Dissertation

The Asymptotic Behaviour of the Riemann Mapping Function at Analytic Cusps

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Notation

\( \mathbb{N} := \{1, 2, \ldots\} \) set of natural numbers

\( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) set of nonnegative integers

\( \mathbb{R} \) field of real numbers

\( \mathbb{R}^* := \mathbb{R} \setminus \{0\} \) set of real numbers without 0

\( \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\} \) set of positive real numbers

\( \mathbb{R}_{\leq 0} := \{x \in \mathbb{R} \mid x \leq 0\} \) set of negative real numbers

\( \mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\} \) field of complex numbers

\( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) set of complex numbers without 0

\( \mathbb{C}_- := \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) slit plane

\( \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\} \) upper half plane

\( \mathbb{H}_- := \{z \in \mathbb{C} \mid \text{Im}(z) < 0\} \) lower half plane

\( \mathbb{E} := \{z \in \mathbb{C} \mid |z| < 1\} \) open Euclidian unit disk with radius 1 and center 0

\( \mathbb{E}_1 := \{z \in \mathbb{C} \mid |z - 1| < 1\} \) open Euclidean unit disk with radius 1 and center 1

\( B(a, r) := \{z \in \mathbb{C} \mid |z - a| < r\} \) open Euclidean disk with radius \( r > 0 \) and center \( a \in \mathbb{C} \)

\( \overline{B(a, r)} := \{z \in \mathbb{C} \mid |z - a| \leq r\} \) closed Euclidean disk with radius \( r > 0 \) and center \( a \in \mathbb{C} \)

\( \overline{A} \) closure of a set \( A \subset \mathbb{C} \)

\( \overset{\circ}{A} \) interior of a set \( A \subset \mathbb{C} \)

\( \partial A := \overline{A} \setminus \overset{\circ}{A} \) boundary of a set \( A \subset \mathbb{C} \)
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<td>Re(z), Im(z)</td>
<td>real and imaginary part of $z \in \mathbb{C}$</td>
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<tr>
<td>Re(f), Im(f)</td>
<td>real and imaginary part of a complex-valued function $f$</td>
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<td>$</td>
<td>z</td>
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<td>graph(f)</td>
<td>graph of a function $f$</td>
</tr>
<tr>
<td>$\mathbb{R}[x_1, \ldots, x_n]$</td>
<td>real polynomial ring in $n$ variables</td>
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<tr>
<td>arg(z) $\in (-\pi, \pi]$</td>
<td>standard argument of $z \in \mathbb{C}^*$</td>
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<td>dist(w, z) $:=</td>
<td>w - z</td>
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Chapter 1

Introduction

1.1 Motivation

Geometric function theory is a branch of complex analysis and studies geometric properties of analytic functions. While complex analysis has its roots in the 19th century, the origins of geometric function theory can be traced back to the beginning of the 20th century. Among other topics, it is dealing with conformal maps and analytic continuation. One of the fundamental theorems in this branch of mathematics is the Riemann Mapping Theorem. It states the existence of a conformal map between a simply connected proper domain of the complex plane and the unit disk. In fact, this theorem was the beginning of the study of complex analysis from a geometric point of view and the names of many mathematicians are linked to the history of it. In 1851, Riemann stated the theorem in his PhD thesis "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse", see Riemann [18], and sketched a proof for bounded domains with piecewise smooth boundary based on the Dirichlet principle. In 1870, however, Weierstraß showed that the Dirichlet principle was not valid for domains with general boundary and concluded that there may not exist a minimum as Riemann assumed in his proof. The criticism of Weierstraß on the Dirichlet principle was weakened in 1904 as David Hilbert proved the Dirichlet principle to the extent required by Riemann and so the validity of Riemann’s proof of the theorem was restored. In the
meantime many new methods were developed for proving the Riemann Mapping Theorem. For instance, in 1870 a student of Weierstraß, Hermann Amandus Schwarz, gave a proof of the Riemann Mapping Theorem for the case of simply connected domains with piecewise analytic boundary by showing that the Green’s function exists under these assumptions, see Schwarz [20]. Another well known proof was performed by Osgood in 1900. He showed the theorem for simply connected domains with an arbitrary boundary by proving the existence of the Green’s function corresponding to the domain, see Osgood [14]. A common way to prove the Riemann Mapping Theorem nowadays is through the use of the Theorem of Montel [6, Theorem IV.4.9 on p. 236]. For an overview on the history of the proofs for the Riemann Mapping Theorem we refer to Gray [7].

An important question that arises in the context of the Riemann Mapping Theorem is the following: what can we say about the extendability of a conformal map to the boundary? Constantin Carathéodory, a mathematician of Greek origin, worked on topics related to this question and achieved pioneering results. In 1913, he proved that a conformal map of a domain which is bounded by a Jordan arc onto the unit disk has a continuous and bijective continuation to the boundaries. This conjecture was given earlier by William Fogg Osgood.

The topic of this work originates from the field of geometric function theory and is concerned with the asymptotic behaviour of conformal mappings at the boundary. This means we want to describe their behaviour by well-known, elementary functions. Motivated by real analytic geometry we assume that the boundary is piecewise analytic. There are two cases we have to distinguish: the behaviour at regular boundary points and the behaviour at singular boundary points. If we have a regular boundary point, we can determine the asymptotic behaviour of the mapping function at this point by the Schwarz Reflection at analytic arcs, a technique for analytic continuation. If we have a singular boundary point, we have to distinguish analytic corners, where the opening angle of two regular analytic arcs is greater than 0, and analytic cusps, where the opening angle vanishes. The asymptotic behaviour of conformal mapping functions at analytic corners has been already studied extensively. For instance, Leon Lichtenstein
and Stephan Emanuel Warschawski focussed on the behaviour of the mapping function and its derivatives at analytic corners, see Lichtenstein [13] and Warschawski [22]. While Lichtenstein concentrated on a special case regarding the asymptotic behaviour of the derivatives, Warschawski generalised these results.

A much stronger result than Lichtenstein and Warschawski was stated by Russell Sherman Lehman in 1957. He showed that the mapping function can be developed in a certain generalised power series, see Lehman [11]. This result was used to investigate geometric statements of the Riemann Mapping Theorem related to real analytic geometry: on the basis of Lehman’s result, Tobias Kaiser investigated in 2009, see [8], the o-minimal content of the Riemann Mapping Theorem by proving that the mapping functions of semianalytic domains without cusps are in many cases definable in an o-minimal structure. O-minimal structures generalise the geometry of algebraic character. In the early 80’s Lou van den Dries et al. isolated one property of ordered structures that leads to tame behaviour, the so-called o-minimality property. Important concepts and functions from analysis can be realised in o-minimal structures which are defined by nice tameness- and finitness-properties. A natural question arising from this “tame” point of view is if it is also possible to make a statement for domains with analytic cusps similar to the one Kaiser proved for domains with analytic corners by using the generalised power series.

While there are many results on the asymptotic behaviour at analytic corners, there are only a few articles that deal with the asymptotic behaviour of mapping functions at analytic cusps. One is from Warschawski who works in a more general setting than that of analytic cusps. A second result was published in 2010 by Kaiser [9] where he investigated the behaviour of the mapping function at analytic cusps for domains with small perturbation of angles and gave upper bounds for its derivatives. However, to this day there is still a knowledge gap and many questions related to the asymptotic behaviour of such mapping functions at analytic cusps remain unanswered. Therefore, the goal of this work is to discuss the asymptotic behaviour at analytic cusps in a more general setting than that of domains with small perturbation of angles. Furthermore,
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we investigate the behaviour of its derivatives, its inverse, and of the derivatives of its inverse. Based on these results we also give a conjecture on the asymptotic power series expansion for the inverse function.

1.2 Outline and Research Contributions

In Chapter 2 we initiate basic terminology of complex analysis required later on in the thesis. The well known Riemann Mapping Theorem that gives us the existence of a conformal map of a simply connected proper domain of the complex plane onto the unit disk is presented in Section 2.2. We recall Carathéodory’s important Prime End Theorem on the extension of conformal maps to the boundary in Section 2.3. It is one of the main topics in geometric function theory to understand the behaviour of such maps at the boundary. In this work, we concentrate on piecewise analytic boundaries which means that the boundary of our simply connected proper domain of the complex plane consists of a finite number of analytic arcs. Section 2.3 contains a classification for the different boundary points, namely singular and regular boundary points, and introduces analytic corners and analytic cusps. The next part of this chapter, Section 2.4 covers the asymptotic behaviour of a Riemann map at regular boundary points: we recall two techniques for analytic continuation, namely the Schwarz Reflection Principle and the Schwarz Reflection at analytic arcs, and by using the latter one can show that a given mapping function behaves like $z$ at a regular boundary point. Section 2.5 focusses on the behaviour of a Riemann map at an analytic corner which means that the opening angle between two regular analytic arcs is greater than 0. We present the results of Lichtenstein, Warschawski, and Lehman on the asymptotic behaviour of the Riemann map of a simply connected domain with an analytic corner onto the upper half plane and on estimates for its derivatives. Especially, the result of Lehman is fundamental since he showed that the mapping function can be developed in a certain generalised power series. On the basis of this result Kaiser proved that the mapping functions of semianalytic domains without cusps are in many cases definable in an o-minimal
1.2 Outline and Research Contributions

structure. Section 2.6 gives an overview on the basic concepts of o-minimal structures, presents some concrete examples for the latter, and recalls the above mentioned theorem of Kaiser.

Chapter 3 starts with general premises, see Section 3.1 where we introduce basic notation. Moreover, we see that by applying a translation we can focus on the case that we have a simply connected proper domain of the complex plane with an analytic cusp at 0. In Section 3.2, we present the results of Kaiser for the asymptotic behaviour of a Riemann map in the situation that the domain has small perturbation of angles. After applying a preliminary transformation to our domain, we investigate the asymptotic behaviour of the Riemann map for the case that one of the two boundary arcs in some neighbourhood of the cusp at 0 coincides with the positive real axis, see Section 3.3. We denote such a domain by $\Omega$. In Section 3.4, we present a theorem of Warschawski on the modulus and the argument of a conformal map of a simply connected proper domain of the complex plane onto the unit disk with center 1 in a neighbourhood of 0. Furthermore, we give a remark on the estimates for the modulus and the argument of the conformal map if the latter domain is mapped onto the upper half plane instead of mapping it onto the unit disk with center 1. By adapting this theorem to our case, we can therefore give estimates for the modulus and the argument of our Riemann map $\varphi : \Omega \to \mathbb{H}$ with $\varphi(0) = 0$ at the analytic cusp at 0. We introduce the asymptotic tuple of $\Omega$ which consists of geometric invariants depending on the shape of the domain $\Omega$ at the cusp. As a first main result we prove the following corollary.

**Corollary**

*We have*

\[(a)\]

\[|\varphi(z)| = c \exp \left( |z|^{-d} \sum_{n=0}^{d-1} b_n |z|^n + a \log |z| + o(1) \right)\]

*where $c \in \mathbb{R}_{>0}$ is a constant and $(b_0, \ldots, b_{d-1}, a) \in \mathbb{R}^{d+1}$ is the asymptotic tuple of $\Omega$.\]
Chapter 1 Introduction

\[\text{(b)}\]
\[
\arg(\varphi(z)) = \pi \arg(z)|z|^{-d} \left( \sum_{n=0}^{d} c_n |z|^n \right) + o(1).
\]

Based on this result we prove in Section 3.5 the following theorem on the asymptotic behaviour of \(\varphi\) at 0.

**Theorem A**

We have
\[
\varphi(z) \sim z^a \exp \left( \frac{b_0}{z^d} + \frac{b_1}{z^{d-1}} + \cdots + \frac{b_{d-1}}{z} \right)
\]
at 0 on \(\Omega\) where \((b_0, \ldots, b_{d-1}, a)\) is the asymptotic tuple of \(\Omega\).

Moreover, we give an example and discuss the special situation where \(a = 0\). That is particularly interesting as this is the case when \(\Omega\) has small perturbation of angles.

Furthermore, we determine in Section 3.6 the asymptotic behaviour of the derivatives \(\varphi^{(n)}\) for \(n \in \mathbb{N}\).

**Theorem B**

Let \(n \in \mathbb{N}_0\). Then
\[
\varphi^{(n)}(z) \sim \varphi(z)z^{-n(d+1)} \sim z^{a-n(d+1)} \exp \left( \frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z} \right)
\]
at 0 on \(\Omega\) where \((b_0, \ldots, b_{d-1}, a)\) is the asymptotic tuple of \(\Omega\).

By using these results, we subsequently study in Section 3.7 the behaviour of the inverse mapping function \(\psi : \mathbb{H} \to \Omega\) of the Riemann map \(\varphi\). Moreover, we have again \(\psi(0) = 0\).

**Theorem C**

We have at 0 on \(\mathbb{H}\)
\[
\psi(z) \sim \left( -\frac{\pi}{a_d d \log(|z|)} \right)^{\frac{1}{3}}.
\]
In Section 3.8 we investigate the behaviour of the derivatives of the inverse and obtain the following theorem.

**Theorem D**

*Let* $n \in \mathbb{N}$. Then

$$
\psi^{(n)}(z) \sim (\psi(z))^{d+1}z^{-n} \sim \left(-\frac{1}{\log(z)}\right)^{\frac{1}{d}+1}z^{-n}
$$

*at 0 on* $\mathbb{H}$.

Finally, Chapter 4 concludes the thesis by giving an outlook on future work. In continuation of the research presented in this thesis, it is the goal to derive a generalised power series for the mapping function $\varphi$ analogously to the result of Lehman for analytic corners. We therefore present a conjecture on the asymptotic power series expansion of $\psi$ at 0 in a special case and give an idea of the proof. By verifying this conjecture it would be immediately possible to determine the asymptotic power series expansion of $\varphi$. 
Chapter 2

Preliminaries

2.1 Basic Definitions and Lemmas

In this section we present some basic definitions and lemmas required later on in the thesis. If not stated otherwise we follow the definitions as given in Ahlfors [1], Freitag and Busam [6], Kaiser [9], Lang [10], and Pommerenke [16].

Definition 2.1.1
Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$. The function $f$ is called analytic at $z_0 \in U$ if there exists a power series

$$
\sum_{n=0}^{\infty} a_n(z - z_0)^n
$$

and some $r > 0$ such that the series converges absolutely for $z \in B(z_0, r)$, and such that we have

$$
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n
$$

for all $z \in B(z_0, r)$. The function $f$ is called analytic in $U$ if $f$ is analytic at every $z \in U$.

Definition 2.1.2
Let $U \subset \mathbb{C}$ be an open, nonempty set and let $a, b \in \mathbb{R}$ with $a < b$. A curve in $U$ is a
continuous map of the form

\[
\gamma : [a, b] \rightarrow U, \ t \mapsto \gamma(t).
\]

Moreover, \(\gamma(a)\) is called the \textbf{beginning point} and \(\gamma(b)\) the \textbf{endpoint}. A curve \(\gamma\) is called \textbf{closed} if \(\gamma(a) = \gamma(b)\).

**Definition 2.1.3**

We say that a curve \(\gamma : [a, b] \rightarrow \mathbb{C}\) is \textbf{regular} at \(t_0 \in [a, b]\) if \(\gamma\) is differentiable at \(t_0\) and \(\gamma'(t_0) \neq 0\). The curve \(\gamma\) is called \textbf{regular} if it is regular at every point. We say that \(\gamma\) is \textbf{singular} at \(t_0\) if \(\gamma\) is differentiable at \(t_0\) and \(\gamma'(t_0) = 0\).

**Definition 2.1.4**

A curve \(\gamma : [a, b] \rightarrow \mathbb{C}\) is called a \textbf{simple closed curve} or a \textbf{Jordan curve} if \(\gamma\) is closed and \(\gamma(s) \neq \gamma(t)\) for all \(s, t \in [a, b]\) with \(s < t\) and \(t - s < b - a\).

**Definition 2.1.5**

A curve \(\gamma : [a, b] \rightarrow \mathbb{C}\) is called \textbf{real analytic} if for each point \(t_0 \in [a, b]\) there exists a convergent power series expansion

\[
\gamma(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n
\]

for all \(t\) sufficiently close to \(t_0\).

**Definition 2.1.6**

A curve is called \textbf{smooth} if it is continuously differentiable.

**Definition 2.1.7**

A curve \(\gamma : [a, b] \rightarrow \mathbb{C}\) is called \textbf{piecewise smooth} if there is a partition \(a = a_0 < a_1 < \cdots < a_n = b\) of the interval \([a, b]\) such that the restrictions \(\gamma_\nu := \gamma|_{[a_\nu, a_{\nu+1}]}\) are smooth for \(0 \leq \nu < n\).
Definition 2.1.8
Let \( \gamma : [a, b] \to \mathbb{C} \) be a curve. The trace of the curve \( \gamma \) is denoted by \( \text{tr}(\gamma) := \gamma([a, b]) \subset \mathbb{C} \) and is called an arc (parameterised by \( \gamma \)).

Definition 2.1.9
An arc \( \Gamma \) is called regular at \( P \in \Gamma \) if there exists locally a parameterisation \( \gamma \) of \( \Gamma \) with \( \gamma(t_0) = P \) such that \( \gamma \) is regular at \( t_0 \). Otherwise \( \Gamma \) is singular at \( P \). An arc \( \Gamma \) is called regular if \( \Gamma \) is regular at every point.

