

Universität Passau
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Dissertation

**Strongly Asymptotically Optimal Methods for the
Pathwise Global Approximation of Stochastic Differential
Equations with Coefficients of Super-linear Growth**

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1

Introduction

1.1 Problem Description and Main Results

Let $T \in (0, \infty)$, let $d, m \in \mathbb{N}$, and consider a d -dimensional stochastic differential equation (in short: SDE)

$$\begin{aligned} dX(t) &= \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t), \quad t \in [0, T], \\ X(0) &= \xi, \end{aligned} \tag{1.1}$$

with drift coefficient $\mu : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, diffusion coefficient $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$, m -dimensional Brownian motion W , and random initial value ξ such that (1.1) has a pathwise unique strong solution $X := (X(t))_{t \in [0, T]}$. As in most cases this solution is not given explicitly, one is interested in approximations of this stochastic process. In this thesis, we are concerned with strong approximation of the solution, meaning we aim at approximating X pathwise. Moreover, we address this problem globally in time and not only, for instance, at the final time point T .

We consider approximation methods that are based (in a measurable way) on the evaluation of ξ and on finitely many sequential evaluations of W . For such an approximation \hat{X} , we measure its error via

$$e_{q, \infty}(\hat{X}) := \left(\mathbb{E} \left[\sup_{t \in [0, T]} \max_{i \in \{1, \dots, d\}} |X_i(t) - \hat{X}_i(t)|^q \right] \right)^{1/q},$$

and

$$e_{q, p}(\hat{X}) := \left(\mathbb{E} \left[\left(\int_0^T \sum_{i=1}^d |X_i(t) - \hat{X}_i(t)|^p dt \right)^{q/p} \right] \right)^{1/q},$$

for $p, q \in [1, \infty)$. Here, the error measure $e_{q, \infty}$ quantifies the q th mean supremum distance between X and \hat{X} , and the error measure $e_{q, p}$ quantifies the q th mean L_p distance between X and \hat{X} . Depending on the regarded error criterion, the realizations of \hat{X} are interpreted as elements of the space of continuous functions $C([0, T]; \mathbb{R}^d)$ or

of the space of p times integrable functions $L_p([0, T]; \mathbb{R}^d)$, $p \in [1, \infty)$. In each case, we measure the cost of \widehat{X} by the average number of evaluations of the underlying Brownian motion W employed in its construction.

For $S = C([0, T]; \mathbb{R}^d)$ and $S = L_p([0, T]; \mathbb{R}^d)$, $p \in [1, \infty)$, we study the classes of adaptive approximations $(\mathbb{X}_N^{\text{ad}}(S))_{N \in \mathbb{N}}$ and the classes of equidistant approximations $(\mathbb{X}_N^{\text{eq}}(S))_{N \in \mathbb{N}}$, which are given as follows. For each $N \in \mathbb{N}$, the set $\mathbb{X}_N^{\text{ad}}(S)$ denotes the class of all approximations with realizations in S that are based only on the evaluation of ξ and on at most N sequential evaluations of W on average, and the set $\mathbb{X}_N^{\text{eq}}(S)$ denotes the class of all approximations with realizations in S that are based only on the evaluation of ξ and of $W(T/N), W(2T/N), \dots, W(T)$.

With respect to both error criteria $e_{q, \infty}$ and $e_{q, p}$, our primary objective is to find approximations that are strongly asymptotically optimal in the respective classes of adaptive and of equidistant approximations. To clarify this, fix $* \in \{\text{ad}, \text{eq}\}$ and $p \in [1, \infty)$ for the moment. For the error $e_{q, \infty}$, we seek approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ that satisfy $\widehat{X}_N \in \mathbb{X}_N^*(C([0, T]; \mathbb{R}^d))$ for every $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \frac{e_{q, \infty}(\widehat{X}_N)}{\inf \left\{ e_{q, \infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(C([0, T]; \mathbb{R}^d)) \right\}} = 1$$

for certain $q \in [1, \infty)$. Analogously, for the error $e_{q, p}$, we seek approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ that satisfy $\widehat{X}_N \in \mathbb{X}_N^*(L_p([0, T]; \mathbb{R}^d))$ for every $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \frac{e_{q, p}(\widehat{X}_N)}{\inf \left\{ e_{q, p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(L_p([0, T]; \mathbb{R}^d)) \right\}} = 1$$

for certain $q \in [1, \infty)$. In both limits above, we follow the convention $0/0 := 1$ if necessary.

Regarding the supremum error, strong asymptotic optimality has already been established in the case that the coefficients of the SDE (1.1) are globally Lipschitz continuous and of at most linear growth (each with respect to the state variable). Müller-Gronbach (2002a) showed that specific Euler–Maruyama type schemes perform strongly asymptotically optimal in this situation. More precisely, the author showed strong asymptotic optimality for, on the one hand, a sequence $(\widehat{E}_N^{\text{ad}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Euler–Maruyama schemes on suitably constructed adaptive time discretizations in the classes $(\mathbb{X}_N^{\text{ad}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and for, on the other hand, a sequence $(\widehat{E}_N^{\text{eq}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Euler–Maruyama schemes on equidistant time discretizations in the classes $(\mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. In this thesis, we generalize these results to SDEs whose coefficients may grow super-linearly. More precisely, we show under rather mild assumptions on the SDE (1.1), notably polynomial growth conditions on its coefficients, strong asymptotic optimality for a sequence $(\widehat{X}_N^{\text{ad}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated so-called tamed Euler schemes on suitably constructed adaptive time discretizations in the classes $(\mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and for a sequence $(\widehat{X}_N^{\text{eq}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated tamed Euler schemes on equidistant time discretizations in the classes $(\mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$.

Regarding the L_p error, strong asymptotic optimality has already been established in the case that the coefficients of the SDE (1.1) as well as their partial derivatives with respect to the state variable are globally Lipschitz continuous and of at most linear growth (each with respect to the state variable). Müller-Gronbach (2002b) showed that specific Milstein type schemes perform strongly asymptotically optimal in this situation. More precisely, the author showed strong asymptotic optimality for, on the one hand, a sequence $(\widehat{M}_N^{\text{ad}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Milstein schemes on suitably constructed adaptive time discretizations in the classes $(\mathbb{X}_N^{\text{ad}}(L_p([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and for, on the other hand, a sequence $(\widehat{M}_N^{\text{eq}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated Milstein schemes on equidistant time discretizations in the classes $(\mathbb{X}_N^{\text{eq}}(L_p([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. In this thesis, we generalize these results to SDEs whose coefficients may grow super-linearly. More precisely, we show under rather mild assumptions on the SDE (1.1), notably polynomial growth conditions on its coefficients and their partial derivatives, strong asymptotic optimality for a sequence $(\widehat{Y}_N^{\text{ad}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated so-called tamed Milstein schemes on suitably constructed adaptive time discretizations in the classes $(\mathbb{X}_N^{\text{ad}}(L_p([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and for a sequence $(\widehat{Y}_N^{\text{eq}})_{N \in \mathbb{N}}$ of piecewise-linearly interpolated tamed Milstein schemes on equidistant time discretizations in the classes $(\mathbb{X}_N^{\text{eq}}(L_p([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$.

Fix $* \in \{\text{ad}, \text{eq}\}$ and $p \in [1, \infty]$ for the moment, and put $S = C([0, T]; \mathbb{R}^d)$ if $p = \infty$ and $S = L_p([0, T]; \mathbb{R}^d)$ otherwise. A common way of proving strong asymptotic optimality of specific approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ in the classes $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$ is as follows: In a first step, one establishes an asymptotic lower bound for the N th minimal errors in the given classes, i.e., one shows

$$\liminf_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot \inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\} \geq C_{q,p}^*$$

for some $C_{q,p}^* \in (0, \infty)$ and $\gamma : \mathbb{N} \rightarrow (0, \infty)$. In a second step, one aims at showing a matching asymptotic upper bound for the errors of the given approximations, i.e., one tries to obtain

$$\limsup_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot e_{q,p}(\widehat{X}_N) \leq C_{q,p}^*.$$

In these notes, we maintain the strategy above and pursue this approach for both the supremum error criterion and the L_p error criterion.

To illustrate our contribution to the field of research, we analyze the SDE relating to the Heston-3/2-model originating from mathematical finance. A scalar version of this SDE is given by

$$\begin{aligned} dX(t) &= \alpha \cdot X(t) \cdot (\beta - |X(t)|) dt + |X(t)|^{3/2} dW(t), \quad t \in [0, 1], \\ X(0) &= \xi, \end{aligned} \tag{1.2}$$

with parameters $d = m = 1$ and $\alpha, \beta, \xi \in (0, \infty)$. We exemplarily focus on the supremum error criterion. In Theorem 3.13, we show for certain constellations of the parameters α and β that the asymptotic lower bound

$$\liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(C([0, 1]; \mathbb{R}^d)) \right\} \geq C_{q,\infty}^* \tag{1.3}$$

holds for various values of $q \in [1, \infty)$ and for all $* \in \{\text{ad}, \text{eq}\}$ where

$$C_{q,\infty}^{\text{ad}} := 2^{-1/2} \cdot \left(\mathbb{E} \left[\left(\int_0^1 |X(t)|^3 dt \right)^{q/(q+2)} \right] \right)^{(q+2)/(2q)},$$

$$C_{q,\infty}^{\text{eq}} := 2^{-1/2} \cdot \left(\mathbb{E} \left[\sup_{t \in [0,1]} |X(t)|^{3q/2} \right] \right)^{1/q}.$$

Since the coefficients of the autonomous SDE (1.2) are not of at most linear growth, we cannot apply the main theorems in Müller-Gronbach (2002a) to infer that the particular Euler schemes $(\widehat{E}_N^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{E}_N^{\text{eq}})_{N \in \mathbb{N}}$ satisfy a matching asymptotic upper bound and thus are strongly asymptotically optimal in the classes $(\mathbb{X}_N^{\text{ad}}(C([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$, respectively. Even worse, Theorem 1 in Hutzenthaler et al. (2011) implies that for each $q \in [1, \infty)$ the associated errors $e_{q,\infty}(\widehat{E}_N^{\text{ad}})$ and $e_{q,\infty}(\widehat{E}_N^{\text{eq}})$ tend to infinity as N tends to infinity. In contrast, we show in Corollary 3.15 that—in the same setting as for (1.3)—the tamed Euler schemes $(\widehat{X}_N^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{X}_N^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_N^{\text{ad}}(C([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$, respectively.

The remainder of this dissertation is organized as follows. In Chapter 2, we rigorously define the classes of adaptive and of equidistant approximations, and formally explain strong asymptotic optimality in these classes afterwards. Chapter 3 is devoted to strongly asymptotically optimal approximations with respect to the errors $e_{q,\infty}$. After a brief literature review, including results in the classical framework of SDEs with globally Lipschitz continuous coefficients, we give new results in the setting of SDEs whose coefficients may grow super-linearly. In Chapter 4, we consider the error $e_{q,p}$. As before, we first give an overview of already existing results and then present our findings for SDEs with super-linearly growing coefficients. Limitations of our work are discussed in Chapter 5, and we moreover provide a brief outlook on open problems therein. The Appendices A and B consist of auxiliary results which are used in the proofs of our main theorems.

1.2 Notations

The following notations are used throughout these notes.

We denote the integer part of $x \in \mathbb{R}$ by $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$. For $y, z \in \mathbb{R}$, we put $y \wedge z := \min\{y, z\}$ and $y \vee z := \max\{y, z\}$. For an arbitrary set M , we define $\#M$ to be the cardinality of M . Moreover, in the case that $M \subseteq \Omega$ for some set Ω , we define $\mathbb{1}_M : \Omega \rightarrow \{0, 1\}$ to be the indicator function of M .

Let $n, k \in \mathbb{N}$ and $p \in (0, \infty)$. For a vector $x \in \mathbb{R}^n$ and $i \in \{1, \dots, n\}$, the i th entry of x is denoted by x_i , the transpose of x is written as x^\top , and we put

$$|x|_p := \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad |x|_\infty := \max_{i \in \{1, \dots, n\}} |x_i|.$$

For a matrix $A \in \mathbb{R}^{n \times k}$ and $(i, j) \in \{1, \dots, n\} \times \{1, \dots, k\}$, the i th row and the j th column of A are denoted by A_i and $A^{(j)}$, respectively, and we put

$$|A|_{p,2} := \left(\sum_{i=1}^n \left(\sum_{j=1}^k A_{i,j}^2 \right)^{p/2} \right)^{1/p}, \quad |A|_{\infty,2} := \max_{i \in \{1, \dots, n\}} \left(\sum_{j=1}^k A_{i,j}^2 \right)^{1/2}.$$

The Euclidean norms of a vector and a matrix are abbreviated by $|\cdot| := |\cdot|_2$ and $|\cdot| := |\cdot|_{2,2}$, respectively.

Let $T \in (0, \infty)$, $n \in \mathbb{N}$, and $p \in [1, \infty)$. We denote the Banach space of continuous functions $f : [0, T] \rightarrow \mathbb{R}^n$ equipped with the norm

$$\|f\|_{C([0,T])} := \sup_{t \in [0,T]} |f(t)|_{\infty}$$

by $(C([0, T]; \mathbb{R}^n), \|\cdot\|_{C([0,T])})$. Moreover, we write $\mathcal{B}(C([0, T]; \mathbb{R}^n))$ for the Borel- σ -algebra on $C([0, T]; \mathbb{R}^n)$ that is induced by $\|\cdot\|_{C([0,T])}$. Next, we denote the Banach space of equivalence classes (with respect to the equivalence relation of almost everywhere equality) of measurable functions $f : [0, T] \rightarrow \mathbb{R}^n$ with $\int_0^T |f(t)|_p^p dt < \infty$ equipped with the norm

$$\|[f]\|_{L_p([0,T])} := \left(\int_0^T |f(t)|_p^p dt \right)^{1/p}$$

by $(L_p([0, T]; \mathbb{R}^n), \|\cdot\|_{L_p([0,T])})$. Moreover, we write $\mathcal{B}(L_p([0, T]; \mathbb{R}^n))$ for the Borel- σ -algebra on $L_p([0, T]; \mathbb{R}^n)$ that is induced by $\|\cdot\|_{L_p([0,T])}$. For ease of notation, we will write $\|f\|_{L_p([0,T])}$ instead of $\|[f]\|_{L_p([0,T])}$ in the sequel. Furthermore, we also put

$$\|g\|_{L_r([0,T])} := \left(\int_0^T |g(t)|_r^r dt \right)^{1/r}$$

for a measurable function $g : [0, T] \rightarrow \mathbb{R}^n$ and $r \in (0, 1)$.

Let $T \in (0, \infty)$, $d \in \mathbb{N}$, and $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. For each component f_i , $i \in \{1, \dots, d\}$, of f , we denote by

$$\nabla_t f_i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad (t, x) \mapsto \frac{\partial}{\partial t} f_i(t, x),$$

its partial derivative with respect to the time variable and by

$$\nabla f_i : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{1 \times d}, \quad (t, x) \mapsto \left(\frac{\partial}{\partial x_1} f_i(t, x), \dots, \frac{\partial}{\partial x_d} f_i(t, x) \right),$$

its partial derivatives with respect to the state variable, each in case that these partial derivatives exist. Moreover, we put

$$\nabla f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \quad (t, x) \mapsto \begin{pmatrix} \nabla f_1(t, x) \\ \vdots \\ \nabla f_d(t, x) \end{pmatrix},$$

in such a situation. Furthermore, we denote the Hessian of a twice partially differentiable function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\text{Hg} : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \quad x \mapsto \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} g(x) & \frac{\partial^2}{\partial x_1 \partial x_2} g(x) & \cdots & \frac{\partial^2}{\partial x_1 \partial x_d} g(x) \\ \frac{\partial^2}{\partial x_2 \partial x_1} g(x) & \frac{\partial^2}{\partial x_2^2} g(x) & \cdots & \frac{\partial^2}{\partial x_2 \partial x_d} g(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_d \partial x_1} g(x) & \frac{\partial^2}{\partial x_d \partial x_2} g(x) & \cdots & \frac{\partial^2}{\partial x_d^2} g(x) \end{pmatrix}.$$

Let $n \in \mathbb{N}$, $p \in (0, \infty)$, $q \in [1, \infty)$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a random vector $Z : \Omega \rightarrow \mathbb{R}^n$ on $(\Omega, \mathcal{F}, \mathbb{P})$, we put

$$\|Z\|_{L_p(\Omega)} := (\mathbb{E}[|Z|^p])^{1/p}.$$

Furthermore, for a sequence Z_0, Z_1, Z_2, \dots of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, we define

$$\begin{aligned} Z_N \xrightarrow[N \rightarrow \infty]{\text{a.s.}} Z_0 & \quad \Leftrightarrow \quad \mathbb{P}\left(\left\{\lim_{N \rightarrow \infty} Z_N = Z_0\right\}\right) = 1, \\ Z_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} Z_0 & \quad \Leftrightarrow \quad \forall \varepsilon \in (0, \infty) : \lim_{N \rightarrow \infty} \mathbb{P}(\{|Z_N - Z_0| \geq \varepsilon\}) = 0, \end{aligned}$$

and, in the case that $\|Z_N\|_{L_q(\Omega)} < \infty$ holds for all $N \in \mathbb{N} \cup \{0\}$,

$$Z_N \xrightarrow[N \rightarrow \infty]{L_q} Z_0 \quad \Leftrightarrow \quad \lim_{N \rightarrow \infty} \|Z_N - Z_0\|_{L_q(\Omega)} = 0.$$

1.3 Setting

Throughout this thesis, we require the following setting.

Let $T \in (0, \infty)$, let $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a normal filtration $(\mathcal{F}(t))_{t \in [0, T]}$, let $W : [0, T] \times \Omega \rightarrow \mathbb{R}^m$ be a standard $(\mathcal{F}(t))_{t \in [0, T]}$ -Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mu : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be $(\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))$ - $\mathcal{B}(\mathbb{R}^d)$ -measurable, let $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ be $(\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d))$ - $\mathcal{B}(\mathbb{R}^{d \times m})$ -measurable, and let $\xi : \Omega \rightarrow \mathbb{R}^d$ be $\mathcal{F}(0)$ - $\mathcal{B}(\mathbb{R}^d)$ -measurable and satisfy $\mathbb{E}[|\xi|^2] < \infty$.

We study the d -dimensional stochastic differential equation with drift coefficient μ , diffusion coefficient σ , driving Brownian motion W , and initial value ξ which is written as

$$\begin{aligned} dX(t) &= \mu(t, X(t)) dt + \sigma(t, X(t)) dW(t), \quad t \in [0, T], \\ X(0) &= \xi. \end{aligned} \tag{1.4}$$

An $(\mathcal{F}(t))_{t \in [0, T]}$ -adapted stochastic process $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ with continuous trajectories is called a (*pathwise* or *strong*) *solution* of the SDE (1.4) if

$$\mathbb{P}\left(\left\{\int_0^T |\mu(t, X(t))| + |\sigma(t, X(t))|^2 dt < \infty\right\}\right) = 1$$

and for all $t \in [0, T]$ it holds that

$$\mathbb{P}\left(\left\{X(t) = \xi + \int_0^t \mu(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dW(s)\right\}\right) = 1.$$

Moreover, a solution $(X(t))_{t \in [0, T]}$ of the SDE (1.4) is said to be (*pathwise*) *unique* if for any other solution $(\bar{X}(t))_{t \in [0, T]}$ we have

$$\mathbb{P}\left(\left\{\forall t \in [0, T] : X(t) = \bar{X}(t)\right\}\right) = 1.$$

If the coefficients μ and σ do not depend on the time variable, i.e., if we have $\mu(s, x) = \mu(t, x)$ and $\sigma(s, x) = \sigma(t, x)$ for all $s, t \in [0, T]$ and $x \in \mathbb{R}^d$, then the SDE (1.4) is said to be *autonomous*. For ease of notation, we will write $\mu(x)$ and $\sigma(x)$ instead of $\mu(t, x)$ and $\sigma(t, x)$, respectively, in such a case.

2

Strong Approximations of Stochastic Differential Equations and Strong Asymptotic Optimality

In this chapter, we introduce all the mathematical objects that are needed for the definition of strong asymptotic optimality of a given sequence of strong approximations in a given sequence of classes of approximations with regard to a given error criterion.

In Section 2.1, we formally define what is meant by a (strong) approximation for the pathwise approximation problem we address in this thesis. Moreover, we introduce appropriate cost and error criteria for such approximations. In Section 2.2, we present the main classes of approximations that will be considered in the further analysis, namely, the classes of adaptive and of equidistant approximations. In Section 2.3, we first give the notion of N th minimal errors in specific classes. The concepts of optimality and strong asymptotic optimality are discussed afterwards, and we illustrate these ideas by an example. Subsequently, we show that asymptotic upper and lower error bounds provide a way of proving strong asymptotic optimality. Finally, we indicate how strongly asymptotically optimal approximations can be used for the benchmarking of approximations.

Throughout this chapter, we require that a unique solution $X := (X(t))_{t \in [0, T]}$ of the SDE (1.4) exists. Moreover, let (S, \mathcal{A}) be a measurable space given by $(S, \mathcal{A}) = (C([0, T]; \mathbb{R}^d), \mathcal{B}(C([0, T]; \mathbb{R}^d)))$ or $(S, \mathcal{A}) = (L_p([0, T]; \mathbb{R}^d), \mathcal{B}(L_p([0, T]; \mathbb{R}^d)))$ for some $p \in [1, \infty)$. We will specifically address the space $C([0, T]; \mathbb{R}^d)$ in Chapter 3 and the spaces $L_p([0, T]; \mathbb{R}^d)$, $p \in [1, \infty)$, in Chapter 4.

2.1 Strong Approximations

It is well-known that the solution X almost surely satisfies

$$X = f(\xi, W)$$

for some measurable function $f : \mathbb{R}^d \times C([0, T]; \mathbb{R}^m) \rightarrow C([0, T]; \mathbb{R}^d)$, see, for instance, Corollary 5.3.23 in Karatzas & Shreve (1998). To exclude trivial problem cases, it is therefore reasonable to not allow strong approximations to use entire trajectories of the driving Brownian motion. We thus restrict ourselves to such approximations that are

based only on partial information about the trajectories of W . In this thesis, we focus more precisely on strong approximations that solely depend in any measurable way on the evaluation of ξ and on finitely many sequential evaluations of W .

For the subsequent definition of strong approximations, we follow the ideas of Müller-Gronbach (2002a) and of Hefter et al. (2019) to a great extent. A (*strong*) *approximation* $\widehat{X} : \Omega \rightarrow S$ for the solution of the SDE (1.4) is determined by three sequences

$$\psi := (\psi_k)_{k \in \mathbb{N}}, \quad \chi := (\chi_k)_{k \in \mathbb{N}}, \quad \varphi := (\varphi_k)_{k \in \mathbb{N}},$$

of measurable mappings

$$\begin{aligned} \psi_k &: \mathbb{R}^d \times (\mathbb{R}^m)^{k-1} \rightarrow (0, T], \\ \chi_k &: \mathbb{R}^d \times (\mathbb{R}^m)^k \rightarrow \{\text{STOP}, \text{GO}\}, \\ \varphi_k &: \mathbb{R}^d \times (\mathbb{R}^m)^k \rightarrow S, \end{aligned}$$

for $k \in \mathbb{N}$. Here, the sequence ψ is used to obtain the sequential evaluation sites for W in $(0, T]$, the sequence χ determines when to stop the evaluation of W , and the sequence φ is used to get the outcome of \widehat{X} once the evaluation of W has stopped. More precisely, fix $\omega \in \Omega$ and let $x := \xi(\omega)$ and $w := W(\omega)$ be the corresponding realizations of ξ and W , respectively. We start the evaluation of W at the time point $\psi_1(x)$. After k steps, we are given the data $D_k(\omega) := (x, y_1, \dots, y_k)$ where

$$y_1 := w(\psi_1(x)), \quad \dots, \quad y_k := w(\psi_k(x, y_1, \dots, y_{k-1})),$$

and we decide whether to stop or to proceed with the evaluation of W according to the value of $\chi_k(D_k(\omega))$. The total number of evaluations of W is given by

$$\nu(\omega) := \min \{k \in \mathbb{N} \mid \chi_k(D_k(\omega)) = \text{STOP}\}. \quad (2.1)$$

To exclude non-terminating evaluations of W , we require $\nu < \infty$ almost surely. We obtain the realization of the approximation \widehat{X} by

$$\widehat{X}(\omega) := \varphi_{\nu(\omega)}(D_{\nu(\omega)}(\omega))$$

in the case that $\nu(\omega) < \infty$ and arbitrarily otherwise. For technical reasons, we assume without loss of generality that for all $k, \ell \in \mathbb{N}$ with $k \neq \ell$, for all $x \in \mathbb{R}^d$, and for all $y \in (\mathbb{R}^m)^{\max\{k, \ell\}-1}$ it holds that

$$\psi_k(x, y_1, \dots, y_{k-1}) \neq \psi_\ell(x, y_1, \dots, y_{\ell-1}).$$

Hereinafter, we denote by $\mathbb{X}(S)$ the set of all approximations which are of the form as described above. Note that for an approximation $\widehat{X} \in \mathbb{X}(C([0, T]; \mathbb{R}^d))$ we have $[\widehat{X}] \in \mathbb{X}(L_p([0, T]; \mathbb{R}^d))$ for every $p \in [1, \infty)$; for ease of notation, we will drop the distinction between $[\widehat{X}]$ and \widehat{X} in the sequel.

Next, we specify and explain the cost and error criteria that are used in the further course of these notes. To this end, fix an approximation $\widehat{X} := \widehat{X}^{(\psi, \chi, \varphi)} \in \mathbb{X}(S)$ with determining sequences ψ , ξ , and φ . We measure the *cost* $c(\widehat{X})$ of \widehat{X} by

$$c(\widehat{X}) := \mathbb{E}[\nu]$$

where ν is as in (2.1), i.e., we are interested only in the average number of evaluations of W employed in \widehat{X} . For instance, we do not take into account any expenses for arithmetic operations or for function evaluations, such as evaluations of μ or σ . Moreover, we measure the *error* $e_{q,p}(\widehat{X})$ of \widehat{X} as follows. Let $q \in [1, \infty)$. In the case $(S, \mathcal{A}) = (L_p([0, T]; \mathbb{R}^d), \mathcal{B}(L_p([0, T]; \mathbb{R}^d)))$ for some $p \in [1, \infty)$, we consider the L_p distance between respective realizations of X and \widehat{X} and take the L_q average over all these distances, i.e., we set

$$e_{q,p}(\widehat{X}) := \left\| \left\| X - \widehat{X} \right\|_{L_p([0, T])} \right\|_{L_q(\Omega)}. \quad (2.2)$$

Analogously, we put

$$e_{q,\infty}(\widehat{X}) := \left\| \left\| X - \widehat{X} \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \quad (2.3)$$

if $(S, \mathcal{A}) = (C([0, T]; \mathbb{R}^d), \mathcal{B}(C([0, T]; \mathbb{R}^d)))$. Note that, in contrast to various other error criteria considered in the literature, we measure the distance between X and \widehat{X} globally in time and not only at the final time point T .

Observe that the error measures introduced above satisfy the following easy-to-prove monotonicity properties. For all $q, \tilde{q}, p \in [1, \infty)$ with $q \leq \tilde{q}$ we have

$$e_{q,\infty}(\widehat{X}) \leq e_{\tilde{q},\infty}(\widehat{X}) \quad (2.4)$$

for all $\widehat{X} \in \mathbb{X}(C([0, T]; \mathbb{R}^d))$ and

$$e_{q,p}(\widehat{X}) \leq e_{\tilde{q},p}(\widehat{X}) \quad (2.5)$$

for all $\widehat{X} \in \mathbb{X}(L_p([0, T]; \mathbb{R}^d))$. Moreover, for all $q, p, \tilde{p} \in [1, \infty)$ with $p \leq \tilde{p}$ we have

$$e_{q,p}(\widehat{X}) \leq (T \cdot d)^{1/p} \cdot e_{q,\infty}(\widehat{X}) \quad (2.6)$$

for all $\widehat{X} \in \mathbb{X}(C([0, T]; \mathbb{R}^d))$ and

$$e_{q,p}(\widehat{X}) \leq (T \cdot d)^{1/p-1/\tilde{p}} \cdot e_{q,\tilde{p}}(\widehat{X}) \quad (2.7)$$

for all $\widehat{X} \in \mathbb{X}(L_{\tilde{p}}([0, T]; \mathbb{R}^d))$ where one applies the Hölder inequality to show (2.7).

2.2 The Classes of Adaptive and of Equidistant Approximations

In this subsection, we specify the two principal classes of approximations that are studied in this thesis, namely, the *classes of adaptive approximations* $(\mathbb{X}_N^{\text{ad}}(S))_{N \in \mathbb{N}}$ and the *classes of equidistant approximations* $(\mathbb{X}_N^{\text{eq}}(S))_{N \in \mathbb{N}}$. To this end, fix $N \in \mathbb{N}$ for the moment. First, the class $\mathbb{X}_N^{\text{ad}}(S)$ consists of all approximations that are based on the evaluation of ξ and on at most N sequential evaluations of W on average, i.e., we define

$$\mathbb{X}_N^{\text{ad}}(S) := \left\{ \widehat{X} \in \mathbb{X}(S) \mid c(\widehat{X}) \leq N \right\}.$$

Second, the class $\mathbb{X}_N^{\text{eq}}(S)$ consists of all approximations that are based on the evaluation of ξ and on the evaluation of W at exactly the equidistant sites kT/N , $k \in \{1, \dots, N\}$, i.e., we define

$$\begin{aligned} \mathbb{X}_N^{\text{eq}}(S) := \left\{ \widehat{X} \in \mathbb{X}(S) \mid \chi_1 = \dots = \chi_{N-1} = \text{GO}, \chi_N = \chi_{N+1} = \dots = \text{STOP}, \right. \\ \psi_k = kT/N \text{ for all } k \in \{1, \dots, N\}, \\ \left. \text{and } \psi_k \text{ is constant for each } k \in \{N+1, N+2, \dots\} \right\}. \end{aligned}$$

It is easy to see that $\mathbb{X}_N^{\text{eq}}(S) \subseteq \mathbb{X}_N^{\text{ad}}(S)$ and

$$\begin{aligned} \mathbb{X}_N^{\text{eq}}(S) = \left\{ u(\xi, W(T/N), W(2T/N), \dots, W(T)) \mid \right. \\ \left. u : \mathbb{R}^d \times (\mathbb{R}^m)^N \rightarrow S \text{ is measurable} \right\}. \end{aligned} \quad (2.8)$$

Note that the classes of adaptive and of equidistant approximations incorporate a large variety of important approximations. In particular, classical approximations like Euler–Maruyama type schemes corresponding to suitably chosen adaptive time discretizations (e.g., appropriate versions of the schemes presented in Fang & Giles (2018), Kelly & Lord (2018a,b), Hofmann et al. (2000a,b, 2001), and Müller-Gronbach (2002a,b)) or to equidistant time discretizations lie in the respective classes. Additionally, observe that these classes also contain even possibly non-implementable approximations like conditional expectations of the form $\mathbb{E}[X \mid (\xi, W(T/N), W(2T/N), \dots, W(T))]$, $N \in \mathbb{N}$, cf. the representation (2.8).

The classes $(\mathbb{X}_N^{\text{ad}}(S))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(S))_{N \in \mathbb{N}}$ are clearly not the only classes which may be studied. For example, the authors in Hofmann et al. (2001) also consider the classes $(\mathbb{X}_N^{\text{sn}}(S))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{ss}}(S))_{N \in \mathbb{N}}$ which are defined by

$$\begin{aligned} \mathbb{X}_N^{\text{sn}}(S) := \left\{ \widehat{X} \in \mathbb{X}(S) \mid \chi_k \text{ is constant for each } k \in \mathbb{N}, \right. \\ \left. \text{and } \nu = \min\{k \in \mathbb{N} \mid \chi_k = \text{STOP}\} \leq N \right\} \end{aligned}$$

and

$$\mathbb{X}_N^{\text{ss}}(S) := \left\{ \widehat{X} \in \mathbb{X}(S) \mid \psi_k \text{ and } \chi_k \text{ are constant for each } k \in \mathbb{N}, \right. \\ \left. \text{and } \nu = \min\{k \in \mathbb{N} \mid \chi_k = \text{STOP}\} \leq N \right\}$$

for $N \in \mathbb{N}$. The class $\mathbb{X}_N^{\text{sn}}(S)$ comprises all approximations that use the same number (at most N) of evaluations of W for each realization, and the class \mathbb{X}_N^{ss} comprises all approximations that not only use the same number but even exactly the same (at most N) evaluation sites of W for each realization. Note that

$$\mathbb{X}_N^{\text{eq}}(S) \subseteq \mathbb{X}_N^{\text{ss}}(S) \subseteq \mathbb{X}_N^{\text{sn}}(S) \subseteq \mathbb{X}_N^{\text{ad}}(S)$$

holds for each $N \in \mathbb{N}$. In this thesis, we (almost without exception) focus on the classes $(\mathbb{X}_N^{\text{eq}}(S))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{ad}}(S))_{N \in \mathbb{N}}$ as these constitute the edge cases in the chain of subset relations above and cover, in our opinion, the most interesting approximations appearing in practice.

2.3 Strong Asymptotic Optimality

Throughout this section, let $*$ \in $\{\text{ad}, \text{sn}, \text{ss}, \text{eq}\}$, $p \in [1, \infty]$, and $q \in [1, \infty)$. Moreover, let

$$(S, \mathcal{A}) = \begin{cases} (L_p([0, T]; \mathbb{R}^d), \mathcal{B}(L_p([0, T]; \mathbb{R}^d))), & \text{if } p < \infty, \\ (C([0, T]; \mathbb{R}^d), \mathcal{B}(C([0, T]; \mathbb{R}^d))), & \text{else.} \end{cases}$$

Given the sequence $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$, we are interested in approximations that perform extraordinarily well in these classes with respect to the error $e_{q,p}$. This leads us to the concepts of N th minimal errors and, building on that, the notions of optimality and strong asymptotic optimality.

Fix $N \in \mathbb{N}$ for the moment. We call the quantity $\inf \{e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S)\}$ the N th minimal error in the classes $(\mathbb{X}_M^*(S))_{M \in \mathbb{N}}$, and an approximation $\widehat{X}_N \in \mathbb{X}_N^*(S)$ is said to be *optimal in $\mathbb{X}_N^*(S)$* if it achieves the corresponding N th minimal error, i.e., if it satisfies

$$e_{q,p}(\widehat{X}_N) = \inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\}.$$

Unfortunately, optimal approximations are known only in exceptional cases. On this account, we relax the notion of optimality and switch to asymptotic optimality instead. We call approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ *strongly asymptotically optimal in $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$* if $\widehat{X}_N \in \mathbb{X}_N^*(S)$ holds for each $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} \frac{e_{q,p}(\widehat{X}_N)}{\inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\}} = 1,$$

following the convention $0/0 := 1$ if necessary.

Clearly, optimality in each class implies strong asymptotic optimality.

To illustrate the concepts above, we study a simple SDE in the upcoming example and thereby check whether piecewise-linear interpolations of W at equidistant time points are (strongly asymptotically) optimal in certain classes.

Example 2.1. *Let $d = m = 1$. We consider the SDE*

$$\begin{aligned} dX(t) &= 1 dW(t), \quad t \in [0, 1], \\ X(0) &= 0, \end{aligned} \tag{2.9}$$

as well as the error measure $e_{2,2}$. Clearly, the process $(W(t))_{t \in [0,1]}$ is the unique solution of the SDE (2.9). Now let $N \in \mathbb{N}$ be fixed for the moment. We study the piecewise-linear interpolation $\widehat{W}_N^{\text{eq}} : \Omega \rightarrow C([0, 1]; \mathbb{R})$ of W at the equidistant sites k/N , $k \in \{0, \dots, N\}$, which is given by

$$\begin{aligned} \widehat{W}_N^{\text{eq}}(0) &:= 0, \\ \widehat{W}_N^{\text{eq}}(t) &:= ((k+1) - tN) \cdot W(k/N) + (tN - k) \cdot W((k+1)/N), \end{aligned}$$

for all $k \in \{0, \dots, N-1\}$ and for all $t \in (k/N, (k+1)/N]$. By using the representation (2.8), we immediately find $\widehat{W}_N^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(L_2([0, 1]; \mathbb{R}))$. Straight-forward calculations show that

$$e_{2,2}(\widehat{W}_N^{\text{eq}}) = (6N)^{-1/2} = \inf \left\{ e_{2,2}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(L_2([0, 1]; \mathbb{R})) \right\}.$$

Moreover, it holds that

$$\inf \left\{ e_{2,2}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{sn}}(L_2([0, 1]; \mathbb{R})) \right\} = (6N+2)^{-1/2},$$

see, for instance, Theorem 6.3 in Lee (1986). Hence, the approximation $\widehat{W}_N^{\text{eq}}$ is optimal in $\mathbb{X}_N^{\text{eq}}(L_2([0, 1]; \mathbb{R}))$ and not optimal in $\mathbb{X}_N^{\text{sn}}(L_2([0, 1]; \mathbb{R}))$; yet, the approximations $(\widehat{W}_N^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in both the classes $(\mathbb{X}_N^{\text{eq}}(L_2([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{sn}}(L_2([0, 1]; \mathbb{R})))_{N \in \mathbb{N}}$. \diamond

A common way to prove strong asymptotic optimality is to establish asymptotic lower and upper error bounds with matching asymptotic constants. More precisely, given classes $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$ and approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ with $\widehat{X}_N \in \mathbb{X}_N^*(S)$ for each $N \in \mathbb{N}$, one aims at showing

$$\liminf_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot \inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\} \geq c \tag{2.10}$$

and

$$\limsup_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot e_{q,p}(\widehat{X}_N) \leq C \tag{2.11}$$

for some constants $c, C \in (0, \infty)$ and for some $\gamma : \mathbb{N} \rightarrow (0, \infty)$. The inequality (2.10) constitutes an *asymptotic lower bound* for the N th minimal errors in the classes $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$ with *asymptotic constant* c and (informally written) *convergence rate* $\gamma(N)$. Analogously, the inequality (2.11) constitutes an *asymptotic upper bound* for the errors of the approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ with *asymptotic constant* C and (informally written) *convergence rate* $\gamma(N)$. If one accomplishes $c = C$ in the situation above, then the approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ are indeed strongly asymptotically optimal in the classes $(\mathbb{X}_N^*(S))_{N \in \mathbb{N}}$. In this case, the corresponding asymptotic constant c is said to be *sharp*.

Observe that combining N th minimal errors and strong asymptotic optimality can serve as a tool for benchmarking approximations. Fix $N \in \mathbb{N}$ and consider an approximation $\widehat{Y} \in \mathbb{X}_N^*(S)$. Clearly, this approximation performs all the better compared to other approximations from the class $\mathbb{X}_N^*(S)$, the closer the fraction

$$\frac{e_{q,p}(\widehat{Y})}{\inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\}} \quad (2.12)$$

is to 1. Informally speaking, the quantity (2.12) indicates how far the approximation \widehat{Y} deviates from being optimal in $\mathbb{X}_N^*(S)$. Unfortunately, the N th minimal error occurring in (2.12) is typically hard to determine and is hence estimated as follows. Provided that we are given further approximations $(\widehat{X}_M)_{M \in \mathbb{N}}$ that are known to be strongly asymptotically optimal in $(\mathbb{X}_M^*(S))_{M \in \mathbb{N}}$, we first deduce that

$$\frac{e_{q,p}(\widehat{X}_N)}{\inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(S) \right\}}$$

is close to 1 for N sufficiently large. We then replace the denominator in (2.12) by $e_{q,p}(\widehat{X}_N)$ and consider the fraction

$$\frac{e_{q,p}(\widehat{Y})}{e_{q,p}(\widehat{X}_N)}$$

instead of (2.12). Under the premise that N is sufficiently large, this new quantity constitutes a value that can be compared to similarly obtained values corresponding to different approximations from the class $\mathbb{X}_N^*(S)$.

