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DISSERTATION

Curvature Detection by Integral Transforms

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Abstract

In various fields of image analysis, determining the precise geometry of occurrent edges, e.g. the contour of an object, is a crucial task. Especially the curvature of an edge is of great practical relevance. In this thesis, we develop different methods to detect a variety of edge features, among them the curvature.

We first examine the properties of the parabolic Radon transform and show that it can be used to detect the edge curvature, as the smoothness of the parabolic Radon transform changes when the parabola is tangential to an edge and also, when additionally the curvature of the parabola coincides with the edge curvature. By subsequently introducing a parabolic Fourier transform and establishing a precise relation between the smoothness of a certain class of functions with the decay of the Fourier transform, we show that the smoothness result for the parabolic Radon transform can be translated into a change of the decay rate of the parabolic Fourier transform.

Furthermore, we introduce an extension of the continuous shearlet transform which additionally utilizes shears of higher order. This extension, called the Taylorlet transform, allows for a detection of the position and orientation, as well as the curvature and other higher order geometric information of edges. We introduce novel vanishing moment conditions of the form $\int_{\mathbb{R}} g(\pm t^k) t^m \, dt = 0$ which enable a more robust detection of the geometric edge features and examine two different constructions for Taylorlets. Lastly, we translate the results of the Taylorlet transform in $\mathbb{R}^2$ into $\mathbb{R}^3$ and thereby allow for the analysis of the geometry of object surfaces.

Danksagung

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CHAPTER 1

Introduction

The automatization of processes plays a major role for the technological progress of today's society. For this venture, it is often necessary to process information in a fast and stable fashion. In the field of autonomous driving, for instance, the recognition of road marks, traffic signs and other road users is of pivotal importance. In medical imaging, it is desirable to design systems that are capable of autonomous detection of cancerous tissue. But also many applications in other fields show the importance of object recognition and thus, good and stable detectors are crucial.

Since the concept of an edge, i.e., the occurrence of a rapid change of colour along a curve, already plays a prominent role in the mammalian visual system [HW62, HW68], it is no surprise that the detection of edges also is a major part in most object recognition algorithms, e.g. [Mar76, MH80]. Many secondary features of edges are also often used in the field of computer vision. Among them the curvature of an edge is of special interest, see e.g. [DZM$^+$07, MEO11, FB14]. The great significance of the edge curvature to the human capability of recognizing objects has already been discovered in the 1950s in psychological studies [Att54, AA56].

There is also a mathematical reason for the usefulness of curvature, as every planar C^2-curve is uniquely determined by its curvature profile up to translations and rotations. Thus, the curvature provides a natural and meaningful measure to categorize shapes and object contours.

The structure of an edge curvature estimator is usually divided into two parts. In the first step, an edge detector is applied and subsequently, a curvature estimator is employed on the newly found edge. For the purpose of detecting edges there exists a plethora of different methods, e.g. the classical Canny edge detector [Can87] and multiscale approaches such as the method of wavelet modulus maxima [MH92] or shearlet-based algorithms [YLEK09]. The numerical computation of the curvature could be handled by naively using the curvature formula for a C^2-curve $\gamma : I \to \mathbb{R}^2$ on an interval I:

$$\kappa_\gamma(t) = \frac{\det\left(\gamma'(t), \gamma''(t)\right)}{\|\gamma'(t)\|_2^3} \quad \text{for all } t \in I.$$

As a calculation based on this formula is numerically unstable due to the presence of derivatives

and a division, usually other methods are utilized. Either a purely discrete approach is applied by defining a discrete curvature, e. g. [CMT01], or the discretely given curve is artificially made continuous, e. g. by scale-space methods [MB03] or by interpolation [FJ89].

A drawback of the approach to first detect edges and then estimate the curvature is its numerical instability, as the deviation of the detected edge from the actual edge can increase the error of the curvature estimation step immensely. The main goal of this thesis is to overcome this problem by a localization of the procedure. We have developed methods that are able to determine the local orientation and curvature of the edge without knowledge of the complete edge.

Such an approach has already been successfully used, e. g. in [LPS16]. In this thesis, we will present two different methods that follow the idea of localization. The first one is described in chapter 3 and utilizes the parabolic Radon transform, i. e., the integral of a function over parabolae, and the thereof derived parabolic Fourier transform. The second method is called the Taylorlet transform and chapters 4 and 5 are devoted to this approach. It is based on the continuous shearlet transform and additionally uses shears of higher order to allow for a detection of the curvature. Both methods yield detection results that are based on differences of the decay rate of the respective integral transform.

The main challenge of our approach is to find conditions for the analyzed function and in case of the Taylorlet transform also for the analyzing function, the Taylorlet, that enforce the differences in the decay rate we need for detecting the edge orientation and curvature. Similar results on the decay rate already exist for the continuous wavelet transform, e. g. [MH92], and the continuous shearlet transform in 2D [GL09, Gro11, KP15] and in 3D [GL11].

A specific challenge for the Taylorlet transform is to find a construction that yields an analyzing function satisfying all the necessary conditions to ensure a well-defined integral transform yielding the desired decay rates.

1.1   Outline

The thesis provides different mathematical perspectives on the task of curvature detection. The focus of chapter 2 is the conformal monogenic signal curvature. We first introduce the underlying concepts of this notion - the Riesz transform, the monogenic scale-space and the isophote curvature. In [FWS11] it has been claimed that the conformal monogenic signal curvature equals the isophote curvature. We will show that this claim does not hold.

The third chapter is devoted to the parabolic Radon transform, the related parabolic Fourier transform and their connection to the edge curvature. We will introduce the concept of the wavefront set as a mathematical formulation of the orientation and position of an edge. After a brief presentation of the classical Radon transform and its connection to the wavefront set, we generalize this notion by introducing the parabolic Radon transform. We subsequently show that it can be used to detect the edge curvature, as the smoothness of the parabolic Radon transform changes when the parabola is tangential to an edge and also, when additionally the curvature of the parabola coincides with the edge curvature. By introducing the parabolic Radon

transform and establishing a precise relation between the smoothness of a certain class of functions with the decay of the Fourier transform, we show that the smoothness result for the parabolic Radon transform can be translated into a change of the decay rate of the parabolic Fourier transform.

The main focus of chapter 4 is to introduce the novel Taylorlet transform and analyze its main properties. It is an extension of the continuous shearlet transform that additionally utilizes shears of higher order like the bendlet transform [LPS16]. The goal of this chapter is to show that this new transform allows for a detection of the position and orientation, as well as the curvature and other higher order geometric information of edges. We introduce novel vanishing moment conditions of the form $\int_{\mathbb{R}} g \left(\pm t^k \right) t^m \, dt = 0$ for $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and a restrictiveness condition to ensure certain decay rates of the Taylorlet transform that facilitate the detection of the sought-for edge features. Furthermore, we will present two methods for the construction of Taylorlets that satisfy the required conditions. The first one is based on the idea that vanishing moments can be generated by taking derivatives and utilizing a method to produce vanishing moments of higher order from classical vanishing moments. The second method relies on an operator-valued approach to q-calculus. We finish the chapter with a numerical example.

The goal of chapter 5 is to extend the notion of the Taylorlet transform and the main results of the previous chapter to the third dimension. We start by introducing the multivariate Taylor series expansion and the Hankel transform. Both concepts are needed for the proof of the main result that is given subsequently. The detection result deviates from its two-dimensional pendant, as in the third dimension an additional case occurs, when the sheared version of the surface is locally hyperbolic. For this case, a special set of conditions, the hyperbolic restrictiveness has to be additionally imposed on the Taylorlet. Afterwards, we will prove that a construction based on the q-calculus approach in the previous chapter yields a Taylorlet that satisfies all of the necessary conditions to guarantee certain decay rates of the Taylorlet transform. In the last section of this chapter, we present a fast algorithm for the detection of the edge curvature.

We conclude the thesis with a brief summary and a discussion of open problems for future research in chapter 6.

1.2   Publications

Parts of this thesis have already been published:

- The results of the sections 4.2, 4.3.1, 4.4 and 4.5 with the exception of subsection 4.5.3 have been published in [Fin19].

- Subsection 4.3.2 from the beginning up to Theorem 4.23 with the exception of Theorem 4.16 and Corollary 4.17 have been published in [FK19].

1.3 Notation and basic results

This chapter introduces the basic terminology and important results that we will use throughout the thesis.

1.3.1 Symbols

In this subsection we introduce the symbols we will use in this thesis.

Sets of numbers

$\mathbb{N} := \{1, 2, 3, \ldots\}$	Set of natural numbers
$\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$	Set of natural numbers including 0
$\mathbb{Z}$	Set of integers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{R}_\pm$	Set of positive / negative real numbers
$\mathbb{C}$	Field of complex numbers

Function and distribution spaces

L^p	Space of p-th power integrable functions
C	Space of continuous functions
C^n	Space of n times continuously differentiable functions
$C^\infty = \mathcal{E} := \bigcap_{n \in \mathbb{N}} C^n$	Space of smooth functions
$C_c^\infty = \mathcal{D}$	Space of compactly supported smooth functions
$\mathcal{S}$	Space of Schwartz functions
$\mathcal{E}'$	Space of compactly supported distributions
$\mathcal{D}'$	Space of distributions
$\mathcal{S}'$	Space of tempered distributions

Geometry

$e_\theta := (\cos\theta, \sin\theta)^T$ Unit vector of the angle θ

$x_\theta := \langle x, e_\theta \rangle$ Inner product of x and e_θ

$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ 2D rotation of the angle θ

$\omega^\perp := \{x \in \mathbb{R}^n : \langle x, \omega \rangle = 0\}$ Orthogonal complement of ω

Integral transforms

$\mathcal{F}$ or $\hat{\ }$ $\hat{f}(\xi) = \int\limits_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi \rangle} \, dx$ Fourier transform

$\mathbf{H}$ $\mathbf{H}f(x) = \frac{1}{\pi} \cdot \text{P.V.} \int\limits_{\mathbb{R}} \frac{f(y)}{x-y} \, dy$ Hilbert transform

$\mathbf{R}$ $\mathbf{R}_j f(x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \cdot \text{P.V.} \int\limits_{\mathbb{R}^n} \frac{y_j \cdot f(x-y)}{\|y\|^{n+1}} \, dy$ Riesz transform

$\mathcal{R}$ $\mathcal{R}f(\omega, s) = \int\limits_{\omega^\perp} f(s\omega + y) \, dy$ Radon transform

$\mathcal{P}$ $\mathcal{P}f(x, \theta, a) = \int\limits_{\mathbb{R}} f\left(x + R_\theta \cdot \left(\frac{a}{2}t^2, t\right)^T\right) dt$ Parabolic Radon transform

$\mathcal{Q}$ $\mathcal{Q}_{v,\theta,a}f(\omega) = \int\limits_{\mathbb{R}} \mathcal{P}f\left(R_\theta \cdot (u, v)^T, \theta, a\right) e^{-iu\omega} \, du$ Parabolic Fourier transform

$\mathbf{Fr}$ $\mathbf{Fr}_\tau f(x) = \frac{1}{\tau} \cdot \int\limits_{\mathbb{R}} f(x-y) \cdot \exp\left(i \cdot \frac{y^2}{2\tau^2}\right) dy$ Fresnel transform

$\mathcal{W}$ $\mathcal{W}_\psi f(a, b) = a^{-1/2} \cdot \int\limits_{\mathbb{R}} f(x) \overline{\psi((x-b)/a)} \, dx.$ Wavelet transform

$\mathcal{SH}$ $\mathcal{SH}_\psi f(a, s, t) = \int\limits_{\mathbb{R}^2} f(x) \psi_{a,s,t}(x) \, dx$ Shearlet transform

$\mathcal{T}$ $\mathcal{T}_\tau f(a, s, t) = \int\limits_{\mathbb{R}^2} f(x) \tau_{a,s,t}(x) \, dx$ Taylorlet transform

$\mathcal{H}$ $\mathcal{H}_v f(\rho) = \int\limits_0^\infty f(r) J_v(\rho r) \rho \, d\rho$ Hankel transform

Asymptotic relations

$f = \mathcal{O}(g)$ $\Leftrightarrow \limsup\limits_{x \to a} \frac{|f(x)|}{|g(x)|} < \infty$ $|f|$ is asymptotically bounded by g

$f \sim g$ $\Leftrightarrow f = \mathcal{O}(g) \wedge g = \mathcal{O}(f)$ f is asymptotically similar to g

$f = o(g)$ $\Leftrightarrow \lim\limits_{x \to a} \frac{f(x)}{g(x)} = 0$ f is asymptotically dominated by g

$f = \Omega(g)$ $\Leftrightarrow g = \mathcal{O}(f)$ f is an asymptotic upper bound for g

$f = \omega(g)$ $\Leftrightarrow g = o(f)$ f asymptotically dominates g

Topology

Let $A \subset X$.

$\overline{A}$	Closure
$A^c := X \setminus A$	Complement

1.3.2 Curvature

As the main goal of this thesis is the computation of curvature, we give a quick introduction into the differential geometry of planar curves and of surfaces with a strong focus on their curvature. This subsection follows the course of [Bär10].

Curvature of planar curves

Definition 1.1 (Parametrized Curve)**.** A *parametrized curve* is a differentiable map $\gamma : I \to \mathbb{R}^n$ of an open interval $I \subset \mathbb{R}$ to $\mathbb{R}^n$.

Definition 1.2 (Regular curve)**.** A *regular curve* is a parametrized curve whose first derivative does not vanish, i.e.,

$$\gamma'(t) \neq 0 \quad \text{for all } t \in I.$$

Definition 1.3 (Tangent and normal vector)**.** Let $\gamma : I \to \mathbb{R}^2$ be a regular curve. Its first derivative $\gamma'(t)$ is called *tangent vector* and the vector

$$n(t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \gamma'(t)$$

is called *normal vector*.

Definition 1.4 (Parametrization by arc-length)**.** A regular curve $\gamma : I \to \mathbb{R}^n$ is called *parametrized by arc-length*, if $\|\gamma'(t)\| = 1$ for all $t \in I$.

Definition 1.5 (Curvature)**.** Let $\gamma \in C^2\left(I, \mathbb{R}^2\right)$ be parametrized by arc-length and let N denote its normal vector. Then the parameter $\kappa_\gamma : I \to \mathbb{R}$ is called the *curvature* of γ, if it satisfies

$$\gamma''(t) = \kappa_\gamma(t) \cdot N(t) \quad \text{for all } t \in I.$$

As not every curve is parametrized by arc-length, it is beneficial to have a more general formula for the curvature.

Theorem 1.6 (Curvature formula)**.** *Let $\gamma \in C^2\left(I, \mathbb{R}^2\right)$ be a regular curve. Then,*

$$\kappa_\gamma(t) = \frac{\det\left(\gamma'(t), \gamma''(t)\right)}{\|\gamma'(t)\|_2^3} \quad \textit{for all } t \in I.$$

Especially in chapter 4, we deal with curves that can be described as graph of a function. Naturally, we are interested in determining their curvature.

Corollary 1.7 (Curvature of a graph)**.** *Let $f \in C^2(I, \mathbb{R})$. Then, its* graph

$$\gamma : I \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

has the curvature

$$\kappa_\gamma(t) = \frac{f''(t)}{\left(1 + \left[f'(t)\right]^2\right)^{3/2}} \quad \text{for all } t \in I.$$

Curvature of surfaces in $\mathbb{R}^3$

Definition 1.8 (Regular surface and local parametrization)**.** A set $S \subset \mathbb{R}^3$ is called *regular surface*, if for all $p \in S$ there exists an open neighborhood $V \subset \mathbb{R}^3$, an open set $U \subset \mathbb{R}^2$ and a map $F \in C^1\left(U, \mathbb{R}^3\right)$ such that

   (i) $F(U) = S \cap V$ and the map $F : U \to S \cap V$ is homeomorphic,

   (ii) the Jacobian $J_F(u)$ is of rank 2 for all $u \in U$.

The triple (U, F, V) is called *local parametrization* of S.

In two dimensions, it was possible to describe the tangent via a single vector. In three dimensions, the tangent vectors form a plane.

Definition 1.9 (Tangential plane)**.** Let $S \subset \mathbb{R}^3$ be a regular surface and let $p \in S$. Then, the set

$$T_p S = \left\{X \in \mathbb{R}^3 : \exists\, \varepsilon > 0 \,\exists\, \text{parametrized curve } \gamma : (-\varepsilon, \varepsilon) \to S \text{ such that } \gamma(0) = p,\ \gamma'(0) = X\right\}$$

is called *tangential plane* of S in p.

Definition 1.10 (Differential map)**.** Let $S_1, S_2 \subset \mathbb{R}^3$ be regular surfaces, let $f : S_1 \to S_2$ be a smooth map and $p \in S_1$. The *differential* of f at p is the map

$$d_p f : T_p S_1 \to T_p S_2,$$

which is given by the rule: for $X \in T_p S_1$ and a sufficiently small $\varepsilon > 0$ let $\gamma : (-\varepsilon, \varepsilon) \to S_1$ with $\gamma(0) = p$ and $\gamma'(0) = X$ and set

$$d_p f(X) := \frac{d}{dt}\left(f \circ \gamma\right)\Big|_{t=0} \in T_{f(p)} S_2.$$

Definition 1.11 (First fundamental form)**.** Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$ and let (U, F, V) be a local parametrization of S. The *first fundamental form* of S in p is given by

$$g_p : T_p S \times T_p S \to \mathbb{R}, \quad (X, Y) \mapsto \langle X, Y \rangle.$$

The matrix representation of g_p with respect to the local parametrization (U, F, V) is given by the maps

$$g_{ij} : U \to \mathbb{R}, \quad u \mapsto g_p\left(\partial_i F(u), \partial_j F(u)\right), \quad i, j \in \{1, 2\}.$$

Definition 1.12 (Normal field and orientability)**.** Let $S \subset \mathbb{R}^3$ be a regular surface. A *normal field* on S is a map $N : S \to \mathbb{R}^3$ such that $N(p) \perp T_p S$ for all $p \in S$. A normal field N on S is said to be a *unit normal field*, if $\|N(p)\| = 1$ for all $p \in S$. The regular surface S is called *orientable*, if there exists a continuous unit normal field on S.

Definition 1.13 (Weingarten map)**.** Let $S \subset \mathbb{R}^3$ be a regular, orientable surface with unit normal field N and let $p \in S$. The endomorphism

$$W_p : T_p S \to T_p S, \quad X \mapsto -d_p N(X)$$

is called the *Weingarten map*.

Proposition 1.14. *[Bär10, Proposition 3.5.5] Let $S \subset \mathbb{R}^3$ be a regular, orientable surface. Then, its Weingarten map is self-adjoint with respect to the first fundamental form.*

Definition 1.15 (Second fundamental form)**.** Let $S \subset \mathbb{R}^3$ be a regular, orientable surface with unit normal field N and let $p \in S$. The *second fundamental form* of S in p is given by

$$h_p : T_p S \times T_p S, \quad (X, Y) \mapsto g_p\left(W_p(X), Y\right).$$

The matrix representation of h_p with respect to the local parametrization (U, F, V) is given by the maps

$$h_{ij} : U \to \mathbb{R}, \quad u \mapsto g_p\left(\partial_{ij} F(u), N(p)\right), \quad i, j \in \{1, 2\}.$$

Proposition 1.16. *[Bär10] Let $S \subset \mathbb{R}^3$ be a regular, orientable surface and let g, h denote the matrix representation of first and second fundamental form. Then, the matrix representation of its Weingarten map is given by*

$$w = h \cdot g^{-1}.$$

Definition 1.17 (Gauss, mean and principal curvature and principal directions)**.** Let $S \subset \mathbb{R}^3$ be a regular, orientable surface and let $p \in S$. Its *Gauss curvature* is given by

$$K(p) = \det\left(W_p\right).$$

The *mean curvature* of S in p is defined as

$$H(p) = \frac{1}{2}\text{Spur}(W_p).$$

The eigenvalues of the Weingarten map W_p are called *principal curvatures* and the related eigenvectors are known as the *principal directions*.

Due to Proposition 1.14, the Weingarten map is self-adjoint. Thus, its matrix representation has an orthonormal basis of eigenvectors which are the principal directions. Hence, they are orthogonal to each other.

In chapter 5, we will handle surfaces that are given as graphs of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Hence, it is crucial to have a formula for the Weingarten map to obtain the different kinds of curvature.

Proposition 1.18. *[Weingarten map of a graph] Let $U \subset \mathbb{R}^2$ be an open set, $f \in C^2(U, \mathbb{R})$ and let S denote the graph of f. Then, the matrix representation of the Weingarten map for $(x, y) \in U$ is given by*

$$w(x, y) = \frac{1}{\left(1 + \|\nabla f(x, y)\|^2\right)^{3/2}} \cdot (Hf)(x, y) \cdot \begin{pmatrix} 1 + \left[\partial_y f(x, y)\right]^2 & -\partial_x f(x, y) \cdot \partial_y f(x, y) \\ -\partial_x f(x, y) \cdot \partial_y f(x, y) & 1 + \left[\partial_x f(x, y)\right]^2 \end{pmatrix},$$

where Hf denotes the Hessian matrix of f.

Proof. In this proof, we will often omit the arguments of the functions, when they are obvious. As S is the graph of f, a parametrization is given by

$$F : U \to S, \quad (x, y) \longmapsto \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}.$$

Hence, we obtain the first and second derivatives

$$\partial_x F = \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}, \partial_y F = \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix}, \partial_{xx} F = \begin{pmatrix} 0 \\ 0 \\ \partial_{xx} f \end{pmatrix}, \partial_{xy} F = \begin{pmatrix} 0 \\ 0 \\ \partial_{xy} f \end{pmatrix}, \partial_{yy} F = \begin{pmatrix} 0 \\ 0 \\ \partial_{yy} f \end{pmatrix}.$$

The matrix representation of the first fundamental form is given by

$$g = \begin{pmatrix} 1 + \left[\partial_x f\right]^2 & \partial_x f \cdot \partial_y f \\ \partial_x f \cdot \partial_y f & 1 + \left[\partial_y f\right]^2 \end{pmatrix}$$

and its inverse reads

$$g^{-1} = \frac{1}{1 + \|\nabla f\|^2} \cdot \begin{pmatrix} 1 + \left[\partial_y f\right]^2 & -\partial_x f \cdot \partial_y f \\ -\partial_x f \cdot \partial_y f & 1 + \left[\partial_x f\right]^2 \end{pmatrix}.$$

The unit normal field can be determined to be

$$N = \frac{\partial_x F \times \partial_y F}{\|\partial_x F \times \partial_y F\|} = \frac{1}{\sqrt{1 + \|\nabla f\|^2}} \cdot \begin{pmatrix} -\partial_x f \\ -\partial_y f \\ 1 \end{pmatrix}.$$

Hence, we can utilize the formula for the matrix representation of the second fundamental form to obtain

$$h = \frac{1}{\sqrt{1 + \|\nabla f\|^2}} \cdot \begin{pmatrix} \partial_{xx}f & \partial_{xy}f \\ \partial_{xy}f & \partial_{yy}f \end{pmatrix} = \frac{1}{\sqrt{1 + \|\nabla f\|^2}} \cdot Hf.$$

Inserting the matrix representations of the first and second fundamental form into the formula of Proposition 1.16 yields

$$w = h \cdot g^{-1} = \frac{1}{\left(1 + \|\nabla f\|^2\right)^{3/2}} \cdot Hf \cdot \begin{pmatrix} 1 + \left[\partial_y f\right]^2 & -\partial_x f \cdot \partial_y f \\ -\partial_x f \cdot \partial_y f & 1 + \left[\partial_x f\right]^2 \end{pmatrix}.$$

$\square$

1.3.3   Fourier transform

This transform is especially important for many of the ventures in this thesis, as it allows one to quantify smoothness in terms of decay rate. Since there exist many different conventions on the Fourier transform, we will fix the associated notations in this subsection, as well as the constants appearing in the most common Fourier results. Both the notation and the theorems are taken from [Kat04, Chapter VI], which treats the Fourier transform on the real line. Although the Fourier transform in this subsection is defined on $\mathbb{R}^n$ rather than on $\mathbb{R}$, the same arguments can be applied for the proofs as in the one-dimensional case.

Definition 1.19 (Fourier transform). For $f : \mathbb{R}^n \to \mathbb{C}$, its Fourier transform is

$$\hat{f} : \mathbb{R}^n \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}^n} f(x) \cdot e^{-i \cdot \langle x, \omega \rangle} \, dx,$$

whenever $\hat{f}$ is well defined. We will often denote the Fourier transform as $\mathcal{F}$, if the function to be transformed is overly long.

The next theorem states the most basic properties of the Fourier transform.

Theorem 1.20 (Basic properties of the Fourier transform). *[Kat04, Chapter VI, Theorem 1.1] Let* $f, g \in L^1(\mathbb{R}^n)$. *Then for almost all* $\omega \in \mathbb{R}^n$,

(i) $\widehat{f + g}(\omega) = \hat{f}(\omega) + \hat{g}(\omega)$,

(ii) *and for all* $\alpha \in \mathbb{C}$, $\widehat{\alpha f}(\omega) = \alpha \cdot \hat{f}(\omega)$,

(iii) $\widehat{\bar{f}}(\omega) = \overline{\hat{f}(-\omega)}$,

(iv) *and for all* $y \in \mathbb{R}^n$, $\widehat{T_y f}(\omega) = \hat{f}(\omega) \cdot e^{-i \cdot \langle \omega, y \rangle}$,

(v) $|\hat{f}(\omega)| \le \|f\|_{L^1}$ *and* $\hat{f} \in L^\infty(\mathbb{R}^n)$,

(vi) and for all $\lambda > 0$, $\widehat{D_\lambda f}(\omega) = \frac{1}{\lambda} \cdot \hat{f}\left(\frac{\omega}{\lambda}\right)$.

The following theorem shows the uniform continuity of the Fourier transform of an L^1-function.

Theorem 1.21. *[Kat04, Chapter VI, Theorem 1.2] Let $f \in L^1(\mathbb{R}^n)$. Then $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$.*

For the subsequent theorem we introduce the concept of the convolution.

Definition 1.22 (Convolution)**.** Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution of f and g is

$$f * g : \mathbb{R}^n \to \mathbb{C}, \quad x \mapsto \int_{\mathbb{R}^n} f(y) \cdot g(x-y) \, dy,$$

whenever it is well-defined.

The following theorem allows for a multiplicative representation of a convolution in the Fourier domain.

Theorem 1.23 (Convolution theorem)**.** *[Kat04, Chapter VI, Theorem 1.3] Let $f, g \in L^1(\mathbb{R}^n)$. Then, $f * g \in L^1(\mathbb{R}^n)$ and $\widehat{f * g} = \hat{f} \cdot \hat{g}$.*

In the next definition we will introduce iterated integrals, which will be made use of in the subsequent theorem.

Definition 1.24 (Iterated integral)**.** Let $j \in \mathbb{N}_0$, $x \in \mathbb{R}$ and $f \in L^1(\mathbb{R})$. We then denote

$$I_+ f(x) = \int_{-\infty}^x f(y) \, dy \quad \text{and} \quad I_- f(x) = \int_x^\infty f(y) \, dy.$$

The iterated integral is then inductively defined by

$$I_\pm^{j+1} f = I_\pm \circ I_\pm^j f.$$

For $g \in L^1(\mathbb{R}^n)$ and $k \in \{1, \ldots, n\}$, we denote the partial iterated integrals by

$$I_{+,k} g(x) = \int_{-\infty}^{x_k} g\left(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n\right) \, dy$$
$$\text{and} \quad I_{-,k} g(x) = \int_{x_k}^\infty g\left(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_n\right) \, dy.$$

Again, the associated iterated integral is defined inductively by

$$I_{\pm,k}^{j+1} g = I_{\pm,k} \circ I_{\pm,k}^j g.$$

This definition allows for an examination of the Fourier transform of iterated integrals.

Theorem 1.25. *[Kat04, Chapter VI, Theorem 1.5] Let $f \in L^1(\mathbb{R}^n)$ and let $k \in \{1, \ldots, n\}$. Then, if $I_{\pm,k} f \in L^1(\mathbb{R}^n)$ we have*

$$\widehat{I_{\pm,k} f}(\omega) = \frac{1}{i\omega_k} \cdot \hat{f}(\omega).$$

A similar result can be proved for the derivative, as can be seen in the next theorem.

Theorem 1.26. *[Kat04, Chapter VI, Theorem 1.6] Let $f \in L^1(\mathbb{R}^n)$ and let $x_k \cdot f(x) \in L^1(\mathbb{R}^n, dx)$ for all $k \in \{1, \ldots, n\}$. Then, $\hat{f} \in C^1(\mathbb{R}^n)$ and*

$$\partial_{\omega_k}\hat{f}(\omega) = -i \cdot \mathcal{F}\left(x_k \cdot f(x)\right)(\omega)$$

for all $\omega \in \mathbb{R}^n$.

The next theorem gives conditions for approximate identities, i. e., families of L^1-functions which act in their limit as the identity of the convolution.

Theorem 1.27 (Approximate Identity)**.** *[Kat04, Chapter VI, Theorem 1.10] Let $\{k_\lambda\}_{\lambda \in \mathbb{N}} \subset L^1(\mathbb{R})$ such that*

1. *$\int_{\mathbb{R}} k_\lambda(x)\, dx = 1$,*

2. *$\|k_\lambda\|_{L^1} = \mathcal{O}(1)$ for $\lambda \to \infty$,*

3. *$\lim_{\lambda \to \infty} \int_{|x| > \delta} |k_\lambda(x)|\, dx = 0$ for all $\delta > 0$.*

Then, for all $f \in L^1(\mathbb{R})$,

$$\lim_{\lambda \to \infty} \|f - k_\lambda * f\|_{L^1} = 0.$$

The next theorems yields an inversion formula of the Fourier transform.

Theorem 1.28 (Inversion of the Fourier transform)**.** *[Kat04, Chapter VI, Theorem 1.11] Let $f \in L^1(\mathbb{R}^n)$. Then for almost all $x \in \mathbb{R}^n$,*

$$f(x) = \frac{1}{(2\pi)^n} \cdot \lim_{\lambda \to \infty} \int_{[-\lambda, \lambda]^n} \prod_{k=1}^n \left(1 - \frac{|\omega_k|}{\lambda}\right) \cdot \hat{f}(\omega) \cdot e^{i\langle x, \omega\rangle}\, d\omega.$$

By utilizing the Hahn-Banach theorem, it is possible to extend the definition of the Fourier transform to $L^2(\mathbb{R}^n)$. This extension is often called *Plancherel transform* and is an isometry (up to a constant), as the following theorem shows. However, in this thesis we will refer to this extension as Fourier transform.

Theorem 1.29 (Plancherel's theorem)**.** *[Kat04, Section VI.3] Let $f, g \in L^2(\mathbb{R}^n)$. Then,*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi \quad and$$

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \cdot \left\|\hat{f}\right\|_{L^2}.$$

Of pivotal importance to the theory of Fourier transforms are the *Schwartz space* and its dual, the *space of tempered distributions*, which are introduced in the following definition.

Definition 1.30. The Schwartz space is defined as

$$\mathcal{S}(\mathbb{R}^n) = \left\{ \phi \in C^\infty(\mathbb{R}^n) \;\middle|\; \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial_x^\beta \phi(x) \right| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n \right\},$$

and its dual space, the space of tempered distributions reads $\mathcal{S}'(\mathbb{R}^n)$.

The significance of the Schwartz space for the Fourier transform lies in the following statement.

Theorem 1.31. *[Hör83, Theorem 7.1.5] The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an automorphism.*

As the Fourier transform is an $\mathcal{S}(\mathbb{R}^n)$-automorphism, it is possible to extend the definition of the Fourier transform to $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions.

Definition 1.32. [Fourier transform on $\mathcal{S}'$] Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Its Fourier transform is then defined as

$$\tilde{u}(\phi) = u(\hat{\phi}) \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

For this variation of the Fourier transform, the proof of a uniqueness theorem is especially easy.

Theorem 1.33 (Uniqueness theorem)**.** *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\tilde{u} \equiv 0$. Then, $u \equiv 0$.*

Proof.

$$
\begin{aligned}
\tilde{u} \equiv 0 \quad &\Rightarrow \quad \tilde{u}(\phi) = 0 \quad \forall \, \phi \in \mathcal{S}(\mathbb{R}^n) \\
&\Rightarrow \quad u(\hat{\phi}) = 0 \quad \forall \, \phi \in \mathcal{S}(\mathbb{R}^n) \\
&\stackrel{\text{Theorem 1.31}}{\Rightarrow} u(\phi) = 0 \quad \forall \, \phi \in \mathcal{S}(\mathbb{R}^n) \\
&\Rightarrow \quad u \equiv 0.
\end{aligned}
$$

$\square$

CHAPTER 2

Conformal monogenic signal curvature

In this chapter we introduce the conformal monogenic signal and some of its secondary parameters used in image processing. We focus on the conformal monogenic signal curvature introduced in [FWS11]. This particular value of a signal is determined by means of the Riesz transform and stereographic projections. It has been claimed to equal the isophote curvature of the signal. We will show that this is not the case.

This chapter is divided into three sections. In the first one, we introduce the basic methods which are utilized later. The second section describes the concept of a conformal monogenic signal and shows some of its most prominent features. In the final section we will consider a very common class of examples in which the conformal monogenic signal curvature is not well defined.

2.1 Underlying concepts

In this section, we will introduce the basic notions which are needed for the definition of the conformal monogenic signal.

In the first place, we consider the *analytic and monogenic signals,* which extend a given signal $f : \mathbb{R}^n \to \mathbb{R}$ by n additional components. These are determined by the Hilbert and the Riesz transform of the original signal, respectively, and enable a decomposition of the emerging signal into polar coordinates which contain valuable signal information.

In the second subsection we introduce the concept of scale-space, which is based on the convolution with a smoothing kernel. We will focus on the *Poisson scale-space* and its special relation to the monogenic signal.

In the last subsection we discuss the *isophote curvature* which is the local curvature of the level curve of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Here we take a closer look on its practical importance and show ways to determine this curvature.

2.1.1   Analytic and monogenic signals

Due to the omnipresence of different communication devices in today's society we can hardly overestimate the importance of signal processing. Thereby the signal analysis and the robustness of the utilized methods against various sources of noise play a pivotal role.

One of the pioneers of information theory and signal processing was Denis Gabor, to whom we owe - among other famous inventions - the idea of the analytic signal. He noticed that by adding to a real valued signal $f : \mathbb{R} \to \mathbb{R}$ the imaginary unit times its Hilbert transform, the amplitudes belonging to the negative frequencies of the signal are suppressed without loss of information of the signal [Gab46]. Furthermore, a decomposition of the analytic signal into polar coordinates facilitates an analysis of the amplitude and the phase of the signal.

In order to further investigate these ideas, we first introduce the Hilbert transform.

Definition 2.1 (Hilbert transform)**.** Let $f \in L^2(\mathbb{R})$. Its Hilbert transform is defined as

$$\mathbf{H}f(t) = \frac{1}{\pi} \cdot \text{P.V.} \int_{\mathbb{R}} \frac{f(\tau)}{t-\tau} \, d\tau = \frac{1}{\pi} \cdot \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{f(t-\tau)}{\tau} \, d\tau.$$

Gabor utilized the Hilbert transform [Gab46] to define the *analytic signal* which denotes the complex valued signal $f_a = f + i \cdot \mathbf{H}f$.

Definition 2.2 (Analytic signal, instantaneous amplitude, phase and frequency)**.** Let $f \in L^2(\mathbb{R}, \mathbb{R})$. Its *analytic signal* is defined as

$$f_a = f + i\mathbf{H}f.$$

The *instantaneous amplitude* reads

$$A_f = \sqrt{|f|^2 + |\mathbf{H}f|^2},$$

the *instantaneous phase* is defined as

$$\varphi_f = \text{atan2}(\Re(f), \Im(f)),$$

and the *instantaneous frequency* as

$$\omega_f = \varphi_f'.$$

We will show that the Hilbert transform commutes with certain operators which will be introduced in the next definition.

Definition 2.3. Let D_a denote a dilation, T_b a translation and R_ρ a rotation operator, so that

$$\begin{aligned}
D_a, T_b, R_\rho : L^2(\mathbb{R}^n) &\to L^2(\mathbb{R}^n), \\
D_a f(x) &= f(ax), \quad a > 0, \\
T_b f(x) &= f(x-b), \quad b \in \mathbb{R}^n, \\
R_\rho f(x) &= f(\rho^{-1}x), \quad \rho \in SO(n).
\end{aligned}$$

Proposition 2.4 (Properties of the Hilbert transform)**.** *Let $f \in L^2(\mathbb{R})$. It holds that*

(i) $\|\mathbf{H}f\|_2 < \infty$,

(ii) $\mathbf{H}(\mathbf{H}f) = -f$,

(iii) $\mathbf{H}T_b f = T_b\mathbf{H}f$ *for all* $b \in \mathbb{R}$,

(iv) $\mathbf{H}D_a f = D_a\mathbf{H}f$ *for all* $a > 0$,

(v) $\mathcal{F}(\mathbf{H}f)(\omega) = -i \cdot \mathrm{sgn}(\omega) \cdot \hat{f}(\omega)$ *for almost all* $\omega \in \mathbb{R}$.

(vi) $\|\mathbf{H}\|_{\mathrm{op}:L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq 1$.

Proof. $(i) - (v)$: See [Ste70, p. 55].
(vi): Application of (v) and Plancherel's theorem. $\qquad\square$

Property (v) in the upper proposition can also be used to prove the aforementioned vanishing negative frequencies of the analytic signal.

Lemma 2.5. *Let $f \in L^2(\mathbb{R})$. Then*

$$\hat{f}_a(\omega) = \begin{cases} 0, & \textit{for almost all } \omega < 0, \\ 2\hat{f}(\omega), & \textit{for almost all } \omega > 0. \end{cases}$$

Proof. Application of Proposition 2.4 (v). $\qquad\square$

As the analytic signal became a classical tool in signal analysis, Felsberg and Sommer generalized this concept to higher dimensions [FS01]. This way, the processing of e. g. 2D signals as images can also profit from a similar representation. To this end, the Riesz transform, a natural extension of the Hilbert transform to higher dimensions, was employed.

Definition 2.6 (Riesz transform, monogenic signal)**.** Let $n \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n, \mathbb{R})$. The Riesz transform is defined as follows:

$$\mathbf{R}f = \begin{pmatrix} \mathbf{R}_1 f \\ \vdots \\ \mathbf{R}_n f \end{pmatrix},$$

where for $j \in \{1, \dots, n\}$ we have

$$\mathbf{R}_j f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{y_j}{\|y\|^{n+1}} f(x-y) \, dy \quad \text{for almost all } x \in \mathbb{R}^n.$$

The monogenic signal f_m of a function $f \in L^2(\mathbb{R}^n)$ is defined as

$$f_m = (f, \mathbf{R}_1 f, \dots, \mathbf{R}_n f).$$

The Riesz transform exhibits some remarkable similarities to the Hilbert transform.

Theorem 2.7 (Properties of the Riesz transform)**.** *The Riesz transform is determined up to a constant by the following properties:*

$$
\begin{aligned}
T_b \mathbf{R}_j &= \mathbf{R}_j T_b && \text{for all } b \in \mathbb{R}^2, \\
D_a \mathbf{R}_j &= \mathbf{R}_j D_a && \text{for all } a > 0, \\
\mathbf{R}_j R_\rho &= \sum_{k=1}^{n} \rho_{j,k} R_\rho \mathbf{R}_k && \text{for all } \rho \in SO(2).
\end{aligned}
$$

Proof. See [Ste70, Ch III, Prop 2].   □

In addition to the upper properties, the Riesz transform allows for a similar Fourier representation as the Hilbert transform, which is shown in the next lemma.

Lemma 2.8. *For $f \in L^2(\mathbb{R}^n)$, the Fourier transform of its Riesz transform reads*

$$
\mathcal{F}\mathbf{R}f(\xi) = -i \frac{\xi}{\|\xi\|} \mathcal{F}f(\xi) \quad \text{for almost all } \xi \in \mathbb{R}^n.
$$

Analogous to the analytic signal, the monogenic signal allows for a decomposition into meaningful quantities in spherical coordinates.

Definition 2.9 (Instantaneous amplitude, phase and orientation)**.** Let $f \in L^2(\mathbb{R}^2)$. Then the instantaneous amplitude, phase and orientation of f are defined as

$$
A_f = \sqrt{|f|^2 + |\mathbf{R}_1 f|^2 + |\mathbf{R}_2 f|^2},
$$
$$
\varphi_f = \arctan\left(\frac{f}{\sqrt{|\mathbf{R}_1 f|^2 + |\mathbf{R}_2 f|^2}} \right),
$$
$$
\text{and} \quad \theta_f = \operatorname{atan2}(\mathbf{R}_1 f, \mathbf{R}_2 f)
$$

respectively.

2.1.2   Poisson scale-space

A major issue in image processing and image analysis is the handling of differently sized objects in an image. For this purpose, the scale space theory was developed which is motivated by the biological functioning of the sense of sight. This theory offers a mathematical framework for the consistent treatment of objects of various scales and employs a smoothing kernel to this end. The most widely used scale space is based on the Gaussian kernel and was introduced by Witkin in 1983 [Wit83].

Definition 2.10 (Gaussian scale-space)**.** Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Its *scale-space representation* Φ is defined as a convolution with the Gaussian kernel

$$g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}, \ g_s(x) = \frac{1}{(2\pi s)^{n/2}} \cdot e^{-|x|^2/2s},$$

so that

$$\Phi(f, x, s) = (f * g_s)(x)$$

for all $x \in \mathbb{R}^n$ and for all $s > 0$. The parameter s is called the *scale*.

Remark 2.11. Since $g_s \in \mathcal{S}(\mathbb{R}^n)$ for all $s > 0$, the Fourier convolution theorem and the decay rates of $\hat{f}$ and $\widehat{g_s}$ yield that the scale-space representation $\Phi(f, x, s)$ is smooth with respect to the parameter x, i.e., $f * g_s \in C^\infty(\mathbb{R}^n)$.

As the next theorem shows, the Gaussian scale-space – like the Hilbert transform – allows for a unique characterization by demanding a set of conditions which are usually called *scale-space axioms*. While there are many different choices of axioms which uniquely determine the scale space to depend on a Gaussian kernel, we chose to use the set of conditions which was originally introduced by Florack et al. in one dimension [FtHRKV92] and extended to several dimensions by Pauwels et al. [PVGFM95] .

Theorem 2.12 (Uniqueness theorem)**.** *Let*

$$\phi : L^1(\mathbb{R}^n) \times \mathbb{R}_+ \to L^1(\mathbb{R}^n \times \mathbb{R}^n)$$

and let

$$\Phi : L^1(\mathbb{R}^n) \times \mathbb{R}_+ \to L^1(\mathbb{R}^n), \quad (f, s) \mapsto \int_{\mathbb{R}^n} \phi(f, s)(\cdot, y) \, dy$$

fulfill the following conditions:

 I. Linearity*: $\Phi(\lambda f + g, x, s) = \lambda \Phi(f, x, s) + \Phi(g, x, s)$ for all $\lambda, s > 0$, $f, g \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.*

 II. Shift invariance*: $\Phi(T_y f, x, s) = \Phi(f, x - y, s)$ for all $x, y \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$ and $s > 0$.*

 III. Rotational invariance*: $\Phi(R_\rho f, x, s) = \Phi(f, \rho \cdot x, s)$ for all $\rho \in SO(n)$, $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $s > 0$.*

 IV. Semi-group structure*: There exists a mapping $S : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\Phi(\Phi(f, x, s), x, t) = \Phi(f, x, S(s, t))$$

 for all $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $s, t > 0$.

 V. Scale-invariance*: There exists a mapping $T : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\Phi(D_a f, x, s) = \Phi(f, ax, T(s, a))$$

 for all $a, s > 0$, $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

VI. Separability*: There exist mappings*

$$\phi_1, \ldots, \phi_n : L^1(\mathbb{R}^n) \times \mathbb{R}_+ \to L^1(\mathbb{R}^n \times \mathbb{R})$$

and

$$\Phi_1, \ldots, \Phi_n : L^1(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}_+ \to L^1(\mathbb{R}^n), \quad (f, x, s) \mapsto \int_{\mathbb{R}} \phi_i(f, s)(x, y_i) \, dy_i$$

such that

$$\Phi(f, x, s) = \Phi_n(\Phi_{n-1}(\ldots(\Phi_1(f, x, s), x, s), \ldots), x, s)$$

for all $x \in \mathbb{R}^n$ *and* $s > 0$.

Then there exist $\alpha, c > 0$ *such that*

$$\Phi(f, x, s) = c \cdot \int_{\mathbb{R}^n} f(x - y) e^{-\alpha \cdot \|y\|^2 / s} dy$$

Proof. See [PVGFM95].   □

By leaving out the axiom (VI) of Theorem 2.12, the rest of the conditions allow for a greater variety of convolution kernels, as the following lemma shows.

Lemma 2.13. *Let*

$$\Phi : L^1(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}_+ \to L^1(\mathbb{R}^n), \quad (f, x, s) \mapsto \int_{\mathbb{R}^n} \phi(f, x, y, s) \, dy$$

fulfill all conditions of Theorem 2.12 except for VI. Then there exist $\alpha, c, p > 0$ *such that*

$$\Phi(f, x, s) = \int_{\mathbb{R}^n} f(x - y) g_s(y) \, dy,$$

whence

$$\hat{g}_s(\xi) := c e^{-\alpha s |\xi|^p}.$$

Proof. See [PVGFM95].   □

The choice $p = 2$ delivers the only separable kernel - the Gaussian. If we choose $p = 1$ instead and normalize the convolution kernel, we obtain the *Poisson kernel on the upper half-space*

$$P(x, s) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{s}{(|x|^2 + s^2)^{\frac{n+1}{2}}} \quad \text{for all} \quad x \in \mathbb{R}^n, \ s > 0.$$

Felsberg and Sommer exploited the fact that the Poisson kernel also generates a linear, rotation- and scale-invariant scale-space, and introduced the *monogenic scale-space* on this basis [FS04]. They utilized the close relation between Poisson kernel and the Riesz transform, since the latter produces the flux of the Poisson scale-space, as the following theorem shows.

Theorem 2.14. *Let $H_+^n := \{x \in \mathbb{R}^n : x_n > 0\}$ and let $x' = (x_1, \ldots, x_{n-1})$ for $x \in \mathbb{R}^n$. Furthermore let*

$$\mathbb{H}^2(H_+^n) := \left\{ f : H_+^n \to \mathbb{R} \,\middle|\, \sup_{x_n > 0} \left(\int_{\mathbb{R}^{n-1}} |f(x)|^2 \, dx' \right)^{\frac{1}{2}} < \infty \right\}.$$

Let $h \in \mathbb{H}^2(H_+^n) \cap C^2(H_+^n)$ be a harmonic function, i.e., $\Delta h \equiv 0$, and $g = \nabla h$ is the corresponding gradient field fulfilling the boundary condition

$$g_n(x', 0) = f(x'). \tag{1}$$

Then

$$g_j(x', s) = \mathbf{R}_j (f * P(\,\cdot\,, s))(x') \quad \text{for all} \quad j \in \{1, \ldots, n-1\}, \; x' \in \mathbb{R}^{n-1}, \; s > 0,$$

and

$$g_n(x', s) = (f * P(\,\cdot\,, s))(x') \quad \text{for all} \quad x' \in \mathbb{R}^{n-1}, \; s > 0.$$

Proof. See [Ste70, Ch III, Thm 3] and [FS04]. □

Thus, the Poisson scale-space can be considered as the natural scale-space for monogenic signals.

2.1.3 Isophote Curvature

Iso-surfaces, i.e., surfaces on which a multivariate mapping is constant, are to be found in diverse fields of science. Usually encountered as 1D curves in a bivariate mapping, their arguably most well known example is the contour line in topographic maps. Also in other areas such as metereology, engineering and astronomy they play a major role as isobars, isotherms and isophotes, respectively. In this subsection, we focus on the latter, a line of constant brightness.

Definition 2.15 (Isophote). Let $f \in C(\mathbb{R}^n, \mathbb{R})$ and $c \in \mathbb{R}$. Its isophotes are sets of the form

$$\Lambda_{f,c} := \{x \in \mathbb{R}^n : f(x) = c\}.$$

Since the isophote lines of an image carry a large part of its structure, they are used in object recognition, e.g. in [LHR05]. Especially the isophote curvature is utilized to this end, e.g. for eye tracking in [VG08], or it is used for image enhancement, as in [MS98].

The isophote curvature can be determined in the way presented in the following proposition.

Proposition 2.16 (Isophote curvature). *[Gol05, Prop 3.1] Let $f \in C(\mathbb{R}^2, \mathbb{R})$, $c \in \mathbb{R}$ and*

$$\gamma_{f,c} : I \subset \mathbb{R} \to \Lambda_{f;c}, \quad t \mapsto \gamma_{f;c}(t).$$

Then the curvature of the isophote $\gamma_{f;c}$ in the point $x = \gamma_{f;c}(t)$, for $t \in I$, can be computed as

$$\kappa_f(x) = -\frac{\partial_{11} f(x)[\partial_2 f(x)]^2 - 2 \cdot \partial_{12} f(x)\partial_1 f(x)\partial_2 f(x) + [\partial_1 f(x)]^2\partial_{22} f(x)}{\left([\partial_1 f(x)]^2 + [\partial_2 f(x)]^2\right)^{\frac{3}{2}}}.$$

Proof. W. l. o. g. the isophote curve $\gamma = \gamma_{f;c} : I \to \Lambda_{f;c}$ is parametrized by arc-length. Then the isophote curvature can be computed as

$$\kappa_f\big(\gamma(s)\big) = \langle N(s), \ddot{\gamma}(s)\rangle \quad \text{for all } s \in I,$$

where N is the unit normal vector of γ. Naturally the equation

$$\langle N, \dot{\gamma}\rangle = 0$$

holds. By taking the derivative of both sides of the upper equation, we obtain

$$\langle \dot{N}, \dot{\gamma}\rangle + \langle N, \ddot{\gamma}\rangle = 0 \Rightarrow \kappa_f \circ \gamma = \langle N, \ddot{\gamma}\rangle = -\langle \dot{N}, \dot{\gamma}\rangle. \tag{2}$$

The normal vector can be determined implicitly by taking the derivative of

$$f \circ \gamma \equiv c.$$

Hence, we obtain

$$\frac{d}{ds} f \circ \gamma = \langle \nabla f \circ \gamma, \dot{\gamma}\rangle \equiv 0.$$

Hence, the normal vector can be written as

$$N = \frac{\nabla f \circ \gamma}{\|\nabla f \circ \gamma\|}.$$

By inserting this into (2), we obtain by the chain rule that

$$\begin{aligned}
\kappa_f \circ \gamma &= -\langle \dot{N}, \dot{\gamma}\rangle \\
&= -\left\langle \nabla\left(\frac{\nabla f \circ \gamma}{\|\nabla f \circ \gamma\|}\right) \cdot \dot{\gamma}, \dot{\gamma}\right\rangle \\
&= -\left\langle \frac{\|\nabla f \circ \gamma\| \cdot (Hf \circ \gamma) - \nabla\|\nabla f \circ \gamma\| \cdot \nabla f \circ \gamma^T}{\|\nabla f \circ \gamma\|^2} \cdot \dot{\gamma}, \dot{\gamma}\right\rangle \\
&= -\frac{1}{\|\nabla f \circ \gamma\|} \cdot \langle (Hf \circ \gamma) \cdot \dot{\gamma}, \dot{\gamma}\rangle, \tag{3}
\end{aligned}$$

where Hf denotes the Hesse matrix of f. Since $\dot{\gamma}$ is the unit tangent vector and therefore orthogonal to N, we can rewrite it as

$$\dot{\gamma} = \pm \frac{1}{\|\nabla f \circ \gamma\|} \cdot \begin{pmatrix} -\partial_2 f \circ \gamma \\ \partial_1 f \circ \gamma \end{pmatrix}.$$

By inserting this into (3), we get the desired result, as the uncertainty of the sign cancels out through $\dot{\gamma}$ appearing twice. $\qquad\qquad\square$

2.2 The conformal monogenic signal

In [FWS11] the authors imposed a spherical geometry on the monogenic signal. By considering the Riesz transform of the stereographic projection of a signal in the monogenic scale-space they defined the *conformal monogenic signal*. To this end, they used a stereographic projection which maps the sphere without north pole.

We now follow their idea in [FWS11], introduce the main ingredients and consider their main example. However, we will see, that this approach to the searched isophote curvature bears various traps, including convergence issues.

Main example
Let
$$f_m(x) = a\cos(k|x - m| + \phi) \quad \text{for all } x \in \mathbb{R}^2,$$
where $m \in \mathbb{R}^2$, $a, k \in \mathbb{R}_+$ and $\phi \in [0, 2\pi)$.

In [FWS11], a spherical geometry was imposed on the monogenic signal of f_m. There, the *conformal monogenic signal* was defined by considering the Riesz transform of the stereographic projection of a signal in the monogenic scale-space.

Definition 2.17 (Stereographic projection)**.** Let
$$\mathbb{S}^2 := S^2_{\frac{1}{2}}\left(\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}\right) \setminus \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

be a sphere in $\mathbb{R}^3$ with radius $\frac{1}{2}$ and center $c = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ without north pole. The stereographic pro-

jection is defined by
$$\mathbb{S} : \mathbb{S}^2 \to \mathbb{R}^2, \quad u \mapsto \frac{1}{1 - u_3} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The inverse of the stereographic projection then reads

$$
\mathbb{S}^{-1} : \mathbb{R}^2 \to \mathbb{S}^2, \quad x \mapsto \frac{1}{1+|x|^2} \cdot \begin{pmatrix} x_1 \\ x_2 \\ |x|^2 \end{pmatrix}.
$$

For the study in [FWS11], the circular signal f_m and its embedding in the sphere $\mathbb{S}^2$ and in the Poisson-scale space were used. In addition, two variants of the conformal monogenic signal were considered. In the following, we state the formal definitions.

Definition 2.18. Let f_m be as in the main example. Denote

$$
g_m^x : \mathbb{R}^3 \to \mathbb{R}, \quad u \mapsto \begin{cases} f_m(\mathbb{S}(u) + x) & \text{for } u \in \mathbb{S}^2, \\ 0 & \text{else,} \end{cases}
$$

the spherical embedding of f_m. Then, the integral

$$
g_m^{x,s}(u) = \mathcal{P}_s[g_m^x](u) = \int_{\mathbb{R}^3} g_m^x(v) p_s(u-v) dm(v) \quad \text{for almost all } u \in \mathbb{S}^2 \tag{4}
$$

is called the Poisson scale-space embedding of g_m^x. Here $dm(v)$ denotes the Radon measure supported on the sphere $\mathbb{S}^2$, where c is the center of $\mathbb{S}^2$ and $\delta(\cdot)$ denotes the delta distribution:

$$
dm(v) = \delta\left(|v-c| - \frac{1}{2}\right) dv.
$$

Note that in [FWS11], the translation of $\frac{1}{2}$ is not printed, but certainly meant.

Now, we can simplify the integral (4) by restriction to the support of g_m^x. Hence,

$$
g_m^{x,s}(u) = \int_{\mathbb{S}^2} g_m^x(v) p_s(u-v) d\sigma(v), \quad \text{for all } u \in \mathbb{S}^2,
$$

where $d\sigma$ denotes the surface measure of $\mathbb{S}^2$.

Remark 2.19. It is possible to define the Poisson scale-space embedding $g_m^{x,s}$ on the complete three-dimensional space $\mathbb{R}^3$, but in [FWS11, p. 314] a restriction to $\mathbb{S}^2$ was chosen:

"...$g_m^{x,s}$ is supported on a two-dimensional surface in $\mathbb{R}^3$, ... "

The *conformal monogenic signal* was introduced as 3D monogenic signal of $g_m^{x,s}$. In fact, in [FWS11, equation (88)], there are two definitions of the conformal monogenic signal. The first one reads as follows.

Definition 2.20 (Conformal monogenic signal 1)**.** [FWS11, Definition 2, (88)] Let f_m be as defined in the main example. Then, for $s > 0$ in omitting the parameter m,

$$f_s^1(x) = \begin{pmatrix} g_m^{x,s}(0) \\ \mathbf{H}^1[g_m^{x,s}](0) \\ \mathbf{H}^2[g_m^{x,s}](0) \\ \mathbf{H}^3[g_m^{x,s}](0) \end{pmatrix} \quad \text{for almost all } x \in \mathbb{R}^2$$

is called *conformal monogenic signal.*

Remark 2.21. It is unclear, how the Riesz transform $\mathbf{H}^i[g_m^{x,s}]$ is exactly defined in the original paper [FWS11]. Since $g_m^{x,s}$ is supported on a two-dimensional surface due to Remark 2.19, which is a Lebesgue null set, the Riesz transform of $g_m^{x,s}$ ought to be 0. A similar case appears in [FWS11, p. 314] for the Radon transform. There the problem of vanishing integral transform was swiftly circumnavigated by considering a Radon measure on the lower dimensional manifold $\mathbb{S}^2$.

"Let **P** denote a plane in $\mathbb{R}^3$ with $\mathbf{C} = \mathbf{P} \cap \mathbb{S}^2$ such that $\mathbf{C} \neq \emptyset$. If we integrate over the plane **P**, we actually want to integrate $g_m^{x,s}$ over the circle **C**. Since this is a Lebesgue null set with respect to the standard Lebesgue measure in the plane we have to introduce an alternate measure. Instead, the Radon transform has to be understood with respect to the Radon measure $\delta(\mathbf{C}(u))du$ where $\mathbf{C}(u) = 0 \Leftrightarrow u \in \mathbf{C} \dots$"

We assume that the same reasoning applies in the case of the Riesz transform and the convolution with the Riesz kernel takes place on the sphere $\mathbb{S}^2$. Thus, for $i \in \{1,2,3\}$

$$\mathbf{H}^i[g_m^{x,s}](u) = \int_{\mathbb{S}^2} g_m^{x,s}(v) h^i(u - v) d\sigma(v) \quad \text{for almost all } x \in \mathbb{R}^2,$$

where as above $d\sigma(v)$ denotes the surface measure of $\mathbb{S}^2$.

The conjugated Poisson kernels were introduced in order to give an alternative definition of the *conformal monogenic signal.* [FWS11, Definition 2, (88)]

Definition 2.22 (Conjugated Poisson kernel and conformal monogenic signal 2)**.** Let

$$q_s^i(x) = \mathbf{H}^i[p_s](x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{s}{(|x|^2 + s^2)^{\frac{n+1}{2}}} \quad \text{for all } x \in \mathbb{R}^2, \ s > 0, \ i \in \{1,2,3\}.$$

For a signal $f \in L^\infty(\mathbb{R}^2)$, we define the linear operator

$$\mathbf{Q}_s^i[f] = q_s^i * f$$

and the second version of the *conformal monogenic signal* [FWS11, Definition 2, (88)]

$$
f_s^2(x) = \begin{pmatrix} \mathcal{P}_s[g_m^x](0) \\ \mathbf{Q}_s^1[g_m^x](0) \\ \mathbf{Q}_s^2[g_m^x](0) \\ \mathbf{Q}_s^3[g_m^x](0) \end{pmatrix} \quad \text{for almost all } x \in \mathbb{R}^2
$$

in omitting the parameter m. In this case, the operator $\mathbf{Q}_s^i$ with the same reasoning as in Remark 2.21 denotes an integral operator on the sphere $\mathbb{S}^2$

$$
\mathbf{Q}_s^i[g_m^x](u) = \int_{\mathbb{S}^2} g_m^x(v) q_s^i(u - v) d\sigma(v).
$$

In [FWS11, Definition 2] it reads that $f_s^1 = f_s^2$. In the following sections we will show that this does not hold.

As a last ingredient, we state the definition of the *conformal monogenic signal curvature* based on the conformal monogenic signal. We will distinguish between the curvatures related to the two different conformal monogenic signals in order to treat the cases seperately.

Definition 2.23 (Conformal monogenic signal curvature)**.** Let f_m be as in the main example. Then, for $j \in \{1, 2\}$, $s > 0$ and $x \in \mathbb{R}^2$,

$$
\kappa_{cms}^j(f_m)(s, x) := \frac{2 f_{s,4}^j(x)}{\sqrt{|f_{s,2}^j(x)|^2 + |f_{s,3}^j(x)|^2}} \tag{5}
$$

is called the *conformal monogenic signal curvature* for the monogenic signal f_s^j, whenever κ_{cms}^j is finite.

Equipped with these formal foundations, as they are given in [FWS11], we will show the following three items in the two subsequent sections:

1. The conformal monogenic signal curvature $\kappa_{cms}^1(f_m)(s, x)$ is not well defined. In fact, its numerator
$$
2 f_{s,4}^1(x) = \mathbf{H}^3[g_m^{x,s}](0) = \lim_{\varepsilon \to 0} \int_{\mathbb{S}^2 \setminus B_\varepsilon} g_m^{x,s}(v) h^3(u - v) d\sigma(v)
$$
diverges for all (x, s) in a set with positive Lebesgue measure.

2. In general, the two definitions for the conformal monogenic signal are not equal. I. e., $f_s^1 \neq f_s^2$.

3. The conformal monogenic signal curvature $\kappa_{cms}^2(f_m)(s, x)$ is not well defined either. Its denominator
$$
\sqrt{|f_{s,2}^j(x)|^2 + |f_{s,3}^j(x)|^2}
$$

vanishes on infinitely many circles centered at the origin.

These findings have the consequence that the isophote curvature - unfortunately - cannot be obtained by this approach.

2.3 The divergence of the monogenic signal f_s^1

In this section, we will show that the conformal monogenic signal f_s^1 as defined in Definition 2.20 is not finite and thus, the according conformal monogenic signal curvature $\kappa_{cms}(f)$ is not well defined. To this end, we will show that the third component of the Riesz transform on the sphere in general is not finite.

Theorem 2.24. *Let $g \in C(\mathbb{S}^2) \cap L^\infty(\mathbb{S}^2)$. If $g(0) \neq 0$, then $\mathbf{H}^3[g](0)$ is not finite.*

Proof. We have

$$\mathbf{H}^3[g](0) = \frac{1}{\pi^2} \cdot \text{P.V.} \int_{\mathbb{S}^2} \frac{0 - v_3}{|0 - v|^4} \cdot g(v) \, d\sigma(v).$$

By substituting $v = \mathbf{S}^{-1}(y) = \frac{1}{1+|y|^2} \begin{pmatrix} y_1 \\ y_2 \\ |y|^2 \end{pmatrix}$, we obtain the integration measure $d\sigma(v) = \frac{1}{(1+|y|^2)^2} \, dy$

and $|v|^2 = \frac{|y|^2}{1+|y|^2}$. After this substitution, we can handle the integral in the real plane instead of the sphere $\mathbb{S}^2$. Together with $v_3 = \frac{|y|^2}{1+|y|^2}$ this delivers

$$\mathbf{H}^3[g](0) = -\frac{1}{\pi^2} \cdot \text{P.V.} \int_{\mathbb{R}^2} \frac{g(\mathbf{S}^{-1}(y))}{|y|^2(1+|y|^2)} \, dy$$
$$= -\frac{1}{\pi^2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon} \frac{g(\mathbf{S}^{-1}(y))}{|y|^2(1+|y|^2)} \, dy.$$

Since $\mathbf{S}^{-1}(0) = 0$ and $g(0) \neq 0$ and due to the continuity of g there exist $\delta > 0$ and $c > 0$ such that

$$|g(\mathbf{S}^{-1}(y))| \geq c \quad \text{for all } y \in B_\delta. \tag{6}$$

Hence, we obtain

$$\mathbf{H}^3[g](0) = -\frac{1}{\pi^2} \cdot \lim_{\varepsilon \to 0} \Bigg(\underbrace{\int_{\mathbb{R}^2 \setminus B_\delta} \frac{g(\mathbf{S}^{-1}(y))}{|y|^2(1+|y|^2)} dy}_{=: I} + \underbrace{\int_{B_\delta \setminus B_\varepsilon} \frac{g(\mathbf{S}^{-1}(y))}{|y|^2(1+|y|^2)} dy}_{=: J} \Bigg).$$

Since $g \in C(\mathbb{S}^2) \cap L^\infty(\mathbb{S}^2)$, we also have $g \circ \mathbf{S}^{-1} \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Hence, the integral I is finite, as the pole of the integrand in the origin is cut out by a ball with radius δ. Because g does not

change its sign on B_δ we obtain for the integral J that

$$
\begin{aligned}
|J| &= \int_{B_\delta \setminus B_\varepsilon} \frac{|g(\mathbf{S}^{-1}(y))|}{|y|^2 (1 + |y|^2)} \, dy \\
&\overset{(6)}{\geq} \int_{B_\delta \setminus B_\varepsilon} \frac{c}{|y|^2 (1 + |y|^2)} \, dy \\
&= \int_\varepsilon^\delta \frac{c}{r^2 (1 + r^2)} \, r \, dr \\
&\geq \frac{c}{1 + \delta^2} \int_\varepsilon^\delta \frac{1}{r} \, dr \\
&= \frac{c}{1 + \delta^2} \cdot \left[\log \delta - \log \varepsilon \right].
\end{aligned}
$$

Hence,

$$
|\mathbf{H}^3[g](0)| = \frac{1}{\pi^2} \cdot \lim_{\varepsilon \to 0} |I + J| = \infty.
$$

Consequently, $\mathbf{H}^3[g](0)$ is not finite.

$\square$

Whenever we have $g_m^{x,s}(0) \neq 0$, the conformal monogenic signal $f_s^1(x)$ is not finite due to Theorem 2.24. Hence, the related conformal monogenic signal curvature $\kappa_{cms}^1(f_m)(s, x)$ is not well defined. In particular, there exist f_m as in the main example such that $g_m^{x,s}(0) \neq 0$. To see this, consider

$$
\begin{aligned}
g_m^{x,s}(0) &= \frac{1}{\pi^2} \int_{\mathbb{S}^2} f_m(x + \mathbf{S}(v)) \cdot \frac{s}{(|v|^2 + s^2)^2} \, d\sigma(v) \\
&= \frac{1}{\pi^2} \int_{\mathbb{R}^2} f_m(x + y) \cdot \frac{s}{\left(\frac{|y|^2}{1 + |y|^2} + s^2 \right)^2} \frac{dy}{(1 + |y|^2)^2} \\
&= \frac{1}{\pi^2} \int_{\mathbb{R}^2} f_m(x - y) \cdot \frac{s}{\left(s^2 + (1 + s^2)|y|^2 \right)^2} \, dy.
\end{aligned}
$$

The last integral actually is a convolution with a positive bell-shaped radially symmetric kernel.

2.4   The non-equality of the conformal monogenic signal curvature and the isophote curvature

In this section, we will show that Definition 2.20 and Definition 2.22 produce different conformal monogenic signals. Moreover, we will show that the conformal monogenic signal curvature κ_{cms}^2 is not well defined. To this end, we will represent the convolution with the conjugated Poisson kernel of a spherically embedded signal as an operation on the original planar signal, and this for all bounded signals f.

Lemma 2.25. *Let $f \in L^\infty(\mathbb{R}^2)$, $g^x(u) = f(S(u)+x)$ for all $x \in \mathbb{R}^2$ and $u \in \mathbb{S}^2$, let $s > 0$ and $i \in \{1,2,3\}$. Then,*

$$\mathbf{Q}_s^i[g^x](0) := \int_{\mathbb{S}^2} g^x(v) q_s^i(0-v) d\sigma(v) = \int_{\mathbb{R}^2} f(x-y) \tilde{q}_s^i(y) dy \quad \text{for almost all } x \in \mathbb{R}^2,$$

where

$$\tilde{q}_s^1(y) = \frac{y_1}{\pi^2(1+|y|^2)\left(s^2+(1+s^2)|y|^2\right)^2},$$

$$\tilde{q}_s^2(y) = \frac{y_2}{\pi^2(1+|y|^2)\left(s^2+(1+s^2)|y|^2\right)^2},$$

$$\tilde{q}_s^3(y) = -\frac{|y|^2}{\pi^2(1+|y|^2)\left(s^2+(1+s^2)|y|^2\right)^2}$$

for all $y \in \mathbb{R}^2$.

