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Complexity Analysis of Quantizations of Multidimensional Stochastic Differential Equations

Tina Klaus

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Erstgutachter:	Prof. Dr. Thomas Müller-Gronbach, Universität Passau
Zweitgutachter:	Prof. Dr. Andreas Rößler, Universität zu Lübeck

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1 Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $(B, \|\cdot\|_B)$ be a separable Banach space endowed with its Borel σ-algebra $\mathcal{B}(B)$, and let $X : \Omega \to B$ be $\mathcal{A}$-$\mathcal{B}(B)$-measurable. For $N \in \mathbb{N}$ the *quantization problem for X of level N* consists in approximating X by a $\mathcal{A}$-$\mathcal{B}(B)$-measurable mapping $\widetilde{X} : \Omega \to B$ whose range $\operatorname{ran}(\widetilde{X})$ satisfies $|\operatorname{ran}(\widetilde{X})| \leqslant N$. Any such random element $\widetilde{X}$ is called an *N-quantization of X* (in short: *quantization* of X). For $s \in [1, \infty)$ the associated *quantization error of order s* is defined by

$$e^{(s)}(X, \widetilde{X}, B) := \left(\mathbb{E}\left[\|X - \widetilde{X}\|_B^s\right]\right)^{1/s}, \tag{1}$$

and the *Nth minimal quantization error*

$$e_N^{(s)}(X, B) := \inf\left\{e^{(s)}(X, \widetilde{X}, B) \;\middle|\; \widetilde{X} \text{ is an } N\text{-quantization of } X\right\}$$

of order s (in short: *minimal quantization error of X*) is the minimal error that can be achieved by any N-quantization of X.

This dissertation is devoted to the quantization problem for X being the solution process of an autonomous d-dimensional stochastic differential equation (abridged by SDE) of the following type

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \qquad t \in [0, 1],$$

with an r-dimensional driving Brownian motion W, and we consider X as a random element with values in the space $L_p([0,1]; \mathbb{R}^d)$, $p \in [1, \infty)$, and $\mathcal{C}([0,1]; \mathbb{R}^d)$. Our main objective is to study the asymptotic behavior of the Nth minimal quantization error when the size N of the quantizations tends to infinity, and we provide sharp lower and upper error bounds along with the corresponding asymptotic constants. In particular, the results regarding the sharp asymptotic constants are new. If $d > 1$, the lower bounds hold for those N-quantizations of X which belong to the classes of *product-quantizations* of X. If $d = 1$, the lower bounds even hold for any N-quantization of X. Especially, if $d = 1$ and the dimension r of the driving Brownian motion W satisfies $r \geqslant 2$, our results generalize already existing results on the sharp asymptotics of the Nth minimal quantization error of solutions of scalar SDEs, where the term *scalar* refers to the case $d = r = 1$.

As part of our analysis, we present a semi-constructive method which yields sequences $(\widetilde{X}_N)_{N \in \mathbb{N}}$ of N-quantizations of X that satisfy

$$\limsup_{N \to \infty} \frac{e^{(s)}(X, \widetilde{X}_N, B)}{e_N^{(s)}(X, B)} \leqslant \delta$$

for a $\delta \in [1, \infty)$. Such sequences of quantizations are called *asymptotically optimal*, and, if $\delta = 1$, they are called *strongly asymptotically optimal*. Our method generalizes the quantization procedure developed for scalar SDEs presented in [MGR13]. In special cases our method is even constructive and easy to implement.

A further objective of this dissertation is to study the asymptotics of the Nth minimal quantization error of one-dimensional Itô processes in the spaces $L_p([0,1];\mathbb{R})$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R})$. More precisely, we derive new sharp upper and lower error bounds together with corresponding asymptotic constants. Subsequently, we use those results to derive the sharp asymptotics of the Nth minimal quantization error of solutions of such one-dimensional SDEs which are driven by a multidimensional Brownian motion. Furthermore, we provide a lower bound for the Nth minimal quantization error w.r.t. to certain product-quantizations of multidimensional Itô processes in the space $\mathcal{C}([0,1];\mathbb{R}^d)$. The main purpose of both results, the one on one-dimensional Itô processes and the one on multidimensional Itô processes, is to serve as auxiliary results for the analysis on quantization of multidimensional SDEs.

Our analysis of the asymptotic behavior of the Nth minimal quantization error of a solution X of an SDE is based on the general concept of information-based complexity, see [TWW88]. A problem formulation in terms of information-based complexity comprises a set F whose elements are called the *problem elements*, a normed vector space G, and an operator $S : F \to G$, which is called the *solution operator* that specifies the approximation problem under consideration. Elements $S(f)$, $f \in F$, are called *solution elements*. For $f \in F$ one aims at computing an approximation $U(f)$ of $S(f)$, and the distance of the approximation $U(f)$ to the solution element $S(f)$ is measured with respect to a prescribed error criterion. These notions are transferred to the quantization problem for X in following way. The problem elements to be approximated are the (in general infinitely many) trajectories of X, which lie in the space $\mathcal{C}([0,1];\mathbb{R}^d)$, the solution operator associated to our approximation problem is given by the identity operator on $\mathcal{C}([0,1];\mathbb{R}^d)$, approximations of the trajectories of X are given by the (finitely many) paths of a quantization $\widetilde{X}$ of X, and the error criterion is set in (1).

Moreover, the *ε-complexity* (in short: *complexity*) corresponding to our approximation problem denoted by $\mathrm{comp}(\varepsilon)$ is given by the minimal *computational cost* to obtain a quantization of X with prescribed accuracy $\varepsilon > 0$, i.e.,

$$\mathrm{comp}(\varepsilon) = \inf\left\{ \mathrm{cost}(\widetilde{X}) \;\middle|\; \widetilde{X} \text{ is a quantization of } X,\ \left(\mathbb{E}\left[\|X - \widetilde{X}\|_B^s\right]\right)^{1/s} \leqslant \varepsilon \right\},$$

where $\mathrm{cost}(\widetilde{X})$, the computational cost to construct the paths and probability weights of a quantization $\widetilde{X}$, is determined by

(i) all required function evaluations, for instance, of the coefficients a and b, which specify the SDE under consideration, and possibly of their partial derivatives, and

(ii) all required arithmetic operations.

By assuming that the construction of a path and its probability weight of a quantization $\widetilde{X}$ requires at least one function evaluation or one arithmetic operation, where each of them is performed at an a priori fixed finite unit cost $c > 0$, we obtain

$$c \cdot |\operatorname{ran}(\widetilde{X})| \leqslant \mathrm{cost}(\widetilde{X}).$$

On the other hand, the computational cost to construct the paths and probability weights of the elements of our sequences $(\widetilde{X}_N)_{N\in\mathbb{N}}$ of N-quantizations of X is close to the size of the quantization in the sense that

$$\mathrm{cost}(\widetilde{X}_N) \leqslant C \cdot \ln|\operatorname{ran}(\widetilde{X}_N)| \cdot |\operatorname{ran}(\widetilde{X}_N)|$$

where C denotes a positive real constant which does neither depend on N nor on the underlying SDE. Therefore, we roughly measure the computational cost to construct the paths and probability weights of a

quantization $\widetilde{X}$ of X by the size of $\operatorname{ran}(\widetilde{X})$. We add that we follow the procedure most common in the literature by formulating our results in terms of the sizes of quantizations rather than in terms of complexity.

A Concise Historical Overview of the Literature

We provide a concise overview of the literature in the field of quantization. In the course of this, we mainly focus on results on the asymptotics of the minimal quantization error and on constructive approaches to quantization, which are connected to the topic of this thesis. Further references of interest to our purposes will be drawn attention to at appropriate places throughout the thesis.

Quantization has been extensively investigated in various contexts since the late 1940's initiated by the development of pulse-code modulation, which was the first digital technique for conveying an analog information signal such as telephone speech over an analog channel such as a telephone wire. Especially, the papers by Shannon [Sha48], Oliver, Pierce and Shannon [OPS48], and Bennett [Ben48] are considered as seminal works for the developments in the field of quantization. Gray and Neuhoff provide an elaborate and comprehensive overview of the history and practice of quantization in their paper [GN98], which roughly comprises the period between the late 1930's and the late 1990's.

From a probabilistic point of view the following numerical integration problem is a classical problem, which, among other things, motivated the study of quantization. Let X be a random element with values in a separable Banach space $(B, \|\cdot\|_B)$, and let $T : B \to \mathbb{R}$ be a Borel measurable and integrable mapping. In many fields of application, for instance, mathematical finance, one wishes to approximate $\mathbb{E}[T(X)]$. Utilizing the notion of quantization yields an approach to obtain an approximation of $\mathbb{E}[T(X)]$. More precisely, one approximates X by an N-quantization $\widetilde{X}$ of X for a desired $N \in \mathbb{N}$ and uses

$$\mathbb{E}\big[T(\widetilde{X})\big] = \sum_{a \in \operatorname{ran}(\widetilde{X})} T(a) \cdot P\big(\{\widetilde{X} = a\}\big)$$

as an approximation for the sought quantity $\mathbb{E}[T(X)]$. The numerical computation of the above formula is feasible if one can evaluate the function T and the distribution $P_{\widetilde{X}}$ is known. Hence for practical purposes one is not interested only in the support points of a quantization but also in the corresponding probability weights.

Typical examples for the Banach space $(B, \|\cdot\|_B)$ most commonly treated in the literature are $(\mathbb{R}^d, \|\cdot\|)$ where $\|\cdot\|$ denotes any norm on $\mathbb{R}^d$, $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$, and $(L_p([0,1];\mathbb{R}^d), \|\cdot\|_{L_p([0,1];\mathbb{R}^d)})$ for $p \in [1,\infty)$. In the finite-dimensional case one speaks of *vector quantization* whereas the infinite-dimensional case is referred to as *functional quantization*.

In the context of vector quantization we refer to the standard monograph by Graf and Luschgy [GL00], which comprises a large variety of results on vector quantization from a mathematical point of view. Results regarding the sharp asymptotics of the minimal quantization error in finite-dimensional spaces for random elements with a non-singular distribution are due to Zador in 1963 [Zad63], Bucklew and Wise in 1982 [BW82], and Graf and Luschgy in 2000 [GL00]. The book by Graf and Luschgy additionally contains an investigation of the asymptotics of the minimal quantization error for several classes of random elements with a singular distribution, see [GL00, Chapter III].

In contrast to vector quantization, the topic of functional quantization has only been intensively treated since the early 2000's. In the field of functional quantization the asymptotic behavior of the minimal quantization error as well as methods for constructive quantization were first studied for Gaussian process. The derivation of upper bounds for the minimal quantization error of certain centered Gaussian processes with values in a separable Banach space was first treated by Fehringer in 2001 in his dissertation [Feh01]. Therein the asymptotic behavior of the *small ball function* corresponding to the distribution of

the respective process is utilized as one of the main tools. Fehringer also provided a lower bound for the minimal quantization error of Brownian motions in the Hilbert space $L_2([0,1];\mathbb{R})$. Combined with the upper bounds he obtained the sharp rate of convergence of $(\ln N)^{-1/2}$ for the minimal quantization error for Brownian motions in the space $L_2([0,1];\mathbb{R})$. However, the analysis in [Feh01] does not provide sharp asymptotic constants. The results of Fehringer were extended in [DFMS03] by, among other things, the provision of lower bounds for the minimal quantization error of centered Gaussian processes in separable Banach spaces. But, again, the results in [DFMS03] do not incorporate sharp asymptotic constants.

In 2002, parallel to the aforementioned works and by employing a different method, Luschgy and Pagès also derived the sharp rate of convergence of $(\ln N)^{-1/2}$ for the minimal quantization error for a large class of centered Gaussian processes in the Hilbert space $L_2([0,1];\mathbb{R})$, see [LP02]. They pursued a constructive approach based on the Karhunen-Loève expansion of the respective process and on quantization of normally distributed random variables. In particular, in the case of Brownian motions and Brownian bridges one is in full knowledge of the eigenfunctions and eigenvalues in the Karhunen-Loève expansion. We will provide a more detailed presentation of this approach in Chapter 2.

In [LP04] Luschgy and Pagès complemented their work in [LP02] by the provision of the corresponding sharp asymptotic constants but still in the Hilbert space framework only. Independently of the work by Luschgy and Pagès, Dereich derived the same results as in [LP04] in his dissertation [Der03] by using similar techniques.

Regarding a survey on constructive approaches to quantization of stochastic process and applications to mathematical finance we refer the reader to [PP09], in which, to a great extend, only Gaussian processes with values in a Hilbert space are considered. For applications of quantization to numerical methods in mathematical finance one may also consult the papers [PJ03], [PPP04] and [PP05], for example.

In the context of constructive quantization of stochastic processes in other frameworks than in a Hilbert space setting, we mention the following two approaches. In [Wil08] Wilbertz developed a constructive method for quantization of Brownian motions in the space $\mathcal{C}([0,1];\mathbb{R})$ based on spline approximations. In [LP08] Luschgy and Pagès developed a different constructive approach for quantization of stochastic processes X in the space $L_p([0,1];\mathbb{R}^d)$ for $p \in [1,\infty)$. They exploit the mean regularity of X, and the main tool in the construction is the expansion of X in terms of the Haar basis and the assumption that optimal quantizations of the corresponding coefficients are available. In particular, they obtain upper bounds for the minimal quantization error of one-dimensional Itô processes and multidimensional SDEs but in both cases without further specified asymptotic constants.

Dereich and Scheutzow fully resolved the sharp asymptotics of the minimal quantization error for a large class of Gaussian processes in the spaces $L_p([0,1];\mathbb{R})$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R})$ in 2006 in [DS06]. In [DS06] they derive the true rate of convergence of $(\ln N)^{-1/2}$ along with corresponding sharp asymptotic constants. According to the authors the results presented in the aforementioned paper served as preparatory work to derive the sharp asymptotics of the minimal quantization error of one-dimensional SDEs driven by a one-dimensional Brownian motion in the spaces $L_p([0,1];\mathbb{R})$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R})$, which were published by Dereich in 2008 in [Der08a] and [Der08b]. Under rather mild smoothness assumptions on the coefficients of the SDE it is found that the asymptotic behavior of the minimal quantization error is related to the asymptotics of the minimal quantization error of the driving Brownian motion and to the local regularity of the diffusion coefficient of the SDE. We will return to the papers [Der08a] and [Der08b] in greater detail in Chapter 4.

Prior to the work by Dereich, Luschgy and Pagès were the first who treated the complexity of quantizations of solutions of SDEs in their work [LP06] published in 2006. It was a first attempt to provide an upper bound for the minimal quantization error as well as to construct sequences of asymptotically optimal quantizations of the solution X of a one-dimensional SDE. As a core ingredient in their con-

struction Luschgy and Pagès use quantizations of the driving Brownian motion, which is similar to our method. But they employ a different method to obtain N-quantizations of X. More precisely, they solve N deterministic ordinary differential equations (abridged by ODEs) in order to obtain an N-quantization of X. Moreover, unlike our method, one of the key assumptions in [LP06] is strict positivity of the diffusion coefficient of the SDE. Due to this assumption they are in the position to apply the Lamperti transform, which is used to transform the solution X into a diffusion process which satisfies a new SDE whose diffusion coefficient is constant 1. The work in [LP06] is extended to multidimensional SDEs by Pagès and Sellami in [PS11] by again solving a system of ODEs related to the considered SDE and, instead of applying the Lamperti transform, by utilizing rough path theory.

Regarding a survey of results on the asymptotic behavior of the minimal quantization error of, on the one hand, random elements with values in finite-dimensional spaces and, on the other hand, stochastic processes, we refer the reader to [Der09].

In 2009 in [CDMGR09] Creutzig, Dereich, Müller-Gronbach and Ritter derived lower and upper bounds for the minimal quantization error of solutions of multidimensional SDEs which proved the true rate of convergence to be $(\ln N)^{-1/2}$. In the aforementioned paper this result serves as an auxiliary statement in the study of an infinite-dimensional quadrature problem, namely, the study of numerical integration of Lipschitz continuous functionals defined on a Banach space by means of deterministic and randomized algorithms. We stress that the results in [CDMGR09, Proposition 3] on the minimal quantization error do not incorporate further specified asymptotic constants. Moreover, the regularity assumptions imposed on the coefficients of the SDE in the aforementioned paper are stronger compared the ones imposed in our analysis.

Last but not least, we want to draw the reader's attention to the work by Müller-Gronbach and Ritter [MGR13] published in 2013. Therein they present a fully constructive method for quantization of a solution of a scalar SDE in the path spaces $L_p([0,1];\mathbb{R})$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R})$. This work is of great significance to this dissertation since one of our goals is to generalize the method presented in [MGR13] to the case of multidimensional SDEs driven by a multidimensional Brownian motion.

Synopsis of the Thesis and Main Results

This dissertation is organized as follows. Chapters 2 and 3 are more of a preliminary character. In Chapter 2 we provide preliminary results on quantization. First, we rigorously introduce the basic definitions and properties of quantizations already hinted at above, and, in addition to that, we present two alternative approaches to quantization. Subsequently, we focus on the existence of *optimal quantizations*, i.e., quantizations that achieve the minimal quantization error. Afterwards we present in greater detail selected results on the asymptotic behavior of the minimal quantization error already mentioned in the above literature overview, for instance, Zador's theorem and the results of Dereich and Scheutzow published in [DS06]. Additionally, we introduce several (semi-)constructive approaches to quantization. In particular, we provide a method which yields asymptotically optimal sequences of quantizations of Lévy Areas.

In Chapter 3 we first settle the main setting considered in this thesis. Afterwards we introduce the notion of strong solutions of SDEs along with standard results on existence of strong solutions. Subsequently, we introduce two classical time-discrete strong Itô-Taylor approximation schemes for strong solutions of SDEs, namely the Euler-Maruyama scheme and the Milstein Scheme. Both will play a prominent role in the succeeding chapters. In either case we additionally provide a time-continuous approximation scheme along with upper error bounds as well as results on the finiteness of certain moments.

Chapter 4 is mainly devoted to quantization of one-dimensional and multidimensional Itô processes. First, we derive the sharp asymptotics of the minimal quantization error of one-dimensional Itô processes in the spaces $L_p([0,1];\mathbb{R})$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R})$. To our knowledge the results on the lower error

bounds and on the sharp asymptotic constants are new. At this point we throw the work done by DEREICH in [Der08a] and [Der08b] into sharp relief, since it contributed greatly to the derivation of our results. More precisely, the key ideas and techniques employed in DEREICH's are carried over to prove the main result of Section 4.1, namely Theorem 4.1.2. In Section 4.2 we utilize the results obtained in Section 4.1 in order to derive new results on the sharp asymptotics of the minimal quantization error of such one-dimensional SDEs which are driven by a multidimensional Brownian motion. Those results are obtained under rather mild assumptions on the coefficients of the SDE. The last section of Chapter 4, Section 4.3, is devoted to quantization of multidimensional Itô processes. In the main result of this section, namely Theorem 4.3.2, we provide a new lower bound for the minimal quantization error w.r.t. certain product-quantizations in the space $\mathcal{C}([0,1];\mathbb{R}^d)$. The lower bound is derived by, on the one hand, utilizing the key ideas and (slightly modified) techniques employed in [Der08b], and, on the other hand, by employing further different arguments. As already mentioned, the main purpose of Chapter 4 is to provide auxiliary results for the analysis of the asymptotics of the minimal quantization error of solutions of multidimensional SDEs.

Chapter 5 is concerned with quantization of the solution $X = (X^1, \dots, X^d)'$ of a d-dimensional SDE

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW(t), \qquad t \in [0,1],$$

driven by an r-dimensional Brownian motion W, and we provide a method which yields (strongly) asymptotically optimal sequences $(\widetilde{X}_N)_{N\in\mathbb{N}}$ in the classes of product-quantizations of X in the space $(L_p([0,1];\mathbb{R}^d), \|\cdot\|_{L_p([0,1];\mathbb{R}^d)})$, $p \in [1,\infty)$, and in the space $\mathcal{C}([0,1];\mathbb{R}^d)$, on the one hand, equipped with the supremum norm $\|\cdot\|_\infty$, and, on the other hand, equipped with the norm $|||\cdot|||_s : \mathcal{C}([0,1];\mathbb{R}^d) \to [0,\infty)$ defined by

$$|||f|||_s = \left(\sum_{k=1}^{d} \|f^k\|_\infty^s\right)^{1/s},$$

where $f = (f^1, \dots, f^d)'$ and $s \in [1,\infty)$ is the moment parameter considered in the quantization error criterion (1). Clearly, $|||\cdot|||_s$ indeed defines a norm on $\mathcal{C}([0,1];\mathbb{R}^d)$. Moreover, $|||\cdot|||_s$ is equivalent to the supremum norm $\|\cdot\|_\infty$, and hence $(\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)$ is a separable Banach space. Although $|||\cdot|||_s$ is not the norm one would naturally choose when working with the space $\mathcal{C}([0,1];\mathbb{R}^d)$, it seems to be quite well-fitted when considering product-quantizations. If the dimension d of the SDE satisfies $d = 1$, our constructed sequences of quantizations of X are even strongly asymptotically optimal in the classes of all possible quantizations of X. As already mentioned, our construction generalizes the quantization procedure for solutions of scalar SDEs developed in [MGR13], and it basically consists of two steps. As a first step, we apply a coarse-level quantization which consists of quantizations of finite-dimensional projections of X. The construction of the coarse-level quantization is built-up by the time-discrete d-dimensional Milstein scheme, by quantizations of standard normally distributed random variables, and by quantizations of Lévy Areas. Especially the Lévy Areas occurring in the multidimensional Milstein scheme pose an additional challenge when considering multidimensional SDEs. As a second step, we apply a fine-level quantization which takes into account the local regularity of the components X^k of X. The techniques employed in the fine-level quantization are similar to asymptotically optimal step-size control for strong approximation of SDEs, see, for instance, [HMGR01], [MG02a], and [MG02b]. For technical reasons we use a different refinement strategy for each of the above mentioned Banach spaces. In the case of quantization in $L_p([0,1];\mathbb{R}^d)$ we separately quantize Brownian bridges defined on subintervals of $[0,1]$ by applying a sequence of quantizations of Brownian bridges on $[0,1]$ as one of the main tools. In the case of quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$ equipped with the supremum norm $\|\cdot\|_\infty$, instead of separately applying

quantizations to Brownian bridges, we quantize in one go a weighted combination of Brownian bridges. To this end, one of the core ingredients is a sequence of strongly asymptotically optimal quantizations of Brownian motions on $[0,1]$. In the case of quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$ equipped with the norm $|||\cdot|||_s$, we built-up a fine-level quantization by building blocks and ideas from both the $L_p([0,1];\mathbb{R}^d)$-case and the $\mathcal{C}([0,1];\mathbb{R}^d)$-case considered beforehand. In each case the fine-level quantization is crucial for the overall performance of the quantization of X. We add that in the case of quantization in $L_p([0,1];\mathbb{R}^d)$ we confine ourselves to the case where the moment parameter in the quantization error criterion (1) equals the parameter p. Our constructions are of a semi-constructive type since, in general, the determination of the distribution of the quantizations remains an open problem. However, in special cases our method is constructive and easy to implement, and the computational cost to determine the paths and corresponding probability weights of the quantizations $\widetilde{X}_N$ is proportional to $\ln N \cdot N$. The computational cost takes into account all required function evaluations of the coefficients of the SDE and their partial derivatives as well as all required arithmetic operations to carry out the algorithm. We point out that it is reasonable to assume that the support points and probability weights of all quantizations employed in our construction can be obtained in precomputational steps, see [Wil08] as well as the website

`http://www.quantize.maths-fi.com`

for downloads. Therefore, we do not have to take into account the cost to construct them in our analysis of the computational cost.

Chapter 6 contains the proofs of the main results of Chapter 5, namely, of Theorem 5.2.4, Theorem 5.3.4 and Theorem 5.4.1.

In Chapter 7 we close this thesis with some final remarks and a collection of open problems.

Basic Notation

We collect basic notation used throughout this dissertation. Further notations, where most of them are used only at specific places within the text, will be introduced when needed. We also refer the reader to the notation index attached at the end of this thesis.

- $\mathbb{N}$ denotes the set of natural numbers without 0, and $\mathbb{Z}$ denotes the set of integers. Moreover, for $z \in \mathbb{Z}$ we use $\mathbb{N}_z = \{n \in \mathbb{Z} \mid n \geqslant \max\{z, 0\}\}$ to denote the set of all non-negative integers being greater or equal than z.
- For a set A we use $|A| \in \{0\} \cup \mathbb{N} \cup \{\infty\}$ to denote the (possibly infinite) number of points in A.
- Let A, B be non-empty sets, and let $T : A \to B$. We use $\operatorname{ran}(T)$ to denote the image set of T, i.e., $\operatorname{ran}(T) = \{T(a) \mid a \in A\}$.
- Let $(D, \|\cdot\|_D)$ be a normed $\mathbb{R}$-vector space. By $\mathcal{B}(D)$ we denote the Borel σ-algebra on D.
- For $v \in \mathbb{R}^d$ we use $\|v\|_p$ to denote the ℓ_p-norm, $p \in [1,\infty)$, of v, and we use $\|v\|_\infty$ to denote the max-norm of v. Moreover, for a matrix $V = (v_{i,j})$, V_i and $V^{(j)}$ stand for the ith row and jth column of V, respectively, and similarly to the ℓ_p-norm on $\mathbb{R}^d$ we put $\|V\|_p := \big(\sum_{i,j} |v_{i,j}|^p\big)^{1/p}$ for $p \in [1,\infty)$.
- Let $a, b \in \mathbb{R}$ with $a < b$, let $d \in \mathbb{N}$, and let $p \in (0,\infty)$. We use $L_p([a,b];\mathbb{R}^d)$ to denote the vector space of all equivalence classes of Borel measurable functions $f : [a,b] \to \mathbb{R}^d$ such that

$$\int_a^b \|f\|_p^p \, dt < \infty.$$

We equip the space $L_p([a,b];\mathbb{R}^d)$ with the (quasi-)norm $\|\cdot\|_{L_p([a,b];\mathbb{R}^d)} : L_p([a,b];\mathbb{R}^d) \to [0,\infty)$ given by

$$\|[f]\|_{L_p([a,b];\mathbb{R}^d)} = \left(\int_a^b \|f(t)\|_p^p \, dt\right)^{1/p}.$$

We mostly write $L_p^d[a,b]$ and $\|\cdot\|_{L_p^d[a,b]}$ instead of $L_p([a,b];\mathbb{R}^d)$ and $\|\cdot\|_{L_p([a,b];\mathbb{R}^d)}$, respectively. If $d=1$, we use the even shorter notation $L_p[a,b]$ and $\|\cdot\|_{L_p[a,b]}$. We follow the literature and identify functions with their respective equivalence classes. Additionally, recall that $(L_p^d[a,b], \|\cdot\|_{L_p^d[a,b]})$ is a separable Banach space for every $p \in [1,\infty)$.

Moreover, $\mathcal{C}([a,b];\mathbb{R}^d)$ denotes the set of all continuous functions $f : [a,b] \to \mathbb{R}^d$. If not stated otherwise, we equip this space with the supremum norm $\|\cdot\|_\infty$ given by

$$\|f\|_\infty = \max_{k=1,\dots,d} \sup_{t\in[a,b]} |f^k(t)|$$

for $f = (f^1,\dots,f^d)' \in \mathcal{C}([a,b];\mathbb{R}^d)$. If $d=1$, we use the shorter notation $\mathcal{C}[a,b]$. Recall that $(\mathcal{C}([a,b];\mathbb{R}^d), \|\cdot\|_\infty)$ is a separable Banach space.

Furthermore, for $\alpha \in (0,1]$ we use $\mathcal{C}^\alpha([a,b];\mathbb{R}^d)$ to denote the set of all α-Hölder continuous functions which consists of all functions $f : [a,b] \to \mathbb{R}^d$ such that

$$|f|_{\alpha,a,b} := \sup_{a\leqslant s<t\leqslant b} \frac{\|f(t)-f(s)\|_2}{|t-s|^\alpha} < \infty,$$

and we equip this space with the norm $\|\cdot\|_{\alpha,a,b} : \mathcal{C}^\alpha([a,b];\mathbb{R}^d) \to [0,\infty)$ given by

$$\|f\|_{\alpha,a,b} = \|f\|_\infty + |f|_{\alpha,a,b}.$$

As above, if $d=1$ we use a shorter notation, namely $\mathcal{C}^\alpha[a,b]$. Additionally, if $a=0$ and $b=1$, we abbreviate $\|\cdot\|_\alpha := \|\cdot\|_{\alpha,0,1}$ as well as $|\cdot|_\alpha := |\cdot|_{\alpha,0,1}$. Last but not least, note the well known fact that $(\mathcal{C}^\alpha([a,b];\mathbb{R}^d), \|\cdot\|_{\alpha,a,b})$ is a Banach space.

- Let $d \in \mathbb{N}$, and let $f = (f^1,\dots,f^d)' : \mathbb{R}^d \to \mathbb{R}^d$ be partially differentiable. By ∇f^k we denote the gradient of f^k, and we use $\nabla f : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ to denote the Jacobean matrix of f, i.e.,

$$\nabla f = \begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^d}{\partial x_1} & \cdots & \frac{\partial f^d}{\partial x_d} \end{pmatrix}.$$

- In the whole thesis ln stands for the natural logarithm, and occasionally we make use of the Landau symbol o.

Preliminaries on Stochastic Processes

In this subsection, let $d \in \mathbb{N}$, let $(\Omega,\mathcal{A},P)$ be a probability space, and let $a,b \in \mathbb{R}$ with $a<b$.

A family $X = (X(t))_{t\in[a,b]}$ of $\mathcal{A}$-$\mathcal{B}(\mathbb{R}^d)$-measurable mappings $X(t) : \Omega \to \mathbb{R}^d$ is called an **$\mathbb{R}^d$-valued stochastic process (with time interval $[a,b]$)**. For fixed $\omega \in \Omega$ the function $X(\cdot,\omega) : [a,b] \to \mathbb{R}^d$ given by $X(t,\omega) = X(t)(\omega)$ is called a **sample path** or **trajectory** of X associated with ω, and we

call X **continuous** if all sample paths of X are continuous. Moreover, X is called **measurable** if the mapping $\psi_X : [a,b] \times \Omega \to \mathbb{R}^d$ defined by

$$\psi_X(t,\omega) = X(t)(\omega)$$

is $\mathcal{B}([a,b]) \otimes \mathcal{A}$-$\mathcal{B}(\mathbb{R}^d)$-measurable. All stochastic processes of interest in this thesis possess continuous sample paths, which guarantees measurability, see, for instance, [KS88, Remark 1.14].

In this thesis we mostly interpret measurable stochastic processes as random elements with values in a function space such as the space of (equivalence classes) of *p-integrable functions* $L_p^d[a,b]$, $p \in [1,\infty)$ or the space of continuous functions $\mathcal{C}([a,b];\mathbb{R}^d)$. The following results motivates this viewpoint on stochastic processes. For proofs we refer the reader to [PZ14, Proposition 3.18 and Proposition 3.19].

Lemma 1.1
Let $X = (X(t))_{t\in[a,b]}$ be a measurable $\mathbb{R}^d$-valued stochastic process on $(\Omega, \mathcal{A}, P)$.

a) Assume that X is continuous. Then the mapping $\widetilde{X} : \Omega \to \mathcal{C}([a,b];\mathbb{R}^d)$ defined by $\widetilde{X}(\omega) = X(\cdot,\omega)$ is $\mathcal{A}$-$\mathcal{B}\big(\mathcal{C}([a,b];\mathbb{R}^d)\big)$-measurable.

b) The mapping $\overline{X} : \Omega \to L_p^d[a,b]$ defined by

$$\overline{X}(\omega) = \begin{cases} \big[X(\cdot,\omega)\big], & \text{if } X(\cdot,\omega) \text{ is } p\text{-integrable}, \\ 0, & \text{else}, \end{cases}$$

is $\mathcal{A}$-$\mathcal{B}\big(L_p^d[a,b]\big)$-measurable.

Hence saying that we interpret X as a random element with values in $L_p^d[a,b]$ or $\mathcal{C}([a,b],\mathbb{R}^d)$ means that we are actually considering the associated random elements $\widetilde{X}$ and $\overline{X}$, respectively. For the rest of this thesis we abuse notation and write X instead of $\widetilde{X}$ and $\overline{X}$.

2 | Preliminaries on Quantization

In this chapter we present those definitions and results in the field of *quantization* of a random element Z which are important to our purposes. In particular, we present selected constructive approaches to quantization of specific random elements, which will be employed in the subsequent chapters of this thesis. This chapter is organized as follows. In Section 1 we first introduce basic definitions with regard to quantization of Z. Subsequently, we pay special attention to so-called *Voronoi quantizations* of Z since this concept will be exploited at several places within the text. We close the first section by presenting those properties of the *Nth minimal quantization error* which are most important to our purposes.

In Section 2 we provide selected results on the existence of *optimal quantizations* in the context of those Banach spaces which will play a crucial role in this thesis.

In the last section, Section 3, our focus is on the asymptotic behavior of the Nth minimal quantization error when N tends to infinity. We commence by presenting the main result in the finite-dimensional case, namely, Zador's theorem. Afterwards we outline a method which yields an *asymptotically optimal* sequence of quantizations of standard normally distributed random variables. Additionally, we collect further properties of the resulting sequence of quantizations, which will prove beneficial in the following chapters. Subsequent to that, we focus on quantization of Lévy areas. More precisely, we present a different construction leading to a sequence of quantizations of the following stochastic Itô integral $\int_0^1 W_1(s)\,dW_2(s)$ where (W_1, W_2) is a two-dimensional Brownian motion on $[0,1]$. Both aforementioned constructions of sequences of quantizations will be employed in Chapter 5. In the last part of Section 3 we present the sharp asymptotics of the Nth minimal quantization error of Brownian motions and Brownian bridges on $[0,1]$ where both of them are viewed as random elements in the spaces $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ and $(L_p[0,1], \|\cdot\|_{L_p[0,1]})$ for $p \in [1,\infty)$. Additionally, we outline a constructive approach to quantization of a Brownian bridge B on $[0,1]$ which is based on the Karhunen-Loève expansion of B and on quantizations of standard normally distributed random variables.

We add that the topics of quantization of SDEs and, more general, quantization of Itô processes are postponed to the Chapters 4 and 5.

2.1. Basic Definitions and Basic Properties of Quantizations

In the whole section let $s \in [1,\infty)$, let $(\Omega, \mathcal{A}, P)$ be a probability space, let $(B, \|\cdot\|_B)$ be a separable $\mathbb{R}$-Banach space endowed with its Borel σ-field $\mathcal{B}(B)$, and let $Z : \Omega \to B$ be $\mathcal{A}$-$\mathcal{B}(B)$-measurable. Furthermore, to avoid trivial cases we require $|\operatorname{ran}(Z)| = \infty$, and we assume the following integrability condition to hold:

$$\mathbb{E}\big[\|Z\|_B^s\big] < \infty. \tag{2.1.1}$$

Remark 2.1.1
Let $\overline{Z} : \Omega \to B$ be $\mathcal{A}$-$\mathcal{B}(B)$-measurable. The separability assumption on B then guarantees that $Z + \overline{Z}$ is again $\mathcal{A}$-$\mathcal{B}(B)$-measurable. In contrast to that, the sum of two random elements with values in a non-separable Banach space need not be a random element. But since the random element which is added to Z in the upcoming Definition 2.1.2 takes only finitely many values, measurability of their sum is not an issue. Furthermore, we point out that the integrability condition (2.1.1) ensures that all expected values in the upcoming Definition 2.1.2 are finite.

Definition 2.1.2
Let $N \in \mathbb{N}$.

a) A random element $\widetilde{Z} : \Omega \to B$ with $|\operatorname{ran}(\widetilde{Z})| \leqslant N$ is called an ***N*-quantization of *Z***.

b) For an N-quantization $\widetilde{Z}$ of Z the **quantization error (of order *s*)** is defined by

$$e^{(s)}(Z, \widetilde{Z}, B) := \Big(\mathbb{E}\Big[\|Z - \widetilde{Z}\|_B^s\Big]\Big)^{1/s}.$$

Moreover, the ***N*th minimal quantization error of *Z* (of order *s*)** is given by

$$e_N^{(s)}(Z, B) := \inf\big\{e^{(s)}(Z, \widetilde{Z}, B) \;\big|\; \widetilde{Z} \text{ is an } N\text{-quantization of } Z\big\}.$$

c) An N-quantization $\widetilde{Z}$ of Z is called an ***N*-optimal quantization of *Z* (of order *s*)** if

$$e^{(s)}(Z, \widetilde{Z}, B) = e_N^{(s)}(Z, B).$$

Notation
For $N \in \mathbb{N}$ we denote the set which contains all N-optimal quantizations of Z of order s by $C_N^{(s)}(Z, B)$.

Remark 2.1.3

a) The set $C_N^{(s)}(Z, B)$ might be empty, see Remark 2.2.4 c).

b) In the case that Z has finite range itself, the task of approximating Z by an N-quantization $\widetilde{Z}$ is trivial if N is sufficiently large. In that case a suitable choice is $\widetilde{Z} = Z$. This is the reason why we assumed $|\operatorname{ran}(Z)| = \infty$ at the beginning.

c) For applications of quantizations to, for instance, numerical integration one seeks for such quantizations of Z which can be implemented. But, unfortunately, in general the probability weights corresponding to a quantization of Z are hard to compute. We will come back to this problem at certain places within this thesis.

Apart from the approach to quantization of a random element presented in Definition 2.1.2, we outline two different approaches to quantization, and we will see in Lemma 2.1.4 that those alternative approaches are equivalent to the already presented one.

The first alternative approach to the notion of quantization is connected to the application of quantizations in the field of signal processing and source coding. Roughly speaking, the main task in this context consists in processing a signal with (possibly) infinitely many outcomes via a communication channel with limited bandwidth. To this end, the original signal has to be transferred into a signal which takes only

finitely many values. Hence, from this point of view, one obtains an N-quantization of Z by composing Z with a Borel measurable function $f : B \to B$ with $|\operatorname{ran}(f)| \leqslant N$ such that the resulting N-quantization of Z is of the form $\widetilde{Z} = f(Z)$. From now on, we call such a function f an ***N*-quantizer**.

The second alternative approach to the concept of quantization consists in constructing a set $\alpha \subseteq B$ with $1 \leqslant |\alpha| \leqslant N$ such that the sth mean distance of Z and α is as small as possible. From now on, we call any set $\alpha \subseteq B$ with $1 \leqslant |\alpha| \leqslant N$ an ***N*-codebook**.

The following lemma shows that for all $N \in \mathbb{N}$ the minimal errors induced by those two alternative approaches to the concept of quantization are the same as the Nth minimal quantization error of Z defined in Definition 2.1.2 b). Consequently, this lemma justifies switching between those three approaches when constructing quantizations of random elements, and therefore we might choose that approach which is the most advantageous in the respective situation.

Lemma 2.1.4
For all $N \in \mathbb{N}$ it holds

$$\begin{aligned} e_N^{(s)}(Z, B) &= \inf\Big\{ \Big(\mathbb{E}\Big[\|Z - f(Z)\|_B^s\Big]\Big)^{1/s} \;\Big|\; f : B \to B \text{ Borel measurable}, |\operatorname{ran}(f)| \leqslant N \Big\} \\ &= \inf\Big\{ \Big(\mathbb{E}\Big[\min_{a\in\alpha} \|Z - a\|_B^s\Big]\Big)^{1/s} \;\Big|\; \alpha \subseteq B,\ 1 \leqslant |\alpha| \leqslant N \Big\}. \end{aligned}$$

In the proof of the previous lemma we will employ the notion of Voronoi partitions of B induced by finite subsets of B. For the convenience of the reader we provide a definition of the notion of Voronoi partitions at this point.

Definition 2.1.5
Let $\alpha \subseteq B$ with $1 \leqslant |\alpha| < \infty$. We call $\{V_\alpha(a) \mid a \in \alpha\}$ a **Voronoi partition of *B* induced by *α*** if the following is satisfied:

(i) For all $a \in \alpha$ it holds $V_\alpha(a) \in \mathcal{B}(B)$.

(ii) For all $a \in \alpha$ it holds $V_\alpha(a) \subseteq \big\{x \in B \,\big|\, \|x - a\|_B = \min_{b\in\alpha} \|x - b\|_B\big\}$.

(iii) For all $a, a' \in \alpha$ with $a \neq a'$ it holds $V_\alpha(a) \cap V_\alpha(a') = \varnothing$.

(iv) $\bigcup_{a\in\alpha} V_\alpha(a) = B$.

Moreover, the sets $V_\alpha(a)$, $a \in \alpha$, are called **Voronoi cells** of the Voronoi partition $\{V_\alpha(a) \mid a \in \alpha\}$.

Remark 2.1.6
For every $\alpha \subseteq B$ with $1 \leqslant |\alpha| < \infty$ there exists at least one Voronoi partition of B induced by α.

Indeed, we denote the elements of α by a_i for $i = 1, \ldots, |\alpha|$, and we abbreviate

$$W_\alpha(a_i) := \big\{x \in B \,\big|\, \|x - a_i\|_B = \min_{b\in\alpha} \|x - b\|_B\big\}$$

for all $i = 1, \ldots, |\alpha|$. Now, put $V_\alpha(a_1) := W_\alpha(a_1)$ and

$$V_\alpha(a_i) := W_\alpha(a_i) \setminus \bigcup_{j=1}^{i-1} W_\alpha(a_j)$$

for all $i = 2, \ldots, |\alpha|$. Then it is easy to see that $\big\{V_\alpha(a_i) \;\big|\; i = 1, \ldots, |\alpha|\big\}$ is a Voronoi partition of B induced by α.

Proof of Lemma 2.1.4:
Let $N \in \mathbb{N}$. First, let $f : B \to B$ be a Borel measurable mapping with $|\operatorname{ran}(f)| \leqslant N$. Then $\widetilde{Z} := f(Z)$ is an N-quantization of Z, and we have

$$e_N^{(s)}(Z,B) \leqslant e^{(s)}(Z,\widetilde{Z},B) = \Big(\mathbb{E}\Big[\|Z - f(Z)\|_B^s\Big]\Big)^{1/s}.$$

Therefore,

$$e_N^{(s)}(Z,B) \leqslant \inf\Big\{ \Big(\mathbb{E}\Big[\|Z - f(Z)\|_B^s\Big]\Big)^{1/s} \;\Big|\; f : B \to B \text{ Borel measurable}, |\operatorname{ran}(f)| \leqslant N\Big\}.$$

Secondly, let $\widetilde{Z} : \Omega \to B$ be an N-quantization of Z. Then $\beta := \widetilde{Z}(\Omega)$ is an N-codebook, and hence

$$\inf\Big\{ \Big(\mathbb{E}\Big[\min_{a\in\alpha}\|Z - a\|_B^s\Big]\Big)^{1/s} \;\Big|\; \alpha \subseteq B,\, 1 \leqslant |\alpha| \leqslant N\Big\} \leqslant \Big(\mathbb{E}\Big[\min_{b\in\beta}\|Z - b\|_B^s\Big]\Big)^{1/s} \leqslant \Big(\mathbb{E}\Big[\|Z - \widetilde{Z}\|_B^s\Big]\Big)^{1/s},$$

which leads to

$$\inf\Big\{ \Big(\mathbb{E}\Big[\min_{a\in\alpha}\|Z - a\|_B^s\Big]\Big)^{1/s} \;\Big|\; \alpha \subseteq B,\, 1 \leqslant |\alpha| \leqslant N\Big\} \leqslant e_N^{(s)}(Z,B).$$

It remains to prove

$$\begin{aligned} &\inf\Big\{ \Big(\mathbb{E}\Big[\|Z - f(Z)\|_B^s\Big]\Big)^{1/s} \;\Big|\; f : B \to B \text{ Borel measurable}, |\operatorname{ran}(f)| \leqslant N\Big\} \\ &\leqslant \inf\Big\{ \Big(\mathbb{E}\Big[\min_{a\in\alpha}\|Z - a\|_B^s\Big]\Big)^{1/s} \;\Big|\; \alpha \subseteq B,\, 1 \leqslant |\alpha| \leqslant N\Big\}. \end{aligned} \tag{2.1.2}$$

Let $\alpha \subseteq B$ with $1 \leqslant |\alpha| \leqslant N$, and let $\mathcal{V}_\alpha = \{V_\alpha(a) \mid a \in \alpha\}$ be a Voronoi partition of B induced by α. Moreover, we define $\pi_{\mathcal{V}_\alpha} : B \to B$ by

$$\pi_{\mathcal{V}_\alpha}(b) = \sum_{a\in\alpha} a \cdot \mathbb{1}_{V_\alpha(a)}(b). \tag{2.1.3}$$

Due to (i) and (iii) in Definition 2.1.5, and due to the choice of α, the mapping $\pi_{\mathcal{V}_\alpha}$ is well-defined and Borel measurable with $|\operatorname{ran}(\pi_{\mathcal{V}_\alpha})| \leqslant N$. Additionally, in view of Definition 2.1.5 (ii)--(iv), it holds

$$\begin{aligned} \|Z(\omega) - \pi_{\mathcal{V}_\alpha}(Z(\omega))\|_B &= \sum_{b\in\alpha} \|Z(\omega) - b\|_B \cdot \mathbb{1}_{V_\alpha(b)}(Z(\omega)) \\ &= \sum_{b\in\alpha} \min_{a\in\alpha} \|Z(\omega) - a\|_B \cdot \mathbb{1}_{V_\alpha(b)}(Z(\omega)) \\ &= \min_{a\in\alpha} \|Z(\omega) - a\|_B \end{aligned}$$

for all $\omega \in \Omega$. Therefore,

$$\inf\Big\{ \Big(\mathbb{E}\Big[\|Z - f(Z)\|_B^s\Big]\Big)^{1/s} \;\Big|\; f : B \to B \text{ Borel measurable}, |\operatorname{ran}(f)| \leqslant N\Big\} \;\leqslant \Big(\mathbb{E}\Big[\min_{a\in\alpha}\|Z - a\|_B^s\Big]\Big)^{1/s}.$$

This implies (2.1.2), which finishes the proof. □

Remark 2.1.7
Let $N \in \mathbb{N}$, and let $\alpha \subseteq B$ with $1 \leqslant |\alpha| \leqslant N$. Furthermore, let $\mathcal{V}_\alpha = \{V_\alpha(a) \mid a \in \alpha\}$ be a Voronoi partition of B induced by α. The mapping $\pi_{\mathcal{V}_\alpha}$ defined in (2.1.3) is called **nearest-neighbor projection associated to $\mathcal{V}_\alpha$**. Note that for every Borel measurable mapping $f : B \to B$ with $f(B) \subseteq \alpha$ it holds

$$\mathbb{E}\big[\|Z - f(Z)\|_B^s\big] \geqslant \mathbb{E}\Big[\min_{b\in\alpha} \|Z - b\|_B^s\Big] = \mathbb{E}\big[\|Z - \pi_{\mathcal{V}_\alpha}(Z)\|_B^s\big].$$

Thus, $\pi_{\mathcal{V}_\alpha}(Z)$ yields the best approximation of Z compared to all approximations of Z by a random element of the form $f(Z)$ where $f : B \to B$ is a Borel measurable mapping with values in α.

We close this section by presenting selected well known properties of the Nth minimal quantization error. To underline the importance of assuming the integrability condition (2.1.1) to hold, on the one hand, and considering *separable* Banach spaces, on the other hand, we also incorporate a proof.

Proposition 2.1.8

a) For all $N \in \mathbb{N}$ it holds $e_N^{(s)}(Z, B) \geqslant e_{N+1}^{(s)}(Z, B)$.

b) It holds $\lim_{N\to\infty} e_N^{(s)}(Z, B) = 0$.

c) Let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space, and let $S : B \to E$ be a bounded linear operator. Then, for all $N \in \mathbb{N}$,

$$e_N^{(s)}(S(Z), E) \leqslant \|S\|_{\mathrm{op}} \cdot e_N^{(s)}(Z, B).$$

Proof:

a) This statement follows directly from the definition of the Nth minimal quantization error of Z.

b) Since B is separable, there exists a set $\widetilde{B} = \{y_n \mid n \in \mathbb{N}\} \subseteq B$ such that $\mathrm{cl}(\widetilde{B}) = B$. W.l.o.g. we may assume $y_1 = 0$. Lemma 2.1.4 leads to

$$0 \leqslant \big(e_N^{(s)}(Z, B)\big)^s = \inf\Big\{\mathbb{E}\Big[\min_{a\in\alpha} \|Z - a\|_B^s\Big] \;\Big|\; \alpha \subseteq B,\, 1 \leqslant |\alpha| \leqslant N\Big\} \leqslant \mathbb{E}\Big[\min_{i=1,\dots,N} \|Z - y_i\|_B^s\Big] \quad (2.1.4)$$

for all $N \in \mathbb{N}$. Additionally, the fact that $\widetilde{B}$ is dense in B yields

$$\lim_{N\to\infty} \min_{i=1,\dots,N} \|Z - y_i\|_B = 0.$$

Furthermore, for all $N \in \mathbb{N}$ the random variable $\min_{i=1,\dots,N} \|Z - y_i\|_B^s$ is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable, and

$$\min_{i=1,\dots,N} \|Z - y_i\|_B^s \leqslant \|Z - y_1\|_B^s = \|Z\|_B^s.$$

Also recall that we have assumed the integrability condition (2.1.1) to hold. Thus, all assumptions of Lebesgue's dominated convergence theorem are satisfied, which then leads to

$$\lim_{N\to\infty} \mathbb{E}\Big[\min_{i=1,\dots,N} \|Z - y_i\|_B^s\Big] = \mathbb{E}\Big[\lim_{N\to\infty} \min_{i=1,\dots,N} \|Z - y_i\|_B^s\Big] = 0.$$

Together with (2.1.4) we finally arrive at

$$\lim_{N\to\infty} e_N^{(s)}(Z, B) = 0.$$

c) Keep in mind that we assumed the integrability condition (2.1.1) to hold, and then combine Lemma 2.1.4 with the proof of Lemma 1 in [GLP07].

□

2.2. Existence of Optimal Quantizations

In this section we present selected results on existence of N-optimal quantizations since those results justify certain assumptions to be made in the subsequent chapters. The results provided in this section are due to Graf, Luschgy and Pagès, see [GLP07].

As in the previous section, let $s \in [1, \infty)$, let $(B, \|\cdot\|_B)$ be a separable $\mathbb{R}$-Banach space endowed with its Borel σ-field $\mathcal{B}(B)$, and let Z be a B-valued random element with $|\operatorname{ran}(Z)| = \infty$ which satisfies the integrability condition (2.1.1), i.e.,

$$\mathbb{E}\big[\|Z\|_B^s\big] < \infty.$$

Recall that the dual B' and bidual B'' of B are given by

$$B' = L(B, \mathbb{R}) \quad \text{and} \quad B'' = L(B', \mathbb{R}),$$

respectively. Equipped with the respective operator norm $\|\cdot\|_{\mathrm{op}}$ each of them is again a Banach space. Here we abuse notation and use the same symbol for both norms.

Before we present the main theorems of this section, we first need to clarify what is meant if B is said to be 1-*complemented in its bidual* B'', since this property is one of the prerequisites in the upcoming theorems.

Definition 2.2.1
The Banach space B is said to be **1-complemented in its bidual** B'' if there exists a linear projection $P : B'' \to \iota(B)$ such that

$$\|P\|_{\mathrm{op}} \leqslant 1$$

where $\iota : B \to B''$ denotes the canonical embedding of B into B''.

In the following remark we present a sufficient condition for a Banach space to be 1-complemented in its bidual space. For further sufficient conditions which guarantee this property we refer to [GLP07, Corollary 1].

Remark 2.2.2
Assume that B is reflexive. Then the canonical embedding operator $\iota : B \to B''$ is an isometric isomorphism, and in particular it holds $\iota(B) = B''$. Now consider the identity mapping $\mathrm{id}_{B''}$ on B''. Then $\mathrm{id}_{B''}$ is a linear projection, and it is easy to see that $\|\mathrm{id}_{B''}\|_{\mathrm{op}} \leqslant 1$. Consequently, B is 1-complemented in its bidual B''. To sum up, every reflexive Banach space is 1-complemented in its bidual space.

In the following theorem we summarize some of the main results in [GLP07] on the existence of N-optimal quantizations of Z.

Theorem 2.2.3 *([GLP07, Theorem 1, Proposition 2])*
Assume that B is 1-complemented in its bidual B''. Then, for all $N \in \mathbb{N}$,

$$C_N^{(s)}(Z, B) \neq \varnothing.$$

Remark 2.2.4

a) All $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$-spaces with $p \in [1,\infty)$ are 1-complemented in their bidual spaces. Indeed, for $p \in (1,\infty)$, the space $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$ is reflexive and thus Remark 2.2.2 applies. The space $(L_1^d[0,1], \|\cdot\|_{L_1^d[0,1]})$, however, is not reflexive. For more details on why $(L_1^d[0,1], \|\cdot\|_{L_1^d[0,1]})$ satisfies the assumptions of Theorem 2.2.3, we refer to [GLP07, Corollary 1 (i) and the Example on p. 32]. Thus, any random element with values in one of those spaces admits for all $N \in \mathbb{N}$ at least one N-optimal quantization, and hence in those spaces the infimum in Definition 2.1.2 b) actually stands as a minimum.

b) Assume $(B, \|\cdot\|_B) = (\mathbb{R}^d, \|\cdot\|)$ where $\|\cdot\|$ denotes any norm on $\mathbb{R}^d$. Since $\mathbb{R}^d$ is a reflexive Banach space, Remark 2.2.2 yields that Theorem 2.2.3 incorporates the case of finite-dimensional Banach spaces as well. Regarding existence results on optimal quantizations in this context we also refer the reader to [GL00, 4.12 Theorem].

In particular, if $d = 1$, even more is known. Under certain additional assumptions not only does an N-optimal quantization exist but it is also unique. More precisely, if P_Z is **strongly unimodal**, i.e., P_Z is an absolutely continuous distribution with Lebesgue density h such that $I := \{x \in \mathbb{R} \mid h(x) > 0\}$ is an open interval and $\ln \circ h_{|I}$ is concave, then $\left|C_N^{(s)}(Z, \mathbb{R})\right| = 1$ for all $N \in \mathbb{N}$, see [GL00, 5.1 Theorem].

c) Theorem 2.2.3 is not applicable to the case $(B, \|\cdot\|_B) = (\mathcal{C}([0,1]; \mathbb{R}^d), \|\cdot\|_\infty)$ since this space is not 1-complemented in its bidual space, see the counterexample [GLP07, p. 42]. Therein a real-valued stochastic process on $[0,1]$ with continuous paths is presented which does not possess a 1-optimal quantization in the space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$.

However, in the upcoming Theorem 2.2.7 we will see that for all $N \in \mathbb{N}$ there exists a random element with values in the space of all bounded, Borel measurable functions which takes exactly N values and which achieves the minimal quantization error in the space $(\mathcal{C}([0,1]; \mathbb{R}^d), \|\cdot\|_\infty)$.

Example 2.2.5
We consider the Banach space $(\mathbb{R}, |\cdot|)$, and we assume $Z \sim N(0,1)$. Then, P_Z is absolutely continuous with Lebesgue density $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp(-x^2/2).$$

Note that $I = \{x \in \mathbb{R} \mid \varphi(x) > 0\} = \mathbb{R}$, and for $x \in I$ we have

$$\ln(\varphi(x)) = -\ln\left(\sqrt{2\pi}\right) - \frac{x^2}{2}.$$

Clearly, $\ln \circ \varphi_{|I}$ is concave, and thus due to Remark 2.2.4 b), for all $s \in [1,\infty)$ and all $N \in \mathbb{N}$, there exists exactly one N-optimal quantization $\widetilde{Z}_N$ of Z of order s.

Remark 2.2.6
In [PJ03] Pagès and Printems present numerical methods to determine N-optimal quantizations of one- and multidimensional standard normally distributed random variables in the case $s = 2$. In the one-dimensional case they use Newton's method in order to obtain approximations of N-optimal quantizations. Whereas in the case of quantization in the space $\mathbb{R}^d$, $d \in \mathbb{N}_2$, stochastic gradient methods are employed. The support points and their corresponding probability weights of the approximations of N-optimal quantizations determined with the just described methods are available for downloads at the website

`http://quantize.maths-fi.com`.

In the one-dimensional case the database comprises optimal quantizations up to a size of $N = 5999$. In the multidimensional case it comprises optimal quantizations up to a size of $N = 1450$ for dimensions between $d = 2$ and $d = 10$.

For the next theorem we put

$$\mathcal{M}^b([0,1];\mathbb{R}^d) := \{f : [0,1] \to \mathbb{R}^d \mid f \text{ is Borel measurable and bounded}\},$$

and we equip this space with the supremum norm $\|\cdot\|_\infty$. It is well known that $\big(\mathcal{M}^b([0,1];\mathbb{R}^d), \|\cdot\|_\infty\big)$ is a Banach space.

Theorem 2.2.7 *([GLP07, Proposition 1, Theorem 3, Theorem 4])*

(i) For all $N \in \mathbb{N}$ it holds

$$e_N^{(s)}\big(Z, \mathcal{C}([0,1];\mathbb{R}^d)\big) = e_N^{(s)}\big(Z, \mathcal{M}^b([0,1];\mathbb{R}^d)\big).$$

(ii) For all $N \in \mathbb{N}$ there exists a $\mathcal{M}^b([0,1];\mathbb{R}^d)$-valued random element $\widetilde{Z}_N$ with $|\operatorname{ran}(\widetilde{Z}_N)| = N$ such that

$$e_N^{(s)}\big(Z, \mathcal{C}([0,1];\mathbb{R}^d)\big) = \Big(\mathbb{E}\Big[\|Z - \widetilde{Z}_N\|_\infty^s\Big]\Big)^{1/s}.$$

2.3. Asymptotic Behavior of the Minimal Quantization Error

In the foregoing section we treated the question of whether optimal quantizations of a random element Z with values in a separable $\mathbb{R}$-Banach space $(B, \|\cdot\|_B)$ exist. But even if one knows that such quantizations exist, they are in general hard to determine. Thus, in most cases, one confines oneself to an analysis of the asymptotic behavior of the Nth minimal quantization error of Z.