Example 2.2. *Reconsider the SDE (2.9) as well as the approximations $(\widehat{W}_M^{\text{eq}})_{M \in \mathbb{N}}$ from Example 2.1, and fix $N = 2^{10}$. We aim at assigning a benchmark value to the specific approximation $\widehat{W}_{2^9}^{\text{eq}}$ in the class $\mathbb{X}_{2^{10}}^{\text{ss}}(L_2([0, 1]; \mathbb{R}))$. In the absence of a concrete value of $\inf \{ e_{2,2}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_{2^{10}}^{\text{ss}}(L_2([0, 1]; \mathbb{R})) \}$, we follow the approach above. Since the approximations $(\widehat{W}_M^{\text{eq}})_{M \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_M^{\text{sn}}(L_2([0, 1]; \mathbb{R})))_{M \in \mathbb{N}}$ (see Example 2.1), it is easy to conclude that these approximations are also strongly asymptotically optimal in the classes $(\mathbb{X}_M^{\text{ss}}(L_2([0, 1]; \mathbb{R})))_{M \in \mathbb{N}}$. Hence, the score*

$$\frac{e_{2,2}(\widehat{W}_{2^9}^{\text{eq}})}{e_{2,2}(\widehat{W}_{2^{10}}^{\text{eq}})} = \frac{(6 \cdot 2^9)^{-1/2}}{(6 \cdot 2^{10})^{-1/2}} = 2^{1/2} (= 1.4142 \dots)$$

can be compared to respective values of other approximations from $\mathbb{X}_{2^{10}}^{\text{ss}}(L_2([0, 1]; \mathbb{R}))$. \diamond

Note that the benchmarking idea above requires certain errors to be known, see Example 2.2. If we do not have access to the exact values of these errors, one might try to estimate them, e.g., via Monte Carlo simulations.

3 Strongly Asymptotically Optimal Approximations with respect to the Supremum Error

In this chapter, we study the error criterion $e_{q,\infty}$, which is defined by (2.3), and we seek approximations that are strongly asymptotically optimal in the corresponding classes of adaptive and of equidistant approximations, i.e., in $(\mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. On the one hand, for SDEs with globally Lipschitz continuous coefficients, Müller-Gronbach (2002a) showed that specific Euler–Maruyama schemes relating to adaptive and to equidistant time discretizations are strongly asymptotically optimal in their respective classes. On the other hand, for SDEs with super-linearly growing coefficients, the main theorem in Hutzenthaler et al. (2011) implies that the errors of these particular approximations tend to infinity as the numbers of discretization sites tend to infinity. In the present chapter, we generalize the results of the first mentioned reference such that SDEs of the latter type are incorporated. More precisely, we show under rather mild assumptions on the underlying SDE, notably polynomial growth conditions on its coefficients, that specific tamed Euler schemes relating to adaptive and to equidistant time discretizations are strongly asymptotically optimal in the aforementioned classes. To illustrate our findings, we numerically analyze the SDE associated with the Heston–3/2–model originating from mathematical finance.

In Section 3.1, we introduce the assumptions that will be imposed on the underlying SDE in the subsequent analysis. Moreover, we show interdependencies among these conditions as well as results on existence-and-uniqueness and moment bounds of the solution of the SDE. In Section 3.2, we provide a brief literature review on results regarding the error $e_{q,\infty}$. In particular, we state well-known results in the classical setting of SDEs with globally Lipschitz continuous coefficients. In Section 3.3, we first present a continuous-time tamed Euler scheme. Building upon this scheme, we construct an equidistant and an adaptive tamed Euler scheme in full details afterwards. In Section 3.4, we state the main results of this chapter, i.e., strong asymptotic optimality of the previously constructed adaptive and equidistant tamed Euler schemes in their respective classes. In Section 3.5, we illustrate our findings via a numerical experiment. To this end, we revisit the introductory example from Section 1.1, namely, the SDE relating to the Heston–3/2–model. In Section 3.6, we carry out the proofs of our main results.

Throughout this chapter, we fix $(S, \mathcal{A}) = (C([0, T]; \mathbb{R}^d), \mathcal{B}(C([0, T]; \mathbb{R}^d)))$ as the underlying measurable space as per Chapter 2. In the case that the SDE (1.4) has a unique solution $(X(t))_{t \in [0, T]}$, we put

$$\begin{aligned} C_{q, \infty}^{\text{ad}} &:= 2^{-1/2} \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{L_2([0, T])} \right\|_{L_{2q/(q+2)}(\Omega)} \in [0, \infty], \\ C_{q, \infty}^{\text{eq}} &:= (T/2)^{1/2} \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \in [0, \infty], \end{aligned} \quad (3.1)$$

for $q \in [1, \infty)$. The quantities $C_{q, \infty}^{\text{ad}}$ and $C_{q, \infty}^{\text{eq}}$ will turn out to be the sharp asymptotic constants for the N th minimal errors in the classes $(\mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively. It is easy to see that $C_{q, \infty}^{\text{ad}} \leq C_{q, \infty}^{\text{eq}}$ holds for all $q \in [1, \infty)$.

3.1 Assumptions

In the further course of this thesis, we will impose several conditions on the initial value and on the coefficients of the SDE (1.4). For $r \in [0, \infty)$ and $\varphi \in \{\mu, \sigma\}$, we introduce the following assumptions:

Assumption (I_r). *The initial value ξ satisfies $\mathbb{E}[|\xi|^r] < \infty$.*

Assumption (locL). *The coefficients μ and σ satisfy a local Lipschitz condition with respect to the state variable, i.e., for all $M \in \mathbb{N}$ there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^d$ with $\max\{|x|, |y|\} \leq M$ it holds that*

$$\max \left\{ |\mu(t, x) - \mu(t, y)|, |\sigma(t, x) - \sigma(t, y)| \right\} \leq C \cdot |x - y|.$$

Assumption (H). *The coefficients μ and σ are Hölder-1/2-continuous with respect to the time variable with a Hölder bound that is linearly growing in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $s, t \in [0, T]$ and for all $x \in \mathbb{R}^d$ it holds that*

$$\max \left\{ |\mu(s, x) - \mu(t, x)|, |\sigma(s, x) - \sigma(t, x)| \right\} \leq C \cdot |s - t|^{1/2} \cdot (1 + |x|).$$

Assumption (K_r). *The coefficients μ and σ satisfy a so-called “Khasminskii-type condition”, i.e., there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x \in \mathbb{R}^d$ it holds that*

$$2 \cdot x^\top \cdot \mu(t, x) + (r - 1) \cdot |\sigma(t, x)|^2 \leq C \cdot (1 + |x|)^2.$$

Assumption (M_r). *The coefficients μ and σ satisfy a so-called “monotonicity condition”, i.e., there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^d$ it holds that*

$$2 \cdot (x - y)^\top \cdot (\mu(t, x) - \mu(t, y)) + (r - 1) \cdot |\sigma(t, x) - \sigma(t, y)|^2 \leq C \cdot |x - y|^2.$$

Assumption (pG $_r^\varphi$). *The coefficient φ grows at most polynomially in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x \in \mathbb{R}^d$ it holds that*

$$|\varphi(t, x)| \leq C \cdot (1 + |x|)^r.$$

Assumption (pL $_r^\varphi$). *The coefficient φ is Lipschitz continuous with respect to the state variable with a Lipschitz bound that is polynomially growing in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^d$ it holds that*

$$|\varphi(t, x) - \varphi(t, y)| \leq C \cdot |x - y| \cdot (1 + |x| + |y|)^r.$$

During the last decades, it was common practice to essentially require the Assumptions (pL $_0^\mu$) and (pL $_0^\sigma$), i.e., to assume that the drift and diffusion coefficients are globally Lipschitz continuous (with respect to the state variable), see, e.g., Maruyama (1955), Milstein (1995), Kloeden & Platen (1992), Mao (2007). Only recently, one has begun to study SDEs that do not satisfy these stringent assumptions, such as SDEs with super-linearly growing or discontinuous coefficients.

The following easy-to-prove remark comprises various relations between the assumptions introduced above.

Remark 3.1. (i) *Let (I $_r$) be satisfied for some $r \in [0, \infty)$. Then (I $_s$) is satisfied for all $s \in [0, r]$.*

(ii) *Let (K $_r$) be satisfied for some $r \in [0, \infty)$. Then (K $_s$) is satisfied for all $s \in [0, r]$.*

(iii) *Let (M $_r$) be satisfied for some $r \in [0, \infty)$. Then (M $_s$) is satisfied for all $s \in [0, r]$.*

(iv) *Let (pG $_r^\varphi$) be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [0, \infty)$. Then (pG $_s^\varphi$) is satisfied for all $s \in [r, \infty)$.*

(v) *Let (pL $_r^\varphi$) be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [0, \infty)$. Then (pL $_s^\varphi$) is satisfied for all $s \in [r, \infty)$.*

(vi) *Let (K $_r$) and (pG $_s^\mu$) be satisfied for some $r \in (1, \infty)$ and $s \in [0, \infty)$. Then (pG $_{\max\{1, (s+1)/2\}}^\sigma$) is satisfied.*

(vii) *Let (M $_r$) and (pL $_s^\mu$) be satisfied for some $r \in (1, \infty)$ and $s \in [0, \infty)$. Then (pL $_{s/2}^\sigma$) is satisfied.*

(viii) *Let (pL $_r^\mu$) and (pL $_s^\sigma$) be satisfied for some $r, s \in [0, \infty)$. Then (locL) is satisfied.*

(ix) *Let (pG $_1^\mu$) and (pG $_1^\sigma$) be satisfied. Then (K $_r$) is satisfied for all $r \in [0, \infty)$.*

(x) *Let (pL $_0^\mu$) and (pL $_0^\sigma$) be satisfied. Then (M $_r$) is satisfied for all $r \in [0, \infty)$.*

(xi) *Let (H) and (pL $_r^\varphi$) be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [0, \infty)$. Then (pG $_{r+1}^\varphi$) is satisfied.*

(xii) *If the SDE (1.4) is autonomous, then (H) is satisfied.*

It is well-known that the Assumptions (I_a) , (locL) , and (K_a) for some $a \in [2, \infty)$ ensure the existence of a unique solution $(X(t))_{t \in [0, T]}$ of the SDE (1.4) that also satisfies

$$\sup_{t \in [0, T]} \mathbb{E} \left[|X(t)|^a \right] < \infty, \quad (3.2)$$

see, for instance, Theorem 4.1 of Chapter 2 in Mao (2007). If, additionally, the Assumption (pG_r^σ) is satisfied for some $r \in [1, \infty)$ with $a \geq 2r$, then it holds that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X(t)|^{a-2r+2} \right] < \infty,$$

see Proposition A.5. Furthermore, in the case that the SDE (1.4) possesses a unique solution and the Assumptions (I_a) , (pG_1^μ) , and (pG_1^σ) are satisfied for some $a \in [2, \infty)$, there exists $C \in (0, \infty)$ such that for all $s, t \in [0, T]$ it holds that

$$\|X(s) - X(t)\|_{L_a(\Omega)} \leq C \cdot |s - t|^{1/2}, \quad (3.3)$$

see, for instance, Theorem 4.3 of Chapter 2 in Mao (2007).

3.2 Literature Review

We now provide a concise overview of results concerning the considered pathwise approximation problem, mainly with regard to the error $e_{q, \infty}$. This selection does not claim to be complete or exhaustive, but should give enough background to motivate the central objective of this chapter.

The remainder of this section is organized as follows. We start our discussion by presenting a result on lower bounds for the N th minimal errors in the classes of adaptive approximations which holds under quite general assumptions. This allows us to deduce asymptotic lower bounds for the N th minimal errors in the classes of adaptive and of equidistant approximations, but with unspecified asymptotic constants only. Yet, in the special case of SDEs with globally Lipschitz continuous coefficients, one entirely knows the asymptotics of these N th minimal errors. As a next step, we are interested in approximations which possess the same convergence rate as the N th minimal errors in the previous results or, even better, which are strongly asymptotically optimal in the aforementioned classes. For SDEs with globally Lipschitz continuous coefficients, specific equidistant and adaptive variants of the Euler–Maruyama scheme achieve the desired strong asymptotic optimality. However, the errors of these particular approximations tend to infinity once the drift or the diffusion coefficient grows super-linearly. In search of strongly asymptotically optimal approximations for SDEs of the latter type, we come into contact with so-called tamed Euler schemes for which we find upper error bounds of the right convergence rate.

At first, lower error bounds for the pathwise approximation of SDEs have been extensively studied for the case of coefficients that are globally Lipschitz continuous, see, e.g., Cambanis & Hu (1996), Hofmann et al. (2000*a, b*, 2001), and Müller-Gronbach (2002*a, b*).

Recently, the authors in Hefter et al. (2019) established lower bounds for the N th minimal errors in the classes of adaptive approximations that hold under rather mild assumptions on the underlying SDE. In particular, the coefficients of the SDE (1.4) are required to have sufficient regularity only locally, in a small neighborhood of the initial value.

Proposition 3.2. *Let $d = m = 1$ and let the SDE (1.4) have a unique solution $(X(t))_{t \in [0, T]}$. Moreover, let $t_0 \in [0, T]$, $T_0 \in (t_0, T]$, and let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval such that for all $i, j \in \{0, 1\}$ the partial derivatives $\mu^{(i, j)}$ and $\sigma^{(i, j)}$ exist on $[t_0, T_0] \times I$ and are continuous, for all $(t, x) \in [t_0, T_0] \times I$ it holds that $\sigma(t, x) \neq 0$, and it holds that $\mathbb{P}(\{X(t_0) \in I\}) > 0$. Then for all $q \in [1, \infty)$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\inf \left\{ e_{q, \infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R})) \right\} \geq C \cdot (\log(N)/N)^{1/2}. \quad (3.4)$$

Proof. Theorem 9 in Hefter et al. (2019) shows (3.4) for the particular error $e_{1, \infty}$. The general case then follows from the monotonicity property (2.4). \square

From the preceding proposition, we are able to immediately deduce asymptotic lower bounds for both the N th minimal errors in the classes of adaptive and of equidistant approximations. Yet, the corresponding asymptotic constants are unspecified in this situation which does not enable us to resolve whether these constants are sharp or not.

However, in the special case of an SDE with globally Lipschitz continuous coefficients, Müller-Gronbach (2002a) determined the exact asymptotics (including sharp asymptotic constants) of the N th minimal errors in the classes of adaptive and of equidistant approximations.

Proposition 3.3. *Let $T = 1$ and let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ it holds that*

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q, \infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(C([0, 1]; \mathbb{R}^d)) \right\} = C_{q, \infty}^{\text{rad}}$$

and

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q, \infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R}^d)) \right\} = C_{q, \infty}^{\text{eq}}.$$

Proof. See Theorem 3 in Müller-Gronbach (2002a) for the adaptive case. The equidistant case is not explicitly shown in this reference, but follows from the asymptotic upper bound given by Lemma 11 and the asymptotic lower bound given by Lemma 8 (slightly modified towards N th minimal errors) stated therein. \square

Next, we turn to upper bounds for the errors of specific adaptive and equidistant approximations, and we seek in particular those approximations that have the same convergence rate $(\log(N)/N)^{1/2}$ as the N th minimal errors studied above. During the last decades, upper error bounds for the pathwise approximation of SDEs have been established mostly in the case of globally Lipschitz continuous coefficients, see, e.g., the seminal works by Maruyama (1955) and Milstein (1995) or the book of Kloeden & Platen (1992), which contains upper error bounds for various strong Itô–Taylor approximations such as the Euler–Maruyama or Milstein schemes.

First of all, we study the Euler–Maruyama schemes, which were initially introduced in Maruyama (1955) and can be regarded as an extension of the classical Euler schemes for ordinary differential equations towards SDEs. We begin with the definition of the continuous-time Euler–Maruyama schemes. To this end, let $N \in \mathbb{N}$ and consider the equidistant time discretization

$$t_\ell^{(N)} := \ell T/N, \quad \ell \in \{0, \dots, N\}. \quad (3.5)$$

The *continuous-time Euler–Maruyama scheme* $\tilde{E}_N : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{E}_N(0) &:= \xi, \\ \tilde{E}_N(t) &:= \tilde{E}_N(t_\ell^{(N)}) + \mu(t_\ell^{(N)}, \tilde{E}_N(t_\ell^{(N)})) \cdot (t - t_\ell^{(N)}) \\ &\quad + \sigma(t_\ell^{(N)}, \tilde{E}_N(t_\ell^{(N)})) \cdot (W(t) - W(t_\ell^{(N)})), \end{aligned}$$

for all $\ell \in \{0, \dots, N-1\}$ and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)}]$. Observe that, since entire trajectories of the driving Brownian motion are used in the construction above, this scheme is not an approximation in the sense of Section 2.1, and we thus find $\tilde{E}_N \notin \mathbb{X}(C([0, T]; \mathbb{R}^d))$. It is well-known that the continuous-time Euler–Maruyama schemes converge strongly to the solution of the SDE (1.4) with order (at least) $1/2$ if the drift and diffusion coefficients are globally Lipschitz continuous, as the following proposition clarifies.

Proposition 3.4. *Let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \tilde{E}_N\|_{C([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1/2}.$$

Proof. See, for instance, Proposition 14 in Faure (1992). \square

Note that we do not use the term $e_{q, \infty}(\tilde{E}_N)$ in the proposition above since the error measure $e_{q, \infty}$ is only defined on $\mathbb{X}(C([0, T]; \mathbb{R}^d))$.

To obtain equidistant approximations that are based on the continuous-time Euler–Maruyama schemes, we consider the linear interpolations of these schemes at the equidistant sites (3.5). For $N \in \mathbb{N}$, the *equidistant Euler–Maruyama scheme* $\hat{E}_N^{\text{eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\hat{E}_N^{\text{eq}}(t_\ell^{(N)}) := \tilde{E}_N(t_\ell^{(N)})$$

for all $\ell \in \{0, \dots, N\}$ and linearly interpolated between these time points. By suitably choosing sequences ψ , χ , and φ as per Section 2.1, we obtain $\hat{E}_N^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d))$. Under the same assumptions as in the preceding proposition, one also knows upper bounds for the errors of these equidistant Euler–Maruyama schemes.

Proposition 3.5. *Let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N} \setminus \{1\}$ it holds that*

$$e_{q, \infty}(\hat{E}_N^{\text{eq}}) \leq C \cdot (\log(N)/N)^{1/2}.$$

Proof. See Proposition 16 in Faure (1992). \square

From the result above, we can deduce an asymptotic upper bound for the errors of the equidistant Euler–Maruyama schemes but, again, only with unspecified asymptotic constant. In contrast, the following proposition shows the exact asymptotics (including sharp asymptotic constant) of the errors of these approximations in the case that the coefficients of the SDE (1.4) are globally Lipschitz continuous.

Proposition 3.6. *Let $T = 1$ and let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ it holds that*

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot e_{q,\infty}(\widehat{E}_N^{\text{eq}}) = C_{q,\infty}^{\text{eq}}.$$

Proof. See Theorem 2 in Müller-Gronbach (2002a). \square

Having considered the Euler–Maruyama schemes at equidistant sites only so far, we next turn to variants of these schemes that are based on adaptive time discretizations. For this purpose, we study the following adaptive Euler–Maruyama schemes, which were introduced in Müller-Gronbach (2002a). Fix $T = 1$ and let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers satisfying

$$\lim_{N \rightarrow \infty} \frac{k_N}{N} = 0 = \lim_{N \rightarrow \infty} \frac{N}{k_N \cdot \log(N)}. \quad (3.6)$$

Fix $N \in \mathbb{N}$ for the moment and put

$$\mathcal{A}_{k_N} := \left(\frac{1}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \sigma(t_\ell^{(k_N)}, \widetilde{E}_{k_N}(t_\ell^{(k_N)})) \right|_{\infty,2}^2 \right)^{1/2}.$$

Let $q \in [1, \infty)$. For each $\ell \in \{0, \dots, k_N - 1\}$, we consider the random discretization

$$t_\ell^{(k_N)} = \tau_{\ell,0}^{(k_N)} < \tau_{\ell,1}^{(k_N)} < \dots < \tau_{\ell,\eta_\ell+1}^{(k_N)} = t_{\ell+1}^{(k_N)}$$

of $[t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]$ where

$$\eta_\ell := \mathbb{1}_{\{\mathcal{A}_{k_N} > 0\}} \cdot \left[N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} \cdot \frac{\left| \sigma(t_\ell^{(k_N)}, \widetilde{E}_{k_N,r}(t_\ell^{(k_N)})) \right|_{\infty,2}^2}{\sum_{\iota=0}^{k_N-1} \left| \sigma(t_\iota^{(k_N)}, \widetilde{E}_{k_N,r}(t_\iota^{(k_N)})) \right|_{\infty,2}^2} \right]$$

and

$$\tau_{\ell,\kappa}^{(k_N)} := t_\ell^{(k_N)} + \frac{1}{k_N} \cdot \frac{\kappa}{\eta_\ell + 1}$$

for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$. The *adaptive Euler–Maruyama scheme* $\widehat{E}_{N,q}^{\text{ad}} : \Omega \rightarrow C([0, 1]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \widehat{E}_{N,q}^{\text{ad}}(\tau_{\ell,\kappa}^{(k_N)}) &:= \widetilde{E}_{k_N}(t_\ell^{(k_N)}) + \mu(t_\ell^{(k_N)}, \widetilde{E}_{k_N}(t_\ell^{(k_N)})) \cdot (\tau_{\ell,\kappa}^{(k_N)} - t_\ell^{(k_N)}) \\ &\quad + \sigma(t_\ell^{(k_N)}, \widetilde{E}_{k_N}(t_\ell^{(k_N)})) \cdot (W(\tau_{\ell,\kappa}^{(k_N)}) - W(t_\ell^{(k_N)})) \end{aligned} \quad (3.7)$$

for all $\ell \in \{0, \dots, k_N - 1\}$ and for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$ and linearly interpolated between all these time points. By suitably choosing sequences ψ , χ , and φ as per Section 2.1, we obtain $\widehat{E}_{N,q}^{\text{ad}} \in \mathbb{X}_{[c(\widehat{E}_{N,q}^{\text{ad}})]}^{\text{ad}}(C([0, 1]; \mathbb{R}^d))$ if the cost of this approximation satisfy $c(\widehat{E}_{N,q}^{\text{ad}}) < \infty$. Müller-Gronbach (2002a) established the exact asymptotics of the errors of the adaptive Euler–Maruyama schemes defined above in the case that the underlying SDE possesses globally Lipschitz continuous coefficients.

Proposition 3.7. *Let $T = 1$ and let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ it holds that*

$$\lim_{N \rightarrow \infty} \left(c(\widehat{E}_{N,q}^{\text{ad}}) / \log(c(\widehat{E}_{N,q}^{\text{ad}})) \right)^{1/2} \cdot e_{q,\infty}(\widehat{E}_{N,q}^{\text{ad}}) = C_{q,\infty}^{\text{ad}}.$$

Proof. See Theorem 1 in Müller-Gronbach (2002a). □

Since both the convergence rates and the corresponding asymptotic constants match in the Propositions 3.3, 3.6, and 3.7, we obtain strong asymptotic optimality of the adaptive and of the equidistant Euler–Maruyama schemes in the classes of adaptive and of equidistant approximations, respectively.

Corollary 3.8. *Let $T = 1$ and let the Assumptions (I_a) , (H) , (pL_0^μ) , and (pL_0^σ) be satisfied for some $a \in [2, \infty)$. Then for all $q \in [1, a]$ with $C_{q,\infty}^{\text{ad}} > 0$ it holds that*

$$\lim_{N \rightarrow \infty} \frac{e_{q,\infty}(\widehat{E}_{N,q}^{\text{ad}})}{\inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_{[c(\widehat{E}_{N,q}^{\text{ad}})]}^{\text{ad}}(C([0, 1]; \mathbb{R}^d)) \right\}} = 1$$

and

$$\lim_{N \rightarrow \infty} \frac{e_{q,\infty}(\widehat{E}_N^{\text{eq}})}{\inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R}^d)) \right\}} = 1;$$

i.e., the approximations $(\widehat{E}_{N,q}^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{E}_N^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_{[c(\widehat{E}_{N,q}^{\text{ad}})]}^{\text{ad}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively.

Proof. This result is an immediate consequence of the Propositions 3.3, 3.6, and 3.7. □

In view of all the previous considerations, our primary goal in this chapter is as follows:

We aim at extending the results of Müller-Gronbach (2002a)—especially the ones given in Corollary 3.8—to SDEs whose coefficients are allowed to grow super-linearly. In particular, we seek strongly asymptotically optimal approximations in the classes of adaptive and of equidistant approximations in such a setting.

To begin with, it is natural to ask whether the equidistant and the adaptive Euler–Maruyama schemes introduced above preserve strong asymptotic optimality in their respective classes even for SDEs whose coefficients grow super-linearly. We answer this question to the negative in the following proposition.

Proposition 3.9. *Let $d = m = T = 1$ and let the SDE (1.4) be autonomous and have a unique solution $(X(t))_{t \in [0,1]}$. Moreover, assume that $\mathbb{P}(\{\sigma(\xi) \neq 0\}) > 0$ and that there exist $C \in [1, \infty)$, $\alpha, \beta \in (1, \infty)$ with $\alpha < \beta$ such that for all $x \in \mathbb{R}$ with $|x| \geq C$ it holds that $\max\{|\mu(x)|, |\sigma(x)|\} \geq C^{-1} \cdot |x|^\beta$ and $\min\{|\mu(x)|, |\sigma(x)|\} \leq C \cdot |x|^\alpha$. Then for all $q \in [1, \infty)$ with $\|X(1)\|_{L_q(\Omega)} < \infty$ it holds that*

$$\lim_{N \rightarrow \infty} \|X(1) - \tilde{E}_N(1)\|_{L_q(\Omega)} = \infty \quad (3.8)$$

and

$$\lim_{N \rightarrow \infty} e_{q,\infty}(\hat{E}_N^{\text{eq}}) = \infty = \lim_{N \rightarrow \infty} e_{q,\infty}(\hat{E}_{N,q}^{\text{ad}}).$$

Proof. See Theorem 2.1 in Hutzenthaler et al. (2011) for a proof of (3.8). Moreover, we have

$$\hat{E}_N^{\text{eq}}(1) = \tilde{E}_N(1) = \hat{E}_{N,q}^{\text{ad}}(1)$$

for all $N \in \mathbb{N}$. Combining this with (3.8) shows

$$\infty = \lim_{N \rightarrow \infty} \|X(1) - \tilde{E}_N(1)\|_{L_q(\Omega)} \leq \lim_{N \rightarrow \infty} e_{q,\infty}(\hat{E}_N^{\text{eq}})$$

and

$$\infty = \lim_{N \rightarrow \infty} \|X(1) - \tilde{E}_N(1)\|_{L_q(\Omega)} \leq \lim_{N \rightarrow \infty} e_{q,\infty}(\hat{E}_{N,q}^{\text{ad}}).$$

□

The preceding proposition emphasizes that the previously constructed equidistant and adaptive Euler–Maruyama schemes are not suitable to globally approximate SDEs with super-linearly growing coefficients, which constitute a special case of SDEs whose coefficients are not globally Lipschitz continuous. In recent years, especially in the last decade, approximations that converge strongly to the solution even if the coefficients of the considered SDE are non-globally Lipschitz continuous have experienced increased attention. For instance, we mention implicit schemes (see, e.g., Higham et al. (2002)), tamed schemes (see Hutzenthaler et al. (2012), Gan & Wang (2013), Sabanis (2016), Kumar & Sabanis (2019), Sabanis & Zhang (2018)), truncated schemes (see Mao (2015), Guo et al. (2018)), projected schemes (see Beyn et al. (2016, 2017)), and balanced schemes (see Tretyakov & Zhang (2013)). For the particular case of SDEs with discontinuous coefficients, Euler–Maruyama type schemes are addressed in Leobacher & Szölgényi (2018), Ngo & Taguchi (2017), and Müller-Gronbach & Yaroslavtseva (2020), where the last mentioned reference also provides upper bounds for the specific error criterion $e_{q,\infty}$. In Hefter & Herzwurm (2017), the authors studied a particular SDE for which the solution is the square of a one-dimensional Bessel-process, and they gave upper error bounds for certain squared piecewise-constantly interpolated projected Euler schemes. Moreover, Fang & Giles (2018) constructed an adaptive timestepping strategy which

lets the related piecewise-constantly interpolated Euler–Maruyama schemes convergence strongly with order (at least) $1/2$ for SDEs with one-sided Lipschitz continuous drift coefficient and globally Lipschitz continuous diffusion coefficient; Kelly & Lord (2018a,b) extended this timestepping idea towards SDEs whose diffusion coefficients may also be non-globally Lipschitz continuous. We stress that the asymptotic constants obtained from all the previously mentioned references are (up to exceptional cases) unspecified.

To achieve the primary goal of this chapter, we follow the approach of so-called tamed Euler schemes as a basis for the construction of strongly asymptotically optimal approximations in the setting of SDEs with super-linearly growing coefficients.

A first type of tamed Euler schemes was initially proposed in Hutzenthaler et al. (2012) where the authors introduced the following continuous-time tamed Euler schemes. For this purpose, let the SDE (1.4) be autonomous for the moment. For $N \in \mathbb{N}$, the *continuous-time tamed Euler scheme* $\tilde{X}_N^{\text{HJK}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{X}_N^{\text{HJK}}(0) &:= \xi, \\ \tilde{X}_N^{\text{HJK}}(t) &:= \tilde{X}_N^{\text{HJK}}(t_\ell^{(N)}) + \frac{\mu(\tilde{X}_N^{\text{HJK}}(t_\ell^{(N)}))}{1 + T/N \cdot |\mu(\tilde{X}_N^{\text{HJK}}(t_\ell^{(N)}))|} \cdot (t - t_\ell^{(N)}) \\ &\quad + \sigma(\tilde{X}_N^{\text{HJK}}(t_\ell^{(N)})) \cdot (W(t) - W(t_\ell^{(N)})), \end{aligned}$$

for all $\ell \in \{0, \dots, N-1\}$ and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$, and the *equidistant tamed Euler scheme* $\hat{X}_N^{\text{HJK,eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\hat{X}_N^{\text{HJK,eq}}(t_\ell^{(N)}) := \tilde{X}_N^{\text{HJK}}(t_\ell^{(N)})$$

for all $\ell \in \{0, \dots, N\}$ and linearly interpolated between these time points. In the case of an SDE whose diffusion coefficient is still required to be globally Lipschitz continuous and whose drift coefficient merely has to satisfy a so-called “one-sided Lipschitz condition”, the continuous-time and the equidistant tamed Euler schemes above converge strongly to the solution. Even more, these schemes admit analogous upper error bounds as the continuous-time and the equidistant Euler–Maruyama schemes as stated in the Propositions 3.4 and 3.5.

Proposition 3.10. *Let the SDE (1.4) be autonomous and let the Assumptions (I_a) , (M_1) , and (pL_0^g) be satisfied for all $a \in [2, \infty)$. Moreover, let μ be continuously differentiable and assume that there exists $r \in (0, \infty)$ such that for all $x \in \mathbb{R}^d$ it holds that $|\nabla \mu(x)| \leq r \cdot (1 + |x|)^r$. Then for all $q \in [1, \infty)$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N} \setminus \{1\}$ it holds that*

$$\left\| \|X - \tilde{X}_N^{\text{HJK}}\|_{C([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1/2} \quad (3.9)$$

and

$$e_{q, \infty}(\hat{X}_N^{\text{HJK,eq}}) \leq C \cdot (\log(N)/N)^{1/2}. \quad (3.10)$$

Proof. See Theorem 1.1 in Hutzenthaler et al. (2012) for a proof of (3.9) and see Corollary 6.1 in Hutzenthaler et al. (2013) for a proof of (3.10). \square

Inspired by the work of Hutzenthaler et al. (2012), Sabanis (2016) constructed a further type of continuous-time tamed Euler schemes. For $N \in \mathbb{N}$ and $r \in [0, \infty)$, the *continuous-time tamed Euler scheme* $\tilde{X}_{N,r}^S : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{X}_{N,r}^S(t) := & \xi + \int_0^t \frac{\mu(s, \tilde{X}_{N,r}^S(\lfloor sN \rfloor / N))}{1 + N^{-1/2} \cdot |\tilde{X}_{N,r}^S(\lfloor sN \rfloor / N)|^r} ds \\ & + \int_0^t \frac{\sigma(s, \tilde{X}_{N,r}^S(\lfloor sN \rfloor / N))}{1 + N^{-1/2} \cdot |\tilde{X}_{N,r}^S(\lfloor sN \rfloor / N)|^r} dW(s), \end{aligned} \quad (3.11)$$

for all $t \in [0, T]$. Again, these schemes converge strongly with order (at least) $1/2$ but now even for SDEs whose diffusion coefficients may grow super-linearly.

Proposition 3.11. *Let the Assumptions (I_a) , (H) , (K_a) , (M_b) , and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Then for all $q \in [1, \min\{b, a/(2r + 1)\})$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \tilde{X}_{N,r}^S\|_{C([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1/2}.$$

Proof. See Theorem 3 in Sabanis (2016). □

Our main idea is now to modify the tamed Euler schemes of Sabanis (2016) in such a way that we can derive equidistant and adaptive variants that are strongly asymptotically optimal in their respective classes. These new approximations will be presented in full details in the next section.

3.3 The Equidistant and the Adaptive Tamed Euler Schemes

In the following, we introduce two variants of so-called tamed Euler schemes that are based on equidistant and on adaptive time discretizations, respectively. The crucial ingredient for both approximations is a continuous-time tamed Euler scheme which, on the one hand, is suitably close to the solution of the SDE (1.4) and which, on the other hand, possesses a simple recursive structure that will be exploited in the further analysis. These equidistant and adaptive tamed Euler schemes will turn out to be strongly asymptotically optimal in the classes of equidistant and of adaptive approximations, respectively.

We point out that the continuous-time tamed Euler scheme presented here is heavily inspired by the one introduced in Sabanis (2016). The reason we do not use the latter is that our approach is more convenient for our analysis; in particular, our scheme satisfies the desired recursion (3.12) below. Nevertheless, observe that both schemes coincide in the case that the SDE (1.4) is autonomous and $T = 1$.

3.3.1 The Continuous-time Tamed Euler Schemes

Let $N \in \mathbb{N}$ and $r \in [0, \infty)$, and recall the equidistant time discretization (3.5). The *continuous-time tamed Euler scheme* $\tilde{X}_{N,r} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{X}_{N,r}(0) &:= \xi, \\ \tilde{X}_{N,r}(t) &:= \tilde{X}_{N,r}(t_\ell^{(N)}) + \frac{\mu(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \cdot (t - t_\ell^{(N)}) \\ &\quad + \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \cdot (W(t) - W(t_\ell^{(N)})), \end{aligned} \quad (3.12)$$

for all $\ell \in \{0, \dots, N-1\}$ and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$.

Note that this process satisfies almost surely for all $t \in [0, T]$

$$\begin{aligned} \tilde{X}_{N,r}(t) &= \xi + \int_0^t \frac{\mu(\lfloor sN/T \rfloor \cdot T/N, \tilde{X}_{N,r}(\lfloor sN/T \rfloor \cdot T/N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\lfloor sN/T \rfloor \cdot T/N)|^r} ds \\ &\quad + \int_0^t \frac{\sigma(\lfloor sN/T \rfloor \cdot T/N, \tilde{X}_{N,r}(\lfloor sN/T \rfloor \cdot T/N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\lfloor sN/T \rfloor \cdot T/N)|^r} dW(s); \end{aligned} \quad (3.13)$$

cf. the construction (3.11).

Since entire trajectories of the driving Brownian motion are used in the construction of the continuous-time tamed Euler scheme, this scheme is not an approximation in the sense of Section 2.1, and we thus find $\tilde{X}_{N,r} \notin \mathbb{X}(C([0, T]; \mathbb{R}^d))$.

In Section A.3, we gather useful properties of the continuous-time tamed Euler schemes, namely, moment bounds and strong convergence with order (at least) $1/2$.

3.3.2 The Equidistant Tamed Euler Schemes

Next, based on the continuous-time tamed Euler scheme, we construct approximations which use not whole paths of the driving Brownian motion but evaluate W only at equidistant sites.

As before, let $N \in \mathbb{N}$ and $r \in [0, \infty)$. The *equidistant tamed Euler scheme* $\hat{X}_{N,r}^{\text{eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\hat{X}_{N,r}^{\text{eq}}(t_\ell^{(N)}) := \tilde{X}_{N,r}(t_\ell^{(N)})$$

for all $\ell \in \{0, \dots, N\}$ and linearly interpolated between these time points.

By suitably choosing sequences ψ , χ , and φ as per Section 2.1, we obtain $\hat{X}_{N,r}^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d))$. Clearly, the total number of evaluations of W employed in the approximation $\hat{X}_{N,r}^{\text{eq}}$ is given by N for each realization, which immediately yields $c(\hat{X}_{N,r}^{\text{eq}}) = N$.

3.3.3 The Adaptive Tamed Euler Schemes

The following construction of the adaptive tamed Euler schemes is heavily inspired by the corresponding construction of the adaptive Euler–Maruyama schemes $\widetilde{E}_{N,q}^{\text{ad}}$ presented in Section 3.2.

Note that, under suitable regularity assumptions on the coefficients of the SDE (1.4), its solution $(X(t))_{t \in [0,T]}$ satisfies

$$\mathbb{E} \left[|X_i(t + \delta) - X_i(t)|^2 \mid X(t) \right] = \sum_{j=1}^m |\sigma_{i,j}(t, X(t))|^2 \cdot \delta + o(\delta)$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$. Hence, the paths of each component X_i of the solution of the considered SDE are, in the root mean square sense and conditioned on $X(t)$, locally Hölder–1/2–continuous with Hölder constant $(\sum_{j=1}^m |\sigma_{i,j}(t, X(t))|^2)^{1/2}$, and the maximum over $i \in \{1, \dots, d\}$ of all these constants is given by $|\sigma(t, X(t))|_{\infty,2}$. For this reason, it is more beneficial to evaluate W more often in regions where the value of $|\sigma(t, X(t))|_{\infty,2}$ is large and vice versa.

Motivated by this idea, we construct our adaptive tamed Euler scheme in two steps. First, we use equidistant time steps to roughly approximate the solution and thereby obtain estimates for the conditional Hölder constants at these sites. Second, we refine our approximation by taking into account the local smoothness of the solution. More precisely, we distribute additional evaluation sites between those equidistant time points for which the corresponding maximum of the estimated Hölder constant is large in proportion to the other time points.

Let $r \in [0, \infty)$ and let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers satisfying the limits (3.6). Fix $N \in \mathbb{N}$ and put

$$\mathcal{A}_{k_N} := \left(\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \right|_{\infty,2}^2 \right)^{1/2}. \quad (3.14)$$

Let $q \in [1, \infty)$. For each $\ell \in \{0, \dots, k_N - 1\}$, we consider the random discretization

$$t_\ell^{(k_N)} = \tau_{\ell,0}^{(k_N)} < \tau_{\ell,1}^{(k_N)} < \dots < \tau_{\ell,\eta_\ell+1}^{(k_N)} = t_{\ell+1}^{(k_N)} \quad (3.15)$$

of $[t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]$ where

$$\eta_\ell := \mathbb{1}_{\{\mathcal{A}_{k_N} > 0\}} \cdot \left[N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} \cdot \frac{\left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \right|_{\infty,2}^2}{\sum_{i=0}^{k_N-1} \left| \frac{\sigma(t_i^{(k_N)}, \widetilde{X}_{k_N,r}(t_i^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_i^{(k_N)})|^r} \right|_{\infty,2}^2} \right] \quad (3.16)$$

and

$$\tau_{\ell,\kappa}^{(k_N)} := t_\ell^{(k_N)} + \frac{T}{k_N} \cdot \frac{\kappa}{\eta_\ell + 1}$$

for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$. The *adaptive tamed Euler scheme* $\widehat{X}_{N,r,q}^{\text{ad}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \widehat{X}_{N,r,q}^{\text{ad}}(\tau_{\ell,\kappa}^{(k_N)}) &:= \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}) + \frac{\mu(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \cdot (\tau_{\ell,\kappa}^{(k_N)} - t_\ell^{(k_N)}) \\ &\quad + \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \cdot (W(\tau_{\ell,\kappa}^{(k_N)}) - W(t_\ell^{(k_N)})) \end{aligned}$$

for all $\ell \in \{0, \dots, k_N - 1\}$ and for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$, and linearly interpolated between all these time points; cf. the construction (3.7).