Proof. Let $i \in \{1,2,3\}$ and $s > 0$. Then, we have

$$\mathbf{Q}_s^i[g^x](0) = \frac{1}{\pi^2} \cdot \int_{\mathbb{S}^2} \frac{0-v_i}{(|-v|^2+s^2)^2} \cdot f(S(v)+x) \, d\sigma(v).$$

By substituting $v = S^{-1}(z)$, we obtain the integration measure $d\sigma(v) = \frac{1}{(1+|z|^2)^2} dz$ and $|v|^2 = \frac{|z|^2}{1+|z|^2}$. This yields

$$\mathbf{Q}_s^i[g^x](0) = -\frac{1}{\pi^2} \cdot \int_{\mathbb{R}^2} \frac{v_i(z)}{(|z|^2+s^2(1+|z|^2))^2} \cdot f(x+z) \, dz,$$

The desired result now follows by inserting the components of

$$v(z) = S^{-1}(z) = \frac{1}{1+|z|^2} \cdot \begin{pmatrix} z_1 \\ z_2 \\ |z|^2 \end{pmatrix}$$

and substituting $z = -y$. $\qquad\qquad\square$

With this result and Theorem 2.24, we can now prove that the concepts of the conformal monogenic signal in Definition 2.20 and Definition 2.22 are different for the main example f_m.

Lemma 2.26. *Let*

$$f_m(x) = a\cos(k|x-m|+\phi) \quad \text{for all } x \in \mathbb{R}^2,$$

where $m \in \mathbb{R}^2$, $a, k \in \mathbb{R}_+$ and $\phi \in [0, 2\pi)$. Then, the related conformal monogenic signal

$$f_s^2(x) = \begin{pmatrix} \mathcal{P}_s[g_m^x](0) \\ \mathbf{Q}_s^1[g_m^x](0) \\ \mathbf{Q}_s^2[g_m^x](0) \\ \mathbf{Q}_s^3[g_m^x](0) \end{pmatrix}$$

has finite entries for all $s > 0$ and for all $x \in \mathbb{R}^2$.

Proof. The signal f_m is bounded and continuous on $\mathbb{R}^2$ by construction: $f_m \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$. Hence, for g_m^x as in Definition 2.20, together with Lemma 2.25

$$\mathbf{Q}_s^i[g_m^x](0) = \int_{\mathbb{R}^2} f_m(x - y)\tilde{q}_s^i(y)dy \quad \text{for almost all } x \in \mathbb{R}^2.$$

Furthermore, $\tilde{q}_s^i \in L^1(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ for all $s > 0$ and $i \in \{1, 2, 3\}$. Hence, $(f_m * \tilde{q}_s^i) \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ due to Young's convolution inequality [Gra08] and so f_s^2 exists everywhere.   □

This result makes f_s^2 a well-defined candidate for further inquiries. However, as we will see in the following theorem, the denominator of the conformal monogenic signal curvature vanishes on infinitely many circles around the origin.

Theorem 2.27. *Let $f(x) = \cos(|x|)$ for all $x \in \mathbb{R}^2$ and let $g^x(u) = f(\mathbf{S}(u) + x)$ for all $u \in \mathbb{S}^2$ and $x \in \mathbb{R}^2$. Then there exists $\delta > 0$ such that for all $s \in (0, \delta)$ and for all $k \in \mathbb{N}$ there exists $\rho \in \left((2k-1) \cdot \frac{\pi}{2}, (2k+1) \cdot \frac{\pi}{2}\right)$ such that the denominator of the conformal monogenic signal curvature,*

$$\sqrt{|\mathbf{Q}_s^1[g^x](0)|^2 + |\mathbf{Q}_s^2[g^x](0)|^2} = 0 \quad \text{for all } x \in S_\rho^1(0).$$

Proof. First we introduce the operator

$$\mathbf{Q}_s^c := \mathbf{Q}_s^1 + i \cdot \mathbf{Q}_s^2.$$

With this new operator we can rewrite the denominator of $\kappa_{cms}[f](s, x)$ as

$$\sqrt{|\mathbf{Q}_s^1[g^x](0)|^2 + |\mathbf{Q}_s^2[g^x](0)|^2} = |\mathbf{Q}_s^c[g^x](0)|.$$

We will now check for which $x \in \mathbb{R}^2$ we have $\mathbf{Q}_s^c[g^x](0) = 0$.

$$\begin{aligned} \mathbf{Q}_s^c[g^x](0) &= \mathbf{Q}_s^1[g^x](0) + i \cdot \mathbf{Q}_s^2[g^x](0) \\ &= \int_{\mathbb{R}^2} \cos(|x - y|) \cdot \frac{y_1 + iy_2}{\pi^2(1 + |y|^2)\left(s^2 + (1 + s^2)|y|^2\right)} \, dy. \end{aligned}$$

In the next step, we transform x and y into polar coordinates, i.e.

$$x = \rho \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad y = r \cdot \begin{pmatrix} \cos(\varphi + \theta) \\ \sin(\varphi + \theta) \end{pmatrix},$$

with $r, \rho > 0$ and $\theta, \varphi \in [0, 2\pi)$. With this substitution, we obtain

$$
\begin{aligned}
&\mathbf{Q}_s^c[g^x](0) \\
&= \frac{1}{\pi^2(1+s^2)^2} \int_0^\infty \int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \frac{re^{i(\varphi+\theta)}}{(1+r^2)\left(\frac{s^2}{1+s^2} + r^2\right)} \, d\varphi \, r \, dr \\
&= \frac{e^{i\theta}}{\pi^2(1+s^2)^2} \int_0^\infty \int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \frac{r^2 e^{i\varphi}}{(1+r^2)\left(\frac{s^2}{1+s^2} + r^2\right)} \, d\varphi \, dr \\
&= \frac{e^{i\theta}}{\pi^2(1+s^2)^2} \cdot \left(\int_0^\infty \int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \frac{r^2 \cos\varphi}{(1+r^2)\left(\frac{s^2}{1+s^2} + r^2\right)} \, d\varphi \, dr \right. \\
&\quad \left. + i \cdot \int_0^\infty \int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \frac{r^2 \sin\varphi}{(1+r^2)\left(\frac{s^2}{1+s^2} + r^2\right)} \, d\varphi \, dr \right).
\end{aligned}
$$

The upper equation can be simplified as the following integral over φ vanishes:

$$
\begin{aligned}
&\int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \sin\varphi \, d\varphi \\
&= \int_0^\pi \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \sin\varphi \, d\varphi + \int_\pi^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \sin\varphi \, d\varphi \\
&= \int_0^\pi \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \sin\varphi \, d\varphi \\
&\quad + \int_0^\pi \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot (-\sin\varphi) \, d\varphi = 0.
\end{aligned}
$$

Thus we obtain

$$
\mathbf{Q}_s^c[g^x](0) = \frac{e^{i\theta}}{\pi^2(1+s^2)^2} \cdot \int_0^\infty \underbrace{\int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \frac{r^2 \cos\varphi}{(1+r^2)\left(\frac{s^2}{1+s^2} + r^2\right)^2} \, d\varphi}_{=: \ell_{s,\rho}(r)} \, dr.
$$

In order to prove the existence of $\rho > 0$ such that $\mathbf{Q}_s^c[g^x](0) = 0$, we exploit the fact that

$$e^{-i\theta} \mathbf{Q}_s^c[g^x] \in \mathbb{R} \quad \text{for } x = \rho \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{for all } \rho > 0, \; \theta \in [0, 2\pi).$$

Using this fact, we will show that there exist $\rho_+, \rho_- > 0$ such that for $x_+ = \rho_+ \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $x_- =$

$\rho_- \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ we have

$$e^{-i\theta} \mathbf{Q}_s^c[g^{x_+}](0) > 0 \quad \text{and} \quad e^{-i\theta} \mathbf{Q}_s^c[g^{x_-}](0) < 0.$$

and apply the intermediate value theorem. To show this, we introduce the function

$$w_{r,\rho}(\varphi) := \cos\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) \cdot \cos\varphi$$

and prove that there exist $\rho_+, \rho_- > 0$ and $\varepsilon > 0$ such that

$$\int_0^{2\pi} w_{r,\rho_+}(\varphi)\, d\varphi > 0 \quad \text{and} \quad \int_0^{2\pi} w_{r,\rho_-}(\varphi)\, d\varphi < 0 \quad \text{for all } r \in (0, \varepsilon).$$

For this purpose, we first observe the behavior of the integral for $r = 0$:

$$\int_0^{2\pi} w_{0,\rho}(\varphi)\, d\varphi = \int_0^{2\pi} \cos\rho \cdot \cos\varphi\, d\varphi = 0. \tag{7}$$

So we have to consider the first derivative with respect to r in order to determine the sign of the integral for small r. With the Leibniz integral rule and the dominated convergence theorem we obtain

$$\partial_r \int_0^{2\pi} w_{r,\rho}(\varphi)\, d\varphi \bigg|_{r=0} = \int_0^{2\pi} \partial_r w_{r,\rho}(\varphi)\, d\varphi \bigg|_{r=0}$$

$$= \int_0^{2\pi} (-r - \rho\cos\varphi)\cos\varphi \cdot \text{sinc}\left(\sqrt{r^2 - 2r\rho\cos\varphi + \rho^2}\right) d\varphi \bigg|_{r=0}$$

$$= -\int_0^{2\pi} \rho\cos^2\varphi \cdot \text{sinc}\,\rho\, d\varphi = -\pi\sin\rho. \tag{8}$$

Now let $\rho_+, \rho_- > 0$ such that $\sin\rho_+ < 0$ and $\sin\rho_- > 0$. Then there exists $\varepsilon > 0$ such that

$$\int_0^{2\pi} w_{r,\rho_+}(\varphi)\, d\varphi > 0 \quad \text{and} \quad \int_0^{2\pi} w_{r,\rho_-}(\varphi)\, d\varphi < 0 \quad \text{for all } r \in (0, \varepsilon).$$

Let $\rho_+ > 0$ be chosen as above. Then the function

$$\ell_{s,\rho_+}(r) = \int_0^{2\pi} w_{r,\rho_+}(\varphi)\, d\varphi \cdot \frac{r^2}{(1 + r^2)\left(\frac{s^2}{1+s^2} + r^2\right)^2}$$

is positive and monotonically increasing for $s \to 0$ for all $r \in (0, \varepsilon)$. We thus obtain by applying the monotone convergence theorem that

$$
\begin{aligned}
\lim_{s \to 0} \int_0^\varepsilon \ell_{s,\rho_+}(r) \, dr &= \int_0^\varepsilon \lim_{s \to 0} \ell_{s,\rho_+}(r) \, dr \\
&= \int_0^\varepsilon \int_0^{2\pi} \cos\left(\sqrt{r^2 - 2r\rho_+ \cos\varphi + \rho_+^2}\right) \cdot \frac{\cos\varphi}{(1 + r^2)r^2} \, d\varphi \\
&= \int_0^\varepsilon \int_0^{2\pi} w_{r,\rho_+}(\varphi) \, d\varphi \, \frac{1}{r^2(1 + r^2)} \, dr
\end{aligned}
$$

Due to (7) and (8), the integrand $\ell_{s,\rho_+}(r)$ has a pole of order 1 in the origin and hence

$$
\lim_{s \to 0} \int_0^\varepsilon \ell_{s,\rho_+}(r) \, dr = \infty.
$$

By splitting up the integral that defines $\mathbf{Q}_s^c[g^x](0)$

$$
\begin{aligned}
e^{-i\theta} \pi^2 (1 + s^2)^2 \cdot \mathbf{Q}_s^c[g^{x_+}](0) &= \int_0^\infty \ell_{s,\rho_+}(r) \, dr \\
&= \int_0^\varepsilon \ell_{s,\rho_+}(r) \, dr + \int_\varepsilon^\infty \ell_{s,\rho_+}(r) \, dr
\end{aligned}
$$

for $x_+ = \rho_+ \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, we obtain

$$
\lim_{s \to 0} e^{-i\theta} \pi^2 (1 + s^2)^2 \cdot \mathbf{Q}_s^c[g^{x_+}](0) = \underbrace{\lim_{s \to 0} \int_0^\varepsilon \ell_{s,\rho_+}(r) \, dr}_{=\infty} + \lim_{s \to 0} \int_\varepsilon^\infty \ell_{s,\rho_+}(r) \, dr.
$$

In order to prove that $\lim_{s \to 0} \mathbf{Q}_s^c[g^{x_+}](0) = \infty$, it suffices to show that

$$
\left| \lim_{s \to 0} \int_\varepsilon^\infty \ell_{s,\rho_+}(r) \, dr \right| < \infty.
$$

This can be proved by considering

$$
\begin{aligned}
\left| \lim_{s \to 0} \int_\varepsilon^\infty \ell_{s,\rho_+}(r) \, dr \right| &\leq \int_\varepsilon^\infty \int_0^{2\pi} \left| \cos\left(\sqrt{r^2 - 2r\rho_+ \cos\varphi + \rho_+^2}\right) \cdot \frac{\cos\varphi}{r^2(1 + r^2)} \right| \, d\varphi \, dr \\
&\leq 2\pi \int_\varepsilon^\infty \frac{1}{r^2(1 + r^2)} \, dr < \infty.
\end{aligned}
$$

Hence, we have shown that $\lim_{s \to 0} \mathbf{Q}_s^c[g^{x_+}](0) = \infty$. Thus, there exists $\tilde{\delta} > 0$ such that

$$
\mathbf{Q}_s^c[g^{x_+}](0) > 0 \quad \text{for all } s \in (0, \tilde{\delta}).
$$

Similarly we obtain that there exists $\delta' > 0$ such that

$$\mathbf{Q}_s^c[g^{x_-}](0) < 0 \quad \text{for all } s \in (0, \delta')$$

for $x_- = \rho_- \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\sin\rho_- > 0$. By choosing $\delta := \min\{\tilde{\delta}, \delta'\}$, we get

$$\mathbf{Q}_s^c[g^{x_+}](0) > 0 \quad \text{and} \quad \mathbf{Q}_s^c[g^{x_-}](0) < 0 \quad \text{for all } s \in (0, \delta).$$

Since the value of $e^{-i\theta}\mathbf{Q}_s^c[g^x](0)$ is independent of θ, the same applies to $|\mathbf{Q}_s^c[g^x](0)|$. Hence, by applying the intermediate value theorem, we obtain that there exists $\rho_0 > 0$ between ρ_+ and ρ_- such that for $x_0 = \rho_0 \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ we have

$$\mathbf{Q}_s^c[g^{x_0}](0) = 0 \quad \text{for all } \theta \in [0, 2\pi).$$

Consequently, for all $s \in (0, \delta)$ and for all $k \in \mathbb{N}$ there exists $\rho_0 \in \left((2k-1) \cdot \frac{\pi}{2}, (2k+1) \cdot \frac{\pi}{2}\right)$ such that

$$\mathbf{Q}_s^c[g^x](0) = 0 \quad \text{for all } x \in S_{\rho_0}^1(0).$$

$\square$

Due to these issues with the approach of the conformal monogenic signal curvature, we looked for other methods to determine curvature in images.

Edge curvature and the parabolic Fourier transform

The focus of this chapter lies on the parabolic Radon transform, the thereof derived parabolic Fourier transform and their connection to the local curvature of edges. We begin with a mathematical treatment of the subject of edges by giving an overview over microlocal analysis with a focus on the central concept of the wavefront set. Subsequently, we introduce the classical Radon transform with its most fundamental properties and show a relation to the wavefront set.

In the next section we consider the parabolic Radon transform as an extension of the classical Radon transform and show its most important properties. In the following section, we prove that the local curvature of the edges of the characteristic function of so called admissible sets can be determined by the smoothness of its parabolic Radon transform.

Via an analogue of the Fourier slice theorem, we introduce the parabolic Fourier transform in the fourth section and highlight some of its properties including its inherent relation to the Fresnel transform. We subsequently show that the local curvature of an edge of admissible functions can be detected by observing the decay rate of its parabolic Fourier transform. In order to prove this result, we show a connection between the smoothness of a special class of functions with isolated singularities and the decay rate of its Fourier transform.

3.1 Edges and their curvature

The human visual system possesses phenomenal capacities of recognizing objects in the surrounding environment and putting them into a meaningful context. Consequently, the research of the natural processes taking place in the visual cortex and their conversion into implementable algorithms for artificial object recognition was and still is of great interest. Among other research, the influential pioneer work of Hubel and Wiesel [HW62, HW68] on the visual system of cats rsp. monkeys already indicated that a great portion of the neurons (so called

simple cells) react to the stimulus of lines at certain angles. Hence, they have the functioning of edge detectors and provide means of recognizing rapid changes in an image along differently oriented lines. For their contributions to the research of the information processing of the visual system, Hubel and Wiesel received the Nobel Prize in Medicine in 1981.

Subsequent to their studies, many research groups in information sciences developed image processing algorithms resembling the functioning of the visual system. Thereby they incorporated edge detectors in a preprocessing step (leading to a *raw primal sketch*) to highlight e. g. the boundaries of an object, thus making it easier to categorize and recognize objects [Mar76, MH80]. Indeed, edge detection still plays an important role in today's research in the fields of computer vision and image processing.

In the research of the significance of edges for the human visual system, it has been found that the curvature of edges also plays a role in the recognition of objects [Att54, AA56]. The importance of edge curvature was subsequently also discovered in artificial object recognition and many algorithms in this field indeed incorporate the curvature of an edge, e. g. [DZM⁺07, MEO11, FB14].

In order to allow for a mathematical treatment of edges, we first need to properly define what exactly is meant by this term. To this end, we generally interpret an image as a function $f : \mathbb{R}^2 \to \mathbb{R}$, thus treating it as a gray scale image. In this context, edges are curves consisting of singularities of f. For a precise terminology of this subject, we summarize some of the basic principles of microlocal analysis, following the course of [Hör83] and [Rud91].

Definition 3.1 (Rapid decay)**.** Let $X \subset \mathbb{C}^n$ be an open unlimited set and let $f : \mathbb{C}^n \to \mathbb{C}$. We say that f decays rapidly in X if

$$\forall\, N \in \mathbb{N} \ \exists\, C, L > 0 : \ |f(x)| \le C \cdot (1 + |x|)^{-N} \ \text{ for all } x \in X \ \Big|\ |x| \ge L.$$

Definition 3.2 (Function and distribution spaces)**.** We will denote the space of *smooth functions* as $\mathcal{E}(\mathbb{R}^n) := C^\infty(\mathbb{R}^n)$ and its dual space, the space of *compactly supported distributions* as $\mathcal{E}'(\mathbb{R}^n)$. Furthermore, the space of *smooth functions with compact support* will be denoted as

$$\mathcal{D}(\mathbb{R}^n) = \big\{ f \in \mathcal{E}\left(\mathbb{R}^n\right) \,|\, \mathrm{supp}(f) \text{ is compact} \big\}$$

and the corresponding dual space as $\mathcal{D}'(\mathbb{R}^n)$. Furthermore, the space of *locally integrable functions* is defined as

$$L^1_{\mathrm{loc}}(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \ \Big|\ \int_K |f(x)|\, dx < \infty \ \forall\, K \subset \mathbb{R}^n \text{ compact} \right\}.$$

For $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the mapping

$$\Lambda_h : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}, \quad \phi \mapsto \langle h, \phi \rangle.$$

A distribution which can be represented in the form Λ_h, is called a *regular distribution*.

Theorem 3.3. *For $h \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the map $\Lambda_h \in \mathcal{D}'(\mathbb{R}^n)$.*

Proof. See [Rud91, Theorem 6.8]. □

For the relation of the Fourier transform with $\mathcal{S}$ and $\mathcal{S}'$, we refer the reader to Section 1.3.2.

The Fourier transform of a compactly supported distribution can be defined pointwise.

Theorem 3.4. *The Fourier transform of a distribution $u \in \mathcal{E}'(\mathbb{R}^n)$ is a regular distribution, i.e. $\tilde{u} = \Lambda_{\hat{u}}$, where*

$$\hat{u}(\xi) = u(e_\xi) \quad \forall\, \xi \in \mathbb{R}^n$$

with $e_\xi : \mathbb{R}^n \to \mathbb{R}, x \mapsto e^{-i\langle x, \xi \rangle}$.

Proof. See [Hör83, Theorem 7.1.14]. □

The main idea in this proof is that the function $e_\xi : \mathbb{R}^n \to \mathbb{R}, x \mapsto e^{-i\langle x, \xi \rangle}$ is in $\mathcal{E}(\mathbb{R}^n)$ and hence $u(e^{-i\langle \cdot, \xi \rangle})$ is finite for all $\xi \in \mathbb{R}^n$.

The definition of the Fourier transform for tempered distributions now allows the introduction of the *singular support* which describes the set of all singularities.

Definition 3.5 (Singular Support)**.** The singular support of $f \in \mathcal{D}'(\mathbb{R}^n)$ is defined as

$$\operatorname{sing\,supp}(f) := \left\{ x \in \mathbb{R}^n \mid \forall\, \phi \in \mathcal{D}(\mathbb{R}^n) \text{ s. t. } \phi(x) \neq 0 : \widehat{\Lambda_\phi f} \text{ does not decay rapidly in } \mathbb{R}^n \right\}.$$

Since $\phi \in \mathcal{D}(\mathbb{R}^n)$ has compact support, $\Lambda_\phi f \in \mathcal{E}'(\mathbb{R}^n)$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$, $f \in \mathcal{D}'(\mathbb{R}^n)$, as $\Lambda_\phi f$ is compactly supported, as well. Hence, the Fourier transform $\widehat{\Lambda_\phi f}$ in the upper expression is defined pointwise due to Theorem 3.4.

We will now illustrate the concept of the singular support with two examples.

Example 3.6. Let $\delta \in \mathcal{D}'(\mathbb{R}^n)$ be the *Dirac distribution* defined by

$$\delta(\phi) = \phi(0) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n).$$

The singular support of δ can be determined via Definition 1.32:

$$\widehat{\Lambda_\phi \delta}(\xi) = \Lambda_\phi \delta(e_\xi) = \delta(\phi \cdot e_\xi) = \phi(0) \quad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n).$$

Thus, $\widehat{\Lambda_\phi \delta}$ decays rapidly if and only if $\phi(0) = 0$. Consequently,

$$\operatorname{sing\,supp}(\delta) = \{0\}.$$

Example 3.7. Let $\omega \in S^{n-1}$ and let $\delta^\omega \in \mathcal{D}'(\mathbb{R}^n)$ be defined by

$$\delta^\omega(\phi) = \int_{\omega^\perp} \phi(x)\,dx \quad \forall\, \phi \in \mathcal{D}(\mathbb{R}^n),$$

where $\omega^\perp$ denotes the orthogonal complement of ω. The singular support of δ^ω can be determined by

$$\widehat{\Lambda_\phi \delta^\omega}(\xi) = \Lambda_\phi \delta^\omega(e_\xi) = \delta^\omega(\phi \cdot e_\xi) = \int_{\omega^\perp} \phi(x) e^{-i\langle x, \xi\rangle} dx.$$

By decomposing $\xi \in \mathbb{R}^n$ into $\xi = \zeta + t\omega$ with $t \in \mathbb{R}$ and $\zeta \in \omega^\perp$, we can rewrite this Fourier transform into

$$\widehat{\Lambda_\phi \delta^\omega}(\xi) = \int_{\omega^\perp} \phi(x) e^{-i\cdot\langle x, \zeta + t\omega\rangle} dx = \int_{\omega^\perp} \phi(x) e^{-i\cdot\langle x, \zeta\rangle} dx = \widehat{\Lambda_\phi \delta^\omega}(\zeta).$$

Hence, $\widehat{\Lambda_\phi \delta^\omega}(\zeta + t\omega)$ decays rapidly for all $t \in \mathbb{R}$ if and only if $\widehat{\Lambda_\phi \delta^\omega}(\zeta) = 0$. If there exists an $x \in \omega^\perp$ such that $\phi(x) \neq 0$, there is a $\zeta \in \omega^\perp$ such that $\widehat{\Lambda_\phi \delta^\omega}(\zeta) \neq 0$ due to the uniqueness theorem of the Fourier transform. Then, $\widehat{\Lambda_\phi \delta^\omega}$ does not decay rapidly in $\mathbb{R}^n$. Consequently,

$$\operatorname{sing\,supp}(\delta^\omega) = \omega^\perp.$$

As we also want to discern between different orientations of edges, we have to introduce a directional component of the smoothness. This can be done by observing the decay rate of the Fourier transform of a function on a cone.

Definition 3.8 (Conic neighborhood, $\mathbb{R}^n_\times$)**.** Let $\mathbb{R}^n_\times = \mathbb{R}^n \setminus \{0\}$ and $\xi \in \mathbb{R}^n_\times$. We call $\Gamma \subset \mathbb{R}^n$ a conic neighborhood of ξ, if Γ is a neighborhood of ξ and $c \cdot \Gamma \subset \Gamma$ for all $c > 0$.

In order to get a basic understanding of the orientation of edges, we now introduce the notion of the *frequency set*. The idea behind it is that basically the Fourier transform translates the smoothness of a function into a more or less fast decay rate. Hence, in order to obtain a directionality of the smoothness, we can take a look at the decay rate of the Fourier transform in certain directions.

Definition 3.9 (Frequency Set)**.** Let $f \in \mathcal{E}'(\mathbb{R}^n)$. Then, the frequency set of f is defined as

$$\Sigma(f) := \left\{ \xi \in \mathbb{R}^n_\times \mid \forall \text{ conic neighborhoods } \Gamma \text{ of } \xi : \ \hat{f} \text{ does not decay rapidly on } \Gamma \right\}.$$

Note that $f \in \mathcal{E}'(\mathbb{R}^n)$ is necessary in the definition of the frequency set to ensure that $\hat{f}$ is defined pointwise. A fundamental property of the frequency set is described by the following lemma.

Lemma 3.10. *Let $f \in \mathcal{E}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then,*

$$\Sigma\left(\Lambda_\phi f\right) \subset \Sigma(f).$$

Proof. See [Hör83, Lemma 8.1.1]. □

In other words, the multiplication with a smooth function cannot introduce new directions to the frequency set. We will use this lemma to introduce the localized frequency set.

Definition 3.11 (Localized Frequency Set)**.** Let $f \in \mathcal{D}'(\mathbb{R}^n)$. The localized frequency set of f at $x \in \mathbb{R}^n$ is defined as

$$\Sigma_x(f) := \bigcap_{\substack{\phi \in \mathcal{D}(\mathbb{R}^n) \\ \phi(x) \neq 0}} \Sigma(\Lambda_\phi f).$$

Now we have the necessary terminology to define the crucial concept of the *wavefront set*.

Definition 3.12 (Wavefront Set)**.** Let $f \in \mathcal{D}'(\mathbb{R}^n)$. The wavefront set of f is defined as

$$\mathrm{WF}(f) := \left\{ (x, \xi) \in \mathrm{sing\,supp}(f) \times \mathbb{R}^n_\times : \xi \in \Sigma_x(f) \right\}.$$

In order to get a feeling for the wavefront set we now look at the two examples that we have already used for the illustration of the singular support.

Example 3.13. Let $\delta \in \mathcal{D}'(\mathbb{R}^n)$ be the Dirac distribution. From Example 3.6, we already know that

$$\mathrm{sing\,supp}(\delta) = \{0\}.$$

Hence, we obtain that

$$\mathrm{WF}(\delta) \subset \{0\} \times \mathbb{R}^n_\times.$$

In order to precisely determine the wavefront set of δ, we identify the localized frequency set $\Sigma_0(f)$. So let $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi(0) \neq 0$. Since $\Lambda_\phi \delta \in \mathcal{E}'(\mathbb{R}^n)$, we have $\widehat{\Lambda_\phi \delta}(\xi) = (\Lambda_\phi \delta)(e_\xi)$ for all $\xi \in \mathbb{R}^n$ due to Theorem 3.4. Hence, we obtain that

$$\widehat{\Lambda_\phi \delta}(\xi) = \delta(\phi e_\xi) = \phi(0) \neq 0 \quad \text{by requirement} \quad \forall \xi \in \mathbb{R}^n_\times.$$

We thus get $\mathrm{WF}(\delta) = \{0\} \times \mathbb{R}^n_\times$.

Example 3.14. Let $\omega \in S^{n-1}$ and let $\delta^\omega \in \mathcal{D}'(\mathbb{R}^n)$ be defined as in Example 3.7. From this example we already know that

$$\mathrm{sing\,supp}(\delta^\omega) = \omega^\perp.$$

Thus, we get that

$$\mathrm{WF}(\delta^\omega) \subset \omega^\perp \times \mathbb{R}^n_\times.$$

In order to obtain the precise form of $\mathrm{WF}(\delta^\omega)$, we fix $x_0 \in \mathrm{sing\,supp}(\delta^\omega) = \omega^\perp$ and determine the localized frequency set $\Sigma_{x_0}(\delta^\omega)$. To this end, let $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\phi(x_0) \neq 0$. By examining the restriction of ϕ to the hyper plane $\omega^\perp$, the uniqueness theorem Theorem 1.33 yields that there exists $\zeta \in \omega^\perp$ such that

$$\widehat{\Lambda_\phi \delta^\omega}(\zeta) = \int_{\omega^\perp} \phi(x) \cdot e^{-i\langle x, \zeta \rangle} \, dx = \left(\phi\big|_{\omega^\perp} \right)^\wedge (\zeta) \neq 0.$$

By choosing $\xi_t := \zeta + t \cdot \omega$ for $t \in \mathbb{R}$, we obtain

$$\widehat{\Lambda_\phi \delta^\omega}(\xi_t) = \int_{\omega^\perp} \phi(x) \cdot e^{-i\langle x, \zeta + t\omega \rangle} \, dx = \int_{\omega^\perp} \phi(x) \cdot e^{-i\langle x, \zeta \rangle} \, dx \neq 0 \quad \forall t \in \mathbb{R}. \tag{1}$$

Now let $\Gamma \subset \mathbb{R}^n$ be a conical neighborhood of ω and $-\omega$. As

$$\lim_{t \to \pm\infty} \frac{\xi_t}{\|\xi_t\|} = \pm\omega,$$

there exists $T > 0$ such that $\xi_t \in \Gamma$ for all $|t| > T$. Due to (1), $\widehat{\Lambda_\phi \delta^\omega}$ does not decay rapidly on Γ. As $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ was chosen arbitrarily, we obtain

$$\text{span}(\omega) \subset \Sigma_{x_0}(\delta^\omega).$$

Since $x_0 \in \omega^\perp$ was chosen arbitrarily, as well, we have

$$\omega^\perp \times \text{span}(\omega) \subset \text{WF}(\delta^\omega).$$

We now prove that $\omega^\perp \times \text{span}(\omega) = \text{WF}(\delta^\omega)$ by showing that for $x_0 \in \omega^\perp$ and $\xi \in \mathbb{R}^n \setminus \text{span}(\omega)$,

$$(x_0, \xi) \notin \text{WF}(\delta^\omega).$$

As $\xi \notin \text{span}(\omega)$, there exists $\zeta \in \omega^\perp \setminus \{0\}$ and $t \in \mathbb{R}$ such that $\xi = \zeta + t \cdot \omega$. Hence, there exists $\varepsilon > 0$ such that $\|\zeta\| > 2\varepsilon \cdot |t|$. Now let Γ be a conic neighborhood of ξ such that

$$\Gamma \cap \left\{ \zeta' + t' \cdot \omega \in \mathbb{R}^n \,\middle|\, \zeta' \in \omega^\perp \wedge t' \in \mathbb{R} \wedge |\zeta'| \leq \varepsilon \cdot |t| \right\} = \emptyset. \tag{2}$$

Let $\{\xi_k\}_{k \in \mathbb{N}} \subset \Gamma$ such that $\|\xi_k\| \xrightarrow{k \to \infty} \infty$. After an application of the decomposition

$$\xi_k = \zeta_k + t_k \cdot \omega, \quad \text{where } \zeta_k \in \omega^\perp, t_k \in \mathbb{R} \quad \text{for all } k \in \mathbb{N},$$

(2) yields

$$\|\xi_k\|^2 = \|\zeta_k\|^2 + |t_k|^2 < \left(1 + \frac{1}{\varepsilon^2}\right) \cdot \|\zeta_k\|^2.$$

Thus, $\|\zeta_k\| \xrightarrow{k \to \infty} \infty$. Since

$$\widehat{\Lambda_\phi \cdot \delta^\omega}(\xi_k) = \widehat{\Lambda_\phi \cdot \delta^\omega}(\zeta_k) = \left(\phi\Big|_{\omega^\perp}\right)(\zeta_k),$$

and $\phi\Big|_{\omega^\perp} \in \mathcal{D}(\omega^\perp)$, the function $\widehat{\Lambda_\phi \cdot \delta^\omega}$ decays rapidly in Γ. Hence,

$$\xi \notin \Sigma_{x_0}(\delta^\omega).$$

As $\xi \notin \text{span}(\omega)$ was chosen arbitrarily, we can conclude that

$$\Sigma_{x_0}(\delta^\omega) = \text{span}(\omega)$$

and thus

$$\text{WF}(\delta^\omega) = \omega^\perp \times \text{span}(\omega).$$

We will now establish a relation between the wavefront set of a function and its Radon transform. To this end, we introduce the Radon transform, which is widely used in applications as computer tomography and electron microscopy to model the image formation of an object in parallel scanning geometry [NW01, 3.1.1].

Definition 3.15 (Cylinder, Radon transform)**.** [NW01, 2.1] Let $n \in \mathbb{N}$. Then, the unit cylinder is defined as

$$Z^n := S^{n-1} \times \mathbb{R}$$

and the Schwartz space on the cylinder Z^n reads

$$\mathcal{S}(Z^n) := \left\{ g \in C^\infty(Z^n) : \forall \, k, \ell \in \mathbb{N}_0 \sup_{(\omega, s) \in Z^n} |s|^k \partial_s^\ell g(\theta, s) < \infty \right\}.$$

Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\omega \in S^{n-1}$ and $s \in \mathbb{R}$. Furthermore, let $\omega^\perp$ denote the hyper plane through the origin, perpendicular to ω. The Radon transform of f is then defined as

$$\mathcal{R}f(\omega, s) = \int_{\omega^\perp} f(s\omega + y) \, dy.$$

The following theorem shows that the Radon transform bijectively maps Schwartz functions in $\mathbb{R}^n$ onto Schwartz functions on the cylinder.

Theorem 3.16 (The Schwartz theorem)**.** *[Hel10] The Radon transform on $\mathbb{R}^n$ is a linear one-to-one mapping of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(Z^n)$.*

The mapping properties of the Radon transform allow us to derive the well know Fourier slice theorem which establishes a between Radon and Fourier transform.

Theorem 3.17. *[NW01] Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\omega \in S^{n-1}$. Then,*

$$\hat{f}(\rho \cdot \omega) = \left(\mathcal{R}f(\omega, \bullet) \right)^\wedge (\rho) \quad \text{for all } \rho \in \mathbb{R}.$$

In the next two propositions, we will prove that the Radon transform has similar mapping properties for $L^1(\mathbb{R}^n)$ and that the Fourier slice theorem can be extended to $L^1(\mathbb{R}^n)$.

Proposition 3.18. *The Radon transform $\mathcal{R} : L^1(\mathbb{R}^n) \to L^1(Z^n)$ is linear and continuous.*

Proof. Let σ be the surface measure on S^{n-1}. Then,

$$\int_{S^{n-1}} \int_{\mathbb{R}} \left| \mathcal{R}f(\omega, s) \right| \, ds \, d\sigma(\omega) = \int_{S^{n-1}} \int_{\mathbb{R}} \left| \int_{\omega^\perp} f(s\omega + y) \, dy \right| \, ds \, d\sigma(\omega)$$

$$\leq \int_{S^{n-1}} \int_{\mathbb{R}} \int_{\omega^\perp} \left| f(s\omega + y) \right| \, dy \, ds \, d\sigma(\omega)$$

$$= \int_{S^{n-1}} \|f\|_{L^1} \, d\sigma(\omega) = \sigma(S^{n-1}) \cdot \|f\|_{L^1}.$$

$\square$

Proposition 3.19. *Let $f \in L^1(\mathbb{R}^n)$, $\omega \in S^{n-1}$. Then,*

$$\hat{f}(\rho \cdot \omega) = \left(\mathcal{R}f(\omega, \bullet)\right)^{\wedge}(\rho) \quad \textit{for all } \rho \in \mathbb{R}.$$

Proof. Proposition 3.18 yields that $\mathcal{R}f(\omega, \bullet) \in L^1(\mathbb{R})$. Hence, $\hat{f}$ and $\left(\mathcal{R}f(\omega, \bullet)\right)^{\wedge}$ are bounded and uniformly continuous due to Theorem 1.20 and Theorem 1.21.

Thus, for all $\rho \in \mathbb{R}$ and $\omega \in S^{n-1}$,

$$\begin{aligned}
\left(\mathcal{R}f(\omega, \bullet)\right)^{\wedge}(\rho) &= \int_{\mathbb{R}} \mathcal{R}f(\omega, s)e^{-i\rho s}\, ds \\
&= \int_{\mathbb{R}} \int_{\omega^{\perp}} f(s\omega + y)\, dy\, e^{-i\rho s}\, ds \\
&= \int_{\mathbb{R}^n} f(x)e^{-i\langle x, \rho\omega\rangle}\, dx \\
&= \hat{f}(\rho\omega).
\end{aligned}$$

$\square$

By utilizing Proposition 3.19 and using that $\phi \cdot f \in L^1(\mathbb{R}^n)$ for all $f \in L^1_{\text{loc}}$, $\phi \in \mathcal{D}(\mathbb{R}^n)$, we can establish a connection between the wavefront set and the Radon transform. By rewriting the definition of the wavefront set using the Fourier slice theorem, we can say that $(x, \xi) \notin WF(f)$ for an $f \in L^1_{\text{loc}}$, if and only if there exists $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi(x) \neq 0$ and a neighborhood U of ω_0 on S^{n-1} such that for all $N \in \mathbb{N}$ there is a constant $C > 0$ with

$$\left|\left(\mathcal{R}f(\omega, \bullet)\right)^{\wedge}(\tau)\right| \leq C \cdot (1 + \tau^2)^{-N} \quad \text{for all } \omega \in U, \tau \in \mathbb{R}.$$

In other words, $\left(\mathcal{R}(\phi f)(\omega, s)\right)$ has to be sufficiently smooth in the variable s around $\omega = \omega_0$ [Gro11, Section 2.2].

Our original interest lies in the curvature of an edge. As its orientation can be determined by the Radon transform, i.e., by observing integrals over lines, we will now examine the possibility to determine the edge curvature by observing integrals over parabolae. This leads us to the parabolic Radon transform.

3.2 Parabolic Radon transform

The parabolic Radon transform plays a role in reflection seismology [GS09], which is used in fields such as petrol prospecting, geophysical sciences and oceanography. In this application, especially the inversion is interesting and for special cases of the parabolic Radon transform, an inversion formula is known. In [Cor81], Cormack derives an inversion formula for the Radon transform on a family of planar curves which - for given $\alpha > 0$, $p > 0$ and $\varphi \in \mathbb{T}$ - can be described

in polar coordinates by

$$r^\alpha \cdot \cos\left(\alpha(\theta - \varphi)\right) = p^\alpha, \quad \text{for } r > 0, \ |\theta - \varphi| \le \frac{\pi}{2\alpha}.$$

For $\alpha = \frac{1}{2}$, these curves are parabolae whose vertex lies in the origin. In [DVOS98], the authors consider the isofocal parabolic Radon transform - a Radon transform over a family of parabolae whose focus lies in the origin. With the notation $x_\theta = \langle x, e_\theta \rangle$ for $\theta \in \mathbb{T}$ and for $r > 0$, these parabolae can be represented as

$$\left\{ x \in \mathbb{R}^2 : \ x_\theta = \frac{x_{\theta + \pi/2}^2}{4r} - r \right\}$$

and the vertex lies in the point $-re_\theta$. By showing a relation to the classical Radon transform and utilizing a Fourier series expansion, the authors show an inversion formula.

For the purpose of detecting edge curvature, we will need at least three degrees of freedom: the opening direction of the parabola, a translation parameter in the opening direction and the curvature. So, unfortunately, we cannot fall back on either of these two established parabolic Radon transforms, as they only offer two degrees of freedom. Hence, we introduce a more general parabolic Radon transform, described in the following definition.

Definition 3.20 (Parabola and parabolic Radon transform). Let $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}$ and $a \in \mathbb{R}$. Then, the parabola with vertex in x, opening direction θ and vertex curvature a is denoted by

$$P_{x,\theta,a} = \left\{ x + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} : \ t \in \mathbb{R} \right\}.$$

The left and right arm of the parabola are denoted as

$$P_{x,\theta,a}^- = \left\{ x + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} : \ t \le 0 \right\} \quad \text{and} \quad P_{x,\theta,a}^+ = \left\{ x + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} : \ t \ge 0 \right\},$$

respectively. For a function $f : \mathbb{R}^2 \to \mathbb{R}$, its parabolic Radon transform is defined as

$$\mathcal{P}f(x,\theta,a) = \int_\mathbb{R} f\left(x + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) dt \quad \text{for all } x \in \mathbb{R}^2, \theta \in \mathbb{T}, a \in \mathbb{R},$$

whenever the integral is well defined. When the focus lies on the translation variable x, it is sometimes written as

$$\mathcal{P}_{\theta,a} f(x) = \mathcal{P}f(x,\theta,a) \quad \text{for all } x \in \mathbb{R}^2, \theta \in \mathbb{T}, a \in \mathbb{R}.$$

We will first introduce new function spaces which impose decay conditions, to examine some of the mapping properties of the PRT.

Definition 3.21. For $\alpha \in \mathbb{R}$ we define

$$\mathcal{B}^\alpha\left(\mathbb{R}^n\right) := \left\{f:\mathbb{R}^n \to \mathbb{C}: \|f\|_{\mathcal{B}^\alpha} < \infty\right\},$$

where

$$\|f\|_{\mathcal{B}^\alpha} := \left\|\left(1 + \|\cdot\|^\alpha\right)f\right\|_{L^\infty}.$$

Furthermore,

$$\tilde{\mathcal{B}}^\alpha\left(\mathbb{R}^n\right) := \left\{f:\mathbb{R}^n \to \mathbb{C} \,\middle|\, \exists \varepsilon > 0 : \|f\|_{\mathcal{B}^{\alpha+\varepsilon}} < \infty\right\}.$$

In the following lemma we will show some mapping properties of the parabolic Radon transform and derive its continuity on $\mathcal{B}^{1+\delta}$.

Lemma 3.22. *The parabolic Radon transform maps* $\mathcal{P}: \tilde{\mathcal{B}}^1 \to L^\infty\left(\mathbb{R}^2 \times \mathbb{T} \times \mathbb{R}^+\right)$. *Furthermore,* $\mathcal{P}$ *is a continuous operator on* $\mathcal{B}^{1+\varepsilon}\left(\mathbb{R}^2\right)$ *for all* $\varepsilon > 0$.

Proof. Let $f \in \tilde{\mathcal{B}}^1\left(\mathbb{R}^2\right)$. Then, by definition there exist constants $\varepsilon, C > 0$ such that

$$|f(x)| \le \frac{C}{1 + \|x\|^{1+\varepsilon}}.$$

Hence,

$$\left|\mathcal{P}f(x,\theta,a)\right| \le \int_{\mathbb{R}} \frac{C\,dt}{1 + \|x + R_\theta \cdot \left(\frac{a}{2}t^2, t\right)^T\|^{1+\varepsilon}}$$

$$\le \int_{\mathbb{R}} \frac{C\,dt}{1 + \left|\|x\| - \sqrt{\frac{a^2}{4}t^4 + t^2}\right|^{1+\varepsilon}}$$

We substitute $r = \|x\| - \sqrt{\frac{a^2}{4}t^4 + t^2}$. Then, we have

$$\left|\frac{dt}{dr}\right| = \frac{a\,|r - \|x\||}{\sqrt{2\left(\sqrt{1 + a^2\,(r - \|x\|)^2} - 1\right)\left(1 + a^2\,(r - \|x\|)^2\right)}} \le 1$$

for all $a, r > 0$, $x \in \mathbb{R}^2$. Consequently, we get

$$\left|\mathcal{P}f(x,\theta,a)\right| \le \int_{\mathbb{R}} \frac{C\,dr}{1 + r^{1+\varepsilon}} < \infty.$$

For all $\varepsilon > 0$ and $f \in \mathcal{B}^{1+\varepsilon}\left(\mathbb{R}^2\right)$, we obtain

$$\left|\mathcal{P}f(x,\theta,a)\right| \le \int_{\mathbb{R}} \frac{\|f\|_{\mathcal{B}^{1+\varepsilon}}\,dr}{1 + r^{1+\varepsilon}} = C_\varepsilon\,\|f\|_{\mathcal{B}^{1+\varepsilon}},$$

where $C_\varepsilon < \infty$ for all $\varepsilon > 0$. Hence, $\mathcal{P}$ is continuous on $\mathcal{B}^{1+\varepsilon}\left(\mathbb{R}^2\right)$ for $\varepsilon > 0$. $\square$

We can now represent the parabolic Radon transform as a convolution.

Lemma 3.23. *For $\theta \in \mathbb{T}$, $a \in \mathbb{R} \setminus \{0\}$, let*

$$g_{\theta,a} \in \mathcal{S}'(\mathbb{R}^2), \quad x \mapsto \delta\left(x_\theta + \frac{a}{2}x_{\theta+\pi/2}^2\right).$$

Let $\varepsilon > 0$ and $f \in \mathcal{B}^{1+\varepsilon}(\mathbb{R}^2)$. Then, the parabolic Radon transform of f can be represented as

$$\mathcal{P}_{\theta,a}f(x) = \left(g_{\theta,a} * f\right)(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Proof. First, let $f \in \mathcal{S}\left(\mathbb{R}^2\right)$. Then,

$$\begin{aligned}
\mathcal{P}f(y,\theta,a) &= \int_{\mathbb{R}} f\left(y + R_\theta \cdot \left(\frac{at^2}{2}, t\right)^T\right) dt \\
&= \int_{\mathbb{R}^2} f(x)\delta\left((x-y)_\theta - \frac{a(x-y)_{\theta+\frac{\pi}{2}}^2}{2}\right) dx \\
&= \left(f * g_{\theta,a}\right)(y),
\end{aligned}$$

where $g_{\theta,a}(x) = \delta\left(x_\theta + \frac{a}{2} \cdot x_{\theta+\pi/2}^2\right)$ and $x_\theta = \langle x, e_\theta \rangle$. We can easily see from Lemma 3.22 that $g_{\theta,a} \in \mathcal{S}'\left(\mathbb{R}^2\right)$. Since $\mathcal{P}$ is continuous on $\mathcal{B}^{1+\varepsilon}\left(\mathbb{R}^2\right)$ and $\mathcal{S}\left(\mathbb{R}^2\right)$ is dense in $\mathcal{B}^m\left(\mathbb{R}^2\right)$ for every $m \geq 0$ (a direct consequence of the norm of $\mathcal{S}$), the convolutional representation is also valid for every $f \in \mathcal{B}^{1+\varepsilon}\left(\mathbb{R}^2\right)$. □

In the next lemma, we will determine the Fourier transform of the convolution kernel of the PRT.

Lemma 3.24. *Let $\theta \in \mathbb{T}$ and $a \in \mathbb{R} \setminus \{0\}$. The Fourier transform of $g_{\theta,a}$ in the sense of tempered distributions is given by*

$$\hat{g}_{\theta,a}(\xi) = \sqrt{\frac{2\pi}{|a\xi_\theta|}} e^{i \cdot \text{sgn}(a\xi_\theta)\pi/4} \cdot e^{-i\xi_{\theta+\pi/2}^2/(2a\xi_\theta)}.$$

Proof. As the Fourier transform is an automorphism on $\mathcal{S}'(\mathbb{R}^2)$ [Wal94, 11. X], we have $\hat{g}_{\theta,a} \in \mathcal{S}'\left(\mathbb{R}^2\right)$. We determine $\hat{g}_{\theta,a}$ via the definition of the Fourier transform of a tempered distribution:

$$\left\langle \hat{g}_{\theta,a}, \phi \right\rangle = \left\langle g_{\theta,a}, \hat{\phi} \right\rangle \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^2).$$

Now let ϕ be an arbitrary element of $\boldsymbol{\mathcal{S}}(\mathbb{R}^2)$. We then obtain

$$
\begin{aligned}
\langle \hat{g}_{\theta,a}, \phi \rangle &= \langle g_{\theta,a}, \hat{\phi} \rangle \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \delta\left(x_\theta + \frac{a}{2} \cdot x_{\theta+\pi/2}^2\right) \cdot \hat{\phi}(x) \, dx_\theta \, dx_{\theta+\pi/2} \\
&= \int_{\mathbb{R}} \hat{\phi}\left(R_\theta \cdot \begin{pmatrix} -\frac{a}{2} \cdot x_{\theta+\pi/2}^2 \\ x_{\theta+\pi/2} \end{pmatrix}\right) dx_{\theta+\pi/2}.
\end{aligned}
$$

Representing the upper integral over $\mathbb{R}$ as the result of a limit process and inserting the definition of $\hat{\phi}$ yields

$$
\begin{aligned}
\langle \hat{g}_{\theta,a}, \phi \rangle &= \lim_{r,R\to\infty} \int_{-r}^{R} \hat{\phi}\left(R_\theta \cdot \begin{pmatrix} -\frac{a}{2} \cdot v^2 \\ v \end{pmatrix}\right) dv \\
&= \lim_{r,R\to\infty} \int_{-r}^{R} \int_{\mathbb{R}^2} \phi(y) e^{i\left(\frac{a}{2} y_\theta \cdot v^2 - y_{\theta+\pi/2} \cdot v\right)} \, dy \, dv \\
&= \lim_{r,R\to\infty} \int_{\mathbb{R}^2} \phi(y) \cdot \int_{-r}^{R} e^{i\left(\frac{a}{2} y_\theta \cdot v^2 - y_{\theta+\pi/2} \cdot v\right)} \, dv \, dy \\
&= \lim_{r,R\to\infty} \int_{\mathbb{R}^2} \phi(y) e^{-i \cdot \frac{y_{\theta+\pi/2}^2}{2ay_\theta}} \cdot \int_{-r}^{R} e^{i \cdot \frac{a}{2} y_\theta \left(v - \frac{y_{\theta+\pi/2}}{ay_\theta}\right)^2} \, dv \, dy \\
&= \int_{\mathbb{R}^2} \phi(y) e^{-i \cdot \frac{y_{\theta+\pi/2}^2}{2ay_\theta}} \cdot \lim_{r,R\to\infty} \int_{-r}^{R} e^{i \cdot \frac{a}{2} y_\theta \left(v - \frac{y_{\theta+\pi/2}}{ay_\theta}\right)^2} \, dv \, dy.
\end{aligned} \tag{3}
$$

In the last step we used $\phi \in \boldsymbol{\mathcal{S}}(\mathbb{R}^2)$ and the dominated convergence theorem. We proceed by determining the limit process in the upper integrals:

$$
\begin{aligned}
\lim_{r,R\to\infty} \int_{-r}^{R} e^{i \cdot b(v-c)^2} \, dv &= \lim_{R\to\infty} \int_{-(R+c)}^{R-c} e^{i \cdot bw^2} \, dw \\
&= \frac{1}{\sqrt{|b|}} \cdot \lim_{r,R\to\infty} \int_{-\sqrt{|b|}(r+c)}^{\sqrt{|b|}(R-c)} e^{i \cdot \mathrm{sgn}(b) u^2} \, du \\
&= \frac{1}{\sqrt{|b|}} \cdot \lim_{r,R\to\infty} \int_{-\sqrt{|b|}(r+c)}^{\sqrt{|b|}(R-c)} \left[\cos\left(u^2\right) + i \cdot \mathrm{sgn}(b) \sin\left(u^2\right)\right] \, du.
\end{aligned}
$$

The remaining limit process results in the well known Fresnel integrals

$$
\lim_{R\to\infty} \int_{0}^{R} \cos\left(t^2\right) \, dt = \int_{0}^{R} \sin\left(t^2\right) \, dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}
$$

due to [AS⁺72, 7.3]. Hence,

$$\lim_{r,R\to\infty}\int_{-r}^{R} e^{i\cdot b(v-c)^2}\, dv = \frac{1}{\sqrt{|b|}}\cdot\sqrt{\frac{\pi}{2}}\cdot(1+i\operatorname{sgn}(b)) = \sqrt{\frac{\pi}{|b|}}\cdot e^{i\operatorname{sgn}(b)\cdot\frac{\pi}{4}}.$$

Inserting this result into (3) yields

$$\langle \hat{g}_{\theta,a},\phi\rangle = \int_{\mathbb{R}^2}\phi(y)\cdot\sqrt{\frac{2\pi}{|ay_\theta|}}e^{i\operatorname{sgn}(ay_\theta)\frac{\pi}{4}}\cdot e^{-i\cdot\frac{y_{\theta+\pi/2}^2}{2ay_\theta}}\, dy.$$

Thus we obtain the desired form of $\hat{g}_{\theta,a}$. $\qquad\qquad\square$

As we already determined the Fourier transform of $g_{\theta,a}$, we are now interested in deriving sufficient conditions for the existence of the Fourier transform of $\mathcal{P}f(x,\theta,a)$ at least in a weak sense, i. e. $\mathcal{P}f(\cdot,\theta,a)\in\mathcal{S}'\left(\mathbb{R}^2\right)$. The next lemma gives a condition which ensures this.

Lemma 3.25. *Let $f\in L^p\left(\mathbb{R}^2\right)$, $1\le p<2$, $\theta\in\mathbb{T}$ and $a\ne 0$. Then*

$$\mathcal{P}f(\cdot,\theta,a)\in\mathcal{S}'\left(\mathbb{R}^2\right).$$

This is in general not true for $f\in L^2\left(\mathbb{R}^2\right)$.

Proof. We will start in the Fourier domain and show that the supposed Fourier transform of $\mathcal{P}f$ is an element of the dual Schwartz space for $f\in L^p\left(\mathbb{R}^2\right)$, $1\le p<2$. Then its inverse Fourier transform and thus the PRT itself also is in $\mathcal{S}'$. Let $Q(\xi):=\hat{g}_{\theta,a}(\xi)\cdot\hat{f}(\xi)$. We want to show that

$$|\langle Q,\phi\rangle|<\infty \quad \forall\phi\in\mathcal{S}\left(\mathbb{R}^2\right).$$

We can rewrite the scalar product as $\langle Q,\phi\rangle = \langle\hat{f},\overline{\hat{g}_{\theta,a}}\cdot\phi\rangle$. Since $|\hat{g}_{\theta,a}(\xi)| = \frac{c}{\sqrt{|\xi_\theta|}}$, we can derive that $\hat{g}_{\theta,a}\cdot\phi\in L^p\left(\mathbb{R}^2\right)$ for $1\le p<2$. Hence, $|\langle Q,\phi\rangle|<\infty$ for all $\hat{f}\in L^q\left(\mathbb{R}^2\right)$, $2<q\le\infty$.
We will now rush over some mapping properties of the Fourier operator $\mathcal{F}$. For further details, we refer the reader to [Kat04]. Due to the Hausdorff-Young inequality [Kat04, VI.3.2] for $f\in L^1\cap L^2\left(\mathbb{R}^n\right)$, $1\le p\le 2$ and $q=\frac{p}{p-1}$, we have

$$\|\hat{f}\|_{L^q}\le C_p\cdot\|f\|_{L^p}.$$

By extending $\mathcal{F}$ from $L^1\cap L^2$ to L^p, $1\le p\le 2$, we obtain the following mapping property:

$$\mathcal{F}\colon L^p\left(\mathbb{R}^n\right)\to L^q\left(\mathbb{R}^n\right),$$

where again $1\le p\le 2$ and $q=\frac{p}{p-1}$.
Hence, the condition $f\in L^p\left(\mathbb{R}^2\right)$ for some $p\in[1,2)$ ensures that $\hat{f}\in L^q\left(\mathbb{R}^2\right)$ with $2<q\le\infty$ and thus $Q\in\mathcal{S}'\left(\mathbb{R}^2\right)$. Consequently, its inverse Fourier transform is defined and we have $\mathcal{F}^{-1}Q(x)=\mathcal{P}f(x,\theta,a)\in\mathcal{S}'\left(\mathbb{R}^2\right)$.

For a given $\theta \in \mathbb{T}$ our L^2-counterexample is

$$\hat{f}(\xi) := \frac{e^{-\xi_{\theta+\pi/2}^2}}{\sqrt{|\xi_\theta| \cdot (1 + [\log|\xi_\theta|]^2)}}.$$

That this function is an element of $L^2\left(\mathbb{R}^2\right)$ can be seen by looking at the ξ_θ-dependent part

$$\int_\mathbb{R} \left(\frac{1}{\sqrt{|t| \cdot (1 + [\log|t|]^2)}}\right)^2 dt = 2 \int_0^\infty \frac{dt}{t \cdot (1 + [\log t]^2)} = 2 \int_\mathbb{R} \frac{ds}{1 + s^2} = 2\pi < \infty,$$

where we applied the substitution $s = \log t$ in the second step. In a similar fashion we can show that $\hat{g}_{\theta,a} \cdot \hat{f}$ is locally not integrable and hence not in $\mathcal{S}'\left(\mathbb{R}^2\right)$.

$$\int_{-\varepsilon}^\varepsilon \frac{1}{\sqrt{|t|}} \cdot \frac{1}{\sqrt{|t| \cdot (1 + [\log|t|]^2)}} \, dt = 2 \int_0^\varepsilon \frac{dt}{t\sqrt{1 + [\log t]^2}} = 2 \int_{-\log\varepsilon}^\infty \frac{ds}{\sqrt{1 + s^2}} = \infty$$

for all $\varepsilon > 0$. $\qquad\square$

The important question of an inversion formula for the parabolic Radon transform is answered by the next theorem.

Theorem 3.26. *Let $f \in L^p\left(\mathbb{R}^2\right)$, $1 \le p < 2$, $\theta \in \mathbb{T}$ and $a \ne 0$. Then*

$$2\pi|a| \cdot \mathcal{P}_{\theta,-a}\left(-\frac{\partial^2}{\partial x_\theta^2}\right)^{\frac{1}{2}} \mathcal{P}_{\theta,a} f = f,$$

where $\mathcal{P}_{\theta,a}f(x) := \mathcal{P}f(x,\theta,a)$. This inversion formula is to be understood in the $\mathcal{S}'$-sense.

Proof. Due to Lemma 3.25, the Fourier transforms of the left side of the equation we want to show, is well defined in the $\mathcal{S}'$-sense. Hence,

$$\mathcal{F}\left(\mathcal{P}_{\theta,-a}\left(-\frac{\partial^2}{\partial x_\theta^2}\right)^{\frac{1}{2}} \mathcal{P}_{\theta,a} f\right)(\xi)$$

$$= \hat{g}_{\theta,-a}(\xi) \cdot |\xi_\theta| \cdot \hat{g}_{\theta,a}(\xi) \cdot \hat{f}(\xi)$$

$$= \sqrt{\frac{2\pi}{|a\xi_\theta|}} \cdot e^{i \cdot \text{sgn}(-a)\frac{\pi}{4}\text{sgn}(\xi_\theta)} \cdot e^{-i \cdot \frac{\xi_{\theta+\pi/2}^2}{2(-a)\xi_\theta}} \cdot |\xi_\theta| \cdot \sqrt{\frac{2\pi}{|a\xi_\theta|}} \cdot e^{i \cdot \text{sgn}(a)\frac{\pi}{4}\text{sgn}(\xi_\theta)} \cdot e^{-i \cdot \frac{\xi_{\theta+\pi/2}^2}{2a\xi_\theta}} \cdot \hat{f}(\xi)$$

$$= \frac{2\pi}{|a|} \hat{f}(\xi)$$

for almost all $\xi \in \mathbb{R}^2$. By applying the inverse Fourier transform to this result, we obtain that

$$\mathcal{P}_{\theta,-a}\left(-\frac{\partial^2}{\partial x_\theta^2}\right)^{\frac{1}{2}} \mathcal{P}_{\theta,a}f(x) = \frac{1}{2\pi|a|} f(x)$$

in the sense of tempered distributions. $\qquad\square$

3.3 Smoothness results

The parabolic Radon transform is particularly interesting for our venture, since it allows for the detection of the edge curvature of certain functions. To this end, we consider a class of sets which is related to the cartoon-like images introduced by Candes and Donoho [CD04]. This class is described in the following definition.

Definition 3.27 (Admissible set)**.** A set $A \subset \mathbb{R}^2$ is called admissible, if

1. A is an open, connected set with a C^3-boundary ∂A,

2. for all $x \in \partial A$, there exists $\varepsilon > 0$ such that one of the following two statements hold:

    (a) For all $u \in \mathbb{R}$, $\left| \left(P^+_{x+u \cdot e_\theta, \theta, \kappa} \right) \cap B_\varepsilon(x) \cap \partial A \right| \leq 1$ and $\left| \left(P^-_{x+u \cdot e_\theta, \theta, \kappa} \right) \cap B_\varepsilon(x) \cap \partial A \right| \leq 1$,

    (b) $P_{x, \theta, \kappa} \cap B_\varepsilon(x) = \partial A \cap B_\varepsilon(x)$,

    where θ is the angle indicating the inner normal vector of A in x and κ is the associated curvature, and

3. for all $\omega \in S^1$ and all compact sets $K \subset \mathbb{R}^2$, there exist only finitely many $x \in \partial A \cap K$ such that ω is orthogonal to ∂A in x.
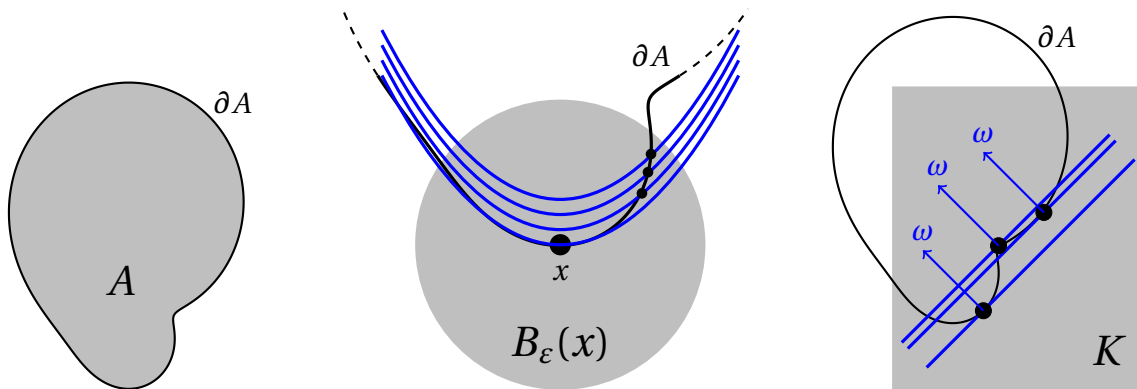
Fig. 3.1: Visualization of the conditions of an admissible set
Left: Condition 1 - A is an open, connected set with a C^3-boundary,
Middle: Condition 2 (a) - in a neighborhood of $x \in \partial A$, the boundary ∂A and the left or right arm of the osculating parabola have at most one intersection,
Right: Condition 3 - for all directions ω and compact sets K, there exist only finitely many points on ∂A, where ω is perpendicular to ∂A.

The definition of admissible sets seems to be rather technical and unintuitive. It is needed for Proposition 3.30 in order to obtain a smoothness result for the parabolic Radon transform. By representing the boundary ∂A locally as graph of a function, we obtain a useful characterization of these functions describing the boundary.

Lemma 3.28. *The following two statements are equivalent:*

1. *$A \subset \mathbb{R}^2$ is admissible.*

2. *$A \subset \mathbb{R}^2$ is open and connected. Furthermore, for all $x^0 \in \partial A$ there exists an $\varepsilon > 0$ and a function $q \in C^3((-\varepsilon, \varepsilon))$ such that*

   (a) *$q(0) = 0$ and $q'(0) = 0$,*

   (b) *$x^0 + R_\theta \cdot \begin{pmatrix} q(t) \\ t \end{pmatrix} \in \partial A$ for all $t \in (-\varepsilon, \varepsilon)$, where $\theta \in \mathbb{T}$ is the angle corresponding to the inner normal vector of A in x^0,*

   (c) *the function $q_\kappa : (-\varepsilon, \varepsilon) \to \mathbb{R}, t \mapsto q(t) - \frac{\kappa}{2} \cdot t^2$ is one-to-one on $(-\varepsilon, 0]$ and $[0, \varepsilon)$ or $q_\kappa \equiv 0$ on $(-\varepsilon, \varepsilon)$ and*

   (d) *0 is the only root of q' in $(-\varepsilon, \varepsilon)$.*

Proof. <u>1. $\Rightarrow$ 2.</u> :

If A is admissible, A is open and connected.

<u>Existence of $q \in C^3((-\varepsilon, \varepsilon))$:</u>

A has a C^3-boundary which can be described by a positively oriented curve $\gamma \in C^3([0,1], \mathbb{R}^2)$. Let $x^0 := \gamma(0)$. Since θ is the angle of the inner normal vector of A in x^0, $\theta - \frac{\pi}{2}$ indicates the associated tangential direction. Hence, we can divide the curve γ into

$$\gamma(t) = x^0 + \gamma_\theta(t) \cdot e_\theta + \gamma_{\theta - \frac{\pi}{2}}(t) \cdot e_{\theta - \frac{\pi}{2}} \quad \text{for all } t \in [0, 1],$$

$$\text{where} \quad \gamma_\theta(0) = 0$$

$$\text{and} \quad \gamma_{\theta - \frac{\pi}{2}}(0) = 0.$$

Since the normal direction θ is orthogonal to the tangential direction, we obtain the derivatives

$$\dot{\gamma}(t) = \dot{\gamma}_\theta(t) \cdot e_\theta + \dot{\gamma}_{\theta - \frac{\pi}{2}}(t) \cdot e_{\theta - \frac{\pi}{2}} \quad \text{for all } t \in [0, 1],$$

$$\text{where} \quad \dot{\gamma}_\theta(0) = 0$$

$$\text{and} \quad \dot{\gamma}_{\theta - \frac{\pi}{2}}(0) \neq 0.$$

Since γ is positively oriented, $\dot{\gamma}_{\theta - \frac{\pi}{2}}(0) > 0$. Due to the continuity of $\dot{\gamma}$, there exists $\delta > 0$ such that $\dot{\gamma}_{\theta - \frac{\pi}{2}}(t) > 0$ for all $t \in (-\delta, \delta)$. Thus, $\gamma_{\theta - \frac{\pi}{2}}$ is strictly increasing on $(-\delta, \delta)$ and hence invertible. Then there exists $\varepsilon' > 0$ such that $(-\varepsilon', \varepsilon') \subset \gamma_{\theta - \frac{\pi}{2}}((-\delta, \delta))$. We can now define the function

$$q : (-\varepsilon', \varepsilon') \to \mathbb{R}, \quad t \mapsto \gamma_\theta\left(\gamma_{\theta - \frac{\pi}{2}}^{-1}(t)\right).$$

Since $\gamma \in C^3([0, 1], \mathbb{R}^2)$ and $\dot{\gamma}_{\theta - \frac{\pi}{2}}(t) > 0$ for all $t \in (-\delta, \delta)$, we obtain $q \in C^3((-\varepsilon', \varepsilon'))$.