Recall that in Proposition 2.1.8 we have already shown that, for all $s \in [1,\infty)$,

$$\lim_{N\to\infty} e_N^{(s)}(Z, B) = 0.$$

Now two research questions naturally arise:

- How fast does $(e_N^{(s)}(Z,B))_{N\in\mathbb{N}}$ converge to 0?
- How to construct (and implement) *asymptotically optimal sequences* $(\widetilde{Z}_N)_{N\in\mathbb{N}}$ of N-quantizations of Z?

Before we try to give answers to those questions, we first need to clarify what is meant by an *asymptotically optimal sequence of N-quantizations*.

Definition 2.3.1
Let $s \in [1,\infty)$, let $(B, \|\cdot\|_B)$ be a separable $\mathbb{R}$-Banach space, and let Z be a B-valued random element with $|\operatorname{ran}(Z)| = \infty$. A sequence $(\widetilde{Z}_N)_{N\in\mathbb{N}}$ of N-quantizations of Z is called **asymptotically optimal (of order s)** if there exists a $\delta \in [1,\infty)$ such that

$$\limsup_{N\to\infty} \frac{e^{(s)}\big(Z, \widetilde{Z}_N, B\big)}{e_N^{(s)}\big(Z, B\big)} \leqslant \delta.$$

If the previous inequality holds with $\delta = 1$, the sequence $(\widetilde{Z}_N)_{N\in\mathbb{N}}$ is called **strongly asymptotically optimal (of order s)**.

Analogously, we call a sequence $(f_N)_{N\in\mathbb{N}}$ of N-quantizers $f_N : B \to B$ strongly asymptotically optimal (of order s) if

$$\limsup_{N\to\infty} \frac{e^{(s)}\big(Z, f_N(Z), B\big)}{e_N^{(s)}\big(Z, B\big)} \leqslant 1.$$

2.3.1. Finite-dimensional Vector Quantization

In this subsection we consider the Banach space $(\mathbb{R}^d, \|\cdot\|)$ where, for now, $\|\cdot\|$ denotes any norm on $\mathbb{R}^d$.

Zador's Theorem

We present the main result on the sharp asymptotics of the minimal quantization error in the context of finite-dimensional vector quantization, namely, Zador's theorem. It was first stated by Zador (1963) [Zad63] in the case $s = 2$ for random vectors with an absolutely continuous distribution. The theorem in its general form, i.e., for any $\mathbb{R}^d$-valued random vector, is due to Bucklew and Wise (1982) [BW82]. But Bucklew and Wise also treat the case $s = 2$, only. Here we present the version of Zador's theorem as stated in [GL00, 6.2 Theorem], which is formulated for any $\mathbb{R}^d$-valued random vector and any moment parameter $s \in [1, \infty)$.

Theorem 2.3.2
Let $s \in [1, \infty)$, let Z be an $\mathbb{R}^d$-valued random vector with distribution P_Z, and let

$$P_Z = P_Z^{\mathrm{cont}} + P_Z^{\mathrm{sing}}$$

be the Lebesgue decomposition of P_Z with respect to the d-dimensional Lebesgue measure λ_d where P_Z^{cont} denotes the absolutely continuous and P_Z^{sing} the singular part of P_Z. Furthermore, let $h : \mathbb{R}^d \to [0, \infty)$ be a Lebesgue density of P_Z^{cont}, and suppose that $\mathbb{E}\big[\|Z\|^{s+\delta}\big] < \infty$ for a $\delta > 0$. Then there exists a constant $c(s, d) \in (0, \infty)$ depending only on the moment parameter s, the dimension d and the norm $\|\cdot\|$ such that

$$\lim_{N\to\infty} N^{1/d} \cdot e_N^{(s)}(Z, \mathbb{R}^d) = c(s, d) \cdot \left(\int_{\mathbb{R}^d} |h(x)|^{d/(d+s)}\, dx\right)^{(d+s)/ds}. \tag{2.3.1}$$

Remark 2.3.3

a) The exact value of the constant $c(s, d)$ in the previous theorem is known only in very special cases. For instance, if $d = 1$ and $\|\cdot\| = |\cdot|$, it holds

$$c(s, 1) = \frac{1}{2 \cdot (s + 1)^{1/s}}$$

for all $s \in [1, \infty)$, see, for example, [GL00, Lemma 2.9 combined with Remark 8.10 (c)].

b) The moment condition $\mathbb{E}\big[\|Z\|^{s+\delta}\big] < \infty$ in Theorem 2.3.2 ensures that

$$\left(\int_{\mathbb{R}^d} |h(x)|^{d/(d+s)}\, dx\right)^{(d+s)/ds} < \infty.$$

For a proof see [GL00, 6.3 Remark]. Moreover, we refer the reader to the example [GL00, Example 6.4], which shows that the moment condition in Theorem 2.3.2 cannot be weakened.

c) If

$$\int_{\mathbb{R}^d} |h(x)|^{d/(d+s)}\, dx > 0,$$

Theorem 2.3.2 yields that the Nth minimal quantization error $e_N^{(s)}(Z, \mathbb{R}^d)$ goes to 0 as fast as $N^{-1/d}$ when N tends to infinity. If $h = 0$ a.e., (2.3.1) implies only $e_N^{(s)}(Z, \mathbb{R}^d) = o(N^{-1/d})$.

Quantization of $N(0,1)$-distributed Random Variables

In this subsection we assume $d = 1$ and $\|\cdot\| = |\cdot|$. Moreover, let Z be a real-valued random variable with $Z \sim N(0,1)$. We outline a construction which yields a strongly asymptotically optimal sequence of N-quantizations of Z. In [GL00, Chapter 7] one finds a more detailed discussion on this construction, which may be applied to a wider class of real-valued random elements than only to standard normally distributed ones, see also Remark 2.3.5 at the end of this subsection.

Let $s \in [1, \infty)$, and let $N \in \mathbb{N}$. Moreover, for $i = 1, \ldots, N$ let $z_{(2i-1)/2N}$ denote the $(2i-1)/2N$-quantile of the standard normal distribution and put

$$a_{i,N}^{(s)} := \sqrt{1+s} \cdot z_{\frac{2i-1}{2N}}. \tag{2.3.2}$$

Then $a_{1,N}^{(s)} < a_{2,N}^{(s)} < \cdots < a_{N,N}^{(s)}$. In addition to that, we define

$$A_N^{(s)} := \{a_{1,N}^{(s)}, \ldots, a_{N,N}^{(s)}\}$$

and

- $m_{i,N}^{(s)} := 1/2 \cdot \left(a_{i,N}^{(s)} + a_{i+1,N}^{(s)}\right)$ for $i = 1, \ldots, N-1$ as well as $m_{0,N}^{(s)} := -\infty$ and $m_{N,N}^{(s)} := \infty$,
- $V_{A_N^{(s)}}(a_{i,N}^{(s)}) := \left(m_{i-1,N}^{(s)}, m_{i,N}^{(s)}\right]$ for $i = 1, \ldots, N-1$, and
- $V_{A_N^{(s)}}(a_{N,N}^{(s)}) := \left(m_{N-1,N}^{(s)}, m_{N,N}^{(s)}\right)$.

Then, due to construction,

$$\mathcal{V}_{A_N^{(s)}} := \left\{ V_{A_N^{(s)}}(a_{i,N}^{(s)}) \;\middle|\; i = 1, \ldots, N \right\} \tag{2.3.3}$$

is a Voronoi-partition of $\mathbb{R}$ with associated nearest-neighbor projection $\pi_{\mathcal{V}_{A_N^{(s)}}}$ given by

$$\pi_{\mathcal{V}_{A_N^{(s)}}} = \sum_{i=1}^{N} a_{i,N}^{(s)} \cdot \mathbb{1}_{V_{A_N^{(s)}}(a_{i,N}^{(s)})}. \tag{2.3.4}$$

Now by defining

$$\widetilde{Z}_N^{(s)} := \pi_{\mathcal{V}_{A_N^{(s)}}}(Z)$$

we obtain an N-quantization of Z.

In [GL00, Section 7.3] it is proven in a more general context that the sequence $(\widetilde{Z}_N^{(s)})_{N\in\mathbb{N}}$ satisfies

$$\lim_{N\to\infty} N \cdot e^{(s)}\big(Z, \widetilde{Z}_N^{(s)}, \mathbb{R}\big) = \frac{1}{2\cdot(s+1)^{1/s}} \cdot \left(\int_{\mathbb{R}} |\varphi(x)|^{1/(1+s)}\, dx\right)^{(1+s)/s}$$

where φ is the density of the standard normal distribution.

In the following lemma we collect further properties of the sequence $(\widetilde{Z}_N^{(s)})_{N\in\mathbb{N}}$, which will prove beneficial in Chapter 5.

Lemma 2.3.4

a) For all $N \in \mathbb{N}$ it holds $\mathbb{E}\big[\widetilde{Z}_N^{(s)}\big] = 0$.

b) For all $\tilde{s} \in [1, \infty)$ it holds

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[\big|\widetilde{Z}_N^{(s)}\big|^{\tilde{s}}\big] < \infty.$$

c) There exists a constant $c(s) \in (0, \infty)$ depending only on s such that

$$e^{(s)}\big(Z, \widetilde{Z}_N^{(s)}, \mathbb{R}\big) \leqslant c(s) \cdot N^{-1}$$

for all $N \in \mathbb{N}$.

Proof:

a) This statement follows directly from the definition of the sets in the Voronoi partitions $\mathcal{V}_{A_N^{(s)}}$, $N \in \mathbb{N}$, and the symmetry of P_Z.

b) For a proof we refer to [MGR10, Section 3.1].

c) As already mentioned above, the sequence $(\widetilde{Z}_N^{(s)})_{N\in\mathbb{N}}$ of N-quantizations of Z is strongly asymptotically optimal of order s, i.e., it holds

$$\lim_{N\to\infty} N \cdot e^{(s)}\big(Z, \widetilde{Z}_N^{(s)}, \mathbb{R}\big) = \frac{1}{2\cdot(s+1)^{1/s}} \cdot \left(\int_{\mathbb{R}} |\varphi(x)|^{1/(1+s)}\, dx\right)^{(1+s)/s}.$$

Thus, there exists a constant $c(s) \in (0, \infty)$ depending only on s such that

$$e^{(s)}\big(Z, \widetilde{Z}_N^{(s)}, \mathbb{R}\big) \leqslant c(s) \cdot N^{-1}$$

for all $N \in \mathbb{N}$.

□

Remark 2.3.5

a) For $N \in \mathbb{N}$ it holds

$$P\Big(\big\{\widetilde{Z}_N^{(s)} = a_{i,N}^{(s)}\big\}\Big) = P\Big(\big\{Z \in V_{A_N^{(s)}}(a_{i,N}^{(s)})\big\}\Big)$$

for all $i = 1, \ldots, N$. Since $P_Z = N(0,1)$, the above probability weights can easily be computed. Hence the above presented method for quantization of standard normally distributed random variables is fully constructive, and the algorithm is easy to implement.

b) As already mentioned at the beginning of this subsection, the above presented method for quantization of standard normally distributed random variables is just a specific example of a quantization procedure which is applicable to a whole class of real-valued random variables.

Indeed, let $s \in [1, \infty)$, and let Y be a real-valued random variable with an absolutely continuous distribution P_Y such that P_Y possesses a Lebesgue density h which satisfies

(i) $I = \{x \in \mathbb{R} \mid h(x) > 0\}$ is an open interval, and

(ii) h is continuous on I.

Moreover, we require $\mathbb{E}\big[|Y|^{s+\delta}\big] < \infty$ for a $\delta > 0$. Then the previous moment condition and (i) imply

$$0 < \int_{\mathbb{R}} \big|h(t)\big|^{1/(1+s)}\, dt < \infty.$$

We add that (ii) is a technical assumption on h which is exploited in the proofs in [GL00, Section 7.3]. Furthermore, we define $h_s : \mathbb{R} \to \mathbb{R}$ by

$$h_s(x) = \frac{\big(h(x)\big)^{1/(1+s)}}{\int_{\mathbb{R}} \big(h(t)\big)^{1/(1+s)}\, dt},$$

and by P_s we denote the absolutely continuous distribution on $\mathbb{R}$ with Lebesgue density h_s. Next, for $N \in \mathbb{N}$, let $b_{i,N}^{(s)}$ be the $(2i-1)/2N$-quantile of the distribution P_s for every $i = 1, \ldots, N$. Moreover, for $N \in \mathbb{N}$ we put

$$\beta_N^{(s)} := \{b_{1,N}^{(s)}, \ldots, b_{N,N}^{(s)}\},$$

we define the Voronoi cells of an associated Voronoi partition $\mathcal{V}_{\beta_N^{(s)}}$ in the same way as the Voronoi cells of the Voronoi partition in (2.3.3), and we define $\pi_{\mathcal{V}_{\beta_N^{(s)}}}$ analogously to (2.3.4). Then, as proven in [GL00, Section 7.3], the sequence $(\widetilde{Y}_N^{(s)})_{N\in\mathbb{N}}$ with

$$\widetilde{Y}_N^{(s)} := \pi_{\mathcal{V}_{\beta_N^{(s)}}}(Y)$$

is strongly asymptotically optimal of order s.

If $Y \sim N(0,1)$, it is easy to see that P_Y possesses a Lebesgue density which satisfies (i) and (ii) above, and additionally the required moment condition on Y is satisfied. Moreover, it holds $P_s = N(0, s+1)$, see [GL00, Table 7.1], and hence $b_{i,N}^{(s)} = a_{i,n}^{(s)}$ for all $i = 1, \ldots, N$ and $N \in \mathbb{N}$ where $a_{i,N}^{(s)}$ are the numbers defined in (2.3.2). Thus, the construction for quantization of standard normally distributed random variables presented at the beginning of this subsection is just a special case of this more general procedure.

c) At this point, we also refer to [MGR13, Section 3.1]. Therein they present a different constructive method for quantization of standard normally distributed random variables which yields a sequence of N-quantizations that also satisfies a)--c) in Lemma 2.3.4. In contrast to the construction presented here, the method in [MGR13, Section 3.1] is neither a Voronoi quantization nor does it employ quantiles of the standard normal distribution.

Quantization of Lévy Areas

Let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,1]}$ which satisfies the **usual conditions**, i.e., $\mathcal{F}$ is right-continuous and

$$\{A \subseteq \Omega \mid \exists N \in \mathcal{A} : A \subseteq N \wedge P(N) = 0\} \subseteq \mathcal{F}(0).$$

Moreover, let $W = (W_1, W_2)$ be a two-dimensional $\mathcal{F}$-Brownian motion on $[0,1]$ defined on $(\Omega, \mathcal{A}, P)$.

The statements presented in the next lemma will prove beneficial in the remaining part of this thesis. In particular, b) and c) will play a role in the construction of a sequence of quantizations of the Itô integral $\int_0^1 W_1(s)\, dW_2(s)$.

Lemma 2.3.6

a) It holds

$$\mathbb{E}\left[\int_0^1 W_1(s)\, dW_2(s)\right] = 0.$$

b) For every $p \in [1, \infty)$ it holds

$$\mathbb{E}\left[\left|\int_0^1 W_1(s)\, dW_2(s)\right|^p\right] < \infty.$$

c) It holds

$$\int_0^1 W_1(s)\, dW_2(s) \overset{\mathcal{L}}{=} \int_0^1 \big(-W_1(s)\big)\, dW_2(s).$$

d) Let $s, t \in [0,1]$ with $s < t$. Then

$$\int_s^t \big(W_1(u) - W_1(s)\big)\, dW_2(u)$$

is independent of $\mathcal{F}(s)$.

Proof:

a) This statement is well known, and therefore a proof is omitted.

b) It suffices to carry out the proof for $p \in [2, \infty)$. Applying the Burkholder-Davis-Gundy inequality,

Jensen's inequality, and Fubini's theorem, yields the existence of a constant $c \in (0, \infty)$ such that

$$\begin{aligned}\mathbb{E}\left[\Big|\int_0^1 W_1(s)\,dW_2(s)\Big|^p\right] &\leqslant \mathbb{E}\left[\sup_{t\in[0,1]}\Big|\int_0^t W_1(s)\,dW_2(s)\Big|^p\right]\\ &\leqslant c\cdot\mathbb{E}\left[\left(\int_0^1 |W_1(s)|^2\,ds\right)^{p/2}\right]\\ &\leqslant c\cdot\int_0^1 \mathbb{E}\big[|W_1(s)|^p\big]\,ds\\ &\leqslant c\cdot\mathbb{E}\big[\sup_{s\in[0,1]}|W_1(s)|^p\big].\end{aligned}$$

Now, applying the well-known fact that $\mathbb{E}\big[\sup_{s\in[0,1]}|W_1(s)|^p\big] < \infty$ finishes the proof of b).

c) We put

$$X := \int_0^1 W_1(s)\,dW_2(s) \qquad \text{and} \qquad Y := -X.$$

Moreover, for $n \in \mathbb{N}$ let $\pi_n = \{t_0, \dots, t_n\}$ be the discretization of $[0,1]$ given by

$$t_i = \frac{i}{n}$$

for all $i = 0, \dots, n$. Note that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[0,1]$ with

$$\lim_{n\to\infty}\max_{k=1,\dots,n}|t_k - t_{k-1}| = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Furthermore, we define sequences $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$ of real-valued random variables by

$$X_n = \sum_{k=1}^n W_1(t_{k-1})\cdot\big(W_2(t_k) - W_2(t_{k-1})\big) \quad \text{and} \quad Y_n = -X_n.$$

Then, $X_n \overset{L_2}{\to} X$ as well as $Y_n \overset{L_2}{\to} Y$, see, for example, [Kuo06, Theorem 4.7.1.], and hence $X_n \overset{\mathcal{L}}{\to} X$ as well as $Y_n \overset{\mathcal{L}}{\to} Y$.

For all $n \in \mathbb{N}$ there exists a Borel measurable mapping $\psi_n : \mathcal{C}([0,1];\mathbb{R}^2) \to \mathbb{R}$ such that

$$X_n = \psi_n\big(W_1, W_2\big) \quad \text{and} \quad Y_n = \psi_n\big(-W_1, W_2\big). \tag{2.3.5}$$

Additionally, note that the process $-W_1$ is a Brownian motion on $[0,1]$ which, on top of that, is independent of W_2. Thus, $(-W_1, W_2)$ is a two-dimensional Brownian motion on $[0,1]$, and hence $(W_1, W_2) \overset{\mathcal{L}}{=} (-W_1, W_2)$. Together with (2.3.5) we obtain $X_n \overset{\mathcal{L}}{=} Y_n$ for all $n \in \mathbb{N}$. Consequently, all assumptions of Lemma C.1 are satisfied, which finally leads to $X \overset{\mathcal{L}}{=} Y$.

d) We abbreviate

$$I := \int_s^t \big(W_1(u) - W_1(s)\big)\,dW_2(u),$$

and we show that I and $\mathbb{1}_A$ are independent for all $A \in \mathcal{F}(s)$. In view of Proposition C.2 the latter holds if and only if

$$\varphi_{(I,\mathbb{1}_A)}(s,t) = \varphi_I(s) \cdot \varphi_{\mathbb{1}_A}(t)$$

for all $(s,t) \in \mathbb{R}^2$ and all $A \in \mathcal{F}(s)$.

We fix $A \in \mathcal{F}(s)$, and for $n \in \mathbb{N}$ let $\pi_n = \{t_0, \dots, t_n\}$ be the discretization of $[s,t]$ with

$$t_i = s + \frac{i}{n} \cdot (t-s)$$

for all $i = 0, \dots, n$. Then $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[s,t]$ with

$$\lim_{n\to\infty} \max_{k=1,\dots,n} |t_k - t_{k-1}| = \lim_{n\to\infty} \frac{1}{n} \cdot (t-s) = 0.$$

Furthermore, we put

$$I_n := \sum_{k=1}^{n} \big(W_1(t_{k-1}) - W_1(s)\big) \cdot \big(W_2(t_k) - W_2(t_{k-1})\big)$$

for all $n \in \mathbb{N}$. Then $I_n \xrightarrow{L_2} I$, and hence $I_n \xrightarrow{\mathcal{L}} I$, which implies

$$\lim_{n\to\infty} \varphi_{I_n}(t) = \varphi_I(t) \tag{2.3.6}$$

for all $t \in \mathbb{R}$. Moreover, $I_n \xrightarrow{L_2} I$ implies

$$\lim_{n\to\infty} \mathbb{E}\big[\|(I_n, \mathbb{1}_A) - (I, \mathbb{1}_A)\|_2^2\big] = \lim_{n\to\infty} \mathbb{E}\big[|I_n - I|^2\big] = 0,$$

and hence $(I_n, \mathbb{1}_A) \xrightarrow{\mathcal{L}} (I, \mathbb{1}_A)$. Therefore,

$$\lim_{n\to\infty} \varphi_{(I_n,\mathbb{1}_A)}(s,t) = \varphi_{(I,\mathbb{1}_A)}(s,t) \tag{2.3.7}$$

for all $(s,t) \in \mathbb{R}^2$. Additionally, it is easy to see that, for all $n \in \mathbb{N}$, I_n and $\mathbb{1}_A$ are independent. Now by combining (2.3.7), Proposition C.2, and (2.3.6), we finally end up with

$$\varphi_{(I,\mathbb{1}_A)}(s,t) = \lim_{n\to\infty} \varphi_{(I_n,\mathbb{1}_A)}(s,t) = \lim_{n\to\infty} \big(\varphi_{I_n}(s) \cdot \varphi_{\mathbb{1}_A}(t)\big) = \big(\lim_{n\to\infty} \varphi_{I_n}(s)\big) \cdot \varphi_{\mathbb{1}_A}(t) = \varphi_I(s) \cdot \varphi_{\mathbb{1}_A}(t)$$

for all $(s,t) \in \mathbb{R}^2$, which finishes the proof of d).

□

For the rest of this subsection we put $Z := \int_0^1 W_1(s)\, dW_2(s)$, and we outline a method which yields a sequence $(\widetilde{Z}_N)_{N\in\mathbb{N}}$ of quantizations of Z where each quantization $\widetilde{Z}_N$ has finite range of roughly N points. The method is also employed in [MGRY15] and it is based on a more general construction due to [DV11]. In [DV11] Dereich and Vormoor present a quantization procedure for such $\mathbb{R}^d$-valued random vectors which have finite moments of any order. Due to Lemma 2.3.6 b) Z has finite moments of any order, and hence the quantization procedure presented in [MGRY15] is a suitable choice for quantization of Z.

Let $\gamma \in (1, \infty)$, and let $N \in \mathbb{N}$. We put

$$J := \lfloor N^{1/\gamma} \rfloor,$$

and for $j = 1, \ldots, J$ we define

$$N_j = \left\lfloor \frac{N}{j^\gamma} \right\rfloor.$$

Moreover, for $k = 1, \ldots, N_j + 1$ and $j = 1, \ldots, J$ we put

$$b_{j,k} := j - 1 + \frac{k-1}{N_j},$$

and we consider the mapping $T_N^{(\gamma)} : \mathbb{R} \to \mathbb{R}$ defined by

$$T_N^{(\gamma)}(x) = \sum_{j=1}^{J} \sum_{k=1}^{N_j} \Big(b_{j,k} \cdot \mathbb{1}_{[b_{j,k}, b_{j,k+1})}(x) - b_{j,k} \cdot \mathbb{1}_{(-b_{j,k+1}, -b_{j,k}]}(x) \Big). \tag{2.3.8}$$

To sum up, this construction is built-up as follows. For $j = 1, \ldots, J$ the intervals $[j-1, j)$ and $(-j, -(j-1)]$ are partitioned into equidistant subintervals of length $1/N_j$. Based on this, the function $T_N^{(\gamma)}$ maps a point $x \in (-J, J)$ to the left endpoint of the respective subinterval x lies in if $x \geqslant 0$, whereas x is mapped to the right endpoint of the respective subinterval x lies in if $x < 0$. Moreover, $T_N^{(\gamma)}$ maps all points in $(-\infty, -J] \cup [J, \infty)$ to 0.

Now by defining

$$\widetilde{Z}_N^{(\gamma)} := T_N^{(\gamma)}(Z)$$

we obtain a quantization of Z.

In the following lemma we collect selected properties of the sequence $(\widetilde{Z}_N^{(\gamma)})_{N \in \mathbb{N}}$, which will play a role in Chapter 5.

Lemma 2.3.7

a) For all $N \in \mathbb{N}$ it holds

$$T_N^{(\gamma)}(\mathbb{R}) \subseteq [-N^{1/\gamma}, N^{1/\gamma}]$$

and

$$\big| \operatorname{ran}\big(T_N^{(\gamma)}\big) \big| = 2 \cdot \sum_{j=1}^{J} N_j - 1 \leqslant 2 \cdot N \cdot \frac{\gamma}{\gamma - 1}.$$

b) For all $N \in \mathbb{N}$ it holds $\mathbb{E}\big[\widetilde{Z}_N^{(\gamma)}\big] = 0$.

c) For all $s \in [1, \infty)$ and all $p \in (1 + (1+\gamma) \cdot s, \infty)$ there exists a constant $c(p, s) \in (0, \infty)$ depending only on p and s such that

$$\mathbb{E}\Big[|Z - \widetilde{Z}_N^{(\gamma)}|^s \Big] \leqslant c(p,s) \cdot \max\big\{1, \mathbb{E}\big[|Z|^p\big]\big\} \cdot N^{-s}$$

for all $N \in \mathbb{N}$.

d) For all $s \in [1, \infty)$ it holds

$$\sup_{N \in \mathbb{N}} \mathbb{E}\big[|\widetilde{Z}_N^{(\gamma)}|^s\big] < \infty.$$

Proof:

a) Let $N \in \mathbb{N}$. The definition of the mapping $T_N^{(\gamma)}$ directly implies

$$T_N^{(\gamma)}(\mathbb{R}) \subseteq [-N^{1/\gamma}, N^{1/\gamma}] \quad \text{and} \quad \big|\operatorname{ran}\big(T_N^{(\gamma)}\big)\big| = 2 \cdot \sum_{j=1}^{J} N_j - 1.$$

Moreover, since $\gamma > 1$ the over-harmonic series $\sum_{j=1}^{\infty} 1/j^\gamma$ converges to $\zeta(\gamma)$ where ζ denotes the Riemann zeta function. Thus, by the definition of the numbers N_j and by the standard estimate $\zeta(\gamma) < \gamma/(\gamma - 1)$, see, for instance, [IR90, Proof of Proposition 16.1.2.], we conclude that

$$2 \cdot \sum_{j=1}^{J} N_j - 1 \leqslant 2 \cdot N \cdot \sum_{j=1}^{J} \frac{1}{j^\gamma} \leqslant 2 \cdot N \cdot \zeta(\gamma) \leqslant 2 \cdot N \cdot \frac{\gamma}{\gamma - 1},$$

which finishes the proof of a).

b) Let $N \in \mathbb{N}$. Due to the definition of the function $T_N^{(\gamma)}$, and due to the symmetry of P_Z, see Lemma 2.3.6 c), we obtain

$$\begin{aligned}
\mathbb{E}\big[\widetilde{Z}_N^{(\gamma)}\big] &= \sum_{j=1}^{J} \sum_{k=1}^{N_j} b_{j,k} \cdot \Big(P\big(\{Z \in [b_{j,k}, b_{j,k+1})\}\big) - P\big(\{Z \in (-b_{j,k+1}, -b_{j,k}]\}\big)\Big) \\
&= \sum_{j=1}^{J} \sum_{k=1}^{N_j} b_{j,k} \cdot \Big(P\big(\{Z \in [b_{j,k}, b_{j,k+1})\}\big) - P\big(\{Z \in [b_{j,k}, b_{j,k+1})\}\big)\Big) \\
&= 0.
\end{aligned}$$

c) For a proof we refer the reader to [MGRY15, Lemma 5.1.].

d) Let $s \in [1, \infty)$, and choose $p \in (1 + (1 + \gamma) \cdot s, \infty)$. Due to c) there exists a constant $c(p, s) \in [1, \infty)$ depending only on p and s such that

$$\mathbb{E}\big[|\widetilde{Z}_N^{(\gamma)}|^s\big] \leqslant c(p,s) \cdot \Big(\mathbb{E}\big[|Z|^s\big] + \max\big\{1, \mathbb{E}\big[|Z|^p\big]\big\} \cdot N^{-s}\Big) \leqslant c(p,s) \cdot \Big(\mathbb{E}\big[|Z|^s\big] + \max\big\{1, \mathbb{E}\big[|Z|^p\big]\big\}\Big) \tag{2.3.9}$$

for all $N \in \mathbb{N}$. Moreover, Lemma 2.3.6 b) yields

$$\max\big\{\mathbb{E}\big[|Z|^s\big], \mathbb{E}\big[|Z|^p\big]\big\} < \infty.$$

Therefore, together with (2.3.9) we arrive at

$$\sup_{N \in \mathbb{N}} \mathbb{E}\big[|\widetilde{Z}_N^{(\gamma)}|^s\big] < \infty.$$

□

Remark 2.3.8
Let $N \in \mathbb{N}$. The probability weights corresponding to the quantization $\widetilde{Z}_N^{(\gamma)}$ are hard determine. It is well known that the distribution P_Z is an absolutely continuous distribution, see, for instance, [RW01]. But to our knowledge there is, so far, no closed form expression of the distribution function F_Z of Z to be found in the literature. Hence, for us, it remains an open problem how to determine or at least how to apprioximate the probability weights corresponding to $\widetilde{Z}_N^{(\gamma)}$. Therefore, the presented quantization procedure for Z is only of a semi-constructive type.

2.3.2. Quantization of Brownian Motions and Brownian Bridges

In this subsection we present the results of Dereich and Scheutzow in [DS06] on the sharp asymptotics of the Nth minimal quantization error of one-dimensional Brownian motions and Brownian bridges on $[0,1]$ both of them interpreted as random elements with values in the Banach spaces $\big(L_p[0,1], \|\cdot\|_{L_p[0,1]}\big)$, $p \in [1,\infty)$, and $\big(\mathcal{C}[0,1], \|\cdot\|_\infty\big)$. But first, for the convenience of the reader, we recall the definition of a Brownian bridge.

Definition 2.3.9
Let $a, b \in [0,\infty)$ with $a < b$. A real-valued stochastic process $B = (B(t))_{t\in[a,b]}$ is called a **Brownian bridge on $[a,b]$** if

(i) B is a Gaussian process with continuous paths,

(ii) for all $t \in [a,b]$ it holds $\mathbb{E}[B(t)] = 0$, and

(iii) for all $s,t \in [a,b]$ it holds

$$\mathrm{Cov}[B(s),B(t)] = \frac{\big(\min(s,t)-a\big)\cdot\big(b-\max(s,t)\big)}{b-a}.$$

Remark 2.3.10

a) An attribute peculiar to a Brownian bridge B on $[a,b]$ resulting from (ii) and (iii) in the previous definition is the fact that $B(a) = 0 = B(b)$ a.s., which motivates the term *bridge*.

b) One obtains a prominent example of a Brownian bridge on $[0,1]$ by the following construction. Let $W = (W(t))_{t\in[0,1]}$ be a one-dimensional Brownian motion, and let $B = (B(t))_{t\in[0,1]}$ be defined by

$$B(t) = W(t) - t\cdot W(1).$$

It is easy to see that the process B is a Brownian bridge on $[0,1]$.

Theorem 2.3.11 *([DS06, Theorem 1.1, Theorem 1.3])*
Let W be a one-dimensional Brownian motion on $[0,1]$, and let B be a Brownian bridge on $[0,1]$.

(i) Let $p \in [1,\infty)$. Then there exists a constant $\kappa_p \in (0,\infty)$ such that, for all $s \in [1,\infty)$,

$$\lim_{N\to\infty} \sqrt{\ln N}\cdot e_N^{(s)}\big(W, L_p[0,1]\big) = \lim_{N\to\infty} \sqrt{\ln N}\cdot e_N^{(s)}\big(B, L_p[0,1]\big) = \kappa_p.$$

(ii) There exists a constant $\kappa_\infty \in (0,\infty)$ such that, for all $s \in [1,\infty)$,

$$\lim_{N\to\infty} \sqrt{\ln N}\cdot e_N^{(s)}\big(W, \mathcal{C}[0,1]\big) = \lim_{N\to\infty} \sqrt{\ln N}\cdot e_N^{(s)}\big(B, \mathcal{C}[0,1]\big) = \kappa_\infty.$$

Remark 2.3.12

a) For $p = 2$ it is known that $\kappa_2 = \sqrt{2}/\pi$, see, for example, [Der03, Section 6.4] and [LP04, Section 3]. Whereas for all other $p \in [1, \infty)$ one has only estimates for κ_p. More precisely, we put

$$\lambda(p) := \inf \left\{ \int_{-\infty}^{\infty} |x|^p \cdot \big(\varphi(x)\big)^2 \, dx + \frac{1}{2} \cdot \int_{-\infty}^{\infty} \big(\varphi'(x)\big)^2 \, dx \right\}$$

where the infimum is taken over all weakly differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \big(\varphi(x)\big)^2 \, dx = 1.$$

Then,

$$\kappa_p \in \Big[c(p), \sqrt{8} \cdot c(p) \Big]$$

where $c(p) := 2^{1/p} \cdot \sqrt{p} \cdot \big(\lambda(p)/(2+p)\big)^{(2+p)/(2p)}$, see [Der09, Section 3.2].

b) The exact value of the constant κ_∞ is unknown. One only has the estimate

$$\kappa_\infty \in \Big[\pi/\sqrt{8}, \pi \Big],$$

see, for instance, [DFMS03].

Constructive Quantization of Brownian Bridges in $L_2[0,1]$

We close this section by outlining a constructive method for quantization of Brownian bridges on $[0,1]$ in the Hilbert space $L_2[0,1]$ w.r.t. the moment parameter $s = 2$. Throughout this subsection, by $\langle \cdot, \cdot \rangle_{L_2[0,1]}$ we denote the inner product on $L_2[0,1]$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $B = (B(t))_{t \in [0,1]}$ be a Brownian bridge defined on $(\Omega, \mathcal{A}, P)$, which we interpret as a random element with values in the space $L_2[0,1]$. The method is based on the Karhunen-Loève expansion of B and on quantizations of standard normally distributed random variables, and it is a specific example of a more general construction which is mostly employed for quantization of Gaussian processes with values in a separable Hilbert space. For more detailed and more general accounts on Karhunen-Loève expansions and their application in the context of quantization of Gaussian processes one might consult, for example, [LP02], [Der03], [PP05], the survey [PP09], and the website

`http://www.quantize.maths-fi.com`

for downloads. But since we utilize the method only in the context of quantization of a Browinian bridge on $[0,1]$ in the space $L_2[0,1]$, we confine ourselves to this case here.

We commence by considering the linear operator $\Gamma_B : L_2[0,1] \to L_2[0,1]$ given by

$$(\Gamma_B f)(t) = \int_0^1 f(s) \cdot \mathbb{E}\left[B(t) \cdot B(s)\right] ds, \qquad t \in [0,1].$$

The operator Γ_B is called the **covariance operator of B**. Due to definition 2.3.9 (ii) it holds

$$\mathbb{E}\left[B(t) \cdot B(s)\right] = \min(s,t) - s \cdot t$$

for all $s, t \in [0,1]$, and hence we have

$$(\Gamma_B f)(t) = \int_0^1 f(s) \cdot \big(\min(s,t) - s \cdot t \big)\, ds$$

for $t \in [0,1]$ and $f \in L_2[0,1]$. The covariance operator Γ_B belongs to a wide class of integral operators which are called Fredholm integral operators. In the following lemma we collect further properties of Γ_B.

Lemma 2.3.13
The operator Γ_B is compact, positive and self-adjoint.

Proof:
First, note that the kernel $k : [0,1]^2 \to \mathbb{R}$ of the Fredholm integral operator Γ_B is given by

$$k(s,t) = \min(s,t) - s \cdot t.$$

It is easy to see, that k is continuous. The fact that a Fredholm integral operator with a continuous kernel is compact then yields that Γ_B is compact. Secondly, the statement that Γ_B is self-adjoint and positive can be easily verified by direct calculation. For a proof we refer to [Tou08]. □

Due to the previous lemma all prerequisites of the spectral theorem for compact, self-adjoint operators on Hilbert spaces are satisfied, by means of which one derives that for P-a.a. $\omega \in \Omega$ it holds

$$B(\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \cdot \xi_n(\omega) \cdot e_n \tag{2.3.10}$$

where convergence holds in the space $L_2[0,1]$ and

(i) e_n, $n \in \mathbb{N}$, are the eigenfunctions of Γ_B,

(ii) $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues corresponding to the eigenfunctions e_n, $n \in \mathbb{N}$, and

(iii) $(\xi_n)_{n\in\mathbb{N}}$ is the sequence of i.i.d. standard normally distributed random variables given by

$$\xi_n := \frac{1}{\sqrt{\lambda_n}} \cdot \langle e_n, B \rangle_{L_2[0,1]}$$

for $n \in \mathbb{N}$.

It is well known that the eigenfunctions $e_n : [0,1] \to \mathbb{R}$ and corresponding eigenvalues λ_n of Γ_B are given by

$$e_n(t) = \sqrt{2} \cdot \sin(\pi n \cdot t), \qquad t \in [0,1],$$

and $\lambda_n = (\pi n)^{-2}$, respectively, see, for instance, [LP02, Example 3.3.].

Definition 2.3.14
The series on the right-hand side in (2.3.10) is called **Karhunen-Loève expansion of B**.

Remark 2.3.15
Due to (2.3.10) there exists a set $\Omega^* \in \mathcal{A}$ with $P(\Omega^*) = 1$ such that

$$B(\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \cdot \xi_n(\omega) \cdot e_n$$

for all $\omega \in \Omega^*$. Now consider the mapping $\overline{B} : \Omega \to L_2[0,1]$ defined by

$$\overline{B}(\omega) = \begin{cases} B(\omega), & \text{if } \omega \in \Omega^*, \\ 0, & \text{else.} \end{cases}$$

Since $B \overset{P\text{-a.s.}}{=} \overline{B}$, we have $B \overset{\mathcal{L}}{=} \overline{B}$. Hence from now on we may assume equality in (2.3.10) without changing the distribution of B, and we abuse notation and write B instead of $\overline{B}$.

Let $N \in \mathbb{N}$. The following procedure yields an N-quantization of B. The idea is simply to take a truncation index $d(N) \in \mathbb{N}$ as well as numbers $N_1, \ldots, N_{d(N)} \in \mathbb{N}$ such that $N_1 \cdots N_{d(N)} \leqslant N$, and to apply an N_n-optimal quantization $\widetilde{\xi}_n^{(N_n)}$ to the random variable ξ_n for $n = 1, \ldots, d(N)$. Then by defining

$$\widetilde{B}_N^{(N_1,\ldots,N_{d(N)})} := \sum_{n=1}^{d(N)} \sqrt{\lambda_n} \cdot \widetilde{\xi}_n^{(N_n)} \cdot e_n \tag{2.3.11}$$

we obtain an N-quantization of B. We denote the set which contains all N-quantizations of B that are of the form as in (2.3.11) by $\mathcal{K}_N$.

Remark 2.3.16

a) Ideally, for $N \in \mathbb{N}$, one would prefer those N-quantizations of B in $\mathcal{K}_N$ that induce the least possible error among all N-quantizations of B that are of the form as in (2.3.11). More precisely, for $N \in \mathbb{N}$ one seeks for an N-quantization $\widetilde{B}_{N,\mathrm{opti}} \in \mathcal{K}_N$, if it exists, such that

$$\mathbb{E}\left[\|B - \widetilde{B}_{N,\mathrm{opti}}\|_{L_2[0,1]}^2\right] = \inf\left\{\mathbb{E}\left[\|B - \widetilde{B}\|_{L_2[0,1]}^2\right] \;\middle|\; \widetilde{B} \in \mathcal{K}_N\right\}. \tag{2.3.12}$$

In [PP05, Section 4] it is shown that for all $N \in \mathbb{N}$ such a quantization $\widetilde{B}_{N,\mathrm{opti}}$ exists, and hence the infimum in (2.3.12) actually stands as a minimum. Moreover, due to [PP05, Proposition 3] the sequence $\left(\widetilde{B}_{N,\mathrm{opti}}\right)_{N\in\mathbb{N}}$ of N-quantizations of B is asymptotically optimal of order 2.

For $N \in \mathbb{N}$ we call the truncation index corresponding to $\widetilde{B}_{N,\mathrm{opti}}$, which we donte by $d(N)_{\mathrm{opti}}$, **optimal truncation index**. The set $\{N_1, \ldots, N_{d(N)_{\mathrm{opti}}}\}$ such that $N_1 \cdots N_{d(N)_{\mathrm{opti}}} \leqslant N$ corresponding to $\widetilde{B}_{N,\mathrm{opti}}$ is called **optimal product-decomposition** of N. A database which contains tables with numerical values for $d(N)_{\mathrm{opti}}$ along with a corresponding optimal product-decomposition $\{N_1, \ldots, N_{d(N)_{\mathrm{opti}}}\}$ of N is available for downloads at the above mentioned website for $N = 1$ up to $N = 11519$. For further details on how the aforementioned database is derived, we refer the reader to [PP05]. Figure 2.3.1 displays the paths of a quantization $\widetilde{B}_{33,\mathrm{opti}}$ with $d(33)_{\mathrm{opti}} = 2$ and with optimal product-decomposition $33 = 11 \cdot 3$.

b) Let $N \in \mathbb{N}$, and let $d(N) \in \mathbb{N}$ as well as $N_1, \ldots, N_{d(N)} \in \mathbb{N}$ such that $N_1 \cdots N_{d(N)} \leqslant N$. Moreover, for all $n = 1, \ldots, d(N)$ let $\widetilde{\xi}_n^{(N_n)}$ be the N_n-optimal quantization of ξ_n, and let $\widetilde{B}_N^{(N_1,\ldots,N_{d(N)})}$ be the corresponding N-quantization of B which is of the form as in (2.3.11).

Each $\alpha \in \operatorname{ran}\left(\widetilde{B}_N^{(N_1,\ldots,N_{d(N)})}\right)$ corresponds to a vector $(a_1, \ldots, a_{d(N)}) \in \times_{n=1}^{d(N)} \operatorname{ran}\left(\widetilde{\xi}_n^{(N_n)}\right)$ such that

$$\alpha = \sum_{n=1}^{d(N)} \sqrt{\lambda_n} \cdot a_n \cdot e_n.$$

Since the random variables $\xi_1, \ldots, \xi_{d(N)}$ are independent, we have

$$P\Big(\big\{\widetilde{B}_N^{(N_1,\ldots,N_{d(N)})} = \alpha\big\}\Big) = \prod_{n=1}^{d(N)} P\Big(\big\{\widetilde{\xi}_n^{(N_n)} = a_n\big\}\Big).$$

Additionally, recall that approximations of optimal quantizations of one-dimensional standard normally distributed random variables along with their corresponding probability weights are available at the above mentioned website for downloads, see also Remark 2.2.6. Thus, the above presented quantization procedure for Brownian bridges on $[0,1]$ is fully constructive, and moreover the algorithm is easy to implement.

c) An alternative approach to obtain an N-quantization by utilizing the Karhunen-Loève expansion of B is the following. For $N \in \mathbb{N}$ and a truncation index $d(N)$ one applies an N-optimal quantization to the whole multidimensional standard normally distributed random vector $(\xi_1, \ldots, \xi_{d(N)})'$, instead of separately applying quantizations to the random variables ξ_n for $n = 1, \ldots, d(N)$. Since this approach is not of interest to our purposes, we will not go into greater detail at this point. For a detailed account on this alternative approach to quantization of B we refer the reader to, for example, [Der03] and [LP04].

d) For a constructive approach to quantization of Gaussian processes in the space $L_p[0,1]$ we refer the reader to [LP08]. Therein the authors use the expansion of the respective process in terms of the Haar basis as one of the main tools.

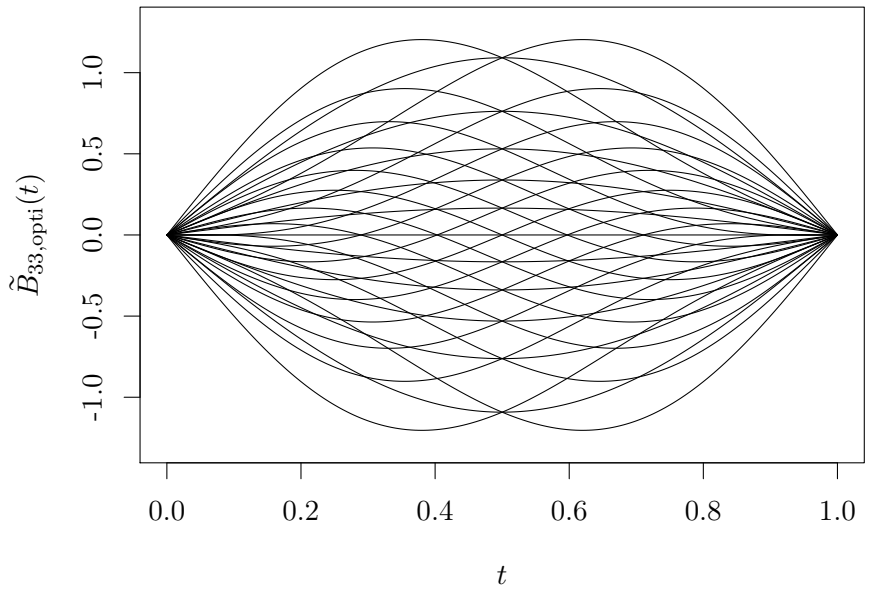


Figure 2.3.1.: Paths of a quantization $\widetilde{B}_{33,\text{opti}}$ of a Brownian bridge B on $[0,1]$

3 Preliminaries on Stochastic Differential Equations

In this chapter we provide preliminaries on SDEs which are required in this thesis. The chapter is organized as follows. In Section 1 we settle the main setting for the remaining part of this thesis.

Subsequent to this, in Section 2, we introduce the notion of a *strong solution* of an SDE. Additionally, we present basic results on existence and uniqueness of strong solutions. We conclude the second section with a remark on diffusion processes and their interrelationship with strong solutions of SDEs.

In the last section, in Section 3, we present two time-discrete strong approximation schemes for strong solutions of SDEs, namely the *Euler scheme* and the *Milstein scheme*, along with a corresponding time-continuous process in each case.

3.1. Main Setting of the Thesis

3.1.1. Basic Setting

Let $d, r \in \mathbb{N}$, and let $(\Omega, \mathcal{A}, P)$ be a complete probability space, i.e.,

$$\{A \subseteq \Omega \mid \exists N \in \mathcal{A} : A \subseteq N \wedge P(N) = 0\} \subseteq \mathcal{A}.$$

We consider a d-dimensional system of autonomous stochastic differential equations of the following type

$$\begin{aligned} dX(t) &= a(X(t))\,dt + b(X(t))\,dW(t), \quad t \in [0,1], \\ X(0) &= x_0, \end{aligned} \tag{3.1.1}$$

with initial condition $x_0 = (x_0^1, \ldots, x_0^d)' \in \mathbb{R}^d$, r-dimensional driving Brownian motion

$$W = (W_1, \ldots, W_r)'$$

defined on $(\Omega, \mathcal{A}, P)$, Borel measurable drift

$$a = (a_1, \ldots, a_d)' : \mathbb{R}^d \to \mathbb{R}^d$$

and Borel measurable diffusion coefficient

$$b = (b_{k,j})_{\substack{k=1,\ldots,d, \\ j=1,\ldots,r}} : \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

Note that (3.1.1) is just a short notation for the following system of stochastic integral equations:

$$\begin{aligned}
X^1(t) &= x_0^1 + \int_0^t a_1\big(X(s)\big)\,ds + \sum_{j=1}^{r}\int_0^t b_{1,j}\big(X(s)\big)\,dW_j(s)\\
X^2(t) &= x_0^2 + \int_0^t a_2\big(X(s)\big)\,ds + \sum_{j=1}^{r}\int_0^t b_{2,j}\big(X(s)\big)\,dW_j(s)\\
&\vdots\\
X^d(t) &= x_0^d + \int_0^t a_d\big(X(s)\big)\,ds + \sum_{j=1}^{r}\int_0^t b_{d,j}\big(X(s)\big)\,dW_j(s)
\end{aligned}$$

for $t \in [0,1]$.

Additionally, we equip $(\Omega, \mathcal{A}, P)$ with a specific filtration, which is convenient when considering strong solutions of SDEs. More precisely, we take the natural filtration $\mathcal{F}^W = (\mathcal{F}^W(t))_{t\in[0,1]}$ generated by the Brownian motion W, which is given by

$$\mathcal{F}^W(t) = \sigma\big(\{W(s) \mid s \in [0,t]\}\big)$$

for $t \in [0,1]$, and we put

$$\mathcal{N}\big(\mathcal{F}^W\big) := \big\{A \subseteq \Omega \,\big|\, \exists N \in \mathcal{F}^W(1) : A \subseteq N \wedge P(N) = 0\big\}.$$

Then, by utilizing $\mathcal{F}^W$ and $\mathcal{N}\big(\mathcal{F}^W\big)$, we define

$$\mathcal{F}(t) := \sigma\big(\mathcal{F}^W(t) \cup \mathcal{N}\big(\mathcal{F}^W\big)\big) \tag{3.1.2}$$

for $t \in [0,1]$. Note that since we assumed that $(\Omega, \mathcal{A}, P)$ is complete, it holds $\mathcal{F}(t) \subseteq \mathcal{A}$ for all $t \in [0,1]$, and hence $\mathcal{F} := (\mathcal{F}(t))_{t\in[0,1]}$ is a filtration on $(\Omega, \mathcal{A}, P)$. Furthermore, $\mathcal{F}$ is augmented, right-continuous, and W is a $\mathcal{F}$-Brownian motion see, for instance, [KS88, Theorem 2.7.9 and the proof of Proposition 2.7.7].

3.1.2. Additional Technical Assumptions

We impose the following additional technical assumptions on the coefficients a and b of the SDE (3.1.1):

For $f \in \{a, b^{(1)}, \dots, b^{(r)}\}$ we assume that

(C1) f is continuously differentiable, and

(C2) there exists a constant $K \in (0,\infty)$ such that

(i) $\sup\limits_{x\in\mathbb{R}^d} \big\|\nabla f(x)\big\|_2 \leqslant K$, and

(ii) for all $x, y \in \mathbb{R}^d$ it holds $\big\|\nabla f(x) - \nabla f(y)\big\|_2 \leqslant K \cdot \|x - y\|_2$.

In particular, assumption **(C1)** ensures that all partial derivatives of the functions $a, b^{(1)}, \dots, b^{(r)}$ exist, and, in addition to that, due to **(C2)** the partial derivatives of $a, b^{(1)}, \dots, b^{(r)}$ are uniformly bounded and globally Lipschitz continuous.

Remark 3.1.1

a) The assumptions **(C1)** and **(C2)** are stronger compared to the so-called **standard assumptions** on the coefficients of an SDE, which in case of an autonomous SDE require that both the drift and diffusion coefficient are globally Lipschitz continuous. In case of a non-autonomous SDE the standard assumptions additionally incorporate the assumption that both a and b are of at most linear growth. See also the following remarks in b).

b) In the proof of the upcoming Corollary 3.2.5 we will show that **(C1)** and **(C2)** (i) imply that both a and b are globally Lipschitz continuous. This in turn, since a and b are time-independent, yields that a and b are of at most linear growth, see also Remark 3.2.4.

c) Due to **(C2)** (i) the derivatives ∇a, $\nabla b^{(j)}$, $j = 1, \dots, r$, are uniformly bounded, and together with Taylor's theorem we obtain the existence of a constant $C \in (0, \infty)$ such that for all $x, y \in \mathbb{R}^d$ it holds

$$\|a(x) - a(y) - \nabla a(y) \cdot (x - y)\|_2 \leqslant C \cdot \|x - y\|_2^2$$

as well as

$$\max_{j=1,\dots,r} \|b^{(j)}(x) - b^{(j)}(y) - \nabla b^{(j)}(y) \cdot (x - y)\|_2 \leqslant C \cdot \|x - y\|_2^2.$$

3.2. Basic Facts on Strong Solutions of SDEs

In this section, we assume the setting in Section 3.1.1, and we introduce the notion of a *strong solution* of an SDE. Additionally, we provide basic results on existence and uniqueness of strong solutions under standard assumptions on the coefficients of the SDE. The material presented in this section along with further facts on SDEs can be found, for example, in Karatzas and Shreve [KS88]. For an introduction to the field of stochastic analysis we also refer the reader to the monographs by, for instance, Arnold [Arn74], Protter [Pro05] and Mao [Mao07].

Definition 3.2.1
A **strong solution** of the SDE (3.1.1), on the given probability space $(\Omega, \mathcal{A}, P)$ and with respect to the Brownian motion W and initial condition x_0, is an $\mathbb{R}^d$-valued stochastic process $X = (X(t))_{t \in [0,1]}$ with the following properties:

(i) X has continuous paths.

(ii) X is adapted to $\mathcal{F}$.

(iii) For all $t \in [0, 1]$ it holds

$$\int_0^t \sum_{k=1}^d \left(\left| a_k\big(X(s)\big) \right| + \sum_{j=1}^r \left| b_{k,j}\big(X(s)\big) \right|^2 \right) ds < \infty \quad P\text{-a.s.}$$

(iv) For all $k = 1, \dots, d$ and $t \in [0, 1]$ it holds

$$X^k(t) = x_0^k + \int_0^t a_k\big(X(s)\big)\, ds + \sum_{j=1}^r \int_0^t b_{k,j}\big(X(s)\big)\, W_j(s) \quad P\text{-a.s.}$$

Remark 3.2.2
According to the previous definition all paths of a strong solution of an SDE are continuous. Hence, in view of the observations presented at the end of the introductory chapter, we may interpret a strong solution as a random element with values in the spaces $\mathcal{C}([0,1];\mathbb{R}^d)$ and $L_p^d[0,1]$ for $p \in [1,\infty)$.

The following theorem states that under standard assumptions on the coefficients of the SDE (3.1.1) there exists a unique strong solution.

Theorem 3.2.3
Assume that there exists a constant $C \in (0,\infty)$ such that for all $x, y \in \mathbb{R}^d$ it holds

$$\max\big\{\|a(x)-a(y)\|_2, \|b(x)-b(y)\|_2\big\} \leqslant C \cdot \|x-y\|_2. \tag{3.2.1}$$

Then there exists a unique (up to indistinguishability) strong solution X of the SDE (3.1.1), and for all $q \in [1,\infty)$ there exists a constant $c \in (0,\infty)$ depending on q, the initial value x_0, and the constant in (3.2.1), such that

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \|X(t)\|_\infty^q\Big] \leqslant c. \tag{3.2.2}$$

Proof:
For a proof we refer the reader to, for example, [KS88, Theorem 2.9]. □

Remark 3.2.4
More generally, one mostly considers non-autonomous SDEs whose initial value is a random vector. In this case one has to include two additional prerequisites in Theorem 3.2.3 in order to obtain the same results stated therein. First, one additionally has to require that a and b are of at most linear growth. Secondly, one has to include an additional assumption on the finiteness of the absolute moments of the initial value, see, for example, [KS88, Theorem 2.9].

The following considerations show that in the context of the setting presented in Section 3.1.1 those two additional assumptions need not be included in Theorem 3.2.3. First, recall that the SDE (3.1.1) is time-independent. Hence assuming a as well as b to be globally Lipschitz continuous implies that both a and b are of at most linear growth. Indeed, let $x \in \mathbb{R}^d$. Inequality (3.2.1) yields

$$\|f(x)\|_2 \leqslant \|f(x)-f(0)\|_2 + \|f(0)\|_2 \leqslant C \cdot \|x\|_2 + \|f(0)\|_2 \leqslant \max\{C, \|f(0)\|_2\} \cdot \big(1+\|x\|_2\big)$$

for $f \in \{a,b\}$. Secondly, since the initial value x_0 of the SDE (3.1.1) is deterministic, all moments $\mathbb{E}\big[|x_0|^q\big]$, $q \in [1,\infty)$, are finite.

Corollary 3.2.5
*The assumptions **(C1)** and **(C2)** (i) guarantee the existence of a unique strong solution X of the SDE (3.1.1).*

Proof:
Let $f \in \{a, b^{(1)}, \ldots, b^{(r)}\}$. We show that f is globally Lipschitz continuous. Then the assertion of the corollary follows by applying Theorem 3.2.3. Due to assumption **(C1)** f satisfies the prerequisites of the

mean value theorem, which together with assumption **(C2)** (i) yields

$$\begin{aligned}\|f(x) - f(y)\|_2 &= \left\| \int_0^1 \nabla f(y + t \cdot (x-y)) \cdot (x-y)\, dt \right\|_2 \\ &\leqslant \int_0^1 \|\nabla f(y + t \cdot (x-y)) \cdot (x-y)\|_2\, dt \\ &\leqslant \int_0^1 \|\nabla f(y + t \cdot (x-y))\|_2 \cdot \|(x-y)\|_2\, dt \\ &\leqslant \sqrt{d} \cdot 1K \cdot \|(x-y)\|_2\end{aligned}$$

for all $x, y \in \mathbb{R}^d$ where K is the constant in **(C2)**. This implies that a and b are globally Lipschitz continuous. □

For the remaining part of this section we assume that there exists a strong solution $X = (X^1, \dots, X^d)'$ of the SDE (3.1.1). If a and b are continuous, X is a diffusion process. In particular, for all $p \in [1, \infty)$, all $k = 1, \dots, d$, all $s, t \in [0,1]$ with $s \leqslant t$, and $P_{X(s)}$-a.a. $x \in \mathbb{R}^d$, we have

$$\mathbb{E}\Big[|X^k(t) - X^k(s)|^p \;\Big|\; X(s) = x \Big] = \mathfrak{m}_p^p \cdot \|b_k(x)\|_2^p \cdot (t-s)^{p/2} + o\big((t-s)^{p/2}\big)$$

where $\mathfrak{m}_p$ denotes the pth root of the pth absolute moment of a one-dimensional standard normally distributed random variable, i.e.,

$$\mathfrak{m}_p = \left(\int_{-\infty}^{\infty} |y|^p / (2\pi)^{1/2} \cdot \exp(-y^2/2)\, dy \right)^{1/p},$$

see [MG02b, Section II.1]. Thus, given $X(s) = x$, the kth component X^k of the solution X is locally Hölder continuous of order $1/2$ in the pth mean sense, and $\mathfrak{m}_p \cdot \|b_k(x)\|_2$ might be called a conditional L_p-Hölder constant for X^k at point x. In view of that, the local smoothness of X^k at time point $t \in [0,1]$ is determined by the size of

$$s_{\mathrm{locH}}\big(X^k(t)\big) := \|b_k\big(X(t)\big)\|_2.$$

Remark 3.2.6
Assumption **(C1)** yields that a and b are continuous. Consequently, assuming the setting in Section 3.1 to hold guarantees that the strong solution X of the SDE (3.1.1) is a diffusion process, and hence the observations prior to this remark hold. We will come back to them in Chapter 5.

