

Optimal Quantization for Dyadic Homogeneous Cantor Distributions

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For a large class of dyadic homogeneous Cantor distributions in \mathbb{R} , which are not necessarily self-similar, we determine the optimal quantizers, give a characterization for the existence of the quantization dimension, and show the non-existence of the quantization coefficient. The class contains all self-similar dyadic Cantor distributions, with contraction factor less than or equal to $\frac{1}{3}$. For these distributions we calculate the quantization errors explicitly.

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1 Introduction

The problem of quantization originally arose in electrical engineering in the context of signal processing and data compression (cf. [5]). Mathematically the problem can be stated as follows. Given a Borel probability distribution ν on \mathbb{R}^d , $n \in \mathbb{N} = \{1, 2, \dots\}$ and $r \geq 1$, determine the set

$$C_{n,r}(\nu) = \{\beta \subset \mathbb{R}^d : \text{card}(\beta) \leq n \text{ and } \Psi_{\beta,r}(\nu) = V_{n,r}(\nu)\}$$

of all n -optimal sets (quantizers) of order r , with

$$\Psi_{\beta,r}(\nu) = \int \min_{b \in \beta} |x - b|^r d\nu(x)$$

and

$$V_{n,r}(\nu) = \inf_{\beta \subset \mathbb{R}^d, \text{card}(\beta) \leq n} \Psi_{\beta,r}(\nu)$$

the optimal quantization error. This problem of optimal quantization was systematically studied by Graf and Luschgy [3]. Under very general assumptions they have shown, that $C_{n,r}(\nu)$ is always non-empty (cf. [3], Theorem 4.12) and that every $\beta \in C_{n,r}(\nu)$ has exactly n elements (cf. [3], Theorem 4.1).

The idea of defining a dimension D_r by means of quantization goes back to Zador. In [11] he defined the quantization dimension

$$D_r = \lim_{n \rightarrow \infty} \frac{r \log(n)}{-\log(V_{n,r}(\nu))},$$

if the limit exists. Later on this concept was investigated by several authors (see, for instance, [3], [8], [9]). If the quantization dimension D_r exists, it can be asked whether the sequence $(n^{\frac{r}{D_r}} V_{n,r}(\nu))_{n \in \mathbb{N}}$ converges, i.e. whether the quantization coefficient exists.

In the present paper we consider dyadic homogeneous Cantor distributions. To this end we first have to define dyadic homogeneous Cantor sets depending on a sequence $(c_k)_{k \in \mathbb{N}} \in]0, \frac{1}{2}]^{\mathbb{N}}$ of contraction factors. The

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corresponding dyadic homogeneous Cantor measure is induced by the natural mass distribution on the dyadic homogeneous Cantor set and completely characterized through the sequence $(c_k)_{k \in \mathbb{N}}$. Recently, Kesseböhmer and Zhu [6] determined the optimal quantizers for these distributions, provided $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{4}$ (cf. [6], Proposition 3.7). Lindsay [7] was the first who used a special example of this type to show that the quantization dimension of order 1 need not to exist (cf. [7], Example 5.5). Later Kesseböhmer and Zhu [6] also gave such an example (cf. [6], Theorem 1.5 (1)).

If $c_k = c$ for every $k \in \mathbb{N}$ then the dyadic homogeneous Cantor distribution becomes self-similar. The special case $c = \frac{1}{3}$ yields the classical Cantor distribution. Under the additional assumption that $r = 2$, Graf and Luschgy solved the quantization problem for this last distribution completely, i.e. they determined the optimal quantizers and the optimal quantization errors (cf. [2], Theorem 5.2). They proved the existence of the quantization dimension (cf. [2], Theorem 6.6) and that the quantization coefficient does not exist (cf. [2], Theorem 6.3).

As mentioned above, Lindsay was the first to study quantization for special non-self-similar Cantor measures. A general theory for the quantization of these measures was developed by Kesseböhmer and Zhu [6]. Many of their results are applicable to the classical Cantor Distribution (cf. [6], Theorem 1.4, Theorem 1.6 (1) and (2) and Proposition 3.1). But their analysis of the n -optimal quantizers of order r for dyadic homogeneous Cantor distributions is limited to $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{4}$ and, therefore, does not include the classical case.

We will generalize some of the results of Kesseböhmer and Zhu to the case that $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{3}$ (Theorem 4.4) and apply them to the classical Cantor distribution. Here we get new results in the case $r \neq 2$. Moreover, we will characterize those dyadic homogeneous Cantor distributions for which the quantization dimension exists (Remark 5.2). If the quantization dimension of the dyadic homogeneous Cantor distribution exists, we give necessary and sufficient conditions for the sequence $(n^{\frac{r}{D_r}} V_{n,r}(\nu))_{n \in \mathbb{N}}$ to be bounded and bounded away from zero. In this situation the quantization coefficient does not exist (Proposition 5.3).

For the self similar case $\mu = \mu_c$ with $c \in]0, \frac{1}{3}]$ we will derive a functional representation for $\lim_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu_c)$ resp. $\overline{\lim}_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu_c)$. Such a characterization is not known for c strictly between $\frac{1}{3}$ and $\frac{1}{2}$.

2 Notation and basic facts

First note that dyadic homogeneous Cantor sets are a special case of homogeneous Moran sets as proposed by Wen et.al. (cf. [1], [10]). Moreover, dyadic homogeneous Cantor sets and the related dyadic homogeneous Cantor distributions were defined by Kesseböhmer and Zhu [6]. We follow the notation in [6] and will verify in this section some simple properties of these distributions.

Let $(c_k)_{k \in \mathbb{N}} \in]0, \frac{1}{2}]^{\mathbb{N}}$ be given. The dyadic homogeneous Cantor set $C((c_k))$ is constructed inductively. Start with $E_0 = [0, 1]$ and let $\mathcal{D}_0 = \{E_0\}$ be the collection of the so-called basic intervals of order 0. Now remove an open interval of length $1 - 2c_1$ in the middle of E_0 , so that $[0, c_1]$ and $[1 - c_1, 1]$ remain. We call the remaining sets, basic intervals of order 1. The union of all basic intervals of order 1 is denoted by E_1 , the collection of all these intervals by \mathcal{D}_1 . Let $k \in \mathbb{N}$ and let E_k be the union of all basic intervals of order k and \mathcal{D}_k the collection of all basic intervals of order k . Let $\pi_k = \prod_{i=1}^k c_i$. By removing an open interval of length $(1 - 2c_{k+1})\pi_k$ in the middle of each basic interval of order k we obtain the collection \mathcal{D}_{k+1} of basic intervals of order $k + 1$, having all the length π_{k+1} . Set $C((c_k)) = \bigcap_{k \in \mathbb{N}} E_k$.

The corresponding homogeneous Cantor measure μ is the unique Borel probability measure on \mathbb{R} , with

$$\mu(F) = 2^{-k} \text{ for every } F \in \mathcal{D}_k \quad (1)$$

and

$$\mu(A) = \inf \left\{ \sum_i \mu(U_i) : A \cap C((c_k)) \subset \bigcup_i U_i, U_i \in \bigcup_{k \geq 1} \mathcal{D}_k \right\} \quad (2)$$

for every Borel subset $A \subset \mathbb{R}$. It has support $C((c_k))$. For the rest of this paper the distribution μ is always the dyadic homogeneous Cantor distribution defined by the sequence $(c_k)_{k \in \mathbb{N}}$ with $0 < c_k \leq \frac{1}{3}$ for every $k \in \mathbb{N}$.

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For every $k \in \mathbb{N}_0$ and $F \in \mathcal{D}_k$ we denote by F_1 the left and by F_2 the right basic interval of order $k + 1$, contained in F . Moreover the midpoints of F, F_1, F_2 are denoted by a, a_1, a_2 resp. We

make canonical use of this notation, for example a_{121} denotes the midpoint of the basic interval $F_{121} \subset F$ of order $k + 3$. For $t \in \mathbb{R}$ define

$$\mathbb{R} \ni x \xrightarrow{\Theta_t} \Theta_t(x) = x + t$$

as the translation by t . Moreover, let

$$\mathbb{R} \ni x \xrightarrow{\Phi_t} \Phi_t(x) = t + (t - x)$$

be the reflection relative to t . For an arbitrary Borel probability distribution ν on \mathbb{R} and Borel measurable set $A \subset \mathbb{R}$ with $\nu(A) > 0$, let $\nu(\cdot | A) = \frac{\nu(A \cap \cdot)}{\nu(A)}$ be the conditional distribution of ν w.r.t. A . Now we can state two simple but useful properties of μ .

Lemma 2.1 *Let $k \in \mathbb{N}_0$. Let $G, H \in \mathcal{D}_{k+2}$ with $\min(G) \leq \min(H)$. Let $t = \frac{1}{2}(\min(G) + \max(H))$ and $s = \min(H) - \min(G)$. Then*

$$\mu(\cdot | H) = \mu(\cdot | G) \circ \Phi_t^{-1}$$

and

$$\mu(\cdot | H) = \mu(\cdot | G) \circ \Theta_s^{-1}.$$

Proof. Since $\text{diam}(H) = \text{diam}(G)$ we get from the definition of t and s that $\Phi_t(G) = H$ and $\Theta_s(G) = H$. Hence, the assertion follows from the construction of μ .

3 Elementary properties of optimal quantizers

For the rest of this paper let $r \geq 1$ and let all distributions be defined on the Borel subsets of \mathbb{R} . Moreover a probability distribution ν is called symmetric with respect to $t \in \mathbb{R}$, if $\nu = \nu \circ \Phi_t^{-1}$.

We begin this section with a result about the 1-optimal quantizers of general symmetric distributions on \mathbb{R} . Then we use these statements to show elementary properties of the optimal quantizers of dyadic homogeneous Cantor measures.

Proposition 3.1 *Let ν be a probability distribution which is symmetric w.r.t. $t \in \mathbb{R}$. Then for $r > 1$ and $b \in \mathbb{R} \setminus \{t\}$,*

$$\int |x - b|^r d\nu(x) > \int |x - t|^r d\nu(x).$$

Proof. Let $b \in \mathbb{R} \setminus \{t\}$. From the proof of Theorem 2.4 in [3] we know that the mapping

$$\mathbb{R} \ni z \rightarrow \Psi_r(z) = \int |x - z|^r d\nu(x)$$

is strictly convex. Hence we have

$$\int |x - t|^r d\nu(x) = \Psi_r(t) < \frac{\Psi_r(t + (b - t)) + \Psi_r(t - (b - t))}{2}.$$

Using the symmetry of ν w.r.t. t , we get $\Psi_r(t + (b - t)) = \Psi_r(t - (b - t))$, which yields

$$\frac{\Psi_r(t + (b - t)) + \Psi_r(t - (b - t))}{2} = \Psi_r(b) = \int |x - b|^r d\nu(x)$$

and completes the proof.

