq F_k$ for all $(i, k) \in T$.

Obviously, the multiple knapsack problem, as stated in section 1, is a canonical generalization of the single 0/1 knapsack problem, where $|M| = 1$. In analogy to the definition of MK let

$$SK(N, f, F) := \text{conv}\{x \in \mathbb{R}^N \mid \sum_{i \in N} f_i x_i \leq F, x_i \in \{0, 1\}, i \in N\}$$

denote the *single knapsack polytope*.

Although a lot of emphasis has been put on studying the facial structure of $SK(N, f, F)$ (see, for example, [?],[?],[?],[?],[?], [?]), MK and generalizations of it have not yet been studied to the same extent. In a few papers we find investigations in this direction. Crowder, Johnson and Padberg [?] consider general 0/1 linear programs with no apparent structure: Let be given a matrix $A \in \mathbb{Q}^{m \times n}$, $a_{ki} \geq 0$ for all $i = 1, \ldots, n, k = 1, \ldots, m$, a vector $b \in \mathbb{Q}^m$ and define $IP := \text{conv}\{x \in \{0, 1\}^n \mid Ax \leq b\}$. With each constraint k the authors associate the single knapsack polytope $SK_k := SK(\{1, \ldots, n\}, (a_{k1}, \ldots, a_{kn})^T, b_k)$. Clearly, $IP \subseteq \bigcap_{k=1}^{m} SK_k$. In [?] large scale 0/1 linear programs are solved by using single knapsack inequalities for the polytopes SK_k in order to chop off fractional solutions that are obtained during the run of a cutting plane algorithm.

Gottlieb and Rao ([?], [?]) study the *generalized assignment problem*, a generalization of the multiple knapsack problem, where every item i may have "a particular weight f_{ik}" for each knapsack k. By setting

$$GAP(N \times M, f, F) := \text{conv}\{x \in \mathbb{R}^{N \times M} \mid \begin{array}{ll} \sum_{i \in N} f_{ik} x_{ik} \leq F_k, & k \in M, \\ \sum_{k \in M} x_{ik} \leq 1, & i \in N, \\ x_{ik} \in \{0, 1\}, & i \in N,\ k \in M\} \end{array}$$

we obtain the straightforward definition of a polytope associated with the generalized assignment problem (abbreviated by GAP).

In [?] and [?] a study of this polytope is addressed. One of the results due to Gottlieb and Rao is that every facet of some single knapsack polytope $SK(N, (f_{ik})_{i \in N}, F_k)$, $k \in M$ is a facet of GAP. In these papers several valid inequalities for GAP are presented. Also, some necessary conditions that a valid inequality must satisfy in order to define a facet of the generalized assignment polytope are given.

In the remainder of this section we sketch some results that are of interest for a polyhedral study of the multiple knapsack polytope.

Lemma 2.1 *The trivial inequality*

$$x_{ik} \geq 0$$

defines a facet of MK for all $i \in N, k \in M$.

If $m \geq 2$, the trivial inequality

$$\sum_{k \in M} x_{ik} \leq 1$$

defines a facet of MK for all $i \in N$. ■

The following lemma states that all nontrivial facets associated with a single knapsack polytope are inherited by MK. This result (in a more general version) is due to Gottlieb and Rao [**?**].

Lemma 2.2 *Let be given $V \subseteq N$ and $k \in M$. Suppose $a^T x \leq \alpha$ is a nontrivial facet-defining inequality of $SK(V, f, F_k)$. Then, $\overline{a}^T x \leq \alpha$ defines a facet of $MK(V \times M, f, F)$, where $\overline{a} \in I\!R^{V \times M}$ and*

$$\overline{a}_{il} := \begin{cases} a_i, & \text{if } l = k,\ i \in V, \\ 0, & \text{otherwise.} \end{cases}$$

■

Let (N, M, f, F) be an instance of the multiple knapsack problem. Suppose, S is a minimal cover with respect to some $k \in M$. Then, the *minimal cover inequality*

$$\sum_{i \in S} x_{ik} \leq |S| - 1$$

defines a facet for the polytope $SK(S, f, F_k)$ ([**?**], [**?**], [W75]). By applying Lemma 2.2 we can conclude that this minimal cover inequality defines a facet for $MK(S \times M, f, F)$. Similarly, let $N' \cup \{z\}, N' \subseteq N, |N'| = n'$ and $z \in N \setminus N'$ be a $(1, d)$ configuration with respect to some knapsack k. The $(1, d)$-*configuration inequality*

$$(n' - d + 1)x_{zk} + \sum_{i \in N'} x_{ik} \leq n'$$

defines a facet for the polytope $SK(N' \cup \{z\}, f, F_k)$ (cf. [**?**]). Again, from Lemma 2.2 we can conclude that this inequality defines a facet for $MK((N' \cup \{z\}) \times M, f, F)$.

The subsequent lemma elucidates two easy, yet important features of facet-defining inequalities for MK. This result is also true for the GAP-polytope and is due to Gottlieb and Rao.

Lemma 2.3 *Let be given an instance (N, M, f, F) of the multiple knapsack problem. Moreover, let $a^T x \leq \alpha$ be a nontrivial facet-defining inequality for some polytope $MK(V \times M, f, F)$, where $V \subseteq N$.*

- *The coefficients a_{ik}, $i \in V, k \in M$ are all non-negative.*
- *There exists some $k \in M$ such that $A_k := \{i \in V \mid a_{ik} > 0\}$ defines a cover with respect to k.* ■

Finally, let us give some remarks on the sequential lifting of inequalities. The concept of lifting is due to Padberg ([**?**]) and consists of the following idea. Given a polytope $P \subseteq [0,1]^N$ with 0/1 vertices and let N denote the index set of variables. Let i be an element of N and suppose, $a^T x \leq \alpha$ is a facet-defining inequality for $P \cap \{x \in \mathbb{R}^N \mid x_i = 0\}$. Set $\gamma_i := \max\{a^T x \mid x \in P, x_i = 1\}$. Then, the inequality $(a + (\alpha - \gamma_i)e_i)^T x \leq \alpha$ defines a facet for the polytope P. In the particular case of minimal cover inequalities and $(1, d)$-configuration inequalities, the coefficients γ_i can be computed in polynomial time. This result is due to Zemel ([**?**]) and is obtained by dualizing the maximization problem above, slightly modifying it and solving this modified program by applying dynamic programming techniques. Due to the particular structure of the optimization problem the running time of the dynamic program is bounded by a polynomial in the size of the input data.

In the following two chapters we present our investigations for the multiple knapsack polytope.

3 Joint Inequalities for the MK Polytope.

From the previous section we conclude that valid (facet-defining) inequalities for the single knapsack polytope can be lifted to valid (facet-defining) inequalities for proper instances of the multiple knapsack problem. However, there exist instances of the multiple knapsack problem where inequalities that combine the coefficients of more than one knapsack define facets. Inequalities of this type will be called *joint inequalities*.

In the following two subsections particular classes of joint inequalities are discussed. Here, the overall organization of the two subsections is as follows. First, we characterize conditions such that the class of inequalities under consideration is valid for MK. For illustration a small example will be given. Finally, necessary and sufficient conditions are given such that the corresponding inequalities define facets for the multiple knapsack polytope of proper instances.

3.1 The Heterogeneous Two-Cover Inequality

In this subsection we present an inequality that involves two covers and two knapsacks. The coefficients of the inequality are not all equal zero or one.

Theorem 3.1.1 *Let be given an instance (N, M, f, F) of the multiple knapsack problem. We choose two indices $k, l \in M, k \neq l$ and assume:*

- *$S \subseteq N$ is a cover with respect to k;*
- *$G \subseteq N \setminus S$ is some subset of items.*

Under these assumptions, the inequality

$$\sum_{i \in S} x_{ik} + \sum_{i \in S \cup G} (|S| - 1) x_{il} \leq |S|(|S| - 1)$$

is called heterogeneous two-cover inequality*. It is valid for $MK((S \cup G) \times M, f, F)$ if and only if for all $\tilde{G} \subseteq G$ and $\tilde{S} \subseteq S$ with $|\tilde{G}| = |\tilde{S}| \geq 1$ the set $S \setminus \tilde{S} \cup \tilde{G}$ is a cover with respect to knapsack l.*

Proof. Suppose, for all $\tilde{G} \subseteq G$ and $\tilde{S} \subseteq S, |\tilde{G}| = |\tilde{S}| \geq 1$, the set $S \setminus \tilde{S} \cup \tilde{G}$ is a cover with respect to knapsack l. Let $x' \in MK((S \cup G) \times M, f, F)$ be an assignment and set $\tilde{S}_k := \{i \in S \mid x'_{ik} = 1\}$, $\tilde{G} := \{i \in G \mid x'_{il} = 1\}$ and $\tilde{S}_l := \{i \in S \mid x'_{il} = 1\}$. If $|\tilde{S}_k| \geq 1$, the condition implies that $|\tilde{G} \cup \tilde{S}_l| \leq |S| - 1$. Since S is a cover, it follows that $|\tilde{S}_k| \leq |S| - 1$. Thus, $\sum_{i \in S} x'_{ik} + \sum_{i \in S \cup G} (|S| - 1) x'_{il} = |\tilde{S}_k| + (|S| - 1)(|\tilde{S}_l| + |\tilde{G}|) \leq |S| - 1 + (|S| - 1)(|S| - 1) = |S|(|S| - 1)$.

In case $|\tilde{S}_k| = \emptyset$, either $\tilde{S}_l = S$ or $|\tilde{S}_l \cup \tilde{G}| \leq |S| - 1$ due to the condition. Thus, the inequality is also valid in this case.

Conversely, suppose the inequality is valid, but the condition is violated, i.e., there exist $\tilde{G} \subseteq G$ and $\tilde{S} \subseteq S$ with $|\tilde{G}| = |\tilde{S}| \geq 1$ such that the set $S \setminus \tilde{S} \cup \tilde{G}$ is not a cover with respect to knapsack l.

Then, the vector x' defined via

$$x'_{iv} := \begin{cases} 1, & i \in \tilde{S}, v = k, \\ 1, & i \in \tilde{G} \cup S \setminus \tilde{S}, v = l, \\ 0, & \text{otherwise}, \end{cases}$$

is an element of $MK((S \cup G) \times M, f, F)$ satisfying $\sum_{i \in S} x'_{ik} + \sum_{i \in S \cup G}(|S|-1)x'_{il} = |\tilde{S}| + (|S|-1)(|S \setminus \tilde{S}| + |\tilde{G}|) = |\tilde{S}| + (|S|-1)|S| \geq 1 + (|S|-1)|S|$. This contradicts the assumption that the inequality is valid. ■

Before studying the problem instances for which the heterogeneous two-cover inequality defines a facet, let us introduce an additional definition.