3.3. Strong Itô-Taylor Approximation Schemes

We assume the setting in Section 3.1.1, and we present two classical methods for strong approximation of solutions of SDEs, namely, the *Euler scheme* and the *Milstein scheme*. Both schemes are based on a fixed discretization of the time interval $[0,1]$ and recursively generate an approximation of X at the discretization points.

The Euler scheme and the Milstein scheme are examples of a wider class of strong approximation schemes that are based on the Itô-Taylor expansion, which was first introduced by Wagner and Platen in 1978 in [WP78]. In this context we also refer the reader to the standard monograph by Kloeden and Platen [KP95] in which, among other things, an overview of strong Itô-Taylor approximation schemes is provided along with corresponding error analyses.

For the remaining part of this chapter, let $m \in \mathbb{N}$ and let

$$0 = t_0 < t_1 < \cdots < t_m = 1 \tag{3.3.1}$$

be a fixed discretization of the time interval $[0,1]$. Moreover, we put

$$\Delta_l := t_{l+1} - t_l$$

for $l = 0, \ldots, m-1$, and

$$\Delta_{\max} := \max_{l=0,\ldots,m-1} \Delta_l.$$

3.3.1. Euler-Maruyama Scheme

The first time-discrete strong approximation scheme to be introduced is the Euler scheme, which is also called Euler-Maruyama scheme and dates back to Maruyama in 1955, see [Mar55]. The Euler scheme is the simplest strong Itô-Taylor approximation scheme.

The **time-discrete *d*-dimensional Euler scheme**, in short **Euler scheme**,

$$\widehat{X}_m^{\mathrm{E}} = \big(\widehat{X}_m^{\mathrm{E}}(t_0), \ldots, \widehat{X}_m^{\mathrm{E}}(t_m)\big)$$

corresponding to the discretization (3.3.1) is defined by

$$\begin{aligned}
&\widehat{X}_m^{\mathrm{E}}(t_0) = x_0,\\
&\widehat{X}_m^{\mathrm{E}}(t_l) = \widehat{X}_m^{\mathrm{E}}(t_{l-1}) + a\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot (t_l - t_{l-1}) + b\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot \big(W(t_l) - W(t_{l-1})\big)
\end{aligned}$$

for $l = 1, \ldots, m$. In particular, its kth component, $k = 1, \ldots, d$, reads

$$\begin{aligned}
&\widehat{X}_m^{\mathrm{E},k}(t_0) = x_0^k,\\
&\widehat{X}_m^{\mathrm{E},k}(t_l) = \widehat{X}_m^{\mathrm{E},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot (t_l - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot \big(W_j(t_l) - W_j(t_{l-1})\big)
\end{aligned}$$

for $l = 1, \ldots, m$.

Additionally, we introduce a time-continuous $\mathbb{R}^d$-valued stochastic process associated to $\widehat{X}_m^{\mathrm{E}}$. The ***d*-dimensional Euler process**

$$X_m^{\mathrm{E}} = \big(X_m^{\mathrm{E},1}, \ldots, X_m^{\mathrm{E},d}\big)'$$

on $[0,1]$ corresponding to the discretization (3.3.1) is defined by

$$X_m^{\mathrm{E},k}(t_0) = x_0^k,$$
$$X_m^{\mathrm{E},k}(t) = \widehat{X}_m^{\mathrm{E},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big)\cdot(t-t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big)\cdot\big(W_j(t)-W_j(t_{l-1})\big)$$

for $t \in (t_{l-1}, t_l]$, $l = 1,\dots,m$ and $k = 1,\dots,d$. Note that, first, all paths of X_m^{E} are continuous, and, secondly, it holds $X_m^{\mathrm{E}}(t_l) = \widehat{X}_m^{\mathrm{E}}(t_l)$ for all $l = 0,\dots,m$.

The results of the following proposition are well known, and therefore a proof is omitted.

Proposition 3.3.1 *([Fau92, Proposition 14])*
Let $q \in [1,\infty)$, and assume that a and b are globally Lipschitz continuous. Then there exists a constant $c \in (0,\infty)$ such that, for all $m \in \mathbb{N}$,

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \|X_m^{\mathrm{E}}(t)\|_\infty^q\Big] \leqslant c$$

as well as

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \|X(t) - X_m^{\mathrm{E}}(t)\|_\infty^q\Big] \leqslant c\cdot\Delta_{\max}^{q/2}.$$

3.3.2. Milstein Scheme

In addition to the setting in Section 3.1.1 we assume that b is partially differentiable. We introduce the time-discrete Milstein scheme, which dates back to Milstein in 1974, see [Mil75].

For $j_1, j_2 = 1,\dots,r$ and $l = 1,\dots,m$ we abbreviate

$$J_{(j_1,j_2)}^l := \int_{t_{l-1}}^{t_l} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\,dW_{j_2}(s).$$

Moreover, for $j = 1,\dots,r$, recall that

$$\nabla b^{(j)} = \begin{pmatrix} \nabla b_{1,j} \\ \vdots \\ \nabla b_{d,j} \end{pmatrix}$$

denotes the $\mathbb{R}^{d\times d}$-valued mapping which consists of all partial derivatives of $b^{(j)}$.

The **time-discrete *d*-dimensional Milstein scheme**, in short **Milstein scheme**,

$$\widehat{X}_m^{\mathrm{M}} = \big(\widehat{X}_m^{\mathrm{M}}(t_0),\dots,\widehat{X}_m^{\mathrm{M}}(t_m)\big)$$

corresponding to the discretization (3.3.1) is defined by

$$\begin{aligned}\widehat{X}_m^{\mathrm{M}}(t_0) &= x_0\\ \widehat{X}_m^{\mathrm{M}}(t_l) &= \widehat{X}_m^{\mathrm{M}}(t_{l-1}) + a\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot(t_l - t_{l-1}) + b\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W(t_l)-W(t_{l-1})\big)\\ &\quad + \sum_{j_1,j_2=1}^{r} \nabla b^{(j_2)} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot J_{(j_1,j_2)}^l\end{aligned}$$

for $l = 1,\dots,m$.

Componentwise this reads

$$\begin{aligned}\widehat{X}_m^{\mathrm{M},k}(t_0) &= x_0^k,\\ \widehat{X}_m^{\mathrm{M},k}(t_l) &= \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot(t_l - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_j(t_l) - W_j(t_{l-1})\big)\\ &\quad + \sum_{j_1,j_2=1}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot J^l_{(j_1,j_2)}\end{aligned} \tag{3.3.2}$$

for $l = 1,\dots m$ and $k = 1,\dots,d$.

Hence, in comparison to the Euler scheme, besides the increments of the driving Brownian motion the Milstein scheme is additionally composed of multiple Itô integrals.

Additionally, we introduce a time-continuous $\mathbb{R}^d$-valued stochastic process associated to $\widehat{X}_m^{\mathrm{M}}$. The ***d*-dimensional Milstein process**

$$X_m^{\mathrm{M}} = \big(X_m^{\mathrm{M},1},\dots,X_m^{\mathrm{M},d}\big)'$$

on $[0,1]$ corresponding to the discretization (3.3.1) is defined by

$$\begin{aligned}X_m^{\mathrm{M},k}(t_0) &= x_0^k,\\ X_m^{\mathrm{M},k}(t) &= \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot(t - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_j(t) - W_j(t_{l-1})\big)\\ &\quad + \sum_{j_1,j_2=1}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\int_{t_{l-1}}^{t}\big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\,dW_{j_2}(s)\end{aligned} \tag{3.3.3}$$

for $t \in (t_{l-1}, t_l]$, $l = 1,\dots,m$ and $k = 1,\dots,d$.

Note that all paths of X_m^{E} are continuous and $X_m^{\mathrm{E}}(t_l) = \widehat{X}_m^{\mathrm{E}}(t_l)$ for all $l = 0,\dots,m$.

The following proposition states that all uniform moments of the Milstein process are uniformly bounded in m, and it provides an upper bound for the qth mean uniform distance of X and X_m^{M} under rather mild assumptions on the coefficients of the SDE (3.1.1).

Proposition 3.3.2
*Let $q \in [1,\infty)$, and we assume **(C1)** as well as **(C2)** to hold. Then there exists a constant $c \in (0,\infty)$ such that, for all $m \in \mathbb{N}$,*

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \|X_m^{\mathrm{M}}(t)\|_\infty^q\Big] \leqslant c \tag{3.3.4}$$

as well as

$$\mathbb{E}\Big[\sup_{t\in[0,1]} \|X(t) - X_m^{\mathrm{M}}(t)\|_\infty^q\Big] \leqslant c\cdot\Delta_{\max}^q. \tag{3.3.5}$$

Proof:
Let $m \in \mathbb{N}$. Throughout the proof c denotes a not further specified positive real constant which may vary from line to line and which might only depend on the moment parameter q, on the dimension d of the SDE (3.1.1), on the dimension r of the driving Brownian motion W, on the initial value x_0, and on the constant in **(C2)**.

Recall that $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,1]}$ is the filtration as defined in (3.1.2). Furthermore, we write X^{M} and $\widehat{X}^{\mathrm{M}}$ instead of X_m^{M} and $\widehat{X}_m^{\mathrm{M}}$, respectively, and for $k = 1, \dots, d$ and $j = 1, \dots, r$ we write $\widehat{a}_k^{l-1}$ and $\widehat{b}_{k,j}^{l-1}$ instead of $a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)$ and $b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)$, respectively. Additionally, we put

- $A^k := \sum_{l=1}^{m} \widehat{a}_k^{l-1} \cdot \mathbb{1}_{(t_{l-1},t_l]}$,
- $B^{k,j} := \sum_{l=1}^{m} \Big[\widehat{b}_{k,j}^{l-1} + \sum_{j_1=1}^{r} \nabla b_{k,j} b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big) \cdot \big(W_{j_1} - W_{j_1}(t_{l-1})\big)\Big] \cdot \mathbb{1}_{(t_{l-1},t_l]}$, and
- $C^{k,j} := \sum_{l=1}^{m} \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big) \cdot \big(W_j - W_j(t_{l-1})\big) \cdot \mathbb{1}_{(t_{l-1},t_l]}$

for $k = 1, \dots, d$ and $j = 1, \dots, r$.

By the definition of X^{M}, see (3.3.3), we have

$$X^{\mathrm{M},k}(t) \overset{P\text{-a.s.}}{=} x_0^k + \int_0^t A^k(s)\,ds + \sum_{j=1}^{r} \int_0^t B^{k,j}(s)\,dW_j(s) \tag{3.3.6}$$

as well as

$$\begin{aligned} X^k(t) - X^{\mathrm{M},k}(t) \overset{P\text{-a.s.}}{=} & \int_0^t \Big(a_k\big(X(s)\big) - A^k(s) - \sum_{j=1}^{r} C^{k,j}(s)\Big)\,ds + \sum_{j=1}^{r} \int_0^t C^{k,j}(s)\,ds \\ & + \sum_{j=1}^{r} \int_0^t \Big(b_{k,j}\big(X(s)\big) - B^{k,j}(s)\Big)\,dW_j(s) \end{aligned} \tag{3.3.7}$$

for all $t \in [0,1]$ and $k = 1, \dots, d$.

For the moment we assume $q \in \big(2\mathbb{N} \cap \mathbb{N}_4\big)$, and we split up the proof into single steps.

Step 1: The triangle inequality and Hölder's inequality imply

$$\begin{aligned} \left|x_0^k + \int_0^t A^k(s)\,ds + \sum_{j=1}^{r} \int_0^t B^{k,j}(s)\,dW_j(s)\right|^q &\leqslant c \cdot \left(|x_0^k|^q + \left|\int_0^t A^k(s)\,ds\right|^q + \sum_{j=1}^{r} \left|\int_0^t B^{k,j}(s)\,dW_j(s)\right|^q\right) \\ &\leqslant c \cdot \left(1 + \int_0^t \big|A^k(s)\big|^q\,ds + \sum_{j=1}^{r} \left|\int_0^t B^{k,j}(s)\,dW_j(s)\right|^q\right) \end{aligned}$$

for all $t \in [0,1]$ and $k = 1, \dots, d$. Thus, (3.3.6) together with the previous inequality, the Burkholder-

Davis-Gundy inequality, Hölder's inequality (recall $q > 2$), and Fubini's theorem, lead to

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}|X^{\mathrm{M},k}(s)|^q\Big] &\leqslant c\cdot\left(1+\mathbb{E}\left[\sup_{s\in[0,t]}\int_0^s\big|A^k(u)\big|^q\,du\right]+\sum_{j=1}^r\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s B^{k,j}(u)\,dW_j(u)\right|^q\right]\right)\\
&\leqslant c\cdot\left(1+\mathbb{E}\left[\int_0^t\big|A^k(u)\big|^q\,du\right]+\sum_{j=1}^r\mathbb{E}\left[\left(\int_0^t|B^{k,j}(u)|^2\,du\right)^{q/2}\right]\right)\\
&\leqslant c\cdot\left(1+\mathbb{E}\left[\int_0^t\big|A^k(u)\big|^q\,du\right]+\sum_{j=1}^r\mathbb{E}\left[\int_0^t|B^{k,j}(u)|^q\,du\right]\right)\\
&= c\cdot\left(1+\int_0^t\Big(\mathbb{E}\Big[\big|A^k(u)\big|^q\Big]+\sum_{j=1}^r\mathbb{E}\Big[|B^{k,j}(u)|^q\Big]\Big)\,du\right)
\end{aligned}$$

for all $t\in[0,1]$ and $k=1,\dots,d$. Hence

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\|X^{\mathrm{M}}(s)\|_\infty^q\Big]\leqslant c\cdot\left(1+\int_0^t\sum_{k=1}^d\Big(\mathbb{E}\Big[\big|A^k(u)\big|^q\Big]+\sum_{j=1}^r\mathbb{E}\Big[|B^{k,j}(u)|^q\Big]\Big)\,du\right)\tag{3.3.8}$$

for all $t\in[0,1]$. Next, observe (3.3.7), and by employing similar arguments which led to (3.3.8), we conclude that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]}\|X(s)-X^{\mathrm{M}}(s)\|_\infty^q\Big]\\
&\leqslant c\cdot\left(\int_0^t\sum_{k=1}^d\Big(\mathbb{E}\Big[\big|a_k\big(X(u)\big)-A^k(u)-\sum_{j=1}^r C^{k,j}(u)\big|^q\Big]+\sum_{j=1}^r\mathbb{E}\Big[\big|b_{k,j}\big(X(u)\big)-B^{k,j}(u)\big|^q\Big]\Big)\,du\right.\\
&\qquad\left.+\sum_{k=1}^d\sum_{j=1}^r\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s C^{k,j}(u)\,du\right|^q\right]\right)
\end{aligned}\tag{3.3.9}$$

for all $t\in[0,1]$.

Step 2: In this step we further estimate the expression on the right hand side of (3.3.8). First, by the fact that a is of at most linear growth, we have

$$\big|A^k(s)\big|^q=\big|\widehat{a}_k^{l-1}\big|^q\leqslant c\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\leqslant c\cdot\big(1+\sup_{u\in[0,s]}\|X^{\mathrm{M}}(u)\|_\infty^q\big)\tag{3.3.10}$$

for all $s\in(t_{l-1},t_l]$, $l=1,\dots,m$ and $k=1,\dots,d$. Secondly, since b is of at most linear growth, and due to **(C2)** (i), we obtain

$$|B^{k,j}(s)|^q\leqslant c\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot\Big(1+\sum_{j_1=1}^r|W_{j_1}(s)-W_{j_1}(t_{l-1})|^q\Big)\tag{3.3.11}$$

for all $s\in(t_{l-1},t_l]$, $l=1,\dots,m$, $k=1,\dots,d$ and $j=1,\dots,r$.

Next, note that for all $l=1,\dots,m$ the random vector $\widehat{X}^{\mathrm{M}}(t_{l-1})$ is $\mathcal{F}(t_{l-1})$-$\mathcal{B}(\mathbb{R}^d)$-measurable and $\sigma\big(\{W(s)-W(t_{l-1})\mid s\in(t_{l-1},t_l]\}\big)$ is independent of $\mathcal{F}(t_{l-1})$. Hence, by (3.3.8), (3.3.10),

(3.3.11), and by employing (C.2) as well as (C.3) in the proof of Lemma C.5, we arrive at

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\big\|X^{\mathrm{M}}(s)\big\|_\infty^q\Big] \leqslant c\cdot\left(1+\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}\|X^{\mathrm{M}}(u)\|_\infty^q\Big]\,ds\right)$$

for all $t\in[0,1]$.

As a next step, we verify that the prerequisites of Gronwall's inequality, see Lemma A.1, are satisfied. To this end, we consider the function $f:[0,1]\to[0,\infty]$ defined by

$$f(t)=\mathbb{E}\Big[\sup_{s\in[0,t]}\big\|X^{\mathrm{M}}(s)\big\|_\infty^q\Big],$$

and we show that f is bounded and Borel measurable. Indeed, since the function f is monotonically increasing, we conclude that f is Borel measurable. It remains to show that f is bounded. Due to the definition of the Milstein scheme, see (3.3.2), due to the properties of a, b and their partial derivatives, since $\widehat{X}^{\mathrm{M}}(t_{l-1})$ is $\mathcal{F}(t_{l-1})$-$\mathcal{B}(\mathbb{R}^d)$-measurable for all $l=1,\dots,m$, due to Lemma 2.3.6 b) and d), due to the properties of W, and since all absolute moments of normally distributed random variables are finite, we obtain

$$\begin{aligned}
\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_l)\|_\infty^q\Big] &\leqslant c\cdot\Bigg(\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)+\sum_{j=1}^r\mathbb{E}\Big[\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot\big|W_j(t_l)-W_j(t_{l-1})\big|^q\Big]\\
&\qquad+\sum_{j_1,j_2=1}^r\mathbb{E}\Big[\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot|J^l_{j_1,j_2}|^q\Big]\Bigg)\\
&= c\cdot\Bigg(\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)+\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)\cdot\sum_{j=1}^r\mathbb{E}\Big[\big|W_j(t_l)-W_j(t_{l-1})\big|^q\Big]\\
&\qquad+\cdot\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)\cdot\sum_{j_1,j_2=1}^r\mathbb{E}\Big[|J^l_{j_1,j_2}|^q\Big]\Bigg)\\
&\leqslant c\cdot\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)
\end{aligned}$$

for all $l=1,\dots,m$. Together with the fact that $x_0\in\mathbb{R}^d$, the previous inequality iteratively implies

$$\max_{l=0,\dots,m}\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_l)\|_\infty^q\Big]<\infty.$$

Thus, combined with (3.3.8), (3.3.10), (3.3.11) and the properties of the normal distribution we have

$$\begin{aligned}
\sup_{t\in[0,1]}f(t) &\leqslant c\cdot\left(1+\int_0^1\sum_{k=1}^d\Big(\mathbb{E}\Big[|A^k(s)|^q\Big]+\sum_{j=1}^r\mathbb{E}\Big[|B^{k,j}(s)|^q\Big]\Big)\,ds\right)\\
&\leqslant c\cdot\Bigg(\sum_{l=1}^m\int_{t_{l-1}}^{t_l}\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)\,ds\\
&\qquad+\sum_{l=1}^m\int_{t_{l-1}}^{t_l}\mathbb{E}\Big[\Big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big)\cdot\Big(1+\sum_{j_1=1}^r|W_{j_1}(s)-W_{j_1}(t_{l-1})|^q\Big)\Big]\,ds\Bigg)\\
&\leqslant c\cdot\Big(1+\max_{l=0,\dots,m}\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_l)\|_\infty^q\Big]\Big)\\
&<\infty.
\end{aligned}$$

Now it follows from Gronwall's inequality that

$$\mathbb{E}\Big[\sup_{t\in[0,1]}\|X^{\mathrm{M}}(t)\|_\infty^q\Big]\leqslant\sup_{t\in[0,1]}\mathbb{E}\Big[\sup_{s\in[0,t]}\big\|X^{\mathrm{M}}(s)\big\|_\infty^q\Big]\leqslant c,$$

which finishes the proof of (3.3.4).

Step 3: It remains to prove (3.3.5). To this end, we further estimate the expression on the right hand side of (3.3.9).

For the moment, we fix $k\in\{1,\dots,d\}$, $j\in\{1,\dots,r\}$, $l\in\{1,\dots,m\}$ and $s\in(t_{l-1},t_l]$. Since b is globally Lipschitz continuous, since the partial derivatives of b are uniformly bounded, and by Remark 3.1.1 c), we obtain

$$\begin{aligned}
&\big|b_{k,j}\big(X(s)\big)-B^{k,j}(s)\big|^q\\
&\leqslant c\cdot\Big(\big|b_{k,j}\big(X(s)\big)-b_{k,j}\big(X^{\mathrm{M}}(s)\big)\big|^q\\
&\qquad+\big|b_{k,j}\big(X^{\mathrm{M}}(s)\big)-b_{k,j}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)-\nabla b_{k,j}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})\big)\big|^q\\
&\qquad+\Big|\nabla b_{k,j}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})\big)-\sum_{j_1=1}^r\nabla b_{k,j}b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_{j_1}(s)-W_{j_1}(t_{l-1})\big)\big|^q\Big)\\
&\leqslant c\cdot\Big(\|X(s)-X^{\mathrm{M}}(s)\|_\infty^q+\|X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})\|_\infty^{2q}\\
&\qquad+\Big|\nabla b_{k,j}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\Big(X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})-\sum_{j_1=1}^r b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_{j_1}(s)-W_{j_1}(t_{l-1})\big)\Big)\Big|^q\Big)\\
&\leqslant c\cdot\Big(\|X(s)-X^{\mathrm{M}}(s)\|_\infty^q+\|X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})\|_\infty^{2q}\\
&\qquad+\sum_{i=1}^d\Big|X^{\mathrm{M},i}(s)-X^{\mathrm{M},i}(t_{l-1})-\sum_{j_1=1}^r b_{i,j_1}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_{j_1}(s)-W_{j_1}(t_{l-1})\big)\Big|^q\Big).
\end{aligned}\tag{3.3.12}$$

By the definition of X^{M}, and by the properties of a, b as well as by the properties of the partial derivatives of b, we have

$$\begin{aligned}
&\sum_{i=1}^d\big|X^{\mathrm{M},i}(s)-X^{\mathrm{M},i}(t_{l-1})-\sum_{j_1=1}^r b_{i,j_1}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_{j_1}(s)-W_{j_1}(t_{l-1})\big)\big|^q\\
&=\sum_{i=1}^d\Big|a_i\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot(s-t_{l-1})+\sum_{j_1,j_2=1}^r\nabla b_{i,j_2}b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\int_{t_{l-1}}^s\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^q\\
&\leqslant c\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot\Big(\Delta_{l-1}^q+\sum_{j_1,j_2=1}^r\Big|\int_{t_{l-1}}^s\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^q\Big)\\
&\leqslant c\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot\Big(\Delta_{\max}^q+\sum_{j_1,j_2=1}^r\sup_{\tilde{s}\in[t_{l-1},s]}\Big|\int_{t_{l-1}}^{\tilde{s}}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^q\Big).
\end{aligned}$$

Hence, together with Lemma 2.3.6 d), inequality (3.3.4), the Burkholder-Davis-Gundy inequality, Minkowski's integral inequality, see Proposition A.3, and the properties of the Brownian motion W, we end up with

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{d}\left|X^{\mathrm{M},i}(s)-X^{\mathrm{M},i}(t_{l-1})-\sum_{j_1=1}^{r}b_{i,j_1}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_{j_1}(s)-W_{j_1}(t_{l-1})\big)\right|^q\right]\\
&\leqslant c\cdot\mathbb{E}\left[\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\big)\cdot\Big(\Delta_{\max}^q+\sum_{j_1,j_2=1}^{r}\sup_{\tilde{s}\in[t_{l-1},s]}\Big|\int_{t_{l-1}}^{\tilde{s}}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^q\Big)\right]\\
&=c\cdot\Big(1+\mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^q\Big]\Big)\cdot\Big(\Delta_{\max}^q+\sum_{j_1,j_2=1}^{r}\mathbb{E}\Big[\sup_{\tilde{s}\in[t_{l-1},s]}\Big|\int_{t_{l-1}}^{\tilde{s}}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^q\Big]\Big)\\
&\leqslant c\cdot\left(\Delta_{\max}^q+\sum_{j_1=1}^{r}\mathbb{E}\Big[\Big(\int_{t_{l-1}}^{s}\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big|^2\,du\Big)^{q/2}\Big]\right)\\
&=c\cdot\left(\Delta_{\max}^q+\sum_{j_1=1}^{r}\left[\left(\mathbb{E}\Big[\Big(\int_{t_{l-1}}^{s}\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big|^2\,du\Big)^{q/2}\Big]\right)^{2/q}\right]^{q/2}\right)\\
&\leqslant c\cdot\left(\Delta_{\max}^q+\sum_{j_1=1}^{r}\left[\int_{t_{l-1}}^{s}\Big(\mathbb{E}\big[\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\big|^q\big]\Big)^{2/q}\,du\right]^{q/2}\right)\\
&\leqslant c\cdot\left(\Delta_{\max}^q+\left[\int_{t_{l-1}}^{s}\Delta_{\max}\,du\right]^{q/2}\right)\\
&\leqslant c\cdot\Delta_{\max}^q.
\end{aligned}
\tag{3.3.13}
$$

Next, the definition of X^{M} and the properties of a and b lead to

$$
\begin{aligned}
&\sum_{i=1}^{d}\left|X^{\mathrm{M},i}(s)-X^{\mathrm{M},i}(t_{l-1})\right|\\
&\leqslant c\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty\big)\cdot\Big(\Delta_{\max}+\sum_{j=1}^{r}\big|W_j(s)-W_j(t_{l-1})\big|+\sum_{j_1,j_2=1}^{r}\Big|\int_{t_{l-1}}^{s}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|\Big).
\end{aligned}
$$

Then, by using similar arguments as in (3.3.13) we derive that

$$
\mathbb{E}\Big[\|X^{\mathrm{M}}(s)-X^{\mathrm{M}}(t_{l-1})\|_\infty^{2q}\Big]\leqslant\sum_{i=1}^{d}\mathbb{E}\Big[|X^{\mathrm{M},i}(s)-X^{\mathrm{M},i}(t_{l-1})|^{2q}\Big]\leqslant c\cdot\Delta_{\max}^q. \tag{3.3.14}
$$

Now by combining (3.3.12)--(3.3.14) we arrive at

$$
\int_0^t\sum_{k=1}^{d}\sum_{j=1}^{r}\mathbb{E}\Big[\big|b_{k,j}\big(X(s)\big)-B^{k,j}(s)\big|^q\Big]\,ds\leqslant c\cdot\left(\Delta_{\max}^q+\int_0^t\mathbb{E}\Big[\sup_{u\in[0,s]}\|X(u)-X^{\mathrm{M}}(u)\|_\infty^q\Big]\,ds\right) \tag{3.3.15}
$$

for all $t\in[0,1]$.

In a similar way one proves that

$$\int_0^t \sum_{k=1}^{d} \mathbb{E}\Big[\big|a_k\big(X(s)\big) - A^k(s) - \sum_{j=1}^{r} C^{k,j}(s)\big|^q\Big]\, ds \leqslant c \cdot \left(\Delta_{\max}^q + \int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]} \|X(u) - X^{\mathrm{M}}(u)\|_\infty^q\Big]\, ds\right) \tag{3.3.16}$$

for all $t \in [0,1]$.

In view of (3.3.9), it remains to further estimate

$$\sum_{k=1}^{d}\sum_{j=1}^{r} \mathbb{E}\left[\sup_{s\in[0,t]} \left|\int_0^s C^{k,j}(u)\, du\right|^q\right].$$

For ease of notation, for $k = 1,\dots,d$ and $j = 1,\dots,r$, we put

$$V^{k,j}(t) := \int_0^t C^{k,j}(s)\, ds$$

for $t \in (0,1]$ as well as $V^{k,j}(0) := 0$. Moreover, for $l = 1,\dots,m$ and $j = 1,\dots,r$ we define real-valued stochastic processes $Z^{l,j} = (Z^{l,j}(t))_{t\in[t_{l-1},t_l]}$ by

$$Z^{l,j}(t) = W_j(t) - W_j(t_{l-1}).$$

Now observe:

(i) Due to the definition of the mappings $C^{k,j}$ we have

$$\begin{aligned} V^{k,j}(t) &= \sum_{i=0}^{l-1} \int_{t_i}^{t\wedge t_{i+1}} \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_i)\big) \cdot \big(W_j(u) - W_j(t_i)\big)\, du \\ &= \sum_{i=0}^{l-1} \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_i)\big) \cdot \int_{t_i}^{t\wedge t_{i+1}} \big(W_j(u) - W_j(t_i)\big)\, du \end{aligned}$$

for all $t \in [t_{l-1}, t_l]$, $l = 1,\dots,m$, $j = 1,\dots,r$ and $k = 1,\dots,d$.

(ii) For all $j = 1,\dots,r$ and $k = 1,\dots,d$, due to the definition of the filtration $\mathcal{F}$ and the definition of the mapping $C^{k,j}$, the process $V^{k,j}$ is adapted to $\mathcal{F}$.

(iii) For all $j = 1,\dots,r$ and $l = 1,\dots,m$ the process $Z^{l,j}$ is a Brownian motion w.r.t to the filtration $(\mathcal{F}(t))_{t\in[t_{l-1},t_l]}$.

(iv) For all $l = 1,\dots,m$ the random vector $\widehat{X}^{\mathrm{M}}(t_{l-1})$ is $\mathcal{F}(t_{l-1})$-$\mathcal{B}(\mathbb{R}^d)$-measurable.

We use these observations to show that every process $V^{k,j}$ is an $\mathcal{F}$-martingale. Indeed, let $s,t \in [0,1]$ with $0 \leqslant s < t \leqslant 1$. Then there exists an $l \in \{1,\dots,m\}$ such that $t \in (t_{l-1}, t_l]$. It is enough to consider the case $s \in (t_{l-1}, t_l]$. The properties of the conditional expected value, see Proposition B.1 a), c) and d), the observations (i)--(iv), the properties of Brownian motions, and Fubini's theorem, lead to

$$\begin{aligned}
\mathbb{E}\Big[V^{k,j}(t)\,\Big|\,\mathcal{F}_s\Big] &\overset{P\text{-a.s.}}{=} \mathbb{E}\left[V^{k,j}(s)\,\middle|\,\mathcal{F}_s\right] + \mathbb{E}\left[\nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\int_s^t Z^{l,j}(u)\,du\,\middle|\,\mathcal{F}_s\right] \\
&\overset{P\text{-a.s.}}{=} V^{k,j}(s) + \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\mathbb{E}\left[\int_s^t Z^{l,j}(u)\,du\,\middle|\,\mathcal{F}_s\right] \\
&\overset{P\text{-a.s.}}{=} V^{k,j}(s) + \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\mathbb{E}\left[\int_s^t Z^{l,j}(u)\,du\right] \\
&= V^{k,j}(s) + \nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)\cdot\int_s^t \mathbb{E}\Big[Z^{l,j}(u)\Big]\,du \\
&= V^{k,j}(s).
\end{aligned}$$

It follows from Doob's martingale inequality

$$\sum_{k=1}^{d}\sum_{j=1}^{r}\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s C^{k,j}(u)\,du\right|^q\right] \leqslant c\cdot\sum_{k=1}^{d}\sum_{j=1}^{r}\mathbb{E}\left[\left|\int_0^1 C^{k,j}(s)\,ds\right|^q\right] = c\cdot\sum_{k=1}^{d}\sum_{j=1}^{r}\mathbb{E}\left[\left|V^{k,j}(t_m)\right|^q\right]. \tag{3.3.17}$$

As a next step, recall that $q\in 2\mathbb{N}$, and note that

$$\big(V^{k,j}(t_l),\widehat{X}^{\mathrm{M}}(t_l)\big) \quad \text{and} \quad \int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds$$

are independent for all $l=0,\dots,m-1$. Thus, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big(V^{k,j}(t_{l+1})\big)^q\Big] \\
&= \mathbb{E}\left[\Big(V^{k,j}(t_l)+\int_{t_l}^{t_{l+1}} C^{k,j}(s)\,ds\Big)^q\right] \\
&= \sum_{\mu=0}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q-\mu}\cdot\Big(\int_{t_l}^{t_{l+1}} C^{k,j}(s)\,ds\Big)^{\mu}\right] \\
&= \sum_{\mu=0}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q-\mu}\cdot\big(\nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_l)\big)\big)^{\mu}\cdot\Big(\int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds\Big)^{\mu}\right] \\
&= \sum_{\mu=0}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q-\mu}\cdot\big(\nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_l)\big)\big)^{\mu}\right]\cdot\mathbb{E}\left[\Big(\int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds\Big)^{\mu}\right] \\
&= \mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q}\right] + \sum_{\mu=2}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q-\mu}\cdot\big(\nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_l)\big)\big)^{\mu}\right]\cdot\mathbb{E}\left[\Big(\int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds\Big)^{\mu}\right]
\end{aligned} \tag{3.3.18}$$

for all $l=0,\dots,m-1$, $k=1,\dots,d$ and $j=1,\dots,r$, and where for the last equality we employed the fact that

$$\mathbb{E}\left[\int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds\right] = \int_{t_l}^{t_{l+1}}\mathbb{E}\big[W_j(s)-W_j(t_l)\big]\,ds = 0.$$

Moreover, by applying Hölder's inequality, and due to the properties of the Brownian motion W, we have

$$\begin{aligned}\mathbb{E}\left[\Big(\int_{t_l}^{t_{l+1}} Z^{l+1,j}(s)\,ds\Big)^{\mu}\right] &\leqslant \mathbb{E}\left[\Big(\int_{t_l}^{t_{l+1}} |W_j(s)-W_j(t_l)|^{\mu}\,ds\Big)\right]\cdot\Delta_l^{\mu-1}\\ &\leqslant \left(\int_{t_l}^{t_{l+1}}\Delta_l^{\mu/2}\,ds\right)\cdot\Delta_l^{\mu-1}\\ &= \Delta_l^{3\mu/2}\leqslant \Delta_l\cdot\Delta_l^{\mu}\end{aligned} \tag{3.3.19}$$

for every $\mu=2,\dots,q$. By combining (3.3.18) and (3.3.19), by using the properties of a and b, and by applying (3.3.4), we arrive at

$$\begin{aligned}&\mathbb{E}\left[\big(V^{k,j}(t_{l+1})^q\big)\right]\\ &\leqslant \mathbb{E}\left[\big|V^{k,j}(t_l)\big|^q\right]+\Delta_l\cdot\sum_{\mu=2}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[\big(V^{k,j}(t_l)\big)^{q-\mu}\cdot\big(\nabla a_k b^{(j)}\big(\widehat{X}^{\mathrm{M}}(t_l)\big)\big)^{\mu}\cdot\Delta_l^{\mu}\right]\\ &\leqslant \mathbb{E}\left[\big|V^{k,j}(t_l)\big|^q\right]+c\cdot\Delta_l\cdot\mathbb{E}\left[\sum_{\mu=0}^{q}\binom{q}{\mu}\cdot\big|V^{k,j}(t_l)\big|^{q-\mu}\cdot\big(1+\|\widehat{X}^{\mathrm{M}}(t_l)\|_{\infty}\big)^{\mu}\cdot\Delta_{\max}^{\mu}\right]\\ &= \mathbb{E}\left[\big|V^{k,j}(t_l)\big|^q\right]+c\cdot\Delta_l\cdot\mathbb{E}\left[\Big(\big|V^{k,j}(t_l)\big|+\big(1+\|\widehat{X}^{\mathrm{M}}(t_l)\|_{\infty}\big)\cdot\Delta_{\max}\Big)^q\right]\\ &\leqslant \mathbb{E}\left[\big|V^{k,j}(t_l)\big|^q\right]\cdot\big(1+c\cdot\Delta_l\big)+c\cdot\Delta_l\cdot\Delta_{\max}^q\end{aligned}$$

for all $l=0,\dots,m-1$. Thus, by employing the discrete version of Gronwall's inequality, see Corollary A.2, we obtain

$$\mathbb{E}\left[\big|V^{k,j}(t_m)\big|^q\right]\leqslant\max_{l=0,\dots,m}\mathbb{E}\left[\big|V^{k,j}(t_l)\big|^q\right]\leqslant c\cdot\Delta_{\max}^q \tag{3.3.20}$$

for all $k=1,\dots,d$ and $j=1,\dots,r$.

To sum up, in view of (3.3.9), (3.3.15), (3.3.16), (3.3.17) and (3.3.20) we derive that

$$\mathbb{E}\Big[\sup_{s\in[0,t]}\big\|X(s)-X^{\mathrm{M}}(s)\big\|_{\infty}^q\Big]\leqslant c\cdot\left(\Delta_{\max}^q+\int_0^t\mathbb{E}\Big[\sup_{u\in[0,s]}\|X(u)-X^{\mathrm{M}}(u)\|_{\infty}^q\Big]\,ds\right)$$

for all $t\in[0,1]$. Similar to Step 2 one shows that $g:[0,1]\to[0,\infty]$ defined by

$$g(t)=\mathbb{E}\Big[\sup_{s\in[0,t]}\big\|X(s)-X^{\mathrm{M}}(s)\big\|_{\infty}^q\Big]$$

is a bounded Borel measurable function, and hence Gronwall's inequality yields

$$\mathbb{E}\Big[\sup_{t\in[0,1]}\big\|X(t)-X^{\mathrm{M}}(t)\big\|_{\infty}^q\Big]\leqslant c\cdot\Delta_{\max}^q,$$

which finishes the proof of (3.3.5).

For arbitrary $q\in[1,\infty)$ we choose $\overline{q}=\min\{n\in(2\mathbb{N}\cap\mathbb{N}_4)\mid n\geqslant q\}$. Then Hölder's inequality and the

results obtained in Step 1 up to Step 3 above imply

$$\mathbb{E}\Big[\sup_{t\in[0,1]}\|X^{\mathrm{M}}(t)\|_\infty^q\Big] = \left(\Big(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X^{\mathrm{M}}(t)\|_\infty^q\Big]\Big)^{1/q}\right)^q \leqslant \left(\Big(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X^{\mathrm{M}}(t)\|_\infty^{\overline{q}}\Big]\Big)^{1/\overline{q}}\right)^q \leqslant c$$

as well as

$$\mathbb{E}\Big[\sup_{t\in[0,1]}\|X(t)-X^{\mathrm{M}}(t)\|_\infty^q\Big] \leqslant \left(\Big(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X(t)-X^{\mathrm{M}}(t)\|_\infty^{\overline{q}}\Big]\Big)^{1/\overline{q}}\right)^q \leqslant c\cdot\Delta_{\max}^q,$$

which finishes the proof of Proposition 3.3.2. □

Remark 3.3.3
The results of Proposition 3.3.2 are well known under much stronger assumptions on the coefficients a and b than those imposed in Proposition 3.3.2, see, for example, [Fau92, Proposition 25] and [KP95, Theorem 10.6.3 and Corollary 10.6.4].

As a consequence of Proposition 3.3.2 and Proposition 3.3.1 we derive an upper bound for the qth mean maximum distance of the Milstein scheme and the Euler scheme corresponding to the discretization (3.3.1), which will prove beneficial in the Chapters 5 and 6.

Corollary 3.3.4
*Let $q\in[1,\infty)$, and we assume **(C1)** as well as **(C2)** to hold. Then there exists a constant $c\in(0,\infty)$ such that, for all $m\in\mathbb{N}$,*

$$\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{M}}(t_l)-\widehat{X}_m^{\mathrm{E}}(t_l)\|_\infty^q\Big] \leqslant c\cdot\Delta_{\max}^{q/2}.$$

Proof:
Let $m\in\mathbb{N}$. Due to Proposition 3.3.1 and Proposition 3.3.2 there exists a constant $c\in(0,\infty)$ not depending on m such that

$$\begin{aligned}
&\left(\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{M}}(t_l)-\widehat{X}_m^{\mathrm{E}}(t_l)\|_\infty^q\Big]\right)^{1/q}\\
&\leqslant \left(\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widehat{X}_m^{\mathrm{M}}(t_l)\|_\infty^q\Big]\right)^{1/q} + \left(\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widehat{X}_m^{\mathrm{E}}(t_l)\|_\infty^q\Big]\right)^{1/q}\\
&\leqslant c\cdot\Delta_{\max}^{1/2}.
\end{aligned}$$

□

We close this chapter by focusing on the special situation where b has the so-called *commutativity property*.

Definition 3.3.5
The diffusion coefficient b is said to have the **commutativity property** if

$$\big(\nabla b^{(j_1)}\cdot b^{(j_2)}\big)(x) = \big(\nabla b^{(j_2)}\cdot b^{(j_1)}\big)(x)$$

for all $x\in\mathbb{R}^d$ and $j_1,j_2=1,\dots,r$.

For instance, the diffusion coefficient of a scalar SDE and the diffusion coefficient of an SDE with additive noise or diagonal noise has the commutativity property.

Now, the following question naturally arises:

Which form does the Milstein scheme take if b has the commutativity property?

First note that by applying Itô's formula we obtain

$$J^l_{(j_1,j_2)} = \int_{t_{l-1}}^{t_l} \left(W_{j_1}(s) - W_{j_1}(t_{l-1})\right) dW_{j_2}(s) \overset{P\text{-a.s.}}{=} \frac{1}{2} \cdot \left(\left(W_{j_1}(t_l) - W_{j_1}(t_{l-1})\right)^2 - \Delta_{l-1}\right)$$

for all $j_1, j_2 = 1, \dots, r$ with $j_1 = j_2$ and all $l = 1, \dots, m$, and

$$\int_{t_{l-1}}^{t_l} \left(W_{j_1}(s) - W_{j_1}(t_{l-1})\right) dW_{j_2}(s) + \int_{t_{l-1}}^{t_l} \left(W_{j_2}(s) - W_{j_2}(t_{l-1})\right) dW_{j_1}(s) \overset{P\text{-a.s.}}{=} \left(W_{j_1}(t_l) - W_{j_1}(t_{l-1})\right) \cdot \left(W_{j_2}(t_l) - W_{j_2}(t_{l-1})\right)$$

for all $l = 1, \dots, m$ and $j_1, j_2 = 1, \dots, r$ with $j_1 \neq j_2$, see, for instance, [KP95, (10.3.6) and (10.3.15)].

Thus, if b has the commutativity property, the kth component, $k = 1, \dots, d$, of the Milstein scheme can be rewritten such that

$$\begin{aligned}
&\widehat{X}^{\mathrm{M},k}_m(t_0) = x_0^k,\\
&\widehat{X}^{\mathrm{M},k}_m(t_l) \overset{P\text{-a.s.}}{=} \widehat{X}^{\mathrm{M},k}_m(t_{l-1}) + a_k\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot (t_l - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \big(W_j(t_l) - W_j(t_{l-1})\big)\\
&\qquad + \frac{1}{2} \cdot \sum_{j=1}^{r} \nabla b_{k,j} b^{(j)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \left(\big(W_j(t_l) - W_j(t_{l-1})\big)^2 - \Delta_{l-1}\right)\\
&\qquad + \sum_{\substack{j_1,j_2=1\\ j_1 \neq j_2}}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot J^l_{(j_1,j_2)}\\
&\quad = \widehat{X}^{\mathrm{M},k}_m(t_{l-1}) + a_k\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot (t_l - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \big(W_j(t_l) - W_j(t_{l-1})\big)\\
&\qquad + \frac{1}{2} \cdot \sum_{j=1}^{r} \nabla b_{k,j} b^{(j)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \left(\big(W_j(t_l) - W_j(t_{l-1})\big)^2 - \Delta_{l-1}\right)\\
&\qquad + \sum_{\substack{j_1,j_2=1\\ j_1 < j_2}}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \left(J^l_{(j_1,j_2)} + J^l_{(j_2,j_1)}\right)\\
&\quad \overset{P\text{-a.s.}}{=} \widehat{X}^{\mathrm{M},k}_m(t_{l-1}) + a_k\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot (t_l - t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \big(W_j(t_l) - W_j(t_{l-1})\big)\\
&\qquad + \frac{1}{2} \cdot \sum_{j=1}^{r} \nabla b_{k,j} b^{(j)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \left(\big(W_j(t_l) - W_j(t_{l-1})\big)^2 - \Delta_{l-1}\right)\\
&\qquad + \sum_{\substack{j_1,j_2=1\\ j_1 < j_2}}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big) \cdot \big(W_{j_1}(t_l) - W_{j_1}(t_{l-1})\big) \cdot \big(W_{j_2}(t_l) - W_{j_2}(t_{l-1})\big)
\end{aligned} \tag{3.3.21}$$

for all $l = 1, \dots, m$. This special form of the Milstein scheme will play a role in Chapter 5.

4 | Quantization of Itô Processes

In this chapter we mainly focus on quantization of Itô processes. The main results of this chapter, see Theorem 4.1.2 and Theorem 4.3.2, will serve as auxiliary results in Chapter 5.

The chapter is organized as follows. In Section 1 we derive new results on the sharp asymptotics of the Nth minimal quantization error of one-dimensional Itô processes in the spaces $L_p[0,1]$, $p \in [1,\infty)$, and $\mathcal{C}[0,1]$.

In Section 2 we apply the results obtained in the first section in order to derive the sharp asymptotics of the Nth minimal quantization error of solutions of such one-dimensional SDEs that are driven by a multidimensional Brownian motion. To our knowledge these results are new, and they generalize the main theorems in [Der08a] and [Der08b].

In the last section, Section 3, we provide a lower bound for the Nth minimal quantization error with respect to certain *product-quantizations* of multidimensional Itô process in the space $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$. The main result of this section, Theorem 4.3.2, will be crucial for deducing the results of Theorem 5.3.5 in Chapter 5.

Before we deal with the above described subjects, we provide a few definitions. Among other things, we clarify at this point what is to be understood by a *product-quantization*. In this context we confine ourselves to those Banach spaces which will be focused on in the remaining part of this thesis.

Let $d \in \mathbb{N}$, and let $p, s \in [1,\infty)$. Recall that $|||\cdot|||_s$ is the norm on the space $\mathcal{C}([0,1];\mathbb{R}^d)$ introduced in the introductory chapter, see page 6. Furthermore, let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $X = (X^1, \dots, X^d)'$ be a random element defined on $(\Omega, \mathcal{A}, P)$ with values in the space $(B, \|\cdot\|_B)$ where $(B, \|\cdot\|_B) \in \big\{(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]}), (\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty), (\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)\big\}$.

Definition 4.0.1
Let $N \in \mathbb{N}$.

a) An N-quantization $\widetilde{X} = (\widetilde{X}^1, \dots, \widetilde{X}^d)'$ of X is called an ***N*-product-quantization of *X*** if there exist real numbers $t_1, \dots, t_d \in [0,1]$ such that

   (i) $\sum_{k=1}^{d} t_k \leqslant 1$ and

   (ii) for all $k = 1, \dots, d$ it holds $|\operatorname{ran}(\widetilde{X}^k)| \leqslant \lfloor N^{t_k} \rfloor$.

   Furthermore, an N-product-quantization $\widetilde{X}$ of X is called an ***N*-uniform-product-quantization of *X*** if $t_k = 1/d$ for all $k = 1, \dots, d$.

b) Let $f : B \to B$ be an N-quantizer and put $(\widetilde{X}^1, \dots, \widetilde{X}^d)' := f\big((X^1, \dots, X^d)'\big)$. If there exist real numbers $t_1, \dots, t_d \in [0,1]$ such that

(i) $\sum_{k=1}^{d} t_k \leqslant 1$ and

(ii) for all $k = 1, \dots, d$ it holds $|\operatorname{ran}(\widetilde{X}^k)| \leqslant \lfloor N^{t_k} \rfloor$,

then f is called an ***N*-product-quantizer**. Moreover, an N-product-quantizer f is called an ***N*-uniform-product-quantizer** if $t_k = 1/d$ for all $k = 1, \dots, d$.

Definition 4.0.1 naturally gives rise to the following classes of quantizations. For $N \in \mathbb{N}$ we define

- $\mathbb{X}_N := \big\{\widetilde{X} : \Omega \to B \mid \widetilde{X} \text{ is an } N\text{-quantization of } X\big\}$,
- $\mathbb{X}_{N,\mathrm{prod}} := \big\{\widetilde{X} : \Omega \to B \mid \widetilde{X} \text{ is an } N\text{-product-quantization of } X\big\}$,
- $\mathbb{X}_{N,\mathrm{prod,uni}} := \big\{\widetilde{X} : \Omega \to B \mid \widetilde{X} \text{ is an } N\text{-uniform-product-quantization of } X\}$.

Clearly,

$$\mathbb{X}_{N,\mathrm{prod,uni}} \subseteq \mathbb{X}_{N,\mathrm{prod}} \subseteq \mathbb{X}_N \tag{4.0.1}$$

for all $N \in \mathbb{N}$. If $d = 1$, we have equality in (4.0.1) for all $N \in \mathbb{N}$.

Next, we introduce the minimal quantization errors associated to the above classes of product-quantizations.

Definition 4.0.2
Let $s \in [1, \infty)$ and $N \in \mathbb{N}$.

a) We call

$$e^{(s)}_{N,\mathrm{prod}}(X, B) := \inf\big\{e^{(s)}\big(X, \widetilde{X}, B\big) \mid \widetilde{X} \in \mathbb{X}_{N,\mathrm{prod}}\big\}$$

the ***N*th minimal quantization error of *X* (of order *s*) w.r.t. product-quantizations of *X***.

b) We call

$$e^{(s)}_{N,\mathrm{prod,uni}}(X, B) := \inf\big\{e^{(s)}\big(X, \widetilde{X}, B\big) \mid \widetilde{X} \in \mathbb{X}_{N,\mathrm{prod,uni}}\big\}$$

the ***N*th minimal quantization error of *X* (of order *s*) w.r.t. *N*-uniform-product-quantizations of *X***.

The following lemma will prove beneficial in the proof of the upcoming Theorem 4.3.2.

Lemma 4.0.3
Let $s \in [1, \infty)$. For all $N \in \mathbb{N}$ it holds

$$e^{(s)}_{N,\mathrm{prod,uni}}(X, B) = \inf\big\{e^{(s)}\big(X, f(X), B\big) \mid f \textit{ is an } N\textit{-uniform-product-quantizer}\big\}.$$

Proof:
This lemma can be proven similarly to Lemma 2.1.4. □

4.1. Quantization of One-dimensional Itô Processes

In this section we derive the sharp asymptotics of the Nth minimal quantization error of one-dimensional Itô processes in the spaces $(L_p[0,1], \|\cdot\|_{L_p[0,1]})$, $p \in [1,\infty)$, and $(\mathcal{C}[0,1], \|\cdot\|_\infty)$. To our knowledge the main result of this section, Theorem 4.1.2, is new.

In this section let $r \in \mathbb{N}$, let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,1]}$ which satisfies the usual conditions, and let $W = (W_1, \dots, W_r)'$ be an r-dimensional $\mathcal{F}$-Brownian motion on $[0,1]$ defined on $(\Omega, \mathcal{A}, P)$.

Let $X = (X(t))_{t\in[0,1]}$ be a real-valued continuous stochastic process such that for all $t \in [0,1]$ it holds

$$X(t) = x_0 + \int_0^t Y(s)\,ds + \sum_{j=1}^r \int_0^t Z_j(s)\,dW_j(s) \qquad P\text{-a.s.} \tag{4.1.1}$$

where we require:

(A1) It holds $x_0 \in \mathbb{R}$.

(A2) $Y = (Y(s))_{s\in[0,1]}$ is a real-valued stochastic process such that

(i) Y is measurable and adapted to $\mathcal{F}$, and

(ii) for all $q \in [1,\infty)$ it holds $\mathbb{E}\big[\|Y\|^q_{L_2[0,1]}\big] < \infty$.

(A3) For all $j = 1, \dots, r$ it holds that $Z_j = (Z_j(s))_{s\in[0,1]}$ is a real-valued stochastic process such that $Z_j(0) \in \mathbb{R}$, Z_j is adapted to $\mathcal{F}$ and the paths of Z_j are γ-Hölder continuous for every $\gamma \in (0, 1/2)$.

(A4) There exists an $\alpha \in (0,1/2)$ such that for all $q \in [1,\infty)$ it holds $\mathbb{E}\big[\big|\|Z(\cdot)\|_2^2\big|_\alpha^q\big] < \infty$ where $Z := \big(Z_1, \dots, Z_r\big)$ and, recall,

$$|f|_\alpha = \sup_{0\leqslant s<t\leqslant 1} \frac{|f(t) - f(s)|}{|t-s|^\alpha}$$

for $f : [0,1] \to \mathbb{R}$.

Remark 4.1.1

a) For all $j = 1, \dots, r$ assumption **(A3)** yields that all paths of Z_j are continuous and thus Z_j is measurable. Furthermore, it is easy to see that **(A3)** implies that all paths of the process $\big(\|Z(s)\|_2^2\big)_{s\in[0,1]}$ lie in the space of α-Hölder continuous functions $\mathcal{C}^\alpha[0,1]$ where α denotes the constant in assumption **(A4)**. Hence we may interpret the process $(\|Z(s)\|_2^2)_{s\in[0,1]}$ as a random element defined on Ω with values in $\mathcal{C}^\alpha[0,1]$. In addition to that, note that the mapping $f \mapsto |f|_\alpha$, $f \in \mathcal{C}^\alpha[0,1]$, is continuous and therefore $\mathcal{B}(\mathcal{C}^\alpha[0,1])$-$\mathcal{B}(\mathbb{R})$-measurable. Thus, for all $q \in [1,\infty)$, the expected value considered in assumption **(A4)** is well-defined.

b) Hölder's inequality and assumption **(A2)** (ii) yield

$$\mathbb{E}\left[\int_0^t |Y(s)|\,ds\right] \leqslant \mathbb{E}\left[\int_0^1 |Y(s)|\,ds\right] \leqslant \mathbb{E}\left[\left(\int_0^1 |Y(s)|^2\,ds\right)^{1/2}\right] < \infty$$

for all $t \in [0,1]$. Thus, for all $t \in [0,1]$, we have

$$\int_0^t |Y(s)|\, ds < \infty \qquad P\text{-a.s.}$$

Hence, in particular, almost all sample paths of the process $\left(\int_0^t Y(s)\, ds\right)_{t\in[0,1]}$ are continuous.

Additionally, the fact that the process Y is adapted to $\mathcal{F}$ leads to the same property for the process $\left(\int_0^t Y(s)\, ds\right)_{t\in[0,1]}$.

c) As already mentioned in a), due to assumption **(A3)**, every process Z_j, $j = 1, \dots, r$, has continuous paths and thus may be interpreted as a random element with values in the space $\mathcal{C}[0,1]$. Furthermore, the assumptions **(A3)** and **(A4)** imply

$$\mathbb{E}\Big[\sup_{t\in[0,1]} |Z_j(t)|^q\Big] < \infty$$

for all $q \in [1,\infty)$ and $j = 1, \dots, r$. Indeed, let $q \in [1,\infty)$ and $j \in \{1, \dots, r\}$. Hölder's inequality, and the assumptions **(A3)** and **(A4)** imply

$$\begin{aligned}
\left(\mathbb{E}\Big[\sup_{t\in[0,1]} |Z_j(t)|^q\Big]\right)^{1/q} &\leqslant \left(\mathbb{E}\Big[\sup_{t\in[0,1]} |Z_j(t)|^{2q}\Big]\right)^{1/2q} \\
&\leqslant \left(\mathbb{E}\Big[\Big(\sup_{t\in[0,1]} \big|\, \|Z(t)\|_2^2 - \|Z(0)\|_2^2 + \|Z(0)\|_2^2 \,\big|\Big)^q\Big]\right)^{1/2q} \\
&\leqslant 2 \cdot \Big(\mathbb{E}\Big[\big|\, \|Z(\cdot)\|_2^2 \big|_\alpha^q\Big] + \mathbb{E}\Big[\|Z(0)\|_2^{2q}\Big]\Big)^{1/2q} \\
&< \infty,
\end{aligned} \tag{4.1.2}$$

where α is the parameter in assumption **(A4)**.

Furthermore, by (4.1.2) we have

$$\mathbb{E}\left[\int_0^t \big(Z_j(s)\big)^2\, ds\right] \leqslant \mathbb{E}\Big[\sup_{t\in[0,1]} |Z_j(t)|^2\Big] < \infty$$

for all $t \in [0,1]$ and $j = 1, \dots, r$. Together with assumption **(A3)** and the construction of the stochastic Itô integral, we conclude that for all $j = 1, \dots, r$ the process

$$\left(\int_0^t Z_j(s)\, dW_j(s)\right)_{t\in[0,1]}$$

is a continuous $\mathcal{F}$-martingale. Thus, the process $M = (M(t))_{t\in[0,1]}$ defined by

$$M(t) = \sum_{j=1}^r \int_0^t Z_j(s)\, dW_j(s)$$

is a continuous $\mathcal{F}$-martingale, and it is well known that its uniquely (up to indistinguishability)

determined quadratic variation process $\langle M \rangle = \big(\langle M \rangle(t)\big)_{t\in[0,1]}$ is given by

$$\langle M \rangle(t) = \int_0^t \|Z(s)\|_2^2 \, ds,$$

see, for example, [KS88, Remark 3.3.30].

d) Due to the remarks in a) and b) the expression on the right-hand side of (4.1.1) is well-defined.

e) The process X has continuous sample paths. Hence we may interpret X as a random element with values in the space $\mathcal{C}[0,1]$ or $L_p[0,1]$ for $p \in [1,\infty)$.

By employing the same key ideas and techniques as in [Der08a] and [Der08b] one obtains the following result.

Theorem 4.1.2

(i) Let $p \in [1,\infty)$. Then for all $s \in [1,\infty)$ it holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e_N^{(s)}(X, L_p[0,1]) = \kappa_p \cdot \left(\mathbb{E}\left[\big\| \, \|Z(\cdot)\|_2 \, \big\|_{L_{2p/(p+2)}[0,1]}^s \right] \right)^{1/s}$$

where κ_p is the constant in Theorem 2.3.11 (i).