Definition 2.1.10
The trace of a Jordan curve is called a Jordan arc.

Definition 2.1.11
An arc \( \Gamma \) is said to be an analytic arc at \( P \in \Gamma \) if \( \Gamma \) has a real analytic parameterisation in some neighbourhood \( U \subset \mathbb{C} \) of \( P \), i.e. there exists a real analytic curve \( \gamma : ]-\epsilon, \epsilon[ \to U \) such that \( \gamma(0) = P \), \( \gamma : ]-\epsilon, \epsilon[ \to \gamma(]-\epsilon, \epsilon[) \) is injective and \( U \cap \Gamma = \gamma(]-\epsilon, \epsilon[) \). An arc \( \Gamma \) is said to be an analytic arc if this property holds for every point \( P \in \Gamma \).

Definition 2.1.12
An arc \( \Gamma \) is said to be piecewise (regular) analytic if it can be decomposed into a finite number of arcs \( \Gamma_1, \ldots, \Gamma_n \), \( n \in \mathbb{N} \), such that there exists for every \( i \in \{1, \ldots, n\} \) a real analytic (regular) curve \( \gamma_i : ]-\epsilon_i, \epsilon_i[ \to \mathbb{C} \) and a compact interval \( I_i \subset ]-\epsilon_i, \epsilon_i[ \) with \( \Gamma_i = \gamma_i(I_i) \).

Definition 2.1.13
A set \( A \subset \mathbb{C} \) is called arcwise connected if for any two points \( w, z \in A \) there is a piecewise smooth curve \( \gamma : [a, b] \to A \) such that
\[ \gamma(a) = w, \quad \gamma(b) = z. \]

Definition 2.1.14
Let \( U \subset \mathbb{C} \) be an open set. \( U \) is called a domain if it is nonempty and arcwise connected.
Definition 2.1.15
A domain $G \subset \mathbb{C}$ is called simply connected if the closed set $\mathbb{C} \setminus G$ has no bounded connected components.

Definition 2.1.16
A map $\varphi : U \to V$ between two open sets $U, V \subset \mathbb{C}$ is called conformal if the following conditions are satisfied

(a) $\varphi$ is bijective,

(b) $\varphi$ is analytic,

(c) $\varphi^{-1}$ is analytic.

The geometrical significance of a conformal map $\varphi$ is that the oriented angle between the traces of two regular curves $\gamma_1$ and $\gamma_2$ at an intersection point $a \in U$ is equal to the oriented angle between the arcs $\text{tr}(\varphi \circ \gamma_1)$ and $\text{tr}(\varphi \circ \gamma_2)$ at their intersection point $\varphi(a)$.

Definition 2.1.17
Given a non-vanishing real or complex power series $h(t) = \sum_{n=0}^{\infty} a_n t^n$ we set

$$\text{ord}(h) := \min\{n \in \mathbb{N}_0 : a_n \neq 0\}$$

which is called the order of the power series. Moreover, we set $\text{ord}(0) := \infty$.

To describe the asymptotic behaviour of a mapping function we need some further notation. Suppose we have a function $f$ and want to consider how quickly the function goes to 0 near 0. Then we would like to find a simple function $g$ which also vanishes at 0 such that $g$ and $f$ are almost equal near 0.

Definition 2.1.18
Let $A \subset \mathbb{C}$ be a nonempty set with $0 \in \overline{A}$, let $f, g : A \to \mathbb{C}$ and let $g(z) \neq 0$ for all $z$ in a neighbourhood of 0. We write
2.1 Basic Definitions and Lemmas

(a) \( f \simeq g \) at 0 on \( A \) if
\[
\lim_{z \to 0} \frac{f(z)}{g(z)} = 1,
\]
(b) \( f \sim g \) at 0 on \( A \) if
\[
\lim_{z \to 0} \frac{f(z)}{g(z)} \in \mathbb{C}^*,
\]
(c) \( f = o(g) \) at 0 on \( A \) if
\[
\lim_{z \to 0} \frac{f(z)}{g(z)} = 0,
\]
(d) \( f = O(g) \) at 0 on \( A \) if there is a \( C > 0 \) such that
\[
|f(z)| \leq C|g(z)|
\]
for all \( z \) in a neighbourhood of 0.

The first two notation, (a) and (b), describe the situation where the function \( f \) is asymptotically similar and asymptotically equivalent, respectively, to \( g \) around 0. Graphically, the \( o \)-notation describes that \( f \) goes much faster to 0 around 0 than \( g \). The well-known \( O \)-notation characterises functions according to their growth rate which is often referred to as the order of the function. A description in terms of \( O \)-notation provides an upper bound on the growth rate of the function. It is commonly used to describe how closely a finite series approximates a given function. The function \( g \) is typically chosen to be as simple as possible and is often used to describe the error term in an approximation to a mathematical function. This means we do not know the exact form of the error term but we have pointed out how small it is near 0. The little \( o \)-notation makes a stronger statement than the corresponding big \( O \)-notation since every function \( f \) that is \( o(g) \) is also \( O(g) \) near 0.

**Remark 2.1.19**

Note that the representation \( f = O(g) \) and \( f = o(g) \), respectively, is a slight abuse of notation which is commonly used to express \( f \in O(g) \) and \( f \in o(g) \), respectively.
Chapter 2 Preliminaries

Lemma 2.1.20
Let \( A \subset \mathbb{C} \) with \( A \neq \emptyset \) and \( 0 \in \overline{A} \) and let \( f, g : A \to \mathbb{C} \) be functions with \( g(z) \neq 0 \) for all \( z \) in a neighbourhood of \( 0 \). If \( f = g + o(1) \) then \( \exp(f) \simeq \exp(g) \).

Proof: We have

\[
    f(z) = g(z) + o(1) \iff f(z) - g(z) = o(1)
    \iff \lim_{z \to 0} (f(z) - g(z)) = 0
    \Rightarrow \lim_{z \to 0} \exp(f(z)) - \exp(g(z)) = 1
    \iff \lim_{z \to 0} \exp(f(z)) = 1
    \iff \exp(f(z)) \simeq \exp(g(z)).
\]

Lemma 2.1.21
Let \( A \subset \mathbb{C} \) be a domain with \( 0 \in \partial A \) and let \( f, g : A \to \mathbb{C} \) be analytic functions with \( g(z) \neq 0 \) for all \( z \) in a neighbourhood of \( 0 \). Furthermore, we assume that \( \lim_{z \to 0} f(z) = 0 = \lim_{z \to 0} g(z) \). Moreover, let \( f(z) \sim g(z) \). Then we have

\[
    \log(f) \simeq \log(g).
\]

Proof: Since \( f(z) \sim g(z) \) we have

\[
    \lim_{z \to 0} \frac{f(z)}{g(z)} \in \mathbb{C}^*.
\]

Thus there exists an analytic function \( \xi : A \to \mathbb{C} \) with \( \lim_{z \to 0} \xi(z) \in \mathbb{C}^* \) such that \( f(z) = g(z)\xi(z) \). Applying the logarithm we get for \( \ell(z) \in \{-1, 0, 1\} \) that

\[
    \log(f(z)) = \log(g(z)\xi(z)) = \log(g(z)) + \log(\xi(z)) + 2\pi i\ell(z).
\]
Since \( \lim_{z \to 0} \log(\xi(z)) \in \mathbb{C} \) and \( \lim_{z \to 0} \frac{1}{\log(g(z))} = 0 \) it follows that

\[
\lim_{z \to 0} \frac{\log(f(z))}{\log(g(z))} = \lim_{z \to 0} \frac{\log(g(z)) + \log(\xi(z)) + 2\pi i l(z)}{\log(g(z))} \\
= \lim_{z \to 0} \left( \frac{\log(g(z))}{\log(g(z))} + \frac{\log(\xi(z)) + 2\pi i l(z)}{\log(g(z))} \right) \\
= 1 + \lim_{z \to 0} \frac{\log(\xi(z)) + 2\pi i l(z)}{\log(g(z))} \\
= 1.
\]

\( \square \)

### 2.2 The Riemann Mapping Theorem

As initially mentioned, one of the most important theorems in geometric function theory is the Riemann Mapping Theorem. This theorem gives the existence of a conformal mapping function of a simply connected domain which is a proper subset of the complex plane to the unit disk.

**Theorem 2.2.1 (Riemann 1851)**

Let \( \Omega \subset \mathbb{C} \) be a simply connected domain. Then there exists a conformal map \( \varphi : \Omega \to \mathbb{D} \).

*Proof:* We refer to Remmert [17, p. 175 f.].

In the following, we refer to such a map as a **Riemann map**. An example can be seen in Figure 2.1.

The Riemann Mapping Theorem implies that any two simply connected proper domains of the complex plane, which can be highly complicated, can be mapped conformally onto each other. While it assures the existence of such a conformal mapping, it is not of much help when we are faced with the practical problem of finding the mapping function which transforms two given domains onto each other. Even if we consider a map from the interior of a circle to the interior of a square, which is a relatively simple conformal
map, we have no explicit formula using only elementary functions. The necessity thus arises to develop special techniques which help with the treatment of a given mapping problem. It is sufficient to adapt these techniques to the case in which one of the domains is the unit disk since if we can map two domains onto the same disk, we can also map them onto each other. Furthermore, it is an equivalent problem to consider a conformal map onto the upper half plane instead of the unit disk which is a composition of a Riemann map with a Möbius transformation. This transformation which maps the unit disk onto the upper half plane is the so-called Cayley transformation, given in the next definition.

**Definition 2.2.2**

(a) The map $h_C : \mathbb{C} \to \mathbb{C}$ defined by

$$h_C(z) = \frac{z - i}{z + i}$$

is called the **Cayley transformation** and maps the upper half plane conformally onto the unit disk, i.e. $h_C(\mathbb{H}) = \mathbb{E}$. 
2.3 Classification of Boundary Points

(b) The map \( h_{C'} : \mathbb{C} \to \mathbb{C} \) given by

\[
h_{C'}(z) = \frac{1 + z}{1 - z}
\]

is the inverse of the Cayley transformation. In particular, note that \( h_{C'}(E) = \mathbb{H} \).

2.3 Classification of Boundary Points

Since we want to investigate the asymptotic behaviour of a Riemann map at special boundary points later on, we have to examine the criteria for extending a Riemann map to the boundary and discuss the different types of boundary points. Osgood first conjectured that a Riemann map can be extended continuously to the boundary if the boundary of the domain is a Jordan arc, we refer to Osgood [15]. As already mentioned, this conjecture was proven by Carathéodory in 1913, see [2], and is often called Carathéodory’s Prime End Theorem. We recall this result below.

**Theorem 2.3.1 (Carathéodory’s Prime End Theorem)**

Let \( \Omega \subsetneq \mathbb{C} \) be a simply connected domain and let \( \partial \Omega \) be a Jordan arc. Then the Riemann map

\[
\varphi : \Omega \to E
\]

extends continuously to the boundary, giving a homeomorphism

\[
\phi : \partial \Omega \to \partial E.
\]

**Proof:** We refer to Carathéodory [2].

This theorem implies that a Riemann map between two simply connected domains in the complex plain whose boundaries are Jordan arcs can be extended to a homeomorphism of their closures. The next step is to have a closer look at the different kinds of boundary points. In the following, let \( \Omega \subsetneq \mathbb{C} \) be a simply connected domain whose boundary \( \partial \Omega \) is a Jordan arc and is piecewise analytic and regular.
Definition 2.3.2
If there exists locally a parameterisation \( \gamma : [a, b] \to \mathbb{C} \) of \( \partial \Omega \), \( t_0 \in [a, b] \) with \( \gamma(t_0) = P \) and \( \gamma \) regular at \( t_0 \) we say that \( \Omega \) has a regular boundary point at \( P \). Otherwise we say that \( \Omega \) has a singular boundary point at \( \gamma(t_0) \).

The opening angle between two regular analytic arcs, intersecting at one point, is the opening angle between the tangents of the arcs at the intersection point, see for example Figure 2.2 where the blue half lines are the tangents. We assume that \( 0 \in \partial \Omega \), otherwise we apply a translation. If \( 0 \) is a singular boundary point, the interior angle of \( \Omega \) at \( 0 \) is the opening angle between the two regular analytic arcs of \( \partial \Omega \) at \( 0 \) which has a nonempty intersection with \( \Omega \). We have to distinguish two cases:

1. \( \Omega \) has an analytic corner at \( 0 \).
2. \( \Omega \) has an analytic cusp at \( 0 \).

Definition 2.3.3
We say that \( \Omega \) has an analytic corner at \( 0 \) if \( 0 \) is a singular boundary point and the boundary at \( 0 \) is locally given by two regular analytic arcs with interior angle \( \pi \alpha \) where \( 0 < \alpha \leq 2 \).

For an example of an analytic corner at \( 0 \), see Figure 2.2.

![Figure 2.2: Analytic corner at 0 with opening angle \( \pi \alpha \) where \( 0 < \alpha \leq 2 \)](image)
2.4 Asymptotic Behaviour at Analytically Regular Boundary Points

**Definition 2.3.4**

We say that Ω has an **analytic cusp** at 0 if the boundary of Ω at 0 is locally given by two regular analytic arcs such that the interior angle at 0 vanishes.

An example of an analytic cusp at 0 is depicted in Figure 2.3.

![Figure 2.3: Analytic cusp at 0](image)

The following holds for analytic corners and analytic cusps at 0, respectively: since the boundary of Ω in a neighbourhood of 0 consists of two regular analytic arcs, which we denote by Γ₁ and Γ₂, there exists some ε > 0 such that the parameterisations of the boundary arcs are analytic on $B(0, \epsilon)$ and such that $\partial \Omega$ close to 0 is given by $\Gamma_1 \cup \Gamma_2$. We denote these parameterisations by $\gamma_1$ and $\gamma_2$ where $\Gamma_1 := \gamma_1([0, \epsilon[)$ and $\Gamma_2 := \gamma_2([0, \epsilon[)$, respectively. Since $\Gamma_1$ and $\Gamma_2$ are regular by assumption, we may assume that $\gamma_1$ and $\gamma_2$ are regular. Thus, $\gamma_1'(0)\gamma_2'(0) \neq 0$ holds. Moreover, we have $\gamma_1(0) = \gamma_2(0) = 0$.

2.4 Asymptotic Behaviour at Analytically Regular Boundary Points

In this section, we investigate the asymptotic behaviour of a Riemann map at an analytically regular boundary point. In this particular case, we can use the Schwarz Reflection at analytic arcs, a technique for analytic continuation to extend the domain of definition of an analytic function.

2.4.1 Schwarz Reflection

First, we recall the so-called Symmetry Principle or Schwarz Reflection Principle which enables one to find a continuation easily and explicitly under rather special conditions.
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**Theorem 2.4.1 (Schwarz Reflection Principle)**

Let $G_+ \subset \mathbb{H}$ be a domain and suppose that the boundary of $G_+$ contains an open interval $I \subset \mathbb{R}$. Let $G_-$ be the reflection of $G_+$ across the real axis, i.e. $G_- := \{ z \in \mathbb{C} | \overline{z} \in G_+ \}$. Furthermore, let $G = G_+ \cup I \cup G_-$. Then the following holds:

(a) If $f$ is analytic on $G_+$ and on $G_-$ and continuous on $G$, then $f$ is analytic on $G$.

(b) If $f$ is a continuous function on $G_+ \cup I$, analytic on $G_+$, and $f$ is real-valued on $I$, then $f$ has an unique analytic continuation $F$ on $G$ and $F$ satisfies

$$F(z) = \overline{f(\overline{z})} \text{ for all } z \in G_-.$$ 

**Proof:** See Lang [10, p. 294 f.]. □

Schwarz was able to generalise his Reflection Principle to more general arcs, namely to regular analytic arcs. We present his theorem in the following and perform a reflection at a regular analytic arc in the proof of Remark 2.4.4.

**Definition 2.4.2**

Let $U \subset \mathbb{C}$ be an open set and let $\Gamma \subset \partial U$ be a regular analytic arc such that $U$ lies on one side of $\Gamma$. We say that an analytic function $f : U \to \mathbb{C}$ has an analytic continuation across the arc $\Gamma$ if there exists an open neighbourhood $V$ of $\Gamma$ such that $f$ has an analytic continuation to $U \cup V$.

**Theorem 2.4.3 (Schwarz Reflection at analytic arcs)**

Let $f : U \to \mathbb{C}$ be analytic on an open set $U \subset \mathbb{C}$. Let $\Gamma$ be a regular analytic arc with $\Gamma \subset \partial U$ such that $U$ lies on one side of $\Gamma$. Assume that $f$ extends to a continuous function on $U \cup \Gamma$ and that $f(\Gamma)$ is contained in a regular analytic arc $\eta$ such that $f(U)$ lies on one side of $\eta$. Then $f$ has an analytic continuation across $\Gamma$. This analytic continuation is called the reflection of $f$ across $\Gamma$.

**Proof:** We refer to Lang [10, p. 299 f.]. □
2.4 Asymptotic Behaviour at Analytically Regular Boundary Points

2.4.2 General Premises

Let $\Omega \subset \mathbb{C}$ be a simply connected domain with piecewise analytic boundary and $0 \in \partial \Omega$. By the Riemann Mapping Theorem 2.2.1 there exists a conformal map $\varphi : \Omega \to \mathbb{H}$. The mapping function $\varphi$ can be extended continuously to the boundary $\partial \Omega$ and thus we can assume without restriction that $\varphi(0) = 0$ after applying a suitable translation.