Definition. Let be given an instance $(N, M, f,\ F)$ of the multiple knapsack problem and let $d \geq 1$ be some integer. We say that a subset of items $S \subseteq N$ is a *d-cover* with respect to some knapsack $k \in M$, if every subset $D \subseteq S$ with $|D| = |S| - d$ satisfies $f(D) \leq F_k$ and $f(D \cup \{s\}) > F_k$ for all $s \in S \setminus D$.

Using this notation, a minimal cover is a 1-cover.

Theorem 3.1.2 *Let be given an instance $(N, M, f,\ F)$ of the multiple knapsack problem. We choose two indices $k, l \in M, k \neq l$ and assume:*

- *$S \subseteq N$ is a minimal cover with respect to k and satisfies $f(S) \leq F_l$;*
- *$G \subseteq N \setminus S, |G| \geq |S|$ is a $(|G| - |S| + 1)$-cover with respect to l.*

Under these assumptions, the heterogeneous two-cover inequality

$$\sum_{i \in S} x_{ik} + \sum_{i \in S \cup G} (|S|-1)x_{il} \leq |S|(|S|-1)$$

defines a facet of the polytope $MK((S \cup G) \times M, f, F)$ if and only if it is valid for $MK((S \cup G) \times M, f, F)$ and, for every $s \in S$, there exists a subset $\tilde{G} \subseteq G$ with $|\tilde{G}| = |S| - 2$ such that $\sum_{i \in \tilde{G}} f_i + f_s \leq F_l$.

Proof. For ease of exposition let us abbreviate the heterogeneous two-cover inequality by $a^T x \leq \alpha$. First, we prove that the conditions given in the theorem are necessary.

Obviously, $a^T x \leq \alpha$ is a valid inequality. Now assume that there exists an item $s \in S$ such that for all subsets $\tilde{G} \subseteq G$, $|\tilde{G}| = |S| - 2$, the relation $f(\tilde{G}) + f_s > F_l$

holds. Under this assumption we show that every element x of $EQ(a^T x \leq \alpha)$ satisfies the equation $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} = |S| - 1$. If this is true, $EQ(a^T x \leq \alpha)$ does not define a facet, a contradiction. Hence, the conditions are necessary, indeed.

Now let us prove that every element $x \in EQ(a^T x \leq \alpha)$ also satisfies $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} = |S| - 1$, if $s \in S$ is an item such that for all subsets $\tilde{G} \subseteq G$, $|\tilde{G}| = |S| - 2$, the relation $f(\tilde{G}) + f_s > F_l$ holds. Let be given some $x \in EQ(a^T x \leq \alpha)$. First, we show that $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} \leq |S| - 1$. Suppose this is not the case. Then we define x' via:

$$x'_{iv} := \begin{cases} x_{il}, & i \in G \cup S \setminus \{s\}, v = l, \\ 1, & i = s, v = k, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, $a^T x' \geq 1 + (|S| - 1)|S| = \alpha + 1$, a contradiction. Now suppose, x satisfies $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} < |S| - 1$. Here, we distinguish the following two cases:

- Case 1. $x_{sl} = 0$.
  Since $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} < |S| - 1$ holds, we obtain $\sum_{i \in S} x_{ik} + \sum_{i \in S \cup G} (|S| - 1) x_{il} < |S| - 1 + (|S| - 1)(|S| - 1) = |S|(|S| - 1)$, which contradicts that $x \in EQ(a^T x \leq \alpha)$.

- Case 2. $x_{sl} = 1$.
  Since $\sum_{i \in \tilde{G}} f_i + f_s > F_l$ holds for all subsets $\tilde{G} \subseteq G$, $|\tilde{G}| = |S| - 2$, we can conclude that $|\{i \in G \mid x_{il} = 1\}| < |S| - 2$. In case $\{i \in S \setminus \{s\} \mid x_{il} = 1\} = \emptyset$, we obtain $a^T x = |\{i \in S \mid x_{ik} = 1\}| + (|S| - 1)(|\{i \in G \mid x_{il} = 1\}| + 1) < |S| - 1 + (|S| - 1)(|S| - 2 + 1) = \alpha$, a contradiction. Otherwise, $\{i \in S \setminus \{s\} \mid x_{il} = 1\} \neq \emptyset$. Since $x_{sl} = 1$, we know that $|\{i \in S \mid x_{ik} = 1\}| < |S| - 1$. Together with $\sum_{i \in G} x_{il} + \sum_{i \in S \setminus \{s\}} x_{il} < |S| - 1$, we obtain $a^T x < |S| - 1 + (|S| - 1)(|S| - 2 + 1) = \alpha$. Thus, x cannot be an element of $EQ(a^T x \leq \alpha)$, which, again, is a contradiction.

This completes the proof that the conditions are necessary.

In order to prove the converse direction, we assume that $EQ(a^T x \leq \alpha) \subseteq EQ(b^T x \leq \beta)$, where $b^T x \leq \beta$ is a facet-defining inequality of $MK((S \cup G) \times M, f, F)$.

Let $\tilde{G} \subseteq G$ with $|\tilde{G}| = |S| - 1$. For every $s \in S$, we define the vector x^s as follows:

$$x^s_{iv} := \begin{cases} 1, & i \in S \setminus \{s\}, v = k, \\ 1, & i \in \tilde{G}, v = l, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, x^s is an element of $EQ(a^T x \leq \alpha)$ and for every $s' \in S \setminus \{s\}$, the vector $x^s - e_{s'k} + e_{sk}$ is also contained in $EQ(a^T x \leq \alpha)$. Thus, $b^T x^s = b^T(x^s - e_{s'k} + e_{sk})$ and, hence, $b_{s'k} = b_{sk}$ for all $s' \in S$. Moreover, $x^s + e_{sj} \in EQ(a^T x \leq \alpha)$ for all $j \in M \setminus \{k, l\}$ which implies $b^T x^s = b^T(x^s + e_{sj})$. This yields $b_{sj} = 0$ for all $j \in M \setminus \{k, l\}$. Since the same argument applies to all $s \in S$, we obtain that $b_{sj} = 0$ for all $s \in S$, $j \in M \setminus \{k, l\}$.

Now, choose $g \in \tilde{G}$ and $h \in G \setminus \tilde{G}$ (the node h exists, since $|\tilde{G}| < |G|$) and notice that $x^s - e_{gl} + e_{hl} \in EQ(a^T x \leq \alpha)$, since G is a $(|G| - |S| + 1)$-cover. Thus, $b^T x^s = b^T(x^s - e_{gl} + e_{hl})$, and therefore, $b_{gl} = b_{hl}$. Using the same argument for different choices of g and h we conclude that $b_{gl} = b_{hl}$ for all $g, h \in G$.

The conditions of the Theorem **??** guarantee that, for every $s \in S$, there exists some $G' \subseteq G$, $|G'| = |S| - 2$ such that $\sum_{i \in G'} f_i + f_s \leq F_l$. We define the vector $x^1 = (x^1_{ij})$ as follows:

$$x^1_{ij} := \begin{cases} 1, & i \in S \setminus \{s\}, j = k, \\ 1, & i \in G' \cup \{s\}, j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, x^1 is an element of $EQ(a^T x \leq \alpha)$. Since G is a $(|G| - |S| + 1)$-cover, the vector $x^1 - e_{sl} + e_{gl}$ belongs to $EQ(a^T x \leq \alpha)$, for all $g \in G \setminus G'$. Thus, $b^T x^1 = b^T(x^1 - e_{sl} + e_{gl})$ and, hence, $b_{sl} = b_{gl}$ for all $g \in G$. Since the same construction applies to all $s \in S$, we conclude that $b_{sl} = b_{gl}$ for all $s \in S$, $g \in G$.

Now, we define the vector x^2 by setting:

$$x^2_{ij} := \begin{cases} 1, & i \in S, j = l, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $x^2 \in EQ(a^T x \leq \alpha)$. Hence, $b^T x^s = b^T x^2$ which yields, $(|S| - 1)b_{sk} + (|S| - 1)b_{gl} = |S| b_{sl}$. Since $b_{gl} = b_{sl}$ for all $s \in S$, $g \in G$, we get $b_{sl} = (|S| - 1)b_{sk}$.

Finally, the vector $x^2 + e_{gj}$ is an element of $EQ(a^T x \leq \alpha)$ for all $g \in G$ and $j \in M \setminus \{l\}$. Thus, $b^T x^2 = b^T(x^2 + e_{gj})$, which implies that $b_{gj} = 0$ for all $g \in G$, $j \in M \setminus \{l\}$.

This shows that the inequalities $a^T x \leq \alpha$ and $b^T x \leq \beta$ are equal up to multiplication with a scalar, which completes the proof. ■

Example 3.1.3 Let be given an instance of the multiple knapsack problem where $N = \{1, 2, 3, 4, 5, 6, 7\}$, $M = \{1, 2\}$, $f = (4, 5, 7, 8, 8, 8, 8)$, $F = (16, 14)$ and consider the corresponding knapsack constraints:

$$\begin{aligned} 4x_{1,1} + 5x_{2,1} + 7x_{3,1} + 8x_{4,1} + 8x_{5,1} + 8x_{6,1} + 8x_{7,1} &\leq 16, \\ 4x_{1,2} + 5x_{2,2} + 7x_{3,2} + 8x_{4,2} + 8x_{5,2} + 8x_{6,2} + 8x_{7,2} &\leq 14. \end{aligned}$$

It is easy to check that the set $S = \{1, 2, 3\}$ is a minimal cover for knapsack 2, and $G = \{4, 5, 6, 7\}$ meets the requirements of Theorems **??** and **??**. Thus, the heterogeneous two-cover inequality $x_{1,2} + x_{2,2} + x_{3,2} + 2x_{1,1} + 2x_{2,1} + 2x_{3,1} + 2x_{4,1} + 2x_{5,1} + 2x_{6,1} + 2x_{7,1} \leq 6$ defines a facet for the corresponding polytope. If we change the weight of item 3 to 9 ($f_3 := 9$), this inequality is still valid, yet not facet-defining, since $f_3 + f_i > F_l = 16$ for all $i \in G$ (see Theorem **??**). If we change the weight of item 4 to 7 ($f_4 := 7$), this heterogeneous two-cover inequality is not even valid any more, since the set $\{1, 2, 4\}$ (choose $\tilde{S} = \{3\}$ and $\tilde{G} = \{4\}$ in Theorem **??**) is not a cover with respect to knapsack 1. ■

This example demonstrates that the properties of a heterogeneous two-cover inequality to be facet-defining or valid are extremely sensitive according to (minor) changes of the input data. A slight modification of the weight of some item may cause that a facet-defining inequality is not even valid any more.