By suitably choosing sequences ψ , χ , and φ as per Section 2.1, we obtain $\widehat{X}_{N,r,q}^{\text{ad}} \in \mathbb{X}_{[c(\widehat{X}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(C([0, T]; \mathbb{R}^d))$ if the cost of this approximation satisfy $c(\widehat{X}_{N,r,q}^{\text{ad}}) < \infty$. Define $\nu_{N,r,q}^{\text{ad}}$ to be the (random) number of evaluations of W employed in the approximation $\widehat{X}_{N,r,q}^{\text{ad}}$. Observe that

$$\nu_{N,r,q}^{\text{ad}} = k_N + \sum_{\ell=0}^{k_N-1} \eta_\ell \leq k_N + N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} \quad (3.17)$$

and

$$\nu_{N,r,q}^{\text{ad}} \geq \max \left\{ k_N, k_N + \mathbf{1}_{\{\mathcal{A}_{k_N} > 0\}} \cdot (N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} - k_N) \right\}. \quad (3.18)$$

3.4 Main Results

The following theorems entirely specify the asymptotics of the N th minimal errors in the classes of adaptive and of equidistant approximations as well as the asymptotics of the errors of the adaptive and of the equidistant tamed Euler schemes. As a consequence, we will conclude strong asymptotic optimality of these approximations in their respective classes. The proofs of all theorems are postponed to Section 3.6. Finally, we compare our results to those given in Müller-Gronbach (2002a) at the end of this section.

Recall the definition (3.1) of the asymptotic constants from the beginning of this chapter. The succeeding remark provides sufficient conditions for the finiteness of these two constants.

Remark 3.12. *Let the Assumptions (I_a) , (locL) , (K_a) , and (pG_r^σ) be satisfied for some $a \in [2, \infty)$ and $r \in [1, \infty)$ with $a \geq \max\{2r, 3r - 2\}$. Then Proposition A.5 implies $C_{(a-2r+2)/r, \infty}^{\text{eq}} < \infty$.*

First, we specify the asymptotics of the N th minimal errors in the classes of adaptive and of equidistant approximations. More precisely, we not only state the convergence rates but also give the sharp asymptotic constants. To this end, recall the definition (3.1) of $C_{q,\infty}^{\text{ad}}$ and $C_{q,\infty}^{\text{eq}}$.

Theorem 3.13. *Let the Assumptions (I_a) , (H) , (K_a) , (M_b) , and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Then for all $q \in [1, \min\{b, a/(2r + 1)\})$ it holds that*

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)) \right\} = C_{q,\infty}^{\text{ad}} \quad (3.19)$$

and

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)) \right\} = C_{q,\infty}^{\text{eq}}. \quad (3.20)$$

Proof. This result is an immediate consequence of the Lemmas 3.22, 3.23, 3.24, and 3.25 given in Section 3.6. \square

Next, we specify the asymptotics of the errors of the adaptive and of the equidistant tamed Euler schemes. Again, we not only state the convergence rates but also provide the sharp asymptotic constants. Right before, we show that the average numbers of observation sites of the adaptive tamed Euler schemes are finite and give their asymptotic behavior.

Theorem 3.14. *Let the Assumptions (I_a) , (H) , (K_a) , (M_b) , and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Then for all $q \in [1, \min\{b, a/(2r + 1)\})$ it holds that*

$$c(\widehat{X}_{N,r,q}^{\text{ad}}) < \infty$$

for each $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} c(\widehat{X}_{N,r,q}^{\text{ad}}) \cdot N^{-1} = \left(2^{1/2} \cdot C_{q,\infty}^{\text{ad}} \right)^{2q/(q+2)}$$

as well as

$$\lim_{N \rightarrow \infty} \left(c(\widehat{X}_{N,r,q}^{\text{ad}}) / \log(c(\widehat{X}_{N,r,q}^{\text{ad}})) \right)^{1/2} \cdot e_{q,\infty}(\widehat{X}_{N,r,q}^{\text{ad}}) = C_{q,\infty}^{\text{ad}}$$

and

$$\lim_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot e_{q,\infty}(\widehat{X}_{N,r}^{\text{eq}}) = C_{q,\infty}^{\text{eq}}.$$

Proof. This result is an immediate consequence of the Lemmas 3.21, 3.22, 3.23, 3.24, and 3.25 given in Section 3.6. \square

Since both the convergence rates and the corresponding asymptotic constants match in the preceding theorems, we obtain strong asymptotic optimality of the adaptive and of the equidistant tamed Euler schemes in their respective classes.

Corollary 3.15. *Let the Assumptions (I_a), (H), (K_a), (M_b), and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Then for all $q \in [1, \min\{b, a/(2r + 1)\})$ with $C_{q,\infty}^{\text{ad}} > 0$ it holds that*

$$\lim_{N \rightarrow \infty} \frac{e_{q,\infty}(\widehat{X}_{N,r,q}^{\text{ad}})}{\inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_{[c(\widehat{X}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(C([0, T]; \mathbb{R}^d)) \right\}} = 1$$

and

$$\lim_{N \rightarrow \infty} \frac{e_{q,\infty}(\widehat{X}_{N,r}^{\text{eq}})}{\inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)) \right\}} = 1;$$

i.e., the approximations $(\widehat{X}_{N,r,q}^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{X}_{N,r}^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_{[c(\widehat{X}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively.

Proof. This result is an immediate consequence of the Theorems 3.13 and 3.14. \square

Remark 3.16. *Note that the assumptions required in the preceding results essentially come from Proposition A.7, which states strong convergence of the continuous-time tamed Euler schemes. In this situation, we also have moment bounds for these schemes as well as for the solution, see the Propositions A.6 and A.5.*

In the next remark, we characterize equality of the asymptotic constants $C_{q,\infty}^{\text{ad}}$ and $C_{q,\infty}^{\text{eq}}$, and briefly comment on the special case when these two constants both are zero.

Remark 3.17. *Let the SDE (1.4) have a unique solution $(X(t))_{t \in [0, T]}$ and let $q \in [1, \infty)$ such that $C_{q,\infty}^{\text{eq}} < \infty$.*

Then we have $C_{q,\infty}^{\text{ad}} = C_{q,\infty}^{\text{eq}}$ if and only if almost surely $|\sigma(t, X(t))|_{\infty, 2} = \|\sigma(0, \xi)\|_{\infty, 2} \|_{L_2(\Omega)}$ holds for all $t \in [0, T]$, cf. Remark 1 in Müller-Gronbach (2002a). In particular, this case applies to SDEs whose diffusion coefficient is constant.

Moreover, we have $C_{q,\infty}^{\text{eq}} = 0$ (and hence $C_{q,\infty}^{\text{ad}} = 0$) if and only if almost surely $\sigma(t, X(t)) = 0$ holds for all $t \in [0, T]$. In such a situation, Theorems 3.13 and 3.14 do not allow to conclude strong asymptotic optimality. More precisely, these results merely show that the N th minimal errors in the classes of adaptive and of equidistant approximations as well as the errors of the adaptive and of the equidistant tamed Euler schemes each are in $o((\log(N)/N)^{1/2})$. In fact, these N th minimal errors are actually zero as the underlying SDE almost surely represents an ordinary differential equation in the considered case.

The preceding findings (Theorem 3.13, Theorem 3.14, and Corollary 3.15) generalize the corresponding results presented in Müller-Gronbach (2002a). In this reference, the coefficients of the SDE (1.4) are required to be globally Lipschitz continuous, whereas in our setting these coefficients may even grow super-linearly.

Remark 3.18. Let $T = 1$ and let the Assumptions $(I_{a^*}), (H)$, (pL_0^μ) , and (pL_0^σ) be satisfied for some $a^* \in [2, \infty)$. Proposition 3.3 then shows that the N th minimal errors in the classes of adaptive and of equidistant approximations satisfy the asymptotics (3.19) and (3.20) for all $q \in [1, a^*]$.

Moreover, in the setting above, Remark 3.1 yields that the Assumption (K_a) is satisfied for all $a \in [0, \infty)$, that the Assumption (M_b) is satisfied for all $b \in [0, \infty)$, and that the Assumption (pL_r^μ) is satisfied for all $r \in [0, \infty)$. By choosing $a = b = a^*$ and $r = 0$, Theorem 3.13 shows that the N th minimal errors in the classes of adaptive and of equidistant approximations satisfy (3.19) and (3.20) for all $q \in [1, a^*]$.

The only difference is, hence, that the choice $q = a^*$ is included in the results of Müller-Gronbach (2002a) whereas this case is not incorporated in our findings.

3.5 Numerical Experiment

We illustrate the results of the preceding section by a numerical experiment. To this end, we consider the introductory SDE (1.2) regarding the Heston–3/2–model with parameters $d = 1$, $m = 1$, $T = 1$, $\alpha = 5$, $\beta = 1$, and $\xi = 1$. Thus, this SDE reads as

$$\begin{aligned} dX(t) &= 5 \cdot X(t) \cdot (1 - |X(t)|) dt + |X(t)|^{3/2} dW(t), \quad t \in [0, 1], \\ X(0) &= 1. \end{aligned} \quad (3.21)$$

It is easy to see that the SDE (3.21) satisfies all the assumptions of our main theorems. More precisely, we have that Assumption (I_a) is satisfied for all $a \in [0, \infty)$, Assumption (H) is satisfied, Assumption (K_a) is satisfied for all $a \in [2, 11]$, Assumption (M_b) is satisfied for all $b \in [2, 6]$, and Assumption (pL_r^μ) is satisfied for all $r \in [1, \infty)$; cf. the Appendix in Sabanis (2016). For the rest of this section, we fix $a = 11$, $b = 6$, $r = 1$, and $q = 2$.

In view of Theorem 3.14, we aim at visualizing that, for large $N \in \mathbb{N}$, the approximation errors $e_{2,\infty}(\widehat{X}_{N,1,2}^{\text{ad}})$ and $e_{2,\infty}(\widehat{X}_{N,1}^{\text{eq}})$ of the adaptive and of the equidistant tamed Euler schemes are close to $C_{2,\infty}^{\text{ad}} \cdot (\log(c(\widehat{X}_{N,1,2}^{\text{ad}})))/c(\widehat{X}_{N,1,2}^{\text{ad}})^{1/2}$ and $C_{2,\infty}^{\text{eq}} \cdot (\log(N)/N)^{1/2}$, respectively.

In doing so, we encounter three different approximation issues, namely, the approximation of the asymptotic constants $C_{2,\infty}^{\text{ad}}$ and $C_{2,\infty}^{\text{eq}}$, of the errors $e_{2,\infty}(\widehat{X}_{N,1,2}^{\text{ad}})$ and $e_{2,\infty}(\widehat{X}_{N,1}^{\text{eq}})$, and of the average number of evaluations $c(\widehat{X}_{N,1,2}^{\text{ad}})$.

Regarding the first approximation issue, we do not know numerically suitable closed-form expressions of the constants $C_{2,\infty}^{\text{ad}}$ and $C_{2,\infty}^{\text{eq}}$, nor of the solution, for the particular SDE (3.21). Therefore, we estimate these constants via Monte Carlo simulations in which we approximate the solution by an equidistant tamed Euler scheme with a sufficiently large number of discretization points. More precisely, we estimate $C_{2,\infty}^{\text{ad}}$ and $C_{2,\infty}^{\text{eq}}$ by

$$\widehat{C}_{2,\infty,M,N}^{\text{ad}} := 2^{-1/2} \cdot \frac{1}{M} \cdot \sum_{m=1}^M \left(\frac{1}{N} \cdot \sum_{\ell=0}^{N-1} |\widehat{X}_{N,1,m}^{\text{eq}}(t_\ell^{(N)})|^3 \right)^{1/2}$$

and

$$\widehat{C}_{2,\infty,M,N}^{\text{eq}} := 2^{-1/2} \cdot \left(\frac{1}{M} \cdot \sum_{m=1}^M \max_{\ell \in \{0, \dots, N\}} |\widehat{X}_{N,1,m}^{\text{eq}}(t_\ell^{(N)})|^3 \right)^{1/2},$$

respectively, where $M, N \in \mathbb{N}$ and where the random vectors

$$(\widehat{X}_{N,1,m}^{\text{eq}}(t_0^{(N)}), \dots, \widehat{X}_{N,1,m}^{\text{eq}}(t_N^{(N)})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{X}_{N,1}^{\text{eq}}(t_0^{(N)}), \dots, \widehat{X}_{N,1}^{\text{eq}}(t_N^{(N)}))$. Observe that for $C_{2,\infty}^{\text{ad}}$, we approximate the Lebesgue integral occurring in its definition by left Riemann sums. Proposition A.7 implies that $\widehat{C}_{2,\infty,M,N}^{\text{ad}}$ and $\widehat{C}_{2,\infty,M,N}^{\text{eq}}$ tend to $C_{2,\infty}^{\text{ad}}$ and $C_{2,\infty}^{\text{eq}}$, respectively, as M and N tend to infinity. Figure 3.1 depicts simulations of $\widehat{C}_{2,\infty,M,2^{27}}^{\text{ad}}$ and $\widehat{C}_{2,\infty,M,2^{27}}^{\text{eq}}$ in dependence of M along with their corresponding 95% CLT-based confidence intervals. Furthermore, we utilize the specific approximation values $C_{2,\infty}^{\text{ad}} \approx 0.7080$ and $C_{2,\infty}^{\text{eq}} \approx 1.7749$ obtained from realizations of $\widehat{C}_{2,\infty,10^4,2^{27}}^{\text{ad}}$ and of $\widehat{C}_{2,\infty,10^4,2^{27}}^{\text{eq}}$, respectively, for the black lines featured in Figure 3.2.

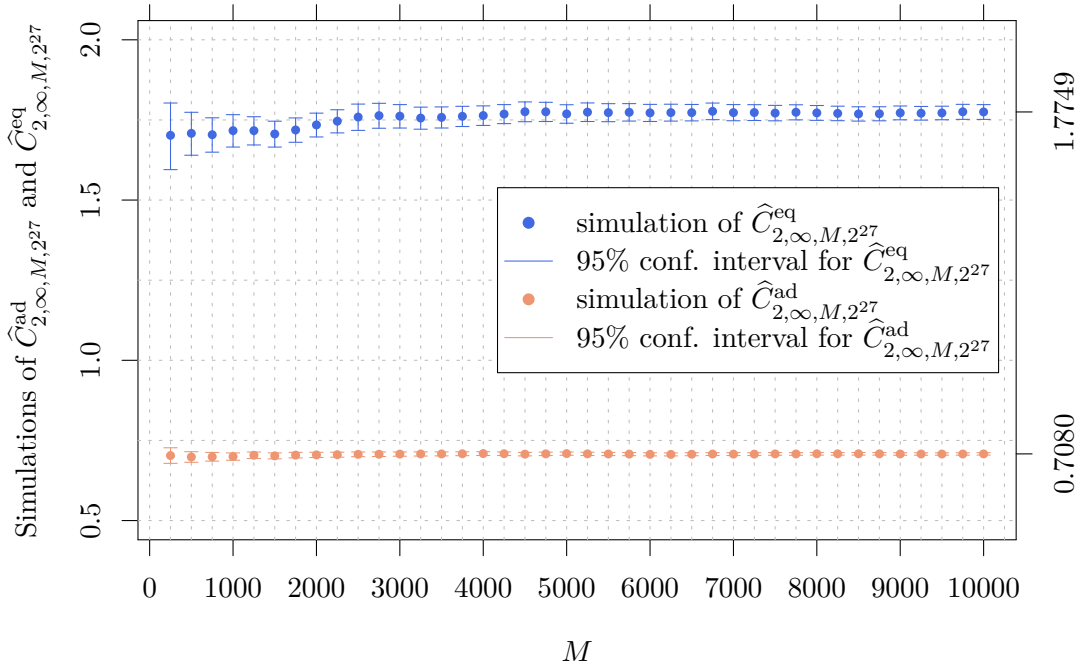


Figure 3.1: Monte Carlo approximations of the asymptotic constants $C_{2,\infty}^{\text{ad}}$ and $C_{2,\infty}^{\text{eq}}$ for the SDE (3.21).

The remaining two approximation issues are addressed simultaneously. Similarly to the approximation of the asymptotic constants, we again estimate the solution by a sufficiently accurate equidistant tamed Euler scheme, and we approximate the errors of the equidistant tamed Euler schemes as well as the errors and the average numbers of evaluations of the adaptive tamed Euler schemes via Monte Carlo simulations. More precisely, for each $N \in \mathbb{N}$ we estimate $e_{2,\infty}(\widehat{X}_{N,1}^{\text{eq}})$, $e_{2,\infty}(\widehat{X}_{N,1,2}^{\text{ad}})$, and $c(\widehat{X}_{N,1,2}^{\text{ad}})$ by

$$\widehat{e}_{2,\infty,M,N_{\max},N}^{\text{eq}} := \left(\frac{1}{M} \cdot \sum_{m=1}^M \max_{\ell \in \{0, \dots, N_{\max}\}} |\widehat{X}_{N_{\max},1,m}^{\text{eq}}(t_{\ell}^{(N_{\max})}) - \widehat{X}_{N,1,m}^{\text{eq}}(t_{\ell}^{(N_{\max})})|^2 \right)^{1/2},$$

$$\widehat{e}_{2,\infty,M,N_{\max},N}^{\text{ad}} := \left(\frac{1}{M} \cdot \sum_{m=1}^M \max_{\ell \in \{0, \dots, N_{\max}\}} |\widehat{X}_{N_{\max},1,m}^{\text{eq}}(t_{\ell}^{(N_{\max})}) - \widehat{X}_{N,1,2,m}^{\text{ad}}(t_{\ell}^{(N_{\max})})|^2 \right)^{1/2},$$

and

$$\widehat{c}_{M,N} := \frac{1}{M} \cdot \sum_{m=1}^M \nu_{N,1,2,m}^{\text{ad}},$$

respectively, where $M, N_{\max} \in \mathbb{N}$ and where the random vectors

$$(\widehat{X}_{N_{\max},1,m}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N_{\max},1,m}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{X}_{N_{\max},1}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N_{\max},1}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})}))$, the random vectors

$$(\widehat{X}_{N,1,m}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N,1,m}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{X}_{N,1}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N,1}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})}))$, the random vectors

$$(\widehat{X}_{N,1,2,m}^{\text{ad}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N,1,2,m}^{\text{ad}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{X}_{N,1,2}^{\text{ad}}(t_0^{(N_{\max})}), \dots, \widehat{X}_{N,1,2}^{\text{ad}}(t_{N_{\max}}^{(N_{\max})}))$, and the random variables

$$\nu_{N,1,2,m}^{\text{ad}}, \quad m \in \{1, \dots, M\},$$

are independent copies of $\nu_{N,1,2}^{\text{ad}}$. For the adaptive tamed Euler schemes, we thereby used $k_N := \lceil N \cdot (\log(N+1))^{-1/2} \rceil$ for all $N \in \mathbb{N}$ on every computation. Numerical estimates $(N, \widehat{e}_{2,\infty,10^4,2^{27},N}^{\text{eq}})$, $N \in \{2^6, 2^8, \dots, 2^{20}\}$, and $(\widehat{c}_{10^3,N}, \widehat{e}_{2,\infty,10^3,2^{27},N}^{\text{ad}})$, $N \in \{2^7, 2^9, \dots, 2^{21}\}$, are visualized in Figure 3.2.

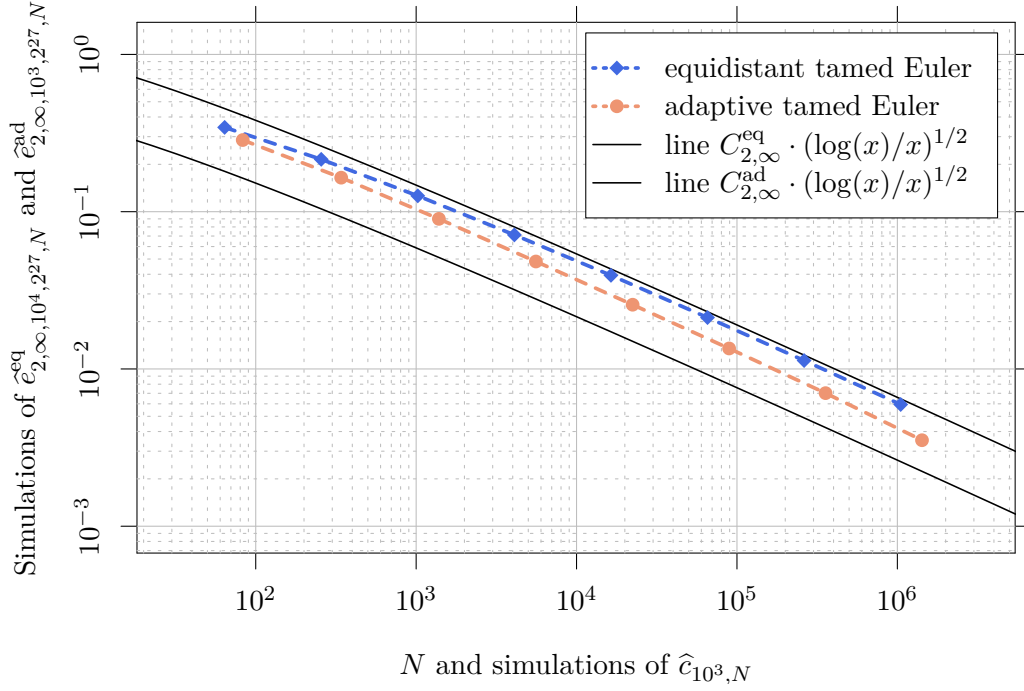


Figure 3.2: Monte Carlo approximations of the errors $e_{2,\infty}(\widehat{X}_{N,1}^{\text{eq}})$ and $e_{2,\infty}(\widehat{X}_{N,1,2}^{\text{ad}})$ versus N and Monte Carlo approximations of the average number of evaluations $c(\widehat{X}_{N,1,2}^{\text{ad}})$ for the SDE (3.21).

3.6 Proofs

In this section, we prove our main theorems from Section 3.4 by establishing asymptotic lower and upper error bounds as indicated in Section 2.3. The structure of the corresponding proofs is to a large extent based on techniques developed in Müller-Gronbach (2002a).

Throughout this section, let the Assumptions (I_a) , (H) , (K_a) , (M_b) , and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Observe that, in this setting, the Assumptions (locL) , (pG_{r+1}^μ) , $(\text{pL}_{r/2}^\sigma)$, and $(\text{pG}_{(r+2)/2}^\sigma)$ are also satisfied; see Remark 3.1. In addition, let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers such that the limits (3.6) hold, and let c denote unspecified positive constants that may vary at every occurrence and that may only depend on T , d , m , and the parameters and constants from the preceding assumptions.

3.6.1 Preliminaries

As a first step, we show that the supremum distance between, on the one hand, the solution inserted in the diffusion coefficient and, on the other hand, the continuous-time

tamed Euler scheme inserted in the tamed diffusion coefficient converges in probability to zero.

Lemma 3.19. *It holds that*

$$\sup_{t \in [0, T]} \left| \sigma(t, X(t)) - \frac{\sigma(t, \tilde{X}_{N,r}(t))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right| \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. Due to the triangle inequality, it suffices to show that

$$\sup_{t \in [0, T]} \left| \sigma(t, X(t)) - \sigma(t, \tilde{X}_{N,r}(t)) \right| \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \quad (3.22)$$

and

$$\sup_{t \in [0, T]} \left| \sigma(t, \tilde{X}_{N,r}(t)) - \frac{\sigma(t, \tilde{X}_{N,r}(t))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right| \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0. \quad (3.23)$$

To this end, we show L_θ convergence of the respective random variables to zero for appropriate values of $\theta \in (0, \infty)$. First, combining the Assumption (pL $_{r/2}^\sigma$), the Cauchy–Schwarz inequality, the Hölder inequality, and the Propositions A.7, A.5, and A.6 yields

$$\begin{aligned} & \left\| \sup_{t \in [0, T]} \left| \sigma(t, X(t)) - \sigma(t, \tilde{X}_{N,r}(t)) \right| \right\|_{L_\theta(\Omega)} \\ & \leq c \cdot \left\| \sup_{t \in [0, T]} |X(t) - \tilde{X}_{N,r}(t)| \cdot \left(1 + |X(t)| + |\tilde{X}_{N,r}(t)|\right)^{r/2} \right\|_{L_\theta(\Omega)} \\ & \leq c \cdot \left\| \sup_{t \in [0, T]} |X(t) - \tilde{X}_{N,r}(t)| \right\|_{L_{2\theta}(\Omega)} \cdot \left\| \sup_{t \in [0, T]} \left(1 + |X(t)| + |\tilde{X}_{N,r}(t)|\right)^{r/2} \right\|_{L_{2\theta}(\Omega)} \\ & \leq c \cdot \left\| \sup_{t \in [0, T]} |X(t) - \tilde{X}_{N,r}(t)| \right\|_{L_{2\theta}(\Omega)} \\ & \quad \cdot \left(1 + \left\| \sup_{t \in [0, T]} |X(t)|^{r/2} \right\|_{L_{2\theta}(\Omega)} + \left\| \sup_{t \in [0, T]} |\tilde{X}_{N,r}(t)|^{r/2} \right\|_{L_{2\theta}(\Omega)} \right) \\ & \leq c \cdot N^{-1/2} \end{aligned}$$

for all $N \in \mathbb{N}$ where $\theta := \min\{b, a/(2r+1)\}/3 \in [2/3, \infty)$. By letting N tend to infinity, we eventually obtain (3.22). Second, combining the Assumption (pG $_{(r+2)/2}^\sigma$), the triangle

inequality, and the Proposition A.6 yields

$$\begin{aligned}
 & \left\| \sup_{t \in [0, T]} \left| \sigma(t, \tilde{X}_{N,r}(t)) - \frac{\sigma(t, \tilde{X}_{N,r}(t))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right| \right\|_{L_\theta(\Omega)} \\
 &= (T/N)^{1/2} \cdot \left\| \sup_{t \in [0, T]} \left| \frac{\sigma(t, \tilde{X}_{N,r}(t)) \cdot |\tilde{X}_{N,r}(t)|^r}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t)|^r} \right| \right\|_{L_\theta(\Omega)} \\
 &\leq (T/N)^{1/2} \cdot \left\| \sup_{t \in [0, T]} |\sigma(t, \tilde{X}_{N,r}(t))| \cdot |\tilde{X}_{N,r}(t)|^r \right\|_{L_\theta(\Omega)} \\
 &\leq c \cdot N^{-1/2} \cdot \left\| \sup_{t \in [0, T]} \left(1 + |\tilde{X}_{N,r}(t)|\right)^{(r+2)/2} \cdot |\tilde{X}_{N,r}(t)|^r \right\|_{L_\theta(\Omega)} \\
 &\leq c \cdot N^{-1/2} \cdot \left\| \sup_{t \in [0, T]} \left(1 + |\tilde{X}_{N,r}(t)|\right)^{(3r+2)/2} \right\|_{L_\theta(\Omega)} \\
 &\leq c \cdot N^{-1/2} \cdot \left(1 + \left\| \sup_{t \in [0, T]} |\tilde{X}_{N,r}(t)| \right\|_{L_{a-r}(\Omega)}\right)^{(3r+2)/2} \\
 &\leq c \cdot N^{-1/2}
 \end{aligned}$$

for all $N \in \mathbb{N}$ where $\theta := 2 \cdot (a - r)/(3r + 2) \in [2, \infty)$. By letting N tend to infinity, we eventually obtain (3.23). \square

The previous result will turn out to be a crucial tool for the proofs of both the asymptotic lower and upper error bounds. More precisely, we will combine convergence in probability with uniform integrability to establish convergence in L_q for $q \in (0, \min\{b, a/(2r + 1)\})$ at certain places within our proofs, such as in the upcoming Lemma 3.21. In contrast, directly applying the Assumption (pL $_{r/2}^\sigma$) and the Hölder inequality as in the calculations above leads also to L_q convergence, but only up to smaller values of q .

Note that the preceding lemma yields the following useful result.

Corollary 3.20. *It holds that*

$$\mathcal{A}_{k_N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{L_2([0, T])}$$

and

$$\max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])}.$$

Proof. It suffices to show

$$\mathcal{A}_{k_N} - \left(\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} |\sigma(t_\ell^{(k_N)}, X(t_\ell^{(k_N)}))|_{\infty,2}^2 \right)^{1/2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \quad (3.24)$$

and

$$\left(\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} |\sigma(t_\ell^{(k_N)}, X(t_\ell^{(k_N)}))|_{\infty,2}^2 \right)^{1/2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])} \quad (3.25)$$

as well as

$$\begin{aligned} & \max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty,2} \\ & - \max_{\ell \in \{0, \dots, N-1\}} |\sigma(t_\ell^{(N)}, X(t_\ell^{(N)}))|_{\infty,2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \end{aligned} \quad (3.26)$$

and

$$\max_{\ell \in \{0, \dots, N-1\}} |\sigma(t_\ell^{(N)}, X(t_\ell^{(N)}))|_{\infty,2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{C([0,T])}. \quad (3.27)$$

First, the convergences (3.24) and (3.26) follow from Lemma 3.19 by using the inverse triangle inequality.

For the other two issues, observe that the function

$$[0, T] \rightarrow \mathbb{R}, \quad t \mapsto |\sigma(t, X(t, \omega))|_{\infty,2},$$

is continuous for every $\omega \in \Omega$ due to the Assumptions (locL) and (H). We thus obtain

$$\left(\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} |\sigma(t_\ell^{(k_N)}, X(t_\ell^{(k_N)}))|_{\infty,2}^2 \right)^{1/2} \xrightarrow[N \rightarrow \infty]{\text{a.s.}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])}$$

and

$$\max_{\ell \in \{0, \dots, N-1\}} |\sigma(t_\ell^{(N)}, X(t_\ell^{(N)}))|_{\infty,2} \xrightarrow[N \rightarrow \infty]{\text{a.s.}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{C([0,T])},$$

which prove (3.25) and (3.27). \square

Next, we show that the average numbers of evaluations of the driving Brownian motion employed in the adaptive tamed Euler schemes are finite and asymptotically equivalent to a constant times N . We obtain, in particular, that each such approximation does indeed lie in one of the classes of adaptive schemes.

Lemma 3.21. *For all $q \in [1, \min\{b, a/(2r+1)\})$ we have*

$$c(\widehat{X}_{N,r,q}^{\text{ad}}) < \infty \quad (3.28)$$

for each $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} c(\widehat{X}_{N,r,q}^{\text{ad}}) \cdot N^{-1} = \left(2^{1/2} \cdot C_{q,\infty}^{\text{ad}}\right)^{2q/(q+2)}. \quad (3.29)$$

Proof. For each $N \in \mathbb{N}$, the estimates (3.17) and (3.18) on the total number of evaluations of W employed in the respective adaptive tamed Euler scheme yield

$$c(\widehat{X}_{N,r,q}^{\text{ad}}) \leq k_N + N \cdot \mathbb{E}\left[\mathcal{A}_{k_N}^{2q/(q+2)}\right] \quad (3.30)$$

and

$$\begin{aligned} c(\widehat{X}_{N,r,q}^{\text{ad}}) &\geq k_N + \mathbb{E}\left[\mathbb{1}_{\{\mathcal{A}_{k_N} > 0\}} \cdot \left(N \cdot \mathcal{A}_{k_N}^{2q/(q+2)} - k_N\right)\right] \\ &= N \cdot \mathbb{E}\left[\mathcal{A}_{k_N}^{2q/(q+2)}\right] + k_N \cdot \mathbb{P}\left(\{\mathcal{A}_{k_N} = 0\}\right) \\ &\geq N \cdot \mathbb{E}\left[\mathcal{A}_{k_N}^{2q/(q+2)}\right]. \end{aligned} \quad (3.31)$$

Combining (3.30), Assumption $(\text{pG}_{(r+2)/2}^\sigma)$, and Proposition A.6 shows (3.28).

Next, note first that we have

$$\mathcal{A}_{k_N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])}$$

due to Corollary 3.20 and that the sequence $(\mathcal{A}_{k_N}^{2q/(q+2)})_{N \in \mathbb{N}}$ is uniformly integrable due to Assumption $(\text{pG}_{(r+2)/2}^\sigma)$ and Proposition A.6. Hence, we obtain

$$\mathcal{A}_{k_N}^{2q/(q+2)} \xrightarrow[N \rightarrow \infty]{L_1} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])}^{2q/(q+2)}. \quad (3.32)$$

By using (3.30), (3.31), (3.32), and the first limit in (3.6), we then conclude (3.29). \square

3.6.2 Asymptotic Lower Bounds

We start by introducing some notation that will be used in this subsection. For all $N \in \mathbb{N}$, for all $\alpha_1, \dots, \alpha_N \in [0, \infty)$, for all $q \in [1, \infty)$, and for independent Brownian bridges B_1, \dots, B_N on $[0, 1]$ we put

$$\mathcal{M}_q(\alpha_1, \dots, \alpha_N) := \mathbb{E}\left[\max_{\ell \in \{1, \dots, N\}} \left(\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|\right)^q\right] \in [0, \infty)$$

and

$$\mathcal{M}_q(N) := \mathcal{M}_q(\underbrace{1, \dots, 1}_{N \text{ times}}).$$

First, we prove an asymptotic lower bound for the N th minimal errors in the classes of adaptive approximations.

Lemma 3.22. *For all $q \in [1, \min\{b, a/(2r+1)\}]$ it holds that*

$$\liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)) \right\} \geq C_{q,\infty}^{\text{ad}}. \quad (3.33)$$

Proof. Fix $N \in \mathbb{N}$ with $N > \exp(2)$ and $\widehat{X}_N \in \mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d))$ for the moment. Due to the inverse triangle inequality and Proposition A.7, it holds that

$$\begin{aligned} e_{q,\infty}(\widehat{X}_N) &= \left\| \|X - \widehat{X}_N\|_{C([0, T])} \right\|_{L_q(\Omega)} \\ &\geq \left\| \|\widetilde{X}_{k_N, r} - \widehat{X}_N\|_{C([0, T])} \right\|_{L_q(\Omega)} - c \cdot k_N^{-1/2}. \end{aligned} \quad (3.34)$$

Let D_N denote the entire data used by the approximation \widehat{X}_N in the sense of Section 2.1, define Ψ_N to be the set of observation sites of the driving Brownian motion employed in \widehat{X}_N , and put $\nu_N := \#\Psi_N$.

As a first step, we show that the distance between $\widetilde{X}_{k_N, r}$ and \widehat{X}_N as above is greater or equal than the respective distance between $\widetilde{X}_{k_N, r}$ and $\mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N]$. Because of the first limit in (3.6), we may actually assume that $\{t_1^{(k_N)}, \dots, t_{k_N}^{(k_N)}\} \subseteq \Psi_N$. Hence, each $\widetilde{X}_{k_N, r}(t_\ell^{(k_N)})$, $\ell \in \{0, \dots, k_N\}$, is measurable with respect to the σ -algebra generated by D_N and we thereby obtain

$$\begin{aligned} &\widetilde{X}_{k_N, r}(t) - \mathbb{E}[\widetilde{X}_{k_N, r}(t) \mid D_N] \\ &= \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N, r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N, r}(t_\ell^{(k_N)})|^r} \cdot (W(t) - \mathbb{E}[W(t) \mid D_N]) \end{aligned} \quad (3.35)$$

for all $\ell \in \{0, \dots, k_N - 1\}$ and for all $t \in (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$. Similarly to the derivations of the Lemmas 1 and 2 in Yaroslavtseva (2017), one shows that for \mathbb{P}^{D_N} -almost all $(x, y) \in \mathbb{R}^d \times \bigcup_{n \in \mathbb{N}} (\mathbb{R}^m)^n$ it holds that

$$\mathbb{P}^{W \mid D_N = (x, y)} = \mathbb{P}^{-W \mid D_N = (x, y)},$$

which along with (3.35) yields

$$\mathbb{P}^{\widetilde{X}_{k_N, r} - \mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N] \mid D_N = (x, y)} = \mathbb{P}^{-\widetilde{X}_{k_N, r} + \mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N] \mid D_N = (x, y)}.$$

We thus conclude that $(\widetilde{X}_{k_N, r} - \mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N], D_N)$ and $(-\widetilde{X}_{k_N, r} + \mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N], D_N)$ are identically distributed. Since, additionally, both \widehat{X}_N and $\mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N]$ are measurable functions of D_N , we consequently find that $\widetilde{X}_{k_N, r} - \widehat{X}_N$ and $2\mathbb{E}[\widetilde{X}_{k_N, r} \mid D_N] - \widetilde{X}_{k_N, r} - \widehat{X}_N$ are also identically distributed.