Remark 3.2 If, in addition, ν has a compact support, then for $r = 1$ the weaker inequality

$$\int |x - b|^r d\nu(x) \geq \int |x - t|^r d\nu(x)$$

still holds for all $b \in \mathbb{R}$.

Now fix $k \in \mathbb{N}_0$ and $F \in \mathcal{D}_k$ for the rest of this section.

Corollary 3.3 *Let a be the midpoint of F . Then for $b \in \mathbb{R} \setminus \{a\}$,*

$$\int_F |x - b|^r d\mu(x) \geq \int_F |x - a|^r d\mu(x). \quad (3)$$

If $r > 1$, then (3) is strict.

Proof. Since $\mu(F) > 0$ it suffices to prove (3) for $\mu(\cdot | F)$ instead of μ . Note, that $\mu(\cdot | F)$ is symmetric w.r.t. a and has a compact support. Hence, the assertion follows directly from Proposition 3.1 and Remark 3.2.

Remark 3.4 Note, that the part for $r > 1$ of Corollary 3.3 has already been proved in [6], Lemma 3.2.

For the rest of this section let $n \geq 2$ and $\beta \in C_{n,r}(\mu(\cdot | F))$. By [3], Theorem 4.1 we have $\text{card}(\beta) = n$. Denote $\beta = \{b_1, \dots, b_n\}$ with $b_1 < \dots < b_n$. Let $b_0 = \Phi_{\min(F)}(b_1)$ and $b_{n+1} = \Phi_{\max(F)}(b_n)$. By [3], Remark 4.6 (a) we know, that $\beta \subset F$. Next we will prove some essential properties of β , in particular we will show, that $\beta \subset F_1 \cup F_2$.

Lemma 3.5 *Let $i \in \{1, \dots, n\}$. If $\mu([\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}] \cap F_1) = 0$, then $b_i \in F_2$. If $\mu([\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}] \cap F_2) = 0$, then $b_i \in F_1$.*

Proof. Since $\mu(\mathbb{R} \setminus [0, 1]) = 0$, we get from [3], Theorem 4.1, that $[\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}]$ has strictly positive μ -measure and $\{b_i\} \in C_{1,r}(\mu(\cdot | [\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}]))$.

1. Let $\mu([\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}] \cap F_1) = 0$. Hence, $\mu(\cdot | [\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}])$ is concentrated on $F_2 \cap [\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}]$. Thus, Lemma 2.6 (a) in [3] yields $b_i \in F_2$.

2. As in 1. we obtain $b_i \in F_1$, if $\mu([\frac{b_{i-1}+b_i}{2}, \frac{b_i+b_{i+1}}{2}] \cap F_2) = 0$.

Lemma 3.6 (a) b_1 belongs to F_1 and b_n to F_2 .

(b) If $n \geq 3$, then

$$1 < j := \min\{i \in \{1, \dots, n\} : b_i > \max(F_1)\} \leq n \quad (4)$$

and

(i) $b_{j-1} \leq \max(F_1)$,

(ii) $b_{j+1} \geq \min(F_2)$,

(iii) if $b_j \in F \setminus (F_1 \cup F_2)$, then $j < n$ and $\frac{b_{j-1}+b_j}{2} < \max(F_1)$ and $\frac{b_j+b_{j+1}}{2} > \min(F_2)$.

Proof. By the construction of μ we can assume w.l.o.g. that $k = 0$. Then we have $F = E_0 = [0, 1]$ and $\mu(\cdot | F) = \mu$. We proceed in several steps.

1. We will prove that $V_{2,r}(\mu) \leq 4 \int_{F_{11}} (\frac{c_1}{2} - x)^r d\mu(x)$.

By the symmetry of μ (Lemma 2.1) it is easy to see that

$$V_{2,r}(\mu) \leq \Psi_{\{a_1, a_2\}, r}(\mu) = 4 \int_{F_{11}} (\frac{c_1}{2} - x)^r d\mu(x).$$

2. We will show, that $b_1 < \frac{3}{2}c_1$.

Assume the contrary, i.e. $b_1 \geq \frac{3}{2}c_1$. This yields

$$\begin{aligned} V_{2,r}(\mu) &\geq V_{n,r}(\mu) = \Psi_{\beta, r}(\mu) \geq \int_{F_1} (b_1 - x)^r d\mu(x) \geq \int_{F_1} (\frac{3}{2}c_1 - x)^r d\mu(x) \\ &= \int_{F_{11}} (\frac{3}{2}c_1 - x)^r d\mu(x) + \int_{F_{12}} (\frac{3}{2}c_1 - x)^r d\mu(x). \end{aligned}$$

Note, that $\frac{3}{2}c_1 - x \geq \frac{c_1}{2}$ for every $x \in F_{12}$ and $\frac{c_1}{2} - x \leq \frac{c_1}{2}$ for every $x \in F_{11}$, and, therefore,

$$\int_{F_{12}} \left(\frac{3}{2}c_1 - x\right)^r d\mu(x) \geq \left(\frac{c_1}{2}\right)^r \mu(F_{12}) = \left(\frac{c_1}{2}\right)^r \mu(F_{11}) \geq \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x).$$

Thus we get

$$V_{2,r}(\mu) \geq \int_{F_{11}} \left(\frac{3}{2}c_1 - x\right)^r d\mu(x) + \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x).$$

Since $r \geq 1$ it follows that for every $x \in F_{11}$,

$$\left(\frac{3}{2}c_1 - x\right)^r = 3^r \left(\frac{c_1}{2} - \frac{1}{3}x\right)^r \geq 3 \left(\frac{c_1}{2} - x\right)^r.$$

Because the last inequality is strict for every $x \in F_{11} \setminus \{0\}$ we obtain

$$V_{2,r}(\mu) > 4 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x),$$

a contradiction to 1.

3. We will show, that $b_1 \leq c_1$.

Assume the contrary. By 2. we have

$$c_1 < b_1 < \frac{3}{2}c_1. \tag{5}$$

If $\frac{b_1+b_2}{2} < 1 - c_1$, then Lemma 3.5 would imply $b_1 \leq c_1$. Hence we have

$$1 - c_1 \leq \frac{b_1 + b_2}{2}. \tag{6}$$

Combining (5) and (6) we get

$$\begin{aligned} b_2 &\geq 2 - 2c_1 - b_1 > 2 - \frac{7}{2}c_1 \\ &= 1 - \frac{c_1}{2} + 1 - 3c_1 \geq 1 - \frac{c_1}{2}. \end{aligned}$$

We conclude that

$$\begin{aligned} V_{2,r}(\mu) &\geq \Psi_{\beta,r}(\mu) \geq \int_{F_{11}} (b_1 - x)^r d\mu(x) + \int_{F_{12}} (b_1 - x)^r d\mu(x) \\ &+ \int_{F_{21} \cap [0, \frac{b_1+b_2}{2}]} (x - b_1)^r d\mu(x) + \int_{F_{21} \cap [\frac{b_1+b_2}{2}, 1]} (b_2 - x)^r d\mu(x) \end{aligned} \tag{7}$$

Lemma 2.1 yields $\mu(\cdot | F_{21}) = \mu(\cdot | F_{11}) \circ \Theta_{1-c_1}^{-1}$ resp. $\mu(\cdot | F_{12}) = \mu(\cdot | F_{11}) \circ \Theta_{c_1(1-c_2)}^{-1}$. Thus, inequality (7) turns into

$$\begin{aligned} V_{2,r}(\mu) &\geq \int_{F_{11}} (b_1 - x)^r d\mu(x) + \int_{F_{11}} (b_1 - c_1(1 - c_2) - x)^r d\mu(x) \\ &+ \int_{F_{11} \cap [0, \frac{b_1+b_2}{2} - (1-c_1)]} (x - (b_1 - (1 - c_1)))^r d\mu(x) \\ &+ \int_{F_{11} \cap [\frac{b_1+b_2}{2} - (1-c_1), c_1]} (b_2 - (1 - c_1) - x)^r d\mu(x). \end{aligned} \tag{8}$$

Let $x \in F_{11}$. Since $r \geq 1$ we have

$$\begin{aligned} & \frac{1}{2}[(b_1 - c_1(1 - c_2) - x)^r + (x - (b_1 - (1 - c_1)))^r] \\ & \geq \left[\frac{1}{2}(b_1 - c_1 + c_1c_2 - x + x - b_1 + 1 - c_1)\right]^r \\ & > \left(\frac{1}{2} - c_1\right)^r \geq \left(\frac{c_1}{2} - x\right)^r. \end{aligned} \tag{9}$$

Using (6), we get similarly

$$\begin{aligned} & \frac{1}{2}[(b_1 - c_1(1 - c_2) - x)^r + (b_2 - (1 - c_1) - x)^r] \\ & \geq \left[\frac{1}{2}(b_1 - c_1 + c_1c_2 - x + b_2 - 1 + c_1 - x)\right]^r \\ & \geq \left(\frac{b_1 + b_2}{2} - \frac{1}{2} - x\right)^r \geq \left(1 - c_1 - \frac{1}{2} - x\right)^r \end{aligned}$$

Since $c_1 \leq \frac{1}{3}$ we obtain

$$\begin{aligned} & \left(1 - c_1 - \frac{1}{2} - x\right)^r \\ & = \left(\frac{1}{2} - c_1 - x\right)^r \geq \left(\frac{3}{2}c_1 - c_1 - x\right)^r = \left(\frac{c_1}{2} - x\right)^r. \end{aligned} \tag{10}$$

Using the combination of (9) and (10), inequality (8) turns into

$$V_{2,r}(\mu) \geq \int_{F_{11}} (b_1 - x)^r d\mu(x) + 2 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x).$$

Since $b_1 > c_1$ and $c_1 - x > c_1 - 2x$ for every $x \in F_{11} \setminus \{0\}$ we deduce

$$\begin{aligned} V_{2,r}(\mu) & \geq \int_{F_{11}} (c_1 - x)^r d\mu(x) + 2 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) \\ & > 2^r \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) + 2 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) \geq 4 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x), \end{aligned}$$

a contradiction to 1.