Remark 3.1.4 Theorems **??** and **??** can be slightly modified in order to apply to the generalized assignment polytope. This is due to the fact that the (necessary and sufficient) conditions in these theorems only involve properties of the weights of the items with respect to *one* particular knapsack. For instance, the corresponding counterpart of Theorem **??** can be formulated as follows:

Let be given an instance (N, M, f, F) of the generalized assignment problem. We choose two indices $k, l \in M, k \neq l$ and assume:

$S \subseteq N$ is a cover with respect to k and $G \subseteq N \setminus S$ is some subset of items.

The inequality

$$\sum_{i \in S} x_{ik} + \sum_{i \in S \cup G} (|S| - 1)x_{il} \leq |S|(|S| - 1)$$

is valid for $GAP((S \cup G) \times M, f, F)$ if and only if for all $\tilde{G} \subseteq G$ and $\tilde{S} \subseteq S$ with $|\tilde{G}| = |\tilde{S}| \geq 1$ the set $S \setminus \tilde{S} \cup \tilde{G}$ is a cover with respect to knapsack l.

Theorem **??** can be formulated appropriately, in order to apply to the generalized assignment problem. ■

3.2 The Multiple Cover Inequality

In [?], it was observed that, given a set $S \subseteq N$ and $J \subseteq M$ with $f(S) > F(J)$, the inequality

$$\sum_{j \in J} \sum_{i \in S} x_{ij} \leq |S| - 1$$

is valid for the polytope MK. If $|J| \geq 2$ a set of items S with the property $f(S) > F(J)$ is called a *multiple cover* with respect to J and the corresponding inequality *multiple cover inequality*. A set of items S is called a *minimal multiple cover* with respect to J, if $f(S) > F(J)$ and, for all $s \in S$, there exists a valid assignment of all items in $S \setminus \{s\}$ to the knapsacks in J. However, the multiple cover inequality does not always define a facet of MK as the following example shows.

Example 3.2.1 Let be given an instance of the multiple knapsack problem where $N = \{1,2,3,4,5\}$, $M = \{1,2\}$, $f = (3,4,5,5,7)$, $F = (8,7)$ and consider the corresponding knapsack constraints:

$$\begin{aligned} 3x_{1,1} + 4x_{2,1} + 5x_{3,1} + 5x_{4,1} + 7x_{5,1} &\leq 8, \\ 3x_{1,2} + 4x_{2,2} + 5x_{3,2} + 5x_{4,2} + 7x_{5,2} &\leq 7. \end{aligned}$$

The set $S = \{2,4,5\}$ is a minimal multiple cover for $J = \{1,2\}$. The multiple cover inequality $x_{2,1} + x_{2,2} + x_{4,1} + x_{4,2} + x_{5,1} + x_{5,2} \leq 2$ is clearly valid for the corresponding polytope, but does not define a facet, since it is the sum of the two valid inequalities $x_{2,1} + x_{4,1} + x_{5,1} \leq 1$ and $x_{2,2} + x_{4,2} + x_{5,2} \leq 1$. ■

In the remainder of this subsection we focus on necessary and sufficient conditions such that the multiple cover inequality is facet-defining. Before treating the general case we elucidate the conditions for the special case of the multiple knapsack problem where the knapsack capacity values are all equal. More formally, let $MUKP$ (*multiple uniform knapsack problem*) denote all instances of the multiple knapsack problem such that $F_k = F_l$ for all $k, l \in M$. Given $A \subseteq N$ and $B \subseteq M$, we define $MUK(A \times B, f, F)$ as the corresponding polytope.

Theorem 3.2.2 *Let (N, M, f, F) be an instance of the multiple uniform knapsack problem. Let $S \subseteq N$ be a minimal multiple cover for some $J \subseteq M$. Then, the multiple cover inequality*

$$\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1$$

defines a facet of $MUK(S \times J, f, F)$ *if and only if there exists an item* $i \in S$ *and a valid assignment* x' *of the items in* $S \setminus \{i\}$ *to the knapsacks in* J *such that* $|B_k(x')| \neq |B_l(x')|$ *for some* $k, l \in J, k \neq l$.

Proof. We first prove that the condition is sufficient. Set $a = \sum_{i \in S} \sum_{j \in J} e_{ij}$ and $\alpha = |S| - 1$. The inequality $a^T x \leq \alpha$ is clearly valid. Let us prove that it defines a facet of $MUK(S \times J, f, F)$. Suppose that $b^T x \leq \beta$ defines a facet of $MUK(S \times J, f, F)$ such that $EQ(a^T x \leq \alpha) \subseteq EQ(b^T x \leq \beta)$. Let i_0 be an index such that $f_{i_0} = \min\{f_i \mid i \in S\}$ and let x^1 denote a valid assignment of the items in $S \setminus \{i_0\}$ to the knapsacks in J. Obviously, x^1 is in $EQ(a^T x \leq \alpha)$. Also, notice that for all $k \in J$ and $i \in B_k(x^1)$, the vector $x^1 - e_{ik} + e_{i_0 k}$ is an element of $EQ(a^T x \leq \alpha)$. Thus, $b^T x^1 = b^T(x^1 - e_{ik} + e_{i_0 k})$, yielding $b_{ik} = b_{i_0 k}$. Moreover, since the capacities of the knapsacks are all equal, we can exchange the items of every pair of knapsacks and repeat the same arguments as above. Summing up, we conclude that, for every $k \in J$, there exists a constant c_k such that $b_{ik} = c_k$ for all $i \in S$.

In order to prove that $c_k = c_l$ for $k \neq l, k, l \in J$, let $i \in S$ be an item and let x' denote a valid assignment of the items in $S \setminus \{i\}$ to the knapsacks in J such that $|B_k(x')| \neq |B_l(x')|$ for some $k, l \in J, k \neq l$ as required in the condition. Since all knapsacks have the same capacity, we can construct a valid assignment $x'' = (x''_{ij})$ via:

$$x''_{ij} := \begin{cases} x'_{il}, & \text{for all } i \in S, j = k, \\ x'_{ik}, & \text{for all } i \in S, j = l, \\ x'_{ij}, & \text{otherwise.} \end{cases}$$

Clearly, x' and x'' belong to the face $EQ(a^T x \leq \alpha)$. Thus, $b^T x' = b^T x''$, yielding

$$|B_k(x')| c_k + |B_l(x')| c_l = |B_k(x')| c_l + |B_l(x')| c_k.$$

This implies $c_k = c_l$. Due to the uniform knapsack capacities, we can apply this construction for all other knapsacks and, finally obtain that $b^T x \leq \beta$ and $a^T x \leq \alpha$ are equal up to multiplication with a scalar, which completes the first part of the proof.

It remains to be shown that the condition is necessary. Suppose it is not satisfied, i.e., for all $x \in EQ(a^T x \leq \alpha)$ and $k, l \in J, k \neq l$, $|B_k(x)| = |B_l(x)|$ holds. In this case, all $x \in EQ(a^T x \leq \alpha)$ satisfy the equation $\sum_{i \in S} x_{ik} = \frac{|S|-1}{|J|}$, for all $k \in J$. Thus, the inequality cannot be facet-defining. ■

In the remaining part of this subsection we will treat the general case where arbitrary knapsack capacities are given. Unfortunately, it turns out that necessary and sufficient conditions for the multiple cover inequality to define a facet are rather complicated and involve many (probably) unavoidable technicalities.

Suppose, we are given a minimal multiple cover S for the knapsacks in $J \subseteq M$. Let us assume that for every $i \in S$ and $k \in J$ there exists a valid assignment $x \in EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \leq |S|-1)$ such that $x_{ik} = 1$. Otherwise, $EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \leq |S|-1)$ is a subset of $\{x \in \mathbb{R}^{S\times J} \mid x_{ik} = 0\}$, which implies that $\sum_{i\in S}\sum_{j\in J} x_{ij} \leq |S|-1$ cannot define a facet.

For the exposition of the next lemma let us further assume that the items are ordered such that $f_1 \leq \ldots \leq f_{|S|}$. For a valid assignment $x' \in MK(S \times J, f, F)$ we define

$$j(x', k) := \max\{j \mid j \in B_k(x')\}.$$

For a given $i \in S$, we set $j_i^k := \max\{j(x', k) \mid x'$ is a valid assignment of $S \setminus \{i\}$ to the knapsacks in $J\}$. Let be given a set X of vectors. We define

$$\mathrm{diff}(X) := \{x - y \mid x, y \in X\}$$

to be the difference set of X and

$$\mathrm{lin}(X) := \{\sum_{i=1}^{v} \lambda_i x_i \mid x_1, \ldots, x_v \in X, \lambda_1, \ldots, \lambda_v \in \mathbb{R}, v \in \mathbb{N}\}$$

to be the linearity space of X.

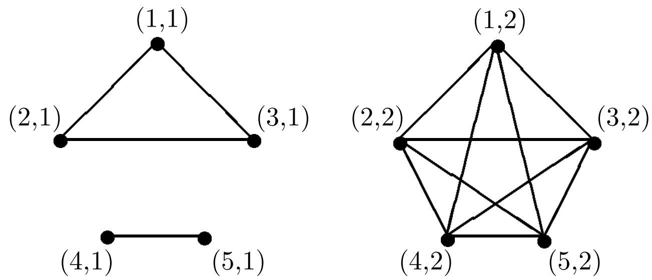
Let $S \subseteq N$ be a minimal multiple cover with respect to $J \subseteq M$. We define the *exchange-graph* $G = (V, E)$ in the following way. The node set V corresponds to the set $S \times J$ and two nodes (i_1, k_1) and (i_2, k_2), $i_1 < i_2$, are adjacent in the exchange-graph if $k_1 = k_2$ and $j_{i_1}^{k_1} \geq i_2$. Let d denote the number of connected components of G (note that $d \geq |J|$) and (V_l, E_l), $l = 1, \ldots, d$ the corresponding components.