Therefore, we obtain

$$\begin{aligned}
 & \left\| \left\| \tilde{X}_{k_N, r} - \hat{X}_N \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \\
 &= 1/2 \cdot \left(\left\| \left\| \tilde{X}_{k_N, r} - \hat{X}_N \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \right. \\
 & \quad \left. + \left\| \left\| -2\mathbb{E}[\tilde{X}_{k_N, r} | D_N] + \tilde{X}_{k_N, r} + \hat{X}_N \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \right) \\
 & \geq 1/2 \cdot \left\| \left\| \tilde{X}_{k_N, r} - \hat{X}_N - 2\mathbb{E}[\tilde{X}_{k_N, r} | D_N] + \tilde{X}_{k_N, r} + \hat{X}_N \right\|_{C([0, T])} \right\|_{L_q(\Omega)} \\
 &= \left\| \left\| \tilde{X}_{k_N, r} - \mathbb{E}[\tilde{X}_{k_N, r} | D_N] \right\|_{C([0, T])} \right\|_{L_q(\Omega)}.
 \end{aligned} \tag{3.36}$$

Now let

$$i^* : [0, T] \times \mathbb{R}^d \rightarrow \{1, \dots, d\}, \quad (t, x) \mapsto \min \left\{ i \in \{1, \dots, d\} \mid |\sigma(t, x)|_{\infty, 2} = |\sigma_i(t, x)|_2 \right\}.$$

Conditioned on D_N , the observation sites Ψ_N are fixed and for every $j \in \{1, \dots, m\}$ the Gaussian process $(W_j(t) - \mathbb{E}[W_j(t) | D_N])_{t \in [0, T]}$ consists of ν_N independent Brownian bridges between the discretization points. Hence, we obtain with Lemma A.1

$$\begin{aligned}
 & \mathbb{E} \left[\left\| \tilde{X}_{k_N, r} - \mathbb{E}[\tilde{X}_{k_N, r} | D_N] \right\|_{C([0, T])}^q \mid D_N \right] \\
 &= \mathbb{E} \left[\max_{i \in \{1, \dots, d\}} \max_{\ell \in \{0, \dots, k_N - 1\}} \sup_{t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]} \left| \sum_{j=1}^m \frac{\sigma_{i, j}(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N, r}(t_\ell^{(k_N)})|^r} \right. \right. \\
 & \quad \left. \left. \cdot (W_j(t) - \mathbb{E}[W_j(t) | D_N]) \right|^q \mid D_N \right] \\
 & \geq \mathbb{E} \left[\max_{\ell \in \{0, \dots, k_N - 1\}} \sup_{t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]} \left| \sum_{j=1}^m \frac{\sigma_{i^*(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)})), j}(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N, r}(t_\ell^{(k_N)})|^r} \right. \right. \\
 & \quad \left. \left. \cdot (W_j(t) - \mathbb{E}[W_j(t) | D_N]) \right|^q \mid D_N \right] \tag{3.37} \\
 &= \mathbb{E} \left[\max_{\ell \in \{0, \dots, k_N - 1\}} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N, r}(t_\ell^{(k_N)})|^r} \right|_{\infty, 2}^q \right. \\
 & \quad \left. \cdot \sup_{t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]} |W_1(t) - \mathbb{E}[W_1(t) | D_N]|^q \mid D_N \right] \\
 & \geq \left(\frac{1}{\nu_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N, r}(t_\ell^{(k_N)})|^r} \right|_{\infty, 2}^2 \cdot \frac{T}{k_N} \right)^{q/2} \cdot \mathcal{M}_q(\nu_N) \\
 &= \nu_N^{-q/2} \cdot \mathcal{A}_{k_N}^q \cdot \mathcal{M}_q(\nu_N)
 \end{aligned}$$

almost surely where \mathcal{A}_{k_N} is defined as in (3.14). Put

$$\delta_N := \max \left\{ 1, \sum_{\substack{\ell \in \{0, \dots, k_N - 1\} \\ \sigma(t_\ell^{(k_N)}, \tilde{X}_{k_N, r}(t_\ell^{(k_N)})) \neq 0}} \left(\#(\Psi_N \cap (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})) + 1 \right) \right\}. \quad (3.38)$$

Clearly, we have $\delta_N \leq \nu_N$ and consequently

$$\left\| \nu_N^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\nu_N))^{1/q} \right\|_{L_q(\Omega)} \geq \left\| \delta_N^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N)) \right\|_{L_q(\Omega)}. \quad (3.39)$$

Now consider the monotonically increasing and concave function

$$\lambda : [0, \infty) \rightarrow [0, \infty), \quad x \mapsto \begin{cases} x/\log(x), & \text{if } x > \exp(2), \\ x/2, & \text{else.} \end{cases}$$

The monotonicity of this function along with the inequality $\mathbb{E}[\nu_N] \leq N$ and the Jensen inequality gives

$$N/\log(N) = \lambda(N) \geq \lambda(\mathbb{E}[\nu_N]) \geq \mathbb{E}[\lambda(\nu_N)] \geq \mathbb{E}[\lambda(\delta_N)] \geq \mathbb{E}[\mathbf{1}_{\{\delta_N > \exp(2)\}} \cdot \delta_N / \log(\delta_N)].$$

By using this result and applying the Hölder inequality, we arrive at

$$\begin{aligned} & (N/\log(N))^{1/2} \cdot \left\| \delta_N^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N))^{1/q} \right\|_{L_q(\Omega)} \\ & \geq \left\| \left(\mathbf{1}_{\{\delta_N > \exp(2)\}} \cdot \delta_N / \log(\delta_N) \right)^{1/2} \right\|_{L_2(\Omega)} \cdot \left\| \delta_N^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N))^{1/q} \right\|_{L_q(\Omega)} \\ & \geq \left\| \left(\mathbf{1}_{\{\delta_N > \exp(2)\}} \cdot \delta_N / \log(\delta_N) \right)^{1/2} \cdot \delta_N^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N))^{1/q} \right\|_{L_{2q/(q+2)}(\Omega)} \quad (3.40) \\ & \geq \left\| (\log(\delta_N))^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N))^{1/q} \right. \\ & \quad \left. \cdot \mathbf{1}_{\{\delta_N > \exp(2)\} \cap \{\|\sigma(\cdot, X(\cdot))\|_{\infty, 2} \|C([0, T])\| > 0\}} \right\|_{L_{2q/(q+2)}(\Omega)}. \end{aligned}$$

Combining (3.34), (3.36), (3.37), (3.39), (3.40), and the second limit in (3.6) yields

$$\begin{aligned} & \liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q, \infty}(\hat{X}) \mid \hat{X} \in \mathbb{X}_N^{\text{ad}}(C([0, T]; \mathbb{R}^d)) \right\} \\ & \geq \liminf_{N \rightarrow \infty} \left\| \alpha^{(N)} \right\|_{L_{2q/(q+2)}(\Omega)} \end{aligned} \quad (3.41)$$

where

$$\alpha^{(N)} := (\log(\delta_N))^{-1/2} \cdot \mathcal{A}_{k_N} \cdot (\mathcal{M}_q(\delta_N))^{1/q} \cdot \mathbf{1}_{\{\delta_N > \exp(2)\} \cap \{\|\sigma(\cdot, X(\cdot))\|_{\infty, 2} \|C([0, T])\| > 0\}}.$$

Next, we use the subsequence argument that is provided by Lemma B.1 to infer (3.33) from (3.41). First of all, Corollary 3.20 and Lemma 3.19 yield

$$\mathcal{A}_{k_N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \quad (3.42)$$

and

$$\sup_{t \in [0, T]} \left| \left| \sigma(t, X(t)) \right|_{\infty, 2} - \left| \frac{\sigma(t, \tilde{X}_{k_N, r}(t))}{1 + (T/k_N)^{1/2} \cdot |\tilde{X}_{k_N, r}(t)|^r} \right|_{\infty, 2} \right| \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0. \quad (3.43)$$

Now let $(\alpha^{(N_{\kappa_n})})_{\kappa_n \in \mathbb{N}}$ be a subsequence of $(\alpha^{(N)})_{N \in \mathbb{N}}$. In view of (3.42) and (3.43), there exists a subsequence $(\mathcal{A}_{k_{N_{\kappa_n}}})_{n \in \mathbb{N}}$ of $(\mathcal{A}_{k_{N_{\kappa}}})_{\kappa \in \mathbb{N}}$ such that

$$\mathcal{A}_{k_{N_{\kappa_n}}} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \quad (3.44)$$

and

$$\sup_{t \in [0, T]} \left| \left| \sigma(t, X(t)) \right|_{\infty, 2} - \left| \frac{\sigma(t, \tilde{X}_{k_{N_{\kappa_n}}, r}(t))}{1 + (T/k_{N_{\kappa_n}})^{1/2} \cdot |\tilde{X}_{k_{N_{\kappa_n}}, r}(t)|^r} \right|_{\infty, 2} \right| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (3.45)$$

Recall the definition (3.38) of δ_N and observe that the function

$$[0, T] \rightarrow \mathbb{R}, \quad t \mapsto |\sigma(t, X(t, \omega))|_{\infty, 2},$$

is continuous for every $\omega \in \Omega$ due to the Assumptions (locL) and (H). By (3.45) and the just mentioned continuity, it holds that

$$\begin{aligned} & \mathbb{P} \left(\left\{ \lim_{n \rightarrow \infty} \delta_{N_{\kappa_n}} = \infty \right\} \cap \left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right) \\ &= \mathbb{P} \left(\left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right). \end{aligned} \quad (3.46)$$

Clearly, we have

$$\begin{aligned} & \mathbb{P} \left(\left\{ \liminf_{n \rightarrow \infty} \alpha^{(N_{\kappa_n})} \geq 2^{-1/2} \cdot \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \right\} \right. \\ & \quad \left. \cap \left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} = 0 \right\} \right) \\ &= \mathbb{P} \left(\left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} = 0 \right\} \right). \end{aligned} \quad (3.47)$$

Moreover, the limit (3.44), Lemma A.2(iii), and the equation (3.46) yield

$$\begin{aligned}
& \mathbb{P} \left(\left\{ \liminf_{n \rightarrow \infty} \alpha^{(N_{\kappa_n})} \geq 2^{-1/2} \cdot \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \right\} \right. \\
& \quad \left. \cap \left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right) \\
&= \mathbb{P} \left(\left\{ \liminf_{n \rightarrow \infty} \alpha^{(N_{\kappa_n})} \geq 2^{-1/2} \cdot \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \right\} \right. \\
& \quad \left. \cap \left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right. \\
& \quad \left. \cap \left\{ \lim_{n \rightarrow \infty} \mathcal{A}_{k_{N_{\kappa_n}}} = \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \right\} \right) \\
&\geq \mathbb{P} \left(\left\{ \lim_{n \rightarrow \infty} \delta_{N_{\kappa_n}} = \infty \right\} \cap \left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right. \\
& \quad \left. \cap \left\{ \lim_{n \rightarrow \infty} \mathcal{A}_{k_{N_{\kappa_n}}} = \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])} \right\} \right) \\
&= \mathbb{P} \left(\left\{ \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{C([0, T])} > 0 \right\} \right)
\end{aligned} \tag{3.48}$$

Combining (3.47) and (3.48), we conclude that

$$\liminf_{n \rightarrow \infty} \alpha^{(N_{\kappa_n})} \geq 2^{-1/2} \cdot \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{\infty, 2} \right\|_{L_2([0, T])}$$

holds almost surely. Consequently, Fatou's lemma gives

$$\liminf_{n \rightarrow \infty} \left\| \alpha^{(N_{\kappa_n})} \right\|_{L_{2q/(q+2)}(\Omega)} \geq C_{q, \infty}^{\text{rad}}.$$

Finally, employing Lemma B.1 finishes the proof of this lemma. \square

Next, we prove an asymptotic lower bound for the N th minimal errors in the classes of equidistant approximations.

Lemma 3.23. *For all $q \in [1, \min\{b, a/(2r + 1)\})$ it holds that*

$$\liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q, \infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)) \right\} \geq C_{q, \infty}^{\text{eq}}. \tag{3.49}$$

Proof. Fix $N \in \mathbb{N}$ and $\widehat{X}_N \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d))$ for the moment, and consider the data $D_N := (\xi, W(t_1^{(N)}), \dots, W(t_N^{(N)}))$.

Similarly to the estimates (3.34), (3.36), and (3.37) in the proof of Lemma 3.22, one successively shows that

$$e_{q, \infty}(\widehat{X}_N) \geq \left\| \left\| \widetilde{X}_{N, r} - \widehat{X}_N \right\|_{C([0, T])} \right\|_{L_q(\Omega)} - c \cdot N^{-1/2}, \tag{3.50}$$

$$\left\| \|\tilde{X}_{N,r} - \hat{X}_N\|_{C([0,T])} \right\|_{L_q(\Omega)} \geq \left\| \|\tilde{X}_{N,r} - \mathbb{E}[\tilde{X}_{N,r} | D_N]\|_{C([0,T])} \right\|_{L_q(\Omega)}, \quad (3.51)$$

and that

$$\begin{aligned} & \mathbb{E} \left[\left\| \tilde{X}_{N,r} - \mathbb{E}[\tilde{X}_{N,r} | D_N] \right\|_{C([0,T])}^q \middle| D_N \right] \\ & \geq \mathbb{E} \left[\max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|^q \right. \\ & \quad \cdot \left. \sup_{t \in [t_\ell^{(N)}, t_{\ell+1}^{(N)}]} |W_1(t) - \mathbb{E}[W_1(t) | D_N]|^q \middle| D_N \right] \\ & = (T/N)^{q/2} \cdot \mathcal{M}_q(\alpha_0^{(N)}, \dots, \alpha_{N-1}^{(N)}) \end{aligned} \quad (3.52)$$

holds almost surely where for the last equality we scaled each interval $[t_\ell^{(N)}, t_{\ell+1}^{(N)}]$, $\ell \in \{0, \dots, N-1\}$, to $[0, 1]$ and where

$$\alpha_\ell^{(N)} := \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2}$$

for all $\ell \in \{0, \dots, N-1\}$.

Combining (3.50), (3.51), and (3.52) yields

$$\begin{aligned} & \liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q, \infty}(\hat{X}) \mid \hat{X} \in \mathbb{X}_N^{\text{eq}}(C([0, T]; \mathbb{R}^d)) \right\} \\ & \geq T^{1/2} \cdot \liminf_{N \rightarrow \infty} \left\| (\log(N))^{-1/2} \cdot \mathcal{M}_q^{1/q}(\alpha_0^{(N)}, \dots, \alpha_{N-1}^{(N)}) \right\|_{L_q(\Omega)}. \end{aligned} \quad (3.53)$$

Next, we again use the subsequence argument that is provided by Lemma B.1 to infer (3.49) from (3.53). First of all, Corollary 3.20 gives

$$\alpha_\ell^{(N)} := \max_{\ell \in \{0, \dots, N-1\}} \alpha_\ell^{(N)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])}. \quad (3.54)$$

Now let $(\alpha^{(N_\kappa)})_{\kappa \in \mathbb{N}}$ be a subsequence of $(\alpha^{(N)})_{N \in \mathbb{N}}$. In view of (3.54), there exists a subsequence $(\alpha^{(N_{\kappa_n})})_{n \in \mathbb{N}}$ of $(\alpha^{(N_\kappa)})_{\kappa \in \mathbb{N}}$ such that

$$\alpha^{(N_{\kappa_n})} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])}.$$

Hence, Lemma A.2(iii) yields

$$(\log(N_{\kappa_n}))^{-1/2} \cdot \mathcal{M}_q^{1/q}(\alpha_0^{(N_{\kappa_n})}, \dots, \alpha_{N_{\kappa_n}-1}^{(N_{\kappa_n})}) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 2^{-1/2} \cdot \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])}.$$

Consequently, Fatou's lemma gives

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left\| (\log(N_{\kappa_n}))^{-1/2} \cdot \left(\mathcal{M}_q(\alpha_0^{(N_{\kappa_n})}, \dots, \alpha_{N_{\kappa_n}-1}^{(N_{\kappa_n})}) \right)^{1/q} \right\|_{L_q(\Omega)} \\ & \geq 2^{-1/2} \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0, T])} \right\|_{L_q(\Omega)}. \end{aligned}$$

Finally, employing Lemma B.1 finishes the proof of this lemma. \square

3.6.3 Asymptotic Upper Bounds

We start by introducing some notation that will be used in this subsection. For all $N \in \mathbb{N}$, for all $\alpha_1, \dots, \alpha_N \in [0, \infty)$, for all $q \in [1, \infty)$, and for independent Brownian bridges B_1, \dots, B_N on $[0, 1]$ we put

$$\mathcal{G}_q(\cdot; \alpha_1, \dots, \alpha_N) : [0, \infty) \rightarrow [0, 1], \quad u \mapsto \mathbb{P}\left(\left\{\max_{\ell \in \{1, \dots, N\}} \left(\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|\right)^q > u\right\}\right),$$

and

$$\mathcal{G}_q(\cdot; N) := \mathcal{G}_q(\cdot; \underbrace{1, \dots, 1}_{N \text{ times}}).$$

First, we prove an asymptotic upper bound for the errors of the adaptive tamed Euler schemes.

Lemma 3.24. *For all $q \in [1, \min\{b, a/(2r + 1)\})$ it holds that*

$$\limsup_{N \rightarrow \infty} \left(c(\widehat{X}_{N,r,q}^{\text{ad}}) / \log(c(\widehat{X}_{N,r,q}^{\text{ad}})) \right)^{1/2} \cdot e_{q,\infty}(\widehat{X}_{N,r,q}^{\text{ad}}) \leq C_{q,\infty}^{\text{ad}}.$$

Proof. Fix $N \in \mathbb{N}$ with $1 < k_N \leq N$ for the moment. Due to the triangle inequality and Proposition A.7, it holds that

$$\begin{aligned} e_{q,\infty}(\widehat{X}_{N,r,q}^{\text{ad}}) &= \left\| \|X - \widehat{X}_{N,r,q}^{\text{ad}}\|_{C([0,T])} \right\|_{L_q(\Omega)} \\ &\leq \left\| \|\widetilde{X}_{k_N,r} - \widehat{X}_{N,r,q}^{\text{ad}}\|_{C([0,T])} \right\|_{L_q(\Omega)} + c \cdot k_N^{-1/2}. \end{aligned} \quad (3.55)$$

Note that for all $\ell \in \{0, \dots, k_N - 1\}$ and for all $t \in (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$ we have

$$\widetilde{X}_{k_N,r}(t) - \widehat{X}_{N,r,q}^{\text{ad}}(t) = \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \cdot (W(t) - \widehat{W}_N^{\text{ad}}(t))$$

where $\widehat{W}_N^{\text{ad}} : \Omega \rightarrow C([0, T]; \mathbb{R}^m)$ denotes the piecewise-linear interpolation of W at the adaptive time points (3.15). Recall the definitions (3.14) and (3.16) of \mathcal{A}_{k_N} and η_ℓ , respectively. Observe that for all $\ell \in \{0, \dots, k_N - 1\}$ it holds that

$$\eta_\ell + 1 \geq \mathbf{1}_{\{\mathcal{A}_{k_N}=0\}} + \mathbf{1}_{\{\mathcal{A}_{k_N}>0\}} \cdot \frac{N \cdot T}{k_N} \cdot \mathcal{A}_{k_N}^{-4/(q+2)} \cdot \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \right|_{\infty, 2}^2,$$

which for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, k_N - 1\}$, and for all $\kappa \in \{0, \dots, \eta_\ell\}$ implies

$$\begin{aligned} &\left((\tau_{\ell, \kappa+1}^{(k_N)} - \tau_{\ell, \kappa}^{(k_N)}) \cdot \sum_{j=1}^m \left(\frac{\sigma_{i,j}(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \right)^2 \right)^{1/2} \\ &\leq \mathcal{A}_{k_N}^{2/(q+2)} \cdot N^{-1/2}. \end{aligned} \quad (3.56)$$

Put $D_{k_N} := (\xi, W(t_1^{(k_N)}), \dots, W(t_{k_N}^{(k_N)}))$. Conditioned on D_{k_N} , the observation sites (3.15) are fixed and for every $j \in \{1, \dots, m\}$ the Gaussian process $(W_j(t) - \widehat{W}_{N,j}^{\text{ad}}(t))_{t \in [0, T]}$ consists of $\nu_{N,r,q}^{\text{ad}}$ independent Brownian bridges between the discretization points. Hence, Lemma A.3 applied with

$$\beta := \mathcal{A}_{k_N}^{2/(q+2)} \cdot N^{-1/2}$$

(where we keep (3.56) in mind) and

$$z := (\beta^2 \cdot \log(\nu_{N,r,q})/2)^{q/2}$$

gives

$$\begin{aligned} & \mathbb{E} \left[\left\| \widetilde{X}_{k_N,r} - \widehat{X}_{N,r,q}^{\text{ad}} \right\|_{C([0,T])}^q \middle| D_{k_N} \right] \\ &= \mathbb{E} \left[\max_{i \in \{1, \dots, d\}} \max_{\substack{\ell \in \{0, \dots, k_N-1\} \\ \kappa \in \{0, \dots, \eta_\ell\}}} \sup_{t \in [\tau_{\ell, \kappa}^{(k_N)}, \tau_{\ell, \kappa+1}^{(k_N)}]} \left| \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(k_N)}, \widetilde{X}_{k_N,r}(t_\ell^{(k_N)}))}{1 + (T/k_N)^{1/2} \cdot |\widetilde{X}_{k_N,r}(t_\ell^{(k_N)})|^r} \right. \right. \\ & \quad \left. \left. \cdot (W_j(t) - \widehat{W}_{N,j}^{\text{ad}}(t)) \right|^q \middle| D_{k_N} \right] \quad (3.57) \\ &\leq (\beta^2 \cdot \log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2} + d \cdot \int_{(\beta^2 \cdot \log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2}}^{\infty} \mathcal{G}_q(u; \underbrace{\beta, \dots, \beta}_{\nu_{N,r,q}^{\text{ad}} \text{ times}}) du \end{aligned}$$

almost surely. By using integration by substitution in the case $\beta \neq 0$, we further obtain

$$\begin{aligned} & (\beta^2 \cdot \log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2} + d \cdot \int_{(\beta^2 \cdot \log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2}}^{\infty} \mathcal{G}_q(u; \underbrace{\beta, \dots, \beta}_{\nu_{N,r,q}^{\text{ad}} \text{ times}}) du \\ &= (\beta^2 \cdot \log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2} \\ & \quad \cdot \left(1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^{\infty} \mathcal{G}_q(u \cdot (\log(\nu_{N,r,q}^{\text{ad}}))^{-q/2}; \nu_{N,r,q}^{\text{ad}}) du \right) \\ &= \mathcal{A}_{k_N}^{2q/(q+2)} \cdot N^{-q/2} \cdot (\log(\nu_{N,r,q}^{\text{ad}})/2)^{q/2} \cdot I_{\nu_{N,r,q}^{\text{ad}}} \quad (3.58) \end{aligned}$$

almost surely where

$$I_{\nu_{N,r,q}^{\text{ad}}} := \left(1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^{\infty} \mathcal{G}_q(u \cdot (\log(\nu_{N,r,q}^{\text{ad}}))^{q/2}; \nu_{N,r,q}^{\text{ad}}) du \right).$$

Clearly, equation (3.58) holds also true in the case $\beta = 0$. We then deduce from (3.57) and (3.58) that

$$\begin{aligned} & \left(c(\widehat{X}_{N,r,q}^{\text{ad}}) / \log(c(\widehat{X}_{N,r,q}^{\text{ad}})) \right)^{1/2} \cdot \left\| \widetilde{X}_{k_N,r} - \widehat{X}_{N,r,q}^{\text{ad}} \right\|_{C([0,T])} \Big\|_{L_q(\Omega)} \\ &\leq \left(\frac{c(\widehat{X}_{N,r,q}^{\text{ad}})}{\log(c(\widehat{X}_{N,r,q}^{\text{ad}}))} \cdot \frac{\log(N)}{2N} \right)^{1/2} \cdot \left\| \left(\frac{\log(\nu_{N,r,q}^{\text{ad}})}{\log(N)} \right)^{1/2} \cdot \mathcal{A}_{k_N}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}}^{1/q} \right\|_{L_q(\Omega)}. \quad (3.59) \end{aligned}$$

Our main task will now be to show that the limit of the right hand side of (3.59) is bounded above by $C_{q,\infty}^{\text{ad}}$ as N tends to infinity. To this end, note first that

$$\mathcal{A}_{k_N} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])} \quad (3.60)$$

due to Corollary 3.20 and that the sequence $(\mathcal{A}_{k_N}^{2q/(q+2)})_{N \in \mathbb{N}}$ is uniformly integrable due to Assumption $(\text{pG}_{(r+2)/2}^\sigma)$ and Proposition A.6. Hence, we obtain

$$\mathcal{A}_{k_N}^{2q/(q+2)} \xrightarrow[N \rightarrow \infty]{L_1} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])}^{2q/(q+2)}. \quad (3.61)$$

Now we separately analyze the asymptotics of the two relevant terms appearing in the right hand side of (3.59). Similarly to the proof of (3.29), one shows

$$\lim_{N \rightarrow \infty} \frac{c(\widehat{X}_{N,r,q}^{\text{ad}})}{\log(c(\widehat{X}_{N,r,q}^{\text{ad}}))} \cdot \frac{\log(N)}{N} = \left\| \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])} \right\|_{L_{2q/(q+2)}(\Omega)}^{2q/(q+2)} \quad (3.62)$$

by using (3.17), (3.18), (3.6), and (3.61). Second, observe that the estimate (3.17) and the inequality $\log(1+x) \leq x$ for all $x \in (-1, \infty)$ yield

$$\begin{aligned} & \left\| \left(\frac{\log(\nu_{N,r,q}^{\text{ad}})}{\log(N)} \right)^{1/2} \cdot \mathcal{A}_{k_N}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}}^{1/q} \right\|_{L_q(\Omega)} \\ & \leq \left\| \left(1 + \frac{\mathcal{A}_{k_N}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot \mathcal{A}_{k_N}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}}^{1/q} \right\|_{L_q(\Omega)} \end{aligned} \quad (3.63)$$

for all $N \in \mathbb{N}$ such that $1 < k_N \leq N$. Furthermore, note that $\nu_{N,r,q}^{\text{ad}}$ tends to infinity as N tends to infinity due to (3.18). Hence, Lemma A.2(ii) immediately yields

$$I_{\nu_{N,r,q}^{\text{ad}}} \xrightarrow[N \rightarrow \infty]{\text{a.s.}} 1. \quad (3.64)$$

Combining (3.60) and (3.64) gives

$$\left(1 + \frac{\mathcal{A}_{k_N}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot \mathcal{A}_{k_N}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}}^{1/q} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])}^{2/(q+2)}.$$

Moreover, the sequence

$$\left(\left(1 + \frac{\mathcal{A}_{k_N}^{2q/(q+2)}}{\log(N)} \right)^{q/2} \cdot \mathcal{A}_{k_N}^{2q/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}} \right)_{N \in \mathbb{N}}$$

is uniformly integrable due to Assumption $(\text{pG}_{(r+2)/2}^\sigma)$, Proposition A.6, and Lemma A.2(i). Hence, we obtain

$$\left(1 + \frac{\mathcal{A}_{k_N}^{2q/(q+2)}}{\log(N)} \right)^{1/2} \cdot \mathcal{A}_{k_N}^{2/(q+2)} \cdot I_{\nu_{N,r,q}^{\text{ad}}}^{1/q} \xrightarrow[N \rightarrow \infty]{L_q} \left\| |\sigma(\cdot, X(\cdot))|_{\infty,2} \right\|_{L_2([0,T])}^{2/(q+2)}. \quad (3.65)$$

Finally, combining (3.55), (3.59), (3.62), (3.63), (3.65), and the second limit in (3.6) finishes the proof of this lemma. \square

Next, we prove an asymptotic upper bound for the errors of the equidistant tamed Euler schemes.

Lemma 3.25. *For all $q \in [1, \min\{b, a/(2r + 1)\})$ it holds that*

$$\limsup_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot e_{q,\infty}(\widehat{X}_{N,r}^{\text{eq}}) \leq C_{q,\infty}^{\text{eq}}.$$

Proof. Fix $N \in \mathbb{N} \setminus \{1\}$ for the moment and put $D_N := (\xi, W(t_1^{(N)}), \dots, W(t_N^{(N)}))$.

Analogously to the respective parts in the proof of Lemma 3.24, one shows that

$$e_{q,\infty}(\widehat{X}_{N,r}^{\text{eq}}) \leq \left\| \|\widetilde{X}_{N,r} - \widehat{X}_{N,r}^{\text{eq}}\|_{C([0,T])} \right\|_{L_q(\Omega)} + c \cdot N^{-1/2} \quad (3.66)$$

and that

$$\begin{aligned} & \mathbb{E} \left[\left\| \|\widetilde{X}_{N,r} - \widehat{X}_{N,r}^{\text{eq}}\|_{C([0,T])}^q \mid D_N \right\| \right] \\ &= \mathbb{E} \left[\max_{i \in \{1, \dots, d\}} \max_{\ell \in \{0, \dots, N-1\}} \sup_{t \in [t_\ell^{(N)}, t_{\ell+1}^{(N)}]} \left| \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(N)}, \widetilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\widetilde{X}_{N,r}(t_\ell^{(N)})|^r} \right. \right. \\ & \quad \left. \left. \cdot \left(W_j(t) - \widehat{W}_{N,j}^{\text{eq}}(t) \right)^q \mid D_N \right] \right] \end{aligned} \quad (3.67)$$

holds almost surely where $\widehat{W}_N^{\text{eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^m)$ denotes the piecewise-linear interpolation of W at the equidistant time points (3.5). Observe that for all $\ell \in \{0, \dots, N-1\}$ and for all $i \in \{1, \dots, d\}$ it holds that

$$\begin{aligned} & \left((t_{\ell+1}^{(N)} - t_\ell^{(N)}) \cdot \sum_{j=1}^m \left(\frac{\sigma_{i,j}(t_\ell^{(N)}, \widetilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\widetilde{X}_{N,r}(t_\ell^{(N)})|^r} \right)^2 \right)^{1/2} \\ & \leq (T/N)^{1/2} \cdot \left| \frac{\sigma(t_\ell^{(N)}, \widetilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\widetilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2}. \end{aligned} \quad (3.68)$$

Similarly to the proofs of (3.57) and (3.58), Lemma A.3 applied with

$$\beta := (T/N)^{1/2} \cdot \max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \widetilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\widetilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2}$$

(where we keep (3.68) in mind) and

$$z := (\beta^2 \cdot \log(N)/2)^{q/2}$$

gives

$$\begin{aligned}
& \mathbb{E} \left[\max_{i \in \{1, \dots, d\}} \max_{\ell \in \{0, \dots, N-1\}} \sup_{t \in [t_\ell^{(N)}, t_{\ell+1}^{(N)}]} \left| \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right. \right. \\
& \qquad \qquad \qquad \left. \left. \cdot (W_j(t) - \widehat{W}_{N,j}^{\text{eq}}(t)) \right|^q \middle| D_N \right] \\
& \leq (\beta^2 \cdot \log(N)/2)^{q/2} + d \cdot \int_{(\beta^2 \cdot \log(N)/2)^{q/2}}^\infty \mathcal{G}_q(u; \underbrace{\beta, \dots, \beta}_{N \text{ times}}) du \tag{3.69} \\
& = (\beta^2 \cdot \log(N)/2)^{q/2} \cdot \left(1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{-q/2}; N) du \right) \\
& = (T/N)^{q/2} \cdot \max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2}^q \cdot (\log(N)/2)^{q/2} \cdot I_N
\end{aligned}$$

almost surely where

$$I_N := \left(1 + d \cdot 2^{q/2} \cdot \int_{2^{-q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; N) du \right).$$

Thus, we conclude from (3.67) and (3.69) that

$$\begin{aligned}
& (N/\log(N))^{1/2} \cdot \left\| \tilde{X}_{N,r} - \widehat{X}_{N,r}^{\text{eq}} \right\|_{C([0,T])} \Big\|_{L_q(\Omega)} \\
& \leq (T/2)^{1/2} \cdot \left\| \max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right| \right\|_{\infty, 2} \Big\|_{L_q(\Omega)} \cdot I_N^{1/q}. \tag{3.70}
\end{aligned}$$

As a final step, we show that the right hand side of (3.70) tends to $C_{q,\infty}^{\text{eq}}$ as N tends to infinity. To this end, note first that

$$\max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0,T])}$$

due to Corollary 3.20 and that the sequence

$$\left(\max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2}^q \right)_{N \in \mathbb{N}}$$

is uniformly integrable due to Assumption (pG $_{(r+2)/2}^\sigma$) and Proposition A.6. Hence, we obtain

$$\max_{\ell \in \{0, \dots, N-1\}} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{X}_{N,r}(t_\ell^{(N)}))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(t_\ell^{(N)})|^r} \right|_{\infty, 2} \xrightarrow[N \rightarrow \infty]{L_q} \left\| |\sigma(\cdot, X(\cdot))|_{\infty, 2} \right\|_{C([0,T])}. \tag{3.71}$$

Moreover, Lemma A.2(ii) immediately yields

$$\lim_{N \rightarrow \infty} I_N = 1. \quad (3.72)$$

Finally, combining (3.66), (3.70), (3.71), and (3.72) finishes the proof of this lemma. \square

4 Strongly Asymptotically Optimal Approximations with respect to the L_p Error

In this chapter, we study the error criterion $e_{q,p}$, which is defined by (2.2), and we seek approximations that are strongly asymptotically optimal in the corresponding classes of adaptive and of equidistant approximations, i.e., in $(\mathbb{X}_N^{\text{ad}}(L_p([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(L_p([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. Almost without exception, we focus on the special case $p = q$ as otherwise the analysis becomes significantly more difficult. We briefly comment on the general case in a stand-alone section. On the one hand, for SDEs whose coefficients as well as their partial derivatives with respect to the state variable are globally Lipschitz continuous, Müller-Gronbach (2002b) showed that specific Milstein schemes relating to adaptive and to equidistant time discretizations are strongly asymptotically optimal in their respective classes. On the other hand, for SDEs with super-linearly growing coefficients, the errors of these particular approximations tend to infinity as the numbers of discretization sites tend to infinity. In the present chapter, we generalize the results of the mentioned reference such that SDEs of the latter type are incorporated. More precisely, we show under rather mild assumptions on the underlying SDE, notably polynomial growth conditions on its coefficients and their partial derivatives, that specific tamed Milstein schemes relating to adaptive and to equidistant time discretizations are strongly asymptotically optimal in the aforementioned classes. To illustrate our findings, we numerically analyze an exemplary SDE.

In Section 4.1, we introduce several assumptions that are required in addition to the ones given in Section 3.1. Moreover, we show interdependencies among the new as well as among the new and the old conditions. In Section 4.2, we provide a concise literature review on results regarding the error $e_{q,q}$. In particular, we state well-known results in the classical setting of SDEs for which both the coefficients and their partial derivatives with respect to the state variable are globally Lipschitz continuous. In Section 4.3, we first present a continuous-time tamed Milstein scheme. Building upon this scheme, we construct an equidistant and an adaptive tamed Milstein scheme in full details afterwards. In Section 4.4, we state the main results of this chapter, i.e., strong asymptotic optimality of the previously constructed adaptive and equidistant tamed Milstein schemes in their respective classes. In Section 4.5, we discuss the case $e_{q,p}$ for $p \neq q$ which

turns out to be considerably harder to analyze than the special case $e_{q,q}$. In Section 4.6, we illustrate our findings via a numerical experiment. To this end, we study an SDE whose diffusion coefficient grows quadratically in the state variable. In Section 4.7, we carry out the proofs of our main theorems.

Throughout this chapter, we consider

$$(S, \mathcal{A}) \in \left\{ (L_p([0, T]; \mathbb{R}^d), \mathcal{B}(L_p([0, T]; \mathbb{R}^d))) \mid p \in [1, \infty) \right\}$$

as the underlying measurable space as per Chapter 2. In the case that the SDE (1.4) has a unique solution $(X(t))_{t \in [0, T]}$, we put

$$\begin{aligned} C_{q,q}^{\text{ad}} &:= \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{2q/(q+2)}([0, T])} \right\|_{L_{2q/(q+2)}(\Omega)} \in [0, \infty], \\ C_{q,q}^{\text{eq}} &:= \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot T^{1/2} \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \in [0, \infty], \end{aligned} \quad (4.1)$$

for $q \in [1, \infty)$ where

$$\mathfrak{m}_q := \left(\int_{\mathbb{R}} |x|^q / (2\pi)^{1/2} \cdot \exp(-x^2/2) \, dx \right)^{1/q} \quad (4.2)$$

and

$$\mathfrak{g}_q := \left(\int_0^1 x^{q/2} \cdot (1-x)^{q/2} \, dx \right)^{1/q}. \quad (4.3)$$

The quantities $C_{q,q}^{\text{ad}}$ and $C_{q,q}^{\text{eq}}$ will turn out to be the sharp asymptotic constants for the N th minimal errors in the classes $(\mathbb{X}_N^{\text{ad}}(L_q([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively. It is easy to see that $C_{q,q}^{\text{ad}} \leq C_{q,q}^{\text{eq}}$ holds for all $q \in [1, \infty)$.

4.1 Assumptions

In addition to the assumptions introduced in Section 3.1, we present further conditions on the coefficients of the SDE (1.4) here. For $r \in [-1/2, \infty)$ and $\varphi \in \{\mu, \sigma\}$, we introduce the following assumptions:

Assumption (C). *The diffusion coefficient σ satisfies the so-called ‘‘commutativity condition’’, i.e., σ has partial derivatives with respect to the state variable and for all $j_1, j_2 \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that*

$$\nabla \sigma^{(j_1)}(t, x) \cdot \sigma^{(j_2)}(t, x) = \nabla \sigma^{(j_2)}(t, x) \cdot \sigma^{(j_1)}(t, x).$$

Assumption (LLG $^\varphi$). *The coefficient φ is globally Lipschitz continuous with respect to the time variable with a Lipschitz bound that grows at most linearly in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $s, t \in [0, T]$ and for all $x \in \mathbb{R}^d$ it holds that*

$$|\varphi(s, x) - \varphi(t, x)| \leq C \cdot |s - t| \cdot (1 + |x|).$$

Assumption ($\text{pG}_r^{\nabla\mu}$). Each component of the drift coefficient μ has partial derivatives with respect to the state variable that grow at most polynomially in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|\nabla\mu_i(t, x)| \leq C \cdot (1 + |x|)^r.$$

Assumption ($\text{pG}_r^{\nabla\sigma}$). Each component of the diffusion coefficient σ has partial derivatives with respect to the state variable that grow at most polynomially in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $j \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|\nabla\sigma_{i,j}(t, x)| \leq C \cdot (1 + |x|)^r.$$

Assumption ($\text{pL}_r^{\nabla\mu}$). Each component of the drift coefficient μ has partial derivatives with respect to the state variable that are Lipschitz continuous with respect to the state variable with a Lipschitz bound that grows at most polynomially in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $t \in [0, T]$, and for all $x, y \in \mathbb{R}^d$ it holds that

$$|\nabla\mu_i(t, x) - \nabla\mu_i(t, y)| \leq C \cdot |x - y| \cdot (1 + |x| + |y|)^r.$$

Assumption ($\text{pL}_r^{\nabla\sigma}$). Each component of the diffusion coefficient σ has partial derivatives with respect to the state variable that are Lipschitz continuous with respect to the state variable with a Lipschitz bound that grows at most polynomially in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $j \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x, y \in \mathbb{R}^d$ it holds that

$$|\nabla\sigma_{i,j}(t, x) - \nabla\sigma_{i,j}(t, y)| \leq C \cdot |x - y| \cdot (1 + |x| + |y|)^r.$$

Assumption ($\text{B}^{\nabla_t\mu}$). Each component of the drift coefficient μ is continuously differentiable with a partial derivative with respect to the time variable that is uniformly bounded in the time variable and that grows at most linearly in the state variable, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|\nabla_t\mu_i(t, x)| \leq C \cdot (1 + |x|).$$

Assumption ($\text{B}^{\nabla_t\sigma}$). Each component of the diffusion coefficient σ is continuously differentiable with a partial derivative with respect to the time variable that is uniformly bounded in time, i.e., there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $j \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|\nabla_t\sigma_{i,j}(t, x)| \leq C \cdot (1 + |x|).$$

The following easy-to-prove remark comprises various relations between the assumptions above and the ones given in Section 3.1.

Remark 4.1. (i) *If $d = m = 1$, then (C) is satisfied.*

(ii) *If the SDE (1.4) is autonomous, then (LLG^μ) and (LLG^σ) are satisfied.*

(iii) *Let the SDE (1.4) be autonomous and let $(\text{pL}_r^{\nabla\varphi})$ be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [-1/2, \infty)$. Then $(\text{B}^{\nabla\varphi})$ is satisfied.*

(iv) *Let $(\text{B}^{\nabla\varphi})$ be satisfied for some $\varphi \in \{\mu, \sigma\}$. Then (LLG^φ) is satisfied.*

(v) *Let $(\text{pL}_r^{\nabla\varphi})$ and $(\text{B}^{\nabla\varphi})$ be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [-1/2, \infty)$. Then $(\text{pG}_{r+1}^{\nabla\varphi})$ is satisfied.*

(vi) *Let $(\text{pG}_r^{\nabla\varphi})$ be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [0, \infty)$. Then (pL_r^φ) is satisfied.*

(vii) *Let (pL_r^φ) and (LLG^φ) be satisfied for some $\varphi \in \{\mu, \sigma\}$ and $r \in [0, \infty)$. Then $(\text{pG}_{r+1}^\varphi)$ is satisfied.*

For useful properties of the solution of the SDE (1.4), such as existence-and-uniqueness or moment bounds, we refer the reader to the end of Section 3.1.

4.2 Literature Review

We now provide a concise overview of results concerning the considered pathwise approximation problem, mainly with regard to the error $e_{q,q}$. This selection does not claim to be complete or exhaustive, but should give enough background to motivate the central objective of this chapter.

The remainder of this section is organized as follows. We start our discussion by presenting a result on lower bounds for the N th minimal errors in the classes of adaptive approximations which holds under quite general assumptions. This allows us to deduce asymptotic lower bounds for the N th minimal errors in the classes of adaptive and of equidistant approximations, but with unspecified asymptotic constants only. Yet, in the special case of SDEs whose coefficients as well as their (partial) derivatives (with respect to the state variable) are globally Lipschitz continuous, one entirely knows the asymptotics of these N th minimal errors. As a next step, we are interested in approximations which possess the same convergence rate as the N th minimal errors in the previous results or, even better, which are strongly asymptotically optimal in the aforementioned classes. For SDEs whose coefficients as well as their derivatives are globally Lipschitz continuous, specific equidistant and adaptive variants of the Milstein scheme achieve the desired strong asymptotic optimality. However, the errors of these particular approximations tend to infinity once the drift or the diffusion coefficient grows super-linearly. In search of strongly asymptotically optimal approximations for SDEs of the latter type, we come into contact with so-called tamed Milstein schemes for which we find upper error bounds of the right convergence rate.

