Asymptotic order of quantization for Cantor distributions in terms of Euler characteristic, Hausdorff and Packing measure

Wolfgang Kreitmeier

Department of Informatics and Mathematics, Measure and Integration, University of Passau, D-94030 Passau, Germany.

Abstract

For homogeneous one-dimensional Cantor sets, which are not necessarily self-similar, we show under some restrictions that the Euler exponent equals the quantization dimension of the uniform distribution on these Cantor sets. Moreover for a special sub-class of these sets we present a linkage between the Hausdorff and the Packing measure of these sets and the high-rate asymptotics of the quantization error.

Key words: homogeneous Cantor set, Euler characteristic, Euler exponent, quantization dimension, quantization coefficient, Hausdorff dimension, Hausdorff measure, Packing dimension, Packing measure

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1 Introduction and notation

The problem of quantization for probability measures has its origin in electrical engineering technologies such as data compression and signal processing (cf. [13]). The main issue is to find an optimal approximation of a given distribution by a discrete one, containing at most a fixed number of supporting points. The distance between approximation and original distribution, measured in terms of a suitable metric, is called quantization error. Although the attention of research was originally focused on distributions, which are absolutely continuous with respect to the Lebesgue measure, the quantization problem was also investigated during the last years for probabilities supported on fractal sets (see e.g. [10,20,21,23,30,38]). A main aspect of research concerns the

Email address: wkreitmeier@gmx.de (Wolfgang Kreitmeier).
quantitative behaviour of the quantization error, if the number of supporting points of the approximation tends to infinity. Introduced by Zador [35], a characteristic of this high-rate asymptotics of the quantization error is the notion of quantization dimension. Independently of these studies to the problem of quantization, Llorente and Winter [25] investigated the topological structure of (self-similar) fractal sets by means of Euler characteristic. In their work, a dimension is introduced with the notion of Euler exponent. As a third aspect, the Hausdorff and Packing measure concentrated on fractals resp. the dimensions based on these measures were also studied intensively (see e.g [2,7,17,19,31]).

The aim of this paper is to present new coherences between these three aspects in the theory of fractals for the special case of one-dimensional homogeneous Cantor sets. Beside of the known linkages between quantization, Hausdorff resp. Packing dimension (cf. [2,10,15,20,29,38]), we will show that the Euler exponent equals the quantization dimension under certain restrictions (cf. Theorem 2.5, Theorem 2.7). Moreover for a special sub-class of these one-dimensional homogeneous Cantor sets we will present a linkage between the Hausdorff and the Packing measure of these sets and the high-rate asymptotics of the quantization error (cf. Theorem 2.10). The paper is organized as follows. The rest of this section is dedicated to an exact definition of the above mentioned concepts of Euler exponent, Hausdorff resp. Packing dimension and Quantization of probabilities. In Section 2 one-dimensional homogeneous Cantor sets and the uniform distribution concentrated on such sets will be defined. Afterwards the main results of this paper, as described above, are stated and proved. Section 3 contains open problems and concluding remarks naturally arising out of this work.

1.1 Euler exponent for fractals

Although Fractal sets and the related distributions supported on these sets have been studied in detail during the last decades (see e.g. [5,19,27] and the references therein), topological aspects were only sparsely investigated. Recently, Llorente and Winter [25] presented a notion of Euler characteristic for a fractal set $F$ on $\mathbb{R}^d$ with $d \in \mathbb{N} = \{1, 2, \ldots\}$. With $\varepsilon > 0$ and the Euclidean norm $\| \cdot \|$ they considered for the $\varepsilon$—neighbourhood

$$F_\varepsilon := \{ x \in \mathbb{R}^d : \inf_{y \in F} \| x - y \| \leq \varepsilon \}$$

(1)

of $F$ the classical Euler characteristic $\chi(F_\varepsilon)$. If $F_\varepsilon$ is polyconvex, i.e. consists of a finite union of convex sets, then $\chi(F_\varepsilon)$ always exists, is identical with the number of connected components of $F_\varepsilon$ and coincides with the classical cell complex definiton from algebraic topology (cf. [25, section 4] and the references
therein). Based on this they defined
\[ \kappa = \kappa(F) := \inf \{ t \geq 0 : \epsilon^t \chi(F_\epsilon) \text{ is bounded as } \epsilon \to 0 \} \]
as the Euler exponent of \( F \). In this paper we use the following definitions.

**Definition 1.1** Let \( K = \{ t \geq 0 : \epsilon^t \chi(F_\epsilon) \text{ is bounded as } \epsilon \to 0 \} \). The number
\[ \bar{\kappa}(F) := \begin{cases} \inf K, & \text{if } K \neq \emptyset \\ \infty, & \text{if } K = \emptyset \end{cases} \tag{2} \]
is called upper Euler exponent of \( F \). The number
\[ \underline{\kappa}(F) := \sup \{ t \geq 0 : \epsilon^t \chi(F_\epsilon) \to \infty \text{ as } \epsilon \to 0 \} \]
is called lower Euler exponent of \( F \). If both values coincide, we call \( \kappa = \bar{\kappa} = \underline{\kappa} \) the Euler exponent of \( F \).

Clearly, \( \underline{\kappa} \leq \bar{\kappa} \), which justifies the definition.

### 1.2 Hausdorff and Packing dimension

An important and difficult issue in the study of a (fractal) set \( F \subset \mathbb{R}^d \) is the determination of its Hausdorff measure and the related Hausdorff dimension, which we will define now. Let \( \epsilon > 0 \) and \( I \subset \mathbb{N} \). The collection of sets \( (U_i)_{i \in I} \) is an \( \epsilon \)-cover of \( F \), if \( F \) is covered by the union of all \( U_i \) and each set \( U_i \) does have at most diameter \( \epsilon \), i.e.
\[ \text{diam}(U_i) = \sup \{ \| x - y \| : x, y \in U_i \} \leq \epsilon. \]

**Definition 1.2** For \( s \geq 0 \) let
\[ \mathcal{H}_s^\epsilon(F) = \inf \{ \sum_{i \in I} \text{diam}(U_i)^s : (U_i)_{i \in I} \text{ is an } \epsilon\text{-cover of } F \}. \]
We call \( \mathcal{H}_s^\epsilon(F) = \lim_{\epsilon \to 0} \mathcal{H}_s^\epsilon(F) \) the \( s \)-dimensional Hausdorff measure of \( F \).

It is easy to check, that \( \mathcal{H}_s^\epsilon(F) \) is non-increasing with \( s \) and that \( \mathcal{H}_t^\epsilon(F) > 0 \) for a \( t > 0 \) implies \( \mathcal{H}_s^\epsilon(F) = \infty \) for all \( 0 \leq s < t \).