Example 3.2.3 Let be given an instance of the multiple knapsack problem where $N = \{1, 2, 3, 4, 5\}$, $M = \{1, 2\}$, $f = (5, 5, 5, 8, 9)$, $F = (10, 17)$ and consider the corresponding knapsack constraints:

$$\begin{aligned} 5x_{1,1} + 5x_{2,1} + 5x_{3,1} + 8x_{4,1} + 9x_{5,1} \leq\ & 10, \\ 5x_{1,2} + 5x_{2,2} + 5x_{3,2} + 8x_{4,2} + 9x_{5,2} \leq\ & 17. \end{aligned}$$

The set $S = \{1, 2, 3, 4, 5\}$ is a minimal multiple cover with respect to $J = \{1, 2\}$. The corresponding exchange-graph has three components and is shown below.

i	1	2	3	4	5
j_i^1	3	3	2	5	4
j_i^2	5	5	5	5	4



Lemma 3.2.4 *For every component l in the exchange graph G, there exist $|V_l| - 1$ linearly independent vectors belonging to* $\text{lin}(\text{diff}(EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)))$. *Moreover, every nonzero component of any of these vectors is a nonzero component of χ^{V_l}.*

Proof. Let $l \in \{1, \ldots, d\}$ and T_l be a spanning tree in $G_l = (V_l, E_l)$. Let $\{(i_1, k), (i_2, k)\}$ be an edge in the tree T_l. W. l. o. g. we assume that $i_1 < i_2$. Thus, $j^k_{i_1} \geq i_2$. Let x' be a valid assignment of $S \setminus \{i_1\}$ to the knapsacks in J such that $\sum_{i \in S} \sum_{j \in J} x'_{ij} = |S| - 1$ and $x_{j^k_{i_1} k} = 1$ (there exists some by definition of $j^k_{i_1}$). We distinguish the following two cases.

Case 1: $i_2 \in B_k(x')$. Since $f_{i_1} \leq f_{i_2}$, the vector $x'' := x' - e_{i_2 k} + e_{i_1 k}$ also satisfies $\sum_{i \in S} \sum_{j \in J} x''_{ij} = |S| - 1$. Thus, $x'' - x' = e_{i_1 k} - e_{i_2 k}$ is an element of $\text{lin}(\text{diff}(EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)))$.

Case 2: $i_2 \notin B_k(x')$. Suppose i_2 is assigned to knapsack v. Since $f_{i_1} \leq f_{i_2} \leq f_{j^k_{i_1}}$ we obtain that $x^2 = x' - e_{j^k_{i_1} k} + e_{i_1 k}$, $x^3 = x' - e_{i_2 v} + e_{i_2 k} + e_{i_1 v} - e_{j^k_{i_1} k}$ and $x^4 = x' - e_{i_2 v} + e_{i_1 v}$ are valid assignments with $x^2, x^3, x^4 \in EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)$. Thus, the vector $(x^2 - x') - ((x^3 - x') - (x^4 - x')) = -e_{j^k_{i_1} k} + e_{i_1 k} + e_{i_2 v} - e_{i_2 k} - e_{i_1 v} + e_{j^k_{i_1} k} - e_{i_2 v} + e_{i_1 v} = e_{i_1 k} - e_{i_2 k}$ is an element of $\text{lin}(\text{diff}(EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)))$.

This implies that, for every edge $\{(i_1, k), (i_2, k)\}$ of the tree, the vector $e_{i_1 k} - e_{i_2 k}$ is contained in $\text{lin}(\text{diff}(EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)))$. Obviously, the nonzero components of these vectors are nonzero components of χ^{V_l}. Since T_l is a tree, it is not difficult to see that this set of vectors is linearly independent. ■

Consider any of the components $G_l = (V_l, E_l)$, $l \in \{1, \ldots, d\}$ and let T_l be the spanning tree in G_l used in the proof of Lemma **??**. Furthermore, let t_l be some node of V_l and set $T := \bigcup_{l=1}^{d} \{t_l\}$. In the following, T is called the set of *special nodes*. From the proof of Lemma **??** we know that for every edge $e = uv$ of the tree the vector $e_u - e_v$ is an element of $\text{lin}(\text{diff}(EQ(\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1)))$. For a node $w \in V_l$ denote by $P(w)$ the unique path from t_l to w.

For $w \in V_l \setminus \{t_l\}$, set $\mu_w := \sum_{uv \in P(w)} \sigma(uv)\,(e_u - e_v)$ where $\sigma(uv) = 1$, if $|P(u)| < |P(v)|$, and $\sigma(uv) = -1$, otherwise. Obviously, μ_w is an element of $\mathrm{lin}(\mathrm{diff}(EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)))$ and $(\mu_w)_{ik} = 1$, if $(i,k) = t_l$, $(\mu_w)_{ik} = -1$, if $(i,k) = w$, and $(\mu_w)_{ik} = 0$, otherwise. Set $D := \{\mu_w \mid w \in V_l \setminus \{t_l\}$ for some $l \in \{1,\dots,d\}\ \}$. It is clear that D is a set of linearly independent vectors, and $|D| = \sum_{l=1}^{d}(|V_l| - 1) = |S||J| - d$.

For a vector $x \in \mathbb{R}^{S\times J}$ we introduce the symbol $(x)^l \in \mathbb{R}^{V_l}$ to denote the subvector of x corresponding to the variables $(i,k) \in V_l$.

For an assignment x' we define the *cardinality vector* $g(x') \in \mathbb{N}^d$ by setting

$$g_l(x') := |\{(i,k) \in V_l \mid x'_{ik} = 1\}|, \text{ for } l = 1,\dots,d.$$

Let $\{x^1,\dots,x^c\}$ be a maximal set of valid assignments such that $x^i \in EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)$ for $i = 1,\dots,c$ and $\{g(x^i) \mid i = 1,\dots,c\}$ is linearly independent. Set $D' = D \cup \{x^i - x^1 \mid i = 2,\dots,c\}$. For every $i \in \{2,\dots,c\}$ and $z = x - y \in D$, the vectors $x^i - x^1$ and z are linearly independent, since x and y have identical cardinality vectors. Thus, D' is a set of $|S||J| - d + c - 1$ linearly independent vectors. In the next theorem we show that the set D' is a basis of $\mathrm{lin}(\mathrm{diff}(EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)))$. As a corollary we obtain that the multiple cover inequality defines a facet if and only if $c = d$.

Theorem 3.2.5 *D' is a basis of* $\mathrm{lin}(\mathrm{diff}(EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)))$.

Proof. Let be given $x, y \in EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)$ and denote by $T = \{t_i \mid i = 1,\dots,l\}$ the set of special nodes. Set

$$z^i = x^i - x^1 - \sum_{\{v \in V\setminus T \mid (x^i - x^1)_v = -1\}} \mu_v + \sum_{\{v \in V\setminus T \mid (x^i - x^1)_v = 1\}} \mu_v.$$

for $i = 2,\dots,c$. It is easy to see that each of the vectors $z^i = (z^i)^l$, $l = 1,\dots,d$, is of the form

$$(z^i)^l_w = \begin{cases} g_l(x^i) - g_l(x^1), & \text{if } w = t_l,\\ 0, & \text{otherwise.}\end{cases}$$

In the same way we obtain a vector $\tau \in \mathbb{R}^{S\times J}$ by setting

$$\tau = x - y - \sum_{\{v \in V\setminus T \mid (x-y)_v = -1\}} \mu_v + \sum_{\{v \in V\setminus T \mid (x-y)_v = 1\}} \mu_v$$

where $\tau = (\tau)^l$ for all $l = 1,\dots,d$ is of the form

$$(\tau)^l_w = \begin{cases} g_l(x) - g_l(y), & \text{if } w = t_l,\\ 0, & \text{otherwise.}\end{cases}$$

Due to the choice of $x^1, \dots, x^c$ there exists $\lambda_2, \dots, \lambda_c$ such that

$$\tau = \sum_{i=2}^{c} \lambda_i z^i.$$

This implies that D' is a basis of $\mathrm{lin}(\mathrm{diff}(EQ(\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1)))$. ■

Corollary 3.2.6 *Let (N, M, f, F) be an instance of the multiple knapsack problem. The inequality*

$$\sum_{i\in S}\sum_{j\in J} x_{ij} \le |S|-1$$

defines a facet of $MK(S \times J, f, F)$ if and only if there exist valid assignments $x^1, \dots, x^d$ such that $x^i \in EQ(\sum_{j\in J} x_{ij} \le |S|-1)$, $i = 1, \dots, d$ and $\{g(x^i) \mid i = 1, \dots, d\}$ is a linearly independent set. ■

Let us now comment this and the previous results. Lemma **??** states that the dimension of $EQ(\sum_{i\in S}\sum_{j\in J} x_{ij})$ is greater or equal than $|S||J| - d$. Theorem **??** guarantees that, if we find b valid assignments of $EQ(\sum_{i\in S}\sum_{j\in J} x_{ij})$ whose cardinality vectors are linearly independent, the dimension of the face is at least $|S||J| - d + b$. Hence, in order to prove that a multiple cover inequality defines a facet, we must find d assignments that are elements of the face and whose cardinality vectors are linearly independent. This task is still nontrivial, yet simplifies the original task, since, instead of looking for $|S||J|+1$ affinely independent vectors on the face, it suffices to find d ones, whose cardinality vectors are linearly independent. In the remainder of this section we present some applications of Corollary **??**.

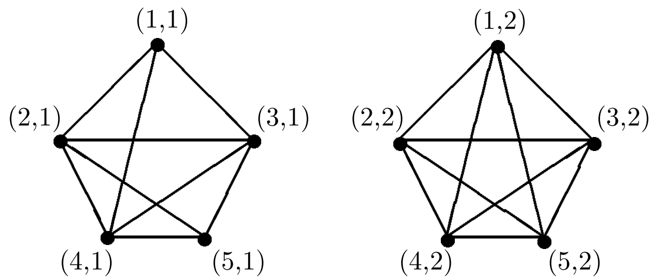
Example 3.2.7 For the multiple cover introduced in Example **??** one can check that there do not exist $d = 3$ linearly independent cardinality vectors. For every valid assignment x that satisfies $\sum_{i=1}^{5} x_{i1} + x_{i2} = 4$, the cardinality vector is either $(2, 0, 2)^T$ or $(0, 1, 3)^T$. By applying Corollary **??**, the corresponding multiple cover inequality does not define a facet in this case. ■

Example 3.2.8 Let be given an instance of the multiple knapsack problem where $N = \{1, 2, 3, 4, 5\}$, $M = \{1, 2\}$, $f = (1, 2, 2, 2, 3)$, $F = (4, 5)$ and consider the corresponding knapsack constraints:

$$\begin{aligned} x_{1,1} + 2x_{2,1} + 2x_{3,1} + 2x_{4,1} + 3x_{5,1} &\le 4, \\ x_{1,2} + 2x_{2,2} + 2x_{3,2} + 2x_{4,2} + 3x_{5,2} &\le 5. \end{aligned}$$

The set $S = \{1, 2, 3, 4, 5\}$ is a minimal multiple cover with respect to $J = \{1, 2\}$. The exchange graph is shown below.

i	1	2	3	4	5
j_i^1	4	5	5	5	4
j_i^2	5	5	5	5	4



In this example, there exist valid assignments x and y where $x_{1,1} = x_{2,1} = 1$, $x_{3,2} = x_{4,2} = 1$, $x_{5,1} = x_{5,2} = 0$ and $y_{1,2} = y_{2,2} = y_{3,2} = 1$, $y_{4,1} = 1$, $y_{5,1} = y_{5,2} = 0$. The cardinality vectors $g(x) = (2, 2)^T$ and $g(y) = (1, 3)^T$ are linearly independent. By applying Corollary **??** we can conclude that the inequality $\sum_{i=1}^{5} \sum_{j=1}^{2} x_{ij} \leq 4$ defines a facet of the corresponding polytope. ■

The following example deals with a special case of minimal multiple cover inequalities. Here, it is easy to see that the conditions in Corollary **??** are satisfied.

