

# OPERATIONAL AND FINANCIAL RISK MANAGEMENT UNDER COMMODITY PRICE UNCERTAINTY

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*Für Manfred, Vera und Theresa*



*Unfortunately, in many companies, the CFO is handling financial risk,  
the CEO is handling strategic risk, and the COO is handling operational risk,  
but no one is looking at all those risks as one.*

Jim Loucks



# ABSTRACT

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The combined impact of changing global demand and supply dynamics, extensive trading and speculation as well as global recessionary fears, has led to an environment of unprecedented volatility in worldwide commodity markets. As a result, effective risk management has become an increasingly important topic on the agenda of top management in a broad range of industries. While practical evidence shows that successful firms integrate both operational decision making and financial hedging in a firm-wide, coordinated risk management strategy, this entails managerial challenges. On the one hand, quantifying a firm's exposure to raw material cost risk necessitates a sound understanding of the stochastic commodity market dynamics. On the other hand, once the exposure to different sources of risk is understood, executives face an intricate optimization problem over their operational and financial decision variables with the ultimate goal to reduce profit variability, while maintaining attractive business opportunities. In this thesis, the topic of operational and financial risk management is investigated from three different perspectives.

In Chapter 2, a four-factor maximal affine stochastic volatility model of commodity prices is developed, which is consistent with many stylized characteristics of storable commodity markets as well as the historical term structure of commodity futures and option prices. Based on this model, we provide new insights with respect to the structural dynamics of commodity markets and the pricing and hedging of commodity derivatives. As the stochastic model used to describe the uncertain evolution of commodity prices can have important implications also in the valuation and risk management of real assets, a realistic commodity price model is a prerequisite for the integrated risk management models outlined in the subsequent chapters of this dissertation.

Given the previously developed intuition for commodity market dynamics, we model the integrated operational and financial risk management problem of a stylized, single-product industrial firm in Chapter 3. The firm faces risk in the price of commodity inputs and price sensitive, stochastic demand. Within this setting, the firm seeks to maximize inter-temporal utility under downside risk aversion over a multi-period time horizon by dynamically choosing physical procurement volumes, unit selling prices, and a futures hedge. We provide a flexible, simulation-based optimization algorithm, which allows us to solve the firm's decision problem under realistic, multi-factor commodity price dynamics involving uncertainty in the interest rate and convenience yield as well as stochastic volatility. Based on this model, we characterize the firm's optimal operating policy and investigate a range of topics including: (a) the value of managerial flexibility and the economic cost of restrictive supply contracts; (b) the importance of accounting for the stochastic nature of costs, interest rates, convenience yields, and volatility in risk management; (c) parameter and estimation risk; (d) the impact of risk aversion and hedging on the distribution of cash flows; and (e) the sensitivity of expected performance to key input parameters.

As opposed to the case of a single-product firm, integrated procurement risk management in a large,

multi-divisional organization does not only require the above mentioned cross-functional coordination between, for example, the purchasing, sales, and finance department but also involves a cross-divisional coordination of actions in order to effectively target the firm-wide net risk exposure. To capture the specific aspects of integrated risk management in this type of setting, we extend the above model to a two-product firm in Chapter 4. Within this model, each of the two divisions are subject to cost and demand risk, which can be respectively correlated. Moreover, we allow for dynamic cross-selling to capture the potential complementarity/substitutability of items. The firm has access to futures, call, and put options associated with each of the commodity input markets for financial hedging. Under an inter-temporal mean-variance utility function, we are able to provide analytic solutions to the firm's dynamic procurement, pricing, and financial hedging problem. Based on a complementary numerical study, we analyze the impact of risk correlations and unilateral changes in the market environment of one division on the entire firm. Moreover, we discuss the impact of hedging and risk aversion on optimal policy and assess the effectiveness of different operational and financial hedging strategies for risk reduction.



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## ABBREVIATIONS

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AIC	Akaike information criterion
BFGS	Broyden-Fletcher-Goldfarb-Shanno
BIC	Bayesian information criterion
COMEX	New York Commodities Exchange
DP	dynamic pricing policy
DSRA	downside risk aversion
EKF	extended Kalman filter
FP	fixed price policy
GARCH	generalized autoregressive conditional heteroscedasticity
GBM	geometric Brownian motion
HJM	Heath, Jarrow, and Morton
HP	Hewlett-Packard
ITM	in-the-money
LL	log-likelihood
LME	London Metal Exchange
LSM	least squares Monte Carlo
LSMO	least squares Monte Carlo optimization
MAE	mean absolute pricing error
MC	multi-contract hedging policy
NYMEX	New York Mercantile Exchange
ODE	ordinary differential equation
OTM	out-of-the-money
OU	Ornstein-Uhlenbeck

PRM	procurement risk management
RMSE	root-mean-square error
RN	risk neutrality
SC	stochastic convenience yield
SDE	stochastic differential equation
SM	stochastic long-term mean
SMV	stochastic mean and volatility
SQP	sequential quadratic programming
SRA	symmetric risk aversion
SRC	stochastic interest rate and convenience yield
SRCV	stochastic interest rate, convenience yield, and volatility
SV	stochastic volatility
USV	unspanned stochastic volatility
VaR	value at risk
WTI	West Texas Intermediate

# 1

## INTRODUCTION

---

Commodity price volatility constitutes a major risk in many industries that can jeopardize the success of entire companies if left unmanaged. Regardless of whether it concerns firms engaged in the extraction and marketing of primary commodities, refiners relying on the spread between uncertain input and output prices, or industrial firms procuring materials for the downstream production process, an effective management of the risks associated with volatile commodity prices is of paramount importance. Well-known examples of the consequences caused by ill-advised procurement risk management (PRM) include, among others, the case of Metallgesellschaft (Edwards & Canter, 1995) and the \$1 billion loss on precious metals inventory and forward contracts posted by Ford in 2001 (Hu, Munson, & Fotopoulos, 2012). While commodity price volatility alone can jeopardize even the best planned business strategy, firms are often exposed to additional risks such as the uncertainty about future end-product demand (Nagali et al., 2008). Traditionally, these different types of risk are managed and hedged separately (Kouvelis, Li, & Ding, 2013). Accordingly, demand risk is treated by means of inventory management and cost volatility is hedged through derivatives in the financial market. However, as operational and financial risk management decisions are not generally separable, this approach creates problems, particularly in large, multi-divisional organizations (Fisher & Kumar, 2010).

In contrast to this traditional, disjointed view on PRM, practical evidence from Tevelson, Ross, and Paraniakas (2007), PwC (2009), and Fisher and Kumar (2010) suggests that effective PRM strategies are commonly characterized by a more holistic view on risk exposure as well as an integrated use of operational and financial hedging instruments spanning the entire organization. Instead of conducting isolated risk management efforts in individual divisions, successful companies target the firm-wide net risk exposure and integrate operational decision making with financial hedging activities. This can involve, for example, the coordination of the firm's pricing, inventory management, and financial hedging decisions in order to enhance expected profits, while reducing firm-wide profit variability. Although, investors can hedge against risks, such as commodity price volatility, through their own portfolio diversification, effective hedging conducted by the firm can not only provide protection from financial distress but also create value by allowing the firm to reduce sourcing and inventory related costs, focus on innovation and new product development, and exploit attractive investment opportunities (see, for example, C. W. Smith & Stulz, 1985; Froot, Scharfstein, & Stein, 1993; Allayannis & Weston,

2001; MacKay & Moeller, 2007). A popular example of a successful PRM program can be found at Hewlett-Packard (HP) (Nagali et al., 2008; Xiao, Yang, & Zhang, 2015). Launched in the early 2000s, this program unifies a portfolio sourcing approach involving spot and forward purchasing, a dynamic pricing policy that takes into account the current state of costs and inventories as well as a financial hedging strategy. Besides providing planning security to HP's management the program generated \$425 million in cost savings over the 6-year period following its introduction.

However, while the above example corroborates the value of effective risk management in practice, managing the combined exposure to multiple sources of uncertainty resembles a critical management challenge in today's fast changing business environment. On the one hand, managers must understand the complex nature of commodity market dynamics in order to gauge the firm's exposure to cost volatility and to take appropriate countermeasures. In fact, erroneous assumptions about the stochastic dynamics of raw material prices can lead to suboptimal decision making and poor investment behavior (Schöne, 2014; Tsekrekos, Shackleton, & Wojakowski, 2012). On the other hand, deciding on the jointly optimal operational and financial risk management policy resembles an intricate optimization problem under multiple sources of uncertainty involving several trade-offs, numerous decision variables across the organization, and a multi-period planning horizon. Although, different variants of PRM programs are more commonly adopted in practice (PwC, 2009), there is yet little research that substantiates and further cultivates these industry practices based on empirical analysis, quantitative modeling, and optimization.

To address this research gap, this thesis contributes three self-contained studies, which are organized along the overriding objective to improve our understanding of jointly optimal operational and financial risk management in the context of an industrial firm. We choose the following course of investigation. First, an arbitrage-free, stochastic model of storable commodity prices is presented, which is consistent with empirical spot and derivative price dynamics. Second, we develop and analyze a quantitative model of the integrated dynamic procurement, pricing, and financial hedging problem of a stylized single-product firm facing raw material cost uncertainty and demand risk. The model is formulated such that sophisticated multi-factor commodity price models can be accounted for. Third, we extend the previously discussed single-product model to a two-product setting, which allows us to investigate not only the interplay between different functional decisions, such as pricing, hedging, and inventory management but also the implications of integrated decision making across multiple divisions with different, interdependent risks. We will provide a more detailed introduction to the topics of commodity price modeling and integrated PRM in the following subsections and summarize the main contributions to existing research of this thesis.

## 1.1 Commodity price dynamics

Managing storable commodity price risk necessitates a sound understanding of the stochastic dynamics present in commodity spot and derivative markets. In particular, different assumptions about the model used to describe the uncertain price dynamics can have significant implications for valuation results, hedging policies, and the optimal operation of real assets. With respect to the valuation and hedging of financial derivatives, this follows, among others, from Lo and Wang (1995), Bakshi, Cao, and Zhiwu

(1997), and Hillard and Reis (1998). In the context of capital investment valuation, the critical role of stochastic process selection is confirmed by Tsekrekos et al. (2012) and Schöne (2014).

In contrast to other asset classes such as stocks and bonds, commodity price dynamics exhibit a number of distinctive characteristics, which have to be accounted for in this context. First, commodity prices tend to mean-revert to an equilibrium level determined by the long-run marginal cost (Pindyck, 1999). In times of high prices, high-cost producers of the commodity can profitably enter into the market and thereby increase supply and reduce prices. On the contrary, inefficient producers will be forced out of the market during times of low prices, which decreases supply and ultimately drives prices back up. While this argument is economically intuitive, mean reversion to a constant equilibrium level is generally statistically rejected over long time spans of several decades or even centuries (see, e.g., Dixit & Pindyck, 1994; Pindyck, 1999). Instead, it has been suggested that the equilibrium price level itself might follow a stochastic, non-stationary process, which may capture persistent changes in the global demand and supply of a commodity (Tang, 2012). Apart from the stochastic nature of raw material spot prices, also the interest rate, convenience yield, and storage cost are important, stochastically moving quantities that affect the term structure of futures prices and the expected future spot price. Finally, the volatility of commodity prices tends to vary stochastically over time, which has important implications for the valuation of financial and real options involving the managerial flexibility to take future actions contingent upon the state of commodity prices.

In the context of PRM, these stylized characteristics may be particularly relevant. First, the spot price distribution directly affects the expectation and variance of procurement costs. Second, the level of interest rates and the net convenience yield determine economic inventory holding costs as well as the fair price of futures contracts, which are potentially part of the firm's financial hedging strategy. Third, the presence of stochastic volatility does not only influence the probability of extreme events in the raw material market but also affects the value of managerial flexibility and financial options. In order to account for these considerations, we develop a four-factor stochastic volatility model of storable commodity prices in Chapter 2, which can serve as an input to the subsequently studied PRM models and, further, allows us to make the following two specific contributions to the existing literature on commodity price modeling:

- (i) The four-factor maximal affine stochastic volatility model presented in this thesis is among the first models that allow for more than three independent sources of risk associated with commodity price dynamics. This allows the model to capture many specific characteristics of commodity markets including mean reversion to a stochastic long-term level, persistent shocks in the long-term evolution of the spot price, stochastic volatility in the spot price and the convenience yield, a time varying correlation structure of the state variables, the Samuelson (1965) effect, time-varying degrees of backwardation as well as time-varying risk premia. The model is highly tractable, allowing for quasi-analytical pricing formulas for futures and European options using transform techniques. It also allows for the valuation of exotic derivatives and real options using simulation methods.

- (ii) Complementary to the analytical results of Chapter 2, an extensive empirical study of the copper market is presented, in which the model is evaluated based on time series data of 37,126 futures and 12,166 corresponding American call option prices over a seven-year period. Based on this empirical study, it is shown that the model delivers significant improvements in terms of pricing and hedging commodity derivatives compared to benchmarks. Moreover, the study reveals that three independent stochastic factors may be sufficient to achieve high pricing accuracy with respect to either futures or option contracts, while a fourth factor is necessary to achieve comparably low pricing errors with respect to both contract types at the same time. Finally, it is shown that volatility contains important unspanned components in the copper market, implying that an option portfolio cannot be fully hedged by solely trading in the underlying futures contracts.

## 1.2 Integrated risk management

As stated in the introductory discussion, effective PRM strategies commonly integrate operational decision making, such as inventory management and pricing, with financial hedging activities, while adopting a firm-wide, all-encompassing definition of risk exposure. However, the necessary degree of coordination across both functional and divisional units poses managerial challenges amid volatile procurement costs and stochastic demand. For instance, consider the following example of an industrial firm facing an unexpected increase in raw material costs. Suppose the firm responds to the cost increase by raising prices. Under the assumption of price elastic demand, this decision can cause a decline in unit sales, higher than expected inventory levels at the end of this period, and a corresponding increase in associated inventory carrying cost. Since excess inventories may be deployed to satisfy future demand, necessary raw material procurement quantities in the present and future business period will decline. As a result, the exposure to future cost volatility is reduced, requiring an adjustment of the volume of traded hedging instruments. Finally, lower procurement and production levels may be associated with a loss in economies of scale, resulting in higher unit manufacturing costs. Now, should prices be raised in the first place? This type of decision problem may be further complicated by the presence of multiple, potentially interdependent business units with different product offerings. For example, cross-divisional correlations in demand or cost risk may provide natural hedging opportunities, which have to be recognized in jointly optimal decision making. Moreover, changes in the market environment in one division, may entail a necessary revision of optimal policies in all divisions, due to, for instance, interrelations with firm-wide profit variance or the presence of cross-selling effects.

In Chapter 3, a stylized model of a risk-averse industrial firm is developed, in which the firm sources raw material from a volatile commodity spot market, manufactures a single end-product that is sold to price sensitive customers under stochastic demand. Within this multi-period framework, we solve the firm's dynamic procurement, pricing, and financial hedging problem under multi-factor commodity price dynamics. Based on this model, we make the following specific contributions to the existing literature:

- (i) The model and analysis presented in Chapter 3, represent the first research that addresses the joint pricing, inventory management, and financial hedging problem of a risk-averse firm facing stochastic costs and demand. In this context, a flexible, Monte Carlo-based optimization algorithm



is presented, which allows to solve the firm's inter-temporal decision problem under arbitrary multi-factor cost dynamics.

- (ii) Complementary to the theoretical results of this chapter, a numerical study is conducted, which provides novel insights into: (a) optimal PRM policies and the value of managerial flexibility; (b) the sensitivity of optimal policies and expected performance to different commodity price models and parameter risk; (c) the impact of risk aversion on the distribution of cash flows; and (d) the sensitivity of results to a variety of critical input parameters.

As opposed to PRM in the previously considered single-product firm, risk management in large, multi-divisional enterprises does not only involve cross-functional decision coordination between, for instance, the procurement, sales, and finance department but can further necessitate cross-divisional information exchange and a firm-wide alignment of actions (Fisher & Kumar, 2010). Chapter 4 is, therefore, dedicated to optimal PRM in a two-product firm. While it is only a minor extension to allow for  $n$ -products, this specification is sufficient to allow for an investigation of the PRM specific trade-offs involved in the management of a large organization. While the general setup of the two-product model is similar to the single-product model, it features cost and demand risk in each of the two divisions, which can be respectively correlated. Additionally, we account for the possibility of stockout-based cross-demand between individual divisions to capture the potential substitutability/complementarity of the offered items. The firm dynamically maximizes inter-period, mean-variance utility over a multi-period horizon by choosing the jointly optimal vectors of stocking quantities and unit selling prices as well as a variance minimizing hedging portfolio consisting of futures, call, and put option contracts tradeable in each of the associated commodity markets. Based on this model, we make the following main contributions to existing literature in Chapter 4:

- (i) This is the first research investigating the jointly optimal procurement, pricing, and financial hedging problem of a multi-product, risk-averse firm and provides novel insights with respect to the interplay between multi-divisional operations, firm-wide net risk exposure, and optimal hedging in different markets. Within this framework, we provide analytic solutions to the firm's operational and financial decision problem.
- (ii) To supplement the analytical results of this chapter, a numerical study is provided, which investigates the impact of cross-divisional risk correlations and unilateral changes in the market environment of one division on the optimal firm-wide PRM policy. We then assess the impact of risk aversion and financial hedging on profits, utility, and optimal policies and provide a comparison of variance reduction under different operational and financial hedging strategies.



# 2

## A FOUR-FACTOR STOCHASTIC VOLATILITY MODEL OF COMMODITY PRICES

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*The content presented in this chapter is based on Schöne and Spinler (2015).*

The number of factors driving the uncertain dynamics of commodity prices has been a central consideration in financial literature. While the majority of empirical studies relies on the assumption that up to three factors are sufficient to explain all relevant uncertainty inherent in commodity spot, futures, and option prices, evidence from Trolle and Schwartz (2009b) and Hughen (2010) indicates a need for additional risk factors. In this chapter, we propose a four-factor maximal affine stochastic volatility model that allows for three independent sources of risk in the futures term structure and an additional, potentially unspanned stochastic volatility process. The model principally integrates the insights from Hughen (2010) and Tang (2012) and nests many well-known models in the literature. It can account for several stylized facts associated with commodity dynamics such as mean reversion to a stochastic level, stochastic volatility in the convenience yield, a time-varying correlation structure, and time-varying risk-premia. In-sample and out-of-sample tests indicate a superior model fit to futures and options data as well as lower hedging errors compared to three-factor benchmark models. The results also indicate that three factors are not sufficient to model the joint dynamics of future and option prices accurately.

### 2.1 Introduction

Stochastic commodity price models play a central role in the valuation and hedging of commodity-linked financial and real assets. The critical importance of carefully specifying the process followed by the underlying asset is highlighted in a number of studies. In the context of financial options, these include Lo and Wang (1995), Bakshi et al. (1997), and Hillard and Reis (1998). With respect to real investments, Tsekrekos et al. (2012) and Schöne (2014) show that valuation results and optimal investment behavior

depend critically on the stochastic commodity price process driving the uncertain cash flow stream of an asset.

In the wake of erratic price swings observed in many commodity markets during the last decade and the stellar growth in the volume of outstanding commodity derivatives during this period, also the valuation and risk-management of commodity-linked investments has gained greatly in importance.<sup>1</sup> However, despite the now abundant literature concerned with commodity price models, the topic is a subject of ongoing debate. This has to do with the stepwise discovery of numerous stylized facts associated with commodity price dynamics and the resulting challenge to address the new insights with more consistent, tractable, and intuitive pricing models for futures, options, and real assets.

In this context, a central consideration has been the number of factors assumed to drive the uncertain spot price dynamics. In the early stages, one-factor models dominated the scientific landscape in commodity price modeling. For instance, Schwartz (1982), Brennan and Schwartz (1985), Siegel, Smith, and Paddock (1987), and Cortazar and Schwartz (1993) assumed that all the uncertainty of price movements could be captured by a geometric Brownian motion (GBM). Similarly, Laughton and Jacoby (1993), Ross (1997), and Schwartz (1997), advocated one-factor mean-reverting Ornstein-Uhlenbeck (OU) models, which have the more favorable implication that the volatility of futures returns declines with contract maturity - a common property in commodity markets known as the Samuelson (1965) effect.

Common to all one-factor models, however, is the unrealistic implication that the returns of all futures in the term structure are perfectly correlated and that the degree of backwardation is time-invariant. Thus, the well-known two-factor models of Gibson and Schwartz (1990), Schwartz (1997), and the formally equivalent model of Schwartz and Smith (2000) emerged as more realistic alternatives to one-factor processes. To further improve pricing performance, several articles have advocated three-factor models. These include, for instance, the constant volatility models in Cortazar and Schwartz (1994), Schwartz (1997), Clewlow and Strickland (1999), Cortazar and Schwartz (2003), Casassus and Collin-Dufresne (2005), and Tang (2012) as well as the stochastic volatility models in Deng (2000), Richter and Sørensen (2003), Geman and Nguyen (2005), Hikspoors and Jaimungal (2008), Lutz (2010), Hughen (2010), and Liu and Tang (2011).

Yet, in spite of the success of these models, several studies indicate that more than three factors may be required to explain not only the cross-sectional dynamics of the futures curve but also the term structure of volatility and option prices. For instance, Cortazar and Naranjo (2006) show that a fourth factor is needed to fit the volatility term structure of crude oil futures and Trolle and Schwartz (2009b) find, based on the Heath, Jarrow, and Morton (1992) (HJM) framework, that the best fit to futures and options data is in a four-factor model featuring six independent sources of risk. In this model, three factors are associated with the futures curve and two factors capture the variation in implied volatility of

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<sup>1</sup> According to the Bank of International Settlements (2007), the number of outstanding exchange-traded commodity derivative contracts almost tripled from 2002 to 2005. During this period, the number of outstanding copper futures and options increased by 41% and 140%, respectively. Even more impressive is the development in the over-the-counter market, where the number of outstanding commodity derivative contracts reached \$6.4 trillion in mid-2006, about 14 times the level of 1998.

futures options across time, maturity, and moneyness. A more implicit argument for commodities being driven by more than three sources of uncertainty follows from Hughen (2010). In particular, his three factor stochastic volatility model exhibits the best fit to futures prices when these are allowed to depend on all three state variables, including the volatility process. In this configuration, however, the historical estimate of the volatility state variable is unrealistic and model fit to options data is unsatisfactory. When the model is restricted such that volatility is unspanned by futures, both the estimate of the volatility state variable and the fit to options data improve. However, in this setting futures pricing performance deteriorates to the level of a two-factor model. Hughen (2010) concludes that a fourth factor may be needed to successfully model the joint dynamics of futures and option prices.

While these articles suggest that a general pricing model for commodity futures and options necessitates three factors associated with the futures curve and at least one additional, potentially unspanned stochastic volatility (USV) factor, this possibility has been little explored by previous research. In fact, the only model consistent with this postulated requirement is the HJM-type model of Trolle and Schwartz (2009b).

To shed further light on the merits of allowing for an additional source of uncertainty, we make two main contributions to existing research in this chapter. First, a four-factor maximal affine stochastic volatility model is developed, which appears to be the first maximal stochastic volatility model for commodity prices that is not of the HJM-type but allows for more than three independent sources of uncertainty.<sup>2</sup> The model accommodates many stylized facts associated with commodities. These include mean reversion to a stochastic long-term level, persistent shocks in the long-term evolution of the spot price, stochastic volatility in the spot price and the convenience yield, a time varying correlation structure of the state variables, the Samuelson effect, time-varying degrees of backwardation as well as time-varying risk premia. The model is highly tractable as it leads to quasi-analytical formulas for futures and European options using transform techniques. It can account for the typical lag between the maturity of an option and the underlying futures contract and allows for the valuation of complex derivatives and real options using, for instance, the method of Longstaff and Schwartz (2001).<sup>3</sup> The model nests many well-known stochastic processes including the models of Gibson and Schwartz (1990), Richter and Sørensen (2003), Hikspoors and Jaimungal (2007), and Tang (2012).

The second contribution of this chapter is the empirical implementation of the model in the copper market using time series consisting of 37,126 futures observations and 12,166 corresponding American call option prices over the period from March 2006 to December 2013. In this context two research questions are addressed. First, does the four-factor model proposed in this chapter deliver a superior pricing performance for the joint dynamics of futures and option prices compared to successful three-factor models present in the literature? Second, do option prices contain important information not fully spanned by futures contracts in the copper market?<sup>4</sup>

<sup>2</sup> Collin-Dufresne, Goldstein, and Jones (2008) discuss models of order  $N > 3$  in the context of yield curve modeling. The four-factor commodity price model of Cortazar and Naranjo (2006) does not allow for stochastic volatility and the jump-diffusion model of Yan (2002) is not maximal.

<sup>3</sup> According to Schwartz and Miltersen (1998), accounting for the time lag between the maturity of the option contract and the underlying futures can have a significant effect on option prices.

<sup>4</sup> This question is raised by Trolle and Schwartz (2009b), who find evidence for unspanned stochastic volatility in commodity

The main results can be summarized in two statements. First, the four-factor model developed in this chapter delivers considerable improvements over the three-factor benchmark models in terms of pricing and hedging commodity derivatives. Moreover, three factors are not sufficient to price futures and options jointly but may yield a good fit to either type of contract, depending on model parameterization. Second, volatility contains important unspanned components in the copper market, implying that an option portfolio cannot be fully hedged by solely trading in the underlying futures.

The remainder of this chapter is organized as follows. In Section 2.2, the model is presented and formulas for the valuation of commodity futures and options are developed. The dataset and empirical model implementation as well as pricing and hedging results are discussed in Section 2.3. Section 2.4 concludes.

## 2.2 Commodity price model

It is assumed that the commodity spot price ( $S_t$ ) is driven by four independent sources of uncertainty. This assumption is based on two arguments. First, it follows from the introductory discussion that three factors are necessary to explain the cross-sectional variation in futures prices. Second, Trolle and Schwartz (2009b) find that 85.6% of the instantaneous variance of the spot price and 96.2% of the instantaneous variance in the cost of carry can be explained by one additional volatility factor. As the potential gain in pricing accuracy from a second volatility factor has to be traded-off against higher model complexity, a four-factor model is assumed to resemble the most practical compromise for the purpose of this thesis.

The log-spot price, denoted by  $x_t := \ln S_t$ , fluctuates around a non-stationary long-term mean  $\theta_t$  and  $q_t$  is a mean-reverting temporary factor with a similar interpretation as the net convenience yield.<sup>5</sup> The variance process of  $S_t$  is denoted by  $v_t$ . For easier reference, we will refer to the model as the stochastic mean and volatility (SMV) model. Under the risk-neutral probability measure  $\mathbb{Q}$ , the stochastic differential equation (SDE) of the state vector

$$X_t = [x_t \quad \theta_t \quad q_t \quad v_t]^\top \quad (2.1)$$

can be written in matrix notation as

$$dX_t = [A + BX_t] dt + \Sigma_t dW_t^{\mathbb{Q}}, \quad (2.2)$$

where the instantaneous variance and covariance are given by

$$\Sigma_t \Sigma_t^\top = \Omega_0 + \Omega_1 v_t, \quad (2.3)$$

and  $A$ ,  $B$ ,  $\Omega_0$ , and  $\Omega_1$  contain constants.<sup>6</sup> The notation used in this paper is closely related to Hugen

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markets. However, their empirical study is limited to the case of crude oil.

<sup>5</sup> The relationship between  $q_t$  and the net convenience yield of the commodity is discussed more formally later in this section. Note that a similar approach is adopted in Tang (2012).

<sup>6</sup> We assume the existence of an equivalent risk-neutral probability measure, implying the absence of arbitrage according to the first fundamental theorem of asset pricing (see, e.g., Duffie, 1996). The measure is generally unique and markets are complete if a sufficient number of futures are traded (Casassus & Collin-Dufresne, 2005).

(2010), who also provides several examples reconciling the matrix form adopted in this chapter with the more traditional form of SDEs in the context of several well-known pricing models. Substituting  $A$ ,  $B$ ,  $\Omega_0$ , and  $\Omega_1$  into Equation 2.2 yields four SDEs corresponding to the traditional writing style.

Given a probability space and filtration generated by a standard Brownian Motion  $W_t^{\mathbb{Q}}$  in  $\mathbb{R}^4$ , the commodity spot price is an Itô process with respect to  $W_t^{\mathbb{Q}}$ . It is well-known (see, e.g., Björk, 2009) that in the absence of arbitrage, the risk-neutral drift of  $S_t$  must satisfy  $\mathbb{E}_t^{\mathbb{Q}}[dS_t] = (r - \delta_t) S_t dt$ , where  $r$  denotes the risk-free interest rate and  $\delta_t$  refers to the net convenience yield, which accounts for the "interim benefits accruing to the physical owners of a commodity as a rate" (Routledge, Seppi, and Spatt (2000, p. 1314)). The instantaneous variance of Equation 2.2 needs to satisfy

$$\Sigma_{t,11}^2 dW_{t,1}^{\mathbb{Q}} + \Sigma_{t,12}^2 dW_{t,2}^{\mathbb{Q}} + \Sigma_{t,13}^2 dW_{t,3}^{\mathbb{Q}} + \Sigma_{t,14}^2 dW_{t,4}^{\mathbb{Q}} = v_t. \quad (2.4)$$

In the terminology of Duffie and Kan (1996), models of the form in Equation 2.2, in which  $x_t$  and  $\Sigma_t \Sigma_t^{\top}$  are affine functions of the state variables, are referred to as affine models of the spot price. In particular, a four-factor model in which the instantaneous variance is driven by a single state variable ( $v_t$ ) belongs to the class of  $\mathbb{A}_1(4)$  models. The model is 'maximal' in terms of Dai and Singleton (2000) and Collin-Dufresne et al. (2008) in the sense that the maximum number of risk-neutral parameters exists, which are also identifiable from contingent claims prices.<sup>7</sup> The risk-neutral drift in the SMV model is determined by  $A$  and  $B$ :

$$A = \begin{bmatrix} 0 \\ \mu_2 \\ \kappa_{33} \mu_3 \\ \kappa_{44} \mu_4 \end{bmatrix}, \quad B = \begin{bmatrix} -\kappa_{11} & \kappa_{11} & -1 & \kappa_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa_{33} & 0 \\ 0 & 0 & 0 & -\kappa_{44} \end{bmatrix}. \quad (2.5)$$

This specification agrees with many theoretical and empirical insights about commodity prices. In this regard, two aspects deserve further discussion.

First,  $\kappa_{11}$  creates a co-integration relationship between  $x_t$  and  $\theta_t$  such that the spot price may exhibit mean reversion towards the stochastic long-term level  $\theta_t$  at a mean reversion speed  $\kappa_{11} > 0$ . This agrees with graphical evidence presented in Casassus and Collin-Dufresne (2005, p. 2298), Tang (2012, p. 782), and Section 2.3.1 of this chapter, where it is shown that the price of the short-term futures - an approximation of the spot price - tends to fluctuate around the price of the long-term futures, which can also be interpreted as the expected long-term price level  $\theta_t$ . The process of the long-term mean is chosen to be non-stationary, implying that also the system of  $x_t$  and  $\theta_t$  is non-stationary. This assumption is justified, among others, by Pindyck and Rubinfeld (1991) and Pindyck (1999), who show that more than 100 years of data are necessary to reject a unit root in crude oil and copper prices and fail to reject a random walk for coal, natural gas, and lumber over the same period. Moreover, Cashin, Liang, and

<sup>7</sup> The concept of 'maximality' of term structure models in the context of commodity markets is also discussed, for instance, in Casassus and Collin-Dufresne (2005), Hughen (2010), and Liu and Tang (2011).

McDermott (2000) and Bloch, Fraser, and MacDonald (2012) find that commodity price shocks are typically persistent.<sup>8</sup> However, although the system is non-stationary, the model dynamics allow for mean reversion in the futures term structure, which does not require mean reversion to a constant mean (Tang, 2012).<sup>9,10</sup>

Second, allowing the drift of  $x_t$  to be a function of the state vector allows for a richer unconditional correlation structure between the spot price, convenience yield, and volatility, compared to many earlier models, where the state variables may only have an instantaneous correlation in the Brownian motion terms (Casassus & Collin-Dufresne, 2005).<sup>11</sup> This is desirable given the empirically documented interaction between these state variables. For instance, Routledge et al. (2000) show in a competitive rational expectations model that convenience yields should be high in times of low commodity storage in the economy. As prices are typically high in times of supply shortages (Wu & Chen, 2010), it follows that the theory of storage predicts a positive relationship between spot prices and the convenience yield. To see how the positive level dependence between the spot price and the convenience yield is captured by the SMV model, we use the arguments of Tang (2012). In particular, recap that the risk-neutral drift of the spot price has to satisfy the following relationship

$$r - \delta_t - \frac{1}{2}\sigma_t^2 = \kappa_{11}(\theta_t - x_t) - q_t + \kappa_{14}v_t \quad \Leftrightarrow \quad \delta_t = r - \frac{1}{2}\sigma_t^2 - \kappa_{11}(\theta_t - x_t) + q_t - \kappa_{14}v_t.$$

By equating the drift of the SMV model with the required risk-neutral dynamics of the spot price, it is evident that a positive relationship between the net convenience yield and the log-spot price is established if  $\kappa_{11} > 0$ . Hence, the SMV model allows for an unconditional correlation between the spot price and the convenience yield in a similar way as the models in Casassus and Collin-Dufresne (2005) and Huguen (2010), where this effect is accomplished by allowing the drift of the convenience yield process to depend on the log-spot price.

With respect to the interaction between the spot price and volatility, Geman and Nguyen (2005) find a negative relationship between volatility and world inventory in the soybean market. Assuming, again, that prices are high in times of a relative scarcity of a commodity, we expect a positive relationship between spot prices and volatility.<sup>12</sup> The drift of the SMV model captures this possibility if  $\kappa_{14} > 0$ . An alternative interpretation for the role of  $\kappa_{14}$  follows from Litzenberger and Rabinowitz (1995) and Ng and Pirrong (1994), who find that the degree of backwardation in oil and metal futures is positively related to volatility. If this effect dominates, we expect  $\kappa_{14} < 0$ , since an increase in volatility will lower

<sup>8</sup> The assumption of persistent shocks in the long-term mean is relaxed by Korn (2005), who finds that this may have important implications for the valuation of long-term futures.

<sup>9</sup> In particular, Bessembinder, Coughenour, Seguin, and Smoller (1995) show that shocks to commodity spot prices are not fully absorbed by the futures curve, indicating that investors expect parts of these shocks to be reversed in the future. Yet, this does not necessarily imply stationarity of the spot price process in the sense of unit-root or variance ratio tests (Schöne, 2014).

<sup>10</sup> Previous studies discussing models featuring mean reversion to a non-stationary level include Pilipovic (1997), Hikspoors and Jaimungal (2007), Lutz (2010), and Tang (2012). On the contrary, Deng (2000), Geman and Nguyen (2005), Hikspoors and Jaimungal (2008), and Realdon (2013) allow mean reversion in the long-term level.

<sup>11</sup> Examples of such models can be found in Brennan (1991), Gibson and Schwartz (1990), and Schwartz (1997).

<sup>12</sup> This is often referred to as the inverse leverage effect, which is discussed in more detail in Section 2.3.1.



the risk-neutral drift of  $x_t$  and, hence, the expected future spot price. This corresponds to a lower futures price and a higher degree of backwardation. In either case, the presence of an unconditional correlation structure between  $x_t$ ,  $q_t$ , and  $v_t$  enhances model flexibility and complements the instantaneous variance and covariance structure described in the following paragraph.

The instantaneous variance and covariance structure of the SMV model closely follows Hugen (2010) and is specified by the following set of parameters

$$\Omega_0 = \begin{bmatrix} 0 & s_{12} & s_{13} & \vartheta\sigma_{14} \\ s_{12} & s_{22} & s_{23} & \vartheta\sigma_{24} \\ s_{13} & s_{23} & s_{33} & \vartheta\sigma_{34} \\ \vartheta\sigma_{14} & \vartheta\sigma_{24} & \vartheta\sigma_{34} & \vartheta\sigma_{44} \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} \end{bmatrix}. \quad (2.6)$$

Accordingly, the SMV model allows for stochastic volatility in all state variables and a full, time-varying instantaneous correlation structure associated with the entire state vector. This takes into account, among others, three well-documented empirical observations. First, spot price volatility is highly stochastic in most commodity markets (e.g., Pindyck, 2004; and Schöne, 2014), which coincides with fat tails in the distribution of commodity returns (e.g., Kat & Oomen, 2007) and pronounced implied volatility smiles in options data (e.g., Eydeland & Wolyniec, 2003). Second, the volatility of the convenience yield is stochastic (Liu & Tang, 2011). Third, Routledge et al. (2000) predict that the correlation between the spot price and the convenience yield needs to be time-varying in a competitive rational expectations model.

### 2.2.1 Identifiability and admissibility

In order for all model parameters to be identifiable from contingent claims prices and to ensure that the SDE in Equation 2.2 is admissible, i.e., that it has a unique solution for any initial value of the state vector, restrictions need to be imposed on the parameters.

From Dai and Singleton (2000) and Collin-Dufresne et al. (2008) it follows that a model of the form  $\mathbb{A}_1(4)$  admits 22 identifiable parameters under the risk-neutral measure. In particular, we have the following four restrictions on the number of identifiable parameters. First, two parameters may define the drift of the variance process. Second, five parameters may determine the drift of the spot price process. Third, three covariance terms may exist between the variance process and the remaining state variables. Fourth, 12 parameters may determine the variance and covariance terms of the Gaussian state variables ( $x_t$ ,  $\theta_t$ , and  $q_t$ ). These conditions are satisfied and there is no ambiguity in the dynamics of the state vector.<sup>13</sup> However, it is worth noting that with respect to the fourth condition, the formulation of  $\Sigma_t \Sigma_t^\top = \Omega_0 + \Omega_1 v_t$  differs from the canonical form in Collin-Dufresne et al. (2008), as the volatility of the variance process is not standardized in the SMV model. Instead, the variance process coincides with the variance of the first state variable ( $\Omega_0(1, 1) = 0$  and  $\Omega_1(1, 1) = 1$ ) and has an instantaneous variance of  $\sigma_{44}$ . Hence, the SMV model has 10 variance/covariance terms of the Gaussian state variables and the

<sup>13</sup> See Collin-Dufresne et al. (2008) and Hugen (2010) for details.

additional parameters  $\sigma_{44}$  and  $\nu$ . In total, there are 22 risk-neutral parameters as permitted.

The existence of a unique solution to Equation 2.2 in stochastic volatility models necessitates that  $v_t$  does not stochastically move out of the region, where the instantaneous process variance is positive and  $\Sigma\Sigma^\top = \Omega_0 + \Omega_1 v_t$  is positive definite. Following Hughen (2010), necessary and sufficient constraints on the parameters are the Feller conditions

$$2\kappa_{44} (\mu_4 + \vartheta) \geq \sigma_{44}, \quad 2\hat{\kappa}_{44} (\hat{\mu}_4 + \vartheta) \geq \sigma_{44}, \quad \vartheta \leq 0, \quad (2.7)$$

and the expressions  $\Omega_1$  as well as  $\Omega_0 - \Omega_1 \vartheta$  need to be positive semidefinite, i.e., the smallest eigenvalue must be non-decreasing in both expressions. The second term in Equation 2.7 refers to the model dynamics under the physical probability measure that will be discussed in Section 2.2.5.

## 2.2.2 Futures prices

It is well-known (e.g., Cox, Ingersoll, & Ross, 1981) that the futures price at time  $t$  with maturity  $T-t = \tau$ , is equivalent to the risk-neutral expectation of the future spot price

$$F_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}} [S_{t+\tau}] = \mathbb{E}_t^{\mathbb{Q}} [e^{x_{t+\tau}}] \quad (2.8)$$

and that the futures price is the solution to the following Feynman-Kac equation with boundary condition  $F_{T,0} = S_T$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial X} (A + BX) + \frac{1}{2} \text{Tr} \left( \frac{\partial^2 F}{\partial X^2} (\Omega_0 + \Omega_1 v) \right). \quad (2.9)$$

From Duffie and Kan (1996), it follows that the solution to affine term structure models of the form in Equation 2.9 is given by

$$\ln F(x_t, \theta_t, q_t, v_t, \tau) = \alpha(\tau) + \beta_1(\tau) x_t + \beta_2(\tau) \theta_t + \beta_3(\tau) q_t + \beta_4(\tau) v_t. \quad (2.10)$$

Substituting Equation 2.10 into Equation 2.9 yields

$$\begin{aligned} \alpha' + \beta' X_t &= \beta A + \beta B X_t + \frac{1}{2} \text{Tr} \left( \beta^\top \beta (\Omega_0 + \Omega_1 v_t) \right) \\ &= \beta A + \beta B X_t + \frac{1}{2} \left( \beta \Omega_0 \beta^\top + \beta \Omega_1 \beta^\top v_t \right), \end{aligned} \quad (2.11)$$

where  $\beta = [\beta_1(\tau) \ \beta_2(\tau) \ \beta_3(\tau) \ \beta_4(\tau)]$ . Separation of variables leads to the following system of ordinary differential equations (ODE)

$$\alpha' = \beta A + \frac{1}{2} \beta \Omega_0 \beta^\top \quad (2.12)$$

$$\beta'_1 = \beta B_{(1)} \quad (2.13)$$

$$\beta'_2 = \beta B_{(2)} \quad (2.14)$$

$$\beta'_3 = \beta B_{(3)} \quad (2.15)$$

$$\beta'_4 = \beta B_{(4)} + \frac{1}{2} \beta \Omega_1 \beta^\top, \quad (2.16)$$

where  $B_{(j)}$  refers to the  $j$ th column of  $B$ . This system is of the Riccati type and uniquely determines  $\alpha$ ,  $\beta$ , and  $F$ . As the futures price at maturity must coincide with the spot price, we have the following boundary conditions:  $\beta_1 = 1$  and  $\alpha(0) = \beta_2(0) = \beta_3(0) = \beta_4(0) = 0$ . This is quickly verified by substituting the boundary conditions into Equation 2.10, which gives  $F(x_T, \theta_T, q_T, v_T, 0) = S_T$ . The solution to the system in Equation 2.12-2.16 can be obtained using a standard Runge-Kutta algorithm.

### 2.2.3 Option prices

In commodity markets, options are typically written on futures contracts and not the physical underlying. Thus, we will focus on futures options in this section. By applying Itô's lemma to Equation 2.10, we obtain the following risk-neutral dynamics of the futures price and the volatility state variable

$$dF_t = F_t \beta(\tau) \Sigma_t dW_t \quad (2.17)$$

$$dv_t = (\mu_4 + \kappa_{44} v_t) dt + \vec{e}_4 \Sigma_t dW_t, \quad (2.18)$$

where  $\vec{e}_4 = [0 \ 0 \ 0 \ 1]$ . Following Heston (1993), we can represent the value of a European call option  $C$  on a futures contract  $F$  with strike  $K$ , futures maturity  $\tau_1$ , option maturity  $\tau_0$ , and discount function  $P(t, \tau_0)$  in the form of Black (1976)

$$C(t, \tau_0, \tau_1, F, v, K) = P(t, \tau_0) (F \Pi_1 - K \Pi_2). \quad (2.19)$$

$\Pi_1$  and  $\Pi_2$  can be calculated using the following formula from Heston (1993)

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-i\phi \ln[K]} f_j(x, v, \tau_0; \phi)}{i\phi} \right] d\phi, \quad j = 1, 2 \quad (2.20)$$

where  $f(\cdot; \phi)$  refers to the model's characteristic function.<sup>14, 15</sup> It is known that  $f(\cdot)$  is exponential-affine and takes the form

$$f(x, v, \tau_0, \tau_1; \phi) = e^{a_j(\tau_0, \tau_1) + b_j(\tau_0, \tau_1)v + i\phi x}. \quad (2.21)$$

<sup>14</sup> The discount function denotes the time- $t$  price of a zero-coupon bond with a face value of one. In the empirical model implementation discussed in Section 2.3, the discount function  $P(t, \tau_0)$  is obtained by fitting a Nelson and Siegel (1987) curve to the LIBOR/Swap curve on each trading day. The curve consists of 1-month, 3-month, 6-month, and 12-month LIBOR rates and the 2-year swap rate. A similar approach is used by Trolle and Schwartz (2009b).

<sup>15</sup> The pricing formula of Heston (1993) was later extended, for instance, by Bakshi and Madan (2000) and Duffie, Pan, and Singleton (2000).

Using the approach of Heston (1993) and his appendix, we obtain the affine coefficients  $a_j$  and  $b_j$  as solutions to the following pair of ordinary differential equations

$$a_j = \frac{1}{2} \left[ (2u_j - 1) i\phi - \phi^2 \right] \beta \Omega_0 \beta^\top + \left[ (u_j + i\phi) \beta \Omega_0 \vec{e}_4 + \mu_4 \right] b_j + \frac{1}{2} \sigma_{44} \vartheta b_j^2 \quad (2.22)$$

$$b_j = \frac{1}{2} \left[ (2u_j - 1) i\phi - \phi^2 \right] \beta \Omega_1 \beta^\top + \left[ (u_j + i\phi) \beta \Omega_1 \vec{e}_4 - \kappa_{44} \right] b_j + \frac{1}{2} \sigma_{44} b_j^2, \quad (2.23)$$

where  $u_1 = 1$ ,  $u_2 = 0$ , and  $a(0) = b(0) = 0$ . In the remainder of this chapter, the integral in Equation 2.20 is evaluated using a Gauss-Legendre quadrature with 30 integration points and the integral is truncated at 400.<sup>16</sup>

#### 2.2.4 Unspanned stochastic volatility

An important aspect in the valuation and risk management of commodity options is the question whether options depend on risk not fully spanned by futures contracts. Thus, this section is devoted to briefly discuss the implications of unspanned stochastic volatility for the SMV model.