At first, lower error bounds for the pathwise approximation of SDEs have been extensively studied for the case of coefficients that are globally Lipschitz continuous, see, e.g., Cambanis & Hu (1996), Hofmann et al. (2000a,b, 2001), and Müller-Gronbach (2002a,b). Recently, the authors in Hefter et al. (2019) established lower bounds for the N th minimal errors in the classes of adaptive approximations that hold under rather mild assumptions on the underlying SDE. In particular, the coefficients of the SDE (1.4) are required to have sufficient regularity only locally, in a small neighborhood of the initial value.

Proposition 4.2. *Let $d = m = 1$ and let the SDE (1.4) have a unique solution $(X(t))_{t \in [0, T]}$. Moreover, let $t_0 \in [0, T)$, $T_0 \in (t_0, T]$, and let $\emptyset \neq I \subseteq \mathbb{R}$ be an open interval such that for all $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$ the partial derivatives $\mu^{(i,j)}$ and $\sigma^{(i,j)}$ exist on $[t_0, T_0] \times I$ and are continuous, for all $(t, x) \in [t_0, T_0] \times I$ it holds that $\sigma(t, x) \neq 0$, and it holds that $\mathbb{P}(\{X(t_0) \in I\}) > 0$. Then for all $p, q \in [1, \infty)$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\inf \left\{ e_{q,p}(\hat{X}) \mid \hat{X} \in \mathbb{X}_N^{\text{ad}}(L_p([0, T]; \mathbb{R})) \right\} \geq C \cdot N^{-1/2}. \quad (4.4)$$

Proof. Theorem 12 in Hefter et al. (2019) shows (4.4) for the particular error $e_{1,p}$. The general case then follows from the monotonicity property (2.5). \square

From the preceding proposition, we are able to immediately deduce asymptotic lower bounds for both the N th minimal errors in the classes of adaptive and of equidistant approximations. Yet, the corresponding asymptotic constants are unspecified in this situation which does not enable us to resolve whether these constants are sharp or not.

However, in the special case of an SDE for which the coefficients and their derivatives are globally Lipschitz continuous, Müller-Gronbach (2002b) determined the exact asymptotics (including sharp asymptotic constants) of the N th minimal errors in the classes of adaptive and of equidistant approximations.

Proposition 4.3. *Let $T = 1$ and let the Assumptions (C), (I_a), (pL₀^μ), (pL₀^σ), (pL₀^{∇μ}), (pL₀^{∇σ}), (LLG^μ), and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ it holds that*

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\hat{X}) \mid \hat{X} \in \mathbb{X}_N^{\text{ad}}(L_q([0, 1]; \mathbb{R}^d)) \right\} = C_{q,q}^{\text{ad}}$$

and

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\hat{X}) \mid \hat{X} \in \mathbb{X}_N^{\text{eq}}(L_q([0, 1]; \mathbb{R}^d)) \right\} = C_{q,q}^{\text{eq}}.$$

Proof. See Theorem 5(i) and (iv) of Chapter III in Müller-Gronbach (2002b). \square

Next, we turn to upper bounds for the errors of specific adaptive and equidistant approximations, and we seek in particular those approximations that have the same convergence rate $N^{-1/2}$ as the N th minimal errors studied above. For the equidistant and adaptive variants of the Euler–Maruyama scheme (as well as for the respective

tamed Euler schemes) from Chapter 3, we can, in general, only guarantee the convergence rate $(\log(N)/N)^{1/2}$. We thus switch from the Euler–Maruyama scheme to another classical Itô–Taylor approximation, namely, the Milstein scheme.

For the rest of this section, we require all components of σ to have partial derivatives with respect to the state variable.

We begin with the definition of the continuous-time Milstein schemes. To this end, let $N \in \mathbb{N}$ and recall the equidistant time discretization (3.5). The *continuous-time Milstein scheme* $\widetilde{M}_N : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \widetilde{M}_{N,i}(0) &:= \xi_i, \\ \widetilde{M}_{N,i}(t) &:= \widetilde{M}_{N,i}(t_\ell^{(N)}) + \mu_i(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot (t - t_\ell^{(N)}) \\ &\quad + \sum_{j=1}^m \sigma_{i,j}(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot (W_j(t) - W_j(t_\ell^{(N)})) \\ &\quad + \sum_{j_1, j_2=1}^m (\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot J_{j_1, j_2}^{(\ell, N)}(t), \end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)}]$ where

$$J_{j_1, j_2}^{(\ell, N)}(t) := \int_{t_\ell^{(N)}}^t W_{j_1}(s) - W_{j_1}(t_\ell^{(N)}) dW_{j_2}(s). \quad (4.5)$$

Note that \widetilde{M}_N is well-defined in the sense that its realizations are in fact continuous; for this, we use that the Itô integral appearing in the construction above is continuous. If the Assumption (C) is satisfied, then this scheme reduces to the more common form

$$\begin{aligned} \widetilde{M}_{N,i}(0) &= \xi_i, \\ \widetilde{M}_{N,i}(t) &= \widetilde{M}_{N,i}(t_\ell^{(N)}) + \mu_i(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot (t - t_\ell^{(N)}) \\ &\quad + \sum_{j=1}^m \sigma_{i,j}(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot (W_j(t) - W_j(t_\ell^{(N)})) \\ &\quad + \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m (\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \widetilde{M}_N(t_\ell^{(N)})) \cdot I_{j_1, j_2}^{(\ell, N)}(t), \end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)}]$ where

$$I_{j_1, j_2}^{(\ell, N)}(t) := \begin{cases} (W_{j_1}(t) - W_{j_1}(t_\ell^{(N)})) \cdot (W_{j_2}(t) - W_{j_2}(t_\ell^{(N)})), & \text{if } j_1 \neq j_2, \\ (W_{j_1}(t) - W_{j_1}(t_\ell^{(N)}))^2 - (t - t_\ell^{(N)}), & \text{else.} \end{cases} \quad (4.6)$$

Observe that, since entire trajectories of the driving Brownian motion and iterated Itô integrals are used in the construction above, this scheme is not an approximation in the sense of Section 2.1, and we thus find $\widetilde{M}_N \notin \mathbb{X}(L_p([0, T]; \mathbb{R}^d))$ for every $p \in [1, \infty)$.

It is well-known that the continuous-time Milstein schemes converge strongly to the solution of the SDE (1.4) with order (at least) 1 if the drift and diffusion coefficients as well as their partial derivatives with respect to the state variable are globally Lipschitz continuous, as the following proposition clarifies.

Proposition 4.4. *Let the Assumptions (I_a) , (pL_0^μ) , (pL_0^σ) , $(pL_0^{\nabla\mu})$, $(pL_0^{\nabla\sigma})$, (LLG^μ) , and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \widetilde{M}_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1}. \quad (4.7)$$

Proof. Let $q \in [1, a/2]$. As a first step, one shows the existence of some $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\sup_{t \in [0, T]} \|X(t) - \widetilde{M}_N(t)\|_{L_q(\Omega)} \leq C \cdot N^{-1}. \quad (4.8)$$

A proof for (4.8) is given by Proposition 1 of Chapter V in Müller-Gronbach (2002b) for the special case $T = 1$; for general $T \in (0, \infty)$, one adopts the idea of proof therein accordingly. Finally, we derive (4.7) from the estimate (4.8) by means of Fubini's theorem. \square

Note that we do not use the term $e_{q,q}(\widetilde{M}_N)$ in the proposition above since the error measure $e_{q,q}$ is only defined on $\mathbb{X}(L_q([0, T]; \mathbb{R}^d))$.

To obtain equidistant approximations that are based on the continuous-time Milstein schemes, we consider the linear interpolation of these schemes at the equidistant sites (3.5). For $N \in \mathbb{N}$, the *equidistant Milstein scheme* $\widehat{M}_N^{\text{eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\widehat{M}_N^{\text{eq}}(t_\ell^{(N)}) := \widetilde{M}_N(t_\ell^{(N)})$$

for all $\ell \in \{0, \dots, N\}$ and linearly interpolated between these time points. If the Assumption (C) is satisfied, then we obtain $\widehat{M}_N^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(L_p([0, T]; \mathbb{R}^d))$ for all $p \in [1, \infty)$ by suitably choosing sequences ψ , χ , and φ as per Section 2.1 due to (4.6). Under the same assumptions as in the preceding proposition and additionally extended by the commutativity condition, one also knows upper bounds for the errors of these equidistant Milstein schemes.

Proposition 4.5. *Let the Assumptions (I_a) , (pL_0^μ) , (pL_0^σ) , $(pL_0^{\nabla\mu})$, $(pL_0^{\nabla\sigma})$, (LLG^μ) , and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1/2}.$$

Proof. This result seems to be well-known, but unfortunately we did not find a reference that suits our setting perfectly. For the convenience of the reader, we therefore provide a proof of this proposition here.

Let $q \in [1, a/2]$. Observe that, in the setting of this proposition, the Assumptions (pG_1^μ) and (pG_1^σ) are also satisfied, see Remark 4.1(vii). Hence, we are able to utilize in particular the estimate (3.3) given in Section 3.1. Combining the triangle inequality and the inequalities (3.3) and (4.8) yields the existence of some $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$ it holds that

$$\begin{aligned}
& \|X(t) - \widehat{M}_N^{\text{eq}}(t)\|_{L_q(\Omega)} \\
&= \left\| X(t) - \left[\frac{t_{\ell+1}^{(N)} - t}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot \widetilde{M}_N(t_\ell^{(N)}) + \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot \widetilde{M}_N(t_{\ell+1}^{(N)}) \right] \right\|_{L_q(\Omega)} \\
&\leq \|X(t) - \widetilde{M}_N(t_\ell^{(N)})\|_{L_q(\Omega)} + \|X(t) - \widetilde{M}_N(t_{\ell+1}^{(N)})\|_{L_q(\Omega)} \\
&\leq \|X(t) - X(t_\ell^{(N)})\|_{L_q(\Omega)} + \|X(t_\ell^{(N)}) - \widetilde{M}_N(t_\ell^{(N)})\|_{L_q(\Omega)} \\
&\quad + \|X(t) - X(t_{\ell+1}^{(N)})\|_{L_q(\Omega)} + \|X(t_{\ell+1}^{(N)}) - \widetilde{M}_N(t_{\ell+1}^{(N)})\|_{L_q(\Omega)} \\
&\leq C \cdot N^{-1/2}.
\end{aligned}$$

By using Fubini's theorem along with this estimate, we then obtain

$$\begin{aligned}
\left\| \|X - \widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} &= \left(\sum_{\ell=0}^{N-1} \int_{t_\ell^{(N)}}^{t_{\ell+1}^{(N)}} \|X(t) - \widehat{M}_N^{\text{eq}}(t)\|_{L_q(\Omega)}^q dt \right)^{1/q} \\
&\leq T^{1/q} \cdot C \cdot N^{-1/2}
\end{aligned}$$

for all $N \in \mathbb{N}$, which finishes the proof of this proposition. \square

From the result above, we can deduce an asymptotic upper bound for the errors of the equidistant Milstein schemes but, again, only with unspecified asymptotic constant. In contrast, the following proposition shows the exact asymptotics (including sharp asymptotic constant) of the errors of these approximations in the case that the coefficients of the SDE (1.4) as well as their derivatives are globally Lipschitz continuous.

Proposition 4.6. *Let $T = 1$ and let the Assumptions (C) , (I_a) , (pL_0^μ) , (pL_0^σ) , $(\text{pL}_0^{\nabla\mu})$, $(\text{pL}_0^{\nabla\sigma})$, (LLG^μ) , and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ it holds that*

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot e_{q,q}(\widehat{M}_N^{\text{eq}}) = C_{q,q}^{\text{eq}}.$$

Proof. See Theorem 4(iv) of Chapter III in Müller-Gronbach (2002b). \square

Having considered the Milstein schemes at equidistant sites only so far, we next turn to variants of these schemes that are based on adaptive time discretizations. For this purpose, we study the following adaptive Milstein schemes, which were introduced in Müller-Gronbach (2002b). Fix $T = 1$ and let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers satisfying

$$\lim_{N \rightarrow \infty} \frac{k_N}{N} = 0 = \lim_{N \rightarrow \infty} \frac{N^{1/2}}{k_N}. \quad (4.9)$$

Fix $N \in \mathbb{N}$ for the moment and let $q \in [1, \infty)$. For each $\ell \in \{0, \dots, k_N - 1\}$, we consider the random discretization

$$t_\ell^{(k_N)} = \tau_{\ell,0}^{(k_N)} < \tau_{\ell,1}^{(k_N)} < \dots < \tau_{\ell,\eta_\ell+1}^{(k_N)} = t_{\ell+1}^{(k_N)} \quad (4.10)$$

of $[t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]$ where

$$\eta_\ell := \left\lfloor N \cdot 1/k_N \cdot \left| \sigma(t_\ell^{(k_N)}, \widetilde{M}_{k_N}(t_\ell^{(k_N)})) \right|_{q,2}^{2q/(q+2)} \right\rfloor$$

and

$$\tau_{\ell,\kappa}^{(k_N)} := t_\ell^{(k_N)} + \frac{1}{k_N} \cdot \frac{\kappa}{\eta_\ell + 1}$$

for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$. Let $\widehat{W}_N^{\text{ad}}$ denote the linear interpolation of W at the adaptive time points (4.10), i.e., let

$$\begin{aligned} \widehat{W}_N^{\text{ad}}(0) &:= 0, \\ \widehat{W}_N^{\text{ad}}(t) &:= \frac{\tau_{\ell,\kappa+1}^{(k_N)} - t}{\tau_{\ell,\kappa+1}^{(k_N)} - \tau_{\ell,\kappa}^{(k_N)}} \cdot W(\tau_{\ell,\kappa}^{(k_N)}) + \frac{t - \tau_{\ell,\kappa}^{(k_N)}}{\tau_{\ell,\kappa+1}^{(k_N)} - \tau_{\ell,\kappa}^{(k_N)}} \cdot W(\tau_{\ell,\kappa+1}^{(k_N)}), \end{aligned}$$

for all $\ell \in \{0, \dots, k_N - 1\}$, for all $\kappa \in \{0, \dots, \eta_\ell\}$, and for all $t \in (\tau_{\ell,\kappa}^{(k_N)}, \tau_{\ell,\kappa+1}^{(k_N)})$. The *adaptive Milstein scheme* $\widehat{M}_{N,q}^{\text{ad}} : [0, 1] \times \Omega \rightarrow \mathbb{R}^d$ is given by

$$\begin{aligned} \widehat{M}_{N,q}^{\text{ad}}(t) &:= \widetilde{M}_{k_N}(t_\ell^{(k_N)}) + \mu(t_\ell^{(k_N)}, \widetilde{M}_{k_N}(t_\ell^{(k_N)})) \cdot (t - t_\ell^{(k_N)}) \\ &\quad + \sigma(t_\ell^{(k_N)}, \widetilde{M}_{k_N}(t_\ell^{(k_N)})) \cdot \left(\widehat{W}_N^{\text{ad}}(t) - W(t_\ell^{(k_N)}) \right), \\ \widehat{M}_{N,q}^{\text{ad}}(1) &:= \widetilde{M}_{k_N}(1) \end{aligned} \quad (4.11)$$

for all $\ell \in \{0, \dots, k_N\}$ and for all $t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$. If the Assumption (C) is satisfied, then we obtain $\widehat{M}_{N,q}^{\text{ad}} \in \mathbb{X}_{[c(\widehat{M}_{N,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, 1]; \mathbb{R}^d))$ by suitably choosing sequences ψ , χ , and φ as per Section 2.1 in case the cost of this approximation satisfy $c(\widehat{M}_{N,q}^{\text{ad}}) < \infty$. Müller-Gronbach (2002b) established the exact asymptotics of the errors of the adaptive Milstein schemes defined above for SDEs whose coefficients as well as their derivatives are globally Lipschitz continuous.

Proposition 4.7. *Let $T = 1$ and let the Assumptions (C), (I_a) , (pL_0^μ) , (pL_0^σ) , $(pL_0^{\nabla\mu})$, $(pL_0^{\nabla\sigma})$, (LLG^μ) , and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ it holds that*

$$\lim_{N \rightarrow \infty} (c(\widehat{M}_{N,q}^{\text{ad}}))^{1/2} \cdot e_{q,q}(\widehat{M}_{N,q}^{\text{ad}}) = C_{q,q}^{\text{ad}}.$$

Proof. See Theorem 4(i) of Chapter III in Müller-Gronbach (2002b). \square

Since both the convergence rates and the corresponding asymptotic constants match in the Propositions 4.3, 4.6, and 4.7, we obtain strong asymptotic optimality of the adaptive and of the equidistant Milstein schemes in the classes of adaptive and of equidistant approximations, respectively.

Corollary 4.8. *Let $T = 1$ and let the Assumptions (C), (I_a) , (pL_0^μ) , (pL_0^σ) , $(pL_0^{\nabla\mu})$, $(pL_0^{\nabla\sigma})$, (LLG^μ) , and (LLG^σ) be satisfied for some $a \in [4, \infty)$. Then for all $q \in [1, a/2]$ with $C_{q,q}^{\text{ad}} > 0$ it holds that*

$$\lim_{N \rightarrow \infty} \frac{e_{q,q}(\widehat{M}_{N,q}^{\text{ad}})}{\inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_{[c(\widehat{M}_{N,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, 1]; \mathbb{R}^d)) \right\}} = 1$$

and

$$\lim_{N \rightarrow \infty} \frac{e_{q,q}(\widehat{M}_N^{\text{eq}})}{\inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(L_q([0, 1]; \mathbb{R}^d)) \right\}} = 1;$$

i.e., the approximations $(\widehat{M}_{N,q}^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{M}_N^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_{[c(\widehat{M}_{N,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(L_q([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively.

Proof. This result is an immediate consequence of the Propositions 4.3, 4.6, and 4.7. \square

In view of all the previous considerations, our primary goal in this chapter is as follows:

We aim at extending the results of Müller-Gronbach (2002b)—especially the ones given in Corollary 4.8—to SDEs whose coefficients as well as their partial derivatives with respect to the state variable are allowed to grow super-linearly. In particular, we seek strongly asymptotically optimal approximations in the classes of adaptive and of equidistant approximations in such a setting.

To begin with, it is natural to ask whether the equidistant and the adaptive Milstein schemes introduced above preserve strong asymptotic optimality in their respective classes even for SDEs whose coefficients grow super-linearly. We answer this question (at least for the equidistant case) to the negative in the following proposition.

Proposition 4.9. *Let $d = m = 1$ and let the SDE (1.4) be autonomous and have a unique solution $(X(t))_{t \in [0, T]}$. Moreover, assume that $\mathbb{P}(\{\sigma(\xi) \neq 0\} \cap \{\nabla \sigma(\xi) \neq 0\}) > 0$ and assume that there exist $C \in [1, \infty)$, $\alpha, \beta \in (1, \infty)$ with $\alpha < \beta$ such that for all $x \in \mathbb{R}$ with $|x| \geq C$ it holds that $|\mu(x)| \geq C^{-1} \cdot |x|^\beta$, $|\sigma(x)| \leq C \cdot |x|^\alpha$, and $|(\sigma \cdot \nabla \sigma)(x)| \leq C \cdot |x|^\alpha$. Then for all $q \in [1, \infty)$ with $\| \|X\|_{L_q([0, T])} \|_{L_q(\Omega)} < \infty$ it holds that*

$$\lim_{N \rightarrow \infty} e_{q, q}(\widehat{M}_N^{\text{eq}}) = \infty. \quad (4.12)$$

Proof. Under the given assumptions, the Lemmas 4.13 and 4.14 in Hatzesberger (2014) yield the existence of some $c \in (1, \infty)$ such that for all $N \in \mathbb{N}$ with $N > 4T$ there exists $\Omega_N \in \mathcal{F}$ with

$$\mathbb{P}(\Omega_N) \geq c^{(-N^c)} \quad (4.13)$$

such that for all $\omega \in \Omega_N$ it holds that

$$\min \left\{ |\widetilde{M}_N(t_{N-1}^{(N)}, \omega)|, |\widetilde{M}_N(t_N^{(N)}, \omega)| \right\} \geq 2^{(\alpha^{N-2})}. \quad (4.14)$$

Now let $q \in [1, \infty)$ such that $\| \|X\|_{L_q([0, T])} \|_{L_q(\Omega)} < \infty$. Observe that the inverse triangle inequality gives

$$\begin{aligned} e_{q, q}(\widehat{M}_N^{\text{eq}}) &= \left\| \|X - \widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \\ &\geq \left\| \|\widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} - \left\| \|X\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \\ &\geq \left\| \|\widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_1(\Omega)} - \left\| \|X\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \end{aligned} \quad (4.15)$$

for all $N \in \mathbb{N}$. Moreover, by the construction of the equidistant Milstein schemes, we have

$$\begin{aligned} \left\| \|\widehat{M}_N^{\text{eq}}\|_{L_q([0, T])} \right\|_{L_1(\Omega)} &= \mathbb{E} \left[\left(\int_0^T |\widehat{M}_N^{\text{eq}}(t)|^q dt \right)^{1/q} \right] \\ &\geq \mathbb{E} \left[\left(\int_{t_{N-1}^{(N)}}^{t_N^{(N)}} |\widehat{M}_N^{\text{eq}}(t)|^q dt \right)^{1/q} \right] \\ &\geq (T/N)^{1/q} \cdot \mathbb{E} \left[\inf_{t \in [t_{N-1}^{(N)}, t_N^{(N)}]} |\widehat{M}_N^{\text{eq}}(t)| \right] \\ &= (T/N)^{1/q} \cdot \mathbb{E} \left[\min \left\{ |\widetilde{M}_N(t_{N-1}^{(N)})|, |\widetilde{M}_N(t_N^{(N)})| \right\} \right] \end{aligned} \quad (4.16)$$

for all $N \in \mathbb{N}$. Next, employing the estimates (4.13) and (4.14) shows that

$$\begin{aligned} \mathbb{E} \left[\min \left\{ |\widetilde{M}_N(t_{N-1}^{(N)})|, |\widetilde{M}_N(t_N^{(N)})| \right\} \right] &\geq \mathbb{E} \left[\mathbf{1}_{\Omega_N} \cdot \min \left\{ |\widetilde{M}_N(t_{N-1}^{(N)})|, |\widetilde{M}_N(t_N^{(N)})| \right\} \right] \\ &\geq \mathbb{E} \left[\mathbf{1}_{\Omega_N} \cdot 2^{(\alpha^{N-2})} \right] \\ &= \mathbb{P}(\Omega_N) \cdot 2^{(\alpha^{N-2})} \\ &\geq c^{(-N^c)} \cdot 2^{(\alpha^{N-2})} \end{aligned} \quad (4.17)$$

holds for all $N \in \mathbb{N}$ with $N > 4T$. Combining the inequalities (4.15), (4.16), and (4.17) then yields

$$e_{q,q}(\widehat{M}_N^{\text{eq}}) \geq (T/N)^{1/q} \cdot c^{(-N^c)} \cdot 2^{(\alpha^{N-2})} - \left\| \|X\|_{L_q([0,T])} \right\|_{L_q(\Omega)}$$

for all $N \in \mathbb{N}$ with $N > 4T$. By letting N tend to infinity, we eventually obtain (4.12). \square

We conjecture that a divergence result analogous to the one given by (4.12) also holds true for the adaptive Milstein schemes.

The preceding observations emphasize that the previously constructed equidistant and adaptive Milstein schemes are not suitable to globally approximate SDEs with super-linearly growing coefficients. Due to the monotonicity property (2.6), we immediately obtain upper error bounds for various approximations, in particular tamed schemes, mentioned at the corresponding position in Section 3.2.

To achieve the primary goal of this chapter, we follow the approach of so-called tamed Milstein schemes as a basis for the construction of strongly asymptotically optimal approximations in the setting of SDEs with super-linearly growing coefficients.

Inspired by the tamed Euler scheme introduced in Hutzenthaler et al. (2012), the authors in Gan & Wang (2013) constructed a tamed Milstein scheme in a very similar way. For this purpose, let the SDE (1.4) be autonomous for the moment. For $N \in \mathbb{N}$, the *continuous-time tamed Milstein scheme* $\tilde{Y}_N^{\text{GW}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{Y}_{N,i}^{\text{GW}}(0) &:= \xi_i, \\ \tilde{Y}_{N,i}^{\text{GW}}(t) &:= \tilde{Y}_{N,i}^{\text{GW}}(t_\ell^{(N)}) + \frac{\mu_i(\tilde{Y}_N^{\text{GW}}(t_\ell^{(N)}))}{1 + T/N \cdot |\mu(\tilde{Y}_N^{\text{GW}}(t_\ell^{(N)}))|} \cdot (t - t_\ell^{(N)}) \\ &\quad + \sum_{j=1}^m \sigma_{i,j}(\tilde{Y}_N^{\text{GW}}(t_\ell^{(N)})) \cdot (W_j(t) - W_j(t_\ell^{(N)})) \\ &\quad + \sum_{j_1, j_2=1}^m (\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(\tilde{Y}_N^{\text{GW}}(t_\ell^{(N)})) \cdot J_{j_1, j_2}^{(\ell, N)}(t), \end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$ where $J_{j_1, j_2}^{(\ell, N)}(\cdot)$ is defined as in (4.5). In the case of an SDE whose diffusion coefficient is still required to be globally Lipschitz continuous and whose drift coefficient has to satisfy a so-called “one-sided Lipschitz condition”, the continuous-time tamed Milstein schemes above converge strongly to the solution. Even more, these schemes admit analogous upper error bounds as the continuous-time Milstein schemes as stated in Proposition 4.4.

Proposition 4.10. *Let the SDE (1.4) be autonomous, let $\mu, \sigma^{(1)}, \dots, \sigma^{(m)}$ be twice continuously differentiable, and let the Assumptions (C), (I_a), (M₁), (pL₀^σ), and (pG_r^{∇μ}) be satisfied for all $a \in [2, \infty)$ and for some $r \in [1, \infty)$. Moreover, assume that there exists $C \in (0, \infty)$ such that $|(\nabla\sigma^{(j_2)} \cdot \sigma^{(j_1)})(x) - (\nabla\sigma^{(j_2)} \cdot \sigma^{(j_1)})(y)| \leq C \cdot |x - y|$ holds for all $j_1, j_2 \in \{1, \dots, m\}$ and for all $x, y \in \mathbb{R}^d$, that there exist $C, s \in [1, \infty)$ such that $|\mathbf{H}\mu_i(x)| \leq C \cdot (1 + |x|)^s$ holds for all $i \in \{1, \dots, d\}$ and for all $x \in \mathbb{R}^d$, and that there exists $C \in [1, \infty)$ such that $|\mathbf{H}\sigma_{i,j}(x)| \leq C$ holds for all $i \in \{1, \dots, d\}$, for all $j \in \{1, \dots, m\}$, and for all $x \in \mathbb{R}^d$. Then for all $p, q \in [1, \infty)$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \tilde{Y}_N^{\text{GW}}\|_{L_p([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1}.$$

Proof. This result is a direct consequence of Theorem 3.2 in Gan & Wang (2013). \square

Another tamed Milstein scheme, this time based on the corresponding tamed Euler scheme given in Sabanis (2016), is analyzed in Kumar & Sabanis (2019). Let the SDE (1.4) be autonomous for the moment. For $N \in \mathbb{N}$ and $r \in [0, \infty)$, the *continuous-time tamed Milstein scheme* $\tilde{Y}_{N,r}^{\text{KS}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{Y}_{N,r,i}^{\text{KS}}(t) = & \xi_i + \int_0^t \frac{\mu_i(\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N))}{1 + N^{-1} \cdot |\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N)|^{2r}} ds \\ & + \sum_{j_2=1}^m \int_0^t \frac{\sigma_{i,j_2}(\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N))}{1 + N^{-1} \cdot |\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N)|^{2r}} \\ & + \sum_{j_1=1}^m \frac{(\nabla\sigma_{i,j_2} \cdot \sigma^{(j_1)})(\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N))}{1 + N^{-1} \cdot |\tilde{Y}_{N,r}^{\text{KS}}(\lfloor sN \rfloor / N)|^{2r}} \\ & \cdot (W_{j_1}(s) - W_{j_1}(\lfloor sN \rfloor / N)) dW_{j_2}(s) \end{aligned} \quad (4.18)$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$. Again, these schemes converge strongly with order (at least) 1 but now even for SDEs whose diffusion coefficients may grow super-linearly.

Proposition 4.11. *Let the SDE (1.4) be autonomous and let the Assumptions (I_a), (K_a), (M_b), (pL_{r-1}^{∇μ}), and (pL_{(r-2)/2}^{∇σ}) be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Then for all $q \in [1, b) \cap [1, a/(3r + 1)]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \tilde{Y}_{N,r}^{\text{KS}}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1}.$$

Proof. By using Theorem 2.1 in Kumar & Sabanis (2019), this proposition can be shown in a similar manner to Proposition 4.4. \square

Our main idea is now to modify the tamed Milstein schemes of Kumar & Sabanis (2019) in such a way that we can derive equidistant and adaptive variants that are strongly asymptotically optimal in their respective classes. These new approximations will be presented in full details in the next section.

4.3 The Equidistant and the Adaptive Tamed Milstein Schemes

In the following, we introduce two variants of so-called tamed Milstein schemes that are based on equidistant and on adaptive time discretizations, respectively. The crucial ingredient for both approximations is a continuous-time tamed Milstein scheme which, on the one hand, is suitably close to the solution of the SDE (1.4) and which, on the other hand, possesses a simple recursive structure that will be exploited in the further analysis. These equidistant and adaptive tamed Milstein schemes will turn out to be strongly asymptotically optimal in the classes of equidistant and of adaptive schemes, respectively.

We point out that the continuous-time tamed Milstein scheme presented here is heavily inspired by the one introduced in Kumar & Sabanis (2019). The reason we do not use the latter is that our approach is more convenient for our analysis; in particular, our scheme satisfies the desired recursion (4.19) below. Nevertheless, observe that both schemes coincide in the case that the SDE (1.4) is autonomous and $T = 1$.

Throughout this section, we require all components of σ to have partial derivatives with respect to the state variable.

4.3.1 The Continuous-time Tamed Milstein Schemes

Let $N \in \mathbb{N}$ and $r \in [0, \infty)$, and recall the equidistant time discretization (3.5). The *continuous-time tamed Milstein scheme* $\tilde{Y}_{N,r} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\begin{aligned} \tilde{Y}_{N,r,i}(0) &:= \xi_i, \\ \tilde{Y}_{N,r,i}(t) &:= \tilde{Y}_{N,r,i}(t_\ell^{(N)}) + \frac{\mu_i(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot (t - t_\ell^{(N)}) \\ &\quad + \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot (W_j(t) - W_j(t_\ell^{(N)})) \\ &\quad + \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot J_{j_1, j_2}^{(\ell, N)}(t), \end{aligned} \tag{4.19}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$ where $J_{j_1, j_2}^{(\ell, N)}(\cdot)$ is defined as in (4.5).

Note that $\tilde{Y}_{N,r}$ is well-defined in the sense that its realizations are in fact continuous; for this, we use that the Itô integral appearing in the construction above is continuous.

Moreover, this process satisfies almost surely for all $t \in [0, T]$ and for all $i \in \{1, \dots, d\}$

$$\begin{aligned} \tilde{Y}_{N,r,i}(t) &= \xi_i + \int_0^t \frac{\mu_i(\lfloor sN/T \rfloor \cdot T/N, \tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N))}{1 + T/N \cdot |\tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N)|^{2r}} ds \\ &\quad + \sum_{j_2=1}^m \int_0^t \frac{\sigma_{i,j_2}(\lfloor sN/T \rfloor \cdot T/N, \tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N))}{1 + T/N \cdot |\tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N)|^{2r}} \\ &\quad + \sum_{j_1=1}^m \frac{(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(\lfloor sN/T \rfloor \cdot T/N, \tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N))}{1 + T/N \cdot |\tilde{Y}_{N,r}(\lfloor sN/T \rfloor \cdot T/N)|^{2r}} \\ &\quad \cdot (W_{j_1}(s) - W_{j_1}(\lfloor sN/T \rfloor \cdot T/N)) dW_{j_2}(s); \end{aligned}$$

cf. the construction (4.18). If the Assumption (C) is satisfied, then the continuous-time tamed Milstein scheme reduces to the form

$$\begin{aligned} \tilde{Y}_{N,r,i}(0) &= \xi_i, \\ \tilde{Y}_{N,r,i}(t) &= \tilde{Y}_{N,r,i}(t_\ell^{(N)}) + \frac{\mu_i(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot (t - t_\ell^{(N)}) \\ &\quad + \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot (W_j(t) - W_j(t_\ell^{(N)})) \\ &\quad + \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, N)}(t), \end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$ where $I_{j_1, j_2}^{(\ell, N)}(\cdot)$ is defined as in (4.6).

Since entire trajectories of the driving Brownian motion and iterated Itô integrals are used in the construction of the continuous-time tamed Milstein scheme, this scheme is not an approximation in the sense of Section 2.1, and we thus find $\tilde{Y}_{N,r} \notin \mathbb{X}(L_p([0, T]; \mathbb{R}^d))$ for every $p \in [1, \infty)$.

In Section A.4, we gather useful properties of the continuous-time tamed Milstein schemes, namely, moment bounds and strong convergence with order (at least) 1.

4.3.2 The Equidistant Tamed Milstein Schemes

Next, based on the continuous-time tamed Milstein scheme, we construct approximations which use not whole paths of the driving Brownian motion but evaluate W only at equidistant sites.

As before, let $N \in \mathbb{N}$ and $r \in [0, \infty)$. The *equidistant tamed Milstein scheme* $\hat{Y}_{N,r}^{\text{eq}} : \Omega \rightarrow C([0, T]; \mathbb{R}^d)$ is given by

$$\hat{Y}_{N,r}^{\text{eq}}(t_\ell^{(N)}) := \tilde{Y}_{N,r}(t_\ell^{(N)})$$

for all $\ell \in \{0, \dots, N\}$ and linearly interpolated between these time points.

If the Assumption (C) is satisfied, then we obtain $\widehat{Y}_{N,r}^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(L_p([0, T]; \mathbb{R}^d))$ for all $p \in [1, \infty)$ by suitably choosing sequences ψ , χ , and φ as per Section 2.1. Clearly, the total number of evaluations of W employed in the approximation $\widehat{Y}_{N,r}^{\text{eq}}$ is given by N for each realization in this situation, which immediately yields $c(\widehat{Y}_{N,r}^{\text{eq}}) = N$.

4.3.3 The Adaptive Tamed Milstein Schemes

The following construction of the adaptive tamed Milstein schemes is heavily inspired by the corresponding construction of the adaptive Milstein schemes $\widehat{M}_{N,q}^{\text{ad}}$ presented in Section 4.2.

Note that, under suitable regularity assumptions on the coefficients of the SDE (1.4) and under the commutativity condition (C), its solution $(X(t))_{t \in [0, T]}$ satisfies

$$\mathbb{E} \left[|X(t + \delta) - X(t)|_q^q \mid X(t) \right] = \mathbf{m}_q^q \cdot |\sigma(t, X(t))|_{q,2}^q \cdot \delta^{q/2} + o(\delta^{q/2})$$

for all $q \in [1, \infty)$ and for all $t \in [0, T]$. Hence, the paths of the solution of the considered SDE are, in the L_q mean sense and conditioned on $X(t)$, locally Hölder–1/2–continuous with Hölder constant $|\sigma(t, X(t))|_{q,2}$. For this reason, it is more beneficial to evaluate W more often in regions where the value of $|\sigma(t, X(t))|_{q,2}$ is large and vice versa.

Motivated by this idea, we construct our adaptive tamed Milstein scheme in two steps. First, we use equidistant time steps to roughly approximate the solution and thereby obtain estimates for the conditional Hölder constants at these sites. Second, we refine our approximation by taking into account the local smoothness of the solution. More precisely, we distribute the more additional evaluation sites between those equidistant time points the larger the corresponding estimated Hölder constant is.

Let $r \in [0, \infty)$ and let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers satisfying the limits (4.9). Fix $N \in \mathbb{N}$ for the moment and let $q \in [1, \infty)$. For each $\ell \in \{0, \dots, k_N - 1\}$, we consider the random discretization

$$t_\ell^{(k_N)} = \tau_{\ell,0}^{(k_N)} < \tau_{\ell,1}^{(k_N)} < \dots < \tau_{\ell,\eta_\ell+1}^{(k_N)} = t_{\ell+1}^{(k_N)} \quad (4.20)$$

of $[t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)}]$ where

$$\eta_\ell := \left\lceil N \cdot T/k_N \cdot \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right\rceil$$

and

$$\tau_{\ell,\kappa}^{(k_N)} := t_\ell^{(k_N)} + \frac{T}{k_N} \cdot \frac{\kappa}{\eta_\ell + 1}$$

for all $\kappa \in \{0, \dots, \eta_\ell + 1\}$. Let $\widehat{W}_N^{\text{ad}}$ denote the linear interpolation of W at the adaptive time points (4.20), i.e., let

$$\begin{aligned} \widehat{W}_N^{\text{ad}}(0) &:= 0, \\ \widehat{W}_N^{\text{ad}}(t) &:= \frac{\tau_{\ell,\kappa+1}^{(k_N)} - t}{\tau_{\ell,\kappa+1}^{(k_N)} - \tau_{\ell,\kappa}^{(k_N)}} \cdot W(\tau_{\ell,\kappa}^{(k_N)}) + \frac{t - \tau_{\ell,\kappa}^{(k_N)}}{\tau_{\ell,\kappa+1}^{(k_N)} - \tau_{\ell,\kappa}^{(k_N)}} \cdot W(\tau_{\ell,\kappa+1}^{(k_N)}), \end{aligned} \quad (4.21)$$

for all $\ell \in \{0, \dots, k_N - 1\}$, for all $\kappa \in \{0, \dots, \eta_\ell\}$, and for all $t \in (\tau_{\ell, \kappa}^{(k_N)}, \tau_{\ell, \kappa+1}^{(k_N)})$.

The *adaptive tamed Milstein scheme* $\widehat{Y}_{N,r,q}^{\text{ad}} : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ is given by

$$\begin{aligned} \widehat{Y}_{N,r,q}^{\text{ad}}(t) &:= \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}) + \frac{\mu(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \cdot (t - t_\ell^{(k_N)}) \\ &\quad + \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \cdot (\widehat{W}_N^{\text{ad}}(t) - W(t_\ell^{(k_N)})), \\ \widehat{Y}_{N,r,q}^{\text{ad}}(T) &:= \widetilde{Y}_{k_N,r}(T) \end{aligned}$$

for all $\ell \in \{0, \dots, k_N\}$ and for all $t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$; cf. the construction (4.11).

If the Assumption (C) is satisfied, then we obtain $\widehat{Y}_{N,r,q}^{\text{ad}} \in \mathbb{X}_{[c(\widehat{Y}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, T]; \mathbb{R}^d))$ by suitably choosing sequences ψ , χ , and φ as per Section 2.1 in case the cost of this approximation satisfy $c(\widehat{Y}_{N,r,q}^{\text{ad}}) < \infty$. Define $\nu_{N,r,q}^{\text{ad}}$ to be the (random) number of evaluations of W employed in the approximation $\widehat{Y}_{N,r,q}^{\text{ad}}$. Observe that

$$\nu_{N,r,q}^{\text{ad}} = k_N + \sum_{\ell=0}^{k_N-1} \eta_\ell \leq k_N + N \cdot T/k_N \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \quad (4.22)$$

and

$$\nu_{N,r,q}^{\text{ad}} \geq \max \left\{ k_N, N \cdot T/k_N \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right\}. \quad (4.23)$$

4.4 Main Results

The following theorems entirely specify the asymptotics of the N th minimal errors in the classes of adaptive and of equidistant approximations as well as the asymptotics of the errors of the adaptive and of the equidistant tamed Milstein schemes. As a consequence, we will conclude strong asymptotic optimality of these schemes in their respective classes. The proofs of all theorems are postponed to Section 4.7. Finally, we compare our results to those given in Müller-Gronbach (2002b) at the end of this section.

Recall the definition (4.1) of the asymptotic constants from the beginning of this chapter. The succeeding remark provides sufficient conditions for the finiteness of these two constants.