Example 3.2.9 Given an instance (N, M, f, F) of the multiple knapsack problem. Let S be a subset of items and let J denote a set of knapsacks with the property that $f(S) > F(J)$. For the ease of notation we assume that $S = \{1, \ldots, s\}$, $J = \{1, \ldots, j\}$ and that $f_1 \leq f_2 \leq \ldots \leq f_s$. Suppose there exists a valid assignment z of the items in $S \setminus \{1, s\}$ to the knapsacks in J such that $\sum_{\{i \in S | z_{ik} = 1\}} f_i + f_1 \leq F_k$ for all $k \in J$. Moreover, we assume that, for every $k \in J$, there exists a valid assignment x of the items in $S \setminus \{1\}$ to the knapsacks in J such that $x_{sk} = 1$. Under these assumptions, the multiple cover inequality defines a facet of the polytope $MK(S \times J, f, F)$.

Proof. For every $k \in J$ there exists a valid assignment of the items in $S \setminus \{1\}$ to the knapsacks in J such that $x_{sk} = 1$. Since $f_1 = \min\{f_i \mid i \in S\}$ the exchange graph consists of $|J|$ components, the l^{th} component being a complete graph induced by $V_l = \{(i, l) \mid i = 1, \ldots, s\}$ for all $l = 1, \ldots, |J|$. Now, let x be the valid assignment of the items in $S \setminus \{1, s\}$ to the knapsacks in J such that $\sum_{\{i \in S | z_{ik} = 1\}} f_i + f_1 \leq F_k$ for all $k \in J$. Clearly, $x^k := x + e_{1k}$ is an element of $EQ(\sum_{k \in J} \sum_{i \in S} x_{ik} \leq |S| - 1)$ for all $k \in J$. Thus, the set of cardinality vectors $\{g(x^1), \ldots, g(x^{|J|})\}$ is a linearly independent set of dimension $|J|$ which, by Corollary **??**, implies that the multiple cover inequality defines a facet in this case. ■

4 Extension of Facet-defining Inequalities for MK

In the previous section we dealt with two particular classes of inequalities for MK and focused on conditions for the inequalities to be valid or facet-defining. Let us now present a general procedure that allows the extension of particular classes of inequalities. In this section we first state the theorems in full generality. For illustration the corresponding theorem is subsequently applied to particular examples.

The first two theorems deal with the "extension" of particular facet-defining inequalities. Let be given an instance (N, M, f, F) of the multiple knapsack problem, sets $A_u \subseteq N, B_u \subseteq M$ for $u = 1, \ldots, t$, and suppose, $a^T x \leq \alpha$ is a facet-defining inequality for $MK(\bigcup_{u=1}^t A_u \times B_u, f, F)$. We now choose sets $T_1, \ldots, T_r$ of mutually disjoint items of $N \setminus \bigcup_{u=1}^t A_u$ and $k_1, \ldots, k_r$ pairwise disjoint elements of $M \setminus \bigcup_{u=1}^t B_u$. Provided, the inequality $a^T x \leq \alpha$ has a certain property, we can extend this inequality and prove that it is valid (facet-defining) for a proper subpolytope of MK if and only if some conditions are satisfied.

Theorem 4.1 *Let be given an instance (N, M, f, F) of the multiple knapsack problem, sets $A_u \subseteq N$, $B_u \subseteq M$ for $u = 1, \ldots, t$ (we set $A := \bigcup_{u=1}^t A_u \subseteq N$, $B := \bigcup_{u=1}^t B_u \subseteq M$) and an inequality $a^T y \leq \alpha$ that is facet-defining for $MK(\bigcup_{u=1}^t A_u \times B_u, f, F)$ and that satisfies the following additional requirement:*

($\star$) *For all $\tilde{A} \subseteq A$ with $|\tilde{A}| \geq 2$ the following holds: every assignment $y \in MK((\bigcup_{u=1}^t A_u \times B_u,\ f, F)$ with $y_{ik} = 0$ for all $i \in \tilde{A}$, $k \in \{l \in B \mid (i, l) \in \cup_{u=1}^t A_u \times B_u\}$, satisfies $a^T y \leq \alpha - |\tilde{A}| + 1$.*

We choose a positive integer $r \leq \min\{|N \setminus A|, |M \setminus B|\}$, nonempty sets $T_1, \ldots, T_r$ that are mutually disjoint subsets of $N \setminus A$, and a subset $\{k_1, \ldots, k_r\}$ of $M \setminus B$.

Let us further define $\beta_v := \max\{|G| \mid G \subseteq T_v, f(G) \leq F_{k_v}\}, v = 1, \ldots, r$. Under these assumptions, the inequality

(??) $$a^T x + \sum_{v=1}^{r} \sum_{i \in A \cup T_v} x_{ik_v} \le \alpha + \sum_{v=1}^{r} \beta_v$$

is valid for the polytope $MK((\bigcup_{u=1}^t A_u \times B_u) \cup (\bigcup_{v=1}^r (T_v \cup A) \times \{k_v\}), f, F)$ *if and only if* $\tilde{T}_v \cup \{i\}$ *is a cover with respect to* k_v *for all* $i \in A$, $\tilde{T}_v \subseteq T_v$, $|\tilde{T}_v| = \beta_v$ *and* $v = 1, \ldots, r$.

Proof. For the ease of notation let us refer to inequality (**??**) by $b^T x \le \beta$ and set $Q := \bigcup_{i=1}^t A_i \times B_i$. Without loss of generality we assume that $B = B_i = B_j$ for all $i, j \in \{1, \ldots, t\}$.

Let us first prove that the condition is necessary. Suppose, there exists an index $v \in \{1, \ldots, r\}$, an item $i_0 \in A$ and a set $\tilde{T}_v \subseteq T_v$, $|\tilde{T}_v| = \beta_v$ such that $\tilde{T}_v \cup \{i_0\}$ is not a cover with respect to k_v. Let $\tilde{T}_w \subseteq T_w, w \in \{1, \ldots, r\} \setminus \{v\}$, $|\tilde{T}_w| = \beta_w$, $f(\tilde{T}_w) \le F_{k_w}$.

Since $a^T y \le \alpha$ defines a facet, there exists an assignment $y' \in EQ(MK(Q, f, F), a^T y \le \alpha)$ with $y'_{i_0 k} = 0$ for all $k \in B$. Set $x' = (x'_{ik})$ via:

$$x'_{ik} := \begin{cases} y'_{ik}, & \text{for all } (i,k) \in Q, \\ 1, & \text{for all } i \in \tilde{T}_w, k = k_w, w \in \{1, \ldots, r\} \setminus \{v\}, \\ 1, & \text{for all } i \in \tilde{T}_v, k = k_v, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\tilde{T}_v \cup \{i_0\}$ does not define a cover with respect to knapsack k_v, $x' + e_{i_0 k_v}$ is a valid assignment yielding $b^T(x' + e_{i_0 k_v}) = a^T y' + \sum_{w \in \{1,\ldots,r\} \setminus \{v\}} \beta_w + \beta_v + 1 > \alpha + \sum_{v=1}^r \beta_v$. This implies that the condition is necessary, indeed.

In order to prove the converse direction, let us assume that the inequality is not valid for the polytope $MK(Q \cup \bigcup_{v=1}^r (T_v \cup A) \times \ \{k_v\}, f, F)$, i.e., there exists an assignment $x \in MK(Q \cup \bigcup_{v=1}^r ((T_v \cup A) \times \{k_v\}), f, F)$ with $b^T x > \beta$. Set $\tilde{T}_v := \{i \in T_v \mid x_{ik_v} = 1\}$ and $A_v := \{i \in A \mid x_{ik_v} = 1\}, v = 1, \ldots, r$. Since the inequality $a^T y \le \alpha$ holds for all $y \in MK(Q, f, F)$, there exists some $v \in \{1, \ldots, r\}$ satisfying $|A_v \cup \tilde{T}_v| > \beta_v$. Let $V \subseteq \{1, \ldots, r\}$ denote the subset of knapsacks with $|A_v \cup \tilde{T}_v| > \beta_v$ for all $v \in V$. Due to the condition, every subset of T_v of cardinality β_v and one element from A defines a cover with respect to k_v ($v \in V$). This implies, that $|\tilde{T}_v| \le \beta_v - 1$, thus forcing A_v to be greater or equal than two, which holds for all $v \in V$. Moreover, $A_v \cap A_w = \emptyset$ for all $v, w \in V, v \neq w$. Summing up, due to requirement ($\star$) we obtain

$$\begin{aligned} b^T x &\leq \alpha - \sum_{v \in V} |A_v| + 1 + \sum_{v \in \{1,\dots,r\} \setminus V} \beta_v + \sum_{v \in V} (|\tilde{T}_v| + |A_v|) \\ &\leq \alpha - \sum_{v \in V} |A_v| + 1 + \sum_{v \in \{1,\dots,r\} \setminus V} \beta_v + \sum_{v \in V} (\beta_v + |A_v|) - |V| \\ &= \beta + 1 - |V| \leq \beta. \end{aligned}$$

This contradicts the assumption that x is a point violating the inequality. Thus, the inequality is valid, which completes the proof. ■

In the subsequent theorem, necessary and sufficient conditions are given such that the extended inequality defines a facet.