(ii) For all $s \in [1,\infty)$ it holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e_N^{(s)}\big(X, \mathcal{C}[0,1]\big) = \kappa_\infty \cdot \left(\mathbb{E}\left[\big\| \, \|Z(\cdot)\|_2 \, \big\|_{L_2[0,1]}^s \right] \right)^{1/s}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

Proof:
As a first step, one carries over all auxiliary statements along with their proofs presented in [Der08a, Sections 2 and 3] and [Der08b, Sections 2 to 7]. In the course of this, apply the assumptions **(A1)--(A4)** as well as (4.1.2) in Remark 4.1.1. Subsequent to this, one derives the assertions of the theorem by mimicking the main proofs in the aforementioned papers, see [Der08a, pp. 948-951] and [Der08b, pp. 934-936]. □

4.2. Quantization of One-dimensional SDEs

We apply the results obtained in Theorem 4.1.2 to derive the sharp asymptotics of the minimal quantization error of solutions of such one-dimensional SDEs that are driven by a multidimensional Brownian motion. Since we solely work with autonomous SDEs in this thesis, we also confine ourselves to such SDEs here.

Let $r \in \mathbb{N}$, let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,1]}$ be a filtration which satisfies the usual conditions. We consider a one-dimensional autonomous SDE

$$\begin{aligned} dX(t) &= a(X(t))\, dt + \sum_{j=1}^r b_j(X(t))\, dW_j(t), \qquad t \in [0,1], \\ X(0) &= x_0, \end{aligned} \tag{4.2.1}$$

with initial condition $x_0 \in \mathbb{R}$, r-dimensional driving $\mathcal{F}$-Brownian motion $W = (W_1, \dots, W_r)'$, and Borel measurable functions $a : \mathbb{R} \to \mathbb{R}$ and $b = (b_1, \dots, b_r) : \mathbb{R} \to \mathbb{R}^{1\times r}$ such that (4.2.1) has a strong solution $X = (X(t))_{t\in[0,1]}$ and such that the processes

$$\big(a(X(s))\big)_{s\in[0,1]} \quad \text{as well as} \quad \big(b_j(X(s))\big)_{s\in[0,1]},\ j = 1, \dots, r,$$

have the respective properties in **(A2)--(A4)** introduced in the previous section.

The following result, which is a direct consequence of Theorem 4.1.2, generalizes Theorem 1.1. in [Der08a] and Theorem 1.1. in [Der08b].

Proposition 4.2.1

(i) Let $p \in [1, \infty)$. Then for all $s \in [1, \infty)$ it holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e_N^{(s)}\big(X, L_p[0,1]\big) = \kappa_p \cdot \left(\mathbb{E}\left[\, \| \, \|b(X(\cdot))\|_2 \, \|^s_{L_{2p/(p+2)}[0,1]}\right]\right)^{1/s}$$

where κ_p is the constant in Theorem 2.3.11 (i).

(ii) For all $s \in [1, \infty)$ it holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e_N^{(s)}\big(X, \mathcal{C}[0,1]\big) = \kappa_\infty \cdot \left(\mathbb{E}\left[\, \| \, \|b(X(\cdot))\|_2 \, \|^s_{L_2[0,1]}\right]\right)^{1/s}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

Remark 4.2.2
Assume that a and b satisfy the assumptions **(C1)** and **(C2)** (i). Then there exists a strong solution X of the SDE (4.2.1), see Corollary 3.2.5, and we will show in Theorem 5.2.6 that the processes $\big(a(X(s))\big)_{s\in[0,1]}$ and $\big(b_j(X(s))\big)_{s\in[0,1]}$, $j = 1, \dots, r$, have the respective properties in **(A2)--(A4)**. Hence in the setting of Section 3.1 the statements in Proposition 4.2.1 hold.

4.3. A Lower Bound for Product-Quantizations of Multidimensional Itô Processes

In this section we derive a lower bound for the Nth minimal quantization error w.r.t. N-uniform-product-quantizations of multidimensional Itô processes in the space $(\mathcal{C}([0,1]; \mathbb{R}^d), \|\cdot\|_\infty)$. To this end, for technical reasons, we consider Itô processes defined on the time interval $[0, \infty)$.

In this section let $d, r \in \mathbb{N}$, let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}(t))_{t\in[0,\infty)}$ which satisfies the usual conditions, and let $W = (W_1, \dots, W_r)'$ be an r-dimensional $\mathcal{F}$-Brownian motion on $[0, \infty)$ defined on $(\Omega, \mathcal{A}, P)$.

Let $X = (X^1, \dots, X^d)'$ be an $\mathbb{R}^d$-valued continuous stochastic process on $[0, \infty)$ such that for all $k = 1, \dots, d$ and $t \in [0, \infty)$ it holds

$$X^k(t) = x_0^k + \int_0^t Y^k(s)\, ds + \sum_{j=1}^r \int_0^t Z_j^k(s)\, dW_j(s), \qquad P\text{-a.s.}$$

where we require:

(B1) For all $k = 1, \dots, d$ it holds $x_0^k \in \mathbb{R}$.

(B2) For all $k = 1, \dots, d$ it holds that $Y^k = (Y^k(s))_{s\in[0,\infty)}$ is a real-valued stochastic process such that

(i)) Y^k is measurable, adapted to $\mathcal{F}$, and for all $t \in [0, \infty)$ it holds

$$\int_0^t |Y^k(s)|\, ds < \infty \qquad P\text{-a.s.},$$

and

(ii)) Y^k satisfies **(A2)** (ii).

(B3) For all $k = 1, \dots, d$ and $j = 1, \dots, r$ it holds that $Z_j^k = (Z_j^k(s))_{s\in[0,\infty)}$ is a real-valued stochastic process such that $Z_j^k(0) \in \mathbb{R}$, Z_j^k is measurable and adapted to $\mathcal{F}$, and the process $\left(\int_0^t Z_j^k(s)\, dW_j(s)\right)_{t\in[0,\infty)}$ is well-defined and a continuous $\mathcal{F}$-martingale. Moreover, we require that for all $k = 1, \dots, d$ and all $j = 1, \dots, r$ the process $(Z_j^k(s))_{s\in[0,1]}$ has γ-Hölder continuous paths for all $\gamma \in (0, 1/2)$.

(B4) For all $k = 1, \dots, d$ the process $Z^k := \left(Z_1^k, \dots, Z_r^k\right)$ satisfies **(A4)** as well as

$$\lim_{t\to\infty} \int_0^t \|Z^k(s)\|_2^2\, ds = \infty. \tag{4.3.1}$$

Remark 4.3.1

a) By employing the assumptions **(B2)**--**(B4)** one derives results analogous to those in Remark 4.1.1. In particular, we point out that by assumption **(B3)** we have that for all $k = 1, \dots, d$ the process $M^k = (M^k(t))_{t\in[0,\infty)}$ defined by

$$M^k(t) = \sum_{j=1}^r \int_0^t Z_j^k(s)\, dW_j(s)$$

is a continuous $\mathcal{F}$-martingale, and its quadratic variation process $\langle M^k\rangle = \left(\langle M^k\rangle(t)\right)_{t\in[0,\infty)}$ is given by

$$\langle M^k\rangle(t) = \int_0^t \|Z^k(s)\|_2^2\, ds.$$

Furthermore, for all $k = 1, \dots, d$ assumption **(B4)** ensures $\lim_{t\to\infty}\langle M^k\rangle(t) = \infty$. Note that without changing M^k on the time interval $[0, 1]$ one can always ensure (4.3.1) to hold by changing Z^k outside $[0, 1]$ in a suitable way. Indeed, for $j = 1, \dots, r$ define a stochastic process $(\overline{Z}_j^k(t))_{t\in[0,\infty)}$ by

$$\overline{Z}_j^k(t) = \begin{cases} Z_j^k(t), & \text{if } t \in [0, 1], \\ t, & \text{else.} \end{cases}$$

Then the process $\overline{Z}^k := (\overline{Z}_1^k, \dots, \overline{Z}_r^k)$ satisfies

$$\int_0^t \|\overline{Z}^k(s)\|_2^2\, ds = \int_0^t \|Z^k(s)\|_2^2\, ds$$

for all $t \in [0,1]$, and it is easy to see that

$$\lim_{t\to\infty} \int_0^t \|\overline{Z}^k(s)\|_2^2 \, ds = \infty.$$

But for simplicity we a priori assumed (4.3.1) to hold.

b) For all $k = 1, \dots, d$ and all $q \in [1, \infty)$ it holds $\mathbb{E}\left[\left(\langle M^k \rangle(1)\right)^q\right] < \infty$. Indeed, let $k \in \{1, \dots, d\}$, and let $q \in [1, \infty)$. Similar to Remark 4.1.1 b), and by applying the assumptions **(B3)** and **(B4)**, we conclude that there exists a constant $c(q) \in (0, \infty)$ depending only on q such that

$$\mathbb{E}\left[\left(\langle M^k \rangle(1)\right)^q\right] = \mathbb{E}\left[\left(\int_0^1 \sum_{j=1}^r |Z_j^k(s)|^2 \, ds\right)^q\right] \leqslant c(q) \cdot \sum_{j=1}^r \mathbb{E}\Big[\sup_{t\in[0,1]} \left|Z_j^k(t)\right|^{2q}\Big] < \infty. \tag{4.3.2}$$

We will utilize this fact in the proof of the upcoming Proposition 4.3.5.

We will employ large parts of the key ideas and techniques presented in [Der08b] in order to derive the upcoming Theorem 4.3.2, which is the main result of this section. Unfortunately, our proof of that theorem incorporates only the case where the components of $M := (M^1, \dots, M^d)'$ are mutually orthogonal in the sense that for every $i, k \in \{1, \dots, d\}$ with $i \neq k$ and every $t \in [0, \infty)$ it holds $\langle M^i, M^k \rangle(t) = 0$ where $\langle M^i, M^k \rangle$ denotes the cross-variation process of M^i and M^k. Nevertheless, we expect the results in Theorem 4.3.2 to hold also in the general case, but a rigorous proof of this conjecture remains an open problem.

Now, for the rest of this section we impose the following additional assumption on the processes $Z^1, \dots, Z^d$:

(B5) In addition to the assumptions **(B3)** and **(B4)** we assume that the processes $Z^1, \dots, Z^d$ are of a form such that for every $i, k \in \{1, \dots, d\}$ with $i \neq k$ and every $t \in [0, \infty)$ it holds $\langle M^i, M^k \rangle(t) = 0$.

Theorem 4.3.2
For all $s \in [1, \infty)$ it holds

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod},\mathrm{uni}}\big(X, \mathcal{C}([0,1]; \mathbb{R}^d)\big) \geqslant \kappa_\infty \cdot \sqrt{d} \cdot \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big\| \, \|Z^k(\cdot)\|_2 \, \big\|^s_{L_2[0,1]}\right]\right)^{1/s}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

In the following, we outline the basic strategy for the proof of Theorem 4.3.2. For $k = 1, \dots, d$ and $t \in [0, \infty)$ we put

$$A^k(t) := x_0^k + \int_0^t Y^k(s) \, ds.$$

Then for $k = 1, \dots, d$ and $t \in [0, \infty)$ it holds

$$X^k(t) = A^k(t) + M^k(t) \qquad P\text{-a.s.}$$

Next, since for all $k = 1, \dots, d$ the process M^k is a continuous martingale with $\lim_{t\to\infty} \langle M^k \rangle(t) = \infty$ and due to assumption **(B5)**, the theorem of F. B. Knight, see Proposition C.3 in the Appendix, yields

that we can represent M as a d-dimensional stochastic process consisting of d independent time-changed Brownian motions. More precisely, for $k = 1, \ldots, d$ and $s \in [0, \infty)$ consider the stopping time

$$T^k(s) := \inf\big\{t \in [0,\infty) \,\big|\, \langle M^k\rangle(t) \geqslant s\big\}.$$

Then the stochastic process $B = (B(s))_{s\in[0,\infty)} = (B^1(s), \ldots, B^d(s))'_{s\in[0,\infty)}$ with $B^k(s) = M^k\big(T^k(s)\big)$ for $k = 1, \ldots, d$ and $s \in [0, \infty)$ is a Brownian motion with respect to its completed natural filtration which we denote by $\mathcal{F}^B = \big(\mathcal{F}^B(s)\big)_{s\in[0,\infty)}$.

Now we proceed in roughly two steps. First, for all $k = 1, \ldots, d$ we construct a sequence $(\hat{\psi}^k_n)_{n\in\mathbb{N}}$ of approximations of $\langle M^k\rangle$ on $[0, 1]$ such that for every $n \in \mathbb{N}$ the process $\hat{\psi}^k_n$ has monotonically increasing, continuous sample paths. Secondly, one the one hand, we apply a specific quantization procedure to the process $\big(B^1(\hat{\psi}^1_n(s)), \ldots, B^d(\hat{\psi}^d_n(s))\big)_{s\in[0,1]}$ for all $n \in \mathbb{N}$, and, on the other hand, we quantize the process $(A^1, \ldots, A^d)$. We add that the procedure for quantization of $(A^1, \ldots, A^d)$ is hidden in the statements of Theorem 4.3.6. For further details we refer the reader to [Der08b, Proof of Theorem 7.1.].

4.3.1. Auxiliary Statements for the Proof of Theorem 4.3.2

As already mentioned, for proving Theorem 4.3.2 we employ large parts of the key ideas and auxiliary statements in [Der08b]. More precisely, to avoid redundancy, we carry over only those auxiliary statements stated in the aforementioned paper which make our proof self-contained. Furthermore, we additionally employ arguments in the main proof in Section 4.3.2 which are different to those used in [Der08b].

We proceed as follows. The required auxiliary statements are presented in the following subsections, and afterwards we carry out the main proof of Theorem 4.3.2.

An Auxiliary Lemma on Quantization of Brownian Motions

Lemma 4.3.3
Let $s \in [1, \infty)$, let $T \in (0, \infty)$, and let B be a one-dimensional Brownian motion on $[0, T]$. Moreover, let the operator $S : \mathcal{C}[0, T] \to \mathcal{C}[0, 1]$ be defined by

$$(Sf)(x) = \frac{1}{\sqrt{T}} \cdot f(T \cdot x), \qquad x \in [0, 1].$$

Then the following holds:

a) S is a bounded linear operator with $\|S\|_{\mathrm{op}} = 1/\sqrt{T}$.

b) The process $S(B)$ is a one-dimensional Brownian motion on $[0, 1]$.

c) For all $N \in \mathbb{N}$ it holds

$$e^{(s)}_N\big(S(B), \mathcal{C}[0, 1]\big) \leqslant \frac{1}{\sqrt{T}} \cdot e^{(s)}_N\big(B, \mathcal{C}[0, T]\big).$$

Proof:

a) Clearly, S is linear. Moreover, it holds

$$\|Sf\|_\infty = \frac{1}{\sqrt{T}} \cdot \sup_{x\in[0,1]} |f(T \cdot x)| = \frac{1}{\sqrt{T}} \cdot \sup_{x\in[0,T]} |f(x)| = \frac{1}{\sqrt{T}} \cdot \|f\|_\infty$$

for all $f \in \mathcal{C}[0,T]$. Hence S is bounded with

$$\|S\|_{\mathrm{op}} = \frac{1}{\sqrt{T}}.$$

b) The fact that $S(B)$ is a one-dimensional Brownian motion on $[0,1]$ follows directly from the scaling properties of Brownian motions.

c) Note that $\mathbb{E}\left[\|B\|_\infty^s\right] < \infty$, and therefore Proposition 2.1.8 and a) yield

$$e_N^{(s)}\big(S(B), \mathcal{C}[0,1]\big) \leqslant \|S\|_{\mathrm{op}} \cdot e_N^{(s)}\big(B, \mathcal{C}[0,T]\big) = \frac{1}{\sqrt{T}} \cdot e_N^{(s)}\big(B, \mathcal{C}[0,T]\big)$$

for all $N \in \mathbb{N}$.

□

A Sequence of Approximations of $\langle M^k \rangle$

Recall that for all $k = 1, \dots, d$ assumption **(B4)** yields the existence of a parameter $\alpha^k \in (0, 1/2)$ such that for all $q \in [1, \infty)$ it holds

$$\mathbb{E}\left[\left|\|Z^k(\cdot)\|_2^2\right|_{\alpha^k}^q\right] < \infty. \tag{4.3.3}$$

In this subsection, for every $k = 1, \dots, d$, we construct a sequence of approximations of $(\langle M^k \rangle(t))_{t\in[0,1]}$, which depends on the parameter α^k. For the construction we use a technique proposed at the beginning of Section 4 in [Der08b].

Fix $k \in \{1, \dots, d\}$, let $n \in \mathbb{N}$, and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be the discretization of $[0,1]$ given by

$$t_i = \frac{i}{n}$$

for $i = 0, \dots, n$. Moreover, we consider the set

$$\mathbb{I}(\alpha^k, n) := \left\{ j \cdot n^{-(1+\alpha^k)} \;\middle|\; j \in \mathbb{N}_0 \cap \left[0, n^{2\cdot(1+\alpha^k)}\right] \right\},$$

and we put $j_{\max} := \max\left\{ j \in \mathbb{N}_0 \mid j \in [0, n^{2\cdot(1+\alpha^k)}] \right\}$ as well as

- $C_{\mathbb{I}(\alpha^k,n)}(0) := \left[0, 1/2 \cdot n^{-(1+\alpha^k)}\right)$,
- $C_{\mathbb{I}(\alpha^k,n)}(j) := \left[j \cdot n^{-(1+\alpha^k)} - 1/2 \cdot n^{-(1+\alpha^k)}, j \cdot n^{-(1+\alpha^k)} + 1/2 \cdot n^{-(1+\alpha^k)}\right)$ for $j = 1, \dots, j_{\max} - 1$,
- and $C_{\mathbb{I}(\alpha^k,n)}(j_{\max}) := \left[j_{\max} \cdot n^{-(1+\alpha^k)} - 1/2 \cdot n^{-(1+\alpha^k)}, \infty\right)$.

By utilizing the intervals $C_{\mathbb{I}(\alpha^k,n)}(j)$, $j = 0, \dots, j_{\max}$, we define $\pi_{\mathbb{I}(\alpha^k,n)} : \mathbb{R} \to \mathbb{R}$ by

$$\pi_{\mathbb{I}(\alpha^k,n)}(x) = \sum_{j=0}^{j_{\max}} \frac{j}{n^{1+\alpha^k}} \cdot \mathbb{1}_{C_{\mathbb{I}(\alpha^k,n)}(j)}(x).$$

Now let $\hat{\psi}_n^k = \big(\hat{\psi}_n^k(t)\big)_{t\in[0,1]}$ be the real-valued stochastic process which is defined by

$$\hat{\psi}_n^k(t_i) = \pi_{\mathbb{I}(\alpha^k,n)}\big(\langle M^k\rangle(t_i)\big)$$

for $i = 0,\dots,n$ and

$$\hat{\psi}_n^k(t) = \hat{\psi}_n^k(t_i)\cdot(t_{i+1}-t)\cdot n + \hat{\psi}_n^k(t_{i+1})\cdot(t-t_i)\cdot n$$

for $t \in (t_i, t_{i+1})$ and $i = 0,\dots,n-1$.

Remark 4.3.4

a) For all $n \in \mathbb{N}$ and $k = 1,\dots,d$ the process $\hat{\psi}_n^k$ has continuous and monotonically increasing sample paths, where the latter is a consequence of the fact that the process $\langle M^k\rangle$ has monotonically increasing paths.

b) Let $n \in \mathbb{N}$. By the construction of the process $\hat{\psi}_n^k$, and by the definition of the set $\mathbb{I}(\alpha^k, n)$, it holds

$$\hat{\psi}_n^k(1) \leqslant j_{\max}\cdot n^{-(1+\alpha^k)} \leqslant n^{1+\alpha^k} \leqslant n^{1+\alpha_{\max}}$$

for all $k = 1,\dots,d$ where $\alpha_{\max} := \max_{k=1,\dots,d}\alpha^k$. We will utilize this fact in the proof of Theorem 4.3.2.

The following proposition serves as one of the auxiliary statements in the proof of Theorem 4.3.2.

Proposition 4.3.5
Let $k \in \{1,\dots,d\}$. For every $q \in [1,\infty)$ there exists a constant $c(q) \in (0,\infty)$ depending only on q such that, for all $n \in \mathbb{N}$,

$$\left(\mathbb{E}\Big[\sup_{t\in[0,1]}\big|\langle M^k\rangle(t) - \hat{\psi}_n^k(t)\big|^q\Big]\right)^{1/q} \leqslant c(q)\cdot n^{-(1+\alpha^k)}.$$

Moreover, for all $n \in \mathbb{N}_3$, it holds

$$\ln|\operatorname{ran}(\hat{\psi}_n^k)| \leqslant 6\cdot n\cdot\ln n. \tag{4.3.4}$$

Proof:
We show only (4.3.4). The remaining part of the proof is carried out by mimicking the proof of Proposition 4.1. in [Der08b]. In the course of this, apply (4.3.2) and (4.3.3).

By the definition of $\hat{\psi}_n^k$, and by the definition of the set $\mathbb{I}(\alpha^k,n)$, we have

$$\ln|\operatorname{ran}(\hat{\psi}_n^k)| \leqslant \ln\big(|\mathbb{I}(\alpha^k,n)|^n\big) = n\cdot\ln(|\mathbb{I}(\alpha^k,n)) \leqslant n\cdot\ln(2\cdot n^{2\cdot(1+\alpha^k)})$$

for all $n \in \mathbb{N}$. Together with the fact that $\alpha^k < 1/2$ we obtain

$$\ln|\operatorname{ran}(\hat{\psi}_n^k)| \leqslant n\cdot\ln 2 + n\cdot 2\cdot(1+\alpha^k)\cdot\ln n \leqslant 3\cdot n\cdot(1+\ln n) \leqslant 6\cdot n\cdot\ln n$$

for all $n \in \mathbb{N}_3$. □

The Main Auxiliary Theorem

In this section we provide the main auxiliary result. We prove only the result in a). The statements in b)--d) can be proven analogously to their corresponding statements presented in [Der08b, Theorem 7.1.].

Recall that, for $k = 1, \dots, d$ and $n \in \mathbb{N}$, $\hat{\psi}_n^k$ is the approximation of $(\langle M^k \rangle(t))_{t\in[0,1]}$ constructed in the previous subsection, which depends on the parameter α^k. Moreover, we define $\alpha_{\min} := \min_{k=1,\dots,d} \alpha^k$, and for $n \in \mathbb{N}$ we put $\hat{\psi}_n := \big(\hat{\psi}_n^1, \dots, \hat{\psi}_n^d\big)'$.

Theorem 4.3.6
Let $\gamma \in \big((1+\alpha_{\min})^{-1}, 1\big)$, and let $(n_N)_{N\in\mathbb{N}}$ be the sequence in $\mathbb{N}$ such that $n_1 = 1$ and $n_N = \lceil (\ln N)^\gamma \rceil$ for $N \geqslant 2$. Then for every $s \in [1, \infty)$ there exist a sequence of d-dimensional Brownian motions $\big(\overline{W}_N\big)_{N\in\mathbb{N}} = \big((\overline{W}_N^1, \dots, \overline{W}_N^d)'\big)_{N\in\mathbb{N}}$ on $[0,\infty)$, a sequence $(R_N)_{N\in\mathbb{N}} = \big((R_N^1, \dots, R_N^d)'\big)_{N\in\mathbb{N}}$ of $\mathbb{R}^d$-valued, continuous stochastic processes on $[0,1]$, as well as a sequence $(T_N)_{N\in\mathbb{N}} = \big((T_N^1, \dots, T_N^d)'\big)_{N\in\mathbb{N}}$ of Borel measurable mappings $T_N : \mathcal{C}([0,1];\mathbb{R}^d) \to \mathcal{C}([0,1];\mathbb{R}^d)$ such that the following holds:

a) For all $N \in \mathbb{N}$ the Brownian motion $\overline{W}_N$ is independent of $\hat{\psi}_{n_N}$.

b) For all $k = 1, \dots, d$ and $t \in [0,\infty)$ it holds

$$X^k(t) = \overline{W}_N^k\big(\hat{\psi}_{n_N}^k(t)\big) + R_N^k(t), \qquad P\text{-a.s.}$$

c) For $N \in \mathbb{N}$ we put $(\widetilde{R}_N^1, \dots, \widetilde{R}_N^d)' := T_N\big((R_N^1, \dots, R_N^d)'\big)$.

(i) There exist a real number $\delta_1 \in (0,\infty)$, a constant $c_1 \in (0,\infty)$, and an index $N_1 \in \mathbb{N}_2$ such that, for all $N \geqslant N_1$ and $k = 1, \dots, d$,

$$\left(\mathbb{E}\Big[\sup_{t\in[0,1]} \big|R_N^k - \widetilde{R}_N^k\big|^s\Big]\right)^{1/s} \leqslant c_1 \cdot \big(\ln N\big)^{-(1/2+\delta_1)}.$$

(ii) There exist a real number $\delta_2 \in (0,1)$, a constant $c_2 \in \mathbb{N}$, and an index $N_2 \in \mathbb{N}_2$ such that, for all $N \geqslant N_2$ and $k = 1, \dots, d$,

$$\ln\big|\operatorname{ran}\big(\widetilde{R}_N^k, \hat{\psi}_{n_N}^k\big)\big| \leqslant c_2 \cdot \Big\lceil \big(\ln N\big)^{\delta_2} \Big\rceil \cdot \ln\Big(\Big\lceil \big(\ln N\big)^{\delta_2} \Big\rceil\Big).$$

Proof:
Let $s \in [1,\infty)$. We show that there exists a sequence of d-dimensional Brownian motions on $[0,\infty)$ that satisfies a). To prove the statements b)--d) one first carries over the auxiliary statements in [Der08b, Sections 4 to 6]. Subsequently, one carries out the main proof of the statements b)--d) by mimicking the proof of the analogue statements in [Der08b, Theorem 7.1.].

To prove the existence of a sequence of d-dimensional Brownian motions on $[0,\infty)$ that satisfies a), we use results in the field of enlargements of filtrations presented in [ADI07] and [JY85], which are also employed in [Der08b, Section 6]. Recall that $\mathcal{F}^B$ is the completed filtration generated by the d-dimensional Brownian motion $B = \big(B(t)\big)_{t\in[0,\infty)}$. For $N \in \mathbb{N}$ we consider the right-continuous initial enlargement $\mathcal{G}_{n_N} = (\mathcal{G}_{n_N}(t))_{t\in[0,\infty)}$ of $\mathcal{F}^B$ given by

$$\mathcal{G}_{n_N}(t) := \bigcap_{s>t} \sigma\Big(\mathcal{F}^B(s) \cup \sigma\big(\hat{\psi}_{n_N}\big)\Big).$$

Then the results presented in [ADI07, Lemma 1, Proposition 1 and Theorem 3] imply that for $k = 1, \dots, d$ and $N \in \mathbb{N}$ the process B^k can be written as

$$B^k(t) = \overline{W}_N^k(t) + \int_0^t \beta_N^k(s)\, ds \qquad P\text{-a.s.},$$

for $t \in [0, \infty)$, where $(\beta_N^k(t))_{t\in[0,\infty)}$ is a real-valued, predictable stochastic process adapted to $\mathcal{G}_{n_N}$ and $\overline{W}_N^k = (\overline{W}_N^k(t))_{t\in[0,\infty)}$ is given by

$$\overline{W}_N^k(t) = B^k(t) - \int_0^t \beta_N^k(s)\, ds$$

for $t \in [0, \infty)$. Also see [ADI07, Example 2].

As a next step, we show that for all $N \in \mathbb{N}$ the process $\overline{W}_N = (\overline{W}_N^1, \dots, \overline{W}_N^d)'$ is a d-dimensional Brownian motion on $[0, \infty)$ w.r.t to the filtration $\mathcal{G}_{n_N}$. For the moment we fix $N \in \mathbb{N}$. Now, as stated in [ADI07, Theorem 3], for every $k = 1, \dots, d$ the process $\overline{W}_N^k$ is a continuous local $\mathcal{G}_{n_N}$-martingale. Moreover, note that for every $k = 1, \dots, d$ the process $\left(\int_0^t \beta_N^k(s)\, ds\right)_{t\in[0,\infty)}$ is of bounded variation, and recall that $B^1, \dots, B^d$ are independent one-dimensional Brownian motions. Together with standard facts on cross-variation processes we arrive at

$$\langle \overline{W}_N^i, \overline{W}_N^k \rangle(t) = \delta_{ik} t \tag{4.3.5}$$

for $i, k \in \{1, \dots, d\}$ and $t \in [0, \infty)$ where δ_{ik} denotes Kronecker's delta. Thus all prerequisites of Lévy's characterization of d-dimensional Brownian motions are satisfied, and we may conclude that $\overline{W}_N$ is a d-dimensional Brownian motion w.r.t $\mathcal{G}_{n_N}$. It remains to show that for all $N \in \mathbb{N}$ the Brownian motion $\overline{W}_N$ is independent of $\hat{\psi}_{n_N}$. Indeed, let $N \in \mathbb{N}$. On the one hand, since $\overline{W}_N$ is a $\mathcal{G}_{n_N}$-Brownian motion, $\overline{W}_N$ is independent of $\mathcal{G}_{n_N}(0)$. On the other hand, due to the definition of the filtration $\mathcal{G}_{n_N}$, $\hat{\psi}_{n_N}$ is measurable with respect to $\mathcal{G}_{n_N}(0)$. Hence $\overline{W}_N$ is independent of $\hat{\psi}_{n_N}$, which finishes the proof of a). □

4.3.2. Proof of Theorem 4.3.2

We carry out the main proof.

Proof of Theorem 4.3.2:
Let $s \in [1, \infty)$. Recall that $\alpha_{\max} = \max_{k=1,\dots,d} \alpha^k$ where α^k is the parameter in Proposition 4.3.5. Moreover, put $\alpha_{\min} := \min_{k=1,\dots,d} \alpha^k$, fix $\gamma \in ((1 + \alpha_{\min})^{-1}, 1)$, let $(n_N)_{N\in\mathbb{N}}$ be the sequence in $\mathbb{N}$ such that $n_1 = 1$ and $n_N = \lceil (\ln N)^\gamma \rceil$ for $N \geqslant 2$, and for $N \in \mathbb{N}$ let $(\overline{W}_N)_{N\in\mathbb{N}}$ be the sequence of d-dimensional Brownian motions on $[0, \infty)$, let $(R_N)_{N\in\mathbb{N}}$ be the sequences of continuous stochastic processes on $[0, 1]$ with values in $\mathbb{R}^d$, and let $(T_N)_{N\in\mathbb{N}}$ be the sequences of Borel measurable mappings $T_N = (T_N^1, \dots, T_N^d)' : \mathcal{C}([0, 1]; \mathbb{R}^d) \to \mathcal{C}([0, 1]; \mathbb{R}^d)$ in Theorem 4.3.6. Again, for $N \in \mathbb{N}$, we put $(\widetilde{R}_N^1, \dots, \widetilde{R}_N^d)' := T_N\big((R_N^1, \dots, R_N^d)'\big)$.

We split up the proof into single steps.

Step 1: In this step we make a few preliminary observations. Let $k \in \{1, \dots, d\}$, let $N \in \mathbb{N}$, and for the moment we fix $\omega \in \Omega$. For $t \in \big[0, \hat{\psi}_{n_N}^k(1)(\omega)\big]$ we put

$$\hat{\varphi}_{n_N}^k(t)(\omega) := \inf\big\{s \in [0, \infty) \mid \hat{\psi}_{n_N}^k(s)(\omega) \geqslant t\big\}.$$

Recall that the function $s \mapsto \hat{\psi}^k_{n_N}(s)(\omega)$, $s \in [0,1]$, is monotonically increasing and continuous. Hence the function $t \mapsto \hat{\varphi}^k_{n_N}(t)(\omega)$, $t \in \big[0, \hat{\psi}^k_{n_N}(1)(\omega)\big]$, is its monotonically increasing, continuous pseudo-inverse. In particular, it holds

$$\hat{\psi}^k_{n_N}\big(\hat{\varphi}^k_{n_N}(t)(\omega)\big)(\omega) = t \tag{4.3.6}$$

for all $t \in \big[0, \hat{\psi}^k_{n_N}(1)(\omega)\big]$. Together with Theorem 4.3.6 b) the observation in (4.3.6) leads to

$$\overline{W}^k_N(t)(\omega) = X^k\big(\hat{\varphi}^k_{n_N}(t)(\omega)\big)(\omega) - R^k_N\big(\hat{\varphi}^k_{n_N}(t)(\omega)\big)(\omega) \tag{4.3.7}$$

for all $t \in \big[0, \hat{\psi}^k_{n_N}(1)(\omega)\big]$.

Step 2: First, recall that for all $N \in \mathbb{N}$ and $k = 1, \dots, d$ we have $\hat{\psi}^k_{n_N}(1) \leqslant n_N^{(1+\alpha_{\max})}$. Secondly, for $N \in \mathbb{N}$ let $S_N : \mathcal{C}([0,1];\mathbb{R}^d) \to \mathcal{C}([0,1];\mathbb{R}^d)$ be an N-uniform-product-quantizer of $X_1 := (X(t))_{t\in[0,1]}$, and we put $(\widetilde{X}^1_N, \dots, \widetilde{X}^d_N)' = S_N\big(X_1\big)$. Then, due to Definition 4.0.1 b), we have

$$\big|\operatorname{ran}(\widetilde{X}^k_N)\big| \leqslant \big\lfloor N^{1/d} \big\rfloor \tag{4.3.8}$$

for all $k = 1, \dots, d$ and $N \in \mathbb{N}$.

Now, let $k \in \{1, \dots, d\}$, and let $N \in \mathbb{N}$. We utilize $\widetilde{X}^k_N$ and $\widetilde{R}^k_N$ and define a real-valued stochastic process $\widetilde{W}^k_N$ on $[0, n_N^{(1+\alpha_{\max})}]$ by

$$\widetilde{W}^k_N(t) := \widetilde{X}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) - \widetilde{R}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) \tag{4.3.9}$$

for $t \in \big[0, n_N^{(1+\alpha_{\max})}\big]$. Now combining (4.3.7) and (4.3.9) leads to

$$\begin{aligned}
&\left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|\overline{W}^k_N\big(t \wedge \hat{\psi}^k_{n_N}(1)\big) - \widetilde{W}^k_N\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\Big|^s\Big]\right)^{1/s} \\
&\leqslant \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|X^k\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) - \widetilde{X}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big)\Big|^s\Big]\right)^{1/s} \\
&\quad + \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|R^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) - \widetilde{R}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big)\Big|^s\Big]\right)^{1/s}.
\end{aligned} \tag{4.3.10}$$

Furthermore, for all $k = 1, \dots, d$, $t \in \big[0, n_N^{(1+\alpha_{\max})}\big]$ and $\omega \in \Omega$ it holds $\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)(\omega)\big)(\omega) \in [0,1]$. Hence we have

$$\Big|X^k\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) - \widetilde{X}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big)\Big| \leqslant \max_{k=1,\dots,d} \sup_{u\in[0,1]} \big|X^k(u) - \widetilde{X}^k_N(u)\big|$$

and

$$\Big|R^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big) - \widetilde{R}^k_N\big(\hat{\varphi}^k_{n_N}\big(t \wedge \hat{\psi}^k_{n_N}(1)\big)\big)\Big| \leqslant \max_{k=1,\dots,d} \sup_{u\in[0,1]} \big|R^k_N(u) - \widetilde{R}^k_N(u)\big|.$$

Therefore, together with (4.3.10), we obtain

$$\begin{aligned}&\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,n_N^{(1+\alpha_{\max})}]}\Big|\overline{W}_N^k\big(t\wedge\hat{\psi}_{n_N}^k(1)\big)-\widetilde{W}_N^k\big(t\wedge\hat{\psi}_{n_N}^k(1)\big)\Big|^s\Big]\right)^{1/s}\\&\leqslant\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,1]}\big|X^k(t)-\widetilde{X}_N^k(t)\big|^s\Big]\right)^{1/s}+\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,1]}\big|R_N^k(t)-\widetilde{R}_N^k(t)\big|^s\Big]\right)^{1/s}.\end{aligned}\tag{4.3.11}$$

Step 3: In this step we estimate

$$\limsup_{N\to\infty}\sqrt{\ln N}\cdot\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,1]}\big|R_N^k(t)-\widetilde{R}_N^k(t)\big|^s\Big]\right)^{1/s}.$$

By Theorem 4.3.6 c) there exist a real number $\delta_1\in(0,\infty)$, a constant $c_1\in(0,\infty)$ and an index $N_1\in\mathbb{N}_2$ such that, for all $N\geqslant N_1$,

$$\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,1]}\big|R_N^k(t)-\widetilde{R}_N^k(t)\big|^s\Big]\right)^{1/s}\leqslant c_1\cdot\big(\ln N\big)^{-(1/2+\delta_1)}.$$

Thus,

$$\limsup_{N\to\infty}\sqrt{\ln N}\cdot\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,1]}\big|R_N^k(t)-\widetilde{R}_N^k(t)\big|^s\Big]\right)^{1/s}=0.\tag{4.3.12}$$

Step 4: In this step we further estimate

$$\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\sup_{t\in[0,n_N^{(1+\alpha_{\max})}]}\Big|\overline{W}_N^k\big(t\wedge\hat{\psi}_{n_N}^k(1)\big)-\widetilde{W}_N^k\big(t\wedge\hat{\psi}_{n_N}^k(1)\big)\Big|^s\Big]\right)^{1/s}.$$

To this end, for $x=(x^1,\dots,x^d)'\in\mathbb{R}^d$, we put

$$k_*(x):=\min\big\{k\in\{1,\dots,d\}\;\big|\;|x^k|=\max_{i=1,\dots,d}|x^i|\big\}.$$

Moreover, we will employ the following observations:

- Let $k\in\{1,\dots,d\}$. Theorem 4.3.6 d) combined with (4.3.8) as well as (4.3.9) implies the existence of a real number $\delta_2\in(0,1)$, a constant $c_2\in\mathbb{N}$ and an index $N_2\in\mathbb{N}_2$ such that for all $N\geqslant N_2$ we have

$$\big|\operatorname{ran}\big(\widetilde{W}_N^k\big)\big|\leqslant\big|\operatorname{ran}\big(\widetilde{X}_N^k,\widetilde{R}_N^k,\hat{\psi}_{n_N}^k\big)\big|\leqslant\big\lfloor N^{1/d}\big\rfloor\cdot m_N^{c_2\cdot m_N}\tag{4.3.13}$$

  where $m_N:=\big\lceil\big(\ln N\big)^{\delta_2}\big\rceil$.

- It is easy to see that

$$\lim_{N\to\infty}\frac{m_N\cdot\ln m_N}{\ln N}=0.\tag{4.3.14}$$

- For the moment we fix $N \in \mathbb{N}$. By Theorem 4.3.6 a) we have that

$$\overline{W}_N = \big(\overline{W}_N^1, \dots, \overline{W}_N^d\big)' \quad \text{is independent of} \quad \hat{\psi}_{n_N}(1) = \big(\hat{\psi}_{n_N}^1(1), \dots, \hat{\psi}_{n_N}^d(1)\big)'.$$

  Furthermore, by the definition of $\big(\widetilde{X}_N^1, \dots, \widetilde{X}_N^d\big)'$ and $\big(\widetilde{R}_N^1, \dots, \widetilde{R}_N^d\big)'$, and due to (4.3.7) there exists a Borel measurable mapping $\Phi_N : \mathcal{C}([0,1];\mathbb{R}^d) \to \mathcal{C}([0,1];\mathbb{R}^d)$ such that $\widetilde{W}_N = \Phi_N(\overline{W}_N)$. In particular, combining the previous considerations leads to the fact that $\big(\overline{W}_N, \widetilde{W}_N\big)$ is independent of $\hat{\psi}_{n_N}(1)$.

Now by using the above observations, by applying Proposition B.2, Lemma 4.3.3, (4.3.13), and Proposition 2.1.8 a), we obtain, for all $N \geqslant N_2$,

$$\begin{aligned}
&\left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|\overline{W}_N^k\big(t \wedge \hat{\psi}_{n_N}^k(1)\big) - \widetilde{W}_N^k\big(t \wedge \hat{\psi}_{n_N}^k(1)\big)\Big|^s \;\Big|\; \hat{\psi}_{n_N}(1) = v\Big]\right)^{1/s} \\
&= \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|\overline{W}_N^k\big(t \wedge v^k\big) - \widetilde{W}_N^k\big(t \wedge v^k\big)\Big|^s\Big]\right)^{1/s} \\
&= \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,v^k]} \Big|\overline{W}_N^k(t) - \widetilde{W}_N^k(t)\Big|^s\Big]\right)^{1/s} \\
&\geqslant \left(\mathbb{E}\Big[\sup_{t\in[0,v^{k_*(v)}]} \Big|\overline{W}_N^{k_*(v)}(t) - \widetilde{W}_N^{k_*(v)}(t)\Big|^s\Big]\right)^{1/s} \\
&\geqslant \sqrt{v^{k_*(v)}} \cdot e^{(s)}_{|\operatorname{ran}(\widetilde{W}_N^{k_*(v)})|}\big(\mathcal{W}, \mathcal{C}[0,1]\big) \\
&\geqslant \sqrt{\max_{k=1,\dots,d} v^k} \cdot e^{(s)}_{\lfloor N^{1/d}\rfloor \cdot m_N^{c_2 \cdot m_N}}\big(\mathcal{W}, \mathcal{C}[0,1]\big)
\end{aligned}$$

for $P_{\hat{\psi}_{n_N}(1)}$-a.a. $v = (v^1, \dots, v^d)' \in [0,\infty)^d$ where $\mathcal{W}$ denotes a Brownian motion on $[0,1]$. Thus, together with Theorem 2.3.11 (ii) we arrive at

$$\begin{aligned}
&\liminf_{N\to\infty} \sqrt{\ln N} \cdot \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \sup_{t\in[0,n_N^{(1+\alpha_{\max})}]} \Big|\overline{W}_N^k\big(t \wedge \hat{\psi}_{n_N}^k(1)\big) - \widetilde{W}_N^k\big(t \wedge \hat{\psi}_{n_N}^k(1)\big)\Big|^s\Big]\right)^{1/s} \\
&\geqslant \kappa_\infty \cdot \sqrt{d} \cdot \liminf_{N\to\infty} \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\hat{\psi}_{n_N}^k(1)\big)^{s/2}\right]\right)^{1/s}
\end{aligned} \tag{4.3.15}$$

where κ_∞ is the constant in the aforementioned theorem, and where we used the fact that

$$\liminf_{N\to\infty} \sqrt{\frac{\ln N}{\ln\big(\lfloor N^{1/d}\rfloor \cdot m_N^{c_2 \cdot m_N}\big)}} \geqslant \liminf_{N\to\infty} \sqrt{\frac{\ln N}{(\ln N)/d + c_2 \cdot m_N \cdot \ln m_N}} \geqslant \sqrt{d}.$$

We employed (4.3.14) to derive the last inequality in the above detached formula.

Step 5: We combine the results of Step 1 up to Step 4. Recall that, for $N \in \mathbb{N}$, $\widetilde{X}_N$ is an arbitrary N-uniform-product-quantization of X_1. Then, by Lemma 4.0.3, (4.3.11), (4.3.12) and (4.3.15), we derive that

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod,uni}}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \geqslant \kappa_\infty \cdot \sqrt{d} \cdot \liminf_{N\to\infty} \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\hat{\psi}^k_{n_N}(1)\big)^{s/2}\right]\right)^{1/s}.$$

It remains to further estimate

$$\liminf_{N\to\infty} \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\hat{\psi}^k_{n_N}(1)\big)^{s/2}\right]\right)^{1/s}.$$

The inverse triangle inequality implies

$$\left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\hat{\psi}^k_{n_N}(1)\big)^{s/2}\right]\right)^{1/s} \geqslant \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\langle M^k\rangle(1)\big)^{s/2}\right]\right)^{1/s} - \sum_{k=1}^{d} \left(\mathbb{E}\left[\big|\hat{\psi}^k_{n_N}(1) - \langle M^k\rangle(1)\big|^{s/2}\right]\right)^{1/s}$$

for all $N \in \mathbb{N}$. Moreover, by Proposition 4.3.5 we have

$$\begin{aligned}
\sum_{k=1}^{d} \left(\mathbb{E}\Big[\big|\hat{\psi}^k_{n_N}(1) - \langle M^k\rangle(1)\big|^{s/2}\Big]\right)^{1/s} &\leqslant \sum_{k=1}^{d} \left(\mathbb{E}\Big[\sup_{t\in[0,1]} \big|\hat{\psi}^k_{n_N}(t) - \langle M^k\rangle(t)\big|^{s/2}\Big]\right)^{1/s} \\
&\leqslant c \cdot \sum_{k=1}^{d} \frac{1}{\big(n_N\big)^{(1+\alpha^k)/2}} \\
&\leqslant c \cdot \frac{1}{\big(n_N\big)^{(1+\alpha_{\min})/2}} \\
&\leqslant c \cdot \frac{1}{\big(\ln N\big)^{\gamma\cdot(1+\alpha_{\min})/2}}
\end{aligned}$$

for all $N \in \mathbb{N}$ such that $n_N \geqslant 3$, and where c denotes a positive real constant not depending on N which may vary from line to line. Now due to the choice of γ, we have $\gamma \cdot (1 + \alpha_{\min})/2 > 0$. Hence we end up with

$$\liminf_{N\to\infty} \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big(\hat{\psi}^k_{n_N}(1)\big)^{s/2}\right]\right)^{1/s} \geqslant \left(\mathbb{E}\left[\max_{k=1,\dots,d} \big\| \|Z^k(\cdot)\|_2 \big\|^s_{L_2[0,1]}\right]\right)^{1/s},$$

which finishes the proof.

□

5 | Quantization of Multidimensional SDEs

In this chapter we provide a method which yields (strongly) asymptotically optimal sequences of quantizations of a strong solution $X = (X^1, \dots, X^d)'$ of a d-dimensional system of SDEs, and we consider the spaces $L_p^d[0,1]$, $p \in [1,\infty)$, and $\mathcal{C}([0,1];\mathbb{R}^d)$. Our construction generalizes a quantization procedure for scalar SDEs developed in [MGR13], and it basically consists of two steps, namely, a *coarse-level* quantization and a *fine-level* quantization. We apply both of them separately to each component X^k of X such that we end up with overall product-quantizations of X. In particular, the fine-level quantization will turn out to be a decisive ingredient in the construction since it will be crucial for the overall performance of the quantization.

The chapter is organized as follows. In Section 1 we construct a coarse-level quantization of X. Section 2 deals with quantization of X in the space $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$. The section closes with a discussion, which includes, among other things, the analysis of the computational cost and an example. Section 3 is devoted to quantization of X in the space $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$. The last section of this chapter, Section 4, is concerned with quantization of X in the space $(\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)$, $s \in [1,\infty)$, where $|||\cdot|||_s$ is the norm introduced in the introductory chapter.

General assumptions:

(i) Throughout the chapter we assume the setting in Section 3.1 with $r \in \mathbb{N}_2$.

(ii) To avoid trivial cases we assume that the continuous $\mathbb{R}^r$-valued process $\big(b_k(X(t))\big)_{t\in[0,1]}$ is not indistinguishable from the constant zero process for all $k = 1, \dots, d$, where X is the unique strong solution of the SDE (3.1.1).

(iii) For $m \in \mathbb{N}$ let $0 = t_0 < \cdots < t_m = 1$ be the equidistant discretization of the time interval $[0,1]$ given by

$$t_l = \frac{l}{m} \tag{5.0.1}$$

for all $l = 0, \dots, m$.

5.1. A Coarse-level Quantization

Let $m \in \mathbb{N}$. We aim at constructing a quantization of the $\mathbb{R}^{d\times(m+1)}$-valued random matrix

$$\underline{X_m} := \big(X(t_0), \dots, X(t_m)\big).$$

Due to assumption **(C1)** all partial derivatives of the diffusion coefficient b of the SDE (3.1.1) exist. Hence we are in the position to employ the d-dimensional Milstein scheme corresponding to the discretization (5.0.1) whose kth component, $k = 1, \dots, d$, is given by $\widehat{X}_m^{\mathrm{M},k}(t_0) = x_0^k$ and

$$\widehat{X}_m^{\mathrm{M},k}(t_l) = \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot m^{-1} + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big(W_j(t_l) - W_j(t_{l-1})\big)$$
$$+ \sum_{j_1,j_2=1}^{r} \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot J^l_{(j_1,j_2)}$$

for $l = 1,\dots,m$ where

$$J^l_{(j_1,j_2)} = \int_{t_{l-1}}^{t_l} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(s).$$

From now on, we abbreviate

$$Y_l^j := m^{1/2}\cdot\big(W_j(t_l) - W_j(t_{l-1})\big) \tag{5.1.1}$$

for $j = 1,\dots,r$ and $l = 1,\dots,m$, and

$$I^l_{(j_1,j_2)} := m\cdot J^l_{(j_1,j_2)} \tag{5.1.2}$$

for $j_1, j_2 = 1,\dots,r$ and $l = 1,\dots,m$. Recall that for all $l = 1,\dots,m$ and $j_1, j_2 = 1,\dots,r$ Itô's formula implies

$$J^l_{(j_1,j_2)} + J^l_{(j_2,j_1)} \overset{P\text{-a.s.}}{=} \begin{cases} \big(W_{j_1}(t_l) - W_{j_1}(t_{l-1})\big)^2 - m^{-1}, & \text{if } j_1 = j_2,\\ m^{-1}\cdot Y_l^{j_1}\cdot Y_l^{j_2}, & \text{if } j_1 \neq j_2. \end{cases} \tag{5.1.3}$$

Thus, in view of (5.1.1)--(5.1.3), we rewrite the Milstein scheme such that for all $k = 1,\dots,d$ and $l = 1,\dots,m$ we end up with

$$\begin{aligned}
\widehat{X}_m^{\mathrm{M},k}(t_l) &\overset{P\text{-a.s.}}{=} \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot m^{-1} + m^{-1/2}\cdot\sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot Y_l^j\\
&\quad + \frac{1}{2}\cdot m^{-1}\cdot\sum_{j=1}^{r}\nabla b_{k,j} b^{(j)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big((Y_l^j)^2 - 1\big)\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot I^l_{(j_1,j_2)}\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1>j_2}}^{r}\nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big(Y_l^{j_1}\cdot Y_l^{j_2} - I^l_{(j_2,j_1)}\big)\\
&= \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot m^{-1} + m^{-1/2}\cdot\sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot Y_l^j\\
&\quad + \frac{1}{2}\cdot m^{-1}\cdot\sum_{j=1}^{r}\nabla b_{k,j} b^{(j)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot\big((Y_l^j)^2 - 1\big)\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\Big(\nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big) - \nabla b_{k,j_1} b^{(j_2)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\Big)\cdot I^l_{(j_1,j_2)}\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\nabla b_{k,j_1} b^{(j_2)}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)\cdot Y_l^{j_1}\cdot Y_l^{j_2}.
\end{aligned} \tag{5.1.4}$$

Note that

$$Y_l^j \sim N(0,1)$$

for all $j = 1, \dots r$ and $l = 1, \dots, m$, and

$$I^l_{(j_1,j_2)} \overset{\mathcal{L}}{=} \int_0^1 W_1(s)\, dW_2(s)$$

for all $j_1, j_2 = 1, \dots, r$ with $j_1 < j_2$ and all $l = 1, \dots, m$. Hence, in view of (5.1.4), it stands to reason to employ quantizers for one-dimensional standard normally distributed random variables and quantizers for the Itô integral $\int_0^1 W_1(s)dW_2(s)$ as additional building blocks in our construction.

Let $q \in [1, \infty)$. For quantizing the standardized increments of the driving Brownian motion we choose a sequence $\big(\overline{T}_n^{(q)}\big)_{n\in\mathbb{N}}$ of n-quantizers $\overline{T}_n^{(q)} : \mathbb{R} \to \mathbb{R}$ for standard normally distributed random variables, and for all $j = 1, \dots, r$ and $l = 1, \dots, m$ we put

$$\widetilde{Y}^{(q),j}_{l,n} := \overline{T}_n^{(q)}(Y_l^j) \tag{5.1.5}$$

for all $n \in \mathbb{N}$ and we assume that:

(N1) For all $n \in \mathbb{N}$ it holds $\mathbb{E}\big[\widetilde{Y}^{(q),j}_{l,n}\big] = 0$.

(N2) For all $s \in [1, \infty)$ it holds $\sup_{n\in\mathbb{N}} \mathbb{E}\big[|\widetilde{Y}^{(q),j}_{l,n}|^s\big] < \infty$.

(N3) There exists a constant $c_1(q) \in (0, \infty)$ depending only on q such that

$$e^{(q)}\big(Y_l^j, \widetilde{Y}^{(q),j}_{l,n}, \mathbb{R}\big) \leqslant c_1(q) \cdot n^{-1}$$

for all $n \in \mathbb{N}$.

In Section 2.3.1 we have already presented a method which yields such a sequence $(\overline{T}_n^{(q)})_{n\in\mathbb{N}}$.

Let $\gamma \in (2, \infty)$. For quantizing the standardized Itô integrals we take the sequence of mappings $(T_n^{(\gamma)})_{n\in\mathbb{N}}$ where $T_n^{(\gamma)} : \mathbb{R} \to \mathbb{R}$ is defined as in (2.3.8), and we put

$$\widetilde{I}^{l,n}_{(j_1,j_2)} := T_n^{(\gamma)}(I^l_{(j_1,j_2)})$$

for all $n \in \mathbb{N}$, all $j_1, j_2 = 1, \dots, r$ with $j_1 < j_2$ and all $l = 1, \dots, m$. By Lemma 2.3.7 the sequence $(\widetilde{I}^{l,n}_{(j_1,j_2)})_{n\in\mathbb{N}}$ has the following properties:

(I1) For all $n \in \mathbb{N}$ it holds $\mathbb{E}\big[\widetilde{I}^{l,n}_{(j_1,j_2)}\big] = 0$.

(I2) For all $s \in [1, \infty)$ it holds $\sup_{n\in\mathbb{N}} \mathbb{E}\big[|\widetilde{I}^{l,n}_{(j_1,j_2)}|^s\big] < \infty$.

(I3) There exists a constant $c_2(q) \in (0, \infty)$ depending only on q such that

$$\Big(\mathbb{E}\big[|I^l_{(j_1,j_2)} - \widetilde{I}^{l,n}_{(j_1,j_2)}|^q\big]\Big)^{1/q} \leqslant c_2(q) \cdot n^{-1}$$

for all $n \in \mathbb{N}$.

(I4) For all $n \in \mathbb{N}$ it holds $\big|\operatorname{ran}\big(\widetilde{I}^{l,n}_{(j_1,j_2)}\big)\big| \leqslant 4 \cdot n$.

Now we replace the standardized increments of the driving Brownian motion as well as the standardized Itô integrals in (5.1.4) by their just chosen respective quantizations. More precisely, let $n \in \mathbb{N}$, and we

define

$$\begin{aligned}
\widetilde{X}^{(q),k}_{m,n}(t_0) &= x_0^k,\\
\widetilde{X}^{(q),k}_{m,n}(t_l) &= \widetilde{X}^{(q),k}_{m,n}(t_{l-1}) + a_k\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big)\cdot m^{-1} + m^{-1/2}\cdot\sum_{j=1}^{r} b_{k,j}\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big)\cdot \widetilde{Y}^{(q),j}_{l,n}\\
&\quad + \frac{1}{2}\cdot m^{-1}\cdot\sum_{j=1}^{r}\nabla b_{k,j}b^{(j)}\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big)\cdot\big((\widetilde{Y}^{(q),j}_{l,n})^2-1\big)\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\Big(\nabla b_{k,j_2}b^{(j_1)}\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big) - \nabla b_{k,j_1}b^{(j_2)}\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big)\Big)\cdot \widetilde{I}^{l,n}_{(j_1,j_2)}\\
&\quad + m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\nabla b_{k,j_1}b^{(j_2)}\big(\widetilde{X}^{(q)}_{m,n}(t_{l-1})\big)\cdot \widetilde{Y}^{(q),j_1}_{l,n}\cdot\widetilde{Y}^{(q),j_2}_{l,n}
\end{aligned}$$

for $l=1,\dots,m$ and $k=1,\dots,d$.

By the above construction we have obtained a quantization

$$\underline{\widetilde{X}^{(q)}_{m,n}} := \big(\widetilde{X}^{(q)}_{m,n}(t_0),\dots,\widetilde{X}^{(q)}_{m,n}(t_m)\big) \quad \text{of} \quad \underline{X_m} = \big(X(t_0),\dots,X(t_m)\big)$$

with

$$\big|\operatorname{ran}\big(\underline{\widetilde{X}^{(q),k}_{m,n}}\big)\big| \leqslant \Big(4^{\binom{r}{2}}\cdot n^{r+\binom{r}{2}}\Big)^m, \tag{5.1.6}$$

for all $k=1,\dots,d$, where we used the fact that $\big|\{(i,j)\in\{1,\dots,r\}^2 \mid i<j\}\big| = \binom{r}{2}$ as well as **(I4)**. From now on, we call the $\mathbb{R}^{d\times(m+1)}$-valued random matrix $\underline{\widetilde{X}^{(q)}_{m,n}}$ **Milstein quantization of X (of level n)**.

The following proposition contains the main result of this section, whose proof is postponed to Section 6.1.

Proposition 5.1.1
Let $q\in[1,\infty)$. Then there exists a constant $c\in(0,\infty)$ such that, for all $m,n\in\mathbb{N}$,

$$\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widetilde{X}^{(q)}_{m,n}(t_l)\|_\infty^q\Big] \leqslant c\cdot\big(m^{-1}+n^{-1}\big)^q.$$

Remark 5.1.2
If b has the commutativity property, the Milstein scheme is of the form as in (3.3.21). In this case it suffices to employ only quantizations of standard normally distributed random variables in the coarse-level quantization.