<u>(a):</u>

We can conclude that

$$q(0) = \gamma_\theta \left(\gamma_{\theta - \frac{\pi}{2}}^{-1}(0) \right) = \gamma_\theta(0) = 0, \tag{4}$$

$$q'(0) = \frac{\dot{\gamma}_\theta \left(\gamma_{\theta - \frac{\pi}{2}}^{-1}(0) \right)}{\dot{\gamma}_{\theta - \frac{\pi}{2}} \left(\gamma_{\theta - \frac{\pi}{2}}^{-1}(0) \right)} = \frac{\dot{\gamma}_\theta(0)}{\dot{\gamma}_{\theta - \frac{\pi}{2}}(0)} = 0. \tag{5}$$

Hence, (a) is fulfilled.

<u>(b)</u>:

As θ is the angle corresponding to the inner normal vector, the intersection $A \cap B_{\varepsilon'}(x^0)$ can be described as:

$$\partial A \cap B_{\varepsilon'}(x^0) = \left\{ x \in B_{\varepsilon'}(x^0) : x_\theta - x_\theta^0 = q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) \right\}. \tag{6}$$

Thus, (b) is fulfilled.

<u>(c)</u>:

Since A is admissible, one of the following two statements hold for an $\tilde{\varepsilon} > 0$:

(i) For all $u \in \mathbb{R}$, $\left| \left(P_{x^0 + u \cdot e_\theta, \theta, \kappa}^+ \right) \cap B_{\tilde{\varepsilon}}(x^0) \cap \partial A \right| \leq 1$ and $\left| \left(P_{x^0 + u \cdot e_\theta, \theta, \kappa}^- \right) \cap B_{\tilde{\varepsilon}}(x^0) \cap \partial A \right| \leq 1$,

(ii) $P_{x^0, \theta, \kappa} \cap B_{\tilde{\varepsilon}}(x^0) = \partial A \cap B_{\tilde{\varepsilon}}(x^0)$,

where κ is the local curvature of ∂A in x^0. Let $\varepsilon := \min\{\varepsilon', \tilde{\varepsilon}\}$.

Now (6) yields

$$P_{x^0 + u \cdot e_\theta, \theta, \kappa}^+ \cap B_\varepsilon(x^0) \cap \partial A$$

$$= \left\{ x \in B_\varepsilon(x^0) : x_\theta - x_\theta^0 = q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) \wedge \exists\, t \geq 0 : x = x^0 + R_\theta \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix} \right\}$$

$$= \left\{ x \in B_\varepsilon(x^0) : x_{\theta - \frac{\pi}{2}} > x_{\theta - \frac{\pi}{2}}^0 \wedge x_\theta - x_\theta^0 = q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) \wedge x_\theta - x_\theta^0 = u + \frac{\kappa}{2} \cdot \left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right)^2 \right\}$$

$$= \left\{ x \in B_\varepsilon(x^0) : x_{\theta - \frac{\pi}{2}} > x_{\theta - \frac{\pi}{2}}^0 \wedge x_\theta - x_\theta^0 = q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) \wedge q_\kappa\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) = u \right\}, \tag{7}$$

where $q_\kappa(t) = q(t) - \frac{\kappa}{2} \cdot t^2$ for all $t \in (-\varepsilon, \varepsilon)$. Similarly,

$$P_{x^0 + u \cdot e_\theta, \theta, \kappa}^- \cap B_\varepsilon(x^0) \cap \partial A$$

$$= \left\{ x \in B_\varepsilon(x^0) : x_{\theta - \frac{\pi}{2}} < x_{\theta - \frac{\pi}{2}}^0 \wedge x_\theta - x_\theta^0 = q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) \wedge q_\kappa\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0 \right) = u \right\}. \tag{8}$$

If (i) holds, then for all $u \in \mathbb{R}$, there can be at most one $t_+ \in [0, \varepsilon)$ such that $q_\kappa(t_+) = u$ due to (7), and at most one $t_- \in (-\varepsilon, 0]$ such that $q_\kappa(t_+) = u$ due to (8). Hence, q_κ is one-to-one on $[0, \varepsilon)$ and $(-\varepsilon, 0]$.

If (ii) holds, then $\frac{\kappa}{2} \cdot t^2 = q(t)$ for all $t \in (-\varepsilon, \varepsilon)$. Hence, $q_\kappa \equiv 0$ on $(-\varepsilon, \varepsilon)$.

<u>(d)</u>:

Due to property 3 of Definition 3.27, for all $\omega \in S^1$ and all compact sets K, there exist only finitely many points $x \in \partial A \cap K$ such that ω is orthogonal to ∂A in x. By fixing $x^0 \in \partial A$ and choosing the compact set to be $K = \overline{B_{\varepsilon'}(x^0)}$, we can especially conclude that there are only finitely many points $x \in \partial A \cap K$ which have the same tangential direction e_θ as x^0. By translating this into the setting of the function q whose graph locally represents ∂A, we can conclude that there are only finitely many roots of q' in $(-\varepsilon', \varepsilon')$. Hence, we can choose $\varepsilon \in (0, \varepsilon']$ small enough such that 0 is the only root of q' in $(-\varepsilon, \varepsilon)$. Thus, (d) is fulfilled.

<u>2. $\Rightarrow$ 1. :</u>

<u>Definition 3.27, property 1:</u>

As A is open and connected and since ∂A can be locally represented by C^3-maps q, these maps form an atlas of ∂A. Hence, ∂A is a C^3-manifold.

<u>Definition 3.27, property 2:</u>

Due to (c), one of the two following statements holds:

   (i) q_κ is one-to-one on $(-\varepsilon, 0]$ and on $[0, \varepsilon)$.

   (ii) $q_\kappa \equiv 0$ on $(-\varepsilon, \varepsilon)$.

If (i) holds, then

$$\left|\left\{t \in (-\varepsilon, 0]: \ q_\kappa(t) = u\right\}\right| \leq 1 \quad \text{and} \quad \left|\left\{t \in [0, \varepsilon): \ q_\kappa(t) = u\right\}\right| \leq 1. \tag{9}$$

Now (7) and (8) yield that for all $u \in \mathbb{R}$,

$$\left|P^+_{x^0 + u \cdot e_\theta, \theta, \kappa} \cap B_\varepsilon(x^0) \cap \partial A\right| \leq 1 \quad \text{and} \quad \left|P^-_{x^0 + u \cdot e_\theta, \theta, \kappa} \cap B_\varepsilon(x^0) \cap \partial A\right| \leq 1$$

Hence, 2 (a) of Definition 3.27 holds.

If (ii) is true, then $q(t) = \frac{\kappa}{2} \cdot t^2$ for all $t \in (-\varepsilon, \varepsilon)$. Consequently, $P_{x^0, \theta, \kappa} \cap B_{\tilde{\varepsilon}}(x^0) = \partial A \cap B_{\tilde{\varepsilon}}(x^0)$. Hence, 2 (b) of Definition 3.27 is fulfilled.

<u>Definition 3.27, property 3:</u>

In order to prove that property 3 of Definition 3.27 holds, we assume the contrary - that there exists a compact set $K \subset \mathbb{R}^2$ and $\omega \in S^1$ such that there exist infinitely many points $\{x_i\}_{i \in \mathbb{N}} \subset \partial A$ in which ∂A is orthogonal to ω. Then there exists an accumulation point $\bar{x}$ of the sequence $\{x_i\}_{i \in \mathbb{N}}$. By choosing $\bar{x}$ as the starting point to locally define a function q fulfilling the conditions 2 (a) - (d) of this lemma. Hence, there exists $\varepsilon > 0$ such that 0 is the only root of q' on $(-\varepsilon, \varepsilon)$. Therefore, on the closed ball $\overline{B_\varepsilon(\bar{x})}$, $\bar{x}$ is the only point on ∂A in which ω is orthogonal to ∂A. This contradicts the conclusion that $\bar{x}$ has to be an accumulation point of the sequence $\{x_i\}_{i \in \mathbb{N}}$. Thus, property 3 holds true.

$\square$

Remark 3.29. In order to get a better understanding for condition 2 (c) in Lemma 3.28, we give an example of a smooth function $q \in C^\infty(\mathbb{R})$ which is not one-to-one in any neighborhood of 0. Let

$$q : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} \sin\left(\frac{1}{t}\right) \cdot \exp\left(-\frac{1}{t^2}\right), & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Due to the factor $\sin\left(\frac{1}{t}\right)$, q oscillates in every neighborhood of the origin. Yet, the second factor $\exp\left(-\frac{1}{t^2}\right)$ ensures that $q \in C^\infty(\mathbb{R})$ through its root of order ∞ in the origin.

Proposition 3.30. *Let $A \subset \mathbb{R}^2$ be an admissible set and let $f = \mathbb{1}_A$ be the indicator function of A. Furthermore, let $x^0 \in \partial A$, let θ indicate the angle of the inner normal vector of A in x^0 and let κ be the local curvature of ∂A in x^0. Then the following statement holds:*

For all $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all functions $\phi \in C_c^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(\phi) \subset B_\varepsilon(x^0)$, $\phi(\mathbb{R}^2) \subset [0,\infty)$ and $\phi(x^0) \neq 0$,

$$\mathcal{P}_{\theta,a}(\phi f)(x^0 + u \cdot e_\theta) - \mathcal{P}_{\theta,a}(\phi f)(x^0) \begin{cases} \sim \sqrt{|u|} & \text{for } \frac{u}{\kappa-a} \searrow 0, \quad a \neq \kappa, \\ = \mathcal{O}(u) & \text{for } \frac{u}{\kappa-a} \nearrow 0, \quad a \neq \kappa, \\ = \Omega\left(\sqrt[3]{|u|}\right) & \text{for } u \searrow 0 \text{ or } u \nearrow 0, \quad a = \kappa. \end{cases}$$

Proof. Let $x^0 \in \partial A$, let θ be the angle corresponding to the inner normal vector of A in x^0 and let κ be the associated curvature. Due to Lemma 3.28, there exists $\varepsilon' > 0$ and a function $q \in C^3\left((-\varepsilon',\varepsilon')\right)$ such that $q(0) = 0$ and

$$x^0 + R_\theta \cdot \begin{pmatrix} q(t) \\ t \end{pmatrix} \in \partial A \quad \text{for all } t \in (-\varepsilon',\varepsilon'). \tag{10}$$

For $a \in \mathbb{R}$, we now introduce the function

$$q_a \in C^3\left((-\varepsilon',\varepsilon')\right), \quad t \mapsto q(t) - \frac{a}{2} \cdot t^2. \tag{11}$$

Due to 2 (c) of Lemma 3.28, q_κ is one-to-one on $(-\varepsilon',0]$ and $[0,\varepsilon')$ or $q_\kappa \equiv 0$.

Since $\kappa = q''(0)$, we obtain for each $a \neq \kappa$, that

$$q_a''(0) = q''(0) - a \neq 0. \tag{12}$$

Since q_a'' is continuous, there exists $\varepsilon_2 \in (0,\varepsilon']$ such that q_a'' does not change its sign on $(-\varepsilon_2,\varepsilon_2)$. Consequently, q' is monotonous on $(-\varepsilon_2,\varepsilon_2)$. As $q_a'(0) = 0$ due to (5), there exists $\varepsilon_3 \in (0,\varepsilon_2]$ such that the sign of q_a' is constant on $(-\varepsilon_3,0)$ and on $(0,\varepsilon_3)$. Thus, q_a is monotonous and thus one-to-one on $(-\varepsilon_3,0)$ and on $(0,\varepsilon_3)$.

Let $\varepsilon := \min\{\varepsilon', \varepsilon_3\}$ and let $\phi \in C_c^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}(\phi) \subset B_\varepsilon(x^0)$, $\phi(\mathbb{R}^2) \subset [0, \infty)$ and $\phi(x^0) \neq 0$. Then,

$$
\mathcal{P}_{\theta, a}(\phi f)(x^0 + u \cdot e_\theta) = \int_\mathbb{R} (\phi \mathbb{1}_A)\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} t^2 \\ t \end{pmatrix}\right) dt
$$

$$
= \int_{-\varepsilon}^\varepsilon \left(\phi \mathbb{1}_{A \cap B_\varepsilon(x^0)}\right)\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} t^2 \\ t \end{pmatrix}\right) dt. \tag{13}
$$

Since θ is the angle corresponding to the inner normal vector of A in x^0, (10) yields

$$
A \cap B_\varepsilon(x^0) = \left\{ x \in B_\varepsilon(x^0) : x_\theta - x_\theta^0 > q\left(x_{\theta - \frac{\pi}{2}} - x_{\theta - \frac{\pi}{2}}^0\right) \right\}.
$$

Utilizing (13), we obtain

$$
\mathcal{P}_{\theta, a}(\phi f)(x^0 + u \cdot e_\theta) = \int_{-\varepsilon}^\varepsilon \left(\phi \mathbb{1}_{A \cap B_\varepsilon(x^0)}\right)\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt
$$

$$
= \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u + \frac{a}{2} \cdot t^2 - q(t)\right) dt
$$

$$
= \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u - q_a(t)\right) dt. \tag{14}
$$

The constant $\varepsilon > 0$ is chosen such that q_a is one-to-one on $(-\varepsilon, 0]$ and $[0, \varepsilon)$, with the exception of $a = \kappa$, where it is possible that $q_\kappa \equiv 0$. In the latter case, (14) yields

$$
\mathcal{P}_{\theta, \kappa}(\phi f)(x^0 + u \cdot e_\theta) = \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u - \underbrace{q_\kappa(t)}_{=0}\right) dt
$$

$$
= \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}(u) \, dt
$$

$$
= \mathbb{1}_{\mathbb{R}_+}(u) \cdot \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt. \tag{15}
$$

Since $\phi(t) \geq 0$ for all $t \in \mathbb{R}^2$ and $\phi(x) \neq 0$, we have

$$
\left. \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt \right|_{u=0} > 0.
$$

Due to the factor of $\mathbb{1}_{\mathbb{R}_+}(u)$, (15) yields that

$$\mathcal{P}_{\theta,\kappa}(\phi f)\left(x^0 + u \cdot e_\theta\right) - \mathcal{P}_{\theta,\kappa}(\phi f)\left(x^0 - u \cdot e_\theta\right) \sim 1 = \Omega\left(\sqrt[3]{|u|}\right) \quad \text{for } u \to 0.$$

If $q_a \not\equiv 0$, q_a is one-to-one in $(-\varepsilon, 0]$ and $[0, \varepsilon)$. In order to be able to exploit this injectivity, we split the integral in (14) into:

$$\mathcal{P}_{\theta,a}(\phi f)\left(x^0 + u \cdot e_\theta\right) = \mathcal{P}_{\theta,a}^+(\phi f)\left(x^0 + u \cdot e_\theta\right) + \mathcal{P}_{\theta,a}^-(\phi f)\left(x^0 + u \cdot e_\theta\right) \quad \text{for all } u \in \mathbb{R}. \qquad (16)$$

Using (14), we obtain for the right branch of the parabolic Radon transform:

$$\mathcal{P}_{\theta,a}^+(\phi f)\left(x^0 + u \cdot e_\theta\right) = \int_0^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u - q_a(t)\right) \, dt. \qquad (17)$$

We now have to divide between two cases:

1. $q_a(t) > 0$ for all $t \in (0, \varepsilon)$,

2. $q_a(t) < 0$ for all $t \in (0, \varepsilon)$.

Since we will now deal with the inverse of q_a, we will denote the inverse of q_a on the positive side $(0, \varepsilon)$ by $q_{a,+}^{-1}$ and the inverse of q_a on the negative side $(-\varepsilon, 0)$ by $q_{a,-}^{-1}$.

In case 1, we obtain that

$$u > q_a(t) \Leftrightarrow q_{a,+}^{-1}(u) > t \quad \text{for } t \in (0, \varepsilon), \ u \in q_a\left((0, \varepsilon)\right).$$

In case 2, we get

$$u > q_a(t) \Leftrightarrow q_{a,+}^{-1}(u) < t \quad \text{for } t \in (0, \varepsilon), \ u \in q_a\left((0, \varepsilon)\right).$$

<u>Case 1: $q_a(t) > 0$ for all $t \in (0, \varepsilon)$:</u>

We will now examine (17), assuming that we encounter case 1 and $u \in q_a\left((0, \varepsilon)\right)$, i.e. especially, u>0:

$$
\begin{aligned}
\mathcal{P}_{\theta,a}^+(\phi f)\left(x^0 + u \cdot e_\theta\right) &= \int_0^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u - q_a(t)\right) \, dt \\
&= \int_0^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(q_{a,+}^{-1}(u) - t\right) \, dt \\
&= \int_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) \, dt \qquad (18) \\
&= \underbrace{\phi\left(x^0\right)}_{\neq 0} \cdot q_{a,+}^{-1}(u) + \mathcal{O}(1) \quad \text{for } u \searrow 0.
\end{aligned}
$$

If $u \leq 0$, we can conclude that

$$\mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 + u \cdot e_\theta \right) = \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \cdot \mathbb{1}_{\mathbb{R}_+} \big(\underbrace{u}_{\leq 0} - \underbrace{q_a(t)}_{>0} \big) \, dt = 0. \qquad (19)$$

Hence, we can combine the cases $u > 0$ and $u \leq 0$ to get

$$\mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 \right) \begin{cases} = \phi \left(x^0 \right) \cdot q_{a,+}^{-1}(u) + \mathcal{O}(1), & u \searrow 0, \\ = 0, & u \nearrow 0. \end{cases} \qquad (20)$$

Case 2: $q_a(t) < 0$ for all $t \in (0, \varepsilon)$:

In case 2 we obtain by assuming that $u \in q_a((0,\varepsilon))$, i.e. especially $u < 0$:

$$\mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 + u \cdot e_\theta \right) = \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \cdot \mathbb{1}_{\mathbb{R}_+} \left(u - q_a(t) \right) \, dt$$

$$= \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \cdot \mathbb{1}_{\mathbb{R}_+} \left(t - q_{a,+}^{-1}(u) \right) \, dt$$

$$= \int_{q_{a,+}^{-1}(u)}^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt \qquad (21)$$

$$= \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt - \int_0^{q_{a,+}^{-1}(u)} \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt$$

$$= \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt - \underbrace{\phi \left(x^0 \right)}_{\neq 0} \cdot q_{a,+}^{-1}(u) + o(1) \quad \text{for } u \nearrow 0.$$

For $u \geq 0$, we obtain:

$$\mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 + u \cdot e_\theta \right) = \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \cdot \mathbb{1}_{\mathbb{R}_+} \big(\underbrace{u}_{>0} - \underbrace{q_a(t)}_{<0} \big) \, dt$$

$$= \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt \qquad (22)$$

$$= \int_0^\varepsilon \phi \left(x^0 + R_\theta \cdot \begin{pmatrix} \frac{a}{2} \cdot t^2 \\ t \end{pmatrix} \right) \, dt + \mathcal{O}(u) \quad \text{for } u \searrow 0.$$

By combining the two cases $u < 0$ and $u \geq 0$, we obtain

$$\mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a}^+ (\phi f) \left(x^0 \right) \begin{cases} = \mathcal{O}(u), & u \searrow 0, \\ = -\phi \left(x^0 \right) \cdot q_{a,+}^{-1}(u) + \mathcal{O}(1), & u \nearrow 0. \end{cases} \quad (23)$$

By using the same arguments, we obtain for the left branch of the parabolic Radon transform:

1. If $q_a(t) > 0$ for all $t \in (-\varepsilon, 0)$, then

$$\mathcal{P}_{\theta,a}^- (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a}^- (\phi f) \left(x^0 \right) \begin{cases} = -\phi \left(x^0 \right) \cdot q_{a,-}^{-1}(u) + \mathcal{O}(1), & u \searrow 0, \\ = 0, & u \nearrow 0. \end{cases} \quad (24)$$

2. If $q_a(t) < 0$ for all $t \in (-\varepsilon, 0)$, then

$$\mathcal{P}_{\theta,a}^- (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a}^- (\phi f) \left(x^0 \right) \begin{cases} = \mathcal{O}(u), & u \searrow 0, \\ = \phi \left(x^0 \right) \cdot q_{a,-}^{-1}(u) + \mathcal{O}(1), & u \nearrow 0. \end{cases} \quad (25)$$

By combining the possible cases for the right (20), (23) and left branch (24), (25) of the parabolic Radon transform, we obtain a total of four cases:

I. $q_a(t) > 0$ for all $t \in (0, \varepsilon)$ and $q_a(t) > 0$ for all $t \in (-\varepsilon, 0)$:

$$\mathcal{P}_{\theta,a} (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a} (\phi f) \left(x^0 \right) \begin{cases} = \phi \left(x^0 \right) \cdot [\underbrace{q_{a,+}^{-1}(u)}_{>0 \text{ for } u>0} - \underbrace{q_{a,-}^{-1}(u)}_{<0 \text{ for } u>0}] + \mathcal{O}(1), & u \searrow 0, \\ = 0, & u \nearrow 0. \end{cases}$$

II. $q_a(t) > 0$ for all $t \in (0, \varepsilon)$ and $q_a(t) < 0$ for all $t \in (-\varepsilon, 0)$:

$$\mathcal{P}_{\theta,a} (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a} (\phi f) \left(x^0 \right) \begin{cases} = \phi \left(x^0 \right) \cdot q_{a,+}^{-1}(u) + \mathcal{O}(1), & u \searrow 0, \\ = \phi \left(x^0 \right) \cdot q_{a,-}^{-1}(u) + \mathcal{O}(1), & u \nearrow 0. \end{cases}$$

III. $q_a(t) < 0$ for all $t \in (0, \varepsilon)$ and $q_a(t) > 0$ for all $t \in (-\varepsilon, 0)$:

$$\mathcal{P}_{\theta,a} (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a} (\phi f) \left(x^0 \right) \begin{cases} = \phi \left(x^0 \right) \cdot q_{a,-}^{-1}(u) + \mathcal{O}(1), & u \searrow 0, \\ = -\phi \left(x^0 \right) \cdot q_{a,+}^{-1}(u) + \mathcal{O}(1), & u \nearrow 0. \end{cases}$$

IV. $q_a(t) < 0$ for all $t \in (0, \varepsilon)$ and $q_a(t) < 0$ for all $t \in (-\varepsilon, 0)$:

$$\mathcal{P}_{\theta,a} (\phi f) \left(x^0 + u \cdot e_\theta \right) - \mathcal{P}_{\theta,a} (\phi f) \left(x^0 \right) \begin{cases} = \mathcal{O}(u), & u \searrow 0, \\ = \phi \left(x^0 \right) \cdot [- \underbrace{q_{a,+}^{-1}(u)}_{>0 \text{ for } u<0} + \underbrace{q_{a,-}^{-1}(u)}_{<0 \text{ for } u<0}] + \mathcal{O}(1), & u \nearrow 0. \end{cases}$$

For $a \neq \kappa$, we have $q_a(t) \sim \frac{\kappa - a}{2} \cdot t^2$ for $t \to 0$, and hence

$$q_{a,\pm}^{-1}(u) \sim \pm \sqrt{\frac{2u}{\kappa - a}} \quad \text{for} \quad \frac{u}{\kappa - a} \searrow 0. \tag{26}$$

For $a = \kappa$, it results that $q_\kappa(t) = \mathcal{O}\left(t^3\right)$ for $t \to 0$. Thus,

$$q_{\kappa,\pm}^{-1}(u) = \Omega\left(\sqrt[3]{u}\right) \quad \text{for } u \searrow 0 \text{ or } u \nearrow 0 \text{ or both.} \tag{27}$$

By combining these two asymptotics (26) and (27) for $q_{a,\pm}^{-1}$ with the asymptotics I, II, III and IV of $\mathcal{P}_{\theta,a}\left(\phi f\right)\left(x^0 + u \cdot e_\theta\right) - \mathcal{P}_{\theta,a}\left(\phi f\right)\left(x^0\right)$, we obtain the desired result.

<div align="right">□</div>

3.4   Parabolic Fourier transform

In the last section, we showed that we can characterize the local curvature of an admissible set by the smoothness of the parabolic Radon transform. In order to obtain a measure which is easier to handle, we can apply the Fourier transform to the parabolic Radon transform and observe its decay rate instead of the smoothness.

Definition 3.31 (Parabolic Fourier transform). Let $f \in L^1\left(\mathbb{R}^2\right)$. Its parabolic Fourier transform is defined as

$$\mathcal{Q}_{v,\theta,a} f(\omega) := \int_{\mathbb{R}} \mathcal{P}_{\theta,a} f\left(R_\theta \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right) e^{-iu\omega} \, du \quad \text{for all } a, v, \omega \in \mathbb{R}, \theta \in \mathbb{T}.$$

We will often abbreviate the parabolic Fourier transform as PFT.

3.4.1   Basic properties of the parabolic Fourier transform

The first lemma shows the absolute continuity and boundedness of the parabolic Fourier transform. Furthermore, it allows us to rewrite the parabolic Fourier transform of a function $f \in L^1(\mathbb{R}^2)$ in terms of an integral of f directly instead of an integral of its parabolic Radon transform.

Lemma 3.32. *Let $f \in L^1(\mathbb{R}^2)$. Then,*

1. *$\mathcal{Q}_{v,\theta,a} f$ is absolutely continuous and bounded for all $a, v \in \mathbb{R}, \theta \in \mathbb{T}$,*

2. *$\mathcal{Q}_{v,\theta,a} f(\omega) = \int\limits_{\mathbb{R}^2} f(x) \cdot \exp\left(-i\omega \left[x_\theta - \frac{a}{2} \cdot \left(x_{\theta + \frac{\pi}{2}} - v\right)^2\right]\right) dx$ for all $a, v, \omega \in \mathbb{R}, \theta \in \mathbb{T}$.*

Proof.

1. We will first show that for all $a, v \in \mathbb{R}, \theta \in \mathbb{T}$, the parabolic Radon transform satisfies

$$\mathcal{P}_{\theta,a} f\left(R_\theta \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right) \in L^1(\mathbb{R}, du).$$

By using $f \in L^1(\mathbb{R}^2)$, we obtain for $a, v \in \mathbb{R}, \theta \in \mathbb{T}$,

$$\int_\mathbb{R} \left| \mathcal{P}_{\theta,a} f\left(R_\theta \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right) \right| du$$

$$= \int_\mathbb{R} \left| \int_\mathbb{R} f\left(R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ v + t \end{pmatrix}\right) dt \right| du$$

$$\leq \int_\mathbb{R} \int_\mathbb{R} \left| f\left(R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ v + t \end{pmatrix}\right) \right| dt\, du \qquad \left(\text{substitute } u = w - \frac{a}{2} \cdot t^2,\ t = s - v\right)$$

$$= \int_\mathbb{R} \int_\mathbb{R} \left| f\left(R_\theta \cdot \begin{pmatrix} w \\ s \end{pmatrix}\right) \right| ds\, dw$$

$$= \|f\|_{L^1} < \infty.$$

By utilizing Theorem 1.20 and Theorem 1.21, we obtain the desired result.

2. Similarly, we determine that

$$\mathcal{Q}_{v,\theta,a} f(\omega) = \int_\mathbb{R} \mathcal{P}_{\theta,a} f\left(R_\theta \cdot \begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{-iu\omega}\, du$$

$$= \int_\mathbb{R} \int_\mathbb{R} f\left(R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ v + t \end{pmatrix}\right) dt \cdot e^{-iu\omega}\, du \qquad \left(\text{substitute } u = x_\theta - \frac{a}{2} \cdot t^2\right)$$

$$= \int_\mathbb{R} \int_\mathbb{R} f\left(R_\theta \cdot \begin{pmatrix} x_\theta \\ v + t \end{pmatrix}\right) dt \cdot e^{-i\left(x_\theta - \frac{a}{2} \cdot t^2\right)\omega}\, dx_\theta \qquad \left(\text{substitute } t = x_{\theta + \frac{\pi}{2}} - v\right)$$

$$= \int_{\mathbb{R}^2} f(x) \cdot \exp\left(-i\left[x_\theta - \frac{a}{2} \cdot \left(x_{\theta + \frac{\pi}{2}} - v\right)^2\right]\omega\right) dx.$$

$\square$

The representation of the parabolic Fourier transform in Lemma 3.32 shows that this transform can be viewed as a Fourier transform in the direction e_θ and as a convolution with the function

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto e^{\frac{i}{2} \cdot a\omega t^2}$$

in the direction $e_{\theta + \frac{\pi}{2}}$. The latter is innately connected to the *Fresnel transform*. It was introduced as an approximation formula for the diffraction pattern in the near field of a wave propagating through an aperture [Ers06] and plays an important role in digital holography [CBD99, (1.9)], [LBU03]. A brief look into the theory of the Fresnel transform helps us to find an inversion formula for the parabolic Fourier transform.

Definition 3.33 (Fresnel transform)**.** [LBU03] Let $\tau > 0$ and let

$$k_\tau : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{\tau} \cdot \exp\left(i \cdot \frac{x^2}{2\tau^2}\right).$$

For $f \in L^1(\mathbb{R})$, the Fresnel transform of f is defined as

$$\mathbf{Fr}_\tau f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \left(f * k_\tau\right)(x).$$

Lemma 3.34 (Inversion of the Fresnel transform)**.** *Let $f \in L^1(\mathbb{R})$ and let $\tau > 0$. Then,*

$$f(x) = \frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{-R}^{R} \left(\mathbf{Fr}_\tau f\right)(y) \cdot \overline{k_\tau(x - y)} \, dy \quad \text{for almost all } x \in \mathbb{R}.$$

Proof. For almost all $x \in \mathbb{R}$, we obtain

$$\frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{-R}^{R} \left(\mathbf{Fr}_\tau f\right)(y) \cdot \overline{k_\tau(x - y)} \, dy = \frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{-R}^{R} \int_{\mathbb{R}} f(z) \cdot k_\tau(y - z) \, dz \cdot \overline{k_\tau(x - y)} \, dy$$

$$= \frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{\mathbb{R}} f(z) \cdot \underbrace{\int_{-R}^{R} k_\tau(y - z) \cdot \overline{k_\tau(x - y)} \, dy}_{=K_R(x,z)} \, dz,$$

where the change of integration in the last equality is achieved through Fubini's theorem. In order to prove the desired result, we will determine the integral $K_R(x, z)$:

$$K_R(x, z) = \int_{-R}^{R} k_\tau(y - z) \cdot \overline{k_\tau(x - y)} \, dy$$

$$= \frac{1}{\tau^2} \cdot \int_{-R}^{R} \exp\left(\frac{i}{2\tau^2} \cdot (y - z)^2\right) \cdot \exp\left(-\frac{i}{2\tau^2} \cdot (x - y)^2\right) \, dy$$

$$= \frac{1}{\tau^2} \cdot \int_{-R}^{R} \exp\left(\frac{i}{2\tau^2} \cdot \left(z^2 - x^2 + 2(x - z)y\right)\right) \, dy$$

$$= \frac{1}{\tau^2} \cdot \exp\left(\frac{i}{2\tau^2} \cdot \left(z^2 - x^2\right)\right) \cdot \int_{-R}^{R} \exp\left(\frac{i}{\tau^2} \cdot \left((x - z)y\right)\right) \, dy$$

$$= \frac{1}{\tau^2} \cdot \exp\left(\frac{i}{2\tau^2} \cdot \left(z^2 - x^2\right)\right) \cdot 2R \cdot \operatorname{sinc}\left(\frac{(x - z) \cdot R}{\tau^2}\right).$$

Due to [AS$^+$72, 4.3.142], $\int_{\mathbb{R}} \mathrm{sinc}(x)\, dx = \pi$. Hence, the function

$$S_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{\varepsilon}{\pi} \cdot \mathrm{sinc}\,(\varepsilon \cdot x)$$

is an approximate identity. Since $f \in L^1(\mathbb{R})$, Theorem 1.27 yields

$$\frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{-R}^{R} \left(\mathbf{Fr}_\tau f \right)(y) \cdot \overline{k_\tau(x-y)}\, dy$$

$$= \frac{1}{2\pi} \cdot \lim_{R \to \infty} \int_{\mathbb{R}} f(z) \cdot \int_{-R}^{R} k_\tau(y-z) \cdot \overline{k_\tau(x-y)}\, dy\, dz$$

$$= \exp\left(-\frac{i \cdot x^2}{2\tau^2} \right) \cdot \lim_{R \to \infty} \int_{-R}^{R} f(z) \cdot \exp\left(\frac{i \cdot z^2}{2\tau^2} \right) \cdot \underbrace{\frac{R}{\pi \tau^2} \cdot \mathrm{sinc}\left(\frac{(x-z) \cdot R}{\tau^2} \right)}_{= S_{\tau^2/R}(x-z)}\, dz$$

$$= \exp\left(-\frac{i \cdot x^2}{2\tau^2} \right) \cdot f(x) \cdot \exp\left(\frac{i \cdot x^2}{2\tau^2} \right) = f(x)$$

for almost all $x \in \mathbb{R}$. □

With the help of this result, we can now show the inversion of the parabolic Fourier transform.

Theorem 3.35 (Inversion of the parabolic Fourier transform)**.** *Let $f \in L^1\left(\mathbb{R}^2\right)$, $a \in \mathbb{R} \setminus \{0\}$, $\theta \in \mathbb{T}$. Then, for almost all $x \in \mathbb{R}^2$,*

$$f(x) = \frac{1}{4\pi^2} \cdot \lim_{\lambda, R \to \infty} \int_{-\lambda}^{\lambda} \int_{-R}^{R} \mathcal{Q}_{v, \theta, a} f(\omega) |a\omega| \left(1 - \frac{|x_\theta|}{\lambda} \right) \exp\left(i\omega \cdot \left[x_\theta + \frac{a}{2} \left(x_{\theta + \frac{\pi}{2}} - v \right)^2 \right] \right)\, dv\, d\omega.$$

Proof. By utilizing statement 2 of Lemma 3.32, we can represent the parabolic Fourier transform as

$$\mathcal{Q}_{v, \theta, a} f(\omega) = \int_{\mathbb{R}^2} f(x) \cdot \exp\left(-i\omega \left[x_\theta - \frac{a}{2} \cdot \left(x_{\theta + \frac{\pi}{2}} - v \right)^2 \right] \right)\, dx$$

$$= \int_{\mathbb{R}} \mathcal{F}_{e_\theta} f \left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta + \frac{\pi}{2}} \end{pmatrix} \right) \cdot \exp\left(i \cdot \frac{a\omega}{2} \cdot \left(x_{\theta + \frac{\pi}{2}} - v \right)^2 \right)\, dx_{\theta + \frac{\pi}{2}}$$

$$= \int_{\mathbb{R}} \mathcal{F}_{e_\theta} f \left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta + \frac{\pi}{2}} \end{pmatrix} \right) \cdot \exp\left(\mathrm{sgn}(a\omega) \cdot \frac{i}{2} \cdot \left[\sqrt{|a\omega|} \cdot \left(x_{\theta + \frac{\pi}{2}} - v \right) \right]^2 \right)\, dx_{\theta + \frac{\pi}{2}}$$

for all $\omega \in \mathbb{R}$. By introducing the function,

$$k_{\pm, \tau} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \frac{1}{\tau} \cdot \exp\left(\pm i \cdot \frac{t^2}{2\tau^2} \right),$$

we can rewrite the parabolic Fourier transform for $a, \omega \neq 0$ as

$$\mathcal{Q}_{v,\theta,a}f(\omega) = \frac{1}{\sqrt{|a\omega|}} \cdot \left(\mathcal{F}_{e_\theta}f\left(R_\theta \cdot \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \right) \right) * k_{\text{sgn}(a\omega),|a\omega|^{-\frac{1}{2}}} \left(x_{\theta+\frac{\pi}{2}} \right). \tag{28}$$

We will now distinguish between three cases: $a\omega > 0$, $a\omega < 0$ and $a\omega = 0$.

$\underline{a\omega > 0}$:

(28) yields

$$\mathcal{Q}_{v,\theta,a}f(\omega) = \frac{1}{\sqrt{a\omega}} \cdot \mathbf{Fr}_{(a\omega)^{-\frac{1}{2}}} \left(\mathcal{F}_{e_\theta}f\left(R_\theta \cdot \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \right) \right)(v). \tag{29}$$

By applying Lemma 3.34, we obtain that for almost all $x_{\theta+\frac{\pi}{2}}, \omega \in \mathbb{R}$,

$$\mathcal{F}_{e_\theta}f\left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta+\frac{\pi}{2}} \end{pmatrix} \right) = \frac{1}{2\pi} \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathbf{Fr}_{(a\omega)^{-\frac{1}{2}}} \left(\mathcal{F}_{e_\theta}f\left(R_\theta \cdot \begin{pmatrix} \omega \\ v \end{pmatrix} \right) \right) \cdot \overline{k_{(a\omega)^{-\frac{1}{2}}}\left(x_{\theta+\frac{\pi}{2}} - v \right)} \, dv$$

$$\overset{(29)}{=} \frac{1}{2\pi} \cdot \sqrt{a\omega} \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a}f(\omega) \overline{k_{(a\omega)^{-\frac{1}{2}}}\left(x_{\theta+\frac{\pi}{2}} - v \right)} \, dv$$

$$= \frac{1}{2\pi} \cdot a\omega \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a}f(\omega) \cdot \exp\left(i \cdot \frac{a\omega}{2} \cdot \left(x_{\theta+\frac{\pi}{2}} - v \right)^2 \right) \, dv. \tag{30}$$

$\underline{a\omega < 0}$:

In this case, (28) yields

$$\mathcal{Q}_{v,\theta,a}f(\omega) = \frac{1}{\sqrt{-a\omega}} \cdot \left(\mathcal{F}_{e_\theta}f\left(R_\theta \cdot \begin{pmatrix} \omega \\ \cdot \end{pmatrix} \right) \right) * \overline{k_{(-a\omega)^{-\frac{1}{2}}}}(v). \tag{31}$$

As we want to retrieve the original function f from $\mathcal{Q}_{v,\theta,a}$, we will now show the inversion of a convolution with $\overline{k_\tau}$:

$$\frac{1}{2\pi} \cdot \lim_{R\to\infty} \int_{-R}^{R} \left(f * \overline{k_\tau} \right)(y) k_\tau(x-y) \, dy$$

$$= \overline{\frac{1}{2\pi} \cdot \lim_{R\to\infty} \int_{-R}^{R} \left(\overline{f} * k_\tau \right)(y) \cdot \overline{k_\tau}(x-y) \, dy}$$

$$\overset{\text{Lemma 3.34}}{=} \overline{\overline{f(x)}} = f(x) \quad \text{for almost all } x \in \mathbb{R}.$$

By applying the upper equality, we obtain that for almost all $x_{\theta+\frac{\pi}{2}}, \omega \in \mathbb{R}$,

$$
\mathcal{F}_{e_\theta} f\left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta+\frac{\pi}{2}} \end{pmatrix}\right) = \frac{1}{2\pi} \cdot \lim_{R\to\infty} \int_{-R}^{R} \left(\mathcal{F}_{e_\theta} f\left(R_\theta \cdot \begin{pmatrix} \omega \\ \cdot \end{pmatrix}\right) * \overline{k_{(-a\omega)^{-\frac{1}{2}}}}\right)(v) \cdot k_{(-a\omega)^{-\frac{1}{2}}}\left(x_{\theta+\frac{\pi}{2}} - v\right) dv
$$

$$
\overset{(31)}{=} \frac{1}{2\pi} \cdot \sqrt{-a\omega} \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a} f(\omega) \cdot k_{(-a\omega)^{-\frac{1}{2}}}\left(x_{\theta+\frac{\pi}{2}} - v\right) dv
$$

$$
= -\frac{1}{2\pi} \cdot a\omega \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a} f(\omega) \cdot \exp\left(i \cdot \frac{a\omega}{2} \cdot \left(x_{\theta+\frac{\pi}{2}} - v\right)^2\right) dv. \qquad (32)
$$

$\underline{a\omega = 0:}$

Since $a \in \mathbb{R} \setminus \{0\}$, we can conclude $\omega = 0$. Then, due to statement 2 of Lemma 3.32,

$$
\mathcal{Q}_{v,\theta,a} f(0) = \int_{\mathbb{R}^2} f(x) \cdot \exp\left(-i\omega \left[x_\theta - \frac{a}{2} \cdot \left(x_{\theta+\frac{\pi}{2}} - v\right)^2\right]\right) dx \bigg|_{\omega=0} = \int_{\mathbb{R}^2} f(x) \, dx = \hat{f}(0). \qquad (33)
$$

$\underline{\text{Merger:}}$

By combining (30) and (32), we obtain for $\omega \neq 0$,

$$
\mathcal{F}_{e_\theta} f\left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta+\frac{\pi}{2}} \end{pmatrix}\right) = \frac{1}{2\pi} \cdot |a\omega| \cdot \lim_{R\to\infty} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a} f(\omega) \cdot \exp\left(i \cdot \frac{a\omega}{2} \cdot \left(x_{\theta+\frac{\pi}{2}} - v\right)^2\right) dv. \qquad (34)
$$

Due to (33), $\left|\mathcal{Q}_{v,\theta,a} f(0)\right| = \left|\hat{f}(0)\right| < \infty$. Hence, by applying the inverse Fourier transform we obtain for almost all $x \in \mathbb{R}^2$,

$$
f(x) = \frac{1}{2\pi} \cdot \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \mathcal{F}_{e_\theta} f\left(R_\theta \cdot \begin{pmatrix} \omega \\ x_{\theta+\frac{\pi}{2}} \end{pmatrix}\right) \cdot \left(1 - \frac{|x_\theta|}{\lambda}\right) e^{i\omega x_\theta} \, d\omega
$$

$$
\overset{(34)}{=} \frac{1}{4\pi^2} \cdot \lim_{\lambda,R\to\infty} \int_{-\lambda}^{\lambda} \int_{-R}^{R} \mathcal{Q}_{v,\theta,a} f(\omega) |a\omega| \left(1 - \frac{|x_\theta|}{\lambda}\right) \exp\left(i\omega \cdot \left[x_\theta + \frac{a}{2}\left(x_{\theta+\frac{\pi}{2}} - v\right)^2\right]\right) dv \, d\omega.
$$

$\qquad\quad \square$

As we are interested in determining the curvature of the singular support of the indicator function of an admissible set, we will investigate a direct relation between the smoothness of a function and the decay rate of its Fourier transform in the following subsection.

3.4.2 Singularities and the decay of the Fourier transform

In this subsection, we will show a quite detailed connection between the singularities of a function and the decay rate of its Fourier transform. The journey will start at a well known theorem

of Zygmund establishing a decay rate of the Fourier coefficients of a periodic function in dependence of its module of continuity. On this basis, we will prove similar results for the integral Fourier transform. Furthermore, we will show the special role that isolated singularities play in this setting and we will prove a characterization of a function's modulus of continuity when assuming a certain decay rate of its Fourier transform as well as a vice-versa characterization.

We have already introduced the concept of the singular support which we will now specify with the definition of the singular support of certain orders.

Definition 3.36 (Singular support of order n)**.** Let $f \in \mathcal{S}'(\mathbb{R})$. We say that f has a singularity at the point $x \in \mathbb{R}$, if $x \in \mathrm{sing\,supp}(f)$. For $n \in \mathbb{N}_0$, the *singular support of order n* is defined as

$$S_n(f) := \left\{ x \in \mathbb{R} : \ \forall \phi \in C_c^\infty(\mathbb{R}) \mid \phi(x) \neq 0 : \ \phi f \notin C^n(\mathbb{R}) \right\}.$$

In order to classify the singularities of a function, we need a measure which allows a precise differentiation between the various kinds of singularities that can appear. Our measure of choice is the modulus of continuity.

Definition 3.37 (Central difference and modulus of continuity)**.** Let $f : \mathbb{R} \to \mathbb{R}$. Then for $h > 0$ the *central difference* of f is
$$\Delta_h f(x) = f(x+h) - f(x-h).$$

For $n \in \mathbb{N}$, the *nth order central difference* of f is recursively defined as

$$\Delta_h^n f(x) = \Delta_h \left(\Delta_h^{n-1} f \right)(x).$$

The *nth order modulus of continuity* of f is

$$\omega_n(h; f) := \sup_{h' \leq h} \sup_{x \in \mathbb{R}} |\Delta_{h'}^n f(x)|.$$

Whenever it is clear which function we refer to, we will write $\omega_n(h)$ instead of $\omega_n(h; f)$.

In order to give a precise estimation of the modulus of continuity, we are now introducing slowly and regularly varying functions.

Definition 3.38 (Slowly and regularly varying functions)**.** A function $f : (0, \infty) \to (0, \infty)$ is called *regularly varying with exponent $\alpha \in \mathbb{R}$* if

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha$$

for all $\lambda > 0$. A regularly varying function with exponent 0 is also called *slowly varying*.

A typical slowly varying function is e. g. $L(x) = \max\{1, \log(x)\}$.
We now use the newly introduced terminology to define the Lipschitz space.

Definition 3.39 (Lipschitz space)**.** [Cli91] Let $r : (0, \infty) \to (0, \infty)$ be a regularly varying function and $s(t) = r\left(\frac{1}{t}\right)$. Then for an interval $I \subset \mathbb{R}$, the *Lipschitz space* $\text{Lip}(s, I)$ is defined as

$$\text{Lip}(s, I) := \left\{ f : I \to \mathbb{R} \mid \exists\, C > 0 : \omega_1(h; f) \leq C \cdot s(h) \right\}.$$

For $s(t) = t^\alpha$, we write

$$\text{Lip}_\alpha(I) := \text{Lip}(s, I).$$

Remark 3.40. Due to Karamata every regularly varying function f can be represented by an $\alpha \in \mathbb{R}$ and a slowly varying function L in the following way:

$$f(x) = x^\alpha L(x). \quad \text{[BGT89]}$$

Furthermore, for all regularly varying functions s with the property $s(t) = o(t)$ for $t \to 0$, we obtain

$$f \in \text{Lip}\,(s, I) \Rightarrow f \text{ is constant.}$$

The next theorem of Zygmund shows a basic connection between the modulus of continuity of periodic functions and the decay of their Fourier coefficients.

Theorem 3.41 (Zygmund)**.** [Zyg02] *Let $f \in L^1(\mathbb{T})$. Then for all $k \in \mathbb{Z} \setminus \{0\}$ the following inequality holds:*

$$|\hat{f}_k| \leq \frac{1}{2} \cdot \omega_1\left(\frac{\pi}{2|k|}\right).$$

Proof. For all $k \in \mathbb{Z} \setminus \{0\}$ we obtain by shifting f by $\frac{\pi}{k}$:

$$\hat{f}_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} f\left(x - \frac{\pi}{k}\right) e^{-ikx} dx.$$

By building the arithmetic mean of the two representations of the Fourier coefficient $\hat{f}_k$ we obtain

$$\hat{f}_k = \frac{1}{4\pi} \int_0^{2\pi} \Delta_{\frac{\pi}{2k}} f\left(x - \frac{\pi}{2k}\right) e^{-ikx} dx = \frac{1}{4\pi i} \int_0^{2\pi} \Delta_{\frac{\pi}{2k}} f(x) e^{-ikx} dx$$

and hence

$$|\hat{f}_k| \leq \frac{1}{4\pi} \int_0^{2\pi} |\Delta_{\frac{\pi}{2k}} f(x)| dx \leq \frac{1}{2} \cdot \omega_1\left(\frac{\pi}{2|k|}\right).$$

$\square$

As the next proposition shows, this result can be transferred to the continuous Fourier transform and to higher order differences.

Proposition 3.42. *Let $f \in L^1(\mathbb{R})$ such that* $\mathrm{supp}(f) \subset [a, b]$. *Then, for all $n \in \mathbb{Z}_+$ the following inequality holds for all $\xi \in \mathbb{R} \setminus \{0\}$*

$$|\hat{f}(\xi)| \leq \frac{b-a}{2^{n+1}\pi} \cdot \omega_n\left(\frac{\pi}{|2\xi|}\right).$$

Proof. Since $\mathrm{supp}(f) \subset [a, b]$, we have

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_a^b f(x) e^{-ix\xi} dx.$$

Now we define a periodic function g by

$$g(x) = \sum_{k \in \mathbb{Z}} f(x + k(b - a))$$

and rewrite the Fourier transform of f as

$$\hat{f}(\xi) = \frac{1}{2\pi} \int_a^b g(x) e^{-ix\xi} dx$$
$$= \frac{1}{4\pi i} \int_a^b \Delta_{\frac{\pi}{2\xi}} g(x) e^{-ix\xi} dx$$

by applying the same trick as in the proof of Zygmund's theorem. A recursive application of the trick delivers

$$\hat{f}(\xi) = \frac{1}{(2i)^n \cdot 2\pi} \int_a^b \Delta_{\frac{\pi}{2\xi}}^n g(x) e^{-ix\xi} dx$$

Hence, we obtain

$$|\hat{f}(\xi)| \leq \frac{1}{2^{n+1}\pi} \int_a^b \left|\Delta_{\frac{\pi}{2\xi}}^n g(x)\right| dx \leq \frac{b-a}{2^{n+1}\pi} \cdot \omega_n\left(\frac{\pi}{2|\xi|}\right).$$

$\square$

The last two results especially imply that for a compact interval I and for all $f \in \mathrm{Lip}_\alpha(I)$ we have $\hat{f}(\xi) = \mathcal{O}\left(|\xi|^{-\alpha}\right)$. The following classical example shows the sharpness of this implication.

Example 3.43 (Weierstraß function)**.** Let $0 < \alpha < 1$ and $b > 1$. Then the Weierstraß function f_α is defined by

$$f_\alpha(x) = \sum_{k=0}^{\infty} b^{-\alpha k} \cos\left(b^k x\right).$$

For all $\alpha \in (0, 1)$ it holds that $f_\alpha \in \mathrm{Lip}_\alpha(\mathbb{T})$ [Zyg02]. Furthermore, by choosing $b = 2$, we obtain

$$\hat{f}(\xi) = \begin{cases} |k|^{-\alpha}, & \text{if } |k| = 2^n, \\ 0, & \text{else.} \end{cases}$$

Hence, we can not expect a better result than $\hat{f}(\xi) = \mathcal{O}\left(|\xi|^{-\alpha}\right)$ for $f \in \mathrm{Lip}_\alpha(I)$.

The most prominent property of the Weierstraß function is its that it is nowhere differentiable. Although the continuous, but nowhere differentiable functions over an interval are, w. r. t. the topology of uniform convergence, dense in the set of continuous functions [Ban31], this property is rarely seen in practical situations. When we, for instance, take a look at the unit step function $g(x) = \mathbb{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(x)$, we can see that its modulus of continuity is $\omega_1 \equiv 1$ and that its Fourier transform is $\hat{g}(\xi) = \text{sinc}(\xi) = \mathcal{O}(|\xi|^{-1})$. Hence, the Fourier transform decays faster than predicted by the proposition above. As the next theorem shows, the reason for the faster decay is the fact that its singularities are isolated.

Theorem 3.44. *Let $f \in L^1(\mathbb{R})$, $n \in \mathbb{Z}_+$ and $|S_{n+1}(f)| < \infty$. Furthermore, let there be an $\varepsilon(x) > 0$ for every $x \in S_{n+1}(f)$ such that $f^{(n+1)}$ does not change its sign on $(x - \varepsilon(x), x)$ and $(x, x + \varepsilon(x))$. Then there exists a constant $C > 0$ such that for all $\xi \in \mathbb{R} \backslash \{0\}$ we have*

$$|\hat{f}(\xi)| \leq \frac{C}{|\xi|} \cdot \omega_n\left(\frac{\pi}{2|\xi|}\right).$$

Proof. Since the C^{n+1}-singularities are isolated from each other, we will build a partition of unity which separates them so we can scrutinize them individually. Therefore, let $K := |S_{n+1}(f)|$ and let

$$\varepsilon := \frac{1}{2} \cdot \min_{m \in \{1, \dots, K\}} \varepsilon(x_m) > 0. \tag{35}$$

Hence, we have

$$\min_{k, m \in \{1, \dots, K\}, k \neq m} |x_m - x_k| \geq 2\varepsilon. \tag{36}$$

Let $\{\phi_m\}_{m \in \{1, \dots, K\}} \cup \{\psi\}$ be a partition of unity on $\mathbb{R}$ such that

$$\ker(1 - \phi_m) = \left[x_m - \frac{\varepsilon}{2}, \ x_m + \frac{\varepsilon}{2}\right],$$
$$\text{supp}(\phi_m) = [x_m - \varepsilon, \ x_m + \varepsilon].$$

for all $m \in \{1, \dots, K\}$. Meanwhile, ψ takes care of all the leftover spots which are not covered by the ϕ_m. Due to Equation (35) and Equation (36) the supports of the ϕ_m are disjoint and $f^{(n+1)}(y)$ does not change sign on either of the intervals $(x_m - \varepsilon, x_m)$ and $(x_m, x_m + \varepsilon)$. Now we can rewrite the Fourier transform of f as

$$\hat{f}(\xi) = \sum_{m=1}^{K} \widehat{\phi_m f}(\xi) + \widehat{\psi f}(\xi)$$
$$= \frac{1}{2\pi} \sum_{m=1}^{K} \int_{x_m - \varepsilon}^{x_m + \varepsilon} f(x)\phi_m(x) e^{-ix\xi} dx + \widehat{\psi f}(\xi)$$

for all $\xi \in \mathbb{R}$. As all points in which f is not $n+1$ times continuously differentiable are completely covered by the ϕ_m, we have $\psi f \in C^{n+1}(\mathbb{R})$. Since according to the conditions $f \in L^1(\mathbb{R})$, we in conclusion obtain that $\widehat{\psi f}(\xi) = o(|\xi|^{-(n+1)})$. Now we get back to the remaining summands

$\widehat{\phi_m f}(\xi)$, apply Zygmund's trick iteratively as in Proposition 3.42 and get

$$\widehat{\phi_m f}(\xi) = \frac{1}{(2i)^n \cdot 2\pi} \int_{x_m - \varepsilon - \frac{n\pi}{2|\xi|}}^{x_m + \varepsilon + \frac{n\pi}{2|\xi|}} \Delta^n_{\frac{\pi}{2|\xi|}} (\phi_m f)(x) e^{-ix\xi} dx.$$

Now we assume $|\xi| > \frac{2n\pi}{\varepsilon}$, i.e. $\frac{n\pi}{|\xi|} \in \left(0, \frac{\varepsilon}{2}\right)$, and divide the interval $\left[x_m - \varepsilon - \frac{n\pi}{2|\xi|},\ x_m + \varepsilon + \frac{n\pi}{2|\xi|}\right]$ into the five parts $J_1 = \left[x_m - \varepsilon - \frac{n\pi}{2|\xi|},\ x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}\right]$, $I_1 = \left[x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2},\ x_m - \frac{n\pi}{2|\xi|}\right]$, $I' = \left[x_m - \frac{n\pi}{2|\xi|},\ x_m + \frac{n\pi}{2|\xi|}\right]$, $I_2 = \left[x_m + \frac{n\pi}{2|\xi|},\ x_m - \frac{n\pi}{2|\xi|} + \frac{\varepsilon}{2}\right]$ and $J_2 = \left[x_m - \frac{n\pi}{2|\xi|} + \frac{\varepsilon}{2},\ x_m + \varepsilon + \frac{n\pi}{2|\xi|}\right]$.

J_1 and J_2:

For J_1 and J_2 we obtain due to the differentiability of f on these intervals that

$$\left| \int_{J_k} \Delta^n_{\frac{\pi}{2|\xi|}} (\phi_m f)(x) e^{-ix\xi} dx \right| = o\left(|\xi|^{-(n+1)}\right) \quad \text{for } \xi \to \pm\infty$$

for $k \in \{1, 2\}$.

I':

On the middle interval I' we have $\phi_m \equiv 1$ and hence we can estimate

$$\left| \int_{I'} \Delta^n_{\frac{\pi}{2|\xi|}} (\phi_m f)(x) e^{-ix\xi} dx \right| \le \int_{x_m - \frac{n\pi}{2|\xi|}}^{x_m + \frac{n\pi}{2|\xi|}} \left| \Delta^n_{\frac{\pi}{2|\xi|}} f(x) \right| dx \le \frac{n\pi}{|\xi|} \cdot \omega_n\left(\frac{\pi}{2|\xi|}; f\right).$$

I_1 and I_2:

For the two remaining intervals I_1 and I_2, we cannot apply the same differentiability argument as for J_1 and J_2, since I_1 and I_2 become arbitrarily close to the singularity in x_m for $\xi \to \pm\infty$. Hence, we use a different approach, which utilizes the fact that $f^{(n+1)}$ does not change its sign on $(x_m - \varepsilon, x_m + \varepsilon)$.

Due to the definition of ϕ_m, we have $\phi_m \equiv 1$ on the intervals I_1 and I_2 as well. We will now estimate the integral over I_1. By applying a partial integration, we obtain

$$\left| \int_{I_1} \left[\Delta^n_{\frac{\pi}{2|\xi|}} f(x) \right] e^{-ix\xi} dx \right|$$

$$= \frac{1}{|\xi|} \cdot \left| \left[\Delta^n_{\frac{\pi}{2|\xi|}} f(x) e^{-ix\xi} \right]_{x = x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x = x_m - \frac{n\pi}{2|\xi|}} - \int_{x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x_m - \frac{n\pi}{2|\xi|}} \Delta^n_{\frac{\pi}{2|\xi|}} f'(x) e^{-ix\xi} dx \right|$$

$$\le \frac{1}{|\xi|} \cdot \left(\left| \Delta^n_{\frac{\pi}{2|\xi|}} f(x) \right|_{x = x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x = x_m - \frac{n\pi}{2|\xi|}} + \int_{x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x_m - \frac{n\pi}{2|\xi|}} \left| \Delta^n_{\frac{\pi}{2|\xi|}} f'(x) \right| dx \right)$$

$$\le \frac{\varepsilon}{2|\xi|} \cdot \omega_n\left(\frac{n\pi}{|\xi|}; f\right) + \frac{1}{|\xi|} \cdot \int_{x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x_m - \frac{n\pi}{2|\xi|}} \left| \Delta^n_{\frac{\pi}{2|\xi|}} f'(x) \right| dx \tag{37}$$

For the estimation of the leftover integral, we want to show that $\Delta^n_{\frac{\pi}{2|\xi|}} f'(x)$ does not change its sign on the domain of integration and consequently, we can draw the absolute value out of

the integral without changing the result. Due to the conditions of the theorem, $f^{(n+1)}$ does not change its sign on $\left(x_m - \frac{\varepsilon}{2}, x_m\right)$. W.l.o.g. we will assume $f^{(n+1)}(x)$ is non-negative on the interval $\left(x_m - \frac{\varepsilon}{2}, x_m\right)$. For $x \in \left(x_m + h - \frac{\varepsilon}{2}, x_m - h\right)$, we utilize the fundamental theorem of calculus:

$$\Delta_h f^{(n)}(x) = \int_{x-h}^{x+h} \underbrace{f^{(n+1)}(y)}_{\geq 0} \, dy \geq 0.$$

By recursively applying this argument, we obtain for all $x \in \left(x_m + n \cdot h - \frac{\varepsilon}{2}, x_m - n \cdot h\right)$ that

$$\Delta_h^n f'(x) = \int_{x-h}^{x+h} \underbrace{\Delta_h^{n-1} f''(y)}_{\geq 0} \, dy \geq 0.$$

As the conditions are fulfilled for $h = \frac{\pi}{2|\xi|}$ and $x \in I_1$, we obtain

$$
\begin{aligned}
\int_{x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x_m - \frac{n\pi}{2|\xi|}} \left| \Delta_{\frac{\pi}{2|\xi|}}^n f'(x) \right| dx &= \left| \int_{x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}}^{x_m - \frac{n\pi}{2|\xi|}} \Delta_{\frac{\pi}{2|\xi|}}^n f'(x) dx \right| \\
&= \left| \Delta_{\frac{\pi}{2|\xi|}}^n f\left(x_m - \frac{n\pi}{2|\xi|}\right) - \Delta_{\frac{\pi}{2|\xi|}}^n f\left(x_m + \frac{n\pi}{2|\xi|} - \frac{\varepsilon}{2}\right) \right| \\
&\leq 2 \cdot \omega_n\left(\frac{\pi}{2|\xi|}; f\right).
\end{aligned}
$$
(38)

By combining Equation (37) and Equation (38), we can acquire

$$\left| \int_{I_1} \Delta_{\frac{\pi}{2|\xi|}}^n \left(\phi_m f\right)(x) e^{-ix\xi} dx \right| \leq \frac{4+\varepsilon}{2|\xi|} \cdot \omega_n\left(\frac{\pi}{2|\xi|}; f\right).$$

For the integral over I_2 we can get the same estimation analogously. Now adding up over the five intervals I', I_1, I_2, J_1 and J_2 delivers

$$|\widehat{\phi_m f}(\xi)| \leq \frac{4+\varepsilon+\pi}{|\xi|} \cdot \omega_n\left(\frac{\pi}{2|\xi|}\right) + o\left(|\xi|^{-(n+1)}\right) \quad \text{for } \xi \to \pm\infty$$

for $|\xi| > \frac{2n\pi}{\varepsilon}$. By building the sum over all $m \in \{1, \ldots, K+1\}$, we obtain for all $\xi \in \mathbb{R} \setminus \{0\}$ and some constant $C > 0$ that

$$|\hat{f}(\xi)| \leq \frac{C}{|\xi|} \cdot \omega_n\left(\frac{\pi}{2|\xi|}\right).$$

$\square$

Although the condition that the sign of the nth order difference of the derivative does not change in a neighborhood of a singularity seems quite technical, it is nevertheless necessary as the following example shows.

Example 3.45. Let $f(x) = \sin\left(\frac{1}{x}\right)$. Due to the highly oscillatory nature of the function around the singular point $x = 0$, two points $x_{\max}$ and $x_{\min}$ such that $\sin\left(\frac{1}{x_{\max}}\right) = 1$ and $\sin\left(\frac{1}{x_{\min}}\right) = -1$ can be found in every neighborhood of $x = 0$. Hence, the modulus of continuity is $\omega \equiv 2$. Due to the

isolated singularity in the point $x = 0$, one might expect that its Fourier transform behaves like $\hat{f}(\xi) = \mathcal{O}\left(|\xi|^{-1}\right)$, but instead of this we have

$$\hat{f}(\xi) = -i \cdot \mathrm{sgn}(\xi) \cdot \frac{J_1\left(2\sqrt{|\xi|}\right)}{2\sqrt{|\xi|}} \sim |\xi|^{-\frac{3}{4}} \cdot \cos\left(\frac{\pi}{4} + \sqrt{\xi}\right) \quad \text{for } \xi \to \pm\infty. \quad \text{[PEB54, 2.7 (6)]}$$

Naturally the question arises whether the decay of the Fourier transform also implies a certain behavior of the modulus of continuity. This connection is specified by the next theorem.

Theorem 3.46. *Let $f \in L^1(\mathbb{R})$ and let there exist $n \in \mathbb{Z}_+$, $\alpha > 0$, $C > 0$ and a slowly varying monotonic function $L : (0, \infty) \to (0, \infty)$ such that*

$$|\hat{f}(\xi)| \le \frac{C}{|\xi|^{n+\alpha}} \cdot L(|\xi|).$$

Then there exists constants $\varepsilon, C' > 0$ such that the following inequality holds:

$$\omega_n(h; f) \le C' h^{n-1} \cdot \max\left\{h, h^\alpha \cdot L\left\{\frac{1}{h}\right\}\right\} \quad \text{for all } h \in (0, \varepsilon).$$

Proof. Let $x \in \mathbb{R}$ and $h > 0$. Then

$$\Delta_h f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\left(e^{i(x+h)\xi} - e^{i(x-h)\xi}\right) d\xi$$
$$= 2i \int_{\mathbb{R}} \hat{f}(\xi) \sin(h\xi) e^{ix\xi} d\xi.$$

By recursively applying Δ_h, we get

$$\Delta_h^n f(x) = (2i)^n \int_{\mathbb{R}} \hat{f}(\xi) \sin^n(h\xi) e^{ix\xi} d\xi$$

for all $n \in \mathbb{N}$. Hence, by inserting the inequality from the assumption, we obtain for the absolute value

$$|\Delta_h^n f(x)| \le 2^n \int_{\mathbb{R}} |\hat{f}(\xi)| \cdot |\sin(h\xi)|^n d\xi$$

$$\le 2^{n+1} C \cdot \left[\underbrace{\int_0^1 |\hat{f}(\xi)| \cdot |\sin(h\xi)|^n d\xi}_{=:I_1} + \underbrace{\int_1^{\frac{1}{h}} \frac{|\sin(h\xi)|^n}{\xi^{n+\alpha}} \cdot L(\xi) d\xi}_{=:I_2} + \underbrace{\int_{\frac{1}{h}}^\infty \frac{|\sin(h\xi)|^n}{\xi^{n+\alpha}} \cdot L(\xi) d\xi}_{=:I_3} \right].$$

Summand I_1:

Since $|\sin(h\xi)| \le |h\xi|$ and on the domain of integration for I_1 we have $|\xi| \le 1$, we get

$$I_1 \le h^n \cdot \int_0^1 |\hat{f}(\xi)| d\xi.$$

As $f \in L^1(\mathbb{R})$, we have $\hat{f} \in L^\infty(\mathbb{R})$. Hence, we can estimate

$$I_1 \le \frac{h^n}{2\pi} \cdot \|f\|_{L^1}. \tag{39}$$

Summand I_2:

Since $|\text{sinc}(x)| \le 1$ for all $x \in [0,1]$ and on the domain of integration for I_2, $h\xi \in [0,1]$ holds, we can estimate

$$I_2 \le h^n \cdot \int_1^{\frac{1}{h}} \frac{1}{\xi^\alpha} \cdot L(\xi) d\xi.$$

Since L is slowly varying, there exists an $\varepsilon > 0$ such that the function

$$\ell(t) := \frac{t^{1-\alpha-\delta}}{L\left(\frac{1}{t}\right)}$$

is monotonically decreasing on $(0, \varepsilon]$ with $\delta > \max\{0, 1-\alpha\}$. Since for the second integral $\frac{1}{\xi} \le h$, we obtain for $h < \varepsilon$ that

$$\ell\left(\frac{1}{\xi}\right) \ge \ell(h)$$

on its domain of integration. Hence, by multiplying the integrand with $\ell\left(\frac{1}{\xi}\right) \Big/ \ell(h)$, we obtain

$$I_2 \le h^{n-1+\alpha+\delta} \cdot L\left(\frac{1}{h}\right) \cdot \int_1^{\frac{1}{h}} \frac{d\xi}{\xi^{1-\delta}} = \frac{h^{n-1+\alpha}}{\delta} \cdot \left[1 - h^\delta\right] \cdot L\left(\frac{1}{h}\right). \tag{40}$$

Summand I_3:

Similarly, by utilizing $|\sin(x)| \le 1$ for all $x \in \mathbb{R}$ for I_3, we obtain the estimation

$$I_3 \le \int_{\frac{1}{h}}^\infty \frac{1}{\xi^{n+\alpha}} \cdot L(\xi) d\xi.$$

As L is slowly varying, there exists an $\varepsilon > 0$ such that the function

$$m(t) := t^{\alpha-\delta} \cdot L\left(\frac{1}{t}\right)$$

is monotonically increasing on $(0, \varepsilon]$ for $\delta \in (0, \alpha)$. Since on the domain of integration in I_3, we have $\frac{1}{\xi} \le h$, we obtain for $h < \varepsilon$:

$$m\left(\frac{1}{\xi}\right) \le m(h) \quad \text{for all } \xi \ge \frac{1}{h}.$$

Hence, by multiplying the integrand with $m(h) \big/ m\left(\frac{1}{\xi}\right)$, we obtain

$$I_3 \leq h^{\alpha - \delta} \cdot L\left(\frac{1}{h}\right) \cdot \int_{\frac{1}{h}}^{\infty} \frac{d\xi}{\xi^{n+\delta}} = \frac{h^{n-1+\alpha}}{n-1+\delta} \cdot L\left(\frac{1}{h}\right). \tag{41}$$

By combining (39), (40) and (41), we get that there exists a $C' > 0$ such that the following estimation holds

$$|\Delta_h^n f(x)| \leq C' h^{n-1} \cdot \max\left\{h, h^\alpha L\left(\frac{1}{h}\right)\right\}.$$

Due to the asymptotic property of L we again obtain that there exists an $\varepsilon > 0$ such that $h^\alpha L\left(\frac{1}{h}\right)$ increases monotonically for $h \in (0, \varepsilon]$. Hence, we get for the nth order modulus of continuity

$$\omega_n(h; f) = \sup_{h' \leq h} \sup_{x \in \mathbb{R}} |\Delta_{h'}^n f(x)| = \sup_{x \in \mathbb{R}} |\Delta_h^n f(x)| \leq C' h^{n-1} \cdot \max\left\{h, h^\alpha L\left(\frac{1}{h}\right)\right\}$$

for all $h \in (0, \varepsilon]$. □

The last two theorems can be combined to obtain the following one-to-one correspondence between the decay of the Fourier transform and the modulus of continuity of a sufficient order on a certain function class.