4. We will proof that $b_n \geq 1 - c_1$.

This follows from 3. by symmetry of μ (see Lemma 2.1).

5. We will finish the proof.

From 3. and 4. we get $b_1 \in F_1$, $b_n \in F_2$ and $1 < j \leq n$. If $n = 2$, then $\beta \cap (F \setminus (F_1 \cup F_2)) = \emptyset$.

Let $n \geq 3$. Clearly, $j < n$. From the definition of j one gets $b_{j-1} \leq \max(F_1)$. If we assume $b_{j+1} < \min(F_2)$, Lemma 3.5 would imply $b_j \in F_1$, a contradiction to $b_j > \max(F_1)$. Hence, $b_{j+1} \geq \min(F_2)$. Finally let $b_j \in F \setminus (F_1 \cup F_2)$. Due to $b_n \in F_2$ we have $j < n$.

Again, Lemma 3.5 yields $\frac{b_{j-1} + b_j}{2} < \max(F_1)$ and $\frac{b_j + b_{j+1}}{2} > \min(F_2)$.

Before we can continue to investigate the properties of β we need a result for symmetric probability distributions, which is also interesting in itself.

Lemma 3.7 *Let $r > 1$, $t \in \mathbb{R}$ and let ν be a non-atomic probability distribution which is symmetric w.r.t. $t \in \mathbb{R}$. Let $\{a, b, c\} \subset \mathbb{R}$ with $a < b = \Phi_t(c) < t < c$. If $[a, c]$ has positive ν -measure, then every element of an 1-optimal quantizer of $\nu(\cdot \mid [a, c])$ is smaller or equal to t . Equality only holds, if $\nu([a, b]) = 0$.*

Proof. W.l.o.g. we may assume $t = 0$. Let $r > 1$ and $\{t'\} \in C_{1,r}(\nu(\cdot | [a, c]))$.

1. We will show, that $t' = 0$, if and only if $\nu([a, b]) = 0$.

Let $t' = 0$. Since $r > 1$, one gets from [3], Lemma 2.5, that

$$0 = - \int_{[a,0[} |x|^{r-1} d\nu(x) + \int_{]0,c]} |x|^{r-1} d\nu(x).$$

Due to $\nu(\{z\}) = 0$ for every $z \in \mathbb{R}$ and the symmetry of ν w.r.t. 0 we derive

$$\begin{aligned} 0 &= \int_{[a,b]} |x|^{r-1} d\nu(x) + \int_{[\Phi_0(c),0]} |x|^{r-1} d\nu(x) - \int_{]0,c]} |x|^{r-1} d\nu(x) \\ &= \int_{[a,b]} |x|^{r-1} d\nu(x) \geq \nu([a, b]) \cdot |b|^{r-1}, \end{aligned}$$

which yields $\nu([a, b]) = 0$. On the other hand, if $\nu([a, b]) = 0$, we get $\nu([a, c]) = \nu([b, c])$. By Proposition 3.1 we derive $t' = 0$.

2. We will show, that $t' \leq 0$.

Assume the contrary. From the symmetry of ν and Proposition 3.1 we get

$$\begin{aligned} \int_{[a,c]} |x - t'|^r d\nu(x) &= \int_{[a,b]} |x - t'|^r d\nu(x) + \int_{[b,c]} |x - t'|^r d\nu(x) \\ &> \int_{[a,b]} |x - t'|^r d\nu(x) + \int_{[b,c]} |x|^r d\nu(x). \end{aligned}$$

Due to $t' > 0$ and $b < 0$ we have for every $x \in [a, b]$ that $|x - t'| > |x|$. This yields

$$\int_{[a,c]} |x - t'|^r d\nu(x) > \int_{[a,c]} |x|^r d\nu(x),$$

a contradiction to the optimality of t' .

Remark 3.8 By the same arguments as in the proof of Lemma 3.7 one can show for $b = \Phi_t(c) < a < t < c$, that every element of an 1-optimal set of $\nu(\cdot | [a, c])$ is greater or equal to t , provided $\nu([a, c]) > 0$. Moreover, equality holds in this case if and only if $\nu([b, a]) = 0$.

Now we continue with a result for the 3-optimal quantizers of $\mu(\cdot | F)$. We denote the midpoint of F by a .

Lemma 3.9 Let $r > 1$ and $n = 3$. If $b_2 \in [a, \max(F) - \pi_{k+1}]$ then

$$\max(F_1) - \frac{b_1 + b_2}{2} \leq \frac{b_2 + b_3}{2} - \min(F_2).$$

Proof. By the construction of μ we can assume w.l.o.g. that $F = [0, 1]$. Thus we have $\max(F_1) = c_1$ and $\min(F_2) = 1 - c_1$. Assume that

$$c_1 - \frac{b_1 + b_2}{2} > \frac{b_2 + b_3}{2} - (1 - c_1). \quad (11)$$

We will show, that $b_2 < \frac{1}{2} = a$. From (11) we obtain $\frac{b_1 + b_2}{2} < 1 - \frac{b_2 + b_3}{2} = \Phi_{\frac{1}{2}}(\frac{b_2 + b_3}{2})$. From the construction of μ we know, that μ is non-atomic. By Lemma 3.6 (b) (iii) we obtain $\frac{b_2 + b_3}{2} > 1 - c_1 > \frac{1}{2}$, which yields $\Phi_{\frac{1}{2}}(\frac{b_2 + b_3}{2}) < \frac{1}{2}$. Since

$$\mu = \mu(\cdot | F) = \frac{1}{4}[\mu(\cdot | F_{11}) + \mu(\cdot | F_{12}) + \mu(\cdot | F_{21}) + \mu(\cdot | F_{22})],$$

Lemma 2.1 yields $\mu \circ \Phi_{\frac{1}{2}}^{-1} = \mu$ i.e. μ is symmetric w.r.t. $\frac{1}{2}$. By [3], Theorem 4.1 we know, that

$\mu([\frac{b_1 + b_2}{2}, \frac{b_2 + b_3}{2}]) > 0$ and $\{b_2\}$ is an 1-optimal quantizer of order r for $\mu(\cdot | [\frac{b_1 + b_2}{2}, \frac{b_2 + b_3}{2}])$. By Lemma 3.7,

we have $b_2 \leq \frac{1}{2}$. It remains to show, that the assumption $b_2 = \frac{1}{2}$ leads to a contradiction. Assume $b_2 = \frac{1}{2}$. Since $r > 1$, we know from [3], Theorem 2.4 and Theorem 4.1, that $\{b_1\}$ is the unique 1-optimal quantizer of $\mu(\cdot \mid [0, \frac{b_1+b_2}{2}])$ resp. $\{b_3\}$ is the unique 1-optimal quantizer of $\mu(\cdot \mid [\frac{b_2+b_3}{2}, 1])$. By Lemma 3.7 we have $\mu(\cdot \mid [\frac{b_1+b_2}{2}, \Phi_{\frac{1}{2}}(\frac{b_2+b_3}{2})]) = 0$,

which yields $\mu(\cdot \mid [0, \frac{b_1+b_2}{2}]) = \mu(\cdot \mid [0, \Phi_{\frac{1}{2}}(\frac{b_2+b_3}{2})])$. Using the symmetry of μ we get from [3], Lemma 2.1 (a), that $\Phi_{\frac{1}{2}}(b_1) = b_3$. Hence, $b_2 - b_1 = b_3 - b_2$, which yields $c_1 - \frac{b_1+b_2}{2} = \frac{b_2+b_3}{2} - (1 - c_1)$, a contradiction to inequality (11).

Before we can state the main result of this section, we need the following inequality.

Lemma 3.10 *Let A, B, C and D be non-negative real numbers.*

If $A + B > C + D$ and $\max(A, B) > \max(C, D)$, then

$$A^r + B^r > C^r + D^r.$$

Proof. Immediate consequence of the convexity of $s \mapsto s^r$.

Proposition 3.11 *For $r > 1$ the set β is contained in $F_1 \cup F_2$ and, moreover, $\beta \cap F_1 \neq \emptyset$ and $\beta \cap F_2 \neq \emptyset$.*

Proof. Again, we can assume w.l.o.g. that $k = 0$ and, therefore, $F = [0, 1]$. Recall, that $\beta = \{b_1, \dots, b_n\}$ with $b_1 < \dots < b_n$. Lemma 3.6 (b) yields $\beta \cap F_1 \neq \emptyset$, $\beta \cap F_2 \neq \emptyset$ and $\text{card}(\beta \cap (F \setminus (F_1 \cup F_2))) \leq 1$. It remains to show, that $\beta \cap (F \setminus (F_1 \cup F_2)) = \emptyset$.

Case 1. $n = 2$

In this case the assertion follows from Lemma 3.6 (a).

Case 2. $n = 3$

We will prove the assertion by contradiction. Assume, that $\beta \cap (F \setminus (F_1 \cup F_2)) \neq \emptyset$. By Lemma 3.5, b_2 is the only element of $\beta \cap (F \setminus (F_1 \cup F_2))$.

Since μ is symmetric w.r.t. $\frac{1}{2}$ we can assume that w.l.o.g. $b_2 \geq \frac{1}{2}$.

