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Advancements in Quantitative Investments: Shedding Light on the  
Risk Parity Portfolio

**Dissertation**

**for obtaining the degree of Doctor of Business and Economics  
(Doctor Rerum Politicarum – Dr. rer. pol.)**

**WHU – Otto Beisheim School of Management**

APRIL 2021

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”In the short-run, the market is a voting machine – reflecting a voter-registration test that requires only money, not intelligence or emotional stability – but in the long-run, the market is a weighing machine.” *Benjamin Graham*





# Acknowledgements

First and foremost, I would like to express my sincere gratitude to Professor Dr. Markus Rudolf, who gave me the opportunity to partake in the life of academic research at the WHU and provided me with the flexibility to conduct my research freely as well as for his consistent support and guidance during this research. His insightful advice is whole-heartedly appreciated, without him this dissertation would not have been possible. A special thanks to Professor Dr. Jürgen Weigand, who agreed to act as my second advisor and whose questions and remarks further enhanced the quality of my research.

I would also like to thank the colleagues at Vescore by Vontobel Asset Management AG for their support and funding, especially Dr. Katrin Baedorf who laid the groundwork for this collaboration and Dr. Joséphine Molleyres for the wonderful assistance and insightful dialogues. I am also thankful to the colleagues at Berenberg Quantitative Asset Management by Joh. Berenberg, Gossler & Co. KG for their support and helpful remarks regarding the current state and challenges of the investment management industry, especially Dr. Denisa Cumova for offering this opportunity and Dr. Christoph Frey for the engaging and fruitful discussions.

Moreover, I would like to thank my colleagues at the Allianz Endowed Chair of Finance for their valuable contributions, especially Tobias Burggraf, Konstantin Ignatov, Dr. Alex Kusen and Dr. Marc Wierzbitzki for providing helpful feedback and their loyal assistance. Due to my colleagues, I enjoyed a kind and productive atmosphere at the chair, which was an extremely beneficial asset.

Finally, I am eternally grateful for the unconditional love and everlasting support of my family, especially my parents who made immense sacrifices to ensure the safety and continued success of our family and are my main source of motivation, without them I would not be where I am today. This dissertation is dedicated to them.



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# List of Abbreviations

<b>AMEX</b>	The American Stock Exchange
<b>BM</b>	60/40 stock/bond benchmark portfolio
<b>CAPM</b>	Capital Asset Pricing Model
<b>CO</b>	Commodities
<b>CORR</b>	Risk parity correlation component portfolio
<b>CVaR</b>	Conditional value at risk
<b>DM</b>	Developed markets
<b>EM</b>	Emerging markets
<b>EW</b>	Equally weighted portfolio
<b>G7</b>	Great seven
<b>GARCH</b>	Generalized autoregressive conditional heteroscedasticity
<b>HHI</b>	Herfindahl-Hirschman Index
<b>LRP</b>	Leveraged risk parity portfolio
<b>MDD</b>	Maximum drawdown
<b>MRC</b>	Marginal risk contribution
<b>MSCI</b>	Morgan Stanley Capital International
<b>MV</b>	Minimum variance portfolio
<b>MVCCM</b>	Minimum variance portfolio with linear covariance matrix shrinkage using the constant correlation model

<b>MVID</b>	Minimum variance portfolio with linear covariance matrix shrinkage using the identity matrix model
<b>MVNLS</b>	Minimum variance portfolio with non-linear covariance matrix shrinkage
<b>NASDAQ</b>	The National Association of Securities Dealers Automated Quotations exchange
<b>NBER</b>	National Bureau of Economic Research
<b>NYSE</b>	The New York Stock Exchange
<b>RCR</b>	Risk contribution ratio
<b>RP</b>	Risk parity portfolio
<b>RPCCM</b>	Risk parity portfolio with linear covariance matrix shrinkage using the constant correlation model
<b>RPID</b>	Risk parity portfolio with linear covariance matrix shrinkage using the identity matrix model
<b>RPNLS</b>	Risk parity portfolio with non-linear covariance matrix shrinkage
<b>TC</b>	Transaction costs
<b>TIMING</b>	Risk parity component timing portfolio
<b>TRC</b>	Total risk contribution
<b>USD</b>	United States dollar
<b>VaR</b>	Value at risk
<b>VIF</b>	Variance inflation factor
<b>VOLA</b>	Inverse volatility portfolio / Risk parity volatility component portfolio

# Introduction

After the stock market crash in 2008, a new portfolio strategy began gaining ever increasing attention in academia and practice. This strategy is referred to as risk parity, which in its simplest definition allocates wealth to equalize the amount of volatility contributed by each asset to the overall portfolio volatility. In practice, risk parity has been shown to outperform not only traditional portfolio weighting schemes such as equal- or value-weighting, but also other weighting techniques such as minimum variance (Chaves et al., 2011; Clarke et al., 2013). In academia, the rise in interest was due to the attractive portfolio risk-return profile achieved by risk parity (Prince, 2011), and its flexibility in terms of matching the risk appetite of investors as it can easily be leveraged to achieve specific portfolio volatility levels (Qian, 2012). Lee (2011) argues that risk parity is the most applicable portfolio technique among risk-based techniques due to: (1) its heuristic framework, which is an ad-hoc deconcentration approach on a risk basis that is similar in notion to the equally-weighted portfolio which is an ad-hoc deconcentration allocation technique on a monetary basis; (2) simple underlying economic intuition and financial interpretation of monetary losses (Qian, 2006). Accordingly, numerous pension plans and investment managers adopted this approach, resulting in over \$175 billion in assets under management as of May 2018 (Kochkodin, 2018).

Nonetheless, risk parity faced a lot of scrutiny (Foresti and Rush, 2010; Inker, 2010; Levell, 2010). Lee (2011) argues that risk parity faces numerous challenges, despite its outperformance of other traditional portfolio strategies from a risk-adjusted performance measures. Perhaps the most important challenge is the lack of a theoretical framework that helps explain and predict its performance in varying market conditions and relative to conventional portfolio benchmarks.

The objective of this dissertation thesis is to investigate the risk parity approach in more detail and contribute to existing literature by exploring the underlying theoretical foundations that result in its observed outperformance in empirical studies.

This dissertation aims to illustrate the relationship of the performance of risk parity to longstanding behavioral biases and market anomalies, draw a clear picture regarding the merits and perils of this approach, and demonstrate how risk parity can be improved.

The first chapter proposes a novel approach to help explain the characteristics of risk parity portfolios. Similar to Asness et al. (2019), we decompose the risk parity portfolio into a volatility and a correlation component portfolio, which respectively rely on the volatility vector and the correlation matrix to equalize the asset total risk contributions and are equivalent to risk parity under specific assumptions. We investigate the relationship between risk parity and its component portfolios and whether risk parity can be replicated using a combination of the component portfolios. Finally, we assess how the risk parity component portfolios can be utilized to improve the risk-adjusted return whilst attaining the properties of risk parity.

The second chapter conducts an empirical analysis of the diversification benefits achieved from utilizing a variety of asset classes bearing different risk-return characteristics for risk parity portfolios and compare them with the benefits present in other conventional heuristic and risk-based portfolio strategies. We further investigate the implications of leveraging risk parity portfolios to achieve a specific portfolio volatility level and analyze the resulting advantages and disadvantages.

The third chapter of this dissertation investigates the effects of shrinkage estimation techniques of asset price moments on the risk-return profile of risk parity. In contrast to mean-variance portfolios, the risk contributions of individual assets in risk parity portfolios are fixed a priori. This additional restriction is commonly found to stabilize empirical portfolio weights in time. We investigate whether the risk-budget for each portfolio asset acts as a natural shrinkage target theoretically and empirically. Furthermore, we conduct an extensive empirical analysis of how risk parity fares in comparison to other prominent weighting strategies as well as asset variance-covariance matrix shrinkage techniques in various settings.

Altogether, the aim of dissertation is not only to contribute to earlier findings in the literature and shed more light on the theoretical properties of risk parity, but hopefully also to help portfolio managers obtain a better understanding of the advantages and disadvantages of risk parity portfolios relative to other weighting schemes.

# Chapter 1

## The Value-Drivers of Risk Parity and its Component Portfolios <sup>1</sup>

### 1.1 Introduction

Risk parity ( $RP$ ) portfolios have been gaining increasing attention in academia and practice<sup>2</sup>.  $RP$  portfolios aim to distribute the portfolio volatility equally among assets, so that each asset contributes the same amount of risk to the overall portfolio volatility. Also,  $RP$  can be leveraged to match investor preferences and to achieve a desired target return or volatility level. Lee (2011) praises the  $RP$  approach among other traditional and risk-based weighting strategies as the most applicable in real markets due to its simplistic approach as an ad-hoc risk deconcentration weighting technique (Amenc et al., 2012) and its financial interpretation in terms of monetary losses (Qian, 2006). However, Lee (2011) argues that despite the aforementioned appealing properties of  $RP$ , it faces several challenges. Most prominently is the lack of an underlying theory or framework that helps explain its performance.

The aim of this study is to shed more light on the performance of  $RP$  by investigating its component portfolios<sup>3</sup>, namely the volatility ( $VOLA$ ) and the correlation ( $CORR$ ) component portfolios, which achieve the characteristics of  $RP$  under specific conditions. The approach is similar to Asness et al. (2019), who decompose the Betting-Against-Beta ( $BAB$ ) factor of Frazzini and Pedersen (2014) into the

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<sup>1</sup>This chapter is based on Alkafri (2020). Risk Parity 2.0. *Institutional Money 2*, 118-126.

<sup>2</sup>For example, the increased interest for risk parity portfolios has led to the launch of several risk parity market indices by HFR in 2017 (O’Dea, 2017) and S&P in 2018 (Liu, 2018)

<sup>3</sup>We dub these portfolios the component portfolios since they attempt to achieve the characteristics of  $RP$  by relying on either the volatility component or the correlation component of the covariance matrix, which  $RP$  utilizes to equally distribute risk.

Betting-Against-Volatility (*BAV*) and Betting-Against-Correlation (*BAC*) factors. We assess the relationship of *RP* to its components and provide arguments as to why *RP* is driven the *BAB* factor of Frazzini and Pedersen (2014) and is closely related to its components, *VOLA* and *CORR*, which are in return driven by the the low volatility anomaly and the *BAC* factor of Asness et al. (2019).

Furthermore, we find that *RP* and *VOLA* display similar risk-return characteristics whereas *CORR* exhibits a slight improvement over the equally-weighted portfolio. *RP* does not deliver a significant alpha over its components and almost the entire variation of its returns is essentially capture by *VOLA* and *CORR*. Moreover, we show that *RP* can be replicated by a linear combination of *VOLA* and *CORR* and that utilizing these component portfolios can significantly improve the risk-adjusted performance while maintaining the properties of *RP* to some extent.

The remainder of this chapter is organized as follows. Section 1.2 describes the data set employed in the study. The methodology as well as the value-drivers of *RP*, *VOLA*, and *CORR* are discussed in section 1.3. The empirical results are summarized in section 1.4 and verified in section 1.5, while section 1.6 briefly demonstrates methods of improving *RP* portfolios based on the previous results. Finally, section 1.7 concludes.

## 1.2 Data Description

Data were collected from Datastream and consist of daily total return prices from 01.01.1994 to 30.09.2019. We focus our analysis mainly on an asset allocation perspective as the variation of volatilities and correlations between different asset classes is greater than the variations within the same asset class. As in Rudolf (1994), we restrict the main analysis to three U.S. asset classes, namely equities, bonds, and commodities, to simplify the analysis and ensure tractability as well as the replication of our results by future studies. Equities are represented by the S&P 500 Index, bonds are denoted by the Thomson Reuters U.S. 10-year Government Benchmark Index, and commodities are described by the Bloomberg Commodity Index. The 1-year U.S. Dollar deposit rate is utilized as a proxy for the risk-free asset.

## 1.2. DATA DESCRIPTION

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We employ an out-of-sample rolling-window approach as in DeMiguel et al. (2009). Provided a data set which consists of  $T = 6,717$  business days, we estimate the asset parameters required to construct the portfolios based on the previous  $M = 250$  days. Portfolios are held for  $k = 1$  days and then rebalanced using the previous  $M$  days to estimate the asset moments. Hence, we obtain  $T - M = 6,467$  observations of periodic out-of-sample returns. Table 1.1 reports the summary statistics of the asset class return series.

**Table 1.1:** Summary Statistics of Asset Classes

Asset	Equities	Bonds	Commodities	Risk Free
Return	8.78	4.91	4.86	2.16
Volatility	17.76	7.14	14.78	0.09
Skewness	-9.46	-2.83	-6.07	0.00
Kurtosis	-0.27	-0.13	-0.19	0.28
Sharpe Ratio	0.37	0.38	0.18	0.00

*Notes:* This table provides the return moments and Sharpe Ratios of the U.S. equities, bonds, commodities, and the risk-free rate. The sample period covers 6,717 days from 01.01.1994 to 30.09.2019. All measures are in annual terms.

Equities deliver an average annual return of roughly 9% but is the most volatile (17.76%) of all asset classes. Bonds seem to be the least volatile asset class (7.14%), while delivering a slightly lesser return (4.86%) than commodities, which yield an average annual return of 4.86% with a volatility of 14.78%. Moreover, all assets exhibit negatively skewed and leptokurtic returns, indicating the stark presence of extreme returns that occur during market turbulences. Although commodities exhibit a volatility similar to equities, the returns of commodities are much lower and similar to those of bonds in terms of excess kurtosis. The Sharpe ratios of equities and bonds display a similar risk premium of almost 0.40, while commodities yield a mere 0.18. Lastly, the risk-free rate is positively skewed and platykurtic with an average return of 2.16% and a volatility of 0.09%.

## 1.3 Methodology

### Risk Parity

The traditional view in finance has been coined by the mean-variance framework of Markowitz (1952), and most prominently the Capital Asset Pricing Model (CAPM), which implies a positive linear relationship between an asset's beta and its expected return. This has led to the extensive use of value- and equally-weighted portfolios in practice (Benartzi and Thaler, 2001). Recently, there has been a shift in the literature which dates back to the zero-beta CAPM by Black et al. (1972), who find that the risk-return relationship implied by CAPM becomes flatter when investors face restricted borrowing and lending. Haugen and Heins (1975) confirm a flatter CAPM line and find that in some cases the risk-return relationship can become negative. Blitz and van Vliet (2007) as well as Baker and Haugen (2012) find that low volatility assets outperform high volatility assets, Frazzini and Pedersen (2014) show that low beta assets outperform high beta assets, and Asness et al. (2019) find that assets with lower average correlations to other securities in the market outperform assets with higher average correlations.

As a result, a new paradigm in finance and the practice of portfolio management has emerged, which aims at managing risk budgets of portfolio assets and is referred to as the risk-budgeting approach (Bruder and Roncalli, 2012). Perhaps the most important technique thereof is risk parity ( $RP$ ), which equalizes the amount of risk contributed by each asset to the portfolio volatility.  $RP$  has gained increasing attention in academia and practice since it can be adjusted to match investor risk preferences and leveraged to achieve specific portfolio return or volatility levels.

Numerous studies such as Chaves et al. (2011) and Lee (2011) find that risk-based strategies, such as  $RP$ , outperform traditional weighting techniques such as value- and equal-weighting. However, Lee (2011) argues that although  $RP$  empirically yields a higher risk-adjusted return than traditional weighting schemes, it poses several challenges. Most prominently is the lack of an underlying theory or framework that helps predict its performance relative to its benchmark. Also, unless the optimality conditions of  $RP$  are truly present in capital markets, the parity condition is deemed as merely a starting point when stronger investment views are not present.

Qian (2005) was the first to introduce the concept of  $RP$ , who shows that  $RP$  portfolios are mean-variance efficient if the underlying components have equal Sharpe



ratios and if their returns are not correlated. The author argues that the former condition is appealing since it implies that the expected return is proportional to the risk of each asset class, which is the case according to the Capital Asset Pricing Model (CAPM). Regarding the latter condition, Qian (2005) finds that the historical correlation between stocks and bonds was quite low, albeit not being zero. Thus, the author concludes that *RP* portfolios should be not only efficient in terms of risk allocation but also according to the classical mean-variance framework. Thereafter, Maillard et al. (2010) coined the *RP* weighting technique formally. In the following, the main concepts of *RP* according to Maillard et al. (2010) are summarized.

Let  $\sigma_p = \sqrt{\sum_{i=1}^n w_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n w_i w_j \sigma_{i,j}}$  be the volatility of the portfolio, where  $w_i$  ( $w_j$ ) denotes the weight of asset  $i$  ( $j$ ),  $\sigma_i^2$  describes the return variance of asset  $i$ , and  $\sigma_{i,j}$  represents the covariance between the returns of assets  $i$  and  $j$ . The total risk contribution of asset  $i$  ( $TRC_i$ ) is defined as the product of the weight of asset  $i$  ( $w_i$ ) and its marginal risk contribution ( $MRC_i$ ), where  $MRC_i$  is the first partial derivative of portfolio volatility to the weight of asset  $i$  ( $w_i$ ) and represents the change in portfolio volatility as a result of an infinitesimal increase in the weight of asset  $i$ . Formally, these are defined by

$$MRC_i = \frac{\partial \sigma_p}{\partial w_i} = \frac{w_i \sigma_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^n w_j \sigma_{i,j}}{\sigma_p} = \frac{\sigma_{i,p}}{\sigma_p} \quad (1.1)$$

$$TRC_i = w_i \times MRC_i = \frac{w_i \sigma_{i,p}}{\sigma_p}, \quad (1.2)$$

where  $\sigma_{i,p}$  describes the covariance of asset  $i$  with the portfolio. Accordingly, the following risk decomposition can be obtained<sup>4</sup>

$$\sigma_p = \sum_{i=1}^n TRC_i. \quad (1.3)$$

Hence, portfolio volatility can be expressed as the sum of all *TRCs*. Based on the definitions provided in Equations (1.1)–(1.3), *RP* can be characterized by a state, in which the vector of asset weights ( $\mathbf{w}$ ) is selected such that the *TRCs* of all assets  $1 \leq i \leq n$  are equal, or formally  $TRC_i = TRC_j = \sigma_p/n \quad \forall 1 \leq i, j \leq n$ . This can be realized by the following minimization problem

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<sup>4</sup>Pursuant to the Euler principle, portfolio risk can be decomposed by Value-at-Risk (Gourieroux et al., 2000), the conditional Value-at-Risk or Expected Shortfall (Scaillet, 2004), and by any further homogenous risk measure that is linear (Tasche, 2004).

$$\begin{aligned}
w^* &= \arg \min \sum_{i=1}^n \sum_{j=1}^n (TRC_i - TRC_j)^2 \\
&\text{subject to } \mathbf{1}'\mathbf{w} = 1 \\
&\mathbf{0} \leq \mathbf{w} \leq \mathbf{1},
\end{aligned} \tag{1.4}$$

where  $\mathbf{w}$  is the vector of asset weights, and  $\mathbf{0}$  and  $\mathbf{1}$  are vectors of zeroes and ones, respectively. Moreover, the weights of  $RP$  satisfy the following property<sup>5</sup>

$$w_i = \frac{\beta_{i,p}^{-1}}{\sum_{j=1}^n \beta_{j,p}^{-1}} = \frac{1}{n\beta_{i,p}}, \tag{1.5}$$

where  $\beta_{i,p}$  describes the beta of asset  $i$  to the portfolio. Equation 1.5 shows that  $w_i$  is a function of  $\beta_{i,p}$ , where the weight of asset  $i$  is inversely proportional to its beta.

Frazzini and Pedersen (2014) show that a portfolio which is long in low beta stocks and short in high beta stocks, which they call the Betting-Against-Beta ( $BAB$ ) factor, delivers significant positive abnormal returns. The authors provide an equilibrium model with leverage and margin constraints that asserts these results, where investors purchase high beta stocks as they are expected to deliver higher returns<sup>6</sup>. The increased demand for high beta asset causes them to be overpriced. Meanwhile, low beta assets become relatively underpriced and the resulting mispricing is not arbitrated away as investors face leverage and funding constraints<sup>7</sup>. The authors document significant positive returns for the  $BAB$  factor across several asset classes and countries. Clarke et al. (2013), using a 1,000 stocks example, show that the asset weights of the  $RP$  portfolio are indeed inversely related to the assets' market beta and thereby confirming that Equation 1.5 is fulfilled in real life<sup>8</sup>. Hence,  $RP$  can be viewed as a long-only implementation of the  $BAB$  factor, as it rewards low beta assets and penalizes high beta assets.

A further advantage of risk-based strategies, such as  $RP$ , lies in the input parameters needed to determine asset weights. Chaves et al. (2011) argue that  $RP$  is

<sup>5</sup>For a derivation of Equation 1.5 see Appendix section A.1 and Maillard et al. (2010), p. 63.

<sup>6</sup>The low beta anomaly is present in U.S. corporate bonds, U.S. government bonds, and futures markets. See Derwall et al. (2009) and Frazzini and Pedersen (2014).

<sup>7</sup>The effect of financial constraints on asset prices is not novel and has been demonstrated by several studies. Ashcraft et al. (2010) and Gârleanu and Pedersen (2011) show assets with higher margin required are associated with higher required returns according to Margin CAPM. Gromb and Vayanos (2002) show that financial constraints influence assets' required rate of return, Brunnermeier and Pedersen (2009) link the market liquidity of financial assets to the funding liquidity of investors, and Acharya and Pedersen (2005) show that an asset's market liquidity influences its expected rate of return.

<sup>8</sup>See Clarke et al. (2013), Exhibit 5, p. 47.

superior over mean-variance portfolios since only the covariance matrix is needed to determine  $RP$  weights. Numerous studies such as Merton (1980) and Chopra and Ziemba (1993) have shown that the asset covariance matrix can be estimated more accurately than asset returns based on historical data. Also, Chopra and Ziemba (1993) show that for mean-variance portfolios, estimation errors in variances and covariances have a much lower impact on the portfolio optimization process than estimation errors in asset returns, regardless of the level of investor risk-aversion.

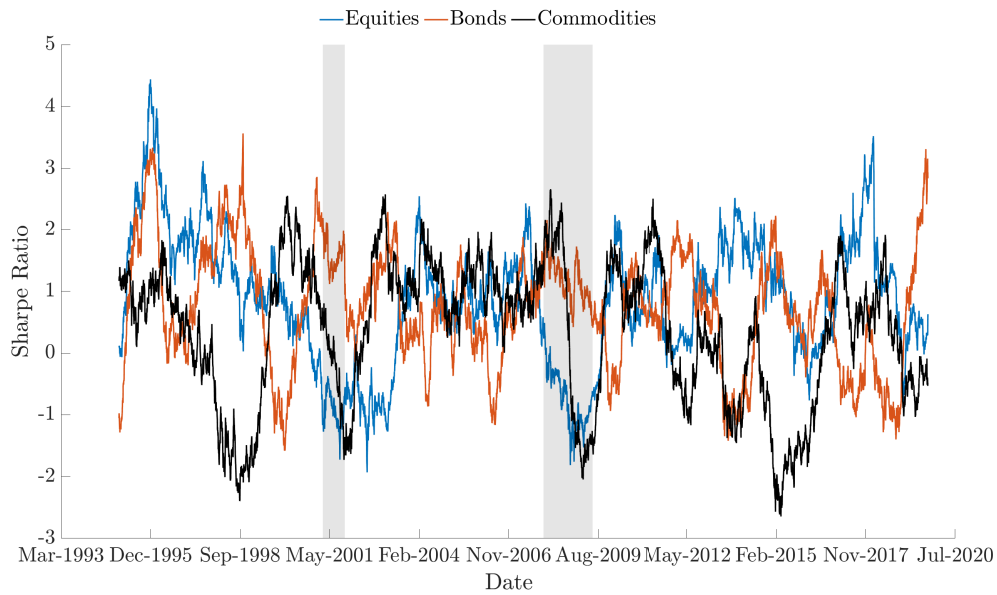
Risk-based strategies are also a special case of volatility weighting strategies that attempt to smooth the overall portfolio volatility over time. Hallerbach (2014) shows that implementing such strategies to smooth the overall portfolio volatility over time leads to a higher information ratio when market volatility varies over time. This result is not novel and has been shown by numerous studies such as Booth and Fama (1992) and Fernholz et al. (1998). Volatility smoothing strategies have also been shown to result in a higher Sharpe Ratio by studies such as Giese (2012), Ilmanen and Kizer (2012), and Kirby and Ostdiek (2012).

Optimality arguments provide another motivation for  $RP$ . Lindberg (2009) shows that equalizing quantities closely related to asset risk contributions solves Markowitz's continuous-time portfolio optimization problem of Merton (1971), if stock prices are driven by Brownian motions with positive drift rates. Hallerbach (2012) shows that risk-based weighting schemes result in a higher Sharpe Ratio and information ratio, and that the overall risk-adjusted performance increases with the degree of volatility smoothing achieved over time<sup>9</sup>. Chaves et al. (2011) confirm these results empirically, where they find that the ex-post risk allocation, as measured by the  $TRC$ , is much more balanced for  $RP$  than equally-weighted or mean-variance portfolios, albeit not being exactly equal across asset classes. Meanwhile, the minimum variance portfolio typically is extremely concentrated in the least volatile asset<sup>10</sup>.

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<sup>9</sup>However, Hallerbach (2012) stresses that implementing volatility weighting strategies requires a strict risk budgeting and monitoring process, especially strategies such as the minimum variance portfolio necessitate setting maximum weights so that the risk of portfolio blow-ups is mitigated when the contemporaneous volatility is somewhat low.

<sup>10</sup>Generally, highly concentrated portfolios have a higher turnover ratio, which translates to higher transaction costs when rebalancing the portfolio and, hence, lower net portfolio returns.

**Figure 1.1:** Rolling Sharpe Ratios of Asset Classes

*Notes:* This figure shows the historical rolling Sharpe ratios of equities, bonds, and commodities. The gray shaded areas indicate U.S. recessions as reported by NBER. The sample period covers the period from 01.01.1994 to 30.09.2019 and consists of 6,399 observations. All estimates are in annual terms.

However, despite numerous arguments in favor of risk parity, the results from Figure 1.1, which depicts the rolling Sharpe ratios of various asset classes, suggest that  $RP$  is not mean-variance efficient, as asset classes exhibit varying Sharpe ratios over time and are seldom equal. As a result, several questions arise such as which components help  $RP$  outperform traditional weighting schemes? And since  $RP$  is not mean-variance efficient, can the risk-return profile be improved by tilting it towards a certain component? To answer these questions, we begin by assessing the properties of the  $RP$  component portfolios, namely the volatility and the correlation component portfolios, which are special cases of  $RP$  and rely on either asset volatilities or correlations to equalize asset  $TRCs$ .

## Components of Risk Parity

Although the covariance matrix is required to determine the *RP* asset weights, the covariance matrix has several drawbacks compared to the volatility vector and the correlation matrix. The asset variance-covariance matrix consists of squared deviations, which exhibit starker changes than volatilities and correlations given higher fluctuations in asset prices. The properties of these statistics, in addition to the rolling annualized returns<sup>11</sup>, are shown in Figure 1.2. Covariances generally fluctuated around zero and were very small in magnitude until the financial crisis of 2008, where they exploded during the crisis and then decreased in magnitude, albeit remaining greatly starker than the changes in covariances before 2008. Meanwhile, the overall changes in volatilities and correlations were more pronounced and remained relatively similar in terms of magnitude throughout the whole sample. In sum, the results imply that weighting schemes that utilize the asset covariance matrix to determine the asset weights might be more prone to huge allocation shifts during turbulent market phases than portfolios that employ either the asset volatility vector or correlation matrix.