Corollary 3.47. *Let $f \in L^1(\mathbb{R})$, $n \in \mathbb{Z}_+$ and $|S_{n+1}(f)| < \infty$. Furthermore, let there be an $\varepsilon(x) > 0$ for every $x \in S_{n+1}(f)$ such that $f^{(n+1)}$ does not change its sign on $(x - \varepsilon(x), x)$ and $(x, x + \varepsilon(x))$. Let $L : (0, \infty) \to (0, \infty)$ be a slowly varying monotonic function and let $\alpha \in (0, 1]$. For the case $\alpha = 1$, we require $L(x) \to \infty$ for $x \to \infty$. Then the two following statements are equivalent:*

(i) $\hat{f}(\xi) = \mathcal{O}\left(|\xi|^{-(n+\alpha)} \cdot L(|\xi|)\right)$ *for* $\xi \to \pm\infty$.

(ii) $\omega_n(h; f) = \mathcal{O}\left(h^{n-1+\alpha} \cdot L\left(\frac{1}{h}\right)\right)$ *for* $h \to 0$.

Moreover, also the following two statements are equivalent:

(iii) $\hat{f}(\xi) = o\left(|\xi|^{-(n+\alpha)} \cdot L(|\xi|)\right)$ *for* $\xi \to \pm\infty$.

(iv) $\omega_n(h; f) = o\left(h^{n-1+\alpha} \cdot L\left(\frac{1}{h}\right)\right)$ *for* $h \to 0$.

Proof. The equivalence of *(i)* and *(ii)* follows directly from Theorem 3.44 and Theorem 3.46.

The proof of *(iii)*⇔*(iv)* follows directly from *(i)*⇔*(ii)* by applying the relation

$$f = o(g) \Leftrightarrow \exists h : (0, \infty) \to (0, \infty) : h = o(g) \wedge f = \mathcal{O}(h).$$

□

3.4.3 Decay result for singularities

In this subsection we will combine the smoothness result of the PRT and the connection be-
tween smoothness and Fourier decay to obtain a certain decay rate for the PFT. It remains to
show that the map

$$u \mapsto \mathcal{P}_{\theta,a}(\phi \mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right)$$

in Proposition 3.30 fulfills the requirements of Corollary 3.47.

Lemma 3.48. *Let $A \subset \mathbb{R}^2$ be an admissible set and let $\mathbb{1}_A$ be the indicator function of A. Further-
more, let $x^0 \in \partial A$, let θ indicate the angle of the inner normal vector of A in x^0 and let κ be the
local curvature of ∂A in x^0. Then, the following three statements hold:*

1. *Let $a \in \mathbb{R}$. For all $\varepsilon > 0$ and additionally $\varepsilon < \frac{2}{|a|}$ if $a \neq 0$, and for all functions $\phi \in C_c^\infty(\mathbb{R}^2)$
   with $\operatorname{supp}(\phi) \subset B_\varepsilon\left(x^0\right), \phi(\mathbb{R}^2) \subset [0, \infty)$ and $\phi\left(x^0\right) \neq 0$, the function*

   $$F : \mathbb{R} \to \mathbb{R}, \quad u \mapsto \mathcal{P}_{\theta,a}\left(\phi \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right)$$

   has the support $\operatorname{supp}(F) \subset (-\varepsilon, \varepsilon)$.

2. *For all $a \in \mathbb{R}$, there exists $\delta, \varepsilon > 0$ such that for all functions $\phi \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(\phi) \subset
   B_\varepsilon\left(x^0\right), \phi(\mathbb{R}^2) \subset [0, \infty)$ and $\phi\left(x^0\right) \neq 0$, the function $u \mapsto \mathcal{P}_{\theta,a}\left(\phi \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right) \in C^2\left((-\delta, 0) \cup (0, \delta)\right)$,
   where δ is independent of the choice of ϕ.*

3. *For all $a \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for all functions $\phi \in C_c^\infty(\mathbb{R}^2)$ fulfilling the condi-
   tions,*

   (i) *there exists $g \in C_c^\infty\left([0, \infty)\right)$, such that $\phi\left(x^0 + x\right) = g\left(\|x\|^2\right)$ for all $x \in \mathbb{R}^2$,*
   (ii) $\operatorname{supp}(g) = [0, \varepsilon^2]$,
   (iii) *g is convex,*
   (iv) *the function $t \mapsto 2g'(t^2) + 4t^2 g''(t^2)$ has exactly one root on $(0, \varepsilon)$,*

   *there exists $\delta > 0$ such that the function $u \mapsto \frac{d^2}{du^2} \mathcal{P}_{\theta,a}\left(\phi \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right)$ does not change its
   sign on $(-\delta, 0)$ and $(0, \delta)$.*

Proof.

1.

By employing the definition of the parabolic Radon transform, we can represent the function F
as

$$F(u) = \int_{\mathbb{R}} \left(\phi \mathbb{1}_A\right)\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{1}{2} a t^2 \\ t \end{pmatrix}\right) dt \quad \text{for all } u \in \mathbb{R}.$$

As ϕ is supported in $B_\varepsilon\left(x^0\right)$, it sufficient to show that

$$\left\|\begin{pmatrix} u + \frac{1}{2}at^2 \\ t \end{pmatrix}\right\| > \varepsilon \quad \text{for all } t \in \mathbb{R}, u \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]. \tag{42}$$

If $a = 0$, this inequality clearly holds, as $\|(u, t)\| \geq |u| > \varepsilon$. For $a \in \mathbb{R} \setminus \{0\}$, we introduce the function $f_u : \mathbb{R} \to \mathbb{R}$, $t \mapsto \left(u + at^2/2\right)^2 + t^2$ for $u \in \mathbb{R}$. In order to show (42), we determine the global minimum of f_u by considering its derivative and prove that it is greater than ε^2.

$$f_u'(t) = 2 \cdot \left(u + at^2/2\right) \cdot at + 2t.$$

We obtain the roots $t = 0$ and $t = \pm\sqrt{-2u/a}$, the latter only if $u/a < 0$. It is easy to see that the global minimum of f_u lies in $t = 0$ for $u/a \geq 0$ and in $t = \pm\sqrt{-2u/a}$ for $u/a < 0$. Hence, we get for the case $u/a \geq 0$ that

$$\min_{t \in \mathbb{R}} f_u(t) = f_u(0) = u^2 > \varepsilon^2 \quad \text{for all } u \in \mathbb{R} \setminus [-\varepsilon, \varepsilon].$$

The second case $u/a < 0$ yields

$$\min_{t \in \mathbb{R}} f_u(t) = f_u\left(\pm\sqrt{-2u/a}\right) = \left|\frac{2u}{a}\right| > \varepsilon^2 \quad \text{for all } u \in \mathbb{R} \setminus [-\varepsilon, \varepsilon],$$

as $\varepsilon < \frac{2}{|a|}$ due to the prerequisites.

2.

Following the results and notations of Lemma 3.28, the boundary ∂A can be represented locally in a neighborhood of any point $x^0 \in \partial A$ as graph of a function $q \in C^3(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ such that

$$x^0 + R_\theta \cdot \begin{pmatrix} q(t) \\ t \end{pmatrix} \in \partial A \quad \text{for all } t \in (-\varepsilon, \varepsilon).$$

Furthermore, the function $q_\kappa(t) := q(t) - \frac{\kappa}{2}t^2$ for all $t \in (-\varepsilon, \varepsilon)$, has to be one-to-one on $(-\varepsilon, 0]$ and $[0, \varepsilon)$ or $q_\kappa \equiv 0$ on $(-\varepsilon, \varepsilon)$.

We first assume that $q_\kappa \equiv 0$. Then we can conclude that

$$\mathcal{P}_{\theta,\kappa}\left(\phi \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right) = \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) \cdot \mathbb{1}_{\mathbb{R}_+}\left(u - \underbrace{q_\kappa(t)}_{=0}\right) dt$$

$$= \mathbb{1}_{\mathbb{R}_+}(u) \cdot \int_{-\varepsilon}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{\kappa}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt.$$

Due to the smoothness of ϕ, the origin is an isolated singularity of $u \mapsto \mathcal{P}_{\theta,\kappa}\left(\phi \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right)$.

We now assume that either $a \neq \kappa$ or $q_\kappa \not\equiv 0$. We will prove that 0 is an isolated singularity of the function $u \mapsto \mathcal{P}_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta)$ by splitting the function up into the left and right branch of the parabolic Radon transform:

$$\mathcal{P}_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta) = \mathcal{P}^+_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta) + \mathcal{P}^-_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta) \quad \text{for all } u \in \mathbb{R}.$$

It is sufficient to show that 0 is an isolated singularity of both branches. We will start by analyzing the right branch.

Right branch $\mathcal{P}^+_{\theta,a}$:

Then, we can consider the following four equations (18), (19), (21) and (22) which display the form of the right branch in a neighborhood of 0:

$$\mathcal{P}^+_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta) = \begin{cases} \int\limits_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt, & \text{for } q_a((0,\varepsilon)) \subset \mathbb{R}_+, u > 0, \\[2mm] 0, & \text{for } q_a((0,\varepsilon)) \subset \mathbb{R}_+, u \leq 0, \\[2mm] \int\limits_{q_{a,+}^{-1}(u)}^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt, & \text{for } q_a((0,\varepsilon)) \subset \mathbb{R}_-, u < 0, \\[2mm] \int\limits_0^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt, & \text{for } q_a((0,\varepsilon)) \subset \mathbb{R}_-, u \geq 0. \end{cases} \quad (43)$$

From these four equations, we can see that the smoothness of $u \mapsto \mathcal{P}^+_{\theta,a}(\phi \mathbb{1}_A)(x^0 + u \cdot e_\theta)$ depends on the smoothness of ϕ and of $q_{a,+}^{-1}$. Since $\phi \in C_c^\infty(\mathbb{R}^2)$, this does not limit the differentiability of the right branch. Therefore, the critical integrals for the smoothness of the right branch are those whose integration limits include $q_{a,+}^{-1}(u)$. Since

$$\int\limits_{q_{a,+}^{-1}(u)}^{\varepsilon} \phi\left(x^0 + R_\theta \begin{pmatrix} u + \frac{a}{2} t^2 \\ t \end{pmatrix}\right) dt = \int\limits_0^{\varepsilon} \phi\left(x^0 + R_\theta \begin{pmatrix} u + \frac{a}{2} t^2 \\ t \end{pmatrix}\right) dt - \int\limits_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \begin{pmatrix} u + \frac{a}{2} t^2 \\ t \end{pmatrix}\right) dt,$$

it is sufficient to examine the following integral:

$$F(u) := \int_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt \quad \text{for } u \in q_a((0,\varepsilon)).$$

An application of Leibniz's integral rule yields its derivative

$$
F'(u) = \left(q_{a,+}^{-1}\right)'(u) \cdot \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right) + \int_0^{q_{a,+}^{-1}(u)} \partial_{e_\theta}\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt
$$

$$
= \frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \cdot \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right) + \int_0^{q_{a,+}^{-1}(u)} \partial_{e_\theta}\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt
$$

$$(44)$$

for all $u \in q_a\left((0,\varepsilon)\right)$. We will now determine the continuity of the second derivative.

Second summand of (44):

Considering the form of the second summand which is very close to that of $F(u)$, taking the derivative of it yields a result of the form of $F'(u)$:

$$
\frac{d}{du} \int_0^{q_{a,+}^{-1}(u)} \partial_{e_\theta}\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt
$$

$$
= \frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \cdot \partial_{e_\theta}\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right) + \int_0^{q_{a,+}^{-1}(u)} \partial_{e_\theta}^2\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt
$$

$$(45)$$

for all $u \in q_a\left((0,\varepsilon)\right)$. Due to Lemma 3.28, 0 is an isolated root of q' and so the origin is also an isolated root of q_a'. Due to the continuity of $q_{a,+}^{-1}$ on $q_a\left((0,\varepsilon)\right)$, the expression $\frac{1}{q_a' \circ q_{a,+}^{-1}}$ is continuous on a neighborhood of 0. Therefore, the derivative of the second summand in (44) is continuous.

First summand of (44):

We will now show that same is true for the first summand in (44). For this purpose, we determine its derivative:

$$
\frac{d}{du}\left(\frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \cdot \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)\right)
$$

$$
= -\frac{q_a''\left(q_{a,+}^{-1}(u)\right)}{\left[q_a'\left(q_{a,+}^{-1}(u)\right)\right]^3} \cdot \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)
$$

$$
+ \frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \cdot \nabla\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)^T \cdot R_\theta \cdot \begin{pmatrix} 1 + a \cdot \frac{q_{a,+}^{-1}(u)}{q_a'\left(q_{a,+}^{-1}(u)\right)} \\ \frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \end{pmatrix} \quad (46)
$$

for all $u \in q_a\left((0,\varepsilon)\right)$. Again, due to the origin being an isolated zero of q_a', this expression is continuous on $q_a\left((0,\varepsilon)\right)$.

Left branch $\mathcal{P}_{\theta,a}^{-}$:

The argumentation for the left branch $u \mapsto \mathcal{P}_{a,\theta}^{-}(\phi\mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right)$ is the same as for the right branch.

Therefore, 0 is an isolated singularity of the function $u \mapsto \mathcal{P}_{a,\theta}(\phi\mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right)$ and so there exists $\delta > 0$ which only depends on $q_a\big((-\varepsilon,\varepsilon)\big)$.

3.

In order to prove the statement, we will examine the second derivative of (43). For this purpose, it is sufficient to analyze the second derivatives of the functions

$$F(u) = \int_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt, \tag{47}$$

$$G(u) = \int_0^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt. \tag{48}$$

Analysis of (47):

The second derivative of F has already been determined in the proof of statement 1. By combining (44), (45) and (46), we obtain

$F''(u)$

$$= \underbrace{\frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)} \cdot \partial_{e_\theta}\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)}_{=:I_1(u)} + \underbrace{\int_0^{q_{a,+}^{-1}(u)} \partial_{e_\theta}^2\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt}_{=:I_2(u)}$$

$$\underbrace{- \frac{q_a''\left(q_{a,+}^{-1}(u)\right)}{\left[q_a'\left(q_{a,+}^{-1}(u)\right)\right]^3} \cdot \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)}_{=:I_3(u)}$$

$$\underbrace{+ \frac{1}{[q_a'\left(q_{a,+}^{-1}(u)\right)]^2} \cdot \nabla\phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot \left[q_{a,+}^{-1}(u)\right]^2 \\ \left[q_{a,+}^{-1}(u)\right] \end{pmatrix}\right)^T \cdot R_\theta \cdot \begin{pmatrix} q_a'\left(q_{a,+}^{-1}(u)\right) + a \cdot q_{a,+}^{-1}(u) \\ 1 \end{pmatrix}}_{=:I_4(u)}.$$

$$\tag{49}$$

We will now determine the asymptotic behavior of the four summands for $u \to 0$:

$$I_1(u) = \mathcal{O}\left(\frac{1}{q_a'\left(q_{a,+}^{-1}(u)\right)}\right), \quad \text{for } u \to 0, \tag{50}$$

$$I_2(u) = \mathcal{O}\left(q_{a,+}^{-1}(u)\right), \quad \text{for } u \to 0, \tag{51}$$

$$I_3(u) = \frac{q_a''\left(q_{a,+}^{-1}(u)\right)}{\left[q_a'\left(q_{a,+}^{-1}(u)\right)\right]^3} \cdot \phi\left(x^0\right) + o\left(\frac{q_a''\left(q_{a,+}^{-1}(u)\right)}{\left[q_a'\left(q_{a,+}^{-1}(u)\right)\right]^3}\right), \quad \text{for } u \to 0, \tag{52}$$

$$I_4(u) = \mathcal{O}\left(\frac{1}{[q_a'\left(q_{a,+}^{-1}(u)\right)]^2}\right), \quad \text{for } u \to 0. \tag{53}$$

Due to the conditions (i) and (iii), g is convex. Hence,

$$g''(t) \geq 0 \quad \text{for all } t \geq 0. \tag{54}$$

So g' is monotonically increasing. Since $g' \in C_c^\infty\left([0,\infty)\right)$, we obtain that

$$g'(t) \leq 0 \quad \text{for all } t \geq 0. \tag{55}$$

Consequently, g is monotonically decreasing. Due to condition (ii), $\text{supp}(g) = \left[0, \varepsilon^2\right]$ and so $g \not\equiv 0$. Hence, $\phi\left(x^0\right) = g(0) > 0$. Thus, due to (52),

$$I_3(u) \sim \frac{q_a''\left(q_{a,+}^{-1}(u)\right)}{\left[q_a'\left(q_{a,+}^{-1}(u)\right)\right]^3}, \quad \text{for } u \to 0. \tag{56}$$

Due to Lemma 3.28, 2 (a), $q'(0) = 0$ and thus $q_a'(0) = 0$ for all $a \in \mathbb{R}$. Therefore, we obtain by Taylor's theorem:

$$q_a'(t) = q_a'(0) + \int_0^t q_a''(\tau)\, d\tau \sim t \cdot q_a''(t), \quad \text{for } t \to 0.$$

An application of this relation to (50), (56) and (53) yields:

$$I_1(u) = \mathcal{O}\left(\frac{1}{q_{a,+}^{-1}(u) \cdot q_a''\left(q_{a,+}^{-1}(u)\right)}\right), \quad \text{for } u \to 0, \tag{57}$$

$$I_3(u) \sim \frac{1}{\left[q_{a,+}^{-1}(u)\right]^3 \cdot \left[q_a''\left(q_{a,+}^{-1}(u)\right)\right]^2}, \quad \text{for } u \to 0, \tag{58}$$

$$I_4(u) = \mathcal{O}\left(\frac{1}{\left[q_{a,+}^{-1}(u)\right]^2 \cdot \left[q_a''\left(q_{a,+}^{-1}(u)\right)\right]^2}\right), \quad \text{for } u \to 0. \tag{59}$$

By comparing (57), (51), (58) and (59), we can see that the term $I_3(u)$ is asymptotically dominant for $u \to 0$. Thus, (49) yields

$$\lim_{u \to 0} \left|F''(u)\right| = \infty. \tag{60}$$

Consequently, there exists a constant $\delta > 0$ such that F'' does not change its sign on $(-\delta, 0)$ and $(0, \delta)$.

Analysis of (48):

We will prove that there exists $\delta > 0$ such that G'' does not change its sign on $(-\delta, 0)$ and $(0, \delta)$ by showing that $G''(0) < 0$. To this end, we apply condition (i) and rewrite G in terms of the function $g\left(\|x\|^2\right) = \phi\left(x^0 + x\right)$ for all $x \in \mathbb{R}^2$:

$$G(u) = \int_0^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt = \int_0^\varepsilon g\left(\left(u + \frac{a}{2} \cdot t^2\right)^2 + t^2\right) dt.$$

We thus obtain

$$G'(u) = \int_0^\varepsilon 2\left(u + \frac{a}{2} \cdot t^2\right) \cdot g'\left(\left(u + \frac{a}{2} \cdot t^2\right)^2 + t^2\right) dt$$

and

$$G''(u) = \int_0^\varepsilon \left[2 \cdot g'\left(\left(u + \frac{a}{2} \cdot t^2\right)^2 + t^2\right) + 4\left(u + \frac{a}{2} \cdot t^2\right)^2 \cdot g''\left(\left(u + \frac{a}{2} \cdot t^2\right)^2 + t^2\right)\right] dt.$$

Hence,

$$\begin{aligned} G''(0) &= \int_0^\varepsilon \left[2 \cdot g'\left(\frac{a^2}{4} \cdot t^4 + t^2\right) + a^2 t^4 \cdot g''\left(\frac{a^2}{4} \cdot t^4 + t^2\right)\right] dt \\ &= \int_0^\varepsilon \left[2 \cdot g'\left(\frac{a^2}{4} \cdot t^4 + t^2\right) + 4 \cdot \left(\frac{a^2}{4} \cdot t^4 + t^2\right) \cdot g''\left(\frac{a^2}{4} \cdot t^4 + t^2\right)\right] dt \\ &\quad - \int_0^\varepsilon 4t^2 \cdot g''\left(\frac{a^2}{4} \cdot t^4 + t^2\right) dt. \end{aligned} \tag{61}$$

We now introduce the function

$$h : [0, \infty) \to \mathbb{R}, \quad t \mapsto g(t^2).$$

Hence,

$$h'(t) = 2t \cdot g'(t) \quad \text{and} \quad h''(t) = 2g'(t^2) + 4t^2 g''(t^2) \quad \text{for all } t \geq 0. \tag{62}$$

With this result, (61 yields

$$G''(0) = \int_0^\varepsilon h''\left(\sqrt{\frac{a^2}{4} \cdot t^4 + t^2}\right) dt - \int_0^\varepsilon 4t^2 \cdot g''\left(\frac{a^2}{4} \cdot t^4 + t^2\right) dt. \tag{63}$$

Due to (54), we obtain for the second summand of (63):

$$- \int_0^\varepsilon 4t^2 \cdot \underbrace{g''\left(\frac{a^2}{4} \cdot t^4 + t^2\right)}_{\geq 0} dt < 0. \tag{64}$$

The left hand side of the upper inequality is negative, since $g'' \geq 0$ and $g'' \not\equiv 0$.

For the analysis of the first summand, we apply the variable substitution $\tau = \sqrt{a^2/4 \cdot t^4 + t^2}$ to obtain

$$\int_0^\varepsilon h'' \left(\sqrt{\frac{a^2}{4} \cdot t^4 + t^2} \right) dt = \int_0^\varepsilon h''(\tau) \cdot \underbrace{\sqrt{\frac{1 + \sqrt{1 + a^2\tau^2}}{2 \cdot (1 + a^2\tau^2)}}}_{=:\psi(\tau)} d\tau. \tag{65}$$

We can check that the function ψ is monotonically decreasing on $[0,\infty)$. Furthermore, due to $\operatorname{supp}(h) = [0,\varepsilon]$, we can observe that

$$\int_0^\varepsilon h''(t) \, dt = \underbrace{h'(\varepsilon)}_{=0} - h'(0) \stackrel{(62)}{=} -\lim_{t \to 0} 2t \cdot g(t) = 0. \tag{66}$$

Due to requirement 2 (iv) of the lemma, there exists exactly one root of h'' on the interior of $\operatorname{supp}(h)$ which we will call t_0. Due to (55), $g'(t) \le 0$ for all $t \ge 0$. Since $g' \not\equiv 0$ and g is convex, we obtain that $g'(0) \ne 0$. Thus, (62) yields

$$h''(0) = 2g'(0) < 0.$$

Consequently,

$$h''(t) < 0 \quad \text{for all } t \in (0, t_0) \quad \text{and} \quad h''(t) > 0 \quad \text{for all } t \in (t_0, \varepsilon).$$

Since additionally ψ is monotonically decreasing, we obtain

$$\begin{aligned}
\int_0^\varepsilon h'' \left(\sqrt{\frac{a^2}{4} \cdot t^4 + t^2} \right) dt &= \int_0^\varepsilon h''(\tau) \cdot \sqrt{\frac{1 + \sqrt{1 + a^2\tau^2}}{2 \cdot (1 + a^2\tau^2)}} \, d\tau \\
&= \int_0^{t_0} \underbrace{h''(\tau)}_{<0} \cdot \psi(\tau) \, d\tau + \int_{t_0}^\varepsilon \underbrace{h''(\tau)}_{>0} \cdot \psi(\tau) \, d\tau \\
&\le \int_0^{t_0} h''(\tau) \cdot \psi(t_0) \, d\tau + \int_{t_0}^\varepsilon h''(\tau) \cdot \psi(t_0) \, d\tau \\
&= \psi(t_0) \cdot \int_0^\varepsilon h(\tau) \, d\tau \stackrel{(66)}{=} 0.
\end{aligned}$$

Combining this result with (63) and (64) yields:

$$G''(0) < 0.$$

Hence, there exists $\delta > 0$ such that G'' does not change its sign on $(-\delta, \delta)$.

By examining (43), we can see that all possible cases for $\mathcal{P}_{\theta,a}^+(\phi \mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right)$ are covered - except for the case, where

$$\mathcal{P}_{\theta,a}^+(\phi \mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right) = \int_{q_{a,+}^{-1}(u)}^\varepsilon \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt.$$

This integral can be decomposed into

$$\int_{q_{a,+}^{-1}(u)}^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt$$

$$= \int_0^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt - \int_0^{q_{a,+}^{-1}(u)} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt$$

$$= G(u) - F(u) \quad \text{for all } u \in q_a\left((0,\varepsilon)\right).$$

Since $\lim\limits_{t\to 0} |F''(t)| = \infty$ due to (60) and $|G''(0)| < \infty$, $F''(t)$ is asymptotically dominant for $t \to 0$. Thus,

$$\lim_{u\to 0} \left| \frac{d^2}{du^2} \int_{q_{a,+}^{-1}(u)}^{\varepsilon} \phi\left(x^0 + R_\theta \cdot \begin{pmatrix} u + \frac{a}{2} \cdot t^2 \\ t \end{pmatrix}\right) dt \right| = \infty.$$

So there exists $\delta > 0$ such that $\frac{d^2}{du^2} \mathcal{P}_{\theta,a}^+(\phi \mathbb{1}_A)\left(x^0 + u \cdot e_\theta\right)$ does not change its sign on $(-\delta, 0)$ and $(0, \delta)$. □

As it is not immediately clear, that a compactly supported function, fulfilling the conditions 2 (i) - (iv) in Lemma 3.48 exists, we give an example of such a function.

Example 3.49. In search for a function fulfilling the requirements of Lemma 3.48, we will fall back to a well known bump function. To this end, let $c \geq 2$ and let

$$\phi_c : \mathbb{R}^2 \to \mathbb{R}, \quad x \longmapsto \begin{cases} \exp\left(\frac{c}{\|x\|^2 - 1}\right), & x \in B_1(0), \\ 0, & \text{else.} \end{cases}$$

Clearly, $\phi_c \in C_c^\infty\left(\mathbb{R}^2\right)$. Moreover, the function

$$g_c : [0,\infty) \to \mathbb{R}, \quad t \longmapsto \begin{cases} \exp\left(\frac{c}{t-1}\right), & t \in [0,1), \\ 0, & \text{else,} \end{cases}$$

fulfills the requirement $\phi(x) = g\left(\|x\|^2\right)$ for all $x \in \mathbb{R}^2$. We can determine the first two derivatives as

$$g_c'(t) = \begin{cases} -\frac{c}{(t-1)^2} \cdot \exp\left(\frac{c}{t-1}\right), & t \in [0,1), \\ 0, & \text{else,} \end{cases}$$

$$g_c''(t) = \begin{cases} \frac{c^2 + 2c(t-1)}{(t-1)^4} \cdot \exp\left(\frac{c}{t-1}\right), & t \in [0,1), \\ 0, & \text{else,} \end{cases}$$

The choice $c \geq 2$ ensures the convexity of g_c, as $g_c''(t) \geq 0$ for all $t \geq 0$, if $c \geq 2$. In order to check for condition 2 (iv), we observe that for all $t \in (0, 1)$ we get:

$$
\begin{aligned}
2g_c'(t^2) + 4t^2 \cdot g_c''(t^2) &= \left(-\frac{2c}{(t^2 - 1)^2} + 4t^2 \cdot \frac{c^2 + 2c(t^2 - 1)}{(t^2 - 1)^4} \right) \cdot \exp\left(\frac{c}{t^2 - 1} \right) \\
&= \frac{-2c(t^2 - 1)^2 + 4c^2 t^2 + 8c(t^2 - 1)}{(t^2 - 1)^4} \cdot \exp\left(\frac{c}{t^2 - 1} \right) \\
&= \frac{-2c \cdot t^4 + \left(12c + 4c^2 \right) \cdot t^2 - 10c}{(t^2 - 1)^4} \cdot \exp\left(\frac{c}{t^2 - 1} \right).
\end{aligned}
$$

In order to search for the roots of the upper function, we look for zeros of the numerator and obtain

$$
-2c \cdot t^4 + \left(12c + 4c^2 \right) \cdot t^2 - 10c = 0 \Leftrightarrow t^2 = c + 3 \pm \sqrt{c^2 + 6c + 4}.
$$

As $c \geq 2$, we obtain for the larger of the two zeros that $v + 3 + \sqrt{c^2 + 6c + 4} > 1$. Thus, we only consider the smaller zero and get

$$
t^2 = c + 3 - \sqrt{c^2 + 6c + 4} = \frac{(c + 3)^2 - (c^2 + 6c + 4)}{c + 3 + \sqrt{c^2 + 6c + 4}} = \frac{5}{c + 3 + \sqrt{c^2 + 6c + 4}} < 1,
$$

as $c \geq 2$. Hence, the equation

$$
2g_c'(t^2) + 4t^2 \cdot g_c''(t^2) = 0
$$

has only one solution on $(0, 1)$ for $c \geq 2$.

The next theorem is the culmination of our result on the dependence of the smoothness of the parabolic Radon transform on the choice of the curvature parameter and the result on the connection between the smoothness and the decay rate of the Fourier transform of a certain class of functions. By combining Proposition 3.30, Corollary 3.47 and Lemma 3.48, we obtain a classification of the decay rate of the parabolic Fourier transform with respect to the curvature parameter.

Theorem 3.50. *Let $A \subset \mathbb{R}^2$ be an admissible set and let $\mathbb{1}_A$ be the indicator function of A. Furthermore, let $x^0 \in \partial A$, let θ indicate the angle of the inner normal vector of A in x^0 and let κ be the local curvature of ∂A in x^0. Moreover, let $c \geq 2$ and*

$$
\phi : \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto
\begin{cases}
\exp\left(\frac{c}{\|x - x_0\|^2 - 1} \right), & \text{for } x \in B_1\left(x^0 \right), \\
0, & \text{else.}
\end{cases}
$$

For $\varepsilon > 0$, we define the dilated version $\phi_\varepsilon := D_{1/\varepsilon}\phi$. Then there exists $\delta > 0$ such that for all $\varepsilon \in (0, \delta)$,

$$
\mathcal{Q}_{x_{\theta + \pi/2}^0, \theta, a}\left(\phi_\varepsilon \mathbb{1}_A \right)(\omega)
\begin{cases}
\in \mathcal{O}\left(|\omega|^{-\frac{3}{2}} \right) \setminus o\left(|\omega|^{-\frac{3}{2}} \right), & \text{for } a \neq \kappa, \\
\notin \mathcal{O}\left(|\omega|^{-\frac{4}{3}} \right), & \text{for } a = \kappa,
\end{cases}
$$

for $\omega \to \pm\infty$.

Proof. We introduce the function

$$F_{\varepsilon,a} : \mathbb{R} \to \mathbb{R}, \quad u \mapsto \mathcal{P}_{\theta,a}\left(\phi_\varepsilon \mathbb{1}_A\right)\left(x^0 + u \cdot e_\theta\right).$$

As A is admissible, Proposition 3.30 yields that

$$\omega_1\left(F_{\varepsilon,a}; h\right) \begin{cases} \sim \sqrt{h} & \text{for } a \neq \kappa, \; h \to 0, \\ \in \Omega\left(\sqrt[3]{h}\right) & \text{for } a = \kappa, \; h \to 0. \end{cases} \tag{67}$$

Due to Example 3.49, ϕ_ε fulfills all requirement of Lemma 3.48. Hence, this lemma yields that for all $a \in \mathbb{R}$ there exist $\delta, \varepsilon > 0$ such that

1. $\mathrm{supp}(F_{\varepsilon,a}) \subset (-\varepsilon, \varepsilon)$,

2. $F_{\varepsilon,a} \in C^2\left((-\delta, 0) \cup (0, \delta)\right)$,

3. $F_{\varepsilon,a}''$ does not change its sign on $(-\delta, 0)$ and $(0, \delta)$.

Due to the properties 1 and 2, there exists $\varepsilon > 0$ such that $F_{\varepsilon,a} \in C^2\left(\mathbb{R} \setminus \{0\}\right)$. Together with property 3, $F_{\varepsilon,a}$ fulfills the requirements of Corollary 3.47. By rewriting (67) as

$$\omega_1\left(F_{\varepsilon,a}; h\right) \begin{cases} \in \mathcal{O}(\sqrt{h}) \setminus o(\sqrt{h}) & \text{for } a \neq \kappa, \; h \to 0, \\ \notin o\left(\sqrt[3]{h}\right) & \text{for } a = \kappa, \; h \to 0, \end{cases}$$

Corollary 3.47 yields the desired result. $\qquad\square$

Taylorlet transform

Parts of this chapter have already been published:

- The results of the sections 4.2, 4.3.1, 4.4 and 4.5 with the exception of subsection 4.5.3 have been published in [Fin19].

- Subsection 4.3.2 from the beginning up to Theorem 4.23 with the exception of Theorem 4.16 and Corollary 4.17 have been published in [FK19].

After having pursued an approach based on the parabolic Radon transform in the last chapter, we will examine a wavelet-like transformation in this chapter. The goal of this chapter is to show that this new transform allows for a detection of the position and orientation, as well as the curvature and other higher order geometric information of edges.

To this end, we give a quick overview over the notion of the continuous shearlet transform and the most important results in the first section.

In the subsequent section, we extend the continuous shearlet transform by shears of higher order to define the Taylorlet transform similar to the bendlet transform [LPS16]. Subsequently, we introduce the basic terminology for this new integral transform, e.g. generalized vanishing moment conditions of the form $\int_{\mathbb{R}} g(\pm t^k) t^m dt = 0$ for $k \in \mathbb{N}$, $m \in \mathbb{N}_0$ and a restrictiveness condition. that guarantee special decay rates for the Taylorlet transform.

The third section is devoted to the construction of functions that satisfy all the wanted conditions. We will present two methods that can be used to construct the desired Taylorlets. The first one is based on the idea that vanishing moments can be generated by taking derivatives as in [MH92] and utilizing a method to produce vanishing moments of higher order from classical vanishing moments. The second method relies on the so called q-calculus which is a discrete version of analysis based on the q-derivative $d_q f(x) = \frac{f(qx) - f(x)}{qx - x}$ for all $x \in \mathbb{R}$, $q > 0$ and for all functions $f : \mathbb{R} \to \mathbb{R}$.

The focus of section 4 is the proof of the main result that ensures certain decay of the Taylorlet transform.

In section 5, we will exploit the emerging differences in the decay rate to devise a detection algorithm for the edge features we are looking for. Furthermore, we examine a numerical example using Taylorlets resulting from both construction methods.

4.1 Continuous shearlet transform

In this section, we present the concept of the continuous shearlet transform which the Taylorlet transform is based upon.

It is hardly possible to introduce shearlets without mentioning wavelets. The discrete form of the wavelet transform is widely used in image processing as a sparse representation system [Mal99, Chapter IX] and is e. g. utilized in the JPEG 2000 image compression standard [CSE00]. As the focus of this thesis is on integral transforms, we will only introduce the continuous wavelet transform in the following definition.

Definition 4.1 (Admissible wavelet, Continuous wavelet transform)**.** Let $\psi \in L^2(\mathbb{R})$ such that

$$0 < c_\psi := \int_{\mathbb{R}} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi|} \, d\xi < \infty. \tag{1}$$

Then, ψ is called *admissible wavelet* and for $f \in L^2(\mathbb{R})$ the *wavelet transform* with respect to ψ is defined as

$$\mathcal{W}_\psi f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a,b) \mapsto |a|^{-1/2} \cdot \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} \, dt.$$

The admissibility condition (1) is needed to ensure the invertibility of the continuous wavelet transform.

Theorem 4.2 (Inversion of the continuous wavelet transform)**.** *Let ψ be an admissible wavelet and for $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ let*

$$\psi_{a,b} \in L^2(\mathbb{R}), \quad \psi_{a,b}(x) = |a|^{-1/2} \cdot \psi\left(\frac{x-b}{a}\right) \quad \text{for almost all } x \in \mathbb{R}.$$

Then, for $f \in L^2(\mathbb{R})$,

$$f(x) = \frac{1}{2\pi c_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_\psi f(a,b) \cdot \psi_{a,b}(x) \, \frac{da}{a^2} \, db \quad \text{for almost all } x \in \mathbb{R}.$$

The continuous wavelet transform exhibits a very beneficial property for the task of edge detection which was discovered by Mallat and Hwang.

Theorem 4.3. *[MH92, Theorem 3] Let $n \in \mathbb{N}$, $\psi \in C_c^n(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \psi(x) x^k \, dx = 0 \quad \text{for all } k \in \{0, \dots, n-1\}.$$

Furthermore, let $f \in L^1([a,b])$ for an open, bounded interval I. If there exists $s_0 > 0$ such that $\left|\mathcal{W}_\psi f\right|$ has no local maxima on $(0,s_0) \times (a,b)$, then for all $\varepsilon > 0$ and $\alpha < n$, f is uniformly α-Lipschitz continuous in $(a - \varepsilon, b + \varepsilon)$.

Hence, it is possible to relate the smoothness of a function f in a point x_0 with the decay behavior of its wavelet transform $\mathcal{W}_\psi f(s,x)$ for x in a neighborhood of x_0 and $s \to 0$. In a nutshell, wavelets are useful tools for edge detection.

For the purpose of detecting not only the position of a singularity along a curve in $\mathbb{R}^2$, but also its orientation, Candes and Donoho established the curvelet transform which utilizes an anisotropic scaling and rotations in addition to the dilation and translation [CD05a, CD05b]. Based on their construction, Guo, Kutyniok and Labate introduced the shearlets which applies shears instead of rotations.

Definition 4.4 (Admissible shearlet, Continuous shearlet transform)**.** Let $\psi \in L^2(\mathbb{R}^2)$ such that

$$0 < \int_{\mathbb{R}^2} \frac{\left|\hat{\psi}(\xi)\right|^2}{|\xi_1|^2}\, d\xi < \infty.$$

Then, ψ is called *admissible shearlet*. [DKM$^+$08] For $a > 0$ and $s \in \mathbb{R}$ let

$$A_a := \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \quad \text{and} \quad S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Then, for $f \in L^2(\mathbb{R}^2)$, $a > 0$, $s \in \mathbb{R}$, $t \in \mathbb{R}^2$, the *continuous shearlet transform* [GKL06] with respect to ψ is defined as

$$\mathcal{SH}_\psi f(a,s,t) = a^{-3/4} \int_{\mathbb{R}^2} f(x)\overline{\psi\left(S_s A_a \cdot (x - t)\right)}\, dx.$$

Of particular importance for the theory of shearlets is the *classical shearlet*.

Definition 4.5 (Classical Shearlet)**.** [KL09] Let $\hat{\psi}_1, \hat{\psi}_2 \in C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\left(\hat{\psi}_1\right) \subset \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right],$$
$$\operatorname{supp}\left(\hat{\psi}_2\right) \subset [-1, 1],$$
$$\hat{\psi}_1 > 0 \text{ on } (-1, 1),$$
$$\int_0^\infty \left|\hat{\psi}_1(a\omega)\right|^2 \frac{da}{a} = 1 \quad \text{for all } \omega \in \mathbb{R},$$
$$\left\|\psi_2\right\|_2 = 1.$$

Then, the function $\psi \in \mathcal{S}(\mathbb{R}^2)$ with

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1) \cdot \hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right) \quad \text{for all } \xi \in \mathbb{R}^2 \text{ with } \xi_1 \neq 0,$$

is called *classical shearlet.* Furthermore, let $\psi^{(v)} \in \mathcal{S}(\mathbb{R}^2)$ such that for for all $\xi \in \mathbb{R}^2$ with $\xi_1 \neq 0$

$$\hat{\psi}^{(v)}(\xi) = \hat{\psi}_1(\xi_2) \cdot \hat{\psi}_2\left(\frac{\xi_1}{\xi_2}\right).$$

The main reason for our interest in shearlets is the *resolution of the wavefront set*, i.e. the wavefront set of a tempered distribution can be determined by its continuous shearlet transform as shown by the next theorem.

Theorem 4.6. *[KL09] Let ψ be a classical shearlet and $f \in \mathcal{S}'(\mathbb{R}^2)$.*

(i) *Let $R = \left\{ t_0 \in \mathbb{R}^2 : \text{for } t \text{ in a neighborhood } U \text{ of } t_0, \left|\mathcal{SH}_\psi f(a,s,t)\right| = \mathcal{O}\left(a^k\right) \text{ and } \left|\mathcal{SH}_{\psi^{(v)}} f(a,s,t)\right| = \mathcal{O}\left(a^k\right) \text{ as } a \to 0, \text{ for all } k \in \mathbb{N}, \text{ with the } \mathcal{O}\text{-terms uniform over } (s,t) \in [-1,1] \times U \right\}$. Then,*

$$\text{sing supp}(f)^c = R.$$

(ii) *Let $D = D_1 \cup D_2$, where $D_1 = \left\{ (t_0, s_0) \in \mathbb{R}^2 \times [-1,1] : \text{for } (s,t) \text{ in a neighborhood } U \text{ of } (s_0, t_0), \left|\mathcal{SH}_\psi f(a,s,t)\right| = \mathcal{O}(a^k) \text{ as } a \to 0, \text{ for all } k \in \mathbb{N}, \text{ with the } \mathcal{O}\text{-term uniform over } (s,t) \in U \right\}$ and $D_2 = \left\{ (t_0, s_0) \in \mathbb{R}^2 \times [1,\infty) : \text{for } \left(\frac{1}{s}, t\right) \text{ in a neighborhood } U \text{ of } (s_0, t_0), \left|\mathcal{SH}_{\psi^{(v)}} f(a,s,t)\right| = \mathcal{O}(a^k) \text{ as } a \to 0, \text{ for all } k \in \mathbb{N}, \text{ with the } \mathcal{O}\text{-term uniform over } \left(\frac{1}{s}, t\right) \in U \right\}$. Then,*

$$\text{WF}(f)^c = D.$$

Hence, the continuous shearlet transform allows for a detection of position and orientation of an edge. The goal of this chapter is to establish a similar relation for the position, orientation, curvature and higher order geometric features of singularities along curves for a class of functions by extending the concept of the shearlets.

4.2   Basic definitions and properties of the Taylorlet transform

The goal of the Taylorlet transform is a precise analytical description of the singular support of the analyzed function f. To this end, we assume that we can represent sing supp(f) locally as the graph of a singularity function $q \in C^\infty(\mathbb{R})$ and describe sing supp(f) by the Taylor coefficients of q. These coefficients can be found by observing the decay rate of the Taylorlet transform. In this way the continuous shearlet transform essentially delivers a local linear approximation to the singular support which can be regarded as a first order Taylor polynomial of the singularity function q. Hence, we will use it as a starting point for the construction of the Taylorlet transform. To this end, we need an extension of the classical shear: we will use a modification of the higher order shearing operators introduced in [LPS16].

Definition 4.7. For $n \in \mathbb{N}_0$ and $s = (s_0, \ldots, s_n)^T \in \mathbb{R}^{n+1}$ the n-th order shearing operator is defined as

$$S_s^{(n)} : \mathbb{R}^2 \to \mathbb{R}^2, \quad S_s^{(n)}(x) := \begin{pmatrix} x_1 + \sum_{\ell=0}^n \frac{s_\ell}{\ell!} \cdot x_2^\ell \\ x_2 \end{pmatrix}.$$

In contrast to [LPS16], here the higher order shearing operator also includes a simple translation along the x_1-axis in the form of s_0. This is included to emphasize the Taylor coefficient perspective on the singular support of the analyzed function.

Furthermore, for $a, \alpha > 0$ we use an α-scaling matrix [LPS16]

$$A_a^{(\alpha)} := \begin{pmatrix} a & 0 \\ 0 & a^\alpha \end{pmatrix}.$$

A central property of analyzing functions of continuous multi-scale transforms is the vanishing moment condition which plays a crucial role for wavelets in order to detect singularities of a certain smoothness [MH92]. For shearlets there exist analogous results [Gro11]. Pursuing a similar goal, the following definition of analyzing Taylorlets incorporates some special vanishing moment properties.

Definition 4.8 (Vanishing moments of higher order, analyzing Taylorlet, restrictiveness)**.** We say that a function $f : \mathbb{R} \to \mathbb{R}$ has M vanishing moments of order n if

$$\int_{\mathbb{R}} f(\pm t^k) t^m dt = 0$$

for all $m \in \{0, \ldots, kM - 1\}$ and for all $k \in \{1, \ldots, n\}$.

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space on $\mathbb{R}^d$ and let $g, h \in \mathcal{S}(\mathbb{R})$ such that g has M vanishing moments of order n. The space of Schwartz functions with infinitely many vanishing moments of order n will be denoted by $\mathcal{S}_n^*(\mathbb{R})$. We call the function

$$\tau = g \otimes h$$

an analyzing Taylorlet of order n with M vanishing moments.

We say τ is restrictive, if additionally

(i) $I_+^j g(0) \neq 0$ for all $j \in \{0, \ldots, M\}$, where $I_+ g(t) = \int_{-\infty}^t g(u) \, du$ and

(ii) $\int_{\mathbb{R}} h(t) dt \neq 0$.

The concept of the restrictiveness is a generalization of the non-vanishing moment conditions employed on certain shearlets in [KP15, section 2.2] for the purpose of edge classification. Furthermore, in Theorem 4.23 we show that for arbitrary $n \in \mathbb{N}$, the set of restrictive analyzing Taylorlets of order n is not empty.

We define the Taylorlet transform as follows.

Definition 4.9 (Taylorlet transform)**.** Let $r, n \in \mathbb{N}$ and let $\tau \in \mathcal{S}(\mathbb{R}^2)$ be an analyzing Taylorlet of order n with r vanishing moments. Moreover, let $\alpha > 0$, $t \in \mathbb{R}$, $a > 0$ and $s \in \mathbb{R}^{n+1}$. We define

$$\tau_{a,s,t}^{(n,\alpha)}(x) := \tau\left(A_{\frac{1}{a}}^{(\alpha)} S_{-s}^{(n)} \begin{pmatrix} x_1 \\ x_2 - t \end{pmatrix} \right) \quad \text{for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

Whenever the values of α and n are clear, we will omit these indices and write $\tau_{a,s,t}$ instead. The Taylorlet transform w. r. t. τ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ is defined as

$$\mathcal{T}_\tau^{(n,\alpha)} f(a, s, t) = \left\langle f, \tau_{a,s,t}^{(n,\alpha)} \right\rangle.$$

In order to properly state the mapping properties of the Taylorlet transform, we now introduce the topologies of the most important function and distribution spaces.

Definition 4.10 (Topologies of some function and distribution spaces)**.** Let X be a topological vector space and let X' be its dual space. A set $B \subset X$ is called bounded, if

$$\sup_{x \in B} |\langle x', x \rangle| < \infty \quad \text{for all } x' \in X'.$$

Let $\mathcal{B}$ denote the set of all bounded subsets of X. The strong dual topology of X' is generated by the following family of semi-norms:

$$\| \cdot \|_B : X' \to [0, \infty), \quad \|x'\|_B = \sup_{x \in B} |\langle x', x \rangle|, \quad \text{where } B \in \mathcal{B}.$$

The topology of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is generated by the seminorms

$$\| \cdot \|_{\alpha, \beta} : \mathcal{S}(\mathbb{R}^d) \to [0, \infty), \quad \|\phi\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^d} \left| x^\alpha \partial_x^\beta \phi(x) \right|, \quad \text{where } \alpha, \beta \in \mathbb{N}_0^d.$$

The space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions is the dual space of $\mathcal{S}(\mathbb{R}^d)$ and will be endowed with the strong dual topology.

The space $C^\infty(\mathbb{R}^d)$ of smooth functions is endowed with the topology generated by the semi-norms

$$\| \cdot \|_{N,K} : C^\infty(\mathbb{R}^d) \to [0, \infty), \quad \|\phi\|_{N,K} = \sup_{|\beta| \leq N} \sup_{x \in K} |\partial_x^\beta \phi(x)|,$$

where $N \in \mathbb{N}_0$ and $K \subset \mathbb{R}^d$ is compact.

In the following proposition we will show some basic properties of the Taylorlet transform.

Proposition 4.11 (Properties of the Taylorlet transform)**.** *Let $\tau \in \mathcal{S}(\mathbb{R}^2)$ be an analyzing Taylorlet of order $n \in \mathbb{N}$. Let $\alpha > 0$ and let $V = \mathbb{R}_+ \times \mathbb{R}^{n+1} \times \mathbb{R}$ denote the parameter space of the Taylorlet transform. Then the following statements hold:*

1. *For all $f \in \mathcal{S}'(\mathbb{R}^2)$ the map $\mathcal{T}_\tau f : V \to \mathbb{C}$ is well-defined.*

2. *For all $f \in \mathcal{S}'(\mathbb{R}^2)$ we have $\mathcal{T}_\tau f \in C^\infty(V)$.*

3. *Let $\mathcal{S}'(\mathbb{R}^2)$ be endowed with the strong dual topology. Then the Taylorlet transform $\mathcal{T}_\tau :$ $\mathcal{S}'(\mathbb{R}^2) \to C^\infty(V)$ is a continuous linear operator.*

4. *Let $f \in L^2(\mathbb{R}^2)$ and let τ have at least one vanishing moment. If $\mathcal{T}_\tau f \equiv 0$, then $f \equiv 0$.*

Proof.   1. Since $f \in \mathcal{S}'(\mathbb{R}^2)$ and $\tau_v \in \mathcal{S}(\mathbb{R}^2)$ for all $v \in V$, the Taylorlet transform

$$\mathcal{T}_\tau f(v) = \langle f, \tau_v \rangle$$

is well-defined.

2. In order to prove that for all $f \in \mathcal{S}'(\mathbb{R}^d)$ the Taylorlet transform of f is smooth, we first show that the parameter derivatives of the Taylorlet are Schwartz functions, that is

$$\partial_v^\beta \tau_v \in \mathcal{S}(\mathbb{R}^2) \quad \text{for all } \beta \in \mathbb{N}_0^{n+3}.$$

A simple computation yields the derivatives of τ_v for all $x \in \mathbb{R}^2$:

$$\partial_a \tau_v(x) = -\frac{x_1 - \sum_{k=0}^n \frac{s_k}{k!} x_2^k}{a^2} \cdot \partial_{x_1} \tau_v(x) - \alpha \cdot \frac{x_2 - t}{a^{\alpha+1}} \cdot \partial_{x_2} \tau_v(x),$$

$$\partial_{s_k} \tau_v(x) = -\frac{x_2^k}{a \cdot k!} \cdot \partial_{x_1} \tau_v(x),$$

$$\partial_t \tau_v(x) = -\frac{1}{a^\alpha} \cdot \partial_{x_2} \tau_v(x).$$

Since the derivatives with respect to v are sums of products of polynomials in x and derivatives of τ_v with respect to x, any derivative with respect to v is a Schwartz function.

For $f \in \mathcal{S}'(\mathbb{R}^2)$, any first order partial derivative of the Taylorlet transform of f can be represented as the limit of a difference quotient, that is

$$\partial_{v_\ell} \mathcal{T}_\tau f(v) = \lim_{h \to 0} \frac{\mathcal{T}_\tau f(v + h e_\ell) - \mathcal{T}_\tau f(v)}{h}$$

for $\ell \in \{1, \ldots, n+3\}$, where e_ℓ denotes the ℓth unit vector. We obtain

$$\begin{aligned}
\partial_{v_\ell} \mathcal{T}_\tau f(v) &= \lim_{h \to 0} \frac{\mathcal{T}_\tau f(v + h e_\ell) - \mathcal{T}_\tau f(v)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \left[\langle f, \tau_{v+he_\ell} \rangle - \langle f, \tau_v \rangle \right] \\
&= \lim_{h \to 0} \Big\langle f, \underbrace{\frac{\tau_{v+he_\ell} - \tau_v}{h}}_{=: \Delta_{v_\ell, h} \tau_v} \Big\rangle.
\end{aligned}$$

We now prove that for all $f \in \mathcal{S}'(\mathbb{R}^2)$, we have

$$\lim_{h \to 0} \langle f, \Delta_{v_\ell, h} \tau_v \rangle = \langle f, \partial_{v_\ell} \tau_v \rangle.$$

This coincides with the weak $\mathcal{S}(\mathbb{R}^2)$-convergence

$$\Delta_{v_\ell, h} \tau_v \rightharpoonup \partial_{v_\ell} \tau_v \quad \text{for } h \to 0.$$

We will show the strong $\mathcal{S}(\mathbb{R}^2)$-convergence of this function sequence, that is, the convergence in the semi-norms $\| \cdot \|_{\alpha, \beta}$ for $\alpha, \beta \in \mathbb{N}_0^2$, which implies the weak convergence. For this, we utilize Taylor's theorem which yields for all $h > 0$ and $x \in \mathbb{R}^2$:

$$\Delta_{v_\ell, h} \tau_v(x) - \partial_{v_\ell} \tau_v(x) = \frac{1}{h} \int_{v_\ell}^{v_\ell + h} \partial_{v_\ell}^2 \tau_{\tilde{v}}(x)(v_\ell - \tilde{v}_\ell) \, d\tilde{v}_\ell, \tag{2}$$

where $\tilde{v} = (v_1, \ldots, v_{\ell-1}, \tilde{v}_\ell, v_{\ell+1}, \ldots, v_{n+3})$. We thus obtain for each $\alpha, \beta \in \mathbb{N}_0^2$

$$
\begin{aligned}
\left\| \Delta_{v_\ell, h} \tau_v - \partial_{v_\ell} \tau_v \right\|_{\alpha, \beta} &= \sup_{x \in \mathbb{R}^2} \left| x^\alpha \partial_x^\beta \left(\Delta_{v_\ell, h} \tau_v(x) - \partial_{v_\ell} \tau_v(x) \right) \right| \\
&\overset{(2)}{=} \sup_{x \in \mathbb{R}^2} \left| x^\alpha \partial_x^\beta \frac{1}{h} \int_{v_\ell}^{v_\ell + h} \partial_{v_\ell}^2 \tau_{\tilde{v}}(x)(v_\ell - \tilde{v}_\ell) \, d\tilde{v}_\ell \right| \\
&\leq \sup_{x \in \mathbb{R}^2} \left| \int_{v_\ell}^{v_\ell + h} x^\alpha \partial_x^\beta \partial_{v_\ell}^2 \tau_{\tilde{v}}(x) \, d\tilde{v}_\ell \right| \\
&\leq h \cdot \sup_{x \in \mathbb{R}^2} \sup_{\tilde{v}_\ell \in [v_\ell, v_\ell + h]} \left| x^\alpha \partial_x^\beta \partial_{v_\ell}^2 \tau_{\tilde{v}}(x) \right|.
\end{aligned}
$$

Since $\partial_v^\gamma \tau_v \in \mathcal{S}(\mathbb{R}^2)$ for all $\gamma \in \mathbb{N}_0^{n+3}$, we have

$$\sup_{x \in \mathbb{R}^2} \sup_{\tilde{v}_\ell \in [v_\ell, v_\ell + h]} \left| x^\alpha \partial_x^\beta \partial_{v_\ell}^2 \tau_{\tilde{v}}(x) \right| < \infty$$

for all $h \in K$, where $K \subset [0, \infty)$ bounded. Hence, we obtain for all $\alpha, \beta \in \mathbb{N}_0^2$:

$$\lim_{h \to 0} \left\| \Delta_{v_\ell, h} \tau_v - \partial_{v_\ell} \tau_v \right\|_{\alpha, \beta} = 0.$$

Consequently, $\partial_{v_\ell} \mathcal{T}_\tau f(v) = \langle f, \partial_{v_\ell} \tau_v \rangle$ for all $\ell \in \{1, \ldots, n+3\}$ and so the partial derivatives exist. This argument can be iterated to show that any derivative of the Taylorlet transform $\mathcal{T}_\tau f$ exists for all $f \in \mathcal{S}'(\mathbb{R}^2)$ and thus $\mathcal{T}_\tau f$ is smooth.

3. The Taylorlet transform $\mathcal{T}_\tau : \mathcal{S}'(\mathbb{R}^2) \to C^\infty(V)$, $\mathcal{T}_\tau f(v) = \langle f, \tau_v \rangle$ is linear due to its definition as linear functional. Since the topology of $C^\infty(V)$ is generated by the seminorms $\| \cdot \|_{N,K}$, it suffices to show that for all $N \in \mathbb{N}_0$, $K \subset V$ compact and $\varepsilon > 0$, there exists a neighborhood $U \subset \mathcal{S}'(\mathbb{R}^2)$ of 0 such that

$$\sup_{|\beta| \leq N} \sup_{v \in K} |\partial_v^\beta \mathcal{T}_\tau f(v)| < \varepsilon \quad \text{for all } f \in U.$$

To this end, let $K \subset V$ be compact and $\varepsilon > 0$. We will now show that the sets

$$T_{\beta,K} = \{\partial_\nu^\beta \tau_\nu : \ \nu \in K\}$$

are bounded for all $\beta \in \mathbb{N}_0^{n+3}$. For this, let $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^2)$. Then,

$$\sup_{\phi \in T_{\beta,K}} |\langle f, \phi \rangle| = \sup_{\nu \in K} \left| \langle f, \partial_\nu^\beta \tau_\nu \rangle \right| < \infty,$$

since $\partial_\nu^\beta \tau_\nu \in \boldsymbol{\mathcal{S}}(\mathbb{R}^2)$ and $K \subset V$ is compact. Consequently, the sets $T_{\beta,K}$ are bounded and so is $T_{N,K} = \bigcap\limits_{|\beta| \leq N} T_{\beta,K}$ for $N \in \mathbb{N}_0$. According to the definition of the strong dual topology, the set $T_{N,K}$ induces a semi-norm in the strong topology of $\boldsymbol{\mathcal{S}}'(\mathbb{R}^2)$ by

$$\| \cdot \|_{T_{N,K}} : \boldsymbol{\mathcal{S}}'(\mathbb{R}^2) \to \mathbb{C}, \quad \|f\|_{T_{N,K}} = \sup_{\phi \in T_{N,K}} |\langle f, \phi \rangle|.$$

Thus, the set

$$U_{N,K,\varepsilon} = \{f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^2) : \|f\|_{T_{N,K}} < \varepsilon\}$$

is open in the strong dual topology of $\boldsymbol{\mathcal{S}}'(\mathbb{R}^2)$ and for all $f \in U_{N,K,\varepsilon}$ we have

$$\sup_{|\beta| \leq N} \sup_{\nu \in K} |\partial_\nu^\beta \boldsymbol{\mathcal{T}}_\tau f(\nu)| = \sup_{|\beta| \leq N} \sup_{\nu \in K} |\langle f, \partial_\nu^\beta \tau_\nu \rangle|$$

$$= \sup_{\phi \in T_{N,K}} |\langle f, \phi \rangle| = \|f\|_{T_{N,K}} < \varepsilon.$$

4. If $\boldsymbol{\mathcal{T}}_\tau f \equiv 0$, we especially have

$$\boldsymbol{\mathcal{T}} f(a, s, t) = 0 \quad \text{for all } a > 0, \ s_0, s_1, t \in \mathbb{R} \text{ and for } s_2 = \ldots = s_n = 0.$$

In this situation, the Taylorlet transform reduces to a shearlet transform utilizing an α-scaling matrix. As shown in [DST12], a shearlet transform of this type offers a reconstruction formula for $f \in L^2(\mathbb{R}^2)$. As τ has at least one vanishing moment of order 1, it is an admissible shearlet and yields the following reconstruction formula for $s_2 = \ldots = s_n = 0$:

$$f(x) = \frac{1}{C_\tau} \int_0^\infty \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \boldsymbol{\mathcal{T}}_\tau f(a, s, t) \tau_{ast}(x) \frac{da}{a^{9/2}} \, dt \, ds_0 \, ds_1 \quad \text{f. a. a. } x \in \mathbb{R}^2.$$

Thus, we obtain $f \equiv 0$.

$\square$

We want to distinguish between the right choice of shearing parameters $s = (s_0, \ldots, s_n)$ which are those corresponding to the Taylor coefficients of q, and the incorrect ones by comparing the respective decay rates of the Taylorlet transform. We will show that, if the choice is incorrect, the

Taylorlet transform decays fast because of the vanishing moments of higher order. The restrictiveness on the other hand makes sure that the Taylorlet transform decays slowly at a correct choice of shearing parameters.

4.3   Construction of a Taylorlet

In the setting of the continuous shearlet transform the vanishing moment property w. r. t. the x_1-direction is inherently given by Definition 4.5 of a classical shearlet ψ, i. e.

$$\hat{\psi}(\xi) = \hat{\psi}_1(\xi_1) \cdot \hat{\psi}_2 \left(\frac{\xi_2}{\xi_1} \right) \quad \text{for all } \xi \in \mathbb{R}^2, \, \xi_1 \neq 0,$$

since $0 \notin \text{supp}(\hat{\psi}_1)$. In the Taylorlet setting, the function g essentially takes over the role of ψ_1. Unfortunately, vanishing moments of g alone are not sufficient for the construction of an analyzing Taylorlet, as we additionally need vanishing moments of $g(\pm t^k)$ for all $k \leq n$. Thus, we cannot rely on a construction via the Fourier transform. So, we will present two different approaches for constructing Taylorlets. The first ansatz utilizes derivatives to produce vanishing moments similar to the method used in [MH92], while we rely on the methodology of q-calculus in the second construction.

4.3.1   Derivative-based construction

We will first present a construction procedure and images of each construction step starting from the exemplary function $\phi(t) = e^{-t^2}$. Later, we prove that the resulting function is a restrictive Taylorlet of order n with M vanishing moments.

General setup

I. We start with an even function $\phi \in \mathcal{S}(\mathbb{R})$ fulfilling

$$\phi^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2 = 0.$$

This condition is necessary for the Taylorlet to be a Schwartz function. For instance, we can choose $\phi(t) = e^{-t^2}$.
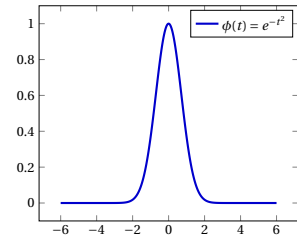
II. Let $v_n \in \mathbb{N}$ be the least common multiple of the numbers $1, \ldots, n$. We define

$$\phi_n(t) := \phi\left(t^{v_n}\right) \quad \text{for all } t \in \mathbb{R}.$$

This function is still in $\mathcal{S}(\mathbb{R})$ and fulfills

$$\phi_n^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2v_n = 0.$$
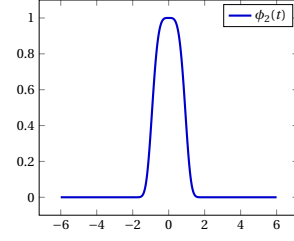
III. We define

$$\phi_{n,M} := \frac{1}{(2Mv_n)!} \cdot \phi_n^{(2Mv_n)}.$$

This function has $2Mv_n$ vanishing moments since each derivative generates one further vanishing moment. Furthermore, the function is also in $\mathcal{S}(\mathbb{R})$ and fulfills

$$\phi_{n,M}^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2v_n = 0.$$
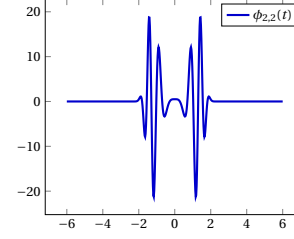
IV. We define

$$\tilde{\phi}_{n,M} = \phi_{n,M}\left(|t|^{\frac{1}{v_n}}\right) \quad \text{for all } t \in \mathbb{R}.$$

The concatenation with the function $|\cdot|^{\frac{1}{v_n}}$ ensures that $\tilde{\phi}_{n,M}$ has vanishing moments of order n. The function $\tilde{\phi}_{n,M}$ fulfills

$$\tilde{\phi}_{n,M}^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2 = 0.$$

So, it is smooth despite the singularity of $|\cdot|^{1/v_n}$ and belongs to $\mathcal{S}(\mathbb{R})$, as well.
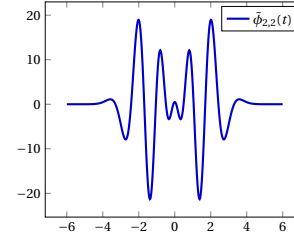
V. For all $t \in \mathbb{R}$, we define

$$g(t) := (1+t)\tilde{\phi}_{n,M}(t).$$

This step guarantees that the necessary properties of g for the restrictiveness are fulfilled. Furthermore, $g \in \mathcal{S}(\mathbb{R})$.
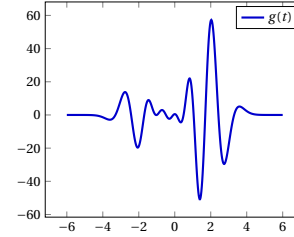
VI. We choose a function $h \in \mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} h(t)dt \neq 0$ and define the Taylorlet $\tau := g \otimes h$. Since $g, h \in \mathcal{S}(\mathbb{R})$, we have $\tau \in \mathcal{S}(\mathbb{R}^2)$.

In the following theorem we will prove some properties of the function τ generated by steps I-VI above.

Theorem 4.12. *Let $M, n \in \mathbb{N}$. The function τ described in the general setup exhibits the following properties:*

   i) $\tau \in \mathcal{S}(\mathbb{R}^2)$.
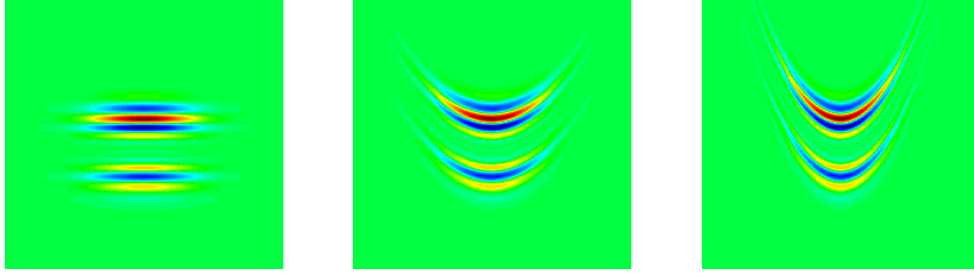
Fig. 4.1: Plots of the Taylorlet example, starting from the function $\phi(t) := e^{-t^2}$, $g(t) := (1+t) \cdot \tilde{\phi}_{2,2}(t)$, $h(t) := e^{-t^2}$, given in (27). Increasing curvature from left to right with $s_2 = 0$ (left), $s_2 = 1$ (center), $s_2 = 2$ (right).

    ii) τ *is an analyzing Taylorlet of order* n *with* $2M - 1$ *vanishing moments.*

    iii) τ *is restrictive.*

Proof.

i) Since $\tau = g \otimes h$ and due to the general setup $h \in \mathcal{S}(\mathbb{R})$, we only need to prove that $g \in \mathcal{S}(\mathbb{R})$. To this end, we show that the Schwartz properties are consecutively passed on to the next function through every step of the general setup.