Theorem 4.2 *Let be given an instance* (N, M, f, F) *of the multiple knapsack problem, sets* $A_u \subseteq N$, $B_u \subseteq M$ *for* $u = 1, \dots, t$ *(we set* $A := \bigcup_{u=1}^t A_u \subseteq N$, $B := \bigcup_{u=1}^t B_u \subseteq M$*) and an inequality* $a^T y \leq \alpha$ *that defines a facet of* $MK(\bigcup_{u=1}^t A_u \times B_u, f, F)$ *and satisfies the following additional requirement:*

($\star$) *For all* $\tilde{A} \subseteq A$ *with* $|\tilde{A}| \geq 2$ *the following holds: every assignment* $y \in MK((\bigcup_{u=1}^t A_u \times B_u,\ f, F)$ *with* $y_{ik} = 0$ *for all* $i \in \tilde{A}$, $k \in \{l \in B \mid (i,l) \in \bigcup_{u=1}^t A_u \times B_u\}$*, satisfies* $a^T y \leq \alpha - |\tilde{A}| + 1$.

We choose a positive integer $r \leq \min\{|N \setminus A|, |M \setminus B|\}$*, nonempty sets* $T_1, \dots, T_r$ *that are mutually disjoint subsets of* $N \setminus A$ *and a subset* $\{k_1, \dots, k_r\}$ *of* $M \setminus B$*. We require that* $T_v \cup \{i\}$ *is a minimal cover with respect to knapsack* k_v *for all* $i \in A$ *and* $v = 1, \dots, r$.

Under these assumptions, the inequality

$$a^T x + \sum_{v=1}^r \sum_{i \in A \cup T_v} x_{ik_v} \leq \alpha + \sum_{v=1}^r |T_v| \qquad (??)$$

defines a facet for $MK(\bigcup_{i=1}^t A_i \times B_i \cup \bigcup_{v=1}^r (T_v \cup A) \times \{k_v\}, f, F)$ *if and only if the following conditions are satisfied:*

for every $v \in \{1, \dots, r\}$ *there exist*

- $\tilde{A} \subseteq A, |\tilde{A}| \geq 2,$
- *an item* $t_v \in T_v$,

- *an assignment* $y \in MK(\bigcup_{u=1}^{t} A_u \times B_u, f, F)$ *such that*

$a^T y = \alpha - |\tilde{A}| + 1$, $y_{ik} = 0$ *for all* $i \in \tilde{A}$, $k \in \{l \in B \mid (i,l) \in \bigcup_{u=1}^{t} A_u \times B_u\}$ *and* $f(T_v \setminus \{t_v\}) + f(\tilde{A}) \leq F_{k_v}$.

Proof. Again, for ease of notation let us refer to inequality (**??**) by $b^T x \leq \beta$, set $Q := \bigcup_{i=1}^{t} A_i \times B_i$ and $\tilde{Q} := Q \cup \bigcup_{v=1}^{r} (T_v \cup A) \times \{k_v\}$. Without loss of generality we assume that $B = B_i = B_j$ for all $i, j \in \{1, \ldots, t\}, i \neq j$. Due to Theorem **??**, the inequality is valid if and only if $T_v \cup \{i\}$ defines a cover with respect to k_v, for all $i \in A$, $v = 1, \ldots, r$. Clearly, this requirement is satisfied and we can conclude that $b^T x \leq \beta$ is valid. In the following we show, that the inequality defines a facet for $MK(\tilde{Q}, f, F)$ if and only if the above conditions are satisfied.

We start by proving that the conditions are sufficient. Suppose, there exists a facet-defining inequality $c^T x \leq \gamma$ such that $EQ(b^T x \leq \beta) \subseteq EQ(c^T x \leq \gamma)$.

Let $y' \in EQ(MK(Q, f, F), a^T y \leq \alpha)$ be an assignment. Obviously, the vector $x' = (x'_{ik})$ defined via:

$$x'_{ik} := \begin{cases} y'_{ik}, & \text{for all } (i,k) \in Q, \\ 1, & \text{for all } i \in T_w, k = k_w, w \in \{1, \ldots, r\}, \\ 0, & \text{otherwise,} \end{cases}$$

is an element of $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$. Since the inequality $a^T y \leq \alpha$ defines a facet for the polytope $MK(Q, f, F)$, we conclude that $\lambda c_{ik} = b_{ik}$ for all $(i,k) \in Q$ for some $\lambda > 0$.

Next, let $i_0 \in A$ and $y'' \in EQ(MK(Q, f, F), a^T y \leq \alpha)$ be an assignment such that $y_{i_0 k} = 0$ for all $k \in B$ (such an assignment must exist, since, otherwise, $EQ(MK(Q, f, F), a^T y \leq \alpha) \subseteq \{y \in \mathbb{R}^Q \mid \sum_{k \in B} y_{i_0 k} = 1\}$). Let be given $v \in \{1, \ldots, r\}$. Since $T_v \cup \{i_0\}$ is a minimal cover for knapsack k_v, the vector $x'' = (x''_{ik})$ defined via:

$$x''_{ik} := \begin{cases} y''_{ik}, & \text{for all } (i,k) \in Q, \\ 1, & \text{for all } i \in T_w, k = k_w, w \in \{1, \ldots, r\} \setminus \{v\}, \\ 1, & \text{for all } i \in T_v \setminus \{t\} \cup \{i_0\}, k = k_v, \\ 0, & \text{otherwise,} \end{cases}$$

is an element of $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$ for all $t \in T_v$. Moreover, $x'' - e_{i_0 k_v} + e_{t k_v}$ is an assignment that is also in $EQ(b^T x \leq \beta)$. Thus, $c^T x'' = c^T(x'' - e_{i_0 k_v} + e_{t k_v})$ for all $t \in T_v$. Since the same argument can be repeated for all $i_0 \in A$ and all $v \in \{1, \ldots, r\}$, we conclude that, for all $v = 1, \ldots, r$, there exists a constant c_v such that $c_{u k_v} = c_v$ for all $u \in A \cup T_v$.

Finally, consider some $v \in \{1, \ldots, r\}$. We know that there exist $\tilde{A} \subseteq A, |\tilde{A}| \geq 2$, an item $t_v \in T_v$ and a valid assignment $y \in MK(Q, f, F)$ such that $f(T_v \setminus \{t_v\}) + f(\tilde{A}) \leq F_{k_v}$, $y \in MK(Q, f, F)$, with $y_{ik} = 0$ for all $i \in \tilde{A}, k \in B$, and $a^T y = \alpha - |\tilde{A}| + 1$. We now define the vector x as follows.

$$x_{ik} := \begin{cases} y_{ik}, & \text{for all } (i,k) \in Q, \\ 1, & \text{for all } i \in T_w, k = k_w, w \in \{1, \ldots, r\} \setminus \{v\}, \\ 1, & \text{for all } i \in T_v \setminus \{t_v\} \cup \tilde{A}, k = k_v, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $b^T x = a^T y + \sum_{w \in \{1,\ldots,r\} \setminus \{v\}} |T_w| + |T_v| - 1 + |\tilde{A}| = \beta$, i.e., x is in $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$. Consider x' as defined above. Since $x' \in EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$, we obtain

$$\begin{aligned} 0 = c^T x - c^T x' &= \lambda\alpha - \lambda|\tilde{A}| + \lambda + \textstyle\sum_{w \in \{1,\ldots,r\} \setminus \{v\}} c_w |T_w| + (T_v - 1 + |\tilde{A}|) c_v - \\ &\quad \lambda\alpha - \textstyle\sum_{w=1}^{r} c_w |T_w| \\ &= (c_v - \lambda)(|\tilde{A}| - 1). \end{aligned}$$

Since $|\tilde{A}| \geq 2$, $c_v = \lambda$ follows. Thus, the inequalities $b^T x \leq \beta$ and $c^T x \leq \gamma$ are equal up to multiplication with a scalar.

It remains to be shown that the conditions are necessary, too.

Assume that the conditions are violated, i.e., there exists some $v \in \{1, \ldots, r\}$ such that, for all $\tilde{A} \subseteq A, |\tilde{A}| \geq 2$, the following holds: every $t_v \in T_v$ satisfies $f(T_v \setminus \{t_v\}) + f(\tilde{A}) > F_{k_v}$, or, every assignment $y \in MK(Q, f, F)$ with $y_{ik} = 0$ for all $i \in \tilde{A}$, $k \in B$ satisfies $a^T x < \alpha - |\tilde{A}| + 1$. In this case, we claim that $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta) \subseteq \{x \in \mathbb{R}^{\tilde{Q}} \mid \sum_{i \in T_v \cup A} x_{ik_v} = |T_v|\}$, yielding that $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$ is not a maximal face of $MK(\tilde{Q}, f, F)$, a contradiction.

Now, let us prove that $EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta) \subseteq \{x \in \mathbb{R}^{\tilde{Q}} \mid \sum_{i \in T_v \cup A} x_{ik_v} = |T_v|\}$. First, suppose there exists some assignment $x \in EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$ with $\sum_{i \in T_v \cup A} x_{ik_v} < |T_v|$. Let $W := \{w \in \{1, \ldots, r\} \setminus \{v\} \mid |\{i \in T_w \cup A \mid x_{ik_w} = 1\}| > |T_w|\}$ and $W' := \{1, \ldots, r\} \setminus W$. Obviously, $W \neq \emptyset$, since $\sum_{i \in T_v \cup A} x_{ik_v} < |T_v|$. Set $\tilde{T}_w := \{i \in T_w \mid x_{ik_w} = 1\}$ and $\tilde{A}_w := \{i \in A \mid x_{ik_w} = 1\}$ for all $w \in W$. Since $T_w \cup \{i\}$ is a cover for all $i \in A$, we conclude that $|\tilde{A}_w| \geq 2$ and $|\tilde{T}_w| \leq |T_w| - 1$. Thus, we obtain that

$$\begin{aligned} b^T x &< \alpha - |\textstyle\bigcup_{w \in W} \tilde{A}_w| + 1 + \sum_{w \in W} (|\tilde{A}_w| + |\tilde{T}_w|) + \sum_{w' \in W' \setminus \{v\}} |T_{w'}| + |T_v| \\ &\leq \alpha + \textstyle\sum_{t=1}^{r} |T_t| + 1 - |W| \leq \beta. \end{aligned}$$

This is a contradiction to $x \in EQ(MK(\tilde{Q}, f, F), b^T x \leq \beta)$. Thus, we conclude that $\sum_{i \in T_v \cup A} x_{ik_v} \geq |T_v|$.