**Definition 1.3** The Hausdorff dimension of \( F \) is defined as
\[ \dim_H(F) = \sup \{ s \geq 0 : \mathcal{H}_s^\epsilon(F) = \infty \} = \inf \{ s \geq 0 : \mathcal{H}_s^\epsilon(F) = 0 \}. \]
Although the Hausdorff dimension was computed for a large class of fractal sets (cf. [15,17,29] and the references therein), the Hausdorff measure has been calculated exactly only for a few fractals so far. For some homogeneous one-dimensional Cantor sets, accurate values were derived (cf. [26,31]). Introduced by Tricot [33], the concept of Packing measure and dimension for fractals was studied by several authors (see e.g. [2,6,7,8] and the references therein). Let $\varepsilon > 0$ and $I \subset \mathbb{N}$. An $\varepsilon$-packing of $F$ is a collection of disjoint balls $(B_i)_{i \in I}$ with diameter at most $\varepsilon$ and midpoints of $B_i$ placed in $F$. We define

$$P^s_\varepsilon(F) = \sup \left\{ \sum_{i=1}^{\infty} \text{diam}(B_i)^s : (B_i)_{i \in I} \text{ is an } \varepsilon\text{-packing of } F \right\}$$

and $P^s_0(F) = \lim_{\delta \to 0} P^s_\delta(F)$. To get a (countable additive) measure we define

$$P^s(F) = \inf \left\{ \sum_{i=1}^{\infty} P^s_0(E_i) : F \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

This Borel probability measure is called $s$-dimensional packing measure of $F$. Similar to the Hausdorff dimension we define the Packing dimension of $F$ by

$$\dim_P(F) = \sup\{s \geq 0 : P^s(F) = \infty\} = \inf\{s \geq 0 : P^s(F) = 0\}.$$

### 1.3 Optimal quantization

Now we introduce and define the optimal quantization of probability distributions. Let $\mu$ be a Borel probability distribution on $\mathbb{R}^d$. For $r \in ]0, \infty[$ and $n \in \mathbb{N}$ the $n$-th (optimal) quantization error of $\mu$ of order $r$ is defined by

$$V_{n,r}(\mu) = \inf \left\{ \int \min_{a \in \alpha} \| x - a \|^r d\mu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where $\text{card}$ denotes cardinality and $\| \cdot \|$ the Euclidean norm. Because it is quite difficult to compute the quantization error, especially in higher dimensions $d$ and for large $n$, one is interested in high rate asymptotics, which is characterized by the well-known concepts of quantization dimension and quantization coefficient (cf. [10]).

**Definition 1.4** We call

$$\overline{D}_r(\mu) := \lim \sup_{n \to \infty} \frac{r \log(n)}{-\log(V_{n,r}(\mu))}$$

the upper and

$$\underline{D}_r(\mu) := \lim \inf_{n \to \infty} \frac{r \log(n)}{-\log(V_{n,r}(\mu))}$$

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the lower quantization dimension of \( \mu \) of order \( r \). If both values coincide, we call the common value quantization dimension of \( \mu \) of order \( r \) and denote it by \( D_r(\mu) \).

**Definition 1.5** If the quantization dimension \( D_r(\mu) \) exists and the sequence \((n^{\frac{1}{D_r(\mu)}}V_{n,r}(\mu))_{n \in \mathbb{N}}\) converges towards a \( Q_r(\mu) > 0 \), we call \( Q_r(\mu) \) the quantization coefficient of \( \mu \) of order \( r \).

If the distribution \( \mu \) is absolute continuous with respect to the \( d \)-dimensional Lebesgue measure and \( \int x^{r+\delta} d\mu(x) < \infty \) for some \( \delta > 0 \), then the quantization dimension exists and equals \( d \). Also the quantization coefficient \( Q_r(\mu) \) exists in this case. This result goes back to Zador [34], respectively Buckley and Wise [4]. A complete proof was given by Graf and Luschgy (cf. [10, Theorem 6.2]). In the case of singular distributions the situation is different. For self-similar distributions, satisfying the so-called open set condition, Graf and Luschgy [11] have shown that the quantization dimension exists. For singular distributions which are not self-similar, Lindsay gave an example for the non-existence of the quantization dimension (cf. [23, Example 5.5]). Later on, the existence of the quantization dimension for distributions on (not necessarily self-similar) Cantor-like sets was systematically studied and characterized by Kesseböhmer and Zhu [20], Kreitmeier [21] and Zhu [38]. For self-similar distributions satisfying the open set condition, the quantization coefficient exists under certain restrictions. Also the non-existence can happen. These facts will be discussed in more detail in section 2.5.

## 2 Homogeneous Cantor sets and homogeneous Cantor distributions in one dimension

### 2.1 Construction and definition

From now on the fractal set \( F \) denotes a homogeneous one-dimensional Cantor set, which we will define now. We adopt the notation used by Qu et.al. [31]. Let \((n_k)_{k \in \mathbb{N}}\) be a sequence of positive integers and \((\epsilon_k)_{k \in \mathbb{N}}\) be a sequence of real numbers. We assume for every \( k \in \mathbb{N} \), that \( n_k \geq 2 \) and \( 0 < n_k \epsilon_k < 1 \). We define \( D_0 = \emptyset \) and

\[ D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\}. \]

Let \( D = \bigcup_{k=0}^{\infty} D_k \) and

\[ \sigma \ast \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m) \]
for $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, $\tau = (\tau_1, \ldots, \tau_m) \in D_m$. Let $I = [0, 1]$. Let $\mathcal{F} = \{I_\sigma : \sigma \in D\}$ be the collection of the closed sub-intervals of $I$ which satisfy

(i) $I_\emptyset = I$

(ii) For any $k \in \mathbb{N}$ and $\sigma \in D_{k-1}$ the sets $I_{\sigma_{si}} (1 \leq i \leq n_k)$ are sub-intervals of $I_{\sigma}$.

The intervals $I_{\sigma_{s1}}, \ldots, I_{\sigma_{sn}}$ are arranged from the left to the right, $I_{\sigma_{s1}}$ and $I_{\sigma}$ have the same left endpoint, $I_{\sigma_{sn}}$ and $I_{\sigma}$ have the same right endpoint, and the lengths of the gaps between any two adjacent sub-intervals are equal.

We denote the length of one of the gaps by $y_k$.

(iii) For any $k \geq 1, 1 \leq j \leq n_k$ and $\sigma \in D_{k-1}$ we have $\text{diam}(I_{\sigma_{sj}}) = c_k \text{diam}(I_{\sigma})$.