First, we observe that with the change of the argument $\phi_n := \phi\big((\cdot)^{\nu_n}\big)$ in step II the Schwartz properties of ϕ remain. Furthermore, we obtain with the condition from step I that

$$\phi_n^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2\nu_n = 0.$$

This property is invariant under the action of step III, i. e.,

$$0 \neq \phi_{n,M}^{(k)}(0) = \phi_n^{(2M\nu_n + k)}(0) \quad \Leftrightarrow \quad k \bmod 2\nu_n = 0$$

for all $M \in \mathbb{N}$. After step IV the function $\tilde{\phi}_{n,M} := \phi_{n,M}\big(|\cdot|^{\frac{1}{\nu_n}}\big)$ is clearly smooth on every set not containing the origin. In order to show the smoothness of $\tilde{\phi}_{n,M}$ in the origin, we use Taylor's theorem to approximate $\phi_{n,M}$ by a Taylor polynomial. We thus obtain that

$$\phi_{n,M}(t) = \sum_{k=0}^{K} \frac{\phi_{n,M}^{(2k\nu_n)}(0)}{(2k\nu_n)!} \cdot t^{2k\nu_n} + o\big(t^{2K\nu_n}\big) \quad \text{for } t \to 0.$$

Hence, we can approximate $\tilde{\phi}_{n,M}$ by a sequence of polynomials, as well.

$$\tilde{\phi}_{n,M}(t) = \phi_{n,M}\big(|t|^{\frac{1}{\nu_n}}\big) = \sum_{k=0}^{K} \frac{\phi_{n,M}^{(2k\nu_n)}(0)}{(2k\nu_n)!} \cdot t^{2k} + o\big(t^{2K}\big) \quad \text{for } t \to 0.$$

Consequently, $\tilde{\phi}_{n,M}$ is smooth and inherits the Schwartzian decay property of $\phi_{n,M}$. Hence, $\tilde{\phi}_{n,M} \in \mathcal{S}(\mathbb{R})$. In the last step we get that

$$g(t) := (1 + t) \cdot \tilde{\phi}_{n,M}(t)$$

is a Schwartz function.

ii) We will prove this statement in three steps. First, we will show that $\phi_{n,M}$ has $2M v_n$ vanishing moments. In a second step we will prove that $\tilde{\phi}_{n,M}$ has $2M$ vanishing moments of order n and in the last part, we show that g has $2M - 1$ vanishing moments of order n.

STEP 1

As shown in the proof of *i)*, $\phi_n, \phi_{n,M} \in \mathcal{S}(\mathbb{R})$. Hence, their Fourier transforms exist and we obtain

$$\widehat{\phi_{n,M}}(\omega) = \left(\phi_n^{(2M v_n)}\right)^{\wedge}(\omega) = (-1)^{M v_n} \omega^{2M v_n} \widehat{\phi_n}(\omega).$$

Consequently, $\widehat{\phi_{n,M}}$ has a root of order at least $2M v_n$ in the origin and hence $\phi_{n,M}$ has at least $2M v_n$ vanishing moments.

STEP 2

We now prove that $\tilde{\phi}_{n,M} := \phi_{n,M}(|\cdot|^{1/v_n})$ has M vanishing moments of order n. To this end, let $k \in \{1, \dots, n\}$. As $\tilde{\phi}_{n,M} \in \mathcal{S}(\mathbb{R})$ due to *i)* and additionially is an even function, we obtain

$$\int_{\mathbb{R}} \tilde{\phi}_{n,M}(\pm x^k) x^m \, dx = 0$$

for all odd $m \in \mathbb{N}$. So we only consider even moments.

$$
\begin{aligned}
&\int_{\mathbb{R}} \tilde{\phi}_{n,M}(\pm x^k) x^{2m} \, dx \\
&= 2 \cdot \int_0^\infty \phi_{n,M}(x^{k/v_n}) \cdot x^{2m} \, dx && \left(\text{substitute } x = y^{v_n/k}\right) \\
&= \frac{2 v_n}{k} \cdot \int_0^\infty \phi_{n,M}(y) \cdot y^{(2m+1) \cdot \frac{v_n}{k} - 1} \, dy \\
&= \frac{2 v_n}{k} \cdot \int_0^\infty \phi_n^{(2M v_n)}(y) \cdot y^{(2m+1) \cdot \frac{v_n}{k} - 1} \, dy && (\text{partial integration}) \\
&= -\frac{2 v_n}{k} \cdot \left[(2m+1) \cdot \tfrac{v_n}{k} - 1\right] \cdot \int_0^\infty \phi_n^{(2M v_n - 1)}(y) \cdot y^{(2m+1) \cdot \frac{v_n}{k} - 2} \, dy && (\text{multiple part. int.}) \\
&= \pm \frac{2 v_n}{k} \cdot \left[(2m+1) \cdot \tfrac{v_n}{k} - 1\right]! \cdot \int_0^\infty \phi_n^{((2kM - 2m - 1) \cdot v_n/k + 1)}(y) \, dy \\
&= \mp \frac{2 v_n}{k} \cdot \left[(2m+1) \cdot \tfrac{v_n}{k} - 1\right]! \cdot \phi_n^{((2kM - 2m - 1) \cdot v_n/k)}(0) = 0.
\end{aligned}
$$

Since $\phi_n^{(\ell)}(0) \neq 0 \Leftrightarrow \ell \bmod 2\nu_n = 0$ and $2kM - 2m - 1 \bmod 2 = 1$, the expression in the last row vanishes.

STEP 3

Now we will show that g has $2M - 1$ vanishing moments of order n.

$$\int_{\mathbb{R}} g\left(t^k\right) t^m dt = \int_{\mathbb{R}} (1 + t^k) \tilde{\phi}_{n,M}(t^k) t^m dt$$
$$= \int_{\mathbb{R}} \tilde{\phi}_{n,M}(t^k) t^m dt + \int_{\mathbb{R}} \tilde{\phi}_{n,M}(t^k) t^{k+m} dt.$$

Due to the result of STEP 2, this expression vanishes if $m + k < 2k \cdot M$. Hence, g has $2M - 1$ vanishing moments of order n and τ is an analyzing Taylorlet of order n with $2M - 1$ vanishing moments.

iii) In order to prove that $\tau = g \otimes h$ is restrictive, it is sufficient to show that

$$I_+^j g(0) \neq 0 \quad \text{for all } j \in \{0, \ldots, 2M - 1\}$$

since $\int_{\mathbb{R}} h(t) dt \neq 0$ is already given in step VI of the general setup. This property will be shown in two steps. First we will prove the sufficiency of

$$\int_0^\infty \tilde{\phi}_{n,M}(t) t^{2m+1} dt \neq 0$$

for all $m \in \{0, \ldots, M - 1\}$. Afterwards we will reduce this property to the already proven property that

$$\phi_{n,M}^{(k)}(0) \neq 0 \quad \Leftrightarrow \quad k \bmod 2\nu_n = 0.$$

STEP 1

$$I_-^j g(0) = \int_0^\infty g(t) t^{j-1} dt$$
$$= \int_0^\infty \tilde{\phi}_{n,M}(t)(1 + t) t^{j-1} dt$$
$$= \int_0^\infty \tilde{\phi}_{n,M}(t) t^{j-1} dt + \int_0^\infty \tilde{\phi}_{n,M}(t) t^j dt.$$

Since $\tilde{\phi}_{n,M}$ is an even function with $2M$ vanishing moments, we obtain for $k < M$ that

$$\int_0^\infty \tilde{\phi}_{n,M}(t) t^{2k} dt = \frac{1}{2} \int_{\mathbb{R}} \tilde{\phi}_{n,M}(t) t^{2k} dt = 0.$$

Hence, we can conclude for the iterated integral of g that

$$I_-^j g(0) = \begin{cases} \int_0^\infty \tilde{\phi}_{n,M}(t) t^{j-1} dt, & \text{if } j \bmod 2 = 0, \\ \int_0^\infty \tilde{\phi}_{n,M}(t) t^j dt, & \text{if } j \bmod 2 = 1. \end{cases}$$

Since g has $2M-1$ vanishing moments, the statements $I_+^j g(0) \neq 0$ and $I_-^j g(0) \neq 0$ are equivalent. Consequently, we obtain that $I_+^j g(0) \neq 0$ for all $j \in \{0, \ldots, 2M-1\}$ is equivalent to

$$\int_0^\infty \tilde{\phi}_{n,M}(t) t^{2m+1} dt \neq 0 \quad \text{for all } m \in \{0, \ldots, M-1\}.$$

STEP 2

$$
\begin{aligned}
\int_0^\infty \tilde{\phi}_{n,M}(t) t^{2m+1} dt \quad &= \quad \int_0^\infty \phi_{n,M}\left(t^{\frac{1}{v_n}}\right) t^{2m+1} dt \\
&\overset{t=u^{v_n}}{=} \quad v_n \cdot \int_0^\infty \phi_{n,M}(u) u^{v_n(2m+1)} \cdot u^{v_n-1} du \\
&= \quad v_n \cdot \int_0^\infty \phi_{n,M}(u) u^{2v_n(m+1)-1} du \\
&= \quad v_n \cdot \int_0^\infty \phi_n^{(2Mv_n)}(u) u^{2v_n(m+1)-1} du \\
&\overset{\text{part. int.}}{=} \quad [2v_n(m+1)-1]! \cdot v_n \cdot \int_0^\infty \phi_n^{(2v_n[M-m-1]+1)}(u) du \\
&= \quad -[2v_n(m+1)-1]! \cdot v_n \cdot \phi_n^{(2v_n[M-m-1])}(0).
\end{aligned}
$$

The last expression does not vanish for any $m \in \{0, \ldots, M-1\}$ because $\phi_n^k(0) \neq 0 \Leftrightarrow k \bmod 2v_n = 0$. Hence,

$$\int_0^\infty \tilde{\phi}_{n,M}(t) t^{2m+1} dt \neq 0 \quad \text{for all } m \in \{0, \ldots, M-1\}.$$

Due to STEP 1 we can conclude that $I_+^j g(0) \neq 0$ for all $j \in \{0, \ldots, M-1\}$. $\square$

Remark 4.13. The sequence $v_n = \text{lcm}\{1, \ldots, n\}$ is innately connected to the second Chebyshev function which plays a crucial role in most proofs of the prime number theorem. The second Chebyshev function is defined as

$$\psi(x) = \sum_{p \in \mathbb{P}, k \in \mathbb{N}: \ p^k \leq x} \log p.$$

Its relation to the sequence v_n is given by the equation $v_n = e^{\psi(n)}$. The second Chebyshev function ψ itself is connected to the prime number theorem. It states that the prime counting function $\pi(x) = |\{p \in \mathbb{P}: \ p \leq x\}|$ exhibits the asymptotics

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$$

This statement can be proven via a relation to ψ, since it can be shown [Apo76, Theorem 4.4] that it is equivalent to ψ having the asymptotic behavior

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1. \tag{3}$$

This property is easier to show and can be proven by a relation to the Riemann Zeta function. On the other hand, (3) also provides the asymptotics for v_n:

$$\lim_{n\to\infty} \frac{\log v_n}{n} = 1 \text{ for } n \to \infty.$$

4.3.2   Construction based on q-calculus

In the previous subsection, we considered a derivative-based approach to the construction of a Taylorlet. While this ansatz allows for arbitrarily many vanishing moments, it falls short of providing a function with infinitely many vanishing moments. In this subsection, we will pursue a different approach which is based on the idea of utilizing dilations for the generation of vanishing moments.

The process of generating vanishing moments by considering linear combinations of dilations of a function yields a structure which can be studied by applying a q-calculus of operator-valued functions. This calculus is a variation of the classical analysis and resembles the finite difference calculus, but uses a multiplicative notation instead. This calculus recently gained interest due to its applications in quantum mechanics. The construction we present has an inherent connection to the q-Pochhammer symbol and the Euler function.

Despite its modern applications in quantum mechanics, the first accounts of q-calculus actually date back to the days of Euler. When he developed the theory of partitions, he introduced the partition function $p : \mathbb{N} \to \mathbb{N}$ with $p(n)$ being the number of distinct ways of representing n as a sum of natural numbers up to ordering. E. g., $p(4) = 5$, as there are five ways to represent 4 as sum of natural numbers:

$$
\begin{aligned}
4 &= 4 \\
&= 3 + 1 \\
&= 2 + 2 \\
&= 2 + 1 + 1 \\
&= 1 + 1 + 1 + 1.
\end{aligned}
$$

Euler found out that the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1 - q^k} = \sum_{n=0}^{\infty} p(n) q^n$$

is the generating function for the partition function [Ern00]. Its reciprocal is also known as Euler's function.

Definition 4.14 (Euler's function)**.** Let $q \in \mathbb{C}$ with $|q| < 1$. Then,

$$\varphi(q) = \prod_{k=1}^{\infty} (1 - q^k)$$

Infinitesimal calculus	q-calculus
$f'(x) = \lim_{q \to 1} \frac{f(qx) - f(x)}{qx - x}$	$d_q f(x) = \frac{f(qx) - f(x)}{qx - x}$
$\frac{d}{dx} x^n = n \cdot x^{n-1}$	$d_q(x^n) = [n]_q \cdot x^{n-1} = \frac{q^n - 1}{q-1} \cdot x^{n-1}$
$n!$	$[n]_q! = \prod_{k=1}^{n} [k]_q$
$\binom{n}{k}$	$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}$
$(a)_n$	$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$

Table 4.3.1: Overview of important q-analogs

is Euler's function.

This function is not to be confused with Euler's totient function which is also denoted by φ, but $\varphi(n)$ there displays the amount of numbers up to n which are relative prime to n.

The concept of q-calculus is similar to that of the finite difference calculus, but the q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$d_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

rather than $D_h f(x) = \frac{f(x+h) - f(x)}{h}$. For a more general overview on q-calculus see [KC02]. Similar to the finite differences, the infinitesimal theory can be obtained by q-calculus via the limit process $q \to 1$.

As an example, the q-derivative of a monomial can be found to be

$$d_q(x^n) = \frac{q^n - 1}{q - 1} \cdot x^{n-1}.$$

The occuring factor is also known as the q-bracket $[n]_q = \frac{q^n - 1}{q - 1}$. This yields a generalization of the classical binomial coefficient

$$\binom{n}{m}_q = \prod_{k=1}^{m} \frac{[n+1-k]_q}{[k]_q}.$$

One of the most central concepts in q-calculus is the analogue of the classical Pochhammer symbol. It is defined as

$$(a; q)_n := \prod_{k=0}^{n-1} \left(1 - aq^k\right).$$

It can be represented in terms of the q-binomial as shown in the following lemma which will be important later in this section.

Lemma 4.15.   *[Ext83, (4.2.3)] Let $x \in \mathbb{R}$, $q > 0$ and $n \in \mathbb{N}_0$. Then*

$$(x; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k \cdot x^k.$$

The q-Pochhammer symbol also is the initial point for a multitude of important functions in q-calculus. Among them, the Euler function can be represented as

$$\varphi(q) = (q; q)_\infty.$$

As we will see in Theorem 4.23, the Euler function can also be expanded as a series of q-Pochhammer symbols. An important fact when dealing with infinite products is the question of convergence, which is answered by the following two results.

Theorem 4.16.   *[Ern00, Theorem 2.2] Let Ω be a region in the complex plane. Suppose $\{f_n\}_{n \in \mathbb{N}_0}$ is a family of holomorphic function in Ω such that*

$$f_n(z) \neq 0 \quad \text{for all } z \in \Omega$$

and the series $\sum_{n \in \mathbb{N}_0} \left| 1 - f_n(z) \right|$ converges uniformly on all compact subsets of Ω. Then, the product

$$f(z) = \prod_{n \in \mathbb{N}_0} f_n(z)$$

converges uniformly on compact subsets of Ω and f is holomorphic in Ω.

Corollary 4.17.   *Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the complex unit disk. Furthermore, let $\alpha, q \in D$, $k \in \mathbb{N}_0$ and let $f : D \to \mathbb{C}$, $z \mapsto (z; q)_\infty$ and $g : D \to \mathbb{C}$, $z \mapsto \left(\alpha \cdot z^k; z \right)_\infty$. Then, f and g are holomorphic on D.*

Proof.

1. For $z \in D$, we can represent f as the product

$$f(z) = \prod_{n=0}^{\infty} \left(1 - q^n z \right).$$

   In the notation of Theorem 4.16, $f_n(z) = 1 - q^n z$. We can now observe that $f_n(z) \neq 0$ for all $n \in \mathbb{N}_0$ and $z \in D$. Moreover, the series

$$\sum_{n=0}^{\infty} \left| 1 - f_n(z) \right| = |z| \cdot \sum_{n=0}^{\infty} |q|^n = \frac{|z|}{1 - |q|}.$$

   converges uniformly, as $|q|, |z| < 1$. Thus, f is holomorphic on D.

2. For $z \in D$, we can represent g as the product

$$g(z) = \prod_{n=0}^{\infty} \left(1 - \alpha \cdot z^{k+n} \right).$$

In the notation of Theorem 4.16, $f_n(z) = 1 - \alpha \cdot z^{k+n}$. We can now observe that $f_n(z) \neq 0$ for all $n \in \mathbb{N}_0$ and $z \in D$. Moreover, the series

$$\sum_{n=0}^{\infty} \left| 1 - f_n(z) \right| = \alpha \cdot |z|^k \cdot \sum_{n=0}^{\infty} |z|^n = \frac{\alpha |z|^k}{1 - |z|}.$$

converges uniformly, as $|\alpha|, |z| < 1$. Thus, g is holomorphic on D.

$\square$

Construction

In order to obtain a function $g \in \mathcal{S}_n^*(\mathbb{R})$, i.e., a Schwartz function whose moments all vanish, it is sufficient to construct a function ψ such that

1. ψ is even,

2. $\psi^{(k)}(0) = c \cdot \delta_{0k}$ for some $c \neq 0$,

3. $\int_0^{\infty} \psi(x) x^k \, dx = 0$ for all $k \in \mathbb{N}_0$.

As the following proposition shows, with such a function ψ, we can construct a function with ∞ many vanishing moments of arbitrary order n.

Proposition 4.18. *Let the function* $\psi \in \mathcal{S}(\mathbb{R})$ *fulfill the conditions 1. and 2. and let* $v_n := \mathrm{lcm}\{1, \ldots, n\}$.

a) *If* $M \in \mathbb{N}$ *and* $\int_0^{\infty} \psi(x) x^k \, dx = 0$ *for all* $k \in \{0, \ldots, M v_n - 1\}$, *the function*

$$g := \psi \circ \sqrt[v_n]{|\cdot|} \in \mathcal{S}(\mathbb{R})$$

has M vanishing moments of order n.

b) *If* ψ *fulfills condition 3., the function*

$$g := \psi \circ \sqrt[v_n]{|\cdot|} \in \mathcal{S}_n^*(\mathbb{R}).$$

Proof.   *a)* The decay conditions of ψ are not changed by the concatenation with $\sqrt[v_n]{|\cdot|}$ and its smoothness is preserved as well due to condition 2. Hence, $g \in \mathcal{S}(\mathbb{R})$. So, it only remains to prove, that g has M vanishing moments of order n. As ψ and g are even Schwartz functions, all odd moments vanish anyway. So we only consider even moments.

$$\int_{\mathbb{R}} g(\pm t^k) t^{2m} dt = \int_{\mathbb{R}} \psi(|t|^{k/v_n}) t^{2m} dt$$

$$= 2 \int_0^{\infty} \psi(t^{k/v_n}) t^{2m} dt$$

$$= 2 \int_0^{\infty} \psi(u) u^{2mv_n/k} \cdot \frac{v_n}{k} u^{v_n/k - 1} du$$

$$= \frac{2v_n}{k} \cdot \int_0^{\infty} \psi(u) u^{(2m+1)v_n/k - 1} du = 0$$

for all $m \in \{0, \ldots, kM - 1\}$, since $k \mid v_n$ for all $k \in \{1, \ldots, n\}$.

b) This follows immediately from a).

<div align="right">□</div>

For the construction of a function ψ with properties 1. - 3. we start with an even bump function $\phi \in C_c^\infty(\mathbb{R})$ and a number $\varepsilon > 0$ such that $\phi|_{[-\varepsilon,\varepsilon]} \equiv 1$. Hence, properties 1. and 2. are already fulfilled, ϕ is a Schwartz function and we only need to gather vanishing moments. This can be achieved for $q \in (0,1)$ by the following function sequence:

$$\phi_{m+1} = \left(\text{Id} - q^{m+1} D_q\right)\phi_m,$$

where for $q > 0$, D_q is the dilation operator with $D_q : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $D_q f(x) = f(qx)$. Each step in this sequence adds one further vanishing moment to the function. Hence, ϕ_m has m vanishing moments, as shown in the next lemma.

Lemma 4.19. *Let $\phi_0 \in \mathcal{S}(\mathbb{R})$ be an even function and let*

$$\phi_{m+1} = \left(\text{Id} - q^{m+1} D_q\right)\phi_m$$

for all $m \in \mathbb{N}_0$. Then $\phi_m \in \mathcal{S}(\mathbb{R})$ and

$$\int_{\mathbb{R}_\pm} \phi_m(x) x^\ell \, dx = 0$$

for all $\ell \in \{0, \ldots, m - 1\}$ and for all $m \in \mathbb{N}_0$.

Proof. The statement can be shown inductively. Obviously, the statement is true for ϕ_0. Now we assume that $\phi_m \in \mathcal{S}(\mathbb{R})$ and $\int_{\mathbb{R}_\pm} \phi_m(x) x^\ell \, dx = 0$ for all $\ell \in \{0, \ldots, m-1\}$. As the Schwartz space is invariant under dilations, $\phi_{m+1} \in \mathcal{S}(\mathbb{R})$. Furthermore, for all $\ell \in \{0, \ldots, m-1\}$, we have

$$\int_{\mathbb{R}_\pm} \phi_{m+1}(x) x^\ell \, dx = \underbrace{\int_{\mathbb{R}_\pm} \phi_m(x) x^\ell \, dx}_{=0} - q^{m+1} \cdot \underbrace{\int_{\mathbb{R}_\pm} \phi_m\left(qx\right) x^\ell \, dx}_{=0}.$$

Furthermore,

$$\begin{aligned}
\int_{\mathbb{R}_\pm} \phi_{m+1}(x) x^m \, dx &= \int_{\mathbb{R}_\pm} \left[\phi_m(x) - q^{m+1}\phi_m\left(qx\right)\right] x^m \, dx \\
&= \int_{\mathbb{R}_\pm} \phi_m(x) x^m \, dx - \int_{\mathbb{R}_\pm} \phi_m\left(qx\right) \cdot \left(qx\right)^m q \, dx \\
&= \int_{\mathbb{R}_\pm} \phi_m(x) x^m \, dx - \int_{\mathbb{R}_\pm} \phi_m(x) x^m \, dx \\
&= 0.
\end{aligned}$$

<div align="right">□</div>

As a consequence of the previous lemma, $\int_0^\infty \phi_m(x)\, dx = 0$ for all $m \in \mathbb{N}_0$. As this contradicts the restrictiveness condition, we cannot immediately use the functions ϕ_m to produce restrictive Taylorlets. We will present a method to achieve this in Lemma 4.27.

The following proposition shows that the function ϕ_m exhibits m vanishing moments, but does so by proving that its Fourier transform has a root of order n in the origin.

Proposition 4.20. *Let $\phi_0 \in L^1(\mathbb{R}, x^n dx)$ and*

$$\phi_{m+1} = \left(\mathrm{Id} - q^{m+1} \cdot D_q \right) \phi_m$$

for all $m \in \mathbb{N}_0$. Then

$$\widehat{\phi_n}(\omega) = \left(1 - q^{-1} \right)^n \cdot \omega^n \cdot d_{q^{-1}}^n \widehat{\phi_0}(\omega) \quad \text{for all } \omega \in \mathbb{R}.$$

Proof. We first look for an appropriate representation of ϕ_n. To this end, we will use Lemma 4.15 and utilize

$$(x; q)_n = \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k \cdot x^k.$$

Since the dilation operator commutes with scalar multiplication, we can write ϕ_n as an operator-valued q-Pochhammer symbol

$$\phi_n = \prod_{m=0}^{n-1} \left(\mathrm{Id} - q^{m+1} D_q \right) \phi_0 = \left(q D_q; q \right)_n \phi_0$$

and obtain that

$$\phi_n = \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k \cdot q^k D_{q^k} \phi_0. \tag{4}$$

Due to [Ern12, (6.98)], for $q > 0$ the nth q-derivative of a function can be represented as

$$d_q^n f(x) = (q-1)^{-n} q^{-\binom{n}{2}} x^{-n} \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k f\left(q^{n-k} x \right).$$

By inserting q^{-1} for q, $f = \hat{\phi}_0$, $x = \omega$ and comparing the upper equation to the equation we want to prove, we can see that it remains to show that

$$\widehat{\phi_n}(\omega) = q^{\binom{n}{2}} \sum_{k=0}^n \binom{n}{k}_{q^{-1}} q^{-\binom{k}{2}} (-1)^{n-k} \widehat{\phi_0}\left(q^{k-n} \omega \right). \tag{5}$$

To this end, we first represent $\binom{n}{k}_q$ in terms of $\binom{n}{k}_{q^{-1}}$. We can write

$$
\begin{aligned}
\binom{n}{k}_q &= \prod_{\ell=1}^{k} \frac{1 - q^{n+1-\ell}}{1 - q^\ell} \\
&= \prod_{\ell=1}^{k} \frac{q^{n+1-\ell}}{q^\ell} \cdot \frac{q^{\ell-n-1} - 1}{q^{-\ell} - 1} \\
&= q^{-2\binom{k+1}{2} + k(n+1)} \cdot \prod_{\ell=1}^{k} \frac{q^{\ell-n-1} - 1}{q^{-\ell} - 1} \\
&= q^{k(n-k)} \cdot \binom{n}{k}_{q^{-1}}.
\end{aligned}
$$

By applying the Fourier transform to equation (4) and inserting the upper equality, we get

$$
\begin{aligned}
\widehat{\phi_n}(\omega) &= \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} (-1)^k \widehat{\phi_0}\left(\frac{\omega}{q^{-k}} \right) \\
&= \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{n-k}{2}} (-1)^{n-k} \widehat{\phi_0}\left(\frac{\omega}{q^{k-n}} \right) \\
&= \sum_{k=0}^{n} \binom{n}{k}_{q^{-1}} q^{k(n-k)} q^{\binom{n-k}{2}} (-1)^{n-k} \widehat{\phi_0}\left(\frac{\omega}{q^{k-n}} \right) \\
&= q^{\binom{n}{2}} \cdot \sum_{k=0}^{n} \binom{n}{k}_{q^{-1}} q^{-\binom{k}{2}} (-1)^{n-k} \widehat{\phi_0}\left(\frac{\omega}{q^{k-n}} \right),
\end{aligned}
$$

where the last equality results from $\binom{a+b}{2} = \binom{a}{2} + ab + \binom{b}{2}$. $\qquad\square$

The next lemma shows that the sequence ϕ_m converges to a function satisfying the properties 1. - 3.

Lemma 4.21. *Let $\phi_0 \in \mathcal{S}(\mathbb{R})$ and let $q \in (0,1)$ and $\varepsilon > 0$ such that $\phi_0|_{[-\varepsilon,\varepsilon]} \equiv 1$ and let*

$$
\phi_{m+1} = \left(\mathrm{Id} - q^{m+1} D_q \right) \phi_m
$$

for all $m \in \mathbb{N}_0$. Then the function sequence ϕ_m converges uniformly to a function $\psi \in \mathcal{S}(\mathbb{R})$ for $m \to \infty$. Furthermore,

$$
\left\| \psi - \phi_m \right\|_\infty \leq \frac{(-q;q)_\infty}{\varphi(q)} \cdot \frac{\left\| \phi_0 \right\|_\infty}{1 + q^{-(m+1)}} \quad \text{for all } m \in \mathbb{N}_0
$$

and ψ fulfills conditions 1. - 3.

Proof. We first show that ϕ_m is a Cauchy sequence w.r.t. the L^∞-norm and hence that it converges uniformly. To this end, let $\ell, m \in \mathbb{N}_0$. Then

$$\begin{aligned}
\|\phi_{m+\ell} - \phi_m\|_\infty &= \left\| \left[\prod \left(\mathrm{Id} - q^{m+1} D_q \right) - \mathrm{Id} \right] \phi_m \right\|_\infty \\
&= \left\| \left[\left(q^{m+1} D_q; q \right)_\ell - \mathrm{Id} \right] \phi_m \right\|_\infty.
\end{aligned}$$

Utilizing Lemma 4.15, we obtain

$$\begin{aligned}
\|\phi_{m+\ell} - \phi_m\|_\infty &= \left\| \left[\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \cdot \binom{\ell}{k}_q \cdot q^{k(m+1)} D_{q^k} - \mathrm{Id} \right] \phi_m \right\|_\infty \\
&= \left\| \left[\sum_{k=1}^{\ell} (-1)^k q^{\binom{k}{2}} \cdot \binom{\ell}{k}_q \cdot q^{k(m+1)} D_{q^k} \right] \phi_m \right\|_\infty \\
&\leq \|\phi_m\|_\infty \cdot \sum_{k=1}^{\ell} q^{\left[\binom{k}{2} + k(m+1) \right]} \cdot \binom{\ell}{k}_q \\
&= \|\phi_m\|_\infty \cdot \sum_{k=1}^{\ell} q^{\left[\binom{k}{2} + k(m+1) \right]} \cdot \prod_{v=0}^{k-1} \frac{1 - q^{v-\ell}}{1 - q^{v+1}} \\
&\leq \|\phi_m\|_\infty \cdot \sum_{k=1}^{\ell} q^{\left[\binom{k}{2} + k(m+1) \right]} \cdot \underbrace{\prod_{v=0}^{\infty} \frac{1}{1 - q^{v+1}}}_{= \frac{1}{\varphi(q)}} \\
&\leq \frac{\|\phi_m\|_\infty}{\varphi(q)} \cdot \sum_{k=1}^{\ell} q^{k(m+1)} \leq \frac{\|\phi_m\|_\infty}{\varphi(q)} \cdot \sum_{k=1}^{\infty} q^{k(m+1)} \\
&= \frac{\|\phi_m\|_\infty}{\varphi(q)} \cdot \frac{1}{1 + q^{-(m+1)}}. \tag{6}
\end{aligned}$$

As the Euler function is strictly decreasing on $[0, 1]$ with $\varphi(0) = 1$ and $\varphi(1) = 0$, it is positive on the interval $(0, 1)$. Hence, it only remains to show that ϕ_m has a uniform upper bound. To this end, we observe that for $m \in \mathbb{N}_0$ we have

$$\begin{aligned}
\|\phi_{m+1}\|_\infty &= \left\| \left(\mathrm{Id} - q^{m+1} D_q \right) \phi_m \right\|_\infty \\
&\leq \|\phi_m\|_\infty + q^{m+1} \|D_q \phi_m\|_\infty \\
&\leq \left(1 + q^{m+1} \right) \cdot \|\phi_m\|_\infty.
\end{aligned}$$

Hence, we can inductively show that

$$\begin{aligned}
\|\phi_{m+1}\|_\infty &\leq \prod_{k=0}^{m} \left(1 + q^{k+1} \right) \cdot \|\phi_0\|_\infty \\
&= \left(-q; q \right)_{m+1} \cdot \|\phi_0\|_\infty.
\end{aligned}$$

Due to Corollary 4.17, for $q \in (0,1)$, the expression $\left(-q;q\right)_m$ converges for $m \to \infty$. By combining this result with (6), we get the desired estimate

$$\left\|\psi - \phi_m\right\|_\infty \leq \frac{\left(-q;q\right)_\infty}{\varphi\left(q\right)} \cdot \frac{\left\|\phi_0\right\|_\infty}{1 + q^{-(m+1)}} \quad \text{for all } m \in \mathbb{N}_0.$$

We now proceed by proving the properties 1. - 3. for ψ.

1. Since ϕ_0 is even and the constructive function sequence consists of linear combinations of dilates of ϕ_0, ψ is even, as well.

2. As $\phi_0|_{[-\varepsilon,\varepsilon]} \equiv 1$, we only have to show that $\psi(0) \neq 0$. We can represent the limit function as

$$\psi = \prod_{k=0}^\infty \left(\text{Id} - q^{k+1} D_q\right)\phi_0.$$

Due to $\phi_0(0) = 1$, we obtain that

$$\psi(0) = \prod_{k=0}^\infty \left(1 - q^{k+1}\right) = \varphi\left(q\right) > 0.$$

3. As shown in Lemma 4.19, $\int_0^\infty \phi_m(x)x^k \, dx = 0$ for all $k \in \{0,\ldots,m-1\}$ and $m \in \mathbb{N}$. Hence, $\int_0^\infty \psi(x)x^k \, dx = 0$ for all $k \in \mathbb{N}_0$. So it remains to prove that $\psi \in \mathcal{S}(\mathbb{R})$. To this end, we define

$$c_{k,\ell,m} := \|x^k \phi_m^{(\ell)}(x)\|_\infty.$$

In order to prove that $\psi \in \mathcal{S}(\mathbb{R})$, we will show that uniform upper bounds in m exist for the $c_{k,\ell,m}$. I. e., for all $k, \ell \in \mathbb{N}$ we determine a $c_{k,\ell} > 0$ such that

$$c_{k,\ell,m} \leq c_{k,\ell} \text{ for all } m \in \mathbb{N}_0.$$

For this purpose we estimate $c_{k,\ell,m+1}$ in terms of $c_{k,\ell,m}$.

$$x^k \cdot \phi_{m+1}^{(\ell)}(x) = x^k \cdot \partial_x^\ell \left(\text{Id} - q^{m+1} D_q\right)\phi_m(x) = x^k \cdot \left(\text{Id} - q^{m+\ell+1} D_q\right)\phi_m^{(\ell)}(x).$$

Hence, we can estimate

$$
\begin{aligned}
c_{k,\ell,m+1} &= \left\| x^k \phi_{m+1}^{(\ell)}(x) \right\|_\infty \\
&= \left\| x^k \cdot \left(\mathrm{Id} - q^{m+\ell+1} D_q \right) \phi_m^{(\ell)}(x) \right\|_\infty \\
&\leq \left\| x^k \phi_m^{(\ell)}(x) \right\|_\infty + q^{m+\ell-k+1} \left\| (qx)^k \phi_m^{(\ell)}(qx) \right\|_\infty \\
&\leq \left(1 + q^{m+\ell-k+1} \right) \cdot c_{k,\ell,m} \\
&\leq \prod_{\nu=0}^{m} \left(1 + q^{\nu+\ell-k+1} \right) \cdot c_{k,\ell,0} \\
&\leq c_{k,\ell,0} \cdot \prod_{\nu=0}^{m} (1 + q^{\ell+1+\nu-k}) \\
&= c_{k,\ell,0} \cdot \left(-q^{\ell+1-k}; q \right)_{m+1} \\
&\leq c_{k,\ell,0} \cdot \left(-q^{\ell+1-k}; q \right)_\infty .
\end{aligned}
$$

The last inequality holds, as for all $a < 0$ and $m, n \in \mathbb{N}$,

$$
(a;q)_{m+n} = \prod_{k=0}^{m+n-1} \left(1 - aq^k \right) = (a;q)_m \cdot \underbrace{\prod_{k=m}^{m+n-1} \left(1 - aq^k \right)}_{>1} > (a;q)_m .
$$

Due to Corollary 4.17, the expression $\left(-q^{\ell+1-k}; q \right)_\infty$ indeed converges for all $q \in (0,1)$. Thus, $c_{k,\ell,m} \leq \left(-q^{\ell+1-k}; q \right)_\infty \cdot c_{k,\ell,0} =: c_{k,\ell}$ for all $m \in \mathbb{N}_0$. Since $\phi_0 \in \mathcal{S}(\mathbb{R})$ by prerequisite, $c_{k,\ell,0}$ is finite for all $k, \ell \in \mathbb{N}_0$.

$\square$

The next lemma gives an explicit representation of ψ as a series of dilates of ϕ_0 by using the q-Pochhammer symbol.

Lemma 4.22. *Let $\phi_0 \in \mathcal{S}(\mathbb{R})$ and let $q \in (0,1)$. Then*

$$
\psi = \lim_{m \to \infty} \phi_m = \sum_{\ell=0}^{\infty} \frac{1}{\left(q^{-1}; q^{-1} \right)_\ell} \cdot D_{q^\ell} \phi_0 .
$$

Proof. We can rewrite the function ψ as

$$
\psi = \prod_{m=0}^{\infty} \left(\mathrm{Id} - q^{m+1} \cdot D_q \right) \phi_0 = \left(q \cdot D_q; q \right)_\infty \phi_0 .
$$

Due to a result of [Eul48, Chapter 16], the following series expansion holds for $|q| < 1$ and for all $z \in \mathbb{C}$:

$$
(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot q^{n(n-1)/2}}{(q;q)_n} \cdot z^n .
$$

Since the dilation operator commutes with the multiplication with constants, we can rewrite the limit function ψ as

$$
\begin{aligned}
\psi &= \left(q \cdot D_q; q\right)_\infty \phi_0 \\
&= \sum_{\ell=0}^\infty \frac{(-1)^\ell \cdot q^{\ell(\ell-1)/2}}{(q;q)_\ell} \cdot q^\ell \cdot D_{q^\ell} \phi_0 \\
&= \sum_{\ell=0}^\infty \frac{(-1)^\ell}{q^{-(\ell+1)\ell/2} \cdot \prod_{k=1}^\ell (1-q^k)} \cdot D_{q^\ell} \phi_0 \\
&= \sum_{\ell=0}^\infty \frac{1}{(q^{-1};q^{-1})_\ell} \cdot D_{q^\ell} \phi_0.
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

The next theorem utilizes all of the previous lemmata to show an explicit formula for the limit function $\psi \in \mathcal{S}^*(\mathbb{R})$.

Theorem 4.23. *Let $q \in (0,1)$, $\varepsilon > 0$ and let $\phi_0 \in C_c^\infty(\mathbb{R})$ be of the form*

$$
\phi_0(x) = \begin{cases}
1 & \text{for } |x| \le \varepsilon, \\
\eta(|x|) & \text{for } |x| \in \left(\varepsilon, \varepsilon q^{-1}\right], \\
0 & \text{for } |x| > \varepsilon q^{-1}.
\end{cases}
$$

where $\eta \in C^\infty\left(\left[\varepsilon, \varepsilon q^{-1}\right]\right)$ is chosen so that $\phi_0 \in C^\infty(\mathbb{R})$. Then, $\psi = \prod_{m=0}^\infty \left(\mathrm{Id} - q^{m+1} D_q\right) \phi_0$ has the explicit representation

$$
\psi(x) = \varphi(q) + \sum_{\ell=0}^\infty \left[\frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot \eta\left(q^\ell \cdot |x|\right) - \sum_{k=0}^\ell \frac{1}{\left(q^{-1};q^{-1}\right)_k} \right] \cdot \mathbb{1}_{\left(\varepsilon q^{-\ell}; \varepsilon q^{-(\ell+1)}\right]}(|x|) \quad \text{for all } x \in \mathbb{R},
$$

where φ is the Euler function.

Proof. Due to the form of ϕ_0 and Lemma 4.22 we obtain that

$$
\begin{aligned}
\psi(x) &= \sum_{\ell=0}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot D_{q^\ell} \phi_0(x) \\
&= \sum_{\ell=0}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot \left[\mathbb{1}_{\left[-\varepsilon q^{-\ell}, \varepsilon q^{-\ell}\right]}(x) + \left(D_{q^\ell}\eta\right)(|x|) \cdot \mathbb{1}_{\left(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}\right]}(|x|) \right] \\
&= \sum_{\ell=0}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot \left(\sum_{k=0}^{\ell-1} \mathbb{1}_{\left(\varepsilon q^{-k}, \varepsilon q^{-(k+1)}\right]}(|x|) + \mathbb{1}_{[-\varepsilon,\varepsilon]}(x) + \eta\left(q^\ell \cdot |x|\right) \cdot \mathbb{1}_{\left(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}\right]}(|x|) \right) \\
&= \sum_{\ell=0}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot \mathbb{1}_{[-\varepsilon,\varepsilon]}(x) + \sum_{k=0}^\infty \mathbb{1}_{\left(\varepsilon q^{-k}, \varepsilon q^{-(k+1)}\right]}(|x|) \cdot \sum_{\ell=k+1}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \\
&\quad + \sum_{\ell=0}^\infty \frac{1}{\left(q^{-1};q^{-1}\right)_\ell} \cdot \eta\left(q^\ell \cdot |x|\right) \cdot \mathbb{1}_{\left(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}\right]}(|x|).
\end{aligned}
$$

By inserting $x = 0$ into the upper equation, we obtain for all $x \in \mathbb{R}$ that

$$\psi(0) = \sum_{\ell=0}^{\infty} \frac{1}{(q^{-1}; q^{-1})_\ell}.$$

By repeating this for the equation

$$\psi(x) = \left[\prod_{m=0}^{\infty} \left(\mathrm{Id} - q^{m+1} \cdot D_q \right) \phi_0 \right](x),$$

we can conclude with $\phi_0(0) = 1$ and $(D_a f)(0) = (\mathrm{Id} f)(0)$ for all $a > 0$ and all $f \in C(\mathbb{R})$ that

$$\sum_{\ell=0}^{\infty} \frac{1}{(q^{-1}; q^{-1})_\ell} = \psi(0) = \left[\prod_{m=0}^{\infty} \left(\mathrm{Id} - q^{m+1} \cdot \mathrm{Id} \right) \phi_0 \right](0) = \prod_{m=0}^{\infty} \left(1 - q^{m+1} \right) = \prod_{m=1}^{\infty} \left(1 - q^m \right) = \varphi(q).$$

This equation together with $\sum_{k=\ell+1}^{\infty} \frac{1}{(q^{-1};q^{-1})_k} = \varphi(q) - \sum_{k=0}^{\ell} \frac{1}{(q^{-1};q^{-1})_k}$ delivers

$$\psi(x) = \varphi(q) \cdot \mathbb{1}_{[-\varepsilon;\varepsilon]}(x) + \sum_{\ell=0}^{\infty} \left[\varphi(q) - \sum_{k=0}^{\ell} \frac{1}{(q^{-1}; q^{-1})_k} + \frac{1}{(q^{-1}; q^{-1})_\ell} \cdot D_{q^\ell} \eta(|x|) \right] \cdot \mathbb{1}_{(q^{-\ell}\varepsilon; q^{-(\ell+1)}\varepsilon]}(|x|)$$

$$= \varphi(q) + \sum_{\ell=0}^{\infty} \left[\frac{1}{(q^{-1}; q^{-1})_\ell} \cdot \eta\left(q^\ell \cdot |x|\right) - \sum_{k=0}^{\ell} \frac{1}{(q^{-1}; q^{-1})_k} \right] \cdot \mathbb{1}_{(\varepsilon q^{-\ell}; \varepsilon q^{-(\ell+1)}]}(|x|).$$

$$\square$$

We can also utilize that $\varphi(x)$ is known for special values of x. For instance, it is known that

$$\varphi\left(e^{-\pi}\right) = \frac{e^{\frac{\pi}{24}} \cdot \Gamma\left(\frac{1}{4}\right)}{2^{\frac{7}{8}} \cdot \pi^{\frac{3}{4}}} \quad \text{[Ber05, p. 326]}.$$

With the choice $q = e^{-\pi}$, we hence obtain the following formula for a function in $\mathcal{S}_1^*(\mathbb{R})$:

$$\psi(x) = \frac{e^{\frac{\pi}{24}} \cdot \Gamma\left(\frac{1}{4}\right)}{2^{\frac{7}{8}} \cdot \pi^{\frac{3}{4}}} + \sum_{\ell=0}^{\infty} \left[\frac{1}{(e^\pi; e^\pi)_\ell} \cdot \eta(e^{-\ell\pi}|x|) - \sum_{k=0}^{\ell} \frac{1}{(e^\pi; e^\pi)_k} \right] \cdot \mathbb{1}_{(\varepsilon e^{\ell\pi}; \varepsilon e^{(\ell+1)\pi}]}(|x|).$$

In order to obtain a Taylorlet which not only fulfills the vanishing moment conditions, but also the restrictiveness, we consider a slightly altered construction procedure which is presented in the following setup.

General setup

1. Let $\phi \in \mathcal{S}(\mathbb{R})$ be a non-negative function.

2. Let $n \in \mathbb{N}$, $v_n := \mathrm{lcm}\{1, \ldots, n\}$ and let $\phi_0(t) := \phi\left(t^{2v_n}\right)$ for all $t \in \mathbb{R}$.

3. Let the function sequence $(\phi_m)_{m \in \mathbb{N}_0}$ be defined iteratively by

$$\phi_{m+1} = \begin{cases} \left(\mathrm{Id} - q^{m+1} D_q\right) \phi_m, & m \bmod 2\nu_n \neq 2\nu_n - 1, \\ \phi_m, & m \bmod 2\nu_n = 2\nu_n - 1. \end{cases}$$

4. Let $\psi := \phi_{2M\nu_n} \circ \sqrt[\nu_n]{|\cdot|}$ for $M \in \mathbb{N} \cup \{\infty\}$.

5. Let $g(t) := (1 + t) \cdot \psi(t)$ for all $t \in \mathbb{R}$.

6. Let $h \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} h(t)\,dt \neq 0$ and let $\tau = g \otimes h$.

We will prove by using two auxiliary results that this construction yields a restrictive Taylorlet with $2M - 1$ vanishing moments of order n. The first of these touches on properties of the function sequence ϕ_m and is described in the following lemma.

Lemma 4.24. *Let $m \in \mathbb{N}_0$ and let ϕ_m be defined as in the general setup. Then, ϕ_m has m vanishing moments. Furthermore, for all $k \in \{0, \ldots, m-1\}$, we have*

$$\int_0^\infty \phi_m(x) \cdot x^k \, dx \begin{cases} = 0, & k \bmod 2\nu_n \neq 2\nu_n - 1, \\ \neq 0, & k \bmod 2\nu_n = 2\nu_n - 1. \end{cases}$$

Proof. The statement that $\int_0^\infty \phi_m(x) \cdot x^k \, dx = 0$ for all $k \in \{0, \ldots, m-1\}$ such that $k \bmod 2\nu_n \neq 2\nu_n - 1$ follows by the same arguments as in Lemma 4.19. Now let the exponent be of the form $k = 2\ell\nu_n - 1$ for $\ell \in \mathbb{N}$. As ϕ_0 is non-negative according to the general setup, we have

$$\int_0^\infty \phi_0(x) \cdot x^{2\ell\nu_n - 1} \, dx > 0 \quad \text{for all } \ell \in \mathbb{N}_0.$$

We will now show inductively that $\int_0^\infty \phi_m(x) \cdot x^{2\ell\nu_n - 1} \, dx \neq 0$ for all $m \in \mathbb{N}_0$. As induction hypothesis, we assume that $\int_0^\infty \phi_m(x) \cdot x^{2\ell\nu_n - 1} \, dx \neq 0$ for an $m \in \mathbb{N}_0$. If $m \bmod 2\nu_n = 2\nu_n - 1$, we obtain that

$$\int_0^\infty \phi_{m+1}(x) \cdot x^{2\ell\nu_n - 1} \, dx = \int_0^\infty \phi_m(x) \cdot x^{2\ell\nu_n - 1} \, dx \neq 0.$$

The case $m \bmod 2\nu_n \neq 2\nu_n - 1$ however yields

$$\begin{aligned}
\int_0^\infty \phi_{m+1}(x) \cdot x^{2\ell\nu_n - 1} \, dx &= \int_0^\infty \left(\mathrm{Id} - q^{m+1} D_q\right) \phi_m(x) \cdot x^{2\ell\nu_n - 1} \, dx \\
&= \int_0^\infty \phi_{m+1}(x) \cdot x^{2\ell\nu_n - 1} \, dx - q^{m+1} \cdot \int_0^\infty \phi_m(qx) \cdot x^{2\ell\nu_n - 1} \, dx \\
&= \int_0^\infty \phi_{m+1}(x) \cdot x^{2\ell\nu_n - 1} \, dx - q^{m+1-2\ell\nu_n} \cdot \int_0^\infty \phi_{m+1}(y) \cdot y^{2\ell\nu_n - 1} \, dy \\
&= (1 - q^{m+1-2\ell\nu_n}) \cdot \int_0^\infty \phi_{m+1}(x) \cdot x^{2\ell\nu_n - 1} \, dx \neq 0,
\end{aligned}$$

as $m \bmod 2\nu_n \neq 2\nu_n - 1$. As $\phi_0(t) := \phi\left(t^{2\nu_n}\right)$ for all $t \in \mathbb{R}$, ϕ_0 is an even function. Since the operators $\mathrm{Id} - q^{m+1} D_q$ preserve the even symmetry, ϕ_m is even, as well. Hence, we can conclude by $\int_0^\infty \phi_m(x) \cdot x^k \, dx = 0$ for $k \in \{0, \ldots, m-1\}$ such that $k \bmod 2\nu_n \neq 2\nu_n - 1$ that

$$\int_{\mathbb{R}} \phi_m(x) \cdot x^k \, dx = 0 \qquad \text{for all } k \in \{0, \ldots, m-1\} \text{ such that} \quad k \bmod 2\nu_n \neq 2\nu_n - 1.$$

Since $k \in \{0, \ldots, m-1\}$ such that $k \bmod 2\nu_n = 2\nu_n - 1$ is necessarily an odd number, the even symmetry of ϕ_m also yields

$$\int_{\mathbb{R}} \phi_m(x) \cdot x^k \, dx = 0 \qquad \text{for all } k \in \{0, \ldots, m-1\} \text{ such that} \quad k \bmod 2\nu_n = 2\nu_n - 1.$$

Thus, ϕ_m has m vanishing moments. $\qquad \square$

The next lemma shows that the results of Lemma 4.24 hold for $m \to \infty$.

Lemma 4.25. *Let $m \in \mathbb{N}_0$ and let ϕ_m be defined as in the general setup. Then the function sequence ϕ_m converges uniformly to a function $\phi_\infty \in \mathcal{S}(\mathbb{R})$ for $m \to \infty$. Furthermore,*

$$\left\| \phi_\infty - \phi_m \right\|_\infty \leq \frac{(-q;q)_\infty}{\varphi(q)} \cdot \frac{\left\| \phi_0 \right\|_\infty}{1 + q^{-(m+1)}} \quad \text{for all } m \in \mathbb{N}_0.$$

The function ϕ_∞ has infinitely many vanishing moments and for all $k \in \mathbb{N}_0$, we have

$$\int_0^\infty \phi_\infty(x) \cdot x^k \, dx \begin{cases} = 0, & k \bmod 2\nu_n \neq 2\nu_n - 1, \\ \neq 0, & k \bmod 2\nu_n = 2\nu_n - 1. \end{cases}$$

Proof. The proof of the uniform convergence of the function sequence ϕ_m to a Schwartz function is analogous to the proof of Lemma 4.21, as the estimates of the latter lemma also hold for the sequence ϕ_m considered in step 3 of the general setup. This also yields the desired inequality for $\left\| \phi_\infty - \phi_m \right\|_\infty$. As $\phi_\infty \in \mathcal{S}(\mathbb{R})$, the moment integrals are well defined and thus

$$\int_0^\infty \phi_\infty(x) \cdot x^k \, dx \begin{cases} = 0, & k \bmod 2\nu_n \neq 2\nu_n - 1, \\ \neq 0, & k \bmod 2\nu_n = 2\nu_n - 1. \end{cases}$$

due to Lemma 4.24. $\qquad \square$

The third lemma describes the pivotal properties of the function ψ.

Lemma 4.26. *Let ψ be defined as in the general setup and let $M \in \mathbb{N} \cup \{\infty\}$. Then, $\psi \in \mathcal{S}(\mathbb{R})$ and ψ has $2M$ vanishing moments of order n or infinitely many vanishing moments of order n, if $M = \infty$. Furthermore,*

$$\int_0^\infty \psi(t) \cdot t^m \, dt = 0 \Leftrightarrow m \in \{0, \ldots, 2M-1\} \wedge m \bmod 2 = 0.$$

Proof. We prove:

1. $\psi \in \mathcal{S}(\mathbb{R})$,

2. $\displaystyle\int_{\mathbb{R}} \psi(\pm t^k) t^m \, dt = 0 \quad \forall \, m \in \{0, \ldots, 2kM - 1\} \quad \forall \, k \in \{1, \ldots, n\}$,

3. $\displaystyle\int_0^\infty \psi(t) \cdot t^m \, dt = 0 \Leftrightarrow m \in \{0, \ldots, 2M - 1\} \wedge m \bmod 2 = 0$.

1. We will now inductively show that for all $m \in \mathbb{N}_0$, ϕ_m can be represented by a Schwartz function $\tilde{\phi}_m$ such that for all $t \in \mathbb{R}$,

$$\phi_m(t) = \tilde{\phi}_m\left(t^{2\nu_n}\right).$$

According to the general setup, $\phi_0(t) = \phi(t^{\nu_n})$ for all $t \in \mathbb{R}$ which marks the base case of the induction. As induction hypothesis let $m \in \mathbb{N}_0$ such that ϕ_m fulfills the upper equation. For $m \in \mathbb{N}_0$ with $m \bmod 2\nu_n = 2\nu_n - 1$, we have $\phi_{m+1} = \phi_m$ due to 3. of the general setup. So ϕ_{m+1} has such a representation, as well. For $m \in \mathbb{N}_0$ with $m \bmod 2\nu_n \neq 2\nu_n - 1$, we obtain

$$\begin{aligned}
\phi_{m+1}(t) &= (\mathrm{Id} - q^{m+1} D_q)\phi_m(t) \\
&= \phi_m(t) - q^{m+1}\phi_m(qt) \\
&= \tilde{\phi}_m\left(t^{2\nu_n}\right) - q^{m+1} \cdot \tilde{\phi}_m\left(q^{2\nu_n} t^{2\nu_n}\right).
\end{aligned}$$

Hence, for $\tilde{\phi}_{m+1} := \tilde{\phi}_m - q^{m+1} \cdot D_{q^{2\nu_n}} \tilde{\phi}_m \in \mathcal{S}(\mathbb{R})$, we have

$$\phi_{m+1}(t) := \tilde{\phi}_{m+1}\left(t^{2\nu_n}\right) \quad \text{for all } t \in \mathbb{R}.$$

Thus, $\psi := \phi_{2M\nu_n}\left(\sqrt[\nu_n]{|\cdot|}\right) = \tilde{\phi}_{2M\nu_n}\left((\cdot)^2\right) \in \mathcal{S}(\mathbb{R})$.

2. According to its definition, ψ is an even function. Hence,

$$\int_{\mathbb{R}} \psi(\pm t^k) t^{2m+1} \, dt = 0$$

for all $k, m \in \mathbb{N}_0$. So, we will only consider the even moments. As $\psi := \phi_{2M\nu_n} \circ \sqrt[\nu_n]{|\cdot|}$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \psi(\pm t^k) t^{2m} \, dt &= \int_{\mathbb{R}} \phi_{2M\nu_n}\left(|t|^{k/\nu_n}\right) t^{2m} \, dt \\
&= 2 \cdot \int_0^\infty \phi_{2M\nu_n}\left(t^{k/\nu_n}\right) t^{2m} \, dt && \left(\text{substitute } t = u^{\nu_n/k}\right) \\
&= 2 \cdot \int_0^\infty \phi_{2M\nu_n}(u) u^{2m\nu_n/k} \cdot \frac{\nu_n}{k} \cdot u^{\nu_n/k-1} \, du \\
&= \frac{2\nu_n}{k} \cdot \int_0^\infty \phi_{2M\nu_n}(u) \cdot u^{2\nu_n \cdot \frac{2m+1}{2k} - 1} \, du = 0.
\end{aligned}$$

Since clearly the exponent $2v_n \cdot \frac{2m+1}{2k} - 1 \not\equiv -1 \mod 2v_n$, Lemma 4.24 and Lemma 4.25, respectively, yield that the upper integral vanishes if $2v_n \cdot \frac{2m+1}{2k} - 1 \leq 2Mv_n - 1$, i.e. $m \leq kM - 1$.

3. By inserting $k = 1$ and m even in the statement of the previous equation and exploiting the evenness of ψ, we obtain that

$$\int_0^\infty \psi(t) t^{2m} \, dt = 0$$

for all $m \in \{0, \dots, M - 1\}$. It is hence sufficient to show that

$$\int_0^\infty \psi(t) t^{2m+1} \, dt \neq 0$$

for all $m \in \{0, \dots, M - 1\}$.

By exploiting the representation $\psi := \phi_{2Mv_n} \circ \sqrt[v_n]{|\cdot|}$, we obtain

$$\int_0^\infty \psi(t) \cdot t^{2m+1} \, dt = \int_0^\infty \phi_{2Mv_n}(\sqrt[v_n]{t}) t^{2m+1} \, dt \qquad \left(\text{substitute } t = u^{v_n}\right)$$

$$= v_n \cdot \int_0^\infty \phi_{2Mv_n}(u) u^{(2m+2) \cdot v_n - 1} \, du \neq 0 \quad \text{for all } m \in \{0, \dots, M - 1\}$$

according to Lemma 4.24 and Lemma 4.25, respectively.

$\square$

Now we can prove that the construction in the general setup indeed produces a restrictive Taylorlet with vanishing moments of order n.

Theorem 4.27. *Let τ be defined as in the general setup and let $M \in \mathbb{N} \cup \{\infty\}$. Then, τ is restrictive and has $2M - 1$ vanishing moments of order n or infinitely many vanishing moments of order n if $M = \infty$.*

Proof. In order to prove the desired statement, we need to show that

1. $\displaystyle\int_{\mathbb{R}} g(\pm t^k) t^m \, dt = 0 \quad \forall \ m \in \{0, \dots, (2M-1) \cdot k - 1\}, \ \forall \ k \in \{1, \dots, n\},$

2. $\displaystyle\int_0^\infty g(t) \cdot t^m \, dx \neq 0 \quad \forall \ m \in \{0, \dots, 2M - 2\}.$

1. According to Lemma 4.26, ψ has $2M$ vanishing moments of order n. Hence,

$$\int_{\mathbb{R}} g(\pm t^k) t^m \, dt = \int_{\mathbb{R}} (1 + t) \cdot \psi(\pm t^k) t^m \, dt = \int_{\mathbb{R}} \psi(\pm t^k) t^m \, dt + \int_{\mathbb{R}} \psi(\pm t^k) t^{m+1} \, dt = 0$$

for all $m \in \{0, \dots, (2M-1) \cdot k - 1\}$ and for all $k \in \{1, \dots, n\}$.

2. We obtain

$$\int_0^\infty g(t) \cdot t^m \, dt = \int_0^\infty (1+t) \cdot \psi(t) \cdot t^m \, dt$$
$$= \int_0^\infty \psi(t) \cdot t^m \, dt + \int_0^\infty \psi(t) \cdot t^{m+1} \, dt.$$

As either the exponent m or the exponent $m+1$ has to be even (and, apparently, the other one odd), Lemma 4.26 yields that integral with the even exponent vanishes and the other does not. Hence,

$$\int_0^\infty g(t) \cdot t^m \, dx \neq 0 \quad \forall \, m \in \{0, \dots, 2M-2\}.$$

$\square$

We will now proceed to show a representation of the function ϕ_∞ which is similar to the result of Theorem 4.23. To this end, we first establish a sum formula for a special q-function.

Lemma 4.28. *Let $a \in \mathbb{N}$, $q \in (0,1)$ and let*

$$F : [-1,1] \to \mathbb{R}, \quad x \mapsto \prod_{m \in \mathbb{N} \setminus (a\mathbb{N})} \left(1 - q^m x\right).$$

Then, there exists a sequence $\{A_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}$ such that for all $x \in [-1,1]$,

$$F(x) = \sum_{n=0}^\infty A_n x^n$$

fulfilling the recurrence relation

$$A_0 = 1,$$
$$A_n = \frac{1}{1 - q^{an}} \cdot \sum_{k=1}^{a-1} \binom{a-1}{k}_q \cdot q^{a(n-k) + \binom{k+1}{2}} \cdot A_{n-k} \qquad \forall \, n \in \mathbb{N},$$

with the convention that $A_{-n} = 0$ for all $n \in \mathbb{N}$.

Proof. The infinite product defining the function F converges due to Corollary 4.17. Now we can observe that for all $q \in (0,1)$, $x \in [-1,1]$,

$$F(x) = \prod_{m \in \mathbb{N} \setminus (a\mathbb{N})} \left(1 - q^m x\right) = \prod_{m=1}^{a-1} \left(1 - q^m x\right) \cdot \prod_{m \in \mathbb{N} \setminus (a\mathbb{N})} \left(1 - q^{a+m} x\right) = (qx; q)_{a-1} \cdot F\left(q^a x\right).$$

Utilizing the series expansion yields

$$\sum_{n=0}^\infty A_n x^n = (qx; q)_{a-1} \cdot \sum_{n=0}^\infty A_n q^{an} x^n.$$

We can now exploit Lemma 4.15 to represent $(qx;q)_{a-1}$ as sum and obtain

$$\sum_{n=0}^{\infty} A_n x^n = \sum_{k=0}^{a-1} \binom{a-1}{k}_q q^{\binom{k+1}{2}} \cdot x^k \cdot \sum_{n=0}^{\infty} A_n q^{an} x^n$$

$$= \sum_{k=0}^{a-1} \binom{a-1}{k}_q q^{\binom{k+1}{2}} \cdot \sum_{n=k}^{\infty} A_{n-k} q^{a(n-k)} x^n.$$

Using the convention that $A_{-n} = 0$ for all $n \in \mathbb{N}$, we can conclude by comparing the coefficients of x^n on both sides that

$$A_n = \sum_{k=0}^{a-1} \binom{a-1}{k}_q q^{\binom{k+1}{2}+a(n-k)} \cdot A_{n-k}$$

$$= \sum_{k=1}^{a-1} \binom{a-1}{k}_q q^{\binom{k+1}{2}+a(n-k)} \cdot A_{n-k} + q^{an} \cdot A_n.$$

By $A_0 = F(0)$, we obtain that $A_0 = 1$. For $n \geq 1$, the upper equation yields

$$A_n = \frac{1}{1-q^{an}} \cdot \sum_{k=1}^{a-1} \binom{a-1}{k}_q q^{\binom{k+1}{2}+a(n-k)} \cdot A_{n-k}.$$

$\square$

The next theorem shows a representation of the function ϕ_∞ which uses Lemma 4.28.

Theorem 4.29. *Let $q \in (0,1)$, $\varepsilon > 0$ and let the sequence $\{A_n\}_{n \in \mathbb{Z}}$ be chosen as in Lemma 4.28 for $a = 2v_n$. Furthermore, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be of the form*

$$\phi_0(x) = \begin{cases} 1 & \text{for } |x| \leq \varepsilon, \\ \eta(|x|) & \text{for } |x| \in \left(\varepsilon, \varepsilon q^{-1}\right], \\ 0 & \text{for } |x| > \varepsilon q^{-1}. \end{cases}$$

where $\eta \in C^\infty\left(\left[\varepsilon, \varepsilon q^{-1}\right]\right)$ is chosen so that $\phi_0 \in C^\infty(\mathbb{R})$. Then, $\phi_\infty = \prod_{m \in \mathbb{N} \setminus (2v_n \mathbb{N})} \left(\mathrm{Id} - q^m D_q\right) \phi_0$ has the explicit representation

$$\phi_\infty(x) = \frac{\varphi(q)}{\varphi(q^{2v_n})} + \sum_{\ell=0}^{\infty} \left[A_\ell \cdot \eta\left(q^\ell \cdot |x|\right) - \sum_{k=0}^{\ell} A_k \right] \cdot \mathbb{1}_{\left(\varepsilon q^{-\ell};\varepsilon q^{-(\ell+1)}\right]}(|x|),$$

where φ is the Euler function.

Proof. We can write the function ϕ_∞ as

$$\phi_\infty = \prod_{m \in \mathbb{N} \setminus (2v_n \mathbb{N})} \left(\mathrm{Id} - q^m \cdot D_q\right) \phi_0.$$

As the dilation operator commutes with the multiplication with constants, we can rewrite the limit function ϕ_∞ due to Lemma 4.28 as

$$\phi_\infty = \prod_{m \in \mathbb{N} \setminus (2\nu_n \mathbb{N})} \left(\mathrm{Id} - q^m \cdot D_q \right) \phi_0$$

$$= \sum_{\ell=0}^{\infty} A_\ell \cdot D_{q^\ell} \phi_0$$

for an appropriate sequence $\{A_\ell\}_{\ell \in \mathbb{N}_0}$ fulfilling the recurrence relations of Lemma 4.28.

Due to the form of ϕ_0 we obtain for all $x \in \mathbb{R}$ that

$$\phi_\infty(x) = \sum_{\ell=0}^{\infty} A_\ell \cdot D_{q^\ell} \phi_0(x)$$

$$= \sum_{\ell=0}^{\infty} A_\ell \cdot \left[\mathbb{1}_{[-\varepsilon q^{-\ell}, \varepsilon q^{-\ell}]}(x) + \left(D_{q^\ell} \eta \right)(|x|) \cdot \mathbb{1}_{(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}]}(|x|) \right]$$

$$= \sum_{\ell=0}^{\infty} A_\ell \cdot \left(\sum_{k=0}^{\ell-1} \mathbb{1}_{(\varepsilon q^{-k}, \varepsilon q^{-(k+1)}]}(|x|) + \mathbb{1}_{[-\varepsilon, \varepsilon]}(x) + \eta \left(q^\ell |x| \right) \cdot \mathbb{1}_{(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}]}(|x|) \right)$$

$$= \sum_{\ell=0}^{\infty} A_\ell \cdot \mathbb{1}_{[-\varepsilon, \varepsilon]}(x) + \sum_{k=0}^{\infty} \mathbb{1}_{(\varepsilon q^{-k}, \varepsilon q^{-(k+1)}]}(|x|) \cdot \sum_{\nu=k+1}^{\infty} A_\nu$$

$$+ \sum_{\ell=0}^{\infty} A_\ell \cdot \eta \left(q^\ell |x| \right) \cdot \mathbb{1}_{(\varepsilon q^{-\ell}, \varepsilon q^{-(\ell+1)}]}(|x|).$$

By inserting $x = 0$, we obtain that

$$\phi_\infty(0) = \sum_{\ell=0}^{\infty} A_\ell.$$

By repeating this for the equation

$$\phi_\infty(x) = \left[\prod_{m \in \mathbb{N} \setminus (2\nu_n \mathbb{N})} \left(\mathrm{Id} - q^m \cdot D_q \right) \phi_0 \right](x),$$

we can conclude with $\phi_0(0) = 1$ and $(D_a f)(0) = (\mathrm{Id} f)(0)$ for all $a > 0$ and all $f \in C(\mathbb{R})$ that

$$\phi_\infty(0) = \left[\prod_{m \in \mathbb{N} \setminus (2\nu_n \mathbb{N})} \left(\mathrm{Id} - q^m \cdot \mathrm{Id} \right) \phi_0 \right](0) = \prod_{m \in \mathbb{N} \setminus (2\nu_n \mathbb{N})} \left(1 - q^m \right) = \prod_{m \in \mathbb{N}} \frac{1 - q^m}{1 - q^{2\nu_n m}} = \frac{\varphi(q)}{\varphi(q^{2\nu_n})}.$$

This equation together with $\sum_{k=\ell+1}^{\infty} A_k = \frac{\varphi(q)}{\varphi(q^{2\nu_n})} - \sum_{k=0}^{\ell} A_k$ delivers

$$\phi_\infty(x) = \frac{\varphi(q)}{\varphi(q^{2\nu_n})} \cdot \mathbb{1}_{[-\varepsilon; \varepsilon]}(x) + \sum_{\ell=0}^{\infty} \left[\frac{\varphi(q)}{\varphi(q^{2\nu_n})} - \sum_{k=0}^{\ell} A_k + A_\ell \cdot D_{q^\ell} \eta(|x|) \right] \cdot \mathbb{1}_{(q^{-\ell} \varepsilon; q^{-(\ell+1)} \varepsilon]}(|x|)$$

$$= \frac{\varphi(q)}{\varphi(q^{2\nu_n})} + \sum_{\ell=0}^{\infty} \left[A_\ell \cdot \eta \left(q^\ell \cdot |x| \right) - \sum_{k=0}^{\ell} A_k \right] \cdot \mathbb{1}_{(\varepsilon q^{-\ell}; \varepsilon q^{-(\ell+1)}]}(|x|).$$

$\varphi(q)$	Precision	Necessary number of terms
$\varphi\left(\frac{1}{2}\right)$	10^{-16}	10
$\varphi\left(\frac{1}{2}\right)$	10^{-10}	8
$\varphi\left(e^{-\pi}\right)$	10^{-16}	5
$\varphi\left(e^{-\pi}\right)$	10^{-10}	4

Table 4.3.2: Approximations of $\varphi(q)$

$\square$

We will now consider the numerics of computing the Euler function. With the series expansion of Theorem 4.23,

$$\varphi(q) = \sum_{k=0}^{\infty} \frac{1}{\left(q^{-1}; q^{-1}\right)_k},$$

we can obtain the following error estimate.