2.4.3 Asymptotic Behaviour

As a consequence of the Schwarz Reflection at analytic arcs, see Theorem 2.4.3, we obtain the following result for the asymptotic behaviour of $\varphi : \Omega \to \mathbb{H}$ at regular boundary points.

Remark 2.4.4

Let 0 be a regular boundary point. Then the Riemann map $\varphi$ is given by a convergent power series at 0 with

$$\varphi(z) \sim z$$

at 0 on $\Omega$.

**Proof:** Let $B(0, \delta)$ with $\delta > 0$ be a neighbourhood of 0 such that $B(0, \delta) \cap \partial \Omega = \Gamma$ is analytic. Thus, there exist real analytic curves $u, v : [-\epsilon, \epsilon] \to \mathbb{R}$ with $(u(0), v(0)) = (0, 0)$ and $\Gamma = (u(t), v(t))$ such that $\Gamma$ can locally be parameterised as

$$\tilde{\gamma} : [-\epsilon, \epsilon] \to \mathbb{C}, \quad \tilde{\gamma}(t) = u(t) + iv(t).$$

Since $\Gamma$ is regular by assumption we may assume that its parameterisation is regular. Moreover, $\Gamma$ is analytic and thus there exists, after shrinking $\epsilon$ if necessary, a complexification of $\tilde{\gamma}$,

$$\gamma : B(0, \epsilon) \to \mathbb{C}, \quad \gamma(z) = u(z) + iv(z)$$

where $\gamma|_{-\epsilon, \epsilon} = \tilde{\gamma}$.
We set \( g(z) := \varphi \circ \gamma(z) \) and therefore we have \( g(t) \in \mathbb{R} \) for \( t \in ]-\epsilon, \epsilon[ \). Hence, \( g \) is a function on \((B(0, \epsilon) \cap \mathbb{H}) \cup ]-\epsilon, \epsilon[\) and analytic on \(B(0, \epsilon) \cap \mathbb{H} \). By the Schwarz Reflection Principle, see Theorem 2.4.1, there exists an unique analytic continuation which we denote by \( G : B(0, \epsilon) \to \mathbb{C} \) of the following form:

\[
G(z) = \begin{cases} 
  g(z) & \text{for } z \in B(0, \epsilon) \cap \mathbb{H}, \\
  g(z) & \text{for } z \in B(0, \epsilon) \cap \mathbb{R}, \\
  \overline{g(z)} & \text{for } z \in B(0, \epsilon) \cap \mathbb{H}^-. 
\end{cases}
\]

Let \( \Omega^* \) be the reflection of \( \Omega \) at \( \Gamma \). Since 0 is a regular boundary point by assumption, we can assume \( \gamma'(0) \neq 0 \) and therefore \( \gamma \) is locally invertible. Shrinking \( \delta \) if necessary, we obtain for the analytic continuation \( \Phi \) of \( \varphi \) that

\[
\Phi(z) = \begin{cases} 
  g(\gamma^{-1}(z)) & \text{for } z \in B(0, \delta) \cap \Omega, \\
  g(\gamma^{-1}(z)) & \text{for } z \in \Gamma, \\
  \overline{g(\gamma^{-1}(z))} & \text{for } z \in B(0, \delta) \cap \Omega^*, \\
  \varphi(z) & \text{for } z \in B(0, \delta) \cap \Omega, \\
  \varphi(z) & \text{for } z \in \Gamma, \\
  \overline{\varphi(\gamma(\varphi^{-1}(z)))} & \text{for } z \in B(0, \delta) \cap \Omega^*. 
\end{cases}
\]

Since \( \Phi \) is analytic on \( B(0, \delta) \) and \( \varphi \) and therefore \( \Phi \) is injective on \( B(0, \delta) \cap \Omega \), we get \( \Phi(z) \sim z \). Hence, \( \varphi(z) \sim z \). An illustration of the construction is shown in Figure 2.4.

\[\square\]

2.5 Asymptotic Behaviour at Analytic Corners

In this chapter, we want to study the results of Lichtenstein, Warschawski, and Lehman for the asymptotic behaviour of the mapping function and its derivatives at an analytic corner. Assuming we have a simply connected proper domain of the complex plane with piecewise analytic boundary and an analytic corner, then one of the fundamental questions is how the Riemann map between this domain and the upper half plane behaves in a neighbourhood of the analytic corner. A partial answer to this issue is given by Lichtenstein and Warschawski who investigated the asymptotic behaviour of such a
2.5 Asymptotic Behaviour at Analytic Corners

Figure 2.4: Visualisation of the Schwarz Reflection at analytic arcs

Riemann map and their derivatives. Lehman even obtained the much stronger result that the mapping function can be developed in a certain generalised power series.

2.5.1 General Premises

Let \( \Omega \subset \mathbb{C} \) be a simply connected domain with piecewise analytic boundary \( \partial \Omega \). We assume that \( 0 \in \partial \Omega \), otherwise we apply a translation. Furthermore, we assume \( \Omega \) has an analytic corner at \( 0 \in \partial \Omega \) with opening angle \( \pi \alpha \) where \( 0 < \alpha \leq 2 \). Due to the Riemann Mapping Theorem \[2.2.1\] there exists a conformal map \( \varphi : \Omega \to \mathbb{H} \). An example for such a map is given in Figure \[2.5\]. By Theorem \[2.3.1\] the mapping function can be extended locally at \( 0 \) to a homeomorphism of their closures. Hence, after applying a
suitable translation we can assume without restriction that $\varphi(0) = 0$.

Figure 2.5: Example of a Riemann map $\varphi$ from $\Omega$ onto the upper half plane $\mathbb{H}$

2.5.2 Asymptotic Behaviour of the Mapping Function and its Inverse

In this section, we want to discuss some basic results of Lichtenstein and Warschawski. In [13] Lichtenstein studied the asymptotic behaviour of conformal mappings and their derivatives of simply connected domains with piecewise analytic boundaries and corners with opening angle $\pi \alpha$ where $0 < \alpha \leq 2$. However, he proved his result on the asymptotic behaviour for the $n$th derivative of such mapping functions, where $n \geq 2$, only for the case that $\alpha$ is irrational. Later Warschawski generalised this theorem in [22] and determined the asymptotic behaviour for the $n$th derivative for an arbitrary $\alpha \in ]0, 2]$.

**Theorem 2.5.1**

We have

$$\varphi(z) \sim z^{\frac{1}{\pi}}$$

at $0$ on $\Omega$.

*Proof:* We refer to Lichtenstein [13] p. 104 ff. \qed

**Theorem 2.5.2**

We have

$$\varphi'(z) \sim z^{\frac{1}{\pi} - 1}$$
2.5 Asymptotic Behaviour at Analytic Corners

at 0 on $\Omega$ and for $n \geq 2$

$$\lim_{z \to 0} \frac{\varphi^{(n)}(z)}{z^{\frac{1}{\alpha} - n}} = c \frac{1}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \ldots \left( \frac{1}{\alpha} - n + 1 \right)$$

where

$$c := \lim_{z \to 0} \frac{\varphi'(z)}{z^{\frac{1}{\alpha} - 1}} \in \mathbb{C}^*.$$

Proof: We refer to Lichtenstein [13] and Warschawski [22].

These results of Lichtenstein and Warschawski can also be deduced from a theorem of Lehman on the asymptotic power series expansion by termwise differentiation. We consider Lehman’s result in more detail in the following section.

2.5.3 Development in a Generalised Power Series

Lehman studied, similar to Lichtenstein and Warschawski, the asymptotic behaviour of the mapping function of a simply connected proper domain of the complex plane with an analytic corner at the origin to the upper half plane at the analytic corner. The main result of Lehman shown in [11] in 1956 was that such a Riemann map can be developed in a certain generalised power series, a much stronger result than that of Lichtenstein and Warschawski. The method he applied for finding the asymptotic expansion is a generalisation of a technique used by Hans Lewy in the case of a straight angle, i.e. $\alpha = 1$, see [12]. Lewy proved that the mapping function has an asymptotic expansion in powers of $z$ and $\log(z)$ if the boundary arcs meet at a straight angle. Wigley showed the existence of an asymptotic development for more general corners, see [23]. Lehman’s results have applications in today’s real geometry: we refer to Chapter 2.6 for o-minimal structures.

Let $\phi_n(z)$ be a sequence of functions such that

$$\lim_{z \to 0} \frac{\phi_{n+1}(z)}{\phi_n(z)} = 0.$$
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A series \( \sum_{n=0}^{\infty} a_n \phi_n(z) \) with complex coefficients \( a_n \) is called an **asymptotic expansion** for \( \varphi(z) \) if the equality

\[
\varphi(z) = \sum_{n=0}^{N} a_n \phi_n(z) + o(\phi_N(z))
\]

holds for every \( N \in \mathbb{N}_0 \) as \( z \to 0 \). In this case, we write

\[
\varphi(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z).
\]

**Theorem 2.5.3 (R. S. Lehman)**

(a) If \( \alpha \) is irrational then \( \varphi \) has at \( 0 \) an asymptotic power series expansion of the form

\[
\sum a_{k,l} z^{k+\frac{l}{\alpha}}
\]

where \( k \geq 0, \ l \geq 1, \ a_{k,l} \in \mathbb{C}, \) and \( a_{0,1} \neq 0 \).

(b) If \( \alpha = \frac{p}{q} \) with \( p, q \) coprime then \( \varphi \) has at \( 0 \) an asymptotic power series expansion of the form

\[
\sum a_{k,l,m} z^{k+\frac{l}{q}} \left( \log(z) \right)^m
\]

where \( k \geq 0, \ 1 \leq l \leq p, \ 0 \leq m \leq \frac{k}{q}, \ a_{k,l,m} \in \mathbb{C}, \) and \( a_{0,1,0} \neq 0 \).

**Proof:** See Lehman [11, p 1439 ff.]. \( \square \)

From this theorem an asymptotic expansion for the inverse function \( \psi : \mathbb{H} \to \Omega \) can be determined immediately. An example for a conformal mapping function \( \psi \) is depicted in Figure 2.6.

**Theorem 2.5.4 (R. S. Lehman)**

Let \( \psi : \mathbb{H} \to \Omega \) be a conformal map with \( \psi(0) = 0 \). Then we have

(a) If \( \alpha \) is irrational then \( \psi \) has at \( 0 \) an asymptotic power series expansion of the form

\[
\sum b_{k,l} z^{k+\alpha l}
\]
where \( k \geq 0, \ l \geq 1, \ b_{k,l} \in \mathbb{C}, \) and \( b_{0,1} \neq 0. \)

(b) If \( \alpha = \frac{p}{q} \) with \( p,q \) coprime then \( \psi \) has at 0 an asymptotic power series expansion of the form

\[
\sum b_{k,l,m} z^{k+l\alpha} (\log(z))^m
\]

where \( k \geq 0, \ 1 \leq l \leq q, \ 0 \leq m \leq \frac{k}{p}, \ b_{k,l,m} \in \mathbb{C}, \) and \( b_{0,1,0} \neq 0. \)

Proof: We refer to Lehman [11, p. 1438 ff.].

2.6 O-minimal Structures

This chapter presents an overview on o-minimal structures which are very important for the fundamental research to understand the correlations between geometry, logic, and analysis. The development of the theory of o-minimal structures began in the early 1980’s, motivated by Tarski’s problem on the decidability of the real exponential field as well as by more general model-theoretic concerns. The model for o-minimal structures is the class of semialgebraic sets. Semialgebraic and subanalytic geometry are characterised by many nice finiteness properties and other tameness features. O-minimal structures are defined by the same finiteness properties but they provide additional structural features: they allow to develop a tame topology and may be seen as an axiomatic treatment of semialgebraic geometry. This means, there are axioms which
allow to perform many constructions inside the structure and an o-minimality axiom which guarantees the tameness of the topology. Although sets definable in an o-minimal structure may have singularities they have nice geometric behaviour. The o-minimality of many remarkable structures was proven and many interesting results were established in the theory of o-minimal structures on the real field $\mathbb{R}$. For thorough studies, we refer the interested reader to Van den Dries [3]. The o-minimal content of the Riemann map of a bounded, simply connected proper domain of the complex plane which is semialgebraic onto the unit disk was proven by Kaiser in [8], provided that the angles at all singular boundary points are irrational multiplies of $\pi$. An essential requirement to be able to derive the proof of this theorem, was the development of the Riemann map at analytic corners in a generalised power series as shown by Lehman.

2.6.1 Definitions

Definition 2.6.1
A set $A \subset \mathbb{R}^n, n \in \mathbb{N}$, is called semialgebraic if it is representable by a finite union of sets of the following kind

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}$$

where $f, g_1, \ldots, g_k \in \mathbb{R}[x_1, \ldots, x_n]$.

Definition 2.6.2
A subset $A \subset \mathbb{R}^n, n \in \mathbb{N}$, is called semianalytic if for each $x_0 \in \mathbb{R}^n$ there are open neighbourhoods $U, V$ of $x_0$ with $\overline{U} \subset V$ and there are real analytic functions $f_i, g_{i,1}, \ldots, g_{i,k}$ on $V, 1 \leq i \leq l$, such that

$$A \cap U = \bigcup_{1 \leq i \leq l} \{x \in U \mid f_i(x) = 0, g_{i,1}(x) > 0, \ldots, g_{i,k}(x) > 0\}.$$
Definition 2.6.3
A subset \( A \subset \mathbb{R}^n \), \( n \in \mathbb{N} \), is called subanalytic if for each \( x_0 \in \mathbb{R}^n \) there is an open neighbourhood \( U \) of \( x_0 \), some \( m \geq n \) and some bounded semianalytic set \( B \subset \mathbb{R}^m \) such that \( A \cap U = \pi_n(B) \) where \( \pi_n : \mathbb{R}^m \to \mathbb{R}^n, (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n) \), is the projection on the first \( n \) coordinates.

Definition 2.6.4
A map is called semialgebraic (semianalytic, subanalytic) if its graph is a semialgebraic (semianalytic, subanalytic) set.

Definition 2.6.5
A set is called globally semianalytic (globally subanalytic) if it is a semianalytic (subanalytic) set after applying the semialgebraic homeomorphism
\[
\mathbb{R}^n \to ]-1,1[^n, x_i \mapsto \frac{x_i}{\sqrt{1 + x_i^2}}
\]
where \( n \in \mathbb{N} \).

Definition 2.6.6
A structure \( M \) (on the real field \( \mathbb{R} \)) is a sequence \( (M_n)_{n \in \mathbb{N}} \) with the following properties:

(a) \( M_n \subset \mathcal{P}(\mathbb{R}^n) \) is a Boolean algebra (i.e. \( \emptyset \in M_n \), if \( A, B \in M_n \), then \( A \cup B \in M_n \) and \( \mathbb{R}^n \setminus A \in M_n \) ) which contains the semialgebraic subsets of \( \mathbb{R}^n \).

(b) If \( A \in M_m \) and \( B \in M_n \) then \( A \times B \in M_{m+n} \).

(c) If \( A \in M_{n+1} \), then \( \pi(A) \in M_n \) where \( \pi \) is the projection on the first \( n \) coordinates.

Definition 2.6.7
A structure \( M \) is called o-minimal if the sets in \( M_1 \) are precisely the finite unions of intervals and points.

Thus, the term “o-minimal” can be explained as follows: the “o” is standing for
“order” and “minimal” means that in dimension one everything can be expressed by the relation “≤”. The tameness of o-minimal structures follows from the latter condition. As a consequence, we obtain that every definable set in any dimension has finitely many components which are definable.

**Definition 2.6.8**

Let $\mathcal{M}$ be a structure on $\mathbb{R}$.

(a) A set $A \subset \mathbb{R}^n$ is **definable** in $\mathcal{M}$ if $A \in M_n$.

(b) A function $f : A \rightarrow B$, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ is **definable** in $\mathcal{M}$ if $\text{graph}(f) \in M_{n+m}$.

**Definition 2.6.9**

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, is called a **restricted analytic function** if there exists a real convergent power series $p$ in $n$ variables which converges on an open neighbourhood of $[-1, 1]^n$ such that

$$f(x) = \begin{cases} p(x) & \text{for } x \in [-1, 1]^n, \\ 0 & \text{for } x \notin [-1, 1]^n. \end{cases}$$

**2.6.2 Examples**

In the following, we give some examples for o-minimal structures on the field $\mathbb{R}$.

**Example 2.6.10**

(a) The semialgebraic sets constitute an o-minimal structure, denoted by $\mathbb{R}$. It is the smallest o-minimal structure.

(b) The structure generated by the restricted analytic functions is o-minimal. It is denoted by $\mathbb{R}_{an}$. The sets definable in $\mathbb{R}_{an}$ are exactly the globally subanalytic sets and the bounded sets in $\mathbb{R}_{an}$ are exactly the bounded subanalytic sets. See Van den Dries and Miller [4].

(c) The structure generated by the exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}$ is o-minimal. It is denoted by $\mathbb{R}_{\exp}$. For more details we refer to Wilkie [24].
(d) \( \mathbb{R}^*_\text{an} \) is the o-minimal structure in which convergent generalised power series are definable. We refer to Van den Dries and Speissegger [5].

### 2.6.3 O-minimal Content of the Riemann Mapping Theorem

Kaiser investigated the o-minimal content of the Riemann Mapping Theorem in [8].