Previous studies have presented mixed evidence as to whether volatility is unspanned in commodity markets. On the one hand, Trolle and Schwartz (2009b) find, in an extensive empirical study, that volatility is indeed mostly unspanned by futures in the crude oil market, implying that options cannot be fully hedged by trading only in their underlying futures. On the other hand, as discussed above, the degree of backwardation in oil and metal futures is positively related to volatility, indicating that futures prices contain at least some information related to volatility.

Contrary to all constant volatility models and many stochastic volatility models, including those of Deng (2000), Richter and Sørensen (2003), Geman and Nguyen (2005), and Hikspoors and Jaimungal (2008), the SMV model can exhibit unspanned stochastic volatility under certain parameter conditions. In particular, we require that the futures price  $F$ , given by Equation 2.10, does not depend on the variance state variable  $v_t$ , which implies that  $\beta_4$  needs to be identically zero. Following the approach in Hughen (2010),  $\beta_4$  will disappear if the function  $M$  (from Equation 2.16) and all its derivatives with respect to  $\tau$  are zero when  $\tau = 0$ .<sup>17</sup>

$$\begin{aligned} M &= \beta B_{(4)} + \frac{1}{2} \beta \Omega_1 \beta^\top = \\ &= \beta_1 \kappa_{14} + \frac{1}{2} \left( \beta_1^2 + \beta_2^2 \sigma_{22} + \beta_3^2 \sigma_{33} \right) + \beta_1 \beta_2 \sigma_{12} + \beta_1 \beta_3 \sigma_{13} + \beta_2 \beta_3 \sigma_{23} \end{aligned} \quad (2.24)$$

Using the boundary conditions of Equation 2.11 and Equation 2.13-2.15, the derivatives of  $M$  can be calculated up to any order. It follows that in the presence of unspanned stochastic volatility  $\kappa_{11} = \sigma_{13} = \sigma_{33} = 0$  and  $\kappa_{14} = -0.5$ . Also the remaining covariance terms related to the third state variable in  $\Omega_1$

<sup>16</sup> More specifically, we use 15 integration points on the interval 0 to 50 and 15 on the interval 50 to 400. A similar approach is adopted by Trolle and Schwartz (2009b).

<sup>17</sup> See Collin-Dufresne and Goldstein (2002) for further reference related to the necessary conditions for unspanned stochastic volatility in term structure models.

( $\sigma_{23}$  and  $\sigma_{34}$ ) will be zero. The restrictions for  $\Omega_1$  and  $\Omega_0 - \Omega_1\vartheta$  to be positive semidefinite remain intact. It is evident that the above restrictions heavily influence the risk-neutral dynamics of the SMV model. When volatility is unspanned, the futures price becomes a function of two state variables and is given by

$$\ln F(x_t, \theta_t, q_t, v_t, \tau) = \alpha(\tau) + x_t + \beta_3(\tau) q_t, \quad (2.25)$$

where  $\alpha(\tau)$  and  $\beta_3(\tau)$  remain analytic functions of time. Options may, additionally, depend on the unspanned stochastic volatility factor.

### 2.2.5 Risk premia

The above discussion refers entirely to the dynamics of the state vector under the risk-neutral probability measure. While this specification is appropriate to price financial securities, we require risk premia for the empirical model implementation and commodity price forecasting. The most common approach in previous empirical research (e.g., Schwartz, 1997; Geman & Nguyen, 2005; Richter & Sørensen, 2003) is to assume that risk premia are constant. However, Casassus and Collin-Dufresne (2005) demonstrate that time-varying risk premia are necessary to capture important mean reversion dynamics in commodity spot prices, which can have substantial consequences for the model-implied holding period return distribution and risk management. Thus, in the SMV model, we follow the approach of Dai and Singleton (2000), Duffee (2002), and Casassus and Collin-Dufresne (2005) and specify the vector of risk premia  $\Lambda = [\lambda_x \ \lambda_\theta \ \lambda_q \ \lambda_v]^\top$  as an affine function of  $X_t$ .

From Girsanov's Theorem it follows (see, e.g., Björk, 2009) that the density transformation from the risk-neutral to the physical measure is given by

$$W_t^{\mathbb{P}} = W_t^{\mathbb{Q}} - \int_0^t \Lambda(X_s) ds, \quad (2.26)$$

where the vector of risk premia is defined as

$$\Lambda = [\hat{A} - A + (\hat{B} - B) X_t] \Sigma^{-1}. \quad (2.27)$$

Substituting Equation 2.26 into Equation 2.2, yields the continuous time model dynamics under the physical probability measure

$$dX_t = [\hat{A} + \hat{B} X_t] dt + \Sigma_t dW_t^{\mathbb{P}}, \quad (2.28)$$

where  $A$  and  $B$  are now replaced by

$$\hat{A} = \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \hat{\kappa}_{33} \hat{\mu}_3 \\ \hat{\kappa}_{44} \hat{\mu}_4 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} -\hat{\kappa}_{11} & \kappa_{11} & -1 & \kappa_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\kappa}_{33} & 0 \\ 0 & 0 & 0 & -\hat{\kappa}_{44} \end{bmatrix}. \quad (2.29)$$

This specification can accommodate, for instance, a negative relationship between commodity spot prices and the spot price risk premium, which is supported by several empirical studies (e.g., Bessembinder et al., 1995).<sup>18</sup> To see this, we can write the risk premium of the log-spot price as

$$\lambda_x = [\hat{\mu}_1 - (\hat{\kappa}_{11} - \kappa_{11}) x_t] \Sigma^{-1}, \quad (2.30)$$

where a negative relationship between  $x_t$  and  $\lambda_x$  exists if  $\hat{\kappa}_{11} > \kappa_{11}$ . By similar arguments, the SMV model can capture, for instance, a negative premium for volatility risk, as discovered by Doran and Ronn (2008) in the natural gas and oil market. Generally, it is possible to allow for an even more flexible specification of  $\Lambda$  by inserting additional parameters into  $\hat{B}$ . In this way, the risk premium of each risk factor may depend not only on the state variable it is directly associated with but also on the other state variables.<sup>19</sup> This possibility, however, is not further explored for two reasons. First, previous studies (e.g., Geman & Nguyen, 2005; and Trolle & Schwartz, 2009b) find that risk premia can typically not be estimated from a series of past derivative prices with statistical significance, suggesting that more elaborate specifications of  $\Lambda$  may lead to little additional and reliable insights. Second, experiments based on the dataset used in this chapter confirm that additional parameters slow down model calibration and seem to add negligible pricing accuracy.

## 2.3 Empirical implementation

In this section, the previously outlined model is evaluated based on empirical data from the market of copper derivatives. We proceed by first presenting the dataset and estimation procedure. Subsequently, calibration results as well as the pricing and hedging performance of the model are discussed.

### 2.3.1 Data

The dataset consists of copper futures and corresponding American call option prices obtained from COMEX. The choice of the copper market as the subject of study is motivated by three aspects. First, copper is considered as one of the most important industrial metals with extensive usage in construction, heating, air conditioning, and wiring applications. Second, the financial market for copper-related products is highly liquid and COMEX copper futures and options are among the most actively traded commodity instruments available.<sup>20</sup> Third, the copper market has been a popular subject in related research (e.g., Schwartz, 1997; Cortazar & Schwartz, 2003; and Tang, 2012). As previous studies find that models with superior performance in the application to copper derivatives are likewise preferable in various other markets (e.g., crude oil and gold), we expect the empirical findings of this section to be similarly indicative of model performance in other important, non-cyclical markets of storable commodities.

<sup>18</sup> This is also one of the main features in Tang (2012), who uses a similar specification of the spot price risk premium.

<sup>19</sup> See Hughen (2010), for a more general specification of risk premia in the context of a three-factor stochastic volatility model.

<sup>20</sup> The number of outstanding copper futures reached 35.5m in 2005. This compares to 93m, 34.5m, and 33.3m for crude oil, gold, and aluminum, respectively (Bank of International Settlements, 2007). According to CME Group (2013), the average volume of daily trades in COMEX copper futures was in excess of 70,000 contracts in 2013.

The sample period lasts from 31 March 2006 to 30 December 2013. The choice of this time horizon reflects three considerations. First, it takes into account the structural changes witnessed by commodity markets over the last decade. In particular, the ascent of commodity prices and volatility beginning in late 2004 are often considered to be symptoms of both structural demand and supply imbalances originating from fast growing demand in Asia and the continuing *financialization* of commodity markets.<sup>21</sup> As these factors are expected to have a persistent impact on commodity price dynamics, it appears sensible to focus on a sample period that already reflects the *new normal* in the copper market and may, thus, yield results that are more indicative of pricing performance in the future. Second, the sample includes times of substantial financial market turmoil surrounding the global financial crisis. As it is useful to gain an understanding of how well pricing models perform even in times of extreme market movements, the financial crisis is kept in the sample. Third, the availability and quality of options data corresponding to the relevant futures contracts improved significantly after the beginning of the sample period.

For both options and futures, we use daily official closing prices, resulting in 1954 trading dates in the sample.<sup>22-23</sup> The futures data consists of 19 generic time series constructed from individual futures contracts with maturities ranging from 1 month to 19 months. This corresponds to a total of 37,126 futures prices in the dataset. The futures are arranged in consecutive order such that  $F_1$  represents a generic series of futures contracts closest to maturity and  $F_{19}$  refers to the series of contracts with the longest maturity. For each series  $F_n$ , the index  $n$  also roughly corresponds to the maturity of the time series in months.<sup>24</sup> In line with common practice (see, e.g., Richter & Sørensen, 2003; Hughen, 2010), maturities are calculated up to the last trading date.<sup>25</sup> For the calibration of the SMV model, every second futures ( $F_{1,3,\dots,19}$ ) is used, while the remaining contracts ( $F_{2,5,\dots,18}$ ) are saved for out-of-sample tests.

Panel A of Figure 2.1 illustrates the price of the shortest and longest maturity futures in the dataset as well as the degree of strong backwardation. The stochastic fluctuations of the short-term futures price around the long-term futures price support the assumption of mean reversion in the spot price to a stochastic equilibrium level in the SMV model, as discussed in Section 2.2. Panel B displays different shapes of the futures term structure on selected dates during the sample period.

The call options data consists of 16 generic time series, which are constructed from individual option contracts. On each trading date, these comprise eight different maturities and two different strike prices for each of the maturities. For model calibration, we use maturities of 1, 5, 9, and 13 months, while maturities of 2, 6, 10, and 14 months are retained for out-of-sample tests. Options with longer maturities are not considered as liquidity tends to decrease with maturity. However, the range of maturities in the sample is sufficient to capture some information with respect to volatility dynamics along the term

<sup>21</sup> See Mayer (2009), Nissanke (2012), and Silvennoinen and Thorp (2013).

<sup>22</sup> The official daily closing price is determined by the closing range used for marking to market all account balances including the calculation of daily gains and losses, margin calls, and invoice prices for deliveries in futures market accounts.

<sup>23</sup> Cortazar and Schwartz (2003) suggest that daily data yields a more complete picture of a model's pricing performance compared to lower frequency (e.g., weekly) data. In particular, they find that two-factor models are able to price the futures term structure accurately on average but can exhibit large pricing errors on individual days.

<sup>24</sup> A graphical illustration of this data organization can be found in Schwartz (1997, p. 935).

<sup>25</sup> The last trading date of a futures contract is approximately one month after the first notice date, whereupon the holder of the futures may be required to take physical delivery of the underlying commodity.

structure. For instance, in line with the Samuelson (1965) effect, Panel F shows decreasing implied Black (1976) (i.e., log-normal) volatilities as maturity increases.

The two strike prices used with each of the maturities are denoted by  $K_1$  and  $K_2$ . Both strikes are adjusted over time to roughly track changes in the spot price. This is shown in Panel C of Figure 2.1. While it would be interesting to observe more than two different strike prices for all maturities on each day to gain a more complete picture of the implied volatility surface, this possibility is not pursued here for simplicity. However, as shown by Panel E and F of Figure 2.1, the dataset contains, nonetheless, important information about implied volatility dynamics for different moneyness levels (defined as the futures price divided by the strike). For instance, Panel F shows a higher average implied volatility for options with strike price  $K_1$  (average moneyness: 1.00), than for options with strike price  $K_2$  (average moneyness: 0.94). This implies an upward sloping volatility skew and reflects that in-the-money calls are on average expensive relative to out-of-the money calls. It is also known as the *inverse leverage effect* and can be explained by a high willingness to pay for protection against rising commodity prices by, e.g., industrial copper consumers. Intuitively, this may be explained by commodities' positive exposure to supply shocks. Liu and Tang (2011) attribute the upward sloping volatility skew for call options on crude oil and copper futures to a positive relationship between spot prices, convenience yields, and volatility, which can also be captured by the SMV model.<sup>26</sup>

Since the option pricing formula derived in Section 2.2.3 is only valid for European options, historical option prices have to be converted to European prices by making an adjustment for the early exercise premium. This is accomplished by, first, fitting a trinomial lattice to the American option price and by then using the inferred volatility to calculate the corresponding European price and the early exercise premium.<sup>27</sup> To minimize the possibility of errors resulting from the approximation of the early exercise premia, we discard all options with moneyness greater than 1.05.<sup>28</sup> Also excluded are OTM options with moneyness below 0.8, as these tend to be less actively traded.<sup>29</sup> After making these adjustments to the dataset, we are left with 12,166 European call option prices. An overview of the distribution of option contracts with respect to moneyness is given in Panel D of Figure 2.1, taking into account the full sample period.

### 2.3.2 Estimation procedure

A common calibration methodology for term structure models with unobservable state variables is the Kalman filter.<sup>30</sup> This approach is particularly well-suited when the estimation involves a complicated model and has desirable econometric properties for the purpose of this thesis (Duffee & Stanton, 2004). As

<sup>26</sup> Liu and Tang (2011) point out that an upward sloping volatility skew is consistent with evidence from Deaton and Laroque (1996), who find that commodity prices exhibit irregular, large, positive price spikes.

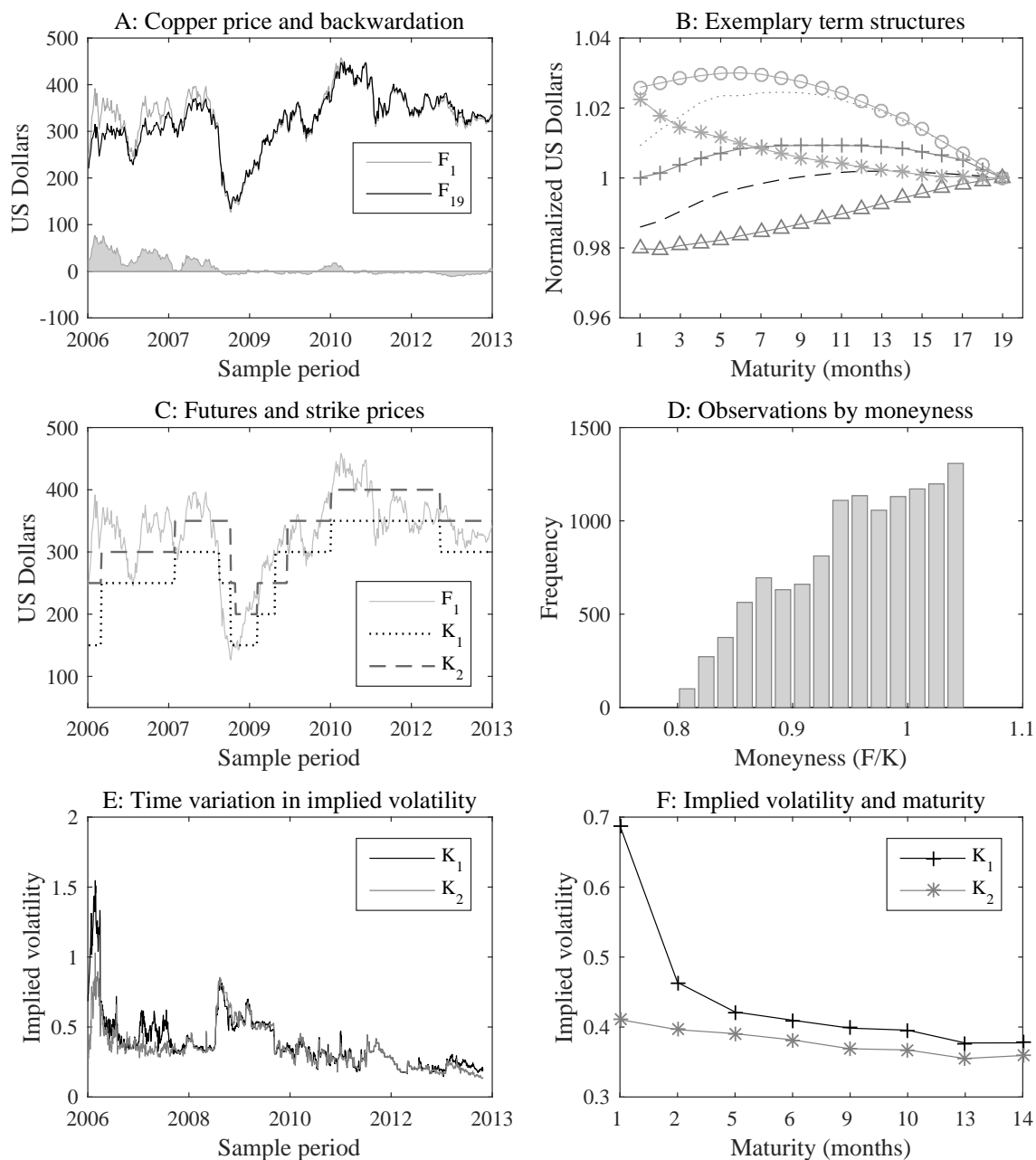
<sup>27</sup> Richter and Sørensen (2003) and Trolle and Schwartz (2009b) use a similar approach.

<sup>28</sup> Trolle and Schwartz (2009b) use an even more restrictive approach and exclude all in-the-money (ITM) options, i.e., with moneyness above one. They point out that out-of-the-money (OTM) options also have the advantage that they tend to be more liquid than ITM options.

<sup>29</sup> Also excluded are options with prices of less than US\$ 0.1.

<sup>30</sup> Previous articles employing the Kalman filter in the calibration of commodity price models with stochastic volatility include Richter and Sørensen (2003), Nielsen and Schwartz (2004), Geman and Nguyen (2005), Trolle and Schwartz (2009b), and Liu and Tang (2011). Further references for constant volatility models are Lo and Wang (1995), Schwartz (1997), Pindyck (1999), Schwartz and Smith (2000), and Cortazar and Naranjo (2006).





**Figure 2.1: Dataset.** Panel A illustrates the evolution of the short-term and long-term copper futures over the sample period (31 March 2006 - 30 December 2013). The histogram shows the degree of strong backwardation calculated as  $F_1 - F_{19}$ , where  $F_1$  is interpreted as a proxy for the spot price. Panel B shows different exemplary term structures, where prices are shown relative to the long-term futures price for better readability. The sample dates corresponding to the curves from top to bottom (referring the left edge of the curves) are: 12.11.07, 21.05.10, 22.07.10, 25.10.10, 02.07.13, 16.12.13. Panel C illustrates how the strike price series  $K_1$  and  $K_2$  are adjusted to track changes in the underlying. Panel D illustrates the distribution of option contracts in the sample w.r.t moneyness ( $F/K$ ). Panel E depicts the implied volatility of the short-term options for  $K_1$  and  $K_2$ . Panel F displays the average implied volatility for all options in the sample against maturity. The average moneyness corresponding to options with strike series  $K_1$  and  $K_2$  are 1.00 and 0.94, respectively.

option prices are non-linear in the volatility state variable and since the SMV model is non-Gaussian, we resort to the extended Kalman filter (EKF) in conjunction with quasi-maximum likelihood estimation.<sup>31</sup> By maximizing the log-likelihood obtained from the Kalman filter recursions, we can infer the risk-neutral parameters from the cross section of derivative prices and the risk premia from the information contained in the time series dimension.

First, the model is cast in state space form, which consists of a transition equation and a measurement equation. The transition equation describes the discrete-time dynamics of the state vector  $X_t$ , while the measurement equation links the unobservable state variables to the observed derivative prices. The transition equation is given by

$$X_{t+1} = \hat{A}\Delta + (I + \hat{B}\Delta) X_t + \omega_{t+1}, \quad \omega_{t+1} \sim \text{i.i.d.}, \quad (2.31)$$

where  $I$  refers to the identity matrix,  $\Delta$  denotes the size of the discrete time steps, and the mean and variance of the disturbance term are:  $\mathbb{E}[\omega_{t+1}] = 0$  and  $\text{Var}[\omega_{t+1}] = \Sigma_t \Sigma_t^\top$ . As the size of the time steps is relatively small given the use of daily data, we follow Schwartz (1997) and use an Euler discretization for the transition equation.<sup>32</sup> The measurement equation can be expressed as

$$Y_t = h(X_t) + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d. } N(0, \xi), \quad (2.32)$$

where  $h$  denotes the pricing function and  $\varepsilon_t$  is a vector of measurement errors. The pricing function is constructed from Equation 2.10, which relates the state vector to the log-futures prices and Equation 2.19, which relates the volatility state variable to the observed option prices. Note that in the option pricing formula, it is assumed that the prevailing price of the underlying futures is known. Alternatively, one might use the Kalman filter prediction for the futures price, however, in this way approximation errors in the futures price can translate into larger option pricing errors, which would, in turn, lead to a distorted picture of the true option pricing performance of the model.

In the setup of the calibration procedure, we closely follow Trolle and Schwartz (2009b). As pointed out in their article, it would be ideal to fit the model directly to implied Black (1976) volatilities instead of option prices, since these are more stable across time, moneyness, and maturity. However, as this requires the numerical inversion of the option pricing formula for each option price, time step, and optimization iteration, it is computationally not desirable. Instead, we scale option prices by their implied Black (1976) vegas, which approximately converts pricing errors in terms of prices to pricing errors in terms of implied volatilities.

Therefore, the vector of  $m$  futures prices,  $F_{t,1}, \dots, F_{t,m}$  and  $n$  call option prices,  $C_{t,1}, \dots, C_{t,n}$ , with corresponding Black (1976) vegas  $V_{t,1}, \dots, V_{t,n}$ , can be written analogously to Trolle and Schwartz

<sup>31</sup> Note that the EKF procedure is theoretically not appropriate in this setting, as the conditional covariance matrix  $\Sigma_t \Sigma_t^\top$  in Equation 2.31 depends on the estimate of the volatility state variable in the Kalman filter recursions (Duan & Simonato, 1999). However, Trolle and Schwartz (2009a) investigate the small-sample properties of this approach and find virtually no biases. Moreover, the EKF has become standard practice in the estimation of stochastic volatility models of the kind studied in this chapter.

<sup>32</sup> Schwartz (1997) finds that using an Euler discretization leads to accurate parameter estimates up to the fourth significant figure, even in the presence of weekly data.

(2009b) as

$$Y_t = (\ln F_{t,1}, \dots, \ln F_{t,m}, C_{t,1}/V_{t,1}, \dots, C_{t,n}/V_{t,n}) . \quad (2.33)$$

The implementation of the measurement equation in the Kalman filter requires the computation of its derivatives with respect to the state vector.<sup>33</sup> In particular, the update of the state estimate relies on the quantity

$$H' = \left. \frac{\partial h(X_t)}{\partial X_t} \right|_{X_t = \hat{X}_{t|t-1}} \quad (2.34)$$

where  $\hat{X}_{t|t-1} = \mathbb{E}_{t-1}[X_t]$ . For the futures pricing part of  $h(X_t)$ ,  $H'$  can be directly computed from Equation 2.10. For the option pricing formula given by Equation 2.19,  $H'$  is equivalent to the option vega, which can be calculated in semi-analytical form using

$$\frac{\partial C}{\partial v} = P(t, \tau_0) \left[ F \frac{\partial \Pi_1}{\partial v} - K \frac{\partial \Pi_2}{\partial v} \right] \quad (2.35)$$

and

$$\frac{\partial \Pi_j}{\partial v} = \frac{1}{\pi} \int_0^\infty \Re \left[ b_j(\tau_0, \tau_1) \frac{e^{-i\phi \ln[K]} f_j(x, v, \tau_0; \phi)}{i\phi} \right] d\phi, \quad j = 1, 2, \quad (2.36)$$

where the notation and computation procedure are adopted from Section 2.2.3. To reduce the number of parameters in the estimation problem, we assume that the measurement errors are cross-sectionally uncorrelated, i.e., that  $\xi$  is diagonal and that one measurement error applies to the entire futures curve and another to all option contracts. As pointed out by Schwartz (1997), the measurement errors may capture, among others, bid-ask spreads, price limits, errors in the data, and non-simultaneity of observations.

For the numerical maximization of the likelihood function, we initially use the direct search Nelder-Mead algorithm as it proves to cope well with discontinuities in the objective function, which may arise from implausible parameter guesses. The result of this optimization run is subsequently passed to the gradient-based sequential quadratic programming (SQP) algorithm with BFGS calculation of the Hessian matrix. The SQP algorithm resembles an efficient optimization method to refine the solution of the direct search method. To minimize the risk of not reaching the global maximum, the procedure is initialized with several plausible initial parameter sets. To speed up calibration, it is useful to precalculate the affine coefficients in Equation 2.12-2.16 and Equation 2.22-2.23 for all relevant maturities in the dataset at the beginning of each evaluation of the likelihood function, as opposed to repeatedly solving both systems of ODEs for each time step in the filter.<sup>34</sup>

<sup>33</sup> Refer to Harvey (1989) for a classical reference for the Kalman filter and Trolle and Schwartz (2009b) for implementation details in the context of commodity term structure models.

<sup>34</sup> Note that in the numerical implementation of the model, we assume that  $s_{12} = s_{13} = s_{23} = 0$ , since experiments reveal that this significantly speeds up calibration as it simplifies meeting the nonlinear constraints on positive definiteness of  $\Sigma_t \Sigma_t^\top$ .

### 2.3.3 Estimation results

The estimated parameters are shown in Table 2.1 for two settings. First, the model is calibrated to futures data only (Setting A) and, second, the model is jointly estimated with futures and options data (Setting B). In both settings, parameter estimates are in line with the expectations discussed in Section 2.2. For instance, since  $\kappa_{11} > 0$ , the spot price reverts to a stochastic long-term level and the drift captures a positive relationship between the spot price level and the convenience yield. The conjectured negative relationship between the spot price risk premium ( $\lambda_x$ ) and  $x_t$  is confirmed, given that  $\hat{\kappa}_{11} > \kappa_{11}$ . The model shows a positive unconditional correlation between the spot price and the variance state variable ( $\kappa_{14} > 0$ ), which may indicate the presence of an inverse leverage effect. At the same time,  $\sigma_{24} < 0$  reflects a negative instantaneous correlation between volatility and the long-term price level, which can also be interpreted as a positive correlation between volatility and the degree of backwardation in the futures curve. With respect to the risk premia, several parameters are not statistically significant, which is typical in the estimation of term structure models (e.g., Geman & Nguyen, 2005; Trolle & Schwartz, 2009b).

While the fundamental interpretations in the SMV model described above are valid in both calibration settings, there are several differences in the parameter estimates depending on the type of contract used in model calibration. To learn more about the relative information content of futures and option prices and to address the question of whether volatility is likely to be unspanned by futures prices in the copper market, the parameter differences and model dynamics in both settings are discussed in more detail hereafter.

Table 2.1 reports the absolute differences between the parameter estimates in both settings ( $|A - B|$ ), where two parameters  $p_A$  and  $p_B$  are considered different from each other at the 95% confidence level if

$$|p_A - p_B| > z\sqrt{(se_A)^2 + (se_B)^2}. \quad (2.37)$$

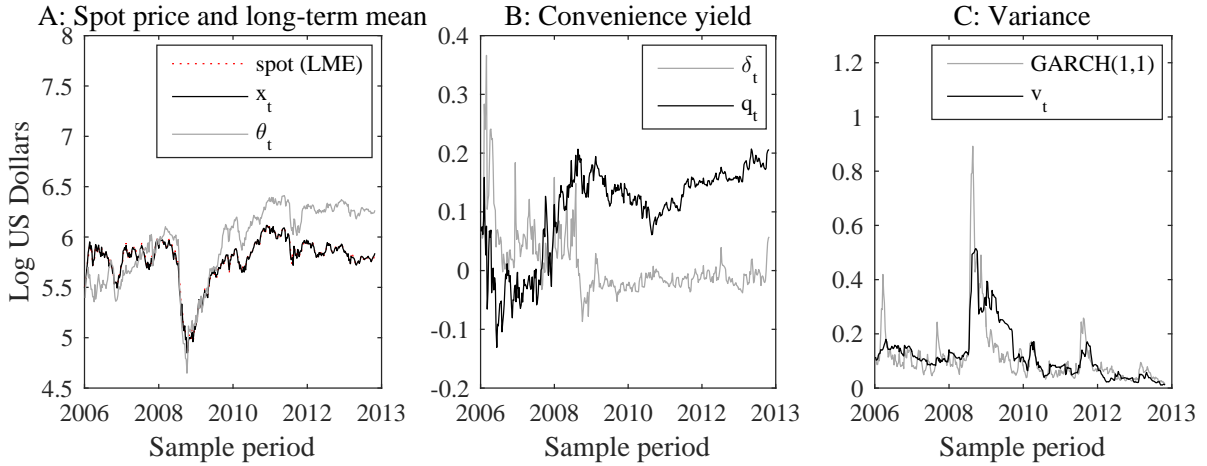
Here,  $z$  denotes the relevant percentile of the standard normal distribution and  $se_A$  and  $se_B$  are standard errors corresponding to the parameter estimates in each setting.<sup>35</sup> This calculation illustrates that there are sizable and statistically significant differences in the parameters, depending on the calibration setting. Hence, the parameter set that yields the best model fit to historical futures prices is significantly different from the parameter set that is optimal to capture the joint dynamics of futures prices and log-normal implied option volatilities. This suggests that option prices contain additional information not fully captured by the futures curve, which may result from the sensitivity of option prices to higher moments of the underlying return distribution, while futures prices simply reflect the risk-neutral expectation of the future spot price. In order to better understand how the dynamics of the SMV model are affected by the presence of option prices in the estimation setting, we compute correlations between the state variables using the parameters from Panel A and B. If the presence of option prices has little effect on a state variable, we would expect a high correlation between the state variable estimates implied by both

<sup>35</sup> The asymptotic covariance matrix and the standard errors are computed from the Hessian matrix of the likelihood function at the optimal parameter set.

**Table 2.1:** Maximum-likelihood parameter estimates of the SMV model for the full-sample period and two calibration settings

Variable	Parameter estimates (2006 - 2013)				A - B
	A: Futures		B: Futures and options		
$\mu_2$	-0.0535	(0.005)*	-0.0581	(0.001)*	0.00
$\mu_3$	0.2940	(0.023)*	0.1503	(0.010)*	0.14*
$\mu_4$	0.1024	(0.007)*	0.0615	(0.000)*	0.04*
$\hat{\mu}_1$	1.0828	(0.599)	5.4826	(1.657)*	4.40*
$\hat{\mu}_2$	0.0600	(0.148)	-0.0557	(0.111)	0.12
$\hat{\mu}_3$	0.3102	(0.042)*	4.9278	(21.240)	4.62
$\hat{\mu}_4$	0.2227	(0.219)	1.6949	(6.308)	1.47
$\kappa_{11}$	0.9766	(0.099)*	0.3630	(0.000)*	0.61*
$\kappa_{14}$	2.1302	(0.259)*	0.0712	(0.000)*	2.06*
$\kappa_{33}$	2.9907	(0.141)*	1.4049	(0.003)*	1.59*
$\kappa_{44}$	0.5995	(0.056)*	0.3080	(0.000)*	0.29*
$\hat{\kappa}_{11}$	1.1374	(0.105)*	1.2970	(0.283)*	0.16
$\hat{\kappa}_{33}$	2.7471	(1.030)*	0.0085	(0.044)	2.74*
$\hat{\kappa}_{44}$	0.2829	(0.201)	0.0343	(0.137)	0.25
$s_{22}$	0.0000	(0.000)	0.0000	(0.000)*	0.00*
$s_{33}$	0.0000	(0.000)	0.0003	(0.000)*	0.00*
$\sigma_{12}$	0.8400	(0.037)*	0.8728	(0.000)*	0.03
$\sigma_{13}$	-0.0405	(0.035)	0.0175	(0.000)*	0.06
$\sigma_{14}$	0.0230	(0.022)	-0.0040	(0.000)*	0.03
$\sigma_{22}$	1.1865	(0.086)*	1.2888	(0.001)*	0.10
$\sigma_{23}$	-0.3350	(0.047)*	0.1901	(0.000)*	0.53*
$\sigma_{24}$	-0.2148	(0.025)*	-0.0842	(0.000)*	0.13*
$\sigma_{33}$	0.4411	(0.037)*	0.1392	(0.000)*	0.30*
$\sigma_{34}$	0.1733	(0.013)*	-0.0018	(0.000)*	0.18*
$\sigma_{44}$	0.1227	(0.009)*	0.0379	(0.000)*	0.08*
$\vartheta$	0.0000	(0.000)	0.0000	(0.000)*	0.00*
$\xi_{\text{futures}}$	0.0000	(0.000)*	0.0000	(0.000)*	0.00*
$\xi_{\text{options}}$	-	-	0.0007	(0.000)*	-
Log-likelihood	117,653		130,898		

Setting A refers to parameters estimated from futures data only. Setting B shows parameters calibrated with both futures and options data. |A - B| indicates the absolute difference between a parameter estimate from A and B. Standard errors are in parenthesis. Significance at the 95% confidence level is indicated by \*.



**Figure 2.2: Estimated state variables.** Panel A shows the estimated spot price  $x_t$  and the long-term mean  $\theta_t$  along with the realized spot price at the London Metal Exchange (LME). Panel B depicts the estimate of  $q_t$  and the historical net convenience yield  $\delta_t$ , implied by the adjacent futures  $F_1$  and  $F_2$ . Panel C illustrates the variance process  $v_t$  and the annualized GARCH(1,1) variance, inferred from daily returns of the short-term futures  $F_1$ . In all panels, every 5th observation is displayed for better readability.

parameter settings, i.e., including and excluding option prices. The results are as follows:  $\rho_x = 0.99$ ,  $\rho_\theta = 0.94$ ,  $\rho_q = 0.08$ ,  $\rho_v = -0.03$ . Moreover, we calculate the correlation between the annualized GARCH(1,1) volatility of daily returns of the short-term futures ( $F_1$ ) and the volatility estimate of the SMV model for both parameter sets. The results are as follows:  $\rho_v^A = 0.09$ ,  $\rho_v^B = 0.81$ . We also impose the parameter conditions for unspanned stochastic volatility derived in Section 2.2.4 and re-estimate the model with futures data. The correlation between the newly estimated volatility state variable and the GARCH(1,1) estimated of realized volatility is:  $\rho_{usv}^A = 0.68$ .

From these findings, it follows that: first, the presence of option prices mainly affects the dynamics of the variance state variable, while the estimates of the spot price and long-term mean are largely unaffected;<sup>36</sup> second, when volatility is spanned and the model is estimated with futures data only, the volatility estimate is unrealistic, suggesting that it captures information that is relevant for pricing the futures curve but does not coincide with volatility. However, when volatility is constrained to be unspanned, it can be estimated with reasonable accuracy from a time series of past futures prices. When the model is jointly estimated with futures and options data, the estimate of volatility most accurately resembles realized GARCH(1,1) and log-normal implied option volatility. However, in this setting  $v_t$  does not need to be completely unspanned. In fact, futures prices depend to some extent on  $v_t$  when the model is jointly estimated in Setting B.<sup>37</sup>

In summary, while volatility can be estimated from the time series information contained in futures

<sup>36</sup> The fact that the dynamics of  $q_t$  change considerably from Setting A to B can be explained by the inter-relation of the volatility process with the drift of the model. For instance, if the dynamics of  $v_t$  change in the presence of options data, this will affect the drift of  $x_t$  via the parameter  $\kappa_{14}$ , which in turn will impact the dynamics of all other state variables and parameters.

<sup>37</sup> Note that futures prices are less dependent on  $v_t$  in Setting B than in Setting A. To see this, we compare the relative magnitude of the term  $\beta_4(\tau)v_t$  in the futures pricing formula (Equation 2.10) for each parameterization.

prices, it cannot be accurately inferred from the futures curve on a given trading day. This is consistent with Trolle and Schwartz (2009b), who find that volatility is largely unspanned by futures in the crude oil market. The results also suggest that the SMV model is best estimated using both futures and options data, as this leads to the most realistic estimate of historical volatility. Under this optimal parameterization, volatility is partly spanned by futures prices in the SMV model.<sup>38</sup> A graphical representation of the estimated state variables using futures and option prices is given in Figure 2.2.

### 2.3.4 Pricing performance

To assess the pricing performance of the SMV model, we consider Akaike's information criterion (AIC), the Bayesian information criterion (BIC), and mean absolute pricing errors (MAE).<sup>39</sup> The SMV model is compared to the three-factor maximal stochastic volatility model of Hughen (2010) (SV) and the three-factor stochastic long-term mean model of Tang (2012) (SM). The choice of these benchmarks is motivated by two aspects. First, the above authors show in their respective articles that both processes deliver superior pricing performance for commodity futures compared to the vast majority of popular models present in the literature. For instance, based on crude oil futures data, Hughen (2010) shows that the SV model improves upon all nested specifications, which include, among others, the models in Gibson and Schwartz (1990), Schwartz and Smith (2000), Richter and Sørensen (2003), Casassus and Collin-Dufresne (2005) (with constant interest rates), Korn (2005), and Cortazar and Naranjo (2006). Also the SM model is shown to yield a superior pricing performance compared to the models in Gibson and Schwartz (1990) and Casassus and Collin-Dufresne (2005) (with constant interest rates). Second, the SMV model principally integrates the instantaneous variance and covariance specification of the SV model and the mean reversion dynamics of the SM model.

The goodness of fit statistics are shown for each of the parameter settings (A and B) discussed in Section 2.3.3. Note that option pricing results are only shown for the SMV and SV model as the constant volatility SM model resembles no sensible benchmark here. The results are reported in Table 2.2, suggesting that the SMV model exhibits a better fit to the data in both settings, justifying the higher number of parameters.<sup>40</sup>

Next, we compute MAEs of fitted and actual futures prices as well as fitted and actual log-normal implied option volatilities. The futures and option pricing results are reported in Table 2.3 and Table 2.4, respectively. Results are shown for an in-sample and out-of-sample setting. In-sample pricing errors are calculated based on the same contracts used for model calibration, while the out-of-sample test uses a set of different contracts (see Section 2.3.1). In each of the settings, we use the Kalman filter to forecast the state vector using information up to the previous day and then price the futures and options term structure based on the predicted state variables. Thus, strictly speaking, neither of the tests is truly in-sample,

<sup>38</sup> Note that the parameter conditions for unspanned stochastic volatility significantly constrain the dynamics of the SMV model so that even if volatility was completely unspanned in reality, it would be unlikely that the conditions emerged as satisfied in a parameter set that is optimal with respect to pricing performance.

<sup>39</sup> The AIC and BIC have the advantage that models under comparison need not be nested, which is not true for, e.g., likelihood ratio tests.

<sup>40</sup> Including the measurement errors from the Kalman filter, the SMV model has 27 parameters in Setting A, compared to 25 and 15 in the SV and SM models, respectively.

**Table 2.2:** Goodness of fit statistics

Model	A: Futures			B: Futures and options		
	LL	AIC	BIC	LL	AIC	BIC
SMV	117,653	-235,252	-235,022	130,898	-261,740	-261,494
SV	111,582	-223,114	-222,901	123,998	-247,944	-247,715
SM	112,219	-224,408	-224,280	-	-	-

Reported are the maximum log-likelihood (LL), Akaike information criterion (AIC), and Bayesian information criterion (BIC) for the two calibration settings and three different models. In Setting A models are estimated with futures data only. In Setting B, models are calibrated with futures and options data. SV refers to the stochastic volatility model of Hughen (2010) and SM denotes the stochastic long-term mean model of Tang (2012). All models are estimated over the full sample period (31 March 2006 to 30 December 2013).

as we only use the predicted and not updated state vector from the Kalman filter recursions to compute pricing errors.<sup>41</sup>

The SMV model exhibits lower futures pricing errors than the SV and SM model for all contract maturities in the in-sample and out-of-sample test. However, the improvement in pricing performance is marginal. On average, the in-sample MAE of the SMV model is 0.17% and 0.27% lower than under the SV and SM model, respectively. The out-of-sample results are similar. This is in line with the introductory discussion, stating that three factors are sufficient to explain nearly all variation in the term structure of futures prices accurately. Hence, the fourth factor in the SMV process adds little accuracy here.

Turning to the errors in actual and fitted log-normal implied volatilities (Table 2.4), the relative advantage of the SMV model over the SV model becomes more visible. The average MAE across moneyness and maturity levels is 72% lower under the SMV model using in-sample data and 61% lower in the out-of-sample setting. A possible explanation for the significantly lower pricing errors in the SMV model is the potential inability of three-factor models to price the term structure of futures and option prices jointly. As discussed in the introduction, this idea is expressed by Hughen (2010), who finds that the SV model is able to either price the futures curve accurately or to provide a good fit to volatility and option prices but that it cannot achieve both objectives at the same time. In his analysis, the result depends on the restrictions imposed on the dynamics of volatility. In particular, if futures prices are allowed to be a function of all three state variables, including the volatility process, the fit to futures data is accurate, however, the model fails to explain volatility dynamics and option prices well. If, instead, volatility is restricted to be unspanned by futures prices, the model fit to volatility and options data improves but this occurs at the expense of the previously low futures pricing errors. In the restricted case, the model allows for only two independent sources of uncertainty in the futures curve, which explains the deterioration in

<sup>41</sup> In both tests it is assumed that parameters do not change over time. To investigate the validity of this assumption, one might calibrate the model to different subsamples and use the obtained parameters to compute pricing errors in different periods (see, e.g., Schwartz, 1997). This analysis is omitted here, since Trolle and Schwartz (2009b) find in a similar context that considering pricing results in different subsamples does not change the performance rank order of compared models.



**Table 2.3:** Mean absolute futures pricing errors

In-sample	SMV		SV		SM	
	MAE	MAE (%)	MAE	MAE (%)	MAE	MAE (%)
F1	4,796	1,545	4,805	1,547	4,810	1,549
F3	4,771	1,536	4,773	1,538	4,776	1,539
F5	4,715	1,518	4,716	1,520	4,724	1,522
F7	4,641	1,498	4,645	1,500	4,652	1,502
F9	4,575	1,481	4,583	1,484	4,588	1,486
F11	4,522	1,470	4,536	1,474	4,539	1,476
F13	4,476	1,462	4,489	1,466	4,494	1,468
F15	4,447	1,460	4,455	1,463	4,461	1,465
F17	4,412	1,456	4,418	1,459	4,425	1,461
F19	4,381	1,453	4,392	1,457	4,395	1,459
Out-of-sample						
F2	4,792	1,544	4,799	1,546	4,800	1,546
F4	4,744	1,527	4,749	1,529	4,755	1,531
F6	4,681	1,509	4,687	1,511	4,693	1,513
F8	4,609	1,489	4,616	1,492	4,622	1,494
F10	4,551	1,476	4,561	1,480	4,563	1,481
F12	4,503	1,467	4,516	1,471	4,517	1,472
F14	4,462	1,461	4,472	1,464	4,476	1,466
F16	4,424	1,456	4,433	1,459	4,437	1,461
F18	4,397	1,455	4,408	1,458	4,412	1,461
In-sample mean	4,574	1,488	4,581	1,491	4,586	1,493
Out-of-sample mean	4,574	1,487	4,582	1,490	4,586	1,492

SV refers to the stochastic volatility model of Hughen (2010). SM denotes the stochastic long-term mean model in Tang (2012). Mean absolute pricing errors (MAE) are calculated as the average across all trading days of the absolute difference in the one day ahead predicted and realized futures price. MAE (%) are computed as the average of daily absolute pricing errors divided by the observed futures prices. Model parameters used to calculate pricing errors are calibrated with futures data only. For the SMV model, these are shown in Part A of Table 2.1. All calculations are based on the full sample period.

model fit.

To see whether this conclusion is equally valid in the context of this dataset and whether the fourth factor in the SMV model can help to overcome the inability of the SV model to capture the joint dynamics of futures and option prices, we report average MAEs of a newly estimated, restricted SV model ( $SV_{USV}$ ), along with the average pricing errors for the SV and SMV model in Table 2.5.<sup>42</sup> Focusing first on the joint calibration results in Part B of the table, three aspects are worth highlighting. First, the SMV model

<sup>42</sup> Please refer to Hughen (2010) for the necessary conditions that ensure unspanned stochastic volatility in the SV model.