Euler Quantization of X

Let $m\in\mathbb{N}$. For technical reasons, at certain places within this chapter, we will not utilize the Milstein quantization of X. Instead, we will consider a quantization of the d-dimensional Euler scheme

$$\widehat{X}^{\mathrm{E}}_m = \big(\widehat{X}^{\mathrm{E}}_m(t_0),\dots,\widehat{X}^{\mathrm{E}}_m(t_m)\big)$$

associated to the discretization (5.0.1), which is built-up analogously to the Milstein quantization. We will go into greater detail about the reasons for this procedure in the next section of this chapter.
Recall that the kth component, $k = 1, \dots, d$, of the d-dimensional Euler scheme is given by

$$\begin{aligned} \widehat{X}_m^{\mathrm{E},k}(t_0) &= x_0^k, \\ \widehat{X}_m^{\mathrm{E},k}(t_l) &= \widehat{X}_m^{\mathrm{E},k}(t_{l-1}) + a_k\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot m^{-1} + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{E}}(t_{l-1})\big) \cdot \big(W_j(t_l) - W_j(t_{l-1})\big) \end{aligned} \tag{5.1.7}$$

for $l = 1, \dots, m$. Let $q \in [1, \infty)$, and let $n \in \mathbb{N}$. In view of (5.1.7) we define

$$\begin{aligned} \widetilde{X}_{m,n}^{\mathrm{E},(q),k}(0) &= x_0^k, \\ \widetilde{X}_{m,n}^{\mathrm{E},(q),k}(t_l) &= \widetilde{X}_{m,n}^{\mathrm{E},(q),k}(t_{l-1}) + a_k\big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{l-1})\big) \cdot m^{-1} + m^{-1/2} \cdot \sum_{j=1}^{r} b_{k,j}\big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{l-1})\big) \cdot \widetilde{Y}_{l,n}^{(q),j} \end{aligned} \tag{5.1.8}$$

for $l = 1, \dots, m$ and $k = 1, \dots, d$, and where $\widetilde{Y}_{l,n}^{(q),j}$ is the quantization in (5.1.5).

From now on, we call the $\mathbb{R}^{d\times(m+1)}$-valued random matrix

$$\underline{\widetilde{X}_{m,n}^{\mathrm{E},(q)}} := \big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_0), \dots, \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_m)\big)$$

Euler quantization of X (of level n).

Proposition 5.1.3
Let $q, s \in [1, \infty)$. Then there exists a constant $c \in (0, \infty)$ such that, for all $m, n \in \mathbb{N}$,

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \|\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^s\Big] \leqslant c$$

as well as

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \|X(t_l) - \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^q\Big] \leqslant c \cdot \big(m^{-1} + n^{-1}\big)^{q/2}.$$

Proof:
Let $m, n \in \mathbb{N}$. In the proof c denotes a not further specified positive real constant which may vary at every occurrence and which does neither depend on m nor on n. Similar to the upcoming Lemmas 6.1.1 and 6.1.2 one proves that

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \|\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^s\Big] \leqslant c$$

and

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \|\widehat{X}_m^{\mathrm{E}}(t_l) - \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^q\Big] \leqslant c \cdot \big(m^{-1} + n^{-1}\big)^{q}. \tag{5.1.9}$$

Together with Proposition 3.3.1 this leads to

$$\begin{aligned} \mathbb{E}\Big[\max_{l=0,\dots,m} \|X(t_l) - \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^q\Big] &\leqslant c \cdot \Big(\mathbb{E}\Big[\sup_{t\in[0,1]} \|X(t) - X_m^{\mathrm{E}}(t)\|_\infty^q\Big] + \mathbb{E}\Big[\max_{l=1,\dots,m} \|\widehat{X}_m^{\mathrm{E}}(t_l) - \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^q\Big]\Big) \\ &\leqslant c \cdot \Big(m^{-q/2} + \big(m^{-1} + n^{-1}\big)^q\Big) \\ &\leqslant c \cdot \big(m^{-1} + n^{-1}\big)^{q/2}. \end{aligned}$$

□

5.2. Quantization in $L_p([0,1];\mathbb{R}^d)$

Throughout the whole section let $p \in [1,\infty)$, and we interpret X as an $L_p^d[0,1]$-valued random element. As a first ingredient we extend the Milstein quantization to a $\mathcal{C}([0,1];\mathbb{R}^d)$-valued random element by means of piecewise linear interpolation, i.e., for $m,n \in \mathbb{N}$ and $q \in [1,\infty)$ we define

$$\widetilde{X}_{m,n}^{(q)}(t) = (t_l - t)\cdot m \cdot \widetilde{X}_{m,n}^{(q)}(t_{l-1}) + (t - t_{l-1})\cdot m \cdot \widetilde{X}_{m,n}^{(q)}(t_l)$$

for $t \in [t_{l-1}, t_l]$ and $l = 1,\dots,m$.

5.2.1. Fine-level Quantization: Quantization of Brownian Bridges

Let $m \in \mathbb{N}$. In this subsection we construct a fine-level quantization which consists of quantizations of Brownian bridges on the subintervals $[t_{l-1}, t_l]$. In the next subsection we will combine the fine-level quantization with the coarse-level quantization constructed in the previous section in order to get a sequence of overall quantizations of X.

For $l = 1,\dots,m$ and $j = 1,\dots,r$ we define a real-valued stochastic process $B_l^j = \big(B_l^j(t)\big)_{t\in[t_{l-1},t_l]}$ by

$$B_l^j(t) = W_j(t) - (t_l - t)\cdot m \cdot W_j(t_{l-1}) - (t - t_{l-1})\cdot m \cdot W_j(t_l). \tag{5.2.1}$$

Note that the processes B_l^j have continuous sample paths. Hence we may interpret B_l^j as an $L_p[t_{l-1}, t_l]$-valued random element.

In the following lemma we collect facts about the processes B_l^j, which will prove beneficial in the remaining part of this thesis. Since those facts are well known, a proof is omitted.

Lemma 5.2.1

a) For all $j = 1,\dots,r$ and $l = 1,\dots,m$ the process B_l^j is a Brownian bridge on $[t_{l-1}, t_l]$.

b) The processes $B_1^1,\dots,B_1^r,\dots,B_m^1,\dots,B_m^r$ are independent.

c) $\big(B_1^1,\dots,B_1^r,\dots,B_m^1,\dots,B_m^r\big)$ is independent of $\big(W(t_0),\dots,W(t_m)\big)$.

The construction of a fine-level quantization is based on the following key observation, see **(I.)**, and on the two succeeding ingredients, see **(II.)**.

(I.) By Lemma 5.2.1, for all $l = 1,\dots,m$, the processes $B_l^1,\dots,B_l^r$ are independent Brownian bridges and thus, in particular, independent Gaussian processes. This implies

$$\sum_{j=1}^{r} \beta_j \cdot B_l^j \overset{\mathcal{L}}{=} \|\beta\|_2 \cdot B_l^1 \tag{5.2.2}$$

for all $\beta = \big(\beta_1,\dots,\beta_r\big)' \in \mathbb{R}^r$ and $l = 1,\dots,m$. In our analysis (5.2.2) will turn out to be one of the key observations. Moreover, (5.2.2) strikes on the idea to design quantizers suited to such processes which are of the form as on the right hand side of (5.2.2) and then apply those quantizers to such processes which are of the form as on the left hand side of (5.2.2). This is one of the key ideas for our construction.

(II.) Consider the following mappings:

- For $K \in \mathbb{N}$ let $S_K : L_p[0,1] \to L_p[0,1]$ be a K-quantizer for a Brownian bridge on $[0,1]$. Regarding constructive approaches which yield quantizers for a Brownian bridge on $[0,1]$ we refer the reader to Section 2.3.2.
- Let $\beta \in \mathbb{R}^r \backslash \{0\}$, and let $l \in \{1,\dots,m\}$. We define $\psi_{\beta,l} : L_p[t_{l-1},t_l] \to L_p[0,1]$ by

$$\psi_{\beta,l}h(t) = \frac{1}{\|\beta\|_2} \cdot m^{1/2} \cdot h(t_{l-1} + m^{-1} \cdot t), \quad t \in [0,1]. \tag{5.2.3}$$

  The operator $\psi_{\beta,l}$ is linear and bijective, and its linear inverse $\psi_{\beta,l}^{-1} : L_p[0,1] \to L_p[t_{l-1},t_l]$ is given by

$$\psi_{\beta,l}^{-1}h(t) = \|\beta\|_2 \cdot m^{-1/2} \cdot h(m \cdot (t - t_{l-1})), \quad t \in [t_{l-1},t_l]. \tag{5.2.4}$$

Lemma 5.2.2
Let $l \in \{1,\dots,m\}$, $\beta \in \mathbb{R}^r \backslash \{0\}$ and let B_l be a Brownian bridge on $[t_{l-1},t_l]$. Then the stochastic process $\big(\psi_{\beta,l}(\|\beta\|_2 \cdot B_l)(t)\big)_{t\in[0,1]}$ is a Brownian bridge on $[0,1]$.

Proof:
This statement follows directly from the scaling property of Brownian bridges. □

As a last step in this subsection we combine the mappings introduced in **(II.)**. For $l = 1,\dots,m$, $K \in \mathbb{N}$ and $\beta \in \mathbb{R}^r$ let $\Phi_{\beta,l,K} : L_p[t_{l-1},t_l] \to L_p[t_{l-1},t_l]$ be defined by

$$\Phi_{\beta,l,K}h = \begin{cases} \psi_{\beta,l}^{-1}\big(S_K(\psi_{\beta,l}h)\big), & \text{if } \|\beta\|_2 \neq 0, \\ 0, & \text{if } \|\beta\|_2 = 0. \end{cases} \tag{5.2.5}$$

Note that for all $l = 1,\dots,m$, $K \in \mathbb{N}$ and $\beta \in \mathbb{R}^r$ the mapping $\Phi_{\beta,l,K}$ is Borel measurable with $\big|\operatorname{ran}\big(\Phi_{\beta,l,K}\big)\big| \leqslant K$.

5.2.2. Overall Quantization

Let $q \in [1,\infty)$, $m \in \mathbb{N}_2$ and $n \in \mathbb{N}$. The strategy to get an overall quantization of each component X^k of X is to apply the quantizers $\Phi_{\beta,l,K}$ constructed in the previous in order to obtain quantizations of Brownian bridges, and, subsequently, to combine the resulting quantizations with the coarse-level quantization.

To this end, we first choose the size K of the quantizers S_K employed in the previous section separately for each component X^k of X by taking into account the local regularity of X^k. Recall that in the pth mean sense the local smoothness of X^k at time point t is determined by the size of

$$s_{\mathrm{locH}}\big(X^k(t)\big) = \Big(\sum_{j=1}^{r} \big(b_{k,j}\big(X(t)\big)\big)^2\Big)^{1/2}, \tag{5.2.6}$$

see Section 3.2. It stands to reason that one chooses a greater K in regions where $s_{\mathrm{locH}}\big(X^k(t)\big)$ is large and vice versa.

In the following, for technical reasons, our construction partly differs from the one in [MGR13], and we use the Euler quantization of X instead of the Milstein quantization in order to get an estimate of the quantity in (5.2.6) at the time points t_l. We briefly justify this procedure. Since in the definition

of the Euler quantization only quantizations of the increments of the Brownian motion W are employed, Lemma 5.2.1 c) then guarantees that

$$\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_0),\dots,\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_m)\big) \quad \text{and} \quad \big(B^1_1,\dots,B^r_1,\dots,B^1_m,\dots,B^r_m\big)$$

are independent. This statement need not hold when replacing $\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_l)$ by the Milstein quantization $\widetilde{X}^{(q)}_{m,n}(t_l)$ since, in general, the multidimensional Milstein scheme contains the multiple Itô integrals $J^l_{(j_1,j_2)}$ with $j_1 \neq j_2$. Now by using the Euler quantiaztion instead of the Milstein quantization we are in the position to utilize Proposition B.2 in our analysis in Chapter 6.

For $k = 1,\dots,d$ and $l = 1,\dots,m$ we estimate the quantity in (5.2.6) at time point t_{l-1} by

$$\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n} := \Big(\sum_{j=1}^{r}\big(b_{k,j}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big)\big)^2\Big)^{1/2}. \tag{5.2.7}$$

Now we utilize (5.2.7) and define

$$\eta^{(p,q),k}_{l,m,n} := \begin{cases} \max\left\{\dfrac{\big(\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n}\big)^{2p/(p+2)}}{\sum_{\nu=1}^{d}\sum_{i=0}^{m-1}\big(\widetilde{s}^{\mathrm{E},\nu,i}_{\mathrm{locH},m,n}\big)^{2p/(p+2)}}, \dfrac{1}{d\cdot m\cdot \ln m}\right\}, & \text{if } \max\limits_{\substack{i=0,\dots,m-1,\\ \nu=1,\dots,d}} \widetilde{s}^{\mathrm{E},\nu,i}_{\mathrm{locH},m,n} > 0, \\ \dfrac{1}{d\cdot m}, & \text{else,} \end{cases} \tag{5.2.8}$$

for $k = 1,\dots,d$ and $l = 1,\dots,m$. Thus, up to the thresholds $(d\cdot m\cdot \ln m)^{-1}$ and $(d\cdot m)^{-1}$ each random variable $\eta^{(p,q),k}_{l,m,n}$ provides an estimate of the local regularity of X^k on the subinterval $[t_{l-1},t_l]$ relative to the local regularity of X over all subintervals. From now on, we will mostly drop the parameters m,n,p,q and write η^k_l instead of $\eta^{(p,q),k}_{l,m,n}$. We also use the shorter notation $\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH}}$ instead of $\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n}$.

We employ the random quantities η^k_l to further specify the size K of the quantizers S_K. Let $M \in \mathbb{N}$. For $l = 1,\dots,m$ and $k = 1,\dots,d$ we choose

$$K_{k,l,M} := \Big\lceil M^{\eta^k_l}\Big\rceil. \tag{5.2.9}$$

Now by defining

$$\widetilde{X}^{(p,q),k}_{m,n,M}(t) = \widetilde{X}^{(q),k}_{m,n}(t) + \Phi_{b_k(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})),l,K_{k,l,M}}\Big(\sum_{j=1}^{r} b_{k,j}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big)\cdot B^j_l\Big)(t) \tag{5.2.10}$$

where

$$b_k\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big) := \big(b_{k,1}(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})),\dots,b_{k,r}(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1}))\big)'$$

for $t \in [t_{l-1},t_l]$, $l = 1,\dots,m$ and $k = 1,\dots,d$, we obtain a quantization $\widetilde{X}^{(p,q)}_{m,n,M} = \big(\widetilde{X}^{(p,q),1}_{m,n,M},\dots,\widetilde{X}^{(p,q),d}_{m,n,M}\big)'$

of $X = \big(X^1, \dots, X^d\big)'$. Together with (5.1.6) we have

$$\begin{aligned}\big|\operatorname{ran}\big(\widetilde{X}^{(p,q),k}_{m,n,M}\big)\big| &\leqslant \Big(4^{\binom{r}{2}} \cdot n^{r+\binom{r}{2}}\Big)^m \cdot \prod_{l=1}^m K_{k,l,M} \\ &\leqslant \Big(4^{\binom{r}{2}} \cdot n^{r+\binom{r}{2}}\Big)^m \cdot \prod_{l=1}^m 2 \cdot M^{\eta_l^k} \\ &\leqslant \Big(4^{1+\binom{r}{2}} \cdot n^{r+\binom{r}{2}}\Big)^m \cdot M^{\sum_{l=1}^m \eta_l^k}\end{aligned}$$

for all $k = 1, \dots, d$, and hence

$$\begin{aligned}\big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big| &\leqslant \prod_{k=1}^d \big|\operatorname{ran}\big(\widetilde{X}^{(p,q),k}_{m,n,M}\big)\big| \\ &\leqslant \Big(4^{d\cdot(1+\binom{r}{2})} \cdot n^{d\cdot(r+\binom{r}{2})}\Big)^m \cdot M^{\sum_{k=1}^d \sum_{l=1}^m \eta_l^k} \\ &\leqslant \Big(4^{d\cdot(1+\binom{r}{2})} \cdot n^{d\cdot(r+\binom{r}{2})}\Big)^m \cdot M^{1+1/\ln m},\end{aligned} \tag{5.2.11}$$

where we used the fact that

$$\sum_{k=1}^d \sum_{l=1}^m \eta_l^k \leqslant \sum_{k=1}^d \sum_{l=1}^m \left(\frac{\big(\tilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH}}\big)^{2p/(p+2)}}{\sum_{\nu=1}^d \sum_{i=0}^{m-1} \big(\tilde{s}^{\mathrm{E},\nu,i}_{\mathrm{locH}}\big)^{2p/(p+2)}} + \frac{1}{d\cdot m\cdot \ln m} \right) = 1 + \frac{1}{\ln m}$$

if $\max\limits_{\substack{i=0,\dots,m-1,\\ \nu=1,\dots,d}} \tilde{s}^{\mathrm{E},\nu,i}_{\mathrm{locH}} > 0$, and

$$\sum_{k=1}^d \sum_{l=1}^m \eta_l^k = \sum_{k=1}^d \sum_{l=1}^m \frac{1}{d\cdot m} = 1 \leqslant 1 + \frac{1}{\ln m}$$

if $\max\limits_{\substack{i=0,\dots,m-1,\\ \nu=1,\dots,d}} \tilde{s}^{\mathrm{E},\nu,i}_{\mathrm{locH}} = 0$.

We consider a sequence of quantizations

$$\widetilde{X}^{(p,q)}_N = \widetilde{X}^{(p,q)}_{m_N,m_N,M_N},$$

$N \in \mathbb{N}$, of X where we choose the parameters m_N and M_N as follows. First, in order to control the size of $\widetilde{X}^{(p,q)}_N$ via (5.2.11) we choose

$$M_N = \max\left\{ \left\lfloor \left(\left(4^{d\cdot(1+\binom{r}{2})} \cdot m_N^{d\cdot(r+\binom{r}{2})} \right)^{-m_N} \cdot N \right)^{\ln m_N/(1+\ln m_N)} \right\rfloor, 1 \right\}. \tag{5.2.12}$$

Secondly, we require

(Lim1) $\lim\limits_{N\to\infty} \dfrac{m_N \cdot \ln m_N}{\ln N} = 0$

and

(Lim2) $\lim\limits_{N\to\infty} \dfrac{m_N^2}{\ln N} = \infty.$

Remark 5.2.3

a) A valid choice of a sequence in $\mathbb{N}$ which satisfies **(Lim1)** and **(Lim2)** is $(m_N)_{N\in\mathbb{N}}$ with $m_1 = 1$ and

$$m_N = \left\lceil (\ln N)^{2/3} \right\rceil, \qquad N \geqslant 2.$$

b) In view of (5.1.6) there exists a constant $c \in (0,\infty)$ such that, for all sufficiently large $N \in \mathbb{N}$,

$$\ln\left(\left|\operatorname{ran}\left(\widetilde{X}^{(q)}_{m_N,m_N}\right)\right|\right) \leqslant c \cdot m_N \cdot \ln(m_N).$$

Hence due to **(Lim1)**, in a logarithmic scale, the size of the coarse-level quantization is asymptotically negligible compared to the size of the overall quantization of X.

c) Note that **(Lim2)** yields

$$\lim_{N\to\infty} \frac{\sqrt{\ln N}}{m_N} = 0.$$

Thus, in view of Proposition 5.1.1 and Theorem 5.2.4 below, the error of the coarse-level quantization tends faster to 0 when N tends to infinity than the error of the overall quantization and is therefore asymptotically negligible. Consequently, the fine-level quantization is a decisive ingredient for the overall performance of our quantization of X.

d) By the construction of the sequence $(\widetilde{X}^{(p,q)}_N)_{N\in\mathbb{N}}$ and due to (5.2.13) in the upcoming Theorem 5.2.4 we have

$$\widetilde{X}^{(p,q)}_N \in \mathbb{X}_{N,\mathrm{prod}}$$

for all sufficiently large $N \in \mathbb{N}$.

Next we state the main result of this section, whose proof is postponed to Section 6.2. We put

$$\overline{\kappa}_p := \limsup_{K\to\infty} \sqrt{\ln K} \cdot \delta_K$$

with $\delta_K := e^{(p)}\big(\overline{B}, S_K(\overline{B}), L_p[0,1]\big)$ where $\overline{B}$ denotes a Brownian bridge on $[0,1]$, and we put

$$C^{(p)} := \left(\mathbb{E}\left[\left(\sum_{k=1}^{d} \left\| \left\| b_k\big(X(\cdot)\big) \right\|_2 \right\|^{2p/(p+2)}_{L_{2p/(p+2)}[0,1]} \right)^{(p+2)/2} \right] \right)^{1/p}.$$

Theorem 5.2.4
Let $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant p\}$. Then

$$\left|\operatorname{ran}\left(\widetilde{X}^{(p,q)}_N\right)\right| \leqslant N \tag{5.2.13}$$

for all sufficiently large $N \in \mathbb{N}$, and

$$\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot e^{(p)}\big(X, \widetilde{X}^{(p,q)}_N, L^d_p[0,1]\big) \right) \leqslant \overline{\kappa}_p \cdot C^{(p)}.$$

Due to the previous theorem the asymptotic behavior of the quantization error of order p corresponding to the sequence $(\widetilde{X}_N^{(p,q)})_{N\in\mathbb{N}}$ depends on the SDE via the constant $C^{(p)}$ and on the sequence $(S_K)_{K\in\mathbb{N}}$ of K-quantizers for a Brownian bridge on $[0,1]$ employed in our construction via $\overline{\kappa}_p$.

Due to Theorem 2.2.3 we may assume that $(S_K)_{K\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order p of K-quantizers for a Brownian bridge on $[0,1]$. Therefore, the preceding theorem together with Theorem 2.3.11 leads to the following result.

Corollary 5.2.5
Let $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant p\}$, and assume that $(S_K)_{K\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order p of K-quantizers $S_K : L_p[0,1] \to L_p[0,1]$ for a Brownian bridge on $[0,1]$. Then,

$$\limsup_{N\to\infty} \left(\sqrt{\ln N}\cdot e^{(p)}\big(X, \widetilde{X}_N^{(p,q)}, L_p^d[0,1]\big)\right) \leqslant \kappa_p \cdot C^{(p)}$$

where κ_p is the constant in Theorem 2.3.11 (i).

We close this subsection with the following theorem, which is derived by applying Theorem 4.1.2 (i) as well as Theorem 5.2.4 and Corollary 5.2.5.

Theorem 5.2.6
It holds

$$\lim_{N\to\infty} \sqrt{\ln N}\cdot e_{N,\mathrm{prod}}^{(p)}\big(X, L_p^d[0,1]\big) = \kappa_p \cdot C^{(p)}$$

where κ_p is the constant in Theorem 2.3.11 (i).

Proof:
In the whole proof c denotes a not further specified positive real constant which may vary from line to line and which does not depend on N.

We prove the upper bound first. Due to Theorem 5.2.4 and Corollary 5.2.5 there exists a sequence $(\widetilde{X}_N)_{N\in\mathbb{N}}$ of quantizations of X such that

$$\limsup_{N\to\infty} \sqrt{\ln N}\cdot e_{N,\mathrm{prod}}^{(p)}\big(X, L_p^d[0,1]\big) \leqslant \limsup_{N\to\infty} \sqrt{\ln N}\cdot e^{(p)}\big(X, \widetilde{X}_N, L_p^d[0,1]\big) \leqslant \kappa_p \cdot C^{(p)}. \tag{5.2.14}$$

Hence it remains to show

$$\liminf_{N\to\infty} \sqrt{\ln N}\cdot e_{N,\mathrm{prod}}^{(p)}\big(X, L_p^d[0,1]\big) \geqslant \kappa_p \cdot C^{(p)}.$$

We split up the proof into single steps.

Step 1: In this step we make a few preliminary observations, and we show that each component X^k of X is an Itô process as required in Section 4.1.

First, by the fact that b is of at most linear growth, and by (3.2.2) in Theorem 3.2.3, we have

$$C^{(p)} = \mathbb{E}\left[\left(\sum_{k=1}^{d} \Big\| \big\|b_k\big(X(\cdot)\big)\big\|_2 \Big\|_{L_{2p/(p+2)}[0,1]}^{2p/(p+2)}\right)^{(p+2)/2}\right] \leqslant c\cdot\left(1+\mathbb{E}\left[\|X\|_\infty^2\right]\right) < \infty. \tag{5.2.15}$$

Secondly, we show that for all $k = 1,\dots,d$ the processes $\big(a_k(X(t))\big)_{t\in[0,1]}$ and $\big(b_{k,j}(X(t))\big)_{t\in[0,1]}$,

$j = 1, \dots, r$, satisfy the respective assumptions in **(A2)**--**(A4)** in Section 4.1 since then we are in the position to apply Theorem 4.1.2 (i). Indeed, let $k \in \{1, \dots, d\}$. Since X is a strong solution of the SDE (3.1.1), X has continuous paths and is adapted to $\mathcal{F}$. In addition to that, due to assumption **(C1)**, the mapping a_k is continuous. Thus, the process $\big(a_k(X(t))\big)_{t\in[0,1]}$ is measurable and adapted to $\mathcal{F}$. Next, we show $\mathbb{E}\big[\|a_k(X(\cdot))\|^s_{L_2[0,1]}\big] < \infty$ for all $s \in [1, \infty)$. It is enough to consider the case $s \in [2, \infty)$. By Jensen's inequality, by the fact that a is of at most linear growth, and by (3.2.2) in Theorem 3.2.3, we have

$$\mathbb{E}\left[\left(\int_0^1 \big|a_k(X(t))\big|^2 \, dt\right)^{s/2}\right] \leqslant \mathbb{E}\left[\int_0^1 \big|a_k(X(t))\big|^s \, dt\right] \leqslant c \cdot (1 + \mathbb{E}\left[\|X\|^s_\infty\right]) < \infty.$$

Hence $\big(a_k(X(t))\big)_{t\in[0,1]}$ has the properties required in **(A2)**.

Next, we verify **(A3)**. Let $j \in \{1, \dots, r\}$. As above one argues that the real-valued process $\big(b_{k,j}(X(t))\big)_{t\in[0,1]}$ is adapted to $\mathcal{F}$. Additionally, since $X(0) = x_0 \in \mathbb{R}$, it holds $b_{k,j}(X(0)) \in \mathbb{R}$. Furthermore, it is well known that the assumptions **(C1)**, **(C2)** and the fact that $x_0 \in \mathbb{R}^d$ imply that almost all paths of X are γ-Hölder continuous for all $\gamma \in (0, 1/2)$. W.l.o.g. we may even assume that all paths of X have this property. Since in addition $b_{k,j}$ is Lipschitz continuous, we obtain that the paths of $\big(b_{k,j}(X(t))\big)_{t\in[0,1]}$ are γ-Hölder continuous for all $\gamma \in (0, 1/2)$. Hence $\big(b_{k,j}(X(t))\big)_{t\in[0,1]}$ has the properties required in **(A3)**.

It remains to verify **(A4)**. Let $s \in [1, \infty)$ and $\alpha \in (0, 1/2)$. It holds

$$\begin{aligned}\big|\, \|b_k(X(\cdot))\|_2^2 \,\big|_\alpha &= \sup_{0\leqslant s<t\leqslant 1} \frac{\big|\, \|b_k(X(t))\|_2 - \|b_k(X(s))\|_2 \,\big| \cdot \big|\, \|b_k(X(t))\|_2 + \|b_k(X(s))\|_2 \,\big|}{|t-s|^\alpha} \\ &\leqslant 2 \cdot \big\|\, \|b_k(X(\cdot))\|_2 \,\big\|_\infty \cdot \big|\, \|b_k(X(\cdot))\|_2 \,\big|_\alpha,\end{aligned}$$

and hence Hölder's inequality implies

$$\mathbb{E}\left[\big|\, \|b_k(X(\cdot))\|_2^2 \,\big|^s_\alpha\right] \leqslant 2^s \cdot \left(\mathbb{E}\left[\big\|\, \|b_k(X(\cdot))\|_2 \,\big\|^{2s}_\infty\right]\right)^{1/2} \cdot \left(\mathbb{E}\left[\big|\, \|b_k(X(\cdot))\|_2 \,\big|^{2s}_\alpha\right]\right)^{1/2}.$$

Utilizing the properties of b as well as Theorem 3.2.3 leads to

$$\left(\mathbb{E}\left[\big\|\, \|b_k(X(\cdot))\|_2 \,\big\|^{2s}_\infty\right]\right)^{1/2} < \infty,$$

and therefore it remains to prove

$$\mathbb{E}\left[\big|\, \|b_k(X(\cdot))\|_2 \,\big|^{2s}_\alpha\right] < \infty.$$

Since b is Lipschitz continuous, it holds

$$\big|\, \|b_k(X(t))\|_2 - \|b_k(X(s))\|_2 \,\big| \leqslant \|b_k(X(t)) - b_k(X(s))\|_2 \leqslant c \cdot \sum_{k=1}^d |X^k(t) - X^k(s)|$$

for all $s, t \in [0, 1]$. Hence

$$\big|\, \|b_k(X(\cdot))\|_2 \,\big|^{2s}_\alpha = \left(\sup_{0\leqslant s<t\leqslant 1} \frac{\big|\, \|b_k(X(t))\|_2 - \|b_k(X(s))\|_2 \,\big|}{|t-s|^\alpha}\right)^{2s} \leqslant c \cdot \sum_{k=1}^d \big|X^k\big|^{2s}_\alpha.$$

Now by Lemma C.6 we arrive at

$$\mathbb{E}\Big[\big|\,\|b_k(X(\cdot))\|_2\,\big|_\alpha^{2s}\Big] \leqslant c\cdot\sum_{k=1}^d \mathbb{E}\Big[\big|X^k\big|_\alpha^{2s}\Big] < \infty,$$

which finishes the proof of **(A4)**.

Step 2: In this step we provide further auxiliary statements. For $N\in\mathbb{N}$ let $\widetilde{X}_N = (\widetilde{X}_N^1,\dots,\widetilde{X}_N^d)'$ be an arbitrary N-product-quantization of X. Then there exist $t_1^{(N)},\dots,t_d^{(N)}\in[0,1]$ such that

$$\sum_{k=1}^d t_k^{(N)} \leqslant 1 \quad\text{and}\quad \big|\operatorname{ran}\big(\widetilde{X}_N^k\big)\big| \leqslant \big\lfloor N^{t_k^{(N)}}\big\rfloor,\ \text{ for } k=1,\dots,d, \tag{5.2.16}$$

for all $N\in\mathbb{N}$. In particular,

$$\big|\operatorname{ran}\big(\widetilde{X}_N^k\big)\big| \leqslant \big\lfloor N^{t_k^{(N)}}\big\rfloor\cdot\big\lfloor\big(\ln N\big)^{2/3}\big\rfloor \tag{5.2.17}$$

for all $k=1,\dots,d$ and $N\in\mathbb{N}_3$. For ease of notation, we abbreviate $m_N^k := \big\lfloor N^{t_k^{(N)}}\big\rfloor\cdot\big\lfloor\big(\ln N\big)^{2/3}\big\rfloor$ for $N\in\mathbb{N}$ and $k=1,\dots,d$. Note that for all $N\in\mathbb{N}$ and $k=1,\dots,d$ we have $m_N^k \geqslant \big\lfloor\big(\ln N\big)^{2/3}\big\rfloor$, and hence

$$\lim_{N\to\infty} m_N^k = \infty \tag{5.2.18}$$

for all $k=1,\dots,d$.

Furthermore, Proposition 2.1.8 a) together with (5.2.17) implies

$$\begin{aligned}
\sqrt{\ln N}\cdot e^{(p)}\big(X,\widetilde{X}_N,L_p^d[0,1]\big) &= \sqrt{\ln N}\cdot\left(\sum_{k=1}^d\Big(e^{(p)}\big(X^k,\widetilde{X}_N^k,L_p[0,1]\big)\Big)^p\right)^{1/p}\\
&= \left(\sum_{k=1}^d\Big(\frac{\ln N}{\ln m_N^k}\Big)^{p/2}\Big(\sqrt{\ln m_N^k}\cdot e^{(p)}\big(X^k,\widetilde{X}_N^k,L_p[0,1]\big)\Big)^p\right)^{1/p}\\
&\geqslant \left(\sum_{k=1}^d\Big(\frac{\ln N}{\ln m_N^k}\Big)^{p/2}\cdot\Big(\sqrt{\ln m_N^k}\cdot e_{m_N^k}^{(p)}\big(X^k,L_p[0,1]\big)\Big)^p\right)^{1/p}
\end{aligned}$$

for all $N\in\mathbb{N}$ such that for all $k=1,\dots,d$ it holds $m_N^k\geqslant 2$. Hence,

$$\sqrt{\ln N}\cdot e_N^{(p)}\big(X,L_p^d[0,1]\big) \geqslant \left(\sum_{k=1}^d\Big(\frac{\ln N}{\ln m_N^k}\Big)^{p/2}\cdot\Big(\sqrt{\ln m_N^k}\cdot e_{m_N^k}^{(p)}\big(X^k,L_p[0,1]\big)\Big)^p\right)^{1/p} \tag{5.2.19}$$

for all $N\in\mathbb{N}$ such that for all $k=1,\dots,d$ it holds $m_N^k\geqslant 2$.

Now by (5.2.18) and Theorem 4.1.2 (i) we have

$$\lim_{N\to\infty}\sqrt{\ln m_N^k}\cdot e_{m_N^k}^{(p)}\big(X^k,L_p[0,1]\big) = \kappa_p\cdot\Big(\mathbb{E}\Big[\big\|\,\|b_k\big(X(\cdot)\big)\|_2\,\big\|_{L_{2p/(p+2)}[0,1]}^p\Big]\Big)^{1/p} \tag{5.2.20}$$

for all $k=1,\dots,d$.

Additionally, observe that, for all $k = 1, \dots, d$, $\big(b_k(X(t))\big)_{t\in[0,1]}$ is a continuous process which we assumed is not indistinguishable from the constant zero process, see assumption (ii) at the beginning of this chapter. Therefore,

$$\Big(\mathbb{E}\Big[\big\|\, \|b_k\big(X(\cdot)\big)\|_2 \,\big\|^p_{L_{2p/(p+2)}[0,1]}\Big]\Big)^{1/p} > 0 \tag{5.2.21}$$

for all $k = 1, \dots, d$.

Step 3: We put

$$L := \liminf_{N\to\infty} \sqrt{\ln N} \cdot e^{(p)}_{N,\mathrm{prod}}\big(X, L^d_p[0,1]\big),$$

and we abbreviate

$$\alpha_N := \sqrt{\ln N} \cdot e^{(p)}_{N,\mathrm{prod}}\big(X, L^d_p[0,1]\big)$$

for $N \in \mathbb{N}$. It is a classical result that there exists a subsequence $(\alpha_{N_l})_{l\in\mathbb{N}}$ of $(\alpha_N)_{N\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} \alpha_{N_l} = L. \tag{5.2.22}$$

Next, we show that there exists a subsequence $(N_{l_m})_{m\in\mathbb{N}}$ of $(N_l)_{l\in\mathbb{N}}$ such that the sequence $(\beta_{N_{l_m}})_{m\in\mathbb{N}}$ given by

$$\beta_{N_{l_m}} := \sum_{k=1}^{d} t_k^{(N_{l_m})} \tag{5.2.23}$$

for $m \in \mathbb{N}$ is convergent. Indeed, first consider the sequence $(t_1^{(N_l)})_{l\in\mathbb{N}}$. By Definition 4.0.1 this sequence is bounded with $t_1^{(N_l)} \in [0,1]$ for all $l \in \mathbb{N}$. Hence there exists a convergent subsequence $(t_1^{(N_{l'})})_{l'\in\mathbb{N}}$ of $(t_1^{(N_l)})_{l\in\mathbb{N}}$ with $v_1 := \lim_{l'\to\infty} t_1^{(N_{l'})} \in [0,1]$. Secondly, we consider the sequence $(t_2^{(N_{l'})})_{l'\in\mathbb{N}}$. Again, this sequence is bounded with $t_2^{(N_{l'})} \in [0,1]$ for all $l' \in \mathbb{N}$. Hence there exists a convergent subsequence $(t_2^{(N_{l''})})_{l''\in\mathbb{N}}$ of $(t_2^{(N_{l'})})_{l'\in\mathbb{N}}$ with $v_2 := \lim_{l''\to\infty} t_2^{(N_{l''})} \in [0,1]$. By repeating these arguments another $(d-2)$-times we obtain a subsequence $(N_{l_m})_{m\in\mathbb{N}}$ of $(N_l)_{l\in\mathbb{N}}$ such that $(t_k^{(N_{l_m})})_{m\in\mathbb{N}}$ converges for every $k = 1, \dots, d$. Thus the sequence $(\beta_{N_{l_m}})_{m\in\mathbb{N}}$ where $\beta_{N_{l_m}}$ is given as in (5.2.23) is convergent with limit $\sum_{k=1}^d v_k$. Furthermore, due to (5.2.16) we have $\beta_{N_{l_m}} \in [0,1]$ for all $m \in \mathbb{N}$ and therefore

$$\sum_{k=1}^{d} v_k \leqslant 1. \tag{5.2.24}$$

As a next step, we show that actually $v_k \in (0,1]$ for all $k = 1, \dots, d$. Indeed, assume that there exists a $k \in \{1, \dots, d\}$ such that $v_k = 0$. Then,

$$\lim_{m\to\infty} \frac{t_k^{(N_{l_m})} \cdot \ln N_{l_m} + \big(\ln N_{l_m}\big)^{2/3}}{\ln N_{l_m}} = 0.$$

This implies

$$\lim_{m\to\infty} \frac{\ln m_{N_{l_m}}^k}{\ln N_{l_m}} = 0,$$

and hence together with (5.2.19)--(5.2.22) we obtain

$$L = \liminf_{m\to\infty} \alpha_{N_{l_m}} \geqslant \kappa_p \cdot \left(\sum_{k=1}^{d} \left(\liminf_{m\to\infty} \frac{\ln N_{l_m}}{\ln m_{N_{l_m}}^k} \right)^{p/2} \cdot \mathbb{E}\left[\left\| \, \|b_k\big(X(\cdot)\big)\|_2 \, \right\|_{L_{2p/(p+2)}[0,1]}^p \right] \right)^{1/p} = \infty,$$

which contradicts (5.2.14) combined with (5.2.15).

Step 4: Due to the results obtained in Step 3, and due to the definition of the sequences $(m_{N_{l_m}}^k)_{m\in\mathbb{N}}$ it holds

$$\left(\liminf_{m\to\infty} \frac{\ln N_{l_m}}{\ln m_{N_{l_m}}^k} \right)^{p/2} \geqslant v_k^{-p/2}$$

for all $k = 1, \dots, d$. Hence, together with (5.2.19), (5.2.20) and (5.2.22) we derive

$$L \geqslant \kappa_p \cdot \left(\sum_{k=1}^{d} \mathbb{E}\left[v_k^{-p/2} \cdot \left\| \, \|b_k\big(X(\cdot)\big)\|_2 \, \right\|_{L_{2p/(p+2)}[0,1]}^p \right] \right)^{1/p}. \tag{5.2.25}$$

For ease of notation, we abbreviate

$$c_k := \left\| \, \|b_k\big(X(\cdot)\big)\|_2 \, \right\|_{L_{2p/(p+2)}[0,1]}$$

for $k = 1, \dots, d$. By the reverse Hölder inequality applied with the exponents $\lambda_1 = -2/p$ and $\lambda_2 = 2/(p+2)$, and by (5.2.24), we have

$$\sum_{k=1}^{d} v_k^{-p/2} \cdot c_k^p \geqslant \left(\sum_{k=1}^{d} c_k^{2p/(p+2)} \right)^{(p+2)/2} \cdot \left(\sum_{k=1}^{d} v_k \right)^{-p/2} \geqslant \left(\sum_{k=1}^{d} c_k^{2p/(p+2)} \right)^{(p+2)/2}.$$

Therefore, in view of (5.2.25), we finally end up with

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e_{N,\mathrm{prod}}^{(p)}\big(X, L_p^d[0,1]\big) \geqslant \kappa_p \cdot \left(\mathbb{E}\left[\left(\sum_{k=1}^{d} \left\| \, \|b_k\big(X(\cdot)\big)\|_2 \, \right\|_{L_{2p/(p+2)}[0,1]}^{2p/(p+2)} \right)^{(p+2)/2} \right] \right)^{1/p},$$

which finishes the proof.

□

Remark 5.2.7
Similar to the proof of Theorem 5.2.6 one shows

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e_{N,\mathrm{prod,uni}}^{(p)}\big(X, L_p^d[0,1]\big) \geqslant \kappa_p \cdot \sqrt{d} \cdot \left(\mathbb{E}\left[\sum_{k=1}^{d} \left\| \, \|b_k\big(X(\cdot)\big)\|_2 \, \right\|_{L_{2p/(p+2)}[0,1]}^p \right] \right)^{1/p}.$$

Now, let $p \in [1,\infty)$, and assume $d \in \mathbb{N}_2$. We compare this lower bound with the one obtained in Theorem 5.2.6, namely

$$\kappa_p \cdot \left(\mathbb{E}\left[\left(\sum_{k=1}^{d} \left\| \left\| b_k\big(X(\cdot)\big) \right\|_2 \right\|_{L_{2p/(p+2)}[0,1]}^{2p/(p+2)} \right)^{(p+2)/2} \right] \right)^{1/p}.$$

To this end, we abbreviate

$$c_k := \left\| \left\| b_k\big(X(\cdot)\big) \right\|_2 \right\|_{L_{2p/(p+2)}[0,1]}$$

for $k = 1, \dots, d$, and, for the moment, we fix $\omega \in \Omega$. By the reverse Hölder inequality we have

$$\sum_{k=1}^{d} v_k^{-p/2} \cdot \big(c_k(\omega)\big)^p \geqslant \left(\sum_{k=1}^{d} \big(c_k(\omega)\big)^{2p/(p+2)} \right)^{(p+2)/2}$$

for all $(v_1, \dots, v_d) \in (0,1]^d$ with $\sum_{k=1}^{d} v_k \leqslant 1$. In particular,

$$d^{p/2} \cdot \sum_{k=1}^{d} \big(c_k(\omega)\big)^p \geqslant \left(\sum_{k=1}^{d} \big(c_k(\omega)\big)^{2p/(p+2)} \right)^{(p+2)/2}.$$

Thus,

$$d^{p/2} \cdot \sum_{k=1}^{d} c_k^p \geqslant \left(\sum_{k=1}^{d} c_k^{2p/(p+2)} \right)^{(p+2)/2},$$

and hence

$$\kappa_p \cdot \sqrt{d} \cdot \left(\mathbb{E}\left[\sum_{k=1}^{d} \left\| \left\| b_k\big(X(\cdot)\big) \right\|_2 \right\|_{L_{2p/(p+2)}[0,1]}^{p} \right] \right)^{1/p} \geqslant \kappa_p \cdot \left(\mathbb{E}\left[\left(\sum_{k=1}^{d} \left\| \left\| b_k\big(X(\cdot)\big) \right\|_2 \right\|_{L_{2p/(p+2)}[0,1]}^{2p/(p+2)} \right)^{(p+2)/2} \right] \right)^{1/p}.$$

Therefore, as one would expect, approximating X by quantizations of the larger classes $\mathbb{X}_{N,\mathrm{prod}}$, $N \in \mathbb{N}$, instead of only taking quantizations from the classes $\mathbb{X}_{N,\mathrm{prod,uni}}$, $N \in \mathbb{N}$, leads to a smaller lower bound.

5.2.3. Discussion

Strong Asymptotically Optimality of the Sequence $(\widetilde{X}_N^{(p,q)})_{N\in\mathbb{N}}$

Let $q \in [1,\infty)$. The following proposition states that the sequence $(\widetilde{X}_N^{(p,q)})_{N\in\mathbb{N}}$ of quantizations of X constructed in the previous section is strongly asymptotically optimal of order p in the classes $\mathbb{X}_{\mathrm{N,prod}}$, $N \in \mathbb{N}$, provided that the sequence $(S_K)_{K\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order p of K-quantizers for a Brownian bridge on $[0,1]$.

Proposition 5.2.8
Let $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant p\}$, and assume that $(S_K)_{K\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of

order p of K-quantizers $S_K : L_p[0,1] \to L_p[0,1]$ for a Brownian bridge on $[0,1]$. Then,

$$\limsup_{N\to\infty} \frac{e^{(p)}\big(X, \widetilde{X}_N^{(p,q)}, L_p^d[0,1]\big)}{e_{N,\mathrm{prod}}^{(p)}\big(X, L_p^d[0,1]\big)} \leqslant 1,$$

and in particular, in case $d = 1$, it holds

$$\limsup_{N\to\infty} \frac{e^{(p)}\big(X, \widetilde{X}_N^{(p,q)}, L_p[0,1]\big)}{e_{N}^{(p)}\big(X, L_p[0,1]\big)} \leqslant 1.$$

Proof:
The first statement is a direct consequence of assumption (ii) at the beginning of this chapter, and of Corollary 5.2.5 as well as Theorem 5.2.6. The second statement is a direct consequence of assumption (ii), and of Proposition 4.2.1 as well as Corollary 5.2.5. □

Computational Cost

Let $q \in [1,\infty)$, $\gamma \in (2,\infty)$, $m \in \mathbb{N}_2$, as well as $n, M \in \mathbb{N}$, and recall that $\widetilde{X}_{m,n,M}^{(p,q)}$ is the quantization of X as constructed in Section 5.2.2.

In this subsection we analyze the computational cost to determine the paths and probability weights of $\widetilde{X}_{m,n,M}^{(p,q)}$ in the space $L_p^d[0,1]$ along with their corresponding probability weights. The analysis is carried out similarly to the analysis of the computational cost in [MGR13, Section 2.5], and it leads to a result analogous to the one derived in the aforementioned paper. We assume that the following quantities, which do not depend on the parameters of the SDE, are available via precomputational steps. Hence they do not have to be taken into account in the analysis of the computational cost.

- We assume that we know the support points $z_i = z_{n,i}$ and their corresponding probability weights $\zeta_i = \zeta_{n,i}$, $i = 1,\dots,n$, of the distribution

  $$P_{\widetilde{Z}_n^{(q)}} = \sum_{i=1}^{n} \zeta_i \cdot \delta_{z_i}$$

  of a sequence of n-quantization $\widetilde{Z}_n^{(q)}$ of a standard normally distributed random variable, which has the properties **(N1)--(N3)**. See Section 2.3.1 for a simple construction.

- Let I be a real-valued random variable with $I \overset{\mathcal{L}}{=} \int_0^1 W_1(s)\,dW_2(s)$. We put $\widetilde{I}_n^{(\gamma)} := T_n^{(\gamma)}(I)$ where $T_n^{(\gamma)}$ is the mapping defined in (2.3.8), and we put $\mathcal{N}(n,\gamma) := |\operatorname{ran}(\widetilde{I}_n^{(\gamma)})|$. By the construction of $T_n^{(\gamma)}$, see Section 2.3.1, we know the support points $\mathfrak{i}_i = \mathfrak{i}_{n,i}$, $i = 1,\dots,\mathcal{N}(n,\gamma)$, of the quantization $\widetilde{Z}_n^{(q)}$. Now consider the set

  $$\Theta_{m,n} := \Big(\{z_1,\dots,z_n\}^r \times \big\{\mathfrak{i}_1,\dots,\mathfrak{i}_{\mathcal{N}(n,\gamma)}\big\}^{\binom{r}{2}}\Big)^m.$$

  This set contains vectors which are built-up by m blocks of equal length $r + \binom{r}{2}$ where each block consists of support points of quanizations of standard normally distributed random variables and of support points of quantizations of I. The vectors in $\Theta_{m,n}$ are used to obtain the Milstein quantization. For now, we assume that for every vector in the set $\Theta_{m,n}$ one knows the corresponding probability weight. We further comment on this assumption in the remark at the end of this subsection.

- We assume that for $K \in \mathbb{N}$ we know the distribution

$$P_{S_K(\overline{B})} = \sum_{i=1}^{K} \beta_{K,i} \cdot \delta_{\mathfrak{b}_{K,i}}$$

of the K-quantizations $S_K(\overline{B})$ of a Brownian bridge $\overline{B}$ on $[0,1]$ where, for $i = 1, \dots, K$, $\beta_{K,i}$ denotes the probability weight corresponding to the path $\mathfrak{b}_{K,i}$ of the quantization $S_K(\overline{B})$. The discussion in Section 2.3.2 justifies this assumption.

Given the coefficients a and b as well as the initial value x_0 of the SDE we have to compute the following constituents of $\widetilde{X}^{(p,q),k}_{m,n,M}$, $k = 1, \dots, d$, where, for now, we assume that all probability weights occurring in the analysis below can be computed. We comment on this assumption in further detail at the end of this subsection.

For $I = 1, \dots, \left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m$ we choose a vector $\theta^{(I)}_{m,n} \in \Theta_{m,n}$, and we compute the following:

(I.) For every $k = 1, \dots, d$ we first compute

(i) the knot $x^{k,(I)} = (x_0^{k,(I)}, \dots, x_m^{k,(I)})$ of the coarse-level quantization $(\widetilde{X}^{(q),k}_{m,n}(t_0), \dots, \widetilde{X}^{(q),k}_{m,n}(t_m))$ corresponding to $\theta^{(I)}_{m,n}$, and

(ii) the knot $x^{\mathrm{E},k,(I)} = (x_0^{\mathrm{E},k,(I)}, \dots, x_{m-1}^{\mathrm{E},k,(I)})$ of the Euler quantization $(\widetilde{X}^{\mathrm{E}(q),k}_{m,n}(t_0), \dots, \widetilde{X}^{\mathrm{E},(q),k}_{m,n}(t_{m-1}))$ corresponding to $\theta^{(I)}_{m,n}$.

Note that for all $k = 1, \dots, d$ the probability weight of $x^{k,(I)}$ is given by the probability weight corresponding to $\theta^{(I)}_{m,n}$ which we denote by $\xi^{(I)}$.

(II.) Secondly, for all $k = 1, \dots, d$ compute the values $(b_{k,j}(x_0^{\mathrm{E},(I)}), \dots, b_{k,j}(x_{m-1}^{\mathrm{E},(I)}))$, $j = 1, \dots, d$, along with the associated values $s_{l-1}^{\mathrm{E},k,(I)}$ of the random variables $\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH}}$ as defined in (5.2.7) for $l = 1, \dots, m$.

(III.) Now fix $k \in \{1, \dots, d\}$. Thirdly, we compute

(i) the size

$$K^{(I)}_{k,l} = K^{(I)}_{k,l,M} = \left\lceil M^{\eta_l^{k,(I)}} \right\rceil$$

of the fine-level quantization on the subinterval $[t_{l-1}, t_l]$ for $l = 1, \dots, m$ where

$$\eta_l^{k,(I)} = \begin{cases} \max\left\{ \dfrac{\left(s_{l-1}^{\mathrm{E},k,(I)}\right)^{2p/(p+2)}}{\sum_{\nu=1}^{d}\sum_{i=0}^{m-1}\left(s_i^{\mathrm{E},\nu,(I)}\right)^{2p/(p+2)}}, \dfrac{1}{d \cdot m \cdot \ln m} \right\}, & \text{if } \max\limits_{\substack{i=0,\dots,m-1,\\ \nu=1,\dots,d}} s_i^{\mathrm{E},\nu,(I)} > 0, \\ \dfrac{1}{d \cdot m}, & \text{else,} \end{cases}$$

and

(ii) for $J^k = (J_1^k, \dots, J_m^k) \in K_k^{(I)} := \{1, \dots, K^{(I)}_{k,1}\} \times \cdots \times \{1, \dots, K^{(I)}_{k,m}\}$ we compute the probability weights

$$\rho^{(I,J^k)} = \prod_{l=1}^{m} \beta_{K^{(I)}_{k,l}, J_l^k}.$$

Note that J^k is a vector of length m which consists of the numberings corresponding to those paths of the fine-level quantization on the respective lth subinterval $[t_{l-1}, t_l]$, $l = 1, \dots, m$, which are concatinated such that one obtains a function defined on the whole interval $[0,1]$.

Let $k \in \{1, \dots, d\}$. For the distribution of $\widetilde{X}^{(p,q),k}_{m,n,M}$ we have

$$P_{\widetilde{X}^{(p,q),k}_{m,n,M}} = \sum_{I=1}^{\left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m} \sum_{J^k \in K_k^{(I)}} \gamma^{(I,J^k)} \cdot \delta_{\mathcal{Y}^{(I,J^k)}} \tag{5.2.26}$$

where, in view of (5.2.2), (5.2.5) and (5.2.10),

$$\mathcal{Y}^{(I,J^k)}(t) = \begin{cases} x_{l-1}^{k,(I)} \cdot (t_l - t) \cdot m + x_l^{k,(I)} \cdot (t - t_{l-1}) \cdot m + \psi^{-1}_{s_{l-1}^{\mathrm{E},k,(I)},l}\big(\mathfrak{b}_{K^{(I)}_{k,l},J^k_l}\big)(t), & \text{if } s_{l-1}^{\mathrm{E},k,(I)} \neq 0, \\ x_{l-1}^{k,(I)} \cdot (t_l - t) \cdot m + x_l^{k,(I)} \cdot (t - t_{l-1}) \cdot m, & \text{if } s_{l-1}^{\mathrm{E},k,(I)} = 0, \end{cases} \tag{5.2.27}$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, d$, and where $\gamma^{(I,J^k)}$ denotes the probability weight corresponding to the path $\mathcal{Y}^{(I,J^k)}$. We further comment on the weights $\gamma^{(I,J^k)}$ in the remark at the end of this subsection. For the distribution of $\widetilde{X}^{(p,q)}_{m,n,M}$ we have

$$P_{\widetilde{X}^{(p,q)}_{m,n,M}} = \sum_{I=1}^{\left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m} \sum_{J \in K^{(I)}} \gamma^{(I,J)} \cdot \delta_{\mathcal{Y}^{(I,J)}}$$

where $K^{(I)} = K_1^{(I)} \times \cdots \times K_d^{(I)}$, where $\mathcal{Y}^{(I,J)} = (\mathcal{Y}^{(I,J^1)}, \dots, \mathcal{Y}^{(I,J^d)})'$ for $J = (J^1, \dots, J^d)' \in K^{(I)}$, and where $\gamma^{(I,J)}$ denotes the probability weight corresponding to the path $\mathcal{Y}^{(I,J)}$. We further comment on the weights $\gamma^{(I,J)}$ in the remark at the end of this subsection.

We use the real number model to analyze the computational cost, i.e., we assume that all evaluations of the coefficients a, b and their partial derivatives as well as all arithmetic operations are carried out at cost one. Regarding an introduction to the real number model we refer the reader to, for example, [Nov95].

The total computational cost, denoted by $\mathrm{cost}(m,n,M)$, to construct the paths $\mathcal{Y}^{(I,J)}$ in $L_p^d[0,1]$ and their corresponding probability weights $\gamma^{(I,J)}$ is then given by the total number of all required evaluations of a, b and their partial derivatives as well as by all required arithmetic operations to carry out the algorithm. By

$$\big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big| \leqslant \sum_{I=1}^{\left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m} |K^{(I)}|$$

we have

$$d \cdot m \cdot \big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big| \leqslant \mathrm{cost}(m,n,M) \leqslant c \cdot d \cdot m \cdot \big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big|, \tag{5.2.28}$$

where the constant c is the number of all required function evaluations and arithmetic operations to carry out the algorithm. Note that c does not depend on the parameters of the SDE. By (5.2.28) we obtain

$$\mathrm{cost}(m,n,M) \leqslant c \cdot d \cdot \big(\ln\big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big|\big) \cdot \big|\operatorname{ran}\big(\widetilde{X}^{(p,q)}_{m,n,M}\big)\big|$$

for all $n \geqslant 2$.

In particular, by Theorem 5.2.4 we have

$$\operatorname{cost}(m_N, m_N, M_N) \leqslant c \cdot d \cdot \left(\ln\left|\operatorname{ran}\left(\widetilde{X}_N^{(p,q)}\right)\right|\right) \cdot \left|\operatorname{ran}\left(\widetilde{X}_N^{(p,q)}\right)\right| \leqslant c \cdot d \cdot \ln N \cdot N$$

for all sufficiently large $N \in \mathbb{N}$. Hence the computational cost to construct the paths and corresponding probability weights of $\widetilde{X}_N^{(p,q)}$ is close to the size of the quantization.

In the following remark we further comment on the probability weights $\xi^{(I)}$ of the coarse-level quantization and on the probability weights $\gamma^{(I,J)}$ of the overall quantization.

Remark 5.2.9

a) Let $k \in \{1, \dots, d\}$. For $I = 1, \dots, \left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m$ let $\theta_{m,n}^{(I)} \in \Theta_{m,n}$, and let $x^{k,(I)}$ be the knot of the kth component of the Milstein quantization corresponding to $\theta_{m,n}^{(I)}$. For us, in general, it remains an open problem to determine, or at least approximate, the probability weight $\xi^{(I)}$ of $x^{k,(I)}$. More precisely, it holds $\theta_{m,n}^{(I)} = (\theta_{m,n}^{(I),1}, \dots, \theta_{m,n}^{(I),m})$ where $\theta_{m,n}^{(I),l} \in \{z_1, \dots, z_n\}^r \times \left\{\mathfrak{i}_1, \dots, \mathfrak{i}_{\mathcal{N}(n,\gamma)}\right\}^{\binom{r}{2}}$ for $l = 1, \dots, m$, i.e., $\theta_{m,n}^{(I)}$ consists of m blocks, where the lth block $\theta_{m,n}^{(I),l}$ contains support points of the quantizations $\widetilde{Y}_{l,n}^{(q),j}$, $j = 1, \dots, r$, and $\widetilde{I}_{(j_1,j_2)}^{(l,n)}$, $j_1, j_2 = 1, \dots, r$ with $j_1 < j_2$. For $l = 1, \dots, m$ we use $\xi^{(I),l}$ to denote the probability weight corresponding to $\theta_{m,n}^{(I),l}$. Clearly,

$$\xi^{(I)} = \prod_{l=1}^{m} \xi^{(I),l}.$$

In general it is hard to determine the weights $\xi^{(I),l}$, $l = 1, \dots, m$, since, to us, the joint distribution of the random variables Y_1^j, $j = 1, \dots, r$, and $I_{(j_1,j_2)}^1$, $j_1, j_2 = 1, \dots, r$ with $j_1 < j_2$, is unknown.

In the very special case that b has the commutativity property, we know the probability weights $\xi^{(I)}$. In this case, in the lth step of the Milstein scheme, one quantizes only the independent standard normally distributed random variables $Y_l^1, \dots, Y_l^r$, see Remark 5.1.2. Hence if b has the commutativity property, $\xi^{(I),l}$ is the product of the respective probability weights corresponding to the quantizations of $Y_l^1, \dots, Y_l^r$.

b) Since the random vectors $\left(Y_1^1, \dots, Y_m^1, \dots, Y_1^r, \dots, Y_m^r, I_{(1,2)}^1, \dots, I_{(r-1,r)}^1, \dots, I_{(1,2)}^m, \dots, I_{(r-1,r)}^m\right)$ and $\left(B_1^1, \dots, B_1^r, \dots, B_m^1, \dots, B_m^r\right)$ are interdependent, the weights $\gamma^{(I,J^k)}$ in (5.2.26) can in general not be calculated by simply multiplying the corresponding weights $\xi^{(I)}$ and $\rho^{(I,J^k)}$.