First, we will prove that for every $x \in F_{11}$ we have

$$\min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r \geq 2\left(\frac{c_1}{2} - x\right)^r \quad (12)$$

By Lemma 3.6 (b) (iii) and Lemma 3.9 we obtain

$$0 < c_1 - \frac{b_1 + b_2}{2} \leq \frac{b_2 + b_3}{2} - (1 - c_1). \quad (13)$$

Let $x \in F_{11}$ with $0 \leq x \leq c_1 - \frac{b_1+b_2}{2}$. Since $c_1 - x \geq \frac{b_1+b_2}{2}$ we get

$$\min_{i=1,2} |c_1 - x - b_i| = b_2 - c_1 + x. \quad (14)$$

Using (13) we have $1 - c_1 + x \leq 1 - c_1 + \frac{b_2+b_3}{2} - (1 - c_1) = \frac{b_2+b_3}{2}$, which yields

$$\min_{i=2,3} |1 - c_1 + x - b_i| = 1 - c_1 + x - b_2. \quad (15)$$

Since $r \geq 1$ the equalities (14) and (15) together with $x \leq \frac{c_1}{2}$ imply that

$$\begin{aligned} & \min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r \\ &= (b_2 - c_1 + x)^r + (1 - c_1 + x - b_2)^r \\ &\geq 2\left(\frac{1}{2}(b_2 - c_1 + x + 1 - c_1 + x - b_2)\right)^r \\ &= 2\left(\frac{1}{2}(1 - 2c_1 + 2x)\right)^r \\ &\geq 2\left(\frac{c_1}{2} + x\right)^r \geq 2\left(\frac{c_1}{2} - x\right)^r. \end{aligned}$$

Now let $x \in F_{11}$ with $c_1 - \frac{b_1+b_2}{2} \leq x \leq \frac{b_2+b_3}{2} - (1-c_1)$. We have $c_1 - x \leq \frac{b_1+b_2}{2}$. By Lemma 3.6 (b) (iii) we get $\frac{b_1+b_2}{2} < c_1$, which yields

$$b_1 < 2c_1 - b_2 \leq 2c_1 - \frac{1}{2} \leq \frac{c_1}{2}, \quad (16)$$

and, therefore

$$b_1 < \frac{c_1}{2} < \frac{2}{3}c_1 \leq c_1(1-c_2) \leq c_1 - x.$$

Hence we obtain

$$\min_{i=1,2} |c_1 - x - b_i| = c_1 - x - b_1. \quad (17)$$

On the other hand, $1 - c_1 + x \leq \frac{b_2+b_3}{2}$. Hence, equation (15) holds in this case as well. Combining (15) and (17) we obtain

$$\begin{aligned} & \min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r \\ & \geq 2\left(\frac{1}{2}(1 - b_1 - b_2)\right)^r = 2\left(\frac{1}{2} - \frac{b_1 + b_2}{2}\right)^r. \end{aligned}$$

Recall, that $x \leq \max(F_{11}) < \frac{c_1}{2}$. Together with $\frac{b_1+b_2}{2} < c_1$ and $\frac{3}{2}c_1 \leq \frac{1}{2}$ we get

$$\begin{aligned} & \min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r \\ & > 2\left(\frac{3}{2}c_1 - c_1\right)^r = 2\left(\frac{c_1}{2}\right)^r \geq 2\left(\frac{c_1}{2} - x\right)^r \end{aligned} \quad (18)$$

Finally let $x \in F_{11}$ and $\frac{b_2+b_3}{2} - (1-c_1) < x$. Obviously, equation (17) also holds in this case. Since $r > 1$, we know from [3], Theorem 2.4 and Theorem 4.1, that $\mu\left(\left[\frac{b_2+b_3}{2}, 1\right]\right) > 0$ and $\{b_3\}$ is the unique 1-optimal quantizer of $\mu(\cdot | \left[\frac{b_2+b_3}{2}, 1\right])$. Since $\mu(\cdot | [1 - c_1, 1]) = \frac{1}{2}[\mu(\cdot | F_{21}) + \mu(\cdot | F_{22})]$ we obtain from Lemma 2.1 that $\mu(\cdot | [1 - c_1, 1])$ is symmetric w.r.t. $1 - \frac{c_1}{2}$. Note, that $\mu(\cdot | [1 - c_1, 1])$ is non-atomic. Moreover, $\mu\left([1 - c_1, \frac{b_2+b_3}{2}]\right) > 0$, because otherwise Lemma 3.5 would imply $b_2 \in F_1$, a contradiction. By Remark 3.8 we get $b_3 > 1 - \frac{c_1}{2}$. Therefore,

$$b_3 > 1 - \frac{c_1}{2} > 1 - \frac{2}{3}c_1 \geq 1 - c_1 + c_1c_2 = 1 - c_1 + \max(F_{11}) \geq 1 - c_1 + x. \quad (19)$$

Hence it follows from (17) that

$$\min_{i=2,3} |1 - c_1 + x - b_i| = b_3 - (1 - c_1 + x). \quad (20)$$

Using (17) and (20) we derive

$$\begin{aligned} & \min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r \\ & \geq 2\left(\frac{1}{2}(c_1 - b_1 - (1 - b_3) + c_1 - 2x)\right)^r. \end{aligned}$$

Using (16) we get $c_1 - b_1 > \frac{c_1}{2}$. By (19) we have $1 - b_3 < \frac{c_1}{2}$. We obtain for every $x \in F_{11}$

$$\min_{i=1,2} |c_1 - x - b_i|^r + \min_{i=2,3} |1 - c_1 + x - b_i|^r > 2\left(\frac{c_1}{2} - x\right)^r \quad (21)$$

and (12) is proved. Since $\max(F_{11}) = c_1c_2 < \frac{1}{4} \leq \frac{b_1+b_2}{2} < \min(F_2)$

and $\min(F_2) < \frac{b_2+b_3}{2} < \frac{(1-c_1)+1}{2} < 1 - c_1c_2 = \min(F_{22})$ we deduce

$$\begin{aligned} V_{3,r}(\mu) &= \int_{F_{11}} |x - b_1|^r d\mu(x) + \int_{F_{12}} \min_{i=1,2} |x - b_i|^r d\mu(x) \\ &+ \int_{F_{21}} \min_{i=2,3} |x - b_i|^r d\mu(x) + \int_{F_{22}} |x - b_3|^r d\mu(x) \end{aligned}$$

Applying Lemma 2.1, we get

$$\begin{aligned}
V_{3,r}(\mu) &= \int_{F_{11}} |x - b_1|^r d\mu(x) + \int_{\Phi_{\frac{c_1}{2}}^{-1}(F_{12})} \min_{i=1,2} |\Phi_{\frac{c_1}{2}}(x) - b_i|^r d\mu(x) \\
&+ \int_{\Phi_{\frac{c_1}{2}}^{-1} \circ \Phi_{\frac{1}{2}}^{-1}(F_{21})} \min_{i=2,3} |\Phi_{\frac{1}{2}} \circ \Phi_{\frac{c_1}{2}}(x) - b_i|^r d\mu(x) + \int_{F_{22}} |x - b_3|^r d\mu(x) \\
&= \int_{F_{11}} |x - b_1|^r d\mu(x) + \int_{F_{22}} |x - b_3|^r d\mu(x) \\
&+ \int_{F_{11}} \min_{i=1,2} |c_1 - x - b_i|^r d\mu(x) + \int_{F_{11}} \min_{i=2,3} |1 - c_1 + x - b_i|^r d\mu(x).
\end{aligned}$$

Corollary 3.3 yields

$$\begin{aligned}
V_{3,r}(\mu) &\geq \int_{F_{11}} |x - a_{11}|^r d\mu(x) + \int_{F_{22}} |x - a_{22}|^r d\mu(x) \\
&+ \int_{F_{11}} \min_{i=1,2} |c_1 - x - b_i|^r d\mu(x) + \int_{F_{11}} \min_{i=2,3} |1 - c_1 + x - b_i|^r d\mu(x).
\end{aligned}$$

We know, that inequality (12) is strict on a subset of F_{11} with positive μ measure. Hence we obtain

$$V_{3,r}(\mu) > \int_{F_{11}} |x - a_{11}|^r d\mu(x) + \int_{F_{22}} |x - a_{22}|^r d\mu(x) + 2 \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x).$$

From Lemma 2.1 we get $\int_{F_{11}} |x - a_{11}|^r d\mu(x) = \int_{F_{21}} |x - a_{21}|^r d\mu(x)$ resp. $\int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) = \int_{F_{12}} \left(x - \frac{c_1}{2}\right)^r d\mu(x)$. Thus we conclude

$$\begin{aligned}
V_{3,r}(\mu) &> \int_{F_{21}} |x - a_{21}|^r d\mu(x) + \int_{F_{22}} |x - a_{22}|^r d\mu(x) \\
&+ \int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) + \int_{F_{12}} \left(x - \frac{c_1}{2}\right)^r d\mu(x).
\end{aligned}$$

By $a_1 = \frac{c_1}{2}$ we have $\int_{F_{11}} \left(\frac{c_1}{2} - x\right)^r d\mu(x) + \int_{F_{12}} \left(x - \frac{c_1}{2}\right)^r d\mu(x) = \int_{F_1} |x - a_1|^r d\mu(x)$. Hence we deduce

$$\begin{aligned}
V_{3,r}(\mu) &> \int_{F_{21}} |x - a_{21}|^r d\mu(x) + \int_{F_{22}} |x - a_{22}|^r d\mu(x) + \int_{F_1} |x - a_1|^r d\mu(x) \\
&= \Psi_{\{a_1, a_{21}, a_{22}\}}(\mu) \geq V_{3,r}(\mu),
\end{aligned}$$

a contradiction.

Case 3. $n \geq 4$

We will prove the assertion by contradiction.