**Table 2.4:** Mean absolute errors in implied option volatilities

Moneyness	Model	In-sample				Out-of-sample			
		C1	C5	C9	C13	C2	C6	C10	C14
0.80-0.85	SMV	4,063	1,529	1,802	1,358	8,341	3,458	2,906	3,169
	SV	9,076	10,336	9,676	8,037	20,259	9,519	8,017	7,472
0.85-0.90	SMV	3,429	1,900	1,478	1,699	6,824	2,734	2,377	2,681
	SV	6,915	7,483	6,953	6,934	17,078	6,537	5,717	6,156
0.90-0.95	SMV	3,342	1,457	1,455	1,965	5,601	2,482	2,553	2,892
	SV	7,270	5,884	5,836	6,121	15,953	5,223	5,197	5,635
0.95-1.00	SMV	2,866	1,420	1,305	1,545	3,548	2,278	2,206	2,479
	SV	7,730	6,369	5,927	5,779	11,284	5,665	5,157	5,125
1.00-1.05	SMV	3,203	1,617	1,365	1,479	3,814	2,162	1,942	1,987
	SV	8,571	6,515	6,123	5,916	12,124	6,094	5,634	5,617

Reported are the mean absolute differences between fitted and actual log-normal implied volatilities in percentages. Results are shown for different maturities ( $C1, \dots, C14$ ) and moneyness levels. Moneyness is defined as the futures price divided by the strike. Results are computed with parameters estimated with futures and options data. For the SMV model, these are shown in Part B of Table 2.1. All calculations are based on the full sample period.

achieves the best fit to both futures prices and option volatilities throughout.<sup>43</sup> Second, the unrestricted SV model shows lower futures pricing errors than the restricted version. Third, the restricted SV model yields a better fit to options data than the unrestricted version.<sup>44</sup>

These results are perfectly in line with the findings of Huguen (2010) and, furthermore, indicate that the four-factor SMV model is general enough to capture the joint dynamics of both instruments without compromise. Comparing the results from Setting A and B, it is evident that the close fit of the SMV model to futures prices does almost not deteriorate once options are added to the estimation. In particular, in the in-sample test, the average futures pricing error increases just marginally by  $4.577/4.574 - 1 = 0.1\%$ .

In summary, the results support the introductory conjecture that four independent sources of uncertainty are necessary to capture the joint dynamics of futures prices and implied option volatilities. Since a good model fit to the futures curve fully utilizes three independent sources of uncertainty, a three-factor model does not permit any additional capacity to capture the volatility dynamics in option prices, which do not seem to fully coincide with the dynamics relevant for futures pricing. The SMV model also improves over the SV and SM model when futures prices are considered in isolation and yields a better fit to the implied volatility dynamics of option prices than the SV and  $SV_{USV}$  model. These results are

<sup>43</sup> For the crude oil market, Trolle and Schwartz (2009b) report a root-mean-square error (RMSE) of actual and predicted log-normal implied volatilities of 1.31% for their five-factor HJM model ( $SV2gen$ ). For a simplified alternative model ( $SV1$ ) with only one stochastic volatility factor that may be partially spanned by futures contracts, they report a RMSE of 2.1%. While these results are not directly comparable to this study, due to the heterogeneity of datasets, they indicate that the SMV model delivers roughly similar pricing errors compared the (only) other four and five-factor models present in commodity derivatives literature.

<sup>44</sup> The volatility estimates in the SV and  $SV_{USV}$  model have correlations with the realized GARCH(1,1) volatility of daily returns of the short-term futures ( $F_1$ ) of 0.18 and 0.81, respectively.

**Table 2.5:** Summary of mean absolute errors

Calibration data:	A: Futures		B: Futures and options			
	Futures		Futures		Options	
	IS	OOS	IS	OOS	IS	OOS
Models						
SMV	4.574	4.574	4.577	4.577	2.014	3.322
SV	4.581	4.582	4.635	4.642	7.173	8.473
SV <sub>USV</sub>	4.640	4.618	4.654	4.635	2.376	3.674

Reported are the average mean absolute pricing errors for futures and options in two settings. Setting A uses model parameters inferred from futures data only. Setting B is based on parameters calibrated from futures and option prices. Pricing errors for futures are defined as the average MAE across maturities as reported at the bottom of Table 2.3. For options, pricing errors are defined as the average MAE computed across all moneyness levels and maturities reported in Table 2.4. IS and OOS denote in-sample and out-of-sample results, respectively. SV<sub>USV</sub> denotes the SV model, restricted to exhibit unspanned stochastic volatility. All calculations are based on the full sample period.

similar in the in-sample and out-of-sample test.

### 2.3.5 Hedging performance

This section provides a comparison of the relative hedging performance of the SMV and SV model.<sup>45</sup> At first, we construct an equally weighted portfolio of call options satisfying the selection criteria discussed in Section 2.3.1. On each trading day, this portfolio comprises between four and eight call options with maturities ranging from 1-13 months in the in-sample case and 2-14 months in the out-of-sample case. In addition, three hedging portfolios are formed. First, H1 denotes a portfolio consisting of two equally weighted futures with maturities corresponding to the shortest and longest maturity in the option portfolio. H1 is rebalanced daily, to render the option portfolio delta neutral. Call option deltas in the SMV model are calculated as follows:  $\Delta^C = P(t, \tau_0) \Pi_1$ . Second, H2 comprises the instruments of H1 and an additional call option with a maturity corresponding to the shortest maturity in the option portfolio. H2 is rebalanced daily to eliminate both delta and gamma risk in the option portfolio. Option gammas in the SMV model are calculated according to

$$\Gamma = P(t, \tau_0) \frac{\partial \Pi_1}{\partial F} = \frac{P(t, \tau_0)}{\pi F} \int_0^\infty \Re \left[ e^{-i\phi \ln[K]} f_1(x, v, \tau_0; \phi) \right] d\phi. \quad (2.38)$$

Third, H3 comprises H2 and a second call option with a maturity corresponding to the longest maturity in the option portfolio. H3 is rebalanced to neutralize delta, gamma, and vega risk in the option portfolio. Option vegas in the SMV model are computed as discussed in Section 2.3.2.

The mean absolute daily changes in the values of the unhedged and hedged portfolios, are reported in Table 2.6. Both in-sample and out-of-sample results indicate that the SMV model leads to lower hedging errors, which can be explained by its superior fit to implied log-normal option volatilities and

<sup>45</sup> Trolle and Schwartz (2009b) use a similar hedging case study to assess the presence of unspanned stochastic volatility in the crude oil market.

**Table 2.6:** Hedging performance of the SMV and SV model

Portfolio	In-sample			Out-of-sample		
	SMV	SV	SMV-SV (%)	SMV	SV	SMV-SV (%)
U	8,44	8,44	0,00	8,80	8,80	0,00
U+H1	2,29	2,44	-6,41	2,43	2,51	-3,44
U+H2	1,23	1,29	-4,97	1,06	1,09	-2,63
U+H3	0,53	0,58	-9,24	0,61	0,62	-2,59
Mean (ex. U)	1,35	1,44	-6,34	1,37	1,41	-3,11

Reported are the mean absolute daily changes in the value of 4 different portfolios. U denotes an unhedged portfolio of call options meeting the selection criteria from Section 2.3.1. H1 includes a delta-hedge using 2 different, equally weighted, futures with maturities corresponding to the shortest and longest option maturity. H2 is rebalanced to hedge delta and gamma risk, using H1 and a short-term call option. H3 is rebalanced to eliminate delta, gamma, and vega risk, using H2 and a long-term call option. All calculations are based on the full sample period.

the resulting accuracy in calculating hedge ratios. If the SMV model were the true generating process of option and futures prices, daily changes in hedged portfolio values would be closer to zero, however, as we do not hedge continuously and due to the presence of idiosyncratic noise in option prices, a considerable hedging error remains (Trolle & Schwartz, 2009b).

## 2.4 Conclusion

In this chapter, we propose a four-factor maximal affine stochastic volatility model, which allows for three independent sources of uncertainty associated with the futures term structure and an additional, potentially unspanned stochastic volatility factor. Moreover, the model features mean reversion to a stochastic long-term level, a time-varying instantaneous correlation structure between all state variables as well as unconditional correlations between the spot price, convenience yield, and volatility state variable. The model is empirically implemented using time series of 37,126 futures and 12,166 corresponding call option prices, observed between March 2006 and December 2013.

The empirical results of this study lead to three conclusions. First, the SMV model exhibits a superior pricing performance compared to the three-factor models of Hughen (2010) and Tang (2012). Since these benchmark models are known to yield a superior fit to futures data in various commodity markets, compared to many commodity price processes, the pricing results of the SMV model are considerable. Second, the findings support the introductory hypothesis that a universal model for commodity futures and options will require more than three independent sources of uncertainty to accurately capture the joint dynamics of both contract types. Third, in line with the findings of Trolle and Schwartz (2009b) in the context of crude oil derivatives, we find that copper options contain important information that is not fully spanned by futures prices. This implies that volatility cannot be accurately inferred from the futures curve alone and option prices cannot be fully hedged by trading in the underlying instruments only. This conclusion is also supported by the results of a simple hedging study.

The SMV model can be extended in a number of ways. For instance, stochastic interest rates may be included for the valuation of long-term futures and real options (see, e.g., Schwartz, 1997; Casassus & Collin-Dufresne, 2005), the spot price process can be augmented with jumps to better fit the short-term implied volatility smile (see, e.g., Deng, 2000; Villaplana, 2003; Casassus & Collin-Dufresne, 2005; and Realdon, 2013), and deterministic seasonality functions can be incorporated in the spot price and variance process for the application to seasonal commodity markets such as soybeans (see, e.g., Fackler & Roberts, 1999; Lucia & Schwartz, 2002; Richter & Sørensen, 2003; and Geman & Nguyen, 2005).

With respect to the empirical part of this chapter, it would be interesting to consider different contracts, markets, and benchmarks. For instance, an assessment of model fit to long-term futures may yield new insights with respect to model applicability in the context of long-term capital investment appraisal and real options valuation. It also needs to be clarified whether the SMV model is similarly powerful in other major commodity markets such as crude oil or gold. Finally, it would be useful to directly compare the performance of the SMV model to different benchmarks such as the HJM-type models in Trolle and Schwartz (2009b). These issues are left to future research.



# 3

## DYNAMIC PROCUREMENT, PRICING, AND HEDGING UNDER COST AND DEMAND RISK

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*The content presented in this chapter is based on Schöne, Spinler, and Birge (2016b). Partial results from this chapter contributed to the conference presentation of Schöne (2015) and the presentation of Spinler (2016).*

In this chapter, we consider a periodic review, single-item, multi-period inventory model in the presence of demand risk and multi-factor commodity price dynamics. Within this framework, we characterize the jointly optimal strategy for the firm's purchasing, pricing, and financial hedging decision under risk aversion. The model is solved using a flexible, Monte Carlo-based optimization algorithm that is novel to the literature on inventory management. Our analysis leads to the following main conclusions: (a) adopting an integrated pricing, procurement, and financial hedging policy can increase expected utility and profits, while reducing downside risk; (b) even a risk-averse firm may benefit from an increase in commodity price volatility in terms of both utility and profits under operational flexibility. The opposite is true for demand risk in the presence of production lead time. However, the negative impact of demand risk on utility and profits is mitigated by operational flexibility. (c) different assumptions about the stochastic cost process can have significant implications for valuation results, optimal policies, and risk management.

### 3.1 Introduction

In today's fast changing market environment, firms are often exposed to multiple sources of uncertainty. On the one hand, commodity price volatility may constitute a major source of risk related to production costs, while revenues are, on the other hand, frequently subject to volatile demand. As the combined risk resulting from both cost and demand risk can pose a significant threat to firms' profitability, effective, cross-functional hedging strategies have become an important topic on the agenda of top management (Fisher & Kumar, 2010). Traditionally, cost and demand risks are hedged separately; risks related to

the evolution of commodity prices are hedged by derivatives and demand risk by physical inventory (Kouvelis et al., 2013). However, as pointed out by the authors, inventory and hedging decisions are not necessarily independent but need to be considered and optimized jointly. More recently, Xiao et al. (2015) suggest that a successful PRM strategy may not only span inventory and hedging decisions but can, additionally, benefit from dynamic pricing of the end product depending on the current state of costs and inventories. By adopting an integrated PRM approach of this kind, Hewlett-Packard realized over \$425 million in cost savings over a 6-year period.

However, while the benefits of an integrated pricing, procurement, and hedging strategy are practically evident, it remains a non-trivial question how the different cross-functional decisions should be optimally coordinated under uncertainty. As an illustration of the complex interdependence of decisions within an integrated PRM strategy, consider the following example of an industrial firm, facing an unexpected increase in raw material prices. Suppose the firm responds to the cost increase by raising prices. Under the assumption of price elastic demand, this decision can cause a decline in unit sales, higher than expected inventory levels at the end of this period, and a corresponding increase in associated inventory carrying cost. Since excess inventories may be deployed to satisfy future demand, necessary raw material procurement quantities in the present and future business period will decline. As a result, the exposure to future cost volatility is reduced, requiring an adjustment of the volume of traded hedging instruments. Finally, lower procurement and production levels may be associated with a loss in economies of scale, resulting in higher unit costs. Now, should prices be raised in the first place?

While this example illustrates the complexity of integrated PRM decisions faced by many firms in practice, previous research on jointly optimal inventory management, financial hedging, and pricing policies is sparse. In fact, Xiao et al. (2015) are the first to study an integrated pricing, multi-sourcing (spot and forward), and inventory control model of a risk-neutral firm under stochastic demand and volatile procurement costs.

In the present study, we extend this model in several directions, allowing for risk aversion, financial hedging, and multi-factor cost dynamics. We make the following three main contributions to existing research: (1) the first study to address the joint pricing, inventory management, and hedging problem of a risk-averse firm facing stochastic costs and demand; (2) a flexible, Monte Carlo-based optimization algorithm that allows to solve the firm's decision problem under multi-factor cost dynamics; (3) an extensive numerical study to (a) characterize optimal PRM policies and the value of managerial flexibility, (b) assess the influence of different stochastic commodity price models and input uncertainty on valuation results, policies, and risks, (c) quantify the impact of risk aversion on the distribution of profits, and (d) determine the sensitivity of valuation results to model inputs.

We consider the following fundamental setting: a stylized industrial firm sources a traded commodity, which is processed into a final product under lead time, stored, and sold at the profit maximizing price to meet uncertain, price elastic demand. The firm procures physical material from the spot market and can enter into futures contracts as a financial hedge against future spot price risk. The firm dynamically optimizes inter-period utility over a multi-period time horizon and is averse to downside risk, which refers to the possibility that future profit expectations are not met. We assume here that the firm has an



incentive for risk reduction as in the theoretical models of C. W. Smith and Stulz (1985) and Froot et al. (1993). While we do not explicitly capture such value-enhancing mechanisms in our models, they are consistent with empirical evidence (e.g., Allayannis & Weston, 2001; and MacKay & Moeller, 2007) that supports the use of hedging. The fundamental setting studied in this chapter is also closely related to business practice and was developed in close consultation with risk management executives at the multi-national chemical company BASF.

Our findings show that a dynamic PRM strategy involving pricing, inventory management, and hedging decisions leads to an increase in both expected utility and profits, while reducing downside risk, compared to operating strategies that are characterized by a lower degree of managerial flexibility. In line with Xiao et al. (2015), we find that the present value of expected profits is increasing in cost volatility. However, the risk-averse firm may not always prefer higher cost volatility due to the associated risk of losses. When a futures hedge can be implemented, even a risk-averse firm can benefit from cost volatility. In contrast, utility and profits are decreasing in demand volatility but this effect is mitigated by dynamic pricing. In this context, the strategic value of dynamic pricing can be increasing in demand volatility for a firm that is currently operating under a fixed price strategy. With respect to commodity price dynamics, we find that different models can have significant implications for valuation results and operating policies. Neglecting the stochastic nature of interest rates, the convenience yield, and volatility can increase downside risk and profit variance. Moreover, our findings suggest that model calibration error can affect operating policies significantly. For instance, in the context of a standard Wiener process, managers should be careful not to underestimate the drift and volatility parameters as this can lead to excessive hedging and eventually to higher downside risk.

## 3.2 Literature review

This research is closely related to the literature on inventory management in the presence of stochastic costs as well as operational and financial hedging. It is also linked to research on the joint pricing and inventory management decision in the presence of stochastic demand. A third related stream of literature is concerned with commodity price modeling, derivatives pricing as well as the valuation and optimal operation of commodity-linked real assets. We will briefly review the literature from each of these fields below.

An excellent survey of earlier studies on the integration of operational and financial risk management can be found in Kleindorfer (2009). Berling and Martínez-de-Albéniz (2011) provide an overview of research focusing on the influence of stochastic purchase prices on optimal inventory policies and Haksöz and Seshadri (2007) give a comprehensive literature survey of spot market operations, purchasing, and the valuation of procurement contracts. Studies focusing on the joint optimization of inventory or capacity and hedging decisions in a single-period setting include Gaur and Seshadri (2005), Caldentey and Haugh (2006), Ding, Dong, and Kouvelis (2007), and Chod, Rudi, and Van Mieghem (2010). Multi-period problems of integrated operating and hedging decisions can be found in X. Chen, Sim, Simchi-Levi, and Peng (2007), Zhu and Kapuscinski (2011), and Kouvelis et al. (2013). The latter of these articles is closely related to the present study regarding a number of aspects. The authors consider the joint

optimization of inventory levels and financial hedging of a risk-averse firm with mean-variance, inter-period utility. The firm faces stochastic costs and demand and has access to the spot, futures, and options market. Selling prices are, however, assumed to be fixed and costs follow a standard one-factor model.

In contrast to the literature on operational and financial risk management under stochastic costs, research on joint pricing and inventory management no longer assumes fixed selling prices. Instead, ordering costs are non-stochastic. An extensive review of this literature stream is given by X. Chen and Simchi-Levi (2012) and a more recent overview can be found in Xiao et al. (2015). Among the studies cited therein, X. Chen and Simchi-Levi (2004a) should be mentioned here. The authors analyze a finite horizon joint pricing and inventory control problem with fixed set-up costs and show the optimality of an  $(s, S, p)$  policy. Here, inventory is managed according to an  $(s, S)$  policy and optimal prices depend on the initial inventory position in each period. In Section 3.6, we extend these findings to the case of stochastic costs and show that optimal prices not only depend on the inventory position but also on the state of costs.

The body of literature on commodity price modeling and the valuation of financial and real assets has become extensive. Related articles on commodity price processes and the valuation of commodity derivatives include Gibson and Schwartz (1990), Schwartz (1997), Schwartz and Smith (2000), Richter and Sørensen (2003), Casassus and Collin-Dufresne (2005), Geman and Nguyen (2005), Trolle and Schwartz (2009b), Hughen (2010), and Tang (2012). The detailed consideration of commodity price dynamics in this chapter is motivated by a number of studies, highlighting the critical importance of stochastic process assumptions in valuation models. For instance, J. E. Smith and McCardle (1999) find that mean reversion in oil and gas prices has a substantial influence on the valuation and operation of oil and gas investments. Tsekrekos et al. (2012) compare the commodity price processes outlined in Schwartz (1997) and find that process choice has important implications for theoretical asset values and optimal decision rules. A similar conclusion is reached by Schöne (2014), who provides an empirical analysis of commodity price dynamics in different markets, suitable stochastic processes of spot price dynamics, and their implications for the valuation and operation of a mining project with multiple embedded real options. More recently, Secomandi, Lai, Margot, Scheller-Wolf, and Seppi (2015) study the valuation and hedging of cash flows generated by commodity- and energy-conversion assets and find that even small errors in the stochastic price model can have a disproportionately larger impact on storage valuation and hedging decisions. Goel (2007) and Goel and Gutierrez (2011) consider the Schwartz and Smith (2000) model in the context of optimal procurement and distribution policies, exploiting information from the term structure of commodity prices. Finally, Boogert and de Jong (2008, 2011) study the valuation of gas storage under single- and multi-factor commodity price dynamics. Their work is methodologically related to our research with respect to the valuation of cash flows under operational flexibility and multi-factor commodity price dynamics.

### 3.3 Single-product model

In this section, we model the dynamic pricing, inventory management, and financial hedging problem of a risk-averse industrial firm under uncertain raw material costs and stochastic, price elastic demand.

The sequence of events in our model is as follows. At the beginning of each period, the firm observes the current commodity price  $s_t$  and the inventory level  $i_t$ , carried over from the previous period. Given this information, the firm chooses the quantity of physical material to buy from the spot market  $b_t$ , the unit selling price of the end product  $p_t$ , and the futures hedge  $f_t$ . After these decisions are made, the periodic demand shock occurs and profits are realized.

With respect to the production process, raw material is directly supplied to the manufacturing process without prior storage and final products are available for sale after a lead time of one period. Thus, there is no raw material inventory in our model and speculative raw material storage is ruled out. This is motivated by four main arguments: first, under the assumption that the firm's subjective storage costs are in line with the market implied cost of carry, the expected return on physical inventory is zero; second, in the absence of transaction costs, the payoff from holding physical inventory corresponds to trading a futures contract and can, therefore, be replicated in the financial market; third, in practice, many firms source raw material in the form of preprocessed parts and may not hold raw material inventory at all; fourth, we assume immediate supply of raw material from the spot market so that raw material storage does not reduce lead time. With respect to the futures hedge, it is assumed that the firm does not take physical delivery of the futures contracts, which are only used as financial hedging instruments. Due to the one-period production time lag, period  $t = T - 1$  is the last time when raw material is bought and period  $t = T - 2$  is the last time when financial hedging takes place. Thus, one cycle of the decision problem spans three time periods. In the exposition below, we denote stochastic variables by uppercase letters and their realizations by corresponding lower case letters.

Let  $\Pi_t^O(p_t, b_t | i_t, s_t, \psi_t)$  denote the firm's random economic operating profit at the end of period  $t$ , given an initial inventory level of final products  $i_t$ , the commodity price realization  $s_t$ , and the observed demand shock  $\psi_t$ . The operating profit in period  $t$  is given by

$$\Pi_t^O(p_t, b_t | i_t, s_t, \psi_t) = p_t q_t - b_t (s_t + w_t) - h_t (i_t + b_t), \quad (3.1)$$

where  $q_t = \min\{d_t, i_t\}$  refers to the number of units sold, the demand is given by  $d_t = a - \eta p_t + \psi_t$ , and  $\psi_t$  is a realization of the standard normal demand shock  $\Psi \sim N(0, \sigma_d)$ . Demand is bounded from below by zero and unmet demand is lost and not backlogged. Note that alternative demand specifications include, for instance, the multiplicative model discussed in X. Chen and Simchi-Levi (2004a) or a model of correlated demand and commodity prices in the sense of Gaur and Seshadri (2005). For a detailed comparison of additive and multiplicative demand in the context of dynamic pricing, we refer to Petruzzi and Dada (1999). The initial inventory level in period  $t + 1$  is equivalent to the ending period inventory at time  $t$  so that  $i_{t+1} = i_t - q_t + b_t$ . To account for limited storage capacity, we impose  $b_t \leq i_{max} - i_t$ , where  $i_{max}$  refers to the capacity of the firm's storage facility.  $w_t$  denotes unit labor costs, reflecting the non-stochastic part of production costs, and  $h_t$  is the economic unit holding cost of inventory, which is defined as  $h_t = s_t (r_t + \varphi)$ .  $r_t$  denotes the current risk free interest rate and  $\varphi$  is a percentage reflecting inventory storage and handling costs.<sup>1</sup> At the end of the optimization horizon ( $t = T$ ), excess inventory

<sup>1</sup> Handling costs may not necessarily be proportional to the commodity price in all practical applications. However, it is straightforward to account for inventory holding costs as an absolute cost per item instead.

can be sold for a salvage value  $SV = (i_T - q_T) \Phi s_T$ , where  $\Phi$  denotes the percentage of the market value of excess inventory that is received as a salvage value. The cash flow from financial hedging is given by

$$\Pi_{t+1}^F (S_{t+1}, f_t) = (S_{t+1} - \mathbb{E}[S_{t+1} | s_t]) f_t, \quad (3.2)$$

where  $f_t$  denotes the number of futures contracts entered in the current period. The futures price reflects the risk-neutral expectation of the future spot price so that  $\mathbb{E}[\Pi_{t+1}^F(\cdot) | s_t] = 0$ . Total periodical end-of-period profits are given as the sum of economic operating profits and the proceeds from financial hedging activities

$$\Pi_t^{OF} (p_t, b_t | i_t, s_t, f_{t-1}, \psi_t) = \Pi_t^O (p_t, b_t | i_t, s_t, \psi_t) + \Pi_t^F (s_t, f_{t-1}). \quad (3.3)$$

As discussed in the introduction, firms may have an incentive for risk reduction, even when investors are well-diversified. For example, in the presence of market frictions, such as taxes or costs of financial distress, risk management enhances firm value (C. W. Smith & Stulz, 1985). Moreover, hedging can help to ensure the availability of sufficient internal funds for investment, research, and development. If the cost of external funding exceeds the cost of internal funds, risk management has a positive effect on firm value (Froot et al., 1993). In our model, we capture such incentives for risk reduction implicitly by introducing risk aversion through a corporate utility function. This is in line with, among others, Eppen, Martin, and Schrage (1989), Caldentey and Haugh (2006), and Kouvelis et al. (2013). For instance, in a similar setting Kouvelis et al. (2013) introduce an inter-period mean-variance utility function of the form  $U = \mathbb{E}[\cdot] - (\lambda/2) V[\cdot]$ , where  $\mathbb{E}[\cdot]$  and  $V[\cdot]$  refer to the expectation and variance of profits, respectively. In this study, we propose a slightly different version of the above utility function, where firms derive disutility from downside risk relative to an expectation, rather than the symmetric variance of future profits. As, for instance, Fisher and Kumar (2010) point out that firms may destroy substantial value by hedging away upside potential in the attempt to gain protection against downside risk, it appears desirable to work with a utility function that does not penalize risk symmetrically. This definition of utility is also in line with the above stated theoretical arguments supporting the value enhancing nature of risk management. For instance, the downside risk of missing profit targets is directly related to the risk of facing a lack of internal funds and expected costs of financial distress. Moreover, downside risk is priced in the cross-section of stock returns with evident implications for firms' cost of capital and market value (Ang, Chen, & Xing, 2006). Measures of downside risk have also been utilized in a variety of settings, including, for example, capacity planning (Eppen et al., 1989) as well as dynamic pricing and inventory management (Y. Chen, Xu, & Zhang, 2009; Xu, 2010). Finally, our utility function is closely related to the widely used mean-semivariance and mean-variance criteria.

The firm's utility associated with a random stream of profits  $\Pi$  in a given period  $t$ , is given recursively by  $U_t = \Pi_t[\cdot] - \lambda D_t[\cdot] + \mathbb{E}[U_{t+1}]$ . Similarly to Kouvelis et al. (2013), expectations, denoted by  $\mathbb{E}$ , are defined under the assumption of the "partially complete" market introduced by J. E. Smith and Nau (1995). Accordingly, the probability measure associated with  $\mathbb{E}$  is a combination of the risk-neutral probability measure on commodity spot prices and the firm's subjective probability measure on demands.

$D$  denotes a downside deviation from a dynamically updated expectation, and  $\lambda \geq 0$  refers to the absolute level of downside risk aversion. Formally, we define the firm's recursive utility at the beginning of each period  $t \in [t, T]$  as

$$U_t(p_t, b_t, f_t \mid i_t, s_t, f_{t-1}) = \mathbb{E}[\Pi_t^{OF}] - \lambda D_t + \mathbb{E}[U_{t+1} \mid s_t], \quad (3.4)$$

where

$$D_t = \left\{ \mathbb{E} \left[ \sum_{k=t}^T e^{-r_t(k-t)\Delta} \Pi_k^{O*} \mid s_{t-1} \right] - \mathbb{E} \left[ \sum_{k=t}^T e^{-r_t(k-t)\Delta} \Pi_k^{OF} \mid s_t \right], 0 \right\}^+ \quad (3.5)$$

and  $^+$  denotes the maximum. Before turning to an interpretation of the utility function, we further clarify the availability of information at the time of decision making. While the above expressions for  $\Pi^O$  and  $\Pi^{OF}$  refer to the end of a given period to simplify the exposition,  $U_t$  denotes utility at the beginning of period  $t$ , when decision making takes place. As stated above, the firm chooses  $p_t$ ,  $b_t$ , and  $f_t$  at the beginning of each period, given the level of inventories, the futures position, and the current state of the commodity price process  $s_t$ , but without knowledge of the realized demand shock  $\psi_t$  that occurs after the decisions are made. If we were to assume knowledge of  $\psi_t$  at the time of decision making, the model would exhibit a deterministic relationship between unit selling prices and resulting demand from the perspective of the firm – a situation that defies reality. Thus, at the time of decision making, the firm takes expectations with respect to both the future commodity price  $S_{t+1}$ , conditional on  $s_t$ , and the demand shock  $\Psi$  that occurs in the current period.

In a given period, the firm's utility is determined by three components: first, the expected profits from operations and hedging activities in the current period  $\Pi_t^{OF}$ ; second, the potential disutility from a decline in the present value of operating profits below projections ( $D_t$ ); third, expected future utility  $U_{t+1}$ . Intuitively,  $D_t$  can be interpreted as follows. In each period, the firm formulates an expectation about expected operating profits in the future, given the currently observable commodity price and inventory level. This expectation takes into account all currently available information and the optimality of future decisions. In the next period, a revised expectation is formulated about current and future profits, given the realization of commodity price uncertainty  $s_t$  and the new state of inventories. The apriori expectation based on  $s_{t-1}$  is compared to the updated expectation, resulting in disutility whenever the current outlook of present and future profits is below the expectation that was set in the previous period. The apriori profit expectation against which the updated profit outlook is compared is denoted by  $\Pi^{O*}$ , where  $*$  indicates the optimality of  $\Pi^O$  with respect to current and future operating activities of the firm.  $\Pi^{O*}$  is best obtained in a separate optimization over  $\Pi^O$ . Under the resulting utility function, the role of hedging is to limit the downside risk inherent in future operating profits that is caused by commodity price and demand volatility. As  $D_t$  refers to profit expectations related to the entire remaining planning horizon rather than standalone periodical profits, we ensure that decisions are inter-temporally optimal. To clarify this statement, consider the following example. If commodity prices are low in a given period, it may be value maximizing to make a high invest in inventory, allowing for higher unit sales in the future. However, the inventory investment in the current period corresponds to an unexpectedly high

cash outflow, which decreases  $\Pi_t^O$  below previous expectations  $\mathbb{E}[\Pi_t^O | s_{t-1}]$ . Thus, if downside risk were measured based on periodical profits, the inventory investment would cause disutility despite its positive effect on expected future profits and the value of the firm. By considering not only current but also expected future profits at the time of decision making, this inter-temporal inconsistency is avoided. The effect of future downside risk on utility is recursively accounted for.

### 3.4 Commodity price dynamics

Following the existing literature on commodity price modeling and the valuation of financial and real assets under uncertainty, we acknowledge that different assumptions about the stochastic process used in a valuation model can affect expectations and policies significantly. With respect to the procurement and hedging model studied in this chapter, the commodity price process assumes a critical role in several regards. First, different stochastic processes may imply diverse distributions of future spot prices, which determine raw material procurement costs. Second, theoretical futures prices are affected by different assumptions about spot price and cost of carry dynamics, which may influence the optimal hedging strategy of the firm. Third, the procurement model features option-like characteristics as the firm is able to adjust selling prices, procurement quantities, and hedging activities contingent upon the future realization of commodity prices. As the value of such real options depends on the price distribution, it follows that optimal policies and valuation estimates may differ under diverse stochastic processes.

To account for the complex nature of commodity markets in a PRM context, we propose the following four-factor model of the risk-neutral spot price dynamics:  $dS = (r - q) S dt + \sigma_s S dW_s + \sqrt{v} S dW_{sv}$ ,  $dr = \kappa_r (\theta_r - r) dt + \sigma_r dW_r$ ,  $d\delta = \kappa_\delta (\theta_\delta - \delta) dt + \sigma_q dW_q$ ,  $dv = \kappa_v (\theta_v - v) dt + \sigma_v \sqrt{v} dW_v$ . Given a probability space and filtration generated by a standard Brownian motion  $W_t$  in  $\mathbb{R}^4$ , the spot price is an Itô process with respect to  $W_t$ .  $S_t$  refers to the commodity spot price,  $r_t$  and  $\delta_t$  denote the instantaneous risk-free interest rate and net convenience yield, respectively, and  $v_t$  is a variance process. Note that, for simplicity, we make an exception from our notational convention and denote the stochastic variables  $r$ ,  $\delta$ , and  $v$  by lowercase letters. The instantaneous correlation structure is defined as follows:  $\text{Cov}(dW_s, dW_r) = \rho_{sr} dt$ ,  $\text{Cov}(dW_s, dW_q) = \rho_{sq} dt$ ,  $\text{Cov}(dW_{sv}, dW_v) = \rho_{sv} dt$ ,  $\text{Cov}(dW_r, dW_q) = \rho_{rq} dt$ . We will refer to this process as the SRCV model in analogy to its four factors. The model is closely related to, among others, the well-known two- and three-factor models of Schwartz (1997) as well as the three- and four-factor models in Richter and Sørensen (2003), Hughen (2010) and Yan (2002). The choice of this specification is motivated by two main arguments. First, by allowing for both stochastic interest rates and convenience yields we are able to infer discount rates and inventory carrying costs directly from futures and bond prices. Second, stochastic volatility is important to explain option prices (Bakshi et al., 1997). As it is well-known that volatility is generally stochastic in commodity markets (see, e.g., Hughen, 2010) and due to the option-like flexibility in the procurement and hedging model, we include volatility risk in the model.

Besides the most general four-factor SRCV process, several nested models are discussed. In particular, we define the SRC process as an SRCV model without stochastic volatility and the SC model as an SRC process with constant interest rate. These models are equivalent to the three- and two-factor model of

Schwartz (1997).  $S$  refers to a one-factor model with constant drift and volatility, which is equivalent to a GBM that is commonly used in the operations management literature to model stochastic costs.

We implement the SRCV process and nested models empirically using weekly prices of WTI crude oil futures from NYMEX with maturities of 1, 3, 12, and 24 months over the time period from January 3, 2000 until December 29, 2014. This leaves us with a total of 3,132 futures price observations. The model is also calibrated to the term structure of interest rates over this period, which is represented by the 1-, 3-, and 12-months LIBOR rates and the 2-year swap rate, which are converted to discount bond prices. Details of the empirical implementation are relegated to Appendix A.2.<sup>2</sup>

### 3.5 Numerical evaluation

In this section, we discuss the implementation of the PRM model. We adopt a numerical optimization approach, which is highly flexible in the sense that it allows for multi-factor commodity market dynamics and a realistic treatment of inventory.<sup>3</sup> We proceed by, first, explaining the dynamic optimization algorithm and, thereafter, provide a detailed discussion of the least squares estimation procedure, which is used to obtain conditional expectations of future utility in the dynamic program.

#### 3.5.1 Dynamic program

In solving the procurement and hedging model, we set up a dynamic program in conjunction with least squares estimation of the continuation value in order to accommodate multi-factor commodity price dynamics. This approach is based on the least squares Monte Carlo (LSM) algorithm of Longstaff and Schwartz (2001) and has close linkages with the valuation method applied by Boogert and de Jong (2008, 2011) in the context of gas storage valuation. We will refer to this valuation algorithm as least squares Monte Carlo optimization (LSMO). Our goal is to find the jointly optimal policy for spot procurement ( $b_t$ ), pricing ( $p_t$ ), and hedging ( $f_t$ ) such that the firm's utility satisfies the following dynamic program:

$$U_t(\cdot) = \max_{p_t \geq 0, 0 \leq b_t \leq I_{max} - I_t, f_t \geq 0} \left\{ u_t(\cdot) + e^{-r_t \Delta} \mathbb{E}[U_{t+1}(\cdot) \mid x_t] \right\} \quad (3.3)$$

for all  $t \in [t, T]$ , where  $u_t$  is defined as the periodic utility excluding the continuation value

$$u_t(\cdot) = \mathbb{E} \left[ \Pi_t^{OF} \right] - \lambda D_t \quad (3.4)$$

and  $D_t$  follows from Equation 3.5. As we work with multi-factor models of commodity price dynamics, we formulate the expectation of future utility conditional on today's state of the commodity price process more generally as  $\mathbb{E}[U_{t+1}(\cdot) \mid x_t]$ , where expectations are taken conditional on  $x_t$ , which denotes the vector of all stochastic state variables in a given process. For instance, in the case of the four-factor SRCV

<sup>2</sup> Note that the model is not limited to an application in the crude oil market. Rather, we choose the crude oil market in our numerical study as it is relevant in a wide business context and has particular relevance for the risk management at chemical firms such as BASF.

<sup>3</sup> In contrast to the numerical setup used in this section, analytical tractability typically requires myopic decisions, which can, in turn, necessitate restrictive assumptions with respect to the treatment of leftover inventory (see, e.g., Kouvelis et al., 2013).

**Algorithm 1:** Least squares Monte Carlo optimization*Initialization:*Generate  $N_x$  commodity price paths  $x_t$  with  $N_t = T/\Delta$  time stepsLet  $\mathcal{X}$  denote a state space with dimension  $N_t \times N_x \times N_i \times N_f$ **Step 1** Initialize terminal contribution**forall** States  $X \in \mathcal{X}_{N_t}$  **do**

$$U_T(p_T^*, b_T^* \equiv 0, f_T^* \equiv 0|X) = \max_{p_T \geq 0, 0 \leq b_T \leq i_{max} - X_i} \{u_T(p_T, b_T \equiv 0, f_T \equiv 0|X)\}$$

**end****Step 2** Backward recursion**for**  $t \leftarrow N_t - 1$  **to** 1 **do**

A. Determine continuation values

**forall** States  $X \in \mathcal{X}_t$  **do**

$$C_t(X) = e^{-r_t \Delta} \mathbb{E} [U_{t+1}(p_{t+1}^*, b_{t+1}^*, f_t = X'_f|X') | X_x]$$

**end**

B. Optimization

**forall** States  $X \in \mathcal{X}_t$  **do**

$$U_t(p_t^*, b_t^*, f_t^* | X) = \max_{p_t \geq 0, 0 \leq b_t \leq i_{max} - X_i, f_t \geq 0} \{u_t(p_t, b_t, f_t | X) + \hat{C}(p_t, b_t, f_t | X)\}, \text{ where}$$

$$\hat{C}(p_t, b_t, f_t | X) = \mathcal{L}^k(\mathcal{I}, \mathcal{F}, C_t(X_x), i_{t+1}(p_t, b_t), f_t)$$

**end****end**

$N_t$  denotes the number of time steps,  $N_x$  denotes the number of price paths,  $N_i$  denotes the number of discretization points for the inventory discretization  $\mathcal{I}$ , and  $N_f$  refers to the number of discretization points for the hedging positions  $\mathcal{F}$ . By  $\mathcal{X}_t$ , we denote the set of all states at the time step  $t$ . Within the set  $\mathcal{X}_t$ ,  $X = \{i_t, x_t, f_{t-1}\}$ , and, for example,  $X_i = i_t$  and  $X_x = x_t$ .  $X'$  refers to the state  $X$  in the subsequent time period  $t + 1$ , which captures the next realization of commodity prices  $x_t$ , but is equivalent to  $X$  with respect to the inventory level  $i$  and futures position  $f$ . The expectation  $\mathbb{E}[\cdot|x]$  is computed using Equation 3.7.  $\mathcal{L}^k(\cdot)$  denotes an interpolation function, where  $k = \{\text{linear, cubic, spline}\}$  indicates the interpolation method used. Thus, for a given state  $X$ ,  $\hat{C}(\cdot)$  denotes the expected continuation value  $C(\cdot)$ , interpolated from the discretization  $\mathcal{I}$  and  $\mathcal{F}$ , given the decisions  $p_t$ ,  $b_t$ , and  $f_t$ . Note that the continuation value does not directly depend on  $p_t$  and  $b_t$  but rather on the ending (beginning) inventory level of the current (next) period, which is a function of  $p_t$  and  $b_t$ . Demand risk  $\Psi$  is omitted in the above representation for simplicity. In the presence of  $\Psi$ , respective expectations have to be taken over  $u_t$  and  $i_{t+1}(p_t, b_t)$ .

model  $x_t = [y_t \ r_t \ \delta_t \ v_t]^\top$ . Besides accounting for stochastic commodity prices, the firm will also need to formulate expectations about the demand shock at the time of decision making. This expectation of  $\mathbb{E}[\Pi_t^{OF}]$  is numerically evaluated using a Gaussian quadrature with 20 integration points.

After generating an appropriate number of commodity price paths using the transition equation of a given commodity price model (please refer to Appendix A.2), the program in Equation 3.3 is solved via backward recursion in a four-dimensional state space, which takes into account the commodity prices, inventory levels, futures positions entered in the previous period, and time. For inventories and futures



positions, we use an appropriate discretization, denoted by  $\mathcal{I}$  and  $\mathcal{F}$ , respectively, which ensures that all feasible actions of the firm lead to states of the system that fall within the domain of the state space. The convergence of the valuation algorithm with respect to the number of price simulations and the inventory discretization is discussed in Section 3.6.1. An overview of the LSMO procedure is given by Algorithm 1.

The program is initialized in the final period  $T$  of the decision horizon as shown by Step 1 of Algorithm 1. Since the firm will neither purchase physical material, due to the production time of one period, nor enter futures contracts in the final period, Equation 3.3 can be simplified to obtain final period utility:

$$U_T(p_T | i_T, s_T, f_{T-1} \equiv 0) = \max_{p_t \geq 0} \{u_T(p_T | i_T, s_T)\}. \quad (3.5)$$

Terminal utility is maximized by setting optimal prices for each realization of the commodity price process, initial inventory level, and futures position traded in the previous period. The numerical optimization procedure used to determine optimal decisions is discussed at the end of this section. In period  $t = T - 1$ , the firm makes both procurement and pricing decisions. However, since there is no physical procurement in the next period, there is no hedging in that period. Note that  $f_{T-1}^* = 0$  is optimal even though the salvage value of inventory is proportional to the commodity price in the final period and  $\lambda \geq 0$ . Intuitively, as the firm effectively holds a long position in physical material, only a short futures position would reduce profit volatility arising from the salvage value in period  $T$ . As we exclude the possibility of short sales,  $f_{T-1}^* = 0$  is the utility maximizing policy. In turn, any policy  $f_{T-1} > 0$  increases profit volatility and downside risk without contributing to expected profits under the risk-neutral measure.

Once the terminal contribution is initialized, we proceed with the backward recursion in Step 2 of Algorithm 1, which is based on a two-step procedure. In Part A, we determine the continuation values  $\mathbb{E}[U_{t+1}(\cdot) | x_t]$ , corresponding to each future state of inventories and hedging positions, conditional on the respective current state. Similarly to the LSM algorithm, this prediction step is performed based on a least squares regression of future utility on the current vector of commodity prices  $x_t$  for each inventory and hedging state. Note that we perform separate regressions for each inventory level and futures hedge, instead of including these in the regression as this leads to more stable regression estimates.<sup>4</sup> The regression-based estimation of conditional expected future utility is further discussed in the subsequent section.

In Part B of Step 2, the utility maximizing policy is chosen such that the sum of current period profits, the downside risk adjustment, and expected future utility is maximized. At time  $t = T - 1, \dots, 1$ , the continuation value depends on the procurement and pricing decision and for periods  $t = T - 2, \dots, 1$  it depends, additionally, on the hedging decision. In particular, the amount of material purchased in the present period directly affects the expected starting inventory level in the following period and unit

<sup>4</sup> This is in line with Boogert and de Jong (2008), who find that the numerical stability of the LSM algorithm deteriorates in the context of gas storage valuation, if volume levels are included in the regression prediction. Instead, they suggest to run separate regressions for each volume level.

selling prices have an impact on expected demand and leftover inventory in the current period, which is carried over to the next period. As the state of future utility is a function of the starting inventory level in a given period, it follows that the pricing and procurement decision affect the continuation value. Similarly, today's hedging decision determines future financial cash flows and expected utility. Whenever a policy  $\{p_t, b_t, f_t\}$  is evaluated that leads to a future inventory level or futures hedge position between two points on the discretization  $\mathcal{I}$  and  $\mathcal{F}$ , the continuation value corresponding to this policy is interpolated using one of the interpolation methods  $k = \{\text{linear}, \text{cubic}, \text{spline}\}$ . Given the current and expected future utility for each evaluated decision  $\{p_t, b_t, f_t\}$ , we select the utility maximizing decision  $\{p_t^*, b_t^*, f_t^*\}$  for each current state of the commodity price, inventory, and hedging position. Next, we continue the recursion.

When the recursion terminates at  $t = 1$ , the utility maximizing policy today ( $t = 0$ ) is implicitly defined by  $U_{t=1}$ . In particular, assuming zero starting inventory and no previous hedging activity, the present value of expected future utility at time  $t = 0$  is given by:

$$\begin{aligned} U_0(b_0, f_0 | s_0) &= \Pi_0^O(b_0 | s_0) + e^{-r_t \Delta} \mathbb{E} \left[ U_1(p_1^*, b_1^*, f_1^* | s_0, i_1 = b_0, f_0) \right] \\ &\approx \Pi_0^O(b_0 | s_0) + \frac{1}{N_x} \sum_{i=1}^{N_x} e^{-r_t \Delta} U_1(p_1^*, b_1^*, f_1^* | s_{i,1}, i_1 = b_0, f_0), \end{aligned} \quad (3.6)$$

where  $*$  denotes the optimality of a decision with respect to future expected utility. As there are no sales in period  $t = 0$ , only  $b_0$  and  $f_0$  have to be chosen. Since  $b_0$  corresponds to the beginning inventory in the next period,  $b_0^*$  and  $f_0^*$  are easily determined as the maximizers of the sum of cash outflows today, resulting from material purchases, and the expected discounted continuation value that is implied by today's procurement and hedging decision. The expectation of future utility with respect to  $x_0$  is simply given as the mean over all commodity price paths in period  $t = 1$ .

The numerical optimization over  $p_t, b_t,$  and  $f_t$  is performed using a grid search method, where  $U_t(\cdot)$  is first evaluated by enumerating over a coarse grid of policies, which is refined in subsequent iterations.  $p_t^*, b_t^*,$  and  $f_t^*$  are selected such that  $U_t$  is jointly maximized. To speed up the optimization procedure, the objective function may be vectorized and evaluated in parallel for all different states at a given time step.