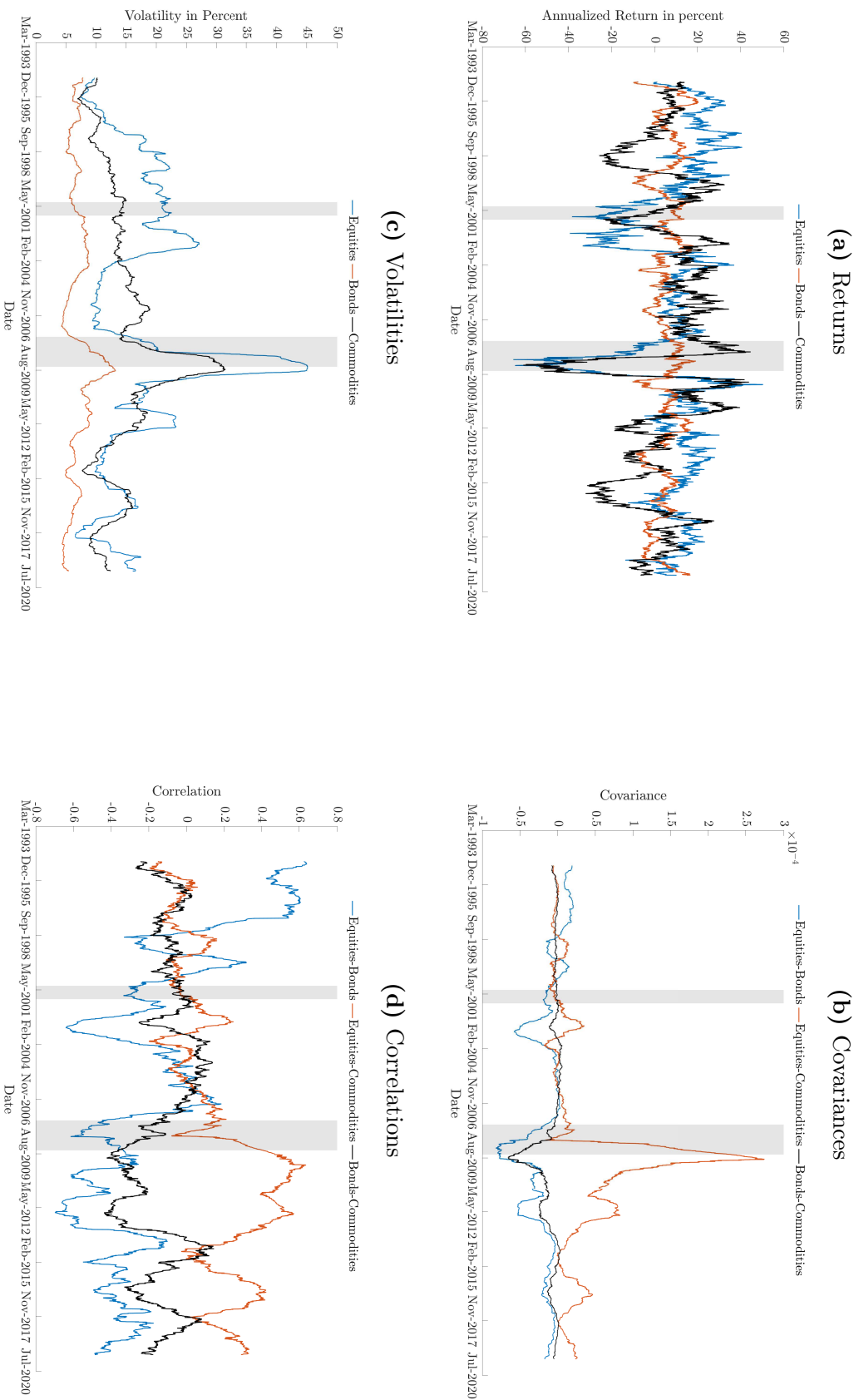
Moreover, not only were the Sharpe ratios of equities, bonds, and commodities generally not equal over time, but the same holds true for asset volatilities, correlations, and covariances. Nevertheless, in certain periods the characteristics of various asset classes cluster around a certain value and/or exhibit somewhat similar patterns. For example, after the financial crisis of 2008, the changes in volatility of equities and commodities were fairly similar, while the ratio of bond volatility to commodity volatility has roughly been stable over time at roughly a factor of two. The correlation of bonds with equities and commodities was generally negative and somewhat similar in magnitude, whereas the correlation of equities and commodities was overall positive and moved in opposite direction of the bond correlations. Since asset volatilities and/or correlations tend to display similar patterns in certain periods of time, allocation techniques that use either the asset volatilities or correlations to determine asset weights are likely to be less volatile with regard to changes in asset weights than strategies employing the covariance matrix.

In the following, we present the volatility (*VOLA*) and the correlation (*CORR*) component portfolios and discuss the market anomalies that are in favor of utilizing these portfolios. Note that both portfolios have been briefly introduced by Maillard et al. (2010), although they mainly investigate the *RP* portfolio.

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<sup>11</sup>We depict the rolling annualized returns for the sake of completion.

Figure 1.2: Rolling Statistics of Asset Classes



Notes: This figure shows the historical rolling returns and volatilities of equities, bonds, and commodities as well as the rolling covariances and correlations among these asset classes. The gray shaded areas indicate U.S. recessions as reported by NBER. The sample period covers the period from 01.01.1994 to 30.09.2019 and consists of 6,717 return observations. All estimates are in annual terms.

## Volatility Component Portfolio

The volatility component portfolio (*VOLA*), also known as the inverse volatility portfolio, relies solely on asset volatilities to equalize asset *TRCs*. Maillard et al. (2010) show that *VOLA* is a special case of *RP* and is equivalent to *RP* under the assumption that all asset correlations are equal<sup>12</sup>. Formally, *VOLA* is defined as

$$w_i = \frac{1/\sigma_i}{\sum_{j=1}^n 1/\sigma_j}, \quad (1.6)$$

where  $w_i$  is the weight of asset  $i$  with  $i = 1, \dots, n$ , while  $\sigma_i$  and  $\sigma_j$  denote the standard deviations of assets  $i$  and  $j$ , respectively. Equation 1.6 shows that the weight of asset  $i$  is inversely related to its volatility. Hence, *VOLA* can be viewed as a long only implementation of the low volatility anomaly as it overweights low volatility stocks and underweights high volatility stocks.

The low volatility anomaly was first documented by Haugen and Heins (1975) and refers to the empirical observation that low volatility assets deliver higher returns than high volatility assets<sup>13</sup>. The low volatility anomaly can be attributed to violations of the assumptions underlying the Capital Asset Pricing Model (*CAPM*) in financial markets. Blitz and van Vliet (2007) as well as Blitz et al. (2014) show that numerous assumptions of *CAPM* are violated in real life such as different lending and borrowing rates, investor margin constraints, investor utility functions, information availability, and investor rationality, which all support the existence and persistence of the low volatility anomaly<sup>14</sup>.

Another rationale for *VOLA* is based on behavioral biases related to the preference for lottery-like stocks, which are positively skewed and highly volatile stocks with a high probability of a huge positive return (lottery-like payoff). Barberis and Huang (2008) develop an equilibrium model based on Kahneman and Tversky's (1992) cumulative prospect theory and find that lottery-like assets can become overpriced and, thus, yield negative average excess returns. The overpricing of lottery-like stocks cannot be fully arbitrated away due to short-selling constraints<sup>15</sup>.

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<sup>12</sup>For more information on the derivation of this results see Appendix section A.2.

<sup>13</sup>The low volatility anomaly is robust across equity markets (Baker and Haugen, 2012), in numerous asset classes including exotic derivatives (Falkenstein, 2009), to industry effects (Asness et al., 2014), and is robust when controlling for the Fama French size factor (Baker and Haugen, 2012), the value factor and the Carhart momentum factor (Blitz and van Vliet, 2007), as well as several other factors such as leverage and trading volume, among others (Ang et al., 2006).

<sup>14</sup>For a thorough literature review on the low volatility anomaly see Blitz et al. (2020).

<sup>15</sup>Mispricing of lottery-like assets increases in economic turmoils (Kumar, 2009), is greater for assets with high limits to arbitrage (Conrad et al., 2014), and is robust when controlling for the

## Correlation Component Portfolio

The correlation component portfolio (*CORR*) equalizes the asset *TRC*s based on the correlation matrix. Maillard et al. (2010) show that *CORR* is a special case of *RP* and is equivalent to *RP* under the assumption that all asset volatilities are equal. The asset weights are defined as follows

$$w_i = \frac{(\sum_{k=1}^n w_k \rho_{i,k})^{-1}}{\sum_{j=1}^n (\sum_{k=1}^n w_k \rho_{j,k})^{-1}}, \quad (1.7)$$

where  $\rho_{j,k}$  ( $\rho_{i,k}$ ) denotes the correlation between assets  $j$  and  $k$  ( $i$  and  $k$ ). *CORR* overweights assets with low average correlations and underweights assets with high average correlations. However, due to the endogeneity of Equation 1.7, the weights of *CORR* need to be obtained through iterative optimization.

*CORR* can be viewed as an implementation of the findings of Asness et al. (2019) that assets with low average correlations outperform assets with high average correlations. Asness et al. (2019) find that a Betting-Against-Correlation (*BAC*) factor, which is long a portfolio of low correlation stocks and short a portfolio of high correlation stocks, delivers significant positive returns and is robust across various markets. *BAC* has a significant CAPM alpha and is robust when controlling for other factors such as the Fama and French (2015) five-factor model. Moreover, Asness et al. (2019) argue that the rationale behind the significant positive returns of the *BAC* factor is analogous to the *BAB* factor as high (low) correlation stocks have high (low) betas when the volatility is held constant. Hence, *BAC* is consistent with equilibrium models with leverage constraints such as the zero-beta CAPM of Black et al. (1972).

## 1.4 Empirical Results

The arguments in favor of *RP* seem to be related to those in favor of its component portfolios, namely *VOLA* and *CORR*. However, despite having the same objective function of equalizing risk contributions, *RP*, *VOLA*, and *CORR* utilize different parameters to determine the portfolio weights. So how does the *RP* portfolio relate to its component portfolios empirically? And does the *RP* portfolio deliver alpha compared to a mix of its components? To answer these questions, we begin by reporting the portfolio risk-return characteristics and correlations in Table 1.2.

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Fama-French three factor model, the Carhart momentum factor, among others (Bali et al., 2011).



**Table 1.2:** Full Sample Portfolio Summary Statistics

(a) Summary Statistics of Portfolios				(b) Portfolio Return Correlations			
Portfolio	RP	VOLA	CORR	Portfolio	RP	VOLA	CORR
Return	5.93	6.04	6.26	RP	1.00	0.99***	0.89***
Volatility	5.65	5.96	7.73	VOLA	0.99***	1.00	0.92***
Sharpe	0.68	0.66	0.54	CORR	0.89***	0.92***	1.00
HHI	40.03	38.15	33.51				

*Notes:* This table provides the out-of-sample portfolio performance measures for the risk parity portfolio (*RP*), the volatility component (*VOLA*), the correlation component (*CORR*) during the period from 01.01.1994 to 30.09.2019, which consists of 6,467 observations. Portfolios are rebalanced on a daily basis and all metrics are in annual terms. \*, \*\*, and \*\*\* indicate that the portfolio correlation is statistically significant from 1 at the 90%, 95%, and 99%, respectively.

Although all portfolio correlations are statistically different from 1 at the 1% significance level, correlations are high and vary between 0.89 and 0.99. The observed high correlations are due to all portfolios having the same objective function, the asset universe consisting of merely three assets, and portfolios being rebalanced on a daily basis. However, the summary statistics of the portfolio returns display a greater level of discrepancy. *RP* not only yields a slightly lower return than *VOLA* and *CORR*, but also a lower volatility. This results in *RP* achieving the highest Sharpe ratio of 0.68. *VOLA* is quite similar to *RP* with a correlation of 0.99 and yields a Sharpe ratio of 0.66. Meanwhile, *CORR* seems quite similar to the equal-weighting technique as its HHI is equal to 33.51% and although it yields the highest return of 6.26%, it yields the lowest Sharpe ratio (0.54) due to the higher exhibited volatility of 7.73%.

Next, to assess the exposure and the relationship of *RP* to *VOLA* and *CORR*, we regress the *RP* portfolio on the component portfolios as follows

$$XR_{RP,t} = \alpha + \beta_1 XR_{VOLA,t} + \epsilon_t \quad (1.8)$$

$$XR_{RP,t} = \alpha + \beta_1 XR_{CORR,t} + \epsilon_t \quad (1.9)$$

$$XR_{RP,t} = \alpha + \beta_1 XR_{VOLA,t} + \beta_2 XR_{CORR,t} + \epsilon_t, \quad (1.10)$$

where  $XR_{RP,t}$ ,  $XR_{VOLA,t}$ , and  $XR_{CORR,t}$  describe the excess returns of *RP*, *VOLA*, and *CORR*, respectively, at time  $t$ .  $\alpha$  is the alpha of *RP* in excess of the component portfolios, while the  $\beta$  denotes the sensitivity of *RP* to the corresponding component portfolio, and  $\epsilon_t$  is the regression estimation error at time  $t$ . A positive (negative)  $\beta$  indicates that *RP* has positive (negative) exposure to the corresponding component

portfolio.

Table 1.3 reports the results of the regression models in Equations (1.8)–(1.10) with Newey-West robust standard errors<sup>16</sup>. The coefficients of *VOLA* and *CORR* are statistically significant at the 1% significance level in all regressions and the  $\bar{R}^2$  is almost equal to one. This suggests that *VOLA* and *CORR* capture a huge proportion of the variation in *RP* returns. *VOLA* has a positive coefficient across all regression models, which implies that *RP* overweights low volatility assets. Meanwhile, *CORR* has a positive coefficient of 0.65 in model (1.9) and a slightly negative coefficient of -0.11 when included with *VOLA*, which implies that *RP* underweights assets that on average share a low correlation with other assets.

**Table 1.3:** Main Regression Results

	1.8	1.9	1.10
Alpha	0.15%	1.13%	0.09%
VOLA	0.93***		1.07***
CORR		0.65***	-0.11***
$\bar{R}^2$	97.12	78.83	97.47

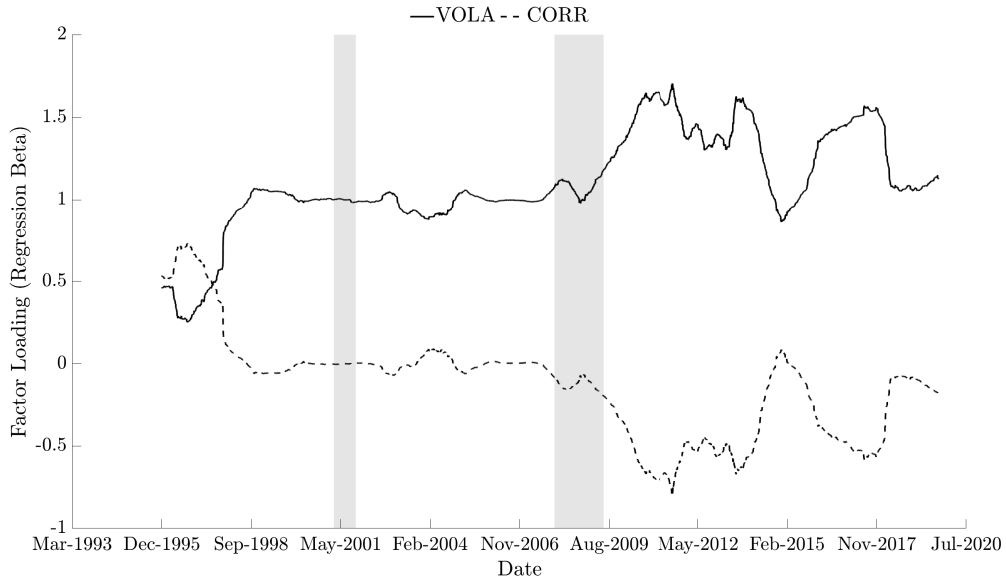
*Notes:* This table reports the results of the regression models of Equations (1.8)–(1.10). The sample period covers 6,467 days from 01.01.1994 to 30.09.2019. Alpha denotes the return in excess of the component portfolios and is in annual terms. Portfolios are rebalanced on a daily basis. Statistically significant measures at the 90%, 95%, and 99% confidence intervals are represented by \*, \*\*, and \*\*\*, respectively.

Moreover, the explanatory power of *VOLA* seems to be higher than *CORR*, as the  $\bar{R}^2$  is higher in models (1.8) than (1.9) and the exposure of *RP* to *VOLA* is generally higher than that of *CORR*. These results reinforce the findings of Asness et al. (2019), who find that the *BAB* factor seems to be mainly driven by the low volatility effect while the *BAC* factor is of secondary importance. Furthermore, the regressions results suggest not only that *VOLA* and *CORR* capture a great proportion of the variance in *RP* returns, but also that a linear combination of the component portfolios might outperform *RP* due to the insignificance of the alpha in model (1.10). Provided this insight, we regress *RP* on *VOLA* and *CORR* as in

<sup>16</sup>We employ the Newey-West robust standard errors to control for heteroscedasticity and autocorrelation, which are typical properties of financial data and present in our sample. Moreover, tests for collinearity were rejected as the highest Variance Inflation Factor (VIF) amounts to 6.84, which is below the conventional threshold of 10. Moreover, the Belsley (1991) collinearity diagnosis also rejects the existence of collinearity with the highest conditional index equal to 5.03, which is well below the threshold value of 30. Results are available upon request.

Equation 1.10 and depict the regression coefficients of the component portfolios in Figure 1.3.

**Figure 1.3:** Rolling Regression Coefficients of the Risk Parity Portfolio



*Notes:* This figure shows the rolling betas when the excess returns of risk parity are regressed on the excess returns of the volatility and the correlation components as in Equation 1.10. The gray shaded areas indicate U.S. recessions as reported by NBER. The sample period covers the period from 01.01.1994 to 30.09.2019 and consists of 6,217 observations. The alpha (regression intercept) is excluded since it is statistically insignificant from zero throughout the sample.

The regression coefficients of *VOLA* and *CORR* change over time but seem to be perfectly negatively correlated, where the sum of both exposures roughly equals one. This suggests that *RP* can indeed be expressed as a sum of its components. The regression coefficient of *VOLA* generally increases over the period analyzed whereas the coefficient of *CORR* decreases. This pattern might be due to the empirical observation that the correlations among asset classes have increased over time, which intuitively would reduce the importance of diversification based on correlations, and hence, lead to a lower exposure to *CORR* and a higher one to *VOLA*.

In summary, our results suggest that the performance of *RP* is closely related to the performance of its component portfolios, namely *VOLA* and *CORR*. *RP* does not deliver a significant alpha over its components and *VOLA* and *CORR* essentially capture the entire variation of the returns of *RP*. Also, the rolling regression results imply that *RP* could be represented by a linear combination of *VOLA* and *CORR*, as their rolling coefficients seem to be perfectly negatively correlated and roughly add up to one.

## 1.5 Robustness Tests

To verify our results, we extend our data set by including all equity and bond markets of the G7 countries, in addition to commodities<sup>17</sup>. Equities, bonds, and commodities are proxied by the corresponding MSCI total return indices, the Thomson Reuters 10 year Government Bond indices, and the Bloomberg Commodity Index, respectively. All assets are denoted in USD. The data set spans the period from 01.01.1999 to 30.09.2019. Table 1.4 reports the summary statistics of  $RP$ ,  $VOLA$ , and  $CORR$ .

**Table 1.4:** G7 Portfolio Summary Statistics

(a) Summary Statistics of Portfolios				(b) Portfolio Return Correlations			
Portfolio	RP	VOLA	CORR	Portfolio	RP	VOLA	CORR
Return	2.87	2.53	2.90	RP	1	0.97***	0.87***
Volatility	4.57	3.87	7.31	VOLA	0.97***	1	0.84***
Sharpe	0.34	0.31	0.22	CORR	0.87***	0.84***	1
HHI	7.96	9.81	6.95				

*Notes:* This table provides the out-of-sample portfolio performance measures for the risk parity portfolio ( $RP$ ), the volatility component ( $VOLA$ ), the correlation component ( $CORR$ ) during the period from 01.01.1999 to 30.09.2019, which consists of 5,159 observations. Portfolios are rebalanced on a daily basis and all metrics are in annual terms. \*, \*\*, and \*\*\* indicate that the portfolio correlation is statistically significant from 1 at the 90%, 95%, and 99%, respectively.

Although the correlations of portfolio returns weaken due to a broader asset universe, the results remain largely unchanged compared to Table 1.2.  $RP$  and  $VOLA$  are very similar, whereas  $CORR$  performs poorest and displays a similar weighting allocation to the equally-weighted portfolio with an HHI of 7.39%.

Furthermore, the results of the Newey-West regression models of Equations (1.8)–(1.10) are reported in Table 1.5<sup>18</sup>. The results are highly similar to those reported in Table 1.3. However, due to the increase in the number of assets, the explanatory power decreases marginally and the coefficient of  $CORR$  turns positive in model (1.10), since exploiting the correlation structure becomes more important with a higher number of assets.

<sup>17</sup>The relationship between  $RP$  and its component portfolios should be weaker in this scenario than in the case of three assets as in section 1.4, since the precision of plug-in estimates, such as historical estimates, in determining the optimal asset weights deteriorates severely by increasing the number of assets (Jobson and Korkie, 1981; Michaud, 1989; Best and Grauer, 1991).

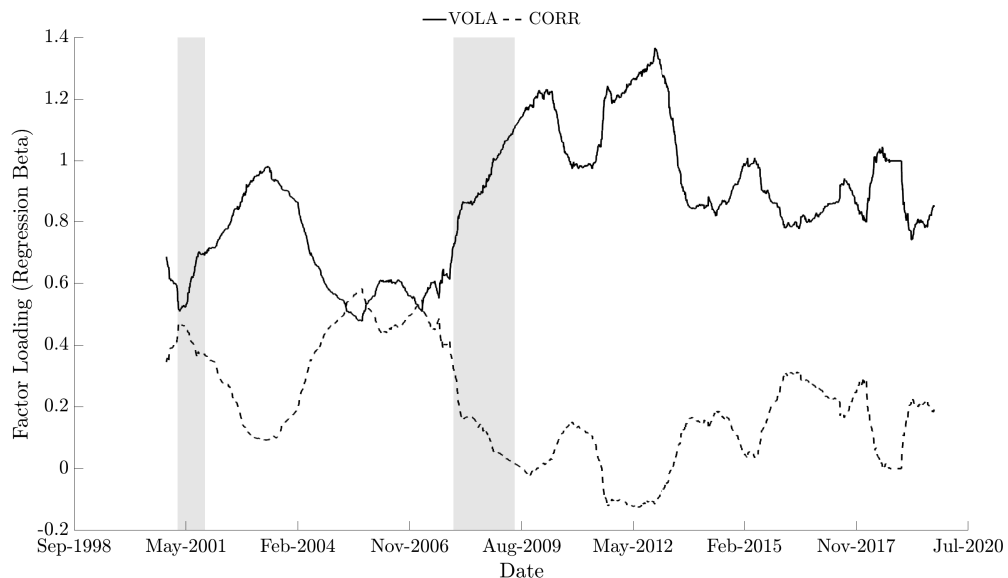
<sup>18</sup>Tests for collinearity were rejected as the highest Variance Inflation Factor (VIF) amounts to 3.48. Moreover, the Belsley (1991) collinearity diagnosis also rejects the existence of collinearity with the highest conditional index equal to 3.44. Results are available upon request.

**Table 1.5:** G7 Portfolios Regression Results

	1.8	1.9	1.10
Alpha	0.17%	0.69%	0.21%
VOLA	1.15***		0.96***
CORR		0.55***	0.12***
$\bar{R}^2$	94.42	76.39	95.43

*Notes:* This table reports the results of the regression models of Equations (1.8)–(1.10). The sample period covers 5,159 days from 01.01.1999 to 30.09.2019. Alpha denotes the return in excess of the component portfolios and is in annual terms. Portfolios are rebalanced on a daily basis. Statistically significant measures at the 90%, 95%, and 99% confidence intervals are represented by \*, \*\*, and \*\*\*, respectively.

Last, we show the rolling regression coefficients of Equation 1.10 in Figure 1.4. The results remain largely unchanged to Table 1.3, although the regression coefficients of *VOLA* and *CORR* are more volatile over time and almost equal during the dot com bubble of 2000-2001 and more prominently the abrupt appreciation of the U.S. Dollar in 2005 (Elwell, 2006), which raised concerns of a currency crisis (Bew, 2005). Altogether, our results seem to be robust when controlling for the number of assets in the asset universe.

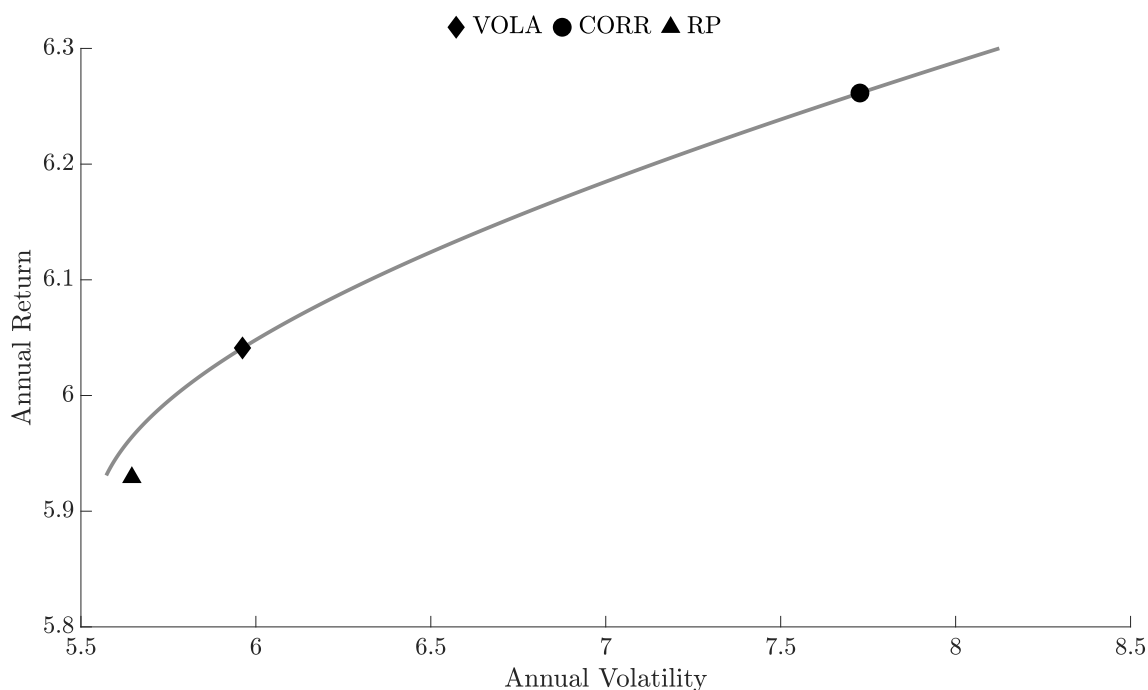
**Figure 1.4:** Rolling Regression Coefficients of G7 Risk Parity Portfolio

*Notes:* This figure shows the rolling betas when the excess returns of risk parity are regressed on the excess returns of the volatility and the correlation components as in Equation 1.10. The gray shaded areas indicate U.S. recessions as reported by NBER. The sample period covers the period from 01.01.1999 to 30.09.2019 and consists of 4,909 observations. The alpha (regression intercept) is excluded as it is statistically insignificant from zero throughout the sample.