For $k \geq 1$ and $\sigma \in D_k$ we call $I_{\sigma}$ a $k$–level set and for $k \geq 0$ we define $F_k = \bigcup_{\sigma \in D_k} I_{\sigma}$ resp. the homogeneous one-dimensional Cantor set $F = \bigcap_{k \geq 0} F_k$.

On this Cantor set $F = F((n_k)_{k \in \mathbb{N}}, (c_k)_{k \in \mathbb{N}})$ a unique probability measure $\mu$ exists with $\mu(F_k) = (n_1 \cdots n_k)^{-1}$ for every $k \in \mathbb{N}$. We call $\mu$ the uniform distribution on $F$ (cf. [20,38]).

2.2 Gap counting function and conditions used

Known from lacunarity analysis for fractals, the gap counting function for $F$ and $\varepsilon > 0$ is defined by (cf. [5, section 7.2])

$$G(\varepsilon) = \text{card}\{I : I \text{ is a closed interval, lies in } [0, 1]\setminus F \text{ and has length } > \varepsilon\}.$$

Recall the $\varepsilon$–parallel set $F_\varepsilon$ of $F$, defined in (1). As already mentioned, the Euler characteristic of this (polyconvex) set $F_\varepsilon$ coincides with the number of its connected components, which can be computed by the number of gaps of $F_\varepsilon$ with length greater than $2\varepsilon$. Thus we have (cf. [25, section 2.4])

$$\chi(F_\varepsilon) = 1 + G(2\varepsilon).$$

(3)

Although likely to be well-known, for the readers convenience the exact value of the gap counting function will be stated and proven in the following proposition. It is used in the sequel to compute via (3) the (upper/lower) Euler exponent of $F$.

Proposition 2.1 Assume that $y_k > y_{k+1}$ for every $k \in \mathbb{N}$. Then

$$G(\varepsilon) = \begin{cases} 
0, & \text{if } \varepsilon \in [y_1, \infty[ \\
\left(\prod_{l=1}^{k} n_l\right) - 1, & \text{if } \varepsilon \in [y_{k+1}, y_k[ \end{cases}.$$

PROOF. From the definition of $F$ we observe, that $G(\cdot)$ is constant on $[y_1, \infty[$ resp. $[y_{k+1}, y_k[$ for every $k \in \mathbb{N}$. Thus it remains to determine $G(y_k)$. Obviously
\(G(y_1) = 0\). Due to \(G(y_k) = G(y_{k-1}) + (n_k - 1)(G(y_{k-1}) + 1)\) for every \(k \geq 2\), the assertion follows immediately by induction over \(k\).

For the proof of our results we need three conditions, also used in [38] for even more general fractal sets. In our setting they take the following special form.

(a) **Bounded distortion** (BD): the Cantor set \(F\) is of bounded distortion, if \(\inf_{k \in \mathbb{N}} c_k > 0\).

(b) **Extra Strong Separation Condition** (ESSC): we say that \(F\) satisfies the extra strong separation condition, if \(\sup_{k \in \mathbb{N}} n_k c_k < 1\).

(c) **Hereditary Condition** (HC): we say that the hereditary condition for \(F\) is satisfied, if \(\sup_{k \in \mathbb{N}} n_k < \infty\).

### 2.3 Comparison of quantization dimension to the Euler exponent

As mentioned in the introduction, the quantization dimension has been already compared with other types of dimensions. First let us outline with the following two theorems a part of these known results, which are needed in this paper.

**Theorem 2.2** ([2,38]) Assume that condition (BD) is satisfied. Then

\[
\dim_P(F) = \inf \{ t \geq 0 : \limsup_{k \to \infty} \prod_{i=1}^{k} n_i c_i^t < \infty \} = \sup_{k \to \infty} \frac{\log(\prod_{i=1}^{k} n_i)}{\log(\prod_{i=1}^{k} c_i)}. \tag{4}
\]

If, additionally, the conditions (HC) and (ESSC) are satisfied, then \(\dim_P(F) = D_r(\mu)\).

**PROOF.** If condition (BD) holds, the equalities for the Packing dimension follow from the work of Baek [2]. Under the conditions (BD), (HC) and (ESSC) it was shown by Zhu [38, Theorem 6], that \(D_r(\mu) = \limsup_{k \to \infty} \frac{\log(\prod_{i=1}^{k} n_i)}{\log(\prod_{i=1}^{k} c_i)}\).

**Remark 2.3** The determination of the Packing dimension by Baek (cf. [2, Corollary 2.3]) was achieved by other methods in earlier works (cf. [8,16]). If the condition (BD) is dropped, it was shown by Feng et.al. (cf. [8, proof of Theorem 3.1]), that

\[
\dim_P(F) = \inf \{ t \geq 0 : y_k G(y_k) \text{ is bounded as } k \to \infty \}
= \limsup_{k \to \infty} \frac{\log(\prod_{i=1}^{k} n_i)}{\log(\prod_{i=1}^{k} c_i) + \log(n_{k+1})}.
\]

Hence it is straightforward to construct examples, showing that equation (4) becomes wrong, if (BD) is dropped.
Theorem 2.4 ([15,29,38]) Assume that condition (ESSC) is satisfied. Then
\[
\dim_H(F) = \sup\{t \geq 0 : \liminf_{k \to \infty} \prod_{l=1}^{k} n_l c_l^t = \infty\} = \liminf_{k \to \infty} \frac{\log(\prod_{l=1}^{k} n_l)}{-\log(\prod_{l=1}^{k} c_l)}. \tag{5}
\]
If, additionally, the conditions (HC) and (BD) are satisfied, then \( \dim_H(F) = D_r(\mu) \).

PROOF. If condition (ESSC) holds, the equalities for the Hausdorff dimension follow from the work of Marion [29] and Hua [15]. Under the conditions (BD), (HC) and (ESSC) it was shown by Zhu [38, Theorem 6], that \( D_r(\mu) = \liminf_{k \to \infty} \frac{\log(\prod_{l=1}^{k} n_l)}{-\log(\prod_{l=1}^{k} c_l)} \).

Now we can compare the Euler exponent with other concepts of fractal dimension.

Theorem 2.5 Assume that \( y_{k+1} < y_k \) for every \( k \in \mathbb{N} \). Then
\[
\pi(F) = \inf\{t \geq 0 : y_k^t G(y_{k+1}) \text{ is bounded as } k \to \infty\}.
\]
Moreover, if condition (BD) is satisfied, then \( \pi(F) = \dim_P(F) \). If, additionally, condition (HC) and (ESSC) are satisfied, then \( \pi(F) = \overline{D}_r(\mu) \).