But if b has the commutativity property, the weights $\gamma^{(I,J^k)}$ can be computed via exactly the same procedure. Indeed, in (3.3.21) we have shown that if b has the commutativity property, the Milstein scheme consists only of the increments of the driving Brownian motion W. Therefore, the Milstein quantization is built-up only by the quantizations $\widetilde{Y}_{l,n}^{(q),j}$, and hence Lemma 5.2.1 c) then guarantees that for $k = 1, \dots, d$ the distribution of $\widetilde{X}_{m,n}^{(p,q),k}$ is given by

$$P_{\widetilde{X}_{m,n,M}^{(p,q),k}} = \sum_{I=1}^{\left(n^r \cdot \mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m} \sum_{J^k \in K_k^{(I)}} \xi^{(I)} \cdot \rho^{(I,J^k)} \cdot \delta_{y^{(I,J^k)}}.$$

For general b determining, or at least approximating, the weights $\gamma^{(I,J^k)}$ still remains an open problem to us.

c) In general, the components of the solution X are interdependent. Hence, even if one knows $P_{\widetilde{X}^{(p,q),k}_{m,n,M}}$ for $k = 1,\dots,d$, it is hard to draw conclusions about $P_{\widetilde{X}^{(p,q)}_{m,n,M}}$. In the very special case that $X^1,\dots,X^d$ are independent the distribution of $\widetilde{X}^{(p,q)}_{m,n,M}$ is given by

$$P_{\widetilde{X}^{(p,q)}_{m,n,M}} = \sum_{I=1}^{\left(n^r\cdot\mathcal{N}(n,\gamma)^{\binom{r}{2}}\right)^m} \sum_{J\in K^{(I)}} \Big(\prod_{k=1}^{d}\gamma^{(I,J^k)}\Big)\cdot\delta_{\mathcal{Y}^{(I,J)}}.$$

The previous remark shows that our construction is of a semi-constructive type, and it is fully constructive in very special cases only.

Example

For the purpose of illustration we provide an example of a rather simple one-dimensional linear SDE driven by a two-dimensional Brownian motion (W_1, W_2), and we construct the paths and corresponding probability weights of a quantization in the path space $L_2[0,1]$. We consider the SDE

$$\begin{aligned} dX(t) &= dt + 2X dW_1(t) - X dW_2(t), \quad t\in[0,1],\\ X(0) &= 2. \end{aligned} \tag{5.2.29}$$

The drift coefficient $a:\mathbb{R}\to\mathbb{R}$ is given by $a(x)=1$, and the diffusion coefficient $b=(b_1,b_2):\mathbb{R}\to\mathbb{R}^{1\times 2}$ is given by $b(x)=(2\cdot x,-x)$. Note that both a and b meet the smoothness assumptions imposed on the coefficients of the SDE in Section 3.1.2.

Regarding the coarse-level quantization we choose the following parameters, where for reasons of clarity and comprehensibility we take small m and n:

$$q=2, \quad m=2 \quad \text{and} \quad n=2.$$

Note that for all $x\in\mathbb{R}$ it holds $b_1'(x)\cdot b_2(x) = -2x = b_1(x)\cdot b_2'(x)$. Thus, b has the commutativity property, and hence, in view of (3.3.21), the Milstein scheme associated to the SDE (5.2.29) is of the following form:

$$\begin{aligned} \widehat{X}_2^{\mathrm{M}}(t_0) &= 2,\\ \widehat{X}_2^{\mathrm{M}}(t_l) &= \widehat{X}_2^{\mathrm{M}}(t_{l-1}) + \frac{1}{2} + \frac{1}{\sqrt{2}}\cdot\widehat{X}_2^{\mathrm{M}}(t_{l-1})\cdot\big(2\cdot Y_l^1 - Y_l^2\big)\\ &\quad + \frac{1}{4}\cdot\widehat{X}_2^{\mathrm{M}}(t_{l-1})\cdot\Big(4\cdot\big((Y_l^1)^2-1\big) + \big((Y_l^2)^2-1\big)\Big) - \widehat{X}_2^{\mathrm{M}}(t_{l-1})\cdot Y_l^1\cdot Y_l^2 \end{aligned}$$

for $l=1,2$, and where Y_l^1 and Y_l^2 are the standard normally distributed random variables defined in (5.1.1).

At each step of the Milstein scheme, except for the starting point t_0, we employ 2-quantizations to each of the independent standard normally distributed random variables Y_l^1 and Y_l^2. This leads to a total number of 16 distinct paths for the coarse-level quantization $\widetilde{X}^{(2)}_{2,2}$, which are displayed in Figure 5.2.1a. For quantizing the random variables Y_l^j we use the method described in Section 2.3.1. Then each support point of the quantization of Y_l^j has a corresponding probability weight of $1/2$, and hence, by Remark 5.2.9 a), we conclude that the probability weight of each path of the coarse-level quantization is $1/16$.

The number of the local refinement of a coarse-level path is determined by its associated values

$\widetilde{X}^{\mathrm{E},(2)}_{2,2}(t_0)$, $\widetilde{X}^{\mathrm{E},(2)}_{2,2}(t_1)$ via the values of the quantities $\widetilde{s}^{\mathrm{E},l-1}_{\mathrm{locH}}$ and η_l for $l = 1, 2$. See Table 5.1 for the numerical values of these quantities in our example. Therein the values are listed bottom-up in the sense that the first row corresponds to the first path from below in Figure 5.2.1a and so on.

Block No.	$\widetilde{X}^{(2)}_{2,2}(1/2)$	$\widetilde{X}^{(2)}_{2,2}(1)$	$\widetilde{X}^{\mathrm{E},(2)}_{2,2}(1/2)$	$\widetilde{X}^{\mathrm{E},(2)}_{2,2}(1)$	$\widetilde{s}^{\mathrm{E},0}_{\mathrm{locH}}$	$\widetilde{s}^{\mathrm{E},1}_{\mathrm{locH}}$	η_1	η_2	$K_{1,8}$	$K_{2,8}$
1	-0.97	-4.64	0.85	3.45	4.47	1.90	0.72	0.72	5	5
	-0.97	-0.39	0.85	2.05	4.47	1.90	0.72	0.72	5	5
	-0.97	0.17	0.85	-0.75	4.47	1.90	0.72	0.72	5	5
	-0.97	1.21	0.85	0.65	4.47	1.90	0.72	0.72	5	5
2	1.19	-0.37	-2.46	0.07	4.47	5.49	0.72	0.72	5	5
	1.19	0.91	-2.46	4.13	4.47	5.49	0.72	0.72	5	5
	1.19	2.57	-2.46	-3.99	4.47	5.49	0.72	0.72	5	5
	1.19	7.76	-2.46	-8.04	4.47	5.49	0.72	0.72	5	5
3	2.33	-1.22	4.15	1.22	4.47	9.28	0.72	0.72	5	5
	2.33	1.30	4.15	-5.64	4.47	9.28	0.72	0.72	5	5
	2.33	2.64	4.15	8.08	4.47	9.28	0.72	0.72	5	5
	2.33	12.87	4.15	14.94	4.47	9.28	0.72	0.72	5	5
4	11.10	-7.66	7.46	1.80	4.47	16.67	0.72	0.79	5	6
	11.10	4.30	7.46	-10.52	4.47	16.67	0.72	0.79	5	6
	11.10	10.68	7.46	14.12	4.47	16.67	0.72	0.79	5	6
	11.10	59.31	7.46	26.44	4.47	16.67	0.72	0.79	5	6

Table 5.1.: Support points of the coarse-level quantization along with the local regularity in the coarse level quantization

For the fine-level quantization we chose $M = 8$, which then specifies the size $K_{1,8}$ and $K_{2,8}$ of the quantization of the Brownian bridges B^1_1 and B^1_2, respectively. See again Table 5.1 for the resulting values in our example. We quantize the Brownian bridges B^1_1 and B^1_2 by employing the method described at the end of Section 2.3.2, where we choose the truncation indices $d(K_{1,8})_{\mathrm{opti}}$ and $d(K_{2,8})_{\mathrm{opti}}$ as in the database available at the website

`http://quantize.maths-fi.com`.

We combine the paths of the coarse-level quantization and the fine-level quantization as in (5.2.27). The paths of the overall quantization $\widetilde{X}^{(2,2)}_{2,2,8}$ are displayed in Figure 5.2.1b. The respective numbers of paths resulting from block 1 up to block 4 in Table 5.1 are 100, 100, 100 and 120 summing up to a total number of 420 paths.

In our example we obtain a total number of 15 distinct probability weights ranging from a minimum value of 0.000493 to a maximum value of 0.005543. For each path of the overall quantization Remark 5.2.9 b) implies that its weight is the product of the weights of the corresponding coarse-level and fine-level paths, where the latter is determined by the corresponding weights of the quantizations of the Brownian bridges B^1_1 and B^1_2. For the first three blocks in Table 5.1 the distinct probability weights of the quantizations of B^1_1 and B^1_2 are 0.1067, 0.2444, 0.2978 in each case. Whereas in the fourth block the distinct probability weights of the quantizations of B^1_1 and B^1_2 are 0.1067, 0.2444, 0.2978 and 0.0740, 0.1810, 0.2450, respectively. To get a better insight on the shape of paths of the quantization

$\widetilde{X}^{(2,2)}_{2,2,8}$, we incorporated four supplementary figures in Figure 5.2.1 where each of them corresponds to one of the four blocks in Table 5.1, see Figures 5.2.1c to 5.2.1f. Additionally, in Figure 5.2.1c to Figure 5.2.1f those paths with probability weight 0.005543 are marked in blue, and the paths with probability weight 0.000493 are marked in light orange.

5.3. Quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$ w.r.t. the Supremum Norm

Throughout this section let $s \in [1,\infty)$, let $\varepsilon > 0$, and let

$$\eta_\varepsilon = \min\big\{n \in \mathbb{N} \,\big|\, d^{1/sn} \leqslant (1+\varepsilon)\big\}. \tag{5.3.1}$$

We aim at constructing a sequence of product-quanizations of the solution X of the SDE (3.1.1) in the space $\mathcal{C}([0,1];\mathbb{R}^d)$ equipped with the supremum norm $\|\cdot\|_\infty$. Recall that for $m,n \in \mathbb{N}$ and $q \in [1,\infty)$ the extension of the Milstein quantization to a $\mathcal{C}([0,1];\mathbb{R}^d)$-valued random element is given by

$$\widetilde{X}^{(q)}_{m,n}(t) = (t_l - t)\cdot m \cdot \widetilde{X}^{(q)}_{m,n}(t_{l-1}) + (t - t_{l-1})\cdot m \cdot \widetilde{X}^{(q)}_{m,n}(t_l)$$

for $t \in [t_{l-1}, t_l]$ and $l = 1,\dots,m$.

5.3.1. Fine-level Quantization: Quantization of a Weighted Combination of Brownian Bridges

Let $m \in \mathbb{N}$, and recall that for $l = 1,\dots,m$ and $j = 1,\dots,r$ the Brownian bridge B^j_l on $[t_{l-1}, t_l]$ is given by

$$B^j_l(t) = W_j(t) - (t_l - t)\cdot m \cdot W_j(t_{l-1}) - (t - t_{l-1})\cdot m \cdot W_j(t_l)$$

for $t \in [t_{l-1}, t_l]$.

In this section we construct a fine-level quantization, which is different from the fine-level quantization used for quantization of X in $L^d_p[0,1]$, see Section 5.2.1. For technical reasons the fine-level quantization presented in this section will not consist in applying quantizations to the Brownian bridges B^j_l separately. Instead, we quantize in one go a weighted combination of the Brownian bridges B^j_l, where we consider deterministic weights first.

More precisely, for $k = 1,\dots,d$ let $\gamma^k = \big(\gamma^k_{l,j}\big) \in \mathbb{R}^{m\times r}$, and we consider the real-valued stochastic process $\overline{B}^{(\gamma^k)} = \big(\overline{B}^{(\gamma^k)}(t)\big)_{t\in[0,1]}$ given by

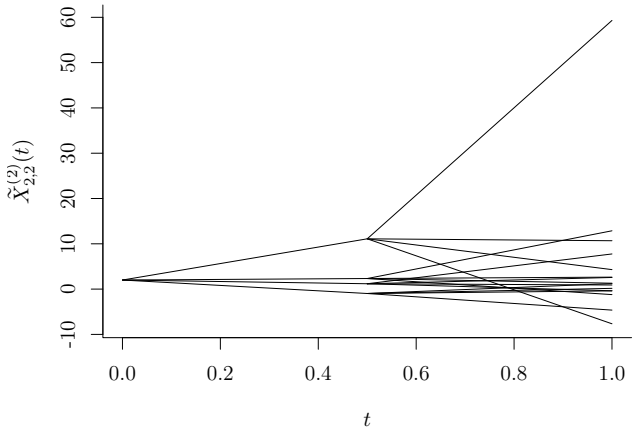
$$\overline{B}^{(\gamma^k)}(t) = \sum_{j=1}^{r} \gamma^k_{l,j} \cdot B^j_l(t)$$

for $t \in [t_{l-1}, t_l]$ and $l = 1,\dots,m$.

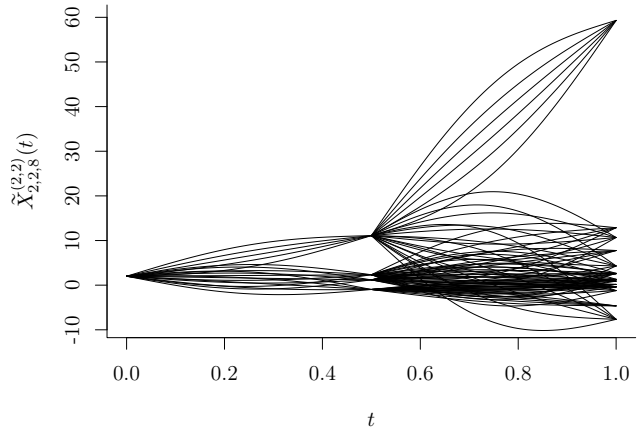
Due to Lemma 5.2.1 b) it holds

$$\overline{B}^{(\gamma^k)} \cdot \mathbb{1}_{[t_{l-1},t_l]} \overset{\mathcal{L}}{=} \|\gamma^k_l\|_2 \cdot B^1_l \tag{5.3.2}$$

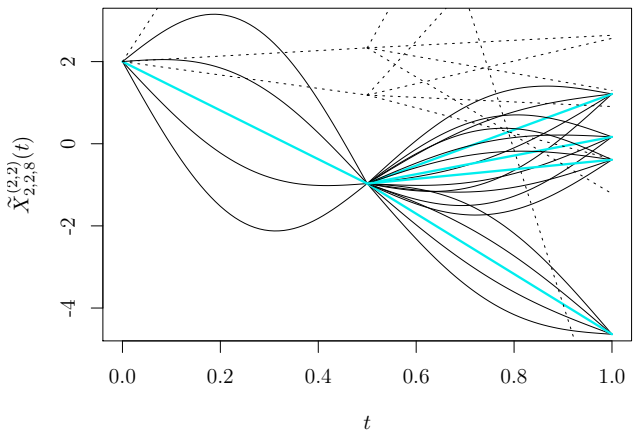
for all $l = 1,\dots,m$ and $k = 1,\dots,d$, where $\gamma^k_l = (\gamma^k_{l,1},\dots,\gamma^k_{l,r})$. Equation (5.3.2) is one of the key observations for our construction.



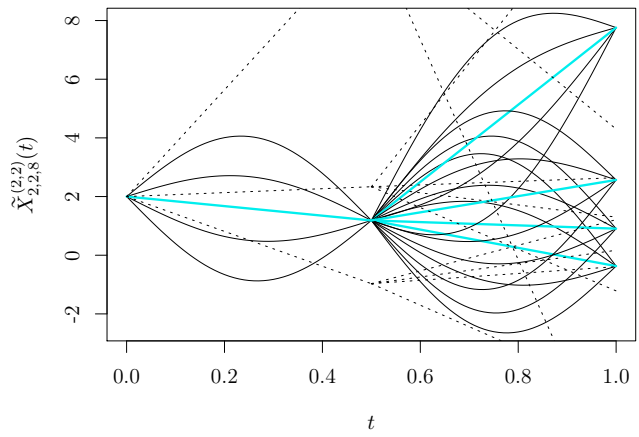
(a) Paths of the coarse-level quantization $\widetilde{X}^{(2)}_{2,2}$



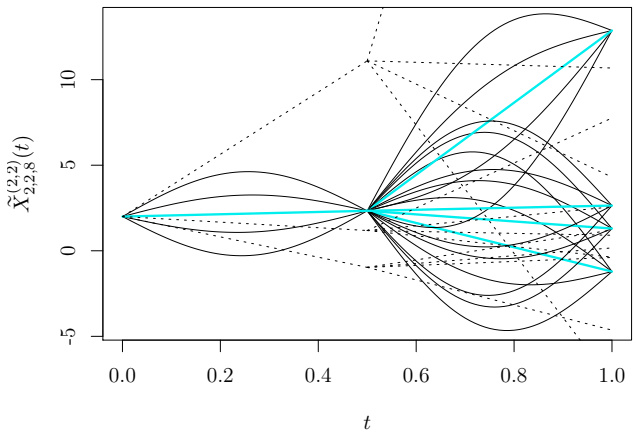
(b) Paths of the overall quantization $\widetilde{X}^{(2,2)}_{2,2,8}$



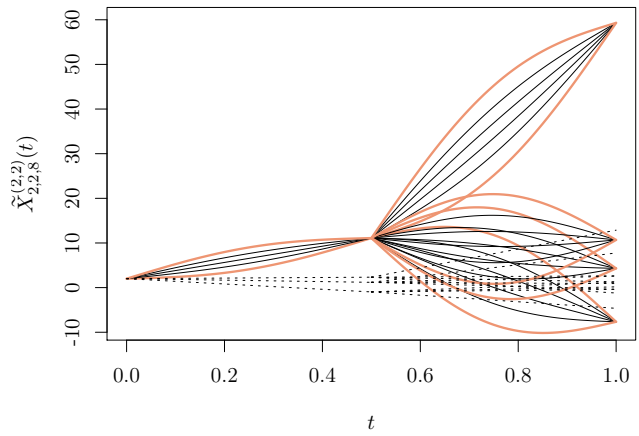
(c) Paths of $\widetilde{X}^{(2,2)}_{2,2,8}$ corresp. to block 1 in Table 5.1



(d) Paths of $\widetilde{X}^{(2,2)}_{2,2,8}$ corresp. to block 2 in Table 5.1



(e) Paths of $\widetilde{X}^{(2,2)}_{2,2,8}$ corresp. to block 3 in Table 5.1



(f) Paths of $\widetilde{X}^{(2,2)}_{2,2,8}$ corresp. to block 4 in Table 5.1

Figure 5.2.1.: Example of a quantization in $L_2[0,1]$

Notation
We define $B_1^{(\gamma^k)} = (B_1^{(\gamma^k)})_{t\in[0,1]}$ by

$$B_1^{(\gamma^k)}(t) = \|\gamma_l^k\|_2 \cdot B_l^1(t)$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$. Moreover, where is does not cause confusion, we write B_l and $B^{(\gamma^k)}$ instead of B_l^1 and $B_1^{(\gamma^k)}$, respectively.

In view of (5.3.2) the main idea is to design (a sequence of) quantizers $\psi^{(\gamma^k)} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ for the process $B_1^{(\gamma^k)}$, which will then be applied to the process $\overline{B}^{(\gamma^k)}$. One of the core ingredients in the construction of the quantizers will be a sequence of strongly asymptotically optimal quantizers for a Brownian motion on $[0,1]$.
For the remaining part of this subsection, let $k \in \{1, \dots, d\}$, and let $\gamma^k = \left(\gamma_{l,j}^k\right) \in \mathbb{R}^{m\times r}$ such that

$$\|\gamma_l^k\|_2 \neq 0 \tag{5.3.3}$$

for all $l = 1, \dots, m$ as well as

$$\sum_{l=1}^{m} \|\gamma_l^k\|_2^2 = m. \tag{5.3.4}$$

We split up our construction into single steps. The first three steps are built-up analogously to the construction for scalar SDEs in [MGR13, Section 2.3].

Step 1: We consider a sequence $Z_1, \dots, Z_m$ of i.i.d. random variables with $Z_1 \sim N(0,1)$ which is independent of W. We put

$$W_0^* := 0 \qquad \text{and} \qquad W_l^* := m^{-1/2} \cdot \sum_{i=1}^{l} Z_i$$

for $l = 1, \dots, m$. Then,

$$W_l^* \overset{\mathcal{L}}{=} W_1(t_l)$$

for all $l = 0, \dots, m$.

Now we add the random variables $W_0^*, \dots, W_m^*$ to the Brownian bridges $B_1, \dots, B_m$ in such a way that we end up with a new Brownian motion. More precisely, we define a real-valued stochastic process $W^* = (W^*(t))_{t\in[0,1]}$ by

$$W^*(t) = B_l(t) + (t_l - t) \cdot m \cdot W_{l-1}^* + (t - t_{l-1}) \cdot m \cdot W_l^* \tag{5.3.5}$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$.

The statement of the following lemma is verified rather easily, and therefore a proof is omitted.

Lemma 5.3.1
The process W^* is a one-dimensional Brownian motion on $[0,1]$.

Step 2: We put $s_0^k := 0$ and

$$s_l^k := \frac{1}{m} \cdot \sum_{i=1}^{l} \|\gamma_i^k\|_2^2$$

for $l = 1, \dots, m$. Then due to (5.3.3) and (5.3.4) we have

$$0 = s_0^k < s_1^k < \cdots < s_m^k = 1.$$

Hence, depending on γ^k, we have introduced a new discretization of the time interval $[0,1]$, and the corresponding piecewise linear time transformation $\tau^k : [0,1] \to [0,1]$ is given by

$$\tau^k(s) = t_{l-1} + \frac{s - s_{l-1}^k}{\|\gamma_l^k\|_2^2}$$

for $s \in [s_{l-1}^k, s_l^k]$ and $l = 1, \dots, m$. In particular, τ^k is bijective and its piecewise linear inverse function $(\tau^k)^{-1} : [0,1] \to [0,1]$ is given by

$$(\tau^k)^{-1}(t) = s_{l-1}^k + \|\gamma_l^k\|_2^2 \cdot (t - t_{l-1})$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$.

Step 3: In this step we construct a one-dimensional Brownian motion on $[0,1]$, whose properties serve as further key observations for our construction of a fine-level quantization. By utilizing the Brownian motion W^*, we define a real-valued stochastic process $W^{(\gamma^k)} = (W^{(\gamma^k)}(s))_{s\in[0,1]}$ via

$$W^{(\gamma^k)}(s) = \sum_{i=1}^{l-1} \|\gamma_i^k\|_2 \cdot \Big(W^*\big(\tau^k(s_i^k)\big) - W^*\big(\tau^k(s_{i-1}^k)\big)\Big) + \|\gamma_l^k\|_2 \cdot \Big(W^*\big(\tau^k(s)\big) - W^*\big(\tau^k(s_{l-1}^k)\big)\Big) \tag{5.3.6}$$

for $s \in [s_{l-1}^k, s_l^k]$ and $l = 1, \dots, m$.

Lemma 5.3.2

a) The process $W^{(\gamma^k)}$ is a one-dimensional Brownian motion on $[0,1]$.

b) The process $W^{(\gamma^k)}$ is independent of $(Y_1^1, \dots, Y_m^1)$.

c) For all $l = 1, \dots, m$ it holds

$$W^{(\gamma^k)}(s_l^k) = m^{-1/2} \cdot \sum_{i=1}^{l} \|\gamma_i^k\|_2 \cdot Z_i. \tag{5.3.7}$$

d) It holds

$$B^{(\gamma^k)}(t) = W^{(\gamma^k)}\big((\tau^k)^{-1}(t)\big) - (t_l - t) \cdot m \cdot W^{(\gamma^k)}\big(s_{l-1}^k\big) - (t - t_{l-1}) \cdot m \cdot W^{(\gamma^k)}\big(s_l^k\big) \tag{5.3.8}$$

for all $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$.

Proof:

a) This statement follows by employing the properties of the Brownian motion W^*.

b) By the definition of W^*, see (5.3.5), there exists a measurable mapping Φ such that

$$W^* = \Phi(B_1, \dots, B_m, Z_1, \dots, Z_m).$$

Moreover, due to Lemma 5.2.1 c), and due to the fact that the random variables $Z_1, \dots, Z_m$ are independent of W, we obtain that $(B_1, \dots, B_m, Z_1, \dots, Z_m)$ is independent of $(Y_1^1, \dots, Y_m^1)$. Hence W^* is independent of $(Y_1^1, \dots, Y_m^1)$, and by the definition of $W^{(\gamma^k)}$, see (5.3.6), we then obtain the assertion in b).

c),d) The statements in c) and d) follow from (5.3.5) and (5.3.6) by direct calculation.

□

Step 4: Observing c) and d) in the previous lemma gives a hint on which ingredients might be employed to obtain a sequence of quantizers for $B^{(\gamma^k)}$. Due to d) the Brownian bridges of $W^{(\gamma^k)}$ on the subintervals $[s_{l-1}^k, s_l^k]$ coincide with $B^{(\gamma^k)}$ up to the time transformation τ^k. Hence in view of (5.3.8) it stands to reason to built-up quantizations of $B^{(\gamma^k)}$ in the following way: First, we apply quantizations to the Brownian motion $W^{(\gamma^k)}$, and, secondly, we apply quantizations to the random variables $W^{(\gamma^k)}(s_l^k)$. In view of (5.3.7), we obtain quantizations of $W^{(\gamma^k)}(s_l^k)$ by applying quantizations of standard normally distributed random variables.

More precisely, we employ the following ingredients as building blocks in our construction:

(I.) Let $q \in [1, \infty)$. We take a sequence $(T_n^{(q)})_{n\in\mathbb{N}}$ of n-quantizers $T_n^{(q)} : \mathbb{R} \to \mathbb{R}$ for a standard normally distributed random variable, which satisfies **(N1)--(N3)**, and for all $n \in \mathbb{N}$ we put $\widetilde{W}_{m,n}^{(\gamma^k,q)}(s_0^k) := 0$ and

$$\widetilde{W}_{m,n}^{(\gamma^k,q)}(s_l^k) := m^{-1/2} \cdot \sum_{i=1}^{l} \|\gamma_i^k\|_2 \cdot \widetilde{Z}_{i,n}^{(q)} \tag{5.3.9}$$

for $l = 1, \dots, m$ where

$$\widetilde{Z}_{i,n}^{(q)} := T_n^{(q)}(Z_i).$$

(II.) Let $(S_{M,\varepsilon})_{M\in\mathbb{N}}$ be a strongly asymptotically optimal sequence of order $s\eta_\varepsilon$ of M-quantizers $S_{M,\varepsilon} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$, where η_ε is the natural number in (5.3.1). At this point, for technical reasons, our construction differs from the one in [MGR13, Section 2.3]. In [MGR13, Section 2.3] the authors choose an arbitrary sequence of M-quantizers for a Brownian motion on $[0,1]$.

Let $n, M \in \mathbb{N}$. We combine the aforementioned building blocks. By defining

$$\widetilde{B}_{m,n,M}^{(\gamma^k,q,\varepsilon)}(t) = S_{M,\varepsilon}\big(W^{(\gamma^k)}\big)\big((\tau^k)^{-1}(t)\big) - (t_l - t) \cdot m \cdot \widetilde{W}_{m,n}^{(\gamma^k,q)}\big(s_{l-1}^k\big) - (t - t_{l-1}) \cdot m \cdot \widetilde{W}_{m,n}^{(\gamma^k.q)}\big(s_l^k\big) \tag{5.3.10}$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$, we obtain a quantization of $B^{(\gamma^k)}$. Note that, in view of (5.3.10) and (5.3.8), there exists a Borel measurable mapping $\psi_{m,n,M}^{(\gamma^k,q,\varepsilon)} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ with

$$\big|\operatorname{ran}\big(\psi_{m,n,M}^{(\gamma^k,q,\varepsilon)}\big)\big| \leqslant n^m \cdot M \tag{5.3.11}$$

such that

$$\widetilde{B}^{(\gamma^k,q,\varepsilon)}_{m,n,M}(t) = \psi^{(\gamma^k,q,\varepsilon)}_{m,n,M}\big(B^{(\gamma^k)}\big)(t) \tag{5.3.12}$$

for all $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$.

5.3.2. Overall Quantization

Let $q \in [1, \infty)$ and $m, n, M \in \mathbb{N}$. We combine the fine-level quantization constructed in the previous subsection with the coarse-level quantization of X in order to obtain an overall product-quantization of X. First we replace the deterministic weights considered in the previous subsection by random weights. In analogy to the construction in [MGR13] we define $\overline{b}_{k,j,m} : \mathbb{R}^d \to \mathbb{R}$ by

$$\overline{b}_{k,j,m}(x) = \operatorname{sgn}\big(b_{k,j}(x)\big) \cdot \max\big(|b_{k,j}(x)|, m^{-1}\big)$$

for $k = 1, \dots, d$ and $j = 1, \dots, r$ where $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ is given by

$$\operatorname{sgn}(y) = \begin{cases} 1, & \text{if } y \geqslant 0, \\ -1, & \text{if } y < 0. \end{cases}$$

Now we take random weights Γ^k by taking into account the local regularity of X^k. More precisely, for $k = 1, \dots, d$, $j = 1, \dots, r$ and $l = 1, \dots, m$ we put

$$\Gamma^{k,(q)}_{l,j,m,n} := \overline{b}_{k,j,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big) \cdot \left(m^{-1} \cdot \sum_{i=0}^{m-1} \big\| \overline{b}_{k,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big) \big\|_2^2 \right)^{-1/2} \tag{5.3.13}$$

where

$$\overline{b}_{k,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big) = \big(\overline{b}_{k,1,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big), \dots, \overline{b}_{k,r,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big)\big).$$

For ease of notation, we write $\Gamma^k_{l,j}$ instead of $\Gamma^{k,(q)}_{l,j,m,n}$. Note that the random matrix $\Gamma^k = (\Gamma^k_{l,j})$ takes values in $\mathbb{R}^{m \times r}$ and satisfies

$$\|\Gamma^k_l\|_2 \neq 0$$

for all $l = 1, \dots, m$ as well as

$$\sum_{l=1}^{m} \|\Gamma^k_l\|_2^2 = m,$$

which is in accordance with the assumptions imposed on the deterministic weights in (5.3.3) and (5.3.4). We add that the random weights Γ^k are different from the ones employed in [MGR13, Section 2.3]. Therein the random weights are built-up by the Milstein quantization whereas, for technical reasons, we employed the Euler quantization.

Now for $k = 1, \dots, d$ we define a quantization $\widetilde{X}^{(q,\varepsilon),k}_{m,n,M}$ of X^k by

$$\widetilde{X}^{(q,\varepsilon),k}_{m,n,M}(t) = \widetilde{X}^{(q),k}_{m,n}(t) + \left(m^{-1} \cdot \sum_{i=0}^{m-1} \big\| \overline{b}_{k,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big) \big\|_2^2 \right)^{1/2} \cdot \psi^{(\Gamma^k,q,\varepsilon)}_{m,n,M}\big(\overline{B}^{(\Gamma^k)}\big)(t) \tag{5.3.14}$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \dots, m$.

Hence, by separately quantizing each component X^k of X, we have obtained a product-quantization $\widetilde{X}^{(q,\varepsilon)}_{m,n,M} = \big(\widetilde{X}^{(q,\varepsilon),1}_{m,n,M}, \dots, \widetilde{X}^{(q,\varepsilon),d}_{m,n,M}\big)'$ of $X = (X^1, \dots, X^d)'$. By (5.1.6) and (5.3.11) we have, for all $k = 1, \dots, d$,

$$\big|\operatorname{ran}\big(\widetilde{X}^{(q,\varepsilon),k}_{m,n,M}\big)\big| \leqslant \Big(4^{\binom{r}{2}} \cdot n^{1+r+\binom{r}{2}}\Big)^m \cdot M.$$

Thus,

$$\big|\operatorname{ran}\big(\widetilde{X}^{(q,\varepsilon)}_{m,n,M}\big)\big| \leqslant \prod_{k=1}^{d} \big|\operatorname{ran}\big(\widetilde{X}^{(q,\varepsilon),k}_{m,n,M}\big)\big| \leqslant \Big(4^{d\cdot\binom{r}{2}} \cdot n^{d\cdot(1+r+\binom{r}{2})}\Big)^m \cdot M^d. \tag{5.3.15}$$

We consider a sequence of quantizations

$$\widetilde{X}^{(q,\varepsilon)}_N = \widetilde{X}^{(q,\varepsilon)}_{m_N,m_N,M_N},$$

$N \in \mathbb{N}$, of X where we choose the parameters m_N and M_N as follows. First, in order to control the size of $\widetilde{X}^{(q,\varepsilon)}_N$ via (5.3.15) we choose

$$M_N := \max\left\{\left\lfloor \left(\left(4^{d\cdot\binom{r}{2}} \cdot m_N^{d\cdot(1+r+\binom{r}{2})}\right)^{-m_N} \cdot N\right)^{1/d} \right\rfloor, 1\right\}. \tag{5.3.16}$$

Moreover, in addition to **(Lim1)** we require

(Lim3) $\displaystyle\lim_{N\to\infty} \frac{\big(m_N^\alpha / \ln m_N\big)^2}{\ln N} = \infty$

for all $\alpha \in (3/4, 1)$. The technical assumption **(Lim3)** is another point in which our construction is distinct from the one in [MGR13].

Remark 5.3.3

a) A valid choice of a sequence in $\mathbb{N}$ which satisfies **(Lim1)** and **(Lim3)** is, again, $(m_N)_{N\in\mathbb{N}}$ with $m_1 = 1$ and

$$m_N = \Big\lceil (\ln N)^{2/3} \Big\rceil, \qquad N \geqslant 2.$$

b) Note that **(Lim3)** implies

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot \frac{\ln m_N}{m_N^\alpha} = 0$$

for all $\alpha \in (3/4, 1)$. Obviously, assumption **(Lim3)** is slightly stronger than **(Lim2)**.

c) By the construction of the sequence $(\widetilde{X}^{(q,\varepsilon)}_N)_{N\in\mathbb{N}}$ and due to (5.3.17) in the upcoming Theorem 5.3.4 we have

$$\widetilde{X}^{(q,\varepsilon)}_N \in \mathbb{X}_{N,\mathrm{prod,uni}}$$

for all sufficiently large $N \in \mathbb{N}$.

The following theorem contains the main result of this section, whose proof is postponed to Section 6.3. We put

$$C^{(\infty,s)} := \sqrt{d} \cdot \left(\mathbb{E}\left[\max_{k=1,\dots,d} \left\| \|b_k(X(\cdot))\|_2 \right\|_{L_2[0,1]}^s \right] \right)^{1/s}.$$

Theorem 5.3.4
Let $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant s\eta_\varepsilon\}$. Then

$$\left| \operatorname{ran}\left(\widetilde{X}_N^{(q,\varepsilon)}\right) \right| \leqslant N \tag{5.3.17}$$

for all sufficiently large $N \in \mathbb{N}$, and

$$\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot e^{(s)}\big(X, \widetilde{X}_N^{(q,\varepsilon)}, \mathcal{C}([0,1];\mathbb{R}^d)\big) \right) \leqslant \kappa_\infty \cdot C^{(\infty,s)} \cdot (1+\varepsilon)$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

The previous theorem and Theorem 4.3.2 lead to the following result on the asymptotic behavior of the Nth minimal quantization error w.r.t to N-uniform-product-quantizations of X in the space $\mathcal{C}([0,1];\mathbb{R}^d)$.

Theorem 5.3.5
For all $s \in [1,\infty)$ it holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod,uni}}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) = \kappa_\infty \cdot C^{(\infty,s)}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

Proof:
First, analogously to the proof of Theorem 5.2.6 one shows that X is an Itô process as required in Section 4.3. Hence by Theorem 4.3.2 we have

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod,uni}}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \geqslant \kappa_\infty \cdot C^{(\infty,s)}.$$

Secondly, for every $\varepsilon > 0$, due to the construction of quantizations of X above, and due to Theorem 5.3.4, there exists a sequence $(\widetilde{X}_N^{(\varepsilon)})_{N\in\mathbb{N}}$ of N-quantizations of X such that

$$\limsup_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}\big(X, \widetilde{X}_N^{(\varepsilon)}, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant \kappa_\infty \cdot C^{(\infty,s)} \cdot (1+\varepsilon).$$

Thus, for every $\varepsilon > 0$ we have

$$\limsup_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod,uni}}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant \kappa_\infty \cdot C^{(\infty,s)} \cdot (1+\varepsilon).$$

Therefore,

$$\limsup_{N\to\infty} \sqrt{\ln N} \cdot e^{(s)}_{N,\mathrm{prod,uni}}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant \inf_{\varepsilon>0} \kappa_\infty \cdot C^{(\infty,s)} \cdot (1+\varepsilon) = \kappa_\infty \cdot C^{(\infty,s)},$$

which finishes the proof. □

Remark 5.3.6
One obtains results on the computational cost to determine the paths and corresponding probability weights of the quantizations $\widetilde{X}^{(q,\varepsilon)}_{m,n,M}$ analogous to those presented in Section 5.2.3 by carrying out an analysis similar to the one in Section 5.2.3.

5.4. Quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$ w.r.t. a Norm Equivalent to the Supremum Norm

Throughout this section, let $s \in [1,\infty)$, and we interpret X as a random element in the space $\mathcal{C}([0,1];\mathbb{R}^d)$ equipped with the norm $|||\cdot|||_s$. By considering the norm $|||\cdot|||_s$ instead of the supremum norm $\|\cdot\|_\infty$ we derive results on the asymptotic behavior of the minimal quantization error of X w.r.t to quantizations in the wider classes $\mathbb{X}_{N,\mathrm{prod}}$ instead of only in the classes $\mathbb{X}_{N,\mathrm{prod,uni}}$.

Except for a few alterations we use the ideas and building blocks presented in the Sections 5.3.1 and 5.3.2 to obtain a sequence of product-quantizations of X in the space $(\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)$. We additionally employ ideas utilized in the case of quantization in $L^d_p[0,1]$.

5.4.1. Fine-level Quantization

Let $q \in [1,\infty)$, and let $m,n \in \mathbb{N}$. We adopt the whole construction presented in Section 5.3.1 except for one alteration. Namely, in Step 4 instead of taking a strongly asymptotically optimal sequence of quantizers for a Brownian motion on $[0,1]$, we choose an arbitrary sequence $(S_K)_{K\in\mathbb{N}}$ of K-quantizers $S_K : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$.

Then the quantization of $B^{(\gamma^k)}$ in (5.3.10) reads

$$\widetilde{B}^{(\gamma^k,q)}_{m,n,K}(t) = S_K\big(W^{(\gamma^k)}\big)\big((\tau^k)^{-1}(t)\big) - (t_l - t)\cdot m \cdot \widetilde{W}^{(\gamma^k,q)}_{m,n}\big(s^k_{l-1}\big) - (t - t_{l-1})\cdot m \cdot \widetilde{W}^{(\gamma^k,q)}_{m,n}\big(s^k_l\big)$$

for $t \in [t_{l-1}, t_l]$, $l = 1,\dots,m$, $k = 1,\dots,d$ and $K \in \mathbb{N}$. As in Section 5.3.1, for all $k = 1,\dots,d$ and $K \in \mathbb{N}$, there exists a Borel measurable mapping $\psi^{(\gamma^k,q)}_{m,n,K} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ with

$$\big|\operatorname{ran}\big(\psi^{(\gamma^k,q)}_{m,n,K}\big)\big| \leqslant n^m \cdot K. \tag{5.4.1}$$

such that

$$\widetilde{B}^{(\gamma^k,q)}_{m,n,K}(t) = \psi^{(\gamma^k,q)}_{m,n,K}\big(B^{(\gamma^k)}\big)(t)$$

for all $t \in [t_{l-1}, t_l]$ and $l = 1,\dots,m$.

5.4.2. Overall Quantization

Let $q \in [1,\infty)$, $m \in \mathbb{N}_2$, $n \in \mathbb{N}$, and for $k = 1,\dots,d$ let $\Gamma^k = (\Gamma^k_{l,j})$ be the random matrix where the entries $\Gamma^k_{l,j}$ are defined as in (5.3.13).

Additionally, we further specify the size K of the quantizers S_K by taking into account the respective

path properties of X^k. More precisely, let $M \in \mathbb{N}$, and for $k = 1, \dots, d$ we put

$$\mu_{m,n}^{(s),k} := \max\left\{ \frac{\left(\widetilde{A}_{m,n}^k\right)^{2s/(s+2)}}{\sum_{i=1}^d \left(\widetilde{A}_{m,n}^i\right)^{2s/(s+2)}}, \frac{1}{d \cdot m \cdot \ln m} \right\} \tag{5.4.2}$$

where

$$\widetilde{A}_{m,n}^i := \left(m^{-1} \cdot \sum_{l=1}^m \left\| \overline{b}_{i,m}\left(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{l-1})\right) \right\|_2^2 \right)^{1/2}.$$

For ease of notation, we write μ^k instead of $\mu_{m,n}^{(s),k}$. Now, similarly to the case of quantization in $L_p^d[0,1]$, we choose the size K dependent on M and μ^k by taking

$$K_{k,M} := \left\lceil M^{\mu^k} \right\rceil. \tag{5.4.3}$$

In analogy to (5.3.14), by defining

$$\widetilde{X}_{m,n,M}^{(q),k}(t) = \widetilde{X}_{m,n}^{(q),k}(t) + \widetilde{A}_{m,n}^k \cdot \psi_{m,n,K_{k,M}}^{(\Gamma^k)}\left(\overline{B}^{(\gamma^k)}\right)(t)$$

for $t \in [t_{l-1}, t_l]$, $l = 1, \dots, m$ and $k = 1, \dots, d$, we obtain a product-quantization

$$\widetilde{X}_{m,n,M}^{(q)} = (\widetilde{X}_{m,n,M}^{(q),1}, \dots, \widetilde{X}_{m,n,M}^{(q),d})' \qquad \text{of} \qquad X = (X^1, \dots, X^d).$$

By the choice of $K_{k,M}$, by (5.4.1), and by (5.1.6), we have

$$\left|\operatorname{ran}\left(\widetilde{X}_{m,n,M}^{(q),k}\right)\right| \leqslant \left(4^{\binom{r}{2}} \cdot n^{1+r+\binom{r}{2}}\right)^m \cdot K_{k,M} \leqslant \left(2 \cdot 4^{\binom{r}{2}} \cdot n^{1+r+\binom{r}{2}}\right)^m \cdot M^{\mu^k}$$

for all $k = 1, \dots, d$. Thus,

$$\begin{aligned} \left|\operatorname{ran}\left(\widetilde{X}_{m,n,M}^{(q)}\right)\right| &\leqslant \prod_{k=1}^d \left|\operatorname{ran}\left(\widetilde{X}_{m,n,M}^{(q),k}\right)\right| \\ &\leqslant \left(2^d \cdot 4^{d\cdot\binom{r}{2}} \cdot n^{d\cdot(1+r+\binom{r}{2})}\right)^m \cdot M^{\sum_{k=1}^d \mu^k} \\ &\leqslant \left(2^d \cdot 4^{d\cdot\binom{r}{2}} \cdot n^{d\cdot(1+r+\binom{r}{2})}\right)^m \cdot M^{1+1/\ln m}. \end{aligned} \tag{5.4.4}$$

We consider a sequence of quantizations

$$\widetilde{X}_N^{(q)} = \widetilde{X}_{m_N,m_N,M_N}^{(q)},$$

$N \in \mathbb{N}$, of X, where we chose the parameters m_N and M_N as follows. In order to control the size of $\widetilde{X}_N^{(q)}$ via (5.4.4) we choose

$$M_N = \max\left\{ \left\lfloor \left(\left(2^d \cdot 4^{d\cdot\binom{r}{2}} \cdot m_N^{d\cdot(1+r+\binom{r}{2})}\right)^{-m_N} \cdot N \right)^{\ln m_N/(1+\ln m_N)} \right\rfloor, 1 \right\}. \tag{5.4.5}$$

Moreover, we require **(Lim1)** as well as **(Lim3)** to hold. Then the statements in Remark 5.3.3 a) and b) stay valid, and instead of the statement in Remark 5.3.3 c), by arguing similarly as stated therein, we have

$$\widetilde{X}_N^{(q)} \in \mathbb{X}_{N,\mathrm{prod}}$$

for all sufficiently large $N \in \mathbb{N}$.

We put

$$\overline{\kappa}_{\infty,s} := \limsup_{K\to\infty} \sqrt{\ln K} \cdot e^{(s)}\big(\overline{W}, S_K(\overline{W}), \mathcal{C}[0,1]\big)$$

where $\overline{W}$ denotes a Brownian motion on $[0,1]$, and we put

$$\mathfrak{C}^{(\infty,s)} := \left(\mathbb{E}\left[\left(\sum_{k=1}^{d} \big\| \|b_k(X(\cdot))\|_2 \big\|_{L_2[0,1]}^{2s/(s+2)} \right)^{(s+2)/2} \right] \right)^{1/s}.$$

The construction leads to the following result.

Theorem 5.4.1
Let $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant s\}$. Then

$$\big|\operatorname{ran}\big(\widetilde{X}_N^{(q)}\big)\big| \leqslant N$$

for all sufficiently large $N \in \mathbb{N}$, and

$$\limsup_{N\to\infty} \Big(\sqrt{\ln N} \cdot e^{(s)}\big(X, \widetilde{X}_N^{(q)}, \mathcal{C}([0,1];\mathbb{R}^d)\big)\Big) \leqslant \overline{\kappa}_{\infty,s} \cdot \mathfrak{C}^{(\infty,s)}.$$

By applying the previous theorem, Theorem 2.2.7, and Theorem 4.1.2 (ii), we obtain the following result on the asymptotic behavior of the Nth minimal quantization error w.r.t. to N-product-quantizations of X in the space $(\mathcal{C}([0,1];\mathbb{R}^d), \|\!|\cdot|\!\|_s)$. We abbreviate $\mathcal{M}^b[0,1] := \mathcal{M}^b([0,1];\mathbb{R})$.

Theorem 5.4.2
It holds

$$\lim_{N\to\infty} \sqrt{\ln N} \cdot e_{N,\mathrm{prod}}^{(s)}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) = \kappa_\infty \cdot \mathfrak{C}^{(\infty,s)}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii).

Proof:
By utilizing Theorem 4.1.2 (ii) one shows

$$\liminf_{N\to\infty} \sqrt{\ln N} \cdot e_{N,\mathrm{prod}}^{(s)}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \geqslant \kappa_\infty \cdot \mathfrak{C}^{(\infty,s)}$$

analogously to the proof of the lower bound in Theorem 5.2.6.

It remains to show

$$\limsup_{N\to\infty} \sqrt{\ln N} \cdot e_{N,\mathrm{prod}}^{(s)}\big(X, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant \kappa_\infty \cdot \mathfrak{C}^{(\infty,s)}.$$

Let $(S_K)_{K\in\mathbb{N}}$ be a sequence of K-quantizers $S_K : \mathcal{M}^b[0,1] \to \mathcal{M}^b[0,1]$ for a Brownian motion on $[0,1]$

which is strongly asymptotically optimal of order s. Note that due to Theorem 2.2.7 (ii) such a sequence exists. Then by carrying out a construction which is built-up analogously to the one leading to Theorem 5.4.1, one obtains that there exists a sequence $(\widetilde{X}_N)_{N\in\mathbb{N}}$ of N-quantizations of X in the space $\mathcal{M}^b([0,1];\mathbb{R}^d)$ such that

$$\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot e^{(s)}\big(X,\widetilde{X}_N,\mathcal{M}^b([0,1];\mathbb{R}^d)\big)\right)\leqslant \overline{\kappa}_{\infty,s}\cdot\mathfrak{C}^{(\infty,s)}.$$

Hence, by Theorem 2.2.7 (i), we end up with

$$\limsup_{N\to\infty}\sqrt{\ln N}\cdot e^{(s)}_{N,\mathrm{prod}}\big(X,\mathcal{C}([0,1];\mathbb{R}^d)\big)=\limsup_{N\to\infty}\sqrt{\ln N}\cdot e^{(s)}_{N,\mathrm{prod}}\big(X,\mathcal{M}^b([0,1];\mathbb{R}^d)\big)\leqslant\kappa_\infty\cdot\mathfrak{C}^{(\infty,s)}.$$

□

We close this section with the following result, which states that the sequence $(\widetilde{X}^{(q)}_N)_{N\in\mathbb{N}}$ is strongly asymptotically optimal of order s provided that the sequence $(S_K)_{K\in\mathbb{N}}$ is strongly asymptotically optimal of order s.

Proposition 5.4.3
Let $q\geqslant\min\{\widetilde{q}\in 2\mathbb{N}\mid\widetilde{q}\geqslant s\}$. If $(S_K)_{K\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order s of K-quantizers $S_K:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$, it holds

$$\limsup_{N\to\infty}\frac{e^{(s)}\big(X,\widetilde{X}^{(q)}_N,\mathcal{C}([0,1];\mathbb{R}^d)\big)}{e^{(s)}_{N,\mathrm{prod}}\big(X,\mathcal{C}([0,1];\mathbb{R}^d)\big)}\leqslant 1,$$

and in particular, in case $d=1$, it holds

$$\limsup_{N\to\infty}\frac{e^{(s)}\big(X,\widetilde{X}^{(q)}_N,\mathcal{C}[0,1]\big)}{e^{(s)}_{N}\big(X,\mathcal{C}[0,1]\big)}\leqslant 1.$$

Proof:
Note that assumption (ii) at the beginning of this chapter ensures $\mathfrak{C}^{(\infty,s)}>0$. Then the statements are a direct consequence of Theorem 5.4.1, Theorem 5.4.2 and Theorem 2.3.11 (ii). □

Remark 5.4.4
One obtains results on the computational cost to determine the paths and corresponding probability weights of the quantizations $\widetilde{X}^{(q)}_{m,n,M}$ analogous to those presented in Section 5.2.3 by carrying out an analysis similar to the one in Section 5.2.3.

6 | Proofs of Chapter 5

In the whole chapter c denotes a not further specified positive real constant which may vary from line to line and which might depend only on the dimension d of the SDE (3.1.1), on the dimension r of the driving Brownian motion W, on the initial value x_0, on the constants in **(C2)**,**(N3)** and **(I3)**, on the conditions **(N2)** and **(I2)**, as well as on all involved moment parameters.

Furthermore, in the whole chapter, for $m, n \in \mathbb{N}$ and $q \in [1, \infty)$, we use the abbreviations

- $\widehat{a}_k^{l-1} := a_k\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)$, $\widehat{b}_{k,j}^{l-1} := b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big)$ and $\widehat{c}_{k,j_1,j_2}^{l-1} := \nabla b_{k,j_2} b^{(j_1)}\big(\widehat{X}^{\mathrm{M}}(t_{l-1})\big)$ as well as
- $\widetilde{a}_k^{l-1} := a_k\big(\widetilde{X}_{m,n}^{(q)}(t_{l-1})\big)$, $\widetilde{b}_{k,j}^{l-1} := b_{k,j}\big(\widetilde{X}_{m,n}^{(q)}(t_{l-1})\big)$ and $\widetilde{c}_{k,j_1,j_2}^{l-1} := \nabla b_{k,j_2} b^{(j_1)}\big(\widetilde{X}_{m,n}^{(q)}(t_{l-1})\big)$

for $k = 1, \dots, d$, $j, j_1, j_2 = 1, \dots, r$ and $l = 1, \dots, m$. Analogously we define

- $\widehat{a}_k^{\mathrm{E},l-1}$, $\widehat{b}_{k,j}^{\mathrm{E},l-1}$ and $\widehat{c}_{k,j_1,j_2}^{\mathrm{E},l-1}$ as well as
- $\widetilde{a}_k^{\mathrm{E},l-1}$, $\widetilde{b}_{k,j}^{\mathrm{E},l-1}$ and $\widetilde{c}_{k,j_1,j_2}^{\mathrm{E},l-1}$

with $\widehat{X}_m^{\mathrm{M}}(t_{l-1})$ and $\widetilde{X}_{m,n}^{(q)}(t_{l-1})$ replaced by $\widehat{X}_m^{\mathrm{E}}(t_{l-1})$ and $\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{l-1})$, respectively.

In the majority of cases, where it does not cause confusion, we also drop the notational indices E and M, which are used to indicate which time-discrete approximation scheme is under consideration, and write $\widehat{X}_m$ instead of $\widehat{X}_m^{\mathrm{M}}$ and $\widehat{X}_m^{\mathrm{E}}$.

6.1. Coarse-level Quantization: Proof of Proposition 5.1.1

6.1.1. Auxiliary Lemmas 1 and 2

In the first main auxiliary lemma we provide a moment bound for the Milstein quantization.

Lemma 6.1.1
Let $s, q \in [1, \infty)$. Then, for all $m, n \in \mathbb{N}$,

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \|\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^s\Big] \leqslant c.$$

Proof:
Let $m, n \in \mathbb{N}$ and, for the moment, we assume $s \in 2\mathbb{N}$. For ease of notation, we drop the parameter q and write $\widetilde{X}_{m,n}(t_l)$ and $\widetilde{Y}_{l,n}^j$ instead of $\widetilde{X}_{m,n}^{(q)}(t_l)$ and $\widetilde{Y}_{l,n}^{(q),j}$, respectively.

We split up the proof into single steps.

Step 1: Let $k \in \{1, \dots, d\}$. Due to the recursive structure of the Milstein quantization we have

$$\begin{aligned}
\widetilde{X}^k_{m,n}(t_l) &= x_0^k + m^{-1} \cdot \sum_{i=0}^{l-1} \widetilde{a}^i_k + m^{-1/2} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \widetilde{b}^i_{k,j} \cdot \widetilde{Y}^j_{i+1,n} + \frac{1}{2} \cdot m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \widetilde{c}^i_{k,j,j} \cdot \big((\widetilde{Y}^j_{i+1,n})^2 - 1 \big) \\
&\quad + m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \Big(\big(\widetilde{c}^i_{k,j_2,j_1} - \widetilde{c}^i_{k,j_1,j_2} \big) \cdot \widetilde{I}^{i+1,n}_{(j_1,j_2)} + \widetilde{c}^i_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{i+1,n} \cdot \widetilde{Y}^{j_2}_{i+1,n} \Big) \\
&= x_0^k + m^{-1} \cdot \sum_{i=0}^{l-1} \widetilde{a}^i_k + m^{-1/2} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \widetilde{b}^i_{k,j} \cdot \widetilde{Y}^j_{i+1,n} \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \widetilde{c}^i_{k,j,j} \cdot \big((\widetilde{Y}^j_{i+1,n})^2 - \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] + \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] - 1 \big) \\
&\quad + m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \Big(\big(\widetilde{c}^i_{k,j_2,j_1} - \widetilde{c}^i_{k,j_1,j_2} \big) \cdot \widetilde{I}^{i+1,n}_{(j_1,j_2)} + \widetilde{c}^i_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{i+1,n} \cdot \widetilde{Y}^{j_2}_{i+1,n} \Big)
\end{aligned}$$

for all $l = 1, \dots, m$.

Next, for $i = 0, \dots, m-1$ we put

$$\widetilde{\vartheta}^k_i := m^{-1} \cdot \widetilde{a}^i_k + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}^i_{k,j,j} \cdot \big(\mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] - 1 \big)$$

and

$$\begin{aligned}
\widetilde{\xi}^k_i &:= m^{-1/2} \cdot \sum_{j=1}^{r} \widetilde{b}^i_{k,j} \cdot \widetilde{Y}^j_{i+1,n} + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}^i_{k,j,j} \cdot \big((\widetilde{Y}^j_{i+1,n})^2 - \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] \big) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \Big(\big(\widetilde{c}^i_{k,j_2,j_1} - \widetilde{c}^i_{k,j_1,j_2} \big) \cdot \widetilde{I}^{i+1,n}_{(j_1,j_2)} + \widetilde{c}^i_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{i+1,n} \cdot \widetilde{Y}^{j_2}_{i+1,n} \Big).
\end{aligned} \tag{6.1.1}$$

Then, for all $l = 0, \dots, m$, it holds

$$\widetilde{X}^k_{m,n}(t_l) = \widetilde{U}^k_l + \widetilde{V}^k_l \tag{6.1.2}$$

where

$$\widetilde{U}^k_l := x_0^k + \sum_{i=0}^{l-1} \widetilde{\vartheta}^k_i \quad \text{and} \quad \widetilde{V}^k_l := \sum_{i=0}^{l-1} \widetilde{\xi}^k_i \tag{6.1.3}$$

for $l = 1, \dots, m$ and $\widetilde{U}^k_0 := x_0^k$ as well as $\widetilde{V}^k_0 := 0$. Moreover, we put

$$\widetilde{V}^{k,*}_l := \max_{j=0,\dots,l} |\widetilde{V}^k_j|$$

for $l = 0, \dots, m$.