## 1.6 On Improving Risk Parity

Provided that *VOLA* and *CORR* explain a great amount of its return variation, we investigate whether *RP* can be improved by utilizing these component portfolios. We construct 2,000 linear combination portfolios consisting of *VOLA* and *CORR* that range in weighting between 150%/−50% to −50%/150%. We depict *RP*, *VOLA*, *CORR*, and their linear combinations in the mean-variance framework in Figure 1.5. *VOLA* and *CORR* deliver a higher annual return and a higher annual volatility, but neither component strictly dominates *RP* in the mean-variance space. However, a linear combination of the components can dominate the *RP* portfolio from a mean-variance perspective. Given that a false combination might lead to worse risk-return properties, how can one utilize the components on an ad-hoc basis? Can we forecast in which market periods would *VOLA* or *CORR* deliver a superior performance? And how can we shift the allocation from *RP* to its components<sup>19</sup>?

**Figure 1.5:** Risk-Return Profiles of Component Linear Combination Portfolios



*Notes:* This figure provides the risk-return properties of portfolios constructed as a linear combination of the *RP* components. The linear combination portfolios range in weighting between −150%/50% to 50%/−150% for *VOLA* and *CORR*, respectively. The sample covers the period from 01.01.1994 to 30.09.2019, which consists of 6,467 observations. Portfolios are rebalanced on a daily basis and all metrics are in annual terms.

<sup>19</sup>Certainly, several methods could be employed, but for brevity, we propose a single and simple approach to show that an improvement based on the component portfolios is possible.

Numerous studies such as Longin and Solnik (2001) and Campbell et al. (2002) have documented that correlations in financial markets vary based on the general market environment. In recessionary periods, correlations tend to increase so that diversifying a portfolio based on correlations becomes more difficult. In expansionary periods, markets exhibit positive returns on average and volatilities decrease, making asset correlations vital for diversifying portfolio risk. Accordingly, if an investor desires to maintain the properties of *RP* whilst improving general risk-return characteristics, an investor can discredit asset correlations and utilize *VOLA* during recessions. Meanwhile, in expansionary phases the investor can employ *CORR* to exploit the correlation structure of financial returns and minimize risk.

Provided the aforementioned arguments and using the NBER data for U.S. recessions, we propose a *RP* component timing strategy (*TIMING*) that employs *VOLA* in recessionary periods and *CORR* in expansionary phases. We assess the risk-return properties of the *RP* portfolio in comparison to its component portfolios as well as the equally-weighted portfolio (*EW*) and the minimum variance portfolio (*MV*). We select these two portfolios as Maillard et al. (2010) show that the volatility of *RP* lies between that of *MV* and *EW*, or formally  $\sigma_{MV} < \sigma_{RP} < \sigma_{EW}$ . Table 1.6 reports the portfolio correlations as well as the portfolio risk-return properties.

All portfolio correlations are quite high, ranging between 0.75 and 0.99, albeit being statistically significantly different from 1. *VOLA* and *RP* portfolios exhibit slightly higher returns and volatilities compared to *MV* while *CORR* shows a marginal improvement over *EW*. This results in a Sharpe ratio below 0.70 for *MV*, *RP*, and *VOLA*, while *EW* and *CORR* yield a mere ratio of 0.52 and 0.54, respectively. Meanwhile, *TIMING* shows a significant improvement over all strategies at the 1% significance level by yielding a Sharpe ratio of 0.73.

Moreover, regarding the distribution of portfolio returns, all strategies exhibit negatively skewed returns, which coincides with the longstanding findings of Moore (1962) and Fama (1965), among others, that financial returns are negatively skewed in big sample sets. Portfolio returns are also leptokurtic, which suggests that all portfolios suffer from extreme negative returns in turbulent market phases. *CORR* yields the highest kurtosis of roughly 9.4, while *RP* and *VOLA* exhibit a kurtosis of roughly 6.8. Meanwhile, *TIMING* exhibits the lowest kurtosis in its portfolio returns. This indicates that the returns of *TIMING* are more stable and experience fewer extreme returns.

Finally, regarding portfolio concentrations as measured by *HHI*, *EW* (33.33%), *CORR* (33.41%), and *TIMING* (33.51%) are the least concentrated portfolios, *MV* (46.75%) is the most concentrated portfolio, while *RP* and *VOLA* yield an *HHI* of 40.03% and 38.15%, respectively. Thus, the results of Table 1.6 suggest that the *RP* components have several benefits, where utilizing them not only helps maintain the appealing properties of *RP*, but also helps in significantly improving portfolio properties.

**Table 1.6:** Risk Parity Component Timing Strategy

(a) Portfolio Return Correlations

Portfolio	EW	MV	RP	VOLA	CORR	TIMING
EW	1	0.71***	0.86***	0.91***	0.99***	0.95***
MV	0.71***	1	0.96***	0.92***	0.75***	0.83***
RP	0.86***	0.96***	1	0.99***	0.89***	0.93***
VOLA	0.91***	0.92***	0.99***	1	0.92***	0.95***
CORR	0.99***	0.75***	0.89***	0.92***	1	0.95***
TIMING	0.95***	0.83***	0.93***	0.95***	0.95***	1

(b) Summary Statistics of Portfolios

Portfolio	EW	MV	RP	VOLA	CORR	TIMING
Return	6.36	5.77	5.93	6.04	6.26	7.03
Volatility	8.23	5.29	5.65	5.96	7.73	6.79
Skewness	-0.37	-0.11	-0.15	-0.20	-0.33	-0.18
Kurtosis	9.36	7.05	6.80	6.84	9.43	5.80
Sharpe	0.52***	0.69***	0.68***	0.66***	0.54***	0.73
HHI	33.33	46.75	40.03	38.15	33.51	34.41

*Notes:* This table provides the out-of-sample portfolio performance measures for the equally-weighted portfolio (*EW*), the minimum variance portfolio (*MV*), the risk parity portfolio (*RP*), the volatility component (*VOLA*), the correlation component (*CORR*), and the risk parity component timing portfolio (*TIMING*) during the period from 01.01.1994 to 30.09.2019, which consists of 6,467 observations. Portfolios are rebalanced on a daily basis and all metrics are in annual terms. \*, \*\*, and \*\*\* indicate that the portfolio correlation is statistically significant from 1 in Table 1.6a and significantly different from the Sharpe ratio of *TIMING* in Table 1.6b at the 90%, 95%, and 99%, respectively.



## 1.7 Conclusion

Risk parity ( $RP$ ) portfolios are highly attractive to investors and have been gaining increasing attention in academia and practice. They aim to equalize the total risk contributions of assets and can be leveraged to match investor preferences such as achieve a desired target return or maintain a specific volatility level. The aim of this study is to analyze the properties of  $RP$  in more detail by investigating its component portfolios, namely the volatility ( $VOLA$ ) and the correlation ( $CORR$ ) component portfolios, which achieve the characteristics of  $RP$  under certain circumstances. The approach is similar to Asness et al. (2019), who decompose the Betting-Against-Beta factor of Frazzini and Pedersen (2014) into the Betting-Against-Volatility and Betting-Against-Correlation factor. We assess the relationship of  $RP$  to its components and provide arguments in favor of  $RP$ , explain how its returns are driven by the  $BAB$  of Frazzini and Pedersen (2014) and is closely related to its components,  $VOLA$  and  $CORR$ , which are driven by the low-volatility anomaly and the  $BAC$  factor of Asness et al. (2019).

$RP$  and  $VOLA$  display similar risk-return characteristics whereas  $CORR$  exhibits a slight improvement over the equally-weighted portfolio ( $EW$ ).  $RP$  delivers a risk-return profile that is not statistically different from the minimum variance ( $MV$ ) portfolio, but achieves a much lower level of portfolio concentration. As a result,  $RP$  might be more attractive to investors as less concentrated portfolios typically result in lower turnover ratios and, hence, higher net returns and as  $RP$  would not require additional constraints to prevent the portfolio from achieving high and unstable weights as is the case for  $MV$  (Michaud (1989)). Moreover,  $RP$  does not deliver a significant alpha over its components and  $VOLA$  and  $CORR$  essentially capture the entire variation of the returns of  $RP$ , where  $RP$  has a high positive exposure to  $VOLA$  and typically a low or even slightly negative exposure to  $CORR$ .

Finally, we show that the rolling regression coefficients of  $VOLA$  and  $CORR$  are almost perfectly negatively correlated and sum up to one, which implies that  $RP$  can be replicated by a linear combination of  $VOLA$  and  $CORR$  and that these component portfolios can be utilized to improve the general risk-return profile  $RP$  while maintaining its properties to some extent. We briefly illustrate an example where we utilize  $VOLA$  in recessionary and  $CORR$  in expansionary periods, which results in a significant improvement in portfolio performance compared to  $RP$ . However, we leave further investigations into methods of improving  $RP$  via its components for future research.

# Chapter 2

## Diversification Benefits for Risk Parity<sup>1</sup>

### 2.1 Introduction

Diversification is paramount for portfolio performance and is often claimed to be the only free lunch in investing. Perold (2012) argues that diversification also protects from suffering huge losses of underperforming assets that may be irreversible. This chapter contributes to the existing body of literature by investigating in the out-of-sample diversification benefits of leveraged and unleveraged risk parity portfolios for a variety of asset classes from a long-only U.S. investor perspective. The chapter focuses on the properties an asset class has to fulfill in order to be beneficial in a risk parity strategy. Moreover, comparisons are also drawn to other popular risk-based strategies, such as minimum variance or inverse volatility. The risk parity strategy uses the asset class variance-covariance information in order to achieve equal risk contribution. This study is related to Qian (2011, 2012), who analytically demonstrates the in-sample benefits of the risk parity strategy.

The notion and benefits of portfolio diversification were first demonstrated in the mean-variance framework of Markowitz (1952), which utilizes differences in the asset covariance matrix to minimize risk for a given level of portfolio return. Numerous studies such as Michaud (1989) show that the risk-return characteristics of mean-variance portfolios are inferior compared to alternative weighting schemes, such as equal-weighting, and are rarely employed in practice. This is due to the estimation errors underlying the forecasts of asset moments, which the mean-variance frame-

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<sup>1</sup>This chapter is based on Alkafri and Molloyres (2019). Diversification Benefits for Risk Parity. *Unpublished working paper.*

work typically maximizes, resulting in subpar portfolios.<sup>2</sup> Recently, numerous studies such as Chaves et al. (2011), Chow et al. (2011), Lee (2011), and Clarke et al. (2013) find that risk-based portfolios, such as risk parity and minimum variance, deliver better risk-adjusted returns than capitalization weighted portfolios or other traditional weighting techniques as for example equally weighted portfolios. Chaves et al. (2011) show that risk parity portfolios perform better than mean-variance optimized portfolios as well as the minimum variance portfolios, yielding a much more stable Sharpe ratio. However, the majority of these studies are conducted in an in-sample manner, which does not accurately represent the decision environment, where the future returns and volatilities of assets are uncertain and forecasts thereof underlie estimation error. This study contrasts those studies by using a more accurate out-of-sample representation of the performance of such portfolios.

Our findings show that risk parity portfolios achieve a superior risk-adjusted return compared with other popular weighting strategies while yielding a lower risk exposure and volatility. Risk parity portfolios also exhibit a stable and smooth change in weights over time. However, they are very susceptible to the properties of its assets, where adding a negative risk premium asset significantly worsens its risk-return profile. Leveraging the risk parity portfolio substantially improves the risk-adjusted performance, which is in line with the findings of Qian (2012). Leveraging risk parity also results in a minor increase in risk exposure and in a much higher portfolio concentration.

The remainder of this chapter is organized as follows: Section 2.2 describes the dataset in terms of construction and risk-return properties. In section 2.3, the methodology and employed portfolio weighting strategies are introduced. The empirical results are presented and discussed in section 2.4, whereas robustness checks are presented in section 2.5. Section 2.6 concludes and draws finishing remarks.

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<sup>2</sup>Chopra and Ziemba (1993) show that negative effects of estimation errors in asset means are greater by a factor of ten as compared to estimation errors of asset variances and covariances.

## 2.2 Data Description

Five asset classes are utilized in this study, which are equities and bonds of developed markets (*DM*), equities and bonds of emerging markets (*EM*), and commodities. Developed markets are represented by the G7 countries, which are Canada, France, Germany, Italy, Japan, the United Kingdom, and the United States of America. Emerging markets are represented by the People’s Republic of China, the Czech Republic, India, Mexico, Poland and South Africa, which are the emerging markets with the largest overall debt outstanding and the longest series of historical data. In order to be independent from the construction methodology of conventional developed and emerging market indices and to allow equal exposure to every market, we equally weight each country within each asset class and market classification. This is because the variation of volatilities and correlations between different asset classes is greater than the variations within the same asset class.

Historical asset prices are collected from Datastream. The dataset spans the period from January 2011 to September 2019 and consists of daily total return prices, which assume the reinvestment of dividends or coupon payments, and adjusts for stock splits or share buybacks. Equity indices are represented by the MSCI stock market indices and are denoted in USD. Bond indices are represented by Thomson Reuters 10-year government bond indices, which are denoted in local currency. Accordingly, bond returns are hedged to USD according to the interest rate parity model. Commodities are represented by the Bloomberg Commodity index and are denominated in USD. Finally, the 1-month U.S. deposit rate is employed as a proxy for the risk-free asset.

The summary statistics of the various asset classes are reported in Table 2.1. The average annualized returns are 4.84% p.a. for developed market equities, 7.55% p.a. for developed markets bonds, 0.80% p.a. for emerging markets equities, and 9.71% p.a. for emerging markets bonds, while commodities yield a negative return of  $-7.27\%$  p.a. The annualized volatility and the Value-at-Risk (*VaR*) measures suggest that bonds are the least risky, followed by equities, whereas commodities are the asset class with the highest risk. All asset classes exhibit negatively skewed and leptokurtic returns, and show similar distributions. Developed equities are the most negatively skewed ( $-0.57$ ) and are heavily influenced by fat tail returns, with a kurtosis of 8.62. Finally, in terms of the Sharpe ratio, commodities yield a negative ratio of  $-0.57$ . Developed and emerging market equities achieve a Sharpe ratio of 0.32 and 0.06, respectively, while bonds achieved the highest Sharpe ratios of 0.64

## 2.2. DATA DESCRIPTION

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and 0.74 for developed and emerging markets.

To gain more insights regarding the potential diversification benefits, we report the correlation coefficients in Table 2.2. Developed and emerging equity markets share a high correlation with each other amounting to 0.76 and a relatively high correlation with commodities of roughly 0.45 and 0.42 for developed and emerging equities, respectively. The correlation between developed and emerging bond markets amounts to 0.75. Developed bond markets share negative correlations with developed ( $-0.53$ ) and emerging ( $-0.41$ ) equities as well as commodities ( $-0.35$ ). Emerging market bonds exhibit a negative correlation with emerging market equities of  $-0.66$ , while experiencing a similar correlation with developed market equities ( $-0.67$ ) and commodities ( $-0.42$ ).

The results from tables 2.1 and 2.2 suggest that emerging market bonds are a promising asset class for improving the out-of-sample risk-return profile of traditional developed market stock/bond portfolios, as they offer a high risk-return ratio while exhibiting low correlations with other asset classes. Despite the low correlation of commodities with bond markets, they might not improve portfolio performance due to the high correlation with equities and the overall negative risk premium.

**Table 2.1:** Summary Statistics of Asset Classes

Asset	DM Equities	DM Bonds	EM Equities	EM Bonds	Commodities
Return	4.84	7.55	0.80	9.71	$-7.27$
Volatility	15.16	11.77	14.14	13.04	12.86
Skewness	$-0.57$	$-0.02$	$-0.47$	0.28	$-0.28$
Kurtosis	8.62	5.76	6.21	5.50	5.39
Sharpe Ratio	0.32	0.64	0.06	0.74	$-0.57$
Value-at-Risk	30.42	19.84	32.10	20.62	37.20

*Notes:* This table provides asset return moments, Sharpe ratios, and the Value-at-Risk of the stock and bond indices as well as the commodity index. The daily sample period covers 2277 days from January 2011 to September 2019. All measures are in annual terms. DM and EM represent developed markets and emerging markets, respectively.

Finally, the results are in line with those of conventional literature as well as in practice. Emerging markets performed very well in the last decade, mainly due to the reforms following the emerging markets' crisis.<sup>3</sup> The poor performance of commodities as well as the relatively higher correlations observed with other asset

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<sup>3</sup>For more information on the performance of emerging markets, especially the outperformance of emerging market bonds, refer to Roberts et al. (2018).

classes of the investigated time period are also in line with conventional literature investigating the benefits of commodities in asset allocation, such as Daskalaki and Skiadopoulos (2011). Also, developed market equities and bonds were affected by the Euro sovereign bond crisis spillovers as well as the recent U.S. trade wars, which both negatively influenced the performance of developed markets.

**Table 2.2:** Correlation Matrix of Asset Classes

Asset	DM Equities	DM Bonds	EM Equities	EM Bonds	Commodities
DM Equities	1.0000	-0.5309	0.7558	-0.6696	0.4547
DM Bonds	-0.5309	1.0000	-0.4080	0.7525	-0.3538
EM Equities	0.7558	-0.4080	1.0000	-0.6581	0.4228
EM Bonds	-0.6696	0.7525	-0.6581	1.0000	-0.4157
Commodities	0.4547	-0.3538	0.4228	-0.4157	1.0000

*Notes:* This table provides asset return correlation coefficients for the sample period, which covers 2277 days from January 2011 to September 2019. DM and EM represent developed markets and emerging markets, respectively.

## 2.3 Methodology

We evaluate the out-of-sample benefits of adding different asset classes to the risk parity strategy and compare them against different asset allocation strategies, which are extensively employed in conventional literature and in practice. To do so, we investigate the performance of a non-leveraged risk parity portfolio ( $RP$ ), as well as a leveraged risk parity portfolio ( $LRP$ ) with a target volatility equal to that of the benchmark (amounting to the historical average volatility of 60/40 mutual funds of roughly 7%). In addition, the strategies include two heuristic weighting techniques, which are the 60/40 stock/bond traditional allocation benchmark ( $BM$ ) and an equally weighted portfolio ( $EW$ ), as well as two risk-based asset allocation models, which are the Markowitz minimum variance portfolio ( $MV$ ) and the inverse volatility weighting technique ( $VOLA$ ).

This study employs a rolling-window out-of-sample approach similar to DeMiguel et al. (2009). Given a time series consisting of  $T = 2277$  observations, we use an estimation window of  $M = 250$  business days to estimate the asset moments that are used as input parameters to determine the portfolio weights of each allocation strategy. Each portfolio is held for  $k = 1$  business days, after which the portfolio is rebalanced by using the previous  $M = 250$  business days as an estimation window. Accordingly, the process continues dropping the earliest observation from

the estimation window while adding the most recent observation, which results in a total of  $T - M = 2025$  periodic out-of-sample observations of portfolio returns. The remainder of this section describes the employed asset allocation strategies in more detail.

### **Risk-Based Asset Allocation Models**

Risk-based strategies rely greatly on assumptions or empirical findings that are counterintuitive to traditional theories in finance literature. Although the Capital Asset Pricing Model (CAPM) implies a positive linear beta-return relationship, numerous studies such as Black et al. (1972) as well as Haugen and Heins (1975) observe a flatter risk-return relationship than that implied by the CAPM. However, more recent studies such as Baker and Haugen (2010) as well as Baker et al. (2011) document a negative relationship. Furthermore, the observed negative relationship is not specific to a certain asset class or market. Blitz and van Vliet (2007) as well as Frazzini and Pedersen (2014) show that the negative relationship is not only present in less efficient financial markets such as emerging markets, but also in developed markets. Frazzini and Pedersen (2014) document the negative beta-return relationship in the U.S. corporate bond market, the U.S. government bond market, and the futures market. Falkenstein (2009) shows that the negative relationship is also present in roughly all major asset classes, including derivatives markets.

Moreover, Baker et al. (2011) show that the low-volatility effect is robust even after controlling for the market capitalization of firms. Asness et al. (2014) show that the anomaly remains present when controlling for industry effects, and Blitz and van Vliet (2007) show that the negative relationship remains significant even when controlling for the Fama French three factors and the Carhart momentum factor.

### **Risk Parity**

We begin by describing the risk parity strategy, which is the focus of this chapter. The roots of risk parity date back to the 1990s, when Bridgewater Associates developed and launched the first risk parity portfolio under the name of the *All-Weather Fund*.<sup>4</sup> Originally, an allocation strategy was in parity when the asset weights were proportional to their corresponding inverse volatility.<sup>5</sup> Later on, Qian (2005) as well as Maillard et al. (2010) developed a more formal definition that incorporates asset

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<sup>4</sup>For more information refer to Prince (2011).

<sup>5</sup>This approach is currently referred to as the inverse volatility or the naive risk parity weighting technique, which is explained in more detail later in this section.

correlations, where a portfolio is in parity when the risk budgets of all assets are equalized, or in other words, when each asset has the same contribution to total portfolio risk. The weights of the risk parity portfolio can be expressed as follows:

$$y^* = \arg \min \sqrt{y' \Sigma y}$$

$$s.t. \begin{cases} \sum_{i=1}^n \ln y_i \geq c \\ y \geq 0 \end{cases}, \quad (2.1)$$

where the optimal weights that sum up to 100% are then determined as  $w_i^* = y_i^* / \sum_{j=1}^n y_j^*$ .  $c$  is an arbitrarily chosen constant and  $\Sigma$  describes the asset variance-covariance matrix. As can be seen from Equation 2.1, risk parity is closely related to the minimum variance portfolio, with an additional diversification constraint.<sup>6</sup> Indeed, Maillard et al. (2010) show that, based on the risk budget determined by  $c$ , the volatility of risk parity portfolio lies between those of the minimum variance and the equally weighted portfolios, or formally  $\sigma_{MV} < \sigma_{RP} < \sigma_{EW}$ .<sup>7</sup>

Moreover, Lee (2011) shows that the risk parity portfolio weights can be expressed as  $w_i = \beta_i^{-1}(w_i) / \sum_{j=1}^n \beta_j^{-1}(w_j) = 1/n\beta_i(w_i)$ , where  $\beta_i$  is a function of  $w_i$ , denotes the beta of asset  $i$  towards the constructed portfolio. As a result, the returns of the risk parity portfolio are driven by low beta assets and strongly related to the betting-against-beta factor of Frazzini and Pedersen (2014), which states that low beta stocks usually outperform high beta stocks. Similarly, the risk parity portfolio is related to the low-volatility anomaly documented by Baker et al. (2011) as well as the betting-against-correlation factor of Asness et al. (2019).

In addition to the unleveraged risk parity portfolio ( $RP$ ), we construct a leveraged risk parity portfolio ( $LRP$ ), which aims to achieve a target portfolio volatility similar to the historical volatility exhibited by multi-asset stock/bond benchmarks of 7%. Lee (2011) argues that although the parity element might be of secondary importance for investors, the ability to match investor risk preferences or to achieve a target level of portfolio return or risk by leveraging the risk parity portfolio (e.g., using futures contracts) makes the strategy highly attractive and unique. Qian (2012) finds that leveraged risk parity portfolios yield a positive diversification return as long as the leverage ratio is not greater than a factor of three of invested wealth. Furthermore, the author argues that turnover and diversification return de-

<sup>6</sup>For more information on the derivation of Equation 2.1 and the arbitrary constant  $c$ , see Maillard et al. (2010), Appendices A and B, p. 68.

<sup>7</sup>For a proof of this characteristic, refer to Maillard et al. (2010), Appendix B, pp. 68-69.



pend greatly on the level of correlations among assets of the asset universe and that a low correlation across assets is paramount for maintaining a low turnover ratio and a positive diversification return for risk parity portfolios.

### Markowitz Minimum Variance

Typically, the mean-variance framework minimizes portfolio risk for a given level of portfolio return. However, in the special case of the minimum variance portfolio, only portfolio risk is minimized while asset returns are completely disregarded. The asset weights of  $MV$  can be expressed as follows:

$$\begin{aligned} \min w' \Sigma w \\ \sum_{i=1}^n w_i = 1 \end{aligned} \tag{2.2}$$

Despite the poor performance of mean-variance portfolios in practice, numerous studies, such as Chopra and Ziemba (1993) as well as Chow et al. (2011), document that  $MV$  outperforms traditional heuristic weighting techniques in equity markets. Chopra and Ziemba (1993) show analytically that better portfolios can be obtained when stocks are assumed to have the same expected return as compared to portfolios based on noisy estimates of the expected returns of individual stocks, which is an implicit assumption underlying the  $MV$  optimization. Moreover,  $MV$  equalizes the risk contributions of assets, but only on a marginal basis.

Nevertheless,  $MV$  has certain drawbacks. As in mean-variance portfolios, minimum variance optimized portfolios are typically highly concentrated in a select number of assets with the lowest volatility and its weights are rather unstable, which results in a high turnover ratio. These shortcomings have been documented by numerous studies such as Clarke et al. (2013) and DeMiguel et al. (2009), who raise doubt as to whether this strategy addresses investor preferences, since relative to a value-weighted portfolio, the  $MV$  strategy shifts concentrations from assets with the highest market capitalizations to the assets with the lowest volatilities. As a result, the authors argue that imposing constraints on  $MV$  weights to penalize assets with the lowest volatilities and avoiding high concentrations might help mitigate these problems and improve portfolio performance.

### **Inverse Volatility (Naive Risk Parity)**

As the name implies, inverse volatility assigns large weights to low-volatility assets. This strategy is also referred to as the naive risk parity strategy, as it equalizes the risk contributions of assets when the correlations among assets are almost identical. Formally, the weights of each asset can be expressed as

$$w_i = \frac{1/\sigma_i}{\sum_{i=1}^n 1/\sigma_i} \quad (2.3)$$

where  $\sigma_i$  is the volatility of asset  $i$ . Each asset is assigned a weight, which corresponds to its inverse volatility divided by the sum of all assets' inverse volatilities.

Anderson et al. (2012) report that the naive risk parity strategy performs well and usually outperforms the equally and value-weighted portfolios as well as the 60/40 stock/bond portfolio. The approach is mainly driven by the low-volatility effect documented by Baker et al. (2011), among others. Nonetheless, unlike the aforementioned risk-based strategies, *VOLA* does not necessarily minimize total portfolio risk since asset covariances are not taken into account.

## **Heuristic Asset Allocation Techniques**

### **60/40 Stock/Bond Portfolio**

The benchmark in this study is a portfolio with a 60 percent allocation in developed market equities and a 40 percent allocation in developed market bonds, which is conventional in finance literature. This weighting technique has received a lot of criticism recently, as studies such as Qian (2011) show that these portfolios are not diversified and even though equities amount to only 60 percent of the total portfolio, they account for roughly 85 percent of the overall portfolio volatility.