Lemma 4.30. *Let $q \in (0,1)$ and $n \in \mathbb{N}$. Then,*

$$\left| \varphi(q) - \sum_{k=0}^{n} \frac{1}{\left(q^{-1}; q^{-1}\right)_k} \right| < \frac{1}{\left| \left(q^{-1}; q^{-1}\right)_{n+1} \right|}.$$

Proof. Due to the series expansion of the Euler function, we obtain

$$\varphi(q) - \sum_{k=0}^{n} \frac{1}{\left(q^{-1}; q^{-1}\right)_k} = \sum_{k=n+1}^{\infty} \frac{1}{\left(q^{-1}; q^{-1}\right)_k}.$$

Since $q \in (0,1)$, the q-Pochhammer symbol

$$\left(q^{-1}; q^{-1}\right)_k = \prod_{\ell=1}^{k} \left(1 - q^{-\ell}\right)$$

alternates its sign with k. Additionally, its absolute value $\left|\left(q^{-1};q^{-1}\right)_k\right|$ exhibits a strictly monotonic growth. W. l. o. g. let n be odd. Then

$$
\left| \varphi(q) - \sum_{k=0}^{n} \frac{1}{\left(q^{-1};q^{-1}\right)_k} \right|
$$

$$
= \left| \sum_{k=n+1}^{\infty} \frac{1}{\left(q^{-1};q^{-1}\right)_k} \right|
$$

$$
= \left| \frac{1}{\left|\left(q^{-1};q^{-1}\right)_{n+1}\right|} + \sum_{k=0}^{\infty} \underbrace{\left(\frac{1}{\left|\left(q^{-1};q^{-1}\right)_{2k+n+3}\right|} - \frac{1}{\left|\left(q^{-1};q^{-1}\right)_{2k+n+2}\right|} \right)}_{<0 \text{ for all } k \in \mathbb{N}} \right| \qquad (n \text{ odd})
$$

$$
< \frac{1}{\left|\left(q^{-1};q^{-1}\right)\right|_{n+1}\right|}.
$$

$\square$

As the q-Pochhammer symbol $\left(q^{-1};q^{-1}\right)_k$ grows faster than $k!$ for $q \in (0,1)$, the estimate is extremely good. Thus, the Euler function can be computed very efficiently. Suppose, we want to calculate the Euler function $\varphi(1/2)$ to a precision of 10^{-16}. We can check that $(2;2)_{10} \approx 1.04 \cdot 10^{16}$ and hence see that using 10 terms is sufficient for a precision of 10^{-16}.

4.4 Detection result

We first introduce the class of feasible functions which is used in the main result.

Definition 4.31 (Feasible function, singularity function)**.** Let δ denote the Dirac distribution. Furthermore, let $j \in \mathbb{N}_0$, $q \in C^\infty(\mathbb{R})$ and let

$$
f(x) := I_\pm^j \delta(x_1 - q(x_2)),
$$

where $I_\pm^j$ is the jth iterated integral. Then f is called a j-feasible function with singularity function q.

The variable j describes the smoothness of f. In terms of Sobolev spaces, we obtain for $j \geq 1$, that $f \in W^{j-1,\infty}(\mathbb{R}^2)$. For instance, by choosing $q(x_2) = x_2^2$ for all $x_2 \in \mathbb{R}$ and $j \geq 1$, we obtain the function

$$
f(x) = \frac{(x_1 - x_2^2)^{j-1}}{(j-1)!} \cdot H\left(\pm\left(x_1 - x_2^2\right)\right),
$$

where $H : \mathbb{R} \to \mathbb{R}, t \mapsto \mathbb{1}_{\mathbb{R}^+}(t)$ is the Heaviside step function.
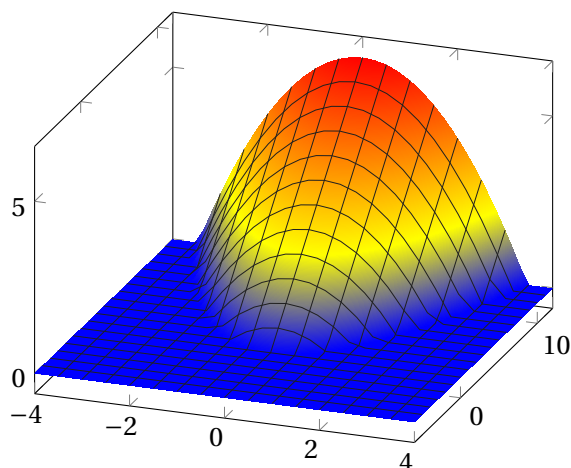
Fig. 4.2: Plot of the 2-feasible function $f(x) = (x_1 - x_2^2) \cdot H(x_1 - x_2^2)$

In order to classify the shearing variables w. r. t. the local properties of the singularity function q, we introduce the concept of the highest approximation order.

Definition 4.32 (Highest approximation order)**.** Let $j, n \in \mathbb{N}_0$ and let f be a j−feasible function with singularity function q. Furthermore, let $t \in \mathbb{R}$ and $k \in \{0, \ldots, n-1\}$. If $s_\ell = q^{(\ell)}(t)$ for all $\ell \in \{0, \ldots, k\}$ and $s_{k+1} \neq q^{(k+1)}(t)$, we say that k is the highest approximation order of the shearing variable $s = (s_0, \ldots, s_n)$ for f in t.

The following theorem states the main result of this chapter and treats the classification of the Taylorlet transform's decay w. r. t. the highest approximation order.

Theorem 4.33. *Let $M, n \in \mathbb{N}$ and let τ be an analyzing Taylorlet of order n with M vanishing moments. Let furthermore $j < M$, $t \in \mathbb{R}$ and let f be a j−feasible function.*

1. *Let $\alpha > 0$. If $s_0 \neq q(t)$, the Taylorlet transform has a decay of*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \mathcal{O}(a^N) \quad for\ a \to 0$$

   for all $N > 0$.

2. *Let $\alpha < \frac{1}{n}$ and let $k \in \{0, \ldots, n-1\}$ be the highest approximation order of s for f in t. Then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{j-1+(M-j)[1-(k+1)\alpha]}\right) \quad for\ a \to 0.$$

3. *Let $\alpha > \frac{1}{n+1}$ and let τ be restrictive. If n is the highest approximation order of s for f in t, then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) \sim a^{j-1} \quad for\ a \to 0.$$

Remark 4.34. The local curvature of the graph of q can be determined with the first and second Taylor coefficient via Corollary 1.7

$$\kappa_q(t) = \frac{q''(t)}{\left(1 + \left[q'(t)\right]^2\right)^{3/2}} \quad \text{for all } t \in \mathbb{R}.$$

Hence, the dependence of the detected features from the moment conditions can be summarized in the following table.

Analyzing function	Moment condition	Detected geometric features
Shearlet $\psi \in \mathcal{S}\left(\mathbb{R}^2\right)$	$\int_{\mathbb{R}} \psi(x_1, x_2) x_1^m dx_1 = 0$ for all $m \in \mathbb{N}_0$	Position and direction of singularities
Taylorlet $\tau = g \otimes h \in \mathcal{S}\left(\mathbb{R}^2\right)$	$\int_{\mathbb{R}} g(t) t^m dt = 0 = \int_{\mathbb{R}} g(\pm t^2) t^m dt$ for all $m \in \mathbb{N}_0$	Position, direction and curvature of singularities
Taylorlet $\tau = g \otimes h \in \mathcal{S}\left(\mathbb{R}^2\right)$	$\int_{\mathbb{R}} g(\pm t^k) t^m dt = 0$ for all $k \in \{1, \ldots, n\}$, $m \in \mathbb{N}_0$	First n+1 Taylor cofficients of the singularity function

Table 4.4.1: Moment conditions and detection results

The strategy for the proof of this theorem consists of multiple reductions to simpler cases. In the proof of Theorem 4.33, we show that it is sufficient to consider 0-feasible functions i.e., functions of the form $f(x) = \delta(x_1 - q(x_2))$ for $x \in \mathbb{R}^2$. In Lemma 4.42 we then prove that all cases of the Taylorlet transform can be reduced to a linear combination of integrals of the form

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z t^k \\ t \end{pmatrix} t^{km+\ell} dt, \tag{7}$$

where $\ell, m \in \mathbb{N}_0$ and $k \in \{1, \ldots, n\}$. In order to obtain the decay rate of the Taylorlet transform, we have to determine the behavior of the integrals (7) for $z \to \pm\infty$. In Lemma 4.40, we show that we can ensure a fast decay of the integrals (7) for $z \to \pm\infty$ by imposing vanishing moment conditions of higher order to the analyzing Taylorlet $\tau \in \mathcal{S}(\mathbb{R}^2)$. Thus the vanishing moments of higher order are the key feature to the fast decay.

Remark 4.35. At this point we want to highlight the role of the vanishing moments of higher order for the classification result. Suppose, we wish to analyze the example function

$$f(x) = \delta\left(x_1 - \frac{c}{2} x_2^2\right)$$

and intend to find out the curvature of its singular support in the origin with the help of the Taylorlet transform. We furthermore assume that the analyzing Taylorlet $\tau = g \otimes h$ is only of order 1, and that g has infinitely many vanishing moments of order 1, but no vanishing moments of order 2, i.e.,

$$\int_{\mathbb{R}} g(x_1) x_1^m dx_1 = 0 \quad \text{for all } m \in \mathbb{N}_0 \quad \text{and} \quad \int_{\mathbb{R}} g(\pm x_1^2) dx_1 \neq 0.$$

Case 1: $s_0 \neq 0$ or $s_1 \neq 0$:

Then, the theory of the shearlet transform delivers for $\alpha \in (0,1)$ that

$$\mathcal{T}^{(2,\alpha)} f(a,s,0) = \int_{\mathbb{R}^2} \tau \left(\begin{matrix} \left[x_1 - \frac{s_2}{2} x_2^2 - s_1 x_2 - s_0 \right] / a \\ x_2 / a^\alpha \end{matrix} \right) \delta \left(x_1 - \frac{c}{2} x_2^2 \right) dx = \mathcal{O}(a^N),$$

for $a \to 0$ for all $N \in \mathbb{N}$. Yet, we would also like to have that

$$\mathcal{T}^{(2,\alpha)} f(a,s,0) = \mathcal{O}(a^N), \quad \text{for } a \to 0$$

for some large $N \in \mathbb{N}$, if $s_0 = 0$, $s_1 = 0$ and $s_2 \neq c$.

Case 2: $s_0 = 0$, $s_1 = 0$ and $s_2 \neq c$:

If g does not have vanishing moments of second order, we obtain for $\alpha \in \left(0, \frac{1}{2}\right)$ that

$$\mathcal{T}^{(2,\alpha)} f(a,s,0) = \int_{\mathbb{R}^2} \tau \left(\begin{matrix} \left[x_1 - \frac{s_2}{2} x_2^2 \right] / a \\ x_2 / a^\alpha \end{matrix} \right) \delta \left(x_1 - \frac{c}{2} x_2^2 \right) dx$$

$$= \int_{\mathbb{R}} \tau \left(\begin{matrix} \frac{c - s_2}{2} \cdot x_2^2 / a \\ x_2 / a^\alpha \end{matrix} \right) dx_2 \qquad (\text{substituting } x_2 = a^\alpha u)$$

$$= a^\alpha \int_{\mathbb{R}} g \left(a^{2\alpha - 1} \cdot \frac{c - s_2}{2} \cdot u^2 \right) h(u) du.$$

By defining the function

$$g_{\pm,2} : \mathbb{R} \to \mathbb{R}, u \mapsto g(\pm u^2)$$

and applying Plancherel's theorem to the last integral, we obtain

$$\mathcal{T}^{(2,\alpha)} f(a,s,0) = \frac{\sqrt{a}}{\sqrt{2|c - s_2|\pi}} \cdot \int_{\mathbb{R}} \hat{g}_{\text{sgn}(c - s_2),2} (\sqrt{2}|c - s_2|^{-\frac{1}{2}} a^{\frac{1}{2} - \alpha} \omega) \hat{h}(\omega) d\omega.$$

Due to $g, h \in \mathcal{S}(\mathbb{R})$, $\alpha < \frac{1}{2}$ and the dominated convergence theorem, we get

$$\lim_{a \to 0} a^{-\frac{1}{2}} \mathcal{T}^{(2,\alpha)} f(a, s, 0) = \frac{1}{\sqrt{2|c - s_2|\pi}} \cdot \int_{\mathbb{R}} \underbrace{\hat{g}_{\mathrm{sgn}(c-s_2),2}(0)}_{= \int_{\mathbb{R}} g(\mathrm{sgn}(c-s_2) u^2) du} \hat{h}(\omega) d\omega$$

$$= \sqrt{\frac{2}{|c - s_2|}} \cdot \int_{\mathbb{R}} g(\mathrm{sgn}(c - s_2) u^2) du \cdot h(0).$$

Hence, for $h(0) \neq 0$, e.g. for $h(u) = e^{-u^2}$ from the example in the General Setup, we obtain that

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0) \sim \sqrt{a} \quad \text{for } a \to 0. \tag{8}$$

Case 3: $s_0 = 0$, $s_1 = 0$ and $s_2 = c$:

We obtain

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0) = \int_{\mathbb{R}} \tau \begin{pmatrix} 0 \\ x_2/a^\alpha \end{pmatrix} dx_2 = a^\alpha g(0) \cdot \int_{\mathbb{R}} h(u) du.$$

Thus, for $g(0) \neq 0$ and $\int_{\mathbb{R}} h(u) du \neq 0$, we have

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0) \sim a^\alpha \quad \text{for } a \to 0. \tag{9}$$

Due to statement 3. of Theorem 4.33, we need $\alpha > \frac{1}{3}$ for the detection of the curvature. Hence, the ratio of the decay rates for $c = s_2$ $\big((9)\big)$ and $c \neq s_2$ $\big((8)\big)$ is $a^{\frac{1}{2} - \alpha}$ and hence at best $a^{\frac{1}{6}}$. Detecting this difference can become difficult in numerical practice without vanishing moments of higher order. As identifying this difference in the decay rates is necessary for the detection of the edge curvature, the task of determining the local edge curvature thus might get numerically unstable.

Remark 4.36. At this point we want to highlight the importance of the restrictiveness for the Taylorlet transform. This property makes sure that a Taylorlet transform of order n decays slowly if the highest approximation order is n. If a Taylorlet lacks the restrictiveness, we can construct an example function whose Taylorlet transform is equal to zero for all $a > 0$ if the highest approximation order is n.

Let $\tau = g \otimes h$ be a Taylorlet of order n with M vanishing moments such that there is a number $j \in \{0, \ldots, M - 1\}$ with

$$\int_0^\infty g(t) t^j dt = 0.$$

Thus, τ is not restrictive. Furthermore let $\alpha \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ and

$$f(x) := x_1^j \mathbb{1}_{\mathbb{R}_+}(x_1).$$

Then we obtain for the Taylorlet transform of f that

$$
\begin{aligned}
\mathcal{T}_{\tau}^{(n,\alpha)} f(a,0,0) &= \int_{\mathbb{R}^2} f(x) \cdot \tau \begin{pmatrix} x_1/a \\ x_2/a^\alpha \end{pmatrix} dx \\
&= a^{1+\alpha} \cdot \int_{\mathbb{R}} (ay_1)^j \mathbb{1}_{\mathbb{R}_+}(y_1) \cdot g(y_1) dy_1 \cdot \int_{\mathbb{R}} h(y_2) dy_2 \\
&= a^{j+1+\alpha} \cdot \underbrace{\int_0^\infty y_1^k \cdot g(y_1) dy_1}_{=0} \cdot \int_{\mathbb{R}} h(y_2) dy_2 = 0.
\end{aligned}
$$

Remark 4.37. Furthermore, we want to emphasize the significance of the choice of α for the Taylorlet transform.

As the general setup involves the least common multiple v_n of the numbers $1, \dots, n$, it is possible that the order of the Taylorlet is higher than originally intended. For instance, consider an analyzing Taylorlet τ of order 5. When built according to the general setup, we have $v_5 = v_6 = 60$. To this end, Theorem 4.23 states that τ is also an analyzing Taylorlet of order 6.

The problems that arise from a wrong choice of α become clear when we consider a case where $\alpha < \frac{1}{6}$, f is a j−feasible function and τ is the analyzing Taylorlet of order 5 (and 6) described above. If the highest approximation order of $s \in \mathbb{R}^6$ is 5, we can treat the Taylorlet transform $\mathcal{T}^{5,\alpha} f(a,s,t)$ like $\mathcal{T}^{6,\alpha} f(a,\sigma,t)$, where $\sigma = (s_0, \dots, s_5, 0)$. We are allowed to do so, because for all $k \in \mathbb{N}_0$ we can write every shearing operator $S_s^{(k)}$ of order k as shearing operator $S_{s'}^{(k+1)}$ where $s' = (s_0, \dots, s_k, 0)$. Let $t \in \mathbb{R}$. If the highest approximation order of σ for f in t is 5, all conditions of case 2. of Theorem 4.33 are met and so the Taylorlet transform has a decay of $\mathcal{O}\left(a^{(M-j)(1-6\alpha)-1}\right)$ for $a \to 0$. This can be significantly faster than the decay of $\sim a^{-1}$ for $a \to 0$ which occurs for the choice of $\alpha > \frac{1}{6}$.

Due to the detection result, the construction of a function $g \in \mathcal{S}_n^*(\mathbb{R})$ is highly desirable, as the corresponding Taylorlet $\tau = g \otimes h$ allows for a very fast detection of the Taylor coefficients of the singularity function. Furthermore, such a Taylorlet simplifies said detection, as shown in the following corollary.

Corollary 4.38. *Let $n \in \mathbb{N}$ and let τ be a restrictive, analyzing Taylorlet of order n with infinitely many vanishing moments in x_1-direction. Let furthermore $\alpha \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, $j \geq 0$ and let $q \in C^\infty(\mathbb{R})$ be the singularity function of*

$$
f(x) = \left[x_1 - q(x_2) \right]^j \cdot \mathbb{1}_{\mathbb{R}_\pm}\left(x_1 - q(x_2) \right).
$$

Then

$$
\mathcal{T}^{(n,\alpha)} f(a,s,t) = \mathcal{O}\left(a^N\right) \quad \text{for } a \to 0
$$

for all $N > 0$, if and only if there exists a $k \in \{0, \dots, n\}$ such that $s_k \neq q^{(k)}(t)$.

In order to prove Theorem 4.33, we need the following auxiliary results.

Lemma 4.39. *Let $f \in C(\mathbb{R})$ such that for all $n \in \mathbb{N}_0$ there exists a constant $c_n \in \mathbb{R}_+$ with*

$$\sup_{t \in \mathbb{R}} \left| t^n \cdot f(t) \right| = c_n < \infty.$$

Then

$$\int_{\mathbb{R} \setminus [-a^\beta, a^\beta]} f(t/a) \, dt = \mathcal{O}(a^N) \quad \text{for } a \to 0$$

for all $\beta < 1$ and $N \in \mathbb{N}$.

Proof. By applying the decay condition, we obtain for $n > 1$

$$\left| \int_{\mathbb{R} \setminus [-a^\beta, a^\beta]} f(t/a) \, dt \right| \leq 2c_n \int_{a^\beta}^{\infty} (a/t)^n \, dt = \frac{2c_n}{n-1} a^{(1-\beta)n+\beta}.$$

Since we can choose $n \in \mathbb{N}_0$ arbitrarily large and since $\beta < 1$, we get the desired result. $\qquad\square$

The next lemma provides a relation between the vanishing moments of order n and the decay rate of integrals over the graph of a monomial. This will become important as the Taylorlet transform of a feasible function can be represented as a sum over integrals of this type.

Lemma 4.40. *Let $M, n \in \mathbb{N}$ and let τ be an analyzing Taylorlet of order n with M vanishing moments. Then for all $\ell, m \in \mathbb{N}_0$ and for all $k \in \{1, \ldots, n\}$, we have*

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} t^{km+\ell} \, dt = \mathcal{O}\left(|z|^{-\left(M+m+\frac{1}{k}\right)} \right) \text{ for } z \to \pm\infty. \tag{10}$$

Proof. The idea is to represent the integral in (14) as a Fourier transform, to utilize the separation approach $\tau = g \otimes h$ and to show the decay result via the Fourier transforms of g and h. We define the function

$$\tilde{\tau}_k(z, \omega) := \int_{\mathbb{R}} \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} e^{-it\omega} \, dt.$$

Then we can rewrite the left side of (14) into

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} t^{km+\ell} \, dt = i^\ell \partial_\omega^\ell \partial_z^m \tilde{\tau}_k(z, 0). \tag{11}$$

For $k \in \mathbb{N}$ we introduce the function

$$g_{\pm,k} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto g(\pm t^k).$$

Due to the vanishing moment property we can conclude that

$$\widehat{g_{\pm,k}}^{(v)}(0) = 0 \text{ for all } v \in \{0, \ldots, kM - 1\}.$$

Consequently, we get the decay rate

$$\widehat{g_{\pm,k}}^{(\nu)}(\omega) = \mathcal{O}(\omega^{kM-\nu}) \quad \text{for } \omega \to 0. \tag{12}$$

We now obtain

$$\tilde{\tau}_k(z,\omega) = \left(\frac{1}{|z|^{1/k}} \left(g_{\mathrm{sgn}(z),k} \right)^{\wedge} \left(\frac{\cdot}{|z|^{1/k}} \right) * \hat{h} \right)(\omega)$$

and hence

$$\partial_\omega^\ell \tilde{\tau}_k(z,0) = \frac{1}{|z|^{1/k}} \int_{\mathbb{R}} \left(g_{\mathrm{sgn}(z),k} \right)^{\wedge} \left(-\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega.$$

We will now check the decay rate of $\partial_\omega^\ell \partial_z^m \tilde{\tau}_k(z,0)$. For this, we observe that

$$\partial_\omega^\ell \partial_z^m \tilde{\tau}_k(z,0) = \partial_z^m \left(|z|^{-1/k} \int_{\mathbb{R}} \left(g_{\pm,k} \right)^{\wedge} \left(-\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega \right)$$

$$= \sum_{\nu=0}^{m} c_\nu |z|^{-[m+(\nu+1)/k]} \int_{\mathbb{R}} \omega^\nu \left[\left(g_{\mathrm{sgn}(z),k} \right)^{\wedge} \right]^{(\nu)} \left(-\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega$$

with $c_\nu \in \mathbb{R}$ for all $\nu \in \{0,\dots,m\}$. By applying (12) and $h \in \mathcal{S}(\mathbb{R})$ we estimate the terms in this equation as

$$\left| |z|^{-[m+(\nu+1)/k]} \int_{\mathbb{R}} \omega^\nu \left[\left(g_{\mathrm{sgn}(z),k} \right)^{\wedge} \right]^{(\nu)} \left(-\frac{\omega}{|z|^{1/k}} \right) \hat{h}^{(\ell)}(\omega) d\omega \right|$$

$$\leq 2|z|^{-[m+(\nu+1)/k]} \int_0^\infty \omega^\nu \cdot \left(\omega |z|^{-1/k} \right)^{kM-\nu} \cdot \frac{1}{1+\omega^{kM+1}} d\omega$$

$$= c \cdot |z|^{-\left(M+m+\frac{1}{k} \right)}$$

for some constant $c > 0$. $\qquad \square$

It is also possible to prove a slightly weaker result without relying on the tensor product form of the Taylorlet.

Lemma 4.41. *Let $M, n \in \mathbb{N}$ and let $\tau \in \mathcal{S}(\mathbb{R}^2)$ such that*

$$\int_{\mathbb{R}} \tau \begin{pmatrix} \pm x_1^k \\ x_2 \end{pmatrix} x_1^\ell dx_1 = 0 \quad \text{for all } \ell \in \{0,\dots,kM-1\} \text{ and for all } k \in \{1,\dots,n\}. \tag{13}$$

Then for all $\ell, m \in \mathbb{N}$, for all $k \in \{1,\dots,n\}$ and $\varepsilon > 0$, we have

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} t^{km+\ell} dt = \mathcal{O}\left(|z|^{\varepsilon - \left(M+m+\frac{1}{k} \right)} \right) \text{ for } z \to \pm\infty. \tag{14}$$

Proof. For $k \in \mathbb{N}$, we introduce the function

$$\tau_{\pm,k}(x) := \tau \begin{pmatrix} \pm x_1^k \\ x_2 \end{pmatrix}.$$

Clearly, $\tau_{\pm,k} \in \mathcal{S}(\mathbb{R}^2)$. We will now show some asymptotics of its Fourier transform, which we will exploit later.

According to the (13), by applying the Fourier transform, we obtain

$$\mathcal{F}\tau_{\pm,k}(\omega) = \mathcal{O}\left(\omega_1^{kM}\right), \quad \text{for } \omega_1 \to 0 \quad \text{for all } \omega_2 \in \mathbb{R}$$

and hence

$$\partial_1^\gamma \partial_2^\ell \mathcal{F}\tau_{\pm,k}(\omega) = \mathcal{O}\left(\omega_1^{kM-\nu}\right), \quad \text{for } \omega_1 \to 0 \quad \text{for all } \omega_2 \in \mathbb{R}. \tag{15}$$

In order to utilize the upper asymptotics we need to rewrite the left hand side of (14) into a Fourier setting. To this end, we apply a Fourier transform w. r. t. the second component to it.

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} t^{km+\ell} dt = \int_{\mathbb{R}} \partial_z^m \tau \begin{pmatrix} zt^k \\ t \end{pmatrix} t^\ell dt$$

$$= \int_{\mathbb{R}} \partial_z^m \tau_{\text{sgn}(z),k} \begin{pmatrix} |z|^{\frac{1}{k}} t \\ t \end{pmatrix} t^\ell dt$$

$$= \frac{1}{2\pi} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_z^m i^\ell \partial_2^\ell \mathcal{F}_2 \tau_{\text{sgn}(z),k} \begin{pmatrix} |z|^{\frac{1}{k}} t \\ \omega \end{pmatrix} e^{i\omega t} d\omega \, dt.$$

Since $\tau_{\pm,k} \in \mathcal{S}(\mathbb{R}^2)$ for all $k \in \mathbb{N}$, we can apply Fubini's theorem to obtain

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} z \cdot t^k \\ t \end{pmatrix} t^{km+\ell} dt = \frac{i^\ell}{2\pi} \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_z^m \partial_2^\ell \mathcal{F}_2 \tau_{\text{sgn}(z),k} \begin{pmatrix} |z|^{\frac{1}{k}} t \\ \omega \end{pmatrix} e^{i\omega t} dt \, d\omega$$

$$= \frac{i^\ell}{2\pi} \cdot \int_{\mathbb{R}} \partial_z^m \partial_2^\ell \left(|z|^{-\frac{1}{k}} \cdot \mathcal{F}\tau_{\text{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}} \omega \\ \omega \end{pmatrix} \right) d\omega.$$

By executing the partial derivative w. r. t. z we obtain

$$\int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} zt^k \\ t \end{pmatrix} t^{km+\ell} dt = \sum_{\nu=1}^m c_\nu \int_{\mathbb{R}} \partial_1^\gamma \partial_2^\ell \mathcal{F}\tau_{\text{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}} \omega \\ \omega \end{pmatrix} \cdot z^{-m} \cdot |z|^{-\frac{\nu+1}{k}} \omega^\nu d\omega \tag{16}$$

for some constants c_ν, $\nu \in \{1, \dots, m\}$.

We will now continue with asymptotic estimates for all summands in (16). For $\nu \in \{1,\dots,m\}$ we have

$$\left| \int_{\mathbb{R}} \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \cdot z^{-m} \cdot |z|^{-\frac{\nu+1}{k}} \omega^\nu \, d\omega \right|$$

$$\leq \int_{\mathbb{R}} \left| \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \right| \cdot |z|^{-m-\frac{\nu+1}{k}} |\omega|^\nu \, d\omega$$

$$= \underbrace{\int_{-|z|^\alpha}^{|z|^\alpha} \left| \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \right| \cdot |z|^{-m-\frac{\nu+1}{k}} |\omega|^\nu \, d\omega}_{=:(I)}$$

$$+ \underbrace{\int_{|\omega|>|z|^\alpha} \left| \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \right| \cdot |z|^{-m-\frac{\nu+1}{k}} |\omega|^\nu \, d\omega}_{=:(II)}$$

for an $\alpha \in \mathbb{R}$. In order to estimate the integral (I), we utilize (15) to obtain that

$$\left| \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \right| \leq C \cdot \left(|z|^{-\frac{1}{k}}\omega \right)^{kM-\nu} \qquad \text{for all } z \in \mathbb{R}\setminus\{0\}, \omega \in \mathbb{R}.$$

Hence,

$$(I) \leq C|z|^{-m-\frac{\nu+1}{k}} \int_{-|z|^\alpha}^{|z|^\alpha} \left(|z|^{-\frac{1}{k}}\omega \right)^{kM-\nu} |\omega|^\nu \, d\omega = \frac{2C}{kM+1-\nu} \cdot |z|^{-m-\frac{\nu+1}{k}} \cdot |z|^{-M+\frac{\nu}{k}} \cdot |z|^{(kM+1)\alpha}.$$

For the estimation of (II) we exploit that $\tau_{\pm,k} \in \mathcal{S}(\mathbb{R}^2)$ to get that for all $p \in \mathbb{N}$ there is a $\gamma_p > 0$ such that

$$\left| \partial_1^\nu \partial_2^\ell \mathcal{F}\tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}}\omega \\ \omega \end{pmatrix} \right| \leq \gamma_p \cdot |\omega|^{-p}.$$

Consequently, we can conclude that

$$(II) \leq |z|^{-m-\frac{\nu+1}{k}} \cdot \gamma_p \cdot \int_{|\omega|>|z|^\alpha} |\omega|^{\nu-p} \, d\omega = \frac{2\gamma_p}{\nu+1-p} \cdot |z|^{-m-\frac{\nu+1}{k}} \cdot |z|^{\alpha(\nu+1-p)}.$$

In order to optimize the asymptotics, we observe that $(II) = \mathcal{O}\left(|z|^{-N} \right)$ for all $N \in \mathbb{N}$, as long as $\alpha > 0$, since p can be chosen arbitrarily large. Consequently, the decay rate of the a summand

in (16) is dominated by (I). As $\alpha > 0$ can be chosen arbitrarily small, we obtain that for all $\varepsilon > 0$,

$$\left| \int_{\mathbb{R}} \partial_1^\nu \partial_2^\ell \mathcal{F} \tau_{\mathrm{sgn}(z),k} \begin{pmatrix} -|z|^{-\frac{1}{k}} \omega \\ \omega \end{pmatrix} \cdot z^{-m} \cdot |z|^{-\frac{\nu+1}{k}} \omega^\nu \, d\omega \right| = \mathcal{O}\left(a^{\varepsilon - (M+m+\frac{1}{k})} \right) \quad \text{for } a \to 0.$$

Hence, the desired result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The next lemma's statement is essentially the same as in Theorem 4.33, but we restrict the choice of analyzed functions to 0-feasible functions.

Lemma 4.42. *Let $M, n \in \mathbb{N}$ and let τ be an analyzing Taylorlet of order n with M vanishing moments. Let furthermore $t \in \mathbb{R}$ and let f be a 0-feasible function.*

1. *Let $\alpha > 0$. If $s_0 \neq q(t)$, the Taylorlet transform has a decay of*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \mathcal{O}\left(a^N \right) \quad \text{for } a \to 0$$

   for all $N > 0$.

2. *Let $\alpha < \frac{1}{n}$ and let $k \in \{0, \ldots, n-1\}$ be the highest approximation order of s for f in t. Then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{M[1-(k+1)\alpha]} \right) \quad \text{for } a \to 0.$$

3. *Let $\alpha > \frac{1}{n+1}$ and let τ be restrictive. If n is the highest approximation order of s for f in t, then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) \sim 1 \quad \text{for } a \to 0.$$

Proof. We restrict ourselves to the case $t = 0$ as all other cases are equivalent to treating a shifted version of f. Furthermore, we note that $f \in \mathcal{S}'(\mathbb{R}^2)$. Hence, the Taylorlet transform $\mathcal{T}^{(n,\alpha)} f(a, s, 0) = \langle \tau_{a,s,0}, f \rangle$ is well defined.

1. The idea is to exploit the special form of f in order to simplify its Taylorlet transform and to use the Schwartz class decay condition of τ in order to estimate the integral.

The structure of f leads to the following form of the Taylorlet transform:

$$\begin{aligned}
\mathcal{T}^{(n,\alpha)} f(a, s, 0) &= \int_{\mathbb{R}^2} \delta(x_1 - q(x_2)) \tau_{a,s,0}(x) \, dx \\
&= \int_{\mathbb{R}} \tau \begin{pmatrix} \left[q(x_2) - \sum_{\ell=0}^n \frac{s_\ell}{\ell!} \cdot x_2^\ell \right] / a \\ x_2 / a^\alpha \end{pmatrix} dx_2 \\
&= \int_{\mathbb{R}} g\left(\tilde{q}(x_2) / a \right) h(x_2 / a^\alpha) \, dx_2,
\end{aligned} \tag{17}$$

where $\tilde{q}(x_2) = q(x_2) - \sum_{k=0}^{n} \frac{s_k}{k!} \cdot x_2^k$. Since $g, h \in \mathcal{S}(\mathbb{R})$, the integrand in the last line fulfills the necessary decay condition of Lemma 4.39. By applying this lemma, we can conclude that

$$\left| \int_{\mathbb{R} \setminus [-a^\beta, a^\beta]} g\left(\tilde{q}(x_2)/a\right) h(x_2/a^\alpha) dx_2 \right|$$

$$\leq \|g\|_{L^\infty} \cdot \left| \int_{\mathbb{R} \setminus [-a^\beta, a^\beta]} h(x_2/a^\alpha) dx_2 \right| \stackrel{\text{Lemma 4.39}}{=} \mathcal{O}(a^N) \quad \text{for } a \to 0$$

for all $N \in \mathbb{N}$ if $\beta < \alpha$. Hence,

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0) = \int_{-a^\beta}^{a^\beta} g\left(\tilde{q}(x_2)/a\right) h(x_2/a^\alpha) dx_2 + \mathcal{O}(a^N) \quad \text{for } a \to 0$$

for all $N > 0$. Due to the conditions of this lemma, $\tilde{q} \in C^\infty(\mathbb{R})$ and $\tilde{q}(0) \neq 0$. Hence, there exists an $\varepsilon > 0$ such that $d := \min_{x_2 \in [-\varepsilon, \varepsilon]} |\tilde{q}(x_2)| > 0$. By employing the boundedness of h and the Schwartz decay condition that for all $M \in \mathbb{N}$ there exists $c_m > 0$ such that $\sup_{x_2 \in \mathbb{R}} |x_2^M \cdot g(x_2)| = c_M < \infty$, we get

$$|\mathcal{T}^{(n,\alpha)} f(a, s, 0)| \leq \|h\|_\infty c_M \int_{-a^\beta}^{a^\beta} \left(\frac{a}{|\tilde{q}(x_2)|} \right)^M dx_2$$

$$\leq 2\|h\|_\infty c_M d^{-M} a^{M+\beta}.$$

Since we can choose M to be arbitrarily large, the result follows immediately.

2. The general idea of this proof is to represent the Taylorlet transform as a sum of integrals of the form (14) and to apply Lemma 4.40 in order to obtain the desired decay rate. To this end, we will divide the proof into four steps.

STEP 1

In the first step we will show that the Taylorlet transform $\mathcal{T}^{(n,\alpha)} f(a, s, 0)$ is an integral over a curve and we will prove that only a small neighborhood of the origin is relevant for the decay of the Taylorlet transform for $a \to 0$.

First, we rewrite (23):

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0) = \int_\mathbb{R} \tau \left(\begin{array}{c} \left[q(x_2) - \sum_{\ell=0}^{n} \frac{s_\ell}{\ell!} \cdot x_2^\ell \right] / a \\ x_2 / a^\alpha \end{array} \right) dx_2$$

$$= \int_\mathbb{R} \tau \left(\begin{array}{c} \tilde{q}(x_2) \cdot x_2^{k+1} / a \\ x_2 / a^\alpha \end{array} \right) dx_2, \tag{18}$$

where $\tilde{q}(x_2) = \begin{cases} x_2^{-(k+1)} \left[q(x_2) - \sum_{\ell=0}^{n} \frac{s_\ell}{\ell!} \cdot x_2^\ell \right] & \text{for } x_2 \neq 0, \\ \frac{1}{(k+1)!} \cdot \left(q^{(k+1)}(0) - s_{k+1} \right) & \text{for } x_2 = 0. \end{cases}$

Since k is the highest approximation order of s for f in $t = 0$, we have $s_{k+1} \neq q^{(k+1)}(0)$. Hence,

$\tilde{q}(0) \neq 0$. Furthermore, we have $\tilde{q} \in C^\infty(\mathbb{R})$ due to the conditions of this Lemma. In order to show that just a small neighborhood of the origin is responsible for the decay of the Taylorlet transform for $a \to 0$, we observe that the integrand in (18) fulfills the decay condition of Lemma 4.39. By applying this lemma, we obtain for $\beta \in \left(0, \frac{1}{k+1}\right)$ and an arbitrary $N \in \mathbb{N}$ that

$$|\mathcal{T}^{(n,\alpha)}f(a,s,0)| = \left| \int_{-a^\beta}^{a^\beta} \tau \begin{pmatrix} \tilde{q}(x_2) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} dx_2 \right| + \mathcal{O}(a^N) \quad \text{for } a \to 0. \tag{19}$$

STEP 2

If we replaced the term $\tilde{q}(x_2)$ in the argument of the integrand by some constant $c \neq 0$, the integral would be a truncated version of the desired form (14). Hence, we could apply Lemma 4.40 to obtain an estimate for the decay of the Taylorlet transform. In order to get closer to this form, we will approximate the integrand by a Taylor polynomial in this step.

Now we expand the integrand of (19) into a Taylor series with respect to the first component in a neighborhood of the point $\tilde{q}(0) \cdot x_2^{k+1}/a$.

$$\begin{aligned} |\mathcal{T}^{(n,\alpha)}f(a,s,0)| &= \left| \int_{-a^\beta}^{a^\beta} \tau \begin{pmatrix} \tilde{q}(x_2) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} dx_2 \right| + \mathcal{O}(a^N) \\ &\leq \sum_{m=0}^{J} \left| \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \cdot \frac{\left[x_2^{k+1}(\tilde{q}(x_2) - \tilde{q}(0)) \right]^m}{a^m \cdot m!} dx_2 \right| \\ &\quad + c^{J+1} \int_{-a^\beta}^{a^\beta} \left(\frac{|x_2|^{k+1}}{a} \right)^{J+1} \cdot |x_2|^{J+1} dx_2 + \mathcal{O}(a^N). \end{aligned} \tag{20}$$

For the last estimate we used that due to the smoothness of $\tilde{q}$ there exists a $c > 0$ such that $|\tilde{q}(x_2) - \tilde{q}(0)| \leq c|x_2|$ for all $x_2 \in [-a^\beta, a^\beta]$. We now prove that it is possible to choose $J \in \mathbb{N}$ such that the rest term in (20) behaves like $\mathcal{O}(a^N)$ for $a \to 0$ for an arbitrary, but fixed $N \in \mathbb{N}$. We have

$$\int_{-a^\beta}^{a^\beta} \left(\frac{|x_2|^{k+1}}{a} \right)^{J+1} \cdot |x_2|^{J+1} dx_2 \sim a^{(J+1)(\beta(k+2)-1)+\beta} \quad \text{for } a \to 0. \tag{21}$$

By restricting the choice of $\beta \in \left(0, \frac{1}{k+1}\right)$ to $\beta \in \left(\frac{1}{k+2}, \frac{1}{k+1}\right)$, we obtain the desired decay rate of $\mathcal{O}(a^N)$ for

$$J = \left\lceil \frac{N-\beta}{\beta(k+2)-1} \right\rceil - 1.$$

STEP 3

In the third step we will expand $\tilde{q}$ in a Taylor series about the origin to obtain a representation of the Taylorlet transform as a sum of truncated versions of integrals of the form (14).

We now expand the term $\tilde{q}(x_2) - \tilde{q}(0)$ about the point $x_2 = 0$.

$$\mathcal{T}^{(n,\alpha)} f(a,s,0)$$

$$= \sum_{m=0}^{J} \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \cdot \left(\frac{x_2^{k+1}}{a} \right)^m \cdot \frac{\left[\tilde{q}(x_2) - \tilde{q}(0) \right]^m}{m!} dx_2 + \mathcal{O}(a^N)$$

$$= \sum_{m=0}^{J} \frac{1}{m!} \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \cdot \frac{x_2^{(k+1)m}}{a^m} \cdot \left[\rho(x_2) + \sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} x_2^\ell \right]^m dx_2$$

$$+ \mathcal{O}(a^N) \quad \text{for } a \to 0, \tag{22}$$

where $\rho(x_2)$ is the rest term of the Taylor series expansion with the property $\rho(x_2) = \mathcal{O}\left(x_2^{L_m+1} \right)$ for $x_2 \to 0$. Now we estimate the summands for each $m \in \{0, \dots, J\}$.

$$\int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \left(\frac{x_2^{k+1}}{a} \right)^m \cdot \left[\rho(x_2) + \sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right]^m dx_2$$

$$= \sum_{v=0}^{m} \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \frac{x_2^{(k+1)m}}{a^m} \binom{m}{v} [\rho(x_2)]^v \left[\sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} x_2^\ell \right]^{m-v} dx_2.$$

Since $\tau \in \mathcal{S}(\mathbb{R}^2)$, $\rho(x_2) = \mathcal{O}(x_2^{L_m+1})$ for $x_2 \to 0$ and $\sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell = \mathcal{O}(x_2)$ for $x_2 \to 0$, for every $v \in \{1, \dots, m\}$ there exist $c_v, a_0 > 0$ such that for all $a < a_0$

$$\left| \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \frac{x_2^{(k+1)m}}{a^m} \binom{m}{v} [\rho(x_2)]^v \left[\sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} x_2^\ell \right]^{m-v} dx_2 \right|$$

$$\leq c_v \cdot a^{-m} \left| \int_{-a^\beta}^{a^\beta} |x_2|^{(k+1)m} |x_2|^{(L_m+1)v} \cdot |x_2|^{m-v} dx_2 \right|$$

$$= \mathcal{O}\left(a^{[(k+2)\beta-1]m + \beta(L_m+2)} \right) \quad \text{for } a \to 0. \tag{23}$$

We now compare the exponent $[(k+2)\beta-1]m + \beta(L_m+2)$ of the decay rate in (22) to the exponent $(J+1)(\beta(k+2)-1) + \beta$ of the decay rate in (21), where the latter decay rate is equal to $\mathcal{O}(a^N)$. By considering that $\beta < \frac{1}{k+1}$, we see that the choice $L_m = J - m$ is sufficient to obtain a decay rate

of $\mathcal{O}(a^N)$ in (22). Hence, for all $m \in \{0, \dots, J\}$ we have

$$\int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \left(\frac{x_2^{k+1}}{a} \right)^m \cdot \left[\rho(x_2) + \sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right]^m dx_2$$

$$= \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \left(\frac{x_2^{k+1}}{a} \right)^m \cdot \left[\sum_{\ell=1}^{L_m} \frac{\tilde{q}^{(\ell)}(0)}{\ell!} \cdot x_2^\ell \right]^m dx_2 + \mathcal{O}(a^N)$$

for $a \to 0$. By inserting this result into (22), we get

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{\ell=m}^{(J-m)m} c_{\ell,m} \int_{-a^\beta}^{a^\beta} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot x_2^{k+1}/a \\ x_2/a^\alpha \end{pmatrix} \cdot x_2^{(k+1)m+\ell} dx_2 + \mathcal{O}(a^N)$$

for $a \to 0$ for appropriate constants $c_{\ell,m} \in \mathbb{R}$. By comparing the summand for $m = 0$ in the equation (22) with the summand for $m = 0$ in the upper equation, we obtain that

$$c_{0,0} = 1. \tag{24}$$

This constant will become important in the proof of statement 3. of this Lemma.

STEP 4

In this final step, we extend the integration limits to $\pm\infty$ and apply Lemma 4.40 to estimate the decay of the Taylorlet transform.

Applying Lemma 4.39 again, we can change back the integration limits to $\pm\infty$ by only adding another $\mathcal{O}(a^N)$-term. Furthermore, we substitute $x_2 = a^\alpha v$ and obtain

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{\ell=m}^{(J-m)m} c_{\ell,m} \int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot a^{(k+1)\alpha-1} v^{k+1} \\ v \end{pmatrix} \cdot (a^\alpha v)^{(k+1)m+\ell} dv$$

$$+ \quad \mathcal{O}(a^N). \tag{25}$$

Finally, we brought the Taylorlet transform into a shape that is fit for an application of Lemma 4.40. Since $\tilde{q}(0) \neq 0$ and $\lim_{a \to 0} a^{(k+1)\alpha - 1} = \infty$, Lemma 4.40 delivers

$$
\begin{aligned}
&|\mathcal{T}^{(n,\alpha)} f(a, s, 0)| \\
&= \mathcal{O}\left(a^{-1} \sum_{m=0}^{J} a^{-m} \sum_{\ell=m}^{(J-m)m} c_{\ell,m} \cdot a^{\alpha\ell} \cdot a^{\alpha(k+1)m} \cdot a^{(r+m)(1-(k+1)\alpha)} \right) \\
&= \mathcal{O}\left(\sum_{m=0}^{J} \sum_{\ell=m}^{(J-m)m} c_{\ell,m} \cdot a^{\alpha\ell + (1-(k+1)\alpha)r - 1} \right) \\
&= \mathcal{O}\left(a^{(1-(k+1)\alpha)r - 1} \right) \quad \text{for } a \to 0.
\end{aligned}
$$

3. For this case we use the same argumentations as in the case *2.* to obtain (25) with the choices of $k = n$ and

$$
\tilde{q}(x_2) = \begin{cases} x_2^{-(n+1)} \cdot \left[q(x_2) - \sum_{\ell=0}^{n} \frac{s_\ell}{\ell!} \cdot x_2^\ell \right], & \text{for } x_2 \neq 0, \\ \frac{1}{(n+1)!} \cdot q^{(n+1)}(0), & \text{for } x_2 = 0. \end{cases}
$$

In spite of the similarities there is a major difference in the situations, namely that

$$
\lim_{a \to 0} a^{(n+1)\alpha - 1} = 0.
$$

Hence, we obtain for the integrals in (25) that

$$
\begin{aligned}
&\lim_{a \to 0} \int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) a^{(n+1)\alpha - 1} u^{n+1} \\ u \end{pmatrix} u^{(n+1)m + \ell} \, du \\
&= \int_{\mathbb{R}} \lim_{a \to 0} g^{(m)}\left(\tilde{q}(0) a^{(n+1)\alpha - 1} u^{n+1} \right) h(u) u^{(n+1)m + \ell} \, du \\
&= g^{(m)}(0) \int_{\mathbb{R}} h(u) u^{(n+1)m + \ell} \, du. \tag{26}
\end{aligned}
$$

We now focus on the powers of a appearing in the summands of (25). For the indices ℓ and m of the double sum's summands in (25) we obtain that

$$
\begin{aligned}
S_{\ell,m}(a) &:= c_{\ell,m} \cdot a^{-(m+1)} \int_{\mathbb{R}} \partial_1^m \tau \begin{pmatrix} \tilde{q}(0) \cdot a^{(k+1)\alpha - 1} v^{k+1} \\ v \end{pmatrix} \cdot (a^\alpha v)^{(k+1)m + \ell} \, dv \\
&= \mathcal{O}\left(a^{[(n+1)\alpha - 1]m + \ell\alpha - 1} \right) \quad \text{for } a \to 0
\end{aligned}
$$

for all $m \in \{0, \dots, J\}$ and $\ell \in \{m, \dots, (J-m)m\}$. Due to the restrictiveness, we have $g(0) \neq 0$ and $\int_{\mathbb{R}} h(u) \, du \neq 0$. Hence, together with (26) and $c_{0,0} = 1$ due to (24) we obtain that

$$
S_{0,0}(a) \sim a^{-1} \cdot c_{0,0} \cdot g(0) \cdot \int_{\mathbb{R}} h(u) \, du \sim a^{-1} \quad \text{for } a \to 0.
$$

Since $(n+1)\alpha - 1 > 0$, $S_{0,0}$ is the slowest decaying summand. Thus,

$$\mathcal{T}^{(n,\alpha)} f(a, s, 0) \sim a^{-1} \quad \text{for } a \to 0.$$

$\square$

With this lemma we are now able to prove Theorem 4.33.

Proof of Theorem 4.33.

The proof strategy is to reduce the case $f(x) = I_{\pm}^{j} \delta(x_1 - q(x_2))$ to the case $f(x) = \delta(x_1 - q(x_2))$ of Lemma 4.42 by partial integration and to show that the resulting iterated integral $I_{\pm}^{j} \tau$ of the Taylorlet τ is a Taylorlet as well.

First, we note that $f = (I_{\pm}^{j} \delta)(\cdot_1 - q(\cdot_2))$ is a tempered distribution for all $j \in \mathbb{N}$. Let $a > 0, s \in \mathbb{R}^{n+1}, t \in \mathbb{R}$. Then the Taylorlet transform $\mathcal{T}^{(n,\alpha)} f(a, s, t)$ is well defined. By partial integration we obtain for $j \geq 1$

$$\begin{aligned}
\mathcal{T}^{(n,\alpha)} f(a, s, t) &= \langle \tau_{a,s,0}, f \rangle \\
&= \int_{\mathbb{R}} \left[a \cdot I_{x_1, \pm} \tau_{a,s,0}(x) \cdot (I_{\pm}^{j} \delta)(x_1 - q(x_2)) \right]_{x_1 = -\infty}^{x_1 = +\infty} dx_2 \\
&\quad + \int_{\mathbb{R}^2} a \cdot I_{x_1, \mp} \tau_{a,s,0}(x) \cdot (I_{\pm}^{j-1} \delta)(x_1 - q(x_2)) dx.
\end{aligned}$$

We now show that the first term disappears. For this we note that $I_{\pm}^{j} \delta(x)$ exhibits only polynomial growth as $|x| \to \infty$. Furthermore, with $\tau = g \otimes h$, only g is altered by the operator $I_{x_1, \pm}$ while h remains the same. Hence, we show that $I_{\pm}^{j} g \in \mathcal{S}(\mathbb{R})$ for all $j < M$. By applying a Fourier transform to the function, we obtain

$$\left(I_{\pm}^{j} g \right)^{\wedge} (\omega) = \frac{\hat{g}(\omega)}{(\pm i \omega)^{j}}.$$

Since g has M vanishing moments and $g \in \mathcal{S}(\mathbb{R})$, we have $\hat{g}(\omega) = \mathcal{O}(\omega^{M})$ for $\omega \to 0$. Consequently, $\frac{\hat{g}(\omega)}{(\pm i \omega)^{j}} \in \mathcal{S}(\mathbb{R})$ and hence also $I_{\pm}^{j} g \in \mathcal{S}(\mathbb{R})$. We thus obtain

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = a \cdot \int_{\mathbb{R}^2} I_{x_1, \mp} \tau_{a,s,0}(x) \cdot (I_{\pm}^{j-1} \delta)(x_1 - q(x_2)) dx.$$

By induction we get

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = a^{j} \cdot \left\langle I_{x_1, \mp}^{j} \tau_{a,s,0}, \delta(x_1 - q(x_2)) \right\rangle.$$

This delivers an additional factor a^{j} to the Taylorlet transform. Now we examine the vanishing moments. By applying partial integration and utilizing $I_{\pm}^{j} g \in \mathcal{S}(\mathbb{R})$ we obtain

$$\left| \int_{\mathbb{R}} (I_{\pm}^{j} g)(\pm t^{k}) t^{m} dt \right| = \left| \int_{\mathbb{R}} g(\pm t^{k}) t^{kj+m} dt \right|.$$

Hence, $I_{\pm}^{j}g$ has $M-j$ vanishing moments of order n and in case 2. with a highest approximation order of k we obtain the decay rate

$$\mathcal{T}^{(n,\alpha)}f(a,s,0) = \mathcal{O}\left(a^{j-1+(M-j)[1-(k+1)\alpha]}\right) \text{ for } a \to 0.$$

It remains to show that the restrictiveness condition of g guarantees $I_{\pm}^{j}g(0) \neq 0$. For this we apply the formula for iterated integrals stating that

$$I_{+}^{j}g(u) = \int_{-\infty}^{u} (u-v)^{j-1}g(v)dv.$$

Hence, we obtain

$$I_{+}^{j}g(0) = (-1)^{j-1}\int_{-\infty}^{0} g(v)v^{j-1}dv = \underbrace{\int_{\mathbb{R}} g(v)v^{j-1}dv}_{=0} + (-1)^{j}\underbrace{\int_{0}^{\infty} g(v)v^{j-1}dv}_{\neq 0} \neq 0.$$

The statement $I_{-}^{j}g(0) \neq 0$ can be proved similarly. Hence, for all $j < r$, the function $I_{x_1,\pm}^{j}\tau$ is a restrictive analyzing Taylorlet of order n with $r-j$ vanishing moments, i. e.,

$$\int_{\mathbb{R}} I_{x_1,\pm}^{j}\tau\begin{pmatrix}0\\u\end{pmatrix}du \neq 0.$$

Consequently, with the additional factor a^{j} we get the decay rate

$$\mathcal{T}^{(n,\alpha)}f(a,s,0) = \mathcal{O}\left(a^{j-1}\right) \text{ for } a \to 0.$$

$\square$

4.5 Numerical examples

In this section we illustrate the main result numerically. To this end, we present a procedure for the detection of the location, the orientation and the curvature of an edge, based on the Taylorlet transform. As an example, we consider the sharp edge of a function

$$f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - q(x_2)), \quad \text{for } x \in \mathbb{R}^2,$$

with $q \in C^{\infty}(\mathbb{R})$ as singularity function.

4.5.1 Detection procedure

Due to Theorem 4.33, the decay rate of the Taylorlet transform changes depending on the highest approximation order of the shearing variable. We can exploit this pattern in a step-by-step

search for consecutive Taylor coefficients of the singularity function q. To this end, we first compute the Taylorlet transform of a function with $\alpha > 1$ and with varying shearing variable s_0 while $s_k = 0$ for all $k \in \{1,\ldots,n\}$. The choice of α and the restrictiveness of the Taylorlet ensure a decay rate of

$$\mathcal{T}^{(0,\alpha)} f(a, s_0, t) \sim 1 \quad \text{for } a \to 0$$

for $s_0 = q(t)$ due to Theorem 4.33.

We then consider the propagation of the local maxima w.r.t. s_0 through the scales. Since the choice $s_0 = q(t)$ leads to the lowest decay rate, we can expect the local maxima near $s_0 = q(t)$ to converge towards this value for decreasing scales in a similar fashion as in the method of wavelet maximum modulus by Mallat and Hwang [MH92]. Subsequently, we fix s_0 to the value $q(t)$, change α such that $\alpha \in \left(\frac{1}{2}, 1\right)$ and search for the matching value of s_1 in the same way as in the preceding step for s_0. Because of the restrictiveness, the Taylorlet transform has a decay rate of $\mathcal{T}^{(1,\alpha)} f(a, s, t) \sim 1$ for $a \to 0$, if additionally $s_1 = \dot{q}(t)$. Due to the vanishing moment condition, the decay rate of the Taylorlet transform is considerably higher for $s_1 \neq \dot{q}(t)$. Hence, the method of maximum modulus is still applicable. With the same argumentation, we can repeat this procedure for all shearing variables s_k with a choice $\alpha \in (\frac{1}{k+1}, \frac{1}{k})$ up to the order of the Taylorlet.

4.5.2 Derivative-based construction

For the implementation of the Taylorlet transform in Matlab we used Taylorlets of order 2 with 3 vanishing moments. They were constructed via the General Setup in subsection 4.3.1 starting from the function $\phi(t) = e^{-t^2}$. Through this procedure we obtain the Taylorlet

$$\tau(x) = g(x_1) \cdot h(x_2), \tag{27}$$

where

$$g(x_1) = \frac{64}{8!} \cdot (1 + x_1) \cdot (315 - 51660\, x_1^2 + 286020\, x_1^4 - 349440\, x_1^6 + 142464\, x_1^8 - 21504\, x_1^{10} + 1024\, x_1^{12}) \cdot e^{-x_1^2},$$

$$h(x_2) = e^{-x_2^2},$$

which is shown in Figure 4.1. To speed up computation time, we employed the one-dimensional adaptive Gauss-Kronrod quadrature `quadgk` in Matlab for the evaluation of the integrals. In order to reduce the Taylorlet transform to a one-dimensional integral, we utilize partial integration, i.e.,

$$\left\langle \tau_{ast}(x), \mathbb{1}_{\mathbb{R}_+}(x_1 - q(x_2)) \right\rangle = \left\langle I_{x_1,+} \tau_{ast}, \delta(x_1 - q(x_2)) \right\rangle = \int_{-\infty}^{\infty} I_{x_1,+} \tau_{ast}(q(t)) dt.$$

Hereby, the antiderivative of τ w.r.t. x_1 can be determined analytically by computing the antiderivative of g, i.e.,

$$\int_{-\infty}^{t} g(x_1)\,dx_1 = -\frac{32}{8!} \cdot e^{-t^2} \cdot \left(-9 - 630\ t - 324\ t^2 + 34020\ t^3 + 25668\ t^4 - 100800\ t^5 - 86784\ t^6 \right.$$
$$\left. + 71040\ t^7 + 65664\ t^8 - 15872\ t^9 - 15360\ t^{10} + 1024\ t^{11} + 1024\ t^{12}\right).$$

4.5.3 Construction based on q-calculus

The Taylorlet we use for the images have 3 vanishing moments of 2nd order in x_1-direction and is of the form suggested in the general setup of subsection 4.3.2 right after Theorem 4.23. To this end, we choose $q = \frac{2}{3}$ and

$$\phi(t) = e^{-2t^2} \quad \text{for all } t \in \mathbb{R}.$$

Hence, we obtain the Taylorlet $\tau = g \otimes h$, where for all $x \in \mathbb{R}^2$,

$$g(x_1) = (1 + x_1) \cdot \left[\prod_{\substack{m=1 \\ 4 \nmid m}}^{12} \left(\mathrm{Id} - q^m D_q\right) \phi_0 \right] \left(\sqrt{|x_1|}\right),$$

$$h(x_2) = e^{-2x_2^2}.$$

4.5.4 Images

In this subsection, we present plots of the Taylorlet transform created with Matlab. Here, the images in tables 4.5.1 and 4.5.2 use Taylorlets generated by the derivative-based construction and the plots in tables 4.5.3 and 4.5.4 employ Taylorlets built via the construction based on q-calculus.

In order to better visualize the local maxima, we normalized the absolute value of the Taylorlet transform in the presented plots such that the maximum value in each scale is 1. Due to this normalization w.r.t. the local maxima on a compact interval regarding the respective shearing variable, discontinuities w.r.t. the dilation parameter can appear (e.g. around $-\log_2 a = 1$ in the bottom right image of table 4.5.1).

Table 4.5.1 contains plots of the Taylorlet transform $\mathcal{T}f(a, s, t)$ of the function $f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - \sin x_2)$ for $t \in \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$. The vertical axis shows the dilation parameter in a binary logarithmic scale while the horizontal axis shows location, slope and parabolic shear. The respective true values can be found in the following table and are indicated by a vertical red line in the plots.