Now, suppose there exists an assignment $x \in EQ(MK(\tilde{Q}, f, F), b^T x \le \beta)$ with $\sum_{i \in T_v \cup A} x_{ik_v} > |T_v|$. Again, define $W := \{w \in \{1, \ldots, r\} \mid |\{i \in T_w \cup A \mid x_{ik_w} = 1\}| > |T_w|\}$, $\tilde{T}_w := \{i \in T_w \mid x_{ik_w} = 1\}$ and $\tilde{A}_w := \{i \in A \mid x_{ik_w} = 1\}$ for all $w \in W$. Clearly, $v \in W$ holds, yielding $|W| \ge 1$. If $|W| \ge 2$, we obtain

$$\begin{array}{rcl} b^T x & \le & \alpha - |\cup_{w \in W} \tilde{A}_w| + 1 + \sum_{w \in W}(|\tilde{A}_w| + |\tilde{T}_w|) + \sum_{w' \in \{1,\ldots,r\} \setminus W} |T_{w'}| \\ & \le & \alpha + \sum_{w=1}^{r} |T_w| - |W| + 1 < \beta. \end{array}$$

It remains the case $W = \{v\}$. Since $T_v \cup \{i\}$ is a cover for knapsack k_v for all $i \in A$, we have $|\tilde{A}_v| \ge 2$ and $|\tilde{T}_v| \le |T_v| - 1$. Define $y \in MK(Q, f, F)$ by setting $y_{ik} = x_{ik}$ for all $(i, k) \in Q$. Due to our assumption we have that $a^T y < \alpha - |\tilde{A}_v| + 1$, or $f(T_v \setminus \{t_v\}) + f(\tilde{A}_v) > F_{k_v}$ for all $t_v \in T_v$. In the first case, we obtain

$$\begin{array}{rcl} b^T x & < & \alpha - |\tilde{A}_v| + 1 + |\tilde{A}_v| + |\tilde{T}_v| + \sum_{w \in \{1,\ldots,r\} \setminus \{v\}} |T_w| \\ & \le & \beta. \end{array}$$

This contradicts the assumption that $x \in EQ(MK(\tilde{Q}, f, F), b^T x \le \beta)$. In the second case, $f(T_v \setminus \{t_v\}) + f(\tilde{A}_v) > F_{k_v}$ for all $t_v \in T_v$, we conclude that $|\tilde{A}_v| \ge 3$ and $|\tilde{T}_v| \le |T_v| - 2$. Then,

$$\begin{array}{rcl} b^T x & \le & \alpha - |\tilde{A}_v| + 1 + |\tilde{A}_v| + |\tilde{T}_v| + \sum_{w \in \{1,\ldots,r\} \setminus \{v\}} |T_w| \\ & \le & \alpha + \sum_{w \in \{1,\ldots,r\} \setminus \{v\}} |T_w| + |T_v| - 2 + 1 \\ & < & \beta. \end{array}$$

This is also a contradiction. ■

In the following we will show some examples where Theorems **??** and **??** can be applied to yield "new" inequalities.

Example 4.3 Given an instance (N, M, f, F) of the multiple knapsack problem. Let $S \subseteq N$ be a minimal cover with respect to some knapsack $k \in M$ and let $M' \subseteq M$ be a subset of knapsacks with $k \in M'$. Then, the inequality

$$\sum_{i \in S} x_{ik} \le |S| - 1$$

defines a facet for the polytope $MK(S \times M', f, F)$. Define $a := \sum_{i \in S} e_{ik}$ and $\alpha := |S| - 1$. Obviously, the inequality $a^T x \le \alpha$ meets requirement $(\star)$ of Theorem **??**. Let us choose a positive integer $r \le \min\{|N \setminus S|, |M \setminus M'|\}$, nonempty sets $T_1, \ldots, T_r$ that are mutually disjoint subsets of $N \setminus S$ and a subset $\{k_1, \ldots, k_r\}$ of $M \setminus M'$. Moreover, we require that $T_v \cup \{i\}$ is a minimal cover with respect to knapsack k_v for all $i \in S$. By applying Theorem **??**, the following inequality

$$(??) \qquad \sum_{i \in S} x_{ik} + \sum_{v=1}^{r} \sum_{i \in S \cup T_v} x_{ik_v} \le |S| - 1 + \sum_{v=1}^{r} |T_v|$$

defines a facet of the polytope $MK((S \times M' \cup (\bigcup_{v=1}^{r}((S \cup T_v) \times \{k_v\})), f, F)$ if and only if for every $v \in \{1, \ldots, r\}$ there exists some $\tilde{A} \subseteq S, |\tilde{A}| = 2$, and an item $t_v \in T_v$ such that $f(T_v \setminus \{t_v\}) + f(\tilde{A}) \leq F_{k_v}$. We call this inequality (**??**) *extended minimal cover inequality.*

It can easily be checked that the extended minimal cover inequality still satisfies requirement ($\star$) of Theorem **??**. Thus, a repeated extension in the "spirit" of Theorem **??** is possible. Moreover, one can convince oneself that, for example, the r-fold repetition of this "extension procedure" (i.e., at each time we extend the original inequality by one set of items T_v and one knapsack k_v) leads to an inequality which is different from the one obtained, if the simultaneous extension by the sets $T_1, \ldots, T_r$ and knapsacks $k_1, \ldots, k_r$ is applied to the original inequality. ■

Example 4.4 Given an instance (N, M, f, F) of the multiple knapsack problem. Let $S \subseteq N$ be a minimal multiple cover with respect to a given set $J \subseteq M$ and suppose, the inequality

$$\sum_{i \in S} \sum_{j \in J} x_{ij} \leq |S| - 1$$

defines a facet for the polytope $MK(S \times J, f, F)$ (see subsection 3.2). Define $a := \sum_{i \in S} \sum_{j \in J} e_{ij}$ and $\alpha := |S| - 1$. For every $\tilde{A} \subseteq S, |\tilde{A}| \geq 2$, there exists a valid assignment y such that $y_{ik} = 0$ for all $k \in J, i \in \tilde{A}$ and $a^T y = |S| - |\tilde{A}| + 1$. Thus, requirement ($\star$) of Theorem **??** is satisfied. Let us choose a positive integer $r \leq \min\{|N \setminus S|, |M \setminus J|\}$, sets $T_1, \ldots, T_r$ that are mutually disjoint subsets of $N \setminus S$ and a subset $\{k_1, \ldots, k_r\}$ of $M \setminus J$. Moreover, we require that $T_v \cup \{i\}$ is a minimal cover with respect to knapsack k_v for all $i \in S$. By applying Theorem **??**, we can conclude that the so-called *extended minimal multiple cover inequality*

$$\sum_{i \in S} \sum_{j \in J} x_{ij} + \sum_{v=1}^{r} \sum_{i \in S \cup T_v} x_{ik_v} \leq |S| - 1 + \sum_{v=1}^{r} |T_v| \tag{??}$$

defines a facet of the polytope $MK((S \times J \cup (\bigcup_{v=1}^{r}((S \cup T_v) \times \{k_v\})), f, F)$ if and only if for every $v \in \{1, \ldots, r\}$ there exists some $\tilde{A} \subseteq S$, $|\tilde{A}| = 2$ and some $t_v \in T_v$ such that $f(T_v \setminus \{t_v\}) + f(\tilde{A}) \leq F_{k_v}$.

Interestingly, the extended minimal multiple cover inequality still satisfies the requirement ($\star$) of Theorem **??**. Thus, the repeated extension of the minimal multiple cover inequality yields an inequality that still allows for a further extension. ■

In order to apply the previous theorems we must require that the inequality $a^T x \leq \alpha$ meets certain properties. We also notice that the minimal cover inequality, the minimal multiple cover inequality and extended versions of both of them fulfill these properties. However, there are several inequalities such as the $(1, d)$-configuration inequality that do not fit into this scheme. In order to develop a "generalization procedure" for inequalities of this type we run into some troubles, since – as one might expect – necessary and sufficient conditions for an extended inequality to be valid or facet-defining, respectively, become more complex, if we do not require in advance, that the inequality to be extended has a particular structure. In the remainder we will present a generalization procedure for extending an arbitrary valid inequality by one additional knapsack and one additional subset of items. The simultaneous extension by arbitrary many knapsacks and (mutually) disjoint subsets of items does no longer lead to handy conditions, so that we decided to omit this. Similarly, a result about the extension of arbitrary facet-defining inequalities in the fashion of Theorem **??** would involve such clumsy conditions that we decided to omit this either. Rather, we present some new classes of facet-defining joint inequalities that cannot be obtained by applying Theorem **??**. However, in order not to be beyond the scope of this paper we skip the proofs and just state the results. For more details we refer the reader to [F93].

Theorem 4.5 *Let be given an instance (N, M, f, F) of the multiple knapsack problem, sets $A_u \subseteq N$, $B_u \subseteq M$ for $u = 1, \ldots, t$ (we set $A := \bigcup_{u=1}^t A_u \subseteq N$, $B := \bigcup_{u=1}^t B_u \subseteq M$, $Q = \cup_{u=1}^t A_u \times B_u$) and an inequality $a^T x \leq \alpha$ that is valid for $MK(Q, f, F)$.*

We choose a set $T \subseteq N \backslash A$ of items and a knapsack $v \in M \backslash B$. For a subset $\tilde{A} \subseteq A$ we define $\mu(\tilde{A}) := \max\{a^T x \mid x \in MK(Q, f, F)$ is an assignment and $x_{ik} = 0$ for all $i \in \tilde{A}, k \in \{l \in B \mid (i, l) \in Q\}\} - (\alpha - |\tilde{A}| + 1)$. Moreover, we set $\beta := \max\{|G| \mid G \subseteq T, f(G) \leq F_v\}$.

Under these assumptions, the inequality

$$a^T x + \sum_{i \in A \cup T} x_{iv} \leq \alpha + \beta \qquad (??)$$

is valid for $MK(Q \cup ((T \cup A) \times \{v\}), f, F)$ if and only if for every $\tilde{A} \subseteq A$, $|\tilde{A}| \geq 1$ the following holds: for every $\tilde{T} \subseteq T$ of cardinality $|\tilde{T}| = \beta - \mu(\tilde{A})$, the set $\tilde{T} \cup \tilde{A}$ defines a cover with respect to v. ■

Example 4.6 Let be given an instance (N, M, f, F) of the multiple knapsack problem. We choose indices $k_1, k_2 \in M, k_1 \neq k_2$ and assume, $S_1 \subseteq N$ is a minimal cover with respect to k_1 and $S_2 \subseteq N$ is a minimal cover with respect to k_2 where $S_1 \cap S_2 \neq \emptyset$. Clearly, the inequality

$$\sum_{i \in S_1} x_{ik_1} + \sum_{i \in S_2} x_{ik_2} \leq |S_1| + |S_2| - 2,$$

is valid for the polytope $MK(N \times M, f, F)$, yet not facet-defining. We choose a set $T \subseteq N \setminus (S_1 \cup S_2)$ and a knapsack $k_3 \in M \setminus \{k_1, k_2\}$. Then, the inequality

$$\sum_{i \in S_1} x_{ik_1} + \sum_{i \in S_2} x_{ik_2} + \sum_{i \in S_1 \cup S_2 \cup T} x_{ik_3} \leq |S_1| + |S_2| + |T| - 2,$$

called *combined cover inequality*, is valid for the polytope $MK(N \times M, f, F)$ if and only if $T \cup \{i\}$ is a cover with respect to k_3 for all $i \in S_1 \cup S_2$.