Assume that $\beta \cap (F \setminus (F_1 \cup F_2)) \neq \emptyset$. By Lemma 3.6 we have $\{b_j\} = \beta \cap (F \setminus (F_1 \cup F_2))$ with j defined in (4) and $\frac{b_{j-1} + b_j}{2} < c_1$. Let $\alpha = \{a_{11}, a_{12}, a_{21}, a_{22}\}$ be the set of midpoints of the sets F_{11}, F_{12}, F_{21} and F_{22} resp. By the symmetry of μ (Lemma 2.1) we get

$$\begin{aligned}
&\int_F \min_{a \in \alpha} |x - a|^r d\mu(x) \\
&= \int_{F_{11} \cup F_{12} \cup F_{21} \cup F_{22}} \min_{a \in \alpha} |x - a|^r d\mu(x) \\
&= 4 \int_{F_{11}} |x - a_{11}|^r d\mu(x) \\
&= 4 \int_{F_{1111} \cup F_{1112} \cup F_{1121} \cup F_{1122}} |x - a_{11}|^r d\mu(x). \tag{22}
\end{aligned}$$

Since $a_{11} = \frac{\pi_2}{2}$ we obtain

$$\max_{x \in F_{1111}} |x - a_{11}| = \frac{\pi_2}{2} = \max_{x \in F_{1122}} |x - a_{11}| \quad (23)$$

and

$$\max_{x \in F_{1112}} |x - a_{11}| = \frac{\pi_2}{2} - \pi_3 + \pi_4 = \max_{x \in F_{1121}} |x - a_{11}|. \quad (24)$$

Combining (22), (23) and (24) we get

$$\begin{aligned} & \int_F \min_{a \in \alpha} |x - a|^r d\mu(x) \\ & \leq 4 \cdot 2\mu(F_{1111}) \left(\left(\frac{\pi_2}{2}\right)^r + \left(\frac{\pi_2}{2} - \pi_3 + \pi_4\right)^r \right) \\ & = \frac{c_1^r}{2} \left(\left(\frac{c_2}{2}\right)^r + \left(\frac{c_2}{2} - c_2c_3 + c_2c_3c_4\right)^r \right) \\ & \leq \frac{c_1^r}{4} (c_2^r + (c_2 - 2c_2c_3 + 2c_2c_3c_4)^r). \end{aligned} \quad (25)$$

Let $x \in F_{122}$. If $\frac{b_{j-1}+b_j}{2} \notin F_{122}$ we have

$$\min_{b \in \beta} |x - b| = b_j - x \geq b_j - c_1 \geq \max(0, b_j - c_1 - \pi_3).$$

If $\frac{b_{j-1}+b_j}{2} \in F_{122}$ and $b_j - c_1 > \pi_3$, then, $b_{j-1} < \min F_{122}$ and

$$\min_{b \in \beta} |x - b| \geq \min(b_j - x, x - b_{j-1}) \geq \min(b_j - c_1, c_1 - \pi_3 - b_{j-1})$$

Since $c_1 \geq \frac{1}{2}(b_{j-1} + b_j)$ and therefore $c_1 - b_{j-1} \geq b_j - c_1$, we obtain

$$\min(b_j - c_1, c_1 - \pi_3 - b_{j-1}) \geq \max(0, b_j - c_1 - \pi_3).$$

If $b_j - c_1 \leq \pi_3$, then

$$\min_{b \in \beta} |x - b| \geq 0 = \max(0, b_j - c_1 - \pi_3).$$

Combining these inequalities we obtain

$$\min_{x \in F_{122}} \min_{b \in \beta} |x - b| \geq \max(0, b_j - c_1 - \pi_3). \quad (26)$$

Similarly one derives

$$\min_{x \in F_{211}} \min_{b \in \beta} |x - b| \geq \max(0, 1 - c_1 - b_j - \pi_3), \quad (27)$$

$$\min_{x \in F_{121}} \min_{b \in \beta} |x - b| \geq \max(0, b_j - c_1 - \pi_2), \quad (28)$$

and

$$\min_{x \in F_{212}} \min_{b \in \beta} |x - b| \geq \max(0, 1 - c_1 - b_j - \pi_2). \quad (29)$$

Combining (26), (27), (28), and (29) we calculate

$$\begin{aligned} V_{n,r}(\mu) &= \int_F \min_{b \in \beta} |x - b|^r d\mu(x) \\ &> \int_{F_{121} \cup F_{122} \cup F_{211} \cup F_{212}} \min_{b \in \beta} |x - b|^r d\mu(x) \\ &\geq \frac{1}{8} \left((\max(0, b_j - c_1 - \pi_3))^r + (\max(0, 1 - c_1 - b_j - \pi_3))^r \right) \\ &\quad + \frac{1}{8} \left((\max(0, b_j - c_1 - \pi_2))^r + (\max(0, 1 - c_1 - b_j - \pi_2))^r \right). \end{aligned}$$

Due to $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$ we get

$$\begin{aligned} V_{n,r}(\mu) &> \frac{1}{8} \left(2 \left(\frac{1 - 2c_1 - 2\pi_3}{2} \right)^r + 2 \left(\frac{1 - 2c_1 - 2\pi_2}{2} \right)^r \right) \\ &\geq \frac{c_1^r}{4} \left(\left(\frac{1}{2} - c_2 c_3 \right)^r + \left(\frac{1}{2} - c_2 \right)^r \right). \end{aligned} \quad (30)$$

Now let $A = \frac{1}{2} - c_2 c_3$ and $B = \frac{1}{2} - c_2$, resp. $C = c_2$ and $D = c_2 - 2c_2 c_3 + 2c_2 c_3 c_4$. Note, that A, B, C and D are non-negative real numbers. Moreover,

$$\begin{aligned} &A + B - (C + D) \\ &= \left(\frac{1}{2} - c_2 c_3 \right) + \left(\frac{1}{2} - c_2 \right) - (c_2 + (c_2 - 2c_2 c_3 + 2c_2 c_3 c_4)) \\ &= 1 - c_2(3 + c_3(2c_4 - 1)) > 1 - 3c_2 \geq 0, \end{aligned}$$

and $\max(A, B) = A > C = \max(C, D)$. By Lemma 3.10 we get

$$\left(\frac{1}{2} - c_2 c_3 \right)^r + \left(\frac{1}{2} - c_2 \right)^r > c_2^r + (c_2 - 2c_2 c_3 + 2c_2 c_3 c_4)^r.$$

Combining the last inequality with (30) and (25) we obtain

$$\int_F \min_{a \in \alpha} |x - a|^r d\mu(x) < \int_F \min_{b \in \beta} |x - b|^r d\mu(x),$$

which contradicts the optimality of β .

Hence in all cases we have $\beta \cap (F \setminus (F_1 \cup F_2)) = \emptyset$ and the proof of Proposition 3.11 is complete.

Remark 3.12 By checking all possibilities, it is easy to see, that

$$\min_{i=1,2} |c_1 - x - b_i| + \min_{i=2,3} |1 - c_1 + x - b_i| \geq c_1 - 2x$$

holds for every $x \in F_{11}$. Moreover, all other arguments in the proof of Proposition 3.11 (especially the application of Corollary 3.3) also work for $r = 1$. Hence, Proposition 3.11 is also valid for $r = 1$.

4 Determination of the optimal quantizers and the optimal quantization error

In this section we will determine the n -optimal quantizers for μ and derive a formula for the quantization error. To this end, we parallel the approach in the proofs of Lemma 3.3 - 3.6 and Proposition 3.7 in [6], making amendments to the case $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$ where necessary.

For a finite non-empty set $\alpha \subset \mathbb{R}$ and $a \in \alpha$ let

$$W(a | \alpha) = \{x \in \mathbb{R} : |x - a| = \min_{b \in \alpha} |x - b|\} \quad (31)$$

be the Voronoi cell generated by $a \in \alpha$. In this section β denotes an arbitrary finite non-empty subset of \mathbb{R} .

Lemma 4.1 *Let $k \in \mathbb{N}$ and $\beta \subset E_{k-1}$. Let $F \in \mathcal{D}_{k-1}$ with $\beta \cap F \neq \emptyset$. Then*

$$\min_{b \in \beta \cap F} |x - b| = \min_{b \in \beta} |x - b| \quad (32)$$

holds for every $x \in F$. If, in addition, $\beta \cap H \neq \emptyset$ for every $H \in \mathcal{D}_{k-1}$, then $\mu(\cdot | \bigcup_{b \in \beta \cap F} W(b | \beta)) = \mu(\cdot | F)$.

Proof. Equation (32) holds since $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$ implies $\min_{x \in F, y \in H} |x - y| \geq \text{diam}(F)$ for every $F, H \in \mathcal{D}_{k-1}$ with $F \neq H$.

Now assume $\beta \cap H \neq \emptyset$ for every $H \in \mathcal{D}_{k-1}$. Combining (31) with (32) we know, that F is a subset of $\bigcup_{b \in \beta \cap F} W(b | \beta)$. Since $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$ and μ is non-atomic, we get $\mu(H \cap \bigcup_{b \in \beta \cap F} W(b | \beta)) = 0$ for every $H \in \mathcal{D}_{k-1} \setminus \{F\}$. Together with $\mu(E_{k-1}) = 1$ one derives $\mu(\cdot | \bigcup_{b \in \beta \cap F} W(b | \beta)) = \mu(\cdot | F)$.

For $n \in \mathbb{N}$ we call a set n -optimal, if it is an n -optimal set of order r for μ .

Lemma 4.2 *For $k, n \in \mathbb{N}$ let $\beta \subset E_{k-1}$ have exactly n elements. Then the following implications hold.*

- (i) *If there exist $G, H \in \mathcal{D}_{k-1}$ with $\text{card}(\beta \cap G) = 0$ and $\text{card}(\beta \cap H) > 2$, then β is not n -optimal.*
- (ii) *If $\beta \cap F \neq \emptyset$ for every $F \in \mathcal{D}_{k-1}$ and if there are $G, H \in \mathcal{D}_{k-1}$ with $\text{card}(\beta \cap G) = 1$ and $\text{card}(\beta \cap H) > 2$, then β is not n -optimal.*

Proof. (i) For $k = 1$ there is nothing to prove. Now let $k > 1$ and let $G, H \in \mathcal{D}_{k-1}$ be as in (i). Set $b_1 = \min(\beta \cap H)$, $b_3 = \max(\beta \cap H)$ and choose $b_2 \in \beta \cap H$ with $b_1 < b_2 < b_3$. For the midpoint a of G define

$$\gamma = (\beta \setminus \{b_2\}) \cup \{a\}.$$

Like in the proof of [6], Lemma 3.4 we get

$$\int_{E_{k-1} \setminus (G \cup H)} \min_{b \in \beta} |x - b|^r d\mu(x) \geq \int_{E_{k-1} \setminus (G \cup H)} \min_{c \in \gamma} |x - c|^r d\mu(x) \quad (33)$$

Since $\beta \cap G = \emptyset$, $\beta \subset E_{k-1}$ and $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$ we have

$$\int_G \min_{b \in \beta} |x - b|^r d\mu(x) \geq \mu(G)(\text{diam}(G))^r = \mu(H)(\text{diam}(H))^r. \quad (34)$$

From the definition of γ we obtain

$$\int_G \min_{c \in \gamma} |x - c|^r d\mu(x) \leq \int_G |x - a|^r d\mu(x) < \left(\frac{1}{2} \text{diam}(G)\right)^r \mu(G). \quad (35)$$

Moreover, we derive

$$\begin{aligned} & \int_H \min_{c \in \gamma} |x - c|^r d\mu(x) - \int_H \min_{b \in \beta} |x - b|^r d\mu(x) \\ & \leq \int_{[b_1, b_3]} (\min(|x - b_1|, |x - b_3|))^r d\mu(x) \\ & \leq \mu(H) \left(\frac{1}{2} \text{diam}(H)\right)^r. \end{aligned} \quad (36)$$

Combining inequalities (34), (35) and (36) yields

$$\begin{aligned} & \int_G \min_{b \in \beta} |x - b|^r d\mu(x) \geq \mu(H)(\text{diam}(H))^r \\ & \geq \mu(G) \left(\frac{1}{2} \text{diam}(G)\right)^r + \mu(H) \left(\frac{1}{2} \text{diam}(H)\right)^r \\ & > \int_G \min_{c \in \gamma} |x - c|^r d\mu(x) + \int_H \min_{c \in \gamma} |x - c|^r d\mu(x) - \int_H \min_{b \in \beta} |x - b|^r d\mu(x). \end{aligned} \quad (37)$$

The combination of (33) and (37) shows, that β is not n -optimal.