### **Equal-Weighting (Naive Portfolio)**

Also referred to as the naive weighting technique or the 1/N portfolio, this weighting scheme simply assigns the same weight to every asset under consideration, or formally  $w_i = 1/n$ . The equally weighted portfolio utilizes the size factor, which is the empirical observation that small market cap stocks outperform its large cap peers, and bets on the mean-reversion in asset prices, as it sells stocks with a positive gain and buys stocks with a negative gain at every rebalancing period.

Benartzi and Thaler (2001) as well as Windcliff and Boyle (2004) document the extensive application of equally weighted portfolios in practice. The wide use of equally weighted portfolios by fund managers is not only due to the heuristic being an extremely robust proxy of the tangency portfolio, but more importantly for being the basis for more recent ad hoc weighting techniques. This strategy does not diversify the portfolio based on utilizing differences in the asset covariance matrix as in Modern Portfolio Theory, but rather deconcentrates the portfolio in an ad hoc manner, in which the initial investment is split in equal monetary amounts across the highest feasible number of assets. The notion of deconcentration paved the way for more recent approaches such as risk parity, which deconcentrates the portfolio on an ad hoc risk basis.

## 2.4 Empirical Results

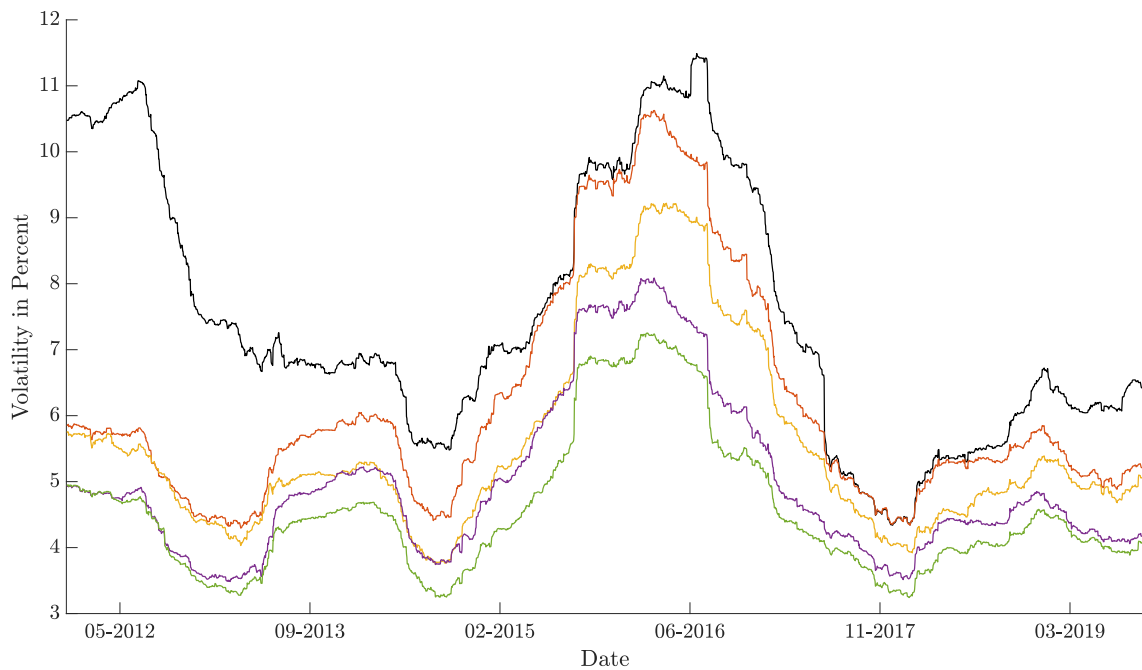
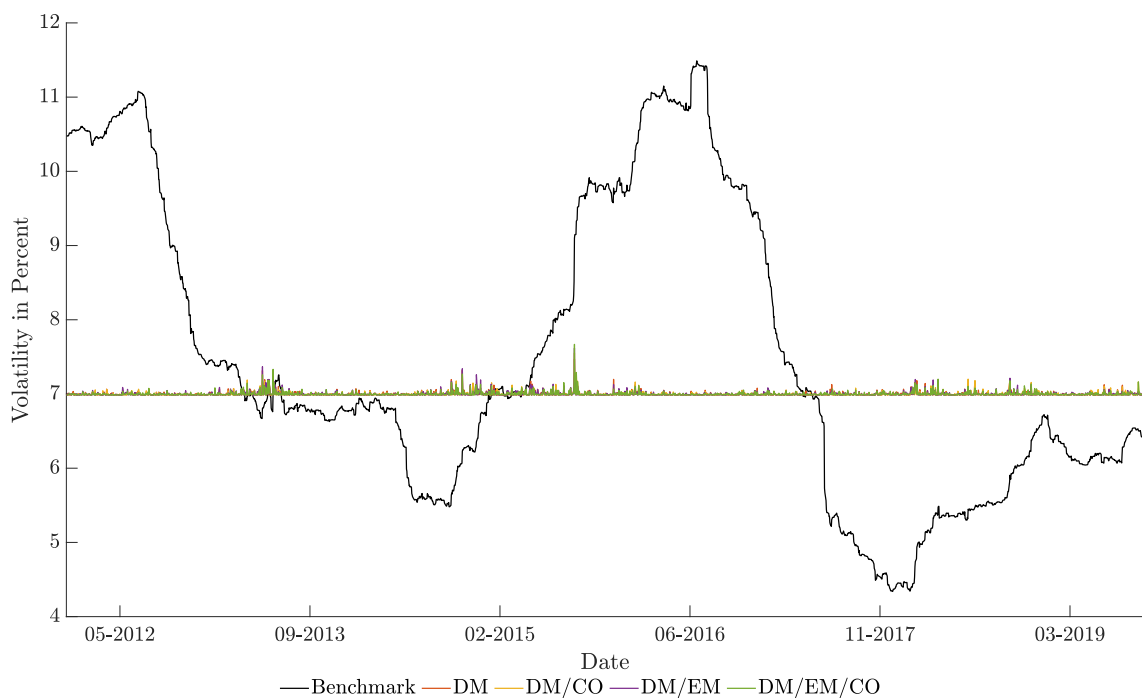
To provide a holistic analysis of the diversification benefits in a comprehensive manner, we divide our results into two subsections, where we begin by assessing the improvements in risk profiles. Later, we investigate the improvements in portfolio performance relative to the benchmark as well as to the other risk-based portfolios. Moreover, since U.S. mutual fund managers typically cannot enter short positions, the main analysis of this study will focus on long-only portfolios.

### Improvement in Risk Profiles

Since risk parity is a risk-based strategy which aims to improve the general risk profile of the portfolio, we begin by depicting the 250 days rolling portfolio volatilities based on the asset covariance matrix of the unleveraged as well as the leveraged risk parity portfolios in Figure 2.1. The portfolios are constructed using different asset classes in comparison to the 60/40 developed markets benchmark. The corresponding changes in portfolio weights over time are shown in Figure 2.2<sup>8</sup>.

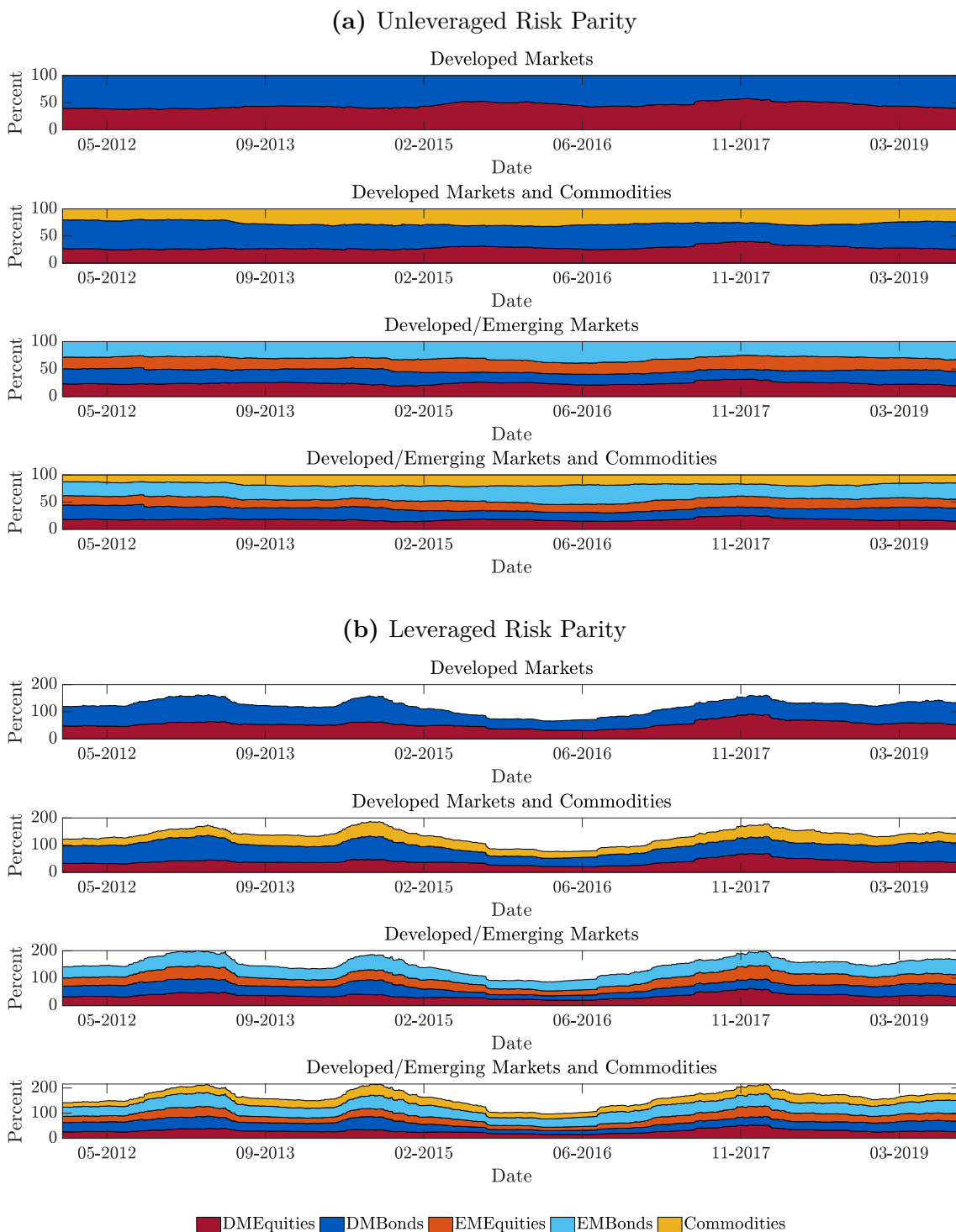
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<sup>8</sup>For completeness, rolling volatilities, portfolio weights over time, and the asset risk contribution ratios to portfolio volatility for every asset class combination of the portfolios presented in section 2.3 are additionally reported in Appendix Appendix B.

**Figure 2.1:** Portfolio Rolling Volatilities**(a)** Unleveraged Risk Parity**(b)** Leveraged Risk Parity

*Notes:* Panel (a) displays the annual rolling volatilities of unleveraged risk parity portfolios compared to the 60/40 stock/bond benchmark, whereas panel (b) exhibits the annual rolling volatilities of risk parity portfolios leveraged to a target overall portfolio volatility of 7%. DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

**Figure 2.2: Risk Parity Weights**



*Notes:* This figure shows the allocation changes over time for unleveraged risk parity portfolios in panel (a) and leveraged risk parity portfolios with a target volatility level of 7% in panel (b). DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

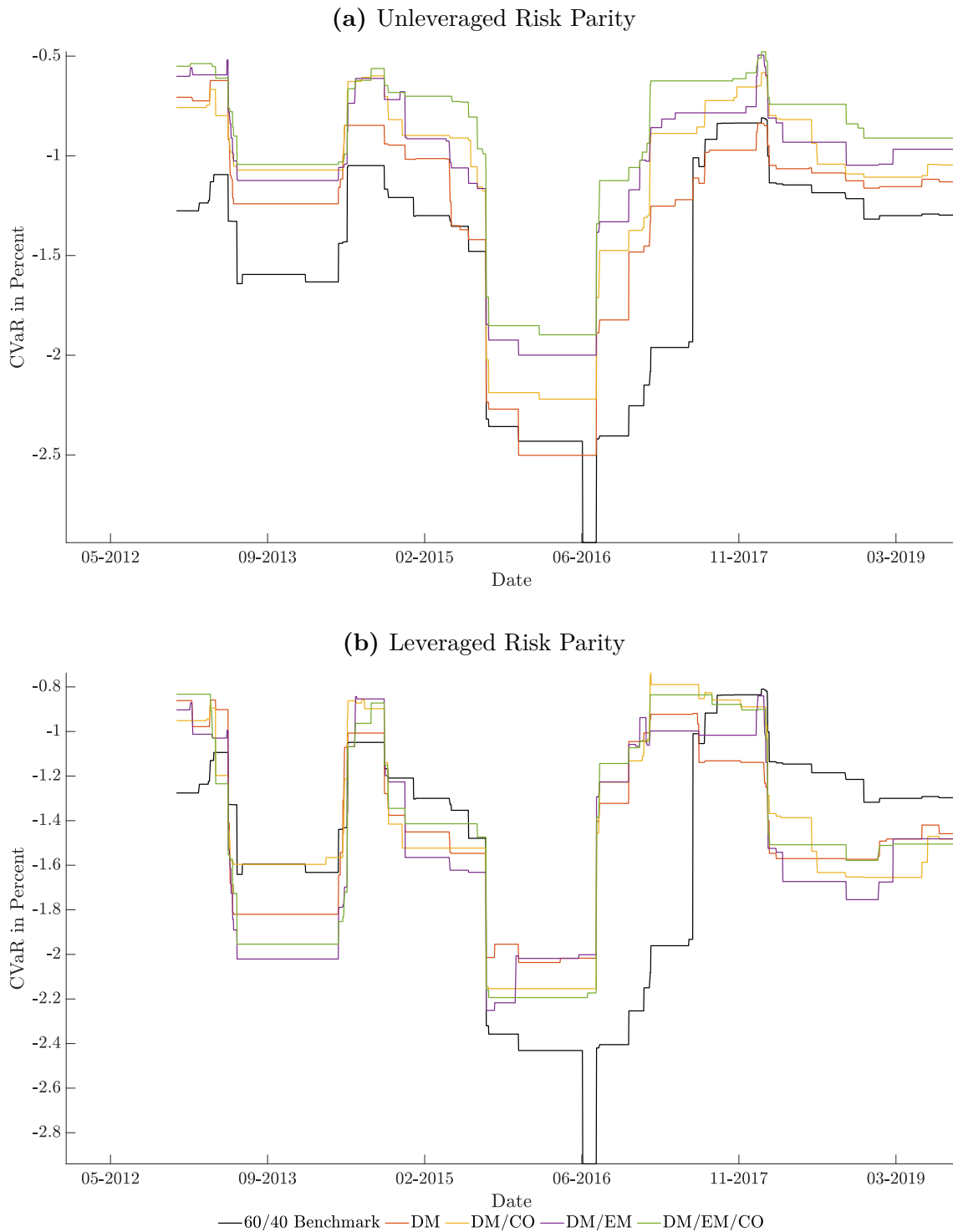
Figure 2.1a shows the rolling volatilities of the unleveraged risk parity portfolios. Generally, risk parity portfolios exhibit a lower portfolio volatility in comparison to the traditional benchmark. During the Euro sovereign crisis between 2011 and 2013, the volatility of the benchmark, ranging between 7% and 11%, greatly exceeds that of risk parity portfolios, while risk parity portfolios yield a somewhat similar volatility varying between 4% and 6%. Afterwards, the changes in the volatilities of the risk parity portfolios and the benchmark become very similar, albeit the benchmark yielding a higher volatility than all risk parity portfolios.

Among the risk parity portfolios, the risk parity portfolio constructed with developed and emerging markets as well as commodities yields the lowest volatility during the Euro sovereign crisis, while the developed markets and commodities portfolio results in the lowest portfolio volatility after 2014.

When risk parity portfolios are leveraged, they achieve their target level of portfolio volatility and all combinations result in a portfolio volatility that is slightly higher than 7%. As can be seen from Figure 2.1b, this is mainly due to the estimation errors underlying the asset variance-covariance matrix forecasts.

Meanwhile, changes in weights for risk parity strategies, both leveraged and unleveraged, are rather smooth over time. Risk parity strategies in all scenarios seem to overweight developed market bonds. This is mainly due to its low volatility compared to other asset classes as well as its low correlations to other asset classes. Moreover, to achieve a target volatility level of 7% that mimics that historical volatility of a traditional 60/40 benchmark, the weights of the strategy are increased proportionally to a leverage factor ranging between one and two, which is well below the maximum leverage threshold identified by Qian (2012) for a positive diversification return.

**Figure 2.3:** Portfolio Rolling Expected Shortfall (CVaR)



*Notes:* Panel (a) displays the rolling expected shortfall (CVaR) of unleveraged risk parity portfolios compared to the 60/40 stock/bond benchmark, whereas panel (b) exhibits the rolling expected shortfall (CVaR) of risk parity portfolios leveraged to a target overall portfolio volatility of 7%. DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

In order to assess the risk exposure variations of risk parity portfolios over time, we show the 250-days rolling expected shortfall, also known as the Conditional-Value-at-Risk (CVaR), of risk parity portfolios as compared to the benchmark in Figure 2.3. Generally, the risk exposures imply a different pattern than that observed for the rolling portfolio volatilities. When risk parity portfolios are not leveraged, as shown in Figure 2.3a, the risk exposure is minimized and constantly smaller in magnitude than that of the benchmark. Moreover, regardless of the initial portfolio constituents, adding commodities to the portfolio results in a decrease of the total risk exposure, whereas including stocks and bonds of emerging markets results in an increase of CVaR.

Figure 2.3b shows the rolling risk exposures of the benchmark as well as leveraged risk parity portfolios as measured by CVaR. The lowest risk exposure is achieved by a developed markets portfolio. Adding commodities to the portfolio results in a higher risk exposure, regardless of the initial portfolio constituents. This might be due to the negative risk premium observed for the commodities index in our sample. Likewise, when stocks and bonds of emerging markets are added to the portfolio, the overall portfolio risk exposure is also increased. As a result, the portfolio consisting of developed and emerging markets as well as commodities has the highest risk exposure. The risk exposures of leveraged risk parity portfolios are higher than their unleveraged counterparts and are higher than the risk exposure of the 60/40 benchmark in certain periods, although they maintain a lower risk exposure on average.

Altogether, the smooth changes in asset weights over time helps unleveraged risk parity portfolios to not only exhibit a lower volatility compared to the benchmark, but also result in a lower risk exposure as measured by the CVaR. Leveraged risk parity portfolios seem to achieve their target volatility level with a high degree of precision by proportionally increasing the asset weights according to their risk budgets, which overall results in leverage ratio below two. However, leveraging the asset weights causes the risk parity portfolio to have a higher level of risk exposure, which in certain periods also exceeds the risk exposure of the 60/40 benchmark.



## Enhancement of Returns and Performance

In this subsection we investigate the performance characteristics of risk parity in comparison to the traditional 60/40 in more detail, then we show the effects of leveraging the risk parity portfolio (to a target volatility level of 7%), and finally compare the performance summary statistics of risk parity with asset allocation strategies presented in section 2.3. Table 2.3 reports the summary statistics of the unleveraged risk parity portfolios in Table 2.3a and the leveraged counterparts in Table 2.3b.

Although unleveraged risk parity portfolios yield a lower volatility than *BM*, they still achieve a higher average return for the developed market portfolio (7.74%). Adding commodities to the portfolio not only decreases the overall portfolio volatility due to its relatively low correlations with other asset classes, but also decreases the portfolio average return due to the negative risk premium exhibited by commodities throughout the sample. Adding emerging markets results in a slight reduction in average return and a minor increase in volatility due to the relatively higher volatilities of emerging markets as compared with developed markets. This causes the developed markets and the developed/emerging markets portfolios to have the highest Sharpe ratios of 1.07 and 1.19, respectively, while portfolios including commodities yield 0.53 and 0.80, thereby underperforming *BM* which yields 0.90.

The returns of all portfolios are negatively skewed. The developed markets portfolio and its counterpart including commodities yield are slightly less skewed than *BM*, whereas the negative skewness of portfolios including emerging markets is more pronounced. Moreover, risk parity portfolios exhibit a higher kurtosis than *BM*, where kurtosis increases the more assets are included in the portfolio, which amounts to a minimum of 6.90 for the developed markets and commodities portfolio and a maximum of 7.81 for the developed/emerging markets and commodities portfolio.

**Table 2.3:** Risk Parity Portfolios Summary Statistics  
(a) Unleveraged Risk Parity Portfolio

Portfolio	BM	DM	DM/CO	DM/EM	DM/EM/CO
Average Return	7.43	7.74	3.90	6.96	4.61
Volatility	7.29	6.41	5.75	5.15	4.70
Sharpe Ratio	0.90	1.07	0.53	1.19	0.80
Skewness	-0.57	-0.49	-0.34	-0.53	-0.42
Kurtosis	6.62	6.98	6.90	7.77	7.81
Beta	1.00	0.78	0.62	0.53	0.45
Alpha		1.76	-1.01	2.68	0.82
t Statistic		1.69*	-0.80	2.19**	0.69
Max Drawdown	14.73	13.55	16.95	8.82	11.14
Value-at-Risk	9.53	7.18	9.48	5.03	6.32
HHI	520.00	50.47	35.30	25.41	20.73

(b) Leveraged Risk Parity Portfolio

Portfolio	BM	DM	DM/CO	DM/EM	DM/EM/CO
Average Return	7.43	9.87	5.36	10.81	7.65
Volatility	7.29	7.21	7.22	7.27	7.26
Sharpe Ratio	0.90	1.25	0.63	1.37	0.94
Skewness	-0.57	-0.32	-0.31	-0.35	-0.34
Kurtosis	6.62	5.03	5.01	5.63	5.59
Beta	1.00	0.85	0.75	0.71	0.67
Alpha		3.53	-0.41	5.41	2.44
t Statistic		2.63**	-0.25	2.95***	1.27
Max Drawdown	14.73	11.81	17.11	11.54	13.85
Value-at-Risk	9.53	6.89	11.42	6.09	9.24
HHI	52.00	73.36	62.67	55.67	53.42

*Notes:* This table provides the out-of-sample portfolio performance measures for the risk parity portfolio and the 60/40 benchmark during the period from January 2011 to September 2019, which consists of 2025 daily portfolio return observations. Portfolios are rebalanced on a daily basis and all measures are annualized. *Beta* and *Alpha* are the  $\beta$  and  $\alpha$  of the CAPM regression, while *t Statistic* denotes the  $\alpha$  student's test statistic. *Max Drawdown* denotes the maximum drawdown of a portfolio. *VaR* denotes the parametric Value-at-Risk at a 99% confidence level during the sample period, while *HHI* represents the Herfindahl-Hirschman Index for portfolio concentration. Values significant at the 90%, 95%, and 99% confidence levels are denoted by \*, \*\*, and \*\*\*, respectively.

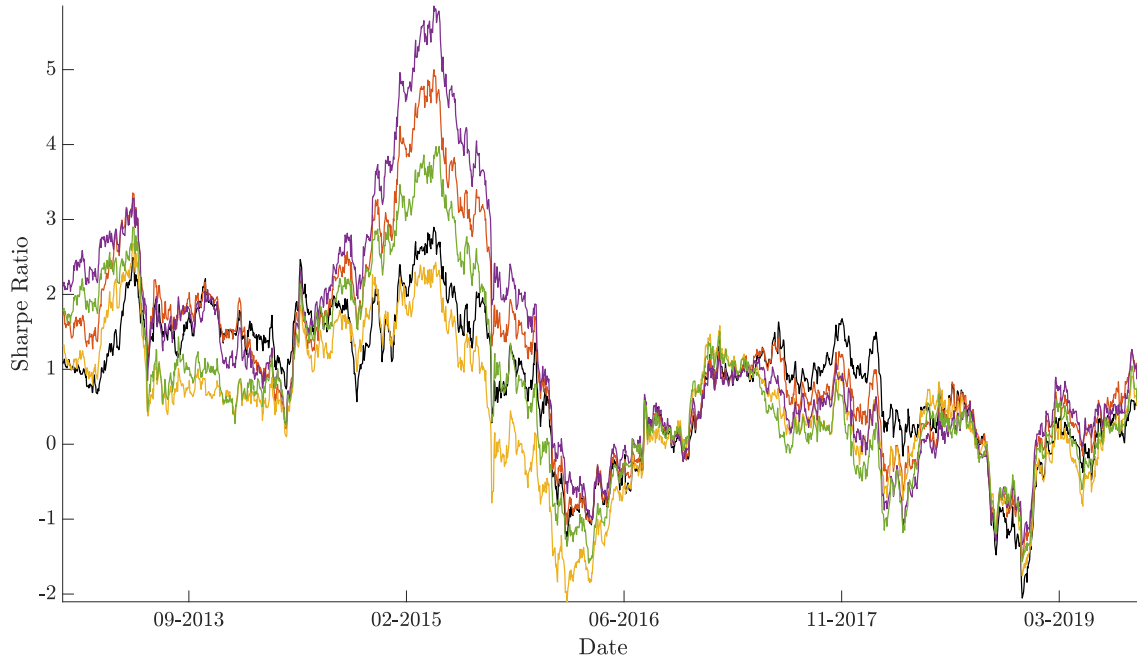
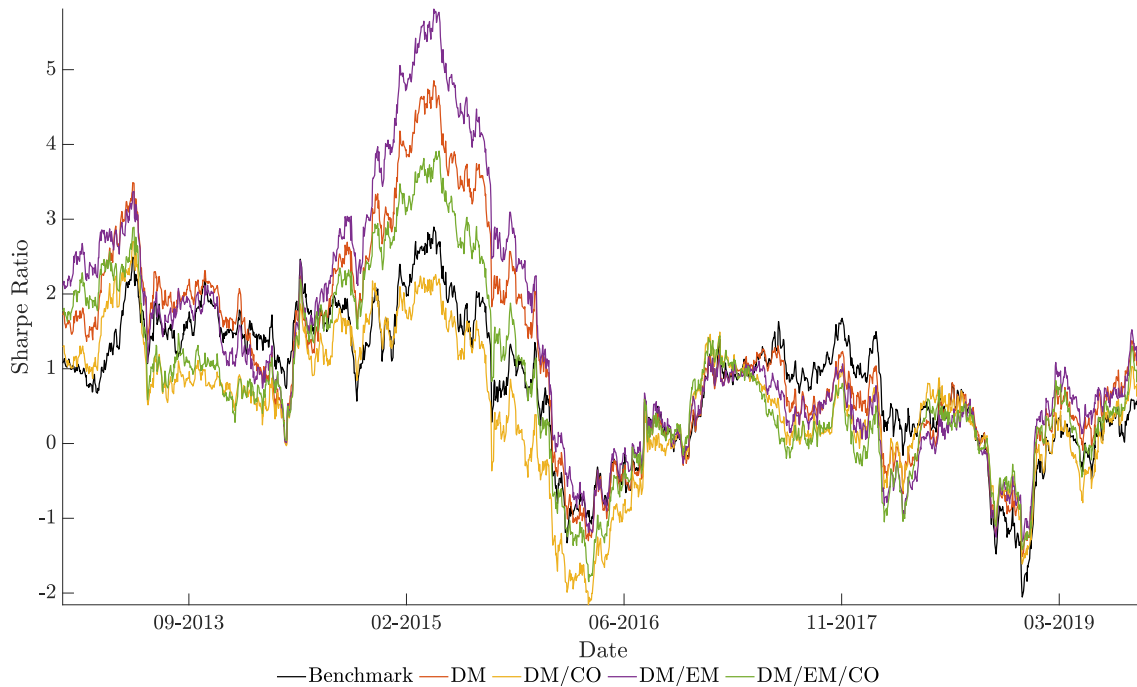
Furthermore, according to the CAPM regression model, the beta of the risk parity portfolios towards the benchmark intuitively decreases by including more assets in the portfolio. All portfolios yield a beta lower than 1, which is highest for the developed markets portfolio with a value of 0.78 and lowest for the developed/emerging markets and commodities portfolio with a beta of 0.45. Also, the portfolios excluding commodities yield significant annual alphas amounting to 1.69% and 2.19% for the developed markets and the developed/emerging markets portfolios, respectively, while portfolios including commodities result in an insignificant alpha of  $-1.01\%$  when added to developed markets and  $0.82\%$  when added to developed/emerging markets.