We now require that $S_1 \cap S_2 = \{s\}$ and that $T \cup \{i\}$ is a minimal cover with respect to k_3 for every $i \in S_1 \cup S_2$. Then, the combined cover inequality defines a facet for $MK((S_1 \times M \setminus \{k_2\}) \cup (S_2 \times M \setminus \{k_1\}) \cup (T \times M \setminus \{k_1, k_2\}), f, F)$ if and only if there exists some $R \subseteq S_1 \cup S_2 \cup T$ with $|R| = |T| + |R \cap S_1| + |(R \setminus \{s\}) \cap S_2| - 1$, $\sum_{i \in R} f_i \leq F_{k_3}$, R has nonempty intersection with both S_1 and S_2 and $R \cap S_1 \neq \{s\} \neq R \cap S_2$. ■

Example 4.7 Let be given an instance (N, M, f, F) of the multiple knapsack problem and suppose, $Q \cup \{s\} \subseteq N$ is some $(1, d)$-configuration with respect to the knapsack $l \in M$. The inequality

$$\sum_{i \in Q} x_{il} + (|Q| - d + 1) x_{sl} \leq |Q|$$

defines a facet for the polytope $MK((Q \cup \{s\}) \times \{l\}, f, F)$. We choose a knapsack $k \in M \setminus \{l\}$ and a subset $T \subseteq N \setminus (Q \cup \{s\})$ of items that satisfies $f(T) \leq F_k$. In order to extend this $(1, d)$-configuration inequality, the conditions of Theorem **??** can be simplified. It is easy to see that the inequality

$$\sum_{i \in Q} x_{il} + (|Q| - d + 1) x_{sl} + \sum_{j \in T \cup (Q \cup \{s\})} x_{jk} \leq |Q| + |T|$$

is valid for the polytope $MK((Q \cup \{s\}) \times \{l\} \cup (T \cup Q \cup \{s\}) \times \{k\}, f, F)$ if and only if

- for all $\tilde{Q} \subseteq Q$ with $|\tilde{Q}| = d - t$, $t \geq 1$ and for every subset $\hat{Q} \subseteq (Q \setminus \tilde{Q}) \cup T$ such that $|\hat{Q}| = |T| + t$, $\hat{Q}$ is a cover with respect to knapsack k;
- $T \cup \{s\}$ is a cover with respect to k.

Now, let $Q \cup \{s\} \subseteq N$ be a $(1, d)$-configuration with respect to some knapsack $l \in M$ and suppose, $T \subseteq N \setminus (Q \cup \{s\})$ satisfies $f(T) \leq F_k$ for some $k \in M \setminus \{l\}$. Moreover, we assume that $T \cup \{s\}$ is a minimal cover with respect to k and for all

$Q' \subseteq Q$, with $|Q'| \le |Q| - d + 1$, $T \cup Q'$ is a $|Q'|$-cover with respect to k. Under these assumptions, the inequality

$$\sum_{i \in Q} x_{il} + (|Q| - d + 1)x_{sl} + \sum_{j \in T \cup (Q \cup \{s\})} x_{jk} \le |Q| + |T|$$

defines a facet for the polytope $MK((Q \cup \{s\}) \times \{l\} \cup (T \cup Q \cup \{s\}) \times \{k\}, f, F)$ if and only if the inequality is valid and (at least) one of the following conditions is satisfied:

i. for some $Q_l \subseteq Q$, $|Q_l| = d - t$, $t > 1$, there exists some set $R \subseteq T \cup (Q \setminus Q_l)$ such that $|R| = |T| + t - 1$ and $f(R) \le F_k$;

ii. there exists some $i \in Q$ and $t \in T$ such that $f(T \setminus \{t\} \cup \{s, i\}) \le F_k$.

■

Remark 4.8 Theorems **??**, **??** and **??** can be slightly modified in order to apply to the generalized assignment polytope. This is due to the fact that the assumptions for any of these theorems as well as the conditions for the corresponding inequalities to be valid or facet-defining, respectively, only depend on properties of the item weights with respect to *one* particular knapsack. For example, in Theorem **??** the condition for the extended inequality to be valid is: "for every $\tilde{T}_v \subseteq T_v$, $|\tilde{T}_v| = \beta_v$ and for all $i \in A$, the set $\tilde{T}_v \cup \{i\}$ defines a cover with respect to k_v". Yet, the property $\tilde{T}_v \cup \{i\}$ defines a cover with respect to k_v only depends on the item weights of the set $\tilde{T}_v \cup \{i\}$ with respect to the particular knapsack k_v. The corresponding counterpart of Theorem **??** can be formulated as follows:

*Let (N, M, f, F) be an instance of the generalized assignment problem and assume that the inequality $a^T y \le \alpha$ is facet-defining for the polytope $GAP(\bigcup_{u=1}^t A_u \times B_u, f, F)$ $(A_u \subseteq N, B_u \subseteq M, u = 1, \dots, t)$ and satisfies the requirement $(\star)$ in Theroem **??**. We set $A := \bigcup_{u=1}^t A_u$, $B := \bigcup_{u=1}^t B_u$ and choose a positive integer $r \le \min\{|N \setminus A|, |M \setminus B|\}$, nonempty sets $T_1, \dots, T_r$ that are mutually disjoint subsets of $N \setminus A$ and a subset $\{k_1, \dots, k_r\}$ of $M \setminus B$.*

We define $\beta_v := \max\{|G| \mid G \subseteq T_v, \sum_{i \in G} f_{ik_v} \le F_{k_v}\}, v = 1, \dots, r$. Then, the inequality

$$a^T x + \sum_{v=1}^{r} \sum_{i \in A \cup T_v} x_{ik_v} \le \alpha + \sum_{v=1}^{r} \beta_v$$

is valid for the polytope $GAP((\bigcup_{u=1}^{t} A_u \times B_u) \cup (\bigcup_{v=1}^{r}(T_v \cup A) \times \{k_v\}), f, F)$ *if and only if* $\tilde{T}_v \cup \{i\}$ *is a cover with respect to* k_v *for all* $i \in A$, $\tilde{T}_v \subseteq T_v$, $|\tilde{T}_v| = \beta_v$ *and* $v = 1, \ldots, r$.

Theorems **??** and **??** can be formulated appropriately, in order to apply to the generalized assignment problem. ■

5 Conclusions

In this paper we studied the multiple knapsack problem from a polyhedral point of view. We presented several classes of valid and facet-defining inequalities. Some of these classes were obtained by applying the theorems of section 4 which, in addition, allow for an iterative generalization of minimal cover or minimal multiple cover inequalities. The inequalities discussed in this paper are the starting point for the development of a cutting plane algorithm which is discussed in our compagnion paper (see [**?**]).

References

[B75] E. Balas, "Facets of the Knapsack Polytope", *Mathematical Programming* 8, 146 - 164 (1975).

[BZ78] E. Balas and E. Zemel, "Facets of the Knapsack Polytope from Minimal Covers", *SIAM Journal on Applied Mathematics* 34, 119 - 148 (1978).

[CJP83] H. Crowder, E. L. Johnson and M. W. Padberg, "Solving Large-Scale Zero-One Linear Programming Problems", *Operations Research* Vol 31, No. 5, 803 - 834 (1983).

[F93] C. E. Ferreira, "A Combinatorial Optimization Investigation for an Application in Computer Systems Design", *PhD. Dissertation*, Technische Universität Berlin, in preparation.

[FMW93] C. E. Ferreira, A. Martin and R. Weismantel, "A Cutting Plane Based Algorithm for the Multiple Knapsack Problem", Konrad-Zuse-Zentrum für Informationstechnik Berlin, Preprint SC 93-7, 1993.

[FGKKMW92] C. E. Ferreira, M. Grötschel, S. Kiefl, C. Krispenz, A. Martin and R. Weismantel, "Some integer programs arising in the design of mainframe computers", Konrad-Zuse-Zentrum für Informationstechnik Berlin, Preprint SC 92-24, 1992.

[GR90] E. S. Gottlieb e M. R. Rao, "The Generalized Assignment Problem: Valid Inequalities and Facets", *Mathematical Programming* 46, 31 - 52 (1990).

[GR90a] E. S. Gottlieb e M. R. Rao, "(1,k)-Configuration Facets for the Generalized Assignment Problem", *Mathematical Programming* 46, 53 - 60 (1990).

[HJP75] P. L. Hammer, E. L. Johnson and U. N. Peled, "Facets of Regular 0-1 Polytopes", *Mathematical Programming* 8, 179 - 206 (1975).

[JMRW92] M. Jünger, A. Martin, G. Reinelt and R. Weismantel, "Quadratic 0/1 Optimization and a Decomposition Approach for the Placement of Electronic Circuits", Konrad-Zuse-Zentrum für Informationstechnik Berlin, Preprint SC 92-10, to appear in Mathematical Programming (1992).

[K72] R. M. Karp, "Reducibility among Combinatorial Problems", in R. E. Miller and J. W. Thatcher (eds.), *Complexity of Computer computations*, Plenum Press, New York, 85 - 103 (1972).

[MT91] S. Martello and P. Toth, *Knapsack Problems - Algorithms and Computer Implementations*, John Wiley and Sons, New York (1991).

[P75] M. W. Padberg, "A Note on 0-1 Programming", *Operations Research* 23, 833 - 837 (1975).

[P80] M. W. Padberg, "(1,k)-Configurations and Facets for Packing Problems", *Mathematical Programming* 18, 94 - 99 (1980).

[We92] R. Weismantel, "Plazieren von Zellen: Analyse und Lösung eines quadratischen 0/1-Optimierungsproblems", *PhD. Dissertation*, Technische Universität Berlin (1992).

[W75] L. A. Wolsey, "Faces of Linear Inequalities in 0-1 Variables", *Mathematical Programming* 8, 165 - 178 (1975).

[W90] L. A. Wolsey, "Valid Inequalities for 0-1 Knapsacks and MIPS with Generalised Upper Bound Constraints", *Discrete Applied Mathematics* 29, 251 - 261 (1990).

[Z86] E. Zemel, "On the Computational Complexity of Facets for the Knapsack Problem", *Working Paper* 713, The Center for Mathematical Studies in Economics and Management Sciences, Northwestern University (1986).