Remark 4.12. *Let the Assumptions (I_a), (locL), (K_a), and (pG_r^σ) be satisfied for some $a \in [2, \infty)$ and $r \in [1, \infty)$ with $a \geq r$. Then Fubini's theorem and the moment estimate (3.2) imply $C_{a/r, a/r}^{\text{eq}} < \infty$.*

First, we specify the asymptotics of the N th minimal errors in the classes of adaptive and of equidistant approximations. More precisely, we not only state the convergence rates but also give the sharp asymptotic constants. To this end, recall the definition (4.1) of $C_{q,q}^{\text{ad}}$ and $C_{q,q}^{\text{eq}}$.

Theorem 4.13. *Let the Assumptions (C), (I_a), (K_a), (M_b), (pL_{r-1}^{∇μ}), (pL_{(r-2)/2}^{∇σ}), (B^{∇tμ}), and (B^{∇tσ}) be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Then for all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(L_q([0, T]; \mathbb{R}^d)) \right\} = C_{q,q}^{\text{ad}} \quad (4.24)$$

and

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d)) \right\} = C_{q,q}^{\text{eq}}. \quad (4.25)$$

Proof. This result is an immediate consequence of the Lemmas 4.24, 4.25, 4.26, and 4.27 given in Section 4.7. \square

Next, we specify the asymptotics of the errors of the adaptive and of the equidistant tamed Milstein schemes. Again, we not only state the convergence rates but also provide the sharp asymptotic constants. Right before, we show that the average numbers of observation sites of the adaptive tamed Milstein schemes are finite and give their asymptotic behavior.

Theorem 4.14. *Let the Assumptions (C), (I_a), (K_a), (M_b), (pL_{r-1}^{∇μ}), (pL_{(r-2)/2}^{∇σ}), (B^{∇tμ}), and (B^{∇tσ}) be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Then for all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$c(\widehat{Y}_{N,r,q}^{\text{ad}}) < \infty$$

for each $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} c(\widehat{Y}_{N,r,q}^{\text{ad}}) \cdot N^{-1} = \left(\mathfrak{m}_q^{-1} \cdot \mathfrak{g}_q^{-1} \cdot C_{q,q}^{\text{ad}} \right)^{2q/(q+2)}$$

as well as

$$\lim_{N \rightarrow \infty} \left(c(\widehat{Y}_{N,r,q}^{\text{ad}}) \right)^{1/2} \cdot e_{q,q}(\widehat{Y}_{N,r,q}^{\text{ad}}) = C_{q,q}^{\text{ad}}$$

and

$$\lim_{N \rightarrow \infty} N^{1/2} \cdot e_{q,q}(\widehat{Y}_{N,r}^{\text{eq}}) = C_{q,q}^{\text{eq}}.$$

Proof. This result is an immediate consequence of the Lemmas 4.23, 4.24, 4.25, 4.26, and 4.27 given in Section 4.7. \square

Since both the convergence rates and the corresponding asymptotic constants match in the preceding theorems, we obtain strong asymptotic optimality of the adaptive and of the equidistant tamed Milstein schemes in their respective classes.

Corollary 4.15. *Let the Assumptions (C), (I_a), (K_a), (M_b), (pL_{r-1}^{∇μ}), (pL_{(r-2)/2}^{∇σ}), (B^{∇tμ}), and (B^{∇tσ}) be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Then for all $q \in [1, b) \cap [1, a/(3r + 1)]$ with $C_{q,q}^{\text{ad}} > 0$ it holds that*

$$\lim_{N \rightarrow \infty} \frac{e_{q,q}(\widehat{Y}_{N,r,q}^{\text{ad}})}{\inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_{[c(\widehat{Y}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, T]; \mathbb{R}^d)) \right\}} = 1$$

and

$$\lim_{N \rightarrow \infty} \frac{e_{q,q}(\widehat{Y}_{N,r}^{\text{eq}})}{\inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d)) \right\}} = 1;$$

i.e., the approximations $(\widehat{Y}_{N,r,q}^{\text{ad}})_{N \in \mathbb{N}}$ and $(\widehat{Y}_{N,r}^{\text{eq}})_{N \in \mathbb{N}}$ are strongly asymptotically optimal in the classes $(\mathbb{X}_{[c(\widehat{Y}_{N,r,q}^{\text{ad}})]}^{\text{ad}}(L_q([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, respectively.

Proof. This result is an immediate consequence of the Theorems 4.13 and 4.14. \square

Remark 4.16. *Note that the assumptions required in the preceding results essentially come from Proposition A.9, which states strong convergence of the continuous-time tamed Milstein schemes. In this situation, we also have moment bounds for these schemes as well as for the solution, see Proposition A.8 and (3.2).*

In the next remark, we characterize equality of the asymptotic constants $C_{q,q}^{\text{ad}}$ and $C_{q,q}^{\text{eq}}$, and briefly comment on the special case when these two constants both are zero.

Remark 4.17. *Let the SDE (1.4) have a unique solution $(X(t))_{t \in [0, T]}$ and let $q \in [1, \infty)$ such that $C_{q,q}^{\text{eq}} < \infty$.*

Then we have $C_{q,q}^{\text{ad}} = C_{q,q}^{\text{eq}}$ if and only if almost surely $|\sigma(t, X(t))|_{q,2} = \|\sigma(0, \xi)|_{q,2}\|_{L_q(\Omega)}$ holds for all $t \in [0, T]$, cf. Remark 6 in Müller-Gronbach (2002b). In particular, this case applies to SDEs whose diffusion coefficient is constant.

Moreover, we have $C_{q,q}^{\text{eq}} = 0$ (and hence $C_{q,q}^{\text{ad}} = 0$) if and only if almost surely $\sigma(t, X(t)) = 0$ holds for almost all $t \in [0, T]$. In such a situation, Theorems 4.13 and 4.14 do not allow to conclude strong asymptotic optimality. More precisely, these results merely show that the N th minimal errors in the classes of adaptive and of equidistant approximations as well as the errors of the adaptive and of the equidistant tamed Milstein schemes each are in $o(N^{-1/2})$. In fact, these N th minimal errors are actually zero as the underlying SDE almost surely represents an ordinary differential equation in the considered case.

To some extent, the preceding findings generalize (at least for autonomous SDEs) the corresponding results presented in Müller-Gronbach (2002b). In this reference, the coefficients of the SDE (1.4) as well as their partial derivatives with respect to the state variable are required to be globally Lipschitz continuous, whereas in our setting these functions may even grow super-linearly.

Remark 4.18. Let the SDE (1.4) be autonomous, let $T = 1$, and let the Assumptions (C), (I_{a^*}) , (pL_0^μ) , (pL_0^σ) , $(pL_0^{\nabla\mu})$, and $(pL_0^{\nabla\sigma})$ be satisfied for some $a^* \in [28, \infty)$. Proposition 4.3 then shows that the N th minimal errors in the classes of adaptive and of equidistant approximations satisfy the asymptotics (4.24) and (4.25) for all $q \in [1, a^*/2]$.

Moreover, in the setting above, Remarks 3.1 and 4.1 yield that the Assumption (K_a) is satisfied for all $a \in [2, \infty)$, that the Assumption (M_b) is satisfied for all $b \in (2, \infty)$, that the Assumption $(pL_{r-1}^{\nabla\mu})$ is satisfied for all $r \in [1, \infty)$, that the Assumption $(pL_{(r-2)/2}^{\nabla\sigma})$ is satisfied for all $r \in [2, \infty)$, and that the Assumptions $(B^{\nabla_t\mu})$ and $(B^{\nabla_t\sigma})$ are satisfied. By choosing $a = b = a^*$ and $r = 2$, Theorem 4.13 shows that the N th minimal errors in the classes of adaptive and of equidistant approximations satisfy (4.24) and (4.25) for all $q \in [1, a^*/7]$.

In summary, we indeed impose weaker conditions on the underlying SDE (except for the moment condition on the initial value); the asymptotics (4.24) and (4.25), though, are obtained only up to smaller values of q . However, if additionally the initial value ξ is deterministic, then we obtain (4.24) and (4.25) in both scenarios for every $q \in [1, \infty)$.

4.5 Problems and Solution Approaches in the Case $p \neq q$

One substantial benefit of taking p equal to q in the definition of $e_{q,p}$ is the possibility to apply Fubini's theorem, which allows to interchange the order of expectation and integration. Therefore, the calculation of an error $e_{q,q}$ reduces to the following tasks: first, determining the q th absolute moments of a certain process at all time points and second, integrating over these expectations afterwards. The following example illustrates a simple application of this idea.

Example 4.19. Reconsider the SDE (2.9) from Section 2.3, which is given by

$$\begin{aligned} dX(t) &= 1 dW(t), \quad t \in [0, 1], \\ X(0) &= 0, \end{aligned}$$

with $d = m = 1$ and which possesses the unique solution $(W(t))_{t \in [0,1]}$. Fix $q \in [1, \infty)$ and $N \in \mathbb{N}$. By using Fubini's theorem, we see that the piecewise-linear interpolation $\widehat{W}_N^{\text{eq}} \in \mathbb{X}_N^{\text{eq}}(L_q([0, 1]; \mathbb{R}))$ of W at the equidistant sites ℓ/N , $\ell \in \{0, \dots, N\}$, satisfies

$$\begin{aligned} e_{q,q}(\widehat{W}_N^{\text{eq}}) &= \left(\mathbb{E} \left[\int_0^1 |W(t) - \widehat{W}_N^{\text{eq}}(t)|^q dt \right] \right)^{1/q} \\ &= \left(\int_0^1 \mathbb{E} [|W(t) - \widehat{W}_N^{\text{eq}}(t)|^q] dt \right)^{1/q} \\ &= \left(\sum_{\ell=0}^{N-1} \int_{\ell/N}^{(\ell+1)/N} \mathbb{E} [|W(t) - \widehat{W}_N^{\text{eq}}(t)|^q] dt \right)^{1/q}. \end{aligned}$$

Recall the definitions of \mathbf{m}_q and \mathbf{g}_q from the beginning of this chapter, and note that the process $(W(t) - \widehat{W}_N^{\text{eq}}(t))_{t \in [0,1]}$ consists of N independent Brownian bridges between the

discretization points. Lemma A.4 gives that

$$\int_{\ell/N}^{(\ell+1)/N} \mathbb{E} \left[|W(t) - \widehat{W}_N^{\text{eq}}(t)|^q \right] dt = \mathfrak{m}_q^q \cdot \mathfrak{g}_q^q \cdot N^{-(q+2)/2}$$

holds for each $\ell \in \{0, \dots, N-1\}$. Hence, we immediately obtain

$$e_{q,q}(\widehat{W}_N^{\text{eq}}) = \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot N^{-1/2}.$$

◇

In the case $p \neq q$, the analysis of an error $e_{q,p}$ proves to be significantly harder than before. Unlike for the special case, we here need to first integrate whole paths of a certain process and then determine absolute moments of these integrals. Already for trivial examples, such as the one above, corresponding calculations become highly challenging to carry out.

Example 4.20. We assume the setting of Example 4.19. Fix $p, q \in [1, \infty)$, $p \neq q$, and $N \in \mathbb{N}$. Without the possibility of applying Fubini's theorem, we merely arrive at

$$\begin{aligned} e_{q,p}(\widehat{W}_N^{\text{eq}}) &= \left(\mathbb{E} \left[\left(\int_0^1 |W(t) - \widehat{W}_N^{\text{eq}}(t)|^p dt \right)^{q/p} \right] \right)^{1/q} \\ &= \left(\mathbb{E} \left[\left(\sum_{\ell=0}^{N-1} \int_{\ell/N}^{(\ell+1)/N} |W(t) - \widehat{W}_N^{\text{eq}}(t)|^p dt \right)^{q/p} \right] \right)^{1/q}. \end{aligned}$$

Up to scaling factors, our task therefore reduces to determining moments of

$$\sum_{\ell=0}^{N-1} \int_0^1 |B_\ell(t)|^p dt \tag{4.26}$$

where B_0, \dots, B_{N-1} are independent Brownian bridges on $[0, 1]$. The mathematical object of an integrated Brownian bridge, which occurs above, is of interest by itself, as the following results demonstrate.

For $p = 1$, Shepp (1982) calculated integer moments of $\int_0^1 |B_0(t)| dt$, and Johnson & Killeen (1983) derived its cumulative distribution function. Unfortunately, these objects are fairly cumbersome as they involve (infinite series of) Airy functions, which complicate their analyses considerably.

For $p = 2$, Tolmatz (2002) obtained a density and integer moments of $\int_0^1 |B_0(t)|^2 dt$. By N times convoluting this density with itself, one in principle obtains a density of (4.26). But as before, the moments mentioned above are only given in an awkward form. More precisely, these values are only given recursively, thereby also containing Bernoulli numbers. ◇

On the one hand, we have just seen that errors $e_{q,p}$ are hard to analyze in the case $p \neq q$. On the other hand, we are able to determine asymptotic lower and upper error bounds at least in certain situations.

First, in the setting of Proposition 4.2, we can immediately deduce asymptotic lower bounds for both the N th minimal errors in the classes $(\mathbb{X}_N^{\text{ad}}(L_p([0, T]; \mathbb{R})))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(L_p([0, T]; \mathbb{R})))_{N \in \mathbb{N}}$ each with convergence rate $N^{-1/2}$ and unspecified asymptotic constant.

Second, an asymptotic lower bound for the N th minimal errors in specific classes with respect to the error $e_{q,q}$ implies an asymptotic lower bound for the N th minimal errors in the respective classes with respect to the error $e_{q,p}$. More precisely, let $* \in \{\text{eq}, \text{ad}\}$ and $p, q \in [1, \infty)$, and assume that there exist $C \in (0, \infty)$ and $\gamma : \mathbb{N} \rightarrow (0, \infty)$ such that

$$\liminf_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot \inf \left\{ e_{q \wedge p, q \wedge p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(L_{q \wedge p}([0, T]; \mathbb{R}^d)) \right\} \geq C.$$

Now observe that the monotonicity properties (2.5) and (2.7) yield

$$e_{q,p}(\widehat{X}) \geq (T \cdot d)^{1/p-1/(q \wedge p)} \cdot e_{q \wedge p, q \wedge p}(\widehat{X})$$

for all $\widehat{X} \in \mathbb{X}(L_p([0, T]; \mathbb{R}^d))$. Combining the last two inequalities, we conclude that

$$\liminf_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot \inf \left\{ e_{q,p}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^*(L_p([0, T]; \mathbb{R}^d)) \right\} \geq (T \cdot d)^{1/p-1/(q \wedge p)} \cdot C.$$

Third, an asymptotic upper bound for the errors of specific approximations with respect to the error $e_{q,q}$ implies an asymptotic upper bound for the errors of these approximations with respect to the error $e_{q,p}$. More precisely, let $* \in \{\text{eq}, \text{ad}\}$ and $p, q \in [1, \infty)$, consider approximations $(\widehat{X}_N)_{N \in \mathbb{N}}$ with $\widehat{X}_N \in \mathbb{X}_N^*(L_{q \vee p}([0, T]; \mathbb{R}^d))$ for each $N \in \mathbb{N}$, and assume that there exist $C \in (0, \infty)$ and $\gamma : \mathbb{N} \rightarrow (0, \infty)$ such that

$$\limsup_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot e_{q \vee p, q \vee p}(\widehat{X}_N) \leq C.$$

Now observe that the monotonicity properties (2.5) and (2.7) yield

$$e_{q,p}(\widehat{X}) \leq (T \cdot d)^{1/p-1/(q \vee p)} \cdot e_{q \vee p, q \vee p}(\widehat{X})$$

holds for all $\widehat{X} \in \mathbb{X}(L_{q \vee p}([0, T]; \mathbb{R}^d))$. Combining the last two inequalities, we conclude that

$$\limsup_{N \rightarrow \infty} (\gamma(N))^{-1} \cdot e_{q,p}(\widehat{X}_N) \leq (T \cdot d)^{1/p-1/(q \vee p)} \cdot C.$$

4.6 Numerical Experiment

We illustrate the results of Section 4.4 by a numerical experiment. To this end, we consider the SDE

$$\begin{aligned} dX(t) &= 13.5 \cdot X(t) \cdot (1 - (X(t))^2) dt + (X(t))^2 dW(t), \quad t \in [0, 1], \\ X(0) &= 5, \end{aligned} \tag{4.27}$$

with $d = 1$ and $m = 1$.

It is easy to see that the SDE (4.27) satisfies all the assumptions of our main theorems. More precisely, we have that Assumption (C) is satisfied, Assumption (I_a) is satisfied for all $a \in [0, \infty)$, Assumption (K_a) is satisfied for all $a \in [2, 28]$, Assumption (M_b) is satisfied for all $b \in [2, 14.5]$, Assumption ($pL_{r-1}^{\nabla\mu}$) is satisfied for all $r \in [2, \infty)$, Assumption ($pL_{(r-2)/2}^{\nabla\sigma}$) is satisfied for all $r \in [2, \infty)$, Assumption ($B^{\nabla\mu}$) is satisfied for all $r \in [0, \infty)$, and Assumption ($B^{\nabla\sigma}$) is satisfied for all $r \in [0, \infty)$; cf. the corresponding part in Section 3.5. For the rest of this section, we fix $a = 28$, $b = 14.5$, $r = 2$, and $q = 2$.

In view of Theorem 4.14, we aim at visualizing that, for large $N \in \mathbb{N}$, the approximation errors $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{ad}})$ and $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{eq}})$ of the adaptive and of the equidistant tamed Milstein schemes are close to $C_{2,2}^{\text{ad}} \cdot (c(\widehat{Y}_{N,2,2}^{\text{ad}}))^{-1/2}$ and $C_{2,2}^{\text{eq}} \cdot N^{-1/2}$, respectively.

In doing so, we encounter three different approximation issues, namely, the approximation of the asymptotic constants $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$, of the errors $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{ad}})$ and $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{eq}})$, and of the average number of evaluations $c(\widehat{Y}_{N,2,2}^{\text{ad}})$.

Regarding the first approximation issue, we do not know numerically suitable closed-form expressions of the constants $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$, nor of the solution, for the particular SDE (4.27). Therefore, we estimate these constants via Monte Carlo simulations in which we approximate the solution by an equidistant tamed Milstein scheme with a sufficiently large number of discretization points. More precisely, we estimate $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$ by

$$\widehat{C}_{2,2,M,N}^{\text{ad}} := 6^{-1/2} \cdot \frac{1}{M} \cdot \frac{1}{N} \cdot \sum_{m=1}^M \sum_{\ell=0}^{N-1} |\widehat{Y}_{N,2,m}^{\text{eq}}(t_\ell^{(N)})|^2$$

and

$$\widehat{C}_{2,2,M,N}^{\text{eq}} := 6^{-1/2} \cdot \left(\frac{1}{M} \cdot \frac{1}{N} \cdot \sum_{m=1}^M \sum_{\ell=0}^{N-1} |\widehat{Y}_{N,2,m}^{\text{eq}}(t_\ell^{(N)})|^4 \right)^{1/2},$$

respectively, where $M, N \in \mathbb{N}$, where the random vectors

$$(\widehat{Y}_{N,2,m}^{\text{eq}}(t_0^{(N)}), \dots, \widehat{Y}_{N,2,m}^{\text{eq}}(t_N^{(N)})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{Y}_{N,2}^{\text{eq}}(t_0^{(N)}), \dots, \widehat{Y}_{N,2}^{\text{eq}}(t_N^{(N)}))$, and where we employ $\mathbf{m}_2 = 1$ and $\mathbf{g}_2 = 6^{-1/2}$. Observe that for both $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$, we approximate the Lebesgue integral occurring in their definitions by left Riemann sums. Proposition A.9 implies that $\widehat{C}_{2,2,M,N}^{\text{ad}}$ and $\widehat{C}_{2,2,M,N}^{\text{eq}}$ tend to $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$, respectively, as M and N tend to infinity. Figure 4.1 depicts simulations of $\widehat{C}_{2,2,M,227}^{\text{ad}}$ and $\widehat{C}_{2,2,M,227}^{\text{eq}}$ in dependence of M along with their corresponding 95% CLT-based confidence intervals. Furthermore, we utilize the specific approximation values $C_{2,2}^{\text{ad}} \approx 0.4419$ and $C_{2,2}^{\text{eq}} \approx 0.5844$ obtained from realizations of $\widehat{C}_{2,2,10^4,227}^{\text{ad}}$ and of $\widehat{C}_{2,2,10^4,227}^{\text{eq}}$, respectively, for the black lines featured in Figure 4.2.

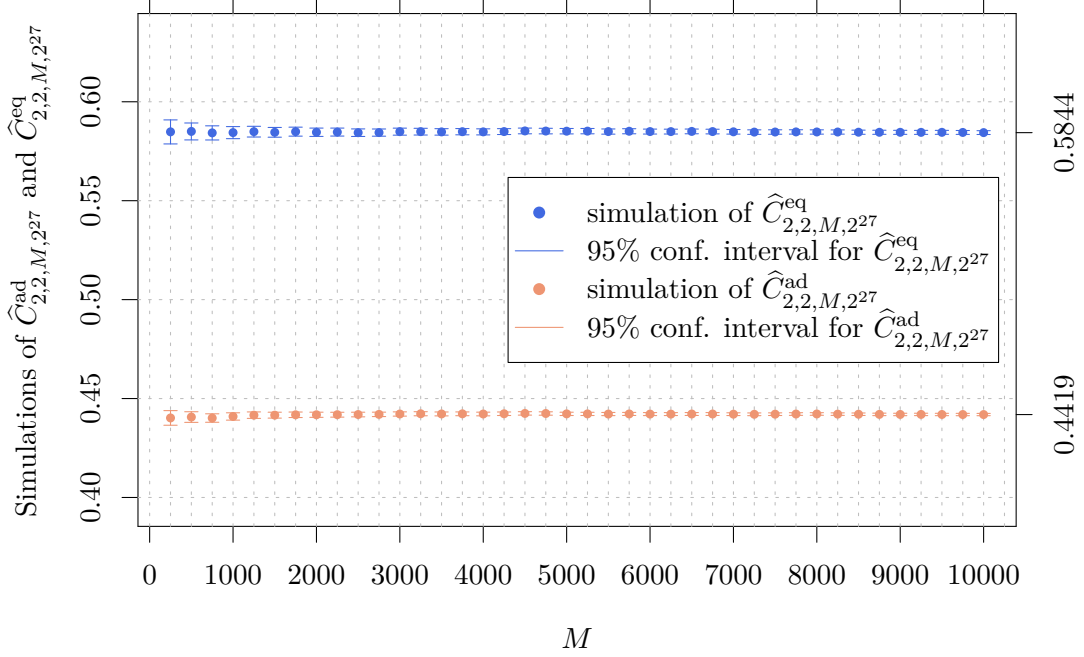


Figure 4.1: Monte Carlo approximations of the asymptotic constants $C_{2,2}^{\text{ad}}$ and $C_{2,2}^{\text{eq}}$ for the SDE (4.27).

The remaining two approximation issues are addressed simultaneously. Similarly to the approximation of the asymptotic constants, we again estimate the solution by a sufficiently accurate equidistant tamed Milstein scheme, and we approximate the errors of the equidistant tamed Milstein schemes as well as the errors and the average numbers of evaluations of the adaptive tamed Milstein schemes via Monte Carlo simulations. More precisely, for each $N \in \mathbb{N}$ we estimate $e_{2,2}(\widehat{Y}_{N,2}^{\text{eq}})$, $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{ad}})$, and $c(\widehat{Y}_{N,2,2}^{\text{ad}})$ by

$$\widehat{e}_{2,2,M,N_{\max},N}^{\text{eq}} := \left(\frac{1}{M} \cdot \frac{1}{N_{\max}} \cdot \sum_{m=1}^M \sum_{\ell=0}^{N_{\max}-1} |\widehat{Y}_{N_{\max},2,m}^{\text{eq}}(t_{\ell}^{(N_{\max})}) - \widehat{Y}_{N,2,m}^{\text{eq}}(t_{\ell}^{(N_{\max})})|^2 \right)^{1/2},$$

$$\widehat{e}_{2,2,M,N_{\max},N}^{\text{ad}} := \left(\frac{1}{M} \cdot \frac{1}{N_{\max}} \cdot \sum_{m=1}^M \sum_{\ell=0}^{N_{\max}-1} |\widehat{Y}_{N_{\max},2,m}^{\text{eq}}(t_{\ell}^{(N_{\max})}) - \widehat{Y}_{N,2,2,m}^{\text{ad}}(t_{\ell}^{(N_{\max})})|^2 \right)^{1/2},$$

and

$$\widehat{c}_{M,N} := \frac{1}{M} \cdot \sum_{m=1}^M \nu_{N,2,2,m}^{\text{ad}},$$

respectively, where $M, N_{\max} \in \mathbb{N}$ and where the random vectors

$$(\widehat{Y}_{N_{\max},2,m}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N_{\max},2,m}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{Y}_{N_{\max},2}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N_{\max},2}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})}))$, the random vectors

$$(\widehat{Y}_{N,2,m}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N,2,m}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{Y}_{N,2}^{\text{eq}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N,2}^{\text{eq}}(t_{N_{\max}}^{(N_{\max})}))$, the random vectors

$$(\widehat{Y}_{N,2,2,m}^{\text{ad}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N,2,2,m}^{\text{ad}}(t_{N_{\max}}^{(N_{\max})})), \quad m \in \{1, \dots, M\},$$

are independent copies of $(\widehat{Y}_{N,2,2}^{\text{ad}}(t_0^{(N_{\max})}), \dots, \widehat{Y}_{N,2,2}^{\text{ad}}(t_{N_{\max}}^{(N_{\max})}))$, and the random variables

$$\nu_{N,2,2,m}^{\text{ad}}, \quad m \in \{1, \dots, M\},$$

are independent copies of $\nu_{N,2,2}^{\text{ad}}$. For the adaptive tamed Milstein schemes, we thereby used $k_N := \lceil N^{9/10} \rceil$ for all $N \in \mathbb{N}$ on every computation. Numerical estimates $(N, \widehat{e}_{2,2,10^4,2^{27},N}^{\text{eq}})$, $N \in \{2^6, 2^8, \dots, 2^{18}\}$, and $(\widehat{c}_{10^3,N}, \widehat{e}_{2,2,10^3,2^{27},N}^{\text{ad}})$, $N \in \{2^6, 2^8, \dots, 2^{18}\}$, are visualized in Figure 4.2.

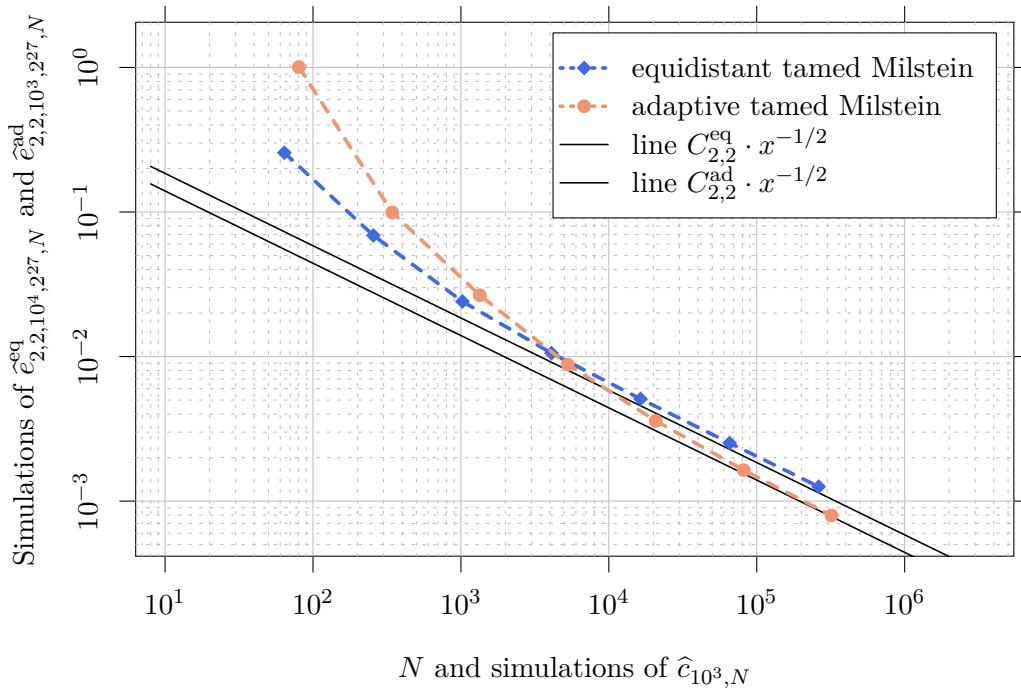


Figure 4.2: Monte Carlo approximations of the errors $e_{2,2}(\widehat{Y}_{N,2}^{\text{eq}})$ and $e_{2,2}(\widehat{Y}_{N,2,2}^{\text{ad}})$ versus N and Monte Carlo approximations of the average number of evaluations $c(\widehat{Y}_{N,2,2}^{\text{ad}})$ for the SDE (4.27).

4.7 Proofs

In this section, we prove our main theorems from Section 4.4 by establishing asymptotic lower and upper error bounds as indicated in Section 2.3. The structure of the corresponding proofs is to a large extent based on techniques developed in Müller-Gronbach (2002b).

Throughout this section, let the Assumptions (C), (I_a), (K_a), (M_b), (pL_{r-1}^{∇μ}), (pL_{(r-2)/2}^{∇σ}), (B^{∇tμ}), and (B^{∇tσ}) be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Observe that, in this setting, the Assumptions (locL), (LLG^μ), (pG_r^{∇μ}), (pL_r^μ), (pG_{r+1}^μ), (LLG^σ), (pG_{r/2}^{∇σ}), (pL_{r/2}^σ), and (pG_{(r+2)/2}^σ) are also satisfied; see Remark 4.1. Moreover, there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $j_1, j_2 \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(t, x)| \leq C \cdot (1 + |x|)^{r+1}. \quad (4.28)$$

In addition, let $(k_N)_{N \in \mathbb{N}}$ be a sequence of natural numbers such that the limits (4.9) hold, and let c denote unspecified positive constants that may vary at every occurrence and that may only depend on T , d , m , and the parameters and constants from the preceding assumptions.

4.7.1 Preliminaries

As a first step, we show that the continuous-time tamed Milstein schemes inserted in the respective tamed diffusion coefficient tend in a certain L_q type sense towards the solution inserted in the diffusion coefficient. This result will turn out to be a crucial tool for the proofs of both the asymptotic lower and upper error bounds.

Lemma 4.21. *For all $q \in [1, 2a/(r+2))$ and for all $\bar{q} \in [2/3, q]$ it holds that*

$$\lim_{N \rightarrow \infty} \left\| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right\|_{L_{\bar{q}}(\Omega)} = \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{\bar{q}}([0,T])} \right\|_{L_{\bar{q}}(\Omega)}.$$

Proof. We divide the proof into the two parts $\bar{q} \in [1, q]$ and $\bar{q} \in [2/3, 1)$.

First, we consider the case $\bar{q} \in [1, q]$. Here, it suffices to show

$$\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right|_{\bar{q}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{\bar{q}}([0,T])} \right\|_{\bar{q}}, \quad (4.29)$$

$$\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left(\left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} - \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right) \right|_{\bar{q}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0, \quad (4.30)$$

$$\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left(\left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} - \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right) \right|_{\bar{q}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0, \quad (4.31)$$

and that

$$\left(\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right|_{\bar{q}} \right)_{N \in \mathbb{N}} \text{ is uniformly integrable,} \quad (4.32)$$

as we obtain

$$\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right|_{\bar{q}} \xrightarrow[N \rightarrow \infty]{L_{\bar{q}}} \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{\bar{q}}([0,T])}$$

in this case.

With regard to (4.29), observe that the function

$$[0, T] \rightarrow \mathbb{R}, \quad t \mapsto |\sigma(t, X(t, \omega))|_{q,2},$$

is continuous for every $\omega \in \Omega$ due to the Assumptions (LLG $^\sigma$) and (pL $_{r/2}^\sigma$). Hence, we obtain

$$\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right|_{\bar{q}} \xrightarrow[N \rightarrow \infty]{\text{a.s.}} \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{\bar{q}}([0,T])}, \quad (4.33)$$

which immediately shows (4.29).

With regard to (4.30) and (4.31), we show L_θ convergence of the respective random variables to zero for appropriate values of $\theta \in (0, \infty)$ in both cases. First, fix $\theta := \min\{b/2, 2a/(9r+3)\} \in [1, \infty)$. Combining the monotonicity property $|\cdot|_{\bar{q}} \leq |\cdot|_1$, the inverse triangle inequality, and the triangle inequality give

$$\begin{aligned} & \left\| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left(\left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} - \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right) \right\|_{\bar{q}, L_\theta(\Omega)} \\ & \leq \left\| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left(\left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} - \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right) \right\|_{1, L_\theta(\Omega)} \\ & \leq \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left\| \sum_{\ell=0}^{N-1} \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} - \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right\|_{L_\theta(\Omega)} \\ & \leq \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \sum_{\ell=0}^{N-1} \left\| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) - \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right\|_{q,2} \Big\|_{L_\theta(\Omega)}. \end{aligned} \quad (4.34)$$

Moreover, Assumption (pL $_{r/2}^\sigma$), the Hölder inequality, Proposition A.9, the triangle in-

equality, the moment estimate (3.2), and Proposition A.8 yield

$$\begin{aligned}
& \left\| \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) - \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right\|_{L_\theta(\Omega)} \\
& \leq c \cdot \left\| \left| X(t_\ell^{(N)}) - \tilde{Y}_{N,r}(t_\ell^{(N)}) \right| \cdot \left(1 + |X(t_\ell^{(N)})| + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right)^{r/2} \right\|_{L_\theta(\Omega)} \\
& \leq c \cdot \left\| X(t_\ell^{(N)}) - \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{3r\theta/2}(\Omega)} \cdot \left\| 1 + |X(t_\ell^{(N)})| + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right\|_{L_{3r\theta/2}(\Omega)}^{r/2} \\
& \leq c \cdot N^{-1} \cdot \left(1 + \left\| X(t_\ell^{(N)}) \right\|_{L_{3r\theta/2}(\Omega)} + \left\| \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{3r\theta/2}(\Omega)} \right)^{r/2} \\
& \leq c \cdot N^{-1}
\end{aligned} \tag{4.35}$$

for all $\ell \in \{0, \dots, N-1\}$. Combining (4.34) and (4.35), we eventually obtain (4.30) as N tends to infinity. Next, fix $\theta := 2a/(5r+2) \in [4, \infty)$. Similarly to the derivation of (4.34), one shows

$$\begin{aligned}
& \left\| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left(\left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} - \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2} \right) \right\|_{L_\theta(\Omega)} \\
& \leq \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \sum_{\ell=0}^{N-1} \left\| \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) - \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2} \right\|_{L_\theta(\Omega)}.
\end{aligned} \tag{4.36}$$

Moreover, Assumption (pG $_{(r+2)/2}^\sigma$), the triangle inequality, and Proposition A.8 yield

$$\begin{aligned}
& \left\| \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) - \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2} \right\|_{L_\theta(\Omega)} \\
& = T/N \cdot \left\| \frac{|\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r} \cdot |\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))|}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right\|_{L_\theta(\Omega)} \\
& \leq T/N \cdot \left\| |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r} \cdot |\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))| \right\|_{L_\theta(\Omega)} \\
& \leq c \cdot N^{-1} \cdot \left\| |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r} \cdot \left(1 + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right)^{(r+2)/2} \right\|_{L_\theta(\Omega)} \\
& \leq c \cdot N^{-1} \cdot \left\| 1 + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right\|_{L_{(5r+2)\theta/2}(\Omega)}^{(5r+2)/2} \\
& \leq c \cdot N^{-1} \cdot \left(1 + \left\| \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{(5r+2)\theta/2}(\Omega)} \right)^{(5r+2)/2} \\
& \leq c \cdot N^{-1}
\end{aligned} \tag{4.37}$$

for all $\ell \in \{0, \dots, N-1\}$. Combining (4.36) and (4.37), we eventually obtain (4.31) as N tends to infinity.

With regard to (4.32), we fix $\check{q} := 2a/(r+2) \in (\bar{q}, \infty)$ first. Combining Lemma B.2, Assumption (pG $_{(r+2)/2}^\sigma$), and Proposition A.8 yields

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left| \left(\frac{T}{N} \right)^{1/\bar{q}} \cdot \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right|_{\bar{q}}^{\check{q}} \right] \\
&= T^{\check{q}/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left(\frac{1}{N} \cdot \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\bar{q}} \right)^{\check{q}/\bar{q}} \right] \\
&\leq T^{\check{q}/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \mathbb{E} \left[\frac{1}{N} \cdot \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\check{q}} \right] \\
&\leq T^{\check{q}/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[\left| \frac{\sigma(t, \tilde{Y}_{N,r}(t))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t)|^{2r}} \right|_{q,2}^{\check{q}} \right] \\
&\leq T^{\check{q}/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[\left| \sigma(t, \tilde{Y}_{N,r}(t)) \right|_{q,2}^{\check{q}} \right] \\
&\leq c \cdot \left(1 + \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[|\tilde{Y}_{N,r}(t)|^{\check{q} \cdot (r+2)/2} \right] \right) \\
&< \infty.
\end{aligned}$$

Thus, we obtain (4.32).

Second, we consider the case $\bar{q} \in [2/3, 1)$. Here, it suffices to show

$$\frac{T}{N} \cdot \left\| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} \xrightarrow[N \rightarrow \infty]{L_1} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{q,2} \right\|_{L_{\bar{q}}([0, T])}^{\bar{q}}, \quad (4.38)$$

$$\frac{T}{N} \cdot \left(\left\| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} - \left\| \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} \right) \xrightarrow[N \rightarrow \infty]{L_1} 0, \quad (4.39)$$

and

$$\frac{T}{N} \cdot \left(\left\| \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} - \left\| \left| \frac{\sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t^{(N)})|^{2r}} \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} \right) \xrightarrow[N \rightarrow \infty]{L_1} 0, \quad (4.40)$$

due to the triangle inequality for $\|\cdot\|_{L_1(\Omega)}$.

With regard to (4.38), observe first that the almost sure convergence (4.33) directly implies

$$\frac{T}{N} \cdot \left\| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} \xrightarrow[N \rightarrow \infty]{\text{a.s.}} \left\| \left| \sigma(\cdot, X(\cdot)) \right|_{q,2} \right\|_{L_{\bar{q}}([0, T])}^{\bar{q}}. \quad (4.41)$$

Moreover, the sequence

$$\left(\frac{T}{N} \cdot \left\| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right\|_{\bar{q}}^{\bar{q}} \right)_{N \in \mathbb{N}} \text{ is uniformly integrable,} \quad (4.42)$$

since Lemma B.2, Assumption (pG $_{(r+2)/2}^\sigma$), and the moment estimate (3.2) yield

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left(\frac{T}{N} \cdot \left| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right|_{\bar{q}} \right)^{1/\bar{q}} \right] \\
&= T^{1/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left(\frac{1}{N} \cdot \sum_{\ell=0}^{N-1} \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} \right)^{1/\bar{q}} \right] \\
&\leq T^{1/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \mathbb{E} \left[\frac{1}{N} \cdot \sum_{\ell=0}^{N-1} \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} \right] \\
&\leq T^{1/\bar{q}} \cdot \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[\left| \sigma(t, X(t)) \right|_{q,2} \right] \\
&\leq c \cdot \left(1 + \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[|X(t)|^{(r+2)/2} \right] \right) \\
&< \infty.
\end{aligned}$$

Combining (4.41) and (4.42), we thus obtain (4.38).