PROOF. From (2) and (3) we get
\[
\pi = \inf\{t \geq 0 : \varepsilon^t G(\varepsilon) \text{ is bounded as } \varepsilon \to 0\}
\]
Applying Proposition 2.1 it follows, that \( \varepsilon^t G(\varepsilon) \) is bounded as \( \varepsilon \to 0 \) if, and only if \( y_k^t G(y_{k+1}) \) is bounded as \( k \to \infty \). This proves the first equation.

Now let \( k \geq 2 \). One calculates (see e.g. [31]) that
\[
y_k = \frac{1 - n_k c_k}{n_k - 1} \prod_{l=1}^{k-1} c_l. \tag{6}
\]

Thus, again from Proposition 2.1 we obtain for every \( k \geq 2 \) and \( t \geq 0 \) that
\[
y_k^t G(y_{k+1}) = \left( \frac{1 - n_k c_k}{n_k - 1} \prod_{l=1}^{k-1} c_l \right)^t \left( \prod_{l=1}^{k} n_l \right)^{t-1} \left( \prod_{l=1}^{k} n_l c_l^t \right) \tag{7}
\]
Note, that (BD) implies
\[
0 < \frac{1 - n_k c_k}{n_k c_k - c_k} < \left( \inf_{l \in \mathbb{N}} c_l \right)^{-1} < \infty. \tag{8}
\]
Moreover we have

\[ \frac{3}{4} \leq 1 - \left( \prod_{l=1}^{k} n_l \right)^{-1} \leq 1. \] (9)

Combining (8) and (9) with (7) we recognize, that \( y_k^l G(y_{k+1}) \) is bounded as \( k \to \infty \), if and only if \( \limsup_{k \to \infty} \prod_{l=1}^{k} n_l c_l^l < \infty \). Applying Theorem 2.2 we deduce \( \pi(F) = \dim_P(F) \) resp. \( \pi(F) = D_r(\mu) \) if, additionally, condition (HC) and (ESSC) are satisfied.

**Remark 2.6** Kessebohmer and Zhu (cf. [20, Lemma 2.4]) have shown for arbitrary Borel probability measures \( \mu \) on \( \mathbb{R}^d \) with compact support \( \text{supp}(\mu) \), that \( \dim_P(\text{supp}(\mu)) \leq D_r(\mu) \). Tricot [32, Theorem 1, \( \Delta_1 = \Delta_5 \) in his notation] already established the identities

\[ \inf \{ t \geq 0 : y_k^l G(y_k) \text{ is bounded as } k \to \infty \} \]

\[ = \inf \{ t \geq 0 : \limsup_{k \to \infty} \prod_{l=1}^{k} n_l c_l^l < \infty \} \]

\[ = \limsup_{k \to \infty} \frac{\log(\prod_{l=1}^{k} n_l)}{-\log(\prod_{l=1}^{k} c_l)} \]

but he omitted the proof and did not make further restrictions on \((c_k)_{k \in \mathbb{N}}\) resp. \((n_k)_{k \in \mathbb{N}}\) beside of \( n_k c_k < 1 \) for every \( k \). Though without any additional restrictions, (10) becomes wrong as demonstrated in Remark 2.3.

**Theorem 2.7** Assume that \( y_{k+1} < y_k \) for every \( k \in \mathbb{N} \). Then

\[ \kappa(F) = \sup \{ t \geq 0 : y_k^l G(y_k) \to \infty \text{ as } k \to \infty \}. \] (11)

Moreover, if condition (ESSC) is satisfied, then \( \kappa(F) = \dim_H(F) \). If, additionally, condition (HC) and (BD) are satisfied, then \( \kappa(F) = D_r(\mu) \).

**PROOF.** By the same arguments as in the proof of Theorem 2.5 one shows, that (11) is true. The remaining assertions follow from Theorem 2.4.

### 2.4 High-rate asymptotics of quantization in terms of Hausdorff and Packing measure

For any \( l \in \mathbb{N} \) we denote \( \mu^{(l)} \) as the uniform distribution of the Cantor set \( F_l = F((n_{k+l-1})_{k \in \mathbb{N}}, (c_{k+l-1})_{k \in \mathbb{N}}) \). For any \( k \in \mathbb{N} \) and \( \sigma \in D_k \) we denote \( I_{\sigma}^{l} \) as the associated \( k \)–level sets. Clearly, \( \mu^{(1)} = \mu \). If \( n_k = 2 \) and \( c_k = c \) for every \( k \) we write \( \mu = \mu_c \) resp. \( I_{\sigma}(c) \) for a \( k \)–level set with \( \sigma \in D_k \). Moreover we denote \( \pi_{k}^{l} = \prod_{j=1}^{k+l-1} c_j \).

**Lemma 2.8** Let \( 0 < c \leq \frac{1}{3} \). Assume that \( r \geq 1 \) and
(a) \( n_k = 2 \) for every \( k \in \mathbb{N} \),
(b) \( 0 < c_k \leq \frac{1}{3} \) for every \( k \in \mathbb{N} \) and
(c) the sequence \((c_k)_{k \in \mathbb{N}}\) converges with \( \lim_{k \to \infty} c_k = c \).

Then
\[
V_{1,r}(\mu^{(l)}) \to V_{i,r}(\mu_c)
\]
as \( l \) tends to infinity.

**PROOF.** We proceed in several steps.
1. We derive an approximation of the quantization error by finite sums.
   Let \( l \in \mathbb{N} \). From [21, Corollary 3.3] we obtain
   \[
   V_{1,r}(\mu^{(l)}) = \int |x - \frac{1}{2}|^r \, d\mu^{(l)}(x)
   \]
   for every \( r \geq 1 \). From the construction and symmetry of \( F_l \) we easily deduce, that for every \( k \in \mathbb{N} \) the relation
   \[
   L_k^l \leq V_{1,r}(\mu^{(l)}) \leq R_k^l
   \]
does hold, with
   \[
   L_k^l := 2 \sum_{\sigma \in D_k, \min(I^l_{\sigma}) > \frac{1}{2}} \left( \min(I^l_{\sigma}) - \frac{1}{2} \right)^r 2^{-k}
   \]
   resp.
   \[
   R_k^l := 2 \sum_{\sigma \in D_k, \max(I^l_{\sigma}) > \frac{1}{2}} \left( \max(I^l_{\sigma}) - \frac{1}{2} \right)^r 2^{-k}.
   \]
   Thus we obtain
   \[
   | V_{1,r}(\mu^{(l)}) - R_k^l | \leq R_k^l - L_k^l
   \]
   \[
   = 2 \sum_{\sigma \in D_k, \min(I^l_{\sigma}) > \frac{1}{2}} 2^{-k} \left( \left( \max(I^l_{\sigma}) - \frac{1}{2} \right)^r - \left( \min(I^l_{\sigma}) - \frac{1}{2} \right)^r \right)
   \]
   \[
   \leq 2^{1-k} \sum_{\sigma \in D_k, \max(I^l_{\sigma}) > \frac{1}{2}} r \left( \max(I^l_{\sigma}) - \frac{1}{2} \right)^{r-1} \left( \max(I^l_{\sigma}) - \min(I^l_{\sigma}) \right)
   \]
   \[
   \leq 2^{1-k} 2^{k-1} r \pi_k^l \leq r \left( \frac{1}{3} \right)^k,
   \]
independent of \( l \). Similar to (12) we obtain with
   \[
   R_k(c) := 2 \sum_{\sigma \in D_k, \max(I^l_{\sigma}(c)) > \frac{1}{2}} \left( \max(I^l_{\sigma}(c)) - \frac{1}{2} \right)^r 2^{-k}
   \]
the relation
\[ | V_{l,r}(\mu_c) - R_k(c) | \leq r \left( \frac{1}{3} \right)^k. \] (13)