In addition, a gradual decrease in portfolio concentration is observed the more asset classes are included in the portfolio, as the HHI is lower than the benchmark in all scenarios. We also observe a much lower VaR for risk parity portfolios. Developed and emerging markets portfolios achieve the lowest VaR of 5.03 while the developed markets and commodities portfolio has the highest VaR of 9.48.

The maximum drawdown of the portfolios is also reduced compared to that of the benchmark of  $14.73\%$ , excluding the developed markets and commodities portfolio which experiences a higher drawdown of  $16.95\%$ . The maximum drawdown is heavily influenced by the risk premium of the assets included in the portfolio: adding commodities, which have a negative risk premium, results in an increase in maximum drawdown while adding emerging markets reduces the maximum drawdown to  $8.82\%$ .

Figure 2.4 shows the rolling Sharpe ratios of risk parity portfolios. Figure 2.4a shows that both stock bond portfolios deliver a higher Sharpe ratio than the benchmark over almost the entire sample period. Meanwhile, the underperformance in terms of the Sharpe ratio observed in Table 2.3 for risk parity portfolios including commodities is mainly due to the period from mid 2013 to late 2016, where both portfolios lag the Sharpe ratio of the benchmark by almost 0.5. When portfolios are leveraged, the Sharpe ratio of all portfolios is shifted slightly upwards. Portfolios that include commodities also underperform the benchmark for the period ranging from mid 2013 to late 2016, albeit by a much lesser difference.

In conclusion, risk parity portfolios are highly sensitive to the properties of their asset classes. They suffer when adding negative risk premium assets, such as in this

**Figure 2.4:** Portfolio Rolling Sharpe Ratios**(a)** Unleveraged Risk Parity**(b)** Leveraged Risk Parity

*Notes:* Panel (a) displays the annual rolling Sharpe ratios of unleveraged risk parity portfolios compared to the 60/40 stock/bond benchmark, whereas panel (b) exhibits the annual rolling Sharpe ratios of risk parity portfolios leveraged to a target overall portfolio volatility of 7%. DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

sample commodities, to its allocation universe, since the strategy invests in those assets to equalize the risk contributions of all assets within the portfolio. Risk parity portfolios yield a lower risk compared to its benchmark in all cases, and when negative risk premium assets are excluded from the asset universe, they outperform the traditional 60/40 benchmark on a variety of measures, such as CVaR and Sharpe ratio, among others.

Nonetheless, proponents of Modern Portfolio Theory might argue that other risk-based strategies would perform better, especially the Markowitz minimum variance portfolio, which is analytically designed to minimize risk and has been documented to yield favorable results by numerous studies. As a result, in Table 2.4, we report the out-of-sample results of the allocation strategies described in section 2.3.

Generally, the patterns exhibited by risk parity portfolios are also exhibited by all other weighting techniques. Adding commodities to the portfolio results in a much lower return and Sharpe ratio due to its negative risk premium in the past decade. Nonetheless, adding commodities also results in a lower annual volatility and VaR, although the magnitude of the decrease is lesser than the decrease in returns.

Moreover, risk-based strategies seem to outperform the equally weighted portfolio with regard to every asset universe combination and every portfolio metric, with the exception of the HHI where the equally weighted portfolio by definition achieves the lowest portfolio concentration. Similarly, *VOLA* performs worst among the risk-based strategies in every measure and across all scenarios, but is the least concentrated portfolio as measured by the HHI. Note that *VOLA* is by definition equivalent to *RP* in Table 2.4a, since the asset universe consists of two asset classes and, hence, only one correlation exists.

*MV* yields slightly better results than *RP*. It results in a slightly higher Sharpe ratio and poses a marginal improvement in terms of VaR, but is much more concentrated than *RP*, where the difference in HHI increases with the number of assets. The relatively higher concentration of minimum variance portfolios is not a new observation and has been documented numerous times. However, *RP* is more prone to assets with a negative risk premium than *MV*, as the Sharpe ratio decreases more when adding commodities. The higher vulnerability of the *RP* is most prominent when commodities are added to developed markets, where the Sharpe ratio of *RP* drops from 1.07 to 0.53, which is still better than that of *VOLA* amounting to 0.35.

**Table 2.4:** Risk-Based Portfolios Summary Statistics

(a) Developed Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	7.52	7.98	7.74	7.74	9.87
Volatility	7.29	6.59	6.43	6.41	6.41	7.21
Sharpe Ratio	0.90	1.01	1.11	1.07	1.07	1.25
VaR	9.53	7.81	6.97	7.18	7.18	6.89
HHI	52.00	50.00	50.80	50.47	50.47	73.36
(b) Developed Markets and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	2.77	4.62	2.99	3.90	5.36
Volatility	7.29	6.42	5.70	6.17	5.75	7.22
Sharpe Ratio	0.90	0.30	0.66	0.35	0.53	0.63
VaR	9.53	12.16	8.65	11.35	9.48	11.42
HHI	52.00	33.33	37.00	33.54	35.30	62.67
(c) Developed and Emerging Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	6.62	7.37	6.91	6.96	10.81
Volatility	7.29	5.36	4.99	5.27	5.15	7.27
Sharpe Ratio	0.90	1.08	1.31	1.15	1.19	1.37
VaR	9.53	5.85	4.24	5.35	5.03	6.09
HHI	52.00	25.00	27.15	25.17	25.41	55.67
(d) Developed, Emerging Markets, and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	3.95	5.01	4.20	4.61	7.65
Volatility	7.29	5.16	4.59	4.99	4.70	7.26
Sharpe Ratio	0.90	0.60	0.91	0.67	0.80	0.94
VaR	9.53	8.05	5.67	7.40	6.32	9.24
HHI	52.00	20.00	21.94	20.11	20.73	53.42

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio strategies described in section 2.3 during the period from January 2011 to September 2019, which consists of 2025 daily portfolio return observations. Portfolios are rebalanced on a daily basis and all measures are annualized. *VaR* denotes the parametric VaR at a 99% confidence level during the sample period, while *HHI* represents the Herfindahl-Hirschman Index for portfolio concentration.

The *LRP* portfolio with a target portfolio volatility level of 7% achieves on average a volatility of 7.20% to 7.30%, which is very close to the one observed by the 60/40 developed markets benchmark of 7.29%. It delivers the highest Sharpe ratio of all weighting techniques, excluding the case with developed markets and commodities, where it performs on par with *MV*. However, the significant increase in Sharpe ratio comes at the price of portfolio concentration, where the HHI of *LRP* is typically the highest due to leverage. Nevertheless, *LRP* still yields a lower VaR than *EW* and *VOLA*, and when commodities are not included, the VaR of *LRP* is lower than *BM* while being not significantly higher than the VaRs of *MV* and *RP*.

The volatility and risk exposure of all unleveraged allocation strategies estimated decrease by expanding the asset universe. Even though the risk parity portfolio generally offers a lower risk exposure and volatility than the majority of the allocation strategies employed in this study, they slightly underperform the minimum variance portfolio in terms of Sharpe ratio. The Sharpe ratio of risk parity portfolios decrease when including asset classes with negative risk premiums, as it is the case for commodities in this sample period. The leveraged risk parity portfolio significantly boosts the performance of the portfolio in terms of return, Sharpe ratio, and maximum drawdown, among others. This is also accompanied by a slight increase in volatility, risk exposure, and most prominently portfolio concentration. Furthermore, risk parity portfolios tend to smoothly change weights over time in both the unleveraged and leveraged versions, which in turn makes the higher portfolio concentration less hazardous in terms of risk exposure.

Additionally, our study indicates that in the past decade, asset classes such as commodities had a negative diversification effect on portfolio performance, and that the best portfolio performance was achieved by a portfolio invested in developed and emerging markets. These results are in line with conventional literature investigating the diversification benefits of commodities such as Bessler and Wolff (2015), who find that commodities do not result in any portfolio gain. The observed outperformance of risk parity portfolios in this study further reinforces the findings of Chaves et al. (2011), who find that risk parity portfolios perform better than conventional asset allocation techniques.

## 2.5 Robustness Tests

Portfolio rebalancing is essential for harvesting diversification return. In other words, diversification and rebalancing are inseparable. But what is the underlying dynamic of rebalancing that leads to a positive diversification return for a typical portfolio? This question is crucial to an understanding of the source of diversification return and in extending the analysis to leveraged long-short portfolios.

### Alternative Asset Universe

Table 2.5 reports the results of all strategies as in section 2.4 using the individual stock and bond country indices as single instruments instead of our constructed market indices. The results remain largely unchanged from those observed in Table 2.4. However, all portfolios exhibit better performance measures provided that the number of assets under consideration increased significantly, which makes the portfolio less prone to the negative risk premium of commodities for example.

### Alternative Rebalancing Frequency

We also vary our rebalancing frequency to monthly portfolio holding periods, which are reported in Table 2.6. The results observed here are very similar to those reported in section 2.4. In fact, when portfolios are rebalanced on a monthly basis, the negative impact of commodities on the performance of  $RP$  and  $LRP$  become even more pronounced. Although  $RP$  and  $LRP$  still outperform all other strategies,  $MV$  yields the highest Sharpe ratio when commodities are included, and slightly outperforms  $RP$  and  $LRP$  for the developed and emerging markets portfolio.

### Alternative Data Periodicity

Our final robustness test involves changing the periodicity of the data. Accordingly, we employ monthly data and estimate the input parameters based on the past 24 months of historical monthly data. Table 2.7 reports the results of employing monthly data series. However, in this setting,  $MV$ , although achieving a relatively low volatility compared to other portfolios, is the worst performing portfolio according to the Sharpe ratio. On the contrary,  $VOLA$  dominates  $MV$  according to the mean-variance framework in every scenario and in some cases performs even better than  $RP$ . Nonetheless, the main patterns of our results remain similar to the previous results of section 2.4:  $LRP$  outperforms all other risk-based strategies as well as  $EW$ , while  $RP$  comes in second with respect to the Sharpe ratio.



**Table 2.5:** Risk-Based Portfolios Summary Statistics with Individual Assets

(a) Developed Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	7.52	6.36	7.25	6.93	11.15
Volatility	7.29	6.59	4.30	5.65	5.05	7.33
Sharpe Ratio	0.90	1.01	1.28	1.13	1.20	1.40
VaR	9.53	7.81	3.64	5.91	4.82	5.91
HHI	7.43	7.14	15.77	8.85	9.22	20.81
(b) Developed Markets and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	6.57	5.16	6.16	5.67	9.66
Volatility	7.29	6.33	4.12	5.43	4.73	7.35
Sharpe Ratio	0.90	0.90	1.05	0.98	1.02	1.20
VaR	9.53	8.15	4.43	6.47	5.33	7.44
HHI	7.43	6.67	13.49	8.13	8.53	21.78
(c) Developed and Emerging Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	6.69	6.78	6.77	6.86	13.95
Volatility	7.29	5.41	3.42	4.61	3.75	7.49
Sharpe Ratio	0.90	1.08	1.74	1.28	1.61	1.75
VaR	9.53	5.89	1.17	3.95	1.85	3.47
HHI	7.43	3.85	7.44	4.66	5.24	20.44
(d) Developed, Emerging Markets, and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	7.43	6.19	6.06	6.14	6.06	12.84
Volatility	7.29	5.28	3.35	4.48	3.59	7.48
Sharpe Ratio	0.90	1.01	1.56	1.18	1.45	1.60
VaR	9.53	6.09	1.73	4.29	2.29	4.57
HHI	7.43	3.70	6.62	4.45	5.02	21.38

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio strategies of section 2.3 during the period from January 2011 to September 2019, which consists of 2025 daily portfolio return observations. Portfolios are rebalanced on a daily basis and all measures are annualized. VaR denotes the parametric VaR at a 99% confidence level during the sample period, while HHI represents the Herfindahl-Hirschman Index for portfolio concentration.

**Table 2.6:** Risk-Based Portfolios Summary Statistics with Monthly Rebalancing

(a) Developed Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.80	6.92	7.32	7.13	7.13	9.02
Volatility	7.51	6.91	7.01	6.90	6.90	8.10
Sharpe Ratio	0.80	0.89	0.93	0.92	0.92	1.02
VaR	10.68	9.15	9.00	8.91	8.91	9.83
HHI	52.00	50.00	50.78	50.46	50.46	72.92
(b) Developed Markets and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.80	2.35	4.11	2.58	3.41	4.64
Volatility	7.51	6.26	5.68	6.15	5.69	7.42
Sharpe Ratio	0.80	0.25	0.58	0.29	0.46	0.52
VaR	10.68	12.21	9.10	11.72	9.83	12.63
HHI	52.00	33.33	36.94	33.53	35.28	62.26
(c) Developed and Emerging Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.80	6.17	6.95	6.47	6.54	10.10
Volatility	7.51	5.29	4.78	5.24	5.08	7.56
Sharpe Ratio	0.80	1.02	1.29	1.08	1.13	1.23
VaR	10.68	6.13	4.17	5.72	5.28	7.48
HHI	52.00	25.00	27.10	25.17	25.40	55.25
(d) Developed, Emerging Markets, and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.80	3.58	4.69	3.84	4.26	7.02
Volatility	7.51	5.00	4.19	4.88	4.41	7.18
Sharpe Ratio	0.80	0.56	0.93	0.63	0.79	0.87
VaR	10.68	8.04	5.06	7.50	6.00	9.68
HHI	52.00	20.00	21.94	20.11	20.72	53.01

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio strategies of section 2.3 during the period from January 2011 to September 2019, which consists of 100 monthly portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. VaR denotes the parametric VaR at a 99% confidence level during the sample period, while HHI represents the Herfindahl-Hirschman Index for portfolio concentration.

**Table 2.7:** Risk-Based Portfolios Summary Statistics with Monthly Data

(a) Developed Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.62	6.81	6.05	6.32	6.32	7.78
Volatility	7.87	7.27	7.28	7.25	7.25	8.04
Sharpe Ratio	0.84	0.93	0.83	0.87	0.87	0.96
VaR	11.69	10.11	10.88	10.54	10.54	10.92
HHI	52.00	50.00	50.88	50.37	50.37	58.67
(b) Developed Markets and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.62	1.96	2.01	1.72	2.25	3.57
Volatility	7.87	6.43	6.16	6.30	6.04	8.35
Sharpe Ratio	0.84	0.30	0.32	0.27	0.37	0.42
VaR	11.69	13.00	12.31	12.93	11.81	15.85
HHI	52.00	33.33	37.08	33.51	35.05	67.10
(c) Developed and Emerging Markets						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.62	6.42	5.12	6.30	5.91	7.46
Volatility	7.87	7.41	7.02	7.33	7.07	8.27
Sharpe Ratio	0.84	0.86	0.73	0.85	0.83	0.90
VaR	11.69	10.81	11.22	10.76	10.54	11.78
HHI	52.00	25.00	32.09	25.31	26.26	33.96
(d) Developed, Emerging Markets, and Commodities						
Portfolio	BM	EW	MV	VOLA	RP	LRP
Return	6.62	3.59	2.39	3.67	2.89	4.77
Volatility	7.87	6.69	5.95	6.63	6.01	8.62
Sharpe Ratio	0.84	0.53	0.40	0.55	0.48	0.55
VaR	11.69	11.97	11.44	11.74	11.09	15.30
HHI	52.00	20.00	28.89	20.22	22.14	44.66

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio strategies of section 2.3 during the period from January 2011 to September 2019, which consists of 78 monthly portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. VaR denotes the parametric VaR at a 99% confidence level during the sample period, while HHI represents the Herfindahl-Hirschman Index for portfolio concentration.

## 2.6 Conclusion

We assess the diversification benefits of risk parity portfolios in the context of multi-asset portfolios consisting of a variety of asset classes bearing different characteristics and compare them to the traditional 60/40 stock/bond portfolio, which is most commonly employed by fund managers. We compare the diversification benefits on a wide variety of measures concerning risk, return, and portfolio concentration. Until now, the majority of studies assessing the diversification benefits of portfolios are in-sample analyses and mainly concerned with the performance of mean-variance portfolios and the corresponding shifts or changes to the efficient frontier. However, this setting does not realistically represent the decision environment that investors and fund managers deal with, as they are unable to accurately forecast the future and their forecasts typically underlie significant estimation errors.

Subsequently, a more accurate reflection of the diversification benefits of portfolio analysis requires an out-of-sample approach. However, existing out-of-sample studies are limited to the mean-variance framework and present contradicting findings. In addition, we compare the diversification benefits of risk parity portfolios with other heuristic weighting schemes and popular risk-based allocation models. The asset allocation strategies include the equal-weighting, minimum variance, and inverse volatility. These portfolios are tested in a wide variety of settings with a broader asset universe, a varying holding period, and a different data periodicity.

In conclusion, risk parity portfolios offer a superior risk-return portfolio profile compared with other popular weighting strategies. They offer a lower risk exposure and volatility than conventional allocation strategies while achieving a relatively high Sharpe ratio and average return. Moreover, the weights of risk parity portfolio are rather stable and change smoothly over time. However, risk parity portfolios are highly sensitive to their assets and adding a negative risk premium asset can cause a significant deterioration in terms of the Sharpe ratio. Leveraging the risk parity portfolio significantly boosts the performance of the portfolio in terms of return and Sharpe ratio, among others. This is also accompanied by a slight increase in volatility, risk exposure, and most prominently portfolio concentration.

Although our findings are highly relevant for practice, investors and fund managers need to exert some caution when implementing these models in practice. Our study has some limitations as we employ a select number of assets, a specific set of allocation models, and a definite sample period. Nevertheless, we feel confident that we

investigated the out-of-sample properties of the risk parity strategy in a holistic sense and compared its benefits with those of its most relevant peer allocation schemes. Furthermore, our analysis relied on historical estimates of asset moments, and as numerous studies such as Chopra and Ziemba (1993) have demonstrated, historical estimates are riddled with estimation errors and have lower forecasting power for future returns. Accordingly, if investors employ forecasting models with a higher predictive power, they will be able to benefit to a higher degree when employing the risk parity portfolio.

# Chapter 3

## Linear Shrinkage in Risk Parity<sup>1</sup>

### 3.1 Introduction

Risk parity ( $RP$ ) is a cross-sectional portfolio allocation technique in which each asset contributes the same amount to the total portfolio volatility. While it says nothing about expected returns, an input most difficult to estimate precisely, it limits the risk contribution of individual assets. This appealing property makes such strategies especially popular among practitioners. Only recently,  $RP$  has gained some interest from academia due to its well documented performance in empirical analysis (Chaves et al., 2011). Some studies such as Clarke et al. (2013) and Lee (2014), among others, find that  $RP$  is superior to traditional strategies such as equal- and value-weighting or other weighting techniques that incorporate other moments of asset prices. Further, Maillard et al. (2010) provide theoretical arguments about  $RP$  and show for example, that a negative relationship exists between  $RP$  asset weights and  $RP$  asset betas, which coincides with the findings of Frazzini and Pedersen (2014) that low beta assets outperform high beta assets.

$RP$  is often compared to the Markowitz (1952) minimum variance ( $MV$ ) portfolio. However, the main input parameter to  $RP$  and  $MV$  strategies is the variance-covariance matrix of the asset returns: a property difficult to estimate when the correlations among assets is high, i.e. in times of market crashes, or when the number of assets increase. A popular method that has been shown to improve the risk-return properties of portfolios when estimates of asset moments contain estimation error (Fabozzi et al., 2007) is called *shrinkage*. This technique tries to

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<sup>1</sup>This chapter is based on Alkafri and Frey (2020). Linear Shrinkage in Risk Parity. Working Paper. The paper was presented at the 1st LUMS / TSM / WHU Doctoral Workshop, Toulouse, France, 2019 and at the Frontiers of Factor Investing Conference, Lancaster, United Kingdom, 2021.

mitigate estimation error by simply averaging over various models. For example, Ledoit and Wolf (2004a) demonstrate how linearly shrinking the asset covariance matrix enhances the performance of portfolios, especially mean-variance optimized portfolios. More recently, Ledoit and Wolf (2017) also propose a non-linear shrinkage technique and demonstrate how the Sharpe ratio of portfolios is increased to a greater extent. The latter shrinkage technique is an important breakthrough in risk-budgeting techniques such as *RP* for numerous reasons. Maillard et al. (2010) show that *RP* can be expressed as a minimum variance portfolio optimization with an additional constraint for equalizing the asset risk budgets. Also, the authors show that the volatility of *RP* lies between the volatility of the minimum variance portfolio and that of the equally-weighted portfolio. This implies that *RP* results from a shrinkage between the minimum variance and the equally-weighted portfolio.

An alternative to overcoming estimation error is to disregard plug-in estimates and employ a heuristic allocation scheme such as equal weighting (*EW*). Given that *EW* disregards asset moments, it is viewed as the most heuristic method of allocation. Still, disregarding the asset characteristics makes certain properties of *EW* portfolios, such as the level of diversification, highly sensitive to the underlying asset universe. For example, if the risk of the underlying assets vary significantly, a high risk concentration and a limited degree of diversification are attained, as the weights of the riskiest and the least risky assets in *EW* are identical. While *EW* strategies are employed on a broad scale by mutual and pension funds (Benartzi and Thaler, 2001; Windcliff and Boyle, 2004), it is viewed as an ad hoc manner of deconcentration by equally spreading the initial wealth across all assets. As a matter of fact, the notion of deconcentration in *EW* has led to the development of *RP* portfolios, which aim to deconcentrate the portfolio from a risk perspective, where each asset contributes to the same amount of risk to the overall portfolio risk. Amenc et al. (2012) argue that such techniques are classified as ad hoc allocation schemes, due to their reliance on the notion of deconcentration and the absence of a theoretical framework.

We investigate the impact of shrinkage estimation techniques for the moments of asset returns in risk parity portfolios. Given the number of assets in the asset universe, the risk contributions of individual assets in *RP* portfolios are fixed a priori. This additional information is commonly found to stabilize empirical portfolio weights over time. Accordingly, we assess whether an asset's risk budget in the *RP* optimization serves as a natural shrinkage target, and whether the parity element of *RP* is beneficial in improving portfolio performance relative to conventional linear

and non-linear shrinkage techniques. We therefore bridge the gap between mean-variance and risk parity portfolios in a unique way. We test our results in asset allocation and high-dimensional portfolios and find that the marginal risk-budget for each portfolio asset indeed serves as a natural shrinkage target, in which  $RP$  weights are a results of shrinking  $MV$  weights towards  $EW$  weights given each asset's risk budget. Hence, we also provide a new perspective on risk parity portfolios. In higher dimensions however, we find that risk parity strategies can not compete with shrinkage solutions to  $MV$  portfolios; a clear indication that parity shrinkage in  $RP$  has its limits.

This study is related to Ardia et al. (2017), who assess the impact of estimation errors in the asset variance-covariance matrix in various risk-based portfolios; among them the  $RP$  strategy. Using Monte Carlo simulation methods, they find that equal-risk-contribution and inverse-volatility weighted portfolio weights are relatively robust to covariance misspecification while the  $MV$  portfolio weights are highly sensitive to errors in both the estimated variances and correlations. While their study only operates in small dimensions with 5 to 30 assets, we investigate the impact of shrinkage estimation techniques also in high dimensions, where  $RP$  portfolios are typically not applied. Accordingly, it is important not only to study where the regularization benefits come from, but also to understand where the limitations arise.

The remainder of this study is organized as follows. Section 3.2 motivates our study by showing the impact of estimation error on  $RP$  and  $MV$  portfolios in a simulation study. Section 3.3 shows that the parity element in  $RP$  acts as a natural shrinkage device. Section 3.4 describes the empirical setup and briefly defines other conventional shrinkage and heuristic weighting techniques in the literature.  $RP$ ,  $MV$ , and the strategies in section 3.4 are assessed in a multi asset allocation setting as well as in high-dimensional equity portfolios in sections 3.5 and 3.6. Section 3.7 concludes.



## 3.2 A Motivational Example

Academia has often focused on mean-variance portfolios (Markowitz, 1952), in which the portfolio weighting is governed by the risk preferences of the investor and the diversification benefits of individual assets. Despite the theoretical elegance and intuitiveness of the mean-variance framework, it did not fare well in practice and resulted in highly concentrated, underperforming, and simply “wrong” portfolios (Michaud, 1989). The observed underperformance of mean-variance optimized portfolios is due to the fact that the deterministic input parameters, namely the asset return vector and the asset covariance matrix, can only be estimated with severe estimation errors.<sup>2</sup> The only applied exception of mean-variance optimized portfolios is *MV*, which only attempts to minimize portfolio risk whilst neglecting asset returns. Numerous studies such as Chopra and Ziemba (1993), Clarke et al. (2006), and Chow et al. (2011) document the outperformance of *MV* in equity markets. However, *MV* has its shortcomings and a main issue is its high concentration in low volatility assets (Chan et al., 1999; Clarke et al., 2011; DeMiguel et al., 2009).