A formal proof of the joint concavity of  $U_t$  with respect to  $p_t, b_t,$  and  $f_t$  is omitted, however, it is numerically established that  $U_t$  is a jointly concave function of the decision variables. When multiple maximizers of  $U_t$  exist, we select the lexicographically smallest solution. In the optimization, the model can be vectorized to a large extent and allows for parallel computing in all dimensions except the backward recursion in time.

### 3.5.2 Estimation of continuation values

As the use of the LSM algorithm is relatively novel in the literature on procurement and inventory control problems, we provide a more detailed description of this method in the context of multi-factor commodity

**Table 3.1:** Parameters

Time horizon (yrs.)	$T$	1	Demand shock volatility	$\sigma_d$	3
Time step (months)	$\Delta$	4	Inventory constraint	$i_{max}$	30
Number of price paths	$N_x$	5,000	Unit storage cost	$\varphi$	5%
Unit labor cost	$w_t$	1.5	Downside risk aversion	$\lambda$	1
Economies of scale	$\gamma$	0.05	Commodity price (t = 0)	$s_0$	8
Salvage value (%)	$\Phi$	60%	Interest rate (t = 0)	$r_0$	0.02
Demand intercept	$a$	60	Convenience yield (t=0)	$\delta_0$	0.05
Price elasticity	$\eta$	5	Variance (t = 0)	$v_0$	0.05

Commodity prices are log-normally distributed under the one-factor model. Calibrated parameters of the one-factor and multi-factor cost processes are given in Appendix A.2. The demand shock is normally distributed:  $\Psi \sim \mathcal{N}(0, \sigma_d)$ .

price dynamics below. Formally, under the LSM algorithm, continuation values are approximated by

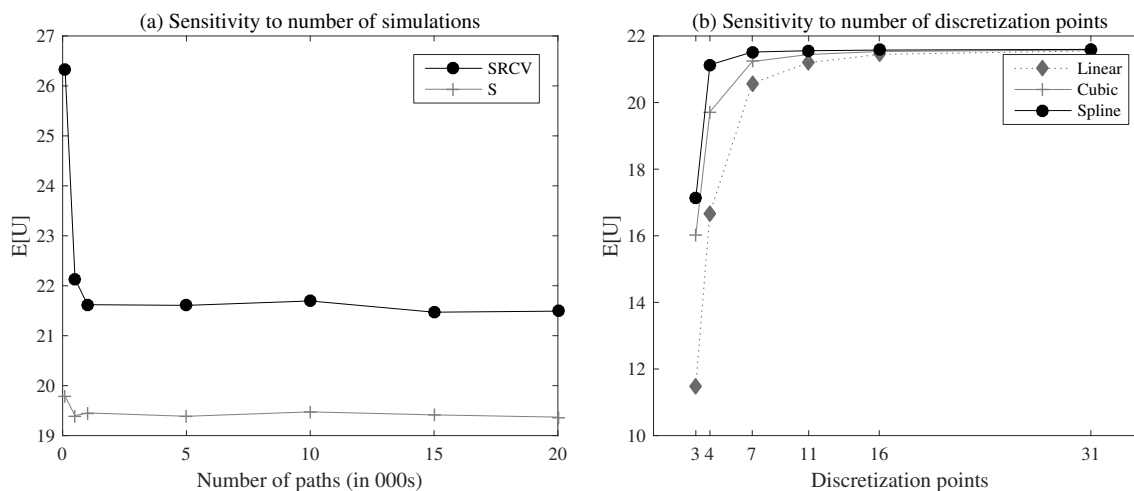
$$C_t = e^{-r_t \Delta} \mathbb{E}[U_{t+1}(\cdot) \mid x_t] \approx \sum_{k=0}^K \beta_{k,t} f_k(x_t), \quad (3.7)$$

where  $f_k$ ,  $k = 0, \dots, K$  refers to a set of basis functions evaluated for all paths  $i = 1, \dots, N_x$  of  $x_{i,t}$  and  $\beta_{k,t}$  denote the corresponding regression coefficients determined by a linear least squares fit of Equation 3.7:

$$\{\beta_{k,t}\}_0^K = \underset{\{\beta_{k,t}\}_0^K}{\operatorname{argmin}} \sum_{i=1}^{N_x} \left[ C_t - \sum_{k=0}^K \beta_{k,t} f_k(x_{i,t}) \right]^2. \quad (3.8)$$

With respect to the choice of basis functions  $f_k$ , the most common approach follows from the original article by Longstaff and Schwartz (2001), suggesting the use of a constant, the first three powers of each state variable, and their first-order cross products. In the case of the four-factor SRCV model, this approach yields a total of 19 basis functions. However, as pointed out by Boogert and de Jong (2011), in a multi-factor setting it is particularly important to ensure that regressions are performed in a stable manner. In their research on gas storage valuation under multi-factor gas price dynamics, the authors report problems with matrix inversion when using a complete set of polynomials of order three, whereas a complete set of polynomials of order two underperformed. In order to obtain a set of basis function that yields both high explanatory power and stable matrix inversion, we follow Boogert and de Jong (2011) and adopt the following procedure.

First, let  $m$  denote the matrix of regressors, where  $m := [1, x, x^2, \dots, x^n]$  so that only the log-spot price is considered as an explanatory variable. We begin to determine the optimal order  $n$  by starting with a high value, which is decreased until the conditional number of  $m^\top m$  crosses a pre-set threshold. In the case of a one-factor model, the resulting specification is used. For a multi-factor model, we next seek to improve the predictive accuracy by adding additional state variables to  $m$ . Other than Boogert and de Jong (2011), we do not use a greedy heuristic but evaluate an ex-ante specified set of meaningful



**Figure 3.1:** Convergence of valuation algorithm.

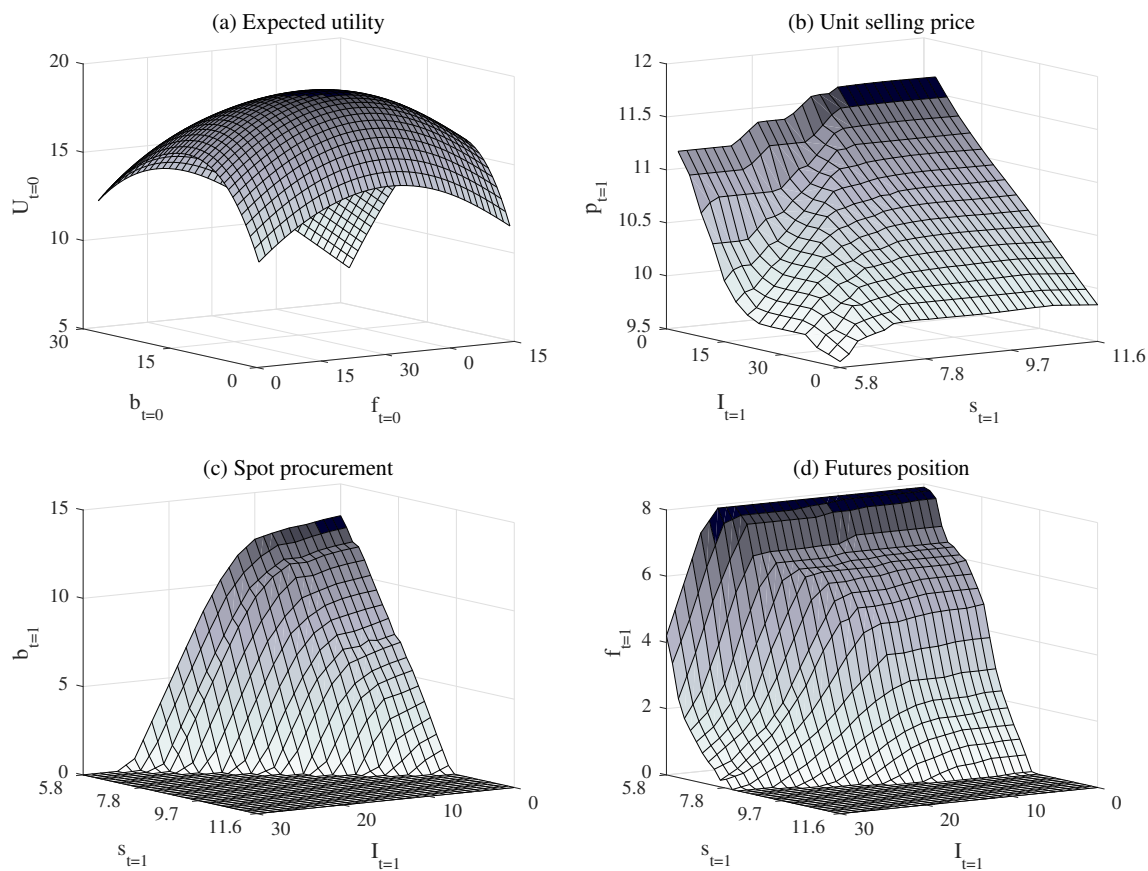
configurations for  $m$ , where the optimal configuration  $m^*$  yields the best least squares fit, while satisfying the restriction on the conditional number of  $m$ . Within the set of possible configurations, we consider different subsets of state variables, powers, and cross products up to order four. The choice of simple powers instead of Laguerre, Hermit, or Chebyshev polynomials is motivated by Boogert and de Jong (2011).

## 3.6 Results

In this section, we report the results of numerical studies on (a) the convergence of the valuation algorithm, (b) optimal policy decisions and the value of flexibility, (c) the influence of different commodity price models on valuation results, (d) the implications of risk aversion, and (e) the sensitivity of results to input parameters. Unless otherwise specified, we use parameters from Table 3.1 and the one-factor model of input costs.

### 3.6.1 Convergence of valuation algorithm

As a first step, we ascertain that valuation results converge to a stable solution and determine the necessary number of commodity price paths and discretization points in the inventory and hedging state space. Figure 3.1a illustrates valuation results for a varying number of commodity price paths. Accordingly, 1,000 simulations are sufficient to reach a stable solution in the present numerical setting. This is true both for the four-factor SRCV process and the one-factor S model. In Figure 3.1b, we investigate the necessary number of discretization points on the inventory and hedging state space to achieve stable valuation estimates under different approximations of the continuation value. The results indicate that seven discretization points may suffice under spline interpolation, allowing for a smaller state space, faster computation, and lower memory usage compared to cubic and spline methods.



**Figure 3.2:**  $U_t$ : expected present value of utility,  $b_t$ : physical spot procurement quantity,  $p_t$ : unit selling price,  $f_t$ : number of futures,  $s_t$ : cost realization in period  $t$ ,  $i_t$ : initial inventory level.

### 3.6.2 Optimal policy

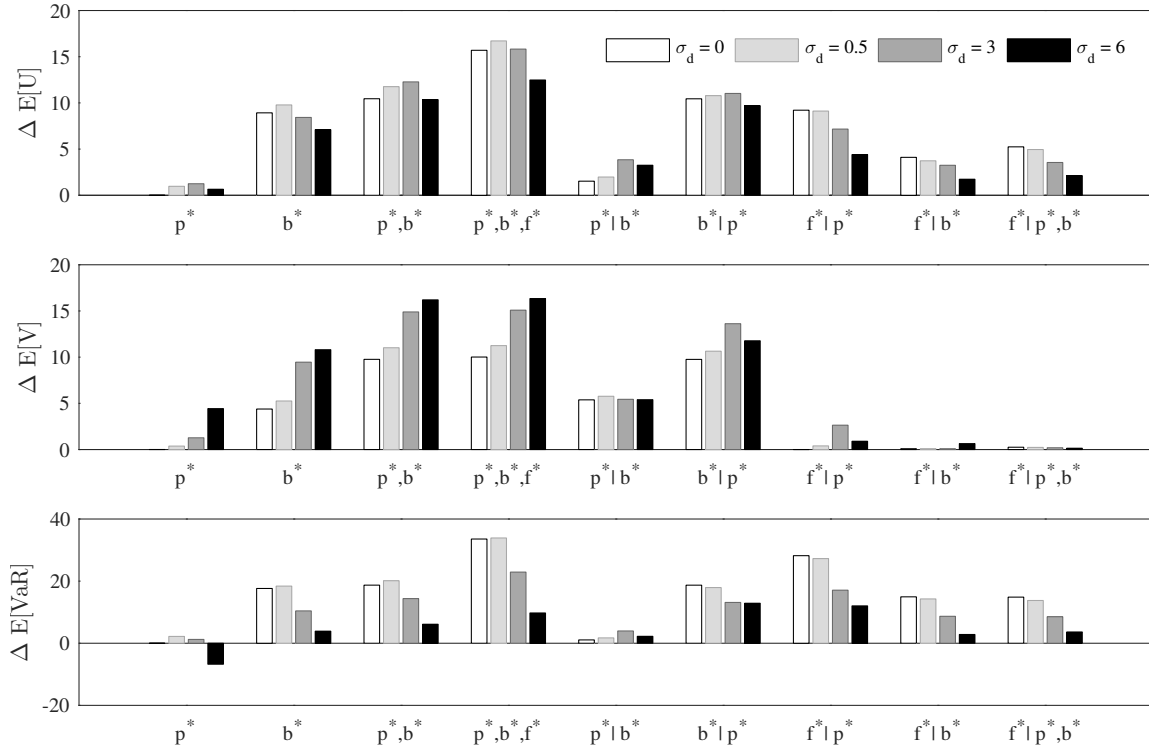
Optimal policy decisions are shown in Figure 3.2. In particular, we illustrate the expected present value of utility as a function of today's hedging and procurement decision in part (a) and the state dependent pricing, purchasing, and hedging policies for an exemplary future period in panels (b), (c), and (d), respectively. Today's optimal procurement and hedging decision is implicitly defined by (a) as the utility maximizing policy. A lower quantity of spot procurement is suboptimal as it leads to lost sales, while a higher quantity creates both high storage costs and an increased risk of overage at the end of the planning horizon. With respect to the hedging decision, it is shown that futures can increase utility; however, excessive hedging beyond the scale of operational needs turns the firm into a speculator on the commodity price, which creates downside risk that decreases utility. The optimal spot purchase quantity for a given futures hedge is a non-linear, decreasing function in the number of futures entered. Thus, physical inventory and financial hedging are strategic substitutes, which is in line with Kouvelis et al. (2013), who demonstrate that in multi-period problems, financial hedging can decrease inventories. As time progresses, the state of commodity prices and the new inventory level become observable. Optimal unit selling prices are decreasing in the current inventory level and initially increasing in the commodity

**Table 3.2:** Valuation results for different degrees of flexibility and demand volatility

Setting	$\sigma_d = 0$			$\sigma_d = 0.5$			$\sigma_d = 3$			$\sigma_d = 6$		
	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$
(1) $\bar{p}, \bar{b}$	11.9	21.7	-7.9	9.5	19.1	-9.7	3.3	8.1	-6.4	0.0	0.0	0.0
(2) $p^*, \bar{b}$	11.9	21.7	-7.9	10.5	19.5	-7.5	4.6	9.4	-5.2	0.7	4.4	-6.8
(3) $\bar{p}, b^*$	20.8	26.1	9.8	19.3	24.4	8.7	11.7	17.6	4.0	7.1	10.8	3.9
(4) $c^+, b^*$	8.9	18.7	6.3	8.6	18.3	6.0	6.0	14.2	2.3	3.9	10.0	0.0
(5) $p^*, b^*$	22.3	31.5	10.8	21.3	30.1	10.4	15.6	23.0	8.0	10.4	16.2	6.1
(6) $p^*, \bar{b}, f^*$	21.1	21.7	20.3	19.6	19.9	19.7	11.7	12.0	11.9	5.1	5.3	5.3
(7) $\bar{p}, b^*, f^*$	24.9	26.2	24.7	23.1	24.5	22.9	15.0	17.7	12.7	8.8	11.4	6.7
(8) $c^+, b^*, f^*$	12.5	19.0	10.2	12.2	18.5	10.1	9.3	14.3	8.3	6.5	10.4	5.1
(9) $p^*, b^*, f^*$	27.6	31.7	25.7	26.3	30.4	24.2	19.1	23.2	16.5	12.5	16.3	9.8

$U$ : present value of expected utility,  $V$ : present value of expected profits corresponding to  $U$ ,  $VaR$ : 5<sup>th</sup> percentile of cash value distribution as an indicator of downside risk. \* shows that the variable is dynamically adjusted in every period,  $\bar{\phantom{x}}$  indicates that the variable is fixed and set once at the beginning of the planning horizon.  $c^+$  refers to cost-plus-pricing. Commodity price volatility is set to  $\sigma_s = 0.3$ , which is denoted in per cent p.a. and not in absolute terms as  $\sigma_d$ .

price. When inventories are low, high prices align the level of demand with supply, creating high margins for the low quantity of units that is available for sale. In turn, it is optimal to promote sales by lowering prices when inventories are high, in order to reduce carrying costs and the risk of excess stock at the end of the planning horizon. In some instances, prices may even be discounted below the level of current production costs as the sales proceeds exceed the expected salvage value of excess stock received in the final period. Unlike under non-stochastic costs (see, e.g., X. Chen & Simchi-Levi, 2004a), the utility maximizing price not only depends on the state of inventory but also exhibits a positive relationship with input costs. An exception to this rule occurs in states that are characterized by a combination of both high costs and inventories, where prices become independent of the cost process. Here, high inventory levels are expected to be sufficient to satisfy future demand or raw material prices are prohibitively high. In either case, the firm will cease to procure material, which renders profits and optimal prices independent of the cost state. The spot procurement and hedging volumes are decreasing in the inventory level and the commodity price. As an illustration of the mechanics behind these policies, consider the following example. For a given state of inventory, low costs correspond to low profit maximizing prices and relatively high current and future expected demand. This drives down current inventories and increases the need for new physical material to satisfy future demand. To gain protection against future cost volatility, the financial hedging volume rises accordingly. In turn, when inventory levels are sufficiently high to satisfy current and future demand, little new material is needed and the exposure to future cost uncertainty is low, resulting in minor hedging activity.



**Figure 3.3: Value of flexibility and demand volatility.**  $\Delta E[U]$  refers to the change in expected utility resulting from managerial flexibility under a given policy. For instance,  $p^*$  represents the strategic value of introducing dynamic pricing from the perspective of a firm that currently operates under a fixed price and fixed quantity policy ( $\bar{p}, \bar{b}$ ).  $\Delta E[V]$  and  $\Delta E[VaR]$  denote the value of flexibility in terms of profits and units of VaR, respectively. This figure corresponds to Table A.1 shown in Appendix A.1.

### 3.6.3 Managerial flexibility

In this section, we compare valuation results under several operating strategies, characterized by different degrees of managerial flexibility. Results are reported for a range of demand and cost volatilities to illustrate the sensitivity of the value of managerial flexibility in different market environments. In order to improve the comparability of results, the same set of random numbers is used in the evaluation of each operating strategy. In Table 3.2, we report the present value of expected utility ( $U$ ) at time  $t = 0$ , the corresponding present value of expected profits ( $V$ ), and the value at risk (VaR) as a measure of downside risk. Here, we define the VaR as the 5<sup>th</sup> percentile of the profit distribution or, in other words, the level of discounted profits that will be exceeded in 95% of cases. Thus, under this interpretation of VaR, high values correspond to low downside risk. The demand volatility levels reported in Table 3.2 ( $\sigma_d = [0.5, 3, 6]$ ) correspond to approx. 4%, 26%, and 52% of average expected periodic demand when the firm chooses  $p$ ,  $b$ , and  $f$  dynamically and when  $\sigma_d = 0$ . Note that  $\sigma_d$  refers to the level of demand shock volatility in absolute terms. With respect to the operating policies, superscript  $-$  indicates that the respective decision variable is fixed, i.e., it is set at the beginning of the planning horizon, such that the expected present value of utility is maximized, and cannot be altered in response to the realization of future uncertainty. In the special case of  $\bar{b}$ , the firm decides on a fixed ordering lot size upfront and

**Table 3.3:** Valuation results for different degrees of flexibility and commodity price volatility

Setting	$\sigma_s = 0.05$			$\sigma_s = 0.15$			$\sigma_s = 0.3$			$\sigma_s = 0.5$		
	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$
(1) $\bar{p}, \bar{b}$	8.2	9.8	5.4	5.4	9.1	-1.5	3.3	8.1	-6.4	0.1	8.1	-17.2
(2) $p^*, \bar{b}$	10.3	11.9	7.4	7.4	11.2	0.0	4.6	9.4	-5.2	1.7	9.7	-14.9
(3) $\bar{p}, b^*$	11.9	12.7	9.6	11.1	13.3	6.6	11.7	17.6	4.0	14.4	23.6	3.6
(4) $c^+, b^*$	11.0	12.4	7.7	8.7	12.8	3.9	6.0	14.2	2.3	5.4	14.3	2.4
(5) $p^*, b^*$	13.3	14.5	10.0	13.3	16.8	8.0	15.6	23.0	8.0	20.1	33.5	8.1
(6) $p^*, \bar{b}, f^*$	11.9	11.9	11.8	11.8	11.9	11.8	11.7	12.0	11.9	11.6	12.9	12.4
(7) $\bar{p}, b^*, f^*$	12.6	12.7	12.4	13.0	13.9	12.6	15.0	17.7	12.7	19.3	24.8	14.2
(8) $c^+, b^*, f^*$	12.0	12.4	11.9	11.5	13.0	11.2	9.3	14.3	8.3	8.3	15.3	7.3
(9) $p^*, b^*, f^*$	14.4	14.5	14.3	15.4	16.8	14.7	19.1	23.2	16.5	26.0	33.9	20.4

$U$ : present value of expected utility ( $t=0$ ),  $V$ : present value of expected profits corresponding to  $U$ ,  $VaR$ : 5<sup>th</sup> percentile of cash value distribution as indicator of downside risk. \* shows that the variable is dynamically adjusted in every period,  $\bar{\phantom{x}}$  indicates that the variable is fixed and set once at the beginning of the planning horizon.  $c^+$  refers to cost-plus-pricing. Volatility of the demand shock is set to  $\sigma_d = 3$  in absolute terms.

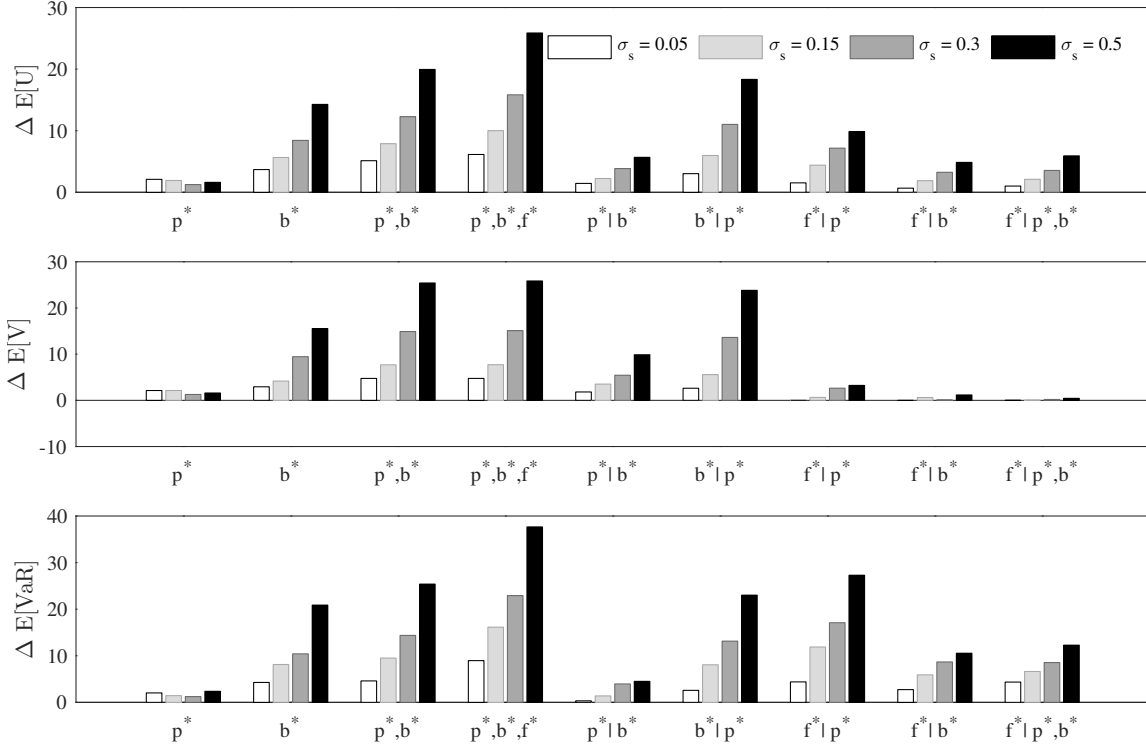
orders either the fixed lot size or, if the inventory constraint is violated, the closest feasible quantity, i.e.,  $b_t = \min\{\bar{b}, i_{max} - i_t\}$ . Superscript \* indicates that a variable is dynamically adjusted to the state of commodity prices and inventory in the sense of Figure 3.2.  $c^+$  refers to a cost-plus-pricing scheme, where a predetermined margin is added to the raw material and production cost to arrive at the periodic selling price in every period. We compare nine different operating strategies: (1)-(5) refer to different pricing and procurement settings without the ability to hedge, while strategies (6)-(9), additionally, involve dynamic hedging with futures.

Table 3.2 shows that dynamic operating strategies, such as (5) and (9) generally outperform strategies that are characterized by a lower degree of flexibility as, for instance, the static policy (1). It stands out that the policies involving cost-plus-pricing in row (4) and (8) perform poorly compared to fixed or dynamic pricing schemes. In fact, the fully static strategy (1) leads to superior expected utility than policy (4) when demand volatility is low. However, when  $\sigma_d$  is relatively high, dynamic purchasing ( $b^*$ ) helps the firm to manage inventory more effectively in response to demand shocks, explaining the superiority of strategy (4) compared to (1). Thus, it follows as a managerial insight that protecting a fixed margin under price elastic, volatile demand and stochastic costs can lead to inferior performance compared to dynamic pricing and even a fixed price policy. Generally, we may notice that  $U$ ,  $V$ , and  $VaR$  are decreasing functions of demand volatility, which increases the risk of excess stock at the end of the planning period. As a result, less material is stored, leading to more frequent stock-outs and lost revenue. Figure 3.3 shows a non-exhaustive overview of the value of managerial flexibility associated with different operating strategies. From left to right, we assess the value of stand-alone pricing ( $p^*$ ) and procurement flexibility ( $b^*$ ) as well as the joint value of flexibility without hedging ( $p^*, b^*$ ) and with hedging ( $p^*, b^*, f^*$ ). Also shown are conditional flexibility values, which refer to the marginal value



of adding flexibility to an existing operating strategy. For instance, the value of introducing dynamic pricing, when the firm currently follows a fixed price but dynamic procurement strategy, is denoted by  $p^*|b^*$ . In some instances, it may be more logical to interpret the value of flexibility as an economic cost of inflexibility. For instance, a manager might need to understand the economic cost of making a fixed price commitment for the coming year. Assuming that the firm follows a dynamic procurement strategy but does not engage in financial hedging, the economic cost of the price commitment corresponds, again, to  $p^*|b^*$ . A number of observations are worth highlighting. First, the cash value of dynamic pricing and purchasing is increasing in the level of demand volatility. Second, dynamic procurement and pricing are strategic complements. For instance, suppose  $\sigma_d = 0$  and the firm procures the same static quantity of material in every period ( $\bar{b}$ ). Then, profit maximizing prices are identical in every period and the ability to revise prices dynamically has no value. When,  $\sigma_d > 0$  dynamic pricing is valuable in so far as it allows to mitigate the effect of unexpected demand shocks. This is depicted in Figure 3.3 for strategy  $p^*$ . However, under a fully dynamic procurement strategy, where optimal inventory levels are affected by cost and demand volatility, utility maximizing prices can differ in every state and period (see Figure 3.2). This leads to a higher value of dynamic pricing than under a static procurement policy as illustrated in Figure 3.3 (comparing  $p^*$  and  $p^*|b^*$ ). Third, consider the value of hedging. Intuitively, futures help to improve VaR but add little to expected profits. Thus, the increase in utility resulting from futures hedging is mainly explained by the reduction in downside risk.

We repeat the above analysis for different levels of commodity price volatility instead of demand volatility and report the results in Table 3.3 and Figure 3.4. Similarly to Xiao et al. (2015), we find that commodity price volatility can lead to an increase in expected profits. As pointed out by the authors this can be explained by the ability of the firm to react to changes in the environment. In this way, a firm is able to extract high profits when costs are low, overcompensating low profits that arise when costs are high. While Table 3.3 shows that the present value of expected profits (V) is monotonically increasing in cost volatility under several strategies, this is not necessarily the case for utility. Here, an increase in cost volatility can be associated with higher downside risk, which is not overcompensated by an increase in profit expectations. This effect is particularly pronounced under the cost-plus-based pricing strategies (4 and 8), where utility is decreasing in volatility throughout. The degree to which downside risk leads to a decrease in utility for higher levels of cost volatility is controlled by the absolute level of downside risk aversion  $\lambda$ . When the firm is able to engage in financial hedging, utility becomes more resistant to cost volatility (e.g., comparing strategy 2 and 6). Under strategies (7) and (9) utility is even monotonically increasing in cost volatility. This positive relationship between utility and cost volatility under financial hedging can be explained by the firm's ability to benefit from an increase in the profit expectation caused by volatility, while limiting the risk of downside losses using futures. In fact, strategies (6), (7), and (9) exhibit an increase in VaR, i.e., a reduction in downside risk, for higher levels of volatility, which is in line with the value of hedging shown in Figure 3.4.



**Figure 3.4: Value of flexibility and cost volatility.**  $\Delta E[U]$  refers to the change in expected utility resulting from managerial flexibility under a given policy. For instance,  $p^*$  represents the strategic value of introducing dynamic pricing from the perspective of a firm that currently operates under a fixed price and fixed quantity policy ( $\bar{p}, \bar{b}$ ).  $\Delta E[V]$  and  $\Delta E[VarR]$  denote the value of flexibility in terms of profits and units of VaR, respectively. This figure corresponds to Table A.2 in Appendix A.1.

### 3.6.4 Commodity price model

In the context of inventory control models, the stochastic behavior of procurement costs are typically modeled based on Markovian one-factor models. However, as discussed above, such models may vastly oversimplify the complex dynamics observed in many commodity markets in reality. In this section, we address the question to what extent the use of more realistic multi-factor commodity price processes may affect valuations, policy rules, and improve expected performance.

Table 3.4 provides a comparison of valuation results for different commodity price models. Valuation results are based on the empirically calibrated parameter set as reported in Appendix A.2. The results suggest that S undervalues the firm, while the SRCV process leads to the highest valuations throughout. In fact, the present value of expected profits (V) under SRCV dynamics exceeds the valuation under S by 15.1%. This difference is considerable given the relatively short one-year planning horizon of this example. Allowing for a stochastic convenience yield and, therefore, a more realistic representation of spot and futures price dynamics, increases the valuation by 8.6% compared to the one-factor model. The additional presence of stochastic interest rates decreases the valuation by 1.2% compared to the SC model. Finally, accounting for stochastic volatility has a positive effect on the present value of expected profits, increasing the valuation by 7.4% compared to the SRC model. This can be attributed

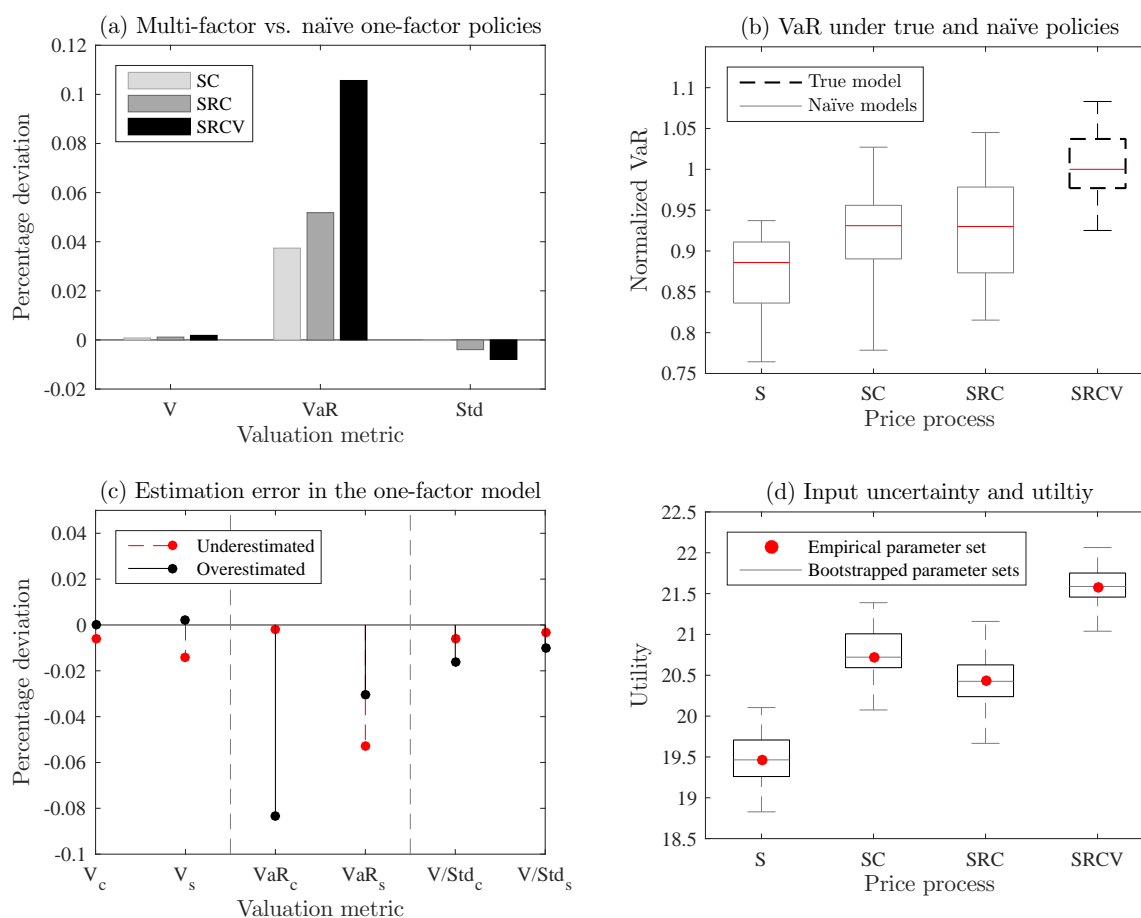
**Table 3.4:** Sensitivity of valuation results to commodity price process

Price process	U	s.e.	V	s.e.	VaR	s.e.
S	19.370	0.055	23.646	0.073	16.645	0.067
SC	20.727	0.062	25.674	0.078	17.238	0.090
SRC	20.427	0.065	25.359	0.075	16.885	0.106
SRCV	21.601	0.077	27.224	0.120	17.680	0.121

*U*: utility, *V*: expected present value of profits, *VaR*: 5<sup>th</sup> percentile of profit distribution. Reported are the mean and standard error of valuation estimates over 10 valuation runs with different random number seeds. Each valuation is based on 10,000 simulated commodity price paths. Parameters of the inventory model and the price processes are reported in Table 3.1 and Appendix A.2, respectively. Note that results for S do not correspond to Table 3.2 and Table 3.3, as different  $\sigma_s$  values are used.

to higher spot price volatility arising under the SRCV model and the ability of the firm to exploit times of high volatility as discussed in Section 3.6.2. However, the increase in utility resulting from stochastic volatility is disproportionately lower (5.7%) due to the higher degree of downside risk resulting from a more volatile cost process.

While Table 3.4 illustrates that valuations can differ considerably under different assumptions about the cost process, it is as yet unclear to what extent simplifying assumptions about cost dynamics lead to suboptimal decision making. To address this question, we consider the following situation in Figure 3.5a. Suppose a naïve firm assumes that commodity prices follow the one-factor (S) model. Based on this assumption, we calculate decision rules for this firm using the empirically calibrated parameter set of the S model. We will refer to this policy as the naïve policy. Next, suppose that in reality commodity prices follow one of the multi-factor models (SC, SRC, or SRCV) rather than the S model. We then conduct simulations of the inventory system, where commodity prices are sampled from respective multi-factor models but naïve policy rules are applied in each state of the system. The goal is to ascertain whether policy rules derived under the commonly used one-factor model lead to good approximations for optimal policies when commodity prices follow more complicated dynamics in reality. Figure 3.5a depicts the percentage improvement in the present value of expected profits (V), downside risk (VaR), and standard deviation of profits (Std) that arise when policy rules are obtained using the true data generating process (SC, SRC, or SRCV) instead of the naïve one-factor model. The results indicate slight improvements in terms of expected profits and profit volatility (+0.2% and -0.8%, respectively, when SRCV is the true model) and significant improvements in terms of downside risk. When costs are sampled from the SRCV model, VaR is 10.6% higher if operating policies are based on the true process instead of the one-factor model. Moreover, note that the reduction in downside risk does not come at the cost of reduced upside potential as the 95<sup>th</sup> percentile of the profit distribution under the true model also exceeds the corresponding naïve percentile (+0.2%). We observe that under the naïve policy, the firm initially purchases too much physical material, while charging low prices. Thereafter, this tendency is reversed and the naïve policy leads to little physical inventory and high prices. Considering the average over all simulated paths, periodic procurement quantities deviate by up to 3.6% from the optimal level, prices



**Figure 3.5: Specification and estimation error.** Panel (a):  $V$  denotes the present value of expected profits,  $VaR$  refers to the 5<sup>th</sup> percentile of the profit distribution, and  $Std$  is the standard deviation of  $V$ . Panel (b):  $VaR$  is normalized with respect to the median of the SRCV output. Panel (c): subscript  $c$  and  $s$  refer to the convenience yield ( $q_0$ ) and standard deviation parameter ( $\sigma_s$ ) of the one-factor model, respectively.  $V_c$  denotes, for instance, the sensitivity of  $V$  with respect to estimation error in  $c$ . Panel (d): boxplot of utility estimates from 100 valuation runs based on bootstrapped parameter sets.

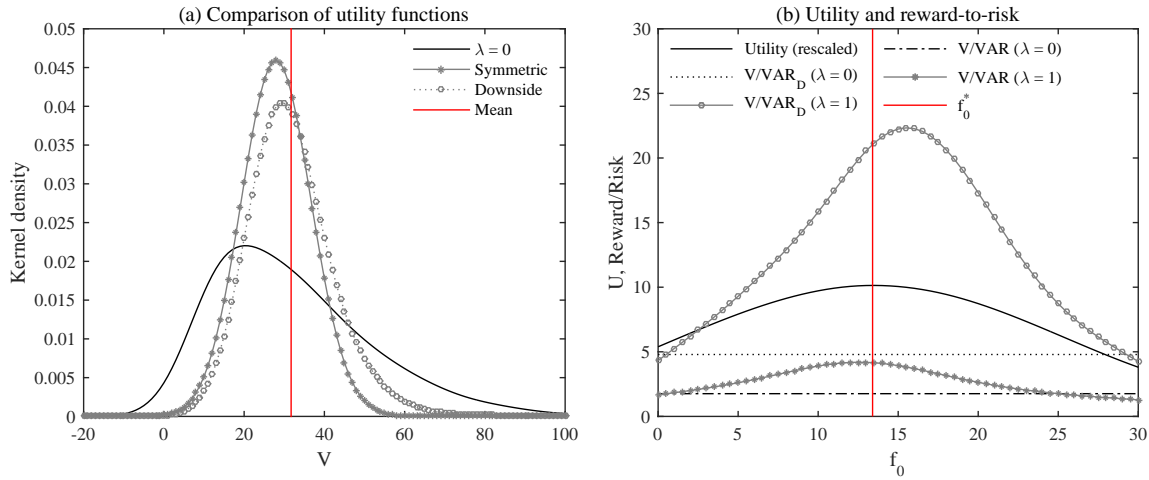
by 1%, and futures positions by 9%. As the planning horizon is increased from one to two years, not only downside risk is significantly higher under the naïve model but also the expected present value of profits is 1.48% lower when SRCV is the true model. This suggests that neglecting the stochastic nature of interest rates, the convenience yield, and volatility in the calculation of optimal policy rules can lead to increased downside risk and profit volatility in the short-term and, additionally, diminish profit expectations over a longer planning horizon. This insight partly contradicts the commonly held belief that a one-factor Wiener process is generally well-suited in the context of short-term inventory control problems (see, e.g., Berling & Rosling, 2005).

In Figure 3.5b, we show an extension of the previous analysis. Instead of comparing policies from different true models (SC, SRC, SRCV) to the naïve policies implied by the S process, we now assume SRCV to be the true data generating process and evaluate the step-wise impact of neglecting stochastic interest rates, convenience yields, and volatility in the calculation of operating policies on downside risk.

The results are based on 10 simulation runs with different random number seeds, where each simulation is based on 50,000 commodity price paths. The boxplot in Figure 3.5b depicts the output of these 10 simulation runs. It is shown that VaR tends to be highest when policies are derived from the true data generating SRCV model. As the stochastic nature of volatility, interest rates, and the convenience yield is neglected in the calculation of operating policies using the SRC, SC, and S model, respectively, downside risk steadily increases (VaR decreases). A similar pattern arises for the expected reward-to-risk ratio (V/STD), which decreases monotonically as more sources of risk are neglected and profit volatility rises.

While it is evident that the use of multi-factor models can help to reduce downside risk and profit volatility without sacrificing profit expectations, more complicated models are often prone to calibration error. In order to understand the effect of model estimation error on procurement risk management, we assess the sensitivity of valuation results to estimation error in different models below. First, we investigate the impact of estimation error in the convenience yield ( $q_0$ ) and volatility parameter ( $\sigma_s$ ) of the one-factor model on expected profits, downside risk, and profit volatility. Similarly to the above analysis, we conduct a simulation of the inventory system using commodity prices that are sampled using the true dataset but policy rules derived under incorrect parameters. In particular, we assume 5% and 30% to be the true values for  $q_0$  and  $\sigma_s$ , respectively, and apply a radius of five percentage points to the true values in order to obtain new, over- and underestimated parameter sets. Valuation results based on the mis-estimated and true parameter set are then compared and displayed in Figure 3.5c in terms of percentage deviation from optimal values. It is evident that estimation error mainly impacts VaR, while expected profits and profit volatility remain comparatively unaffected. This result can be explained as follows. An overestimated convenience yield leads to an underestimated drift and futures price, resulting in excessive futures hedging and higher exposure to commodity price volatility. As a result, downside risk and profit volatility increase. The effect of overestimated volatility is similar as it causes the firm to enter too many futures contracts. In turn, when either parameter is underestimated the firm under-hedges, resulting in above optimal commodity price exposure and an increase in downside risk. The magnitude of these effects, however, can vary under different numerical settings. For instance, when the true volatility level is 20% instead of 30%, the effect on VaR is -16.9% instead of -5.3% for the same five percentage points in estimation error.

Finally, we are interested in the potential impact of empirical estimation error on expected utility. To this end, we generate 100 sample paths of weekly data over a time horizon of 10 years with a given commodity price model using the empirically estimated parameter set. Subsequently, the model is re-calibrated to each of the sample paths and the resulting 100 bootstrapped parameter sets are used to revalue the inventory model. Figure 3.5d shows the range of resulting utility estimates along with the valuation estimate implied by the empirically determined parameter set. Accordingly, we do not find multi-factor models to exhibit higher susceptibility to estimation error than the standard Wiener process. In fact, the range of valuation results is smallest under SRCV cost dynamics.



**Figure 3.6: Risk aversion, hedging, and the distribution of profits.** Panel (a):  $\lambda = 1$  for both SRA and DSRA.  $V$  is the present value of profits. Demand is deterministic ( $\sigma_d = 0$ ). The mean refers to the risk-neutral profit distribution (mean = 31.7). Mean values for DSRA and SRA are 31.7 and 28.1, respectively. Panel (b):,  $VAR$  refers to variance and  $VAR_D$  refers to semi-variance, where the threshold is the expected unhedged profit (31.7).  $f_0^*$  is the optimal hedge in  $t = 0$ .

### 3.6.5 Risk aversion and hedging

So far, the argument of downside risk aversion (DSRA) and the utility function suggested in Section 3.3 have been little justified. In this Section, we discuss the impact of DSRA on the shape of the profit distribution and provide a comparison of results to symmetric risk aversion (SRA) and risk neutrality (RN).

Figure 3.6a provides a comparison of the profit distribution arising under optimization with DSRA, SRA, and RN. Here, SRA is obtained as a simple manipulation of Equation 3.5, where  $\{\cdot, 0\}^+$  is replaced by the absolute value of any deviation from expected operating profits ( $|\cdot|$ ). The result of this comparison is twofold. First, one may notice that the probability mass at the left tail of the distribution is considerably lower when the model is optimized under either DSRA or SRA, compared to the case of RN. This reiterates that financial hedging in conjunction with optimization under risk aversion can reduce downside risk significantly. Second, the potential of realizing high profits is preserved to a greater extent under DSRA compared to SRA. However, the degree to which this is the case depends on the value of  $\lambda$ . Generally, low levels of  $\lambda$  coincide with higher expected profits and volatility, whereas high  $\lambda$  values reduce volatility at the expense of expected profits. While different risk aversion levels may be justified depending on corporate risk preferences, we choose  $\lambda = 1$  for the purpose of numerical examples in this study. The influence of optimization under DSRA on the reward-to-risk profile of profits is illustrated by Figure 3.6b. In particular, today's utility maximizing hedging decision ( $f^*$ ) also corresponds approximately to the maximum ratio of expected profits to profit variance ( $V/VAR$ ) and downside variance ( $V/VAR_D$ ). In contrast, if the model is solved under risk-neutrality, the firm will not exploit the benefits of hedging and experience substantially lower profit expectations per unit of volatility risk.