2. Now we give an upper bound for \(| R_{l,k}^l - R_k(c) |\) for every \(k \geq 2\) and \(l \in \mathbb{N}\). Again let \(l \in \mathbb{N}\). First we will show by induction on \(k\), that for every \(\sigma \in D_k\) the relation
\[ | \max(I_{\sigma}^l) - \max(I_{\sigma}(c)) | \leq | c^k - \pi_k^l | + 2 \sum_{j=1}^{k-1} | c^j - \pi_j^l | \] (14)
does hold. If \(k = 2\), then
\[ | \max(I_{\sigma}^l) - \max(I_{\sigma}(c)) | \leq | c^2 - \pi_2^l | + 2 \sum_{j=1}^{2-1} | c^j - \pi_j^l | \]

for every \(\sigma \in D_k\). Now assume, that (14) is true for every \(m \leq k - 1\). Let \(\sigma \in D_k\). Thus, a \(\tau \in D_{k-1}\) and \(i \in \{1, 2\}\) exists, with \(\tau \ast i = \sigma\). With \(\Delta = | \max(I_{\tau}^l) - \max(I_{\tau}(c)) |\) we obtain
\[ | \max(I_{\sigma}^l) - \max(I_{\sigma}(c)) | \leq \max \left( | \max(I_{\tau}^l) - \max(I_{\tau \ast 1}(c)) |, | \max(I_{\tau \ast 2}^l) - \max(I_{\tau \ast 1}(c)) | \right) \]
\[ = \max \left( | \max(I_{\tau \ast 1}^l) - \max(I_{\tau \ast 1}(c)) |, \Delta \right) \]
\[ = \max \left( | \max(I_{\tau}^l) - (1 - c) \text{diam}(I_{\tau}(c)) \right) \]
\[ - \left( \max(I_{\tau}^l) - (1 - c_{k+l-1}) \text{diam}(I_{\tau}^l) \right) |, \Delta \right) \]
\[ \leq \max \left( | \max(I_{\tau}(c)) - \max(I_{\tau}^l) |, | c^{k-1} - \pi_{k-1}^l | + | c^k - \pi_k^l |, \Delta \right) \]
\[ = \Delta + | c^{k-1} - \pi_{k-1}^l | + | c^k - \pi_k^l |. \]

By the induction hypothesis we know, that
\[ \Delta \leq | c^{k-1} - \pi_{k-1}^l | + 2 \sum_{j=1}^{k-2} | c^j - \pi_j^l |. \] (16)

Combining (16) and (15) we deduce (14).

Now let \(\delta = \sup_{m \geq l} | c - c_m | \leq \frac{1}{3}\). Using (14) we get
\[ \max(I^l_\sigma) - \max(I_\sigma(c)) \leq 2 \sum_{j=1}^{k} | \pi^j_l - c^j | \leq 2 \sum_{j=1}^{k} (c + \delta)^j - c^j \]
\[ \leq 2 \sum_{j=1}^{k} j(c + \delta)^{j-1} \delta < 2 \delta \frac{d}{dc} \sum_{j=1}^{\infty} (c + \delta)^j \]
\[ = \frac{2\delta}{(1 - (c + \delta))^2} \leq 18\delta. \]

The definition of \( R_k(c) \) and \( R^l_k \) together with (17) implies
\[ | R_k(c) - R^l_k | \leq 2^{1-k} \sum_{\sigma \in D_k} r \max(I^l_\sigma) - \max(I_\sigma(c)) \leq 2^{1-k} \cdot 2^{k-1} \cdot r \cdot 18\delta = 18r\delta. \]

3. Now we are able to finish the proof.

Let \( \varepsilon > 0 \). Choose \( l_0 \in \mathbb{N} \) such that \( \sup_{m \geq l} | c - c_m | \leq \frac{\varepsilon}{54r} \) for every \( l > l_0 \).
Let \( k_0 \in \mathbb{N} \) such that \( k_0 > \frac{\log(\varepsilon)}{\log(\frac{1}{3})} \). With (12), (13) and (18) we deduce for every \( l > l_0 \) that
\[ | V_{1,r}(\mu^{(l)}) - V_{1,r}(\mu_c) | \leq | V_{1,r}(\mu^{(l)}) - R^l_{k_0} | + | R^l_{k_0} - R_{k_0}(c) | + | R_{k_0}(c) - V_{1,r}(\mu_c) | \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{54r} + \frac{\varepsilon}{3} = \varepsilon, \]

which proves the assertion.

**Remark 2.9** By well-known convergence results (cf. [10, section 4.5]) the assertion of Lemma 2.8 would easily follow, if one could show, that \( \mu^{(l)} \) converges weakly against \( \mu_c \). Due to the Portmanteau-Theorem the weak convergence is equivalent to
\[ \limsup_{l \to \infty} \mu^{(l)}(A) \leq \mu_c(A) \]
for every closed set \( A \subset F \). Unfortunately the author was not able to prove or disprove (19).