One of the main components for the proof is the result of Lehman that the mapping function of a simply connected domain of the complex plane with an analytic corner onto the upper half plane can be developed in a certain generalised power series. Based on the result of Kaiser we get, as an application, the definability of Schwarz-Christoffel maps, and working with circular polygons we get the definability of certain classes of hypergeometric functions in this o-minimal structure, see Kaiser [8].

Let \( \Omega \subseteq \mathbb{C} \) be a bounded, semianalytic domain which is simply connected. Then \( \partial \Omega \) has only finitely many singular boundary points. For such a singular boundary point \( x \in \partial \Omega \) there is a \( k \in \mathbb{N} \) such that for all sufficiently small neighbourhoods \( V \) of \( x \) the set \( \Omega \cap V \) has exactly \( k \) connected components having \( x \) as boundary point. We denote such a connected component by \( C \) and the interior angle of \( \partial C \) at \( x \) by \( \angle_x C \). Furthermore, let

\[
\text{Sing}(\partial \Omega) := \{x \in \partial \Omega \mid x \text{ is a singular boundary point of } \partial \Omega\}
\]

and for \( x \in \text{Sing}(\partial \Omega) \)

\[
\angle(\Omega, x) := \{\angle_x C \mid C \text{ is a connected component of } \Omega \cap V \text{ at } x \text{ and } x \in \text{Sing}(\partial C)\}.
\]

**Theorem 2.6.11 (T. Kaiser)**

Suppose that \( \angle(\Omega, x) \subset \pi(\mathbb{R} \setminus \mathbb{Q}) \) for all \( x \in \text{Sing}(\partial \Omega) \) then the Riemann map \( \varphi : \Omega \to \mathbb{E} \) is definable in an o-minimal structure.

**Proof:** See Kaiser [8] Theorem 3.3 on p. 20 f.]

\[ \square \]
Chapter 3

Asymptotic Behaviour at Analytic Cusps

We consider a simply connected proper domain of the complex plane with piecewise analytic boundary and an analytic cusp. Due to the Riemann Mapping Theorem, see Theorem 2.2.1, we already know that there exists a conformal map from this domain onto the upper half plane. An example for such a Riemann map is illustrated in Figure 3.1 below.

![Figure 3.1: Example of a Riemann map from a simply connected domain with an analytic cusp onto the upper half plane](image)

Due to Carathéodory, see Theorem 2.3.1, we can extend this Riemann map to the boundary. We are especially interested in the behaviour of the mapping function at one important point: the cusp. Under certain conditions on the domain, the so-called small perturbation of angles, Kaiser already investigated the behaviour. After presenting the results for this special case, we determine in this chapter the asymptotic behaviour in
the general case. Moreover, we show that after applying preliminary transformations we can determine the asymptotic behaviour of such a Riemann map in the special case that the analytic cusp is at 0 and that one of the boundary arcs in a neighbourhood of 0 coincides with the positive real axis. To investigate the behaviour, we give estimates for the modulus and the argument of the Riemann map using a theorem of Warschawski. Furthermore, we determine the asymptotic behaviour of its derivatives. By using these results we can also investigate the behaviour of its inverse and, in addition, of the derivatives of the inverse.

3.1 General Premises

Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain which has an analytic cusp. We can assume that the cusp is at 0, otherwise we apply a translation. We denote the boundary curves by \( \gamma_1 \) and \( \gamma_2 \). There exists some \( \epsilon > 0 \) such that \( \gamma_1 \) and \( \gamma_2 \) are analytic on \( B(0, \epsilon) \) and such that \( \partial \Omega \) close to 0 is given by \( \Gamma_1 \cup \Gamma_2 \) where \( \Gamma_1 := \gamma_1([0, \epsilon]) \) and \( \Gamma_2 := \gamma_2([0, \epsilon]) \). Since \( \Gamma_1 \) and \( \Gamma_2 \) are regular by assumption we can assume that the parameterisations \( \gamma_1 \) and \( \gamma_2 \) are regular. Thus, we have \( \gamma_1'(0) \gamma_2'(0) \neq 0 \). Furthermore, we may assume that, after applying a rotation, \( \Omega \) is tangent to \( \mathbb{R}_{\geq 0} \) at 0, i.e. \( \gamma_1'(0) = \gamma_2'(0) > 0 \), as depicted in Figure 3.2.

Now, if necessary, we shrink \( \epsilon \) so we can write \( \gamma_1(t) \) in polar coordinates as

\[
\gamma_1(t) = |\gamma_1(t)| \exp(i\eta_1(t))
\]

where

\[
\eta_1(t) := \arg(\gamma_1(t)) = \arctan\left(\frac{\text{Im}(\gamma_1(t))}{\text{Re}(\gamma_1(t))}\right).
\]

Since \( \gamma_1(0) = 0 \) we have \( \text{ord}(\text{Re}(\gamma_1(t))) \geq 1 \) and \( \text{ord}(\text{Im}(\gamma_1(t))) \geq 1 \). Moreover, we have \( \gamma_1'(0) \in \mathbb{R}_{>0} \) and thus, \( \text{ord}(\text{Im}(\gamma_1(t))) \geq 2 \). Since \( \gamma_1'(0) \neq 0 \) we obtain \( \text{ord}(\text{Re}(\gamma_1(t))) = 1 \). Hence, it follows that \( |\gamma_1(t)| \) and \( \eta_1(t) \) are real analytic on \([-\epsilon, \epsilon]\) and that \( \text{ord}(|\gamma_1(t)|) = 1 \). Therefore, \( |\gamma_1(t)| \) is locally invertible at 0. Now we can param-
3.1 General Premises

Figure 3.2: Visualisation of $\Omega$ after applying a rotation

parameterise $\gamma_1$ close to 0 by the distance from the origin applying $s := |\gamma_1^{-1}|(t)$ and obtain $\gamma_1(s) = s \exp(i\mu(s))$ where $\mu(s)$ is real analytic in a neighbourhood of 0. Thus, we can assume that $\gamma_1$ is given in standard angle form, denoted by saf($\gamma_1$)($z$). This means that the angle between the half lines through 0 and the unique point on the positive real axis and $\Gamma_1$, respectively, with modulus $z$ is in standard position, i.e. its vertex is located at the origin and one ray is on the positive real axis. This standard angle form is given by

$$\text{saf}(\gamma_1)(z) = \gamma_1(z) = z \exp(i\angle_{\gamma_1}(z))$$

where $\angle_{\gamma_1}(z)$ is a real power series convergent on $B(0, \epsilon)$ and $\angle_{\gamma_1}(0) = 0$.

In the same way we can proceed with $\gamma_2$ and we obtain that $\gamma_2$ is also given in standard angle form

$$\gamma_2(z) = z \exp(i\angle_{\gamma_2}(z))$$
where $\triangleleft_{\gamma_2}(z)$ is a real power series convergent on $B(0,\epsilon)$ and $\triangleleft_{\gamma_2}(0) = 0$. We set

$$\triangleleft_{\Omega}(t) := \triangleleft_{\gamma_2}(t) - \triangleleft_{\gamma_1}(t).$$

Moreover, we assume that $\triangleleft_{\Omega}(t)$ is positive for small positive $t$. Otherwise, we may relabel $\gamma_1$ and $\gamma_2$. Geometrically, $\triangleleft_{\Omega}(t)$ is the angle between the half lines through 0 and the unique point on $\Gamma_1$ and $\Gamma_2$, respectively, with modulus $t$. A visualisation is depicted in Figure 3.3.

**Figure 3.3: Visualisation of $\Omega$, $\triangleleft_{\Omega}$, $\triangleleft_{\gamma_1}$, and $\triangleleft_{\gamma_2}$**

**Definition 3.1.1**

Let $\Omega$ and $\triangleleft_{\Omega}(t)$ be as above. Let $d := \text{ord}(\triangleleft_{\Omega}(t)) \in \mathbb{N}$ and let $a_d \in \mathbb{R}_{>0}$ such that $\triangleleft_{\Omega}(t) \simeq a_d t^d$. We call $\text{ot}(\Omega) := d$ the **order of tangency** of $\Omega$ and $\text{ct}(\Omega) := a_d$ the **coefficient of tangency** of $\Omega$. 
3.2 Small Perturbation of Angles

Let $\varphi : \Omega \to \mathbb{H}$ be a Riemann map. By Carathéodory, see Theorem 2.3.1, the given mapping function can be extended continuously to the boundary and hence to the origin with value in $\mathbb{R}$. By applying a suitable translation we can assume without restriction that $\varphi(0) = 0$. The arc $\Gamma_1$ is mapped to the positive real axis and $\Gamma_2$ is mapped to the negative real axis.

**Setting:**

Let $\Omega \subset \subset \mathbb{C}$ be a simply connected domain with an analytic cusp at 0 and let

$$\varphi : \Omega \to \mathbb{H}$$

be a Riemann map with $\varphi(0) = 0$.

3.2 Small Perturbation of Angles

As mentioned above, Kaiser investigates in [9] the asymptotic behaviour of the mapping function at an analytic cusp in a special case, the so-called small perturbation of angles. In this section, we present the basic definitions and his main result. In this article, he also determines the asymptotic behaviour of its inverse and proves $O$-estimates for the $n$th derivatives, where $n \in \mathbb{N}$, of the mapping function and its inverse.

**Definition 3.2.1**

Let $d := \operatorname{ot}(\Omega)$ and $a_d := \operatorname{ct}(\Omega)$. We say that $\Omega$ has **small perturbation of angles** if the following is fulfilled:

(a) $\min\{\operatorname{ord}(\angle_{\gamma_1}(t)), \operatorname{ord}(\angle_{\gamma_2}(t))\} = d$.

(b) $\operatorname{ord}(\angle_{\Omega}(t) - a_d t^d) > 2d$. 

Example 3.2.2
Let \( d \in \mathbb{N} \) and \( a \in \mathbb{R}_{>0} \). For \( 0 < \sigma < \left( \frac{\pi}{2a} \right)^{\frac{1}{d}} \) the domain
\[
\Omega := \{ z \in \mathbb{C} | \ 0 < |z| < \sigma, 0 < \arg(z) < a|z|^d \}
\]
has small perturbation of angles. We have \( d := \text{ot}(\Omega) \) and \( a := \text{ct}(\Omega) \).

Theorem 3.2.3
Assume that \( \Omega \) has small perturbation of angles. Then
\[
\varphi(z) \sim \exp \left( -\frac{\gamma}{z^d} \right)
\]
at 0 on \( \Omega \) where \( d := \text{ot}(\Omega) \) and \( \gamma := \frac{\pi}{\text{ot}(\Omega) \text{ct}(\Omega)} \).

Proof: We refer to Kaiser [9, Theorem 6 on p. 39 ff.].

3.3 Preliminary Transformation
Since \( \gamma_1 \) is a regular parameterisation of \( \Gamma_1 \) we have \( \gamma_1'(0) \neq 0 \). Hence, there is some \( \delta > 0 \) such that \( \gamma_1^{-1} \) exists on \( B(0, \delta) \). Thus, we can apply the preliminary transformation \( \gamma_1^{-1} \) to \( \Omega \) in a neighbourhood of 0 and thereby map the boundary arc \( \Gamma_1 \) to the positive real axis. We denote \( \gamma_1^{-1}(\Omega \cap B(0, \delta)) \) by \( \Omega_{\gamma_1} \). The transformation \( \gamma_1(t) = t \exp (i \angle \gamma_1(t)) \) depends only on the shape of \( \Omega \) at 0 and is therefore an invariant of \( \Omega \). Hence, also \( \gamma_1^{-1} \) is an invariant of \( \Omega \). For an illustration of the transformation \( \gamma_1^{-1} \) see Figure 3.4.

In the following we investigate the asymptotic behaviour of the Riemann map \( \varphi \), as introduced in Section 3.1 for the special case where \( \Omega \) has already been transformed as described above. For simplicity of notation we denote \( \Omega_{\gamma_1} \) by \( \Omega \) for the rest of the work and assume that the setting is as follows.
Setting:
The regular parameterisations of the boundary arcs of $\Omega$ in a neighbourhood of 0 are given by

$$
\gamma_1(t) = t \quad \text{and} \quad \gamma_2(t) = t \exp(i \angle_\Omega(t)).
$$

Thus, $\angle_{\gamma_1}(t) = 0$ and $\angle_\Omega(t) = \angle_{\gamma_2}(t)$. Furthermore, we assume that

$$
\angle_\Omega(t) = \sum_{n=d}^{\infty} a_n t^n
$$

is a real power series with $a_d \neq 0$ and $d \in \mathbb{N}$.

As already mentioned, $\angle_\Omega(t)$ is the angle between the half lines through 0 and the unique point on $\Gamma_2$ and on the positive real axis, respectively, with modulus $t$, see Figure 3.5. In the following we determine the asymptotic behaviour of the Riemann map $\varphi : \Omega \to \mathbb{H}$ with $\Omega$ as stated above. An example for $\varphi$ is depicted in Figure 3.6.

Note that by determining the asymptotic behaviour of $\varphi$ in the above setting, we can also derive the behaviour in the general case as introduced in Section 3.1.
Figure 3.5: Visualisation of $\angle_{\Omega}(t)$

Figure 3.6: Example of a Riemann map $\varphi$ from $\Omega$ onto the upper half plane $\mathbb{H}$
3.4 Estimates for the Modulus and the Argument

In this section, we give estimates for the modulus and the argument of the mapping function \( \varphi : \Omega \to \mathbb{H} \). Therefore, we present a theorem of Warschawski who worked in a more general setting than that of analytic arcs. This theorem, proven in [21], is on the estimates for the modulus and the argument of a Riemann map from a simply connected domain bounded by a Jordan arc onto the unit disk with center 1 in a neighbourhood of a finite boundary point.

Let \( \Theta \subsetneq \mathbb{C} \) be a simply connected domain and let \( \partial \Theta \) be a Jordan arc with \( 0 \in \partial \Theta \). We denote the parameterisations of the boundary arcs of \( \Theta \) at 0 by \( \theta_1 \) and \( \theta_2 \). Moreover, we assume that these curves are given by \( \theta_1(t) = t \exp(i \angle \theta_1(t)) \) and \( \theta_2(t) = t \exp(i \angle \theta_2(t)) \) in a neighbourhood \( B(0, \delta) \) with \( \delta > 0 \). Analogously to Section 3.1, we set \( \angle \Theta(t) := \angle \theta_2(t) - \angle \theta_1(t) \) and further let

\[
\theta(t) := \frac{1}{2}(\angle \theta_1(t) + \angle \theta_2(t)).
\]

Suppose that \( \angle \theta_1(t) \) and \( \angle \theta_2(t) \) are absolutely continuous in any closed interval within \( 0 < t \leq \delta \). Additionally, we assume that \( t \angle \theta_1'(t) \) and \( t \angle \theta_2'(t) \) approach the same limit, \( \tan(\sigma), |\sigma| < \frac{\pi}{2}, \) as \( t \to 0 \). Let \( \zeta : \Theta \to \mathbb{D}_1 \) be a conformal map. An example for such a Riemann map \( \zeta \) is illustrated in Figure 3.7. Due to Carathéodory, see Theorem 2.3.1, we can extend the mapping function continuously to the boundary and thus we can assume \( \zeta(0) = 0 \). Warschawski obtained estimates for the modulus and the argument of \( \zeta \) which are presented in the following theorem.

**Theorem 3.4.1**

Under the above stated hypothesis we have for \( z = t \exp(i \arg(z)) \) where \( t = |z| \) that

(a)

\[
\arg(\zeta(z)) = \pi \frac{\arg(z) - \theta(t)}{\angle \Theta(t)} + o(1)
\]

as \( z \to 0 \) in \( \Theta \), see [21] §19(b) Theorem XI(A)(vi) on p. 328].
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(b) If \(t<\theta_1'(t)\) and \(t<\theta_2'(t)\) are continuous for \(0 \leq t \leq \delta\) and the integrals

\[
\int_0^\delta \frac{\theta_1''(t)}{\theta(t)} dt, \int_0^\delta \frac{\theta_2''(t)}{\theta(t)} dt, \text{ and } \int_0^\delta \frac{(\theta_1'(t))^2}{\theta(t)} dt
\]

converge then there exists some \(c > 0\) such that

\[
|\zeta(z)| = c \exp \left( -\pi \int_0^\delta \frac{1 + (r\theta'(r))^2}{r<\Theta(r)} dr + \pi \frac{\arg(z) - \theta(t)}{<\Theta(t)} \tan(\sigma) + o(1) \right) \tag{2}
\]

as \(z \to 0\) in \(\Theta\). See [21, §19(b) Theorem XI(B) on p. 328]. If in addition the integrals

\[
\int_0^\delta \frac{t}{<\Theta(t)} (\theta_1'(t))^2 dt \text{ and } \int_0^\delta \frac{t}{<\Theta(t)} (\theta_2'(t))^2 dt
\]

converge, then (2) reduces to

\[
|\zeta(z)| = c \exp \left( -\pi \int_0^\delta \frac{dr}{r<\Theta(r)} + o(1) \right)
\]

as \(z \to 0\) in \(\Theta\), see [21, §19(c) Remark on p. 328].

Proof: As stated in Warschawski [21, p. 327 f.], (a) follows from [21, Theorem X on p. 315 and Corollary 1 on p. 323] and (b) follows from [21, Corollary of Theorem VIII on p. 313 and Corollary of Theorem IV on p. 296].