Step 2: In this step we further estimate $|\widetilde{\vartheta}_i^k|$. Since a and b are of at most linear growth, and due to **(C2)** (i), **(N2)** and (6.1.2), we have

$$\begin{aligned}
|\widetilde{\vartheta}_i^k| &= \Big| m^{-1} \cdot \widetilde{a}_k^i + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}_{k,j,j}^i \cdot \Big(\mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2 \big] - 1 \Big) \Big| \\
&\leqslant c \cdot m^{-1} \cdot \Big(|\widetilde{a}_k^i| + \sum_{j=1}^{r} |\widetilde{c}_{k,j,j}^i| \cdot \big| \mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2 \big] + 1 \big| \Big) \\
&\leqslant c \cdot m^{-1} \cdot \big(1 + \| \widetilde{X}_{m,n}(t_i) \|_2 \big) \\
&\leqslant c \cdot m^{-1} \cdot \left(1 + \sum_{k=1}^{d} |\widetilde{X}_{m,n}^k(t_i)| \right) \\
&\leqslant c \cdot m^{-1} \cdot \left(1 + \sum_{k=1}^{d} |\widetilde{U}_i^k| + \sum_{k=1}^{d} \widetilde{V}_i^{k,*} \right)
\end{aligned}$$

for all $i = 0, \dots, m-1$ and $k = 1, \dots, d$. Hence, together with (6.1.3), we conclude that

$$\begin{aligned}
\sum_{k=1}^{d} |\widetilde{U}_{l+1}^k| &= \sum_{k=1}^{d} \Big| \widetilde{U}_l^k + \widetilde{\vartheta}_l^k \Big| \\
&\leqslant \sum_{k=1}^{d} |\widetilde{U}_l^k| + c \cdot m^{-1} \cdot \left(1 + \sum_{k=1}^{d} |\widetilde{U}_l^k| + \sum_{k=1}^{d} \widetilde{V}_l^{k,*} \right) \\
&= \left(\sum_{k=1}^{d} |\widetilde{U}_l^k| \right) \cdot \big(1 + c \cdot m^{-1} \big) + c \cdot m^{-1} \cdot \left(1 + \sum_{k=1}^{d} \widetilde{V}_l^{k,*} \right)
\end{aligned}$$

for all $l = 0, \dots, m-1$. Then, by the discrete version of Gronwall's inequality, see Corollary A.2, we arrive at

$$\max_{j=0,\dots,l} \sum_{k=1}^{d} |\widetilde{U}_j^k| \leqslant c \cdot \left(1 + \sum_{k=1}^{d} \widetilde{V}_l^{k,*} \right) \tag{6.1.4}$$

for all $l = 0, \dots, m$.

Step 3: We combine the previous two steps. By (6.1.2) and (6.1.4) we have

$$\Big(\max_{j=0,\dots,l} |\widetilde{X}_{m,n}^k(t_j)| \Big)^s \leqslant \left(\Big(\max_{j=0,\dots,l} \sum_{k=1}^{d} |\widetilde{U}_j^k| \Big) + \sum_{k=1}^{d} \widetilde{V}_l^{k,*} \right)^s \leqslant c \cdot \left(1 + \sum_{k=1}^{d} \big(\widetilde{V}_l^{k,*} \big)^s \right) \tag{6.1.5}$$

for all $l = 0, \dots, m$ and $k = 1, \dots, d$. Hence, in particular,

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \| \widetilde{X}_{m,n}(t_l) \|_\infty^s \Big] \leqslant \sum_{k=1}^{d} \mathbb{E}\Big[\max_{l=0,\dots,m} |\widetilde{X}_{m,n}^k(t_l)|^s \Big] \leqslant c \cdot \left(1 + \sum_{k=1}^{d} \mathbb{E}\left[\big(\widetilde{V}_m^{k,*} \big)^s \right] \right). \tag{6.1.6}$$

Thus, in order to prove the assertion of the lemma, it remains to show

$$\sum_{k=1}^{d} \mathbb{E}\Big[\big(\widetilde{V}_m^{k,*} \big)^s \Big] \leqslant c.$$

Step 4: Let $k \in \{1, \dots, d\}$. In this step we show that the time-discrete process $(\widetilde{V}_l^k)_{l=0,\dots,m}$ is an $(\mathcal{F}(t_l))_{l=0,\dots,m}$-martingale where $\mathcal{F}(t_l)$ is an element of the filtration $\mathcal{F}$ as defined in (3.1.2), and where, recall, $\widetilde{V}_0^k = 0$ and

$$\widetilde{V}_l^k = \sum_{i=0}^{l-1} \widetilde{\xi}_i^k$$

for $l = 1, \dots, m$. Now, we prove:

(i) The process $(\widetilde{V}_l^k)_{l=0,\dots,m}$ is adapted to $(\mathcal{F}(t_l))_{l=0,\dots,m}$.

(ii) It holds $\mathbb{E}\big[|\widetilde{V}_l^k|\big] < \infty$ for all $l = 0, \dots, m$.

(iii) It holds

$$\mathbb{E}\Big[\widetilde{V}_l^k \,\Big|\, \mathcal{F}(t_{l-1})\Big] \overset{P\text{-a.s.}}{=} \widetilde{V}_{l-1}^k$$

for all $l = 1, \dots, m$.

ad (i): Obviously, $\widetilde{V}_0^k$ is $\mathcal{F}(t_0)$-$\mathcal{B}(\mathbb{R})$-measurable. Next, let $l \in \{1, \dots, m\}$. Due to the definition of $\widetilde{\xi}_i^k$, see (6.1.1), and due to the definition of the filtration $(\mathcal{F}(t_l))_{l=0,\dots,m}$, the random variable $\widetilde{\xi}_i^k$ is $\mathcal{F}(t_{i+1})$-$\mathcal{B}(\mathbb{R})$-measurable for all $i = 0, \dots, l-1$. Moreover, it holds $\mathcal{F}(t_{i+1}) \subseteq \mathcal{F}(t_l)$ for all $i = 0, \dots, l-1$. Thus, $\widetilde{V}_l^k$ is a sum of $\mathcal{F}(t_l)$-$\mathcal{B}(\mathbb{R})$-measurable random variables, and hence $\widetilde{V}_l^k$ is $\mathcal{F}(t_l)$-$\mathcal{B}(\mathbb{R})$-measurable.

ad (ii): Trivially, it holds $\mathbb{E}[|\widetilde{V}_0^k|] = 0 < \infty$. Next, let $l \in \{1, \dots, m\}$. By the triangle inequality we have

$$|\widetilde{V}_l^k| \leqslant \sum_{i=0}^{l-1} |\widetilde{\xi}_i^k|.$$

Moreover, by **(C2)** (i), **(N2)** and **(I2)**, by the properties of b, by the recursive structure of the Milstein quantization, and by $\mathbb{E}\big[\|x_0\|_\infty\big] < \infty$, we conclude that $\mathbb{E}\big[|\widetilde{\xi}_i^k|\big] < \infty$ for all $i = 0, \dots, m-1$. Thus,

$$\mathbb{E}\big[|\widetilde{V}_l^k|\big] < \infty.$$

ad (iii): Let $l \in \{1, \dots, m\}$. Due to the definition of $\widetilde{V}_l^k$, and due to Proposition B.1 a), it holds

$$\mathbb{E}\Big[\widetilde{V}_l^k \,\Big|\, \mathcal{F}(t_{l-1})\Big] \overset{P\text{-a.s.}}{=} \sum_{i=0}^{l-1} \mathbb{E}\Big[\widetilde{\xi}_i^k \,\Big|\, \mathcal{F}(t_{l-1})\Big].$$

First, we consider the case $i \in \{0, \dots, l-2\}$. By the definition of $\mathcal{F}(t_{l-1})$, by the definition of

the Milstein quantization, and by Proposition B.1 a) and c), we have

$$\begin{aligned}
\mathbb{E}\Big[\widetilde{\xi}_i^k \,\Big|\, \mathcal{F}(t_{l-1})\Big] &\overset{P\text{-a.s.}}{=} m^{-1/2} \cdot \sum_{j=1}^{r} \mathbb{E}\Big[\widetilde{b}_{k,j}^i \cdot \widetilde{Y}_{i+1,n}^j \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \mathbb{E}\Big[\widetilde{c}_{k,j,j}^i \cdot \big((\widetilde{Y}_{i+1,n}^j)^2 - \mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2\big]\big) \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \mathbb{E}\Big[\big(\widetilde{c}_{k,j_2,j_1}^i - \widetilde{c}_{k,j_1,j_2}^i\big) \cdot \widetilde{I}_{(j_1,j_2)}^{i+1,n} \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \mathbb{E}\Big[\widetilde{c}_{k,j_1,j_2}^i \cdot \widetilde{Y}_{i+1,n}^{j_1} \cdot \widetilde{Y}_{i+1,n}^{j_2} \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&= \widetilde{\xi}_i^k.
\end{aligned} \tag{6.1.7}$$

Secondly, we consider the case $i = l-1$. Note that $(Y_l^j, I_{(j_1,j_2)}^l)$ is independent of $\mathcal{F}(t_{l-1})$ for all $j, j_1, j_2 = 1, \ldots, r$, and note that $\widetilde{X}_{m,n}(t_{l-1})$ is $\mathcal{F}(t_{l-1})$-$\mathcal{B}(\mathbb{R}^d)$-measurable. Then Proposition B.1 a), c) and d), and the properties **(N1)** and **(I1)**, lead to

$$\begin{aligned}
\mathbb{E}\Big[\widetilde{\xi}_{l-1}^k \,\Big|\, \mathcal{F}(t_{l-1})\Big] &\overset{P\text{-a.s.}}{=} m^{-1/2} \cdot \sum_{j=1}^{r} \mathbb{E}\Big[\widetilde{b}_{k,j}^{l-1} \cdot \widetilde{Y}_{l,n}^j \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \mathbb{E}\Big[\widetilde{c}_{k,j,j}^{l-1} \cdot \big((\widetilde{Y}_{l,n}^j)^2 - \mathbb{E}\big[(\widetilde{Y}_{l,n}^j)^2\big]\big) \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \mathbb{E}\Big[\big(\widetilde{c}_{k,j_2,j_1}^{l-1} - \widetilde{c}_{k,j_1,j_2}^{l-1}\big) \cdot \widetilde{I}_{(j_1,j_2)}^{l,n} \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \mathbb{E}\Big[\widetilde{c}_{k,j_1,j_2}^{l-1} \cdot \widetilde{Y}_{l,n}^{j_1} \cdot \widetilde{Y}_{l,n}^{j_2} \,\Big|\, \mathcal{F}(t_{l-1})\Big] \\
&\overset{P\text{-a.s.}}{=} m^{1/2} \cdot \sum_{j=1}^{r} \widetilde{b}_{k,j}^{l-1} \cdot \mathbb{E}\big[\widetilde{Y}_{l,n}^j\big] + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}_{k,j,j}^{l-1} \cdot \mathbb{E}\Big[(\widetilde{Y}_{l,n}^j)^2 - \mathbb{E}\big[(\widetilde{Y}_{l,n}^j)^2\big]\Big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \big(\widetilde{c}_{k,j_2,j_1}^{l-1} - \widetilde{c}_{k,j_1,j_2}^{l-1}\big) \cdot \mathbb{E}\big[\widetilde{I}_{(j_1,j_2)}^{l,n}\big] \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \widetilde{c}_{k,j_1,j_2}^{l-1} \cdot \mathbb{E}\big[\widetilde{Y}_{l,n}^{j_1}\big] \cdot \mathbb{E}\big[\widetilde{Y}_{l,n}^{j_2}\big] \\
&= 0.
\end{aligned} \tag{6.1.8}$$

Thus, by (6.1.7) and (6.1.8) we arrive at

$$\mathbb{E}\Big[\widetilde{V}_l^k \,\Big|\, \mathcal{F}(t_{l-1})\Big] \overset{P\text{-a.s.}}{=} \sum_{i=0}^{l-2} \widetilde{\xi}_i^k = \widetilde{V}_{l-1}^k,$$

which finishes the proof of (iii) .

Since $(\widetilde{V}_l^k)_{l=0,\dots,m}$ is an $(\mathcal{F}(t_l))_{l=0,\dots,m}$-martingale, the process $(|\widetilde{V}_l^k|)_{l=0,\dots,m}$ is an $(\mathcal{F}(t_l))_{l=0,\dots,m}$-submartingale. Consequently, by applying Doob's maximal inequality for submartingales we obtain

$$\mathbb{E}\left[\big(\widetilde{V}_l^{k,*}\big)^s\right] = \mathbb{E}\Big[\max_{j=0,\dots,l} \big|\widetilde{V}_j^k\big|^s\Big] \leqslant c \cdot \mathbb{E}\left[\big|\widetilde{V}_l^k\big|^s\right] \tag{6.1.9}$$

for all $l = 0, \dots, m$. Therefore, in view of (6.1.6), it remains to show

$$\sum_{k=1}^{d} \mathbb{E}\left[\big|\widetilde{V}_m^k\big|^s\right] \leqslant c,$$

which we get down to in the next step.

Step 5: First, note that since we assumed $s \in 2\mathbb{N}$, it holds

$$\sum_{k=1}^{d} \mathbb{E}\left[|\widetilde{V}_{l+1}^k|^s\right] = \sum_{k=1}^{d} \mathbb{E}\left[\big|\widetilde{V}_l^k\big|^s\right] + \sum_{k=1}^{d} \sum_{\mu=2}^{s} \binom{s}{\mu} \cdot \mathbb{E}\left[\big(\widetilde{V}_l^k\big)^{s-\mu} \cdot \big(\widetilde{\xi}_l^k\big)^{\mu}\right] + s \cdot \sum_{k=1}^{d} \mathbb{E}\left[\big(\widetilde{V}_l^k\big)^{s-1} \cdot \widetilde{\xi}_l^k\right] \tag{6.1.10}$$

for every $l = 0, \dots, m-1$.

We go on by analyzing the last summand in the previous equation first. The definition of $\widetilde{\xi}_l^k$, see (6.1.1), leads to

$$\begin{aligned}
\big(\widetilde{V}_l^k\big)^{s-1} \cdot \widetilde{\xi}_l^k &= m^{-1/2} \cdot \sum_{j=1}^{r} \big(\widetilde{V}_l^k\big)^{s-1} \cdot \widetilde{b}_{k,j}^l \cdot \widetilde{Y}_{l+1,n}^j \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \big(\widetilde{V}_l^k\big)^{s-1} \cdot \widetilde{c}_{k,j,j}^l \cdot \big((\widetilde{Y}_{l+1,n}^j)^2 - \mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\big) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \big(\widetilde{V}_l^k\big)^{s-1} \cdot \big(\widetilde{c}_{k,j_2,j_1}^l - \widetilde{c}_{k,j_1,j_2}^l\big) \cdot \widetilde{I}_{(j_1,j_2)}^{l+1,n} \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \big(\widetilde{V}_l^k\big)^{s-1} \cdot \widetilde{c}_{k,j_1,j_2}^l \cdot \widetilde{Y}_{l+1,n}^{j_1} \cdot \widetilde{Y}_{l+1,n}^{j_2}
\end{aligned}$$

for all $l = 0, \dots, m-1$ and $k = 1, \dots, d$. Again, note that $(Y_{l+1}^j, I_{(j_1,j_2)}^{l+1})$ is independent of $\mathcal{F}(t_l)$ for all $j, j_1, j_2 = 1, \dots, r$ and $l = 0, \dots, m-1$, and note that the random vector $(\widetilde{X}_{m,n}(t_l), \widetilde{V}_l^k)$ is measurable w.r.t. $\mathcal{F}(t_l)$ for all $l = 0, \dots, m-1$ and $k = 1, \dots, d$. Together with the properties

of W and **(N1)** as well as **(I1)** we arrive at

$$\begin{aligned}\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-1}\cdot\widetilde{\xi}_l^k\Big] &= m^{-1/2}\cdot\sum_{j=1}^{r}\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-1}\cdot\widetilde{b}_{k,j}^l\Big]\cdot\mathbb{E}\Big[\widetilde{Y}_{l+1,n}^j\Big]\\ &\quad+\frac{1}{2}\cdot m^{-1}\cdot\sum_{j=1}^{r}\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-1}\cdot\widetilde{c}_{k,j,j}^l\Big]\cdot\mathbb{E}\Big[(\widetilde{Y}_{l+1,n}^j)^2-\mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\Big]\\ &\quad+m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-1}\cdot\big(\widetilde{c}_{k,j_2,j_1}^l-\widetilde{c}_{k,j_1,j_2}^l\big)\Big]\cdot\mathbb{E}\Big[\widetilde{I}_{(j_1,j_2)}^{l+1,n}\Big] \qquad (6.1.11)\\ &\quad+m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-1}\cdot\widetilde{c}_{k,j_1,j_2}^l\Big]\cdot\mathbb{E}\Big[\widetilde{Y}_{l+1,n}^{j_1}\Big]\cdot\mathbb{E}\Big[\widetilde{Y}_{l+1,n}^{j_2}\Big]\\ &= 0\end{aligned}$$

for all $l=0,\dots,m-1$ and $k=1,\dots,d$.

Next, we further estimate the second summand on the right-hand side of (6.1.10). By the properties of b, and by **(C2)** (i), we obtain

$$\begin{aligned}|\widetilde{\xi}_l^k| &\leqslant m^{-1/2}\cdot\sum_{j=1}^{r}\big|\widetilde{b}_{k,j}^l\big|\cdot\big|\widetilde{Y}_{l+1,n}^j\big|+\frac{1}{2}\cdot m^{-1}\cdot\sum_{j=1}^{r}\big|\widetilde{c}_{k,j,j}^l\big|\cdot\big|(\widetilde{Y}_{l+1,n}^j)^2-\mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\big|\\ &\quad+m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\Big(\big|\widetilde{c}_{k,j_2,j_1}^l-\widetilde{c}_{k,j_1,j_2}^l\big|\cdot\big|\widetilde{I}_{(j_1,j_2)}^{l+1,n}\big|+\big|\widetilde{c}_{k,j_1,j_2}^l\big|\cdot\big|\widetilde{Y}_{l+1,n}^{j_1}\big|\cdot\big|\widetilde{Y}_{l+1,n}^{j_2}\big|\Big)\\ &\leqslant c\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)\cdot\Big(m^{-1/2}\cdot\sum_{j=1}^{r}\big|\widetilde{Y}_{l+1,n}^j\big|+m^{-1}\cdot\sum_{j=1}^{r}\big|(\widetilde{Y}_{l+1,n}^j)^2-\mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\big|\\ &\qquad+m^{-1}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\big(\big|\widetilde{I}_{(j_1,j_2)}^{i+1,n}\big|+\big|\widetilde{Y}_{l+1,n}^{j_1}\big|\cdot\big|\widetilde{Y}_{l+1,n}^{j_2}\big|\big)\Big)\end{aligned}$$

for all $l=0,\dots,m-1$ and $k=1,\dots,d$. Thus, for $\mu=2,\dots,s$, we obtain

$$\begin{aligned}\big|\widetilde{\xi}_l^k\big|^\mu &\leqslant c\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^\mu\cdot\Big(m^{-\mu/2}\cdot\sum_{j=1}^{r}\big|\widetilde{Y}_{l+1,n}^j\big|^\mu\\ &\qquad+m^{-\mu}\cdot\sum_{j=1}^{r}\big|(\widetilde{Y}_{l+1,n}^j)^2-\mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\big|^\mu\\ &\qquad+m^{-\mu}\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\big(\big|\widetilde{I}_{(j_1,j_2)}^{i+1,n}\big|^\mu+\big|\widetilde{Y}_{l+1,n}^{j_1}\big|^\mu\cdot\big|\widetilde{Y}_{l+1,n}^{j_2}\big|^\mu\big)\Big)\end{aligned}$$

for all $l=0,\dots,m-1$ and $k=1,\dots,d$.

Then, by **(N2)** and **(I2)**, as well as (6.1.5) and (6.1.9), we arrive at

$$\begin{aligned}\sum_{k=1}^{d}\sum_{\mu=2}^{s}\binom{s}{\mu}\cdot\mathbb{E}\Big[\big(\widetilde{V}_l^k\big)^{s-\mu}\cdot\big(\widetilde{\xi}_l^k\big)^{\mu}\Big] &\leqslant c\cdot m^{-1}\cdot\sum_{k=1}^{d}\mathbb{E}\Big[\sum_{\mu=2}^{s}\binom{s}{\mu}\cdot\big|\widetilde{V}_l^k\big|^{s-\mu}\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu}\Big]\\ &\leqslant c\cdot m^{-1}\cdot\sum_{k=1}^{d}\mathbb{E}\Big[\Big(\big|\widetilde{V}_l^k\big|+1+\|\widetilde{X}_{m,n}(t_l)\|_2\Big)^{s}\Big]\\ &\leqslant c\cdot m^{-1}\cdot\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_l^k\big|^{s}+1+\sum_{k=1}^{d}\max_{j=0,\dots,l}\big|\widetilde{X}_{m,n}^k(t_j)\big|^{s}\Big]\\ &\leqslant c\cdot m^{-1}\cdot\left(1+\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_l^k\big|^{s}\Big]\right)\end{aligned} \tag{6.1.12}$$

for all $l=0,\dots,m-1$. Now combining (6.1.10)--(6.1.12) leads to

$$\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_{l+1}^k\big|^{s}\Big]\leqslant\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_l^k\big|^{s}\Big]\cdot\big(1+c\cdot m^{-1}\big)+c\cdot m^{-1}$$

for all $l=0,\dots,m-1$. By applying the discrete version of Gronwall's inequality we finally arrive at

$$\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_m^k\big|^{s}\Big]\leqslant\max_{l=0,\dots,m}\sum_{k=1}^{d}\mathbb{E}\Big[\big|\widetilde{V}_l^k\big|^{s}\Big]\leqslant c.$$

For general $s\in[1,\infty)$ let $\tilde{s}=\min\{n\in2\mathbb{N}\mid n\geqslant s\}$. Then the results obtained above and Hölder's inequality imply

$$\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widetilde{X}_{m,n}(t_l)\|_\infty^{s}\Big]\leqslant\left(\left(\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widetilde{X}_{m,n}^k(t_l)\|_\infty^{\tilde{s}}\Big]\right)^{1/\tilde{s}}\right)^{s}\leqslant c,$$

which finishes the proof. □

In the next main auxiliary lemma we provide an upper bound for the qth mean maximum distance of

$$\big(\widetilde{X}_{m,n}^{(q)}(t_0),\dots,\widetilde{X}_{m,n}^{(q)}(t_m)\big)\quad\text{and}\quad\big(\widehat{X}_m^{\mathrm{M}}(t_0),\dots,\widehat{X}_m^{\mathrm{M}}(t_m)\big).$$

Lemma 6.1.2
Assume that $q\in2\mathbb{N}$. Then, for all $m,n\in\mathbb{N}$,

$$\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{M}}(t_l)-\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^{q}\Big]\leqslant c\cdot\big(m^{-1}+n^{-1}\big)^{q}.$$

Proof:
Let $m, n \in \mathbb{N}$ and, as in the proof of Lemma 6.1.1, we drop the parameter q and write $\widetilde{X}_{m,n}(t_l)$ and $\widetilde{Y}^j_{l,n}$ instead of $\widetilde{X}^{(q)}_{m,n}(t_l)$ and $\widetilde{Y}^{(q),j}_{l,n}$, respectively.

We split up the proof into single steps.

Step 1: Let $k \in \{1, \dots, d\}$. Due to the recursive structure of the Milstein scheme and the Milstein quantization it holds

$$\begin{aligned}
&\widehat{X}^k_m(t_l) - \widetilde{X}^k_{m,n}(t_l) \\
&\overset{P\text{-a.s.}}{=} m^{-1} \cdot \sum_{i=0}^{l-1} \left(\widehat{a}^i_k - \widetilde{a}^i_k \right) + m^{-1/2} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \left(\widehat{b}^i_{k,j} \cdot Y^j_{i+1} - \widetilde{b}^i_{k,j} \cdot \widetilde{Y}^j_{i+1,n} \right) \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{j=1}^{r} \left(\widehat{c}^i_{k,j,j} \cdot \left((Y^j_{i+1})^2 - 1 \right) - \widetilde{c}^i_{k,j,j} \cdot \left((\widetilde{Y}^j_{i+1,n})^2 + \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] - \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] - 1 \right) \right) \\
&\quad + m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{r} \left(\left(\widehat{c}^i_{k,j_2,j_1} - \widehat{c}^i_{k,j_1,j_2} \right) \cdot I^{i+1}_{(j_1,j_2)} - \left(\widetilde{c}^i_{k,j_2,j_1} - \widetilde{c}^i_{k,j_1,j_2} \right) \cdot \widetilde{I}^{i+1,n}_{(j_1,j_2)} \right) \\
&\quad + m^{-1} \cdot \sum_{i=0}^{l-1} \sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{r} \left(\widehat{c}^i_{k,j_1,j_2} \cdot Y^{j_1}_{i+1} \cdot Y^{j_2}_{i+1} - \widetilde{c}^i_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{i+1,n} \cdot \widetilde{Y}^{j_2}_{i+1,n} \right)
\end{aligned}$$

for all $l = 1, \dots, m$.

Similar to the proof of Lemma 6.1.1 we put

$$\vartheta^k_i := m^{-1} \cdot \left(\widehat{a}^i_k - \widetilde{a}^i_k \right) + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}^i_{k,j,j} \cdot \left(\mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] - 1 \right)$$

and

$$\begin{aligned}
\xi^k_i &:= m^{-1/2} \cdot \sum_{j=1}^{r} \left(\widehat{b}^i_{k,j} \cdot Y^j_{i+1} - \widetilde{b}^i_{k,j} \cdot \widetilde{Y}^j_{i+1,n} \right) \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \left(\widehat{c}^i_{k,j,j} \cdot \left((Y^j_{i+1})^2 - 1 \right) - \widetilde{c}^i_{k,j,j} \cdot \left((\widetilde{Y}^j_{i+1,n})^2 - \mathbb{E}\big[(\widetilde{Y}^j_{i+1,n})^2 \big] \right) \right) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{r} \left(\left(\widehat{c}^i_{k,j_2,j_1} - \widehat{c}^i_{k,j_1,j_2} \right) \cdot I^{i+1}_{(j_1,j_2)} - \left(\widetilde{c}^i_{k,j_2,j_1} - \widetilde{c}^i_{k,j_1,j_2} \right) \cdot \widetilde{I}^{i+1,n}_{(j_1,j_2)} \right) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{r} \left(\widehat{c}^i_{k,j_1,j_2} \cdot Y^{j_1}_{i+1} \cdot Y^{j_2}_{i+1} - \widetilde{c}^i_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{i+1,n} \cdot \widetilde{Y}^{j_2}_{i+1,n} \right)
\end{aligned} \tag{6.1.13}$$

for $i = 0, \dots, m-1$.

Furthermore, we put

$$U_l^k := \sum_{i=0}^{l-1} \vartheta_i^k \quad \text{and} \quad V_l^k := \sum_{i=0}^{l-1} \xi_i^k \tag{6.1.14}$$

for $l = 1, \dots, m$ as well as $U_0^k := 0$ and $V_0^k := 0$, and we abbreviate

$$\begin{aligned} U_l^{k,*} &:= \max_{j=0,\dots,l} |U_j^k| \quad , \quad V_l^{k,*} := \max_{j=0,\dots,l} |V_j^k|, \\ \widehat{X}_l^{k,*} &:= \max_{j=0,\dots,l} |\widehat{X}_m^k(t_j)| \quad , \quad \widetilde{X}_l^{k,*} := \max_{j=0,\dots,l} |\widetilde{X}_{m,n}^k(t_j)| \end{aligned} \tag{6.1.15}$$

for $l = 0, \dots, m$. Then it holds

$$\widehat{X}_m^k(t_l) - \widetilde{X}_{m,n}^k(t_l) \overset{P\text{-a.s.}}{=} U_l^k + V_l^k$$

and hence

$$|\widehat{X}_m^k(t_l) - \widetilde{X}_{m,n}^k(t_l)| \overset{P\text{-a.s.}}{\leqslant} |U_l^k| + V_l^{k,*} \tag{6.1.16}$$

for all $l = 1, \dots, m$.

Step 2: In this step we further estimate $|\vartheta_i^k|$. First note that, by Hölder's inequality, by **(N2)** and **(N3)**, and by the finiteness of all absolute moments of a standard normally distributed random variable, we have

$$\begin{aligned} \Big|1 - \mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2\big]\Big| &= \Big|\mathbb{E}\big[(Y_{i+1}^j)^2\big] - \mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2\big]\Big| \\ &\leqslant \mathbb{E}\big[|Y_{i+1}^j - \widetilde{Y}_{i+1,n}^j| \cdot |Y_{i+1}^j + \widetilde{Y}_{i+1,n}^j|\big] \\ &\leqslant \Big(\mathbb{E}\big[|Y_{i+1}^j - \widetilde{Y}_{i+1,n}^j|^2\big]\Big)^{1/2} \cdot \Big(\mathbb{E}\big[|Y_{i+1}^j + \widetilde{Y}_{i+1,n}^j|\big]^2\Big)^{1/2} \\ &\leqslant c \cdot n^{-1} \end{aligned} \tag{6.1.17}$$

for all $j = 1, \dots, r$ and $i = 0, \dots, m-1$. Then, together with the property that a is globally Lipschitz continuous and the fact that b is of at most linear growth, and by assumption **(C2)** (i), we obtain

$$\begin{aligned} \big|\vartheta_i^k\big| &= \Big|m^{-1} \cdot \big(\widehat{a}_k^i - \widetilde{a}_k^i\big) + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \widetilde{c}_{k,j,j}^i \cdot \big(\mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2\big] - 1\big)\Big| \\ &\leqslant c \cdot m^{-1} \cdot \Big(\big|\widehat{a}_k^i - \widetilde{a}_k^i\big| + \sum_{j=1}^{r} \big|\widetilde{c}_{k,j,j}^i\big| \cdot \big|1 - \mathbb{E}\big[(\widetilde{Y}_{i+1,n}^j)^2\big]\big|\Big) \\ &\leqslant c \cdot m^{-1} \cdot \Big(\|\widehat{X}_m(t_i) - \widetilde{X}_{m,n}(t_i)\|_2 + \big(1 + \|\widetilde{X}_{m,n}(t_i)\|_2\big) \cdot n^{-1}\Big) \end{aligned} \tag{6.1.18}$$

for all $i = 0, \dots, m-1$ and $k = 1, \dots, d$. Hence, in view of (6.1.14)--(6.1.16) and (6.1.18), we conclude that

$$\begin{aligned}
\sum_{k=1}^{d} |U_{l+1}^k| &\leqslant \sum_{k=1}^{d} |U_l^k| + \sum_{k=1}^{d} |\vartheta_l^k| \\
&\leqslant \sum_{k=1}^{d} |U_l^k| + c \cdot m^{-1} \cdot \Big(\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \cdot n^{-1} \Big) \\
&\leqslant \sum_{k=1}^{d} |U_l^k| + c \cdot m^{-1} \cdot \left(\sum_{k=1}^{d} |\widehat{X}_{m,n}^k(t_l) - \widetilde{X}_m^k(t_l)| + \Big(1 + \sum_{k=1}^{d} |\widetilde{X}_{m,n}^k(t_l)|\Big) \cdot n^{-1} \right) \\
&\leqslant \sum_{k=1}^{d} |U_l^k| + c \cdot m^{-1} \cdot \left(\sum_{k=1}^{d} |U_l^k| + \sum_{k=1}^{d} V_l^{k,*} + \Big(1 + \sum_{k=1}^{d} \widetilde{X}_l^{k,*}\Big) \cdot n^{-1} \right) \\
&= \left(\sum_{k=1}^{d} |U_l^k| \right) \cdot \big(1 + c \cdot m^{-1}\big) + c \cdot m^{-1} \cdot \left(\sum_{k=1}^{d} V_l^{k,*} + \Big(1 + \sum_{k=1}^{d} \widetilde{X}_l^{k,*}\Big) \cdot n^{-1} \right)
\end{aligned}$$

for all $l = 0, \dots, m-1$. Now by the discrete version of Gronwall's inequality we end up with

$$\max_{j=0,\dots,l} \sum_{k=1}^{d} |U_j^k| \leqslant c \cdot \left(\sum_{k=1}^{d} V_l^{k,*} + \Big(1 + \sum_{k=1}^{d} \widetilde{X}_l^{k,*}\Big) \cdot n^{-1} \right) \tag{6.1.19}$$

for all $l = 0, \dots, m$.

Step 3: Combining (6.1.16) and (6.1.19) leads to

$$\begin{aligned}
\left(\max_{j=1,\dots,l} |\widehat{X}_m^k(t_j) - \widetilde{X}_{m,n}^k(t_j)| \right)^q &\leqslant \left(\max_{j=0,\dots,l} \sum_{k=1}^{d} |U_j^k| + \sum_{k=1}^{d} V_l^{k,*} \right)^q \\
&\leqslant c \cdot \left(\sum_{k=1}^{d} (V_l^{k,*})^q + n^{-q} \cdot \Big(1 + \sum_{k=1}^{d} (\widetilde{X}_m^{k,*})^q\Big) \right),
\end{aligned}$$

for all $l = 1, \dots, m$ and $k = 1, \dots, d$, and hence, together with Lemma 6.1.1, we arrive at

$$\mathbb{E}\Big[\max_{j=1,\dots,l} |\widehat{X}_m^k(t_j) - \widetilde{X}_{m,n}^k(t_j)|^q \Big] \leqslant c \cdot \left(\sum_{k=1}^{d} \mathbb{E}\left[(V_l^{k,*})^q \right] + \big(m^{-1} + n^{-1}\big)^q \right) \tag{6.1.20}$$

for all $l = 1, \dots, m$ and $k = 1, \dots, d$. Thus, in particular,

$$\begin{aligned}
\mathbb{E}\Big[\max_{l=0,\dots,m} \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_\infty^q \Big] &\leqslant \sum_{k=1}^{d} \mathbb{E}\Big[\max_{l=1,\dots,m} |\widehat{X}_m^k(t_l) - \widetilde{X}_{m,n}^k(t_l)|^q \Big] \\
&\leqslant c \cdot \left(\sum_{k=1}^{d} \mathbb{E}\left[(V_m^{k,*})^q \right] + \big(m^{-1} + n^{-1}\big)^q \right),
\end{aligned} \tag{6.1.21}$$

where in the first inequality we used the fact that $\widehat{X}_m^k(t_0) = x_0^k = \widetilde{X}_{m,n}^k(t_0)$ for all $k = 1, \dots, d$.

Therefore, it remains to prove that

$$\sum_{k=1}^{d} \mathbb{E}\left[(V_m^{k,*})^q\right] \leqslant c \cdot \left(m^{-1} + n^{-1}\right)^q.$$

Step 4: Similar to Step 4 in the proof of Lemma 6.1.1 one shows that $(V_l^k)_{l=0,\dots,m}$ is an $(\mathcal{F}(t_l))_{l=0,\dots,m}$-martingale for all $k = 1, \dots, d$. Then Doob's maximal inequality for submartingales yields

$$\mathbb{E}\left[(V_l^{k,*})^q\right] \leqslant c \cdot \mathbb{E}\left[|V_l^k|^q\right] \tag{6.1.22}$$

for all $l = 0, \dots, m$ and $k = 1, \dots, d$. Thus, in view of (6.1.21), it suffices to show

$$\sum_{k=1}^{d} \mathbb{E}\left[|V_m^k|^q\right] \leqslant c \cdot \left(m^{-1} + n^{-1}\right)^q. \tag{6.1.23}$$

Step 5: In this step we prove (6.1.23). First, since $q \in 2\mathbb{N}$, and by (6.1.14), we have

$$\sum_{k=1}^{d} \mathbb{E}\left[|V_{l+1}^k|^q\right] = \sum_{k=1}^{d} \mathbb{E}\left[|V_l^k|^q\right] + \sum_{k=1}^{d} \sum_{\mu=2}^{q} \binom{q}{\mu} \cdot \mathbb{E}\left[(V_l^k)^{q-\mu} \cdot (\xi_l^k)^\mu\right] + q \cdot \sum_{k=1}^{d} \mathbb{E}\left[(V_l^k)^{q-1} \cdot \xi_l^k\right] \tag{6.1.24}$$

for all $l = 0, \dots, m-1$.

We go on by estimating the last summand in the previous equality first. In view of (6.1.13) we have

$$\begin{aligned}
\left(V_l^k\right)^{q-1} \cdot \xi_l^k &= m^{-1/2} \cdot \sum_{j=1}^{r} \left(V_l^k\right)^{q-1} \cdot \left(\widehat{b}_{k,j}^l \cdot Y_{l+1}^j - \widetilde{b}_{k,j}^l \cdot \widetilde{Y}_{l+1,n}^j\right) \\
&\quad + \frac{1}{2} \cdot m^{-1} \cdot \sum_{j=1}^{r} \left(V_l^k\right)^{q-1} \cdot \left(\widehat{c}_{k,j,j}^l \cdot \left((Y_{l+1}^j)^2 - 1\right) - \widetilde{c}_{k,j,j}^l \cdot \left((\widetilde{Y}_{l+1,n}^j)^2 - \mathbb{E}\big[(\widetilde{Y}_{l+1,n}^j)^2\big]\right)\right) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \left(V_l^k\right)^{q-1} \cdot \left(\left(\widehat{c}_{k,j_2,j_1}^l - \widehat{c}_{k,j_1,j_2}^l\right) \cdot I_{(j_1,j_2)}^{l+1} - \left(\widetilde{c}_{k,j_2,j_1}^l - \widetilde{c}_{k,j_1,j_2}^l\right) \cdot \widetilde{I}_{(j_1,j_2)}^{l+1,n}\right) \\
&\quad + m^{-1} \cdot \sum_{\substack{j_1,j_2=1 \\ j_1<j_2}}^{r} \left(V_l^k\right)^{q-1} \cdot \left(\widehat{c}_{k,j_1,j_2}^l \cdot Y_{l+1}^{j_1} \cdot Y_{l+1}^{j_2} - \widetilde{c}_{k,j_1,j_2}^l \cdot \widetilde{Y}_{l+1,n}^{j_1} \cdot \widetilde{Y}_{l+1,n}^{j_2}\right)
\end{aligned}$$

for all $l = 0, \dots, m-1$ and $k = 1, \dots, d$. Then, by employing similar arguments as in (6.1.11) we arrive at

$$\sum_{k=1}^{d} \mathbb{E}\left[\left(V_l^k\right)^{q-1} \cdot \xi_l^k\right] = 0. \tag{6.1.25}$$

Next, we further estimate the second summand in (6.1.24). To this end, we fix $\mu \in \{2, \dots, q\}$, $l \in \{0, \dots, m-1\}$ and $k \in \{1, \dots, d\}$ for the moment. Then,

$$|\xi_l^k|^\mu \leqslant c \cdot \left(m^{-\mu/2} \cdot \alpha_{1,k,l}^\mu + m^{-\mu} \cdot \alpha_{2,k,l}^\mu + m^{-\mu} \cdot \alpha_{3,k,l}^\mu\right)$$

where

$$\alpha_{1,k,l} := \sum_{j=1}^{r} \big|\widehat{b}^l_{k,j} \cdot Y^j_{l+1} - \widetilde{b}^l_{k,j} \cdot \widetilde{Y}^j_{l+1,n}\big|,$$

$$\alpha_{2,k,l} := \sum_{j=1}^{r} \big|\widehat{c}^l_{k,j,j} \cdot \big((Y^j_{l+1})^2 - 1\big) - \widetilde{c}^l_{k,j,j} \cdot \big((\widetilde{Y}^j_{l+1,n})^2 - \mathbb{E}\big[(\widetilde{Y}^j_{l+1,n})^2\big]\big)\big|,$$

$$\begin{aligned}\alpha_{3,k,l} := \sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r} \Big(&\big|\big(\widehat{c}^l_{k,j_2,j_1} - \widehat{c}^l_{k,j_1,j_2}\big) \cdot I^{l+1}_{(j_1,j_2)} - \big(\widetilde{c}^l_{k,j_2,j_1} - \widetilde{c}^l_{k,j_1,j_2}\big) \cdot \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big| \\ &+ \big|\widehat{c}^l_{k,j_1,j_2} \cdot Y^{j_1}_{l+1} \cdot Y^{j_2}_{l+1} - \widetilde{c}^l_{k,j_1,j_2} \cdot \widetilde{Y}^{j_1}_{l+1,n} \cdot \widetilde{Y}^{j_2}_{l+1,n}\big|\Big).\end{aligned}$$

In the following we further estimate the above three terms separately. First, since b is globally Lipschitz continuous and of at most linear growth we have

$$\begin{aligned}&\big|\widehat{b}^l_{k,j} \cdot Y^j_{l+1} - \widetilde{b}^l_{k,j} \cdot \widetilde{Y}^j_{l+1,n}\big| \\ &= \big|\widehat{b}^l_{k,j} \cdot Y^j_{l+1} - \widetilde{b}^l_{k,j} \cdot Y^j_{l+1} + \widetilde{b}^l_{k,j} \cdot Y^j_{l+1} - \widetilde{b}^l_{k,j} \cdot \widetilde{Y}^j_{l+1,n}\big| \\ &\leqslant |Y^j_{l+1}| \cdot |\widehat{b}^l_{k,j} - \widetilde{b}^l_{k,j}| + |\widetilde{b}^l_{k,j}| \cdot |Y^j_{l+1} - \widetilde{Y}^j_{l+1,n}| \\ &\leqslant c \cdot \Big(|Y^j_{l+1}| \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \cdot |Y^j_{l+1} - \widetilde{Y}^j_{l+1,n}|\Big)\end{aligned}$$

for all $j = 1, \dots, r$, and hence

$$\alpha^\mu_{1,k,l} \leqslant c \cdot \sum_{j=1}^{r} \Big(|Y^j_{l+1}|^\mu \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^\mu + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^\mu \cdot |Y^j_{l+1} - \widetilde{Y}^j_{l+1,n}|^\mu\Big). \tag{6.1.26}$$

Secondly, by the fact that b is globally Lipschitz continuous and of at most linear growth, by **(C2)**, and by (6.1.17), we have

$$\begin{aligned}&\big|\widehat{c}^l_{k,j,j} \cdot \big((Y^j_{l+1})^2 - 1\big) - \widetilde{c}^l_{k,j,j} \cdot \big((\widetilde{Y}^j_{l+1,n})^2 - \mathbb{E}\big[(\widetilde{Y}^j_{l+1,n})^2\big]\big)\big| \\ &\leqslant \big|(Y^j_{l+1})^2 - 1\big| \cdot \big|\widehat{c}^l_{k,j,j} - \widetilde{c}^l_{k,j,j}\big| + \big|\widehat{c}^l_{k,j,j}\big| \cdot \big|(Y^j_{l+1})^2 - (\widetilde{Y}^j_{l+1,n})^2 - \big(1 - \mathbb{E}\big[(\widetilde{Y}^j_{l+1,n})^2\big]\big)\big| \\ &\leqslant c \cdot \Big(\big|(Y^j_{l+1})^2 - 1\big| \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \\ &\qquad + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \cdot \big(\big|(Y^j_{l+1})^2 - (\widetilde{Y}^j_{l+1,n})^2\big| + \big|1 - \mathbb{E}\big[(\widetilde{Y}^j_{l+1,n})^2\big]\big|\big)\Big) \\ &\leqslant c \cdot \Big(\big|(Y^j_{l+1})^2 - 1\big| \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \\ &\qquad + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \cdot \big(\big|(Y^j_{l+1})^2 - (\widetilde{Y}^j_{l+1,n})^2\big| + n^{-1}\big)\Big)\end{aligned}$$

for all $j = 1, \dots, r$. Therefore,

$$\begin{aligned}\alpha^\mu_{2,k,l} \leqslant c \cdot \sum_{j=1}^{r} \Big(&\big|(Y^j_{l+1})^2 - 1\big|^\mu \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^\mu \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^\mu \\ &+ \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^\mu \cdot \big(\big|(Y^j_{l+1})^2 - (\widetilde{Y}^j_{l+1,n})^2\big|^\mu + n^{-\mu}\big)\Big).\end{aligned} \tag{6.1.27}$$

Thirdly, the properties of b imply

$$\begin{aligned}
&\big|\big(\widehat{c}^{\,l}_{k,j_2,j_1} - \widehat{c}^{\,l}_{k,j_1,j_2}\big)\cdot I^{l+1}_{(j_1,j_2)} - \big(\widetilde{c}^{\,l}_{k,j_2,j_1} - \widetilde{c}^{\,l}_{k,j_1,j_2}\big)\cdot \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big| \\
&\leqslant \big|\big(\widehat{c}^{\,l}_{k,j_2,j_1} - \widehat{c}^{\,l}_{k,j_1,j_2}\big) - \big(\widetilde{c}^{\,l}_{k,j_2,j_1} - \widetilde{c}^{\,l}_{k,j_1,j_2}\big)\big|\cdot\big|I^{l+1}_{(j_1,j_2)}\big| + \big|\widetilde{c}^{\,l}_{k,j_2,j_1} - \widetilde{c}^{\,l}_{k,j_1,j_2}\big|\cdot\big|I^{l+1}_{(j_1,j_2)} - \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big| \\
&\leqslant \big|\widehat{c}^{\,l}_{k,j_2,j_1} - \widetilde{c}^{\,l}_{k,j_2,j_1}\big|\cdot\big|I^{l+1}_{(j_1,j_2)}\big| + \big|\widehat{c}^{\,l}_{k,j_1,j_2} - \widetilde{c}^{\,l}_{k,j_1,j_2}\big|\cdot\big|I^{l+1}_{(j_1,j_2)}\big| + \big(\big|\widetilde{c}^{\,l}_{k,j_2,j_1}\big| + \big|\widetilde{c}^{\,l}_{k,j_1,j_2}\big|\big)\cdot\big|I^{l+1}_{(j_1,j_2)} - \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big| \\
&\leqslant c\cdot\Big(\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)\cdot\big|I^{l+1}_{(j_1,j_2)}\big| + \big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)\cdot\big|I^{l+1}_{(j_1,j_2)} - \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big|\Big)
\end{aligned}$$

and

$$\begin{aligned}
&|\widehat{c}^{\,l}_{k,j_1,j_2}\cdot Y^{j_1}_{l+1}\cdot Y^{j_2}_{l+1} - \widetilde{c}^{\,l}_{k,j_1,j_2}\cdot\widetilde{Y}^{j_1}_{l+1,n}\cdot\widetilde{Y}^{j_2}_{l+1,n}| \\
&\leqslant |\widehat{c}^{\,l}_{k,j_1,j_2} - \widetilde{c}^{\,l}_{k,j_1,j_2}|\cdot|Y^{j_1}_{l+1}|\cdot|Y^{j_2}_{l+1}| + |\widetilde{c}^{\,l}_{k,j_1,j_2}|\cdot|Y^{j_1}_{l+1}\cdot Y^{j_2}_{l+1} - \widetilde{Y}^{j_1}_{l+1,n}\cdot\widetilde{Y}^{j_2}_{l+1,n}| \\
&\leqslant |\widehat{c}^{\,l}_{k,j_1,j_2} - \widetilde{c}^{\,l}_{k,j_1,j_2}|\cdot|Y^{j_1}_{l+1}|\cdot|Y^{j_2}_{l+1}| + |\widetilde{c}^{\,l}_{k,j_1,j_2}|\cdot|Y^{j_1}_{l+1}|\cdot|Y^{j_2}_{l+1} - \widetilde{Y}^{j_2}_{l+1,n}| + |\widetilde{c}^{\,l}_{k,j_1,j_2}|\cdot|\widetilde{Y}^{j_2}_{l+1,n}|\cdot|Y^{j_1}_{l+1} - \widetilde{Y}^{j_1}_{l+1,n}| \\
&\leqslant c\cdot\Big(\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)\cdot|Y^{j_1}_{l+1}|\cdot|Y^{j_2}_{l+1}| \\
&\qquad + \big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)\cdot\big(|Y^{j_1}_{l+1}|\cdot|Y^{j_2}_{l+1} - \widetilde{Y}^{j_2}_{l+1,n}| + |\widetilde{Y}^{j_2}_{l+1,n}|\cdot|Y^{j_1}_{l+1} - \widetilde{Y}^{j_1}_{l+1,n}|\big)\Big)
\end{aligned}$$

for all $j_1, j_2 = 1,\dots,r$ with $j_1 < j_2$. Thus,

$$\begin{aligned}
\alpha^{\mu}_{3,k,l} \leqslant\ & c\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu}\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu}\cdot\big|I^{l+1}_{(j_1,j_2)}\big|^{\mu} \\
&+ c\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu}\cdot\big|I^{l+1}_{(j_1,j_2)} - \widetilde{I}^{l+1,n}_{(j_1,j_2)}\big|^{\mu} \\
&+ c\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu}\cdot\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu}\cdot\big|Y^{j_1}_{l+1}\big|^{\mu}\cdot\big|Y^{j_2}_{l+1}\big|^{\mu} \\
&+ c\cdot\sum_{\substack{j_1,j_2=1\\ j_1<j_2}}^{r}\big(1+\|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu}\cdot\big(|Y^{j_1}_{l+1}|^{\mu}\cdot|Y^{j_2}_{l+1} - \widetilde{Y}^{j_2}_{l+1,n}|^{\mu} + |\widetilde{Y}^{j_2}_{l+1,n}|^{\mu}\cdot|Y^{j_1}_{l+1} - \widetilde{Y}^{j_1}_{l+1,n}|^{\mu}\big).
\end{aligned} \tag{6.1.28}$$

Additionally, due to Hölder's inequality, due to **(N2)** and **(N3)**, and due to the properties of standard normally distributed random variables, we obtain

$$\begin{aligned}
\mathbb{E}\Big[\big|(Y^j_{l+1})^2 - (\widetilde{Y}^j_{l+1,n})^2\big|^{\mu}\Big] &\leqslant \mathbb{E}\Big[\big|Y^j_{l+1} - \widetilde{Y}^j_{l+1,n}\big|^{\mu-1}\cdot\big(|Y^j_{l+1}| + |\widetilde{Y}^j_{l+1,n}|\big)^{\mu+1}\Big] \\
&\leqslant c\cdot\Big(\mathbb{E}\Big[\big|Y^j_{l+1} - \widetilde{Y}^j_{l+1,n}\big|^{\mu}\Big]\Big)^{(\mu-1)/\mu}\cdot\Big(\mathbb{E}\Big[\big(|Y^j_{l+1}| + |\widetilde{Y}^j_{l+1,n}|\big)^{\mu\cdot(\mu+1)}\Big]\Big)^{1/\mu} \\
&\leqslant c\cdot n^{-(\mu-1)}
\end{aligned}$$

for all $j = 1,\dots,r$.

Now by combining the previous inequality with (6.1.26)--(6.1.28), by applying **(N2)**, **(N3)** and **(I3)** as well as Lemma 2.3.6 b), by using the fact that all absolute moments of a standard normally distributed random variable are finite, and by utilizing the fact that

$$\begin{aligned} n^{-\mu} \cdot m^{-\mu/2} + n^{-(\mu-1)} \cdot m^{-\mu} &\leqslant m^{-\mu/2} \cdot \left(n^{-\mu} + n^{-(\mu-1)} \cdot m^{-\mu/2}\right) \\ &\leqslant m^{-\mu/2} \cdot \left(n^{-\mu} + n^{-(\mu-1)} \cdot m^{-1}\right) \\ &\leqslant m^{-1} \cdot \left(m^{-1} + n^{-1}\right)^{\mu} \end{aligned} \tag{6.1.29}$$

for all $\mu = 2. \ldots, q$, we derive that

$$\begin{aligned} &\mathbb{E}\left[|V_l^k|^{q-\mu} \cdot |\xi_l^k|^{\mu}\right] \\ &\leqslant c \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(m^{-\mu/2} \cdot \alpha_{1,k,l}^{\mu} + m^{-\mu} \cdot \alpha_{2,k,l}^{\mu} + m^{-\mu} \cdot \alpha_{3,k,l}^{\mu}\right)\right] \\ &\leqslant c \cdot \Big(m^{-\mu/2} \cdot \Big(\mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu}\right] + n^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right]\Big) \\ &\qquad + m^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot \left(n^{-(\mu-1)} + n^{-\mu}\right) \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot n^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right]\Big) \\ &\leqslant c \cdot \Big(m^{-\mu/2} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu}\right] + m^{-\mu/2} \cdot n^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot n^{-(\mu-1)} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right]\Big) \\ &\leqslant c \cdot \Big(m^{-\mu/2} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu}\right] \\ &\qquad + m^{-\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right] \\ &\qquad + m^{-1} \cdot \left(m^{-1} + n^{-1}\right)^{\mu} \cdot \mathbb{E}\left[|V_l^k|^{q-\mu} \cdot \left(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\right)^{\mu}\right]\Big) \end{aligned}$$

for all $\mu = 2, \ldots, q$, $l = 0, \ldots, m-1$ and $k = 1, \ldots, d$.

Thus, together with Hölder's inequality, Proposition 3.3.2, and Lemma 6.1.1 we arrive at

$$
\begin{aligned}
&\sum_{\mu=2}^{q} \binom{q}{\mu} \cdot \mathbb{E}\Big[|V_l^k|^{q-\mu} \cdot |\xi_l^k|^{\mu}\Big] \\
&\leqslant c \cdot \Big(m^{-1} \cdot \mathbb{E}\Big[\sum_{\mu=0}^{q} \tbinom{q}{\mu} \cdot |V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \Big] \\
&\qquad + m^{-2} \cdot \mathbb{E}\Big[\sum_{\mu=0}^{q} \tbinom{q}{\mu} \cdot |V_l^k|^{q-\mu} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{\mu} \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu} \Big] \\
&\qquad + m^{-1} \cdot \mathbb{E}\Big[\sum_{\mu=0}^{q} \tbinom{q}{\mu} \cdot |V_l^k|^{q-\mu} \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^{\mu} \cdot \big(m^{-1} + n^{-1}\big)^{-\mu} \Big] \Big) \\
&= c \cdot \Big(m^{-1} \cdot \mathbb{E}\Big[\Big(|V_l^k| + \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 \Big)^q \Big] \\
&\qquad + m^{-2} \cdot \mathbb{E}\Big[\Big(|V_l^k| + \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \Big)^q \Big] \\
&\qquad + m^{-1} \cdot \mathbb{E}\Big[\Big(|V_l^k| + \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big) \cdot \big(m^{-1} + n^{-1}\big) \Big)^q \Big] \Big) \\
&\leqslant c \cdot \Big(m^{-1} \cdot \mathbb{E}\Big[|V_l^k|^q \Big] + m^{-1} \cdot \mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \\
&\qquad + m^{-2} \cdot \mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^{q-1} \cdot \|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2 \cdot \big(1 + \|\widetilde{X}_{m,n}(t_l)\|_2\big)^q \Big] \\
&\qquad + m^{-1} \cdot \big(n^{-1} + m^{-1}\big)^q \Big) \\
&\leqslant c \cdot \Big(m^{-1} \cdot \mathbb{E}\Big[|V_l^k|^q \Big] + m^{-1} \cdot \mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \\
&\qquad + m^{-2} \cdot \Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \Big)^{(q-1)/q} + m^{-1} \cdot \big(n^{-1} + m^{-1}\big)^q \Big) \\
&\leqslant c \cdot m^{-1} \cdot \Big(\mathbb{E}\Big[|V_l^k|^q \Big] + \mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] + \big(n^{-1} + m^{-1}\big)^q \Big)
\end{aligned}
\tag{6.1.30}
$$

for all $l = 0, \ldots, m-1$ and $k = 1, \ldots, d$, where in the last inequality we employed an analysis similar to the one in (6.1.29), namely,

$$
\begin{aligned}
&m^{-1} \cdot \mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] + m^{-2} \cdot \Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \Big)^{(q-1)/q} \\
&= m^{-1} \cdot \Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] + m^{-1} \cdot \Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \Big)^{(q-1)/q} \Big) \\
&\leqslant m^{-1} \cdot \Big(\Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] \Big)^{1/q} + m^{-1} \Big)^q \\
&\leqslant c \cdot m^{-1} \cdot \Big(\mathbb{E}\Big[\|\widehat{X}_m(t_l) - \widetilde{X}_{m,n}(t_l)\|_2^q \Big] + \big(m^{-1} + n^{-1}\big)^q \Big).
\end{aligned}
$$

Finally, by (6.1.24), (6.1.25), (6.1.30), and by (6.1.20) as well as by (6.1.22), we have

$$\begin{aligned}\sum_{k=1}^{d}\mathbb{E}\left[|V_{l+1}^k|^q\right] &= \sum_{k=1}^{d}\mathbb{E}\left[|V_l^k|^q\right] + \sum_{k=1}^{d}\sum_{\mu=2}^{q}\binom{q}{\mu}\cdot\mathbb{E}\left[(V_l^k)^{q-\mu}\cdot(\xi_l^k)^\mu\right]\\ &\leqslant \sum_{k=1}^{d}\mathbb{E}\left[|V_l^k|^q\right] + c\cdot m^{-1}\cdot\left(\sum_{k=1}^{d}\mathbb{E}\left[\left|V_l^k\right|^q\right] + \mathbb{E}\left[\|\widehat{X}_m(t_l)-\widetilde{X}_{m,n}(t_l)\|_2^q\right] + \left(n^{-1}+m^{-1}\right)^q\right)\\ &\leqslant \sum_{k=1}^{d}\mathbb{E}\left[|V_l^k|^q\right] + c\cdot m^{-1}\cdot\left(\sum_{k=1}^{d}\mathbb{E}\left[\left|V_l^k\right|^q\right] + \left(n^{-1}+m^{-1}\right)^q\right)\\ &= \left(\sum_{k=1}^{d}\mathbb{E}\left[|V_l^k|^q\right]\right)\cdot\left(1+c\cdot m^{-1}\right) + c\cdot m^{-1}\cdot\left(n^{-1}+m^{-1}\right)^q\end{aligned}$$

for all $l=0,\dots,m-1$. Now, by applying the discrete version of Gronwall's inequality, we end up with

$$\sum_{k=1}^{d}\mathbb{E}\left[|V_m^k|^q\right] \leqslant c\cdot\left(n^{-1}+m^{-1}\right)^q,$$

which finishes the proof.

□

6.1.2. Proof of Proposition 5.1.1

To prove Proposition 5.1.1 we employ Proposition 3.3.2 and Lemma 6.1.2.

Proof of Proposition 5.1.1:
Let $m,n\in\mathbb{N}$ and, for the moment, assume $q\in 2\mathbb{N}$. Then Proposition 3.3.2 and Lemma 6.1.2 yield

$$\begin{aligned}\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^q\Big] &\leqslant c\cdot\left(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X(t)-X_m^{\mathrm{M}}(t)\|_\infty^q\Big] + \mathbb{E}\Big[\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{M}}(t_l)-\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^q\Big]\right)\\ &\leqslant c\cdot\left(m^{-q}+\left(m^{-1}+n^{-1}\right)^q\right)\\ &\leqslant c\cdot\left(m^{-1}+n^{-1}\right)^q.\end{aligned} \tag{6.1.31}$$

For arbitrary $q\in[1,\infty)$ we choose $\tilde{q}=\min\{n\in 2\mathbb{N}\mid n\geqslant q\}$. Then Hölder's inequality and (6.1.31) yield

$$\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^q\Big] \leqslant \left(\mathbb{E}\Big[\max_{l=0,\dots,m}\|X(t_l)-\widetilde{X}_{m,n}^{(q)}(t_l)\|_\infty^{\tilde{q}}\Big]\right)^{q/\tilde{q}} \leqslant c\cdot(m^{-1}+n^{-1})^q.$$

□

6.2. Quantization in $L_p([0,1];\mathbb{R}^d)$: Proof of Theorem 5.2.4

In the whole section we consider fixed parameters $p, q \in [1,\infty)$ with $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant p\}$.