**Theorem 2.10** Let \( D \) be the Hausdorff dimension of \( F \) and \( 0 < c \leq \frac{1}{3} \).
Assume that \( r > 1 \) and the conditions (a) - (c) from Lemma 2.8 are satisfied.
Then
\[ \mathcal{H}^D(F) = \liminf_{k \to \infty} 2^k \pi_k^D = (V_{1,r}(\mu_c))^{-\frac{D}{r}} \liminf_{n \to \infty} n (V_{n,r}(\mu))^{\frac{D}{r}} \]  
(20)
and
\[ P^D(F) \geq 2 \limsup_{k \to \infty} 2^k \pi_k^D = \lim_{r \to \infty} \left( (V_{1,r}(\mu_c))^{-\frac{D}{r}} \limsup_{n \to \infty} n (V_{n,r}(\mu))^{\frac{D}{r}} \right). \]  
(21)

**PROOF.** We proceed in several steps.
1. We will show, that
\[ \mathcal{H}^D(F) = \liminf_{k \to \infty} 2^k \pi_k^D = (V_{1,r}(\mu_c))^{-\frac{D}{r}} \liminf_{k \to \infty} 2^k \left( V_{2^k,r}(\mu) \right)^{\frac{D}{r}}. \]  
(22)
Using (6) and \( \sup_{k \in \mathbb{N}} c_k \leq \frac{1}{3} \) we deduce \( y_k > y_{k+1} \) for every \( k \in \mathbb{N} \). Applying [31, Theorem 1] we obtain
\[ \mathcal{H}^D(F) = \liminf_{k \to \infty} 2^k \pi_k^D. \]  
(23)
From [21, Theorem 4.4] we get for every \( k \in \mathbb{N} \) and \( n \in [2^k, 2^{k+1}[ \) the explicit formula
\[ V_{n,r}(\mu) = \frac{\pi_k^r}{2^k}((2^{k+1} - n)V_{1,r}(\mu^{(k+1)}) + (n - 2^k) \cdot c_{k+1} V_{1,r}(\mu^{(k+2)})). \]  
(24)
Thus we have \( V_{2^k,r}(\mu) = \pi_k^r V_{1,r}(\mu^{(k+1)}) \). Therefore Lemma 2.8 implies
\[ (V_{1,r}(\mu_c))^{-\frac{D}{r}} \liminf_{k \to \infty} 2^k \left( V_{2^k,r}(\mu) \right)^{\frac{D}{r}} \]
\[ = (V_{1,r}(\mu_c))^{-\frac{D}{r}} \liminf_{k \to \infty} 2^k \left( \pi_k^r V_{1,r}(\mu^{(k+1)}) \right)^{\frac{D}{r}} \]
\[ = \liminf_{k \to \infty} 2^k \pi_k^D, \]
which proves together with (23) the equation (22).
2. For any convergent subsequence \( \left( n_k \left( V_{n_k,r}(\mu) \right)^{\frac{D}{r}} \right)_{k \in \mathbb{N}} \) of \( \left( n \left( V_{n,r}(\mu) \right)^{\frac{D}{r}} \right)_{n \in \mathbb{N}} \) we will verify, that
\[ (V_{1,r}(\mu_c))^{-\frac{D}{r}} \liminf_{k \to \infty} n_k \left( V_{n_k,r}(\mu) \right)^{\frac{D}{r}} \geq \mathcal{H}^D(F). \]  
(25)
Let \( \left( n_k \left( V_{n_k,r}(\mu) \right)^{\frac{D}{r}} \right)_{k \in \mathbb{N}} \) be a convergent subsequence of \( \left( n \left( V_{n,r}(\mu) \right)^{\frac{D}{r}} \right)_{n \in \mathbb{N}} \).
Thus, also \( \left( n_k \pi_k V_{n_k,r}(\mu) \right)_{k \in \mathbb{N}} \) does converge.
Let \( l_k \) be defined by \( 2^k \leq n_k < 2^{k+1} \) and \( \alpha_k = \frac{n_k}{2^k} \in [1, 2[ \). Using (24), an easy computation yields
\[ n_k \pi_k V_{n_k,r}(\mu) = \left( (2 - \alpha_k) + (\alpha_k - 1)c_{k+1} V_{1,r}(\mu^{(k+2)}) \right) \alpha_k^2 (2^k) \pi_k V_{1,r}(\mu^{(k+1)}). \]
By taking a proper subsequence of \((n_k)\) we can assume w.l.o.g., that 
\[ \lim_{k \to \infty} \alpha_k = \alpha \in [1, 2). \]
Taking into consideration \(\lim_{k \to \infty} c_k = c\) and Lemma 2.8 we deduce

\[
\liminf_{k \to \infty} n_k^\frac{2}{c} V_{n_k,r}(\mu) = \lim_{k \to \infty} n_k^\frac{2}{c} V_{n_k,r}(\mu) = f_{c,r}(\alpha)V_{1,r}(\mu_c) \liminf_{k \to \infty} (2^k)^\frac{2}{c} \pi_k.
\]

with \(f_{c,r}(\alpha) = ((2 - \alpha) + (\alpha - 1)c^r) \alpha^\frac{2}{c}\). The assumptions (a) - (c) imply, that the conditions (BD), (ESSC) and (HC) are satisfied. Thus we obtain from Theorem 2.7, that \(D = -\frac{\log(c)}{\log(2)}\). It is straightforward to see, that \(f_{c,r}(1) = 1\) resp. \(f_{c,r}(x) \geq 1\) for every \(x \in [1, 2]\) (cf. [21, Remark 6.1]). Hence, (25) follows from (26) and (23). Finally, (25) and (22) yield together (20).

3. We will prove inequality (21).

Note that \(f_{c,r}\) reaches its unique maximum at \(\alpha_0 = 2 - \frac{D}{c} + \frac{D}{c(1 + \frac{D}{c})} (1 + D)\) (cf. [21, Remark 6.1]). Similar to the arguments of step 2 we obtain in case of \(\limsup_{k \to \infty} 2^k \pi_k^D \in [0, \infty]\) that

\[
\left( \limsup_{k \to \infty} 2^k \pi_k^D \right)^{-\frac{1}{c}} (V_{1,r}(\mu_c))^{-\frac{D}{c}} \limsup_{n \to \infty} n (V_{n,r}(\mu))^\frac{D}{c} = (f_{c,r}(\alpha_0))^\frac{D}{c} = \left( \frac{2 - c^r}{1 - c^r} + \frac{D}{c} \right) \left( \frac{D}{c} \right) r^{\frac{D}{c}}.\]

Because \(y_k \geq \frac{D}{c}\) for every \(k\), we get from [2, Theorem 3.1] the lower bound

\[
\mathcal{P}^D(F) \geq 2 \limsup_{k \to \infty} 2^k \pi_k^D. \tag{28}
\]

If \(\limsup_{k \to \infty} 2^k \pi_k^D = 0\) or \(\infty\), then \(\limsup_{k \to \infty} n \hat{\pi} V_{n,r}(\mu) = 0\) resp. \(\infty\) (cf. [21, Corollary 4.7]). Hence, from (27) and (28) we obtain (21).