In the following, it is our aim to adapt his result to our case where we map onto the upper half plane instead of mapping onto \(\mathbb{E}_1\). First, we give a remark on the asymptotic behaviour of a Riemann map \(\vartheta : \Theta \to \mathbb{H}\) and obtain estimates for the modulus and the argument of \(\vartheta\) in terms of \(\zeta\), see Remark 3.4.2. Second, we adapt the above theorem to the situation where we map \(\Omega\), having an analytic cusp at 0, to the upper half plane, see Corollary 3.4.6.
Recall that we can continue \( \vartheta \) to the boundary by Carathéodory’s Prime End Theorem \( 2.3.1 \) and therefore we can assume, without loss of generality, that \( \vartheta(0) = 0 \) after applying a suitable translation.

**Remark 3.4.2**

We have

\[
\vartheta(z) \sim \zeta(z)
\]

at 0 on \( \Theta \). Furthermore, we get for the modulus of \( \vartheta \) that

\[
|\vartheta(z)| \sim |\zeta(z)|
\]

and for the argument of \( \vartheta \) that

\[
\arg(\vartheta(z)) = \arg(\zeta(z)) + \frac{\pi}{2} + o(1)
\]

at 0 on \( \Theta \).

**Proof:** Let \( h_{C'} : \mathbb{C} \to \mathbb{C} \) be the inverse of the Cayley transformation, see Definition \( 2.2.2 \).
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Then we have $h_{C'}(E) = \mathbb{H}$. Furthermore, let $g : \mathbb{C} \to \mathbb{C}$ with $g(z) = z - 1$. Thus, $g(E_1) = E$. Then

$$f := h_{C'} \circ g : E_1 \to \mathbb{H}$$

is given by

$$f(z) = i \frac{z}{2 - z}. \quad (3.1)$$

Since the power series of $f$ around 0 is given by

$$\sum_{n=0}^{\infty} \tilde{a}_n z^n$$

where $\tilde{a}_n = \frac{f^{(n)}(0)}{n!}$ we obtain with $\zeta : \Theta \to E_1$ that

$$\vartheta = f \circ \zeta : \Theta \to \mathbb{H}, \quad \vartheta(z) = \sum_{n=0}^{\infty} \tilde{a}_n (\zeta(z))^n.$$ 

Since $\tilde{a}_0 = \frac{f(0)}{0!} = 0$ and

$$\tilde{a}_1 = \frac{f'(0)}{1!} = h_{C'}(-1) = i \frac{2}{2} \neq 0,$$

we get

$$\vartheta(z) \sim \zeta(z).$$

Therefore, we obtain

$$|\vartheta(z)| \sim |\zeta(z)|.$$ 

Moreover, we have with $(3.1)$

$$\arg(\vartheta(z)) = \arg \left( i \frac{\zeta(z)}{2 - \zeta(z)} \right)$$

$$= \frac{\pi}{2} + \arg(\zeta(z)) - \arg(2 - \zeta(z)).$$
Since
\[ \lim_{z \to 0} \arg (2 - \zeta(z)) = 0 \]
we get
\[ \arg(\vartheta(z)) = \arg(\zeta(z)) + \frac{\pi}{2} + o(1). \]

A visualisation for the construction in Remark 3.4.2 is depicted in Figure 3.8

Figure 3.8: Visualisation for Remark 3.4.2

Now, we can determine estimates for the modulus and the argument of \( \varphi : \Omega \to \mathbb{H} \) where \( \Omega, \prec_{\Omega}(t), \prec_{\gamma_1}(t), \text{ and } \prec_{\gamma_2}(t) \) are defined as stated at the end of Section 3.3. For
this proof we need the multiplicative inverse of a Laurent series and thus we recall below the computation rule for multiplicative inversion of a convergent power series, refer also to Freitag and Busam [6, p. 118].

**Remark 3.4.3**

Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

be a convergent power series with \( a_0 \neq 0 \). Then there exists some \( \epsilon > 0 \) such that \( f(z) \neq 0 \) for all \( z \) in \( B(0, \epsilon) \). Let \( g(z) = \frac{1}{f(z)} \) then \( g(z) \) is analytic on \( B(0, \epsilon) \) and thus representable as a power series

\[ g(z) = \sum_{n=0}^{\infty} b_n z^n. \]

The coefficients of the multiplicative inverse power series \( g(z) \) of \( f(z) \) can be computed as follows. Since \( f(z)g(z) = 1 \) we obtain by the Cauchy product formula

\[
\sum_{k=0}^{n} a_k b_{n-k} = \begin{cases} 
1 & \text{for } n = 0, \\
0 & \text{for } n \neq 0. 
\end{cases}
\]

This system of equations can be recursively solved with respect to \( n \).

**Example 3.4.4**

We want to determine the coefficients \( c_n \) of the Laurent series

\[ \sum_{n=0}^{\infty} c_n t^{n-d} = \frac{1}{\angle_{\Omega}(t)} \]

where \( d \) is the order of tangency. Since

\[ \angle_{\Omega}(t) = \sum_{n=d}^{\infty} a_n t^n = t^d \sum_{n=0}^{\infty} a_{d+n} t^n \]

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we obtain
\[ \sum_{n=0}^{\infty} c_n t^{n-d} = \frac{1}{t^d} \sum_{n=0}^{\infty} c_n t^n = \frac{1}{t^d} \sum_{n=0}^{\infty} a_{d+n} t^n. \]

Thus, \( \sum_{n=0}^{\infty} c_n t^n \) is the multiplicative inverse of \( \sum_{n=0}^{\infty} a_{d+n} t^n \)

where the coefficient of tangency \( a_d \neq 0 \), i.e.
\[ \left( \sum_{n=0}^{\infty} c_n t^n \right) \left( \sum_{n=0}^{\infty} a_{d+n} t^n \right) = 1. \]

Therefore, we obtain
\[ \sum_{k=0}^{n} c_k a_{d+n-k} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \]

Hence, we can determine the coefficients \( c_n \) in the following way:
\[ c_n = \begin{cases} \frac{1}{a_d} & \text{for } n = 0, \\ -\sum_{k=0}^{n-1} c_k \frac{a_{d+n-k}}{a_d} & \text{for } n \neq 0. \end{cases} \]

By solving these equations we obtain
\[ c_0 = \frac{1}{a_d}, \]
\[ c_1 = -\sum_{k=0}^{0} c_k \frac{a_{d+1-k}}{a_d} = -c_0 \frac{a_{d+1}}{a_d} = -\frac{a_{d+1}}{a_d^2}, \]
\[ c_2 = -\sum_{k=0}^{1} c_k \frac{a_{d+2-k}}{a_d} = -c_0 \frac{a_{d+2}}{a_d} - c_1 \frac{a_{d+1}}{a_d} = -\frac{a_{d+2}}{a_d^2} - \left( -\frac{a_{d+1}}{a_d^2} \right) a_d \frac{a_{d+1}}{a_d} = \frac{a_{d+1}^2}{a_d^3} - \frac{a_{d+2}}{a_d^3} \]

\[ \ldots \]
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Definition 3.4.5
Let \( c_n \) be the coefficients of the Laurent series

\[
\sum_{n=0}^{\infty} c_n t^{n-d} = \frac{1}{\angle_{\Omega}(t)}
\]

where \( d \) is the order of tangency. Moreover, we set for \( 0 \leq n \leq d - 1 \)

\[
b_n := \frac{\pi c_n}{n-d} \quad \text{and} \quad a := \pi c_d.
\]

We call the tuple \((b_0, \ldots, b_{d-1}, a)\) \(\in \mathbb{R}^{d+1}\) the asymptotic tuple of \(\Omega\).

The elements of the asymptotic tuple \((b_0, \ldots, b_{d-1}, a)\) \(\in \mathbb{R}^{d+1}\) of \(\Omega\) in Definition 3.4.5 are geometric invariants depending on the shape of the domain \(\Omega\) at the cusp since they can be computed by using \(\angle_{\Omega}(t)\), the real power series which describes the cusp. For the rest of the work all statements regarding the asymptotic behaviour of mapping functions focus on the case that \(z \to 0\). The next result can be shown by applying Theorem 3.4.1 of Warschawski and Remark 3.4.2.

Corollary 3.4.6
We have

(a)

\[
|\varphi(z)| = c \exp \left( |z|^{d-1} \sum_{n=0}^{d-1} b_n |z|^n + a \log |z| + o(1) \right)
\]

where \( c \in \mathbb{R}_{>0} \) is a constant and \((b_0, \ldots, b_{d-1}, a)\) is the asymptotic tuple of \(\Omega\).

(b)

\[
\arg(\varphi(z)) = \pi \arg(z) |z|^{-d} \left( \sum_{n=0}^{d} c_n |z|^n \right) + o(1).
\]

Proof:

Case 1: Let \( d > 1 \).

First of all, we have to check the conditions of Theorem 3.4.1. Since \( \angle_{\gamma_1}(t) = 0 \) and
\( \langle \gamma_2(t) = \langle \Omega(t) \simeq a_d t^d \) \) it follows that \( \langle \gamma_1''(t) = 0 \) and \( \langle \gamma_2''(t) = O(t^{d-2}) \). Therefore, the integrals

\[
\int_0^\delta \langle \gamma_1''(t)dt \quad \text{and} \quad \int_0^\delta \langle \gamma_2''(t)dt
\]

(3.2)

converge for small \( \delta > 0 \). Moreover, we have \( \langle \gamma_1'(t) \rangle^2 \simeq a_d^2 t^{2d-2} \) and thus

\[
\frac{\langle \gamma_1'(t) \rangle^2}{\langle \Omega(t) \rangle} \simeq a_d d^2 t^{d-2}.
\]

Therefore,

\[
\frac{\langle \gamma_1'(t) \rangle^2}{\langle \Omega(t) \rangle} = O(t^{d-2})
\]

and, as a consequence, the integral

\[
\int_0^\delta \frac{\langle \gamma_1'(t) \rangle^2}{\langle \Omega(t) \rangle} dt
\]

(3.3)

converges for small \( \delta > 0 \). In addition, the integrals

\[
\int_0^\delta \frac{t}{\langle \Omega(t) \rangle} \langle \gamma_1'(t) \rangle^2 dt \quad \text{and} \quad \int_0^\delta \frac{t}{\langle \Omega(t) \rangle} \langle \gamma_1'(t) \rangle^2 dt
\]

(3.4)

converge for small \( \delta > 0 \) since

\[
\frac{t}{\langle \Omega(t) \rangle} \langle \gamma_1'(t) \rangle^2 = 0 \quad \text{and} \quad \frac{t}{\langle \Omega(t) \rangle} \langle \gamma_1'(t) \rangle^2 \simeq a_d d^2 t^{d-1}
\]

and thus

\[
\frac{t}{\langle \Omega(t) \rangle} \langle \gamma_1'(t) \rangle^2 = O(t^{d-1}).
\]

By using (3.2), (3.3), and (3.4) we get from Theorem 3.4.1 and Remark 3.4.2 the following
estimate for the modulus of $\varphi$

$$|\varphi(z)| = p_1 \exp \left( -\pi \int_{t}^{\delta} \frac{1}{r < \Omega(r)} dr + o(1) \right)$$  \hspace{1cm} (3.5)$$

for some constant $p_1 > 0$ with $t = |z|$. By computing the integral in (3.5) we obtain for $t \to 0$ the desired estimate. For this calculation we set

$$h(t) := -\pi \int_{t}^{\delta} \frac{1}{r < \Omega(r)} dr$$

and

$$\frac{1}{\Omega(t)} := \sum_{n=0}^{\infty} c_n t^{n-d}.$$  \hspace{1cm} (3.6)$$

The coefficients $c_n$ can be determined as in Example 3.4.4. Hence, we get

$$h(t) = -\pi \int_{t}^{\delta} \frac{1}{r < \Omega(r)} dr$$

$$= -\pi \int_{t}^{\delta} \sum_{n=0}^{\infty} c_n r^{n-(d+1)} dr$$

$$= -\pi \left( \int_{t}^{\delta} \sum_{n=0}^{d-1} c_n r^{n-(d+1)} dr + \int_{t}^{\delta} c_d r^{d-(d+1)} dr + \int_{n=d+1}^{\infty} c_n r^{n-(d+1)} dr \right)$$

$$= -\pi \left( \sum_{n=0}^{d-1} c_n \left[ \frac{1}{n-d} t^{n-d} \right]_{t}^{\delta} + c_d \log(t) \right) + \sum_{n=d+1}^{\infty} c_n \left[ \frac{1}{n-d} t^{n-d} \right]_{t}^{\delta}$$

$$= -\pi \left( p_2 - \sum_{n=0}^{d-1} \frac{c_n}{n-d} t^{n-d} - c_d \log(t) - \sum_{n=d+1}^{\infty} \frac{c_n}{n-d} t^{n-d} \right)$$

with a constant $p_2 \in \mathbb{R}$. Since

$$\lim_{t \to 0} \sum_{n=d+1}^{\infty} \frac{c_n}{n-d} t^{n-d} = 0$$

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it follows that

\[ h(t) = -\pi p_2 + \pi \sum_{n=0}^{d-1} \frac{c_n}{n-d} t^{n-d} + \pi c_d \log(t) + o(1). \]

Therefore, we obtain for \(|z| = t\) that

\[
|\varphi(z)| = p_1 \exp \left( h(t) + o(1) \right) \\
= p_1 \exp \left( -\pi p_2 + \pi \sum_{n=0}^{d-1} \frac{c_n}{n-d} t^{n-d} + \pi c_d \log(t) + o(1) \right) \\
= p_3 \exp \left( \pi \sum_{n=0}^{d-1} \frac{c_n}{n-d} t^{n-d} + \pi c_d \log(t) + o(1) \right)
\]

where \(p_3 := p_1 \exp(-\pi p_2)\).

Now we prove the estimate for the argument of \(\varphi\). As in our case \(\theta(t) = \frac{1}{2} \angle \Omega(t)\) it follows with Theorem 3.4.1 and Remark 3.4.2 that

\[
\arg(\varphi(z)) = \pi \arg(z) - \angle \Omega(t) + \frac{\pi}{2} + o(1) \\
= \pi \arg(z) + o(1) \\
= \pi \sum_{n=0}^{\infty} c_n t^{n-d} + o(1).
\]

Moreover, we have

\[
\lim_{t \to 0} \sum_{n=d+1}^{\infty} c_n t^{n-d} = 0
\]

and therefore we obtain

\[
\arg(\varphi(z)) = \pi \sum_{n=0}^{d} c_n t^{n-d} + o(1).
\]

**Case 2:** Let \(d = 1\).

We apply the transformation \(\omega : \mathbb{C} \to \mathbb{C}, \omega(z) = \sqrt{z}\) to \(\Omega\) and we set \(\tilde{\Omega} := \omega(\Omega)\).
Chapter 3 Asymptotic Behaviour at Analytic Cusps

The boundary arcs of the domain $\tilde{\Omega}$ in a neighbourhood of 0 are given by the following parameterisations

$$\tilde{\gamma}_1(t) = \sqrt{t} \quad \text{and} \quad \tilde{\gamma}_2(t) = \sqrt{t} \exp \left( i \frac{1}{2} \angle_{\Omega}(t) \right).$$

Substituting $s := \sqrt{t}$ we obtain the parametric representation for the boundary arcs of $\tilde{\Omega}$ of the form

$$\tilde{\gamma}_1(s) = s \quad \text{and} \quad \tilde{\gamma}_2(s) = s \exp(i \angle_{\tilde{\Omega}}(s))$$

where $\angle_{\tilde{\Omega}}(s) = \frac{1}{2} \angle_{\Omega}(s^2)$. Therefore,

$$\angle_{\tilde{\Omega}}(s) = \sum_{n=\tilde{d}}^{\infty} \tilde{a}_n s^n$$

where $\tilde{d} = 2$. Let $\tilde{\varphi} : \tilde{\Omega} \to \mathbb{H}$. Since $\angle_{\tilde{\Omega}}$ fulfils the condition from Case 1, we obtain the following estimates for the modulus and the argument of the mapping function $\tilde{\varphi}(\omega(z)) = \varphi : \Omega \to \mathbb{H}$:

$$|\varphi(z)| = |\tilde{\varphi}(\omega(z))|$$

$$= \tilde{p}_3 \exp \left( \pi \sum_{n=0}^{\tilde{d}-1} \frac{\tilde{c}_n}{n-\tilde{d}} s^{n-\tilde{d}} + \pi \tilde{c}_{\tilde{d}} \log(s) + o(1) \right)$$

$$= \tilde{p}_3 \exp \left( \pi \sum_{n=0}^{1} \frac{\tilde{c}_n}{n-2} s^{n-2} + \pi \tilde{c}_2 \log(s) + o(1) \right)$$

for some constant $\tilde{p}_3 > 0$ and

$$\arg(\varphi(z)) = \arg(\tilde{\varphi}(\omega(z)))$$

$$= \pi \arg(\omega(z)) \sum_{n=0}^{\tilde{d}} \tilde{c}_n s^{n-\tilde{d}} + o(1)$$

$$= \pi \arg(\omega(z)) \sum_{n=0}^{2} \tilde{c}_n s^{n-2} + o(1)$$
where \( \tilde{c}_n \) are the coefficients of the Laurent series
\[
\sum_{n=0}^{\infty} \tilde{c}_n s^{-n(d-1)} = \frac{1}{s^{\tilde{c}_0}(s)}.
\]