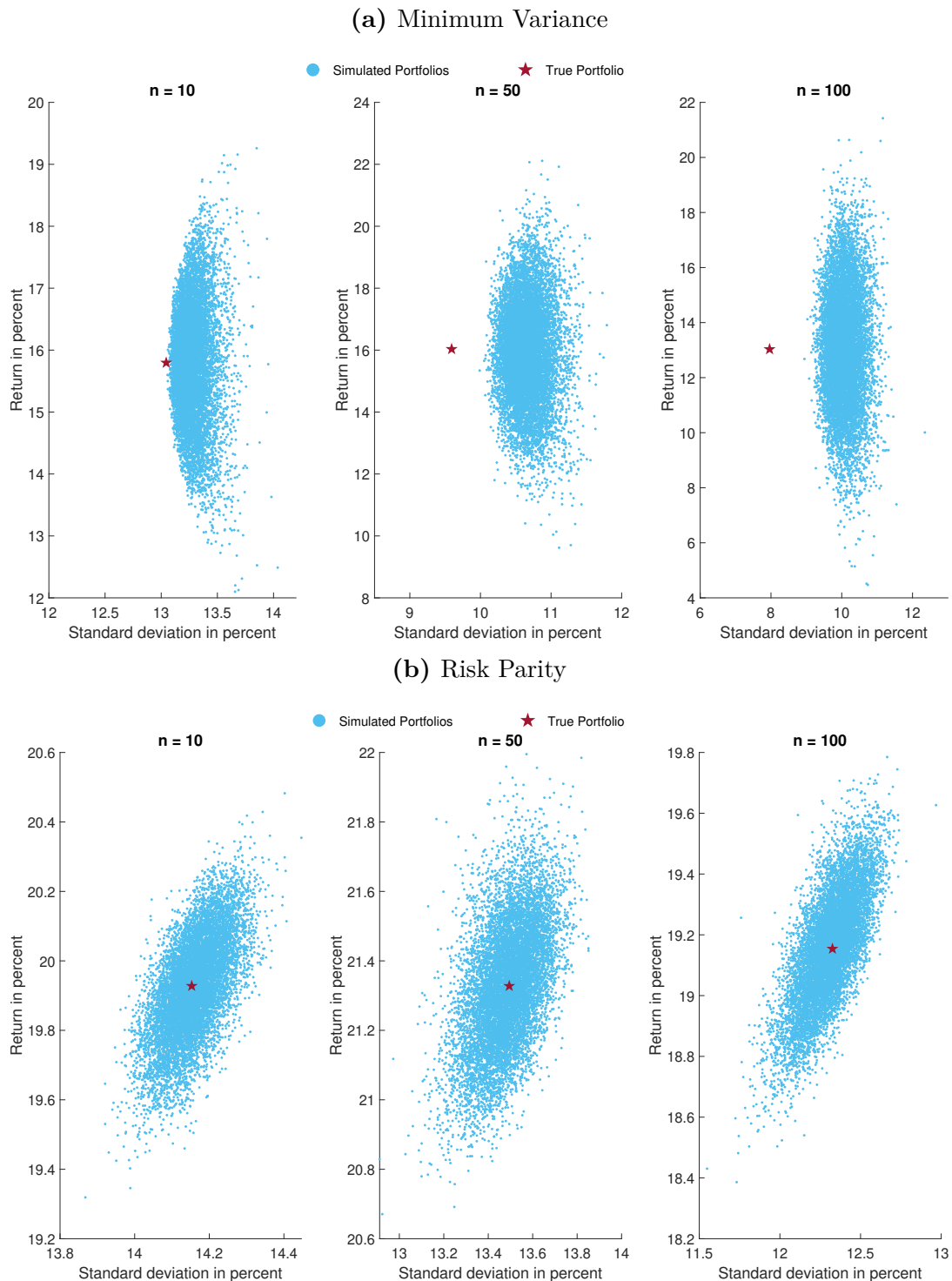
To provide some more empirical intuition for the reader, we replicate the main finding of Jobson and Korkie (1980) to illustrate the difference between *RP* and *MV* in terms of its sensitivity to plug-in estimates. We randomly select  $N = 10, 50, 100$  assets from stocks that are listed in the NYSE, AMEX, and NASDAQ exchanges. In each scenario, we estimate the ex post asset moments and accordingly determine the “true” *RP* and *MV* portfolios and compare them with 10,000 portfolios, which are computed based on 250 hypothetical returns that are simulated using a multivariate normal distribution with return vector ( $\mu$ ) covariance matrix ( $\Sigma$ ) equal to the assets’ “true” ex-post return and covariance matrix. The purpose of the simulation is to evaluate how close these simulated portfolios are compared to the true portfolio when increasing the number of assets held in the portfolio. The results of the simulation experiment are illustrated in Figure 3.1.

Our results are in line with Jobson and Korkie (1980) for *MV*. Simulated *MV* portfolios either yield a higher return with a higher variance or are suboptimal compared with the true *MV* portfolio. Furthermore, the difference in *MV* between the true portfolio and estimations thereof increases with the number of assets. Meanwhile, the simulated *RP* portfolios are heavily concentrated around the true *RP* portfolio, which lies roughly in the center of all simulated portfolios. This observation holds

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<sup>2</sup>See for example Best and Grauer (1991); Chopra and Ziemba (1993); Jobson and Korkie (1980); Michaud (1989)).

Figure 3.1: Portfolio Sharpe Ratio under Estimation Error



*Notes:* This figure illustrates the difference in the risk-return properties between the true portfolio and 10,000 simulated portfolios for the cases of  $N = 10, 50, 100$  assets. The true portfolio is determined by the ex post sample moments, whereas the simulated portfolios are determined by plug-in estimates for 250 simulated returns.

true regardless of the number of assets in the portfolio. The results suggest that  $RP$  can be better estimated using plug-in estimates, as the simulated portfolios lie closely around the true portfolio. Also, simulated  $RP$  portfolios can not only be sub-optimal compared to the true portfolio, but can also dominate the true  $RP$  portfolio in the mean-variance space. Accordingly, the results imply that the  $RP$  technique, by strictly defining the total risk contribution of each asset, induces a structure on the portfolio weights and, hence, can be deemed as a form of portfolio regularization.

### 3.3 Methodology

$RP$  portfolios constitute a middle-ground between  $MV$  and  $EW$ . They maintain the notion of deconcentration as in  $EW$  whilst considering single and joint asset total risk contributions so that all assets contribute equally to portfolio risk. Although  $MV$  equalizes asset risk contributions, it does so merely on a marginal basis. This implies that a minimal change in the weight of any asset in  $MV$  should result in the same ex-ante change in overall portfolio risk. However, the total risk contributions are generally not equal, which causes portfolio risk to be mainly concentrated in a few assets, foregoing the benefits of diversification. Maillard et al. (2010) show that risk parity can be denoted as the minimum variance portfolio with an additional risk budget constraint, formally it can be expressed as

$$\begin{aligned}
 y^* &= \arg \min \sqrt{y' \Sigma y} \\
 \text{s.t. } &\begin{cases} \sum_{i=1}^n \ln y_i \geq c \\ y \geq 0 \end{cases}, \tag{3.1}
 \end{aligned}$$

where the optimal weights  $w^* = \{w_1^*, \dots, w_N^*\}$  that sum up to 100% are then determined as  $w_i^* = y_i^* / \sum_{j=1}^N y_j^* \quad \forall i = 1, \dots, N$ .  $\gamma$  is an arbitrarily chosen constant and  $\Sigma$  describes the asset variance-covariance matrix. Indeed, Maillard et al. (2010) show that, based on the risk budget determined by  $\gamma$ , the volatility of  $RP$  lies between those of  $MV$  and  $EW$ , or formally  $\sigma_{MV} < \sigma_{RP} < \sigma_{EW}$ .<sup>3</sup>

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<sup>3</sup>For more information on the derivation of these properties of  $RP$  as well as the constant  $\gamma$ , refer to Maillard et al. (2010), Appendix B, pp. 68-69.

An alternative to overcoming estimation error is to adjust the plug-in estimates objectively or subjectively, which can be based on the expectations of fund managers, asset pricing theories, or a combination of both (Black and Litterman, 1992). Above all, statistical methods which aim at obtaining estimators that are less prone to sampling errors or fat-tails are employed. Most prominently are shrinkage techniques, which are a routine of averaging various estimators.<sup>4</sup> The notion underlying shrinkage is that portfolios that yield a stable performance regardless of market phases, namely portfolios that are more robust to estimation errors and model risk, are better suited for investors, especially risk-averse investors. This property is paramount in finance, given that financial returns are heavily entailed with extreme returns or fat-tails that typically have a significant impact on the portfolio optimization process and, thus, portfolio performance. Hence, shrinkage techniques reduce the impact of fat-tails and estimation errors, yielding more conservative portfolios that perform well regardless of market conditions (Fabozzi et al., 2007).

Shrinkage techniques date back to James and Stein (1961) and typically consist of: (i) An arbitrary estimator such as historical plug-in estimates, (ii) a structured shrinkage target, and (iii) a parameter that determines the shrinkage intensity between (i) and (ii). James and Stein (1961) show that the benefit of shrinkage techniques is while the shrinkage target is typically biased, it has less variance than the plug-in estimate. Hence, it can be shown that under general conditions a shrinkage intensity exists where the estimation error of the resulting shrinkage estimator is lower than that of the sample plug-in estimate.

## Linear Shrinkage in Risk Parity

Provided our previous arguments, the *RP* strategy can be defined as a simple form of portfolio regularization. While shrinkage and factor approaches reduce dimensionality by reducing the number of estimable parameters, risk parity induces a structure on the portfolio weights by restricting the risk contribution of individual assets to the total portfolio variance.

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<sup>4</sup>Other methods to mitigate the effect of estimation errors include Bayesian techniques (Barry, 1974), Bayesian approach in combination with asset pricing models (Pástor, 2000), robust optimization models (Garlappi et al., 2007), Bayesian robust optimization techniques (Wang, 2005), robust estimation methods (DeMiguel and Nogales, 2009), and employing portfolio constraints (Jagannathan and Ma, 2003).

### 3.3. METHODOLOGY

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Provided  $N$  assets, let  $r = \{r_1, \dots, r_N\}$  be a collection of the  $N$  asset returns and  $w = \{w_1, \dots, w_N\}$  the corresponding portfolio weight vector. The portfolio variance is defined by  $\sigma_p^2(w) = w' \Sigma w = \sum_{i=1}^N w_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{i \neq j} w_i w_j \sigma_{ij}$ , where  $\Sigma$  is the  $N \times N$  variance-covariance matrix of  $r$  with variances  $\sigma_i^2$  and covariances  $\sigma_{ij}$ . The marginal risk contribution of a single asset  $i$  towards the total portfolio is  $\partial \sigma_p(w) / \partial w_i$ . Given that the total portfolio variance is the sum of the individual risk contributions of all assets,  $\sigma_p(w) = \sum_{i=1}^N (w_i \times \partial \sigma_p(w) / \partial w_i)$ , the risk budgeting portfolio is defined by

$$w_{rp} = \left\{ w \in [0, 1]^N : \sum_{i=1}^N w_i = 1, w_i \geq 0, w_i \times (\Sigma w)_i = b_i \times (w' \Sigma w) \right\}, \quad (3.2)$$

where  $w_i$  is the  $i^{\text{th}}$  element of the weight vector,  $(\Sigma w)_i$  denotes the  $i^{\text{th}}$  element of the corresponding vector and  $b_i \geq 0$  is the weight risk budget for asset  $i$  with  $\sum_{i=1}^N b_i = 1$ . For the equally-weighted risk contribution ( $RP$ ) portfolio case in which  $b_i = b_j \forall i, j$ , Maillard et al. (2010) show that  $\sigma_p(w_{mv}) \leq \sigma_p(w_{rp}) \leq \sigma_p(w_{EW})$ .

The important question now is how shrinkage approaches and the risk budgeting restriction are related in a portfolio context. Let us start with the most basic setup of a bivariate  $RP$  portfolio. With  $w = (w_1, 1 - w_1)$  and  $\sigma_1^2$  and  $\sigma_2^2$  denoting the variances of asset 1 and 2, it can be shown that for a long-only case ( $0 \leq w_1 \leq 1$ ), the unique  $RP$  portfolio solution to Equation 3.1 is

$$w_{RP} = \left( \frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}, \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right) = \left( \frac{\sigma_2}{\sigma_1 + \sigma_2}, \frac{\sigma_1}{\sigma_1 + \sigma_2} \right). \quad (3.3)$$

This reveals the enriching property of the  $RP$  portfolio: The higher the variance in asset 2, the higher the weight of asset 1 and vice versa. Further, assuming that assets 1 and 2 are uncorrelated ( $\rho_{12} = 0$ ), then the  $RP$  portfolio coincides with the minimum-variance portfolio ( $w_{RP} = w_{gmv}$ ).<sup>5</sup>

The empirical counterpart with estimated variance-covariance matrix is

$$\hat{w}_{RP} = \left( \frac{\hat{\sigma}_1^{-1}}{\hat{\sigma}_1^{-1} + \hat{\sigma}_2^{-1}}, \frac{\hat{\sigma}_2^{-1}}{\hat{\sigma}_1^{-1} + \hat{\sigma}_2^{-1}} \right) = \left( \frac{\hat{\sigma}_2}{\hat{\sigma}_1 + \hat{\sigma}_2}, \frac{\hat{\sigma}_1}{\hat{\sigma}_1 + \hat{\sigma}_2} \right). \quad (3.4)$$

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<sup>5</sup>It holds that  $w_{gmv} = \left( \frac{\sigma_2 - \sigma_{12}}{\sigma_1 + \sigma_2 - 2\sigma_{12}}, \frac{\sigma_1 - \sigma_{12}}{\sigma_1 + \sigma_2 - 2\sigma_{12}} \right)$  where  $\sigma_{12} = \rho_{12} \sigma_1 \sigma_2$  and  $\rho_{12}$  is the correlation coefficient between assets 1 and 2.

Furthermore, given

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{pmatrix}, \quad (3.5)$$

while for low  $N$  inverting the estimated variance-covariance matrix ( $\hat{\Sigma}$ ) to obtain the  $RP$  portfolio weights is not very costly in terms of estimation errors, it becomes a problem for larger asset universes or more correlated markets. Typically, dimension reduction techniques such as factor modeling or shrinkage towards a shrinkage target are valid attempts to mitigate estimator errors. A traditional shrinkage estimator for the variance-covariance matrix by Ledoit and Wolf (2004a) shrinks the sample variance-covariance matrix estimate towards the identity matrix ( $\mathcal{I}$ ). Given a shrinkage intensity  $\kappa \in [0, 1]$ , the approach yields

$$\begin{aligned} \hat{\Sigma}^* &= \kappa \mathcal{I} + (1 - \kappa) \hat{\Sigma} \\ &= \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} + (1 - \kappa) \begin{pmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{pmatrix} \\ &= \begin{pmatrix} \kappa + (1 - \kappa) \hat{\sigma}_1^2 & (1 - \kappa) \hat{\sigma}_{12} \\ (1 - \kappa) \hat{\sigma}_{12} & \kappa + (1 - \kappa) \hat{\sigma}_2^2 \end{pmatrix} \end{aligned} \quad (3.6)$$

Hence, the  $RP$  portfolio for this case and replacing the traditional variance-covariance matrix with a shrinkage equivalent yields

$$\hat{w}_{RP}^* = \begin{pmatrix} \frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_1^2}}}{\frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_1^2}} + \frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_2^2}}}, \frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_2^2}}}{\frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_1^2}} + \frac{1}{\sqrt{\kappa + (1 - \kappa) \hat{\sigma}_2^2}}} \end{pmatrix}$$

It directly follows that

$$\lim_{\kappa \rightarrow 0} \hat{w}_{RP}^* = \left( \frac{\hat{\sigma}_2}{\hat{\sigma}_1 + \hat{\sigma}_2}, \frac{\hat{\sigma}_1}{\hat{\sigma}_1 + \hat{\sigma}_2} \right) \quad (3.7)$$

$$\lim_{\kappa \rightarrow 1} \hat{w}_{RP}^* = \left( \frac{1}{2}, \frac{1}{2} \right). \quad (3.8)$$

Hence, we see that the shrinkage approach forces the  $RP$  portfolio towards the naive  $EW$  portfolio.

## 3.4 Empirical Setup

### Strategies

Given the results of Section 3.3, the question arises as to how *RP* fares compared to other shrinkage-based allocation schemes. The following section briefly describes the allocation techniques employed in the empirical analysis, which consist of heuristic weighting schemes as well as the most prominent shrinkage techniques in the literature. Table 3.1 provides an overview of all strategies investigated in our study.

**Table 3.1:** List of Employed Portfolio Strategies

Nº	Strategy Definition	Abbreviation
Panel A: Heuristic Weighting Techniques		
1	Equally Weighted Portfolio	<i>EW</i>
2	Inverse Volatility Portfolio	<i>VOLA</i>
Panel B: Minimum Variance Portfolios		
3	Minimum Variance Portfolio	<i>MV</i>
4	Minimum Variance Portfolio with Linear Covariance Matrix Shrinkage using the Identity Matrix Model	<i>MVID</i>
5	Minimum Variance Portfolio with Linear Covariance Matrix Shrinkage using the Constant Correlation Model	<i>MVCCM</i>
6	Minimum Variance Portfolio with Non-Linear Covariance Matrix Shrinkage	<i>MVNLS</i>
Panel C: Risk Parity Portfolios		
7	Risk Parity Portfolio	<i>RP</i>
8	Risk Parity Portfolio with Linear Covariance Matrix Shrinkage using the Identity Matrix Model	<i>RPID</i>
9	Risk Parity Portfolio with Linear Covariance Matrix Shrinkage using the Constant Correlation Model	<i>RPCCM</i>
10	Risk Parity Portfolio with Non-Linear Covariance Matrix Shrinkage	<i>RPNLS</i>

### Inverse Volatility

The inverse volatility strategy<sup>6</sup> (*VOLA*) is a heuristic allocation scheme that assigns weights according to asset volatilities, formally it is defined as

$$w_i = \frac{1/\sigma_i}{\sum_{i=1}^N 1/\sigma_i} \quad (3.9)$$

where  $\sigma_i$  is the volatility of asset  $i$ . The weight of an asset  $i$  equals its inverse volatility divided by the sum of inverse volatilities for all assets. As a result, *VOLA* overweights low volatility stocks and underweights high volatility stocks. Hence, it is related to the low-volatility effect (Baker et al., 2011), which has been shown to outperform other heuristic weighting schemes such as equal- and value-weighting (Anderson et al., 2012). However, the total portfolio volatility is not minimized with *VOLA* since asset covariances are not considered in the weighting process.

### Linear Shrinkage

We also employ the Ledoit and Wolf (2004a,b) linear shrinkage techniques, which can be described as optimally weighted averages of the sample variance-covariance matrix of the asset universe ( $\hat{\Sigma}$ ) and a shrinkage target ( $\hat{F}$ ). The shrinkage approach can be expressed as follows

$$\hat{\Sigma}_{LWLS} = \alpha \hat{F} + (1 - \alpha) \hat{\Sigma},$$

where  $\alpha$  denotes the shrinkage intensity,  $\hat{\Sigma}$  represents the sample variance-covariance matrix, and  $\hat{F}$  describes the shrinkage target.<sup>7</sup> We employ two different models as a shrinkage target, which are the constant correlation model implied variance-covariance matrix (*MVCCM*; *RPCCM*) (Ledoit and Wolf, 2004a) and the variance-covariance matrix implied by a multiple of the identity matrix (*MVID*; *RPID*) (Ledoit and Wolf, 2004b).

### Non-Linear Shrinkage

Recently, a new class of shrinkage methods has emerged, which are referred to as non-linear shrinkage techniques. Unlike linear shrinkage, these techniques impose less structure on the plug in estimates and shrink each estimate by an amount, where the shrinkage intensity varies in a non-linear manner depending on how far they lie

<sup>6</sup>This strategy is also referred to as naive risk parity, as it equalizes the risk contributions of assets when the correlations among assets are (almost) identical. See Maillard et al. (2010).

<sup>7</sup>For more information on the shrinkage targets as well as the derivation of the optimal shrinkage intensity refer to Ledoit and Wolf (2004a,b).



from the shrinkage target. In this study, we will focus on the analytical non-linear shrinkage technique proposed by Ledoit and Wolf (2017) as this is superior in terms of computational power and feasibility as compared with numerical approaches such as Abadir et al. (2014), and Lam (2016), among others. The analytical non-linear shrinkage estimator  $\hat{\Sigma}_{LWNLS}$  is determined by

$$\hat{\Sigma}_{LWNLS} = U_T \hat{\delta}_T^o U_T \text{ with } \hat{\delta}_T^o := \text{Diag} \left( \hat{\phi}_T^o(\lambda_{T,1}), \dots, \lambda_{T,N} \right),$$

where  $U_T := [u_{T,1}, \dots, u_{T,N}]$  is a (full) orthogonal matrix whose columns are the sample eigenvectors  $u_{T,i}$ ,  $\lambda_{T,i}$  denotes the sample eigenvalues, and  $\hat{\phi}_T^o(x)$  represents the optimal shrinkage function.<sup>8</sup> We employ this technique on the risk parity (*RPNLS*) and minimum variance (*MVNLS*) portfolios.

## Data Description

Data were collected from Datastream and consist of daily total return price indices, which are price indices that are adjusted for dividends and stock splits, from 01.01.1994 to 30.09.2019. We employ an out-of-sample rolling-window approach as in DeMiguel et al. (2009). Each data set consists of  $M = 6,783$  business days, where the asset moments are estimated based on the previous  $T = 250$  days. Portfolios are held for  $\theta = 20$  business days then rebalanced using the previous  $T$  business days. We test the portfolios in Table 3.1 in different settings, which are: (1) in multi asset allocation where  $T \gg N$  and the risk-return properties of different asset classes vary to greater extent than the risk-return properties within a single asset class; and (2) highly dimensional<sup>9</sup> equity portfolios where plug-in estimates are quite unreliable (Jobson and Korkie, 1980).

Let  $P_{i,t}$  be the price of asset  $i$  at time  $t$  and  $D_{i,t}$  be the dividend payment of asset  $i$  in period  $t$ , then the total return of asset  $i$  in period  $t + 1$  ( $r_{i,t+1}$ ) is defined as

$$r_{i,t+1} = \frac{P_{i,t+1} + D_{i,t}}{P_{i,t}} - 1. \quad (3.10)$$

Accordingly, the total return of a strategy  $j$  with  $j \in \{1, \dots, 10\}$  and  $N$  assets at time  $t + 1$  is  $r_{j,t+1} = \sum_{i=1}^N (r_{i,t+1} \times \hat{w}_{i,j,t})$ , where  $\hat{w}_{i,j,t}$  represents the weight of asset  $i$  for strategy  $j$  at time  $t$ . At rebalancing, a trade of  $|\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t}|$  occurs for each asset  $i$  held in strategy  $j$ , where  $\hat{w}_{i,j,t+}$  denotes the weight of asset  $i$  for

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<sup>8</sup>For more information on the derivation of  $\hat{\phi}_T^o(x)$  see Ledoit and P ech e (2011), Theorem 3.

<sup>9</sup>Typically, when the number of observations  $T$  is not significantly larger than the number of assets  $N$ , a portfolio is referred to as a high dimensional portfolio.

strategy  $j$  immediately before rebalancing at time  $t + 1$ , which is determined as  $\hat{w}_{i,j,t+} = \hat{w}_{i,j,t} \times (1 + r_{i,t+1})$ . Following DeMiguel et al. (2009), let  $\xi$  represent a proportional transaction cost paid for each asset when rebalancing the portfolio, then the total cost incurred from trading all assets for portfolio  $j$  is obtained by  $\xi \times \sum_{i=1}^N |\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t+}|$ . As a result, the wealth from investing in strategy  $j$  at time  $t + 1$  is determined as

$$W_{j,t+1} = W_{j,t}(1 + r_{j,t+1})(1 - \xi \sum_{i=1}^N |\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t+}|), \quad (3.11)$$

where the return net of transaction costs for strategy  $j$  is given by  $(W_{j,t+1}/W_{j,t}) - 1$  with  $W_{j,t+1}$  and  $W_{j,t}$  describing wealth at time  $t + 1$  and  $t$ , respectively. Moreover, as in DeMiguel et al. (2009), the turnover of strategy  $j$  is defined as the average sum over time of the absolute value of the trades incurred for all assets or formally

$$Turnover_j = \frac{1}{M - T} \sum_{t=1}^{M-T} \sum_{i=1}^N (|\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t+}|). \quad (3.12)$$

Turnover measures the average percentage of total wealth that is traded at each rebalancing period and is related to the transaction costs incurred when implementing a strategy  $j$ . It is noteworthy to mention that a portfolio strategy which is superior in gross terms, that is in the absence of transaction costs, might not be optimal to implement when transaction costs are taken into account.

### 3.5 Applications in Multi Asset Allocation

To demonstrate how  $RP$  fares compared with the aforementioned strategies, we analyze the characteristics of each strategy in an asset allocation setting, where all strategies are rebalanced on a monthly basis. The data set comprises of equities and bonds of all developed countries according to the MSCI classification.<sup>10</sup> Equities are represented by the MSCI total return indices and are denoted in USD. Bond markets are denoted by the Datastream 10 year government bond indices, and are hedged to USD according to the interest rate parity model using the Thomson Reuters spot and 1 month forward rates. Commodities are described by the Bloomberg Commodity Index and are denoted in USD. Table 3.2 reports the risk-return characteristics of  $RP$ ,  $EW$ ,  $MV$ , and the shrinkage techniques highlighted in Table 3.1.

All strategies result in a positive growth of wealth over time. In gross terms,  $MV$

<sup>10</sup>We exclude Hong Kong, Israel, and Singapore due to insufficient historical bond data.

Table 3.2: Global Portfolio Summary Statistics

	<i>EW</i>	<i>VOLA</i>	<i>MV</i>	<i>MVID</i>	<i>MVCCM</i>	<i>MVNLS</i>	<i>RP</i>	<i>RPID</i>	<i>RPCCM</i>	<i>RPNLS</i>
Panel A: General Performance Metrics										
Gross Wealth	232.80	244.61	340.40	242.64	283.96	236.32	290.09	244.61	244.61	299.57
Net Wealth	224.97	235.11	156.81	200.31	190.69	174.69	259.84	235.11	227.94	269.97
TC (\$)	7.82	9.50	183.59	42.33	93.27	61.63	30.26	9.50	16.67	29.60
TC (%)	3.36	3.88	53.93	17.45	32.85	26.08	10.43	3.88	6.81	9.88
Turnover	0.22	0.25	4.94	1.22	2.54	1.93	0.70	0.25	0.45	0.66
HHI	2.44	2.63	30.25	44.49	34.27	19.08	3.45	2.63	3.95	3.06
Panel B: Gross Performance Characteristics										
Annual Return	4.37	4.41	5.27	3.96	4.26	3.62	4.98	4.41	4.41	5.12
Annual Volatility	14.51	13.48	10.26	10.27	6.45	7.48	13.02	13.48	13.48	13.20
Maximum Drawdown	51.12	46.51	23.98	45.77	18.61	22.68	40.62	46.51	46.51	41.04
Skewness	-1.30	-1.14	0.52	0.08	-0.86	-0.81	0.06	-1.14	-1.14	0.05
Kurtosis	13.68	13.06	9.32	5.52	7.61	6.78	9.54	13.06	13.06	8.98
Sharpe Ratio	0.30	0.33	0.51	0.39	0.66	0.48	0.38	0.33	0.33	0.39
Panel C: Net Performance Characteristics										
Annual Return	3.14	3.31	1.74	2.69	2.50	2.16	3.70	3.31	3.19	3.84
Annual Volatility	15.04	13.87	10.18	10.24	6.46	7.54	13.04	13.87	13.87	13.21
Maximum Drawdown	51.24	46.63	35.83	49.34	20.11	24.87	40.90	46.63	46.68	41.30
Skewness	-2.18	-1.91	-0.15	-0.12	-1.04	-0.94	-0.46	-1.91	-1.91	-0.46
Kurtosis	21.44	19.21	8.21	5.44	8.40	7.21	10.04	19.21	19.22	9.62
Sharpe Ratio	0.21	0.24	0.17	0.26	0.39	0.29	0.28	0.24	0.23	0.29

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio allocation schemes summarized in table 3.1. The data set spans the period from January 1994 to December 2019, which consists of 6782 daily portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. *Gross Wealth* is the terminal wealth accumulated over the whole sample period on \$100 investment while *Net Wealth* represents the terminal wealth net of transaction costs (*TC*), which are listed in nominal terms (*TC* (\$)) as well as percentage points of Terminal Gross Wealth (*TC* (%)). *HHI* denotes the Herfindahl-Hirschman Index for portfolio concentration.

yields the highest terminal wealth (\$340.40), followed by *RPNLS* (\$299.57), *RP* (\$290.09), and *MVCCM* (\$283.96), while other shrinkage and heuristic strategies yield a much lower gross wealth of circa \$240. However, *MV* is the most concentrated portfolio with an HHI of 53%. Shrinkage techniques decrease the portfolio concentration, with *MVID* yielding the lowest concentration of 17.45% while *MVCCM* yields the highest HHI of 32.85%. *EW* is the least concentrated portfolio by definition (2.36%), followed by *VOLA* (3.88%), whereas *RP* yields an HHI of 10.43%.