**Table 3.5:** Sensitivity of valuation results to model parameters

$\Delta t$	U	$i_{max}$	U	$\eta$	U	$\gamma$	U	$\Phi$	U
1/2	1.00	5	1.00	4.8	1.00	0.01	1.00	0.10	1.00
1/3	1.53	10	1.52	4.9	0.85	0.05	1.23	0.30	1.02
1/4	1.85	20	1.96	5.0	0.71	0.10	1.42	0.50	1.04
1/6	2.20	30	2.02	5.1	0.59	0.20	1.63	0.80	1.09
1/12	2.41	40	2.03	5.2	0.48	0.50	1.85	1.00	1.14
$\varphi$	U	$c_0$	U	$S_0$	U	$r_0$	U	$\delta_0$	U
0.01	1.00	0.50	1.00	3	1.00	0.01	1.00	0.01	1.00
0.05	0.80	1.00	0.75	5	0.56	0.02	0.90	0.05	1.08
0.10	0.62	1.50	0.54	8	0.10	0.05	0.65	0.10	1.19
0.15	0.48	2.00	0.37	10	0.01	0.08	0.44	0.15	1.30
0.20	0.26	5.00	0.15	15	0.00	0.10	0.33	0.20	1.43

Reported is the sensitivity of expected Utility ( $U$ ) to variation in selected model inputs. For further reference refer to Table 3.1. Utility is normalized to one with respect to the first value of a given input. Commodity prices are assumed to follow the S process with parameters given in Appendix A.2. The sensitivity of results to volatility is addressed in Table A.2 and, thus, not considered here.

### 3.6.6 Parameter sensitivity

In this section, we investigate the sensitivity of valuation results to variation in model inputs. The output is summarized in Table 3.5. In line with expectations, utility ( $U$ ) is increasing in the number of decision making dates per year. For instance, by revising decisions quarterly instead of semi-annually, expected utility can be improved by 85%. The corresponding effect on downside risk is even greater, amounting to a 166% increase in VaR. However, the marginal gain from more frequent decision making is diminishing. For instance, doubling the number of policy revisions per year from 6 to 12, leads to a comparatively low increase in utility of 9.5%.  $U$  is concave in the available storage capacity  $i_{max}$  as low storage capacity constrains periodic sales and strategic material purchasing. The value of marginal storage is decreasing as the capacity becomes sufficient to handle typical demand and cost scenarios. Price elasticity of demand ( $\eta$ ) has a negative relationship with expected utility and profits. An increase in  $\eta$  reduces profit margins and lowers the firm's ability to pass on cost increases. Production economies of scale ( $\gamma$ ) and the salvage value ( $\Phi$ ) both show a positive relationship with utility. However, the impact of  $\Phi$  is relatively low, given the firm's ability to keep excess inventory to a minimum through dynamic pricing. Storage costs ( $\varphi$ ), unit labor costs ( $c_0$ ), and the initial commodity price ( $S_0$ ) exhibit a negative relationship with utility. The impact of  $S_0$  is particularly pronounced, since demand can drop to zero once the product becomes prohibitively costly. The current interest rate  $r_0$  has a negative effect on  $U$  through the commodity price drift, inventory carrying costs, and the discount rate. Finally, the initial convenience yield  $\delta_0$  is positively related to  $U$  as it lowers the spot price drift, expected material costs, and futures prices. However, a high convenience yield may, at the same time, be a symptom of expected physical supply shortages, which are not taken into account in the present model. If an increase in  $\delta_0$  is

associated with a higher probability of material shortages and lost sales, the effect on  $U$  is ambiguous and possibly even negative.

### 3.7 Conclusion

We study a periodic review, single-item, multi-period inventory model in the presence of demand risk and multi-factor commodity price dynamics. This problem formulation is the first to address the joint pricing, purchasing, and financial hedging problem under risk aversion. Also new to the literature on procurement risk management is the desirable feature of our model to allow for multi-factor commodity price dynamics, which are relevant in a multitude of industrial contexts.

This study leads to the following main conclusions. First, adopting an integrated pricing, procurement, and financial hedging policy can increase expected utility and profits substantially, while reducing downside risk compared to operating strategies with a lower degree of flexibility. Second, a risk-averse firm may profit from an increase in commodity price volatility both in terms of utility and profits under a dynamic pricing and procurement strategy. However, while expected profits are monotonically increasing in cost volatility under dynamic procurement strategies, this is not always true for utility due to the corresponding increase in downside risk. We provide the unique insight that even a risk-averse firm may benefit from cost volatility, when a futures hedge can be implemented. The opposite is true for demand risk under production lead time. Since the firm has to make sourcing and production decisions before demand is observed, utility and expected profits are decreasing in demand volatility. However, dynamic procurement and pricing policies can mitigate the negative impact of demand risk. Moreover, the marginal value of introducing pricing and purchasing flexibility can be increasing in both cost and demand risk. Third, we find that different assumptions about the stochastic commodity price process can have significant implications for the valuation of expected profits. In this context, a commonly used one-factor Wiener process can undervalue the firm by as much as 15% over a one-year planning horizon. Moreover, operating policies derived under naïve assumptions about cost dynamics may result in higher downside risk and profit variance compared to optimal policies.

The present model may be extended in a number of ways. For instance, as firms may offer multiple products involving different but potentially correlated input costs, an interesting extension of our model might address the jointly optimal PRM strategy for multiple business units that takes diversification effects into account. Another useful extension is the introduction of financial options as available hedging instruments. In particular, the asymmetric payoff profile of financial options may allow for an increase in utility under downside risk aversion. In this context, an analysis of optimal strike price selection may be a fruitful addition. Finally, time-variation in the price elasticity of demand and demand learning resemble interesting aspects that are left to future research.



# 4

## DYNAMIC PROCUREMENT, PRICING, AND HEDGING IN A TWO-PRODUCT FIRM UNDER COST AND DEMAND RISK

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*The content presented in this chapter is based on Schöne, Spinler, and Birge (2016a).*

In this chapter, we study the joint inventory management, pricing, and financial hedging problem of a risk-averse, two-product firm facing cost and demand risks in each market. Both types of risk are respectively correlated and we account for the possibility of stockout-based cross-demand between individual divisions to capture the potential substitutability/complementarity of items. Physical material is purchased from the spot market and the firm has access to futures, call and put options associated with each input cost for financial hedging. The firm dynamically maximizes inter-period, mean-variance utility over a multi-period horizon. Based on this model, we make two contributions to the existing literature: (i) we provide analytical solutions to the firm's inventory, dynamic pricing, and hedging problem. Whenever a financial hedge is implemented, the inventory and pricing policy is myopic, while the hedging decision is not; (ii) based on a numerical study, we show that cross-divisional risk correlations as well as unilateral changes in cost or demand volatility can affect optimal policies in both divisions and the potential for natural hedging. Moreover, dynamic pricing and financial hedging can contribute to mean profit under the risk-neutral measure, which is due to the real option value of operational flexibility and inventory reduction, respectively. Finally, financial hedging may lower physical inventory and prices in a multi-period setting with a monotonically increasing magnitude in the level of absolute risk aversion.

### 4.1 Introduction

In the wake of unprecedented volatility observed across commodity markets in recent years, thorough risk management practices have become imperative for industrial organizations today. However, understanding and managing a firm's exposure to multiple sources of risk can be a complex task in practice.

This is particularly true for large, multi-divisional organizations facing not only the aggregated risk of input cost volatility in different markets but also the added uncertainty of customer demand risk in a potentially interdependent product portfolio.

According to Fisher and Kumar (2010), treating these different risks in isolation can not only lead to suboptimal risk control but may even increase net risk exposure. For instance, different business units might hedge opposite sides of the same risk and leave the potential for natural hedging inherent in lowly correlated risks unexploited. Moreover, operational flexibility can constitute a powerful means of hedging that is often overlooked when financial hedging decisions are not concerted with other functional units such as the procurement, inventory management, and sales department. For instance, holding physical raw material inventory offers a similar hedge against future cost volatility as a long futures position. Hence, a decision to hold more raw material in response to, for example, higher demand volatility or due to the intent to increase service levels, can necessitate an offsetting decrease in the long futures position of the financial hedging portfolio. In other cases, the sales department may be able to pass an increase in raw material costs partly through to the customer, resulting in a much lower effective exposure to raw material costs than potentially suspected by the risk management department. Moreover, the operational flexibility inherent in dynamic pricing and inventory management may create valuable real options that do not only mitigate a firm's risk exposure but also contribute to profitability (Billington, Johnson, & Triantis, 2002; Xiao et al., 2015). A popular example in this context is Hewlett-Packard, who adopted an integrated risk management strategy involving a portfolio sourcing approach, dynamic pricing, and financial hedging that resulted in \$425 million in cost savings over a 6-year period. More generally, the value-enhancing nature of hedging is confirmed by the theoretical models of C. W. Smith and Stulz (1985) and Froot et al. (1993) as well as empirical and practical evidence (e.g., Allayannis & Weston, 2001; MacKay & Moeller, 2007; and Tevelson et al., 2007).

However, while the economic importance of effective hedging appears undisputed in a wide business context, there is yet little research addressing the integration of operational and financial risk management, particularly in a multi-divisional organization facing multiple, correlated risks. Along these lines, Xiao et al. (2015) are the first to study an inventory control model of a risk-neutral firm with multi-sourcing (spot and forward) and dynamic pricing under cost and demand risk. Schöne et al. (2016b) extend their work accounting for downside risk aversion and multi-factor commodity price dynamics, however, both models focus on a single product firm, leaving the interplay between multi-divisional operations, net risk exposure, and optimal hedging policies to future research.

We address this gap and study the jointly optimal pricing, procurement, and financial hedging policy of a risk-averse, two-product firm seeking to maximize inter-period, mean-variance utility over a multi-period horizon. Each division is exposed to a different input cost risk and sources physical, storable material from the spot market. As financial hedging instruments, the firm has access to futures, call, and put options tradeable in each market. We further allow for stockout-based cross-demand between individual divisions to capture the potential substitutability/complementarity of the offered items, which may create cross-divisional demand interdependencies (e.g., Zhang, Zhang, Zhou, Saigal, & Wang, 2014). This setting closely resembles a multitude of practical business cases such as, for instance, the road bike

industry, where several manufacturers offer both carbon fiber and aluminum frames. While the profits earned by each of these segments are subject to the different raw material costs, well-equipped aluminum bikes are generally viewed as substitutes for low-end carbon fiber bikes, which creates the possibility of dynamic substitution behavior depending on availability. Another example is the electrical equipment industry, where firms such as General Electric offer a wide range of conductors with different raw material cost exposures (for example, the aluminum or copper price) (Onstad, Obayashi, & Shamseddine, 2016; Pryor, Schlobohm, & Brownell, 2008).

To the best of our knowledge, this is the first research to address the topic of integrated operational and financial risk management in a multi-item setting, allowing us to make two contributions to the existing literature: (i) we analytically characterize the firms dynamic inventory and pricing decision, where prices are obtained as function of the jointly optimal stocking factor. Moreover, we adapt the results of Kouvelis et al. (2013) to provide a closed form solution to the firm's multi-contract hedging problem; (ii) we supplement our analytical results with a numerical study to develop further intuition for our findings. This leads to the following four main insights: (a) both the correlation of input costs and demand randomness between divisions can affect profits, utility, and optimal policy. For instance, higher correlations of both types decrease utility but raise inventory and prices of the unhedged firm under stockout-based substitutability of items; (b) also, the optimal financial hedge is affected by the correlation of both cost and demand randomness. The former influences the natural hedge arising from mutually offsetting input costs and, thus, influences the variance minimizing hedging portfolio. The latter has an impact on expected leftover inventory, which serves as an operational hedge that can substitute a long futures position; (c) dynamic pricing and financial hedging become increasingly important means for profit variance reduction as the correlation of input cost rises and the potential for a natural hedge disappears. However, besides variance reduction, operational flexibility in the form of dynamic pricing and financial hedging can contribute to mean profit due to real option value and inventory reduction, respectively; (d) financial hedging may lower physical inventory and prices in a multi-period setting with a monotonically increasing impact in the level of absolute risk aversion. This agrees with the multi-period results of Kouvelis et al. (2013) but is in contrast to the one-period findings of, e.g., Agrawal and Seshadri (2000) and X. Chen et al. (2007).

## 4.2 Literature review

This work relates to four streams of literature: first, the integration of operational and financial risk management under input cost volatility; second, commodity risk management in supply chains; third, joint pricing and inventory management under demand risk; and, fourth, inventory management in the presence of multiple items. An excellent review of the first two themes can be found in Kleindorfer (2009, 2010) and for a comprehensive literature survey of the third stream, we refer to X. Chen and Simchi-Levi (2012). A recent review of the fourth category is provided by Murray, Gosavi, and Talukdar (2012) and Zhang et al. (2014), who address the specific class of multi-product newsvendor problems allowing for interdependent demand due to substitution/cross-selling of items. We will proceed with a non-exhaustive review of these literature streams.

With respect to the integration of operational and financial hedging under stochastic costs, our work is closely related to Kouvelis et al. (2013), who are the first to study the integrated inventory and financial hedging decision in a multi-period setting with cost and demand risk as well as mean-variance inter-period utility. The sourcing decision spans both spot and forward procurement and the firm has access to futures and options as financial hedging instruments. In contrast to our study, they consider a fixed-price, single-item setting, which can clearly neither account for natural hedging opportunities arising from imperfectly correlated procurement costs nor demand interdependencies resulting from cross-selling effects. More recently, the studies of Xiao et al. (2015) and Schöne et al. (2016b) have incorporated dynamic pricing as part of an integrated hedging strategy. Xiao et al. (2015) study the joint dynamic pricing and inventory decision in a single-item periodic review model with dual sourcing (spot and forward) under cost and demand risk. They show that dynamic pricing can mitigate cost and demand risk and allows the risk-neutral firm to benefit from cost volatility in terms of expected profit. Schöne et al. (2016b) extend the work of Xiao et al. (2015) by incorporating risk aversion and multi-factor cost dynamics, showing that even a risk-averse firm can benefit from cost volatility under operational flexibility and financial hedging. However, besides using single-item models, both studies do not allow for financial options as hedging instruments, which have been shown to improve profit variance control when combined with a futures hedge (Kouvelis et al., 2013). Previous, more distantly related work on joint optimization of operational and financial hedging in a multi-period setting appears to be sparse, comprising the work of J. E. Smith and Nau (1995), X. Chen et al. (2007), and Zhu and Kapuscinski (2011), which is reviewed by Kouvelis et al. (2013). Single period models of financial hedging and inventory/capacity investment problems include the work of Gaur and Seshadri (2005), Oum, Oren, and Deng (2006), Ding et al. (2007), Bodily and Palacios (2007), Caldentey and Haugh (2009), Chod et al. (2010). More recently, Okyay, Karaesmen, and Ozekici (2015) consider the financial hedging decision, when the randomness in demand and supply is correlated with the financial market.

In the context of joint pricing and inventory management, the class of newsvendor models has attracted widespread attention as it forms the building block for multi-period inventory and capacity planning as well as supply contract design problems. We will first consider research focusing on the single-period, single-item case. For an excellent review of the early stage research in this branch, we refer to Federgruen and Heching (1999). Among the studies cited therein, Whitin (1955) is the first to study a price-setting newsvendor, where the probability distribution of demand depends on the unit selling price. This formulation is refined by Mills (1959), who explicitly specifies the mean demand as a function of the selling price. Petruzzi and Dada (1999) further explore the implications of additive and multiplicative demand and extend their results to a multi-period setting, where leftover inventory is carried to the subsequent period. In the absence of cross-selling, optimal prices in our model coincide with the single-period, additive demand case in Petruzzi and Dada (1999). Agrawal and Seshadri (2000) extend the work of Petruzzi and Dada (1999) by incorporating risk aversion in the form of a concave utility function of the retailer into the joint pricing and inventory decision. Y. Chen et al. (2009) introduce CVaR as the optimization criterion and Xu (2010) incorporate emergency procurement under a CVaR criterion. A variant of the price-setting newsvendor is studied by Yang, Shi, and Zhao (2011), where the

firm's objective is to maximize the probability of achieving both a profit and a revenue target.

Contrary to the single-period models discussed above, multi-period problems may either: (1) admit a myopic pricing and inventory decision that arise when individual periods are separable, for example, requiring that all leftover inventory is salvaged at the end of a given time period; or (2) treat the pricing decision as a dynamic variable that is altered in response to the realization of inventory levels or input costs over time. In the latter of these classes, the optimal price no longer reflects a single decision but is rather characterized as a decision rule, taking the dynamic state of the system as an input. Besides the studies of Xiao et al. (2015) and Schöne et al. (2016b) cited above, the literature on multi-period inventory models with dynamic pricing and stochastic demand include Federgruen and Heching (1999), the finite and infinite horizon models of X. Chen and Simchi-Levi (2004a, 2004b), Monahan, Petruzzi, and Zhao (2004), Bisi and Dada (2007), and X. Chen et al. (2007), who extend the work of Agrawal and Seshadri (2000) to a multi-period setting, including setup costs and financial hedging. Finally, we mention J. Chen and Bell (2009), who study the impact of customer returns on price and inventory decisions in a single and multi-period setting.

Similarly to the literature on single-item models, research on multi-product settings has become abundant. In our review, we will focus on studies that are related to this paper either through the presence of dynamic pricing or demand interdependencies between items. Within the literature stream on joint pricing and inventory management with multiple items, Aydin and Porteus (2008) provide a model with price-based substitution comparing a variety of consumer choice/demand models with multiplicative uncertainty but without capacity constraints. Dong, Kouvelis, and Tian (2009) study the dynamic pricing and inventory problem with substitute products under a multinomial logit model of consumer choice. We also refer the reader to their literature review on dynamic pricing in a single and multi-item context. J. Shi, Zhang, and Sha (2011) suggest a model with independent demand and additive uncertainty that allows for quantity discounts and multiple capacity constraints. More recently, Murray et al. (2012) integrate both cross-price elasticities and side constraints in a model that yields integer-valued product quantities. Similarly to the single period literature on target setting, C. Shi, Zhao, and Xia (2010) study the price and quantity decision for multiple different divisions that are incentivized by profit targets. Finally, our work is related to the literature on fixed price models featuring stockout-based substitution along the lines of Parlar and Goyal (1984), Khouja, Mehrez, and Rabinowitz (1996), and several later variants, which are reviewed by Zhang et al. (2014). It is also related to the multi-item newsvendor model with cross-selling of Zhang et al. (2014) and the literature on assortment planning with stockout-based substitution, including the work of S. A. Smith and Agrawal (2000), Honhon, Gaur, and Seshadri (2010), and the research cited therein.

### **4.3 Two-product model**

We consider the dynamic pricing, hedging, and inventory decision of a two-product firm facing volatile input costs and price-dependent, random end-product demand. The firm dynamically maximizes inter-period utility over a multi-period planning horizon, which explicitly takes cash flow correlations across periods into account (Alexander & Sobel, 2006; Kouvelis et al., 2013). The two products offered by the

firm are indexed  $i \in \{1, 2\}$  and may be viewed as substitutes or complements. The sequence of events is as follows. At the beginning of each time period  $t = 0, \dots, T$ , the firm observes a vector of current market prices for each of the commodity inputs  $\mathbf{c}_t = [c_{i,t}, c_{j,t}]$ , where  $j \neq i$ , and decides on the vector of raw material purchase quantities  $\mathbf{q}_t = [q_{i,t}, q_{j,t}]$ , end-product selling prices  $\mathbf{p}_t = [p_{i,t}, p_{j,t}]$ , and the hedging strategy vector  $\mathbf{h}_t = [h_{i,t}, h_{j,t}]$ , where  $h_{i,t} = [h_{i,t,1}^k, \dots, h_{i,t,N_i}^k]$  is itself a vector containing the quantities of all  $N_i$  hedging instruments traded in market  $i$  and  $k$  indicates the type of the  $n^{\text{th}}$  contract. Thus, to be clear, we denote by  $\mathbf{h}_t$  a vector containing the concentrated sequences of  $h_{i,t}$  and  $h_{j,t}$ . We use this notation for the concentration of two vectors throughout this paper. The types of hedging instruments available to the firm are further discussed below. Once the above decisions are made, the raw material in each division is processed into end-products according to a make-to-order strategy to meet the uncertain demand. Upon stockout of either product, unsatisfied demand may promote or decrease demand for the second item depending on the substitutability/complementarity of items. Demand that remains unmet after cross-selling is lost and excess inventory is carried over to the subsequent period. In the absence of transaction costs, the model can be rewritten as if the firm always started the next period with zero on-hand inventory. This allows us to treat the periodic raw material stocking decision as being equivalent to the firm's inventory level. We will use this property in our following exposition and further discuss the equivalence of both formulations below.

The random, standalone demand for product  $i$  in a given period is given by  $D_i = y_i(p_i) + \varepsilon_i$ , where  $y_i(p_i) = a_i - b_i p_i$ .  $a_i > 0$  and  $b_i > 0$  are constants and  $\varepsilon$  denotes a random shock, defined on the range  $[A, B]$  with a probability density function  $f(\cdot)$  and cumulative distribution function  $F(\cdot)$ . Further, we denote by  $\boldsymbol{\mu}_d$  the mean vector and by  $\boldsymbol{\Sigma}_d$  the covariance matrix associated with the two shock terms  $\varepsilon_i$ . As customers may view product  $i$  as a substitute or complement of product  $j$ , we introduce the term *effective demand* from Zhang et al. (2014), to account for possible cross-selling effects between the two divisions. Accordingly the effective demand for product  $i$  is given by  $D_i^e = D_i + \beta_{ji} (D_j - q_j)^+$ , where  $(D_j - q_j)^+$  is the shortage of product  $j$  and  $\beta_{ji} \in [-1, 1]$  represents the fraction of unserved demand in division  $j$  shifting to product  $i$ , which we refer to as the cross-selling coefficient. Here, it is implicitly assumed that the fraction of customers willing to shift to division  $i$  when facing a stockout in division  $j$  is independent of the relative product prices in each division. This assumption is relaxed in Section 4.5. If  $\beta_{ji} > 0$ , the two products are viewed as substitutes so that a shortage of product  $j$  leads to an increase in *effective demand* for product  $i$  as a fraction of customers  $\beta_{ji}$ , who would have preferred product  $j$  will switch to product  $i$ . If instead  $\beta_{ji} < 0$  products are complements so that a shortage in one of the products induces a decrease in demand for the second product. If,  $\beta_{ji} = 0$  we have no cross-selling effects. Note that the term cross-selling is used somewhat loosely throughout this chapter and does not refer to the common practice of selling additional products to an existing customer but rather pertains to the stockout-based demand interdependence between divisions and the resulting revenues. The effective

number of units sold by division  $i$  for the different demand/inventory scenarios is given by

$$s_i^e = \begin{cases} D_i & \text{if } D_i \leq q_i, D_j \leq q_j \\ D_i + [q_i - D_i, \beta_{ji} (D_j - q_j)]^- & \text{if } D_i \leq q_i, D_j > q_j \\ q_i & \text{otherwise.} \end{cases} \quad (4.1)$$

The first case in Equation 4.1 describes a situation, where the raw material inventory of each division is sufficient to satisfy individual demand and no cross-selling takes place. The overage and underage in division  $i$  is then given by  $x_i^{o,net} = q_i - s_i^e$ . In the second case, we have overage of product  $i$  and a shortage of product  $j$ . The cross-demand from product  $j$  changes effective sales of product  $i$  by the minimum of cross-demand and the excess inventory of product  $i$  that can be deployed to satisfy the additional demand arriving from division  $j$ . Clearly, if the products are complements, the cross-demand contribution from product  $j$  is negative. However, both effective demand and effective sales are bound from below by zero. If the excess inventory in division  $i$  is sufficient to cover the cross demand from division  $j$ , i.e.,  $q_i - D_i \geq \beta_{ji} (D_j - q_j)$ , the net overage of product  $i$  is given by  $x_i^{o,net} = q_i - [D_i + \beta_{ji} (D_j - q_j)]$ . If, instead, cross-demand exceeds the overage of product  $i$ , we have a net underage of  $x_i^{u,net} = -x_i^{o,net}$  units. The third case summarizes the possibilities that either  $D_i > q_i$  and  $D_j \leq q_j$  or  $D_i > q_i$  and  $D_j > q_j$ . Under both circumstances, sales of product  $i$  are bound from above by the inventory level  $q_i$ , however, in the former case a fraction  $\beta_{ji}$  of the excess demand for product  $i$  will shift to division  $j$  and is reflected in the effective sales of product  $j$  denoted by  $s_j^e$ . To conclude the discussion of demand and sales, note that the divisions are interconnected in two ways. First, the demand shocks  $\varepsilon$  affecting the standalone demand of each product may be correlated by means of a bivariate distribution. Intuitively, this dependency may reflect common demand drivers for both products such as the economic cycle. Second, the effective demand for both products is linked through the cross-selling coefficient to capture the substitutability or complementarity of products from a customer's perspective.

Let  $\mathbb{E} [\Pi_t(\mathbf{q}_t, \mathbf{p}_t, |\mathbf{h}_{t-1}, \mathbf{c}_t)]$  denote the firm's profit in period  $t$ , given the raw material cost vector  $\mathbf{c}_t$  and the previous period hedging decision  $\mathbf{h}_{t-1}$ , when the firm procures  $\mathbf{q}_t$  units of raw material from the spot market, sets end-product prices of  $\mathbf{p}_t$ , and enters into  $\mathbf{h}_t$  financial derivatives. As discussed above, we make the assumption of zero transaction costs associated with the procurement and sale of raw materials, which allows us to treat all excess inventory at the end of a given period as if it was sold back to the spot market and not carried over to the next period. As a result, the firm will start every period with zero on-hand inventory, which preserves the mathematical tractability of the model. This treatment of inventory is similar to Kouvelis et al. (2013). The end-of-period resale price of excess inventory is equivalent to the raw material price at the beginning of the subsequent period and is, therefore, uncertain at the time of procurement. This guarantees the equivalence of our formulation to the case where excess inventory is carried forward. In fact, both the resale value and the risk associated with the market value of inventory are captured by our model. The sales proceeds from returning excess inventory are counted as part of current period profit. Formally, the firm's divisional and total profits are given, respectively,

by

$$\mathbb{E} \left[ \pi_{i,t} \left( q_{i,t}, p_{i,t} | q_{j,t}, \mathbf{h}_{i,t-1}, c_{i,t} \right) \right] = p_{i,t} \mathbb{E}_{\mathcal{E}} \left[ s_{i,t}^e \right] - c_{i,t} q_{i,t} \quad (4.2)$$

$$+ \alpha \theta \mathbb{E}_c \left[ c_{i,t+1} | c_{i,t} \right] \mathbb{E}_{\mathcal{E}} \left[ x_{i,t}^{o,net} \right] - \gamma_i \mathbb{E}_{\mathcal{E}} \left[ x_{i,t}^{u,gross} \right] + \phi_{i,t} \left( \mathbf{h}_{i,t-1}, c_{i,t} \right),$$

$$\mathbb{E} \left[ \Pi_t \left( \mathbf{q}_t, \mathbf{p}_t, | \mathbf{h}_{t-1}, \mathbf{c}_t \right) \right] = \sum_{i=1}^2 \mathbb{E} \left[ \pi_{i,t} \left( q_{i,t}, p_{i,t} | q_{j,t}, \mathbf{h}_{i,t-1}, c_{i,t} \right) \right], \quad (4.3)$$

where  $\mathbb{E}_{\mathcal{E}}$  and  $\mathbb{E}_c$  denote the expectation operators with respect to the demand shocks and raw material cost. In the remainder of our exposition, we will omit the subscripts of the expectation operators. Similarly to Chapter 3, we use the assumption of a "partially complete" market, where the probability measure associated with  $\mathbb{E}$  is a combination of the risk-neutral probability measure on commodity spot prices and the firm's subjective probability measure on demands. For a discussion of the influence of risk-neutral vis-à-vis physical commodity price expectations in a PRM setting, we refer to Kouvelis et al. (2013). The term  $\mathbb{E}[s_i^e]$  reflects the expected revenue of division  $i$  and  $c_{i,t}q_{i,t}$  refers to the total raw material procurement cost of material  $i$  in period  $t$ .  $\alpha = e^{-r\Delta}$  is the period discount factor and  $\Delta$  denotes the length of a time step.  $\theta \in (0, 1)$  captures the economic holding cost of excess inventory. The expected salvage value per unit of overage is given by the expected raw material price at the end of the current period, which is equivalent to the price at the start of the next period:  $\mathbb{E}[c_{i,t+1} | c_{i,t}]$ . As discussed above,  $\mathbb{E}[x_{i,t}^{o,net}]$  denotes the expected quantity of excess inventory net of units cross-sold to the second division.  $\gamma_i \mathbb{E}[x_{i,t}^{u,gross}]$  captures the cost of underage, where  $\gamma_i > 0$  is a per unit penalty cost. The superscript *net* and *gross* indicate whether or not cross-selling is accounted for. Finally,  $\phi_{i,t}(\mathbf{h}_{i,t-1}, c_{i,t})$  is the cash flow from all financial hedging instruments traded in market  $i$ . Here, we generally follow the notation of Kouvelis et al. (2013), which is extended where necessary to capture the presence of multiple markets in our model. As discussed above, we denote by  $h_{i,t-1,n}^k \in \mathbb{R}$  the quantity of hedge  $k = \{f, c, p\}$  traded in market  $i$  at the beginning of period  $t-1$ , where  $n$  is an index to distinguish between individual trades if multiple instruments of the same type  $k$  are used simultaneously in a given market. With respect to the three hedging instruments available to the firm,  $f$  denotes a futures contract and  $c$  and  $p$  refer to a call and put option, respectively.  $\chi_{k,n}(c_{i,t})$  is the payoff function for contract type  $k$ . Note that strike prices traded in each market can be different, as one might indicate by adding an index  $i$  to the payoff functions and strike prices, however, we omit this additional notation for simplicity. The net payoffs for each contract type  $k$  at time  $t$  are:  $\chi_{f,n}(c_{i,t+1}) = c_{i,t+1} - \mathbb{E}[c_{i,t+1} | c_{i,t}]$ ,  $\chi_{c,n}(c_{i,t+1}) = (c_{i,t+1} - K_{c,n})^+ - \mathbb{E}[(c_{i,t+1} - K_{c,n})^+ | c_{i,t}]$ , and  $\chi_{p,n}(c_{i,t+1}) = (K_{p,n} - c_{i,t+1})^+ - \mathbb{E}[(K_{p,n} - c_{i,t+1})^+ | c_{i,t}]$ . All contracts have a maturity of one period. If the firm trades multiple derivatives in a given market, the payoff from financial hedging becomes the sum of cash flows from all individual trades. Thus, as mentioned above, for market  $i$  we have  $\phi_{i,t}(\mathbf{h}_{i,t-1}, c_{i,t}) = h_{i,t-1,1}^f \chi_{f,n}(c_{i,t}) + \sum_{n=1}^{N_{i,c}} h_{i,t-1,n}^c \chi_{c,n}(c_{i,t}) + \sum_{n=1}^{N_{i,p}} h_{i,t-1,n}^p \chi_{p,n}(c_{i,t})$  and  $\Phi_t(\mathbf{h}_{t-1}, \mathbf{c}_t) = \phi_{i,t}(\mathbf{h}_{i,t-1}, c_{i,t}) + \phi_{j,t}(\mathbf{h}_{j,t-1}, c_{j,t})$ . By  $N_{i,c}$  and  $N_{i,p}$ , we denote the number of call and put options traded in market  $i$ , respectively, and  $N_i = 1 + N_{i,c} + N_{i,p}$  is the total number of derivatives



used in market  $i$ . Note that any hedging strategy of the firm will always include the futures contract (Kouvelis et al., 2013).

A convenient notation for the profit function in Equation 4.2 follows from Petruzzi and Dada (1999) and the earlier work of Ernst (1970) and Thowsen (1975). Accordingly, let  $z_i = q_i - y_i(p_i)$ , implying that  $D_i - q_i = \varepsilon_i - z_i$ . If  $\mathbb{E}[\varepsilon_i] = 0$ ,  $z_i$  can be interpreted as the amount of inventory that is held in excess of the expected number of units sold, for a given pricing decision  $p_i$ , which is also referred to as safety stock. It follows that the optimal stocking decision for product  $i$  is given by  $q_i^* = y_i(p_i^*) + z_i^*$ , i.e., the expected number of units sold at the optimal price  $p_i^*$  plus a level of safety stock for protection against demand volatility. We provide the profit function obtained after this substitution step,  $\pi_{i,t}(z_{i,t}, p_{i,t}, h_{i,t} | z_{j,t}, h_{i,t-1}, c_{i,t})$ , along with further explanation in Section B.6 of the Appendix. Similarly to Petruzzi and Dada (1999), we can next rewrite the expected operating profit for product  $i$ , as the sum of a certainty equivalent profit  $P$  (Mills, 1959) and a risky component, referred to as the loss function  $L$  (Silver & Peterson, 1985). While  $P$  is unaffected by demand risk, it remains subject to the uncertain input costs  $c$ . The total periodical profit of division  $i$  ( $\mathbb{E}[\pi_{i,t}(\cdot)]$ ), additionally, includes a third component, which accounts for the cash flow from the financial hedge traded in the previous period. Thus, we have

$$\mathbb{E}[\pi_{i,t}(z_{i,t}, p_{i,t} | z_{j,t}, h_{i,t-1}, c_{i,t})] = P(p_{i,t} | c_{i,t}) - L(z_{i,t}, p_{i,t} | z_{j,t}, c_{i,t}) + \phi_{i,t}(h_{i,t-1}, c_{i,t}), \quad (4.4)$$

where

$$P(p_{i,t} | c_{i,t}) = (p_{i,t} - c_{i,t}) [y_i(p_{i,t}) + \mu_{d,i}] \quad (4.5)$$

and

$$\begin{aligned} L(z_{i,t}, p_{i,t} | z_{j,t}, c_{i,t}) &= [c_{i,t} - \alpha \theta \mathbb{E}[c_{i,t+1} | c_{i,t}]] (\Lambda_{11}^i + \Lambda_{10}^i) + (p_{i,t} + \gamma_i - c_{i,t}) (\Lambda_{00}^i + \Lambda_{01}^i) \\ &\quad - [p_{i,t} - \alpha \theta \mathbb{E}[c_{i,t+1} | c_{i,t}]] X_{ij} \end{aligned} \quad (4.6)$$

Here, we redefine the cross-selling quantity using  $\Lambda$ -notation as  $X_{ij} = (\Lambda_{10}^i, \beta_{ji} \Lambda_{01}^j)^-$ , which is the expected number of units cross-sold from division  $i$  to  $j$ , representing the minimum of cross-demand and available cross-supply or, in other words, the minimum of overage in product  $i$  and cross-demand for product  $i$  from division  $j$ , when there is a shortage in product  $j$ . Similarly,  $X_{ji} = (\Lambda_{10}^j, \beta_{ij} \Lambda_{01}^i)^-$  is the expected number of units cross-sold from division  $j$  to  $i$ , when product  $i$  is out of stock. Before turning to an explanation of  $\Lambda$  in our notation of the loss function, we will provide an intuitive interpretation of  $L$ . Similarly to the standard newsvendor model, the first two terms in Equation 4.6 capture the overage cost of excess raw material inventory, net of its expected salvage value, and the underage costs associated with expected shortages of product  $i$ , respectively. These are unaffected by interdependencies between the two products arising from cross-selling. The third term captures the unit revenue of cross-selling net of the raw material salvage value that is given up when the resource is used in the production process and marketed as an end-product. Cross-selling revenue is interpreted as a negative loss to the firm.



$$x_i^{o,net} = x_i^{o,gross} - X_{ij} \text{ and } x_i^{u,net} = x_i^{u,gross} - X_{ji}.$$

Next, we can simplify the firm's profit function for the final period of the planning horizon  $t = T + 1$ . Facing no demand, the firm neither purchases raw material ( $\mathbf{q}_{T+1} \equiv \mathbf{0}$ ) nor hedges or sets end-product prices. Thus, the profit earned by each division reduces to the hedging payoff resulting from the derivatives traded in the previous period  $T$ . The total profit is then given by

$$\Pi_{T+1}(\mathbf{q}_{T+1} \equiv \mathbf{0} | \mathbf{h}_T, \mathbf{c}_{T+1}) = \sum_{i=1}^2 \phi_{i,T+1}(h_{i,T}, c_{i,T+1}) = \Phi_{T+1}(\mathbf{h}_T, \mathbf{c}_{T+1}). \quad (4.7)$$

As discussed in Section 3.3 of the previous chapter, firms may have various incentives for risk reduction, which are indirectly captured by our model through a corporate utility function. In particular, we follow Kouvelis et al. (2013) and adopt an inter-period mean-variance utility function, which accurately captures cash flow correlations across periods. The use of a total risk measure  $V(\cdot)$ , as opposed to measures of systematic risk, is justified by Hodder and Dincer (1986), who argue that total risk is typically the main concern of managers. The firm's, utility function, conditional on  $\mathbf{c}_t$ , is given by

$$\mathbb{E}[U_t(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)] = \mathbb{E}_t \left[ \sum_{m=t}^{T+1} \alpha^{(m-t)} \Pi_m(\cdot) \right] - \frac{\lambda}{2} V_t \left[ \sum_{m=t}^{T+1} \alpha^{(m-t)} \Pi_m(\cdot) \right], \quad (4.8)$$

where  $\Pi_t$  follows from Equations 4.3 and 4.4.

#### 4.4 Optimal policy

In this section, we characterize the optimal pricing, inventory, and hedging policy for the multi-product firm in a given period  $t$  and discuss an extension of stockout-based cross-demand formulation that accounts for price differences between the two items.

Our goal is to maximize the expected present value of utility with respect to the decision vectors  $\mathbf{z}_t$ ,  $\mathbf{p}_t$ , and  $\mathbf{h}_t$ :

$$\max_{\mathbf{z}_t \in [A, B], \mathbf{p}_t \in \mathbb{R}_{\geq 0}, \mathbf{h}_t \in \mathbb{R}} \mathbb{E}[U_t(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]. \quad (4.9)$$

In order to be able to solve the firm's optimization problem via dynamic programming, it must be accounted for the failure of the iterated expectations property for mean-variance objectives and the resulting time-inconsistency of optimal policy. Using the law of total variance, we can rewrite the firm's conditional utility function from Equation 4.8 in recursive form, which ensures the applicability of dynamic programming and time-consistency (Basak & Chabakauri, 2010). The recursive utility function, conditional on  $\mathbf{c}_t$ , is then given by

$$U_t(\cdot) = \mathbb{E}_t[\Pi_t(\cdot) + \alpha U_{t+1}(\cdot)] - \frac{\lambda}{2} V_t \left[ \mathbb{E}_t[\Pi_t(\cdot)] + \mathbb{E}_{t+1} \left[ \sum_{m=t+1}^{T+1} \alpha^{(m-t)} \Pi_m(\cdot) \right] \right]. \quad (4.10)$$

Note that the current period profit  $\Pi_t$  is part of the expectation and variance term due to the uncertain salvage value of excess inventory that is received at the end of a given period  $t$ . Since the end of period  $t$

coincides with the beginning of period  $t + 1$ , it is only of notational not economical interest whether the salvage value is assigned to the current or next period.

As the firm faces no risks in the final period, Equation 4.10 simplifies to Equation 4.7 at time  $T + 1$ . Note also, while all decision variables are chosen simultaneously, the effective time of  $\mathbf{h}_t$  is in the subsequent period  $t + 1$ , when the hedging contract matures. We proceed with the optimal pricing decision summarized by Lemma 1.

**Lemma 1.** *The optimal pricing decision for division  $i$ , denoted by  $p_{i,t}^*$ , is determined uniquely as a function of the jointly optimal stocking factor  $\mathbf{z}_t$ :*

$$p_{i,t}^*(\mathbf{z}_t^*|c_{i,t}) = p_{i,t}^0(c_{i,t}) - \frac{\Lambda_{00}^i + \Lambda_{01}^i - X_{ij}}{2b_i},$$

where  $p_{i,t}^0(c_{i,t}) = (a_i + b_i c_{i,t} + \mu_{d,i}) / 2b_i$  maximizes the certainty equivalent profit  $P(p_{i,t}|c_{i,t})$  provided by Equation 4.5.

*Proof.* See Appendix B.2.

Petruzzi and Dada (1999) refer to  $p^0$  as the optimal *riskless price*. They also provide a detailed discussion of the relationship between the *riskless price* and the optimal price under demand risk in the context of both additive and multiplicative demand. In the absence of cross-selling, that is, when  $\beta_{ij} = 0$ , the pricing decision of the multi-product firm simplifies to the single-product solution given in Petruzzi and Dada (1999). Note that this holds true even in the presence of correlated demand shocks, i.e., when  $\text{Cov}(\varepsilon_i, \varepsilon_j) \neq 0$ . In particular, if  $\beta_{ji} = 0$ ,  $X_{ij} = 0$  and the quantity  $\Lambda_{00}^i + \Lambda_{01}^i$  is independent of  $z_j$  as the marginal density of  $\varepsilon_i$  is independent of  $\varepsilon_j$  under bivariate normal demand risk.

Next, we establish that the firm's utility, given the vector of optimal prices determined by Lemma 1,  $\mathbb{E}[U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]$ , is unimodal with respect to the stocking decision  $\mathbf{z}_t$ . This is summarized by Proposition 1.

**Proposition 1.** *The jointly optimal inventory and pricing policy of division  $i$  in a multi-product firm is to stock  $q_{i,t}^* = y_i(p_{i,t}^*) + z_{i,t}^*$  units of raw material and set prices  $p_{i,t}^*$  according to Lemma 1. Moreover, there exists a  $\bar{\gamma}_i$  such that under the condition  $\gamma_i > \bar{\gamma}_i$ ,  $\mathbf{z}_t^* = [z_{i,t}^*, z_{j,t}^*]$  is the unique solution to the following system of first order conditions on the interval  $[A, B]$ , for  $i = 1, 2$  and  $j \neq i$*

$$\begin{aligned} \frac{\partial \mathbb{E}[U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}} &: (p_{i,t}^* + \gamma_i - c_{i,t})(\Gamma_{00} + \Gamma_{01}^i) - (c_{i,t} - \alpha \theta \mathbb{E}[c_{i,t+1}|c_{i,t}]) (\Gamma_{11} + \Gamma_{10}^i) \\ &+ \frac{\partial X_{ij}}{\partial z_{i,t}} p_{i,t}^* + \frac{\partial X_{ji}}{\partial z_{i,t}} p_{i,t}^* - \alpha \mathbb{E}[\Omega_3^i | \mathbf{c}_t] \\ &- \lambda \text{Cov}(\Omega_1^i, \Omega_2^i | \mathbf{c}_t) = 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \Omega_1^i &= \sum_{m=t+1}^{T+1} \alpha^{(m-t-1)} \Pi_m \left( \mathbf{z}_m^*, \mathbf{p}_m^*, \mathbf{h}_{m-1}^*, \mathbf{c}_m \right) \\ &+ \theta \left[ c_{i,t+1} \left( \Lambda_{11}^i + \Lambda_{10}^i - X_{ij} \right) + c_{j,t+1} \left( \Lambda_{11}^j + \Lambda_{10}^j - X_{ji} \right) \right], \end{aligned} \quad (4.12)$$

$$\begin{aligned} \Omega_2^i &= \sum_{n=1}^{N_i} \frac{\partial h_{i,t,n}^k}{\partial z_{i,t}} \chi_{k,n} \left( c_{i,t+1} \right) + \sum_{n=1}^{N_j} \frac{\partial h_{j,t,n}^k}{\partial z_{i,t}} \chi_{k,n} \left( c_{j,t+1} \right) \\ &+ \theta c_{i,t+1} \left( \Gamma_{11} + \Gamma_{10}^i \right) - \Omega_3^i, \end{aligned} \quad (4.13)$$

$$\Omega_3^i = \theta \left[ c_{i,t+1} \frac{\partial X_{ij}}{\partial z_{i,t}} + c_{j,t+1} \frac{\partial X_{ji}}{\partial z_{i,t}} \right], \quad (4.14)$$

and  $\partial X_{ij} / \partial z_{i,t}$  and  $\partial X_{ji} / \partial z_{i,t}$  are given by Lemma 3 in Appendix B.2. The hedging related derivatives  $\partial h_{i,t,n}^k / \partial z_{i,t}$  and  $\partial h_{j,t,n}^k / \partial z_{i,t}$  are provided by Lemma 4 of the above appendix.

*Proof.* See Appendix B.3.

We will proceed by first giving an interpretation of the inventory policy given by Proposition 1 and then discuss the uniqueness of  $\mathbf{z}_t^*$  and the relevance of  $\gamma$  in this context.

Equation 4.11 can be interpreted in three parts. The first line is similar to the standard newsvendor solution consisting of the expected overage and underage costs, where the unit salvage value is given by the conditional expectation of the discounted raw material price after accounting for storage costs  $\alpha \theta \mathbb{E} [c_{i,t+1} | c_{i,t}]$ . In the absence of cross-selling and risk aversion ( $\beta_{ij} = 0, \lambda = 0$ ), the remainder of Equation 4.11 is zero. The second line captures the sensitivity of cross-selling revenue with respect to  $z_{i,t}$ , net of the corresponding change in salvage value (represented by  $\Omega_3^i$ ) that occurs, when excess raw material is utilized to serve cross-demand instead of being sold back to the market. The third term is relevant for the risk-averse firm and accounts for the variance-related sensitivity of utility to changes in  $z_{i,t}$ . This term can be both positive or negative, depending on the influence of physical inventory on profit variance and, thus, utility. For instance, in the last time period of the planning horizon  $t = T$ , there is no raw material procurement in the subsequent period  $t = T + 1$ . Thus, an increase in  $z_{i,t}$  will create higher expected leftovers and greater variance of the corresponding salvage value, which exerts a negatively influence on utility. However, in previous time periods  $t < T$ , excess inventory can serve as an operational hedge against volatile future procurement costs. Here, expected profit variance can be decreasing in the amount of physical inventory on hand, resulting in a negative relationship between  $z_{i,t}$  and expected profit variance. Given this intuition, it is never optimal for the risk-neutral firm to choose a negative safety stock at time  $t < T$ ; however, for a sufficiently high degree of absolute risk aversion, the reduction in expected leftovers and profit variance can render  $\mathbf{z}_t^* < 0$ , particularly at the end of the horizon, when there is no procurement in the next period.