(ii) Again, if $k = 1$ there is nothing to prove. So let $k > 1$ and $G, H \in \mathcal{D}_{k-1}$ be as in (ii). Set $\{b_0\} = G \cap \beta$. Let $b_1, b_2, b_3 \in \beta \cap H$ with $b_1 < b_2 < b_3$. Let a_1 and a_2 be the midpoints of G_1 and G_2 and a'_1 and a'_2 the midpoints of H_1 and H_2 respectively. Set

$$\gamma = (\beta \setminus \{b_0, b_1, b_2, b_3\}) \cup \{a_1, a_2, a'_1, a'_2\}.$$

Like in the proof of [6], Lemma 3.5 we get

$$\int_{E_{k-1} \setminus (G \cup H)} \min_{b \in \beta} |x - b|^r d\mu(x) = \int_{E_{k-1} \setminus (G \cup H)} \min_{c \in \gamma} |x - c|^r d\mu(x) \quad (38)$$

By Lemma 4.1 we have

$$\int_G \min_{b \in \beta} |x - b|^r d\mu(x) = \int_G |x - b_0|^r d\mu(x). \quad (39)$$

By Corollary 3.3 we know that

$$\int_G |x - b_0|^r d\mu(x) \geq \int_G |x - a|^r d\mu(x), \quad (40)$$

where a is the midpoint of G . It is easy to show that

$$\int_G |x - a|^r d\mu(x) \geq \frac{1}{2} \mu(G) (\text{diam}(G))^r \left(\left(\frac{1}{2} - c_k \right)^r + \left(\frac{1}{2} - c_k c_{k+1} \right)^r \right). \quad (41)$$

Combining (39) and (40) with (41) we derive

$$\int_G \min_{b \in \beta} |x - b|^r d\mu(x) \geq \frac{1}{2} \mu(G) (\text{diam}(G))^r \left(\left(\frac{1}{2} - c_k \right)^r + \left(\frac{1}{2} - c_k c_{k+1} \right)^r \right). \quad (42)$$

Due to $a_1, a_2 \in \gamma$ we obtain

$$\begin{aligned} \int_G \min_{c \in \gamma} |x - c|^r d\mu(x) &\leq \int_{G_1} |x - a_1|^r d\mu(x) + \int_{G_2} |x - a_2|^r d\mu(x) \\ &\leq \frac{1}{2} \mu(G) (\text{diam}(G))^r \left(\left(\frac{c_k}{2} - c_k c_{k+1} + c_k c_{k+1} c_{k+2} \right)^r + \left(\frac{c_k}{2} \right)^r \right). \end{aligned} \quad (43)$$

In the same way we derive

$$\int_H \min_{c \in \gamma} |x - c|^r d\mu(x) \leq \frac{1}{2} \mu(H) (\text{diam}(H))^r \left(\left(\frac{c_k}{2} - c_k c_{k+1} + c_k c_{k+1} c_{k+2} \right)^r + \left(\frac{c_k}{2} \right)^r \right). \quad (44)$$

Combining (42), (43) and (44) we get

$$\begin{aligned} &\int_{G \cup H} \min_{b \in \beta} |x - b|^r d\mu(x) - \int_{G \cup H} \min_{c \in \gamma} |x - c|^r d\mu(x) \\ &\geq \frac{1}{2} \mu(G) (\text{diam}(G))^r \left[\left(\frac{1}{2} - c_k \right)^r + \left(\frac{1}{2} - c_k c_{k+1} \right)^r \right. \\ &\quad \left. - 2 \left(\frac{c_k}{2} - c_k c_{k+1} + c_k c_{k+1} c_{k+2} \right)^r - 2 \left(\frac{c_k}{2} \right)^r \right] \\ &\geq \frac{1}{2} \mu(G) (\text{diam}(G))^r \left[\left(\frac{1}{2} - c_k \right)^r + \left(\frac{1}{2} - c_k c_{k+1} \right)^r \right. \\ &\quad \left. - (c_k - 2c_k c_{k+1} + 2c_k c_{k+1} c_{k+2})^r - c_k^r \right]. \end{aligned} \quad (46)$$

Now let $A = \frac{1}{2} - c_k$ and $B = \frac{1}{2} - c_k c_{k+1}$ resp. $C = c_k$ and $D = c_k - 2c_k c_{k+1} + 2c_k c_{k+1} c_{k+2}$. Obviously A, B, C and D are non-negative real numbers and we have

$$A > B > C > D.$$

Moreover

$$\begin{aligned} A + B - (C + D) &= \frac{1}{2} - c_k + \frac{1}{2} - c_k c_{k+1} - (c_k - 2c_k c_{k+1} + 2c_k c_{k+1} c_{k+2}) - c_k \\ &= 1 - 3c_k + c_k c_{k+1} (1 - 2c_{k+2}) > 1 - 3c_k \geq 0. \end{aligned}$$

Since $r \geq 1$ we get by Lemma 3.10 that

$$\left(\frac{1}{2} - c_k\right)^r + \left(\frac{1}{2} - c_k c_{k+1}\right)^r - (c_k - 2c_k c_{k+1} + 2c_k c_{k+1} c_{k+2})^r - c_k^r > 0. \quad (47)$$

Combining inequality (47), (46) and (45) we get together with (38)

$$\int \min_{b \in \beta} |x - b|^r d\mu(x) > \int \min_{c \in \gamma} |x - c|^r d\mu(x).$$

Hence β is not n -optimal.

Lemma 4.3 *Let $k \in \mathbb{N}_0, n \in \mathbb{N}$ and $2^k \leq n$. Let β be n -optimal. Then $\beta \subset E_k$ and $F \cap \beta \neq \emptyset$ for every $F \in \mathcal{D}_k$.*

Proof. **1.** Let $k = 0$. By [3], Remark 4.6 (a) we know, that $\beta \subset E_0$. Since $\mathcal{D}_0 = \{E_0\}$ we have $F \cap \beta \neq \emptyset$ for every $F \in \mathcal{D}_0$.

2. Let $k = 1$. Proposition 3.11 yields $\beta \subset E_1$ and $(\beta \cap F) \neq \emptyset$ for every $F \in \mathcal{D}_1$.

3. Let $k \geq 2$ and assume, that for a $j < k$ we have $\beta \subset E_j$. We will show, that $\beta \subset E_{j+1}$ and for every $F \in \mathcal{D}_{j+1}$ we have $\text{card}(\beta \cap F) \geq 1$. Assume, that there is a $G \in \mathcal{D}_j$ with $G \cap \beta = \emptyset$. Then we have a $H \in \mathcal{D}_j$ with $\text{card}(H \cap \beta) > 2$, because otherwise we would get

$$n = \sum_{F \in \mathcal{D}_j} \text{card}(\beta \cap F) \leq 2(2^j - 1) < 2^{j+1} \leq n, \quad (48)$$

a contradiction. By Lemma 4.2 (i) this contradicts the optimality of β . Thus we have $\text{card}(\beta \cap F) \geq 1$ for all $F \in \mathcal{D}_j$. Suppose, that a $G \in \mathcal{D}_j$ contains only one point. Like in (48) we get a $H \in \mathcal{D}_j$ with $\text{card}(H \cap \beta) \geq 3$. By Lemma 4.2 (ii) this contradicts the optimality of β . Therefore, $\text{card}(\beta \cap F) \geq 2$ for all $F \in \mathcal{D}_j$. By [3], Theorem 4.1 we know that for every $F \in \mathcal{D}_j$ the set $\beta \cap F$ is an $\text{card}(\beta \cap F)$ -optimal quantizer for $\mu(\cdot | \bigcup_{b \in \beta \cap F} W(b | \beta))$. By Lemma 4.1 we obtain $\mu(\cdot | \bigcup_{b \in \beta \cap F} W(b | \beta)) = \mu(\cdot | F)$. Hence, Proposition 3.11 and Remark 3.12 yield $\beta \subset E_{j+1}$ and for every $F \in \mathcal{D}_{j+1}$ we have $\text{card}(\beta \cap F) \geq 1$.

4. If one repeats the procedure in 3. till $j = k - 1$, the assertion is proved.

The distribution μ is defined by the sequence $(c_l)_{l=1}^\infty$. For $k \in \mathbb{N}$ let $\mu^{(k)}$ be the dyadic homogeneous Cantor measure defined by the sequence $(c_{l+k-1})_{l=1}^\infty$. Clearly, $\mu^{(1)} = \mu$. Recall $\pi_k = \prod_{i=1}^k c_i$. Now we can state the main result of this paper.