As in Case 1, the coefficients \( \tilde{c}_n \) can be determined as in Example 3.4.4. Since
\[
\tilde{c}_0 = \frac{2}{a_1} = 2c_0, \quad \tilde{c}_1 = 0 \quad \text{and} \quad \tilde{c}_2 = \frac{-2a_2}{a_1^2} = 2c_1.
\]

Using these results and substituting \( s = \sqrt{t} \) back, we end up with
\[
|\varphi(z)| = \tilde{p}_{d} \exp \left( \pi \left( -\frac{\tilde{c}_0}{2} \sqrt{t}^{-2} - \tilde{c}_1 \sqrt{t}^{-1} \right) + \pi \tilde{c}_2 \log(\sqrt{t}) + o(1) \right)
\]
\[
= \tilde{p}_{d} \exp \left( -\pi c_0 t^{-1} + \pi c_1 \log(t) + o(1) \right)
\]
and since \( \arg(\omega(z)) = \arg(\sqrt{z}) = \frac{1}{2} \arg(z) \) we have
\[
\arg(\varphi(z)) = \frac{\pi}{2} \arg(z) \sum_{n=0}^{2} \tilde{c}_n \sqrt{t}^{n-2} + o(1)
\]
\[
= \frac{\pi}{2} \arg(z) \left( \tilde{c}_0 \sqrt{t}^{-2} + \tilde{c}_1 \sqrt{t}^{-1} + \tilde{c}_2 \right) + o(1)
\]
\[
= \frac{\pi}{2} \arg(z) \left( 2c_0 t^{-1} + 2c_1 \right) + o(1)
\]
\[
= \pi \arg(z) \left( c_0 t^{-1} + c_1 \right) + o(1).
\]

In summary, we get
\[
|\varphi(z)| = \begin{cases} 
\tilde{p}_{d} \exp \left( -\pi c_0 t^{-1} + \pi c_1 \log(t) + o(1) \right) & \text{for } d = 1, \\
p_{d} \exp \left( \frac{\pi}{n-d} \sum_{n=0}^{d-1} \frac{c_n}{n} t^{n-d} + \pi c_d \log(t) + o(1) \right) & \text{for } d > 1 
\end{cases}
\]
and

\[ \arg(\varphi(z)) = \begin{cases} 
\pi \arg(z) \left( c_0 t^{-1} + c_1 \right) + o(1) & \text{for } d = 1, \\
\pi \arg(z) \sum_{n=0}^{d} c_n t^{n-d} + o(1) & \text{for } d > 1
\end{cases} \]

which simplifies to

\[ |\varphi(z)| = c \exp \left( t^{-d} \pi \sum_{n=0}^{d-1} \frac{c_n}{n-d} t^n + \pi c_d \log(t) + o(1) \right) \]

for \( d \in \mathbb{N} \) and some constant \( c \in \mathbb{R}_{>0} \) and

\[ \arg(\varphi(z)) = \pi \arg(z) t^{-d} \sum_{n=0}^{d} c_n t^n + o(1) \]

for \( d \in \mathbb{N} \), respectively.

\[ \square \]

3.5 Asymptotic Behaviour of the Riemann map

By using Corollary 3.4.6 and Lemma 2.1.20 we can immediately determine the asymptotic behaviour of the mapping function \( \varphi \).

**Theorem 3.5.1**

We have

\[ \varphi(z) \sim z^a \exp \left( \frac{b_0}{z^d} + \frac{b_1}{z^{d-1}} + \cdots + \frac{b_{d-1}}{z} \right) \]

at 0 on \( \Omega \) where \( (b_0, \ldots, b_{d-1}, a) \) is the asymptotic tuple of \( \Omega \).

**Proof:** Let \( |z| = t \). We set

\[ h(t) := \sum_{n=0}^{d-1} b_n t^{n-d} + a \log(t) \] and \( g(z) := \arg(\varphi(z)) \).
3.5 Asymptotic Behaviour of the Riemann map

We have by Corollary 3.4.6

\[ \log (\varphi(z)) = \log (|\varphi(z)| \exp(i\gamma(z))) \]
\[ = \log (c \exp(h(t) + o(1)) \exp(i\gamma(z))) \]
\[ = \log(c) + \log(\exp(h(t) + o(1))) + \log(\exp(i\gamma(z))) \]
\[ = \log(c) + h(t) + i\gamma(z) + o(1) \]  

(3.7)

where \( c \in \mathbb{R}_{>0} \) is a constant. By fixing

\[ H(z) := \sum_{n=0}^{d-1} b_n z^{n-d} + a \log(z) \]

it follows that

\[ \text{Re}(H(z)) = \sum_{n=0}^{d-1} b_n t^{n-d} \cos((n - d) \arg(z)) + a \log(t) \]

and

\[ \text{Im}(H(z)) = \sum_{n=0}^{d-1} b_n t^{n-d} \sin((n - d) \arg(z)) + a \arg(z). \]

Next, we show that

\[ H(z) = h(t) + i\gamma(z) + o(1). \]  

(3.8)

By the power series expansion of cosine we see that

\[ \cos((n - d) \arg(z)) = 1 + O((\arg(z))^2). \]  

(3.9)

Since \( \angle_{\Omega}(t) = a_d t^d + O(t^{d+1}) \) on \( \Omega \) we have \( 0 \leq \arg(z) \leq a_d t^d + O(t^{d+1}) \) and thus

\[ 0 \leq (\arg(z))^2 \leq a_d^2 t^{2d} + O(t^{2d+1}) \]  

(3.10)

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for $z \in \Omega$ close to 0. By Definition 3.4.5 we have $b_n = \frac{a_n}{n-d}$ and $a = c_d \pi$. Hence, by using the estimates above we obtain

$$
\lim_{z \to 0} (\Re(H(z)) - h(t))
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} b_n t^{n-d} \left( \cos \left( (n - d) \arg(z) \right) - 1 \right) + \pi c_d \log(t) - \pi c_d \log(t) \right)
= \lim_{z \to 0} \sum_{n=0}^{d-1} b_n t^{n-d} \left( 1 + O((\arg(z))^2) - 1 \right)
= \lim_{z \to 0} \sum_{n=0}^{d-1} b_n t^{n-d} \left( O((\arg(z))^2) \right)
= \lim_{z \to 0} \sum_{n=0}^{d-1} b_n t^{n-d} \left( a_d^2 t^{2d} + O(t^{2d+1}) \right)
= 0.
$$

Thus, it follows that

$$
\Re(H(z)) - h(t) = o(1). \quad (3.11)
$$

By the power series expansion of sine we get

$$
\sin((n - d) \arg(z)) = (n - d) \arg(z) + O((\arg(z))^3) \quad (3.12)
$$

and moreover we have on $\Omega$

$$
0 \leq (\arg(z))^3 \leq a_d^3 t^{3d} + O(t^{3d+1}). \quad (3.13)
$$
3.5 Asymptotic Behaviour of the Riemann map

Using these estimates we get

\[
\lim_{z \to 0} (\Im(H(z)) - g(z)) = \lim_{z \to 0} \left( \sum_{n=0}^{d-1} b_n t^{n-d} \sin \left( (n - d) \arg(z) \right) + \pi c_d \arg(z) - \arg(\varphi(z)) \right)
\]

\[
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} b_n t^{n-d} \sin \left( (n - d) \arg(z) \right) + \pi c_d \arg(z) - \pi \arg(z) \sum_{n=0}^{d} c_n t^{n-d} + o(1) \right)
\]

\[
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} \frac{\pi c_n}{n - d} t^{n-d} \left( (n - d) \arg(z) + O((\arg(z))^3) \right) + \pi c_d \arg(z) \right.

- \pi \arg(z) \sum_{n=0}^{d-1} c_n t^{n-d} - \pi c_d \arg(z) + o(1) \bigg)
\]

\[
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} \pi c_n t^{n-d} \left( \arg(z) + O((\arg(z))^3) - \arg(z) \right) + o(1) \right)
\]

\[
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} \pi c_n t^{n-d} \left( O((\arg(z))^3) \right) + o(1) \right)
\]

\[
= \lim_{z \to 0} \left( \sum_{n=0}^{d-1} \pi c_n t^{n-d} \left( \alpha_d^3 t^{3d} + O(t^{3d+1}) \right) + o(1) \right)
\]

\[
= 0.
\]

Hence, we have shown that

\[
\Im(H(z)) - g(z) = o(1). \quad (3.14)
\]

Therefore, by (3.11) and (3.14) we get (3.8) and thus we obtain

\[
\log(\varphi(z)) = \log(\alpha_d^3 t^{3d} + O(t^{3d+1})) + o(1) = \log(\alpha_d^3 t^{3d}) + H(z).
\]

Applying Lemma 2.1.20 shows

\[
\varphi(z) \simeq \exp(\log(\alpha_d^3 t^{3d}) + H(z)) \sim \exp(H(z)). \quad (3.15)
\]
Thus, it follows that

\[ \varphi(z) \sim z^a \exp \left( \frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z} \right) \]

at 0 on \( \Omega \).

Now we are able to determine the asymptotic behaviour of an example mentioned by Kaiser in [9, Remark 8(i)]. This was not possible before since the domain introduced in this example has not small perturbation of angles.

**Example 3.5.2**

Let 

\[ \Omega := \left\{ z \in \mathbb{C} \mid 0 < |z| < \frac{1}{2}, \ 0 < \arg(z) < |z| - |z|^2 \right\} \]

Then \( \Omega \) is a simply connected domain with an analytic cusp at 0 which is tangent to \( \mathbb{R}_{\geq 0} \) as depicted in Figure 3.9.

![Figure 3.9: Visualisation of \( \Omega \)](image)

The boundary curves at 0 are given by \( \gamma_1(t) = t \) and \( \gamma_2(t) = t \exp(i(t - t^2)) \). Let \( \varphi : \Omega \to \mathbb{H} \) be a conformal map with \( \varphi(0) = 0 \). Since \( \zeta_\Omega(t) = t - t^2 \) we get from Theorem 3.5.1 with \( \text{ot}(\Omega) = d = 1 \) that

\[ \varphi(z) \sim z^a \exp \left( \frac{b_0}{z} \right) \]
where $b_0 = \frac{\pi c_0}{a_1} = -\pi c_0$ and $a = \pi c_1$. We have to compute $c_0$ and $c_1$. Since $ct(\Omega) = a_1 = 1$ and $a_2 = -1$ we obtain from Example 3.4.4 that $c_0 = \frac{1}{a_1} = 1$ and $c_1 = -c_0 a_2 = -1$ and therefore we get $b_0 = -\pi$ and $a = \pi$. Finally, it follows that 

$$\varphi(z) \sim z^n \exp \left( -\frac{\pi}{z} \right)$$

at 0 on $\Omega$.

We are now interested in a characterisation for the situation that the logarithmic term of

$$\exp \left( \frac{b_0}{z^d} + \frac{b_1}{z^{d-1}} + \cdots + \frac{b_{d-1}}{z} + a \log(z) \right) \sim \varphi(z)$$

vanishes, i.e. $a = 0$. The following remark shows that the disappearance of the logarithmic term only depends on a finite number of coefficients of the power series $\varsigma_{\Omega(t)}$.

**Remark 3.5.3**

If $a_{d+1} = \ldots = a_{2d} = 0$ we have

$$\varphi(z) \sim \exp \left( -\frac{\pi}{a_d dz^a} \right)$$

at 0 on $\Omega$.

**Proof:** Using Theorem 3.5.1 we see that

$$\varphi(z) \sim z^n \exp \left( \frac{b_0}{z^d} + \frac{b_1}{z^{d-1}} + \cdots + \frac{b_{d-1}}{z} \right)$$

where $(b_0, \ldots, b_{d-1}, a) \in \mathbb{R}^{d+1}$ is the asymptotic tuple of $\Omega$ with $b_n = \frac{\pi c_n}{n^d}$ and $a = \pi c_d$ as in Definition 3.4.5. From Example 3.4.4 we obtain

$$c_n = \begin{cases} \frac{1}{a_d} & \text{for } n = 0, \\ -\sum_{k=0}^{n-1} c_k \frac{a_{d+n-k}}{a_d} & \text{for } n \neq 0 \end{cases}$$
and since \( a_{d+1} = \ldots = a_{2d} = 0 \) we have \( a_{d+n-k} = 0 \) for \( 0 < n \leq d \). Thus, we get

\[
c_n = \begin{cases} 
\frac{1}{a_d} & \text{for } n = 0, \\
0 & \text{for } 0 < n \leq d.
\end{cases}
\]

Therefore, \( b_0 = -\frac{\pi}{a_d} \), \( b_n = 0 \) for \( n \in \{1, \ldots, d-1\} \), and \( a = 0 \) since \( c_d = 0 \). Hence, the claim is proven.

The result of this special case coincides with the result of Theorem 3.2.3. Thus, the logarithmic term vanishes if \( \Omega \) has small perturbation of angles.

### 3.6 Asymptotic Behaviour of the Derivatives

For the following, we are interested in the asymptotic behaviour of the derivatives of \( \varphi \).

We are going to prove that for \( n \in \mathbb{N}_0 \)

\[
\varphi^{(n)}(z) \sim z^{a-n(d+1)} \exp\left(\frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z}\right)
\]

at 0 on \( \Omega \) by using the asymptotic behaviour of \( \varphi \), determined in Theorem 3.5.1 and the following proposition on the behaviour of the derivatives of

\[
z^a \exp\left(\frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z}\right).
\]

**Proposition 3.6.1**

Let \( F : \mathbb{C}^* \to \mathbb{C} \) be given by

\[
F(z) = z^a \exp\left(\frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z}\right)
\]

where \( (b_0, \ldots, b_{d-1}, a) \) is the asymptotic tuple of \( \Omega \). Then we have for \( n \in \mathbb{N} \) at 0 on \( \mathbb{C}^* \) that

\[
F^{(n)}(z) \sim F(z) z^{-(d+1)}.
\]
3.6 Asymptotic Behaviour of the Derivatives

Proof: Let

\[ H(z) := \sum_{k=0}^{d-1} b_k z^{k-d} + a \log(z). \]

By the formula of Faà di Bruno (see Roman [19]) for the derivatives of the composition of two functions and the fact that \( \exp^n(z) = \exp(z) \) for \( n \in \mathbb{N} \) we see that

\[ F^{(n)}(z) = \frac{d^n}{dz^n} (\exp(H(z))) \]

\[ = \sum_{(k_1, \ldots, k_n) \in T_n} \frac{n!}{k_1! \cdots k_n!} \exp(k_1 + \cdots + k_n)(H(z)) \prod_{m=1}^{n} \left( \frac{1}{m!} \frac{d^m}{dz^m} H(z) \right)^{k_m} \]

where \( T_n \) is the set of all \( n \)-tupel \((k_1, \ldots, k_n)\) with \( k_i \in \mathbb{N}_0 \) for \( i \in \{1, \ldots, n\} \) and \( 1k_1 + 2k_2 + \ldots + nk_n = n \). Since

\[ \frac{1}{m!} \frac{d^m}{dz^m} H(z) = \frac{1}{m!} \frac{d^m}{dz^m} \left( \sum_{k=0}^{d-1} b_k z^{k-d} + a \log(z) \right) \]

\[ = \frac{1}{m!} b_0 (-d)(-d-1) \cdots (-d-(m-1)) z^{-d-m} + \ldots \]

\[ + \frac{1}{m!} b_{d-1} (-1)(-2) \cdots (-1-(m-1)) z^{-1-m} \]

\[ + \frac{1}{m!} a (-1)(-2) \cdots (- (m-1)) z^{-m} \]

\[ \sim z^{-(d+m)} \]

for \( m \in \mathbb{N} \) and \( 1k_1 + \ldots + nk_n = n \) we have

\[ \prod_{m=1}^{n} \left( \frac{1}{m!} \frac{d^m}{dz^m} H(z) \right)^{k_m} \sim \prod_{m=1}^{n} z^{-(d+m)k_m} \]

\[ = z^{-(d+1)k_1} \cdots z^{-(d+n)k_n} \]

\[ = z^{-(k_1 + \ldots + k_n)d - (1k_1 + \ldots + nk_n)} \]

\[ = z^{-(k_1 + \ldots + k_n)d - n}. \quad (3.16) \]
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Note that \( k_1 + \ldots + k_n \in \{1, \ldots, n\} \) and \( k_1 + \ldots + k_n = n \) if and only if \( k_1 = n \). Thus, it follows that

\[
F^{(n)}(z) = \sum_{(k_1, \ldots, k_n) \in T_n} \frac{n!}{k_1! \cdot \ldots \cdot k_n!} \exp(H(z)) \prod_{m=1}^{n} \left( \frac{1}{m!} \frac{d^m}{dz^m} H(z) \right)^{k_m} \\
= \exp(H(z)) \sum_{(k_1, \ldots, k_n) \in T_n} \frac{n!}{k_1! \cdot \ldots \cdot k_n!} \prod_{m=1}^{n} \left( \frac{1}{m!} \frac{d^m}{dz^m} H(z) \right)^{k_m} \\
\sim \exp(H(z)) \sum_{(k_1, \ldots, k_n) \in T_n} \frac{n!}{k_1! \cdot \ldots \cdot k_n!} z^{-(k_1+\ldots+k_n)d-n} \\
\sim \exp(H(z)) z^{-n(d+1)} \\
= F(z) z^{-n(d+1)}. 
\]

As already mentioned above, there is some \( \delta > 0 \) such that \( \gamma_2^{-1} \) exists on \( B(0, \delta) \). Let \( \Omega_{\gamma_2} := \gamma_2^{-1}(\Omega \cap B(0, \delta)) \). Then we have the following lemma.