With regard to (4.39), observe that the triangle inequality, the inequality $|x^\alpha - y^\alpha| \leq |x - y|^\alpha$ for all $x, y \in [0, \infty)$ and $\alpha \in (0, 1]$, and the inverse triangle inequality give

$$\begin{aligned}
& \mathbb{E} \left[\left| \frac{T}{N} \cdot \left(\left| \left| \sigma(t^{(N)}, X(t^{(N)})) \right|_{q,2} \right|_{\bar{q}} - \left| \left| \sigma(t^{(N)}, \tilde{Y}_{N,r}(t^{(N)})) \right|_{q,2} \right|_{\bar{q}} \right) \right| \right] \\
&= \frac{T}{N} \cdot \mathbb{E} \left[\left| \sum_{\ell=0}^{N-1} \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} \right|_{\bar{q}} - \sum_{\ell=0}^{N-1} \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right|_{\bar{q}} \right] \\
&\leq \frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \mathbb{E} \left[\left| \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} \right|_{\bar{q}} - \left| \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right|_{\bar{q}} \right] \tag{4.43} \\
&\leq \frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \mathbb{E} \left[\left| \left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) \right|_{q,2} - \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right|_{\bar{q}} \right] \\
&\leq \frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \mathbb{E} \left[\left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) - \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2} \right]_{\bar{q}}.
\end{aligned}$$

Moreover, Assumption (pL $_{r/2}^\sigma$), the Hölder inequality, Proposition A.9, the moment

estimate (3.2), and Proposition A.8 yield

$$\begin{aligned}
& \mathbb{E} \left[\left| \sigma(t_\ell^{(N)}, X(t_\ell^{(N)})) - \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2}^{\bar{q}} \right] \\
& \leq c \cdot \mathbb{E} \left[\left| X(t_\ell^{(N)}) - \tilde{Y}_{N,r}(t_\ell^{(N)}) \right|^{\bar{q}} \cdot \left(1 + |X(t_\ell^{(N)})| + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right)^{r\bar{q}/2} \right] \\
& \leq c \cdot \left\| X(t_\ell^{(N)}) - \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{3r\bar{q}/2}(\Omega)}^{\bar{q}} \cdot \left\| 1 + |X(t_\ell^{(N)})| + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right\|_{L_{3r\bar{q}/2}(\Omega)}^{r\bar{q}/2} \quad (4.44) \\
& \leq c \cdot N^{-\bar{q}} \cdot \left(1 + \left\| X(t_\ell^{(N)}) \right\|_{L_{3r\bar{q}/2}(\Omega)} + \left\| \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{3r\bar{q}/2}(\Omega)} \right)^{r\bar{q}/2} \\
& \leq c \cdot N^{-\bar{q}}
\end{aligned}$$

for all $\ell \in \{0, \dots, N-1\}$. Combining (4.43) and (4.44), we eventually obtain (4.39) as N tends to infinity.

With regard to (4.40), one first shows

$$\begin{aligned}
& \mathbb{E} \left[\left| \frac{T}{N} \cdot \left(\left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2}^{\bar{q}} - \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\bar{q}} \right) \right| \right] \\
& \leq \frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \mathbb{E} \left[\left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) - \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\bar{q}} \right] \quad (4.45) \\
& = \frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \mathbb{E} \left[\left| \frac{T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r} \cdot \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\bar{q}} \right]
\end{aligned}$$

in a similar manner to (4.43). Moreover, Assumption $(\text{pG}_{(r+2)/2}^\sigma)$, the triangle inequality, and Proposition A.8 yield

$$\begin{aligned}
& \mathbb{E} \left[\left| \frac{T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r} \cdot \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^{\bar{q}} \right] \\
& \leq (T/N)^{\bar{q}} \cdot \mathbb{E} \left[\left| |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r\bar{q}} \cdot \left| \sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)})) \right|_{q,2}^{\bar{q}} \right| \right] \\
& \leq c \cdot N^{-\bar{q}} \cdot \mathbb{E} \left[|\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r\bar{q}} \cdot \left(1 + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right)^{\bar{q} \cdot (r+2)/2} \right] \quad (4.46) \\
& \leq c \cdot N^{-\bar{q}} \cdot \left\| 1 + |\tilde{Y}_{N,r}(t_\ell^{(N)})| \right\|_{L_{(5r\bar{q}+2\bar{q})/2}(\Omega)}^{2/(5r\bar{q}+2\bar{q})} \\
& \leq c \cdot N^{-\bar{q}} \cdot \left(1 + \left\| \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{(5r\bar{q}+2\bar{q})/2}(\Omega)} \right)^{2/(5r\bar{q}+2\bar{q})}
\end{aligned}$$

for all $\ell \in \{0, \dots, N-1\}$. Combining (4.45) and (4.46), we eventually obtain (4.40) as N tends to infinity. \square

In the following lemma, we provide an estimate for the absolute moments of the iterated Itô integrals that are employed in the construction of the continuous-time (tamed) Milstein schemes in the case that the commutativity condition is satisfied. For this purpose, recall their definition (4.6).

Lemma 4.22. *Let $j_1, j_2 \in \{1, \dots, m\}$. Then for all $q \in [1, \infty)$ there exists $C \in (0, \infty)$ such that for all $\mathbb{N} \in \mathbb{N}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$ it holds that*

$$\left\| I_{j_1, j_2}^{(\ell, N)}(t) \right\|_{L_q(\Omega)} \leq C \cdot N^{-1}.$$

Proof. Let $\mathbb{N} \in \mathbb{N}$, $\ell \in \{0, \dots, N-1\}$, and $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$. For $j_1 \neq j_2$ the Cauchy–Schwarz inequality gives

$$\left\| I_{j_1, j_2}^{(\ell, N)}(t) \right\|_{L_q(\Omega)} \leq \left\| W_{j_1}(t) - W_{j_1}(t_\ell^{(N)}) \right\|_{L_{2q}(\Omega)} \cdot \left\| W_{j_2}(t) - W_{j_2}(t_\ell^{(N)}) \right\|_{L_{2q}(\Omega)},$$

and for $j_1 = j_2$ the triangle inequality gives

$$\left\| I_{j_1, j_2}^{(\ell, N)}(t) \right\|_{L_q(\Omega)} \leq \left\| (W_{j_1}(t) - W_{j_1}(t_\ell^{(N)}))^2 \right\|_{L_q(\Omega)} + (t - t_\ell^{(N)}).$$

In both cases, the desired estimate immediately follows from estimates on the absolute moments of centered normally distributed random variables. \square

Next, we show that the average numbers of evaluations of the driving Brownian motion employed in the adaptive tamed Milstein schemes are finite and asymptotically equivalent to a constant times N . We obtain, in particular, that each such approximation does indeed lie in one of the classes of adaptive schemes.

Lemma 4.23. *For all $q \in [1, b) \cap [1, a/(3r+1)]$ we have*

$$c(\widehat{Y}_{N,r,q}^{\text{ad}}) < \infty \tag{4.47}$$

for each $N \in \mathbb{N}$ and

$$\lim_{N \rightarrow \infty} c(\widehat{Y}_{N,r,q}^{\text{ad}}) \cdot N^{-1} = \left(\mathfrak{m}_q^{-1} \cdot \mathfrak{g}_q^{-1} \cdot C_{q,q}^{\text{ad}} \right)^{2q/(q+2)}. \tag{4.48}$$

Proof. For each $N \in \mathbb{N}$, the estimates (4.22) and (4.23) on the total number of evaluations of W employed in the respective adaptive tamed Milstein scheme yield

$$c(\widehat{Y}_{N,r,q}^{\text{ad}}) \leq k_N + N \cdot \mathbb{E} \left[\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right] \tag{4.49}$$

and

$$c(\widehat{Y}_{N,r,q}^{\text{ad}}) \geq N \cdot \mathbb{E} \left[\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right]. \tag{4.50}$$

Combining (4.49), Assumption $(\text{pG}_{(r+2)/2}^\sigma)$, and Proposition A.8 shows (4.47).

By using (4.49), (4.50), Lemma 4.21, and the first limit in (4.9), we then conclude (4.48). \square

4.7.2 Asymptotic Lower Bounds

First, we prove an asymptotic lower bound for the N th minimal errors in the classes of adaptive approximations.

Lemma 4.24. *For all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$\liminf_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{ad}}(L_q([0, T]; \mathbb{R}^d)) \right\} \geq C_{q,q}^{\text{ad}}. \quad (4.51)$$

Proof. Fix $N \in \mathbb{N}$ and $\widehat{X}_N \in \mathbb{X}_N^{\text{ad}}(L_q([0, T]; \mathbb{R}^d))$ for the moment. Let D_N denote the entire data used by the approximation \widehat{X}_N in the sense of Section 2.1, define Ψ_N to be the set of observation sites of the driving Brownian motion employed in \widehat{X}_N , and put $\nu_N := \#\Psi_N$. Because of the first limit in (4.9), we may actually assume that $\{t_1^{(k_N)}, \dots, t_{k_N}^{(k_N)}\} \subseteq \Psi_N$. Hence, each $\widetilde{Y}_{k_N, r}(t_\ell^{(k_N)})$, $\ell \in \{0, \dots, k_N\}$, is measurable with respect to the σ -algebra generated by D_N .

Let the process $Z_{k_N} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by

$$Z_{k_N, i}(t) := \sum_{\ell=0}^{k_N-1} \mathbf{1}_{(t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})}(t) \cdot \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \widetilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot \left(I_{j_1, j_2}^{(\ell, k_N)}(t) - \mathbb{E}[I_{j_1, j_2}^{(\ell, k_N)}(t) \mid D_N] \right)$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$.

Due to the inverse triangle inequality and Proposition A.9, it holds that

$$\begin{aligned} e_{q,q}(\widehat{X}_N) &= \left\| \|X - \widehat{X}_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \\ &\geq \left\| \|\widetilde{Y}_{N, r} - Z_{k_N} - \widehat{X}_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \\ &\quad - \left\| \|Z_{k_N}\|_{L_q([0, T])} \right\|_{L_q(\Omega)} - c \cdot k_N^{-1}. \end{aligned} \quad (4.52)$$

As a next step, we show that the norm of Z_{k_N} as above is asymptotically negligible. Combining the triangle inequality, the contraction property of conditional expectations, the triangle inequality again, the independence of $I_{j_1, j_2}^{(\ell, k_N)}(t)$ and $\mathcal{F}(t_\ell^{(k_N)})$, the growth

condition (4.28), Lemma 4.22, and Proposition A.8 yields

$$\begin{aligned}
& \left\| \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot \left(I_{j_1, j_2}^{(\ell, k_N)}(t) - \mathbb{E}[I_{j_1, j_2}^{(\ell, k_N)}(t) | D_N] \right) \right\|_{L_q(\Omega)} \\
& \leq \left\| \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& \quad + \left\| \mathbb{E} \left[\frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \middle| D_N \right] \right\|_{L_q(\Omega)} \\
& \leq \left\| \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& \leq \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& = \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right\|_{L_q(\Omega)} \cdot \left\| I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& \leq c \cdot \sum_{j_1, j_2=1}^m \left(1 + \left\| \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}) \right\|_{L_{q(r+1)}(\Omega)}^{r+1} \right) \cdot k_N^{-1} \\
& \leq c \cdot k_N^{-1}
\end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, k_N - 1\}$, and for all $t \in (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$. Consequently, we obtain by Fubini's theorem that

$$\begin{aligned}
& \left\| \| Z_{k_N} \|_{L_q([0, T])} \right\|_{L_q(\Omega)} \\
& = \left(\sum_{i=1}^d \sum_{\ell=0}^{k_N-1} \int_{t_\ell^{(k_N)}}^{t_{\ell+1}^{(k_N)}} \left\| \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right. \right. \\
& \quad \left. \left. \cdot \left(I_{j_1, j_2}^{(\ell, k_N)}(t) - \mathbb{E}[I_{j_1, j_2}^{(\ell, k_N)}(t) | D_N] \right) \right\|_{L_q(\Omega)}^q dt \right)^{1/q} \\
& \leq c \cdot k_N^{-1}.
\end{aligned} \tag{4.53}$$

Next, we show that the distance between $\tilde{Y}_{k_N, r} - Z_{k_N}$ and \hat{X}_N as in (4.52) is greater or equal than the respective distance between $\tilde{Y}_{k_N, r} - Z_{k_N}$ and $\mathbb{E}[\tilde{Y}_{k_N, r} | D_N]$. Note that, due to the measurability property of $\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})$ mentioned before, we have

$$\begin{aligned}
& \tilde{Y}_{k_N, r, i}(t) - Z_{k_N, i}(t) - \mathbb{E}[\tilde{Y}_{k_N, r, i}(t) | D_N] \\
& = \sum_{j=1}^m \frac{\sigma_{i, j}(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot \left(W_j(t) - \mathbb{E}[W_j(t) | D_N] \right)
\end{aligned} \tag{4.54}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, k_N - 1\}$, and for all $t \in (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$. Similarly to the derivations of the Lemmas 1 and 2 in Yaroslavtseva (2017), one shows that for \mathbb{P}^{D_N} -almost all $(x, y) \in \mathbb{R}^d \times \bigcup_{n \in \mathbb{N}} (\mathbb{R}^m)^n$ it holds that

$$\mathbb{P}^{W|D_N=(x,y)} = \mathbb{P}^{-W|D_N=(x,y)},$$

which along with (4.54) yields

$$\mathbb{P}^{\tilde{Y}_{k_N,r} - Z_{k_N} - \mathbb{E}[\tilde{Y}_{k_N,r}|D_N] | D_N=(x,y)} = \mathbb{P}^{-\tilde{Y}_{k_N,r} + Z_{k_N} + \mathbb{E}[\tilde{Y}_{k_N,r}|D_N] | D_N=(x,y)}.$$

We thus conclude that $(\tilde{Y}_{k_N,r} - Z_{k_N} - \mathbb{E}[\tilde{Y}_{k_N,r} | D_N], D_N)$ and $(-\tilde{Y}_{k_N,r} + Z_{k_N} + \mathbb{E}[\tilde{Y}_{k_N,r} | D_N], D_N)$ are identically distributed. Since, additionally, both \hat{X}_N and $\mathbb{E}[\tilde{Y}_{k_N,r} | D_N]$ are measurable functions of D_N , we consequently find that $\tilde{Y}_{k_N,r} - Z_{k_N} - \hat{X}_N$ and $2\mathbb{E}[\tilde{Y}_{k_N,r} | D_N] - \tilde{Y}_{k_N,r} + Z_{k_N} - \hat{X}_N$ are also identically distributed. Therefore, we obtain

$$\begin{aligned} & \left\| \left\| \tilde{Y}_{k_N,r} - Z_{k_N} - \hat{X}_N \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &= \frac{1}{2} \cdot \left(\left\| \left\| \tilde{Y}_{k_N,r} - Z_{k_N} - \hat{X}_N \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \right. \\ & \quad \left. + \left\| \left\| -2\mathbb{E}[\tilde{Y}_{k_N,r} | D_N] + \tilde{Y}_{k_N,r} - Z_{k_N} + \hat{X}_N \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \right) \\ &\geq \frac{1}{2} \cdot \left\| \left\| \tilde{Y}_{k_N,r} - Z_{k_N} - \hat{X}_N \right. \right. \\ & \quad \left. \left. - 2\mathbb{E}[\tilde{Y}_{k_N,r} | D_N] + \tilde{Y}_{k_N,r} - Z_{k_N} + \hat{X}_N \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &= \left\| \left\| \tilde{Y}_{k_N,r} - Z_{k_N} - \mathbb{E}[\tilde{Y}_{k_N,r} | D_N] \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)}. \end{aligned} \tag{4.55}$$

Now let

$$d_{\ell, k_N} := \#(\Psi_N \cap (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})) + 1$$

for each $\ell \in \{0, \dots, k_N - 1\}$. Conditioned on D_N , the observation sites Ψ_N are fixed and for every $j \in \{1, \dots, m\}$ the Gaussian process $(W_j(t) - \mathbb{E}[W_j(t) | D_N])_{t \in [0, T]}$ consists of ν_N independent Brownian bridges between the discretization points. Hence, we obtain

with Fubini's theorem for conditional expectations and Lemma A.4(ii)

$$\begin{aligned}
& \mathbb{E} \left[\left\| \tilde{Y}_{k_N, r} - Z_{k_N} - \mathbb{E}[\tilde{Y}_{k_N, r} \mid D_N] \right\|_{L_q([0, T])}^q \middle| D_N \right] \\
&= \sum_{\ell=0}^{k_N-1} \sum_{i=1}^d \int_{t_\ell^{(k_N)}}^{t_{\ell+1}^{(k_N)}} \mathbb{E} \left[\left| \sum_{j=1}^m \frac{\sigma_{i,j}(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right. \right. \\
&\quad \left. \left. \cdot (W_j(t) - \mathbb{E}[W_j(t) \mid D_N]) \right|^q \middle| D_N \right] dt \\
&= \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^q \\
&\quad \cdot \int_{t_\ell^{(k_N)}}^{t_{\ell+1}^{(k_N)}} \mathbb{E} \left[|W_1(t) - \mathbb{E}[W_1(t) \mid D_N]|^q \middle| D_N \right] dt \\
&\geq \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^q \cdot \mathbf{m}_q^q \cdot \mathbf{g}_q^q \cdot (T/k_N)^{(q+2)/2} \cdot d_{\ell, k_N}^{-q/2}
\end{aligned} \tag{4.56}$$

almost surely where \mathbf{m}_q and \mathbf{g}_q are as defined in (4.2) and (4.3), respectively. Next, observe that applying the inequality

$$\mathbb{E}[\nu_N] = \mathbb{E} \left[\sum_{\ell=0}^{k_N-1} d_{\ell, k_N} \right] \leq N$$

and two times the Hölder inequality gives

$$\begin{aligned}
& \left(\sum_{\ell=0}^{k_N-1} \mathbb{E} \left[\left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^q \cdot d_{\ell, k_N}^{-q/2} \right] \right)^{2/(q+2)} \cdot N^{q/(q+2)} \\
&\geq \left(\sum_{\ell=0}^{k_N-1} \mathbb{E} \left[\left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^q \cdot d_{\ell, k_N}^{-q/2} \right] \right)^{2/(q+2)} \\
&\quad \cdot \left(\sum_{\ell=0}^{k_N-1} \mathbb{E}[d_{\ell, k_N}] \right)^{q/q+2} \\
&\geq \sum_{\ell=0}^{k_N-1} \left(\mathbb{E} \left[\left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^q \cdot d_{\ell, k_N}^{-q/2} \right] \right)^{2/(q+2)} \cdot (\mathbb{E}[d_{\ell, k_N}])^{q/q+2} \\
&\geq \sum_{\ell=0}^{k_N-1} \mathbb{E} \left[\left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right].
\end{aligned} \tag{4.57}$$

By combining (4.52), (4.53), (4.55), (4.56), and (4.57), we arrive at

$$\begin{aligned}
& N^{1/2} \cdot e_{q,q}(\widehat{X}_N) \\
& \geq \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot \left(\mathbb{E} \left[\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_{q,2}^{2q/(q+2)} \right] \right)^{(q+2)/2q} \\
& \quad - c \cdot N^{1/2} \cdot k_N^{-1}.
\end{aligned} \tag{4.58}$$

Finally, we obtain (4.51) from (4.58) by employing Lemma 4.21 and the second limit in (4.9). \square

Next, we prove an asymptotic lower bound for the N th minimal errors in the classes of equidistant approximations.

Lemma 4.25. *For all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$\liminf_{N \rightarrow \infty} N^{1/2} \cdot \inf \left\{ e_{q,q}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d)) \right\} \geq C_{q,q}^{\text{eq}}. \tag{4.59}$$

Proof. Fix $N \in \mathbb{N}$ and $\widehat{X}_N \in \mathbb{X}_N^{\text{eq}}(L_q([0, T]; \mathbb{R}^d))$ for the moment, and consider the data $D_N := (\xi, W(t_1^{(N)}), \dots, W(t_N^{(N)}))$.

Let the process $Z_N : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by

$$\begin{aligned}
Z_{N,i}(t) := & \sum_{\ell=0}^{N-1} \mathbb{1}_{(t_\ell^{(N)}, t_{\ell+1}^{(N)})}(t) \cdot \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \widetilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\widetilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \\
& \cdot \left(I_{j_1, j_2}^{(\ell, N)}(t) - \mathbb{E}[I_{j_1, j_2}^{(\ell, N)}(t) \mid D_N] \right)
\end{aligned}$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$.

Similarly to the estimates (4.52), (4.53), (4.55), and (4.56) in the proof of Lemma 4.24, one successively shows that

$$e_{q,q}(\widehat{X}_N) \geq \left\| \|\widetilde{Y}_{N,r} - Z_N - \widehat{X}_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} - \left\| \|Z_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} - c \cdot N^{-1}, \tag{4.60}$$

that

$$\left\| \|Z_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \leq c \cdot N^{-1}, \tag{4.61}$$

that

$$\left\| \|\widetilde{Y}_{N,r} - Z_N - \widehat{X}_N\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \geq \left\| \|\widetilde{Y}_{N,r} - Z_N - \mathbb{E}[\widetilde{Y}_{N,r} \mid D_N]\|_{L_q([0, T])} \right\|_{L_q(\Omega)}, \tag{4.62}$$

and that

$$\begin{aligned}
& \mathbb{E} \left[\|\widetilde{Y}_{N,r} - Z_N - \mathbb{E}[\widetilde{Y}_{N,r} \mid D_N]\|_{L_q([0, T])}^q \mid D_N \right] \\
& \geq \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(N)}, \widetilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\widetilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^q \cdot \mathfrak{m}_q^q \cdot \mathfrak{g}_q^q \cdot (T/N)^{(q+2)/2}
\end{aligned} \tag{4.63}$$

holds almost surely.

By combining (4.60), (4.61), (4.62), and (4.63), we arrive at

$$\begin{aligned} & N^{1/2} \cdot e_{q,q}(\widehat{X}_N) \\ & \geq \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot T^{1/2} \cdot \left(\mathbb{E} \left[\frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \right|_q^q \right] \right)^{1/q} - c \cdot N^{-1/2}. \end{aligned} \quad (4.64)$$

Finally, we obtain (4.59) from (4.64) by employing Lemma 4.21. \square

4.7.3 Asymptotic Upper Bounds

First, we prove an asymptotic upper bound for the errors of the adaptive tamed Milstein schemes.

Lemma 4.26. *For all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$\limsup_{N \rightarrow \infty} \left(c(\widehat{Y}_{N,r,q}^{\text{ad}}) \right)^{1/2} \cdot e_{q,q}(\widehat{Y}_{N,r,q}^{\text{ad}}) \leq C_{q,q}^{\text{ad}}. \quad (4.65)$$

Proof. We prove (4.65) by showing

$$\limsup_{N \rightarrow \infty} \left(c(\widehat{Y}_{N,r,q}^{\text{ad}}) \right)^{1/2} \cdot N^{-1/2} \leq \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{2q/(q+2)}([0,T])} \right\|_{L_{2q/(q+2)}(\Omega)}^{q/(q+2)} \quad (4.66)$$

and

$$\limsup_{N \rightarrow \infty} N^{1/2} \cdot e_{q,q}(\widehat{Y}_{N,r,q}^{\text{ad}}) \leq \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot \left\| \left\| |\sigma(\cdot, X(\cdot))|_{q,2} \right\|_{L_{2q/(q+2)}([0,T])} \right\|_{L_{2q/(q+2)}(\Omega)}^{2/(q+2)}. \quad (4.67)$$

First, note that (4.66) is an immediate consequence of (4.48).

Now fix $N \in \mathbb{N}$ for the moment. Let the process $Z_{k_N} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by

$$Z_{k_N,i}(t) := \sum_{\ell=0}^{k_N-1} \mathbb{1}_{[t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})}(t) \cdot \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i,j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \widetilde{Y}_{k_N,r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\widetilde{Y}_{k_N,r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t)$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$.

Due to the triangle inequality and Proposition A.9, it holds that

$$\begin{aligned} e_{q,q}(\widehat{Y}_{N,r,q}^{\text{ad}}) &= \left\| \left\| X - \widehat{Y}_{N,r,q}^{\text{ad}} \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &\leq \left\| \left\| \widetilde{Y}_{k_N,r} - Z_{k_N} - \widehat{Y}_{N,r,q}^{\text{ad}} \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &\quad + \left\| \left\| Z_{k_N} \right\|_{L_q([0,T])} \right\|_{L_q(\Omega)} + c \cdot k_N^{-1}. \end{aligned} \quad (4.68)$$

As a next step, we show that the norm of Z_{k_N} as above is asymptotically negligible. Combining the triangle inequality, the independence of $I_{j_1, j_2}^{(\ell, k_N)}(t)$ and $\mathcal{F}(t_\ell^{(k_N)})$, the growth condition (4.28), Lemma 4.22, and Proposition A.8 yields

$$\begin{aligned}
& \left\| \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& \leq \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& = \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right\|_{L_q(\Omega)} \cdot \left\| I_{j_1, j_2}^{(\ell, k_N)}(t) \right\|_{L_q(\Omega)} \\
& \leq c \cdot \sum_{j_1, j_2=1}^m \left(1 + \left\| \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}) \right\|_{L_{q(r+1)}(\Omega)}^{r+1} \right) \cdot k_N^{-1} \\
& \leq c \cdot k_N^{-1}
\end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, k_N - 1\}$, and for all $t \in (t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$. Similarly to the derivation of (4.53), we thus obtain

$$\left\| \left\| Z_{k_N} \right\|_{L_q([0, T])} \right\|_{L_q(\Omega)} \leq c \cdot k_N^{-1}. \quad (4.69)$$

Next, we show that the norm of $\tilde{Y}_{k_N, r} - Z_{k_N} - \widehat{Y}_{N, r, q}^{\text{ad}}$ as in (4.68) determines the asymptotic constant eventually. Note that for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, k_N - 1\}$, and for all $t \in [t_\ell^{(k_N)}, t_{\ell+1}^{(k_N)})$

$$\tilde{Y}_{k_N, r, i}(t) - Z_{k_N, i}(t) - \widehat{Y}_{N, r, q, i}^{\text{ad}}(t) = \sum_{j=1}^m \frac{\sigma_{i, j}(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \cdot (W_j(t) - \widehat{W}_{N, j}^{\text{ad}}(t))$$

where $\widehat{W}_N^{\text{ad}}$ is defined as in (4.21). Put $D_{k_N} := (\xi, W(t_1^{(k_N)}), \dots, W(t_{k_N}^{(k_N)}))$. Conditioned on D_{k_N} , the observation sites (4.20) are fixed and for every $j \in \{1, \dots, m\}$ the Gaussian process $(W_j(t) - \widehat{W}_{N, j}^{\text{ad}})_{t \in [0, T]}$ consists of $\nu_{N, r, q}^{\text{ad}}$ independent Brownian bridges between

the discretization points. Hence, we obtain with Lemma A.4(i)

$$\begin{aligned}
& \mathbb{E} \left[\left\| \tilde{Y}_{k_N, r} - Z_{k_N} - \widehat{Y}_{N, r, q}^{\text{ad}} \right\|_{L_q([0, T])}^q \middle| D_{k_N} \right] \\
&= \sum_{\ell=0}^{k_N-1} \sum_{i=1}^d \int_{t_\ell^{(k_N)}}^{t_{\ell+1}^{(k_N)}} \mathbb{E} \left[\left| \sum_{j=1}^m \frac{\sigma_{i, j}(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right. \right. \\
&\quad \left. \left. \cdot (W_j(t) - \widehat{W}_{N, j}^{\text{ad}}) \right|^q \middle| D_{k_N} \right] dt \quad (4.70) \\
&= \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^q \cdot \int_{t_\ell^{(k_N)}}^{t_{\ell+1}^{(k_N)}} \mathbb{E} \left[|W_1(t) - \widehat{W}_{N, 1}^{\text{ad}}|^q \middle| D_{k_N} \right] dt \\
&= \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^q \cdot \mathbf{m}_q^q \cdot \mathbf{g}_q^q \cdot \left(\frac{T/k_N}{\eta_\ell + 1} \right)^{q/2} \cdot \frac{T}{k_N}
\end{aligned}$$

almost surely where \mathbf{m}_q and \mathbf{g}_q are as defined in (4.2) and (4.3), respectively. For each $\ell \in \{0, \dots, k_N - 1\}$, we furthermore have

$$\eta_\ell + 1 \geq N \cdot T/k_N \cdot \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^{2q/(q+2)}$$

due to the construction of η_ℓ and consequently

$$\begin{aligned}
& \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^q \cdot \left(\frac{T/k_N}{\eta_\ell + 1} \right)^{q/2} \\
& \leq N^{-q/2} \cdot \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^{2q/(q+2)}. \quad (4.71)
\end{aligned}$$

By combining (4.68), (4.69), (4.70), and (4.71), we arrive at

$$\begin{aligned}
& N^{1/2} \cdot e_{q, q}(\widehat{Y}_{N, r, q}^{\text{ad}}) \\
& \leq \mathbf{m}_q \cdot \mathbf{g}_q \cdot \left(\mathbb{E} \left[\frac{T}{k_N} \cdot \sum_{\ell=0}^{k_N-1} \left| \frac{\sigma(t_\ell^{(k_N)}, \tilde{Y}_{k_N, r}(t_\ell^{(k_N)}))}{1 + T/k_N \cdot |\tilde{Y}_{k_N, r}(t_\ell^{(k_N)})|^{2r}} \right|_{q, 2}^{2q/(q+2)} \right] \right)^{1/q} \\
& \quad + c \cdot N^{1/2} \cdot k_N^{-1}. \quad (4.72)
\end{aligned}$$

Finally, we obtain (4.67) from (4.72) by employing Lemma 4.21 and the second limit in (4.9). \square

Next, we prove an asymptotic upper bound for the errors of the equidistant tamed Milstein schemes.

Lemma 4.27. *For all $q \in [1, b) \cap [1, a/(3r + 1)]$ it holds that*

$$\limsup_{N \rightarrow \infty} N^{1/2} \cdot e_{q, q}(\widehat{Y}_{N, r}^{\text{eq}}) \leq C_{q, q}^{\text{eq}}. \quad (4.73)$$

Proof. Fix $N \in \mathbb{N}$ for the moment and put $D_N := (\xi, W(t_1^{(N)}), \dots, W(t_N^{(N)}))$.

Let the process $Z_N : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be defined by

$$Z_{N,i}(t) := \sum_{\ell=0}^{N-1} \mathbb{1}_{(t_\ell^{(N)}, t_{\ell+1}^{(N)}]}(t) \cdot \frac{1}{2} \cdot \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot \left(I_{j_1, j_2}^{(\ell, N)}(t) - \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)}) \right)$$

for all $i \in \{1, \dots, d\}$ and for all $t \in [0, T]$.

Due to the triangle inequality and Proposition A.9, it holds that

$$\begin{aligned} e_{q,q}(\widehat{Y}_{N,r}^{\text{eq}}) &= \left\| \|X - \widehat{Y}_{N,r}^{\text{eq}}\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &\leq \left\| \|\tilde{Y}_{N,r} - Z_N - \widehat{Y}_{N,r}^{\text{eq}}\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \\ &\quad + \left\| \|Z_N\|_{L_q([0,T])} \right\|_{L_q(\Omega)} + c \cdot N^{-1}. \end{aligned} \quad (4.74)$$

As a next step, we show that the norm of Z_N as above is asymptotically negligible. Combining the triangle inequality (applied twice), the independence of $I_{j_1, j_2}^{(\ell, N)}(t)$ and $\mathcal{F}(t_\ell^{(N)})$ as well as the independence of $I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)})$ and $\mathcal{F}(t_\ell^{(N)})$, the growth condition (4.28), Lemma 4.22, and Proposition A.8 yields

$$\begin{aligned} &\left\| \sum_{j_1, j_2=1}^m \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot \left(I_{j_1, j_2}^{(\ell, N)}(t) - \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)}) \right) \right\|_{L_q(\Omega)} \\ &\leq \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot \left(I_{j_1, j_2}^{(\ell, N)}(t) - \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)}) \right) \right\|_{L_q(\Omega)} \\ &\leq \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot I_{j_1, j_2}^{(\ell, N)}(t) \right\|_{L_q(\Omega)} \\ &\quad + \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \cdot \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)}) \right\|_{L_q(\Omega)} \\ &= \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right\|_{L_q(\Omega)} \cdot \left\| I_{j_1, j_2}^{(\ell, N)}(t) \right\|_{L_q(\Omega)} \\ &\quad + \sum_{j_1, j_2=1}^m \left\| \frac{(\nabla \sigma_{i, j_2} \cdot \sigma^{(j_1)})(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right\|_{L_q(\Omega)} \cdot \frac{t - t_\ell^{(N)}}{t_{\ell+1}^{(N)} - t_\ell^{(N)}} \cdot \left\| I_{j_1, j_2}^{(\ell, N)}(t_{\ell+1}^{(N)}) \right\|_{L_q(\Omega)} \\ &\leq c \cdot \sum_{j_1, j_2=1}^m \left(1 + \left\| \tilde{Y}_{N,r}(t_\ell^{(N)}) \right\|_{L_{q(r+1)}(\Omega)}^{r+1} \right) \cdot N^{-1} \\ &\leq c \cdot N^{-1} \end{aligned}$$

for all $i \in \{1, \dots, d\}$, for all $\ell \in \{0, \dots, N-1\}$, and for all $t \in (t_\ell^{(N)}, t_{\ell+1}^{(N)})$. Similarly to the derivation of (4.53), we thus obtain

$$\left\| \|Z_N\|_{L_q([0,T])} \right\|_{L_q(\Omega)} \leq c \cdot N^{-1}. \quad (4.75)$$

Next, we show that the norm of $\tilde{Y}_{N,r} - Z_N - \hat{Y}_{N,r}^{\text{eq}}$ as in (4.74) determines the asymptotic constant eventually. Similarly to the estimate (4.70), one shows that

$$\begin{aligned} & \mathbb{E} \left[\left\| \tilde{Y}_{N,r} - Z_N - \hat{Y}_{N,r}^{\text{eq}} \right\|_{L_q([0,T])}^q \middle| D_N \right] \\ &= \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^q \cdot \mathfrak{m}_q^q \cdot \mathfrak{g}_q^q \cdot (T/N)^{(q+2)/2} \end{aligned} \quad (4.76)$$

holds almost surely.

By combining (4.74), (4.75), and (4.76), we arrive at

$$\begin{aligned} & N^{1/2} \cdot e_{q,q}(\hat{Y}_{N,r}^{\text{eq}}) \\ & \leq \mathfrak{m}_q \cdot \mathfrak{g}_q \cdot T^{1/2} \cdot \left(\mathbb{E} \left[\frac{T}{N} \cdot \sum_{\ell=0}^{N-1} \left| \frac{\sigma(t_\ell^{(N)}, \tilde{Y}_{N,r}(t_\ell^{(N)}))}{1 + T/N \cdot |\tilde{Y}_{N,r}(t_\ell^{(N)})|^{2r}} \right|_{q,2}^q \right] \right)^{1/q} + c \cdot N^{-1/2}. \end{aligned} \quad (4.77)$$

Finally, we obtain (4.73) from (4.77) by employing Lemma 4.21. \square

5

Final Remarks and Outlook

In this chapter, we outline some limitations of this thesis and thereby provide ideas on future research directions. Moreover, we state two conjectures that are situated in the field of tamed schemes.

First, recall that in the construction of strong approximations, the partial information we use about the driving Brownian motion is given by evaluating W at finitely many time points. In the literature, various other types of such partial information are studied. We mention, for instance, evaluations of partial paths of W (e.g., $(W(t))_{t \in (\delta, T]}$ for some $\delta \in (0, T)$), evaluations of linear continuous functionals of W (e.g., $\int_0^T W(t) dt$), or evaluations of iterated Itô integrals (e.g., $\int_0^T W_1(t) dW_2(t)$). Clearly, the definition of strong approximations and the error analyses change according to which type of partial information about W is presumed.

Next, note that we assumed the commutativity condition in our main results on the pathwise L_p error. When this condition is not satisfied, the analysis becomes significantly more cumbersome, resulting in modified asymptotic constants and, partly, even different convergence rates for specific approximations. We refer the reader to Section III.3 in Müller-Gronbach (2002b) for a detailed discussion on the particular error $e_{2,2}$ in such a situation.

In this thesis, we focused on strongly asymptotically optimal approximations in the classes of adaptive and of equidistant approximations, i.e., the classes $(\mathbb{X}_N^{\text{ad}}(S))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{eq}}(S))_{N \in \mathbb{N}}$ for $S = C([0, T]; \mathbb{R}^d)$ and $S = L_p([0, T]; \mathbb{R}^d)$ with $p \in [1, \infty)$. As already mentioned at the end of Section 2.2, one might also consider other classes, such as the classes $(\mathbb{X}_N^{\text{sn}}(S))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{ss}}(S))_{N \in \mathbb{N}}$ defined therein. In the classical framework of SDEs with globally Lipschitz continuous coefficients, Müller-Gronbach (2002a) obtained the exact asymptotics for the N th minimal errors in the classes $(\mathbb{X}_N^{\text{sn}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$, and Müller-Gronbach (2002b) obtained the exact asymptotics for the N th minimal errors in both the classes $(\mathbb{X}_N^{\text{sn}}(L_q([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$ and $(\mathbb{X}_N^{\text{ss}}(L_q([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. It remains an open problem to analyze these particular classes for SDEs with super-linearly growing coefficients and to close the gap regarding the classes $(\mathbb{X}_N^{\text{ss}}(C([0, 1]; \mathbb{R}^d)))_{N \in \mathbb{N}}$. From our results, nonetheless, we can immediately deduce asymptotic lower and upper error bounds for which the corresponding convergence rates match and the corresponding

asymptotic constants are bounded. More precisely, we can directly infer, for instance,

$$\begin{aligned} C_{q,\infty}^{\text{ad}} &\leq \liminf_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot \inf \left\{ e_{q,\infty}(\widehat{X}) \mid \widehat{X} \in \mathbb{X}_N^{\text{SS}}(C([0, T]; \mathbb{R}^d)) \right\} \\ &\leq \limsup_{N \rightarrow \infty} (N/\log(N))^{1/2} \cdot e_{q,\infty}(\widehat{X}_N^{\text{eq}}) \leq C_{q,\infty}^{\text{eq}} \end{aligned}$$

in the setting of Theorem 3.13.

Finally, we conclude this chapter by presenting two conjectures that arose during the work on the tamed Euler and the tamed Milstein schemes.

First, we have noticed that for our results on strong asymptotic optimality to hold true, the concrete form of such a tamed scheme seems to be irrelevant as long as it satisfies certain properties. We exemplarily discuss the tamed Euler scheme. The essential parts needed in the proof of the Theorems 3.13 and 3.14 are the following: first, the recursive structure (3.12), second, the strong convergence given by Proposition A.7, and third, the moment bounds given by Proposition A.6. We conjecture that for an abstract scheme satisfying the properties above, strong asymptotic optimality does indeed hold true. In this case, the Euler scheme and the tamed Euler scheme are nice examples that fit in such a setting. We intend to further elaborate on this approach in an upcoming paper.

Second, note that we measure the pathwise error of an approximation globally in time throughout this thesis. Another error criterion commonly studied in the literature is the L_q error at the final time point, i.e., the error given by

$$e_q(\widehat{X}) := \|X(T) - \widehat{X}(T)\|_{L_q(\Omega)}$$

for an approximation \widehat{X} and $q \in [1, \infty)$. On the one hand, Müller-Gronbach (2002b) showed that suitable variants of the Wagner–Platen scheme are strongly asymptotically optimal in the case of SDEs whose coefficients, among other assumptions, are required to be globally Lipschitz continuous. On the other hand, for SDEs with super-linearly growing coefficients, Sabanis & Zhang (2018) proposed a tamed Wagner–Platen scheme which, as the continuous-time Wagner–Platen scheme, converges strongly to the solution with order (at least) $3/2$. The question arises, if combining these two ideas leads to strongly asymptotically approximations with regard to the error e_q for SDEs with super-linearly growing coefficients. This approach appears very promising and may constitute the object of future studies.

A On some Properties of Specific Stochastic Processes

In this chapter, we provide useful properties of various stochastic processes occurring throughout these notes.

In Section A.1, we give several results on Brownian bridges that will be used in the respective proofs sections of the Chapters 3 and 4. In Section A.2, we prove boundedness for specific moments of the supremum of the solution of the SDE (1.4). In Section A.3, we consider the continuous-time tamed Euler schemes introduced in Section 3.3. For these processes, moment bounds and strong convergence of order (at least) $1/2$ are shown. In Section A.4, we consider the continuous-time tamed Milstein schemes introduced in Section 4.3. For these processes, moment bounds and strong convergence of order (at least) 1 are shown.

Throughout this chapter, except for Section A.1, we require the setting given in Section 1.3.