Theorem 4.4 *Let $r > 1$ and $n \geq 2$. Let $k \in \mathbb{N}$ and $j \in \mathbb{N}_0$ such that $n = 2^k + j$, $0 \leq j < 2^k$. Let β be n -optimal. Then β consists of the midpoints of some $2^k - j$ basic intervals of order k and the midpoints of the $2j$ basic intervals of order $(k + 1)$ contained in the other j basic intervals of order k . Moreover,*

$$V_{n,r}(\mu) = \frac{\pi_k^r}{2^k} [(2^k - j)V_{1,r}(\mu^{(k+1)}) + j \cdot c_{k+1}^r V_{1,r}(\mu^{(k+2)})]. \quad (49)$$

Proof. By Lemma 4.3 we have $\beta \subset E_k$ and $\text{card}(F \cap \beta) \geq 1$ for every $F \in \mathcal{D}_k$. Assume, that a $G \in \mathcal{D}_k$ exists, with $\text{card}(\beta \cap G) > 2$. Due to $n < 2^{k+1}$ a $H \in \mathcal{D}_k$ exists, with $\text{card}(H \cap \beta) = 1$. By Lemma 4.2 (ii) this contradicts the optimality of β . Hence, for all $F \in \mathcal{D}_k$ we get $1 \leq \text{card}(\beta \cap F) \leq 2$. Let $F \in \mathcal{D}_k$ be arbitrary. Applying [3], Theorem 4.1 in combination with Lemma 4.1, we obtain, that $\beta \cap F$ is an $\text{card}(\beta \cap F)$ -optimal quantizer of order r for $\mu(\cdot | F)$. If $\text{card}(\beta \cap F) = 1$, then we get from Corollary 3.3, that $\beta \cap F$ consists of the midpoint of F . If $\text{card}(\beta \cap F) = 2$, then Lemma 3.6 yields $\text{card}(\beta \cap F_1) = 1$ and $\text{card}(\beta \cap F_2) = 1$. Like in the proof of Lemma 4.1, one gets in this case that $\mu(\cdot | F_i) = \mu(\cdot | \bigcup_{b \in \beta \cap F_i} W(b | \beta))$ for every $i \in \{1, 2\}$. Applying [3], Theorem 4.1, we obtain that for every $i \in \{1, 2\}$ the set $\beta \cap F_i$ is an $\text{card}(\beta \cap F_i)$ -optimal quantizer for $\mu(\cdot | F_i)$. By Corollary 3.3 we get, that $\beta \cap F_1$ consists of the midpoint of F_1 and $\beta \cap F_2$ consists of the midpoint of F_2 . This yields the first part of the assertion. For the optimal quantization error we deduce from the first part that

$$V_{n,r}(\mu) = \sum_{F \in \mathcal{D}_k; \text{card}(\beta \cap F)=1} \mu(F) V_{1,r}(\mu(\cdot | F)) + \sum_{F \in \mathcal{D}_k; \text{card}(\beta \cap F)=2} \mu(F) V_{2,r}(\mu(\cdot | F)). \quad (50)$$

For $l \in \mathbb{N}$ let

$$\mathbb{R} \ni x \xrightarrow{H_l} x \cdot \pi_l \in \mathbb{R}.$$

From the construction of μ we get $\mu(\cdot | H_k([0, 1])) = \mu^{(k+1)} \circ H_k^{-1}$. By Lemma 2.1 we have

$$\mu(\cdot | F) = \mu(\cdot | H_k([0, 1])) \circ \Theta_{\min(F)}^{-1}$$

for every $F \in \mathcal{D}_k$. By [3], Lemma 3.2 (a) we obtain

$$V_{1,r}(\mu(\cdot | F)) = V_{1,r}(\mu^{(k+1)} \circ H_k^{-1}) = \pi_k^r V_{1,r}(\mu^{(k+1)}) \quad (51)$$

for every $F \in \mathcal{D}_k$. Further note, that

$$\begin{aligned} V_{2,r}(\mu(\cdot | F)) &= \frac{1}{2} V_{1,r}(\mu(\cdot | F_1)) + \frac{1}{2} V_{1,r}(\mu(\cdot | F_2)) \\ &= V_{1,r}(\mu(\cdot | F_1)) = V_{1,r}(\mu^{(k+2)} \circ H_{k+1}^{-1}) = \pi_{k+1}^r V_{1,r}(\mu^{(k+2)}) \end{aligned} \quad (52)$$

for every $F \in \mathcal{D}_k$. Using (51) and (52) we derive from equation (50), that

$$\begin{aligned} V_{n,r}(\mu) &= \frac{1}{2^k} [(2^k - j) \pi_k^r V_{1,r}(\mu^{(k+1)}) + j \pi_{k+1}^r V_{1,r}(\mu^{(k+2)})] \\ &= \frac{\pi_k^r}{2^k} [(2^k - j) V_{1,r}(\mu^{(k+1)}) + j \cdot c_{k+1}^r V_{1,r}(\mu^{(k+2)})] \end{aligned}$$

and Theorem 4.4 is proved.

Remark 4.5 For $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{4}$, the first part of Theorem 4.4 was proved by Kesseböhmer and Zhu in [6], Proposition 3.6.

Remark 4.6 For $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{4}$ and $r > 1$, Kesseböhmer and Zhu (cf. [6], Remark 3.8 (1)) mentioned, that $C_{n,r}(\mu) \subset C_{n,1}(\mu)$, but $C_{n,1}(\mu) \setminus C_{n,r}(\mu) \neq \emptyset$. By checking the proof of Theorem 4.4 it is easy to see, that this fact also holds under the weaker assumption $\sup_{k \in \mathbb{N}} c_k \leq \frac{1}{3}$.

Although it seems, that no explicit reference exists for the next result, Kesseböhmer and Zhu derived similar inequalities for $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{4}$ (cf. Proof of Theorem 1.6 (3) in [6]).

Corollary 4.7 *Let $k, n \in \mathbb{N}$ with $2^k \leq n < 2^{k+1}$. Then*

$$2^{-k} [2^{k+1} - n + (n - 2^k) c_{k+1}^r] \left(\frac{1}{6}\right)^r \pi_k^r < V_{n,r}(\mu) < \left(\frac{1}{2}\right)^r \pi_k^r. \quad (53)$$

Proof. Since $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$, we obtain for every $l \in \mathbb{N}$ that $\left(\frac{1}{6}\right)^r < V_{1,r}(\mu^{(l)}) < \left(\frac{1}{2}\right)^r$. Then, inequality (53) is an easy consequence of equation (49).

5 Quantization dimension and quantization coefficient

In this section we will prove a characterization for the existence of the quantization dimension. Moreover, if the quantization dimension exists, we will prove under weak assumptions, that the quantization coefficient does not exist.

The lower resp. upper quantization dimension of μ of order r is defined as

$$\underline{D}_r(\mu) := \liminf_{n \rightarrow \infty} \frac{r \log(n)}{-\log(V_{n,r}(\mu))}$$

resp.

$$\overline{D}_r(\mu) := \limsup_{n \rightarrow \infty} \frac{r \log(n)}{-\log(V_{n,r}(\mu))}.$$

If the two numbers agree, $D_r(\mu) = \underline{D}_r(\mu) = \overline{D}_r(\mu)$ is called the quantization dimension of μ of order r . From [3], Theorem 12.18 one gets, that the quantization dimension always exist for so-called regular probabilities of dimension D , where $D = D_r$. For example the classical Cantor distribution is regular of dimension $D_r = \frac{\log(2)}{\log(3)}$ (cf. [3], Example 12.10).

For general dyadic homogeneous Cantor measures, the quantization dimension need not exist. This was first shown by Lindsay [7], Example 5.5 (see also [6], Theorem 1.5 (1)). Under the condition $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{4}$ Kesseböhmer and Zhu (cf. [6], Theorem 1.6 (3)) proved a characterization for the existence of the quantization dimension. Moreover they showed for $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{2}$, that

$$\overline{D}_r(\mu) = \limsup_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)} \quad (54)$$

(cf. [6], Theorem 1.6 (1)), resp.

$$\underline{D}_r(\mu) \leq \liminf_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)} \quad (55)$$

(cf. [6], Proposition 3.1).

In the following proposition we will sharpen their results under the condition $\sup_{j \in \mathbb{N}} c_j \leq \frac{1}{3}$.

Proposition 5.1

$$\underline{D}_r(\mu) = \liminf_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)} \leq \limsup_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)} = \overline{D}_r(\mu)$$

Proof. Due to (54) and (55) it remains to show, that

$$\underline{D}_r(\mu) \geq \liminf_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)}. \quad (56)$$

For $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}_0$, $j(n) \in [0, 2^{k(n)}[$ with $n = 2^{k(n)} + j(n)$. Since $(V_{n,r}(\mu))_{n \in \mathbb{N}}$ is decreasing, we obtain

$$\begin{aligned} \underline{D}_r(\mu) &= \liminf_{n \rightarrow \infty} \frac{\log(2^{k(n)} + j(n))}{-\frac{1}{r} \log(V_{2^{k(n)} + j(n), r}(\mu))} \geq \liminf_{k \rightarrow \infty} \frac{\log(2^k)}{-\frac{1}{r} \log(V_{2^{k+1}, r}(\mu))} \\ &= \liminf_{k \rightarrow \infty} \left(\frac{\log(2^{k+1})}{-\frac{1}{r} \log(V_{2^{k+1}, r}(\mu))} + \frac{\log(\frac{1}{2})}{-\frac{1}{r} \log(V_{2^{k+1}, r}(\mu))} \right). \end{aligned}$$

Note, that $\lim_{n \rightarrow \infty} V_{n,r}(\mu) = 0$ (cf. [3], Lemma 6.1). This yields

$$\underline{D}_r(\mu) \geq \liminf_{k \rightarrow \infty} \frac{\log(2^k)}{-\frac{1}{r} \log(V_{2^k, r}(\mu))}. \quad (57)$$

Using Corollary 4.7, it is straightforward to check, that

$$\liminf_{k \rightarrow \infty} \frac{\log(2^k)}{-\frac{1}{r} \log(V_{2^k, r}(\mu))} = \liminf_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)}. \quad (58)$$

The combination of (57) and (58) proves inequality (56).

Remark 5.2 As an immediate consequence of Proposition 5.1 one gets, that the quantization dimension $D_r(\mu)$ exists, if and only if $(\frac{1}{k} \sum_{i=1}^k \log(c_i))_{k \in \mathbb{N}}$ converges in \mathbb{R} . If one of these equivalent conditions is satisfied, then

$$D_r(\mu) = \lim_{k \rightarrow \infty} \frac{\log(2)}{-\frac{1}{k} \sum_{i=1}^k \log(c_i)}.$$

If $D_r = D_r(\mu)$ exists and $(n^{\frac{r}{D_r}} V_{n,r}(\mu))_{n \in \mathbb{N}}$ converges towards a value in $]0, \infty[$, we call $\lim_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu)$ the r -th quantization coefficient of the distribution μ . Next we will show, that under weak assumptions this sequence does not converge, i.e. the quantization coefficient does not exist.