As suggested by Proposition 1, the existence of a unique optimal inventory policy  $\mathbf{z}^*$  solving Equation 4.11, requires suitable conditions on  $\gamma_i$ . If  $\beta_{ij} > 0$  and  $\gamma_i < \bar{\gamma}_i$ , where  $\bar{\gamma}_i$  denotes a critical value, Equation 4.11 may have two, practically undesirable solutions. In particular, it can be optimal for the firm to pursue a highly asymmetric procurement strategy, inducing an expected stockout in the less profitable division, while satisfying the resulting price inelastic excess demand with cross-selling from the second division at a potentially higher price. It may also be the case that only the more profitable product is offered. To rule out these possibilities, we may impose the condition that the direct costs associated with an induced stockout in division  $i$  are greater than or equal to the resulting gross cross-selling profit of division  $j$ . That is  $(p_i + \bar{\gamma}_i - c_i) \Lambda_{01}^i \geq [p_j - \alpha\theta\mathbb{E}[c_{j,t+1}|\mathbf{c}_t]] X_{ji} \Leftrightarrow \gamma_i \geq \bar{\gamma}_i = \beta_{ij} [p_j - \alpha\theta\mathbb{E}[c_{j,t+1}|\mathbf{c}_t]] - (p_i - c_i)$ . In the remainder of this study,  $\gamma_i$  is set sufficiently high to guarantee the uniqueness of  $\mathbf{z}_t^*$ . Thus, in our main analysis, the constraint on  $\gamma$  is non-binding and  $\gamma$  can be treated as a parameter. This practice has no effect on the general conclusions of this study. For the purpose of completeness, we provide the optimal operating policy under a binding constraint on  $\gamma_i$  in Corollary 2 of Proposition 1 in Appendix B.4.

Next, we address the question when cross-selling is utility enhancing for the firm. While cross-selling can generate additional revenues, the consumption of raw material for this purpose is not costless. On the one hand, the firm forgoes a salvage value of each unit of raw material that is used up to serve cross-demand. On the other hand, there is an economic value associated with excess inventory that results from its role as an operational hedge against future cost uncertainty. It follows that cross-selling is optimal whenever resulting revenues exceed corresponding costs. We summarize the optimality of cross-selling in Lemma 2.

**Lemma 2.** *It is utility enhancing for the multi-product firm to serve an additional unit of cross-demand in division  $j$  with excess stock  $i$  in a given period  $t < T + 1$ , when the generated revenue exceeds the associated economic costs, i.e., when*

$$p_i - \alpha\theta\mathbb{E}[c_{i,t+1}|\mathbf{c}_t] - \lambda\theta \text{Cov}(c_{i,t+1}, \Omega_1^i|\mathbf{c}_t) > 0, \quad (4.15)$$

where  $\Omega_1^i$  is a decreasing function of  $X_{ij}$ , given by Proposition 1. Hence, the absolute value of  $\text{Cov}(\cdot)$  is decreasing in  $X_{ij}$ . The above condition is always satisfied for the risk-neutral firm and for the risk-averse firm when a single- or multi-contract hedge can be implemented. In the absence of financial hedging, cross-selling may decrease utility of the risk-averse firm if  $-\lambda\theta \text{Cov}(c_{i,t+1}, \Omega_1^i|\mathbf{c}_t) > p_{i,t} - \theta c_{i,t} > 0$ . Since  $\text{Cov}(c_{i,t+1}, \Omega_1^i|\mathbf{c}_t) > 0$  in the final time period  $T$ , this can be true at any time  $t < T$ .

*Proof.* See Appendix B.2.

As stated by this lemma, cross-selling is generally utility enhancing for the hedged firm but the economic implications of resulting inventory reduction have to be understood if a financial hedge is unavailable. In fact, when excess inventory is particularly valuable as an operational hedge against future procurement cost uncertainty, the risk-averse firm should rather hold and trade excess raw material for a salvage value at the end of the current period. Since there is no procurement in the final time period  $t = T + 1$ , excess inventory at time  $T$  cannot be used to offset future purchase cost volatility and, thus, loses its value as

an operational hedge. Instead, reducing excess inventory by means of cross-selling lowers the salvage value and profit variance, resulting in a positive effect on utility. For the same reason, cross-selling is always value enhancing in a single-period setting and for a risk-neutral firm.

Having discussed the optimal pricing and inventory policy, which we refer to as the operational decisions of the firm, we next analyze the optimal corresponding financial hedging strategy. Here, the firm's goal is to minimize the expected variance of future firm-wide profits generated by both divisions. In contrast to a single-product firm, facing only one risky input cost, the multi-product firm's profits are subject to volatility in each market and the possibility of correlated risks. Thus, the optimal hedging policy is not determined as the sum of individual decisions made by each division but rather constitutes a firm-wide integrated strategy that takes cross-divisional interdependencies into account and ultimately addresses the firm's net-exposure to future cost volatility. In our model, risks from cost volatility can arise in two forms. First, the value of leftover raw material in each division is a linear function of the underlying raw material price and fluctuates accordingly. Second, in time periods  $t < T$ , when there is raw material purchasing in the subsequent period, future profits generally exhibit a jointly convex, decreasing shape with respect to raw material costs. The former, linear risk can be perfectly hedged by a short futures position matching the amount of expected leftovers in size. To hedge the latter, possibly non-linear risk, a theoretically optimal hedge consisting of futures contracts alone can only be realized with continuous rebalancing of the hedging position, exploiting the local linearity of profits. In the absence of frequent rebalancing, an optimal hedge should also comprise call and put options with a continuum of strike prices. Finally, in constructing the optimal financial hedge, it should be recognized that the market value of leftover raw material inventory does not only resemble a source of risk but also serves as an operational hedge with a payoff profile equivalent to a long futures position that can mitigate future procurement cost risk. We analyze both a single-contract hedge, where the firm trades only one futures contract in each market and a multi-contract hedge, where the firm can additionally enter into  $N_c^i$  call and  $N_p^i$  put options. Without loss of generality, we require the assumption that these financial contracts are not replicating each other. Our solution and notation for the optimal single- and multi-contract hedging policy are closely related to the multi-contract hedge given in Proposition 5 of Kouvelis et al. (2013), which we extend to the multi-market case. We summarize the optimal hedging policy of the multi-product firm in Proposition 2.

**Proposition 2.** *The jointly optimal financial hedging policy of the multi-product firm at time  $t < T$  is given by*

$$\mathbf{h}_t^* = [\Sigma_t(\mathbf{c}_t)]^{-1} \Psi_t(\mathbf{z}_t^*, \mathbf{c}_t) \quad (4.16)$$

where  $\mathbf{h}_t = [h_{i,t}, h_{j,t}]$  and  $h_{i,t} = [h_{i,t,1}^f, h_{i,t,1}^c, \dots, h_{i,t,N_{i,c}}^c, h_{i,t,1}^p, \dots, h_{i,t,N_{i,p}}^p]$  is the quantity vector associated with instruments traded in market  $i$ .  $\Sigma_t(\mathbf{c}_t)$  is the covariance matrix, conditional on  $\mathbf{c}_t$ , of  $c_{i,t+1}$ ,  $(c_{i,t+1} - K_{c,1})^+$ ,  $\dots$ ,  $(c_{i,t+1} - K_{c,N_{i,c}})^+$ ,  $(K_{p,1} - c_{i,t+1})^+$ ,  $\dots$ ,  $(K_{p,N_{i,p}} - c_{i,t+1})^+ \forall i$ , and  $\Psi_t(\mathbf{z}_t^*, \mathbf{c}_t) = [\Psi_t^i(\mathbf{z}_t^*, \mathbf{c}_t), \Psi_t^j(\mathbf{z}_t^*, \mathbf{c}_t)]$ , where  $\Psi^i(\mathbf{z}_t^*, \mathbf{c}_t) = [\psi_{f,t,1}^i, \psi_{c,t,1}^i, \dots, \psi_{c,t,N_{i,c}}^i, \psi_{p,t,1}^i, \dots, \psi_{p,t,N_{i,p}}^i]$  and, for instance,  $\psi_{c,t,n}^i = -\theta \text{Cov}(X_{c,n}(c_{i,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t) - \text{Cov}(X_{c,n}(c_{i,t+1}), \bar{\Pi}_{t+1} | \mathbf{c}_t)$ .

$x_{i,t}^{o,net} = (\Lambda_{11}^i + \Lambda_{10}^i - X_{ij})$  is the net overage of division  $i$  and  $\bar{\Pi}_{t+1}$  denotes the present value of expected future profits  $\Pi_m(\cdot)$  for  $m = t+1, \dots, T+1$  under optimal policy decisions  $\mathbf{z}_m^*$ ,  $\mathbf{p}_m^*$ , and  $\mathbf{h}_{m-1}^*$ . In the final period  $T$ , the hedging portfolio simplifies to a short futures position in each market:  $\mathbf{h}_{i,T}^* = -\theta x_{i,T}^{o,net}$ .

*Proof.* See Appendix B.3.

Similarly to Kouvelis et al. (2013), we find that the optimal financial hedging strategy in period  $T$  consists only of a short futures position in each market with a volume corresponding the respective amount of expected excess inventory:  $x_{i,T}^{o,net}$ . For periods  $t < T$ , the optimal hedge includes a futures contract and several call and put options in each market, however, the best single-contract hedge is a futures contract. In our model, a multi-contract hedge leads to the same optimal, myopic inventory and pricing policy as a single-contract futures hedge. This is because once a futures contract is traded, inventory loses its role as an operational hedge, which is fully replicated by the futures contract. Thus, allowing for additional options will not alter the stocking and pricing decision. In the absence of financial hedging, the inventory and, hence, the pricing decision are not myopic as they depend on the variance of future profits, which in turn depends on all future optimal decisions.

## 4.5 Price elastic cross-demand

In the above analysis, the stockout-based cross-demand is independent of the relative prices of each product and a fixed fraction of consumers  $\beta_{ij}$  was assumed to shift demand to item  $j \neq i$ , when item  $i$  is unavailable. This leads to cross-demand of  $D_{ij} = \beta_{ij}\Lambda_{01}^i$  units. In line with the assumption of price sensitive customers, however, cross-demand may be alternatively modeled as a decreasing function of the price difference between the two alternatives:  $\beta_{ij}\Lambda_{01}^i - \eta(p_j - p_i)$ . Here,  $\beta_{ij} > 0$  can be interpreted as the fraction of unserved customers for whom product  $j$  resembles a generally feasible alternative. If both products have the same price, all of these customers shift from  $i$  to  $j$ . However, when the substitute product is more expensive ( $p_j > p_i$ ), cross-demand is decreasing in the price difference with a slope  $\eta > 0$ . On the contrary, if  $p_j < p_i$  some additional customers (up to  $(1 - \beta_{ij})\Lambda_{01}^i$ ), for whom the product  $j$  is generally not an attractive alternative, may compromise and still purchase the substitute  $j$  due to its lower price compared to the unavailable primary choice  $i$ . This behavior is consistent with empirical evidence of, for example, Gruen, Corsten, and Bharadwaj (2002), who conduct a worldwide study of consumer responses to stockouts in a retail context. Under the suggested demand model, the expected number of units cross-sold from division  $j$  to  $i$  is now  $X_{ji}^p = [\Lambda_{10}^j, D_{ij}^p]^-$ , reflecting the minimum of overage in product  $j$  that is available for sale and price elastic cross-demand for product  $j$  from division  $i$ , denoted by  $D_{ij}^p$ . To ensure that  $D_{ij}^p$  is economically meaningful, we impose two conditions on cross-demand. First,  $D_{ij}^p$  is bound from above by the expected stockout quantity in a respective division ( $D_{ij}^p \leq \Lambda_{01}^i$ ). This ensures that substitution in our model remains stockout-based and, therefore, that at most every unserved customer will purchase the substitute product. Second, the price differential in  $D_{ij}^p$  can drive cross-demand to zero but cannot change the sign thereof, i.e.,  $\eta(p_j - p_i) \leq \beta_{ij}\Lambda_{01}^i$  for  $i = 1, 2$  and  $j \neq i$ . Taking these conditions into account, we obtain the following piecewise definition for price



elastic cross-demand

$$D_{ij}^p = \begin{cases} \beta_{ij}\Lambda_{01}^i - (\beta_{ij} - 1)\Lambda_{01}^i = \Lambda_{01}^i & \text{if } \eta(p_j - p_i) < (\beta_{ij} - 1) \\ \beta_{ij}\Lambda_{01}^i - \eta(p_j - p_i) & \text{if } (\beta_{ij} - 1) \leq \eta(p_j - p_i) \leq \beta_{ij}\Lambda_{01}^i \\ \beta_{ij}\Lambda_{01}^i - \beta_{ij}\Lambda_{01}^i = 0 & \text{if } \eta(p_j - p_i) > \beta_{ij}\Lambda_{01}^i. \end{cases} \quad (4.17)$$

Generally, if  $p_{i,t}^* \neq p_{j,t}^*$ , the firm's optimal pricing, stocking, and hedging decision deviate from Proposition 1 under price elastic cross-demand. An important difference to the price inelastic base case model arises from the new dependence of excess raw material on the pricing decision, which creates a link between prices, operational hedging, and profit variance of the unhedged firm. We summarize the firm's best operating policy under price elastic cross-demand in the following proposition.

**Proposition 3.** *In the presence of price elastic cross-demand,  $D_{ij}^p$ , the firm's operating policy is as follows. The pricing decision is implicitly given by the subsequent system of first order conditions for  $i = 1, 2$  and  $j \neq i$*

$$\begin{aligned} \frac{\partial \mathbb{E} [U_t^p(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial p_{i,t}} : a_i + \mu_{d,i} - b_i(2p_{i,t} - c_{i,t}) - (\Lambda_{00}^i + \Lambda_{01}^i - X_{ij}^p) \\ + \frac{\partial X_{ij}^p}{\partial p_{i,t}} p_{i,t} + \frac{\partial X_{ji}^p}{\partial p_{i,t}} p_{j,t} - \alpha \mathbb{E} [\Omega_5^i | \mathbf{c}_t] + \lambda \text{Cov}(\Omega_1^i, \Omega_5^i | \mathbf{c}_t) = 0, \end{aligned} \quad (4.18)$$

where  $\Omega_1^i$  is given as part of Proposition 1 with  $X_{ij}$  being replaced by  $X_{ij}^p$  and

$$\Omega_5^i = \theta \left( \frac{\partial X_{ij}^p}{\partial p_{i,t}} c_{i,t+1} + \frac{\partial X_{ji}^p}{\partial p_{i,t}} c_{j,t+1} \right). \quad (4.19)$$

$U_t^p(\cdot)$  denotes the firm's utility function (from Equation 4.8) under price elastic cross-demand, where  $D_{ij}$  is replaced by  $D_{ij}^p$ . The stocking and hedging decision,  $\mathbf{z}_t^*$  and  $\mathbf{h}_t^*$ , are determined in analogy to Proposition 1 and Proposition 2, respectively, where  $X_{ij}$  is replaced by  $X_{ij}^p$ . The partial derivatives of  $X_{ij}^p$  with respect to  $p$  and  $z$  are provided by Lemma 5 in Appendix B.2. To guarantee the uniqueness of  $\mathbf{p}_t^*$ , we require  $\gamma_i > \hat{\gamma}_i$ . Both the critical value  $\hat{\gamma}_i$  and the optimal policy under a binding constraint on  $\hat{\gamma}_i$ , where  $\gamma_i = \hat{\gamma}_i$ , are provided by Corollary 3 in Appendix B.4.

*Proof.* See Appendix B.3.

It is not straightforward to solve the firm's pricing problem explicitly for  $p_{i,t}$  and  $p_{j,t}$  if  $\eta \neq 0$  due to the above discussed interdependence between prices, excess inventory, and profit variance. If the firm is risk-neutral or a futures hedge can be implemented, the variance related part of Equation 4.18 can be disregarded for the pricing decision. We provide the resulting optimal pricing policy in the following corollary of Proposition 3.

**Corollary 1.** *The firm's optimal pricing policy under price elastic cross-demand  $D_{ij}^p$  in the presence of either risk-neutrality ( $\lambda = 0$ ) or any optimal hedging strategy involving the futures contract (Proposition 2) is given by*

$$p_{i,t}^* (\mathbf{z}_t^* | c_{i,t}) = p_{i,t}^0 (c_{i,t}) \frac{2b_i}{2b_i + \eta} - \frac{\Lambda_{00}^i + \Lambda_{01}^i - \beta_{ji} \Lambda_{01}^j - \alpha \mathbb{E} [\Omega_5^i | \mathbf{c}_t]}{2b_i + \eta}, \quad (4.20)$$

where  $\Omega_5^i$  is given as part of Proposition 4.18.

Clearly, if the price elasticity of cross-demand  $\eta = 0$ ,  $D_{ij}^p = D_{ij}$  and the model simplifies to the base case formulation. It follows that Proposition 3 simplifies to Proposition 1.

We will contrast the results obtained under the price elastic specification of cross-demand to the base case model in Section 4.6. As we will show, while price elastic cross-demand reduces the expected cross-selling quantity and revenue, the general findings, managerial implications, and intuition of our model are unaffected by this extension. We will, thus, focus primarily on the simpler base case model in the numerical analysis hereafter. Nonetheless, if the price difference between the offered alternatives  $i$  and  $j$  is significant, the resulting effect on substitution behavior, expected revenues, and optimal stocking levels can become an important factor in practice. This may be particularly relevant in the context of volatile input costs, driving the selling prices of each item. We mention, as an example, the trend of copper/aluminum substitution in several industries, where aluminum-based products have become attractive substitutes for significantly more expensive copper-based alternatives, where the lower conductivity, flexibility, and durability of aluminum is acceptable (see, for example, Onstad et al., 2016).

## 4.6 Numerical study

In this section, we provide a numerical study that complements the analytical results of this paper. We consider the following numerical setup. The planning horizon is set to one year  $T = 1$  and the length of each time step is  $\Delta = 1$ . Thus, our example spans two operating periods, consisting of the initial time period  $t = 0$  and one future period. At time  $t = T + 1$ , all outstanding derivatives mature but there is no operating business activity. The two divisions are characterized by a symmetric set of parameters, allowing us to omit the subscript  $i$  from all parameters. For the base case, the demand curve parameters for each division are set to  $a = 10$ ,  $b = 2$ , and  $\beta_{ij} = \beta_{ji} = 0.8$ . Each demand shock  $\varepsilon$  is an i.i.d normal random variable with mean  $\mu_d = 0$  and standard deviation  $\sigma_d = 1.5$ .  $(\varepsilon_1, \varepsilon_2)$  are jointly bivariate normal with correlation  $\rho_d = 0.5$ . The raw material input costs follow a bivariate geometric Brownian motion (GBM) given by  $c_{i,t+\Delta} = c_{i,t} e^{((r-\delta-\frac{1}{2}\sigma_{c,i}^2)\Delta + \sigma_{c,i}W_{i,t})}$ , where  $c_{1,0} = c_{2,0} = 1.5$ , the risk free interest rate is  $r = 0.05$ , the convenience yield net of storage costs in both markets is  $\delta = -0.10$ , the cost volatility in each market is  $\sigma_c = 0.3$ , and  $\mathbb{E}(dW_{i,t}, dW_{j,t}) = \rho_c \Delta$ , where the instantaneous correlation is  $\rho_c = 0.5$ . Inventory holding costs are set in line with the market to  $\theta = e^{\delta\Delta}$ , unit stockout costs are  $\gamma = 2$ , and the level of absolute risk aversion is  $\lambda = 0.1$ .

As the ability of our model to capture the specific aspects of optimal risk management in a multi-divisional context is a central contribution of this paper, we lay the focus of the following numerical study on the ramifications of cross-divisional effects such as the risk correlations. For a broader sensitivity

analysis in a dynamic pricing, purchasing, and financial hedging model, we refer to Schöne et al. (2016b), who investigate a single-product setting. The remainder of this Section is organized as follows. First, we analyze the impact of demand and cost interdependencies between the two divisions on expected utility, profits, and inventory dynamics in the absence of financial hedging. We then discuss how inventory levels are affected by financial hedging and risk aversion. Finally, we address the influence of different hedging strategies on profit variance. In this stylized two-period model we can compute conditional expectations by integrating over the transition density of the bivariate GBM, where the second period optimization is nested in the first period expectation. When applications require additional time periods, it is straightforward to solve the model by means of dynamic programming.

#### 4.6.1 Impact of risk correlations

The two divisions of the firm are interconnected both through correlated demand and cost risk as well as the possibility of stockout-based cross-selling. Figure 4.2 depicts the impact of variations in the demand and cost correlation on utility, profits, and inventory dynamics when products are substitutes ( $\beta_{ij} = 0.8$ ). With respect to the demand correlation, panel (a) shows a decline in profits and utility for higher levels of  $\rho_d$ , which reflects a loss of cross-selling opportunities. This is explained by panel (c). A higher  $\rho_d$  lowers the probability that a stockout in one division can be absorbed by excess inventory in the second division as both divisions are more likely to face a stockout at the same time. Although, the firm slightly increases inventory levels ( $q^*$ ) in each division to mitigate the higher risk of facing a net stockout, this can not fully compensate the lost revenue from a decline in  $X_{ij}$ . Since utility differs from profits only by a multiple of profit variance, it is clear that  $U_0 < \Pi_0$  for any positive level of profit variance across the range of  $\rho_d$ .

While we do not plot the variance along with utility and profits for better readability, it is interesting to note that variance initially increases and then decreases in  $\rho_d$ . As discussed above, the firm purchases more raw material when  $\rho_d$  increases, leading to a greater exposure of future profits to cost variance. However, at the same time the portion of leftover inventory that is used to serve cross-demand decreases quickly as  $\rho_d$  rises, resulting in higher gross overage and salvage value, which provides an operational hedge against future cost volatility. This effect is responsible for the declining profit variance at higher demand correlations.

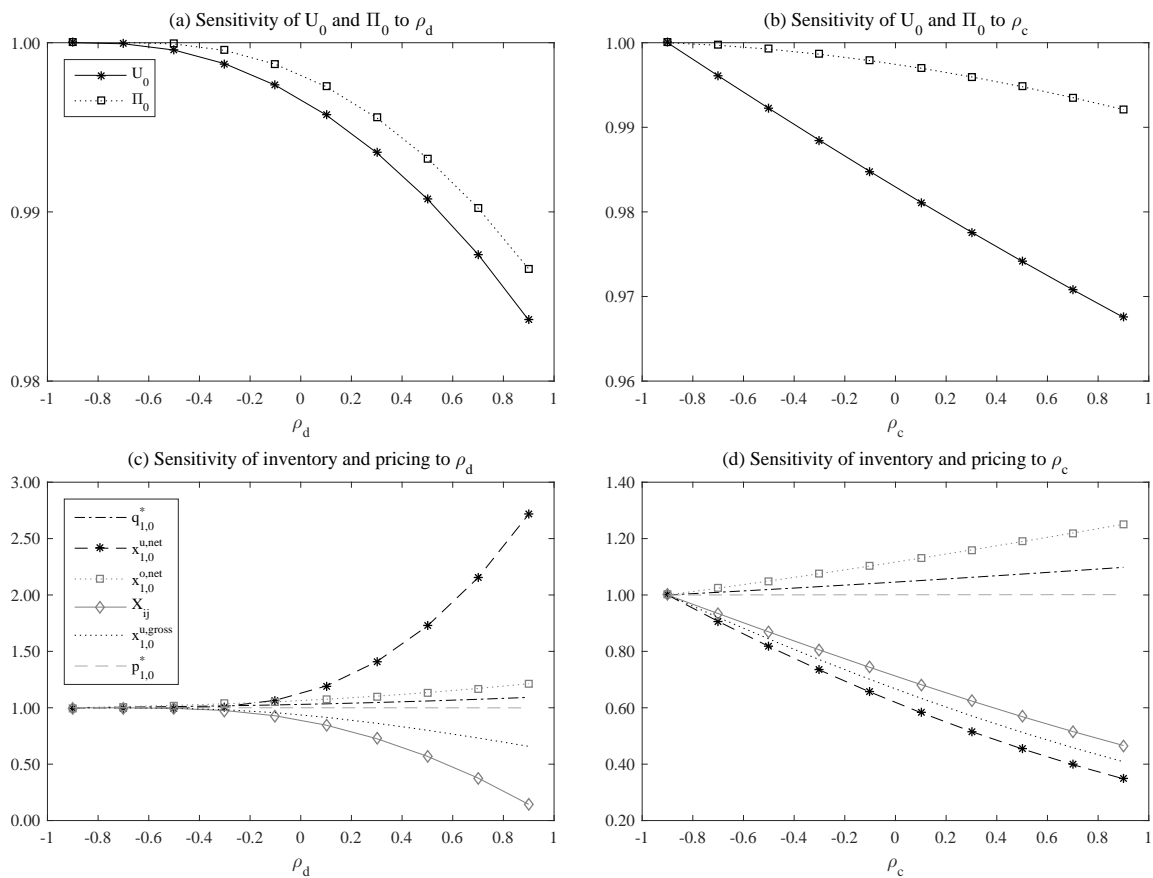
Similarly to the demand correlation,  $\rho_c$  has a negative effect on utility and profits of the unhedged, risk-averse firm. On the one hand, a low correlation of input costs creates a natural hedge for the multi-product firm since situations with high costs and low profits in one division tend to be mitigated by lower costs and higher profits in the second division. Thus, as costs are increasingly correlated, this natural hedge disappears, resulting in a concave increase in overall profit variance and lower utility. In order to counteract this effect, it is optimal to hold more physical inventory and thereby create a higher quantity of expected leftovers, which serve as an operational hedge against future cost volatility. Thus, an increase in input cost correlations raises optimal inventory levels and ultimately leads to a higher service level, reflected in a lower expected net stockout quantity. This is shown by panel (d) of Figure 4.2. However, while this strategy can partly mitigate the increase in profit variance, holding inventory is associated with

an economic cost that reduces expected total profits.

Next, we consider the sensitivity of results, presented by Figure 4.2, to variations in risk aversion and  $\beta_{ij}$ . Under risk-neutrality ( $\lambda = 0$ )  $U_0 = \Pi_0$ . While panel (a) is not further affected, the impact of  $\rho_c$  on utility is significantly weakened. Nonetheless the negative relationship prevails. This is because an increase in  $\rho_c$  leads to a higher risk-neutral probability that the cost processes in both markets move in the same direction, creating a higher likelihood that an extreme scenario occurs. When the expected loss in a high cost scenario outweighs the profit increase in a low cost scenario, expected profits will decrease as the probability of extreme scenarios rises. Finally, the interpretation of results in panel (c) remains unaffected by risk-neutrality and the inventory related variables shown in panel (d) are independent of  $\rho_c$  when  $\lambda = 0$ . Instead of setting  $\lambda = 0$ , we now address the case where the two items are complements with cross-demand coefficient  $\beta_{ij} = -0.8$ . With respect to  $\rho_c$  the sensitivities shown by panel (b) and (d) remain unaffected. However, the impact of  $\rho_d$  on the variables in panels (a) and (c) is reversed with the exception that the cross-selling quantity  $X_{ij}$  remains a decreasing function of  $\rho_d$ . For instance, profits and utility become an increasing function of the demand correlation. Intuitively, when there is little demand for either product, complementarity implies an adverse demand effect for the second product. Thus, when  $\rho_d$  is low, there is a high probability that either of the products shows little demand and negatively affects sales of the second product leading to low overall sales. Finally, in the absence of cross-demand ( $\beta_{ij} = 0$ ), utility, profit, and inventory dynamics reported in panel (a) and panel (b) are independent of  $\rho_d$ . The sensitivity of results to  $\rho_c$  as shown in panels (b) and (d) remains unaffected with the exception that  $X_{ij} = 0$  so that  $x^{u,net} = x^{u,gross}$ . Generally, for a given value of  $\rho_d$ , utility is monotonically increasing in  $\beta_{ij}$ . If items are substitutable ( $\beta_{ij} > 0$ ), higher values of  $\beta_{ij}$  imply that a greater proportion of unsatisfied demand can be addressed with the secondary item, creating additional revenue. On the contrary, with complementary products ( $\beta_{ij} < 0$ ), a stockout in one item adversely affects demand for the second item with a magnitude that is increasing in  $|\beta_{ij}|$ .

## 4.6.2 Risk aversion and financial hedging

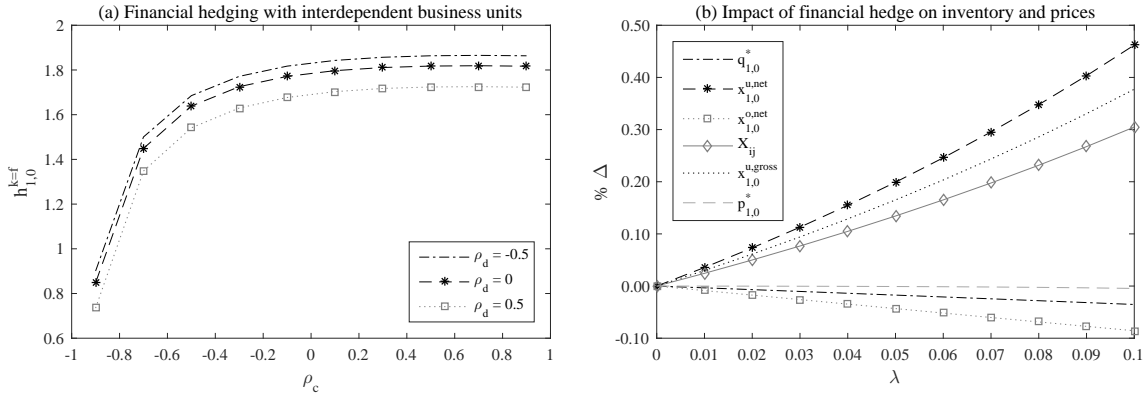
When the firm has access to a futures market associated with each of the raw material inputs, the role of inventory as an operational hedge can be perfectly replicated by a long futures position. Thus, the additional presence of options in the hedging portfolio has no further impact on inventory in our model. Figure 4.3a depicts the optimal futures hedge for different levels of cost and demand correlation. The optimal futures quantity reflects the sum of two components: first, a short position to hedge the risk in the resale value of excess inventory and, second, a long position for protection against future procurement cost volatility. In the presence of financial hedging, the salvage value related short position is independent of  $\rho_c$  and takes a value of -1.74 in this example. The long position is increasing in  $\rho_c$  in order to mitigate the increase in profit variance that occurs when the natural hedge inherent in lowly correlated costs disappears. This also explains the positive and concave net futures position shown by panel (a) of Figure 4.3. Our results further indicate a negative relationship between the demand correlation across business units when  $\beta_{ij} > 0$  and the optimal futures position in each market. As shown by Figure 4.2c, a higher demand correlation leads to an increase in inventory levels  $q$  and expected net overage  $x^{o,net}$ .



**Figure 4.2: Sensitivity of utility, profits, and inventory to risk correlations.** Variables are normalized to one and reflect time period  $t = 0$ . The notation is as follows.  $U$ : expected total utility,  $\Pi$ : expected total profit,  $q^*$ : optimal inventory level,  $x^{u,net}$ : expected net underage (net of cross-selling),  $x^o$ : expected net overage,  $X_{ij}$ : expected cross-selling quantity,  $x_1^{u,gross} = x_1^{u,net} + X_{21}$ : expected overage excluding cross-selling,  $p^*$ : unit selling price.  $U$  and  $\Pi$  reflect the entire firm; the inventory and pricing related variables refer to division  $i = 1$ . In this numerical setup both divisions are symmetric.

Since excess inventory is an operational hedge substituting a long futures position, it follows that the futures hedge should be lower when  $\rho_d$  is high. When  $\beta_{ij} = 0$ , the financial hedging decision is independent of  $\rho_d$  but maintains a similar relationship with  $\rho_c$ .

Part (b) of Figure 4.3 illustrates the impact of financial hedging on inventory levels and prices under different degrees of risk aversion. Generally, inventory levels are most affected by financial hedging when  $\lambda$  is high and remain unaffected under risk-neutrality. Further, we find that hedging can lower prices and reduce inventory in our multi-period model. With respect to the impact on inventory, this is in line with Kouvelis et al. (2013), who point out the difference of this result to earlier research of, e.g., X. Chen et al. (2007) and Zhu and Kapuscinski (2011). However, unlike in any other period, hedging raises inventory levels in the final period, which is consistent with the research on one-period models of, e.g., Ding et al. (2007) and Gaur and Seshadri (2005). As mentioned above, this is attributable to the fact that there is no future procurement at the end of the planning horizon, which could be hedged with

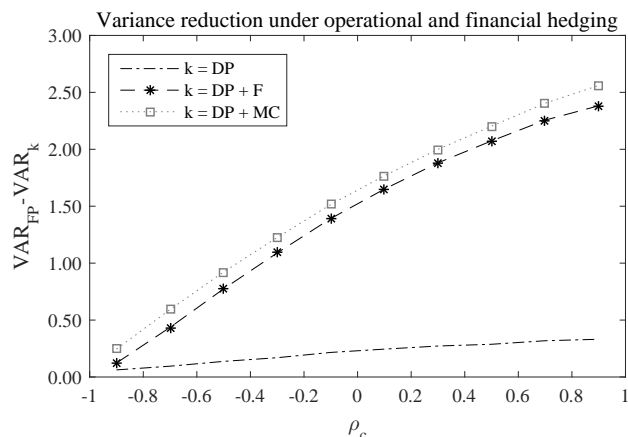


**Figure 4.3: Financial hedging, interdependent business units, and inventory.** All variables are reported for the initial time period  $t = 0$  and product  $i = 1$ , which is equivalent to  $j$  in this numerical setup. The hedging strategy is a single-contract futures hedge. Adding options to the hedging portfolio does not affect inventory/prices. The notation is explained in the footnote of Figure 4.2.

excess inventory. Instead, higher inventory levels lead to an increase in expected salvage value, which drives profit variance. Thus, in the absence of financial hedging the firm would prefer a lower inventory level in the final period.

Without the possibility of hedging, risk aversion generally lowers prices and inventory in the final period when  $\beta_{ij} = 0$ . This is consistent with the single-period models of, e.g., Xu (2010) and Agrawal and Seshadri (2000) under additive demand. However, when  $\beta_{ij} \neq 0$ , asymmetric costs in each division can lead to adverse optimal stocking levels, prices, and cross-selling flows. For higher levels of  $\lambda$  this tendency is amplified such that risk aversion can both increase or decrease prices and inventory levels, depending on the relative procurement cost in each division. For instance, the high cost division chooses a relatively low inventory level and charges higher prices than the low cost division. In this situation, an increase in  $\lambda$  leads to an over-proportionate decrease in optimal inventory in the high cost division and higher expected shortages. It follows from Lemma 1 that an increase in  $\Lambda_{00}^1 + \Lambda_{01}^1 - X_{12}$  leads to lower optimal prices. For the multi-period setting our results differ from the above cited single-period studies as we find that higher values of  $\lambda$  increase optimal prices regardless of cross-selling. This is due to the positive relationship between  $\lambda$  and physical inventory, which is increasingly used as an operational hedge under higher risk aversion. As a result expected underage declines, decreasing  $\Lambda_{00}^1 + \Lambda_{01}^1 - X_{12}$  and, thus, driving up prices.

Next, we address the impact of four different hedging strategies on profit variance and utility for different levels of the cost correlation. First, we consider a fixed price (FP) policy, where the optimal selling price for each product is determined at the beginning of the planning horizon and cannot be altered in response to changes in future procurement costs and stocking levels. However, when costs are prohibitively high, the firm may choose to cease operation. Under the FP policy there is also no financial hedging. Second, the DP strategy refers to dynamic pricing. Similarly to FP, there is no financial hedging but the firm can revise prices in the next period contingent upon the state of input costs and stocking quantities according to Lemma 1. Third, we analyze dynamic pricing in conjunction with

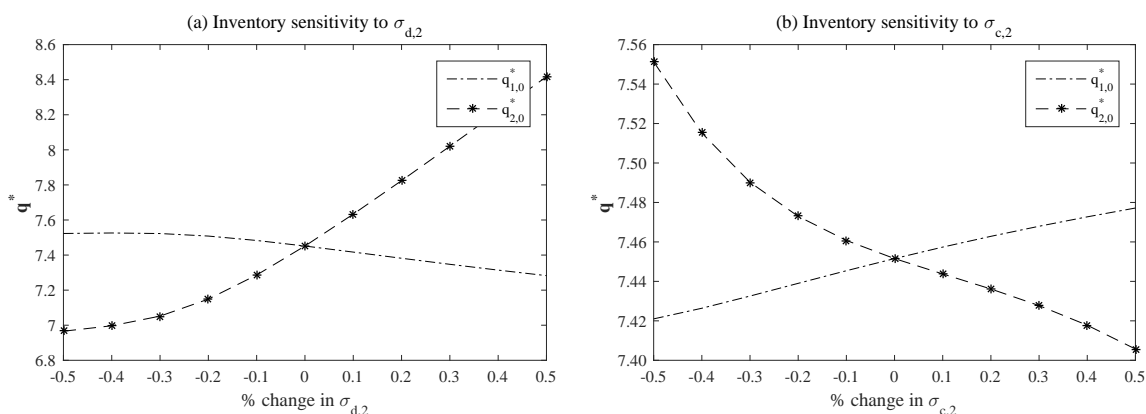


**Figure 4.4: Variance reduction under operational and financial hedging.** Shown is the absolute level of profit variance reduction for different hedging strategies denoted by  $k$  vis-à-vis the fixed price policy (FP).

futures hedging (DP+F) and, fourth, dynamic pricing together with multi-contract hedging (DP+MC). In this strategy, the financial hedge contains one futures and between 35 and 45 call and put options in each market.<sup>1</sup> Figure 4.4 shows a comparison of profit variance levels arising under DP, DP+F, and DP+MC compared to the comparatively static FP policy. We report the absolute level of variance reduction for better readability but also address the level of variance reduction in relative terms below. For an exemplary level of  $\rho_c = 0.5$ , DP reduces profit variance by 13%, DP+F by 94%, and DP+MC by 99.8% relative to the absolute level of profit variance under FP (2.21). All hedges become more important as the cost correlation increases and the natural hedge inherent in the multi-division business disappears. The futures hedge allows to perfectly hedge the variance resulting from the market value of excess inventory and the linear risk in future profits. As future profits are jointly convex in the two input costs, the MC hedge, additionally, provides better control over the non-linear variation in future profits. Further, we observe that including options in a hedging strategy is most valuable when  $\rho_c$  is high. As previously discussed, a futures hedge is optimal in a single-period setting, where the only hedgeable risk results from the salvage value of excess stock. Besides the mitigating effect of the above hedging strategies on profit variance, they can also contribute to mean profit. In this stylized two-period setting and for  $\rho_c = 0.5$ , DP raises profit expectations by 1.3% and DP+F by 2.4%. The benefit of DP over FP is clear, allowing the firm to charge profit maximizing prices in each future cost scenario. While the futures hedge does not directly contribute to mean profit under the risk-neutral measure, it allows the firm to reduce physical inventory previously held as an operational hedge, resulting in lower storage costs and higher profits.

Next, we provide a note on the impact of financial hedging on Figure 4.2, which depicts the sensitivity of utility and inventory dynamics to changes in the demand and cost risk correlations. First, the way  $\rho_d$  influences the firm is little altered by financial hedging so that the general interpretation of panels (a) and (c) remains unchanged. Since the role of physical inventory as a hedging instrument is fully replicated by

<sup>1</sup> The theoretically optimal hedge involving a continuum of strike prices allows for a perfect hedge. Our hedge in this example reduces profit variance by 99.3-99.8%.



**Figure 4.5: Inventory levels and asymmetric divisions.** Shown are the effects of a change in demand and cost volatility, in division  $i = 2$  on inventory levels in both divisions. The percentage change of volatility is measured relative to the base case volatilities  $\sigma_{d,2} = 1.5$  and  $\sigma_{c,2} = 0.3$ . Volatilities in division  $i = 1$  are unchanged:  $\sigma_{d,1} = 1.5$  and  $\sigma_{c,1} = 0.3$ . Both graphs refer to the initial time period  $t = 0$  and hedging is not available ( $\mathbf{h}_t^* = \vec{0}$ ).

a futures position,  $\rho_c$  no longer affects the inventory level and related variables in Figure 4.2d. Depending on the extent to which profit variance can be hedged, utility is equivalent to profit in panel (b).

### 4.6.3 Price elastic cross demand

With respect to the influence of price elastic cross demand on the pricing, inventory, and hedging decision, we find the following. If customers are price sensitive when making a substitution decision, expected cross-demand declines and cross-selling becomes a less effective means of mitigating stockouts. To compensate this effect, the firm chooses a higher safety stock, leading to a reduction in gross underage ( $x^{u,gross}$ ) and a less significant decrease in net underage ( $x^{u,net}$ ). Prices increase and the effect on expected raw material excess stock is positive. This implies an increase in the optimal futures hedging position in the final period/single-period and a possible decrease in any previous period, when excess inventory substitutes a long futures hedge against future input cost volatility.

### 4.6.4 Asymmetric divisions

While it was assumed in the above analysis that both divisions are characterized by an identical set of parameters, we finally address the case of asymmetric divisions with respect to selected parameters. For instance, in line with expectations, we find that under different input costs, the high cost division will choose lower safety stock, higher prices, and a lower futures hedge, leading to a lower service level and a higher expected cross-selling quantity from the low cost to high cost division than vice versa. We also observe that changes in both cost or demand volatility in one division can have implications for optimal inventory levels, prices, and hedging decisions in both divisions. For example, we show the impact of a change in cost and demand volatility on optimal inventory levels in Figure 4.5, when the products are dynamically substituted with coefficient  $\beta_{ij} = 0.8$ , the firm is risk-averse, and financial hedging is not available. As shown by panel (a), an increase in demand volatility in one division tends to decrease inventory in the other division, facing no change in demand volatility. This is attributable to



the higher amount of expected excess inventory in the more volatile division and the resulting effect on profit variance. By reducing inventory, the second division partly mitigates this effect on variance. In the presence of risk-neutrality or financial hedging, also the division facing no volatility increase will choose a slightly higher inventory level, which is driven by an increase in cross-demand originating from the high-volatility division to the low-volatility division. The effect of an asymmetric change in cost volatility on inventory levels is shown by panel (b). For instance, if the cost volatility in one division increases, the inventory level in this division declines in order to mitigate the higher salvage value variance. Due to the dynamic interdependence of demand between divisions and the impact of a change in inventory on profit variance, the optimal strategy for the second division facing no change in cost volatility is to increase inventory in order to absorb the higher stockout-based cross-demand originating from the more volatile division. If a futures hedge can be traded, the higher variance of expected salvage value resulting from additional inventory is hedged by the futures contract and no longer affects the stocking decision. Finally, we note that, while demand volatility exhibits a negative relationship with utility and profits, cost volatility may indeed increase utility of the risk-neutral or hedged firm. This is attributable to managerial flexibility, which allows the firm to react to changes in costs in a way that, in the expectation, high profits generated in low-cost scenarios more than outweigh low profits arising under high cost realizations. This is in line with the results of Xiao et al. (2015) and Schöne et al. (2016b).

## 4.7 Conclusion

In this paper, we study the jointly optimal inventory management, pricing, and financial hedging decision of a risk-averse, two-product firm facing multiple cost and demand risks, which can be correlated across divisions. Our model further accounts for the possibility of stockout-based cross-demand reflecting the substitutability/complementarity of items. Within this framework, we offer new insights with regard to the interplay between operational and financial hedging decisions and provide implicit solutions for the firm's jointly optimal pricing, inventory, and hedging policy.

Our findings, which are summarized in the introduction, generally show that in the presence of interconnected cost and demand risks between divisions, effective risk management does not only require cross-functional decision coordination between, for instance, the procurement and finance department but also necessitates an information exchange and decision alignment across divisions. For instance, to choose an appropriate financial hedge, the risk manager needs to estimate the firm-wide net risk exposure, which depends on the variability and correlation of procurement costs as well as operational hedging measures such as price adjustments and physical raw material inventory in each division. While this insight is consistent with the practical evidence of, for example, Fisher and Kumar (2010), this research is among the first studies to provide a quantitative analysis of the interplay between multiple cost and demand risks in an organization and the optimal operational and financial risk management strategy.

However, while the interdependencies among different organizational divisions should not be overlooked in policy making, we finally summarize a number of cases, where the presence of risk correlations does not affect optimal decision making, allowing managers to treat different organizational divisions as separate units. For instance, in the absence of cross-demand effects, the correlation in demand ran-

domness ( $\rho_d$ ) does no longer influence prices and inventories of either division. Yet, without financial hedging, the cost correlation  $\rho_c$  is still relevant for the inventory levels, which can provide an operational hedge against input cost risk. Once a futures hedge can be implemented, inventory and prices are no longer affected by the correlation of input costs, however, the hedging decision itself remains a function of  $\rho_c$  as it accounts for the natural hedge inherent in future profit, which is driven by the cost correlation. In a single-period setting without cross-demand, both operational and financial hedging decisions are independent of demand and cost correlations, resulting in separable optimal policies for each division. In this case, the operational decision simplifies to a price-setting newsvendor problem (e.g., Petruzzi & Dada, 1999), where the market value of excess inventory can be hedged with an offsetting short position in the futures market.