Shrinkage techniques for *RP* do not necessarily improve portfolio characteristics and generally do not vary significantly from *RP*. *RP* yields slightly better results than its shrinkage counterparts in terms of return, volatility, maximum drawdown, and even turnover. Nevertheless, the variation as mentioned earlier remains marginal and merits no significant improvements. In terms of net Sharpe ratio, only *RPNLS* (0.29) yields a value on par with *RP* (0.28), whilst *RPID* (0.24) and *RPCCM* (0.23) underperform. This indicates that, unlike for the case of *MV*, the regularization implied by the risk budget constraint on the weights of *RP* portfolios is quite stark and no longer enables the portfolio weights of significant further alteration.

Since a higher portfolio concentration typically results in a higher turnover ratio and, thus, higher transaction costs, the results change regarding net terminal wealth. *MV* performs worst due to having the highest turnover ratio (4.94), which translates into transaction costs of roughly 54% of total gross wealth, resulting in a terminal net wealth of merely \$156.81. This result coincides with the findings of Best and Grauer (1991), Michaud (1989), among others, that mean-variance optimized portfolios are highly sensitive to estimation risk, and typically result in high turnover ratios which ultimately leads to a significant underperformance relative to other weighting techniques. Meanwhile, *RP* yields the highest net terminal wealth of roughly \$260, followed by *VOLA* (\$235.11) and *EW* (\$224.97). Prominent shrinkage techniques pose an improvement compared to *MV*, albeit remaining suboptimal compared to *RP* and the heuristic techniques in terms of net terminal wealth.

All portfolio returns are leptokurtic; indicating the existence of fat-tails in the return distribution. This is in line with the longstanding finding that financial returns are heavily entailed with extreme returns or fat-tails. Interestingly, although the traditional view on financial returns is that they are negatively skewed (Fama, 1965) and despite the presence of financial crises in our sample period, not all portfolios exhibit negatively skewed gross returns. *RP* and *MVID* pose virtually no skewness, while

**Figure 3.2:** Global Portfolio Performance Measures over Time



*Notes:* This figure illustrates the differences in gross historical performance, realized Sharpe ratio, realized volatility and realized Conditional Value at Risk (CVaR) between the different portfolio strategies outlined in Table 3.1. The gray shaded areas indicate U.S. recessions as reported by NBER. The results in Panels (b) to (d) are calculated based on a rolling window of 60 monthly portfolio realized return observations.

$MV$  reveals a positive skewness of 0.51. These results imply that in the absence of transaction costs, these allocation strategies are effective in mitigating the extreme negative returns that occur during market turbulences. Nonetheless, all portfolio returns are negatively skewed when taking transaction costs into account.

Regarding the maximum drawdown (MDD), which is the maximum loss during the sample period,  $MVCCM$  and  $MVNLS$  perform best in both gross and net terms.  $MV$  shows a quite low gross MDD (23.98%), but due to the high transaction costs, it results in a net drawdown of 35.83%. Heuristic strategies, namely  $EW$  and  $VOLA$ , exhibit the MDD along with  $MVID$  of roughly 50%, while  $RP$  results in a drawdown of 40%. Finally, albeit  $MV$  yielding the highest Sharpe ratio of 0.66 in gross terms, it yields the lowest ratio in net terms of 0.17.  $MVCCM$  has the highest ratio of 0.39, whereas  $MVNLS$  and  $RP$  result in a ratio of roughly 0.30. The observed outperformance of  $MVCCM$  coincides with the findings of Ledoit and Wolf (2004a), who find that  $MVCCM$  performs best for samples with  $N < 100$  assets.

To better understand the portfolio dynamics we also assess the performance measures over time in Figure 3.1.<sup>11</sup> In terms of historical performance,  $MV$  shows the highest terminal wealth. However, the only strategy to show positive performance during the 2008 financial crisis is  $MVID$ , an indication that naive diversification can be beneficial during market turmoils. While the differences in Sharpe ratio dynamics are not very profound, only  $MV$  based portfolios reduce portfolio volatility significantly. The same is true for the realized CVaR, where the magnitude for the  $MV$  shrinkage and non-shrinkage strategies is much smaller than for  $EW$ ,  $VOLA$ , and the  $RP$  variants.  $RP$  dominates the variance regularization completely in the tails of the portfolio return distribution as the CVaR is identical for all  $RP$  strategies.

### 3.6 Applications in High-Dimensional Portfolios

Next, we assess the strategies in equity portfolios and the effect of dimensionality on portfolio performance, where we randomly select three cases with  $N = 50, 125, 250$  equities from all stocks listed in the NYSE, AMEX, and NASDAQ exchanges. The data set is nominated in USD, consists of total return price indices, and ranges from 01.01.1994 to 31.12.2019. Tables 3.3–3.5 report the results of the out-of-sample performance of each strategy for  $N = 50, 100$ , and 250, respectively.

<sup>11</sup>We also illustrate the portfolio weights over time for each strategy in Table 3.1 in Appendix Appendix C.

Table 3.3: High Dimensional Portfolio Summary Statistics for N=50

	<i>EW</i>	<i>VOLA</i>	<i>MV</i>	<i>MVID</i>	<i>MVCCM</i>	<i>MVNLS</i>	<i>RP</i>	<i>RPID</i>	<i>RPCCM</i>	<i>RPNLS</i>
Panel A: General Performance Metrics										
Gross Wealth	572.67	709.89	557.18	1190.40	1575.46	1325.46	1042.98	683.87	710.02	710.02
Net Wealth	559.83	694.80	541.76	962.15	1468.18	1142.71	903.75	666.26	694.93	693.45
TC (\$)	12.85	15.09	15.42	228.26	107.28	182.75	139.23	17.61	15.09	16.57
TC (%)	2.24	2.13	2.77	19.17	6.81	13.79	13.35	2.57	2.13	2.33
Turnover	0.36	0.34	0.45	3.40	1.13	2.37	2.29	0.42	0.34	0.38
HHI	2.00	2.15	2.08	6.84	9.80	6.97	5.54	2.25	2.15	2.21
Panel B: Gross Performance Characteristics										
Annual Return	8.87	9.13	8.72	10.57	11.72	10.90	9.95	8.92	9.14	9.14
Annual Volatility	19.72	16.92	19.47	13.54	14.16	12.91	12.84	16.52	16.92	16.92
Maximum Drawdown	63.88	57.25	64.22	40.54	34.68	33.37	37.37	57.34	57.25	57.25
Skewness	-1.66	-1.74	-1.68	-1.21	-0.30	-1.13	-1.05	-1.82	-1.74	-1.74
Kurtosis	12.98	13.19	13.64	9.90	6.16	10.53	8.42	14.16	13.19	13.19
Sharpe Ratio	0.45	0.54	0.45	0.78	0.83	0.84	0.77	0.54	0.54	0.54
Panel C: Net Performance Characteristics										
Annual Return	6.67	7.50	6.54	8.76	10.40	9.43	8.52	7.34	7.50	7.50
Annual Volatility	20.98	17.79	20.74	13.83	14.17	13.14	13.05	17.40	17.79	17.79
Maximum Drawdown	63.96	57.33	64.30	41.49	35.00	34.12	38.05	57.42	57.33	57.34
Skewness	-2.76	-2.62	-2.84	-1.70	-0.63	-1.65	-1.43	-2.76	-2.62	-2.62
Kurtosis	23.34	21.43	24.98	12.53	6.78	13.28	9.94	23.37	21.43	21.43
Sharpe Ratio	0.32	0.42	0.32	0.63	0.73	0.72	0.65	0.42	0.42	0.42

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio allocation schemes summarized in table 3.1. The data set spans the period from January 1994 to December 2019, which consists of 6782 daily portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. *Gross Wealth* is the terminal wealth accumulated over the whole sample period on \$100 investment while *Net Wealth* represents the terminal wealth net of transaction costs (*TC*), which are listed in nominal terms (*TC* (\$)) as well as percentage points of Terminal Gross Wealth (*TC* (%)). *HHI* denotes the Herfindahl-Hirschman Index for portfolio concentration.

Table 3.4: High Dimensional Portfolio Summary Statistics for N=100

	<i>EW</i>	<i>VOLA</i>	<i>MV</i>	<i>MVID</i>	<i>MVCCM</i>	<i>MVNLS</i>	<i>RP</i>	<i>RPID</i>	<i>RPCGM</i>	<i>RPNLS</i>
Panel A: General Performance Metrics										
Gross Wealth	713.34	822.52	667.58	863.46	1599.21	1247.03	855.59	769.96	822.63	822.63
Net Wealth	697.79	805.43	648.31	544.59	1463.99	972.73	689.18	749.43	805.54	803.67
TC (\$)	15.55	17.09	19.27	318.87	135.21	274.29	166.41	20.53	17.10	18.97
TC (%)	2.18	2.08	2.89	36.93	8.46	22.00	19.45	2.67	2.08	2.31
Turnover	0.35	0.34	0.47	7.36	1.41	3.97	3.45	0.43	0.34	0.37
HHI	1.00	1.07	1.04	5.16	5.47	4.60	3.32	1.10	1.07	1.08
Panel B: Gross Performance Characteristics										
Annual Return	9.43	9.55	9.17	9.36	11.73	10.54	9.07	9.24	9.55	9.55
Annual Volatility	18.38	16.09	18.31	13.99	13.89	12.06	12.03	15.75	16.09	16.09
Maximum Drawdown	61.02	54.36	61.89	34.97	36.94	28.18	37.20	54.90	54.36	54.36
Skewness	-1.62	-1.65	-1.67	-0.70	0.18	-0.44	-0.79	-1.78	-1.65	-1.65
Kurtosis	12.82	12.73	13.60	6.20	5.80	5.28	5.74	13.95	12.73	12.73
Sharpe Ratio	0.51	0.59	0.50	0.67	0.84	0.87	0.75	0.59	0.59	0.59
Panel C: Net Performance Characteristics										
Annual Return	7.52	8.08	7.24	6.56	10.39	8.81	7.47	7.80	8.08	8.07
Annual Volatility	19.40	16.81	19.37	14.13	13.76	12.07	12.14	16.51	16.81	16.81
Maximum Drawdown	61.09	54.44	61.97	37.56	37.29	28.99	38.00	54.99	54.44	54.45
Skewness	-2.61	-2.46	-2.75	-1.01	-0.09	-0.66	-1.03	-2.66	-2.46	-2.46
Kurtosis	21.91	19.99	23.99	7.36	5.23	5.70	6.48	22.47	19.99	19.98
Sharpe Ratio	0.39	0.48	0.37	0.46	0.75	0.73	0.62	0.47	0.48	0.48

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio allocation schemes summarized in table 3.1. The data set spans the period from January 1994 to December 2019, which consists of 6782 daily portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. *Gross Wealth* is the terminal wealth accumulated over the whole sample period on \$100 investment while *Net Wealth* represents the terminal wealth net of transaction costs (*TC*), which are listed in nominal terms (*TC* (\$)) as well as percentage points of Terminal Gross Wealth (*TC* (%)). *HHI* denotes the Herfindahl-Hirschman Index for portfolio concentration.



Table 3.5: High Dimensional Portfolio Summary Statistics for N=250

	<i>EW</i>	<i>VOLA</i>	<i>MV</i>	<i>MVID</i>	<i>MVCCM</i>	<i>MVNLS</i>	<i>RP</i>	<i>RPID</i>	<i>RPCCM</i>	<i>RPNLS</i>
Panel A: General Performance Metrics										
Gross Wealth	701.17	811.03	1422.36	1505.93	2184.44	701.17	820.75	810.98	810.98	901.11
Net Wealth	685.68	794.10	277.81	1355.57	1306.38	685.68	796.64	794.06	791.73	882.75
TC (\$)	15.49	16.92	1144.54	150.36	878.06	15.49	24.11	16.92	19.25	18.36
TC (%)	2.21	2.09	80.47	9.98	40.20	2.21	2.94	2.09	2.37	2.04
Turnover	0.36	0.34	26.01	1.68	8.20	0.36	0.48	0.34	0.38	0.33
HHI	0.40	0.43	3.49	2.68	2.65	0.40	0.44	0.43	0.43	0.45
Panel B: Gross Performance Characteristics										
Annual Return	9.36	9.49	11.57	11.52	12.60	9.36	9.48	9.49	9.49	10.07
Annual Volatility	18.37	16.05	15.75	14.05	10.94	18.37	15.71	16.05	16.05	17.04
Maximum Drawdown	60.74	54.47	42.21	45.02	32.06	60.74	55.48	54.47	54.47	57.04
Skewness	-1.62	-1.63	-0.32	0.25	-0.58	-1.62	-1.77	-1.63	-1.63	-1.55
Kurtosis	12.24	12.16	8.69	5.48	6.38	12.24	13.25	12.16	12.16	11.82
Sharpe Ratio	0.51	0.59	0.73	0.82	1.15	0.51	0.60	0.59	0.59	0.59
Panel C: Net Performance Characteristics										
Annual Return	7.45	8.02	3.96	10.09	9.95	7.45	8.03	8.02	8.01	8.43
Annual Volatility	19.36	16.74	15.81	13.89	10.95	19.36	16.43	16.74	16.74	17.80
Maximum Drawdown	60.81	54.54	52.77	45.32	33.66	60.81	55.56	54.54	54.55	57.14
Skewness	-2.52	-2.38	-0.92	0.01	-0.83	-2.52	-2.56	-2.38	-2.38	-2.34
Kurtosis	20.00	18.43	11.09	4.95	7.41	20.00	20.55	18.43	18.42	18.27
Sharpe Ratio	0.38	0.48	0.25	0.73	0.91	0.38	0.49	0.48	0.48	0.47

*Notes:* This table provides the out-of-sample portfolio performance measures for the portfolio allocation schemes summarized in table 3.1. The data set spans the period from January 1994 to December 2019, which consists of 6782 daily portfolio return observations. Portfolios are rebalanced on a monthly basis and all measures are annualized. *Gross Wealth* is the terminal wealth accumulated over the whole sample period on \$100 investment while *Net Wealth* represents the terminal wealth net of transaction costs (*TC*), which are listed in nominal terms (*TC* (\$)) as well as percentage points of Terminal Gross Wealth (*TC* (%)). *HHI* denotes the Herfindahl-Hirschman Index for portfolio concentration.

The distribution of portfolio returns remain unchanged to Section 3.5, where almost all strategies exhibit negatively skewed returns. *MV* is the most concentrated portfolio in all cases of  $N$  and exhibits the largest turnover rate, supporting Michaud (1989) that the mean-variance optimization using plug-in estimates results in highly concentrated and unstable portfolios. As the asset universe increases, not only does the concentration and turnover of *MV* increase substantially, but the return of the portfolio also drastically decreases whilst its volatility spikes. Again, this is in line with Michaud (1989) and Best and Grauer (1991), among others, that mean-variance optimized portfolios behave as error maximizers instead of achieving the mean-variance efficient portfolio, thereby deteriorating their risk-adjusted performance. As a result, *MV* performs well only for the case of  $N = 50$  where it yields a Sharpe ratio of 0.63 net of transaction costs, thereby being suboptimal to only the covariance shrinkage strategies of Ledoit and Wolf (2004a,b), whereas its net Sharpe ratio declines with  $N$  yielding the lowest net Sharpe ratio of 0.25 for  $N = 250$ .

Regarding heuristic weighting schemes, *EW* yields stable albeit modest risk-adjusted returns for all  $N$ . As the number of assets increases, the risk-adjusted return of *EW* improves while the turnover remains quite low at around 0.35% of portfolio volume, yielding a higher net Sharpe ratio than *MV* for  $N = 250$ . *VOLA* as well as *RP* display very similar properties, which are yet better than *EW*. The Sharpe ratio of both strategies increases with the number of assets and is almost identical for both strategies, although *VOLA* exhibits a significantly lower turnover ratio of roughly 0.34.

Covariance shrinkage techniques provide mixed results. *MVID* and *MVCCM* yield the highest Sharpe ratios before and after transaction costs for all cases of  $N$ . The difference in risk-adjusted performance is minimal for  $N = 50$  and 100, but the gap widens significantly for  $N = 250$  with *MVID* and *MVCCM* respectively yielding a Sharpe ratio of 0.73 and 0.91 net transaction costs. Meanwhile, *MVNLS* seems to shrink the portfolio towards *EW* as the number of assets increases, where *MVNLS* is identical to *EW* for  $N = 250$  in all portfolio characteristics. These results contradict the findings of Ledoit and Wolf (2019), who using Monte Carlo simulations find that non-linear shrinkage of the covariance matrix performs better than linear shrinkage in high dimensional portfolios. Our results indicate that a more structured shrinkage target, that is linear shrinkage of all parameters to a unified shrinkage target, results in a better risk-adjusted performance for *MV* portfolios than imposing less structure as in individual shrinkage targets via non-linear shrinkage.

Finally, shrinking the covariance matrix in  $RP$  results in a deterioration of portfolio performance for all cases of  $N$ , regardless of the approach. This result is in line with those of Section 3.5 and further reinforces the notion of the inflexibility of the asset weights in  $RP$  portfolios due to the risk budgeting constraint. Altogether, shrinkage techniques seem to exhibit some merits when applied in  $MV$  optimized portfolios, but do not yield significant benefits for variations thereof such as  $RP$ .<sup>12</sup>

## 3.7 Conclusion

In this study we investigate the impact of shrinkage estimation techniques for the variance-covariance matrix of asset returns in risk parity portfolios for different asset universe dimensions. While we provide theoretical and empirical evidence that risk budgeting on the asset level serves as a regularization mechanism, most importantly this chapter combines two strands of literature: Using mean-variance linear and non-linear shrinkage techniques in risk parity portfolios. In this, we provide a unique and novel perspective on risk parity portfolios.

We find that: (i)  $RP$  is a shrinkage variant that works well in lower dimensions; (ii) Combining  $RP$  with shrinkage estimators of the variance-covariance matrix yields even better returns in lower dimensions, but seems suboptimal for larger asset universes; (iii) pure shrinkage estimators for  $MV$  portfolios dominate all  $RP$  variants in higher dimensions, indicating that fixing the risk contribution of individual assets does not mitigate the effects of estimation errors and even dominates regularization effects from more stable variance estimators. A further argument for this is that for large  $N$ , the equal risk budgeting constraint does not decrease portfolio variance.

Future research should consider combinations of mean-variance and risk parity portfolios that specifically account for a better portfolio allocation rule that captures the best of both worlds; for example through an optimal shrinkage intensity similar to DeMiguel et al. (2013). Applications should also include more asset classes such as currencies or private equity, account for fat-tail distribution through a typical GARCH model as in Ardia et al. (2017) or even be concerned with more practical considerations in asset liability and risk management.

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<sup>12</sup>We refrain from including the rolling performance measures for the high dimensional portfolios in this study for parsimony. These results are available upon request from the authors.

# Conclusion

In summary, this dissertation addresses the issues raised by academia and practice regarding the absence of a theoretical framework that explains the properties as well as the risk-return characteristics of risk parity portfolios. We hope that the results of this dissertation project not only answers the questions posed by previous studies in academia, but also helps the investment management industry obtain a better understanding of the strengths and weaknesses of risk parity portfolios.

In the first chapter we find that risk parity is indeed closely related to its component portfolio, namely the volatility and correlation component portfolios, and that it can be replicated by a linear combination of its component portfolios. Furthermore, we show that the outperformance of risk parity, the volatility component, and the correlation component portfolios are driven by the Betting-against-Beta factor (Frazzini and Pedersen, 2014), the volatility effect (Haugen and Heins, 1975; Blitz and van Vliet, 2007), and the Betting-against-Correlation factor (Asness et al., 2019), respectively. Finally, we demonstrate how the component portfolios can be utilized based on varying market conditions to improve the risk-return profile whilst attaining the appealing characteristics of risk parity.

Our findings in the second chapter show that, compared to other strategies, risk parity portfolios yield a superior risk-return profile whilst achieving a lower return volatility and risk exposure. Risk parity portfolios exhibit a low and stable turnover in asset weights, which remains smooth over time and helps reduce transaction costs that occur when rebalancing the portfolio. Nonetheless, the performance of risk parity portfolios is highly sensitive to its constituents and a substantial deterioration in the risk-return characteristics is observed when adding negative risk premium assets to the portfolio. Moreover, leveraging risk parity portfolios to achieve a specific level of portfolio volatility significantly improves its risk-adjusted performance, but also introduces some hazards such as a slight increase in risk exposure and a significant increase in portfolio asset concentration.

The third chapter investigates the impact of shrinkage estimation techniques for the variance-covariance matrix of asset returns in risk parity portfolios for different asset universe dimensions. We provide theoretical and empirical evidence that risk budgeting on the asset level, which is the underlying principal of risk parity portfolios, serves as a regularization mechanism and, hence, we provide a unique and novel perspective on risk parity portfolios. Our results show that: (i) Risk parity is a shrinkage variant that performs well in low dimensional portfolios, where the number of observations of the estimation window are much greater than the number of assets; (ii) Combining risk parity with shrinkage estimation techniques for the variance-covariance matrix could slightly enhance the risk-adjusted return in lower dimensions, but become suboptimal as the number of assets increases; (iii) Pure shrinkage estimators for minimum variance portfolios dominate all risk parity variants in higher dimensions, indicating that fixing the risk contribution of individual assets does not alleviate the impact of estimation errors and dominates regularization effects from more stable variance estimators.

Last but not least, our findings raises some further questions about the risk-return properties of risk parity portfolios, which we hope that they will be addressed by future research.

# Appendix A

## Appendix to Chapter 1

## A.1 The Inverse Beta Relationship of Risk Parity Portfolio Weights

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**Definition: Total Risk Contribution ( $TRC$ )**

Let  $w_i$  be the weight of asset  $i$  in a portfolio with  $i = 1, \dots, n$ ,  $\sigma_{i,p}$  the covariance of asset  $i$  with the portfolio, and  $\sigma_p$  the standard deviation of the portfolio returns, then the Total Risk Contribution of asset  $i$  ( $TRC_i$ ) is defined by

$$TRC_i = \frac{w_i \sigma_{i,p}}{\sigma_p} \quad (\text{A.1})$$

**Definition: Beta of asset  $i$  to Portfolio ( $\beta_{i,p}$ )**

Let  $\sigma_{i,p}$  be the covariance of asset  $i$  with the portfolio and  $\sigma_p^2$  the variance of portfolio returns, then the beta of asset  $i$  to the portfolio is defined by

$$\beta_{i,p} = \frac{\sigma_{i,p}}{\sigma_p^2} \quad (\text{A.2})$$

---

Since the risk parity portfolio equalizes the asset Total Risk Contributions by definition, it satisfies the following condition

$$TRC_i = TRC_j = \frac{\sigma_p}{n} \quad (\text{A.3})$$

By setting Equation A.1 equal to Equation A.3 and solving for  $w_i$  we obtain

$$\begin{aligned} \frac{w_i \sigma_{i,p}}{\sigma_p} &= \frac{\sigma_p}{n} \\ w_i &= \frac{\sigma_p^2}{\sigma_{i,p} n} = \frac{1}{\beta_{i,p} n} = \frac{\beta_{i,p}^{-1}}{n} \end{aligned}$$

Thus, we confirm that the weight of an asset  $i$  in the  $RP$  portfolio is inversely related to the asset's beta to the portfolio  $\beta_{i,p}$  and the total number of assets in the portfolio  $n$ .

Furthermore, provided Equation A.1 and  $\beta_{i,p}$ , we obtain

$$\begin{aligned}
 TRC_i &= \frac{w_i \sigma_{i,p}}{\sigma_p} \\
 TRC_i &= \frac{w_i \sigma_{i,p}}{\sigma_p} \times \frac{\sigma_p}{\sigma_p} \\
 TRC_i &= \frac{w_i \sigma_{i,p} \sigma_p}{\sigma_p^2} \\
 TRC_i &= w_i \beta_{i,p} \sigma_p \\
 \beta_{i,p} &= \frac{TRC_i}{w_i \sigma_p}
 \end{aligned} \tag{A.4}$$

To show that  $\sum_{i=1}^n \beta_{i,p}^{-1} = n$ , we substitute  $\beta_{i,p}$  by Equation A.4

$$\begin{aligned}
 \sum_{i=1}^n \beta_{i,p}^{-1} &= \sum_{i=1}^n \frac{1}{\beta_{i,p}} \\
 &= \sum_{i=1}^n \frac{1}{\left(\frac{TRC_i}{w_i \sigma_p}\right)} \\
 &= \sum_{i=1}^n \frac{w_i \sigma_p}{TRC_i}
 \end{aligned}$$

by substituting Equation A.3 for  $TRC_i$  and provided the constraint in Equation 1.4 that  $\sum_{i=1}^n w_i = 1$

$$\begin{aligned}
 \sum_{i=1}^n \frac{w_i \sigma_p}{TRC_i} &= \sum_{i=1}^n \frac{w_i \sigma_p}{\left(\frac{\sigma_p}{n}\right)} \\
 &= \sum_{i=1}^n w_i n \\
 &= n
 \end{aligned}$$

Thus, we confirm that  $\sum_{i=1}^n \beta_{i,p}^{-1} = n$ .