Proposition 5.3 *Suppose $(\frac{1}{k} \sum_{i=1}^k \log(c_i))_{k \in \mathbb{N}}$ converges in \mathbb{R} . Then D_r exists and*

- (i) $\limsup_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) < \infty$, if and only if $\limsup_{k \rightarrow \infty} 2^{\frac{k}{D_r}} \pi_k < \infty$
- (ii) $\liminf_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) > 0$, if and only if $\liminf_{k \rightarrow \infty} 2^{\frac{k}{D_r}} \pi_k > 0$
- (iii) if $0 < \liminf_{k \rightarrow \infty} 2^{\frac{k}{D_r}} \pi_k$ and $\limsup_{k \rightarrow \infty} 2^{\frac{k}{D_r}} \pi_k < \infty$, then

$$0 < \liminf_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) < \limsup_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) < \infty. \quad (59)$$

Proof. By Remark 5.2 we know that D_r exists. (i) and (ii) follow immediately from Corollary 4.7. (iii) Let $k \geq 2$ and $n_k = 2^k$ resp. $m_k = (\frac{5}{4})2^k$. Applying (49) we get

$$n_k^{\frac{r}{D_r}} V_{n_k,r}(\mu) = (2^k)^{\frac{r}{D_r}} \pi_k^r V_{1,r}(\mu^{(k+1)}) \quad (60)$$

resp.

$$\begin{aligned} m_k^{\frac{r}{D_r}} V_{m_k,r}(\mu) &\geq \left(\left(\frac{5}{4}\right)2^k\right)^{\frac{r}{D_r}} \frac{1}{2^k} [(2^{k+1} - \left(\frac{5}{4}\right)2^k) \pi_k^r V_{1,r}(\mu^{(k+1)})] \\ &= \frac{3}{4} \left(\frac{5}{4}\right)^{\frac{r}{D_r}} (2^k)^{\frac{r}{D_r}} \pi_k^r V_{1,r}(\mu^{(k+1)}). \end{aligned} \quad (61)$$

Note, that

$$\frac{r}{D_r} = \frac{r \lim_{k \rightarrow \infty} \left(-\frac{1}{k} \sum_{i=1}^k \log(c_i)\right)}{\log(2)} \geq \frac{\log(3)}{\log(2)}.$$

Combining this with (60) and (61) we deduce

$$\frac{m_k^{\frac{r}{D_r}} V_{m_k,r}(\mu)}{n_k^{\frac{r}{D_r}} V_{n_k,r}(\mu)} \geq \frac{3}{4} \left(\frac{5}{4}\right)^{\frac{r}{D_r}} \geq \frac{3}{4} \left(\frac{5}{4}\right)^{\frac{\log(3)}{\log(2)}} > 1. \quad (62)$$

Due to the assumptions in (iii) the combination of (62) with (i) and (ii) yields

$$\begin{aligned} 0 &< \liminf_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) \leq \liminf_{k \rightarrow \infty} n_k^{\frac{r}{D_r}} V_{n_k,r}(\mu) \\ &< \limsup_{k \rightarrow \infty} m_k^{\frac{r}{D_r}} V_{m_k,r}(\mu) \leq \limsup_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) < \infty. \end{aligned}$$

Thus (iii) is proved.

Remark 5.4 Clearly (59) holds, if $c_i = c \in]0, \frac{1}{3}]$ for every $i \in \mathbb{N}$. It remains an open question, if $\liminf_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu) < \limsup_{n \rightarrow \infty} n^{\frac{r}{D_r}} V_{n,r}(\mu)$ still holds, if we drop some of the assumptions in Proposition 5.3 (iii).

6 The self-similar case

If $c_k = c \in]0, \frac{1}{2}]$ for every $k \in \mathbb{N}$, the dyadic homogeneous Cantor distribution μ becomes self-similar with contracting parameter c . Graf and Luschgy have shown for this type of measure, that the quantization dimension $D_r(\mu)$ equals $D = D(c) = \frac{\log(2)}{-\log(c)}$ for every $r > 0$ (cf. [4], Remark 5.13(a)).

In this section we obtain for $c \in]0, \frac{1}{3}]$ an explicit formula for the quantization error and can characterize the set of all accumulation points of the sequence $(n^{\frac{r}{D}} V_{n,r}(\mu))_{n \in \mathbb{N}}$.

Remark 6.1 If μ denotes a self-similar Cantor distribution with contracting parameter $c \in]0, \frac{1}{3}]$, then we have $\mu^{(k)} = \mu$ for all k and, therefore, we obtain

$$V_{n,r}(\mu) = \left(\frac{c^r}{2}\right)^k V_{1,r}(\mu)(c^r(n - 2^k) + 2^{k+1} - n) \quad (63)$$

as an immediate consequence of Theorem 4.4. With the mapping $f_c : [1, 2] \rightarrow \mathbb{R}$ defined by

$$f_c(x) = V_{1,r}(\mu)x^{\frac{r}{D}}((2 - c^r) - x(1 - c^r))$$

one gets as in the proof of [2], Theorem 6.3, that $f_c([1, 2])$ equals the set of all accumulation points of the sequence $(n^{\frac{r}{D}}V_{n,r}(\mu))_{n \in \mathbb{N}}$. It is easy to see, that $f_c(1) = V_{1,r}(\mu) = f_c(2)$ and that f'_c has one critical point

$$x_0 := x_0(c) := \frac{2 - c^r}{(1 - c^r)(1 + \frac{D}{r})} \in]1, 2[. \quad (64)$$

Moreover $f'_c(1) > 0$ resp. $f'_c(2) < 0$. Hence f_c attains its maximum at x_0 and we have $f_c([1, 2]) = [f_c(1), f_c(x_0)] = [V_{1,r}(\mu), f_c(x_0)]$. Clearly, the quantization coefficient $\lim_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu)$ does not exist.

Remark 6.2 As a special case, Remark 6.1 yields the optimal quantization for the classical Cantor Distribution ($c_k = \frac{1}{3}$ for all k). For this distribution and $r = 2$, the results outlined in Remark 6.1 were first proved in [2] (Theorem 5.2. and Theorem 6.3). Theorem 5.2. in [2] also describes the n -optimal sets of order 2, and, in this respect, is a special case of Theorem 4.4.

Example 6.3 If we drop the assumption $c \leq \frac{1}{3}$, Remark 6.1 resp. Theorem 4.4 becomes wrong. For example, let μ be self-similar with a contracting parameter $c \in]\frac{5-\sqrt{17}}{2}, \frac{1}{2}[$ and $r = 2$. Let $F = [0, 1]$ and $\alpha = \{a_1, a_{21}, a_{22}\}$ with a_1 the midpoint of F_1 and a_{21}, a_{22} the midpoints of F_{21}, F_{22} . Assume, that $\alpha \in C_{3,2}(\mu)$. Since $c > \frac{5-\sqrt{17}}{2}$ one can prove that

$$\mu(W(a_{21} | \alpha) \cap F_1) > 0. \quad (65)$$

By [3], Theorem 4.1 we obtain $\{a_{21}\} \in C_{1,2}(\mu(\cdot | W(a_{21} | \alpha)))$ and hence

$$a_{21} = E(\mu(\cdot | W(a_{21} | \alpha))). \quad (66)$$

By easy computations one can show, that the expected value of $\mu(\cdot | F_2 \cap W(a_{21} | \alpha))$ equals a_{21} , which differs from $E(\mu(\cdot | W(a_{21} | \alpha)))$, due to equation (65). This contradicts equation (66). Hence we have $\alpha \notin C_{3,2}(\mu)$ and Theorem 4.4 is not true.

Remark 6.4 Let $\mu = \mu_c$ be self-similar with contracting parameter $c \in]0, \frac{1}{2}]$ and $x_0(c)$ as defined in (64). If $c \in]0, \frac{1}{3}]$ we know from Remark 6.1, that

$$\liminf_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu_c) = V_{1,r}(\mu_c) \quad (67)$$

and

$$\limsup_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu_c) = f_c(x_0(c)). \quad (68)$$

Using the representations (67) and (68), one can show that the mappings $c \rightarrow \liminf_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu_c)$ and $c \rightarrow \limsup_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu_c)$ are continuous on $]0, \frac{1}{3}]$. It is easy to prove, that

$$\lim_{c \rightarrow 0} f_c(x_0(c)) = \lim_{c \rightarrow 0} \limsup_{n \rightarrow \infty} n^{\frac{r}{D}}V_{n,r}(\mu_c) = \infty.$$

The mapping $c \rightarrow f_c(x_0(c))$ is also defined for $c \in]\frac{1}{3}, \frac{1}{2}]$. Moreover, one can show, that

$$\lim_{c \rightarrow \frac{1}{2}-0} [f_c(x_0(c)) - V_{1,r}(\mu_c)] > 0.$$

If $c = \frac{1}{2}$ we have $D = 1$ and $\mu_{\frac{1}{2}}$ equals the uniform distribution on $[0, 1]$. The sequence $(n^r V_{n,r}(\mu_{\frac{1}{2}}))_{n \in \mathbb{N}}$ converges in this case (see [3], Theorem 6.2 resp. Example 5.5). Hence we know that

$$\lim_{c \rightarrow \frac{1}{2}-0} [f_c(x_0(c)) - V_{1,r}(\mu_c)] > \limsup_{n \rightarrow \infty} n^r V_{n,r}(\mu_{\frac{1}{2}}) - \liminf_{n \rightarrow \infty} n^r V_{n,r}(\mu_{\frac{1}{2}}) = 0,$$

showing that the set of all accumulation points of the sequence $(n^{\frac{r}{D}} V_{n,r}(\mu))_{n \in \mathbb{N}}$ at $c = \frac{1}{2}$ does not equal the interval $[V_{1,r}(\mu), f_c(x_0)]$ any more. The following questions remain open:

- (i) For which values of $c \in]\frac{1}{3}, \frac{1}{2}]$, does equation (67) resp. (68) hold, if any ?
- (ii) Is $c \rightarrow \liminf_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c)$ resp. $c \rightarrow \limsup_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c)$ continuous on $[\frac{1}{3}, \frac{1}{2}]$?
- (iii) If $c \rightarrow \limsup_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c) - \liminf_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c)$ is not continuous at $c = \frac{1}{2}$ what is

$$\lim_{c \rightarrow \frac{1}{2}-0} [\limsup_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c) - \liminf_{n \rightarrow \infty} n^{\frac{r}{D}} V_{n,r}(\mu_c)],$$

provided it exists ?

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