**Lemma 3.6.2**

We have for \( \Omega_{\gamma_2} \) as introduced above that

\[
\text{ot} \left( \mathcal{A}_{\Omega_{\gamma_2}}(t) \right) = \text{ot} \left( \mathcal{A}_\Omega(t) \right). 
\]

**Proof:** By the general premises we have that the parameterisations of the boundary arcs of \( \Omega \) are given by \( \gamma_1(z) = z \) and \( \gamma_2(z) = z \exp(i \mathcal{A}_\Omega(z)) \) in a neighbourhood of 0 and that \( \text{ot} \left( \mathcal{A}_\Omega(t) \right) = d \). Therefore, the parameterisations of the boundary arcs of \( \Omega_{\gamma_2} \) are given by

\[
\tilde{\gamma}_1(z) = \gamma_2^{-1}(z) \quad \text{and} \quad \tilde{\gamma}_2(z) = z
\]

in a neighbourhood of 0. The standard angle form of \( \tilde{\gamma}_1(z) \) is given by \( \text{saf}(\tilde{\gamma}_1)(z) = z \exp(i \mathcal{A}_{\Omega_{\gamma_2}}(z)) \). By the power series expansion of the inverse we obtain

\[
\gamma_2^{-1}(t) = t - it^{d+1} + O(t^{d+2})
\]

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for some constant $c \in \mathbb{R}^*$ and thus we read off that

$$
\begin{align*}
\text{ord} \left( |\gamma_2^{-1}(t)| \right) &= 1, \\
\text{ord} \left( \text{Re} \left( \gamma_2^{-1}(t) \right) \right) &= 1, \\
\text{ord} \left( \text{Im} \left( \gamma_2^{-1}(t) \right) \right) &= d + 1.
\end{align*}
$$

(3.17)

We have

$$
\gamma_2^{-1}(t) = \omega_1(t) \exp (i \omega_2(t))
$$

where

$$
\omega_1(t) = |\gamma_2^{-1}(t)| \quad \text{and} \quad \omega(t) := \arctan \left( \frac{\text{Im} \left( \gamma_2^{-1}(t) \right)}{\text{Re} \left( \gamma_2^{-1}(t) \right)} \right).
$$

With (3.17) we see that $\text{ord} \left( \omega_1^{-1}(t) \right) = 1$ and $\text{ord} \left( \omega_2(t) \right) = d$. Since $\alpha_{\Omega_2}(t) = \omega_2 \left( \omega_1^{-1}(t) \right)$ we have $\text{ord} \left( \alpha_{\Omega_2}(t) \right) = d = \text{ord} \left( \alpha_{\Omega}(t) \right)$ and the lemma holds. \hfill \Box

**Theorem 3.6.3**

Let $n \in \mathbb{N}_0$. Then

$$
\varphi^{(n)}(z) \sim \varphi(z) z^{-n(d+1)} \sim z^{a-n(d+1)} \exp \left( \frac{b_0}{z^d} + \cdots + \frac{b_{d-1}}{z} \right)
$$

at $0$ on $\Omega$ where $(b_0, \ldots, b_{d-1}, a)$ is the asymptotic tuple of $\Omega$.

**Proof:** We set

$$
H(z) := \sum_{k=0}^{d-1} b_k z^{k-d} + a \log(z)
$$

and

$$
F(z) := \exp(H(z)).
$$

By the preliminaries the boundary arcs of $\Omega$ are given by $\Gamma_1$ and $\Gamma_2$ in some neighbourhood of $0$ with the regular parameterisations $\gamma_1(z) = z$ and $\gamma_2(z) = z \exp(i \alpha_{\Omega}(z))$. Since $\gamma_2'(0) = 1$ we see that $\gamma_2$ is locally invertible. Let $r, s \in \mathbb{R}$ with $0 < s < r$ such that $\gamma_2(z)$ is injective on $B(0, r)$ and that $B(0, s) \subset \gamma_2(B(0, r))$. By the Schwarz Reflection
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Principle via conjugation, see Theorem 2.4.1, we can continue $\varphi$ across the analytic arc $\Gamma_1$. The reflection of $\Omega$ across $\Gamma_1$ is given by

$$\Omega_1 := \{ z \in \mathbb{C} \mid \bar{z} \in \Omega \}.$$  

By the Schwarz Reflection at analytic arcs, see Theorem 2.4.3, there exists an holomorphic extension of $\varphi$ across $\Gamma_2$. We denote the reflection of $\Omega$ across the boundary arc $\Gamma_2$ by

$$\Omega_2 := \{ z \in \mathbb{C} \mid z \in B(0, s) \text{ and } \gamma_2(\gamma_2^{-1}(z)) \in \Omega \}.$$  

Furthermore, let

$$\tilde{\Omega} := (\Omega \cup \Gamma_1 \cup \Omega_2 \cup \Omega_1 \cup \Omega_2) \cap B(0, s).$$

A visualisation of $\tilde{\Omega}$ is depicted in Figure 3.10.

Thus, there exists an holomorphic extension of $\varphi$, which we denote by $\Phi: \tilde{\Omega} \to \mathbb{C}$, in
some neighbourhood of 0. This extension is given by

\[
\Phi(z) = \begin{cases} 
\varphi(z) & \text{for } z \in \Omega \cup \Gamma_1 \cup \Gamma_2, \\
\overline{\varphi(z)} & \text{for } z \in \Omega_1, \\
\varphi\left(\gamma_2 \left( \gamma_2^{-1}(z) \right) \right) & \text{for } z \in \Omega_2.
\end{cases}
\]

Since \( F \) is holomorphic on \( \mathbb{C} \), it also has an holomorphic extension to \( \tilde{\Omega} \) after shrinking \( r \) and \( s \) if necessary. We now prove the following claim on the asymptotic behaviour of \( \Phi \).

**Claim 1:** \( \Phi(z) \sim F(z) \) at 0 on \( \tilde{\Omega} \).

**Proof of Claim 1:** By Theorem 3.5.1 we have \( \varphi(z) \sim F(z) \) at 0 on \( \Omega \cup \Gamma_1 \cup \Gamma_2 \). Since \( F(z) = F(\overline{z}) \) for \( z \in \mathbb{C} \) we have for \( \Phi \) that

\[
\Phi(z) = \overline{\varphi(z)} \sim F(z)
\] (3.18)

at 0 on \( \Omega_1 \). It remains to be shown that \( \Phi(z) \sim F(z) \) at 0 on \( \Omega_2 \). From \( \gamma_2(0) = \gamma_2^{-1}(0) = 0 \) we get at 0 on \( \Omega_2 \) that

\[
\varphi\left(\gamma_2 \left( \gamma_2^{-1}(z) \right) \right) \sim F\left(\gamma_2 \left( \gamma_2^{-1}(z) \right) \right).
\] (3.19)

Therefore, we have to show that

\[
F\left(\gamma_2 \left( \gamma_2^{-1}(z) \right) \right) \sim F(z)
\]

at 0 on \( \Omega_2 \). We have

\[
F(\gamma_2(z)) = \exp \left( \sum_{k=0}^{d-1} b_k (\gamma_2(z))^{k-d} + a \log (\gamma_2(z)) \right).
\]
Since $\gamma_2(z) = z \exp(i \angle_{\Omega}(z))$ we see by the power series expansion of the exponential function that
\[
(\gamma_2(z))^{k-d} = z^{k-d} + O(z^k)
\]
for $k \in \{0, \ldots, d-1\}$. Hence, we obtain
\[
F(\gamma_2(z)) = \exp \left( \sum_{k=0}^{d-1} b_k \left( z^{k-d} + O(z^k) \right) + a \log(z \exp(i \angle_{\Omega}(z))) \right)
\]
\[
= \exp \left( \sum_{k=0}^{d-1} b_k z^{k-d} + \sum_{k=0}^{d-1} b_k O(z^k) \right) + a \log(z) + \log (\exp(i \angle_{\Omega}(z))) + 2\pi i l(z)
\]
\[
= \exp \left( \sum_{k=0}^{d-1} b_k O(z^k) \right) \exp (i \angle_{\Omega}(z)) \exp (2\pi i l(z)) F(z)
\]
where $l(z) \in \{-1, 0, 1\}$. Since $\angle_{\Omega}(z) \sim z^d$ we have at 0 on $\Omega_2$
\[
F(\gamma_2(z)) \sim F(z)
\]
and thus
\[
F(\gamma_2(\gamma_2(z))) \sim F(z).
\] (3.20)

Analogously, we obtain by the power series expansion of the inverse that
\[
F(\gamma_2^{-1}(z)) = \exp \left( H(\gamma_2^{-1}(z)) \right) \sim \exp (H(\bar{z})) = F(\bar{z}).
\] (3.21)

Thus, we see that
\[
F(\gamma_2(\gamma_2^{-1}(z))) \overset{(3.20)}{=} F(\gamma_2^{-1}(z)) \overset{(3.21)}{=} F(\bar{z}).
\]

Hence, by using (3.19) it follows at 0 on $\Omega_2$ that
\[
\Phi(z) = \varphi(\gamma_2(\gamma_2^{-1}(z))) \sim \frac{F(\gamma_2(\gamma_2^{-1}(z)))}{F(\gamma_2^{-1}(z))} \sim F(\bar{z}) = F(z). \] (3.22)
In summary, we have with (3.18) and (3.22) at 0 on $\tilde{\Omega}$ that

$$\Phi(z) \sim F(z).$$

Before we prove the theorem by induction we show that there is some $\rho > 0$ such that $\varphi$ has a holomorphic extension to $B(z, 2\rho |z|^{d+1})$ for $z$ on the boundary arcs in a neighbourhood of 0.

**Claim 2:** There is some $\rho > 0$ such that for all sufficiently small $z \in \Omega \cup \Gamma_1 \cup \Gamma_2$ we have $B(z, 2\rho |z|^{d+1}) \subset \tilde{\Omega}$.

**Proof of Claim 2:** Since $\angle_{\Omega}(t) \sim t^d$ we see that for $t > 0$

$$\text{dist}(t, \Gamma_2) \sim \text{Im}(\gamma_2(t)) = t \sin(\angle_{\Omega}(t)) \sim t^{d+1}$$

at 0. Reflecting at the positive real axis, we find some $\rho_1 > 0$ such that $B(t, 2\rho_1 t^{d+1}) \subset \tilde{\Omega}$ for all sufficiently small $t > 0$. We apply the coordinate transformation $\gamma_2^{-1}$ to $\Omega \cap B(0, \delta)$ and denote the image by $\Omega_{\gamma_2}$. Close to 0 the boundary of $\Omega_{\gamma_2}$ is given by $\tilde{\Gamma}_1 \cup \tilde{\Gamma}_2$ where $\tilde{\Gamma}_1 := \tilde{\gamma}_1(0, \epsilon]$ and $\tilde{\Gamma}_2 := \tilde{\gamma}_2(0, \epsilon]$ for some $\epsilon > 0$ with the parameterisations $\tilde{\gamma}_1(z)$ and $\tilde{\gamma}_2(z) = z$. By Lemma 3.6.2 we have

$$\text{ot}(\angle_{\Omega_{\gamma_2}}(t)) = \text{ot}(\angle_{\Omega}(t)) = d$$

and therefore $\text{dist}(t, \tilde{\Gamma}_2) \sim t^{d+1}$ at 0 for $t > 0$. We denote the reflection of $\Omega_{\gamma_2}$ across $\tilde{\Gamma}_2$, which we obtain through complex conjugation, by $\tilde{\Omega}_{\gamma_2}$. Thus, there exists some $\rho_2 > 0$ such that $B(t, 2\rho_2 t^{d+1}) \subset \left(\Omega_{\gamma_2} \cup \tilde{\Gamma}_2 \cup \tilde{\Omega}_{\gamma_2}\right)$ for all sufficiently small $t > 0$. Since $\gamma_2^{-1}(z) \approx z$ we find, after shrinking $\rho_2$ if necessary, that $B(z, 2\rho_2 |z|^{d+1}) \subset \tilde{\Omega}$ for all $z$ on the trace of $\gamma_2$ which are sufficiently close to 0. Setting $\rho := \min\{\rho_1, \rho_2\}$ we obtain the claim.
For $n \in \mathbb{N}_0$ let $\rho_n := \frac{\rho}{2^n}$ and

$$\tilde{\Omega}_n := \{ z \in \tilde{\Omega} \mid \text{dist}(z, \Omega) < \rho_n |z|^{d+1} \}.$$ 

**Claim 3:** $\Phi(n)(z) \sim F(n)(z)$ at 0 on $\tilde{\Omega}_n$ for all $n \in \mathbb{N}_0$.

**Proof of Claim 3:** We prove this by induction on $n$. The base case $n = 0$ follows with Claim 1. We assume that for an arbitrary, fixed $n \in \mathbb{N}$ we have at 0 on $\tilde{\Omega}_n$

$$\Phi(n)(z) \sim F(n)(z).$$

Hence, there exists an holomorphic function $h : \tilde{\Omega}_n \to \mathbb{C}$ with $h(z) = o(1)$ at 0 on $\tilde{\Omega}_n$ and some constant $p_1 \in \mathbb{C}^*$ such that

$$\Phi(n)(z) = p_1 F(n)(z) + F(n)(z) h(z)$$

on $\tilde{\Omega}_n$. By differentiating we therefore obtain

$$\Phi(n+1)(z) = p_1 F(n+1)(z) + F(n+1)(z) h(z) + F(n)(z) h'(z).$$

Using Proposition [3.6.1](#) we have that

$$F(n+1)(z) \sim F(z) z^{-(n+1)(d+1)}.$$ 

By showing that $h'(z) = o \left( z^{-(d+1)} \right)$ on $\tilde{\Omega}_{n+1}$ we get for a constant $p_2 \in \mathbb{C}^*$

$$\Phi(n+1)(z) = p_2 F(z) z^{-(n+1)(d+1)} + o \left( F(z) z^{-(n+1)(d+1)} \right)$$

and hence at 0 on $\tilde{\Omega}_{n+1}$

$$\Phi(n+1)(z) \sim F(z) z^{-(n+1)(d+1)}.$$ 

Using Claim 2 we get $B(z, \rho_{n+2}|z|^{d+1}) \subset \tilde{\Omega}_n$ for all $z \in \tilde{\Omega}_{n+1}$. Let $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = \rho_{n+2}|z|^{d+1} \exp(it) + z$ be the parameterisation of $\partial B(z, \rho_{n+2}|z|^{d+1})$ for $z \in \tilde{\Omega}_{n+1}$. By
the Cauchy formula we obtain

$$|h'(z)| = \left| \frac{1}{2\pi i} \int_{\partial B(z, \rho_n+2 |z|^{d+1})} \frac{h(w)}{(w-z)^2} \, dw \right|$$

$$\leq \frac{1}{2\pi} \int_{\gamma} \frac{|h(w)|}{|w-z|^2} \, dw$$

$$\leq \frac{1}{2\pi} \max_{|w-z|=\rho_n+2 |z|^{d+1}} |h(w)| \int_{0}^{2\pi} \frac{\left| \gamma'(t) \right|}{\left| \gamma(t) - z \right|^2} \, dt$$

$$= \frac{1}{2\pi} \max_{|w-z|=\rho_n+2 |z|^{d+1}} |h(w)| \int_{0}^{2\pi} \frac{\rho_n+2 |z|^{d+1}}{\rho_n^2+2 |z|^{2(d+1)}} \, dt$$

$$= \frac{1}{\rho_n+2} \max_{|w-z|=\rho_n+2 |z|^{d+1}} |h(w)| \frac{1}{|z|^{(d+1)}}.$$

We set

$$\hat{h}(z) := \frac{1}{\rho_n+2} \max_{|w-z|=\rho_n+2 |z|^{d+1}} |h(w)|.$$

Since $h(z) = o(1)$ it follows that

$$\lim_{z \to 0} h(z) = 0.$$

Therefore, $\hat{h}(z) = o(1)$ and we get $h'(z) = o\left( z^{-(d+1)} \right)$ at 0 on $\tilde{\Omega}_{n+1}$ and furthermore (3.23). Since we have $\varphi(z) = \Phi(z)$ for $z \in \tilde{\Omega}$ we obtain $\varphi(z) \sim F(z)$ at 0 on $\Omega$ and the theorem holds.

**Example 3.6.4**

Let $\Omega$ be as in Example 3.5.2. Then we have $d = 1$ and therefore by Theorem 3.6.3

$$\varphi^{(n)}(z) \sim z^{\pi-2n} \exp\left(-\frac{\pi}{z}\right)$$

at 0 on $\Omega$.  

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3.7 Asymptotic Behaviour of the Inverse

In this section, we want to investigate the asymptotic behaviour of the inverse of the mapping function \( \varphi \) which can be deduced by applying Theorem 3.5.1 and the logarithm. Let \( \psi : \mathbb{H} \rightarrow \Omega \) be the inverse of the Riemann map \( \varphi \). A visualisation of \( \psi \) is shown in Figure 3.11.