Interesting extensions of our model include, for instance, the investigation of different demand models, such as the multinomial logit model of consumer choice, which is widely used in industry. It is also useful to allow for a more flexible cost structure of each offered product so that an item is no longer manufactured from a single commodity input but rather requires a mix of different raw materials with potentially stochastic price evolution. We leave these topics to future research.

# 5

## SUMMARY AND OUTLOOK

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### 5.1 Summary

Driven by the rapid expansion in emerging markets and the vast trading activities of hedge funds, pension funds, and large financial institutions, commodity prices reached new all-time peaks in the recent past, before tumbling to historic lows amid global recessionary fears and political crises. This unprecedented level of volatility observed across commodity markets has contributed to equally dramatic swings in the profitability of firms engaged in the extraction, refinement, or processing of commodities alike (Fisher & Kumar, 2010). As a result, the topic of effective risk management has gained greatly in importance among executives in several industries. According to a survey of manufacturing companies conducted by PwC (2009), 86% of senior executives regard commodity price risk as being important to a company's financial performance, while admitting shortcomings in the management thereof. While procurement risk is traditionally managed by the purchasing function with an emphasis on the availability and quality of supply, practical evidence has shown that firms successfully operating in today's environment adopt a more integrated PRM approach involving the concerted management of inventories, prices, and financial hedging activities (PwC, 2009). However, in line with the above cited survey, the successful implementation of a firm-wide, integrated PRM program entails management challenges. First, quantifying a firm's exposure to raw material cost risk necessitates a sound understanding of the stochastic commodity market dynamics. Second, given an estimate of the firm's exposure to different sources of risk, such as cost and demand volatility, executives face an intricate optimization problem over their operational and financial decision variables with the ultimate goal to reduce profit variance, while maintaining attractive business opportunities. Despite the economic importance of effective risk management in practice, quantitative research on jointly optimal operational and financial PRM strategies is scarce. In fact, to the best of our knowledge, Kouvelis et al. (2013) and Xiao et al. (2015) are the only authors to provide much needed insights at this time. In the attempt to shed further light on the topic of operational and financial risk management under commodity price uncertainty, this thesis contributes three self-contained studies to the existing literature, which extend the available research on commodity price modeling and the above articles in several directions.

In Chapter 2, we develop a four-factor maximal affine stochastic volatility model of commodity prices, which is consistent with various specific characteristics observed in storable, non-seasonal commodity

markets. The model is extensively tested in the copper market using historical data of the futures term structure and corresponding American call options, written on the futures contracts. Based on this study, we assess the model performance with respect to pricing and hedging financial derivatives and provide structural insights into the copper market with respect to, among others, the historical rate of mean reversion, the stochastic evolution of the implied long-term spot price level, and the stochastic behavior of volatility. The model is not only theoretically consistent with many aspects specific to commodity markets but also exhibits superior pricing and hedging performance to different benchmark models in the market of copper derivatives. Therefore, our four-factor model provides a solid basis for both the valuation of financial and real assets as well as the optimization of procurement risk management decisions under commodity price risk.

Turning towards the question of how to optimally align operational decision making, such as inventory management and pricing, with financial hedging activities under commodity price risk in Chapter 3, we develop a mathematical model of a stylized risk-averse industrial firm facing commodity price risk and stochastic, price sensitive demand. The firm obtains physical material from the spot market and has access to the associated futures market for financial hedging. We provide a flexible, simulation-based optimization algorithm to solve the firm's inter-temporal dynamic inventory management, pricing, and hedging problem under realistic commodity price, interest rate, net convenience yield, and volatility dynamics. Optimal policies are obtained as a contingency plan, indicating the optimal decision for any given state of future raw material costs and inventory levels. This representation provides actionable managerial intuition for the interdependence of operational and financial decision making under uncertainty. Moreover, we illustrate the value of managerial flexibility in responding to changes in a risky environment and stress the economic cost of committing to restrictive supply contracts. We highlight the importance of taking into account the stochastic nature of interest rates, the cost of carry, and volatility in the optimization of PRM decisions and discuss the implications of model estimation error. Finally, we provide an extensive sensitivity analysis of expected performance to critical parameters and discuss the effect of risk aversion and financial hedging on the distribution of cash flows.

The single-product model is subsequently extended in Chapter 4 to capture not only the aspect of cross-functional decision coordination between the procurement/inventory management, sales, and finance function but also account for the necessary cross-divisional alignment of actions that facilitates a firm-wide optimization of rewards and risks. Similarly to the single-product model, we consider an industrial context, where raw material is processed into end-products, which are sold to price sensitive customers under stochastic demand. Each of the divisions can face different but respectively correlated cost and demand risks. Moreover, we allow for dynamic cross-selling between the two divisions in order to account for the possible complementarity/substitutability of items. The firm has access to a futures, call, and put options market associated with each of the input commodities and can trade an extensive hedging portfolio for variance reduction in each market. By dynamically choosing the financial hedge, inventory levels, and prices, the firm dynamically maximizes mean-variance inter-period utility over a multi-period time horizon. Within this framework, we are able to obtain analytic solutions to the firm's PRM problem and provide numerical examples to develop managerial intuition. Within this

context, we focus primarily on the specific aspects of PRM with multiple product lines, which extend our insights from Chapter 3. In particular, we analyze the impact of cross-divisional risk correlations on the company's optimal strategy and discuss the effect of unilateral changes in the market environment in one business unit on the entire firm. Moreover, we investigate the impact of risk aversion and financial hedging on expected performance and operational decision making and analyze the effectiveness of different hedging strategies for risk reduction.

While the optimization models presented in Chapter 3 and Chapter 4 of this thesis represent abstractions from the more complex reality, the derived understanding and intuition of the central trade-offs, driving forces, sensitivities, and interdependencies involved in joint operational and financial risk management can, nonetheless, be understood as an important building block for managerial decision making in practice.

## **5.2 Outlook**

Within the context of integrated risk management, we see several fruitful directions for future research. Apart from the desirable extensions outlined in Sections 2.4, 3.7, and 4.7, we emphasize the following four topics for further investigation.

First, the operationalization of the theoretical results developed in this thesis and by other related studies constitutes the foundation for exploiting the potential value inherent in integrated risk management in practice. Thus, the derivation of actionable heuristics from theoretically optimal policies resembles a valuable contribution to existing research that facilitates the transfer of theoretical results into practice. In this context, also the evaluation of theoretical risk management models based on real world business data could enrich the stylized results of this dissertation by providing more realistic estimates of the expected benefits of integrated PRM in practice.

Second, the models presented in Chapter 3 and Chapter 4 can be extended in several directions. For instance, while we focus on the specific implications of commodity price and demand risk in this dissertation, internationally operating firms are certainly exposed to additional sources of risk such as volatility in the foreign exchange market. Since commodities are usually denoted in US\$, European firms may, for example, not only need to account for currency risk in the valuation of their foreign revenues but also depend on the US\$ exchange rate through raw material procurement costs. Therefore, currency risk can contribute to a company's net risk exposure and should be accounted for in an integrated risk management strategy. The idea of capturing the complete risk profile of an organization then leads to the topic of allowing for a suitable correlation structure among the different risks and the mathematically tractable implementation of this complex setting.

Third, another interesting extension of the PRM models of this thesis pertains to the presence of competition. Particularly in the context of price adjustments, managers might fear competitors' reactions to a change in prices and ultimately the risk of triggering a price war. A practical example highlighting the competitive aspects of risk management is the case of Southwest Airlines, which adopted an aggressive financial hedging program between 1999 and 2005 allowing the firm to expand, while competitors were facing bankruptcy (Tevelson et al., 2007). This illustrates the potential reward of hedging and, at the

same time, the risk of foregoing the implementation of a risk management strategy, when the competition is hedged. In order to account for the strategic effects of risk management decisions, competition may be introduced in the models of Chapter 3 and Chapter 4 by accounting for the best response of competing firms in the optimization process.

Fourth, firms have increasingly large amounts of data at their disposal, which creates tremendous opportunities with respect to advertising, revenue management, and forecasting to name but a few examples. While the continuous monitoring of risk exposure is a familiar topic for financial institutions, the availability of increasingly rich operational data in conjunction with financial market information allows the extension of revenue management models (see, e.g., Ferreira, Lee, & Simchi-Levi, 2016) to an integrated PRM framework with information updating based on machine learning techniques. A self-learning, integrated PRM model may have the key advantage that it can cope more flexibly with changes in the market environment, ideally adjusting expectations about costs, volatility, demand, price sensitivities, and substitution behavior as new information becomes available. Particularly, in the context of pervasive uncertainty in the global financial market and the permanent risk of disruptive changes in commodity price and currency dynamics, the capability of flexible and fast adaptation to new circumstances may ultimately lead to better risk control and a competitive advantage.



## APPENDIX TO CHAPTER 3

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### A.1 Valuation of flexibility

**Table A.1:** Valuation of flexibility for different degrees of demand volatility

Setting	$\sigma_d = 0$			$\sigma_d = 0.5$			$\sigma_d = 3$			$\sigma_d = 6$		
	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$
(10) = (2) - (1) = $p^*$	0.0	0.0	0.0	1.0	0.4	2.2	1.3	1.3	1.2	0.7	4.4	-6.8
(11) = (3) - (1) = $b^*$	8.9	4.4	17.6	9.8	5.3	18.4	8.4	9.5	10.4	7.1	10.8	3.9
(12) = (5) - (1) = $p^*, b^*$	10.5	9.8	18.7	11.8	11.0	20.1	12.3	14.9	14.4	10.4	16.2	6.1
(13) = (9) - (1) = $p^*, b^*$	15.7	10.0	33.6	16.7	11.2	33.9	15.8	15.1	22.9	12.5	16.3	9.8
(14) = (5) - (3) = $p^* b^*$	1.5	5.4	1.1	2.0	5.8	1.7	3.8	5.4	4.0	3.3	5.4	2.2
(15) = (5) - (2) = $b^* p^*$	10.4	9.8	18.7	10.8	10.6	17.9	11.0	13.6	13.1	9.7	11.8	12.9
(16) = (6) - (2) = $f^* p^*, \bar{b}$	9.2	0.0	28.2	9.1	0.4	27.3	7.2	2.6	17.1	4.4	0.9	12.0
(17) = (7) - (3) = $f^* \bar{p}, b^*$	4.1	0.1	14.9	3.7	0.1	14.3	3.2	0.1	8.7	1.7	0.6	2.8
(18) = (9) - (5) = $f^* p^*, b^*$	5.2	0.2	14.8	5.0	0.2	13.8	3.6	0.2	8.5	2.1	0.1	3.6

This table corresponds to the results shown in Figure 3.3. Row numbers refer to Table 3.2.  $U$ : value of flexibility measured in units of utility,  $V$ : cash value of flexibility,  $VaR$ : value of flexibility in terms of an increase in the 5<sup>th</sup> percentile of  $V$ .

### A.2 Calibration of commodity price models

In line with the literature on the estimation of term structure models, we use the Kalman filter in conjunction with quasi-maximum likelihood estimation to obtain estimates of the state variables and parameters. Below, we provide analytical futures and bond pricing formulas for the SRCV process, the state space representation for model calibration, and parameter estimates of all models used to generate the results in the main text.

#### A.2.1 Futures and bond prices

Defining  $y(t) = \log[S(t)]$  and applying Itô's lemma to the spot price process yields the following dynamics of the log-spot price

$$dy = \left[ r - \delta - \frac{1}{2} (\sigma_s^2 + \nu) \right] dt + \sigma_s dW_s + \sqrt{\nu} dW_{sv}, \quad (\text{A.1})$$

**Table A.2:** Valuation of flexibility for different degrees of commodity price volatility

Setting	$\sigma_s = 0.05$			$\sigma_s = 0.15$			$\sigma_s = 0.3$			$\sigma_s = 0.5$		
	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$	$U$	$V$	$VaR$
(10) = (2) - (1) = $p^*$	2.1	2.1	2.0	1.9	2.1	1.4	1.3	1.3	1.2	1.6	1.6	2.4
(11) = (3) - (1) = $b^*$	3.7	2.9	4.3	5.6	4.2	8.1	8.4	9.5	10.4	14.3	15.5	20.9
(12) = (5) - (1) = $p^*, b^*$	5.1	4.7	4.6	7.9	7.7	9.5	12.3	14.9	14.4	19.9	25.4	25.4
(13) = (9) - (1) = $p^*, b^*$	6.1	4.7	9.0	10.0	7.7	16.1	15.8	15.1	22.9	25.9	25.8	37.7
(14) = (5) - (3) = $p^* b^*$	1.4	1.8	0.3	2.2	3.5	1.4	3.8	5.4	4.0	5.7	9.9	4.5
(15) = (5) - (2) = $b^* p^*$	3.0	2.6	2.6	6.0	5.6	8.1	11.0	13.6	13.1	18.3	23.8	23.0
(16) = (6) - (2) = $f^* p^*, \bar{b}$	1.5	0.0	4.4	4.4	0.6	11.9	7.2	2.6	17.1	9.9	3.2	27.3
(17) = (7) - (3) = $f^* \bar{p}, b^*$	0.7	0.0	2.7	1.9	0.6	5.9	3.2	0.1	8.7	4.9	1.2	10.5
(18) = (9) - (5) = $f^* p^*, b^*$	1.0	0.0	4.4	2.1	0.0	6.6	3.6	0.2	8.5	5.9	0.4	12.3

This table corresponds to the results shown in Figure 3.4. Row numbers refer to Table 3.3.  $U$ : value of flexibility measured in units of utility,  $V$ : cash value of flexibility,  $VaR$ : value of flexibility in terms of an increase in the 5<sup>th</sup> percentile of  $V$ .

where  $r$  is the instantaneous risk-free interest rate and  $\delta$  refers to the instantaneous net convenience yield. It is well-known that under the above dynamics of the spot price, the futures price must satisfy the following Feynman-Kac partial differential equation

$$\begin{aligned}
& \frac{1}{2} (\sigma_s^2 + v) F_{yy} + \frac{1}{2} \sigma_r^2 F_{rr} + \frac{1}{2} \sigma_\delta^2 F_{\delta\delta} + \frac{1}{2} \sigma_v^2 v F_{vv} \\
& + \rho_{sr} \sigma_s \sigma_r F_{yr} + \rho_{s\delta} \sigma_s \sigma_\delta F_{y\delta} + \rho_{sv} \sigma_v v F_{yv} + \rho_{r\delta} \sigma_r \sigma_\delta F_{r\delta} \\
& + \left[ r - \delta - \frac{1}{2} (\sigma_s^2 + v) \right] F_y + \kappa_r (\theta_r - r) F_r + \kappa_\delta (\theta_\delta - \delta) F_\delta + \kappa_v (\theta_v - v) - F_\tau = 0, \quad (A.2)
\end{aligned}$$

subject to the boundary condition  $F(T, 0) = S(T)$ . It is known that the solution to  $F$  is exponential affine and takes the form

$$F(t, \tau) = \exp [y(t) + \beta_0(\tau) + \beta_r(\tau) r(t) + \beta_\delta(\tau) \delta(t)], \quad (A.3)$$

where  $\tau = t - T$  denotes the maturity of  $F$  at time  $t$ . By substituting  $F$  and separating variables, we obtain the following set of ordinary differential equations for coefficients  $\beta_0(\tau)$ ,  $\beta_r(\tau)$ , and  $\beta_\delta(\tau)$

$$\beta'_0 = \kappa_r \theta_r \beta_r + \kappa_\delta \theta_\delta \beta_\delta + \frac{1}{2} \sigma_r^2 \beta_r^2 + \frac{1}{2} \sigma_\delta^2 \beta_\delta^2 + \sigma_s \sigma_r \rho_{sr} \beta_r + \sigma_s \sigma_\delta \rho_{s\delta} \beta_\delta + \sigma_r \sigma_\delta \rho_{r\delta} \beta_r \beta_\delta \quad (A.4)$$

$$\beta'_r = 1 - \kappa_r \beta_r \quad (A.5)$$

$$\beta'_\delta = -1 - \kappa_\delta \beta_\delta \quad (A.6)$$



with boundary conditions  $\beta_0(0) = 0$ ,  $\beta_r(0) = 0$ , and  $\beta_\delta(0) = 0$ . Note that the futures price does not depend on  $v$ . To see this, suppose counter-factually, that  $F$  is a function of variance, i.e.,  $F = \exp[y_t + \beta_0(\tau) + \beta_r(\tau)r_t + \beta_\delta(\tau)\delta_t + \beta_v(\tau)v_t]$ . Then let  $M = \beta'_v = -\kappa_v\beta_v + \frac{1}{2}\sigma_v^2\beta_v^2 + \sigma_v\rho_{sv}\beta_v$ . However, since  $M = 0$  when  $\tau = 0$  and all derivatives of  $M$  with respect to  $\tau$  are zero,  $\beta_v$  is identically zero. It follows that  $F$  is independent of  $v$ . The solution to Equations (A.4)-(A.6) is given by

$$\beta_r = \frac{r(1 - e^{-\kappa_r\tau})}{\kappa_r} \quad (\text{A.7})$$

$$\beta_\delta = \frac{-(1 - e^{-\kappa_\delta\tau})}{\kappa_\delta} \quad (\text{A.8})$$

$$\begin{aligned} \beta_0 = & \frac{(\kappa_r\theta_r + \rho_{sr}\sigma_s\sigma_r)((1 - e^{-\kappa_r\tau}) - \kappa_r\tau)}{\kappa_r^2} - \frac{\sigma_r^2(4(1 - e^{-\kappa_r\tau}) - (1 - e^{-2\kappa_r\tau}) - 2\kappa_r\tau)}{4\kappa_r^3} \\ & - \frac{(\kappa_\delta\theta_\delta + \rho_{s\delta}\sigma_s\sigma_\delta)((1 - e^{-\kappa_\delta\tau}) - \kappa_\delta\tau)}{\kappa_\delta^2} - \frac{\sigma_\delta^2(4(1 - e^{-\kappa_\delta\tau}) - (1 - e^{-2\kappa_\delta\tau}) - 2\kappa_\delta\tau)}{4\kappa_\delta^3} \\ & + \rho_{r\delta}\sigma_r\sigma_\delta \left[ \frac{(1 - e^{-\kappa_r\tau}) + (1 - e^{-\kappa_\delta\tau}) - (1 - e^{-(\kappa_r + \kappa_\delta)\tau})}{\kappa_r\kappa_\delta(\kappa_r + \kappa_\delta)} \right. \\ & \left. + \frac{\kappa_r^2(1 - e^{-\kappa_\delta\tau}) + \kappa_\delta^2(1 - e^{-\kappa_r\tau}) - \kappa_r\kappa_\delta^2\tau - \kappa_r^2\kappa_\delta\tau}{\kappa_r^2\kappa_\delta^2(\kappa_r + \kappa_\delta)} \right]. \end{aligned} \quad (\text{A.9})$$

The variance of futures returns is a function of maturity given by

$$\sigma_F^2 = \sigma_s^2 + v + \beta_r^2\sigma_r^2 + \beta_\delta^2\sigma_\delta^2 + 2(\beta_r\rho_{sr}\sigma_s\sigma_r + \beta_\delta\rho_{s\delta}\sigma_s\sigma_\delta + \beta_r\beta_\delta\rho_{r\delta}\sigma_r\sigma_\delta), \quad (\text{A.10})$$

which simplifies to the following spot return variance for  $\tau = 0$ :  $\sigma_{F(T,0)}^2 = \sigma_s^2 + v$ . The price of a discount bond under Ornstein-Uhlenbeck interest rate dynamics is given by

$$B(r, \tau) = \exp[\alpha_0(\tau) - \beta_r(\tau)r], \quad (\text{A.11})$$

where  $\alpha_0(\tau)$  is

$$\alpha_0(\tau) = \frac{\theta_r((1 - e^{-\kappa_r\tau}) - \kappa_r\tau)}{\kappa_r} - \frac{\sigma_r^2(4(1 - e^{-\kappa_r\tau}) - (1 - e^{-2\kappa_r\tau}) - 2\kappa_r\tau)}{4\kappa_r^3} \quad (\text{A.12})$$

and  $\beta_r(\tau)$  follows from Equation A.7.

### A.2.2 State space form

The state space representation involves a transition equation of the state vector and a measurement equation, which uniquely determines the state vector from a set of measurements. The transition equation follows from Equation A.1 and the interest rate, convenience yield, and variance process given in the main text. We use the following set of risk premia to obtain the model dynamics under the physical probability measure.  $\lambda := [\lambda_y, \lambda_r, \lambda_\delta, \lambda_v]^\top$ , where  $\lambda_r = (\hat{\theta}_r - \theta_r) / \sigma_r$ ,  $\lambda_\delta = (\hat{\theta}_\delta - \theta_\delta) / \sigma_\delta$ , and  $\lambda_v = (\hat{\theta}_v - \theta_v) / (\sigma_v \sqrt{v})$ . Next, we use an Euler discretization to arrive at the following transition equation

$$X_{t+1} = A + BX_t + \varepsilon_t, \quad (\text{A.13})$$

where

$$X_t = \begin{bmatrix} y_t & r_t & \delta_t & v_t \end{bmatrix}^\top \quad (\text{A.14})$$

$$A = \begin{bmatrix} \lambda_x \sigma_s - \frac{1}{2} \sigma_s^2 & \kappa_r \hat{\theta}_r & \kappa_\delta \hat{\theta}_\delta & \kappa_v \hat{\theta}_v \end{bmatrix}^\top \quad (\text{A.15})$$

$$B = \begin{bmatrix} 1 & \Delta & -\Delta & (\lambda_x - \frac{1}{2}) \Delta \\ 0 & 1 - \kappa_r \Delta & 0 & 0 \\ 0 & 0 & 1 - \kappa_\delta \Delta & 0 \\ 0 & 0 & 0 & 1 - \kappa_v \Delta \end{bmatrix}. \quad (\text{A.16})$$

$\Delta$  refers to the size of the discrete time step and  $\varepsilon_t \sim N(0, \Sigma_t)$  is a Gaussian shock vector with the following instantaneous variance-covariance matrix

$$\Sigma_t \Sigma_t^\top = \begin{bmatrix} \sigma_s^2 + v_t & \rho_{sr} \sigma_s \sigma_r & \rho_{s\delta} \sigma_s \sigma_\delta & \rho_{sv} \sigma_v \\ \rho_{sr} \sigma_s \sigma_r & \sigma_r^2 & \rho_{r\delta} \sigma_r \sigma_\delta & 0 \\ \rho_{s\delta} \sigma_s \sigma_\delta & \rho_{r\delta} \sigma_r \sigma_\delta & \sigma_\delta^2 & 0 \\ \rho_{sv} \sigma_v & 0 & 0 & \sigma_v^2 \end{bmatrix} \Delta \quad (\text{A.17})$$

In order to link the set of state variables to the observable futures and bond prices, we construct the following two measurement equations from Equations A.3 and A.11

$$\ln F_t(\tau_n) = \beta_0(\tau_n) + \begin{bmatrix} 1 & \beta_r(\tau_n) & \beta_\delta(\tau_n) & 0 \end{bmatrix} X_t + \omega_F \quad (\text{A.18})$$

$$\ln B_t(\tau_n) = \gamma_0(\tau_n) + \begin{bmatrix} 0 & -\gamma_r(\tau_n) & 0 & 0 \end{bmatrix} X_t + \omega_B, \quad (\text{A.19})$$

where  $\omega_F$  and  $\omega_B$  are Gaussian noise vectors with variances of  $\xi_F$  and  $\xi_B$ , respectively. The variance

**Table A.3:** Empirical performance of commodity price processes

Process	A: Pricing performance								B: Implied price statistics			
	F1	F3	F12	F24	B1	B3	B12	B24	Mean	Std	Skew.	Kurt.
S	0.74	0.71	-	-	-	-	-	-	7.89	2.41	0.96	4.66
SC	0.38	0.34	0.49	0.39	-	-	-	-	7.72	2.31	0.94	4.66
SRC	0.43	0.34	0.60	0.46	0.04	0.10	0.26	0.16	7.71	2.30	0.94	4.64
SRCV	0.41	0.34	0.60	0.48	0.04	0.09	0.26	0.16	7.69	2.66	1.07	5.02

Panel A: root mean squared pricing errors are computed from futures and bond prices based on the Kalman filter prediction of state variables using all time series information up to the previous week. F1 refers to the one-month futures pricing error and B1 to the one-month discount bond pricing error. Other maturities are denoted by analogy. Panel B: statistics are computed over 100,000 simulated commodity prices after one year. The initial values for the market state vector are:  $s_0 = 8$ ,  $r_0 = 0.02$ ,  $q_0 = 0.05$ , and  $v_0 = 0.05$  and correspond the numerical setting used in the main text.

process ( $v_t$ ) is estimated with the GARCH(1,1) inferred futures return variances corresponding to each futures maturity in the sample. The respective measurement function is constructed from Equation A.10

$$var_t(\tau_n) = \Omega_0(\tau_n) + \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} X_t + \omega_{var}, \quad (\text{A.20})$$

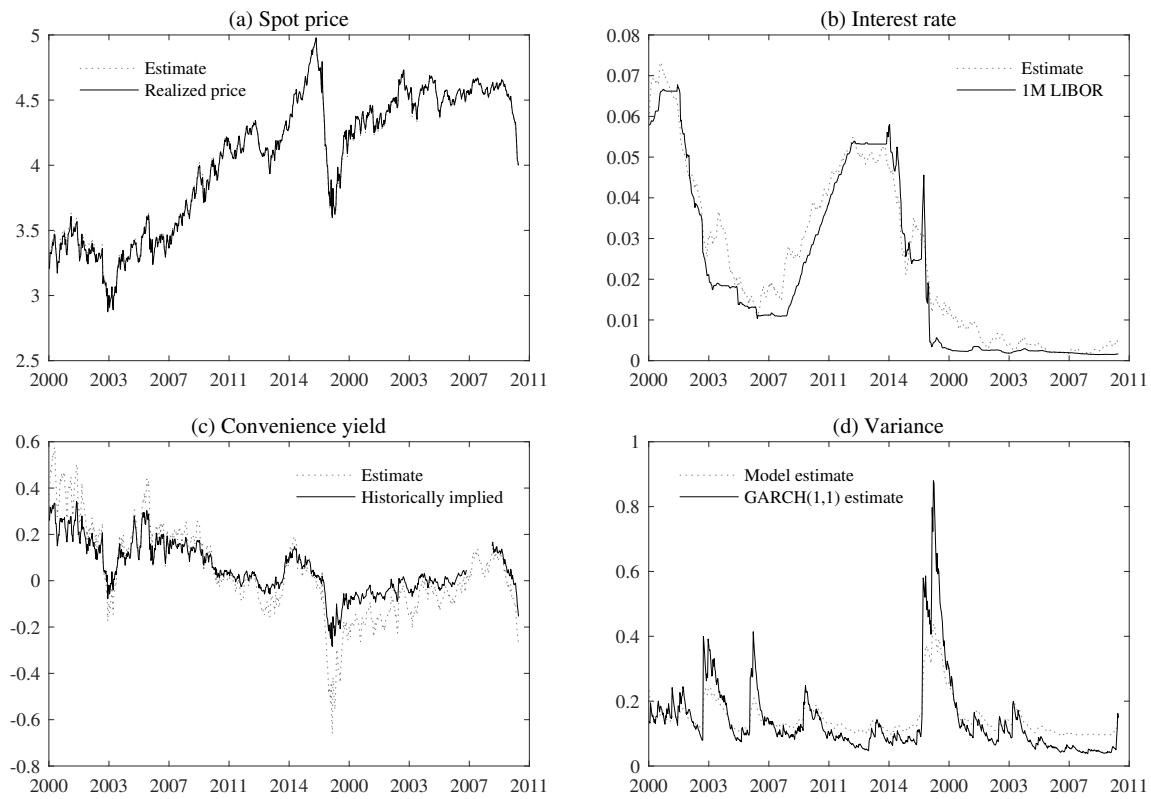
where

$$\Omega_0(\tau_n) = \sigma_s^2 + \beta_r^2 \sigma_r^2 + \beta_q^2 \sigma_q^2 + 2(\beta_r \rho_{sr} \sigma_s \sigma_r + \beta_q \rho_{sq} \sigma_s \sigma_q + \beta_r \beta_q \rho_{rq} \sigma_r \sigma_q) \quad (\text{A.21})$$

and  $\xi_{var}$  is the variance of measurement noise. Alternatively, the variance process may be estimated from option prices, however, we prefer the approach advocated here, as it can be easily implemented without the need for additional data. In order to save parameters, it is assumed that for each series of measurement data, only one measurement error variance applies to all maturities. Once the EKF is implemented, we maximize the likelihood function of the Kalman filter by first applying the Nelder-Mead algorithm and subsequently switching to the gradient-based BFGS algorithm for efficiency.

### A.2.3 Estimation results

A comparison of the pricing performance and implied return distribution of each process to historical observations is given by Table A.3. In panel A, we report the root mean squared pricing errors (RMSE) of realized and one-week ahead predictions of the futures and bond prices. Panel B shows summary statistics describing the commodity price distribution implied by the four different models after one year. Calibrated parameters are reported in Table A.4 and an illustration of the estimated state variables under SRCV dynamics is given by Figure A.1. Note that we calibrate the S process to short-term futures only as the use of long-term futures will lead to an underestimated value of the volatility parameter  $\sigma_s$ . This due to the Samuelson effect, which refers to the empirical observation that futures return variance is declining in contract maturity. As all futures returns in the term structure are perfectly correlated in a one-factor model, long-term futures will lead to a downward bias in the estimated spot volatility. However, at least



**Figure A.1: Estimated and realized state variables of the SRCV model.** (a) shows the spot price estimate and the short-term futures price as an approximation for the spot price. In panel (c), the historically implied convenience yield is calculated using the one-month and three-months futures and the following approximation:  $\delta_t \approx r_t - (\ln(F_t(\tau_2)) - \ln(F_t(\tau_1))) / (\tau_2 - \tau_1)$ .

two futures are considered in order to infer information about the convenience yield.

**Table A.4:** Parameter estimates of commodity price model

Parameter	SRCV		SRC		SC		S	
$\lambda_s$	0.166	(0.263)	0.435	(0.260)	0.349	(0.198)	0.279	(0.259)
$\kappa_r$	0.042	(0.003)*	0.039	(0.003)*	1.000	-	1.000	-
$\theta_r$	0.104	(0.004)*	0.105	(0.004)*	$r_0$	-	$r_0$	-
$\hat{\theta}_r$	0.000	(0.044)	0.000	(0.047)	$r_0$	-	$r_0$	-
$\kappa_\delta$	0.883	(0.018)*	0.856	(0.018)*	1.019	(0.019)*	1.000	-
$\theta_\delta$	0.008	(0.002)*	0.011	(0.004)*	0.000	(0.003)	$\delta_0$	-
$\hat{\theta}_\delta$	0.048	(0.084)	0.008	(0.073)	-0.006	(0.055)	$\delta_0$	-
$\kappa_v$	1.824	(0.331)*	1.000	-	1.000	-	1.000	-
$\hat{\theta}_v$	0.083	(0.012)*	0.000	-	0.000	-	0.000	-
$\sigma_s$	0.311	(0.003)*	0.361	(0.010)*	0.368	(0.008)*	0.297	(0.010)*
$\sigma_r$	0.007	(0.000)*	0.007	(0.000)*	0.000	-	0.000	-
$\sigma_\delta$	0.291	(0.008)*	0.239	(0.009)*	0.275	(0.009)*	0.000	-
$\sigma_v$	0.080	(0.004)*	0.000	-	0.000	-	0.000	-
$\rho_{sr}$	0.170	(0.065)*	0.213	(0.054)*	0.000	-	0.000	-
$\rho_{s\delta}$	0.995	(0.003)*	0.867	(0.013)*	0.862	(0.010)*	0.000	-
$\rho_{sv}$	-0.033	(0.006)*	0.000	-	0.000	-	0.000	-
$\rho_{r\delta}$	0.199	(0.064)*	0.196	(0.057)*	0.000	-	0.000	-
$\xi_F$	0.000	(0.000)*	0.000	(0.000)*	0.000	(0.000)*	0.001	(0.000)*
$\xi_B$	0.000	(0.000)*	0.000	(0.000)*	-	-	-	-
$\xi_{var}$	0.002	(0.000)*	-	-	-	-	-	-
Log-likelihood	34,540		26,396		9,383		3,482	

Reported are the parameter estimates obtained from the Kalman filter and quasi-Maximum likelihood estimation. Standard errors are in parenthesis and \* indicates significance at the 95% level. SRCV is estimated with futures, interest rate, and GARCH(1,1) variance data as discussed above. SRC is estimated with futures and interest rate data. SC and S are estimated with futures data only, where only the 1-month and 3-months contracts are used for the calibration of S. Given the use of different input data for each model, log-likelihood function values are not directly comparable across models.



# B

## APPENDIX TO CHAPTER 4

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### B.1 Definitions

**Definition 1.** The expected overage and underage quantities are given by  $\Lambda_{11}^i = \int_A^{z_1} \int_A^{z_2} (z_i - u_i) f(u_1, u_2) du_1 du_2$ ,  $\Lambda_{10}^i = \int_A^{z_i} \int_{z_j}^B (z_i - u_i) f(u_i, u_j) du_i du_j$ ,  $\Lambda_{01}^i = \int_{z_i}^B \int_A^{z_j} (u_i - z_i) f(u_i, u_j) du_i du_j$ , and  $\Lambda_{00}^i = \int_{z_1}^B \int_{z_2}^B (u_i - z_i) f(u_1, u_2) du_1 du_2$ . Further, we have  $\Lambda_0^i = \int_{z_i}^B (u_i - z_i) f(u_i, z_j) du_i$  and  $\Lambda_1^i = \int_A^{z_i} (z_i - u_i) f(u_i, z_j) du_i$ .

**Definition 2.** The probabilities corresponding to each demand/inventory scenario, denoted by  $\Gamma$ , are simply given by the integral over  $f(\cdot)$  with integration bounds corresponding to  $\Lambda$  given by Definition 1.  $\Gamma_{11} = \int_A^{z_1} \int_A^{z_2} f(u_1, u_2) du_1 du_2$ ,  $\Gamma_{10}^i = \int_A^{z_i} \int_{z_j}^B f(u_i, u_j) du_i du_j$ ,  $\Gamma_{01}^i = \int_{z_i}^B \int_A^{z_j} f(u_i, u_j) du_i du_j$ ,  $\Gamma_{00} = \int_{z_1}^B \int_{z_2}^B f(u_1, u_2) du_1 du_2$ ,  $\Gamma_1^i = \int_A^{z_i} f(u_i, z_j) du_i$ , and  $\Gamma_0^i = \int_{z_i}^B f(u_i, z_j) du_i$ .

### B.2 Lemmas and proofs

**Lemma 3.**  $\partial X_{ij}/\partial z_{i,t}$  and  $\partial X_{ji}/\partial z_{i,t}$  have the following piecewise definitions, reflecting the four cases of possible cross-selling quantities. If  $\beta_{ji}\Lambda_{01}^j \leq \Lambda_{10}^i$  and  $\beta_{ij}\Lambda_{01}^i \leq \Lambda_{10}^j$ ,  $\partial X_{ij}/\partial z_{i,t} = \beta_{ji}\Lambda_0^j$ ,  $\partial X_{ji}/\partial z_{i,t} = -\beta_{ij}\Gamma_{01}^i$ . If  $\beta_{ji}\Lambda_{01}^j \leq \Lambda_{10}^i$  and  $\beta_{ij}\Lambda_{01}^i > \Lambda_{10}^j$ ,  $\partial X_{ij}/\partial z_{i,t} = \beta_{ji}\Lambda_0^j$ ,  $\partial X_{ji}/\partial z_{i,t} = -\Lambda_1^j$ . If  $\beta_{ji}\Lambda_{01}^j > \Lambda_{10}^i$  and  $\beta_{ij}\Lambda_{01}^i \leq \Lambda_{10}^j$ ,  $\partial X_{ij}/\partial z_{i,t} = \Gamma_{10}^i$ ,  $\partial X_{ji}/\partial z_{i,t} = -\beta_{ij}\Gamma_{01}^i$ . If  $\beta_{ji}\Lambda_{01}^j > \Lambda_{10}^i$  and  $\beta_{ij}\Lambda_{01}^i > \Lambda_{10}^j$ ,  $\partial X_{ij}/\partial z_{i,t} = \Gamma_{10}^i$ ,  $\partial X_{ji}/\partial z_{i,t} = -\Lambda_1^j$ . Note that while discontinuous derivatives can generally pose challenges in numerical optimization, our analysis shows that the problem is sufficiently smooth to converge using conventional, gradient based optimization methods.

**Lemma 4.** The derivatives of the optimal hedging quantity (Proposition 2) with respect to the stocking decision  $z_{i,t}$  are as follows

(a) for the multi-contract hedge we can write the derivative as

$$\frac{\partial \mathbf{h}_t^*}{\partial z_{i,t}} = -\theta [\Sigma_t(\mathbf{c}_t)]^{-1} \frac{\partial \mathbf{Y}}{\partial z_{i,t}}, \quad (\text{B.1})$$

where, for an exemplary portfolio of futures and call options, we have

$$\mathbf{Y} = \begin{bmatrix} \text{Cov} \left( \chi_{f,1} (c_{i,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \\ \text{Cov} \left( \chi_{c,1} (c_{i,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \\ \vdots \\ \text{Cov} \left( \chi_{c,N_{i,c}} (c_{i,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \\ \text{Cov} \left( \chi_{f,1} (c_{j,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \\ \text{Cov} \left( \chi_{c,1} (c_{j,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \\ \vdots \\ \text{Cov} \left( \chi_{c,N_{j,c}} (c_{j,t+1}), x_i^{o,net} c_{i,t+1} + x_j^{o,net} c_{j,t+1} | \mathbf{c}_t \right) \end{bmatrix}. \quad (\text{B.2})$$

The term  $\text{Cov} \left( \chi_{k,n} (c_{i,t+1}), \Pi_{t+1}^* | \mathbf{c}_t \right)$  in  $\mathbf{h}_t^*$  is independent of the myopic stocking decision  $z_{i,t}$ , as the proceeds from salvage stock are counted as part of period  $t$  profits.  $\mathbf{Y}$  reflects the covariance between the payoff of each traded hedging instrument and the salvage value of excess inventory. The derivative of the  $n^{\text{th}}$  element of  $\mathbf{Y}$  with respect to  $z_{i,t}$  is given by

$$\frac{\partial y_n}{\partial z_{i,t}} = \mathbb{E} \left[ \chi_{k,n} (c_{i,t+1}) \left[ \theta c_{i,t+1} (\Gamma_{11} + \Gamma_{10}^i) - \Omega_3^i \right] | \mathbf{c}_t \right], \quad (\text{B.3})$$

where  $\Omega_3^i$  follows from Proposition 1. The individual derivatives such as  $\partial h_{i,t}^k / \partial z_{i,t}$  can be easily extracted from the vector valued solution of Equation B.1.

(b) for the single-contract hedge, consisting of one futures position in each market, the term in  $\mathbf{h}_{i,t}^*$  that is affected by  $z_{i,t}$  is  $-\theta x_{i,t}^{o,net}$  so that  $\partial \mathbf{h}_{i,t}^* / \partial z_{i,t} = -\partial \theta x_{i,t}^{o,net} / \partial z_{i,t}$ . The piecewise derivative is defined in accordance with the possible cross-selling scenarios and the resulting net overage quantities  $x_{i,t}^{o,net}$ . If  $\beta_{ji} \Lambda_{01}^j \leq \Lambda_{10}^i$  and  $\beta_{ij} \Lambda_{01}^i \leq \Lambda_{10}^j$ ,  $\partial \mathbf{h}_{i,t} / \partial z_{i,t} = -\theta (\Gamma_{00} + \Gamma_{10}^i - \beta_{ji} \Lambda_0^j)$ ,  $\partial \mathbf{h}_{j,t} / \partial z_{i,t} = -\theta \beta_{ij} \Gamma_{01}^i$ . If  $\beta_{ji} \Lambda_{01}^j \leq \Lambda_{10}^i$  and  $\beta_{ij} \Lambda_{01}^i > \Lambda_{10}^j$ ,  $\partial \mathbf{h}_{i,t} / \partial z_{i,t} = -\theta (\Gamma_{00} + \Gamma_{10}^i - \beta_{ji} \Lambda_0^j)$ ,  $\partial \mathbf{h}_{j,t} / \partial z_{i,t} = -\theta \Lambda_1^j$ . If  $\beta_{ji} \Lambda_{01}^j > \Lambda_{10}^i$  and  $\beta_{ij} \Lambda_{01}^i \leq \Lambda_{10}^j$ ,  $\partial \mathbf{h}_{i,t} / \partial z_{i,t} = -\theta \Gamma_{11}$ ,  $\partial \mathbf{h}_{j,t} / \partial z_{i,t} = -\theta \beta_{ij} \Gamma_{01}^i$ . If  $\beta_{ji} \Lambda_{01}^j > \Lambda_{10}^i$  and  $\beta_{ij} \Lambda_{01}^i > \Lambda_{10}^j$ ,  $\partial \mathbf{h}_{i,t} / \partial z_{i,t} = -\theta \Gamma_{11}$ ,  $\partial \mathbf{h}_{j,t} / \partial z_{i,t} = -\theta \Lambda_1^j$ .

**Lemma 5.** In part (a) and (b) of this lemma, we provide the partial derivatives of expected price elastic cross-demand  $X^p$  with respect to the pricing decision  $p$  and the stocking factor  $z$ , respectively.

(a) the partial derivatives of  $X_{ij}^p$  with respect to  $p$  are  $\partial X_{ij}^p / \partial p_{i,t} = -\eta$  and  $\partial X_{ij}^p / \partial p_{j,t} = \eta$  if  $X_{ij}^p = D_{ji}^p$  and  $(\beta_{ji} - 1) \Lambda_{01}^j \leq \eta (p_i - p_j) \leq \beta_{ji} \Lambda_{01}^j$ . Otherwise,  $\partial X_{ij}^p / \partial p_{i,t} = \partial X_{ij}^p / \partial p_{j,t} = 0$ . For  $X_{ji}^p$  we have  $\partial X_{ji}^p / \partial p_{i,t} = \eta$  and  $\partial X_{ji}^p / \partial p_{j,t} = -\eta$  if  $X_{ji}^p = D_{ij}^p$  and  $(\beta_{ij} - 1) \Lambda_{01}^i \leq \eta (p_j - p_i) \leq \beta_{ij} \Lambda_{01}^i$ . Otherwise,  $\partial X_{ji}^p / \partial p_{i,t} = \partial X_{ji}^p / \partial p_{j,t} = 0$ .

(b) the partial derivatives of  $X_{ij}^p$  with respect to  $z_i$  and  $z_j$  differ from the price inelastic base case derivatives provided by Lemma 3 as follows. If  $X_{ij}^p = \Lambda_{10}^i$ ,  $\partial X_{ij}^p / \partial z_i = \partial X_{ij} / \partial z_i$  and  $\partial X_{ij}^p / \partial z_j =$



$\partial X_{ij}/\partial z_j$ . If, instead,  $X_{ji}^p = D_{ji}^p$ , we have  $\partial X_{ij}^p/\partial z_i = \partial X_{ij}/\partial z_i Z_{ij}^p$  and  $\partial X_{ij}^p/\partial z_j = \partial X_{ij}/\partial z_j Z_{ij}^p$ , where

$$Z_{ij}^p = \begin{cases} 1/\beta_{ji} & \text{if } \eta(p_i - p_j) < (\beta_{ji} - 1)\Lambda_{01}^j \\ 1 & \text{if } (\beta_{ji} - 1)\Lambda_{01}^j \leq \eta(p_i - p_j) \leq \beta_{ji}\Lambda_{01}^j \\ 0 & \text{if } \eta(p_i - p_j) > \beta_{ji}\Lambda_{01}^j. \end{cases} \quad (\text{B.4})$$

Similarly, if  $X_{ji}^p = \Lambda_{10}^j$ ,  $\partial X_{ji}^p/\partial z_i = \partial X_{ji}/\partial z_i$  and  $\partial X_{ji}^p/\partial z_j = \partial X_{ji}/\partial z_j$ . If  $X_{ji}^p = D_{ij}^p$ , we have  $\partial X_{ji}^p/\partial z_i = \partial X_{ji}/\partial z_i Z_{ji}^p$  and  $\partial X_{ji}^p/\partial z_j = \partial X_{ji}/\partial z_j Z_{ji}^p$ , where

$$Z_{ji}^p = \begin{cases} 1/\beta_{ij} & \text{if } \eta(p_j - p_i) < (\beta_{ij} - 1)\Lambda_{01}^i \\ 1 & \text{if } (\beta_{ij} - 1)\Lambda_{01}^i \leq \eta(p_j - p_i) \leq \beta_{ij}\Lambda_{01}^i \\ 0 & \text{if } \eta(p_j - p_i) > \beta_{ij}\Lambda_{01}^i. \end{cases} \quad (\text{B.5})$$

Our analysis shows that the discontinuous demand model and associated piecewise derivatives admit an implementation with gradient-based solution methods as the problem is sufficiently smooth.