## A.2 The Equivalence of the Volatility Component Portfolio to Risk Parity

---

### Definition: The Volatility Component Portfolio (*VOLA*)

Let  $w_i$  is the weight of asset  $i$  with  $i = 1, \dots, n$  and  $\sigma_i$  be the standard deviation of the returns of asset  $i$ . As mentioned in section 1.3, the asset weights of the volatility component portfolio, also known as the inverse volatility strategy, can be defined by

$$w_i = \frac{\sigma_i^{-1}}{\sum_{i=1}^n \sigma_i^{-1}}. \quad (\text{A.5})$$


---

The *VOLA* portfolio corresponds to the *RP* portfolio under the assumption that the correlations among assets are equal. Accordingly, by plugging the asset weights of Equation A.5, the portfolio variance for the case of  $n = 2$  assets can be simplified as follows:

$$\begin{aligned} \sigma_p^2 &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2} \\ &= \left( \frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right)^2 \sigma_1^2 + \left( \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right)^2 \sigma_2^2 \\ &\quad + 2 \left( \frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right) \left( \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right) \sigma_1 \sigma_2 \rho_{1,2} \\ &= \left( \frac{\sigma_2}{\sigma_1 + \sigma_2} \right)^2 \sigma_1^2 + \left( \frac{\sigma_1}{\sigma_1 + \sigma_2} \right)^2 \sigma_2^2 \\ &\quad + 2 \left( \frac{\sigma_2}{\sigma_1 + \sigma_2} \right) \left( \frac{\sigma_1}{\sigma_1 + \sigma_2} \right) \sigma_1 \sigma_2 \rho_{1,2} \\ &= \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1 + \sigma_2)^2} + \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1 + \sigma_2)^2} + 2 \frac{\sigma_1^2 \sigma_2^2 \rho_{1,2}}{(\sigma_1 + \sigma_2)^2} \\ &= \frac{2\sigma_1^2 \sigma_2^2 (1 + \rho_{1,2})}{(\sigma_1 + \sigma_2)^2} \\ &= 2\sigma_1^2 w_1^2 (1 + \rho_{1,2}) = 2\sigma_2^2 w_2^2 (1 + \rho_{1,2}). \end{aligned} \quad (\text{A.6})$$

For  $n = 2$  assets, the portfolio variance is merely twice the squared product of the asset weight and the asset volatility multiplied by one plus the asset correlation. Since  $\sigma_1 w_1 = \sigma_2 w_2$ , the *MRC* of assets 1 and 2 are

$$\begin{aligned} \text{MRC}_1 &= \frac{\partial \sigma_p}{\partial w_1} = \frac{\partial \left( \sqrt{2\sigma_1^2 w_1^2 (1 + \rho_{1,2})} \right)}{\partial w_1} \\ &= \frac{\partial \left( \sqrt{2(1 + \rho_{1,2})} \sigma_1 w_1 \right)}{\partial w_1} = \sqrt{2(1 + \rho_{1,2})} \sigma_1 \\ \text{MRC}_2 &= \frac{\partial \sigma_p}{\partial w_2} = \frac{\partial \left( \sqrt{2\sigma_2^2 w_2^2 (1 + \rho_{1,2})} \right)}{\partial w_2} \\ &= \frac{\partial \left( \sqrt{2(1 + \rho_{1,2})} \sigma_2 w_2 \right)}{\partial w_2} = \sqrt{2(1 + \rho_{1,2})} \sigma_2 \end{aligned} \quad (\text{A.7})$$

And hence it follows directly that

$$\begin{aligned} \text{TRC}_1 &= w_1 \times \frac{\partial \sigma_p}{\partial w_1} = w_1 \sigma_1 \sqrt{2(1 + \rho_{1,2})} \\ &= w_2 \sigma_2 \sqrt{2(1 + \rho_{1,2})} = w_2 \times \frac{\partial \sigma_p}{\partial w_2} = \text{TRC}_2. \end{aligned} \quad (\text{A.8})$$

Deriving similar results in closed-form is not possible for general  $n$ . Hence, to make the case, we assume that all correlations are equal across assets, i.e.  $\rho_{i,j} = \rho \forall i, j$ . Then the total risk contribution of asset  $i$  is given by

$$\text{TRC}_i = \frac{w_i^2 \sigma_i^2 + \rho \sum_{i \neq j} w_i w_j \sigma_i \sigma_j}{\sigma_p} \quad (\text{A.9})$$

The risk parity portfolio is obtained for  $\text{TRC}_i = \text{TRC}_j \forall i, j$  and hence it is

$$\text{TRC}_i = \text{TRC}_j \quad (\text{A.10})$$

$$\Leftrightarrow \frac{w_i^2 \sigma_i^2 + \rho \sum_{i \neq j} w_i w_j \sigma_i \sigma_j}{\sigma_p} = \frac{w_j^2 \sigma_j^2 + \rho \sum_{i \neq j} w_i w_j \sigma_i \sigma_j}{\sigma_p} \quad (\text{A.11})$$

$$\stackrel{*}{\Leftrightarrow} w_i^2 \sigma_i^2 = w_j^2 \sigma_j^2 \quad (\text{A.12})$$

where  $*$  follows by using  $\rho \geq -\frac{1}{n-1}$ . Since the budget constraint requires  $\sum_i w_i = 1$ ,

we find that

$$w_i = \frac{\sigma_i^{-1}}{\sum_{i=1}^n \sigma_i^{-1}} \quad \forall i = 1, \dots, n. \quad (\text{A.13})$$

With this we find for the total risk contribution of asset  $i$   $TRC_i$  that

$$\begin{aligned} TRC_i &= \frac{w_i^2 \sigma_i^2 + \rho \sum_{i \neq j} w_i w_j \sigma_i \sigma_j}{\sigma_p} \\ &= \frac{\left( \frac{\sigma_i^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \right)^2 \sigma_i^2 + \rho \sum_{i \neq j} \frac{\sigma_i^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \frac{\sigma_j^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \sigma_i \sigma_j}{\sqrt{\sum_{k=1}^n \left( \frac{\sigma_k^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \right)^2 \sigma_k^2 + \rho \sum_k \sum_{k \neq l} \frac{\sigma_k^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \frac{\sigma_l^{-1}}{\sum_{i^*=1}^n \sigma_{i^*}^{-1}} \sigma_k \sigma_l}} \\ &= \frac{\left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2} + \rho \sum_{i \neq j} \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}{\sqrt{\sum_{k=1}^n \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2} + \rho \sum_k \sum_{i \neq j} \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}} \\ &= \frac{\left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2} + \rho \times (n-1) \times \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}{\sqrt{n \times \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2} + \rho \times n \times (n-1) \times \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}} \\ &= \frac{(1 + \rho(n-1)) \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}{\sqrt{n(1 + \rho(n-1)) \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-2}}} \\ &= \frac{\sqrt{1 + \rho(n-1)}}{\sqrt{n}} \left( \sum_{i^*=1}^n \sigma_{i^*}^{-1} \right)^{-1} \\ &= TRC_j \quad \forall i, j. \end{aligned}$$

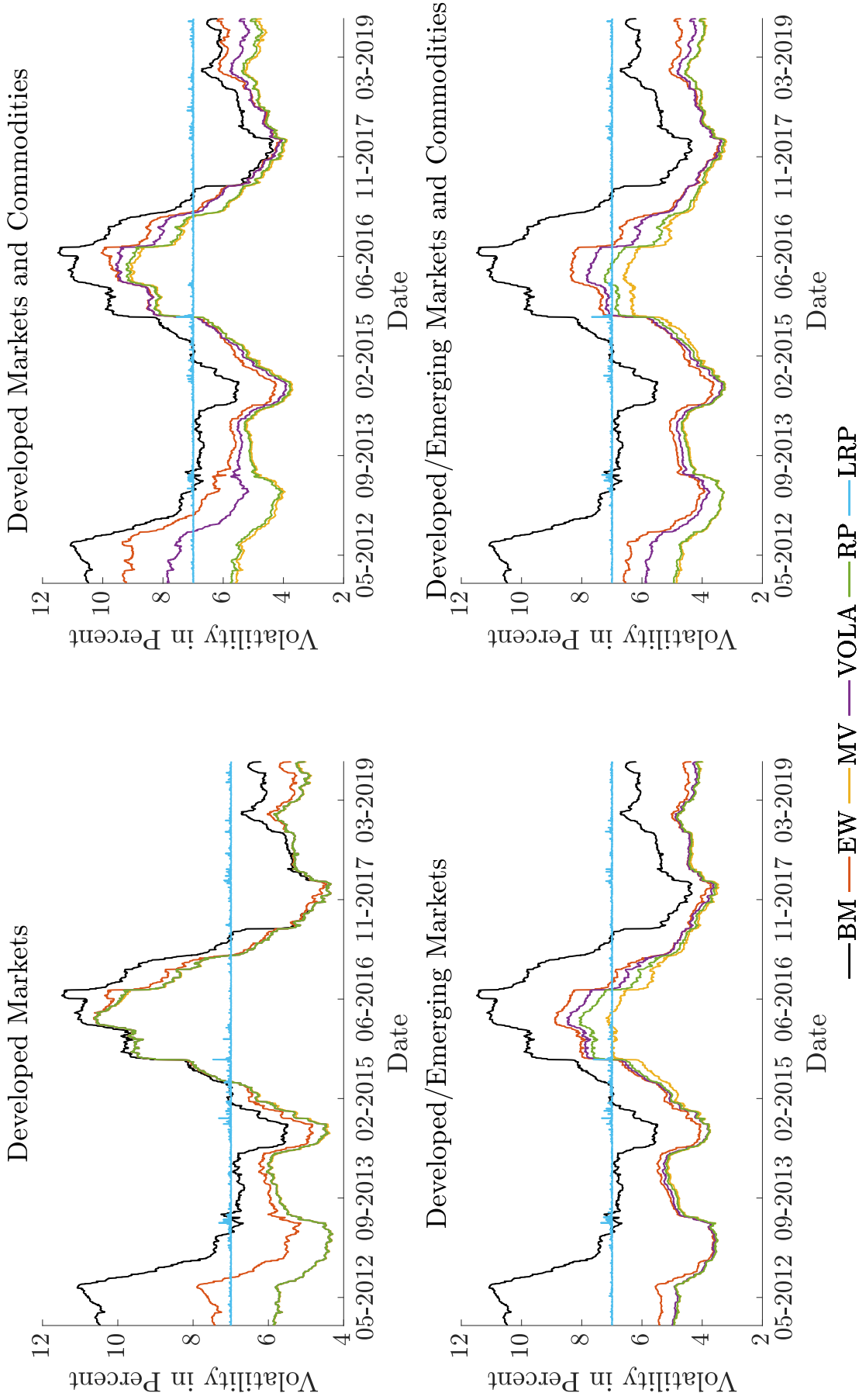
Hence, under the assumption of equal correlations all total risk contributions for the volatility component portfolio (*VOLA*) are identical and, hence, it is equivalent to the risk parity portfolio.

# Appendix B

## Appendix to Chapter 2

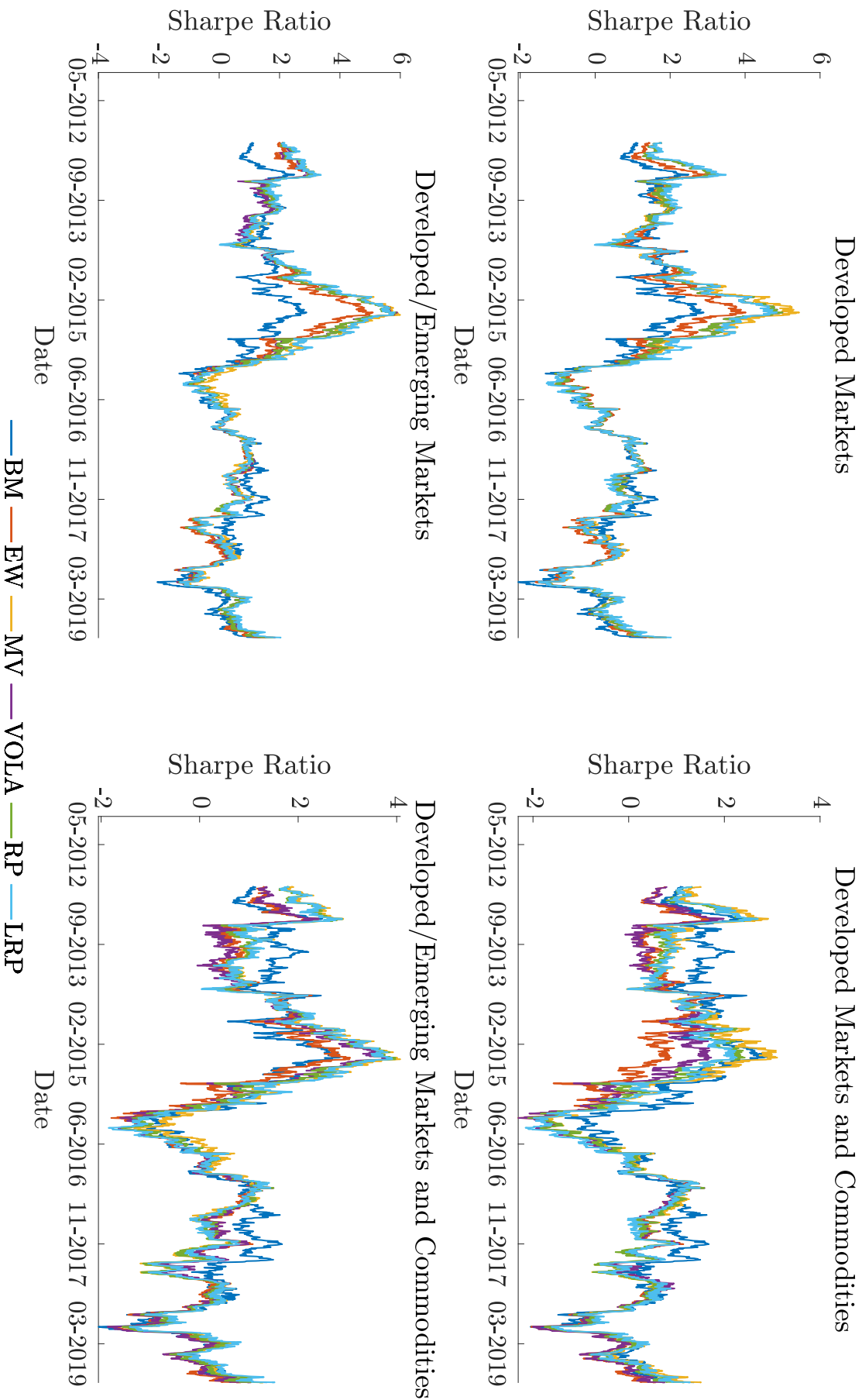
In this appendix we report supplementary results which might be interesting for the reader, but are not essential to our analysis. These mainly consist of the rolling volatility, Sharpe ratio, the weight heat maps for the other portfolios employed in this study, and summary statistics for the asset risk contribution ratios (*RCR*). The risk contribution ratio of an asset denotes the percentage amount of risk contributed by that asset to the total portfolio risk or volatility.

Figure B.1: Portfolio Rolling Volatilities



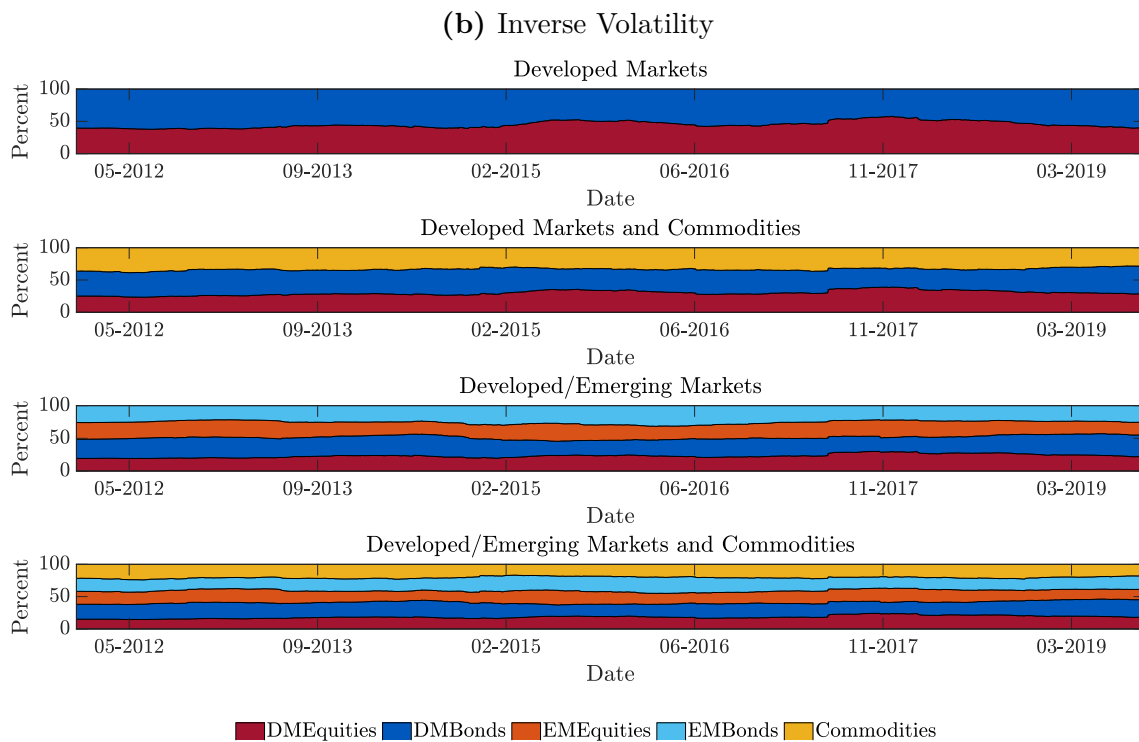
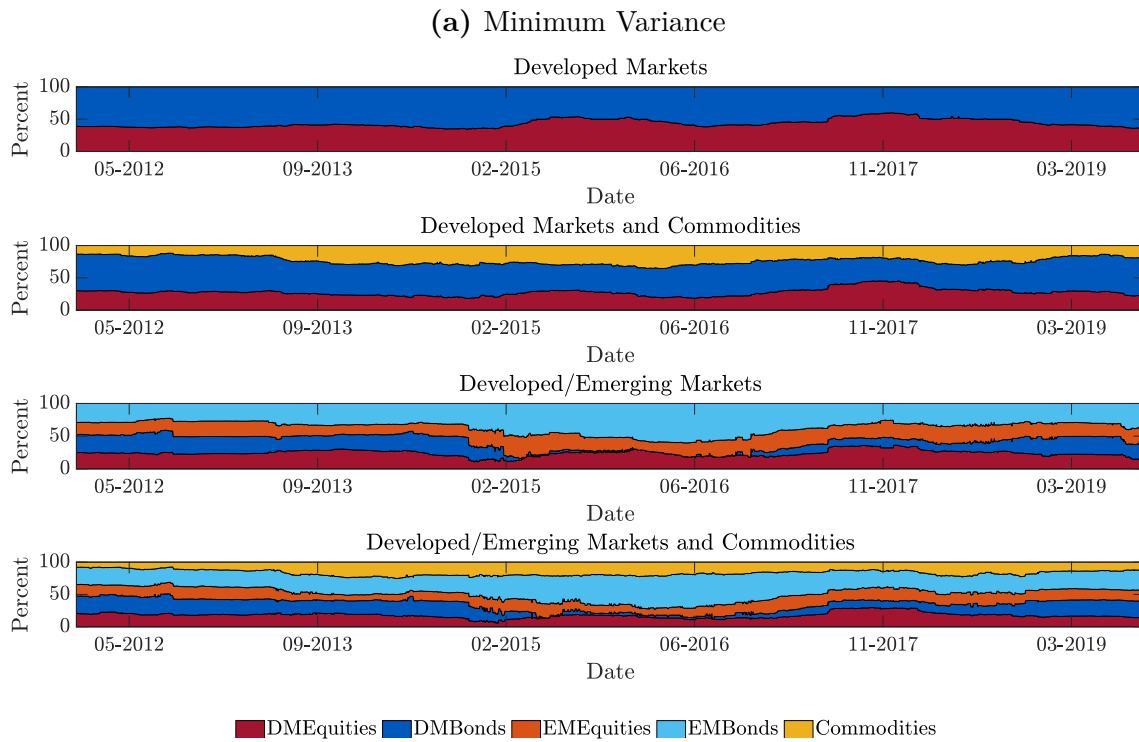
Notes: This figure displays the annual rolling volatilities of the asset allocation strategies described in section 2.3.

Figure B.2: Portfolio Rolling Sharpe Ratios



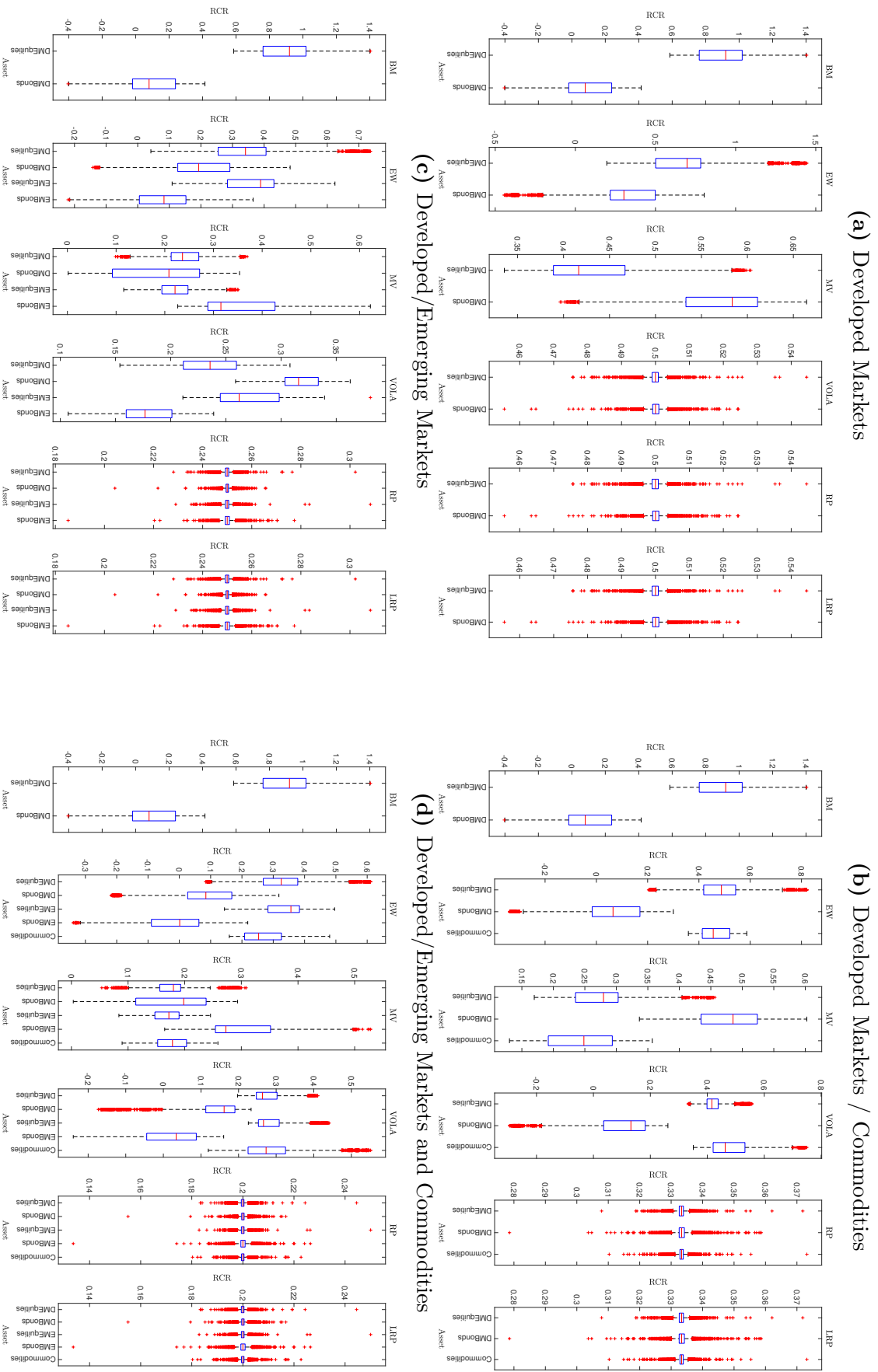
Notes: This figure displays the annual rolling Sharpe ratios of the asset allocation strategies described in section 2.3.

**Figure B.3: Weights of Risk-Based Strategies**



*Notes:* This figure shows the allocation changes over time for the Markowitz minimum variance portfolio in panel (a) and the inverse volatility portfolio in panel (b). DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

**Figure B.4:** Statistics of the Asset Risk Contribution Ratios



*Notes:* This figure shows the statistical properties of the asset risk contribution ratios ( $RCR$ ) in a boxplot for each portfolio. DM, EM, and CO denote developed markets, emerging markets, and commodities, respectively.

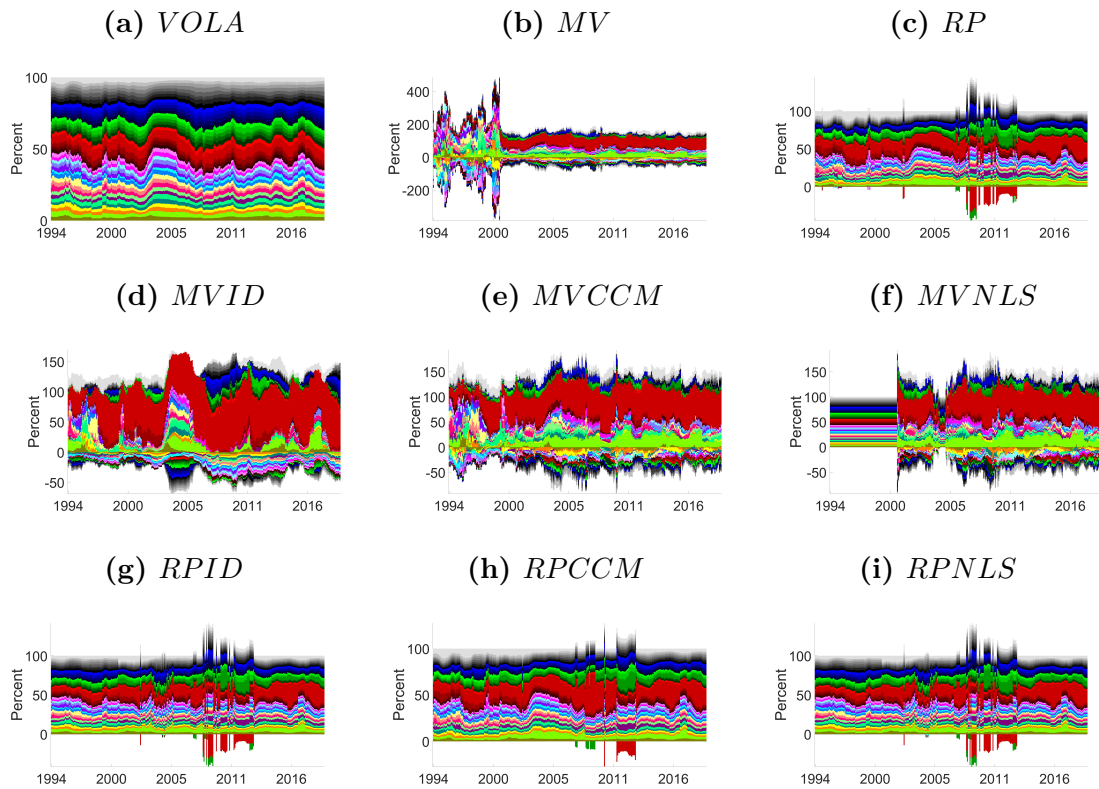


# Appendix C

## Appendix to Chapter 3

In the following we illustrate the portfolio weight changes over time for the portfolios in Section 3.5. *EW* is excluded from the chart as its asset weights are constant over time.

**Figure C.1:** Global Portfolio Weights over Time



*Notes:* This figure shows the allocation changes over time for the different portfolio strategies outlined in Table 3.1. Portfolios are rebalanced on a monthly basis. The data set spans the period from January 1994 to December 2019, which consists of 6782 daily portfolio return observations. Portfolios are rebalanced on a monthly basis.

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