\textbf{Remark 2.11} Inequality (21) can be strict. Consider e.g. the classical middle third Cantor Set, i.e. \(n_k = 2\) and \(c_k = \frac{1}{3}\) for every \(k \in \mathbb{N}\). Here we have \(D = \frac{\log(2)}{\log(3)}\) and (cf. [7]) \(\mathcal{P}^D(F) = 4^D > 2\).

The case \(r = \infty\) leads to the following definition (cf. [10, section 10.1])

\textbf{Definition 2.12} Let \(n \in \mathbb{N}\) and \(r \geq 1\). We define \(e_{n,r}(\mu) = V_{n,r}(\mu)^\frac{1}{r}\) and call

\[
e_{n,\infty}(F) = \inf_{\alpha \in \mathbb{R}^d \atop \text{card}(\alpha) \leq n} \max_{x \in F} \min_{a \in \alpha} \| x - a \|\]

the \(n\)-th covering radius for \(F\).
Due to \( \lim_{r \to \infty} e_{n,r}(\mu) = e_{n,\infty}(F) \) (cf. [10, Lemma 10.1 (b)]) the quantization problem is related to the covering problem.

**Remark 2.13** In case of \( r = \infty \), equation (20) formally turns into

\[
H^D(F) = \lim_{k \to \infty} 2^k \sigma_k^D = 2^D \lim_{n \to \infty} n \left( e_{n,\infty}(F) \right)^D.
\]

But it is not clear, if

\[
\lim_{n \to \infty} n \left( e_{n,\infty}(F) \right)^D = \lim_{r \to \infty} \lim_{n \to \infty} n \left( e_{n,r}(F) \right)^D
\]

is generally true or not. For any nonempty compact set \( K \subset \mathbb{R}^d \), Graf and Luschgy [10, Proposition 11.5] have already shown that

\[
H^D(K) \leq 2^D \lim_{n \to \infty} n \left( e_{n,\infty}(K) \right)^D.
\]

**Example 2.14** Let \( c \in ]0, \frac{1}{2}[. \) Let \( c_1 = c \) and \( c_k = \sqrt{c} \) for every \( k \geq 2 \). By elementary calculations we obtain \( D = \dim_H(E) = \frac{-2 \log(2)}{\log(c)} \) and \( H^D(F) = c^2 \leq \frac{1}{2} < 1, \) resp. \( P^D(F) \geq 2H^D(F) \).

2.5 The self-similar case

Let \( N \in \mathbb{N} \) with \( N \geq 2 \). We assume, that \( n_k = N \) for every \( k \in \mathbb{N} \). If the sequence \( (c_k)_{k \in \mathbb{N}} \) is also constant, i.e. a \( c \in ]0, \frac{1}{N}[ \) exists with \( c_k = c \) for every \( k \in \mathbb{N} \), then the uniform distribution \( \mu \) on \( F \) becomes self-similar.

In general, self-similar sets and distributions are defined also in higher dimensions \( d \in \mathbb{N} \) by an iterated function system (IFS). Let \( S_i, i = 1, \ldots, N \) be similitudes on \( \mathbb{R}^d \) with contracting factor \( c_i \). The non-empty compact set \( F \), characterized by \( \bigcup_{i=1}^N S_i(F) = F \) is called invariant attractor. Moreover a distribution \( \mu \) on \( F \) can be introduced and characterized by \( \sum_{i=1}^N p_i \mu \circ S_i^{-1} = \mu \) with probability vector \( (p_1, \ldots, p_N) \). If we denote \( D \) as the similarity dimension, uniquely defined by \( \sum_{i=1}^N c_i^D = 1 \) and \( p_i = c_i^D \) for every \( i \in \{1, \ldots, N\} \) we call \( \mu \) the uniform distribution on \( F \). A proof of these facts can be found in [19]. As an example, for \( x \in \mathbb{R} \) and \( N = 2 \), the similitudes \( S_1(x) = cx \) and \( S_2(x) = cx + 1 - c \) lead to the same \( F \) and uniform distribution \( \mu \) as inductively constructed in section 2.1.

The IFS satisfies the strong separation condition, if \( S_i(F) \cap S_j(F) = \emptyset \) for every \( i, j \in \{1, \ldots, N\} \) with \( i \neq j \). Moreover, the IFS satisfies the open set condition, if a non-empty open set \( U \) exists, with \( S_i(U) \subset U \) for every \( i = 1, \ldots, N \) and \( S_i(U) \cap S_j(U) = \emptyset \) for every \( i, j \in \{1, \ldots, N\} \) with \( i \neq j \).

It is well-known, that the similarity dimension \( D \) equals the Hausdorff and...
Packing dimension, if the open set condition is satisfied. If the strong separation condition is satisfied and all sets $F_\varepsilon$ are polyconvex, the existence and identity of the Euler exponent with the similarity dimension was shown by Llorente and Winter [25]. The existence and identity of the quantization dimension with the similarity dimension was shown by Graf and Luschgy [11], if the IFS satisfies the open set condition. For the determination of the Packing and Hausdorff measure in some special one-dimensional self-similar cases the reader is referred to several authors [1,7,28,36]. In higher dimensions, the situation becomes even more difficult. The exact value of the Hausdorff measure for the classical Sierpinski gasket is still unknown, but can be approximated arbitrarily well (cf. [3,18]). For a class of generalized Sierpinski triangles and Sierpinski sponges, the Hausdorff measure was calculated exactly (cf. [14,37]). Recently (cf. [24]) it was also shown for higher dimensional self-similar fractals, satisfying the strong separation condition, that the Hausdorff measure equals the inverse of the maximal density of the fractal and the Packing measure equals the inverse of the minimal density of the fractal. In case of $N = 2$ and $d = 1$, the non-convergence of the sequence $(n \prod_{r} V_{n,r}(\mu))_{n \in \mathbb{N}}$, i.e. the non-existence of the quantization coefficient $Q_r(\mu)$, was shown for $c = \frac{1}{3}$ and $r = 2$ by Graf and Luschgy [9] and later also for $c \in [0, \frac{1}{3}]$ and $r > 1$ (cf. [21]). Also for higher-dimensional fractals and the related uniform distributions, the non-existence of the quantization coefficient was shown under special restrictions (cf. [22,30]).

**Definition 2.15** A vector $(t_1, ..., t_N) \in (\mathbb{R}\setminus\{0\})^N$ is called arithmetic, if there exists a real number $t$ and integers $n_1, ..., n_N$ with $t_i = t n_i$ for $i = 1, ..., N$. If no such number exists, the vector is not arithmetic.