A.1 Brownian Bridges

In this section, let B_1, B_2, \dots be a sequence of independent Brownian bridges on $[0, 1]$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the first part of this section, we gather auxiliary results that are employed in the proofs given in Chapter 3. Right before, we introduce some notation that will be used for this purpose. For all $q \in [1, \infty)$, for all $N \in \mathbb{N}$, and for all $\alpha_1, \dots, \alpha_N \in [0, \infty)$ we put

$$\mathcal{G}_q(\cdot; \alpha_1, \dots, \alpha_N) : [0, \infty) \rightarrow [0, 1], \quad u \mapsto \mathbb{P}\left(\left\{\max_{\ell \in \{1, \dots, N\}} \left(\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|\right)^q > u\right\}\right),$$

and

$$\mathcal{M}_q(\alpha_1, \dots, \alpha_N) := \mathbb{E}\left[\max_{\ell \in \{1, \dots, N\}} \left(\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)|\right)^q\right] \in [0, \infty).$$

Moreover, we abbreviate

$$\mathcal{G}_q(\cdot; N) := \mathcal{G}_q(\cdot; \underbrace{1, \dots, 1}_{N \text{ times}}), \quad \mathcal{M}_q(N) := \mathcal{M}_q(\underbrace{1, \dots, 1}_{N \text{ times}}).$$

Lemma A.1. *Let $N \in \mathbb{N}$ and let $\alpha_1, \dots, \alpha_N \in [0, \infty)$. Then for all $q \in [1, \infty)$ it holds that*

$$\mathbb{E} \left[\max_{\ell \in \{1, \dots, N\}} \left(\alpha_\ell \cdot \sup_{t \in [0, 1]} |B_\ell(t)| \right)^q \right] \geq \left(\frac{1}{N} \cdot \sum_{\ell=1}^N \alpha_\ell^2 \right)^{q/2} \cdot \mathcal{M}_q(N).$$

Proof. This lemma is an easy generalization of Lemma 1 in Müller-Gronbach (2002a) regarding non-negative instead of strictly positive scalars. The extension to the general case using the special case is straight-forward and therefore omitted. \square

Lemma A.2. *Let $\alpha \in [0, \infty)$ and for each $N \in \mathbb{N}$ let $\alpha_{1,N}, \dots, \alpha_{N,N} \in [0, \infty)$ such that*

$$\lim_{N \rightarrow \infty} \max_{i \in \{1, \dots, N\}} \alpha_{i,N} = \alpha$$

and

$$\forall \varepsilon \in (0, \infty) : \liminf_{N \rightarrow \infty} N^{-1} \cdot \#\{i \in \{1, \dots, N\} \mid \alpha_{i,N} \geq \alpha - \varepsilon\} > 0.$$

Then the following hold true:

(i) *For each $q \in [1, \infty)$ there exists a constant $C_q \in (0, \infty)$ such that*

$$\sup_{N \in \mathbb{N}} \int_0^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \alpha_{1,N}, \dots, \alpha_{N,N}) du \leq C_q. \quad (\text{A.1})$$

(ii) *For all $q \in [1, \infty)$ it holds that*

$$\lim_{N \rightarrow \infty} \int_{(\alpha^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \alpha_{1,N}, \dots, \alpha_{N,N}) du = 0. \quad (\text{A.2})$$

(iii) *For all $q \in [1, \infty)$ it holds that*

$$\lim_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\alpha_{1,N}, \dots, \alpha_{N,N}))^{1/q} = 2^{-1/2} \cdot \alpha. \quad (\text{A.3})$$

Proof. This lemma is a generalization of Lemma 2 and Corollary 2 in Müller-Gronbach (2002a) regarding non-negative instead of strictly positive scalars.

Let $\delta \in (0, \infty)$ be fixed for the moment, and put $\tilde{\alpha} := \alpha + \delta > 0$ as well as $\tilde{\alpha}_{i,N} := \alpha_{i,N} + \delta > 0$ for all $N \in \mathbb{N}$ and $i \in \{1, \dots, N\}$. In the given setting, we then have

$$\lim_{N \rightarrow \infty} \max_{i \in \{1, \dots, N\}} \tilde{\alpha}_{i,N} = \tilde{\alpha}$$

and

$$\forall \varepsilon \in (0, \infty) : \liminf_{N \rightarrow \infty} N^{-1} \cdot \#\{i \in \{1, \dots, N\} \mid \tilde{\alpha}_{i,N} \geq \tilde{\alpha} - \varepsilon\} > 0.$$

Now let $q \in [1, \infty)$. Then Lemma 2 in Müller-Gronbach (2002a) yields the existence of a constant $C_q \in (0, \infty)$ such that

$$\sup_{N \in \mathbb{N}} \int_0^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du \leq C_q \quad (\text{A.4})$$

and

$$\lim_{N \rightarrow \infty} \int_{(\tilde{\alpha}^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du = 0, \quad (\text{A.5})$$

and Corollary 2 in the same reference gives

$$\lim_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}))^{1/q} = 2^{-1/2} \cdot \tilde{\alpha}. \quad (\text{A.6})$$

Observe that the monotonicity property

$$\mathcal{G}_q(u \cdot (\log(N))^{q/2}; \alpha_{1,N}, \dots, \alpha_{N,N}) du \leq \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du$$

holds true for each $N \in \mathbb{N}$. From this, we can immediately conclude (A.1) from (A.4) as well as

$$\begin{aligned} & \int_{(\alpha^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \alpha_{1,N}, \dots, \alpha_{N,N}) du \\ & \leq \int_{(\alpha^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du \\ & = \int_{(\tilde{\alpha}^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du \\ & \quad + \int_{(\alpha^2/2)^{q/2}}^{(\tilde{\alpha}^2/2)^{q/2}} \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du \\ & \leq \int_{(\tilde{\alpha}^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}) du + (\tilde{\alpha}^2/2)^{q/2} - (\alpha^2/2)^{q/2} \end{aligned}$$

for all $N \in \mathbb{N}$. Utilizing (A.5) hence gives

$$0 \leq \limsup_{N \rightarrow \infty} \int_{(\alpha^2/2)^{q/2}}^\infty \mathcal{G}_q(u \cdot (\log(N))^{q/2}; \alpha_{1,N}, \dots, \alpha_{N,N}) du \leq \frac{\tilde{\alpha}^q - \alpha^q}{2^{q/2}},$$

which proves (A.2) by letting δ tend to 0. At last, we aim to show (A.3). It is easy to see that the monotonicity property

$$(\mathcal{M}_q(\alpha_{1,N}, \dots, \alpha_{N,N}))^{1/q} \leq (\mathcal{M}_q(\tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}))^{1/q}$$

holds true for each $N \in \mathbb{N}$, which along with (A.6) immediately yields

$$\begin{aligned} & \limsup_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\alpha_{1,N}, \dots, \alpha_{N,N}))^{1/q} \\ & \leq \limsup_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}))^{1/q} \\ & = 2^{-1/2} \cdot \tilde{\alpha}. \end{aligned} \quad (\text{A.7})$$

Next, the inverse triangle inequality gives that the monotonicity property

$$(\mathcal{M}_q(\alpha_{1,N}, \dots, \alpha_{N,N}))^{1/q} \geq (\mathcal{M}_q(\tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}))^{1/q} - (\mathcal{M}_q(\delta, \dots, \delta))^{1/q}$$

holds true for each $N \in \mathbb{N}$, which along with two times (A.6) immediately yields

$$\begin{aligned} & \liminf_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\alpha_{1,N}, \dots, \alpha_{N,N}))^{1/q} \\ & \geq \liminf_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\tilde{\alpha}_{1,N}, \dots, \tilde{\alpha}_{N,N}))^{1/q} \\ & \quad - \limsup_{N \rightarrow \infty} (\log(N))^{-1/2} \cdot (\mathcal{M}_q(\delta, \dots, \delta))^{1/q} \\ & = 2^{-1/2} \cdot \tilde{\alpha} - 2^{-1/2} \cdot \delta \\ & = 2^{-1/2} \cdot \alpha. \end{aligned} \tag{A.8}$$

We finally obtain (A.3) from combining (A.7) and (A.8) and letting δ tend to 0. \square

Lemma A.3. *Let $q \in [1, \infty)$, let $N \in \mathbb{N}$, let $0 =: t_0 < t_1 < \dots < t_N := T$, let $\beta_{i,j,\ell} \in \mathbb{R}$ for all $(i, j, \ell) \in \{1, \dots, d\} \times \{1, \dots, m\} \times \{0, \dots, N-1\}$, and for each $(\ell, j) \in \{0, \dots, N-1\} \times \{1, \dots, m\}$ let $B_{j,\ell}$ be a Brownian bridge on $[t_\ell, t_{\ell+1}]$ such that $B_{0,1}, \dots, B_{N-1,m}$ are independent. Then for all $\beta \in [0, \infty)$ with*

$$\beta \geq \max_{i \in \{1, \dots, d\}} \max_{\ell \in \{0, \dots, N-1\}} \left((t_{\ell+1} - t_\ell) \cdot \sum_{j=1}^m \beta_{i,j,\ell}^2 \right)^{1/2},$$

and for all $z \in [0, \infty)$ it holds that

$$\mathbb{E} \left[\max_{i \in \{1, \dots, d\}} \max_{\ell \in \{0, \dots, N-1\}} \sup_{t \in [t_\ell, t_{\ell+1}]} \left| \sum_{j=1}^m \beta_{i,j,\ell} \cdot B_{j,\ell}(t) \right|^q \right] \leq z + d \cdot \int_z^\infty \mathcal{G}_q(u; \underbrace{\beta, \dots, \beta}_{N \text{ times}}) du.$$

Proof. See Lemma 3 in Müller-Gronbach (2002a). \square

In a second step, we provide results on integrated moments of Brownian bridges, which are used in the proofs of Chapter 4. To this end, recall the definitions (4.2) and (4.3) of \mathfrak{m}_q and \mathfrak{g}_q , respectively.

Lemma A.4. (i) *Let $T \in (0, \infty)$, let $a, b \in [0, T]$ with $a < b$, and let B be a Brownian bridge on $[a, b]$. Then for all $q \in [1, \infty)$ it holds that*

$$\int_a^b \mathbb{E} \left[|B(t)|^q \right] dt = \mathfrak{m}_q^q \cdot \mathfrak{g}_q^q \cdot (b-a)^{(q+2)/2}.$$

(ii) *Let $T \in (0, \infty)$, let $a, b \in [0, T]$ with $a < b$, let $N \in \mathbb{N}$, let $a =: \tau_0 < \tau_1 < \dots < \tau_N =: b$, and for each $\ell \in \{0, \dots, N-1\}$ let B_ℓ be a Brownian bridge on $[\tau_\ell, \tau_{\ell+1}]$. Then for all $q \in [1, \infty)$ it holds that*

$$\sum_{\ell=0}^{N-1} \int_{\tau_\ell}^{\tau_{\ell+1}} \mathbb{E} \left[|B_\ell(t)|^q \right] dt \geq \mathfrak{m}_q^q \cdot \mathfrak{g}_q^q \cdot (b-a)^{(q+2)/2} \cdot N^{-q/2}.$$

Proof. This lemma is an easy generalization of the equations (20) and (21) in Chapter III of Müller-Gronbach (2002b). \square

A.2 The Solution Process

In the following, we consider the supremum of the solution of the SDE (1.4) and prove finiteness of specific moments of this random variable under quite weak assumptions.

We thereby use c to denote unspecified positive constants that may vary at every occurrence and that may only depend on T , d , m , and the parameters and constants from the assumptions used in the subsequent proposition.

Proposition A.5. *Let the Assumptions (I_a) , (locL) , (K_a) , and (pG_r^σ) be satisfied for some $a \in [2, \infty)$ and $r \in [1, \infty)$ with $a \geq 2r$. Then it holds that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X(t)|^{a-2r+2} \right] < \infty.$$

Proof. Put $\bar{a} := a - 2r + 2 \in [2, a]$. For each $n \in \mathbb{N}$, observe that the mapping

$$\tau_n : \Omega \rightarrow [0, T], \quad \omega \mapsto T \wedge \inf \left\{ t \in [0, T] \mid n \leq |X(t, \omega)| \right\},$$

is a stopping time that satisfies

$$\sup_{t \in [0, T]} |X(t \wedge \tau_n)| \leq \max \{n, |\xi|\} \tag{A.9}$$

almost surely.

Fix $n \in \mathbb{N}$ for the moment and note that the Assumption (K_a) implies that $(K_{\bar{a}})$ is also satisfied. Employing Itô's formula, the Cauchy–Schwarz inequality, and $(K_{\bar{a}})$ yields that almost surely we have

$$\begin{aligned} & \left(1 + |X(t)|^2\right)^{\bar{a}/2} \\ &= \left(1 + |\xi|^2\right)^{\bar{a}/2} + \frac{\bar{a}}{2} \cdot \int_0^t 2 \cdot \left(1 + |X(s)|^2\right)^{(\bar{a}-2)/2} \cdot X(s)^\top \cdot \mu(s, X(s)) \\ & \quad + (\bar{a} - 2) \cdot \left(1 + |X(s)|^2\right)^{(\bar{a}-4)/2} \cdot \left|X(s)^\top \cdot \sigma(s, X(s))\right|^2 \\ & \quad + \left(1 + |X(s)|^2\right)^{(\bar{a}-2)/2} \cdot \left|\sigma(s, X(s))\right|^2 ds \\ & \quad + \bar{a} \cdot \int_0^t \left(1 + |X(s)|^2\right)^{(\bar{a}-2)/2} \cdot X(s)^\top \cdot \sigma(s, X(s)) dW(s) \\ &\leq \left(1 + |\xi|^2\right)^{\bar{a}/2} + c \cdot \int_0^t \left(1 + |X(s)|^2\right)^{\bar{a}/2} ds \\ & \quad + \bar{a} \cdot \int_0^t \left(1 + |X(s)|^2\right)^{(\bar{a}-2)/2} \cdot X(s)^\top \cdot \sigma(s, X(s)) dW(s) \end{aligned}$$

for all $t \in [0, T]$. Assumption (I_a) , Fubini's theorem, and the moments estimate (3.2)

consequently give

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] \\ & \leq c + \bar{a} \cdot \mathbb{E} \left[\sup_{t \in [0, T]} \int_0^t \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{(\bar{a}-2)/2} \right. \\ & \quad \left. \cdot X(s \wedge \tau_n)^\top \cdot \sigma(s \wedge \tau_n, X(s \wedge \tau_n)) \, dW(s) \right]. \end{aligned} \quad (\text{A.10})$$

Next, observe that the Burkholder–Davis–Gundy inequality and the Cauchy–Schwarz inequality imply

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \int_0^t \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{(\bar{a}-2)/2} \right. \\ & \quad \left. \cdot X(s \wedge \tau_n)^\top \cdot \sigma(s \wedge \tau_n, X(s \wedge \tau_n)) \, dW(s) \right] \\ & \leq \sqrt{32} \cdot \mathbb{E} \left[\left(\int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{\bar{a}-1} \right. \right. \\ & \quad \left. \left. \cdot |\sigma(s \wedge \tau_n, X(s \wedge \tau_n))|^2 \, ds \right)^{1/2} \right]. \end{aligned} \quad (\text{A.11})$$

Moreover, Assumption (pG $^\sigma$) and the inequality $\sqrt{x \cdot y} \leq x/(2\rho) + y\rho/2$ for all $x, y \in [0, \infty)$ and $\rho \in (0, \infty)$ yield

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{\bar{a}-1} \cdot |\sigma(s \wedge \tau_n, X(s \wedge \tau_n))|^2 \, ds \right)^{1/2} \right] \\ & \leq c \cdot \mathbb{E} \left[\left(\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right. \right. \\ & \quad \left. \left. \cdot \int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{\bar{a}/2-1} \cdot \left(1 + |X(s \wedge \tau_n)|^{2r} \right) \, ds \right)^{1/2} \right] \\ & \leq c \cdot \mathbb{E} \left[\left(\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right. \right. \\ & \quad \left. \left. \cdot \int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{a/2} \, ds \right)^{1/2} \right] \\ & \leq \frac{1}{2 \cdot \sqrt{32} \cdot \bar{a}} \cdot \mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] \\ & \quad + c^2 \cdot \sqrt{32} \cdot \bar{a}/2 \cdot \mathbb{E} \left[\int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{a/2} \, ds \right]. \end{aligned} \quad (\text{A.12})$$

Note that

$$\mathbb{E} \left[\int_0^T \mathbf{1}_{\{s \leq \tau_n\}} \cdot \left(1 + |X(s \wedge \tau_n)|^2 \right)^{a/2} \, ds \right] \leq \mathbb{E} \left[\int_0^T \left(1 + |X(s)|^2 \right)^{a/2} \, ds \right] \leq c \quad (\text{A.13})$$

holds, again, due to Fubini's theorem and (3.2). Combining the inequalities (A.10), (A.11), (A.12), and (A.13) shows

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] \leq \frac{1}{2} \cdot \mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] + c.$$

To subtract the first summand of the right hand side from the left hand side, we need to ensure that these quantities are actually not infinite. For this purpose, we employ the inequality (A.9) and Assumption (I_a) to conclude that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] \leq \mathbb{E} \left[\left(1 + \max \{n, |\xi|\}^2 \right)^{\bar{a}/2} \right] < \infty.$$

Hence, we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X(t \wedge \tau_n)|^{\bar{a}} \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} \left(1 + |X(t \wedge \tau_n)|^2 \right)^{\bar{a}/2} \right] \leq c. \quad (\text{A.14})$$

Using Fatou's lemma, we derive from (A.14) that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X(t)|^{\bar{a}} \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |X(t \wedge \tau_n)|^{\bar{a}} \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0, T]} |X(t \wedge \tau_n)|^{\bar{a}} \right] \leq c,$$

which finishes the proof of this proposition. \square

A.3 The Continuous-time Tamed Euler Schemes

As before, we use c to denote unspecified positive constants that may vary at every occurrence and that may only depend on T , d , m , and the parameters and constants from the assumptions used in the respective propositions.

First, we show finiteness of specific moments of the continuous-time tamed Euler schemes.

Proposition A.6. *Let the Assumptions (I_a), (locL), (K_a), and (pG_{r+1}^μ) be satisfied for some $a \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq r + 2$. Then it holds that*

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{X}_{N,r}(t)|^{a-r} \right] < \infty. \quad (\text{A.15})$$

Proof. First of all, note that in the given setting the growth condition (pG_{(r+2)/2}^σ) also holds true. Our main idea of proof is to show (A.15) by means of Gronwall's lemma.

As a first step, observe that for each $N \in \mathbb{N}$ the continuous-time tamed Euler scheme $\tilde{X}_{N,r}$ satisfies

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{X}_{N,r}(t)|^a \right] < \infty \quad (\text{A.16})$$

due to the taming of the drift and the diffusion coefficient in its construction, see Remark 3 in Sabanis (2016). Note, however, that one can not guarantee at the moment that this bound holds uniformly in N .

Next, one establishes the moment bound

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[|\tilde{X}_{N,r}(t)|^a \right] \leq c. \quad (\text{A.17})$$

This is shown in a completely analogous manner to Lemma 2 in Sabanis (2016), and we therefore omit a proof.

We now turn to estimates which allow to apply Gronwall's lemma in a final step. Put $\bar{a} := a - r \in [2, a]$ as well as $\underline{t}_N := \lfloor tN/T \rfloor \cdot T/N$ for $t \in [0, T]$ and $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. First, applying Itô's formula to the Itô process (3.13) yields that almost surely we have

$$\begin{aligned} & \left(1 + |\tilde{X}_{N,r}(t)|^2\right)^{\bar{a}/2} \\ &= \left(1 + |\xi|^2\right)^{\bar{a}/2} \\ & \quad + \frac{\bar{a}}{2} \cdot \int_0^t 2 \cdot \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\mu(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \\ & \quad + (\bar{a} - 2) \cdot \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-4)/2} \cdot \left| \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right|^2 \\ & \quad + \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \cdot \left| \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right|^2 ds \\ & \quad + \bar{a} \cdot \int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} dW(s) \end{aligned}$$

for all $t \in [0, T]$. Note that the Assumption (K_a) implies that $(K_{\bar{a}})$ is also satisfied. By employing the Cauchy-Schwarz inequality and $(K_{\bar{a}})$, we obtain that almost surely it holds

$$\begin{aligned} & \left(1 + |\tilde{X}_{N,r}(t)|^2\right)^{\bar{a}/2} \\ & \leq \left(1 + |\xi|^2\right)^{\bar{a}/2} + c \cdot \int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \cdot \left(1 + |\tilde{X}_{N,r}(\underline{s}_N)|^2\right) ds \\ & \quad + \bar{a} \cdot \int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \\ & \quad \quad \cdot \left(\tilde{X}_{N,r}(s) - \tilde{X}_{N,r}(\underline{s}_N)\right)^\top \cdot \frac{\mu(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} ds \\ & \quad + \bar{a} \cdot \int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2\right)^{(\bar{a}-2)/2} \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} dW(s) \end{aligned}$$

for all $t \in [0, T]$. Now fix $t \in [0, T]$. It follows from the considerations above that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{u \in [0, t]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] \\
& \leq \mathbb{E} \left[(1 + |\xi|^2)^{\bar{a}/2} \right] + c \cdot \mathbb{E} \left[\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \cdot \left(1 + |\tilde{X}_{N,r}(\underline{s}_N)|^2 \right) ds \right. \\
& \quad + \bar{a} \cdot \mathbb{E} \left[\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \right. \\
& \quad \quad \left. \left. \cdot \left| (\tilde{X}_{N,r}(s) - \tilde{X}_{N,r}(\underline{s}_N))^\top \cdot \frac{\mu(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right| ds \right] \right. \\
& \quad \left. + \bar{a} \cdot \mathbb{E} \left[\sup_{u \in [0, t]} \int_0^u \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \right. \right. \\
& \quad \quad \left. \left. \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} dW(s) \right] \right]. \tag{A.18}
\end{aligned}$$

By Young's inequality, we obtain

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \cdot \left(1 + |\tilde{X}_{N,r}(\underline{s}_N)|^2 \right) ds \right] \\
& \leq \int_0^t \mathbb{E} \left[\sup_{u \in [0, s]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] ds. \tag{A.19}
\end{aligned}$$

Moreover, the Cauchy–Schwarz inequality, the triangle inequality, Assumption (pG_{r+1}^μ) , Young's inequality, and (A.17) prove

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \right. \\
& \quad \left. \cdot \left| (\tilde{X}_{N,r}(s) - \tilde{X}_{N,r}(\underline{s}_N))^\top \cdot \frac{\mu(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right| ds \right] \\
& \leq c \cdot \sup_{u \in [0, T]} \mathbb{E} \left[\left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{a/2} \right] \\
& \leq c. \tag{A.20}
\end{aligned}$$

Next, one utilizes the Burkholder–Davis–Gundy inequality and the Cauchy–Schwarz inequality to show

$$\begin{aligned}
& \mathbb{E} \left[\sup_{u \in [0, t]} \int_0^u \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \right. \\
& \quad \left. \cdot \tilde{X}_{N,r}(s)^\top \cdot \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} dW(s) \right] \\
& \leq \sqrt{32} \cdot \mathbb{E} \left[\left(\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\bar{a}-1} \cdot \left| \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right|^2 ds \right)^{1/2} \right]. \tag{A.21}
\end{aligned}$$

The growth condition ($\text{pG}_{(r+2)/2}^\sigma$), the inequality $\sqrt{x \cdot y} \leq x/(2\rho) + y\rho/2$ for all $x, y \in [0, \infty)$ and $\rho \in (0, \infty)$, Young's inequality, and (A.17) furthermore give

$$\begin{aligned}
& \mathbb{E} \left[\left(\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\bar{a}-1} \cdot \left| \frac{\sigma(\underline{s}_N, \tilde{X}_{N,r}(\underline{s}_N))}{1 + (T/N)^{1/2} \cdot |\tilde{X}_{N,r}(\underline{s}_N)|^r} \right|^2 ds \right)^{1/2} \right] \\
& \leq c \cdot \mathbb{E} \left[\left(\int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{\bar{a}-1} \cdot \left(1 + |\tilde{X}_{N,r}(\underline{s}_N)|^2 \right)^{(r+2)/2} ds \right)^{1/2} \right] \\
& \leq c \cdot \mathbb{E} \left[\left(\sup_{u \in [0,t]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right. \right. \\
& \quad \left. \left. \cdot \int_0^t \left(1 + |\tilde{X}_{N,r}(s)|^2 \right)^{(\bar{a}-2)/2} \cdot \left(1 + |\tilde{X}_{N,r}(\underline{s}_N)|^2 \right)^{(r+2)/2} ds \right)^{1/2} \right] \quad (\text{A.22}) \\
& \leq \frac{1}{2 \cdot \sqrt{32} \cdot \bar{a}} \cdot \mathbb{E} \left[\sup_{u \in [0,t]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] \\
& \quad + c \cdot \sup_{u \in [0,T]} \mathbb{E} \left[\left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{a/2} \right] \\
& \leq \frac{1}{2 \cdot \sqrt{32} \cdot \bar{a}} \cdot \mathbb{E} \left[\sup_{u \in [0,t]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] + c.
\end{aligned}$$

Combining (A.16), Assumption (I_a), (A.18), (A.19), (A.20), (A.21), and (A.22) yields

$$\mathbb{E} \left[\sup_{u \in [0,t]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] \leq c + c \cdot \int_0^t \mathbb{E} \left[\sup_{u \in [0,s]} \left(1 + |\tilde{X}_{N,r}(u)|^2 \right)^{\bar{a}/2} \right] ds.$$

Applying Gronwall's lemma finally finishes the proof of this proposition. \square

The next proposition states that the continuous-time tamed Euler schemes converge strongly with order (at least) 1/2.

Proposition A.7. *Let the Assumptions (I_a), (H), (K_a), (M_b), and (pL_r^μ) be satisfied for some $a, b \in [2, \infty)$ and $r \in [0, \infty)$ with $a \geq 4r + 2$. Then for all $q \in [1, \min\{b, a/(2r + 1)\})$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\left\| \|X - \tilde{X}_{N,r}\|_{C([0,T])} \right\|_{L_q(\Omega)} \leq C \cdot N^{-1/2}.$$

Proof. Essentially, the proof of Theorem 3 in Sabanis (2016) carries over here and is therefore omitted. \square

A.4 The Continuous-time Tamed Milstein Schemes

As before, we use c to denote unspecified positive constants that may vary at every occurrence and that may only depend on T , d , m , and the parameters and constants from the assumptions used in the respective propositions.

First, we show finiteness of specific moments of the continuous-time tamed Milstein schemes.

Proposition A.8. *Let the Assumptions (I_a) , (K_a) , (M_b) , $(pL_{r-1}^{\nabla\mu})$, $(pL_{(r-2)/2}^{\nabla\sigma})$, $(B^{\nabla_t\mu})$, and $(B^{\nabla_t\sigma})$ be satisfied for some $a, b \in [2, \infty)$ and $r \in [1, \infty)$. Then it holds that*

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} \left[|\tilde{Y}_{N,r}(t)|^a \right] < \infty.$$

Proof. Observe that, in this setting, the Assumptions (locL) , (LLG^μ) , $(pG_r^{\nabla\mu})$, (pL_r^μ) , (pG_{r+1}^μ) , (LLG^σ) , $(pG_{r/2}^{\nabla\sigma})$, $(pL_{r/2}^\sigma)$, and $(pG_{(r+2)/2}^\sigma)$ introduced in the Sections 3.1 and 4.1 are also satisfied; see Remark 4.1. Moreover, there exists $C \in (0, \infty)$ such that for all $i \in \{1, \dots, d\}$, for all $j_1, j_2 \in \{1, \dots, m\}$, for all $t \in [0, T]$, and for all $x \in \mathbb{R}^d$ it holds that

$$|(\nabla\sigma_{i,j_2} \cdot \sigma^{(j_1)})(t, x)| \leq C \cdot (1 + |x|)^{r+1}.$$

Lemma 3.3 in Kumar & Sabanis (2019) proves Proposition A.8 in the case that the SDE (1.4) is autonomous and $T = 1$. The key ingredients for the proof of the referenced lemma are exactly the conditions given above. With some minor, obvious modifications, one proves the desired result in the non-autonomous case in an analogous manner to the autonomous case with general $T \in (0, \infty)$. Therefore, the proof is omitted. \square

The next proposition states that the continuous-time tamed Milstein schemes converge strongly with order (at least) 1.

Proposition A.9. *Let the Assumptions (I_a) , (K_a) , (M_b) , $(pL_{r-1}^{\nabla\mu})$, $(pL_{(r-2)/2}^{\nabla\sigma})$, $(B^{\nabla_t\mu})$, and $(B^{\nabla_t\sigma})$ be satisfied for some $a \in [2, \infty)$, $b \in (2, \infty)$, and $r \in [1, \infty)$ with $a \geq 12r + 4$. Then for all $q \in [1, b) \cap [1, a/(3r + 1)]$ there exists $C \in (0, \infty)$ such that for all $N \in \mathbb{N}$ it holds that*

$$\sup_{t \in [0, T]} \left\| X(t) - \tilde{Y}_{N,r}(t) \right\|_{L_q(\Omega)} \leq C \cdot N^{-1}.$$

Proof. With some minor, obvious modifications towards the non-autonomous case, the proof of Theorem 2.1 in Kumar & Sabanis (2019) carries over here and is therefore omitted. \square

B

Auxiliary Results

In this chapter, we provide two results from calculus, which are used at several places in the proofs given in the Sections 3.6 and 4.7.

First, we present a lemma containing a simple subsequence argument that is employed in the proofs of the asymptotic lower bounds in Chapter 3.

Lemma B.1. *Let $(a_N)_{N \in \mathbb{N}}$ be a sequence of real numbers and let $C \in \mathbb{R}$. Then the following are equivalent:*

- (i) *It holds that $\liminf_{N \rightarrow \infty} a_N \geq C$.*
- (ii) *For every subsequence $(a_{N_\kappa})_{\kappa \in \mathbb{N}}$ of $(a_N)_{N \in \mathbb{N}}$ there exists a subsequence $(a_{N_{\kappa_n}})_{n \in \mathbb{N}}$ of $(a_{N_\kappa})_{\kappa \in \mathbb{N}}$ such that $\liminf_{n \rightarrow \infty} a_{N_{\kappa_n}} \geq C$.*

Proof. First, assume that $\liminf_{N \rightarrow \infty} a_N \geq C$. Let $(a_{N_\kappa})_{\kappa \in \mathbb{N}}$ be a subsequence of $(a_N)_{N \in \mathbb{N}}$. For all $\kappa \in \mathbb{N}$, we have

$$\inf \{a_m \mid m \geq N_\kappa\} \geq \inf \{a_m \mid m \geq \kappa\},$$

which immediately yields

$$\liminf_{\kappa \rightarrow \infty} a_{N_\kappa} \geq \liminf_{\kappa \rightarrow \infty} a_\kappa \geq C.$$

Now assume that $\alpha := \liminf_{N \rightarrow \infty} a_N < C$. Then for each $\kappa \in \mathbb{N}$ it holds that

$$\inf \{a_m \mid m \geq \kappa\} \leq \alpha < (\alpha + C)/2.$$

Thus, for every $\kappa \in \mathbb{N}$ there exists $N_\kappa \in \mathbb{N}$, $N_\kappa \geq \kappa$, such that $a_{N_\kappa} < (\alpha + C)/2$. Consider the resulting subsequence $(a_{N_\kappa})_{\kappa \in \mathbb{N}}$ of $(a_N)_{N \in \mathbb{N}}$. Let $(a_{N_{\kappa_n}})_{n \in \mathbb{N}}$ be any subsequence of $(a_{N_\kappa})_{\kappa \in \mathbb{N}}$. For all $n \in \mathbb{N}$, we have

$$\inf \{a_{N_{\kappa_m}} \mid m \geq n\} \leq a_{N_{\kappa_n}} < (\alpha + C)/2$$

and consequently obtain

$$\liminf_{n \rightarrow \infty} a_{N_{\kappa_n}} \leq (\alpha + C)/2 < C.$$

□

Next, we state a monotonicity property for power means, which is used in the proof of Lemma 4.21.

Lemma B.2. *Let $N \in \mathbb{N}$ and let $x_1, \dots, x_N \in \mathbb{R}$. Then for all $\alpha, \beta \in (0, \infty)$ with $\alpha \leq \beta$ it holds that*

$$\left(\frac{1}{N} \cdot \sum_{\ell=1}^N |x_\ell|^\alpha \right)^{1/\alpha} \leq \left(\frac{1}{N} \cdot \sum_{\ell=1}^N |x_\ell|^\beta \right)^{1/\beta}.$$

Proof. Without loss of generality, we may assume that the numbers x_1, \dots, x_N are pairwise distinct. Consider a random variable X that is uniformly distributed on $\{x_1, \dots, x_N\}$. Then we have

$$\left(\frac{1}{N} \cdot \sum_{\ell=1}^N |x_\ell|^\alpha \right)^{1/\alpha} = \left(\mathbb{E}[|X|^\alpha] \right)^{1/\alpha} \leq \left(\mathbb{E}[|X|^\beta] \right)^{1/\beta} \leq \left(\frac{1}{N} \cdot \sum_{\ell=1}^N |x_\ell|^\beta \right)^{1/\beta}.$$

□

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Bibliography

- Beyn, W.-J., Isaak, E. & Kruse, R. (2016), ‘Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes’, *J. Sci. Comput.* **67**(3), 955–987.
- Beyn, W.-J., Isaak, E. & Kruse, R. (2017), ‘Stochastic C-stability and B-consistency of explicit and implicit Milstein-type schemes’, *J. Sci. Comput.* **70**(3), 1042–1077.
- Cambanis, S. & Hu, Y. (1996), ‘Exact convergence rate of the Euler–Maruyama scheme, with application to sampling design’, *Stochastics Stochastics Rep.* **59**(3–4), 211–240.
- Fang, W. & Giles, M. B. (2018), Adaptive Euler–Maruyama method for SDEs with non-globally Lipschitz drift, *in* A. B. Owen & P. W. Glynn, eds, ‘Monte Carlo and Quasi-Monte Carlo Methods’, Springer International Publishing, Cham, pp. 217–234.
- Faure, O. (1992), Simulation du mouvement brownien et des diffusions, Thèse de doctorat, Ecole Nationale des Ponts et Chaussées.
- Gan, S. & Wang, X. (2013), ‘The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients’, *J. Differ. Equ. Appl.* **19**(3), 466–490.
- Guo, Q., Liu, W., Mao, X. & Yue, R. (2018), ‘The truncated Milstein method for stochastic differential equations with commutative noise’, *J. Comput. Appl. Math.* **338**, 298–310.
- Hatzesberger, S. (2014), On the divergence of the Euler and the Milstein scheme for stochastic differential equations with superlinearly growing coefficients, Master’s thesis, Universität Passau.
- Hefter, M. & Herzwurm, A. (2017), ‘Optimal strong approximation of the one-dimensional squared Bessel process’, *Commun. Math. Sci.* **15**(8), 2121–2141.
- Hefter, M., Herzwurm, A. & Müller-Gronbach, T. (2019), ‘Lower error bounds for strong approximation of scalar SDEs with non-Lipschitzian coefficients’, *Ann. Appl. Probab.* **29**(1), 178–216.
- Higham, D., Mao, X. & Stuart, A. (2002), ‘Strong convergence of Euler type methods for nonlinear stochastic differential equations’, *SIAM J. Numer. Anal.* **40**, 1041–1063.

- Hofmann, N., Müller-Gronbach, T. & Ritter, K. (2000a), ‘Optimal approximation of stochastic differential equations by adaptive step-size control’, *Math. Comp.* **69**(231), 1017–1034.
- Hofmann, N., Müller-Gronbach, T. & Ritter, K. (2000b), ‘Step size control for the uniform approximation of systems of stochastic differential equations with additive noise’, *Ann. Appl. Probab.* **10**(2), 616–633.
- Hofmann, N., Müller-Gronbach, T. & Ritter, K. (2001), ‘The optimal discretization of stochastic differential equations’, *J. Complexity* **17**(1), 117–153.
- Hutzenthaler, M., Jentzen, A. & Kloeden, P. E. (2011), ‘Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients’, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **467**(2130), 1563–1576.
- Hutzenthaler, M., Jentzen, A. & Kloeden, P. E. (2012), ‘Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients’, *Ann. Appl. Probab.* **22**(4), 1611–1641.
- Hutzenthaler, M., Jentzen, A. & Kloeden, P. E. (2013), ‘Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations’, *Ann. Appl. Probab.* **23**(5), 1913–1966.
- Johnson, B. M. & Killeen, T. (1983), ‘An explicit formula for the c.d.f. of the L_1 norm of the Brownian bridge’, *Ann. Probab.* **11**(3), 807–808.
- Karatzas, I. & Shreve, S. E. (1998), *Brownian Motion and Stochastic Calculus*, 2nd edn, Springer New York.
- Kelly, C. & Lord, G. J. (2018a), ‘Adaptive Euler methods for stochastic systems with non-globally Lipschitz coefficients’, *arXiv:1805.11137*.
- Kelly, C. & Lord, G. J. (2018b), ‘Adaptive time-stepping strategies for nonlinear stochastic systems’, *IMA J. Numer. Anal.* **38**(3), 1523–1549.
- Kloeden, P. E. & Platen, E. (1992), *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin.
- Kumar, C. & Sabanis, S. (2019), ‘On Milstein approximations with varying coefficients: the case of super-linear diffusion coefficients’, *BIT Numer. Math.* **59**, 929–968.
- Lee, D. (1986), ‘Approximation of linear operators on a wiener space’, *Rocky Mountain J. Math.* **16**(4), 641–660.
- Leobacher, G. & Szölgényi, M. (2018), ‘Convergence of the Euler–Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient’, *Numer. Math.* **138**(1), 219–239.

- Mao, X. (2007), *Stochastic Differential Equations and Applications*, 2nd edn, Horwood Publishing Limited, Chichester.
- Mao, X. (2015), ‘The truncated Euler-Maruyama method for stochastic differential equations’, *J. Comput. Appl. Math.* **290**(C), 370–384.
- Maruyama, G. (1955), ‘Continuous Markov processes and stochastic equations’, *Rend. Circ. Mat. Palermo. (2)* **4**, 48–90.
- Milstein, G. N. (1995), *Numerical Integration of Stochastic Differential Equations*, Springer Netherlands.
- Müller-Gronbach, T. (2002a), ‘The optimal uniform approximation of systems of stochastic differential equations’, *Ann. Appl. Probab.* **12**(2), 664–690.
- Müller-Gronbach, T. (2002b), Strong approximation of systems of stochastic differential equations, Habilitation thesis, TU Darmstadt.
- Müller-Gronbach, T. & Yaroslavtseva, L. (2020), ‘On the performance of the Euler—Maruyama scheme for SDEs with discontinuous drift coefficient’, *Ann. Inst. H. Poincaré Probab. Statist.* **56**(2), 1162–1178.
- Ngo, H.-L. & Taguchi, D. (2017), ‘Strong convergence for the Euler—Maruyama approximation of stochastic differential equations with discontinuous coefficients’, *Statist. Probab. Lett.* **125**(C), 55–63.
- Sabanis, S. (2016), ‘Euler approximations with varying coefficients: The case of super-linearly growing diffusion coefficients’, *Ann. Appl. Probab.* **26**(4), 2083–2105.
- Sabanis, S. & Zhang, Y. (2018), ‘On explicit order 1.5 approximations with varying coefficients: the case of super-linear diffusion coefficients’, *J. Complexity* **50**, 84–115.
- Shepp, L. A. (1982), ‘On the integral of the absolute value of the pinned Wiener process’, *Ann. Probab.* **10**(1), 234–239.
- Tolmatz, L. (2002), ‘On the distribution of the square integral of the Brownian bridge’, *Ann. Probab.* **30**(1), 253–269.
- Tretyakov, M. V. & Zhang, Z. (2013), ‘A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications’, *SIAM J. Numer. Anal.* **51**(6), 3135–3162.
- Yaroslavtseva, L. (2017), ‘On non-polynomial lower error bounds for adaptive strong approximation of SDEs’, *J. Complexity* **42**, 1–18.