*Proof of Lemma 1.* The first and second partial derivatives of  $\mathbb{E}[U(\cdot)]$  with respect to  $p_i$  are given by  $\partial \mathbb{E}[U_t(\cdot)]/\partial p_{i,t} = 2b_i(p_{i,t}^0 - p_{i,t}) - (\Lambda_{00}^i + \Lambda_{01}^i) + X_{ij}$  and  $\partial^2 \mathbb{E}[U_t(\cdot)]/\partial p_{i,t}^2 = -2b_i$ , respectively. Rearranging  $\partial \mathbb{E}[U_t(\cdot)]/\partial p_{i,t}$  yields the optimal price for product  $i$  in time period  $t$   $p_{i,t}^*$ . It follows from  $\partial^2 \mathbb{E}[U_t]/\partial p_{i,t}^2$  that  $\mathbb{E}[U_t(\cdot)]$  is concave in  $p_{i,t}$  admitting a unique maximum at  $p_{i,t}^*$ .

*Proof of Lemma 2.* Equation 4.15 is obtained from the partial derivative  $\partial U(\cdot)/\partial X_{ij}$ . Noting that the discounted expected value of the future raw material price under the risk-neutral measure is  $\alpha \mathbb{E}[c_{i,t+1}|c_{i,t}] = e^{-\delta\Delta} c_{i,t}$ , we can simplify Equation 4.15 to  $p_{i,t} - c_{i,t} - \lambda\theta \text{Cov}(c_{i,t+1}, \Omega_1^i | \mathbf{c}_t) > 0$ . As the firm should not consider operating under a negative (or zero) gross margin in our model, it holds that  $p_{i,t} - c_{i,t} > 0$ . Now, we address the statements of Lemma 2. Considering the unhedged firm first, the utility contribution from cross-selling is always positive under risk neutrality ( $\lambda = 0$ ). Under risk aversion ( $\lambda > 0$ ) and  $\theta \text{Cov}(c_{i,t+1}, \Omega_1^i | \mathbf{c}_t) > 0$ , cross-selling is utility improving as it reduces leftover inventory in  $\Omega_1^i$  and, thereby, the value of the covariance term. If instead,  $\theta \text{Cov}(c_{i,t+1}, \Omega_1^i | \mathbf{c}_t) < 0$ , the salvage value of leftover inventory hedges a part of the variance in  $\Pi_{t+1}(\mathbf{z}_{t+1}, \mathbf{p}_{t+1}, |\mathbf{h}_t, \mathbf{c}_{t+1})$ . As cross-selling reduces leftovers it can increase the covariance term (lower absolute value) and decrease utility. When a financial hedge is implemented, the role of inventory as an operational hedge is fully replaced by a futures contract so that the term  $\lambda\theta \text{Cov}(c_{i,t+1}, \Omega_1^i | \mathbf{c}_t)$  can be eliminated from Equation 4.15. It follows that under a positive gross margin cross-selling is always optimal with financial hedging.

*Proofs of Lemmas 3-5.* These lemmas provide derivatives used in the propositions of this paper. A proof of these derivatives is omitted.

### B.3 Proofs of propositions

*Proof of Proposition 1.* First, note that under the jointly optimal inventory and pricing policy,  $p_{i,t} = p_{i,t}^*$  so that in Equation 4.11

$$\frac{\partial U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)}{\partial p_{i,t}} \frac{\partial p_{i,t}^*}{\partial z_{i,t}} + \frac{\partial U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)}{\partial p_{j,t}} \frac{\partial p_{j,t}^*}{\partial z_{i,t}} = 0, \quad (\text{B.6})$$

since  $\partial U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t) / \partial p_{i,t} = 0$  and  $\partial U_t(\mathbf{z}_t, \mathbf{p}_t^*, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t) / \partial p_{j,t} = 0$ , which follows from the definition of  $\mathbf{p}_t^*$  as the utility maximizing price. Thus, the partial derivative of  $U_t(\cdot)$  with respect to  $z_{i,t}$  simplifies to Equation 4.11.

In order to prove the optimality and uniqueness of  $\mathbf{z}_t^*$ , we first establish the concavity of  $U_t(\cdot)$  with respect to  $z_{i,t}$ . We then verify the joint concavity of  $U_t(\cdot)$  with respect to  $\mathbf{z}_t^*$  and discuss the critical stockout penalty  $\bar{\gamma}_i$ . Noting that  $\alpha\theta\mathbb{E}[c_{i,t+1} | \mathbf{c}_t] = c_{i,t}$ , the second partial derivative of  $U_t(\cdot)$  with respect to  $z_{i,t}$  is

$$\begin{aligned} \frac{\partial^2 \mathbb{E}[U_t(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}^2} &= (c_{i,t} - p_{i,t}^* - \gamma_i) (\Gamma_0^j + \Gamma_1^j) \\ &\quad - \lambda [V(\Omega_2^i | \mathbf{c}_t) + \text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t)], \end{aligned} \quad (\text{B.7})$$

where  $\Omega_1^i$  and  $\Omega_2^i$  follow from Proposition 1 and  $\Omega_4^i = \theta c_{i,t+1} (\Gamma_0^j + \Gamma_1^j)$ . Since the firm should only operate under a positive gross margin,  $p_{i,t} > c_{i,t}$ , and  $\gamma_i \geq 0$ , the first part of Equation B.7 is negative. Hence, for the risk-neutral firm it always holds that  $\partial^2 \mathbb{E}[U_t(\cdot)] / \partial z_{i,t}^2 < 0$ . If  $\lambda > 0$ , we first note that  $V(\Omega_2^i | \mathbf{c}_t) > 0$  so that Equation B.7 is only greater than zero if  $\text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t) < (c_{i,t} - p_{i,t} - \gamma_i) (\Gamma_0^j + \Gamma_1^j) / \lambda - V(\Omega_2^i | \mathbf{c}_t) < 0$ , which is never the case in the final period  $T$ , when  $\text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t) > 0$ . Also if a financial hedge is implemented according to Proposition 2, the firm can perfectly hedge the covariance between future costs and profits so that  $\text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t) = 0$ . Finally, for the risk-averse, unhedged firm at time  $t < T$ , the above covariance can be negative. For  $U_t(\cdot)$  to be concave with respect to  $z_i$  in this case, we can rearrange Equation B.7 to show that if  $\xi < 0$ , where  $\xi = (\Omega_2^i | \mathbf{c}_t) + \text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t)$ , it must hold that  $\lambda < (c_{i,t} - p_{i,t} - \gamma_i) (\Gamma_0^j + \Gamma_1^j) / \xi$ . For completeness, if  $\text{Cov}(\Omega_1^i, \Omega_4^i | \mathbf{c}_t) > 0$ , it has to hold that  $\lambda \geq 0$ , which is true by definition. Our analysis shows that the above condition is not restrictive in the sense that it is satisfied for a wide set of realistic parameter values. We can also rewrite this condition in terms of the stockout penalty  $\gamma_i$ , which must satisfy  $\gamma_i > \zeta_i = c_{i,t} - p_{i,t} - \lambda \xi / (\Gamma_0^j + \Gamma_1^j)$ . We find that  $\zeta_i < \bar{\gamma}_i$  is similarly satisfied throughout the numerical study of this paper, where  $\bar{\gamma}_i = \beta_{ij} [p_j - \alpha\theta\mathbb{E}[c_{j,t+1} | \mathbf{c}_t]] - (p_i - c_i)$  follows from Section 4.4. To establish the joint concavity of  $U_t(\cdot)$  with respect to  $z_{i,t}$  and  $z_{j,t}$ , we verify that  $\det(H(z_{i,t}, z_{j,t})) > 0 \forall t$ , which in conjunction with  $\partial^2 \mathbb{E}[U_t(\cdot)] / \partial z_{i,t}^2 < 0$  concludes this proof.

*Proof of Proposition 2.* The first and second partial derivatives of utility with respect to the firm's hedging decision are given by

$$\frac{\partial \mathbb{E} [U_t (\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial \mathbf{h}_t} = -\lambda [\Sigma_t (\mathbf{c}_t) \mathbf{h}_t - \Psi_t (\mathbf{z}_t^*, \mathbf{c}_t)], \quad (\text{B.8})$$

and

$$\frac{\partial^2 \mathbb{E} [U_t (\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial \mathbf{h}_t^2} = -\lambda \Sigma_t (\mathbf{c}_t), \quad (\text{B.9})$$

respectively. Rearranging Equation B.8 yields the solution to  $\mathbf{h}_t^*$  of Proposition 2. As  $\Sigma_t (\mathbf{c}_t)$  is positive semidefinite, the second derivative, reflecting the Hessian matrix of the utility function, is negative semidefinite so that  $U_t (\cdot)$  is concave in  $\mathbf{h}_t$  (Kouvelis et al., 2013).

*Proof of Proposition 3.* The optimal pricing decision under price elastic cross-demand  $D_{ij}^p$  given by Proposition 3 follows from the first partial derivative of  $U_t^p (\cdot)$  with respect to  $p$ . The second partial derivative is given by

$$\frac{\partial^2 \mathbb{E} [U_t^p (\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial p_{i,t}^2} = 2 \left( \frac{\partial X_{ij}^p}{\partial p_{i,t}} - b_i \right) - \lambda V (\Omega_5^i | \mathbf{c}_t), \quad (\text{B.10})$$

where  $\Omega_5^i$  is given as part of Proposition 3. Since  $\partial X_{ij}^p / \partial p_{i,t} \leq 0$  (from Lemma 5),  $b_i > 0$ , and  $V (\cdot) \geq 0$ , Equation B.10 is negative implying that  $U_t^p (\cdot)$  is a concave function of  $p_{i,t}$ , which is uniquely determined as a function of the joint stocking factor  $\mathbf{z}_t^*$ . The proof of joint concavity of  $U_t^p (\cdot)$  with respect to  $\mathbf{z}_t^*$  is analogous to the proof of Proposition 1.

## B.4 Corollaries

**Corollary 2.** *In this corollary of Proposition 1, we discuss the optimal operating (pricing and inventory) policy under the price inelastic specification of cross-demand and a binding constraint on  $\gamma_i$ .*

*In order to obtain a unique solution for  $\mathbf{z}_t^*$  and  $\mathbf{p}_t^*$ , it must hold that  $\gamma_i \geq \bar{\gamma}_i = p_j X_{ji} / \Lambda_{01}^i - (p_i - c_i)$ . If this constraint is binding, we substitute  $\bar{\gamma}_i$  for  $\gamma_i$  into the utility function (Equation 4.8), which we now denote by  $\bar{U} (\cdot)$ . From the partial derivatives of  $\mathbb{E} [\bar{U} (\cdot)]$  with respect to  $p_i$  and  $z_i$ , we obtain the following optimal policies for the pricing and inventory decision*

$$\bar{p}_{i,t}^* (\bar{\mathbf{z}}_t^* | c_{i,t}) = p_{i,t}^* (\mathbf{z}_t^* | c_{i,t}) + \frac{\Lambda_{00}^i + \Lambda_{01}^i - \beta_{ji} (\Lambda_{00}^j + \Lambda_{01}^j)}{2b_i}, \quad (\text{B.11})$$

where  $p_{i,t}^* (\cdot)$  is given by Lemma 1. The joint stocking factor is now obtained as the solution to

$$\frac{\partial \mathbb{E} [\bar{U}_t (\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}} : \frac{\partial \mathbb{E} [U_t (\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}} + (\Gamma_{00} + \Gamma_{01}^i) \left[ c_{i,t} - p_{i,t} - \gamma_i + \beta_{ij} \left( p_{j,t} - \alpha \theta \mathbb{E} [c_{j,t+1} | \mathbf{c}_t] \right) \right], \quad (\text{B.12})$$

where  $\partial \mathbb{E}[U(\cdot)]/\partial z_i$  follows from Proposition 1. Note that the constant  $\gamma_i$ , which is replaced by  $\bar{\gamma}_i$  in  $\mathbb{E}[\bar{U}(\cdot)]$ , disappears in Equation B.12 once  $\partial \mathbb{E}[U(\cdot)]/\partial z_i$  is substituted. The optimal hedging decision is analogous to Proposition 2.

**Corollary 3.** In this corollary of Proposition 3, we address the firm's optimal operating policy under the price elastic specification of cross-demand  $X_{ij}^P = [\Lambda_{10}^i, D_{ji}^P]^-$  and a binding constraint on the stockout penalty.

Similarly to the price inelastic case, uniqueness of  $\mathbf{z}_t^*$  and  $\mathbf{p}_t^*$  under price elastic cross-demand necessitates that the expected cost from an induced stockout in one division is large enough to at least offset the gross profit earned by the second division by serving the resulting cross-demand for the substitute product. Formally, it needs to hold that  $(p_i + \hat{\gamma}_i - c_i) \Lambda_{01}^i \geq [p_j - \mathbb{E}[c_{j,t+1}|\mathbf{c}_t]] X_{ji}^P$ , which leads to a critical value  $\hat{\gamma}_i = [p_j - \alpha \theta \mathbb{E}[c_{j,t+1}|\mathbf{c}_t]] \frac{X_{ji}^P}{\Lambda_{01}^i} - (p_i - c_i)$  and the condition  $\gamma_i \geq \hat{\gamma}_i$ . If the constraint is binding,  $\hat{\gamma}_i$  is substituted for  $\gamma_i$  into the utility function (Equation 4.8), now denoted by  $\hat{U}^P(\cdot)$ . The optimal operating policy is then implicitly characterized by the partial derivatives  $\partial \mathbb{E}[\hat{U}^P(\cdot)]/\partial p_i$  and  $\partial \mathbb{E}[\hat{U}^P(\cdot)]/\partial z_i$ , given by

$$\begin{aligned} \frac{\partial \mathbb{E}[\hat{U}_t^P(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial p_{i,t}} &: \frac{\partial \mathbb{E}[U_t^P(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial p_{i,t}} \\ &- \left[ X_{ij}^P + \frac{\partial X_{ij}^P}{\partial p_{i,t}} (p_{i,t} - \alpha \theta \mathbb{E}[c_{i,t+1}|\mathbf{c}_t]) \right] \frac{\Lambda_{00}^j + \Lambda_{01}^j}{\Lambda_{01}^j} \\ &+ \left[ \Lambda_{01}^i - \frac{\partial X_{ji}^P}{\partial p_{i,t}} (p_{j,t} - \alpha \theta \mathbb{E}[c_{j,t+1}|\mathbf{c}_t]) \right] \frac{\Lambda_{00}^i + \Lambda_{01}^i}{\Lambda_{01}^i} = 0, \end{aligned} \quad (\text{B.13})$$

where  $\partial \mathbb{E}[U^P(\cdot)]/\partial p_i$  follows from Equation 4.18 and

$$\begin{aligned} \frac{\partial \mathbb{E}[\hat{U}_t^P(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}} &: \frac{\partial \mathbb{E}[U_t^P(\mathbf{z}_t, \mathbf{p}_t, \mathbf{h}_t | \mathbf{h}_{t-1}, \mathbf{c}_t)]}{\partial z_{i,t}} \\ &- (p_{i,t} - c_{i,t} - \gamma_i) (\Gamma_{00} + \Gamma_{01}^i) \\ &- (p_{i,t} - \alpha \theta \mathbb{E}[c_{i,t+1}|\mathbf{c}_t]) \left( \frac{\partial X_{ij}^P}{\partial z_{i,t}} - \frac{X_{ij} \Lambda_{01}^j}{\Lambda_{01}^j} \right) \frac{\Lambda_{00}^j + \Lambda_{01}^j}{\Lambda_{01}^j} \\ &- (p_{j,t} - \alpha \theta \mathbb{E}[c_{j,t+1}|\mathbf{c}_t]) \left[ \left( \frac{\partial X_{ji}^P}{\partial z_{i,t}} + \frac{X_{ji} \Gamma_{01}^i}{\Lambda_{01}^i} \right) \frac{\Lambda_{00}^i + \Lambda_{01}^i}{\Lambda_{01}^i} + X_{ji}^P \frac{\Gamma_{00} + \Gamma_{01}^i}{\Lambda_{01}^i} \right] = 0. \end{aligned} \quad (\text{B.14})$$

The partial derivatives of the expected price elastic cross-selling quantity  $X^P$  with respect to  $p$  and  $z$  are given by Lemma 5 in Appendix B.2.

## B.5 Supplement to numerical study

We use the following transition density to compute expectations of the bivariate GBM process:

$$\begin{aligned}
f(\mathbf{c}_t, \mathbf{c}_{t+1}) &= \frac{1}{2\pi\Delta\sqrt{1-\rho_c^2\sigma_{c,1}^2\sigma_{c,2}^2}c_{1,t+1}c_{2,t+1}} \\
&\times \exp\left(-\frac{\log(c_{1,t+1})-\log(c_{1,t})-\mu_{c,1}(\Delta)^2}{2(\sigma_{c,1}^2)(1-\rho_c^2)\Delta}\right) \\
&\times \exp\left(-\frac{\log(c_{2,t+1})-\log(c_{2,t})-\mu_{c,2}(\Delta)^2}{2(\sigma_{c,2}^2)(1-\rho_c^2)\Delta}\right) \\
&\times \exp\left(\frac{(\log(c_{1,t+1})-\log(c_{1,t})-\mu_{c,1}\Delta)(\log(c_{2,t+1})-\log(c_{2,t})-\mu_{c,2}\Delta)\rho_c}{\sigma_{c,1}\sigma_{c,2}(1-\rho_c^2)\Delta}\right), \tag{B.15}
\end{aligned}$$

where  $\mu_{c,i} = r - \delta_i - \frac{1}{2}\sigma_{c,i}^2$ .

## B.6 Derivation of profit function

In this section, we describe the profit function obtained by substituting  $z_i = q_i - y_i(p_i)$  into Equation 4.2 under integral notation. From this representation, we obtain Equation 4.4, which is the profit function used throughout our analysis.

Given the definition of  $z_i$ , we can rewrite the profit of division  $i$  in period  $t$  from Equation 4.2,  $\pi_{i,t}(z_{i,t}, p_{i,t}, h_{i,t} | z_{j,t}, h_{i,t-1}, c_{i,t})$ , as

$$\begin{aligned}
\mathbb{E}[\pi_i(\cdot)] &= \int_A^{z_i} \int_A^{z_j} \left[ p_i [y_i(p_i) + u_i] \right. \\
&\quad \left. + \alpha\theta\mathbb{E}[c_{i,t+1}|c_{i,t}](z_i - u_i) \right] f(u_i, u_j) du_i du_j \\
&\quad + \int_A^{z_i} \int_{z_j}^B \left[ p_i [y_i(p_i) + u_i] + p_i X_{ij} \right. \\
&\quad \left. + \alpha\theta\mathbb{E}[c_{i,t+1}|c_{i,t}](z_i - u_i - X_{ij}) \right] f(u_i, u_j) du_i du_j \\
&\quad + \int_{z_i}^B \int_A^B \left[ p_i [y_i(p_i) + z_i] - \gamma_i(u_i - z_i) \right] f(u_i, u_j) du_i du_j \\
&\quad - c_i [y_i(p_i) + z_i] + \phi_{i,t}(h_{i,t-1}, c_{i,t}), \tag{B.16}
\end{aligned}$$

where  $X_{ij} = [z_i - u_i, \beta_{ji} (u_j - z_j)]^-$ . The three cases (integrals) in Equation B.16 follow from the representation of effective sales in Equation 4.1. In particular, in the first case we have overage in both products, i.e.,  $D_i < q_i$  and  $D_j < q_j$ , which is equivalent to writing  $\varepsilon_i \leq z_i$  and  $\varepsilon_j \leq z_j$ . As the inventory levels of each product are sufficient to satisfy the respective demand, we have no cross-demand for either product, which could arise from a shortage in the second product. Instead, divisional profits are independent and given by the sum of sales revenues and the expected salvage value of excess inventory. Under the second integral, we account for the case where  $\varepsilon_i \leq z_i$  but  $\varepsilon_j > z_j$ , i.e., the situation where the inventory of product  $i$  is sufficient to cover the respective standalone demand  $D_i$ , while division  $j$  is facing a shortage. As discussed above, some of the customers originally interested in product  $j$  may now purchase product  $i$  as a substitute (or purchase neither of the products if they are complementary). Therefore, the profits of division  $i$  are now given by the sum of standalone revenue, the additional sales proceeds from serving the cross-demand, and the salvage value arising from any net leftovers that remain after cross demand is served.

The third case reflects the expected profit when both divisions are facing a stockout. As no cross-selling takes place, divisional profits are independently given as the sum of sales revenue and shortage costs. Note that when  $\beta_{ji} = 0$ , expectations of  $\varepsilon_i$  may simplify to the univariate integral w.r.t  $u_i$  even in the presence of correlated demand shocks  $\text{Cov}(\varepsilon_i, \varepsilon_j) \neq 0$ . This is the case when the marginal densities of the bivariate demand shock are independent of the correlation coefficient as is the case for, e.g., the bivariate normal distribution. However, if cross-selling takes place and demand shocks are correlated, profits of division  $i$  depend on the state of demand  $j$ , requiring the joint evaluation of both demand shock expectations.

It is worth noting that we do not assume a reimbursement of stockout penalty costs in division  $i$  for those units of lost demand that are absorbed by cross-selling from division  $j$ . If desired, this can be easily incorporated by adding the following case to Equation B.16, which then, of course, requires an adjustment of the lower integration boundary in the third integral from  $A$  to  $z_j$

$$\int_{z_i}^B \int_A^{z_j} \left[ p_i [y_i(p_i) + z_i] - \gamma_i [u_i - z_i - X_{ji}] \right] f(u_i, u_j) du_i du_j. \quad (\text{B.17})$$

With this adjustment, shortage penalty costs are incurred based on the net underage quantity after cross-sales  $u_i - z_i - X_{ji}$ , where  $X_{ji} = [z_j - u_j, \beta_{ij} (u_i - z_i)]^-$ . For instance, if the products are substitutes, cross-selling reduces underage costs of division  $i$  by  $\gamma_i X_{ji} > 0$ , which represents the recovered stockout penalty corresponding to the fraction of demand that is lost by division  $i$  but absorbed by division  $j$ . If the two divisions offer complementary products,  $\beta_{ij} < 0$  and  $X_{ji} = \beta_{ij} (u_i - z_i) < 0$  so that division  $i$  incurs additional shortage costs  $\gamma_i \beta_{ij} (u_i - z_i)$ . These reflect the lost sales of product  $j$  induced by the shortage in the complementary product  $i$  and are counted as part of the profit in division  $i$ . For simplicity, we do not further consider the possibility of penalty cost reimbursement in this thesis.

## REFERENCES

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- Agrawal, V., & Seshadri, S. (2000). Impact of uncertainty and risk aversion on price and order quantity in the newsvendor problem. *Manufacturing & Service Operations Management*, 2(4), 410-423.
- Alexander, J. C., & Sobel, M. J. (2006). *Risk neutrality and ordered vector spaces*. (Technical Report TM-808A). Case Western Reserve University, Cleveland.
- Allayannis, G., & Weston, J. P. (2001). The use of foreign currency derivatives and firm market value. *Review of Financial Studies*, 14(1), 243-276.
- Ang, A., Chen, J., & Xing, Y. (2006). Downside risk. *The Review of Financial Studies*, 19(4), 1191-1239.
- Aydin, G., & Porteus, E. L. (2008). Joint inventory and pricing decisions for an assortment. *Operations Research*, 56(5), 1247-1255.
- Bakshi, G., Cao, C., & Zhiwu, C. (1997). Empirical performance of alternative option pricing models. *Journal of Finance*, 52(5), 2003-2049.
- Bakshi, G., & Madan, D. (2000). Spanning and derivative-security valuation. *Journal of Financial Economics*, 55(2), 205-238.
- Bank of International Settlements. (2007). *Quarterly review (March): International banking and financial market developments*.
- Basak, S., & Chabakauri, G. (2010). Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23(8), 2970-3016.
- Berling, P., & Martínez-de-Albéniz, V. (2011). Optimal inventory policies when purchase price and demand are stochastic. *Operations Research*, 59(1), 109-124.
- Berling, P., & Rosling, K. (2005). The effects of financial risks on inventory policy. *Management Science*, 51(12), 1804-1815.
- Bessembinder, H., Coughenour, J. F., Seguin, P. J., & Smoller, M. M. (1995). Mean reversion in equilibrium asset prices: Evidence from the futures term structure. *Journal of Finance*, 50(1), 361-375.
- Billington, C., Johnson, B., & Triantis, A. (2002). A real options perspective on supply chain management in high technology. *Journal of Applied Corporate Finance*, 15(2), 32-43.
- Bisi, A., & Dada, M. (2007). Dynamic learning, pricing, and ordering by a censored newsvendor. *Naval Research Logistics*, 54(4), 448-461.
- Björk, T. (2009). *Arbitrage theory in continuous time*. Oxford: Oxford University Press.
- Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics*, 3, 167-179.
- Bloch, H., Fraser, P., & MacDonald, G. (2012). Commodity prices: How important are real and nominal shocks? *Applied Economics*, 44, 2347-2357.
- Bodily, S., & Palacios, M. (2007). *Integrating financial hedges and operational options in an electric power plant*. Working paper, University of California, Berkeley.
- Boogert, A., & de Jong, C. (2008). Gas storage valuation using a Monte Carlo method. *Journal of Derivatives*, 15(3), 81-98.
- Boogert, A., & de Jong, C. (2011). Gas storage valuation using a multifactor price process. *Journal of Energy Markets*, 4(4), 29-52.
- Brennan, M. J. (1991). The price of convenience and the valuation of commodity contingent claims. In *Stochastic models and option values: Applications to resources, environment and investment problems* (p. 33-71).

- UCLA: Contributions to Economic Analysis, no. 200.
- Brennan, M. J., & Schwartz, E. S. (1985). Evaluating natural resource investments. *Journal of Business*, 58(2), 135-157.
- Caldentey, R., & Haugh, M. (2006). Optimal control and hedging of operations in the presence of financial markets. *Mathematics of Operations Research*, 31(2), 285-304.
- Caldentey, R., & Haugh, M. B. (2009). Supply contracts with financial hedging. *Operations Research*, 57(1), 47-65.
- Casassus, J., & Collin-Dufresne, P. (2005). Stochastic convenience yield implied from commodity futures and interest rates. *Journal of Finance*, 60(5), 2283-2331.
- Cashin, P., Liang, H., & McDermott, J. (2000). How persistent are shocks to world commodity prices? *IMF Staff Papers*, 47(2), 177-217.
- Chen, J., & Bell, P. C. (2009). The impact of customer returns on pricing and order decisions. *European Journal of Operational Research*, 195(1), 280-295.
- Chen, X., Sim, M., Simchi-Levi, D., & Peng, S. (2007). Risk aversion in inventory management. *Operations Research*, 55(5), 828-842.
- Chen, X., & Simchi-Levi, D. (2004a). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research*, 52(6), 887-896.
- Chen, X., & Simchi-Levi, D. (2004b). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research*, 29(3), 698-723.
- Chen, X., & Simchi-Levi, D. (2012). Pricing and inventory management. In R. Phillips & O. Özer (Eds.), *The oxford handbook of pricing management* (p. 784-822). Oxford University Press.
- Chen, Y., Xu, M., & Zhang, Z. (2009). Technical note - a risk-averse newsvendor model under the CVaR criterion. *Operations Research*, 57(4), 1040-1044.
- Chod, J., Rudi, N., & Van Mieghem, J. A. (2010). Operational flexibility and financial hedging: Complements or substitutes? *Management Science*, 56(6), 1030-1045.
- Clewlow, L., & Strickland, C. (1999). *A multi-factor model for energy prices*. Working Paper, University of Technology, School of Finance and Economics, Sydney, Australia.
- CME Group. (2013). *Copper futures and options*. Retrieved from <http://www.cmegroup.com/trading/metals/copper-futures-and-options-fact-card.html>
- Collin-Dufresne, P., & Goldstein, R. S. (2002). Do bonds span the fixed income markets? theory and evidence for unspanned stochastic volatility. *Journal of Finance*, 57(4), 1685-1730.
- Collin-Dufresne, P., Goldstein, R. S., & Jones, C. S. (2008). Identification of maximal affine term structure models. *Journal of Finance*, 63(2), 743-795.
- Cortazar, G., & Naranjo, L. (2006). An n-factor gaussian model of oil futures prices. *Journal of Futures Markets*, 26, 243-268.
- Cortazar, G., & Schwartz, E. (1993). A compound option model of production and intermediate investment. *Journal of Business*, 66(4), 517-540.
- Cortazar, G., & Schwartz, E. S. (1994). The valuation of commodity contingent claims. *The Journal of Derivatives*, 1(4), 27-39.
- Cortazar, G., & Schwartz, E. S. (2003). Implementing a stochastic model for oil futures prices. *Energy Economics*, 25, 215-238.
- Cox, J., Ingersoll, J., & Ross, S. (1981). The relation between forward prices and futures prices. *Journal of Financial Economics*, 9(4), 321-346.
- Dai, Q., & Singleton, K. J. (2000). Specification analysis of affine term structure models. *Journal of Finance*,



- 55(5), 1943-1978.
- Deaton, A., & Laroque, G. (1996). Competitive storage and commodity price dynamics. *Journal of Political Economy*, 104(5), 896-923.
- Deng, S. (2000). *Stochastic models of energy commodity prices and their applications: Mean-reversion with jump and spikes*. Working Paper, University of California Energy Institute, Berkeley, California.
- Ding, Q., Dong, L., & Kouvelis, P. (2007). On the integration of production and financial hedging decisions in global markets. *Operations Research*, 55(3), 470-489.
- Dixit, A. K., & Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton, N.J. Princeton University Press 1994.
- Dong, L., Kouvelis, P., & Tian, Z. (2009). Dynamic pricing and inventory control of substitute products. *Manufacturing & Service Operations Management*, 11(2), 317-339.
- Doran, J., & Ronn, E. (2008). Computing the market price of volatility risk in the energy commodity markets. *Journal of Banking and Finance*, 32(12), 2541-2552.
- Duan, J.-C., & Simonato, J.-G. (1999). Estimating and testing exponential-affine term structure models by kalman filter. *Review of Quantitative Finance and Accounting*, 13(2), 111-135.
- Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *Journal of Finance*, 57(1), 405-443.
- Duffee, G. R., & Stanton, R. (2004). *Estimation of dynamic term structure models*. Working Paper, Haas School of Business, University of California, Berkeley.
- Duffie, D. (1996). *Dynamic asset pricing theory*. Princeton: Princeton University Press.
- Duffie, D., & Kan, R. (1996). A yield-factor model of interest rates. *Mathematical Finance*, 6(4), 379-406.
- Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68, 1343-1376.
- Edwards, F. R., & Canter, M. S. (1995). The collapse of Metallgesellschaft: Unhedgeable risks, poor hedging strategy, or just bad luck? *Journal of Futures Markets*, 15(3), 211-264.
- Eppen, G. D., Martin, R. K., & Schrage, L. (1989). Or practice—A scenario approach to capacity planning. *Operations research*, 37(4), 517-527.
- Ernst, R. L. (1970). *A linear inventory model of a monopolistic firm*. Ph.D. Dissertation, Department of Economics, University of California, Berkeley, CA.
- Eydeland, A., & Wolyniec, K. (2003). *Energy and power risk management: New developments in modeling, pricing, and hedging*. New Jersey: John Wiley & Sons.
- Fackler, P. L., & Roberts, M. C. (1999). A term structure model for agricultural futures. *American Agricultural Economics Association Meetings, Nashville*.
- Federgruen, A., & Heching, A. (1999). Combined pricing and inventory control under uncertainty. *Operations Research*, 47(3), 454-475.
- Ferreira, K. J., Lee, B. H. A., & Simchi-Levi, D. (2016). Analytics for an online retailer: Demand forecasting and price optimization. *Manufacturing & Service Operations Management*, 18(1), 69-89.
- Fisher, B., & Kumar, A. (2010). The right way to hedge. *McKinsey Quarterly (July)*.
- Froot, K. A., Scharfstein, D. S., & Stein, J. C. (1993). Risk management: Coordinating corporate investment and financing policies. *Journal of Finance*, 48(5), 1629-1658.
- Gaur, V., & Seshadri, S. (2005). Hedging inventory risk through market instruments. *Manufacturing & Service Operations Management*, 7(2), 103-120.
- Geman, H., & Nguyen, V.-N. (2005). Soybean inventory and forward curve dynamics. *Management Science*, 51(7), 1076-1091.
- Gibson, R., & Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims.

- Journal of Finance*, 45(3), 959-976.
- Goel, A. (2007). *Integrating commodity markets in the procurement policies for different supply chain structures*. Ph. D. Thesis, University of Texas at Austin.
- Goel, A., & Gutierrez, G. (2011). Multiechelon procurement and distribution policies for traded commodities. *Management Science*, 57(12), 2228-2244.
- Gruen, T. W., Corsten, D. S., & Bharadwaj, S. (2002). *Retail out-of-stocks: A worldwide examination of extent, causes and consumer responses*. Grocery Manufacturers of America, The Food Marketing Institute, and CIES.
- Haksöz, Ç., & Seshadri, S. (2007). Supply chain operations in the presence of a spot market: A review with discussion. *The Journal of the Operational Research Society*, 58(11), 1412-1429.
- Harvey, A. C. (1989). *Forecasting, structural time series models and the Kalman filter*. Cambridge: Cambridge University Press.
- Heath, D., Jarrow, R., & Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60(1), 77-105.
- Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2), 327-343.
- Hikspoors, S., & Jaimungal, S. (2007). Energy spot price models and spread options pricing. *International Journal of Theoretical and Applied Finance*, 10(7), 1111-1135.
- Hikspoors, S., & Jaimungal, S. (2008). Asymptotic pricing of commodity derivatives using stochastic volatility spot models. *Applied Mathematical Finance*, 15, 449-477.
- Hillard, J. E., & Reis, J. (1998). Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot. *Journal of Financial and Quantitative Analysis*, 33, 61-86.
- Hodder, J. E., & Dincer, M. C. (1986). A multifactor model for international plant location and financing under uncertainty. *Computers & Operations Research*, 13(5), 601-609.
- Honhon, D., Gaur, V., & Seshadri, S. (2010). Assortment planning and inventory decisions under stockout-based substitution. *Operations Research*, 58(5), 1364-1379.
- Hu, X., Munson, C., & Fotopoulos, S. (2012). Purchasing decisions under stochastic prices: Approximate solutions for order time, order quantity and supplier selection. *Annals of Operations Research*, 201(1), 287-305.
- Hughen, W. K. (2010). A maximal affine stochastic volatility model of oil prices. *Journal of Futures Markets*, 30(2), 101-133.
- Kat, H. M., & Oomen, R. C. A. (2007). What every investor should know about commodities part i: Univariate return analysis. *Journal of Investment Management*, 5(1), 4-28.
- Khouja, M., Mehrez, A., & Rabinowitz, G. (1996). A two-item newsboy problem with substitutability. *International Journal of Production Economics*, 44(3), 267-275.
- Kleindorfer, P. R. (2009). Integrating physical and financial risk management in supply management. In H. Geman (Ed.), *Risk management in commodity markets: From shipping to agriculturals and energy*. John Wiley & Sons.
- Kleindorfer, P. R. (2010). Hedging commodity risks in supply management. In R. Cont (Ed.), *Encyclopedia of quantitative finance*. John Wiley & Sons.
- Korn, O. (2005). Drift matters: An analysis of commodity derivatives. *Journal of Futures Markets*, 25(3), 211-241.
- Kouvelis, P., Li, R., & Ding, Q. (2013). Managing storable commodity risks: The role of inventory and financial hedge. *Manufacturing & Service Operations Management*, 15(3), 507-521.
- Laughton, D. G., & Jacoby, H. D. (1993). Reversion, timing options, and long-term decision-making. *Financial*

- Management*, 22(3), 225-240.
- Litzenberger, R. H., & Rabinowitz, N. (1995). Backwardation in oil futures markets: Theory and empirical evidence. *The Journal of Finance*(5), 1517.
- Liu, P., & Tang, K. (2011). The stochastic behavior of commodity prices with heteroskedasticity in the convenience yield. *Journal of Empirical Finance*, 18(2), 211-224.
- Lo, A. W., & Wang, J. (1995). Implementing option pricing models when asset returns are predictable. *Journal of Finance*, 50, 87-129.
- Longstaff, F. A., & Schwartz, E. S. (2001). Valuing American options by simulation: A simple least-squares approach. *Review of Financial Studies*, 14(1), 113-147.
- Lucia, J., & Schwartz, E. S. (2002). Electricity prices and power derivatives: Evidence from the Nordic Power Exchange. *Review of Derivatives Research*, 5(1), 5-50.
- Lutz, B. (2010). *Pricing of derivatives on mean-reverting assets*. Berlin: Springer.
- MacKay, P., & Moeller, S. B. (2007). The value of corporate risk management. *The Journal of Finance*, 62(3), 1379-1419.
- Mayer, J. (2009). The growing interdependence between financial and commodity markets. *UNCTAD Discussion Paper 195*.
- Mills, E. S. (1959). Uncertainty and price theory. *Quarterly Journal of Economics*, 73(1), 116-130.
- Monahan, G., Petruzzi, N. C., & Zhao, W. (2004). The dynamic pricing problem from a newsvendor's perspective. *Manufacturing & Service Operations Management*, 6(1), 73-91.
- Murray, C. C., Gosavi, A., & Talukdar, D. (2012). The multi-product price-setting newsvendor with resource capacity constraints. *International Journal of Production Economics*, 138(1), 148-158.
- Nagali, V., Hwang, J., Sanghera, D., Gaskins, M., Pridgen, M., Thurston, T., ... Shoemaker, G. (2008). Procurement risk management (PRM) at Hewlett-Packard company. *Interfaces*, 38(1), 51-60.
- Nelson, C., & Siegel, A. (1987). Parsimonious modeling of yield curves. *Journal of Business*, 60, 473-498.
- Ng, V. K., & Pirrong, S. C. (1994). Fundamentals and volatility: Storage, spreads, and the dynamics of metals prices. *Journal of Business*, 67(2), 203.
- Nielsen, M., & Schwartz, E. (2004). Theory of storage and the pricing of commodity claims. *Review of Derivatives Research*, 7(1), 5-24.
- Nissanke, M. (2012). Commodity market linkages in the global financial crisis: Excess volatility and development impacts. *Journal of Development Studies*, 48(6), 732-750.
- Okyay, H. K., Karaesmen, F., & Ozekici, S. (2015). Hedging demand and supply risks in the newsvendor model. *OR Spectrum*, 37(2), 475-501.
- Onstad, E., Obayashi, Y., & Shamseddine, R. (2016). Auto, power firms save millions swapping copper for aluminum. *Reuters*.
- Oum, Y., Oren, S., & Deng, S. (2006). Hedging quantity risks with standard power options in a competitive wholesale electricity market. *Naval Research Logistics*, 53(7), 697-712.
- Parlar, M., & Goyal, S. K. (1984). Optimal ordering decisions for two substitutable products with stochastic demands. *Opsearch*, 21(1), 1-15.
- Petruzzi, N. C., & Dada, M. (1999). Pricing and the newsvendor problem: A review with extensions. *Operations Research*, 47(2), 183-194.
- Pilipovic, D. (1997). *Valuing and managing energy derivatives*. New York: McGraw-Hill.
- Pindyck, R. S. (1999). The long-run evolution of energy prices. *Energy Journal*, 20(2), 1-27.
- Pindyck, R. S. (2004). Volatility and commodity price dynamics. *Journal of Futures Markets*, 24(11), 1029-1047.
- Pindyck, R. S., & Rubinfeld, D. L. (1991). *Econometric models and economic forecasts*. New York: McGraw-Hill.

- Pryor, L., Schlobohm, R., & Brownell, B. (2008). A comparison of aluminum vs. copper as used in electrical equipment. *General Electric Publication Library*.
- PwC. (2009). *Navigation: Managing commodity risk through market uncertainty*.
- Realdon, M. (2013). Revisiting the pricing of commodity futures and forwards. *Applied Financial Economics*, 23(3), 233-240.
- Richter, M., & Sørensen, C. (2003). *Stochastic volatility and seasonality in commodity futures and options: The case of soybeans*. Working Paper, Copenhagen Business School, Denmark.
- Ross, S. A. (1997). Hedging long run commitments: Exercises in incomplete market pricing. *Economic Notes*, 26(2), 385-419.
- Routledge, B. R., Seppi, D. J., & Spatt, C. S. (2000). Equilibrium forward curves for commodities. *The Journal of Finance*(3), 1297-1338.
- Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(1), 41-49.
- Schöne, M. F. (2014). *Real options valuation: The importance of stochastic process choice in commodity price modelling*. Springer.
- Schöne, M. F. (2015). *Optimal procurement, pricing, and hedging under cost and demand uncertainty*. INFORMS Annual Meeting 2015, Philadelphia.
- Schöne, M. F., & Spinler, S. (2015). *A four-factor stochastic volatility model of commodity prices*. Working Paper, WHU - Otto Beisheim School of Management, Vallendar, Germany.
- Schöne, M. F., Spinler, S., & Birge, J. R. (2016a). *Dynamic procurement, pricing, and hedging in a two-product firm under cost and demand risk*. Working Paper, WHU - Otto Beisheim School of Management, Vallendar, Germany.
- Schöne, M. F., Spinler, S., & Birge, J. R. (2016b). *Dynamic procurement, pricing, and hedging under cost and demand risk*. Working Paper, WHU - Otto Beisheim School of Management, Vallendar, Germany.
- Schwartz, E. S. (1982). The pricing of commodity-linked bonds. *Journal of Finance*, 37(2), 289-300.
- Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3), 923-973.
- Schwartz, E. S., & Miltersen, K. R. (1998). Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. *Journal of Financial and Quantitative Analysis*, 33, 33-59.
- Schwartz, E. S., & Smith, J. (2000). Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7), 893-911.
- Secomandi, N., Lai, G., Margot, F., Scheller-Wolf, A., & Seppi, D. J. (2015). Merchant commodity storage and term-structure model error. *Manufacturing & Service Operations Management*.
- Shi, C., Zhao, X., & Xia, Y. (2010). The setting of profit targets for target oriented divisions. *European Journal of Operational Research*, 206(1), 86-92.
- Shi, J., Zhang, G., & Sha, J. (2011). Jointly pricing and ordering for a multi-product multi-constraint newsvendor problem with supplier quantity discounts. *Applied Mathematical Modelling*, 35(6), 3001-3011.
- Siegel, D., Smith, J., & Paddock, J. (1987). Valuing offshore oil properties with option pricing models. *Midland Corporate Finance Journal*, 5, 22-30.
- Silvennoinen, A., & Thorp, S. (2013). Financialization, crisis and commodity correlation dynamics. *Journal of International Financial Markets, Institutions and Money*, 24, 42-65.
- Silver, E. A., & Peterson, R. (1985). *Decision systems for inventory management and production planning*. John Wiley & Sons.
- Smith, C. W., & Stulz, R. M. (1985). The determinants of firms' hedging policies. *Journal of Financial &*

- Quantitative Analysis*, 20(4), 391-405.
- Smith, J. E., & McCardle, K. F. (1999). Options in the real world: Lessons learned in evaluating oil and gas investments. *Operations Research*, 47(1), 1-15.
- Smith, J. E., & Nau, R. F. (1995). Valuing risky projects: Option pricing theory and decision analysis. *Management Science*, 41(5), 795-816.
- Smith, S. A., & Agrawal, N. (2000). Management of multi-item retail inventory systems with demand substitution. *Operations Research*, 48(1), 50-64.
- Spinler, S. (2016). *Dynamic procurement, pricing, and hedging under cost and demand risk*. Presentation, Massachusetts Institute of Technology, Boston.
- Tang, K. (2012). Time-varying long-run mean of commodity prices and the modeling of futures term structures. *Quantitative Finance*, 12(5), 781-790.
- Tevelson, R., Ross, J., & Paraniakas, P. (2007). *A key to smart sourcing: Hedging against risk*. The Boston Consulting Group.
- Thowsen, G. T. (1975). A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Research Logistics*, 33, 461-476.
- Trolle, A. B., & Schwartz, E. (2009a). A general stochastic volatility model for the pricing of interest rate derivatives. *Review of Financial Studies*, 22(5), 2007-2057.
- Trolle, A. B., & Schwartz, E. S. (2009b). Unspanned stochastic volatility and the pricing of commodity derivatives. *Review of Financial Studies*, 22(11), 4423-4461.
- Tsekrekos, A. E., Shackleton, M. B., & Wojakowski, R. (2012). Evaluating natural resource investments under different model dynamics: Managerial insights. *European Financial Management*, 18(4), 543-577.
- Villaplana, P. (2003). Pricing power derivatives: A two-factor jump-diffusion approach (working paper). *Barcelona, Spain: Universitat Pompeu Fabra*.
- Whitin, T. (1955). Inventory control and price theory. *Management Science*, 2(1), 61-68.
- Wu, O. Q., & Chen, H. (2010). Optimal control and equilibrium behavior of production-inventory systems. *Management Science*, 56(8), 1362-1379.
- Xiao, G., Yang, N., & Zhang, R. (2015). Dynamic pricing and inventory management under fluctuating procurement costs. *Manufacturing & Service Operations Management*, 17(3), 321-334.
- Xu, M. (2010). A price-setting newsvendor model under CVaR decision criterion with emergency procurement. *Journal of Systems Science and Systems Engineering*, 19(1), 85-104.
- Yan, X. S. (2002). Valuation of commodity derivatives in a new multi-factor model. *Review of Derivatives Research*, 5(3), 251-271.
- Yang, S., Shi, C., & Zhao, X. (2011). Optimal ordering and pricing decisions for a target oriented newsvendor. *Omega*, 39(1), 110-115.
- Zhang, R.-Q., Zhang, L.-K., Zhou, W.-H., Saigal, R., & Wang, H.-W. (2014). The multi-item newsvendor model with cross-selling and the solution when demand is jointly normally distributed. *European Journal of Operational Research*, 236(1), 147-159.
- Zhu, W., & Kapuscinski, R. (2011). *Optimal operational versus financial hedging for a risk-averse firm*. Working Paper, Singapore Management University, Singapore.