The self-similar distribution $\mu$ is called non-arithmetic, if the vector

$$(\log(p_1c_1^r), ..., \log(p_Nc_N^r))$$

is not arithmetic. If $\mu$ is not arithmetic and the IFS satisfies the the open set condition, the quantization coefficient $Q_r(\mu)$ exists (cf. [12]). Pötzlberger [30] has also shown this result in the non-arithmetic case under the strong separation condition.

3 Open problems and concluding remarks

In this last section we sum up several conclusions and open questions arising out of the context in this paper.

First one could ask, if the definition of the Euler characteristic $\chi(F_\varepsilon)$ of the $\varepsilon-$parallel set $F_\varepsilon$ of $F$ works also for higher dimensional fractals. Unfortunately, the polyconvexity of $F_\varepsilon$ does not generally hold in higher dimensions.
for strong separated fractals (cf. [25]). Looking again at the already known results regarding the comparison of quantization dimension with Hausdorff and Packing dimension (cf. Theorem 2.2 and 2.4), it is natural to ask if we can replace in (4) the infimum by a minimum resp. in (5) the supremum by a maximum. If yes, this would imply

\[
\limsup_{k \to \infty} \prod_{l=1}^{k} n_l c_l^D < \infty \quad (29)
\]

resp.

\[
\liminf_{k \to \infty} \prod_{l=1}^{k} n_l c_l^D = \infty.
\]

Thus, it is not possible, that a replacement is allowed in both cases simultaneously. It is interesting to note, that (29) is one of the conditions needed in [21, Proposition 5.3 (iii)] to prove the non-existence of the quantization coefficient.

We conclude with a set of open problems.

**Remark 3.1** From Theorem 2.10 we immediately note, that for many one-dimensional homogeneous Cantor sets \( F \) with Hausdorff dimension \( D \) a constant \( M \in [1, \infty] \) exists, with \( P^D(F) = M \cdot \mathcal{H}^D(F) \). Upper and lower bounds for \( M \) were given by Feng [6]. He also raised the unsolved question if \( M = 2^D(2^{1/D} - 1)^D \) is true or not.

**Remark 3.2** It remains an open question, if for homogeneous one-dimensional Cantor sets and their related uniform distributions in the non-dyadic case (i.e. \( n_k > 2 \) for at least one \( k \)) and/or in higher dimensions, Theorem 2.10 still holds true. It suggests itself to conjecture, that under all or a part of the conditions (BD), (ESSC) and (HC) the identity

\[
\liminf_{n \to \infty} n (e_{n,r}(\mu))^D = \liminf_{k \to \infty} \prod_{l=1}^{k} n_l \left( e_{\prod_{l=1}^{k} n_l,r}(\mu) \right)^D. \quad (30)
\]

is valid. Based on this conjecture (30), the known results for the Hausdorff measure of Sierpinski sponges (cf. [37]) resp. Sierpinski gaskets (cf. [3,14,18]) and by using the results in [22], it should be possible to show, that (20) also holds for these sets, if the contraction factors \( (c_k)_{k \in \mathbb{N}} \) are converging and sufficiently small. For other higher dimensional fractal sets \( F \), the situation becomes even more complicated. It remains an open question, for which fractal sets \( F \subset \mathbb{R}^d \) with \( \text{diam}(F) = 1 \), norm exponent \( r \in [1, \infty] \), contracting factors \( (c_k)_{k \in \mathbb{N}} \) with \( \lim_{k \to \infty} c_k = c \) and Hausdorff dimension \( D \), the equation

\[
\mathcal{H}^D(F) = (e_{1,r}(\mu_c))^{-D} \liminf_{n \to \infty} n (e_{n,r}(\mu))^D
\]

is valid.
Remark 3.3 It remains an open question, if (20) also holds for self-similar distributions in the non-arithmetic case under the open set condition. If yes, one could calculate the quantization coefficient in terms of the Hausdorff measure, i.e. the identity
\[ Q_r(\mu) = (V_{1,r}(\mu))^{-1}(\mathcal{H}^D(F))^\frac{r}{r} \]
would hold.

Remark 3.4 Let \( U([0,1]) \) be the uniform distribution on \([0,1]\). We obtain from [10, Example 5.5] that
\[ \lim_{n \to \infty} n e_{n,r} = \frac{1}{2(1+r)^\frac{1}{r}}. \]
On the other hand a direct calculation shows
\[ e_{1,r}(U([0,1])) \mathcal{H}^1(U([0,1])) = \left( \int |x - \frac{1}{2}|^r dU([0,1])(x) \right)^\frac{1}{r} = \frac{1}{2(1+r)^\frac{1}{r}}. \]
Hence, although equation (20) is restricted to \( \sup_{k \in \mathbb{N}} c_k \leq \frac{1}{3} \) it keeps true if \( c_k = \frac{1}{2} \) for every \( k \). Moreover, due to \( \mathcal{H}^1 = \mathcal{P}^1 \) inequality (21) turns into an equality in this case. It remains an open question, for which values \( c \in \left[ \frac{1}{3}, \frac{1}{2} \right] \), if any, equation (20) resp. inequality (21) is still true. The relations (20) and (21) could then help to find an answer for the (unknown) behaviour of the mapping
\[ c \to \limsup_{n \to \infty} n e_{n,r}(\mu_c)^D - \liminf_{n \to \infty} n e_{n,r}(\mu_c)^D \]
in the range of \( \left[ \frac{1}{3}, \frac{1}{2} \right] \) (cf. [21, Remark 6.4]).

Remark 3.5 Motivated by equation (31) it makes sense to conjecture, that relation (20) could be generalized to non-singular distributions with compact support. Unfortunately this is not the case. If \( U([0,1]^2) \) denotes the uniform distribution on \([0,1]^2\) it is well-known (cf. [10, Theorem 8.15]) that
\[ Q_2(U([0,1]^2)) = \frac{5}{18\sqrt{3}} \]
resp.
\[ Q_\infty([0,1]^2) := \lim_{n \to \infty} n e_{n,\infty}([0,1]^2)^D = \left( \frac{2}{3\sqrt{3}} \right)^\frac{1}{2} \]
(cf. [10, p. 148]). Due to \( \mathcal{H}^2([0,1]^2) = \frac{4}{\pi} \) we get
\[ \mathcal{H}^2([0,1]^2) \neq \left( e_{1,r}(U([0,1]^2)) \right)^{-2} \liminf_{n \to \infty} n \left( e_{n,r}(U([0,1]^2)) \right)^2 \]
in the cases \( r = 2 \) and \( r = \infty \). Also if we multiply the right hand side with \( (\text{diam}([0,1]^2))^2 \) the inequality remains.
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