Figure 3.11: Example of a conformal map \( \psi \) from the upper half plane \( \mathbb{H} \) onto \( \Omega \)

**Setting:**

Let \( \psi : \mathbb{H} \rightarrow \Omega \) be a conformal map with \( \psi(0) = 0 \).

**Theorem 3.7.1**

We have

\[
\psi(z) \simeq \left(-\frac{\pi}{a_d d \log(|z|)}\right)^\frac{1}{d}
\]

at 0 on \( \mathbb{H} \).

**Proof:** Let \( \varphi := \psi^{-1} \). By Theorem 3.5.1 we have

\[
\varphi(w) \sim \exp \left( \sum_{n=0}^{d-1} b_n w^{n-d} + a \log(w) \right)
\]

where \((b_0, \ldots, b_{d-1}, a) \in \mathbb{R}^{d+1}\) is the asymptotic tuple of \( \Omega \).
We set

\[ H(w) := \sum_{n=0}^{d-1} b_n w^{n-d} + a \log(w) \] and \( w := \psi(z) \).

Therefore, \( \varphi(w) \sim \exp(H(w)) \) and we obtain

\[ z = \varphi(\psi(z)) \sim \exp(H(\psi(z))). \]

Applying \( \log : \mathbb{H} \rightarrow \mathbb{C} \) we get by Lemma 2.1.21

\[ \log(z) \simeq H(\psi(z)). \]

Since \( \lim_{z \to 0} \psi(z) = 0 \) we see that

\[
H(\psi(z)) = b_0 \psi(z)^{-d} + b_1 \psi(z)^{1-d} + \ldots + b_{d-1} \psi(z)^{1} + a \log(\psi(z)) \\
= \psi(z)^{-d} (b_0 + b_1 \psi(z) + \ldots + b_{d-1} \psi(z)^d) + a \log(\psi(z)) \psi(z)^d \\
\simeq b_0 \psi(z)^{-d}.
\]

Hence,

\[ \log(z) \simeq b_0 \psi(z)^{-d}. \]

Since \( \log(z) \simeq \log|z| \) on \( \mathbb{H} \) and \( b_0 = -\frac{\pi a}{\pi d} \) by Example 3.4.4 we are done.

Note that in case of \( \varphi \) we have an asymptotically similar function and for the inverse \( \psi \) we even have an asymptotically equivalent function. This is due to the fact that in order to obtain the asymptotic behaviour of \( \psi \) we have to apply the logarithm. Moreover, it is also a very interesting fact that the asymptotic behaviour of \( \psi \) seems to be much “simpler” than that of \( \varphi \).

**Example 3.7.2**

Let \( \Omega \) be as in Example 3.5.2. Then we have \( \text{ot}(\Omega) = 1 \) and \( \text{ct}(\Omega) = 1 \) and therefore by
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Theorem 3.7.1 we get
\[ \psi(z) \simeq -\frac{\pi}{\log(z)} \]
at 0 on \( \mathbb{H} \).

3.8 Asymptotic Behaviour of the Derivatives of the Inverse

Next, we investigate the asymptotic behaviour of the derivatives of \( \psi \) which we can determine by using the result for the first derivative of \( \varphi \), see Theorem 3.6.3, the Inverse Function Theorem, the Schwarz Reflection Principle, and the Cauchy formula.

**Theorem 3.8.1**

Let \( n \in \mathbb{N} \). Then
\[ \psi^{(n)}(z) \sim (\psi(z))^{d+1} z^{-n} \sim \left(-\frac{1}{\log(z)}\right)^{\frac{1}{d}+1} z^{-n} \]
at 0 on \( \mathbb{H} \).

**Proof:** From Theorem 3.7.1 we already know that the asymptotic behaviour of \( \psi \) is the following
\[ \psi(w) \simeq \left(-\frac{\pi}{\log(w)}\right)^{\frac{1}{d}} \sim \left(-\frac{1}{\log(w)}\right)^{\frac{1}{d}}. \] (3.24)

By the Inverse Function Theorem we have that
\[ \psi'(z) = \frac{1}{\varphi'(\psi(z))} \] (3.25)
at 0 on \( \mathbb{H} \). Moreover, it follows by Theorem 3.6.3 that
\[ \varphi'(z) \sim \varphi(z) z^{-(d+1)} \] (3.26)
at 0 on \( \Omega \).
3.8 Asymptotic Behaviour of the Derivatives of the Inverse

Hence we obtain

\[
\psi'(z) \overset{\text{(3.25)}}{=} \frac{1}{\varphi'(\psi(z))\psi(z)^{-(d+1)}} = \psi^{(d+1)}(z) \overset{\text{(3.21)}}{=} \left(-\frac{1}{\log(z)}\right)^{1/2 + 1} \frac{1}{z} \quad (3.27)
\]

at 0 on \(\mathbb{H}\). Setting

\[
g(z) := \left(-\frac{1}{\log(z)}\right)^{1/2},
\]

it follows inductively by the product rule that

\[
g^{(n)}(z) = \left(\sum_{k=1}^{n} \tilde{a}_k \left(-\frac{1}{\log(z)}\right)^{1/2 + k}\right) \frac{1}{z^n} \quad (3.28)
\]

for \(n \in \mathbb{N}\), \(\tilde{a}_k \in \mathbb{R}\), and \(\tilde{a}_1 \neq 0\). Hence,

\[
g^{(n)}(z) \sim \left(-\frac{1}{\log(z)}\right)^{1/2 + 1/2} \frac{1}{z^n}. \quad (3.29)
\]

**Claim:** We have \(\psi^{(n+1)}(z) \sim g^{(n+1)}(z)\) for \(n \in \mathbb{N}\).

We prove the claim by showing that for \(n \in \mathbb{N}\) there exists some \(p_{n+1} \in \mathbb{C}^*\) such that

\[
\psi^{(n+1)}(z) = p_{n+1} g^{(n+1)}(z) + o(g^{(n+1)}(z)).
\]

Since by (3.27)

\[
\psi'(z) \sim g'(z)
\]

holds there exists some \(p_1 \in \mathbb{C}^*\) such that

\[
\psi'(z) = p_1 g'(z) + o(g'(z)).
\]

Therefore, there exists an holomorphic function \(h : \mathbb{H} \to \mathbb{C}\) with \(h(z) = o(1)\) such that

\[
\psi'(z) = p_1 g'(z) + g'(z)h(z). \quad (3.30)
\]
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Now let

\[ f(z) := g'(z)h(z). \]

Differentiating (3.30) we obtain

\[ \psi^{(n+1)}(z) = p_1 g^{(n+1)}(z) + f^{(n)}(z) \]

and therefore it suffices to show that \( f^{(n)}(z) = o(g^{(n+1)}(z)) \) and setting \( p_{n+1} := p_1 \) to obtain the claim. By Theorem 2.4.1, the Schwarz Reflection Principle, \( \psi(z) \) has an holomorphic extension to \( U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0} \) where \( U \) is a sufficiently small neighbourhood of 0 and therefore also \( \psi^{(n)}(z) \) has such an holomorphic extension. This extension if given by

\[
\Psi(z) = \begin{cases} 
\psi(z) & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cap \mathbb{H}, \\
\psi(z) & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cap \mathbb{R}, \\
\overline{\psi(z)} & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cap \mathbb{H}_-, 
\end{cases}
\]

and

\[
\Psi(z) = \begin{cases} 
\psi(z) & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0}) \cap \mathbb{H}, \\
\psi(z) & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0}) \cap \mathbb{R}, \\
\gamma_2 \left( \gamma_2^{-1}(\overline{\psi(z)}) \right) & \text{for } z \in (U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0}) \cap \mathbb{H}_-, 
\end{cases}
\]

respectively. The logarithm has also an holomorphic extension to \( U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0} \) and therefore also \( \log^{(n)}(z) \). Thus, \( g \) has an holomorphic extension to \( U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0} \). The same holds for \( h \). Now we show that

\[
\Psi(z) \sim \left( -\frac{1}{\log(z)} \right)^{1/2}
\]

at 0 on \( U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0} \), respectively.
3.8 Asymptotic Behaviour of the Derivatives of the Inverse

Since

\[
\left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}} = \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}
\]

we have

\[
\psi(z) \sim \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}
\]
at 0 on \((U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0}) \cap \mathbb{H}_-\). Moreover we have \(\gamma_2^{-1}(z) \simeq z\) and therefore

\[
\gamma_2^{-1}(\psi(z)) \sim \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}.
\]

Thus, we obtain

\[
\gamma_2^{-1}(\psi(z)) \sim \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}} = \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}.
\]

Since \(\gamma_2(z) \simeq z\) we get

\[
\gamma_2\left(\gamma_2^{-1}(\psi(z))\right) \sim \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}
\]
at 0 on \((U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0}) \cap \mathbb{H}_-\) and thus in summary

\[
\Psi(z) \sim \left( -\frac{1}{\log(z)} \right)^{\frac{1}{d}}
\]
at 0 on \(U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0}\) and \(U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0}\), respectively. Therefore, \(h(z) = o(1)\) on these domains. Hence, for a given \(z \in \mathbb{H} \setminus \{0\}\) with \(|z|\) sufficiently small, \(f\) has an holomorphic extension to \(B(z,|z|)\). Furthermore, let \(\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = \frac{|z|}{2} \exp(it) + z\) be the parameterisation of \(\partial B\left(z, \frac{|z|}{2}\right)\). By the Cauchy formula we have

\[
\left| f^{(n)}(z) \right| = \left| \frac{n!}{2\pi i} \int_{\partial B\left(z, \frac{|z|}{2}\right)} \frac{f(\zeta)}{\left(\zeta - z\right)^{n+1}} d\zeta \right|
\]
Chapter 3 Asymptotic Behaviour at Analytic Cusps

\[ \leq \frac{n!}{2\pi} \int_{\gamma} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} d\zeta \]

\[ \leq \frac{n!}{2\pi} \max_{|w - z| = |z|/2} |f(w)| \int_{0}^{2\pi} \frac{\left|\gamma'(t)\right|}{\left|\gamma(t) - z\right|^{n+1}} dt \]

\[ = \frac{n!}{2\pi} \max_{|w - z| = |z|/2} |f(w)| \int_{0}^{2\pi} \frac{2^{n+1}|z|}{2|z|^{n+1}} dt \]

\[ = 2^n n! \max_{|z|/2} |f(w)| \]

\[ = n! 2^n \max_{|w - z| = |z|/2} |h(w)g'(w)| \]

\[ \leq 2^n n! \max_{|w - z| = |z|/2} |h(w)| \max_{|w - z| = |z|/2} |g'(w)|. \]

We set

\[ \tilde{h}(z) := 2^n n! \max_{|w - z| = |z|/2} |h(w)|. \]

Since \( h(z) = o(1) \) we have \( \tilde{h}(z) = o(1) \). Furthermore,

\[ \max_{|w - z| = |z|/2} |g'(w)| = \max_{|w - z| = |z|/2} \left|\frac{1}{d} \left( -\frac{1}{\log(w)} \right)^{\frac{1}{2} + 1} \frac{1}{w} \right| \]

\[ = \frac{1}{d} \max_{|w - z| = |z|/2} \left|\frac{1}{\log(w)} \right| \left( -\frac{1}{\log(w)} \right)^{\frac{1}{2} + 1} \frac{1}{w} \]

\[ = \frac{1}{d} \max_{|w - z| = |z|/2} \left|\frac{1}{\log(w) + i \arg(w)} \right| \left( -\frac{1}{\log(w)} \right)^{\frac{1}{2} + 1} \frac{1}{w} \]

\[ \leq \frac{1}{d} \max_{|w - z| = |z|/2} \left|\frac{1}{\log(w) + i \arg(w)} \right| \left( -\frac{1}{\log(w)} \right)^{\frac{1}{2} + 1} \frac{1}{w} \]

\[ = \frac{1}{d} \max_{|w - z| = |z|/2} \left|\frac{1}{\log(w) + i \arg(w)} \right| \left( -\frac{1}{\log(w) + i \arg(w)} \right)^{\frac{1}{2} + 1} \frac{1}{w} \]

\[ = \frac{1}{d} \max_{|w - z| = |z|/2} \left|\frac{1}{\log(z/2)} \right| \left( -\frac{1}{\log(z/2)} \right)^{\frac{1}{2} + 1} \frac{1}{z/2} \]

\[ = \left|g' \left( \frac{|z|}{2} \right) \right|. \]
Therefore, we get
\[ |f^{(n)}(z)| \leq \tilde{h}(z) \left| \frac{1}{z^n} g'(\frac{|z|}{2}) \right|. \]
Since
\[ \frac{|z|}{2} \sim |z| \text{ and } \log \left| \frac{z}{2} \right| = \log |z| - \log(2) \simeq \log |z| \]
we have
\[ \left| g'\left(\frac{|z|}{2}\right) \right| \sim \left| g'(|z|) \right|. \]
Moreover, we have \( \log |z| \simeq \log(z) \) and therefore
\[ \left| g'(|z|) \right| \sim \left| g'(z) \right|. \]
Since \( \tilde{h}(z) = o(1) \) we see that
\[ f^{(n)}(z) = o\left( \frac{1}{z^n} g'(z) \right). \] (3.31)
Hence,
\[ f^{(n)}(z) \overset{(3.31)}{=} o\left( \frac{1}{z^n} g'(z) \right) \overset{(3.29)}{=} o\left( \frac{1}{z^{n+1}} \left( -\frac{1}{\log(z)} \right)^{\frac{1}{2}+1} \right) \overset{(3.29)}{=} o(g^{(n+1)}(z)) \]
and we see that
\[ \Psi(z) = p_1 g^{(n+1)}(z) + o\left( g^{(n+1)}(z) \right). \]
Since for \( z \in U \cap \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) and \( z \in U \cap \mathbb{C} \setminus \mathbb{R}_{\geq 0} \), respectively, we have
\[ \Psi(z) = \psi(z) \]
it follows with \( p_1 := p_{n+1} \) that
\[ \psi(z) = p_{n+1} g^{(n+1)}(z) + o\left( g^{(n+1)}(z) \right) \]
and the claim is proven. \( \square \)
Example 3.8.2

Let $\Omega$ be as in Example 3.5.2. Then we have $\omega(\Omega) = 1$ and $\chi(\Omega) = 1$ and therefore by Theorem 3.8.1

$$\psi^{(n)}(z) \sim \left( -\frac{1}{\log(z)} \right)^2 z^{-n}$$

at 0 on $\mathbb{H}$.
Chapter 4

Future Research

In this chapter we present a conjecture on the asymptotic power series expansion of the inverse Riemann map $\psi : \mathbb{H} \to \Omega$ for the special case $\operatorname{ot}(\Omega) = 1$ and give an outline of its proof. At this point we assume that $\psi$ is specified as in Section 3.7.

Conjecture

The mapping function $\psi : \mathbb{H} \to \Omega$ has for $\operatorname{ot}(\Omega) = 1$ an asymptotic power series expansion of the following form

$$\sum_{n=1}^{\infty} p_n \left( \frac{1}{(\log(z))^n} \right)$$

where $p_n \in \mathbb{C}$ and $p_1 \neq 0$.

The idea of the proof is as follows. Let

$$F : \mathbb{H} \to \Omega, \quad F(z) = \psi(-z).$$

Then $F(z) = u(z) + iv(z)$ is a conformal map with $F(0) = 0$. Through the function $F$ the negative real axis is mapped to the positive real axis and the positive real axis is mapped to the arc $\Gamma$ with the parameterisation

$$\gamma(s) = \left( s, \sum_{n=2}^{\infty} \tilde{a}_n s^n \right)$$  \hspace{1cm} (4.1)
where $\tilde{a}_n \in \mathbb{R}$ and $\tilde{a}_2 \neq 0$. A visualisation is depicted in Figure 4.1.

Figure 4.1: Visualisation of the mapping function $F$

Analogously to Lehman [11], we introduce an analytic function $H(z)$ which differs from $F(z)$ only by a single-valued function. Let $0 < A < 1$ and

$$H(z) := \frac{1}{\pi} \int_0^A \frac{\partial v(t,0)}{\partial t} \log \left(1 - \frac{z}{t}\right) dt$$

where $\log \left(1 - \frac{z}{t}\right)$ is the branch which is real for $0 < z < t$. Let

$$p(z) := F(z) - H(z).$$

By proving that $p(z)$ is equal to a power series convergent for $|z| < A$ and estimating the function $H(z)$ we get an estimate for $F(z)$. For the estimation of $H(z)$ we need to determine the asymptotic behaviour of $v'(t)$. Since with (4.1) we have

$$v(t) = \sum_{n=2}^{\infty} \tilde{a}_n (u(t))^n$$
we get by differentiating

\[ v'(t) = u'(t) \sum_{n=2}^{\infty} \tilde{a}_n (u(t))^{n-1}. \]

By determining the asymptotic behaviour of \( u(t) \), which can be derived by the one of \( \psi(z) \), we obtain the asymptotic behaviour of \( v'(t) \). Inserting the latter in the representation of \( H(z) \) shown above, we can estimate \( H(z) \). Repeating this construction we obtain inductively an asymptotic power series expansion for \( F(z) \) and thereby for \( \psi(z) \). We were already able to show some of the steps of the proof but there still remain a couple of open problems especially concerning the estimates for the function \( H(z) \).

By verifying this conjecture it would be moreover possible to derive the asymptotic power series expansion for the general case that \( ot(\Omega) = d \) as well as the one for \( \varphi \).
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