Table 4.5.1: Plots of the Taylorlet transform $\mathcal{T}f(a,s,t)$ using Taylorlets generated by the derivative-based construction for $f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - \sin x_2)$, where $t \in \{0, \frac{\pi}{6}\}$. The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location s_0 (left), the slope s_1 (center) and the parabolic shear s_2 (right). The respective true value is indicated by the vertical red line. The values of α change with s_i: for s_0 we use $\alpha = 1.01$, for s_1 we have $\alpha = 0.51$ and during the search for s_2 we set $\alpha = 0.34$. The Taylorlet transform was computed for points (a, s_i) on a 300×300–grid. We can observe the paths of the local maxima w. r. t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value.
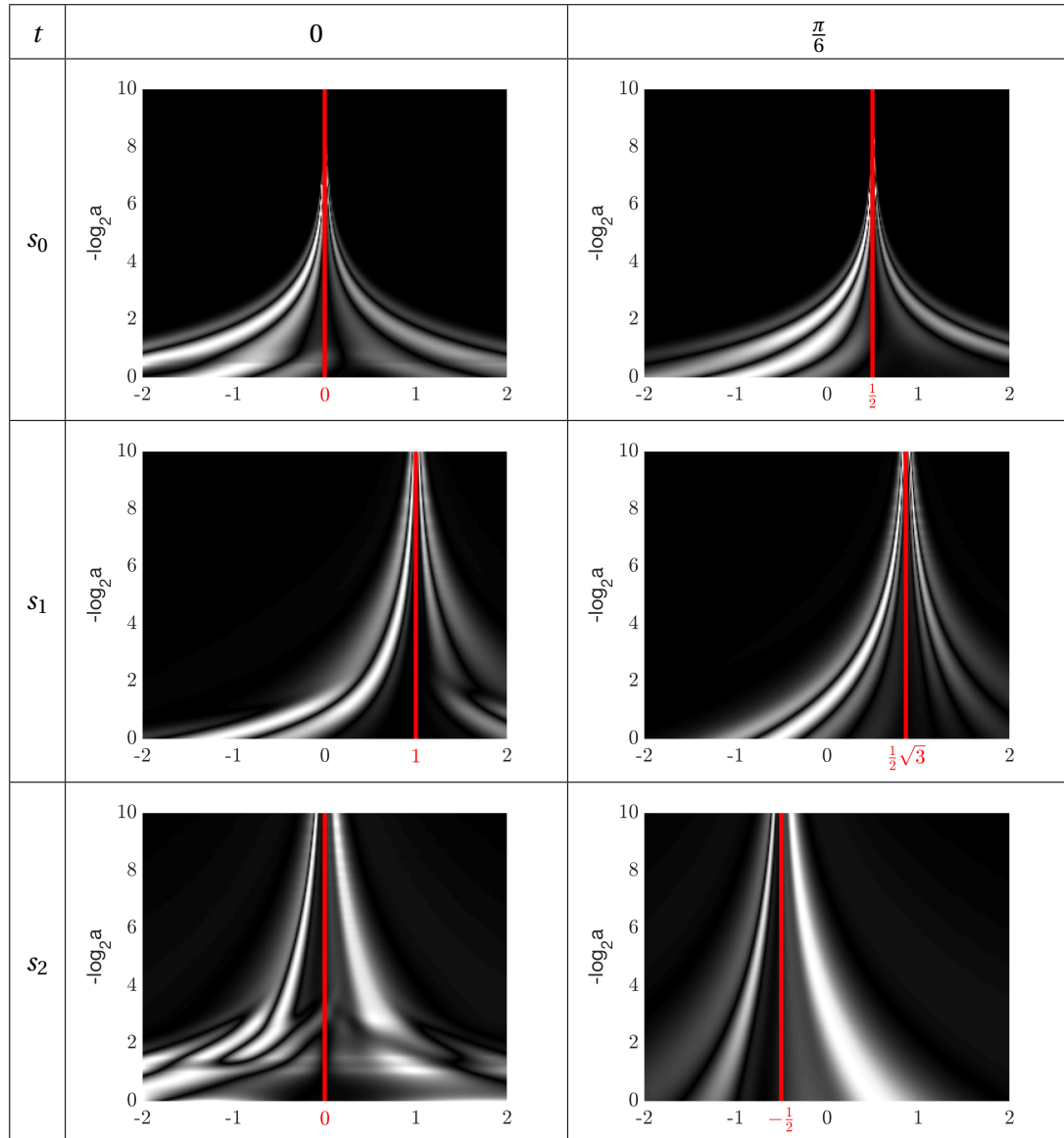
Table 4.5.2: Plots of the Taylorlet transform $\mathcal{T}f(a, s, t)$ using Taylorlets generated by the derivative-based construction for $f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - \sin x_2)$, where $t \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$. The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location s_0 (left), the slope s_1 (center) and the parabolic shear s_2 (right). The respective true value is indicated by the vertical red line. The values of α change with s_i: for s_0 we use $\alpha = 1.01$, for s_1 we have $\alpha = 0.51$ and during the search for s_2 we set $\alpha = 0.34$. The Taylorlet transform was computed for points (a, s_i) on a 300×300−grid. We can observe the paths of the local maxima w.r.t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value.

t	$q(t)$	$q'(t)$	$q''(t)$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$
$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$
$\frac{\pi}{2}$	1	0	-1

The values of α change with s_i. For the detection of the location we use $\alpha = 1.01$, for the slope we have $\alpha = 0.51$ and during the search for the parabolic shear we set $\alpha = 0.34$. We can observe the paths of the local maxima w. r. t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, $\mathcal{T}f(a, s, t)$ decays fast for $a \to 0$, if $s_k \neq q^{(k)}(t)$ and slow for $s_k = q^{(k)}(t)$. Hence, the local maxima of the Taylorlet transform display a fast convergence to the correct value $q^{(k)}(t)$.
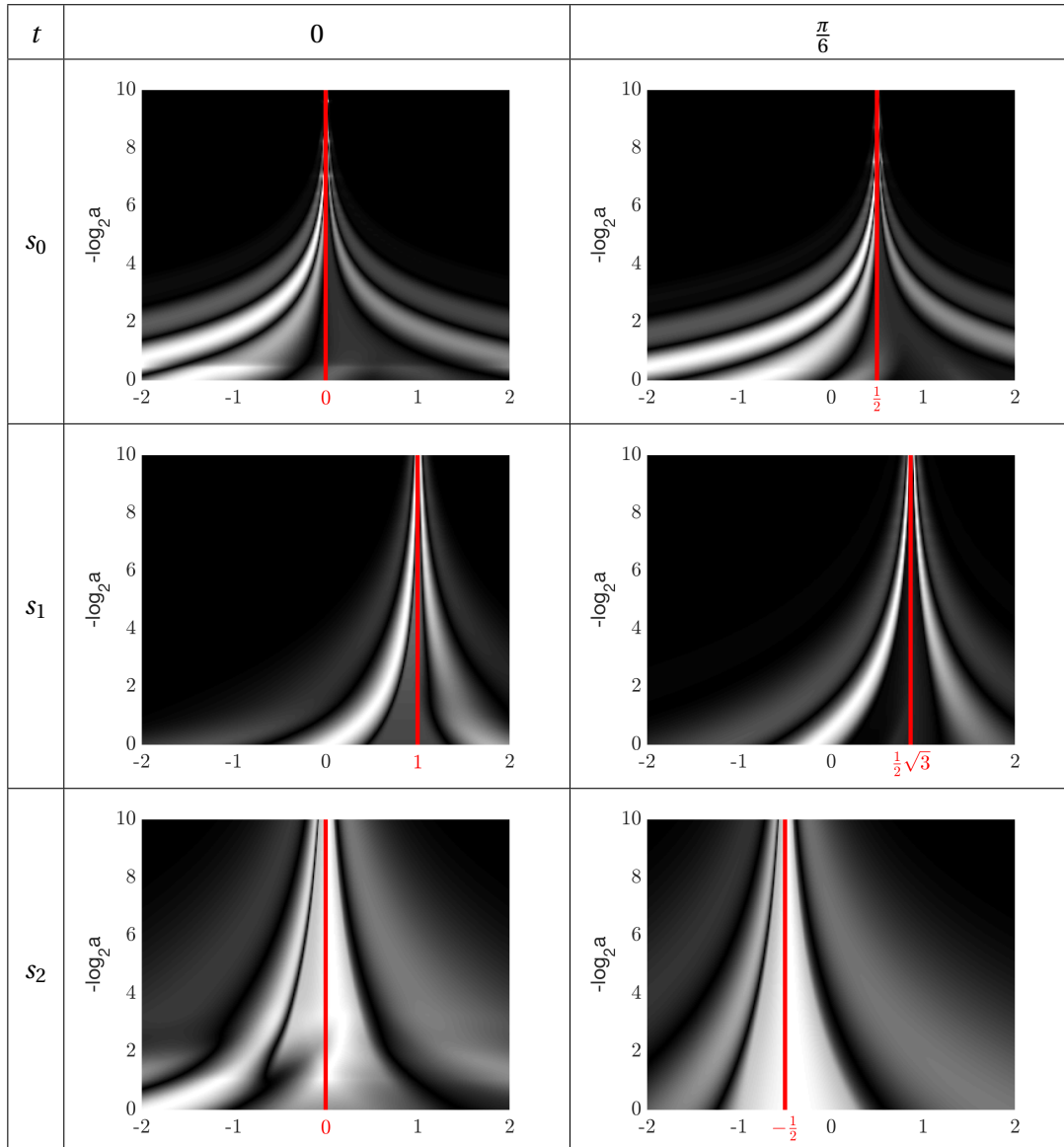
Table 4.5.3: Plots of the Taylorlet transform $\mathcal{T}f(a, s, t)$ using Taylorlets generated by the q-calculus based construction for $f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - \sin x_2)$, where $t \in \{0, \frac{\pi}{6}\}$. The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location s_0 (left), the slope s_1 (center) and the parabolic shear s_2 (right). The respective true value is indicated by the vertical red line. The values of α change with s_i: for s_0 we use $\alpha = 1.01$, for s_1 we have $\alpha = 0.51$ and during the search for s_2 we set $\alpha = 0.34$. The Taylorlet transform was computed for points (a, s_i) on a 300×300–grid. We can observe the paths of the local maxima w.r.t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value.
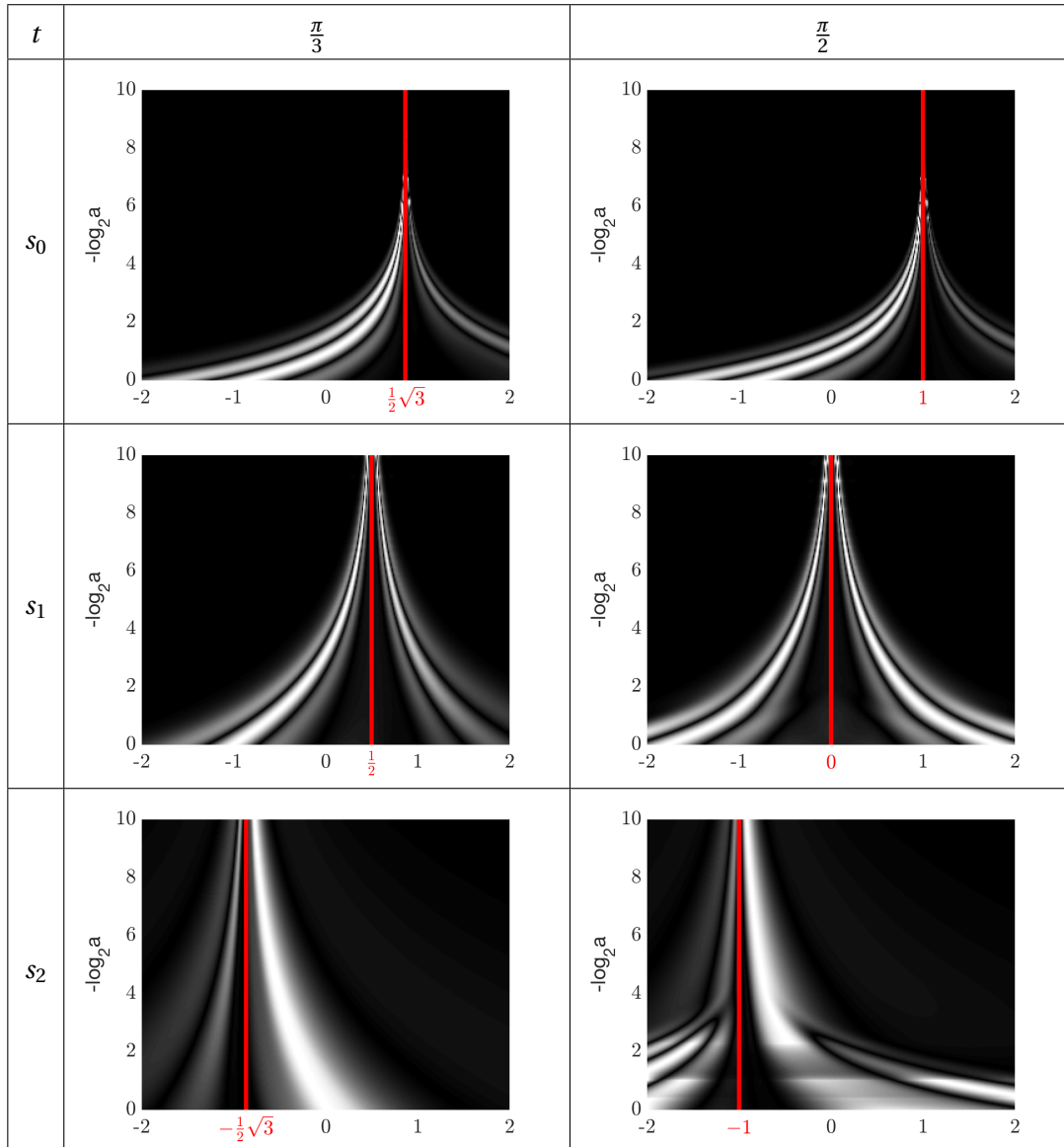
Table 4.5.4: Plots of the Taylorlet transform $\mathcal{T}f(a, s, t)$ using Taylorlets generated by the q-calculus based construction for $f(x) = \mathbb{1}_{\mathbb{R}_+}(x_1 - \sin x_2)$, where $t \in \{\frac{\pi}{3}, \frac{\pi}{2}\}$. The vertical axis shows the dilation parameter in a logarithmic scale $-\log_2 a$. The horizontal axis shows the location s_0 (left), the slope s_1 (center) and the parabolic shear s_2 (right). The respective true value is indicated by the vertical red line. The values of α change with s_i: for s_0 we use $\alpha = 1.01$, for s_1 we have $\alpha = 0.51$ and during the search for s_2 we set $\alpha = 0.34$. The Taylorlet transform was computed for points (a, s_i) on a 300×300–grid. We can observe the paths of the local maxima w. r. t. the respective shearing variable as they converge to the correct related geometric value through the scales. Due to the vanishing moment conditions of higher order, the local maxima display a fast convergence to the correct value.

CHAPTER 5

Extension of the Taylorlet transform to three dimensions

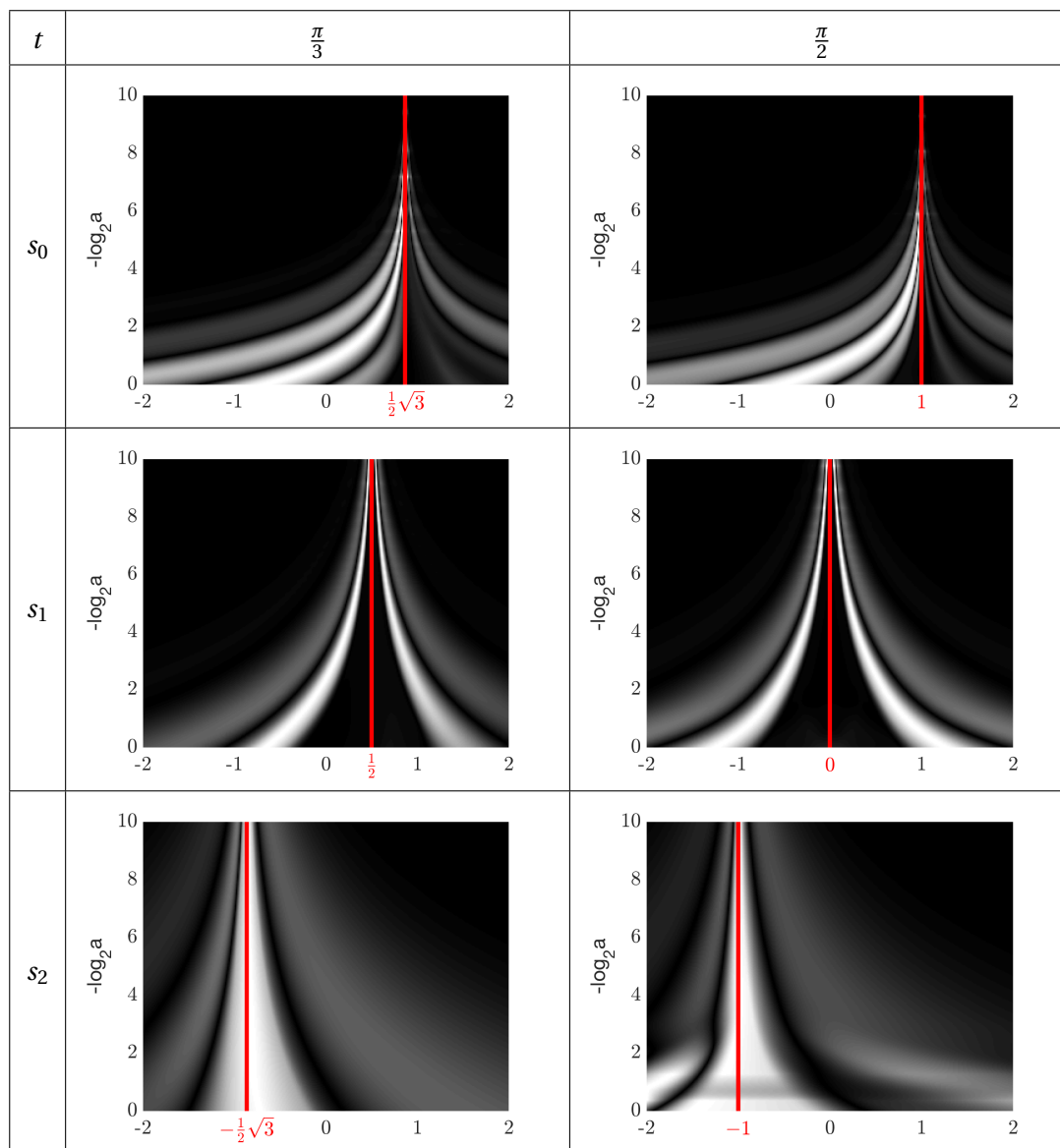Probably the most common extension of mathematical results is a generalization to higher dimensions. In this chapter, we will extend the notion and the detection results of Taylorlets of the previous chapter to the third dimension. There are, of course, some major differences between the two-dimensional and the three-dimensional case. Since the pivotal idea of the Taylorlet transform is the approximation of the singularity function q, the probably greatest disparity is the different Taylor series expansions of $q \in C^\infty(\mathbb{R})$ in the classical case and $q \in C^\infty(\mathbb{R}^2)$ in the three-dimensional case. While the classical case allows for a one-dimensional Taylor series expansion of the form

$$q(u) \sim \sum_k \frac{q^{(k)}(t)}{k!} \cdot (u - t)^k \quad \text{for } u \to t,$$

the three-dimensional case results in a bivariate Taylor series expansion that has the form

$$q(u) \sim \sum_{k_1} \sum_{k_2} \frac{\partial_1^{k_1} \partial_2^{k_2} q(t)}{k_1! \cdot k_2!} \cdot (u_1 - t_1)^{k_1} \cdot (u_2 - t_2)^{k_2} \quad \text{for } u \to t.$$

Hence, we have to handle a multitude of monomials for each polynomial degree $k_1 + k_2$ instead of just one, which complicates the matter considerably.

The structure of this chapter is as follows.

In the first section, we introduce the new terminology for the Taylorlets in three dimensions, the mulitvariate Taylor series expansion and the Hankel transform. The two latter concepts are used in the proof of the detection result.

The second section states the main and the auxiliary results which focus on the curvature of a singularity along a surface and discards all Taylor coefficients of order three or higher. Moreover, the main result in three dimensions deviates from its two-dimensional counterpart, as an additional case occurs, if the sheared version of the singular surface exhibits a locally hyperbolic

geometry. This can be exploited by imposing a new set of conditions - the hyperbolic restrictiveness - on the Taylorlet to obtain a slow decay rate.

The third section gives an overview over the proof strategy and the dependencies of the results and is dedicated to their proof.

Section 4 presents a construction of a three-dimensional Taylorlet that satisfies all conditions to yield the respective decay rates provided by the main result. To this end, we utilize the q-calculus approach that was already applied in the previous chapter.

In the last section of this chapter, we present a fast algorithm for the detection of the edge curvature. We utilize the special decay rate that occurs when the sheared surface is locally hyperbolic, to design a detection algorithm for the curvature that allows for a one-dimensional search space. Paradoxically, it is thus faster than the detection of the orientation which has a two-dimensional search space.

5.1   Basic definitions and notation

For a proper description of three-dimensional Taylorlets, we have to introduce the notions of multi-indices and multivariate Taylor series expansions.

Definition 5.1. A multi-index of dimension $d \in \mathbb{N}$, $d \geq 1$ is a vector $\alpha \in \mathbb{N}_0^d$. The sum of two multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ is understood component-wise:

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_d + \beta_d).$$

The partial order of the multi-indices is also component-wise:

$$\alpha \leq \beta \Leftrightarrow \alpha_k \leq \beta_k \quad \text{for all } k \in \{1, \ldots, d\}.$$

The absolute value of a multi-index α is defined as

$$|\alpha| = \sum_{k=1}^d \alpha_k.$$

For a vector $x \in \mathbb{C}^d$ its αth power is defined as

$$x^\alpha = \prod_{k=1}^d x_k^{\alpha_k}.$$

For $n \in \mathbb{N}$ and a function $f \in C^n(\mathbb{R}^d, \mathbb{R})$, the αth partial derivative of f reads

$$\partial^\alpha f = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} f.$$

The factorial of α is defined as

$$\alpha! = \prod_{k=1}^d \alpha_k!$$

For $n \in \mathbb{N}$, the set of indices with absolute value lower than or equal to n is

$$I(d, n) := \left\{ \alpha \in \mathbb{N}^d : |\alpha| \leq n \right\}.$$

Theorem 5.2 (Multivariate Taylor series expansion)**.** *[Hil13] Let $f \in C^{n+1}(\mathbb{R}^d)$. The nth Taylor polynomial of f about the point $y \in \mathbb{R}^d$ is defined as*

$$T_n f(\cdot\,; y) : \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto \sum_{|\alpha| \leq n} \frac{\partial^\alpha f(y)}{\alpha!} \cdot (x - y)^\alpha.$$

Then the following asymptotic holds true:

$$\left| T_n f(x; y) - f(x) \right| = \mathcal{O}\left(\|x - y\|^{n+1} \right) \quad \textit{for } x \to y.$$

Theorem 5.3 (Polynomial formula)**.** *[Hil13] For $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$, we have*

$$\left(\sum_{i=1}^{d} x_i \right)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} \cdot x^\alpha.$$

Lemma 5.4. *Let $x \in \mathbb{R}^d$ and let $\alpha \in \mathbb{N}_0^d$. Then,*

$$|x^\alpha| \leq \|x\|^{|\alpha|}.$$

Proof. Equivalently, we can show that $|x^\alpha|^2 \leq \|x\|^{2|\alpha|}$. For the left hand side, we obtain

$$|x^\alpha|^2 = \left| \prod_{i=1}^{n} x_i^{2\alpha_i} \right| = \prod_{i=1}^{n} x_i^{2\alpha_i} = x^{2\alpha}.$$

For the right hand side, Theorem 5.3 yields

$$\|x\|^{2|\alpha|} = \left(\sum_{i=1}^{n} x_i^2 \right)^{|\alpha|} \overset{\text{Theorem 5.3}}{=} \sum_{|\beta|=|\alpha|} \frac{|\alpha|!}{\beta!} \cdot x^{2\beta} \geq \frac{|\alpha|!}{\alpha!} \cdot x^{2\alpha} \geq x^{2\alpha} = \left| x^\alpha \right|^2.$$

$\square$

Furthermore, we will introduce the Bessel function and the Hankel transform which we need in the proof of the detection result of the Taylorlet transform.

Definition 5.5 (Bessel function, Hankel transform)**.** Let $v \in \mathbb{C} \setminus \left\{ -\frac{k}{2} : k \in \mathbb{N} \right\}$. Then, the Bessel function of order v is defined as the power series

$$J_v(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}z \right)^{v+2n}}{n! \cdot \Gamma(v + n + 1)}$$

for all $z \in \mathbb{C}$, if $v \geq 0$ and for all $z \in \mathbb{C} \setminus \{0\}$ for $v < 0$ [Wat22, Section 3.1].

For $v > -\frac{1}{2}$, the Hankel transform of order v is defined as

$$\mathcal{H}_v : L^1(\mathbb{R}_+, r\ dr) \to L^\infty(\mathbb{R}_+, d\rho), \quad \mathcal{H}_v f(\rho) = \int_0^\infty f(r) J_v(\rho r) r\ dr \quad \text{[Pou10].}$$

Proposition 5.6 (Properties of the Hankel transform)**.** *Let $f, g \in L^1(\mathbb{R}_+, r\ dr)$ and $v > -\frac{1}{2}$.*

1. $\mathcal{H}_v\left(f\left(a\cdot\right)\right)(\rho) = \frac{1}{a^2} \cdot \mathcal{H}_v\left(\frac{\rho}{a}\right)$ *for all $a, \rho > 0$,*

2. $\int\limits_0^\infty f(r) \cdot g(r)\ r\ dr = \int\limits_0^\infty \mathcal{H}_v f(\rho) \cdot \mathcal{H}_v g(\rho)\ \rho\ d\rho,$

3. *If $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(x) = f(\|x\|)$ for all $x \in \mathbb{R}^2$, then $\hat{F}(\xi) = \mathcal{H}_0 f(\|\xi\|)$ for all $\xi \in \mathbb{R}^2$.*

4. *If $h \in \mathcal{S}(\mathbb{R})$ is even, then $\mathcal{H}_0 h \in \mathcal{S}(\mathbb{R})$ is even.*

Proof. 1. See [Pou10, Section 9.4, 3.].

2. See [Pou10, Section 9.4, 7.].

3. See [Pou10, Section 9.3].

4. Since $h \in \mathcal{S}(\mathbb{R})$ is even, the function $H : \mathbb{R}^2 \to \mathbb{R}$, $x \mapsto h(\|x\|)$ is a radially symmetric Schwartz function. Since $H \in \mathcal{S}(\mathbb{R}^2)$, it follows that $\hat{H} \in \mathcal{S}(\mathbb{R}^2)$. Due to third property, we have

$$\hat{H}(\xi) = \mathcal{H}_0 h(\|\xi\|) \quad \text{for all } \xi \in \mathbb{R}^2.$$

Hence, $\mathcal{H}_0 h \in \mathcal{S}(\mathbb{R})$ is even.

$\square$

Remark 5.7. Property 3 of Proposition 5.6 offers an interesting perspective on the Hankel transform. It can be seen as the two-dimensional Fourier transform acting on radially symmetric functions.

In order to generalize the detection result of the Taylorlet transform from dimension 2 to higher dimensions, we have to adapt the terminology of the Taylorlet transform to higher dimensions.

Definition 5.8 (Higher order shears in three dimensions, scaling matrix)**.** Let $n \in \mathbb{N}$ and let the shearing variable of order n be defined as the map $s : \left\{\beta \in \mathbb{N}_0^2 : |\beta| \le n\right\} \to \mathbb{R}$, $\beta \mapsto s_\beta$. For $x \in \mathbb{R}^3$ we denote

$$x = \begin{pmatrix} x_1 \\ \tilde{x} \end{pmatrix}, \quad \text{where } x_1 \in \mathbb{R} \text{ and } \tilde{x} \in \mathbb{R}^2.$$

The nth order shearing operator is defined as

$$S_s^{(n)} : \mathbb{R}^3 \to \mathbb{R}^3, \quad x \mapsto \begin{pmatrix} x_1 + \sum\limits_{|\beta| \le n} \frac{s_\beta}{\beta!} \cdot \tilde{x}^\beta \\ \tilde{x} \end{pmatrix}.$$

Let $a, \alpha > 0$. The α-scaling matrix is then defined as

$$A_a^{(\alpha)} : \mathbb{R}^3 \to \mathbb{R}^3, \quad x \mapsto \begin{pmatrix} a \cdot x_1 \\ a^\alpha \cdot \tilde{x} \end{pmatrix}.$$

Definition 5.9 (Analyzing Taylorlet, restrictiveness)**.** For dimension 3 let $g \in \mathcal{S}(\mathbb{R})$ such that

$$\int_\mathbb{R} g(r) r^m \, dr = 0 \quad \text{for all } m \in \{0, \dots, M-1\},$$

$$\text{and} \quad \int_0^\infty g(\pm r^2) r^m \, dr = 0 \quad \text{for all } m \in \{0, \dots, 2M-1\}.$$

Moreover, let $\varphi \in \mathcal{S}(\mathbb{R})$ be even such that

$$\int_0^\infty \varphi(r) r^m \, dr = 0 \quad \text{for all } m \in \mathbb{N}, m \geq 2.$$

Let $h \in \mathcal{S}(\mathbb{R}^2)$ such that $h(x) = \varphi(\|x\|)$ for all $x \in \mathbb{R}^2$. We call the function $\tau = g \otimes h$ an analyzing Taylorlet of order 2 in dimension 3 with M vanishing moments.

We say τ is *restrictive*, if additionally

  (i) $g(0) \neq 0$ and

  (ii) $\int_0^\infty \varphi(r) r \, dr \neq 0$.

We call an analyzing Taylorlet τ of order 2 *hyperbolically restrictive*, if

  (i) $\int_0^\infty \left[g(r^2) + g(-r^2) \right] \cdot r \log r \, dr \neq 0$ and

  (ii) $\varphi(0) \neq 0$.

Definition 5.10 (Taylorlet transform)**.** Let $M, n \in \mathbb{N}$ and let $\tau \in \mathcal{S}(\mathbb{R}^3)$ be an analyzing Taylorlet of order n in dimension 3 with M vanishing moments. Let $\alpha > 0$, $t \in \mathbb{R}^2$, $a > 0$ and let $s : \{\beta \in \mathbb{N}_0^2 : |\beta| \leq n\} \to \mathbb{R}, \beta \mapsto s_\beta$. We define

$$\tau_{a,s,t}^{(n,\alpha)}(x) := \tau \left(A_{\frac{1}{a}}^{(\alpha)} S_{-s}^{(n)} \begin{pmatrix} x_1 \\ \tilde{x} - t \end{pmatrix} \right) \quad \text{for all } x = (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^2.$$

The Taylorlet transform w.r.t. τ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \left\langle f, \tau_{a,s,t}^{(n,\alpha)} \right\rangle.$$

Definition 5.11 (Feasible function, singularity function)**.** Let δ denote the Dirac distribution. Let furthermore $j \in \mathbb{N}_0$, $q \in C^\infty(\mathbb{R}^2)$ and let

$$f(x) = I_\pm^j \delta \left(x_1 - q(\tilde{x}) \right).$$

Then f is called a j-feasible function with singularity function q.

Definition 5.12 (Highest approximation order)**.** Let $j, n \in \mathbb{N}_0$ and let f be a j-feasible function with singularity function $q \in C^\infty(\mathbb{R}^2)$. Furthermore, let $t \in \mathbb{R}$, $s : \{\beta \in \mathbb{N}_0^2 : |\beta| \le n\} \to \mathbb{R}$, $\beta \mapsto s_\beta$ and $k \in \{0, \dots, n-1\}$. If $s_\beta = \partial^\beta q(t)$ for all $\beta \in \mathbb{N}_0^2$ with $|\beta| \le k$ and if there exists $\gamma \in \mathbb{N}_0^2$ such that $|\gamma| = k+1$ and $s_\gamma \ne \partial^\gamma q(t)$, we say that k is the highest approximation order of the shearing variable $s = (s_\beta)_{|\beta| \le n}$ for f in t.

Definition 5.13 (Approximation matrix)**.** Let $j \in \mathbb{N}_0$ and let f be a j-feasible function with singularity function $q \in C^\infty(\mathbb{R}^2)$. Furthermore, let $t \in \mathbb{R}$, $s : \{\beta \in \mathbb{N}_0^2 : |\beta| \le n\} \to \mathbb{R}$, $\beta \mapsto s_\beta$. Let

$$q_s(u) = q(u) - \sum_{|\beta| \le 2} \frac{s_\beta}{\beta!} \cdot (u - t)^\beta.$$

Then we call the Hesse matrix

$$A_s(t) = H q_s(t) = H q(t) - S$$

the approximation matrix of s for f in t, where S has the entries $S_{ij} = s_\beta$ with $\beta = e_i + e_j$ for $i, j \in \{1, 2\}$. The approximation matrix describes the Hesse matrix of the graph of the sheared version of q.

5.2   Detection results

With the necessary terminology introduced, we will now continue with the statement of the detection result and prove it afterwards.

Theorem 5.14. *Let $M \in \mathbb{N}$ such that $M \ge 1$ and let $\tilde{\tau} := \tilde{g} \otimes h$ be an analyzing Taylorlet of order 2 in dimension 3 with M vanishing moments. Let furthermore $j \in \mathbb{N}_0$, $t \in \mathbb{R}^2$, let f be a $j-$feasible function, $g := \tilde{g}^{(j)}$ and $\tau := g \otimes h$.*

    I. *Let $\alpha > 0$. If $s_0 \ne q(t)$, the Taylorlet transform has a decay of*

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) = \mathcal{O}(a^N) \quad \textit{for } a \to 0$$

    for all $N > 0$.

    II. *Let $k \in \{0, 1\}$ be the highest approximation order of s for f in t and let $\alpha < \frac{1}{k+1}$.*

        1. *Let $k = 0$. Then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{M(1-\alpha)+j+1+\alpha}\right) \quad \textit{for } a \to 0.$$

        2. *Let $k = 1$.*

            a) *If the approximation matrix $A_s(t)$ is either positive or negative semidefinite, the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{(1-2\alpha)M+j+\frac{1}{2}+\alpha}\right) \quad \textit{for } a \to 0.$$

b) *If $\alpha > \frac{2}{5}$, τ is hyperbolically restrictive and the approximation matrix $A_s(t)$ is indefinite, the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) \sim a^{j+1} \quad \text{for } a \to 0.$$

III. *Let $\alpha > \frac{1}{3}$ and let τ be restrictive. If the highest approximation order of s for f in t is at least 2, then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) \sim a^{j+2\alpha} \quad \text{for } a \to 0.$$

Remark 5.15. According to Theorem 4.33, in two dimensions, one Taylorlet is sufficient to deal with j-feasible functions for all $j \in \{0, \ldots, M-1\}$, where M is the number of higher order vanishing moments of the Taylorlet. In contrast, in three dimensions a new Taylorlet is required for each level of feasibility, as Theorem 5.14 suggests. The reason for this is a conflict between the restrictiveness property in two dimensions and the vanishing moments of the second order in three dimensions. Due to Definition 4.8, if a Taylorlet $\tau = g \otimes h$ is restrictive, we have for all $j \in \{1, \ldots, M\}$:

$$\int_0^\infty g(t) t^{j-1} \, dt = I_+^j g(0) \neq 0.$$

At the same time, however, the vanishing moments of second order in three dimensions would require that for all $m \in \{0, \ldots, M-1\}$

$$0 = \int_0^\infty g\left(\pm t^2\right) t^{2m+1} \, dt = \frac{1}{2} \cdot \int_0^\infty g(\pm u) u^m \, du.$$

As the vanishing moments are needed for the decay rate of the Taylorlet transform, the restrictiveness property cannot be translated directly from two to three dimensions.

In order to prove Theorem 5.14, we need to show some auxiliary results.

Lemma 5.16. *Let $n \in \mathbb{N}_0$ and let $\varphi \in \mathcal{S}(\mathbb{R})$ be even such that*

$$\int_0^\infty \varphi(t) t^m \, dt = 0 \quad \text{for all } m \in \mathbb{N}_0, \ m \geq n.$$

Furthermore, let $\alpha > 0$, $\ell \in \mathbb{N}_0$, $c \in \mathbb{R}$ and let

$$\Phi_\ell : \mathbb{R} \to \mathbb{R}, \quad v \mapsto \int_\mathbb{R} \varphi\left(\sqrt{\alpha^2 (v - c \cdot u)^2 + u^2}\right) u^\ell \, du.$$

Then, $\Phi_\ell \in \mathcal{S}(\mathbb{R})$ and $\Phi_\ell(v) = \mathcal{O}\left(v^{(\ell+1-n)_+}\right)$ for $v \to 0$.

Lemma 5.17. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ be even and let $g \in \mathcal{S}(\mathbb{R})$ have M vanishing moments of order 2. Furthermore, let $m \in \mathbb{N}$, $\gamma \in \mathbb{N}_0^2$ such that $|\gamma| \geq 2m-1$ and let $a_1, a_2 > 0$. Then*

$$\int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi(\|v\|) \cdot v^\gamma \, dv = \mathcal{O}\left(|z|^{-(M+m+1)}\right) \quad \text{for } z \to \pm\infty.$$

Lemma 5.18. *Let* $g \in \mathcal{S}(\mathbb{R})$ *such that*

$$\int_0^\infty g(\pm r^2) \cdot r \, dr = 0 \quad and \quad \int_0^\infty \left[g(r^2) + g(-r^2) \right] \cdot r \log r \, dr \neq 0.$$

Furthermore, let $h \in \mathcal{S}(\mathbb{R}^2)$ *be rotationally symmetric with* $h(0) \neq 0$. *Then,*

$$\int_0^\infty \int_0^\infty g\left(z \cdot (v_1^2 - v_2^2) \right) h(v) \, dv \sim |z|^{-1} \quad for \ z \to \pm\infty.$$

Lemma 5.19. *Let* $\tau \in \mathcal{S}(\mathbb{R}^2)$, $t \in \mathbb{R}^2$ *and let* f *be a* 0*-feasible function with singularity function* $q \in C^\infty(\mathbb{R}^2)$. *Let* $\alpha > 0$ *and let* k *be the highest approximation order of* s *for* f *in* t. *Then, for all* $N \in \mathbb{N}$ *there exist* $J, L \in \mathbb{N}$, *constants* $c_{\gamma,m} \in \mathbb{R}$ *for* $m \in \{0, \dots, J\}$, $\gamma \in \mathbb{N}_0^2$ *such that* $|\gamma| \leq L$, *and a homogeneous polynomial* p_{k+1} *of degree* $k+1$ *such that for* $a \to 0$,

$$\mathcal{T}_\tau^{(2,\alpha)} f(a,s,t) = \sum_{m=0}^J a^{-m} \sum_{|\gamma|=(k+2)m}^L c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} a^{(k+1)\alpha-1} \cdot p_{k+1}(v) \\ v \end{pmatrix} \cdot v^\gamma \, dv + \mathcal{O}(a^N).$$

Furthermore, $c_{0,0} = 1$.

Lemma 5.20. *Let* $M \in \mathbb{N}$ *such that* $M \geq 1$ *and let* τ *be an analyzing Taylorlet of order* 2 *in dimension* 3 *with* M *vanishing moments. Let furthermore* $t \in \mathbb{R}^2$ *and let* f *be a* 0*-feasible function with singularity function* $q \in C^\infty(\mathbb{R}^2)$.

   I. *Let* $\alpha > 0$. *If* $s_0 \neq q(t)$, *the Taylorlet transform has a decay of*

$$\mathcal{T}^{(2,\alpha)} f(a,s,t) = \mathcal{O}(a^N) \quad for \ a \to 0$$

    for all $N > 0$.

   II. *Let* $k \in \{0,1\}$ *be the highest approximation order of* s *for* f *in* t *and let* $\alpha < \frac{1}{k+1}$.

      1. *Let* $k = 0$. *Then the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a,s,t) = \mathcal{O}\left(a^{(1-\alpha)M+1+\alpha} \right) \quad for \ a \to 0.$$

      2. *Let* $k = 1$.

         a) *If the approximation matrix* $A_s(t)$ *is either positive or negative semidefinite, the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a,s,t) = \mathcal{O}\left(a^{(1-2\alpha)M+\frac{1}{2}+\alpha} \right) \quad for \ a \to 0.$$

         b) *If* $\alpha > \frac{2}{5}$, τ *is hyperbolically restrictive and the approximation matrix* $A_s(t)$ *is indefinite, the Taylorlet transform has the decay property*

$$\mathcal{T}^{(2,\alpha)} f(a,s,t) \sim a \quad for \ a \to 0.$$

III. Let $\alpha > \frac{1}{3}$ and let τ be restrictive. If the highest approximation order of s for f in t is at least 2, then the Taylorlet transform has the decay property

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) \sim a^{2\alpha} \quad \text{for } a \to 0.$$

5.3   Proof of the detection results

In order to get an overview over the utilization of the different auxiliary results in the proof, we give a quick remark on the proof strategy.

The idea of the proof is a consecutive reduction to simpler cases. In the proof of Theorem 5.14, we show that it is sufficient to consider 0-feasible functions i. e., distributions of the form

$$f(x) = \delta(x_1 - q(x_2)) \quad \text{for all } x \in \mathbb{R}^2.$$

We subsequently prove in Lemma 5.19 that the Taylorlet transform of a 0-feasible function can be reduced to a linear combination of integrals of the form

$$\int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} z \cdot p(u) \\ u \end{pmatrix} \cdot u^\gamma \, du, \tag{1}$$

where p is a homogeneous polynomial of order $k + 1$ and $|\gamma| \geq m(k + 2)$. Lemma 5.20 makes use of Lemma 5.16 - 5.18 to simplify this integral over $\mathbb{R}^2$ to a one-dimensional integral. Now the desired decay rate can be deduced by exploiting Lemma 4.40, which utilizes the vanishing moments of higher order.

In particular, in the case that the polynomial p is homogeneous of degree 1, the reduction to a one-dimensional integral can be achieved by applying Lemma 5.16. If the polynomial p is homogeneous of degree 2, it can be represented in the form

$$p(u) = u^T A_s(t) u \quad \text{for all } u \in \mathbb{R}^2,$$

where $A_s(t)$ is the approximation matrix of s for f in $t \in \mathbb{R}^2$. By an apt variable substitution of u, the matrix A_s can be diagonalized, which produces two different cases. In the elliptic case, A_s is either positive or negative semidefinite and the integral (1) can be transformed into a one-dimensional integral via Lemma 5.16 and Lemma 5.17. In the hyperbolic case, A_s is indefinite. For the integral, Lemma 5.18 exploits the hyperbolic restrictiveness to yield the wanted asymptotic.

If p is homogeneous of degree 3 or higher, the decay property can be achieved as in Lemma 4.42 by utilizing the restrictiveness.

Proof of Lemma 5.16. This proof consists of two parts. First we show that $\Phi_\ell \in \mathcal{S}(\mathbb{R})$ by representing the derivatives $\Phi_\ell^{(m)}$ as integrals over apt Schwartz functions. We subsequently employ their rapid decay rate to prove that the derivatives $\Phi_\ell^{(m)}$ exhibit a sufficient decay rate to ensure

that $\Phi_\ell \in \mathcal{S}(\mathbb{R})$. In the second part, we prove that $\Phi_\ell(v) = \mathcal{O}\left(v^{(\ell+1-n)_+}\right)$ for $v \to 0$ by showing a connection between $\Phi_\ell^{(m)}(0)$ and integrals of the form $\int_0^\infty \varphi(t)t^m \, dt$. By exploiting the vanishing moments of φ, we can prove that $\Phi_\ell^{(m)}(0) = 0$ for $m \le \ell - n$.

As $\varphi \in \mathcal{S}(\mathbb{R})$ is even, there exists a function $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\varphi(t) = \psi(t^2) \quad \text{for all } t \in \mathbb{R}.$$

Let

$$h : \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto \psi(\|x\|^2).$$

Since every derivative $D^\beta h$ is a sum of derivatives of ψ multiplied by a polynomial of x, we obtain $h \in \mathcal{S}(\mathbb{R}^2)$.

Proof of $\Phi_\ell \in \mathcal{S}(\mathbb{R})$:

We get for Φ_ℓ that

$$\Phi_\ell(v) = \int_{\mathbb{R}} \varphi\left(\sqrt{\alpha^2(v - c \cdot u)^2 + u^2}\right) u^\ell \, du = \int_{\mathbb{R}} h\big(\alpha(v - c \cdot u), u\big) u^\ell \, du.$$

Since $h \in \mathcal{S}(\mathbb{R}^2)$, all derivatives of $h\big(\alpha(v - c \cdot u), u\big) u^\ell$ with respect to v are integrable with respect to u. We thus obtain

$$\Phi_\ell^{(m)}(v) = \frac{d^m}{dv^m} \int_{\mathbb{R}} h\big(\alpha(v - c \cdot u), u\big) u^\ell \, du = \int_{\mathbb{R}} \alpha^m \cdot \partial_1^m h\big(\alpha(v - c \cdot u), u\big) u^\ell \, du.$$

We now introduce the function

$$\tilde{h} : \mathbb{R}^2, \quad (t, x) \mapsto \alpha^m \partial_1^m h(\alpha \cdot t, x) x^\ell.$$

As the Schwartz space is invariant under multiplication with constants, dilation, derivatives and multiplication with polynomials, $\tilde{h} \in \mathcal{S}(\mathbb{R}^2)$. Consequently, for all $N \in \mathbb{N}$ there exists $C > 0$ such that

$$\left|\Phi_\ell^{(m)}(v)\right| = \left|\int_{\mathbb{R}} \tilde{h}(v - c \cdot u, u) \, du\right| \le \int_{\mathbb{R}} \frac{1}{1 + \left[(v - c \cdot u)^2 + \|u\|^2\right]^N} \, du.$$

The upper estimation yields

$$
\begin{aligned}
|\Phi_\ell^{(m)}(v)| &\le \int_{\mathbb{R}} \frac{1}{1 + \left[(v - c \cdot u)^2 + u^2\right]^N} \, du \\
&= \int_{\mathbb{R}} \frac{1}{1 + \left[(1 + c^2) \cdot u^2 - 2cuv + v^2\right]^N} \, du \\
&= \int_{\mathbb{R}} \frac{1}{1 + \left[(1 + c^2) \cdot \left(u - \frac{c}{1+c^2} \cdot v\right)^2 + \frac{v^2}{1+c^2}\right]^N} \, du.
\end{aligned}
$$

By introducing the variables $a = \frac{1}{1+c^2}$ and $b = \frac{c}{1+c^2}$, we obtain

$$|\Phi_\ell^{(m)}(v)| = \int_{\mathbb{R}} \left[1 + \left(a(u-bv)^2 + \frac{v^2}{a}\right)^N\right]^{-1} du.$$

Utilizing $(y+z)^N \geq y^N + z^N$ for all $y, z \geq 0$, we obtain for all $m \geq 1$ that

$$|\Phi_\ell^{(m)}(v)| \leq \int_{\mathbb{R}} \left[1 + \left(a(u-bv)^2 + \frac{v^2}{a}\right)^N\right]^{-1} du \quad \text{(substituting } u = w + bv\text{)}$$

$$= \tilde{C} \cdot \int_{\mathbb{R}} \left[1 + \left(aw^2 + \frac{v^2}{a}\right)^N\right]^{-1} dw$$

$$\leq \tilde{C} \cdot \int_{\mathbb{R}} \left[1 + a^N w^{2N} + \frac{v^{2N}}{a^N}\right]^{-1} dw \quad \left(\text{substituting } w = \sqrt[2N]{1 + \frac{v^{2N}}{a^N}} \cdot y\right)$$

$$= \tilde{C} \cdot \left[1 + \frac{v^{2N}}{a^N}\right]^{\frac{1}{2N}-1} \cdot \underbrace{\int_{\mathbb{R}} \left(1 + a^N y^{2N}\right)^{-1} dy}_{<\infty}$$

$$= C'' \cdot \left(1 + v^{2N}\right)^{\frac{1}{2N}-1},$$

as $a \leq 1$. With the estimate

$$\left(1 + v^{2N}\right)^{\frac{1}{2N}} \leq 2\max\{1, |v|\} \quad \text{for all } v \in \mathbb{R},$$

we obtain the following upper bound for $|\Phi_\ell^{(m)}(v)|$ for all $v \in \mathbb{R}$:

$$|\Phi_\ell^{(m)}(v)| \leq C \cdot \left(1 + v^{2N}\right)^{\frac{1}{2N}-1}$$

$$\leq \frac{2C\max\{1, |v|\}}{1 + v^{2N}} \leq \frac{4C}{1 + |v|^{2N-1}}.$$

As the upper estimate holds for any $m, N \in \mathbb{N}$, we obtain that $\Phi_\ell \in \mathcal{S}(\mathbb{R})$.

Proof of $\Phi_\ell(v) = \mathcal{O}\left(v^{(\ell+1-n)_+}\right)$ for $v \to 0$:

In order to prove the upper asymptotic relation, we first represent the derivatives of Φ_ℓ in terms of the function ψ:

$$\Phi'_\ell(v) = \frac{d}{dv} \int_{\mathbb{R}} \psi\left(\alpha^2 (v - c \cdot u)^2 + u^2\right) u^\ell \, du$$

$$= \int_{\mathbb{R}} \frac{d}{dv} \psi\left(\alpha^2 (v - c \cdot u)^2 + u^2\right) u^\ell \, du$$

$$= 2\alpha^2 \int_{\mathbb{R}} \psi'\left(\alpha^2 (v - c \cdot u)^2 + u^2\right) \cdot (v - c \cdot u) u^\ell \, du. \tag{2}$$

We will now show inductively that for all $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $m \in \mathbb{N}_0$,

$$\partial_v^m \psi \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) = \sum_{k=\lceil m/2 \rceil}^m c_{k,m} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \cdot \left(v - c \cdot u \right)^{2k-m} \tag{3}$$

for some constants $c_{k,m} \in \mathbb{R}$.

Proof of (3):

As we can see by (2), the statement is true for $m = 1$. For greater m we obtain

$$\partial_v^{m+1} \psi \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right)$$

$$= \partial_v \sum_{k=\lceil m/2 \rceil}^m c_{k,m} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \cdot \left(v - c \cdot u \right)^{2k-m}$$

$$= \sum_{k=\lceil m/2 \rceil}^m c_{k,m} \psi^{(k+1)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) 2\alpha^2 \left(v - c \cdot u \right)^{2k+1-m}$$

$$+ \sum_{k=\lceil m/2 \rceil}^m c_{k,m} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \underbrace{\left(2k - m \right)}_{=0 \text{ for } k=m/2} \left(v - c \cdot u \right)^{2k-m-1}.$$

Since the summand in the second sum vanishes if $k = m/2$, for even m the sum starts with the index $\lceil m/2 \rceil + 1 = \lceil (m+1)/2 \rceil$ and for odd m with the starting index is $\lceil m/2 \rceil = \lceil (m+1)/2 \rceil$. Thus, we obtain

$$\partial_v^{m+1} \psi \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right)$$

$$= \sum_{k=\lceil m/2 \rceil+1}^{m+1} 2\alpha^2 c_{k-1,m} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \left(v - c \cdot u \right)^{2k-(m+1)}$$

$$+ \sum_{k=\lceil (m+1)/2 \rceil}^m \left(2k - m \right) c_{k,m} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \left(v - c \cdot u \right)^{2k-(m+1)}$$

$$= \sum_{k=\lceil (m+1)/2 \rceil}^{m+1} c_{k,m+1} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \left(v - c \cdot u \right)^{2k-(m+1)}.$$

Consequently, we get for all $m \in \mathbb{N}_0, m \geq 1$:

$$\Phi_\ell^{(m)}(v) = \partial_v^m \int_{\mathbb{R}} \psi \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) u^\ell \, du$$

$$= \sum_{k=\lceil m/2 \rceil}^m c_{k,m} \int_{\mathbb{R}} \psi^{(k)} \left(\alpha^2 \left(v - c \cdot u \right)^2 + u^2 \right) \left(v - c \cdot u \right)^{2k-m} u^\ell \, du. \tag{4}$$

In order to show that $\Phi_\ell(v) = \mathcal{O}\left(v^{(\ell+1-n)_+}\right)$ for $v \to 0$, we will show that all summands in (4) vanish for $v = 0$ and thus that $\Phi_\ell^{(m)}(0) = 0$ if $m \leq \ell - n$. We obtain

$$
\int_{\mathbb{R}} \psi^{(k)}\left(\alpha^2 (v - c \cdot u)^2 + u^2\right) (v - c \cdot u)^{2k-m} u^\ell \, du \Big|_{v=0}
$$

$$
= (-c)^{2k-m} \cdot \int_{\mathbb{R}} \psi^{(k)}\left((\alpha^2 c^2 + 1) \cdot u^2\right) u^{\ell+2k-m} \, du \qquad \left(\text{substitute } u = \left(\alpha^2 c^2 + 1\right)^{-\frac{1}{2}} t\right)
$$

$$
= \frac{(-c)^{2k-m}}{\left(\alpha^2 c^2 + 1\right)^{\frac{\ell+1-m}{2}+k}} \cdot \int_{\mathbb{R}} \psi^{(k)}\left(t^2\right) t^{\ell+2k-m} \, dt.
$$

Applying k partial integrations yields

$$
\int_{\mathbb{R}} \psi^{(k)}\left(\alpha^2 (v - c \cdot u)^2 + u^2\right) (v - c \cdot u)^{2k-m} u^\ell \, du \Big|_{v=0}
$$

$$
= C \cdot \int_{\mathbb{R}} \psi^{(k)}\left(t^2\right) t^{\ell+2k-m} \, dt
$$

$$
= C' \cdot \int_{\mathbb{R}} \psi\left(t^2\right) \cdot t^{\ell-m} \, dt = C' \cdot \int_{\mathbb{R}} \varphi(t) t^{\ell-m} \, dt
$$

if $\ell \geq m$. As every summand in (4) can be transformed into the upper form, there exists $\tilde{C} \in \mathbb{R}$ such that

$$
\Phi_\ell^{(m)}(0) = \tilde{C} \cdot \int_{\mathbb{R}} \varphi(t) t^{\ell-m} \, dt.
$$

Since $\int_0^\infty \varphi(t) t^p \, dt = 0$ for all $p \in \mathbb{N}_0$, $p \geq n$, we obtain that $\Phi_\ell^{(m)}(0) = 0$ if $\ell - m \geq n$, i.e. $m \leq \ell - n$. As $\Phi_\ell \in \mathcal{S}(\mathbb{R})$, it is especially continuous and thus

$$
\Phi_\ell(v) = \mathcal{O}\left(v^{(\ell+1-n)_+}\right) \quad \text{for } v \to 0.
$$

$\square$

Proof of Lemma 5.17. In this proof we will first reduce the integral in question to a one-dimensional integral of the form

$$
\int_0^\infty g\left(z \cdot \rho^2\right) \Phi_m(\rho) \rho \, d\rho.
$$

We will subsequently apply the Parseval-Hankel theorem and prove the sought-after decay rate of the upper integral by showing that the vanishing moments of second order result in a high decay rate under the Hankel transform.

The substitution $v_i = a_i w_i$ for $i \in \{1, 2\}$ yields

$$
\int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi\left(\|v\|\right) \cdot v^\gamma \, dv
$$

$$
= a_1^{\gamma_1+1} a_2^{\gamma_2+1} \int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \|w\|^2\right) \cdot \varphi\left(\sqrt{a_1^2 w_1^2 + a_2^2 w_2^2}\right) w^\gamma \, dw.
$$

By transforming $w \in \mathbb{R}^2$ to polar coordinates, we obtain

$$\int_{\mathbb{R}^2} g^{(m)} \left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2} \right) \right) \cdot \varphi(\|v\|) \cdot v^\gamma \, dv$$

$$= a_1^{\gamma_1+1} a_2^{\gamma_2+1} \int_0^{2\pi} \int_0^\infty g^{(m)} \left(z\rho^2 \right) \varphi \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \rho^{|\gamma|} \cos^{\gamma_1}(\theta) \sin^{\gamma_2}(\theta) \rho \, d\rho \, d\theta$$

$$= a_1^{\gamma_1+1} a_2^{\gamma_2+1} \int_0^\infty g^{(m)} \left(z\rho^2 \right) \rho^{|\gamma|+1} \cdot \underbrace{\int_0^{2\pi} \varphi \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \cos^{\gamma_1}(\theta) \sin^{\gamma_2}(\theta) \, d\theta}_{=:\Phi(\rho)} \, d\rho.$$

We will now show that $\Phi \in \mathcal{S}(\mathbb{R})$. To this end, we observe that we can estimate its kth derivative in the following way:

$$\left| \Phi^{(k)}(\rho) \right| = \left| \frac{\partial^k}{\partial \rho^k} \int_0^{2\pi} \varphi \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \cos^{\gamma_1}(\theta) \sin^{\gamma_2}(\theta) \, d\theta \right|$$

$$\leq \int_0^{2\pi} \left| \varphi^{(k)} \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \right| \underbrace{\left(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta \right)^{k/2}}_{\leq \max\{a_1, a_2\}^k} \cdot \underbrace{\left| \cos^{\gamma_1}(\theta) \right| \left| \sin^{\gamma_2}(\theta) \right|}_{\leq 1} \, d\theta$$

$$\leq \max\{a_1, a_2\}^k \cdot \int_0^{2\pi} \left| \varphi^{(k)} \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \right| \, d\theta,$$

since $a_1, a_2 > 0$ by prerequisite. As $\varphi \in \mathcal{S}(\mathbb{R})$, for all $k, N \in \mathbb{N}$ there exists $C_{k,N} > 0$ such that

$$|\varphi^{(k)}(\rho)| \leq \frac{C_{k,N}}{\rho^N} \quad \text{for all } \rho \in \mathbb{R}.$$

By applying this inequality, we obtain

$$\left| \Phi^{(k)}(\rho) \right| \leq \max\{a_1, a_2\}^k \cdot \int_0^{2\pi} \left| \varphi^{(k)} \left(\rho \sqrt{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \right) \right| \, d\theta$$

$$\leq \max\{a_1, a_2\}^k \cdot \int_0^{2\pi} C_{k,N} |\rho|^{-N} \cdot \underbrace{\left(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta \right)^{-N/2}}_{\leq \min\{a_1, a_2\}^{-N}} \, d\theta$$

$$\leq 2\pi C_{k,N} \cdot \frac{\max\{a_1, a_2\}^k}{\min\{a_1, a_2\}^N} \cdot |\rho|^{-N} \quad \text{for all } \rho \in \mathbb{R}.$$

Thus, $\Phi \in \mathcal{S}(\mathbb{R})$ and inherits its even symmetry from φ.

We now observe that the integral we want to estimate,

$$\int_{\mathbb{R}^2} g^{(m)} \left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2} \right) \right) \cdot \varphi(\|v\|) \cdot v^\gamma \, dv,$$

vanishes, if a component of γ is odd. Hence, it is sufficient to consider only the case that $|\gamma|$

is even. So there exists $\ell \in \mathbb{N}_0$ such that $|\gamma| = 2\ell$. Then, a consecutive application of partial integration yields

$$\int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi\left(\|v\|\right) \cdot v^\gamma \, dv$$

$$= \int_0^\infty g^{(m)}\left(z \cdot \rho^2\right) \cdot \rho^{2\ell+1} \cdot \Phi(\rho) \, d\rho$$

$$= \int_0^\infty 2\rho z \cdot g^{(m)}\left(z \cdot \rho^2\right) \cdot \frac{\rho^{2\ell+1}}{2z\rho} \cdot \Phi(\rho) \, d\rho$$

$$= -\int_0^\infty g^{(m-1)}\left(z \cdot \rho^2\right) \cdot \frac{d}{d\rho} \frac{\rho^{2\ell}}{2z} \cdot \Phi(\rho) \, d\rho$$

$$= \frac{(-1)^m}{2^m \cdot z^m} \cdot \int_0^\infty g\left(z \cdot \rho^2\right) \cdot \underbrace{\left[\frac{1}{\rho} \frac{d}{d\rho}\right]^m \left(\rho^{2\ell} \cdot \Phi(\rho)\right)}_{=:\Phi_m(\rho)} \cdot \rho \, d\rho$$

As $|\gamma| \geq 2m - 1$ and thus $\ell \geq m$, and $\Phi \in \mathcal{S}(\mathbb{R})$, we obtain $\Phi_m \in \mathcal{S}(\mathbb{R})$ and $\Phi_m(\rho) = \mathcal{O}\left(\rho^{2\ell-2m}\right)$ for $\rho \to 0$. Furthermore, Φ_m is an even function, as the operator $\frac{1}{\rho} \frac{d}{d\rho}$ preserves the even symmetry of a function. We now introduce the function

$$g_{\pm,2} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto g\left(\pm t^2\right).$$

To obtain the asymptotic behavior of the integral $\int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi\left(\|v\|\right) \cdot v^\gamma \, dv$ for $z \to \pm\infty$, we apply the Hankel transform to $g_{\pm,2}$ and Φ_m. Due to statement 1. of Proposition 5.6,

$$\mathcal{H}_0\left(g_{\pm,2}\left(|z|^{\frac{1}{2}}\rho\right)\right)(r) = |z|^{-1} \mathcal{H}_0\left(g_{\pm,2}\right)\left(|z|^{-1/2}r\right).$$

As $g \in \mathcal{S}(\mathbb{R})$ and $\Phi_m \in \mathcal{S}(\mathbb{R})$, statement 2. of Proposition 5.6, the Hankel equivalent of Parseval's theorem, yields

$$\int_{\mathbb{R}^2} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi\left(\|v\|\right) \cdot v^\gamma \, dv$$

$$= \frac{(-1)^m}{(2z)^m} \cdot \int_0^\infty g_{\text{sgn}(z),2}\left(|z|^{\frac{1}{2}} \cdot \rho\right) \cdot \Phi_m(\rho) \cdot \rho \, d\rho$$

$$= \frac{(-1)^m}{(2z)^m} \cdot \int_0^\infty \frac{1}{|z|} \mathcal{H}_0\left(g_{\text{sgn}(z),2}\right)\left(|z|^{-\frac{1}{2}} \cdot r\right) \cdot \mathcal{H}_0\Phi_m(r) \cdot r \, dr.$$

In order to retrieve the asymptotic behavior of the upper integral for $z \to \pm\infty$, we will show that

$$\mathcal{H}_0\left(g_{\text{sgn}(z),2}\right)(r) = \mathcal{O}\left(r^{2M}\right) \quad \text{for } r \to 0. \tag{5}$$

To this end, we observe that

$$\mathcal{H}_0\left(g_{\mathrm{sgn}(z),2}\right)^{(\nu)}(0) = \partial_r^\nu \int_0^\infty g_{\pm,2}\left(\rho\right) J_0(\rho r) \cdot \rho \; d\rho \Big|_{r=0}$$

$$= \int_0^\infty g_{\pm,2}\left(\rho\right) J_0^{(\nu)}(0) \cdot \rho^{\nu+1} \; d\rho = 0 \quad \text{for all } \nu \in \{0,\ldots,2M-2\},$$

as g has M vanishing moments of order 2. Furthermore, $\mathcal{H}_0\left(g_{\mathrm{sgn}(z),2}\right)^{(2M-1)}(0) = 0$, as the Bessel function J_0 is of even symmetry and thus $J_0^{(2M-1)}(0) = 0$.

Moreover, $\mathcal{H}_0\Phi_m \in \mathcal{S}(\mathbb{R})$ due to statement 4. of Proposition 5.6. Together with (5) this yields

$$\left| \int_{\mathbb{R}^k} g^{(m)}\left(z \cdot \left(\frac{v_1^2}{a_1^2} + \frac{v_2^2}{a_2^2}\right)\right) \cdot \varphi\left(\|v\|\right) \cdot v^\gamma \; dv \right|$$

$$\leq \frac{1}{2^m \cdot |z|^{m+1}} \cdot \int_0^\infty \left| \mathcal{H}_0\left(g_{\mathrm{sgn}(z),2}\right)\left(|z|^{-\frac{1}{2}} \cdot r\right) \right| \cdot |\mathcal{H}_0\Phi_m(r)| \cdot r \; dr$$

$$\leq \frac{1}{2^m \cdot |z|^{m+1}} \cdot \int_0^\infty C \cdot |z|^{-\frac{1}{2}(2M)} r^{2M} \cdot \frac{C'}{1+r^{2M+3}} \cdot r \; dr$$

$$= \frac{1}{2^m \cdot |z|^{M+m+1}} \cdot \underbrace{\int_0^\infty \frac{CC' r^{2M+1}}{1+r^{2M+3}} \; dr}_{<\infty} = \mathcal{O}\left(|z|^{-(M+m+1)}\right) \quad \text{for } z \to \pm\infty.$$

$\square$

Proof of Lemma 5.18. We prove this statement in three steps. First, we show that the integral in question can be represented in the form

$$\int_0^\infty \int_0^\infty g\left(z \cdot (v_1^2 - v_2^2)\right) h\left(v\right) \; dv = \int_0^\infty \left[g(zr^2) + g(-zr^2)\right] \Phi(r) \; r \; dr$$

for a function $\Phi : \mathbb{R}_+ \to \mathbb{R}$. Subsequently, we will prove the asymptotic relation

$$\Phi(r) = -h(0) \cdot \log r + \mathcal{O}(1), \quad \text{for } t \to 0.$$

In the third step, we show the asymptotic equivalence

$$\int_0^\infty g(zr^2)\Phi(r) \; r \; dr \sim -h(0) \cdot \int_0^\infty g(zr^2)\log(r) \; r \; dr \quad \text{for } z \to \pm\infty$$

and thus the desired decay result.

STEP 1

We will first divide the integral into two parts:

$$\int_0^\infty \int_0^\infty g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv$$
$$= \int_0^\infty \int_0^{v_1} g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv_2 \ dv_1 + \int_0^\infty \int_{v_1}^\infty g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv_2 \ dv_1.$$

By performing the substitution $v_1 = r\cosh\theta$ and $v_2 = r\sinh\theta$, $dv = r \ dr \ d\theta$ on the first integral, we obtain

$$\int_0^\infty \int_0^{v_1} g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv_2 \ dv_1 = \int_0^\infty \int_0^\infty g(z \cdot r^2) h\left(r \cdot E_\theta^+\right) r \ dr \ d\theta$$

and by substituting $v_1 = r\sinh\theta$ and $v_2 = r\cosh\theta$, $dv = r \ dr \ d\theta$ in the second integral, we get

$$\int_0^\infty \int_{v_1}^\infty g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv_2 \ dv_1 = \int_0^\infty \int_0^\infty g(-z \cdot r^2) h\left(r \cdot E_\theta^-\right) r \ dr \ d\theta,$$

where $E_\theta^+ = \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}$ and $E_\theta^- = \begin{pmatrix} \sinh\theta \\ \cosh\theta \end{pmatrix}$ for all $\theta \in \mathbb{R}$. As h is rotationally symmetric, we have $h(r \cdot E_\theta^+) = h(r \cdot E_\theta^-)$ for all $r > 0$ and $\theta \in \mathbb{R}$. So it suffices to only consider E_θ^+. Since

$$\left| g(z \cdot r^2) h\left(r \cdot E_\theta^+\right) r \right| \leq \frac{C |g(z \cdot r^2)|}{\|E_\theta^+\|} \in L^1\left(\mathbb{R}_+ \times \mathbb{R}_+, dr \ d\theta\right),$$

we can apply Fubini's theorem and first compute the integral with respect to θ:

$$\Phi(r) = \int_0^\infty h\left(r \cdot E_\theta^+\right) \ d\theta.$$

This leads to the formula

$$\int_0^\infty \int_0^\infty g\left(z \cdot (v_1^2 - v_2^2)\right) h(v) \ dv = \int_0^\infty \left[g(zr^2) + g(-zr^2)\right] \Phi(r) \ r \ dr. \tag{6}$$

STEP 2

In the second step, we will determine the asymptotic behavior of $\Phi(r)$ for $r \to 0$.

In order to obtain the behavior of $\Phi(r)$ for $r \to 0$, we apply the substitution $\theta = t - \log r$ to the defining integral:

$$\Phi(r) = \int_0^\infty h\left(r \cdot E_\theta^+\right) \, d\theta$$

$$= \int_{\log r}^\infty h\left(\begin{array}{c} r \cdot \frac{1}{2}\left(e^{t-\log r} + e^{\log r - t}\right) \\ r \cdot \frac{1}{2}\left(e^{t-\log r} - e^{\log r - t}\right) \end{array}\right) \, dt$$

$$= \int_{\log r}^\infty h\left(\begin{array}{c} \frac{1}{2}\left(e^t + r^2 \cdot e^{-t}\right) \\ \frac{1}{2}\left(e^t - r^2 \cdot e^{-t}\right) \end{array}\right) \, dt.$$

By introducing the function

$$T_r : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto \left(\begin{array}{c} \frac{1}{2}\left(e^t + r^2 \cdot e^{-t}\right) \\ \frac{1}{2}\left(e^t - r^2 \cdot e^{-t}\right) \end{array}\right)$$

and applying partial integration, we get

$$\Phi(r) = \int_{\log r}^\infty h\left(T_r(t)\right) \, dt$$

$$= \left[h\left(T_r(t)\right) \cdot t\right]_{t=\log r}^{t=\infty} - \int_{\log r}^\infty \nabla h\left(T_r(t)\right) \cdot T_r'(t) \cdot t \, dt$$

$$= -h\left(T_r(\log r)\right) \cdot \log r - \int_{\log r}^\infty \nabla h\left(T_r(t)\right) \cdot T_r'(t) \cdot t \, dt. \tag{7}$$

For the first summand we obtain

$$-h\left(T_r(\log r)\right) \cdot \log r = -h\left(\begin{array}{c} \frac{1}{2}\left(e^{\log r} + r^2 \cdot e^{-\log r}\right) \\ \frac{1}{2}\left(e^{\log r} - r^2 \cdot e^{-\log r}\right) \end{array}\right) \cdot \log r = -h\left(\begin{array}{c} r \\ 0 \end{array}\right) \cdot \log r.$$

This yields the asymptotic

$$-h\left(T_r(\log r)\right) \cdot \log r = -h(0) \cdot \log r + \mathcal{O}(r \log r), \quad \text{for } r \to 0. \tag{8}$$

In order to get the asymptotic of the second summand in (7), we divide the field of integration into $[\log r, 0]$ and $[0, \infty)$. By utilizing the Cauchy-Schwarz inequality and the decay of $\|\nabla h\|$, we obtain for the integral over $[0, \infty)$:

$$\left| \int_0^\infty \nabla h\left(T_r(t)\right) \cdot T_r'(t) \cdot t \, dt \right| \leq \int_0^\infty \frac{C}{\|T_r(t)\|^3} \cdot \|T_r'(t)\| \cdot t \, dt.$$

For all $r > 0$, $t \in \mathbb{R}$ we have

$$\|T_r'(t)\|^2 = \left\| \begin{pmatrix} \frac{1}{2}\left(e^t - r^2 \cdot e^{-t}\right) \\ \frac{1}{2}\left(e^t + r^2 \cdot e^{-t}\right) \end{pmatrix} \right\|^2 = \frac{1}{2}\left(e^{2t} + r^4 e^{-2t}\right) = \left\| \begin{pmatrix} \frac{1}{2}\left(e^t + r^2 \cdot e^{-t}\right) \\ \frac{1}{2}\left(e^t - r^2 \cdot e^{-t}\right) \end{pmatrix} \right\|^2 = \|T_r(t)\|^2. \qquad (9)$$

Hence, we obtain

$$\left| \int_0^\infty \nabla h\left(T_r(t)\right) \cdot T_r'(t) \cdot t\, dt \right| \le \int_0^\infty \frac{C \cdot \|T_r'(t)\| \cdot t}{\|T_r(t)\|^3}\, dt \overset{(9)}{=} \int_0^\infty \frac{Ct}{\|T_r(t)\|^2}\, dt \overset{(9)}{\le} \int_0^\infty \frac{Ct}{\frac{1}{2} \cdot e^{2t}}\, dt < \infty.$$

For the integral over $[\log r, 0]$ and for $r \in (0, 1)$, we observe that

$$\|T_r'(t)\| \le e^t \Leftrightarrow \sqrt{\frac{1}{2}\left(e^{2t} + r^4 e^{-2t}\right)} \le e^t \Leftrightarrow r^4 e^{-2t} \le e^{2t} \Leftrightarrow \log r \le t.$$

We thus get

$$\left| \int_{\log r}^0 \nabla h\left(T_r(t)\right) \cdot T_r'(t) \cdot t\, dt \right| \le \int_{\log r}^0 \underbrace{\|\nabla h\left(T_r(t)\right)\|}_{\le C} \cdot \underbrace{\|T_r'(t)\|}_{\le e^t} \cdot |t|\, dt$$

$$\le -C \cdot \int_{\log r}^0 t e^t\, dt$$

$$= \left[-C \cdot (t-1)e^t \right]_{t=\log r}^{t=0}$$

$$= C - C \cdot r(1 - \log r) = \mathcal{O}(1) \quad \text{for } r \to 0.$$

In conclusion we have

$$\Phi(r) = -h(0)\log r + \mathcal{O}(1), \quad \text{for } r \to 0. \qquad (10)$$

STEP 3

In the final step, we determine the asymptotic behavior of the integral

$$\int_0^\infty \int_0^\infty g\left(z \cdot \left(v_1^2 - v_2^2\right)\right) h(v)\, dv$$

for $z \to \pm\infty$.

To this end, we will show that in order to determine the asymptotic behavior of the integral in question, it is sufficient to consider the integral only over an arbitrarily small neighborhood of the origin. First, we prove that Φ is bounded away from the origin. For $r > 0$ and the vector

$E_\theta^+ = \begin{pmatrix} \cosh\theta \\ \sinh\theta \end{pmatrix}$ and, we can obtain the following upper bound:

$$
\begin{aligned}
|\Phi(r)| &= \left| \int_0^\infty h\left(r \cdot E_\theta^+\right) \, d\theta \right| \\
&= \int_0^\infty \left| h\left(r \cdot E_\theta^+\right) \right| \, d\theta \\
&\leq \int_0^\infty \frac{C}{r \| E_\theta^+ \|} \, d\theta \leq \frac{C'}{r}.
\end{aligned}
$$

Thus, for all $\varepsilon > 0$, Φ is bounded on every interval (ε, ∞). We now employ (6) and consider the integral over $(\varepsilon\infty)$.

$$
\begin{aligned}
\left| \int_\varepsilon^\infty g\left(zr^2\right) \Phi(r) r \, dr \right| &\leq \int_\varepsilon^\infty \left| g\left(zr^2\right) \right| \cdot |\Phi(r)| \cdot |r| \, dr \\
&\leq \int_\varepsilon^\infty \frac{Cr}{z^N r^{2N}} \, dr = \mathcal{O}\left(|z|^{-N}\right), \quad z \to \pm\infty
\end{aligned}
$$

for arbitrary $N > 0$. Hence,

$$
\int_0^\infty g\left(zr^2\right) \Phi(r) r \, dr = \int_0^\varepsilon g\left(zr^2\right) \Phi(r) r \, dr + \mathcal{O}\left(|z|^{-N}\right) \quad \text{for } z \to \pm\infty.
$$

The variable substitution $r = |z|^{-\frac{1}{2}} \rho$ yields

$$
\int_0^\infty g\left(zr^2\right) \Phi(r) r \, dr = |z|^{-1} \int_0^{\varepsilon\sqrt{|z|}} g\left(\pm\rho^2\right) \Phi\left(|z|^{-\frac{1}{2}}\rho\right) \rho \, d\rho + \mathcal{O}\left(|z|^{-N}\right) \quad \text{for } z \to \pm\infty. \tag{11}
$$

We will now show that the leading asymptotic term of $\Phi(r)$ for $r \to 0$ determines the behavior of the upper integral for $z \to \pm\infty$. Due to (10), the asymptotic relation

$$
\Phi(r) = -h(0) \cdot \log r + \mathcal{O}(1), \quad \text{for } r \to 0
$$

holds. By using similar arguments as for (11), we obtain that

$$
\int_0^\infty g\left(zr^2\right) \log(r) r \, dr = \int_0^\varepsilon g\left(zr^2\right) \log(r) r \, dr + \mathcal{O}\left(|z|^{-N}\right) \quad \text{for } z \to \pm\infty
$$

and thus

$$
\int_0^\infty g\left(\pm\rho^2\right) \log\left(|z|^{-\frac{1}{2}}\rho\right) \rho \, d\rho = \int_0^{\varepsilon\sqrt{|z|}} g\left(\pm\rho^2\right) \log\left(|z|^{-\frac{1}{2}}\rho\right) \rho \, d\rho + \mathcal{O}\left(|z|^{-N}\right) \quad \text{for } z \to \pm\infty.
$$

By using the leading asymptotic of Φ instead of Φ, we get

$$\lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\pm\rho^2\right) \cdot \left(-h(0) \cdot \log\left(|z|^{-\frac{1}{2}}\rho\right)\right) \rho \; d\rho$$

$$= -h(0) \cdot \lim_{z \to \pm\infty} \int_0^{\infty} g\left(\pm\rho^2\right) \log\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho$$

$$= -h(0) \cdot \lim_{z \to \pm\infty} \left[\int_0^{\infty} g\left(\pm\rho^2\right) \log(\rho) \rho \; d\rho - \frac{1}{2} \cdot \log|z| \cdot \underbrace{\int_0^{\infty} g\left(\rho^2\right) \rho \; d\rho}_{=0} \right]$$

$$= -h(0) \cdot \int_0^{\infty} g\left(\pm\rho^2\right) \log(\rho) \rho \; d\rho \neq 0, \tag{12}$$

as $h(0) \neq 0$ and $\int_0^{\infty} g\left(\pm\rho^2\right) \log(\rho) \rho \; d\rho \neq 0$ according to the prerequisites. It now remains to show that indeed

$$\lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\pm\rho^2\right) \cdot \left(-h(0) \cdot \log\left(|z|^{-\frac{1}{2}}\rho\right)\right) \rho \; d\rho = \lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\pm\rho^2\right) \cdot \Phi\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho.$$

To this end, we introduce the function

$$\tilde{\Phi}(r) := \Phi(r) + h(0) \cdot \log r \quad \text{for all } r > 0.$$

Due to (10) it fulfills $\tilde{\Phi}(r) = \mathcal{O}(1)$ for $r \to 0$ and is thus bounded on $(0, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Since the expression

$$\int_0^{\varepsilon\sqrt{|z|}} g\left(\rho^2\right) \log\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho$$

converges for $z \to \pm\infty$ according to (12), the statement

$$\lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\rho^2\right) \Phi\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho = -h(0) \cdot \lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\rho^2\right) \log\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho$$

is equivalent to

$$\lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\rho^2\right) \tilde{\Phi}\left(|z|^{-\frac{1}{2}}\rho\right) \rho \; d\rho = 0.$$

We now proceed by proving this statement for $\varepsilon > 0$ sufficiently small to ensure that $\tilde{\Phi}$ is bounded on $(0, \varepsilon)$. Since, additionally, $g \in \mathcal{S}(\mathbb{R})$, we can see that the integrand has an integrable uniform upper bound:

$$\left| g\left(\rho^2\right) \tilde{\Phi}\left(|z|^{-\frac{1}{2}}\rho\right) \rho \cdot \mathbb{1}_{(0,\varepsilon\sqrt{|z|})}(\rho) \right| \leq \left| g(\rho^2)\rho \right| \cdot \sup_{r \in (0,\varepsilon)} \left| \tilde{\Phi}(r) \right| \quad \text{for all } \rho > 0, z \in \mathbb{R} \setminus \{0\}.$$

We hence may interchange limit and integral and get

$$\lim_{z \to \pm\infty} \int_0^{\varepsilon\sqrt{|z|}} g\left(\rho^2\right) \cdot \tilde{\Phi}\left(|z|^{-\frac{1}{2}}\rho\right) \rho \, d\rho$$
$$= \tilde{\Phi}\left(0\right) \cdot \int_0^\infty g\left(\rho^2\right) \cdot \rho \, d\rho = 0. \tag{13}$$

By combining (11), (12) and (13), we obtain the desired asymptotic result.

$$\square$$

Proof of Lemma 5.19. This proof is divided into four steps. We will exploit the fact that $\tau \in \mathcal{S}\left(\mathbb{R}^2\right)$ in order to restrict (Step 1) or extend (Step 4) the area of integration of the Taylorlet transform according to Lemma 4.39 and utilize Taylor's theorem in order to obtain truncated series of τ (Step 2) and q (Step 3), respectively.

STEP 1

In the first step we will show that the Taylorlet transform $\mathcal{T}^{(2,\alpha)} f(a, s, 0)$ is an integral over a hyper-surface and we will prove that only a small neighborhood of the origin is relevant for the decay of the Taylorlet transform for $a \to 0$.

In order to properly treat the asymptotic behavior of $\mathcal{T} f(a, s, 0)$, we introduce the function

$$q_s : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto q(u) - \sum_{|\beta| \le 2} \frac{s_\beta}{\beta!} \cdot u^\beta.$$

As $q \in C^\infty\left(\mathbb{R}^2\right)$, we have $q_s \in C^\infty\left(\mathbb{R}^2\right)$. Hence, there exists $\tilde{q} \in C^\infty\left(\mathbb{R}^2, \mathbb{R}^2\right)$ such that

$$q_s(u) = q_s(0) + \tilde{q}(u) \cdot u \quad \text{for all } u \in \mathbb{R}^2. \tag{14}$$

Furthermore, there exists a function $Q \in C^\infty\left(\mathbb{R}^2, \mathbb{R}^{2\times2}\right)$ such that

$$q_s(u) = q_s(0) + \nabla q_s(0) \cdot u + \frac{1}{2} \cdot u^T \cdot Q(u) \cdot u \quad \text{for all } u \in \mathbb{R}^2. \tag{15}$$

If 0 is the highest approximation order of s for f in 0, (14) yields

$$q_s(u) = \underbrace{q_s(0)}_{=0} + \tilde{q}(u) \cdot u = \tilde{q}_1(u) \cdot u_1 + \tilde{q}_2(u) \cdot u_2 \quad \text{for all } u \in \mathbb{R}^2,$$

where

$$\tilde{q}(0) = \nabla q_s(0).$$

If 1 is the highest approximation order of s for f in 0, (15) yields

$$q_s(u) = \underbrace{q_s(0)}_{=0} + \underbrace{\nabla q_s(0)}_{=0} \cdot u + \frac{1}{2} \cdot u^T \cdot Q(u) \cdot u = \frac{1}{2} \cdot \sum_{i=1}^{2} \sum_{j=1}^{2} Q_{ij}(u) \cdot u_i \cdot u_j \quad \text{for all } u \in \mathbb{R}^2,$$

where $\quad Q(0) = Hq_s(0).$ (16)

Analogously, for $k \geq 2$ being the highest approximation order of s for f in 0, there exist functions $q_\beta \in C^\infty(\mathbb{R})$ for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = k+1$ such that

$$q_s(u) = \sum_{|\beta|=k+1} q_\beta(u) \cdot u^\beta \quad \text{for all } u \in \mathbb{R}^2.$$

First, we rewrite the Taylorlet transform:

$$\mathcal{T}^{(2,\alpha)} f(a,s,0) = \int_{\mathbb{R}^2} \tau \begin{pmatrix} q_s(u)/a \\ u/a^\alpha \end{pmatrix} du$$

$$= \int_{\mathbb{R}^2} \tau \begin{pmatrix} \sum\limits_{|\beta|=k+1} q_\beta(u) \cdot u^\beta /a \\ u/a^\alpha \end{pmatrix} du,$$ (17)

Since k is the highest approximation order of s for f in $t = 0$, there exists $\beta \in \mathbb{N}_0^2$ with $|\beta| = k+1$ such that $q_\beta(0) \neq 0$.

In order to show that just a small neighborhood of the origin is responsible for the decay of the Taylorlet transform for $a \to 0$, we observe that the integrand in (17) fulfills the decay condition of Lemma 4.39. By applying this lemma, we obtain for $\eta \in \left(0, \frac{1}{k+1}\right)$ and an arbitrary $N \in \mathbb{N}$ that

$$|\mathcal{T}^{(2,\alpha)} f(a,s,0)| = \left| \int_{B_{a^\eta}} \tau \begin{pmatrix} \sum\limits_{|\beta|=k+1} q_\beta(u) \cdot u^\beta /a \\ u/a^\alpha \end{pmatrix} du \right| + \mathcal{O}(a^N) \quad \text{for } a \to 0.$$ (18)

STEP 2

If we replaced the terms $q_\beta(u)$ in the argument of the integrand by some constants c_β, the first argument would be a homogeneous polynomial of order $k+1$ w.r.t. u. In order to achieve this form, we will approximate the integrand by a Taylor polynomial in this step.

Now we expand the integrand of (18) into a Taylor series with respect to the first component in a neighborhood of the point $\sum\limits_{|\beta|=k+1} q_\beta(0) \cdot u^\beta / a$.

$$
|\mathcal{T}^{(2,\alpha)} f(a,s,0)| = \left| \int_{B_{a^\eta}} \tau \begin{pmatrix} \sum\limits_{|\beta|=k+1} q_\beta(u) \cdot u^\beta / a \\ u/a^\alpha \end{pmatrix} du \right| + \mathcal{O}(a^N)
$$

$$
\leq \sum_{m=0}^{J} \left| \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} \sum\limits_{|\beta|=k+1} q_\beta(0) u^\beta / a \\ u/a^\alpha \end{pmatrix} \cdot \frac{1}{a^m \cdot m!} \left(\sum_{|\beta|=k+1} [q_\beta(u) - q_\beta(0)] \cdot u^\beta \right)^m du \right|
$$

$$
+ \frac{\left\| \partial_1^{J+1} \tau \right\|_\infty}{(J+1)!} \cdot \int_{B_{a^\eta}} \frac{1}{a^{J+1}} \left(\sum_{|\beta|=k+1} |q_\beta(u) - q_\beta(0)| \cdot |u^\beta| \right)^{J+1} du + \mathcal{O}(a^N). \tag{19}
$$

Due to Lemma 5.4, $|u^\beta| \leq \|u\|^{|\beta|}$ and since $q_\beta \in C^\infty(\mathbb{R}^2)$ for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = k+1$, there exists a constant $C > 0$ and $a_0 > 0$ such that for all $a < a_0$ and for all $\beta \in \mathbb{N}_0^2$ with $|\beta| = k+1$ we have

$$
\left| q_\beta(u) - q_\beta(0) \right| \leq C \cdot \|u\| \quad \text{for all } u \in B_{a^\eta}.
$$

Hence, we obtain for the rest term in (19):

$$
\frac{\left\| \partial_1^{J+1} \tau \right\|_\infty}{(J+1)!} \cdot \int_{B_{a^\eta}} \frac{1}{a^{J+1}} \cdot \left(\sum_{|\beta|=k+1} |q_\beta(u) - q_\beta(0)| \cdot |u^\beta| \right)^{J+1} du
$$

$$
\leq \frac{\left\| \partial_1^{J+1} \tau \right\|_\infty}{a^{J+1}(J+1)!} \cdot \int_{B_{a^\eta}} \left(\sum_{|\beta|=k+1} C\|u\| \cdot \|u\|^{k+1} \right)^{J+1} du
$$

$$
= \frac{C^{J+1} \left\| \partial_1^{J+1} \tau \right\|_\infty}{a^{J+1}(J+1)!} \cdot \binom{d+k}{k+1}^{J+1} \cdot \int_{B_{a^\eta}} \|u\|^{(k+2)(J+1)} du
$$

$$
= \frac{C'}{a^{J+1}} \cdot \int_0^{a^\eta} r^{(k+2)(J+1)} \cdot r \, dr
$$

$$
= \tilde{C} \cdot a^{\eta[(k+2)(J+1)+2]-(J+1)} \sim a^{(J+1)[(k+2)\eta-1]+2\eta} \quad \text{for } a \to 0.
$$

By choosing $\eta \in \left(\frac{1}{k+2}, \frac{1}{k+1} \right)$ and

$$
J = \left\lceil \frac{N - 2\eta}{\eta(k+2) - 1} \right\rceil - 1,
$$

we obtain the desired decay rate of $\mathcal{O}(a^N)$ for the remainder term in (19) for $a \to 0$.

STEP 3

In the third step we will expand $\sum\limits_{|\beta|=k+1} [q_\beta(u) - q_\beta(0)] \cdot u^\beta$ in a Taylor series about the point $u = 0$.

By introducing the homogeneous polynomial

$$p_{k+1} : \mathbb{R}^2 \to \mathbb{R}, \quad u \mapsto \sum_{|\beta|=k+1} q_\beta(0) \cdot u^\beta \tag{20}$$

of degree $k+1$, we obtain:

$$|\mathcal{T}^{(2,\alpha)} f(a,s,0)|$$

$$\leq \sum_{m=0}^{J} \left| \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \cdot \frac{1}{a^m \cdot m!} \left(\sum_{|\beta|=k+1} \left[q_\beta(u) - q_\beta(0) \right] \cdot u^\beta \right)^m du \right| + \mathcal{O}\left(a^N\right)$$

$$\leq \sum_{m=0}^{J} \left| \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m m!} \left(\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} + \rho_m(u) \right)^m du \right| + \mathcal{O}\left(a^N\right), \tag{21}$$

where ρ_m is the rest term of the Taylor series expansion with the property $\rho_m(u) = \mathcal{O}\left(\|u\|^{L_m+k+2}\right)$ for $u \to 0$. Now we estimate the summands for each $m \in \{0, \dots, J\}$.

$$\int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m \cdot m!} \left(\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} + \rho_m(u) \right)^m du$$

$$= \sum_{\nu=0}^{m} \binom{m}{\nu} \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m \cdot m!} \left(\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} \right)^{m-\nu} \left[\rho_m(u) \right]^\nu du.$$

Since $\tau \in \mathcal{S}(\mathbb{R}^2)$, $\rho_m(u) = \mathcal{O}(\|u\|^{L_m+k+2})$ for $u \to 0$ and $\sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} = \mathcal{O}(\|u\|)$ for $u \to 0$, for every $\nu \in \{1, \dots, m\}$ there exist $c_\nu, a_0 > 0$ such that for all $a < a_0$

$$\left| \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m m!} \binom{m}{\nu} [\rho_m(u)]^\nu \left[\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} \right]^{m-\nu} du \right|$$

$$\leq c_\nu \cdot a^{-m} \int_{B_{a^\eta}} \|u\|^{(k+2)(m-\nu)} \|u\|^{(L_m+k+2)\nu} du$$

$$= \mathcal{O}\left(a^{[(k+2)\eta-1]m+\eta(L_m+2)}\right) \quad \text{for } a \to 0. \tag{22}$$

As $\eta \in \left(\frac{1}{k+2}, \frac{1}{k+1}\right)$, the choice

$$L_m = \left\lceil \frac{N - \left[(k+2)\eta - 1\right] m}{\eta} \right\rceil - 2$$

is sufficient to obtain a decay rate of $\mathcal{O}(a^N)$ in (22). Hence, for all $m \in \{0,\dots,J\}$ we have

$$\int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m m!} \left(\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} + \rho_m(u) \right)^m du$$

$$= \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \frac{1}{a^m m!} \left(\sum_{|\beta|=k+1} u^\beta \sum_{|\delta|=1}^{L_m} q_\beta^{(\delta)}(0) \frac{u^\delta}{\delta!} \right)^m du + \mathcal{O}(a^N)$$

for $a \to 0$. By inserting this result into (21), we get

$$\mathcal{T}^{(2,\alpha)} f(a,s,0)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=m(k+2)}^{m(L_m+k+1)} c_{\gamma,m} \int_{B_{a^\eta}} \partial_1^m \tau \begin{pmatrix} p_{k+1}(u)/a \\ u/a^\alpha \end{pmatrix} \cdot u^\gamma \, du + \mathcal{O}(a^N)$$

for $a \to 0$ for appropriate constants $c_{\ell,m} \in \mathbb{R}$. By comparing the summand for $m = 0$ in the equation (21) with the summand for $m = 0$ in the upper equation, we obtain that

$$c_{0,0} = 1.$$

STEP 4

In this final step, we extend the integration area to $\mathbb{R}^2$ and thus transform it into a form which is fit for an application of Lemma 5.16 - Lemma 5.18 to estimate the decay of the Taylorlet transform.

Applying Lemma 4.39 again, we can change back the integration area to $\mathbb{R}^2$ by only adding another $\mathcal{O}(a^N)$-term. Furthermore, we substitute $u = a^\alpha v$ and obtain

$$\mathcal{T}^{(2,\alpha)} f(a,s,0)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=m(k+2)}^{m(L_m+k+1)} c_{\gamma,m} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} p_{k+1}(a^\alpha v)/a \\ v \end{pmatrix} \cdot (a^\alpha v)^\gamma \cdot a^{2\alpha} \, dv + \mathcal{O}(a^N)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=m(k+2)}^{m(L_m+k+1)} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} a^{(k+1)\alpha-1} \cdot p_{k+1}(v) \\ v \end{pmatrix} \cdot v^\gamma \, dv + \mathcal{O}(a^N).$$

By choosing

$$L := \max_{m \in \{0,\dots,J\}} (L_m + k + 1)m,$$

we obtain the desired form of the sum.

$$\square$$

Proof of Lemma 5.20. We restrict ourselves to the case $t = 0$, as all other cases are equivalent to treating a shifted version of f.

I.

Prerequisites: $\alpha > 0$, $s_0 \neq q(0)$.

The idea is to exploit the special form of f in order to simplify its Taylorlet transform and to use the Schwartz class decay condition of τ in order to estimate the integral.

The structure of f leads to the following form of the Taylorlet transform:

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0) = \int_{\mathbb{R}^3} \delta(x_1 - q(\tilde{x})) \tau_{a,s,0}(x) \, dx$$

$$= \int_{\mathbb{R}^2} \tau \left(\begin{array}{c} \left[q(\tilde{x}) - \sum_{|\gamma| \leq n} \frac{s_\alpha}{\alpha!} \cdot \tilde{x}^\gamma \right] / a \\ \tilde{x} / a^\alpha \end{array} \right) \, d\tilde{x} \tag{23}$$

$$= \int_{\mathbb{R}^2} g\left(\tilde{q}(\tilde{x}) / a \right) \varphi\left(\|\tilde{x}\| / a^\alpha \right) \, d\tilde{x},$$

where $\tilde{q}(\tilde{x}) = q(\tilde{x}) - \sum_{|\gamma| \leq n} \frac{s_\gamma}{\gamma!} \cdot \tilde{x}^\gamma$. Since $g, \varphi \in \mathcal{S}(\mathbb{R})$, the integrand in the last line fulfills the necessary decay condition of Lemma 4.39. By applying this lemma, we can conclude that

$$\left| \int_{\mathbb{R}^2 \setminus B_{a^\beta}} g\left(\tilde{q}(\tilde{x}) / a \right) \varphi\left(\|\tilde{x} / a^\alpha\| \right) \, d\tilde{x} \right|$$

$$\leq \|g\|_{L^\infty} \cdot \left| \int_{\mathbb{R}^2 \setminus B_{a^\beta}} \varphi\left(\tilde{x} / a^\alpha \right) \, d\tilde{x} \right|$$

$$\leq \|g\|_{L^\infty} \cdot \left| \int_{a^\beta}^\infty \varphi\left(r / a^\alpha \right) \cdot r \, dr \right| \overset{\text{Lemma 4.39}}{=} \mathcal{O}\left(a^N \right) \quad \text{for } a \to 0 \quad \text{for all } N \in \mathbb{N},$$

if $\beta < \alpha$. Hence,

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0) = \int_{B_{a^\beta}} g\left(\tilde{q}(\tilde{x}) / a \right) \varphi\left(\|\tilde{x} / a^\alpha\| \right) \, d\tilde{x} + \mathcal{O}(a^N) \quad \text{for } a \to 0$$

for all $N \in \mathbb{N}$. Due to the conditions of this lemma, $\tilde{q} \in C^\infty(\mathbb{R})$ and $\tilde{q}(0) \neq 0$. Hence, there exists an $\varepsilon > 0$ such that $d := \min_{\tilde{x} \in B_\varepsilon} |\tilde{q}(\tilde{x})| > 0$. By employing the boundedness of φ and the Schwartz decay condition for g that for all $N \in \mathbb{N}$ there exists $c_N > 0$ such that $\sup_{\tilde{x} \in \mathbb{R}} \|\tilde{x}^N\| \cdot |g(\tilde{x})| = c_N < \infty$, we get

$$|\mathcal{T}^{(2,\alpha)} f(a, s, 0)| \leq \|\varphi\|_\infty c_N \int_{-a^\beta}^{a^\beta} \left(\frac{a}{|\tilde{q}(\tilde{x})|} \right)^N \, d\tilde{x}$$

$$\leq 2\|\varphi\|_\infty c_N d^{-N} a^{N+\beta}.$$

Since we can choose N to be arbitrarily large, the result follows immediately.

II.

In the proof of this case, we will apply Lemma 5.19 to represent the Taylorlet transform as a finite sum of integrals and we will utilize Lemma 5.16-Lemma 5.18 to the integrals to estimate the decay of the Taylorlet transform.

1.

Prerequisites: Highest approximation order of s for f in 0 is 0 and $\alpha < 1$.

Due to Lemma 5.19, there exists a homogeneous polynomial p_1 of degree 1 such that

$$
\mathcal{T}^{(2,\alpha)} f(a,s,0)
$$

$$
= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=2m}^{L} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} a^{\alpha-1} p_1(u) \\ u \end{pmatrix} u^\gamma \, du + \mathcal{O}\left(a^N\right)
$$

$$
= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=2m}^{L} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{\alpha-1} p_1(u)\right) \varphi\left(\|u\|\right) u^\gamma \, du + \mathcal{O}\left(a^N\right). \tag{24}
$$

Since the highest approximation order of s for f in the origin is 0, $p_1 \neq 0$. Let $p_1(u) = c_1 u_1 + c_2 u_2$. Then w.l.o.g. $c_1 \neq 0$. Now the substitution $u_1 = \frac{1}{c_1} \cdot (v - c_2 u_2)$ yields for any summand in (24):

$$
\int_{\mathbb{R}^2} g^{(m)}\left(a^{\alpha-1} p_1(u)\right) \varphi\left(\|u\|\right) u^\gamma \, du
$$

$$
= \int_{\mathbb{R}} \int_{\mathbb{R}} g^{(m)}\left(a^{\alpha-1} v\right) \varphi\left(\sqrt{\frac{1}{c_1^2} \cdot (v - c_2 u_2)^2 + u_2^2}\right) \cdot \left(\frac{1}{c_1} \cdot (v - c_2 u_2)\right)^{\gamma_1} u_2^{\gamma_2} \, du_2 \, dv
$$

$$
= \frac{1}{c_1^{\gamma_1}} \sum_{\mu=0}^{\gamma_1} \binom{\gamma_1}{\mu} \cdot (-1)^{\gamma_1-\mu} \int_{\mathbb{R}} g^{(m)}\left(a^{\alpha-1} v\right) v^\mu \int_{\mathbb{R}} \varphi\left(\sqrt{\frac{1}{c_1^2} \cdot (v - c_2 u_2)^2 + u_2^2}\right) \cdot (c_2 u_2)^{\gamma_1-\mu} u_2^{\gamma_2} \, du_2 \, dv
$$

$$
= \frac{1}{c_1^{\gamma_1}} \sum_{\mu=0}^{\gamma_1} \binom{\gamma_1}{\mu} \cdot (-c_2)^{\gamma_1-\mu} \int_{\mathbb{R}} g^{(m)}\left(a^{\alpha-1} v\right) v^\mu \int_{\mathbb{R}} \varphi\left(\sqrt{\frac{1}{c_1^2} \cdot (v - c_2 u_2)^2 + u_2^2}\right) \cdot u_2^{|\gamma|-\mu} \, du_2 \, dv \tag{25}
$$

For $\mu \in \{0,\dots,\gamma_1\}$, we now introduce the function

$$
\Phi_\mu : \mathbb{R} \to \mathbb{R}, \quad v \mapsto \int_{\mathbb{R}} \varphi\left(\sqrt{\frac{1}{c_1^2} \cdot (v - c_2 u_2)^2 + u_2^2}\right) u_2^{|\gamma|-\mu} \, du_2.
$$

Since $\int_0^\infty \varphi(r) r^k \, dr = 0$ for all $k \in \mathbb{N}$, $k \geq 2$, Lemma 5.16 yields that $\Phi_\mu \in \mathcal{S}(\mathbb{R})$ and $\Phi_\mu(v) = \mathcal{O}\left(v^{(|\gamma|-\mu-1)_+}\right)$ for $v \to 0$. Hence, there exists $\Psi_\mu \in \mathcal{S}(\mathbb{R})$ such that

$$
\Phi_\mu(v) = v^{(|\gamma|-\mu-1)_+} \cdot \Psi_\mu(v) \quad \text{for all } v \in \mathbb{R}.
$$

This allows us to reduce (25) to

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{\alpha-1}p_1(u)\right)\varphi\left(\|u\|\right)u^\gamma\,du$$

$$= \frac{1}{c_1^{\gamma_1}}\sum_{\mu=0}^{\gamma_1}\binom{\gamma_1}{\mu}\cdot(-c_2)^{\gamma_1-\mu}\int_{\mathbb{R}} g^{(m)}\left(a^{\alpha-1}v\right)\cdot\Psi_\mu(v)\cdot v^{\mu+(|\gamma|-\mu-1)_+}\,dv$$

Since $\mu+(|\gamma|-\mu-1)_+ \geq (|\gamma|-1)_+$ and $|\gamma|\geq 2m$ due to (24), we get $\mu+(|\gamma|-\mu-1)_+ \geq m$ for all $m\in\mathbb{N}_0$. Consequently, we can apply Lemma 4.40 to obtain the following decay rate for the summands in (24):

$$a^{-m}\cdot c_{\gamma,m}\cdot a^{(|\gamma|+2)\alpha}\int_{\mathbb{R}^2} g^{(m)}\left(a^{\alpha-1}p_1(u)\right)\varphi\left(\|u\|\right)u^\gamma\,du = \mathcal{O}\left(a^{(1-\alpha)M+1+(|\gamma|-m+1)\alpha}\right)$$

for $a\to 0$. Since $|\gamma|\geq 2m$, the $(m=0,\gamma=0)$-term dominates asymptotically. Hence,

$$\mathcal{T}^{(2,\alpha)}f(a,s,t) = \mathcal{O}\left(a^{(1-\alpha)M+1+\alpha}\right)\quad\text{for }a\to 0.$$

2.

Prerequisites: Highest approximation order of s for f in 0 is 1 and $\alpha < \frac{1}{2}$.

Due to Lemma 5.19, there exists a homogeneous polynomial p_2 of degree 2 such that

$$\mathcal{T}^{(2,\alpha)}f(a,s,0)$$

$$= \sum_{m=0}^{J} a^{-m}\sum_{|\gamma|=3m}^{L} c_{\gamma,m}\cdot a^{(|\gamma|+2)\alpha}\int_{\mathbb{R}^2}\partial_1^m\tau\begin{pmatrix}a^{2\alpha-1}p_2(u)\\ u\end{pmatrix}u^\gamma\,du + \mathcal{O}\left(a^N\right)$$

$$= \sum_{m=0}^{J} a^{-m}\sum_{|\gamma|=3m}^{L} c_{\gamma,m}\cdot a^{(|\gamma|+2)\alpha}\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1}p_2(u)\right)\varphi\left(\|u\|\right)u^\gamma\,du + \mathcal{O}\left(a^N\right). \tag{26}$$

Due to (20) and (16), the homogeneous polynomial p_2 can be rewritten in the form

$$p_2(u) = \frac{1}{2}\cdot u^T\cdot Hq_s(0)\cdot u = \frac{1}{2}\cdot u^T\cdot A_s(0)\cdot u.$$

By an eigenvalue decomposition of $A_s(0)$ we obtain a matrix $S\in SO(2)$ such that

$$p_2(Su) = u^T S^T A_s(0)Su = \lambda_1\cdot u_1^2 + \lambda_2\cdot u_2^2. \tag{27}$$

We now analyze the summands of (26) by applying the substitution $u = Sv$ with the matrix S from the equation (27):

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi\left(\|u\|\right) u^\gamma \, du$$

$$= \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot p_2(Sv)\right) \cdot \varphi\left(\|Sv\|\right) (Sv)^\gamma \, dv$$

$$\stackrel{(27)}{=} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^2 \lambda_i \cdot v_i^2\right) \cdot \varphi\left(\|v\|\right) \cdot \left(\sum_{k=1}^2 S_{1k} v_k\right)^{\gamma_1} \cdot \left(\sum_{k=1}^2 S_{2k} v_k\right)^{\gamma_2} \, dv$$

$$= \sum_{|\delta|=|\gamma|} C_\delta \cdot \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^2 \lambda_i \cdot v_i^2\right) \cdot \varphi\left(\|v\|\right) v^\delta \, dv. \tag{28}$$

If a component of δ is odd, we get

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^2 \lambda_i \cdot v_i^2\right) \cdot \varphi\left(\|v\|\right) v^\delta \, dv = 0$$

due to the symmetry of the arguments. Hence, we only consider $\delta \in \mathbb{N}_0^2$ such that there exists $v \in \mathbb{N}_0^2$ with $\delta = 2v$.

a)

Prerequisite: $A_s(0)$ is either positive or negative semidefinite.

If the approximation matrix $A_s(0)$ is not indefinite, there exist either only non-positive or only non-negative eigenvalues. To this end, we can divide the indices of the eigenvalues into 3 different sets:

$$I_+ := \{i \in \{1,2\} : \lambda_i > 0\},$$
$$I_- := \{i \in \{1,2\} : \lambda_i < 0\},$$
$$I_0 := \{i \in \{1,2\} : \lambda_i = 0\}.$$

Similarly, we can divide the components of a vector $v \in \mathbb{R}^2$ or a multi-index $\gamma \in \mathbb{N}_0^2$ into three different vectors each:

$$v_+ := (v_i : i \in I_+), \qquad\qquad \gamma_+ := (\gamma_i : i \in I_+),$$
$$v_- := (v_i : i \in I_-), \qquad\qquad \gamma_- := (\gamma_i : i \in I_-),$$
$$v_0 := (v_i : i \in I_0), \qquad\qquad \gamma_0 := (\gamma_i : i \in I_0).$$

As $A_s(0)$ is not indefinite, we either have $I_+ = \emptyset$ or $I_- = \emptyset$. W.l.o.g. let $I_- = \emptyset$. As the highest approximation order of s for f in t is 1, we have $I_+ \neq \emptyset$. So only two cases remain:

(i) $|I_0| = 1$ and $|I_+| = 1$:

In this case, there exists $\lambda_+ > 0$ such that the summands in (28) can be transformed in the following way

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^{2} \lambda_i \cdot v_i^2\right) \cdot \varphi(\|v\|) \, v^\delta \, dv$$

$$= \int_{\mathbb{R}} g^{(m)}\left(a^{2\alpha-1} \lambda_+ v_+^2\right) v_+^{\delta_+} \cdot \int_{\mathbb{R}} \varphi\left(\sqrt{v_+^2 + v_0^2}\right) v_0^{\delta_0} \, dv_0 \, dv_+. \tag{29}$$

By choosing $\alpha = 1$, $c = 0$, Lemma 5.16 yields for the function

$$\Phi : \mathbb{R} \to \mathbb{R}, \quad v_+ \mapsto \int_{\mathbb{R}} \varphi\left(\sqrt{v_+^2 + v_0^2}\right) v_0^{\delta_0} \, dv_0$$

that $\Phi \in \mathcal{S}(\mathbb{R})$ and $\Phi(v) = \mathcal{O}\left(v^{(\delta_0-1)_+}\right)$ for $v \to 0$. Hence, there exists a function $\Psi \in \mathcal{S}(\mathbb{R})$ such that $\Phi(t) = t^{(\delta_0-1)_+} \cdot \Psi(t)$ for all $t \in \mathbb{R}$. Consequently, we can rewrite equation (29) as

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^{2} \lambda_i \cdot v_i^2\right) \cdot \varphi(\|v\|) \, v^\delta \, dv = \int_{\mathbb{R}} g^{(m)}\left(a^{2\alpha-1} \lambda_+ t^2\right) t^{\delta_+ + (\delta_0-1)_+} \cdot \Psi(t) \, dt.$$

As $\delta_+ + (\delta_0 - 1)_+ \geq |\delta| - 1$, $|\delta| = |\gamma|$ due to (28) and $|\gamma| \geq 3m$ due to (26), we obtain

$$\delta_+ + (\delta_0 - 1)_+ \geq m.$$

Thus, Lemma 4.40 yields the following decay rate for the summands in (26):

$$a^{-m} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} p_2(u)\right) \varphi(\|u\|) \, u^\gamma \, du = \mathcal{O}\left(a^{(1-2\alpha)M + (|\gamma|+2-2m)\alpha + \frac{1}{2} - \alpha}\right)$$

for $a \to 0$. Since the sum is asymptotically dominated by the $(m = 0, \gamma = 0)$-term, we obtain the decay rate

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{(1-2\alpha)M + \frac{1}{2} + \alpha}\right) \quad \text{for } a \to 0.$$

(ii) $|I_+| = 2$:

In this case $\delta_+ = \delta$. As $|\delta| = |\gamma| \geq 3m$ due to (26) and (28), we have $|\gamma| \geq 2m - 1$. Hence, the conditions of Lemma 5.17 are fulfilled and we get

$$\int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot \sum_{i=1}^{2} \lambda_i \cdot v_i^2\right) \cdot \varphi(\|v\|) \, v^\delta \, dv = \mathcal{O}\left(a^{(1-2\alpha)(M+m+1)}\right) \quad \text{for } a \to 0.$$

Thus, we obtain the following decay rate for the summands in (26):

$$a^{-m} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} p_2(u)\right) \varphi(\|u\|) \, u^\gamma \, du = \mathcal{O}\left(a^{(1-2\alpha)M + (|\gamma|+2-2m)\alpha + 1 - 2\alpha}\right)$$

for $a \to 0$. Since the sum is again asymptotically dominated by the $(m = 0, \gamma = 0)$-term, we obtain the decay rate

$$\mathcal{T}^{(2,\alpha)} f(a, s, t) = \mathcal{O}\left(a^{(1-2\alpha)M+1}\right) = \mathcal{O}\left(a^{(1-2\alpha)M+\frac{1}{2}+\alpha}\right) \quad \text{for } a \to 0.$$

If $I_+ = \emptyset$ instead, we can use a similar argumentation to obtain the same decay rate as above.

b)

Prerequisites: $\alpha > \frac{2}{5}$, τ is hyperbolically restrictive and $A_s(0)$ is indefinite.

If the approximation matrix $A_s(0)$ is indefinite, there exist positive and negative eigenvalues. Due to (26), we have

$$\mathcal{T} f(a, s, 0) = \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=3m}^{L} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi(\|u\|) u^\gamma \, du + \mathcal{O}\left(a^N\right) \quad (30)$$

for $a \to 0$ for all $N \in \mathbb{N}_0$. First, we will show that all terms but the $(m = 0, \gamma = 0)$-term display a decay rate of $o(a)$ for $a \to 0$. To this end, we observe that

$$\left| a^{-m} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi(\|u\|) u^\gamma \, du \right|$$

$$= |c_{\gamma,m}| \cdot a^{-m+(|\gamma|+2)\alpha} \cdot \int_{\mathbb{R}^2} \left| g^{(m)}\left(a^{2\alpha-1} \cdot p_2(u)\right) \right| \cdot \left| \varphi(\|u\|) u^\gamma \right| \, du$$

$$\leq c_{\gamma,m} \cdot \left\| g^{(m)} \right\|_{L^\infty} \cdot \left\| \varphi(\|\cdot\|)(\cdot)^\gamma \right\|_{L^1} \cdot a^{(|\gamma|+2)\alpha-m}.$$

Because of $|\gamma| \geq 3m$, we obtain

$$(|\gamma| + 2)\alpha - m \geq (3m + 2)\alpha - m = (3\alpha - 1)m + 2\alpha > 1 \quad \text{for all } m \geq 1,$$

as $\alpha > \frac{2}{5}$. Hence,

$$a^{-m} c_{\gamma,m} a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(a^{2\alpha-1} p_2(u)\right) \varphi(\|u\|) u^\gamma \, du = o(a) \quad \text{for } a \to 0, \forall m \geq 1, |\gamma| \geq 3m. \quad (31)$$

for $a \to 0$ for all $m \geq 1$ and $|\gamma| \geq 3m$.

Since $c_{0,0} = 1$ due to Lemma 5.19, the $(m = 0, \gamma = 0)$-term in (30) reads

$$\int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi(\|u\|) \, du.$$

By bringing the homogeneous polynomial $p_2(u)$ into normal form by substituting $u = S \cdot v$ as in (28), we get

$$\int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi(\|u\|) \, du = \int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot \sum_{i=1}^{2} \lambda_i \cdot v_i^2\right) \cdot \varphi(\|v\|) \, dv.$$

W.l. o. g. let $\lambda_1 > 0$ and $\lambda_2 < 0$. By rewriting the eigenvalues into the form $|\lambda_i| = \frac{1}{a_i^2}$, we obtain

$$\int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi\left(\|u\|\right) \, du$$

$$= \int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot \left(\frac{v_1^2}{a_1^2} - \frac{v_2^2}{a_2^2}\right)\right) \varphi\left(\sqrt{v_1^2 + v_2^2}\right) \, dv \quad \text{(substitute } v_i = a_i \cdot w_i)$$

$$= \int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot (w_1^2 - w_2^2)\right) \varphi\left(\sqrt{a_1^2 w_1^2 + a_2^2 w_2^2}\right) \, dw$$

$$= 4 \cdot \int_0^\infty \int_0^\infty g\left(a^{2\alpha-1} \cdot (w_1^2 - w_2^2)\right) \varphi\left(\sqrt{a_1^2 w_1^2 + a_2^2 w_2^2}\right) \, dw.$$

As τ is hyperbolically restrictive and τ has $M \geq 1$ vanishing moments of order 2, we can apply Lemma 5.18 to obtain

$$\int_{\mathbb{R}^2} g\left(a^{2\alpha-1} \cdot p_2(u)\right) \cdot \varphi\left(\|u\|\right) \, du \sim a^{1-2\alpha} \quad \text{for } a \to 0.$$

Inserting this decay rate into the $(m = 0, \gamma = 0)$-term in (30) and considering (31) yields

$$\mathcal{T}f(a, s, 0) \sim a^{(1-2\alpha)+2\alpha} = a \quad \text{for } a \to 0.$$

III.

Prerequisites: $\alpha > \frac{1}{3}$, τ is restrictive and the highest approximation order of s for f in 0 is at least 2.

Since the highest approximation of s for f in 0 is at least 2, we have a polynomial p_3 which is homogeneous of degree 3 or greater or the zero polynomial, due to Lemma 5.19 such that

$$\mathcal{T}^{(2,\alpha)} f(a, s, 0)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=4m}^{L} c_{\gamma,m} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} p_3(a^\alpha v)/a \\ v \end{pmatrix} \cdot (a^\alpha v)^\gamma \cdot a^{2\alpha} \, dv + \mathcal{O}(a^N)$$

$$= \sum_{m=0}^{J} a^{-m} \sum_{|\gamma|=4m}^{L} c_{\gamma,m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)}\left(p_3(a^\alpha v)/a\right) \cdot \varphi(\|v\|) \cdot v^\gamma \, dv + \mathcal{O}(a^N). \qquad (32)$$

Regardless of the exact degree of p_3, we obtain that

$$\lim_{a \to 0} p_3(a^\alpha v)/a = 0 \quad \text{for all } v \in \mathbb{R},$$

since $\alpha > \frac{1}{3}$. Together with $g \in \mathcal{S}(\mathbb{R})$ this yields for the integrals in (32):

$$\lim_{a \to 0} \int_{\mathbb{R}^2} \partial_1^m \tau \begin{pmatrix} p_3(a^\alpha v)/a \\ v \end{pmatrix} \cdot v^\gamma dv = \int_{\mathbb{R}^2} \lim_{a \to 0} g^{(m)} \left(p_3(a^\alpha v)/a \right) \cdot \varphi(\|v\|) \cdot v^\gamma \, dv$$

$$= g^{(m)}(0) \cdot \int_{\mathbb{R}^2} \varphi(\|v\|) v^\gamma \, dv$$

$$= g^{(m)}(0) \cdot \int_0^\infty \int_{S^1} \varphi(r) r^{|\gamma|} \omega^\gamma \, d\sigma_{d-2}(\omega) \, r \, dr$$

$$= C g^{(m)}(0) \cdot \int_0^\infty \varphi(r) r^{|\gamma|+d-2} \, dr. \tag{33}$$

We now focus on the powers of a appearing in the summands of (32). For the indices γ and m of the double sum's summands in (32) we obtain that

$$S_{\gamma,m}(a) := a^{-m} \cdot a^{(|\gamma|+2)\alpha} \int_{\mathbb{R}^2} g^{(m)} \left(p_3(a^\alpha v)/a \right) \cdot \varphi(\|v\|) \cdot v^\gamma dv$$

$$= \mathcal{O}\left(a^{(|\gamma|+2)\alpha - m} \right) \quad \text{for } a \to 0.$$

Since $\alpha > \frac{1}{3}$ and $|\gamma| \geq 4m$, the potentially slowest decaying summand is $S_{0,0}$. We will now check that it is not zero. Due to the restrictiveness $g(0) \neq 0$ and $\int_0^\infty \varphi(r) r \, dr \neq 0$. Hence, together with (33) and $c_{0,0} = 1$ due to Lemma 5.19, we obtain that

$$S_{0,0} \sim a^{2\alpha} \cdot c_{0,0} \cdot g(0) \cdot \int_0^\infty \varphi(r) r \, dr \quad \text{for } a \to 0.$$

Thus,

$$\mathcal{T}^{2,\alpha} f(a, s, 0) \sim a^{2\alpha} \quad \text{for } a \to 0.$$

$\square$

Proof of Theorem 5.14. By partial integration we obtain for $j \geq 1$ that

$$\mathcal{T}^{(n,\alpha)} f(a, s, t) = \int_{\mathbb{R}^d} \tau_{ast}(x) \cdot \left(I_\pm^j \delta \right) (x_1 - q(\tilde{x})) \, dx$$

$$= (-1)^j a^j \int_{\mathbb{R}^d} I_{x_1,\mp}^j \tau_{ast}(x) \delta \left(x_1 - q(\tilde{x}) \right) \, dx$$

$$= (-1)^j a^j \int_{\mathbb{R}^d} I_\mp^j \tilde{g}^{(j)} \left((x_1 - S_s(\tilde{x}))/a \right) h \left((\tilde{x} - t)/a^\alpha \right) \delta \left(x_1 - q(\tilde{x}) \right) \, dx$$

$$= (\mp 1)^j a^j \int_{\mathbb{R}^d} \tilde{g} \left((x_1 - S_s(\tilde{x}))/a \right) h \left((\tilde{x} - t)/a^\alpha \right) \delta \left(x_1 - q(\tilde{x}) \right) \, dx$$

$$= (\mp 1)^j a^j \int_{\mathbb{R}^d} \tilde{\tau}_{ast}(x) \delta \left(x_1 - q(\tilde{x}) \right) \, dx$$

Hence, the integral is reduced to the case of Lemma 5.20. As $\tilde{\tau}$ is an analyzing Taylorlet of order 2 in dimension d with M vanishing moments, Lemma 5.20 yields the wanted decay rates. Due

to the upper equation, the respective decay rates of Lemma 5.20 are only multiplied by a^j to obtain the decay properties in the case of a j-feasible function.

$\square$

5.4 Construction of a three-dimensional Taylorlet

The declared goal of this section is to construct a Taylorlet τ of dimension 3 that additionally satisfies the conditions of all subitems of Theorem 5.14, i. e., it is restrictive and hyperbolically restrictive. For an overview, we give here a list of all required properties for the functions $g, \varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\tau(x) = g(x_1) \cdot \varphi\left(\sqrt{x_2^2 + x_3^2}\right) \quad \text{for all } x \in \mathbb{R}^3 :$$

1. φ is even,

2. $\varphi \in \mathcal{S}(\mathbb{R})$,

3. $\varphi(0) \neq 0$,

4. $\int_0^\infty \varphi(r) r \, dr \neq 0$,

5. $\int_0^\infty \varphi(r) r^m \, dr = 0$ for all $m \geq 2$,

and

6. $g \in \mathcal{S}(\mathbb{R})$,

7. $\int_\mathbb{R} g(r) r^m \, dr = 0$ for all $m \in \{0, \ldots, M-1\}$,

8. $\int_0^\infty g\left(\pm r^2\right) r^m \, dr = 0$ for all $m \in \{0, \ldots, 2M-1\}$,

9. $g(0) \neq 0$,

10. $\int_0^\infty \left[g\left(r^2\right) + g\left(-r^2\right)\right] r \log r \, dr \neq 0$.

Using the q-calculus construction of subsection 4.3.2, we will show in the following theorem that such functions can, indeed, be constructed.

Theorem 5.21. *Let $q \in (0,1)$, $M \geq 2$ and let $\eta \in C_c^\infty(\mathbb{R})$ be even such that there exists $\varepsilon > 0$ with $\eta\big|_{(-\varepsilon, \varepsilon)} \equiv 1$ and $\eta \geq 0$. Then, the functions*

$$\varphi := \prod_{m=2}^\infty \left(\mathrm{Id} - q^{m+1} D_q\right) \eta,$$

$$g := \left(\prod_{m=0}^{M-1} \left(\mathrm{Id} - q^{m+1} D_q\right) \eta\right) \circ \sqrt{|\cdot|},$$

satisfy all of the conditions 1-10.

Proof.

1. As η is even and the operators $\mathrm{Id} - q^{m+1} D_q$ preserve the symmetry, φ is even, as well.

2. For showing this condition, we follow the proof of Lemma 4.21. We introduce the function sequence

$$\eta_k := \prod_{m=2}^{k+1} \left(\mathrm{Id} - q^{m+1} D_q \right) \eta,$$

for all $m \in \mathbb{N}$, i.e.,

$$\eta_{k+1} = \left(\mathrm{Id} - q^{k+3} D_q \right) \eta_k.$$

Additionally, we define

$$c_{k,\ell,m} := \| x^m \eta_k^{(\ell)}(x) \|_\infty.$$

In order to prove that $\lim_{k \to \infty} \eta_k = \phi \in \mathcal{S}(\mathbb{R})$, we will show that uniform upper bounds in k exist for the $c_{k,\ell,m}$. I.e., for all $\ell, m \in \mathbb{N}$ we determine a $c_{\ell,m} > 0$ such that

$$c_{k,\ell,m} \leq c_{\ell,m} \text{ for all } k \in \mathbb{N}.$$

For this purpose we estimate $c_{k+1,\ell,m}$ in terms of $c_{k,\ell,m}$.

$$x^m \cdot \eta_{k+1}^{(\ell)}(x) = x^m \cdot \partial_x^\ell \left(\mathrm{Id} - q^{k+3} D_q \right) \eta_m(x) = x^m \cdot \left(\mathrm{Id} - q^{k+\ell+3} D_q \right) \eta_k^{(\ell)}(x).$$

Hence, we can estimate

$$
\begin{aligned}
c_{k+1,\ell,m} &= \left\| x^m \eta_{k+1}^{(\ell)}(x) \right\|_\infty \\
&= \left\| x^m \cdot \left(\mathrm{Id} - q^{k+\ell+3} D_q \right) \eta_k^{(\ell)}(x) \right\|_\infty \\
&\leq \left\| x^m \eta_k^{(\ell)}(x) \right\|_\infty + q^{k+\ell-m+3} \left\| (qx)^m \eta_k^{(\ell)}(qx) \right\|_\infty \\
&\leq \left(1 + q^{k+\ell-m+3} \right) \cdot c_{k,\ell,m} \\
&\leq \prod_{v=0}^{k} \left(1 + q^{v+\ell-m+3} \right) \cdot c_{0,\ell,m} \\
&\leq c_{0,\ell,m} \cdot \prod_{v=0}^{k} (1 + q^{\ell+3+v-m}) \\
&= c_{0,\ell,m} \cdot \left(-q^{\ell+3-m}; q \right)_{k+1} \\
&\leq c_{0,\ell,m} \cdot \left(-q^{\ell+3-m}; q \right)_\infty.
\end{aligned}
$$

Due to Corollary 4.17, the expression $\left(-q^{\ell+3-m}; q \right)_\infty$ indeed converges for all $q \in (0,1)$. Thus, $c_{k,\ell,m} \leq \left(-q^{\ell+3-m}; q \right)_\infty \cdot c_{0,\ell,m} =: c_{\ell,m}$ for all $k \in \mathbb{N}$. Since $\eta \in \mathcal{S}(\mathbb{R})$ by prerequisite, $c_{0,\ell,m}$ is finite for all $\ell, m \in \mathbb{N}_0$.

3. We obtain that

$$\varphi(0) = \prod_{m=2}^{\infty} \left(1 - q^{m+1}\right) \cdot \underbrace{\eta(0)}_{=1} = \frac{(q,q)_{\infty}}{(1-q)(1-q^2)} > 0.$$

4. Let $T_0 := \int_0^{\infty} \eta(r) r \, dr$. Due to the conditions on η, we have

$$T_0 = \int_0^{\infty} \underbrace{\eta(r)}_{\geq 0} r \, dr \geq \int_0^{\varepsilon} \underbrace{\eta(r)}_{=1} r \, dr = \frac{\varepsilon^2}{2} > 0.$$

For $k \in \mathbb{N}$, we define the functions

$$\eta_k := \prod_{m=2}^{k+1} \left(\mathrm{Id} - q^{m+1}\right) \eta.$$

We will now inductively show that

$$\int_0^{\infty} \eta_k(r) r \, dr = (q;q)_k \cdot T_0$$

and hence, that $\int_0^{\infty} \varphi(r) r \, dr = (q;q)_{\infty} \cdot T_0 > 0$. By defining $\eta_0 := \eta$, we can see that the induction start is fulfilled for $k = 0$. Assuming that the upper statement is fulfilled for $k \in \mathbb{N}_0$, we can utilize the definition of η_k to establish the relation

$$\eta_{k+1} = \left(\mathrm{Id} - q^{k+3} D_q\right) \eta_k.$$

Now we compute

$$\begin{aligned}
\int_0^{\infty} \eta_{k+1}(r) r \, dr &= \int_0^{\infty} \left(\mathrm{Id} - q^{k+3} D_q\right) \eta_k(r) r \, dr \\
&= \int_0^{\infty} \eta_k(r) r \, dr - q^{k+3} \cdot \int_0^{\infty} \eta_k(qr) r \, dr \quad (\text{substitute } r = t/q) \\
&= \int_0^{\infty} \eta_k(r) r \, dr - q^{k+1} \cdot \int_0^{\infty} \eta_k(t) t \, dt \\
&= \left(1 - q^{k+1}\right) \cdot \int_0^{\infty} \eta_k(r) r \, dr \\
&= \left(1 - q^{k+1}\right) \cdot (q;q)_k \cdot T_0 = (q;q)_{k+1} \cdot T_0.
\end{aligned}$$

Hence, the statement $\int_0^{\infty} \eta_k(r) r \, dr = (q;q)_k \cdot T_0$ holds for all $k \in \mathbb{N}_0$ and we obtain

$$\int_0^{\infty} \varphi(r) r \, dr = (q;q)_{\infty} \cdot T_0 > 0.$$

5. Lemma 4.19 and Lemma 4.21 already indicate that

$$\int_0^{\infty} \varphi(r) r^m \, dr = 0 \quad \text{for all } m \geq 2.$$

6. The statement follows immediately via Proposition 4.18.

7. According to Lemma 4.19, the function

$$\prod_{m=0}^{M-1} \left(\mathrm{Id} - q^{m+1} D_q \right) \eta$$

   has M vanishing moments. Consequently, Proposition 4.18 ensures that g has M vanishing moments, as well.

8. Using the same combination of Lemma 4.19 and Proposition 4.18, we can argue that g exhibits the indicated number of vanishing moments of second order.

9. Utilizing the definition of g, inserting the origin into the function yields

$$g(0) = \prod_{m=0}^{M-1} \left(1 - q^{m+1} \right) \cdot \eta(0) = \left(q; q \right)_M > 0.$$

10. The proof strategy is now to show that for the function $\tilde{\eta} := \left(\mathrm{Id} - q D_q \right) \left(\mathrm{Id} - q^2 D_q \right) \eta$, we have that the expression

$$L_0 := \int_0^\infty \tilde{\eta}(r) r \log r \ dr > 0.$$

   Subsequently, we utilize an induction argument ensuring that for any $k \geq 1$,

$$\int_0^\infty \left(\prod_{m=2}^{k+1} \left(\mathrm{Id} - q^{m+1} D_q \right) \tilde{\eta} \right)(r) r \log r \ dr > 0.$$

   Due to its definition and according to Lemma 4.19, we have $\int_0^\infty \tilde{\eta}(r) r \ dr = 0$. We can exploit this relation to show that

$$\begin{aligned}
L_0 &= \int_0^\infty \tilde{\eta}(r) r \log r \ dr \\
&= \int_0^\infty \left(\mathrm{Id} - q D_q \right) \left(\mathrm{Id} - q^2 D_q \right) \eta(r) r \log r \ dr \\
&= \int_0^\infty \eta(r) r \log r \ dr - \left(q + q^2 \right) \cdot \int_0^\infty \eta(qr) r \log r \ dr + q^3 \cdot \int_0^\infty \eta(q^2 r) r \log r \ dr \\
&= \int_0^\infty \eta(r) r \log r \ dr - \left(q^{-1} + 1 \right) \cdot \int_0^\infty \eta(t) t \log(q^{-1} t) \ dt + q^{-1} \cdot \int_0^\infty \eta(t) t \log(q^{-2} t) \ dt \\
&= \underbrace{\left[1 - \left(q^{-1} + 1 \right) + q^{-1} \right]}_{=0} \cdot \int_0^\infty \eta(r) r \log r \ dr + \left[- \left(q^{-1} + 1 \right) \log(q^{-1}) + q^{-1} \log(q^{-2}) \right] \cdot \int_0^\infty \eta(t) t \ dt \\
&= \left(q^{-1} - 1 \right) \log(q^{-1}) \cdot \int_0^\infty \eta(t) t \ dt > 0,
\end{aligned}$$

since $q \in (0,1)$ and $\int_0^\infty \eta(t) t \, dt =: T_0 > 0$, as we already showed in 4. Next, we introduce the functions

$$\tilde{\eta}_k := \prod_{m=2}^{k+1} \left(\mathrm{Id} - q^{m+1} D_q \right) \tilde{\eta},$$

and we will prove inductively that

$$\int_0^\infty \tilde{\eta}_k(r) r \log r \, dr = (q;q)_k \cdot L_0.$$

According to Lemma 4.19, $\int_0^\infty \tilde{\eta}_k(r) r \, dr = 0$ for all $k \in \mathbb{N}_0$, as $\int_0^\infty \tilde{\eta}(r) r \, dr = 0$. We now utilize the relation

$$\tilde{\eta}_{k+1} = \left(\mathrm{Id} - q^{k+3} D_q \right) \tilde{\eta}_k$$

to obtain

$$
\begin{aligned}
&\int_0^\infty \tilde{\eta}_{k+1}(r) r \log r \, dr \\
&= \int_0^\infty \left(\mathrm{Id} - q^{k+3} D_q \right) \tilde{\eta}_k(r) r \log r \, dr \\
&= \int_0^\infty \tilde{\eta}_k(r) r \log r \, dr - q^{k+3} \cdot \int_0^\infty \tilde{\eta}_k(qr) r \log r \, dr \\
&= \int_0^\infty \tilde{\eta}_k(r) r \log r \, dr - q^{k+1} \cdot \int_0^\infty \tilde{\eta}_k(t) t \log(q^{-1} t) \, dt \\
&= \int_0^\infty \tilde{\eta}_k(r) r \log r \, dr - q^{k+1} \cdot \int_0^\infty \tilde{\eta}_k(t) t \log t \, dt - q^{k+1} \log(q^{-1}) \cdot \underbrace{\int_0^\infty \tilde{\eta}_k(t) t \, dt}_{=0} \\
&= \left(1 - q^{k+1} \right) \cdot \int_0^\infty \tilde{\eta}_k(t) t \log t \, dt = (q;q)_{k+1} \cdot L_0 > 0.
\end{aligned}
$$

By observing that the even symmetry of η is preserved by the operators $\mathrm{Id} - q^m D_q$, we see that g is even, as well. Consequently,

$$\int_0^\infty \left[g(r^2) + g(-r^2) \right] r \log r \, dr = 2 \int_0^\infty \tilde{\eta}_{M-2}(r) r \log r \, dr = 2 (q;q)_{M-2} \cdot L_0 > 0.$$

$\square$

5.5 Detection algorithm for higher-dimensional edges

The detection of singularities and their orientation in three dimensions has already been established and well described by Guo and Labate for the case of shearlets [GL11].

For the purpose of finding a detection algorithm for higher-dimensional edges, we assume that we already have the necessary local positional and orientational information and that we are ideally able to discern between the different decay rates of the Taylorlet transform, provided by

Theorem 5.14. By this assumption, we can perfectly distinguish between the three cases:
The matrix $Hq(t) - S$ is

    (i) either positive or negative semidefinite,

   (ii) indefinite,

  (iii) the zero-matrix.

In this scenario, it is our goal to determine the unknown Hessian $Hq(t)$ by choosing suitable matrices S and utilizing the Taylorlet transform to find out which of the three cases (i), (ii) or (iii) applies for $Hq(t) - S$. As the Hessian is a symmetric 2×2-matrix, we need to find 3 variables. As searching for all variables at once results in a 3-dimensional search space, this naive approach is not very cost-efficient. Hence, it is the main focus of this subsection to find a search strategy that allows for consecutive one-dimensional searches with the tools the Taylorlet transform provides. To this end, we describe the notation of the setup hereafter.

Setting

Let $Hq(t) \in \mathrm{Sym}(2, \mathbb{R})$ have the eigenvalues λ_1, λ_2, where $\lambda_1 \geq \lambda_2$. Furthermore, let $D = \mathrm{diag}(\lambda_1, \lambda_2)$ and let $T = (v_1, v_2) \in SO(2)$, where v_1, v_2 are the corresponding eigenvectors to λ_1 and λ_2. Hence, $Hq(t) = TDT^T$.

Detection Algorithm

1. We first choose $S_\lambda = \lambda \cdot \mathrm{Id}$, $\lambda \in \mathbb{R}$. Then,

$$Hq(t) - S_\lambda \text{ is } \begin{cases} \text{positive semidefinite,} & \text{if } \lambda \leq \lambda_2, \\ \text{indefinite,} & \text{if } \lambda \in (\lambda_2, \lambda_1), \\ \text{negative semidefinite,} & \text{if } \lambda \geq \lambda_1. \end{cases}$$

    Thus, we obtain the eigenvalues λ_1 and λ_2.

2. For $\theta \in [0, \pi)$ let

$$S_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

    Then,

$$Hq(t) - S_\theta \begin{cases} \text{is indefinite,} & \text{if } v_1, (\cos\theta, \sin\theta)^T \text{ are linearly independent,} \\ = 0, & \text{if } v_1 \in \mathrm{span}\left((\cos\theta, \sin\theta)^T\right), \end{cases}$$

    and we can find the matrix $Hq(t)$ as $Hq(t) = S_\theta$ for the right choice of θ.

Theorem 5.22. *Let $Hq(t) \in \mathrm{Sym}(2, \mathbb{R})$. Assuming that we can perfectly distinguish between the three cases*

(i) $Hq(t) - S$ is positive or negative semidefinite,

(ii) $Hq(t) - S$ is indefinite,

(iii) $Hq(t) - S = 0$,

the presented detection algorithm yields the matrix $Hq(t)$.

Proof. 1. By choosing $S_\lambda = \lambda \cdot \text{Id}$, $\lambda \in \mathbb{R}$, we obtain the three cases

$$Hq(t) - S_\lambda \text{ is } \begin{cases} \text{positive semidefinite,} & \text{if } \lambda \leq \lambda_2, \\ \text{indefinite,} & \text{if } \lambda \in (\lambda_2, \lambda_1), \\ \text{negative semidefinite,} & \text{if } \lambda \geq \lambda_1. \end{cases}$$

As we can distinguish between the cases of semidefiniteness and indefiniteness of the matrix $Hq(t) - S$ by assumption, we obtain the eigenvalues λ_1 and λ_2 as the values at the borders between the two cases.

2. For $\theta \in \mathbb{R}$, we define the vector $e_\theta = (\cos\theta, \sin\theta)^T$. Choosing

$$S_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{for } \theta \in [0, \pi),$$

yields the two cases that

(a) v_1 and e_θ are linearly independent,

(b) v_1 and e_θ are linearly dependent.

We will now show that in case (a), $Hq(t) - S_\theta$ is indefinite. To this end, we observe that

$$v_1^T (Hq(t) - S_\theta) v_1 = \underbrace{v_1^T Hq(t) v_1}_{=\lambda_1} - \underbrace{v_1^T S_\theta v_1}_{<\lambda_1} > 0,$$

$$e_\theta^T (Hq(t) - S_\theta) e_\theta = \underbrace{e_\theta^T Hq(t) e_\theta}_{<\lambda_1} - \underbrace{e_\theta S_\theta e_\theta}_{=\lambda_1} < 0.$$

In the first inequality, $v_1^T S_\theta v_1 < \lambda_1$, because e_θ the eigenvector of S_θ for the eigenvalue λ_1 and v_1 and e_θ are linearly independent. For the second inequality, the argumentation is analogous. Hence, $Hq(t) - S_\theta$ is indefinite.

In case (b), v_1 and e_θ are linearly dependent and thus $Hq(t)$ and S_θ have the same eigenvalues and the same corresponding eigenspaces. Consequently, $Hq(t) = S_\theta$.

$\square$

Remark 5.23. Proposition 1.18 allows us to compute the matrix representation of the Weingarten map of this graph with the help of the Hesse matrix and the first derivatives to obtain the Gauss and mean curvature of the singular surface.

CHAPTER 6

Conclusion and outlook

In this chapter we take the opportunity to reflect on the shown results. As the solution of a problem naturally gives rise to new question, we will briefly discuss open problems for future research.

After disproving the method of conformal monogenic signal curvature in the second chapter, we turned our attention to the detection of edge curvature. To this end, we studied the properties of the parabolic Radon transform and proved that its smoothness changes, if the parabola we integrate over is tangential to the edge and exhibits the same curvature as the C^3-edge. By examining the Fourier transform of functions with isolated singularities, we discovered a direct connection between the Hölder class of the function and the decay rate of its Fourier transform. Combining these to results allowed us to establish a decay result for the parabolic Fourier transform that can be used to detect the edge curvature.

In the two subsequent chapters we studied the properties of the Taylorlet transform. We extended the continuous shearlet transform by shears of higher order and additionally imposed vanishing moments of higher order and the restrictiveness condition on the analyzing Taylorlet to prove the main result. The latter allows for a Taylor series expansion of the singularity curve, where the Taylor coefficients are obtained by observing the decay rate of the Taylorlet transform. After studying two construction approaches for analyzing Taylorlets, we confirmed the properties of the Taylorlet transform with a numerical example. The fifth chapter was then devoted to the translation of the Taylorlet terminology and results into the third dimension. The main strategy was to reduce every higher-dimensional integral to a one-dimensional integral that is already covered by the theory of the two-dimensional Taylorlets. For the construction of a three-dimensional Taylorlet, we fell back on the q-calculus approach of the previous chapter. Subsequently, we found a new hyperbolic case that cannot occur in the two-dimensional theory and exploited the resulting special decay rate to design a fast detection algorithm for the principal curvatures and directions.

Some new questions have arisen, particularly with regard to the Taylorlet transform. The most evident open problem here is probably an efficient discretization of the Taylorlet transform which can be solved by constructing a discrete Taylorlet frame. As the operations behind the

Taylorlet do not have a group structure, the well known machinery of coorbit theory by Feichtinger and Gröchenig [FG89a, FG89b] cannot be applied for discretization. An alternative approach to this problem would be to look for a construction similar to the design of the curvelet frames [CD05b] that also lack a group structure.

A further question of interest is the generalization of the detection result in three dimensions to higher dimensions, as well as the development of a detection algorithm with a similar speed as in $\mathbb{R}^3$. The biggest challenge for the translation of the main result is the proof of lemma 5.18 in higher dimensions, as the asymptotic analysis of the involved integrals requires a new approach.

As Candes and Donoho proved, the continuous curvelet transform is capable of resolving the wavefront set [CD05a], and Kutyniok and Labate showed that the continuous shearlet transform exhibits the same property [KL09]. Since the Taylorlet transform allows for a full characterization of the first n Taylor coefficients of the singularity function q of a feasible function, it could be beneficial to see whether it gives rise to a generalization of the wavefront set that includes higher Taylor coefficients of a singularity. A possible definition for a generalized wavefront set of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ can be given with a Taylorlet τ with infinitely many vanishing moments of order n and $\alpha \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ as

$$\mathcal{WF}_n(f)^c = \big\{(t, s_0, \ldots, s_n) \in \mathbb{R}^{n+2} : \exists \text{ open neighborhood } U \text{ of } (t, s_0, \ldots, s_n) :$$
$$\mathcal{T}_\tau^{(n,\alpha)} f(a, \cdot, \cdot) \text{ decays superpolynomially fast for } a \to 0 \text{ in } U \text{ globally}\big\}.$$

This concept could enable for a more precise description and analysis of singularities than the definition of the wavefront set allows.

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