6.2.1. Auxiliary Lemma 3: An Auxiliary Process

For $m \in \mathbb{N}$ we define an $\mathbb{R}^d$-valued stochastic process $\overline{X}_m = \big(\overline{X}_m^1, \ldots, \overline{X}_m^d\big)'$ on $[0,1]$ in terms of the time-discrete Milstein scheme and the Brownian bridges B_l^j by

$$\overline{X}_m^k(t) = (t - t_{l-1}) \cdot m \cdot \widehat{X}_m^{\mathrm{M},k}(t_l) + (t_l - t) \cdot m \cdot \widehat{X}_m^{\mathrm{M},k}(t_{l-1}) + \sum_{j=1}^{r} b_{k,j}\big(\widehat{X}_m^{\mathrm{M}}(t_{l-1})\big) \cdot B_l^j(t)$$

for $t \in [t_{l-1}, t_l]$, $l = 1, \ldots, m$ and $k = 1, \ldots, d$.

Lemma 6.2.1
Let $s \in [1,\infty)$. Then, for all $m \in \mathbb{N}$,

$$\sup_{t\in[0,1]} \mathbb{E}\big[\|X(t) - \overline{X}_m(t)\|_\infty^s\big] \leqslant c \cdot m^{-s}.$$

Proof:
Let $m \in \mathbb{N}$. Recall that $X_m^{\mathrm{M}} = (X_m^{\mathrm{M},1}, \ldots, X_m^{\mathrm{M},d})'$ denotes the d-dimensional Milstein process on $[0,1]$ as given in (3.3.3). For ease of notation, we drop the parameter m throughout the proof and write $\overline{X}$ and X^{M} instead of $\overline{X}_m$ and X_m^{M}, respectively. W.l.o.g. we may assume $s \geqslant 2$.

First note that

$$\sup_{t\in[0,1]} \mathbb{E}\Big[\|X(t) - \overline{X}(t)\|_\infty^s\Big] \leqslant \sum_{k=1}^{d} \sup_{t\in[0,1]} \mathbb{E}\Big[|X^k(t) - \overline{X}^k(t)|^s\Big]. \tag{6.2.1}$$

Furthermore, by Proposition 3.3.2, and by the fact that $\overline{X}^k(0) = x_0^k = X^{\mathrm{M},k}(0)$, we obtain

$$\begin{aligned}
\sup_{t\in[0,1]} \mathbb{E}\left[|X^k(t) - \overline{X}^k(t)|^s\right] &= \sup_{t\in[0,1]} \mathbb{E}\left[|X^k(t) - X^{\mathrm{M},k}(t) + X^{\mathrm{M},k}(t) - \overline{X}^k(t)|^s\right] \\
&\leqslant c \cdot \left(\sup_{t\in[0,1]} \mathbb{E}\left[|X^k(t) - X^{\mathrm{M},k}(t)|^s\right] + \sup_{t\in[0,1]} \mathbb{E}\left[|X^{\mathrm{M},k}(t) - \overline{X}^k(t)|^s\right]\right) \qquad (6.2.2) \\
&\leqslant c \cdot \left(m^{-s} + \sup_{t\in(0,1]} \mathbb{E}\left[|X^{\mathrm{M},k}(t) - \overline{X}^k(t)|^s\right]\right)
\end{aligned}$$

for all $k = 1, \ldots, d$.

Next, we fix $k \in \{1, \ldots, d\}$, and for $t \in (0,1]$ we put

$$R^k(t) := \sum_{l=1}^{m} \left(\sum_{j_1,j_2=1}^{r} \widehat{c}_{k,j_2,j_1}^{\,l-1} \cdot \int_{t_{l-1}}^{t} \big(W_{j_1}(u) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(u)\right) \cdot \mathbb{1}_{(t_{l-1},t_l]}(t).$$

Now, for the moment, we fix $t \in (0,1]$. Then there exists an $l \in \{1, \ldots, m\}$ such that $t \in (t_{l-1}, t_l]$. By the definition of the time-discrete Milstein scheme, see (3.3.2), and by the definition of the processes B_l^j, see

(5.2.1), we have

$$\begin{aligned}
\overline{X}^k(t) &= (t-t_{l-1})\cdot m\cdot \widehat{X}^{\mathrm{M},k}(t_l) + (t_l - t)\cdot m\cdot \widehat{X}^{\mathrm{M},k}(t_{l-1}) + \sum_{j=1}^r \widehat{b}_{k,j}^{l-1}\cdot B_l^j(t)\\
&= \widehat{X}^{\mathrm{M},k}(t_{l-1}) + (t-t_{l-1})\cdot \widehat{a}_k^{l-1} + (t-t_{l-1})\cdot m\cdot \sum_{j=1}^r \widehat{b}_{k,j}^{l-1}\cdot \big(W_j(t_l) - W_j(t_{l-1})\big)\\
&\quad + (t-t_{l-1})\cdot m\cdot \sum_{j_1,j_2=1}^r \widehat{c}_{k,j_2,j_1}^{l-1}\cdot J_{(j_1,j_2)}^l\\
&\quad + \sum_{j=1}^r \widehat{b}_{k,j}^{l-1}\cdot \big(W_j(t) - (t_l-t)\cdot m\cdot W_j(t_{l-1}) - (t-t_{l-1})\cdot m\cdot W_j(t_l)\big)\\
&= \widehat{X}^{\mathrm{M},k}(t_{l-1}) + (t-t_{l-1})\cdot \widehat{a}_k^{l-1} + \sum_{j=1}^r \widehat{b}_{k,j}^{l-1}\cdot \big(W_j(t) - W_j(t_{l-1})\big)\\
&\quad + (t-t_{l-1})\cdot m\cdot \sum_{j_1,j_2=1}^r \widehat{c}_{k,j_2,j_1}^{l-1}\cdot J_{(j_1,j_2)}^l.
\end{aligned}$$

Hence

$$\overline{X}^k(t) = \widehat{X}^{\mathrm{M},k}(t_{l-1}) + (t-t_{l-1})\cdot \widehat{a}_k^{l-1} + \sum_{j=1}^r \widehat{b}_{k,j}^{l-1}\cdot \big(W_j(t) - W_j(t_{l-1})\big) + (t-t_{l-1})\cdot m\cdot R^k(t_l),$$

and thus, by the definition of the Milstein process X^{M}, we arrive at

$$X^{\mathrm{M},k}(t) - \overline{X}^k(t) = R^k(t) - (t-t_{l-1})\cdot m\cdot R^k(t_l). \tag{6.2.3}$$

As a next step, we further estimate $\mathbb{E}\big[|R^k(t)|^s\big]$. Since b is of at most linear growth, and due to **(C2)** (i), we have

$$\begin{aligned}
\big|R^k(t)\big|^s &\leqslant c\cdot \sum_{j_1,j_2=1}^r \big|\widehat{c}_{k,j_2,j_1}^{l-1}\big|^s \cdot \Big|\int_{t_{l-1}}^t \big(W_{j_1}(u) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(u)\Big|^s\\
&\leqslant c\cdot \big(1 + \|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^s\big)\cdot \sum_{j_1,j_2=1}^r \Big|\int_{t_{l-1}}^t \big(W_{j_1}(u) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(u)\Big|^s.
\end{aligned}$$

Together with the fact that $\widehat{X}^{\mathrm{M}}(t_{l-1})$ is $\mathcal{F}(t_{l-1})$-$\mathcal{B}(\mathbb{R}^d)$-measurable, Lemma 2.3.6 d), and Proposition 3.3.2, we conclude that

$$\begin{aligned}
\mathbb{E}\Big[\big|R^k(t)\big|^s\Big] &\leqslant c\cdot \Big(1 + \mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}(t_{l-1})\|_\infty^s\Big]\Big)\cdot \sum_{j_1,j_2=1}^r \mathbb{E}\Big[\Big|\int_{t_{l-1}}^t \big(W_{j_1}(u) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(u)\Big|^s\Big]\\
&\leqslant c\cdot \sum_{j_1,j_2=1}^r \mathbb{E}\Big[\Big|\int_{t_{l-1}}^t \big(W_{j_1}(u) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(u)\Big|^s\Big].
\end{aligned} \tag{6.2.4}$$

Moreover, the Burkholder-Davis-Gundy inequality, Minkowski's integral inequality, see Proposition A.3,

and the properties of W imply

$$\begin{aligned}
\mathbb{E}\left[\Big|\int_{t_{l-1}}^{t}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^s\right] &\leqslant \mathbb{E}\left[\sup_{\tilde{s}\in[t_{l-1},t]}\Big|\int_{t_{l-1}}^{\tilde{s}}\big(W_{j_1}(u)-W_{j_1}(t_{l-1})\big)\,dW_{j_2}(u)\Big|^s\right]\\
&\leqslant c\cdot\mathbb{E}\left[\Big|\int_{t_{l-1}}^{t}\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big|^2\,du\Big|^{s/2}\right]\\
&= c\cdot\left[\left(\mathbb{E}\left[\Big|\int_{t_{l-1}}^{t}\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big|^2\,du\Big|^{s/2}\right]\right)^{2/s}\right]^{s/2}\\
&\leqslant c\cdot\left[\int_{t_{l-1}}^{t}\left(\mathbb{E}\left[\big|W_{j_1}(u)-W_{j_1}(t_{l-1})\big|^s\right]\right)^{2/s}du\right]^{s/2}\\
&\leqslant c\cdot\left[\int_{t_{l-1}}^{t}\left(m^{-s/2}\right)^{2/s}du\right]^{s/2}\\
&\leqslant c\cdot\left(m^{-2}\right)^{s/2}\\
&= c\cdot m^{-s}
\end{aligned}\tag{6.2.5}$$

for all $j_1, j_2 = 1,\dots,r$.

Now by combining (6.2.1), (6.2.2), as well as (6.2.3)--(6.2.5), we finally arrive at

$$\sup_{t\in[0,1]}\mathbb{E}\Big[\|X(t)-\overline{X}(t)\|_\infty^s\Big]\leqslant c\cdot\left(m^{-s}+\sum_{k=1}^{d}\sup_{t\in(0,1]}\mathbb{E}\big[|R^k(t)|^s\big]\right)\leqslant c\cdot m^{-s}.$$

□

6.2.2. Auxiliary Lemmas 4 and 5

Recall that, for $K\in\mathbb{N}$,

$$\delta_K = e^{(p)}\big(\overline{B}, S_K(\overline{B}), L_p[0,1]\big)$$

where $\overline{B}$ denotes a Brownian bridge on $[0,1]$ and S_K is a K-quantizer for a Brownian bridge on $[0,1]$.

Furthermore, for $m\in\mathbb{N}_2$, $n, M\in\mathbb{N}$ and $k=1,\dots,d$ we put

$$A^k_{m,n,M} := m^{-(p+2)/2}\cdot\sum_{l=1}^{m}\big(\tilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n}\big)^p\cdot(\delta_{K_{k,l,M}})^p$$

where, recall,

$$\tilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n} = \Big(\sum_{j=1}^{r}\big(b_{k,j}\big(\tilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big)\big)^2\Big)^{1/2},$$

and where the random variables $K_{k,l,M}$ are defined as in (5.2.9). Additionally, we put

$$A_{m,n,M} := \sum_{k=1}^{d} A^k_{m,n,M}.$$

For ease of notation, we write $\widetilde{s}^{k,l-1}$ instead of $\widetilde{s}^{\mathrm{E},k,l-1}_{\mathrm{locH},m,n}$ in the proofs of the following lemmas.

Lemma 6.2.2
For all $m \in \mathbb{N}_2$ and all $n, M \in \mathbb{N}$ it holds

$$\Big(\mathbb{E}\Big[\|\overline{X}_m - \widetilde{X}^{(p,q)}_{m,n,M}\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p} \leqslant c \cdot \big(m^{-1} + n^{-1}\big) + \big(\mathbb{E}[A_{m,n,M}]\big)^{1/p}.$$

Proof:
Let $m \in \mathbb{N}_2$, $n, M \in \mathbb{N}$ and, in addition to the abbreviations already introduced at the beginning of this chapter, we put

$$\widetilde{b}^{\mathrm{E},l-1}_k := \big(\widetilde{b}^{\mathrm{E},l-1}_{k,1}, \ldots, \widetilde{b}^{\mathrm{E},l-1}_{k,r}\big)$$

for $l = 1, \ldots, m$ and $k = 1, \ldots, d$.

We split up the proof into single steps.

Step 1: First, recall that

$$\overline{X}^k_m(t) = (t - t_{l-1}) \cdot m \cdot \widehat{X}^{\mathrm{M},k}_m(t_l) + (t_l - t) \cdot m \cdot \widehat{X}^{\mathrm{M},k}_m(t_{l-1}) + \sum_{j=1}^{r} \widehat{b}^{l-1}_{k,j} \cdot B^j_l(t)$$

and

$$\widetilde{X}^{(p,q),k}_{m,n,M}(t) = \widetilde{X}^{(q),k}_{m,n}(t) + \Phi_{\widetilde{b}^{\mathrm{E},l-1}_k, l, K_{k,l,M}}\Big(\sum_{j=1}^{r} \widetilde{b}^{\mathrm{E},l-1}_{k,j} \cdot B^j_l\Big)(t)$$

for $t \in [t_{l-1}, t_l]$, $l = 1, \ldots, m$ and $k = 1, \ldots, d$.

Secondly, for $k = 1, \ldots, d$ we define real-valued stochastic processes $U^k_1 = (U^k_1(t))_{t\in[0,1]}$ and $U^k_2 = (U^k_2(t))_{t\in[0,1]}$ by

$$\begin{aligned} U^k_1(t) = {} & (t - t_{l-1}) \cdot m \cdot \big(\widehat{X}^{\mathrm{M},k}_m(t_l) - \widetilde{X}^{(q),k}_{m,n}(t_l)\big) + (t_l - t) \cdot m \cdot \big(\widehat{X}^{\mathrm{M},k}_m(t_{l-1}) - \widetilde{X}^{(q),k}_{m,n}(t_{l-1})\big) \\ & + \sum_{j=1}^{r} \Big(\big(\widehat{b}^{l-1}_{k,j} - \widehat{b}^{\mathrm{E},l-1}_{k,j}\big) \cdot B^j_l(t) + \big(\widehat{b}^{\mathrm{E},l-1}_{k,j} - \widetilde{b}^{\mathrm{E},l-1}_{k,j}\big) \cdot B^j_l(t)\Big) \end{aligned}$$

and

$$U^k_2(t) = \sum_{j=1}^{r} \widetilde{b}^{\mathrm{E},l-1}_{k,j} \cdot B^j_l(t) - \Phi_{\widetilde{b}^{\mathrm{E},l-1}_k, l, K_{k,l,M}}\Big(\sum_{j=1}^{r} \widetilde{b}^{\mathrm{E},l-1}_{k,j} \cdot B^j_l\Big)(t)$$

for $t \in [t_{l-1}, t_l]$ and $l = 1, \ldots, m$. Then,

$$\overline{X}_m - \widetilde{X}^{(p,q)}_{m,n,M} = U_1 + U_2$$

where $U_1 = (U^1_1, \ldots, U^d_1)'$ and $U_2 = (U^1_2, \ldots, U^d_2)'$.

Hence

$$\Big(\mathbb{E}\Big[\|\overline{X}_m - \widetilde{X}^{(p,q)}_{m,n,M}\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p} \leqslant \Big(\mathbb{E}\Big[\|U_1\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p} + \Big(\mathbb{E}\Big[\|U_2\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p}. \tag{6.2.6}$$

Step 2: In this step we further estimate $\Big(\mathbb{E}\Big[\|U_1\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p}$. By Fubini's theorem we have

$$\Big(\mathbb{E}\Big[\|U_1\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p} = \left(\sum_{k=1}^{d} \mathbb{E}\left[\sum_{l=1}^{m} \int_{t_{l-1}}^{t_l} |U_1^k(t)|^p \, dt\right]\right)^{1/p} = \left(\sum_{k=1}^{d}\sum_{l=1}^{m} \int_{t_{l-1}}^{t_l} \mathbb{E}\Big[\big|U_1^k(t)\big|^p\Big] \, dt\right)^{1/p}. \tag{6.2.7}$$

Moreover, the properties of b imply

$$\begin{aligned}\big|U_1^k(t)\big|^p \leqslant c \cdot \Big(&\max_{l=0,\dots,m} \|\widehat{X}^{\mathrm{M}}_m(t_l) - \widetilde{X}^{(q)}_{m,n}(t_l)\|^p_\infty + \sum_{j=1}^{r} \|\widehat{X}^{\mathrm{M}}_m(t_{l-1}) - \widehat{X}^{\mathrm{E}}_m(t_{l-1})\|^p_\infty \cdot |B_l^j(t)|^p \\ &+ \sum_{j=1}^{r} \|\widehat{X}^{\mathrm{E}}_m(t_{l-1}) - \widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\|^p_\infty \cdot |B_l^j(t)|^p\Big)\end{aligned} \tag{6.2.8}$$

for all $t \in [t_{l-1}, t_l]$, $l = 1, \dots, m$ and $k = 1, \dots, d$. Note that, for all $j = 1, \dots, r$ and $l = 1, \dots, m$, the Brownian bridge B_l^j is independent of $\mathcal{F}(t_{l-1})$, and note that $(\widehat{X}^{\mathrm{E}}_m(t_{l-1}), \widehat{X}^{\mathrm{M}}_m(t_{l-1}))$ is measurable w.r.t. $\mathcal{F}(t_{l-1})$ for every $l = 1, \dots, m$. Thus, by (6.2.8), by Lemma 6.1.2, by Corollary 3.3.4, and by the choice of q together with (5.1.9) in the proof of Proposition 5.1.3, we conclude

$$\begin{aligned}\mathbb{E}\Big[\big|U_1^k(t)\big|^p\Big] &\leqslant c \cdot \Big(\big(m^{-1} + n^{-1}\big)^p + \sum_{j=1}^{r} \mathbb{E}\Big[\|\widehat{X}^{\mathrm{M}}_m(t_{l-1}) - \widehat{X}^{\mathrm{E}}_m(t_{l-1})\|^p_\infty\Big] \cdot \mathbb{E}\Big[\big|B_l^j(t)\big|^p\Big] \\ &\qquad + \sum_{j=1}^{r} \mathbb{E}\Big[\|\widehat{X}^{\mathrm{E}}_m(t_{l-1}) - \widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\|^p_\infty\Big] \cdot \mathbb{E}\Big[\big|B_l^j(t)\big|^p\Big]\Big) \\ &\leqslant c \cdot \Big(\big(m^{-1} + n^{-1}\big)^p + \big(m^{-1} + n^{-1}\big)^{p/2} \cdot \sum_{j=1}^{r} \mathbb{E}\Big[\big|B_l^j(t)\big|^p\Big]\Big)\end{aligned} \tag{6.2.9}$$

for all $t \in [t_{l-1}, t_l]$, $l = 1, \dots, m$ and $k = 1, \dots, d$. Furthermore, since the processes B_l^j are Brownian bridges on $[t_{l-1}, t_l]$, Definition 2.3.9 yields

$$\mathbb{E}\Big[\big|B_l^j(t)\big|^p\Big] \leqslant c \cdot \big((t - t_{l-1}) \cdot (t_l - t) \cdot m\big)^{p/2} \leqslant c \cdot m^{-p/2} \leqslant c \cdot \big(m^{-1} + n^{-1}\big)^{p/2} \tag{6.2.10}$$

for all $t \in [t_{l-1}, t_l]$, $l = 1, \dots, m$ and $j = 1, \dots, r$.

Hence, in summary, due to (6.2.7), (6.2.9) and (6.2.10) we end up with

$$\Big(\mathbb{E}\Big[\|U_1\|^p_{L^d_p[0,1]}\Big]\Big)^{1/p} \leqslant c \cdot \big(m^{-1} + n^{-1}\big). \tag{6.2.11}$$

Step 3: Let $\beta \in \mathbb{R}^r$, $K \in \mathbb{N}$, and let $l \in \{1,\dots,m\}$. Recall that $\Phi_{\beta,l,K}$ is the mapping as defined in (5.2.5).

First, we assume $\beta \neq 0$. Due to Lemma 5.2.2, due to the definition of the mappings $\psi_{\beta,l}$ and $\psi_{\beta,l}^{-1}$, see (5.2.3) and (5.2.4), respectively, and due to the change of variable rule, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\big\| \|\beta\|_2 \cdot B_l^1 - \Phi_{\beta,l,K}\big(\|\beta\|_2 \cdot B_l^1\big)\big\|_{L_p[t_{l-1},t_l]}^p\right] \\
&= \mathbb{E}\left[\int_{t_{l-1}}^{t_l} \big|\|\beta\|_2 \cdot B_l^1(t) - \Phi_{\beta,l,K}\big(\|\beta\|_2 \cdot B_l^1\big)(t)\big|^p \, dt\right] \\
&= \mathbb{E}\left[\int_{t_{l-1}}^{t_l} \big|\psi_{\beta,l}^{-1}\big(\psi_{\beta,l}\big(\|\beta\|_2 \cdot B_l^1\big)\big)(t) - \Phi_{\beta,l,K}\big(\|\beta\|_2 \cdot B_l^1\big)(t)\big|^p \, dt\right] \\
&= \mathbb{E}\left[\int_{t_{l-1}}^{t_l} \big|\psi_{\beta,l}^{-1}\big(\overline{B}\big)(t) - \psi_{\beta,l}^{-1}\big(S_K(\overline{B})\big)(t)\big|^p \, dt\right] \\
&= \|\beta\|_2^p \cdot \mathbb{E}\left[\int_{t_{l-1}}^{t_l} \Big|m^{-1/2} \cdot \overline{B}(m\cdot(t-t_{l-1})) - m^{-1/2} \cdot S_K\big(\overline{B}\big)(m\cdot(t-t_{l-1}))\Big|^p \, dt\right] \\
&= m^{-p/2} \cdot \|\beta\|_2^p \cdot \mathbb{E}\left[m^{-1} \cdot \int_{t_{l-1}}^{t_l} m \cdot \big|\overline{B}(m\cdot(t-t_{l-1})) - S_K\big(\overline{B}\big)(m\cdot(t-t_{l-1}))\big|^p \, dt\right] \\
&= m^{-(p+2)/2} \cdot \|\beta\|_2^p \cdot \mathbb{E}\left[\int_0^1 \big|\overline{B}(s) - S_K\big(\overline{B}\big)(s)\big|^p \, ds\right] \\
&= m^{-(p+2)/2} \cdot \|\beta\|_2^p \cdot (\delta_K)^p
\end{aligned} \tag{6.2.12}$$

where $\overline{B}$ denotes a Brownian bridge on $[0,1]$.

Secondly, assume $\beta = 0$. Then, due to the definition of the mapping $\Phi_{\beta,l,K}$, it holds

$$\Phi_{\beta,l,K}\big(\|\beta\|_2 \cdot B_l^1\big)(t) = 0$$

for all $t \in [t_{l-1}, t_l]$. Hence, trivially,

$$\mathbb{E}\left[\big\| \|\beta\|_2 \cdot B_l^1 - \Phi_{\beta,l,K}\big(\|\beta\|_2 \cdot B_l^1\big)\big\|_{L_p[t_{l-1},t_l]}^p\right] = m^{-(p+2)/2} \cdot \|\beta\|_2^p \cdot (\delta_K)^p \,. \tag{6.2.13}$$

Step 4: In this step we further analyze $\Big(\mathbb{E}\Big[\|U_2\|_{L_p^d[0,1]}^p\Big]\Big)^{1/p}$. To this end, we consider the $\mathbb{R}^{d\times m}$-valued random matrix

$$\mathcal{V} := \big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_0), \dots, \widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{m-1})\big).$$

In view of (5.1.8) and (5.2.9) it holds that

$$\big(\widetilde{b}_1^{\mathrm{E},l-1}, K_{1,l,M}, \dots, \widetilde{b}_d^{\mathrm{E},l-1}, K_{d,l,M}\big)$$

is measurable with respect to $\sigma(\mathcal{V})$ for all $l = 1,\dots,m$. Moreover, by the definition of the Euler quantization of X together with the definition of the random variables $K_{k,l,M}$ as well as Lemma 5.2.1 c), we get that

$$\big(\widetilde{b}_1^{\mathrm{E},l-1}, K_{1,l,M}, \dots, \widetilde{b}_d^{\mathrm{E},l-1}, K_{d,l,M}\big) \quad \text{and} \quad \big(B_l^1, \dots, B_l^r\big)$$

are independent for all $l = 1,\dots,m$. Thus, for all $k = 1,\dots,d$ and $l = 1,\dots,m$, due to Proposition

B.2, and due to (5.2.2), (6.2.12) as well as (6.2.13), we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\|U_2^k\|^p_{L_p[t_{l-1},t_l]} \,\Big|\, \mathcal{V}=\beta\Big] \\
&= \mathbb{E}\Big[\Big\| \sum_{j=1}^r \widetilde{b}_{k,j}^{\mathrm{E},l-1}\cdot B_l^j - \Phi_{\widetilde{b}_k^{\mathrm{E},l-1},l,K_{k,l,M}}\Big(\sum_{j=1}^r \widetilde{b}_{k,j}^{\mathrm{E},l-1}\cdot B_l^j\Big)\Big\|^p_{L_p[t_{l-1},t_l]} \,\Big|\, \mathcal{V}=\beta\Big] \\
&= \mathbb{E}\Big[\Big\| \sum_{j=1}^r \widetilde{b}_{k,j}^{\mathrm{E},l-1}(\beta)\cdot B_l^j - \Phi_{\widetilde{b}_k^{\mathrm{E},l-1}(\beta),l,K_{k,l,M}(\beta)}\Big(\sum_{j=1}^r \widetilde{b}_{k,j}^{\mathrm{E},l-1}(\beta)\cdot B_l^j\Big)\Big\|^p_{L_p[t_{l-1},t_l]}\Big] \\
&= \mathbb{E}\Big[\Big\| \big\|\widetilde{b}_k^{\mathrm{E},l-1}(\beta)\big\|_2\cdot B_l^1 - \Phi_{\widetilde{b}_k^{\mathrm{E},l-1}(\beta),l,K_{k,l,M}(\beta)}\big(\big\|\widetilde{b}_k^{\mathrm{E},l-1}(\beta)\big\|_2\cdot B_l^1\big)\Big\|^p_{L_p[t_{l-1},t_l]}\Big] \\
&= m^{-(p+2)/2}\cdot \big\|\widetilde{b}_k^{\mathrm{E},l-1}(\beta)\big\|_2^p\cdot\big(\delta_{K_{k,l,M}(\beta)}\big)^p
\end{aligned}$$

for $P_{\mathcal{V}}$-a.a. $\beta\in\mathbb{R}^{d\times m}$ where $\widetilde{b}_{k,j}^{\mathrm{E},l-1}(\beta)$ and $K_{k,l,M}(\beta)$ denote the realizations associated to β of $\widetilde{b}_{k,j}^{\mathrm{E},l-1}$ and $K_{k,l,M}$, respectively. Therefore, we conclude that

$$\begin{aligned}
\Big(\mathbb{E}\Big[\|U_2\|^p_{L_p^d[0,1]}\Big]\Big)^{1/p} &= \left(\sum_{k=1}^d\sum_{l=1}^m \mathbb{E}\Big[\|U_2^k\|^p_{L_p([t_{l-1},t_l])}\Big]\right)^{1/p} \\
&= \left(\sum_{k=1}^d \mathbb{E}\Big[m^{-(p+2)/2}\cdot\sum_{l=1}^m \|\widetilde{b}_k^{\mathrm{E},l-1}\|_2^p\cdot\big(\delta_{K_{k,l,M}}\big)^p\Big]\right)^{1/p} \\
&= \left(\sum_{k=1}^d \mathbb{E}\Big[m^{-(p+2)/2}\cdot\sum_{l=1}^m \big(\widetilde{s}^{k,l-1}\big)^p\cdot\big(\delta_{K_{k,l,M}}\big)^p\Big]\right)^{1/p} \\
&= \left(\sum_{k=1}^d \mathbb{E}\big[A_{m,n,M}^k\big]\right)^{1/p} \\
&= \big(\mathbb{E}\left[A_{m,n,M}\right]\big)^{1/p}.
\end{aligned} \tag{6.2.14}$$

Step 5: Now by combining (6.2.6), (6.2.11) and (6.2.14) we end up with

$$\Big(\mathbb{E}\Big[\|\overline{X}_m-\widetilde{X}^{(p,q)}_{m,n,M}\|^p_{L_p^d[0,1]}\Big]\Big)^{1/p}\leqslant c\cdot\big(m^{-1}+n^{-1}\big)+\big(\mathbb{E}\left[A_{m,n,M}\right]\big)^{1/p},$$

which finishes the proof.

□

For the remaining part of this subsection we consider a sequence $(m_N)_{N\in\mathbb{N}}$ in $\mathbb{N}$ which satisfies **(Lim1)** and **(Lim2)**, and recall that

$$M_N=\max\left\{\left\lfloor\left(\left(4^{d\cdot(1+\binom{r}{2})}\cdot m_N^{d\cdot(r+\binom{r}{2})}\right)^{-m_N}\cdot N\right)^{\ln m_N/(1+\ln m_N)}\right\rfloor,1\right\}$$

for $N\in\mathbb{N}$.

Additionally, recall that

$$\overline{\kappa}_p = \limsup_{K\to\infty} \sqrt{\ln K}\cdot\delta_K$$

and

$$C^{(p)} = \left(\mathbb{E}\left[\left(\sum_{k=1}^{d}\left\|\left\|b_k\big(X(\cdot)\big)\right\|_2\right\|^{2p/(p+2)}_{L_{2p/(p+2)}[0,1]}\right)^{(p+2)/2}\right]\right)^{1/p}.$$

We will employ the next lemma in the proof of the fifth main auxiliary lemma, namely, Lemma 6.2.4.

Lemma 6.2.3
It holds

$$\lim_{N\to\infty}\frac{\ln M_N}{\ln N} = 1.$$

Proof:
First, note that due to **(Lim1)** there exists an index $\mathcal{N}\in\mathbb{N}$ such that

$$\left(\left(4^{d\cdot(1+\binom{r}{2})}\cdot m_N^{d\cdot(r+\binom{r}{2})}\right)^{-m_N}\cdot N\right)^{\ln m_N/(1+\ln m_N)} \geqslant 1$$

for all $N\geqslant\mathcal{N}$. Secondly, due to the definition of M_N, and due to the calculation rules of the logarithm, we have

$$\begin{aligned}
\ln M_N &\leqslant \frac{\ln m_N}{1+\ln m_N}\cdot\ln\left(\left(4^{d\cdot(1+\binom{r}{2})}\cdot m_N^{d\cdot(r+\binom{r}{2})}\right)^{-m_N}\cdot N\right)\\
&= \frac{\ln m_N}{1+\ln m_N}\cdot\left(-m_N\cdot\ln\left(4^{d\cdot(1+\binom{r}{2})}\cdot m_N^{d\cdot(r+\binom{r}{2})}\right)+\ln N\right)\\
&\leqslant \frac{\ln m_N}{1+\ln m_N}\cdot\ln N\\
&\leqslant \ln N
\end{aligned}$$

for all $N\geqslant\mathcal{N}$. Thus,

$$\limsup_{N\to\infty}\frac{\ln M_N}{\ln N}\leqslant 1.$$

Hence it remains to show

$$\liminf_{N\to\infty}\frac{\ln M_N}{\ln N}\geqslant 1.$$

By applying the calculation rules of the logarithm once more, we obtain

$$\begin{aligned}\ln M_N &\geqslant \ln(1/2) + \frac{\ln m_N}{1+\ln m_N} \cdot \left(-m_N \cdot \ln\left(4^{d\cdot(1+\binom{r}{2})} \cdot m_N^{d\cdot(r+\binom{r}{2})} \right) + \ln N \right) \\ &= \ln(1/2) - \frac{\ln m_N}{1+\ln m_N} \cdot m_N \cdot \ln\left(4^{d\cdot(1+\binom{r}{2})} \right) - \frac{\ln m_N}{1+\ln m_N} \cdot m_N \cdot \ln m_N \cdot \left(d\cdot r + d \cdot \binom{r}{2} \right) \\ &\quad + \frac{\ln m_N}{1+\ln m_N} \cdot \ln N \\ &\geqslant \ln(1/2) - \frac{\ln m_N}{1+\ln m_N} \cdot m_N \cdot \ln m_N \cdot \ln\left(4^{d\cdot(1+\binom{r}{2})} \right) - \frac{\ln m_N}{1+\ln m_N} \cdot m_N \cdot \ln m_N \cdot \left(d\cdot r + d \cdot \binom{r}{2} \right) \\ &\quad + \frac{\ln m_N}{1+\ln m_N} \cdot \ln N \end{aligned} \tag{6.2.15}$$

for all $N \geqslant \mathcal{N}$ such that $m_N \geqslant 3$. Moreover, **(Lim2)** implies $\lim_{N\to\infty} m_N = \infty$, and hence

$$\lim_{N\to\infty} \frac{\ln m_N}{1+\ln m_N} = 1.$$

Thus, (6.2.15) together with **(Lim1)** leads to

$$\liminf_{N\to\infty} \frac{\ln M_N}{\ln N} \geqslant 1.$$

□

Lemma 6.2.4
It holds

$$\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot \left(\mathbb{E}\left[A_{m_N, m_N, M_N} \right] \right)^{1/p} \right) \leqslant \overline{\kappa}_p \cdot C^{(p)}.$$

Proof:
We may assume $\overline{\kappa}_p < \infty$ since otherwise there is nothing to prove. Additionally, note that $\overline{\kappa}_p \geqslant \kappa_p > 0$, where κ_p is the constant in Theorem 2.3.11 (i). Moreover, for $N \in \mathbb{N}$ we put

$$\overline{\kappa}_N^{(p)} := \max_{l=1,\dots,m_N} \max_{k=1,\dots,d} \sqrt{\ln K_{k,l,M_N}} \cdot \delta_{K_{k,l,M_N}}.$$

From now on, we use the shorter notation $\overline{\kappa}_N$ instead of $\overline{\kappa}_N^{(p)}$. Note that $\overline{\kappa}_p < \infty$ implies

$$\sup_{N\in\mathbb{N}} \overline{\kappa}_N \leqslant c. \tag{6.2.16}$$

Additionally, for $s \in [1,\infty)$ and $N \in \mathbb{N}$ we put

$$\widetilde{D}_{s,N} := \left(m_N^{-1} \cdot \sum_{l=1}^{m_N} \sum_{k=1}^{d} \left| \widetilde{s}^{k,l-1} \right|^s \right)^{1/s},$$

and we define $D_{s,N}$ analogously to $\widetilde{D}_{s,N}$ with $\widetilde{s}^{k,l-1}$ replaced by $s^{k,l-1} := s_{\mathrm{locH}}\big(X^k(t_{l-1})\big)$.

Furthermore, we abbreviate

$$p^* = 2p/(p+2).$$

We split up the proof into single steps.

Step 1: As a first step we further estimate $\mathbb{E}\big[A_{m_N,m_N,M_N}\big]$. To this end, we fix $N \in \mathbb{N}$ such that $M_N \geqslant 2$. By (5.2.9) and (5.2.8), we have

$$\begin{aligned}
\big(\widetilde{s}^{k,l-1}\big)^p \cdot \big(\ln K_{k,l,M_N}\big)^{-p/2} &\leqslant \big(\widetilde{s}^{k,l-1}\big)^p \cdot \big(\ln M_N^{\eta_l^k}\big)^{-p/2} \\
&\leqslant \big(\ln M_N\big)^{-p/2} \cdot \big(\widetilde{s}^{k,l-1}\big)^p \cdot \left(\frac{\sum_{i=0}^{m_N-1} \sum_{\nu=1}^{d} \big(\widetilde{s}^{\nu,i}\big)^{p^*}}{\big(\widetilde{s}^{k,l-1}\big)^{p^*}} \right)^{p/2} \\
&= \big(\ln M_N\big)^{-p/2} \cdot \left(\sum_{i=0}^{m_N-1} \sum_{\nu=1}^{d} \big(\widetilde{s}^{\nu,i}\big)^{p^*} \right)^{p/2} \cdot \big(\widetilde{s}^{k,l-1}\big)^{p^*}
\end{aligned}$$

for all $l = 1, \ldots, m$ with $\max_{\nu=1,\ldots,d} \widetilde{s}^{\nu,l-1} > 0$ and $k = 1, \ldots, d$. Thus,

$$\begin{aligned}
A_{m_N,m_N,M_N} &= \sum_{k=1}^{d} A^k_{m_N,m_N,M_N} \\
&= m_N^{-(p+2)/2} \cdot \sum_{l=1}^{m_N} \sum_{k=1}^{d} \big(\widetilde{s}^{k,l-1}\big)^p \cdot \big(\ln K_{k,l,M_N}\big)^{-p/2} \cdot \big(\ln K_{k,l,M_N}\big)^{p/2} \cdot (\delta_{K_{k,l,M_N}})^p \\
&\leqslant \big(\ln M_N\big)^{-p/2} \cdot (\overline{\kappa}_N)^p \cdot m_N^{-(p+2)/2} \cdot \left(\sum_{i=0}^{m_N-1} \sum_{\nu=1}^{d} \big(\widetilde{s}^{\nu,i}\big)^{p^*} \right)^{p/2} \cdot \left(\sum_{l=1}^{m_N} \sum_{k=1}^{d} \big(\widetilde{s}^{k,l-1}\big)^{p^*} \right) \\
&= \big(\ln M_N\big)^{-p/2} \cdot \big(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\big)^p,
\end{aligned} \tag{6.2.17}$$

and hence

$$\mathbb{E}\left[A_{m_N,m_N,M_N}\right] \leqslant \big(\ln M_N\big)^{-p/2} \cdot \mathbb{E}\left[\big(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\big)^p \right]. \tag{6.2.18}$$

Step 2: We prove

$$\frac{\limsup_{N\to\infty} \overline{\kappa}_N}{\overline{\kappa}_p} \leqslant 1. \tag{6.2.19}$$

To this end, it is enough to show $\lim_{N\to\infty} \min_{l=1,\ldots,m_N} \min_{k=1,\ldots,d} K_{k,l,M_N} = \infty$.

Let $N \in \mathbb{N}$ such that $m_N \geqslant 3$, and, for the moment, fix $l \in \{1, \ldots, m_N\}$ and $k \in \{1, \ldots, d\}$. By (5.2.9) we have

$$\ln K_{k,l,M_N} \geqslant \ln M_N^{\eta_l^k} = \eta_l^k \cdot \ln M_N.$$

Moreover, if $\max_{\substack{i=0,\dots,m_N-1,\\ \nu=1,\dots,d}} \widetilde{s}^{\nu,i} = 0$, it holds $\eta_l^k = 1/(d \cdot m_N)$ and hence

$$\ln K_{k,l,M_N} \geqslant \frac{1}{d \cdot m_N} \cdot \ln M_N \geqslant \frac{1}{d \cdot m_N \cdot \ln m_N} \cdot \ln M_N.$$

If $\max_{\substack{i=0,\dots,m_N-1,\\ \nu=1,\dots,d}} \widetilde{s}^{\nu,i} > 0$, (5.2.8) yields

$$\ln K_{k,l,M_N} \geqslant \frac{1}{d \cdot m_N \cdot \ln m_N} \cdot \ln M_N.$$

Therefore,

$$K_{k,l,M_N} \geqslant \exp\left(\frac{1}{d \cdot m_N \cdot \ln m_N} \cdot \ln M_N\right)$$

for all $l = 1, \dots, m_N$ and $k = 1, \dots, d$, and hence

$$\min_{l=1,\dots,m_N} \min_{k=1,\dots,d} K_{k,l,M_N} \geqslant \exp\left(\frac{1}{d \cdot m_N \cdot \ln m_N} \cdot \ln M_N\right).$$

Additionally, due to Lemma 6.2.3 we have

$$\lim_{N\to\infty} \frac{\ln M_N}{\ln N} = 1,$$

and thus, together with **(Lim1)**, we obtain

$$\lim_{N\to\infty} \frac{1}{d \cdot m_N \cdot \ln m_N} \cdot \ln M_N = \lim_{N\to\infty} \frac{\ln N}{d \cdot m_N \cdot \ln m_N} \cdot \frac{\ln M_N}{\ln N} = \infty.$$

Consequently,

$$\lim_{N\to\infty} \min_{l=1,\dots,m_N} \min_{k=1,\dots,d} K_{k,l,M_N} = \infty.$$

Step 3: Recall that

$$D_{p^*,N} = \left(m_N^{-1} \cdot \sum_{l=1}^{m_N} \sum_{k=1}^{d} \left\|b_k\big(X(t_{l-1})\big)\right\|_2^{p^*}\right)^{1/p^*}$$

for $N \in \mathbb{N}$. Since X has continuous paths, since b is continuous, see Section 3.2, and due to the fact that the euclidean norm $\|\cdot\|_2$ is continuous, we obtain

$$\lim_{N\to\infty} D_{p^*,N}(\omega) = \lim_{N\to\infty} \left(m_N^{-1} \cdot \sum_{l=1}^{m_N} \sum_{k=1}^{d} \left\|b_k\big(X(t_{l-1})(\omega)\big)\right\|_2^{p^*}\right)^{1/p^*} = \left(\sum_{k=1}^{d} \int_0^1 \left\|b_k\big(X(t)(\omega)\big)\right\|_2^{p^*} dt\right)^{1/p^*} \tag{6.2.20}$$

for all $\omega \in \Omega$.

Moreover, since b is of at most linear growth, and due to (6.2.16), it holds

$$\big(\overline{\kappa}_N \cdot D_{p^*,N}\big)^p \leqslant c \cdot \big(1 + \sup_{s\in[0,1]} \|X(s)\|_\infty^p\big)$$

for all $N \in \mathbb{N}$.

Furthermore, Theorem 3.2.3 guarantees

$$\mathbb{E}\Big[\sup_{s\in[0,1]}\|X(s)\|_\infty^p\Big]<\infty.$$

Hence all assumptions of Fatou's lemma are satisfied, which, together with (6.2.19) and (6.2.20), implies

$$\begin{aligned}\limsup_{N\to\infty}\mathbb{E}\Big[\big(\overline{\kappa}_N\cdot D_{p^*,N}\big)^p\Big]&\leqslant\mathbb{E}\left[\limsup_{N\to\infty}\big(\overline{\kappa}_N\cdot D_{p^*,N}\big)^p\right]\\&\leqslant\mathbb{E}\left[\Big(\limsup_{N\to\infty}\big(\overline{\kappa}_N\big)^p\Big)\cdot\Big(\limsup_{N\to\infty}\big(D_{p^*,N}\big)^p\Big)\right]\\&\leqslant\big(\overline{\kappa}_p\big)^p\cdot\mathbb{E}\left[\limsup_{N\to\infty}\big(D_{p^*,N}\big)^p\right]\\&=\big(\overline{\kappa}_p\cdot C^{(p)}\big)^p.\end{aligned}$$

Step 4: In this step we prove that, depending on the value of p, there exists an $\alpha\in(0,\infty)$ such that

$$\lim_{N\to\infty}\left|\Big(\mathbb{E}\left[\big(\overline{\kappa}_N\cdot\widetilde{D}_{p^*,N}\big)^p\right]\Big)^\alpha-\Big(\mathbb{E}\Big[\big(\overline{\kappa}_N\cdot D_{p^*,N}\big)^p\Big]\Big)^\alpha\right|=0,$$

which then yields

$$\limsup_{N\to\infty}\mathbb{E}\left[\big(\overline{\kappa}_N\cdot\widetilde{D}_{p^*,N}\big)^p\right]\leqslant\limsup_{N\to\infty}\mathbb{E}\Big[\big(\overline{\kappa}_N\cdot D_{p^*,N}\big)^p\Big]\leqslant\big(\overline{\kappa}_p\cdot C^{(p)}\big)^p.\tag{6.2.21}$$

First, assume $p\in[2,\infty)$. Then $1\leqslant p^*<p$, and the inverse triangle inequality as well as the properties of b imply

$$\begin{aligned}\Big|\widetilde{D}_{p^*,N}-D_{p^*,N}\Big|&=\left|\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big|\widetilde{s}^{k,l-1}\big|^{p^*}\right)^{1/p^*}-\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big|s^{k,l-1}\big|^{p^*}\right)^{1/p^*}\right|\\&=\left|\left(\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big|m_N^{-1/p^*}\cdot\widetilde{s}^{k,l-1}\big|^{p^*}\right)^{1/p^*}-\left(\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big|m_N^{-1/p^*}\cdot s^{k,l-1}\big|^{p^*}\right)^{1/p^*}\right|\\&\leqslant\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big|\widetilde{s}^{k,l-1}-s^{k,l-1}\big|^{p^*}\right)^{1/p^*}\\&=\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\sum_{k=1}^{d}\Big|\big\|b_k\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_{l-1})\big)\big\|_2-\big\|b_k\big(X(t_{l-1})\big)\big\|_2\Big|^{p^*}\right)^{1/p^*}\\&\leqslant\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\sum_{k=1}^{d}\big\|b_k\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_{l-1})\big)-b_k\big(X(t_{l-1})\big)\big\|_2^{p^*}\right)^{1/p^*}\\&\leqslant c\cdot\left(m_N^{-1}\cdot\sum_{l=1}^{m_N}\big\|X(t_{l-1})-\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_{l-1})\big\|_\infty^{p^*}\right)^{1/p^*}\\&\leqslant c\cdot\max_{l=0,\dots,m_N}\big\|X(t_l)-\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_l)\big\|_\infty\end{aligned}\tag{6.2.22}$$

for all $N\in\mathbb{N}$. Thus, by the inverse triangle inequality, by $\sup_{N\in\mathbb{N}}\overline{\kappa}_N\leqslant c$, and by the choice of q

together with Proposition 5.1.3, we obtain

$$\begin{aligned}\left|\left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\right)^p\right]\right)^{1/p} - \left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot D_{p^*,N}\right)^p\right]\right)^{1/p}\right| &\leqslant \left(\mathbb{E}\left[\left|\overline{\kappa}_N\right|^p \cdot \left|\widetilde{D}_{p^*,N} - D_{p^*,N}\right|^p\right]\right)^{1/p} \\ &\leqslant c \cdot \left(\mathbb{E}\left[\left|\widetilde{D}_{p^*,N} - D_{p^*,N}\right|^p\right]\right)^{1/p} \\ &\leqslant c \cdot \left(\mathbb{E}\left[\max_{l=0,\dots,m} \left\|X(t_l) - \widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_l)\right\|_\infty^p\right]\right)^{1/p} \\ &\leqslant c \cdot m_N^{-1/2}\end{aligned}$$

for all $N \in \mathbb{N}$. Due to **(Lim2)** we have

$$\lim_{N\to\infty} m_N^{-1/2} = 0,$$

and hence we end up with

$$\lim_{N\to\infty} \left|\left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\right)^p\right]\right)^{1/p} - \left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot D_{p^*,N}\right)^p\right]\right)^{1/p}\right| = 0.$$

Secondly, assume $p \in [1,2)$. Then $p^* < 1$, and hence $p/p^* > 1$. By (6.2.22) we conclude that

$$\left|\left(\widetilde{D}_{p^*,N}\right)^{p^*} - \left(D_{p^*,N}\right)^{p^*}\right| \leqslant \left|\widetilde{D}_{p^*,N} - D_{p^*,N}\right|^{p^*} \leqslant c \cdot \max_{l=0,\dots,m} \left\|X(t_l) - \widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_l)\right\|_\infty^{p^*}.$$

Together with the inverse triangle inequality, $\sup_{N\in\mathbb{N}} \overline{\kappa}_N \leqslant c$, and the choice of q combined with Proposition 5.1.3, we obtain

$$\begin{aligned}\left|\left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\right)^p\right]\right)^{p^*/p} - \left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot D_{p^*,N}\right)^p\right]\right)^{p^*/p}\right| &\leqslant \left(\mathbb{E}\left[\left|\overline{\kappa}_N\right|^p \cdot \left|\left(\widetilde{D}_{p^*,N}\right)^{p^*} - \left(D_{p^*,N}\right)^{p^*}\right|^{p/p^*}\right]\right)^{p^*/p} \\ &\leqslant c \cdot \left(\mathbb{E}\left[\left|\left(\widetilde{D}_{p^*,N}\right)^{p^*} - \left(D_{p^*,N}\right)^{p^*}\right|^{p/p^*}\right]\right)^{p^*/p} \\ &\leqslant c \cdot m_N^{-p^*/2}.\end{aligned}$$

Combined with

$$\lim_{N\to\infty} m_N^{-p^*/2} = 0$$

we finally arrive at

$$\lim_{N\to\infty} \left|\left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\right)^p\right]\right)^{p^*/p} - \left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot D_{p^*,N}\right)^p\right]\right)^{p^*/p}\right| = 0.$$

Step 5: We combine all previous steps. By (6.2.18), Lemma 6.2.3 and (6.2.21) we end up with

$$\begin{aligned}\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot \left(\mathbb{E}[A_{m_N,m_N,M_N}]\right)^{1/p}\right) &\leqslant \limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot \frac{1}{\sqrt{\ln M_N}} \cdot \left(\mathbb{E}\left[\left(\overline{\kappa}_N \cdot \widetilde{D}_{p^*,N}\right)^p\right]\right)^{1/p}\right) \\ &\leqslant \left(\lim_{N\to\infty} \sqrt{\frac{\ln N}{\ln M_N}}\right) \cdot \left(\limsup_{N\to\infty} \mathbb{E}\left[\left(\overline{\kappa}_N \cdot D_{p^*,N}\right)^p\right]\right)^{1/p} \\ &\leqslant \overline{\kappa}_p \cdot C^{(p)}.\end{aligned}$$

□

6.2.3. Proof of Theorem 5.2.4

We prove Theorem 5.2.4 by applying the Lemmas 6.2.1, 6.2.2 and 6.2.4.

Proof of Theorem 5.2.4:
By the aforementioned lemmas and **(Lim2)** we conclude

$$\begin{aligned}
&\limsup_{N\to\infty}\Big(\sqrt{\ln N}\cdot e^{(p)}\big(X,\widetilde{X}_N^{(p,q)},L_p^d[0,1]\big)\Big)\\
&\leqslant \limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\Big(\mathbb{E}\Big[\|X-\overline{X}_{m_N}\|^p_{L_p^d[0,1]}\Big]\Big)^{1/p}\right)+\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\Big(\mathbb{E}\Big[\|\overline{X}_{m_N}-\widetilde{X}_N^{(p,q)}\|^p_{L_p^d[0,1]}\Big]\Big)^{1/p}\right)\\
&\leqslant c\cdot\limsup_{N\to\infty}\frac{\sqrt{\ln N}}{m_N}+\limsup_{N\to\infty}\sqrt{\ln N}\cdot\big(\mathbb{E}[A_{m_N,m_N,M_N}]\big)^{1/p}\\
&\leqslant \overline{\kappa}_p\cdot C^{(p)}.
\end{aligned}$$

The statement on the size of the range of $\widetilde{X}_N^{(p,q)}$ directly follows from (5.2.11) and (5.2.12). □

6.3. Quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$: Proof of Theorem 5.3.4

In the whole section we consider fixed parameters $\varepsilon>0$ and $s,q\in[1,\infty)$ with $q\geqslant\min\{\widetilde{q}\in 2\mathbb{N}\mid \widetilde{q}\geqslant s\eta_\varepsilon\}$, where η_ε the natural number as in (5.3.1).

6.3.1. Auxiliary Lemma 6: The Auxiliary Process Revisited

In this subsection we revisit the auxiliary process introduced in Section 6.2.1. Recall that for $m\in\mathbb{N}$ the d-dimensional stochastic process $\overline{X}_m=\big(\overline{X}^1_m,\dots,\overline{X}^d_m\big)'$ is given by

$$\overline{X}^k_m(t)=(t-t_{l-1})\cdot m\cdot\widehat{X}^{\mathrm{M},k}_m(t_l)+(t_l-t)\cdot m\cdot\widehat{X}^{\mathrm{M},k}_m(t_{l-1})+\sum_{j=1}^r b_{k,j}\big(\widehat{X}^{\mathrm{M}}_m(t_{l-1})\big)\cdot B^j_l(t)$$

for $t\in[t_{l-1},t_l]$, $l=1,\dots,m$ and $k=1,\dots,d$.

Lemma 6.3.1
Let $\tilde{s}\in(2,\infty)$. Then there exists an $\alpha_{\tilde{s}}\in(3/4,1)$ such that, for all $m\in\mathbb{N}_3$,

$$\left(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X(t)-\overline{X}_m(t)\|^{\tilde{s}}_\infty\Big]\right)^{1/\tilde{s}}\leqslant c\cdot\ln m\cdot m^{-\alpha_{\tilde{s}}}.$$

Proof:
Let $m\in\mathbb{N}_3$. We use (3.3.5) in Proposition 3.3.2 and (6.2.3) in the proof of Lemma 6.2.1 to conclude that

$$\left(\mathbb{E}\Big[\sup_{t\in[0,1]}\|X(t)-\overline{X}_m(t)\|^{\tilde{s}}_\infty\Big]\right)^{1/\tilde{s}}\leqslant c\cdot m^{-1}+c\cdot\sum_{k=1}^d\left(\mathbb{E}\Big[\sup_{t\in(0,1]}|R^k(t)|^{\tilde{s}}\Big]\right)^{1/\tilde{s}}\tag{6.3.1}$$

where, recall,

$$R^k(t) = \sum_{l=1}^{m} \left(\sum_{j_1,j_2=1}^{r} \widehat{c}^{\,l-1}_{k,j_2,j_1} \cdot \int_{t_{l-1}}^{t} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(s) \right) \cdot \mathbb{1}_{(t_{l-1},t_l]}(t)$$

for $t \in (0,1]$. The properties of b, Hölder's inequality, and Proposition 3.3.2 imply

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in(0,1]} |R^k(t)|^{\tilde{s}}\Big] \\
&\leqslant c \cdot \sum_{j_1,j_2=1}^{r} \left(1 + \mathbb{E}\Big[\max_{l=0,\dots,m} \|\widehat{X}^{\mathrm{M}}_m(t_l)\|_\infty^{2\tilde{s}}\Big]\right)^{1/2} \cdot \left(\mathbb{E}\left[\max_{l=1,\dots,m} \sup_{t\in[t_{l-1},t_l]} \left|\int_{t_{l-1}}^{t} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(s)\right|^{2\tilde{s}}\right]\right)^{1/2} \\
&\leqslant c \cdot \sum_{j_1,j_2=1}^{r} \left(\mathbb{E}\left[\max_{l=1,\dots,m} \sup_{t\in[t_{l-1},t_l]} \left|\int_{t_{l-1}}^{t} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(s)\right|^{2\tilde{s}}\right]\right)^{1/2}
\end{aligned} \tag{6.3.2}$$

for all $k = 1, \dots, d$. Furthermore, by the Burkholder-Davis-Gundy inequality, by Hölder's inequality, by Fubini's theorem, and by the properties of W, we conclude that

$$\begin{aligned}
&\mathbb{E}\left[\max_{l=1,\dots,m} \sup_{t\in[t_{l-1},t_l]} \left|\int_{t_{l-1}}^{t} \big(W_{j_1}(s) - W_{j_1}(t_{l-1})\big)\, dW_{j_2}(s)\right|^{2\tilde{s}}\right] \\
&\leqslant c \cdot \sum_{l=1}^{m} \mathbb{E}\left[\left(\int_{t_{l-1}}^{t_l} \big|W_{j_1}(s) - W_{j_1}(t_{l-1})\big|^2\, ds\right)^{\tilde{s}}\right] \\
&\leqslant c \cdot \sum_{l=1}^{m} \mathbb{E}\left[\left(\int_{t_{l-1}}^{t_l} \big|W_{j_1}(s) - W_{j_1}(t_{l-1})\big|^{2\tilde{s}}\, ds\right) \cdot \left(\int_{t_{l-1}}^{t_l} 1^{\tilde{s}/(\tilde{s}-1)}\, ds\right)^{\tilde{s}-1}\right] \\
&= c \cdot m^{-(\tilde{s}-1)} \cdot \sum_{l=1}^{m} \int_{t_{l-1}}^{t_l} \mathbb{E}\Big[\big|W_{j_1}(s) - W_{j_1}(t_{l-1})\big|^{2\tilde{s}}\Big]\, ds \\
&\leqslant c \cdot m^{-(\tilde{s}-1)} \cdot m^{-\tilde{s}} \\
&= c \cdot m^{-(2\tilde{s}-1)}
\end{aligned} \tag{6.3.3}$$

for all $j_1, j_2 = 1, \dots, r$. Hence, in view of (6.3.2) and (6.3.3), we arrive at

$$\sum_{k=1}^{d} \left(\mathbb{E}\Big[\sup_{t\in(0,1]} |R^k(t)|^{\tilde{s}}\Big]\right)^{1/\tilde{s}} \leqslant c \cdot m^{-(2\tilde{s}-1)/2\tilde{s}}.$$

Now, put $\alpha_{\tilde{s}} := (2\tilde{s}-1)/2\tilde{s}$. Since $\tilde{s} > 2$, it holds $\alpha_{\tilde{s}} \in (3/4, 1)$. By (6.3.1), and by the fact that $m \geqslant 3$, we finally end up with

$$\left(\mathbb{E}\Big[\sup_{t\in[0,1]} \|X(t) - \overline{X}_m(t)\|_\infty^{\tilde{s}}\Big]\right)^{1/\tilde{s}} \leqslant c \cdot \ln m \cdot m^{-\alpha_{\tilde{s}}}.$$

□

6.3.2. Auxiliary Lemmas 7 and 8

Recall that $(S_{M,\varepsilon})_{M\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order $s\eta_\varepsilon$ of M-quantizers $S_{M,\varepsilon}:\mathcal{C}[0,1]\to\mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$. Moreover, for $M\in\mathbb{N}$ we put

$$\delta_M^{(s\eta_\varepsilon)} := e^{(s\eta_\varepsilon)}\big(\overline{W}, S_{M,\varepsilon}(\overline{W}), \mathcal{C}[0,1]\big)$$

where $\overline{W}$ denotes a Brownian motion on $[0,1]$, and we use the abbreviation

$$\widetilde{A}_{m,n}^k = \left(m^{-1}\cdot\sum_{i=0}^{m-1}\big\|\overline{b}_{k,m}\big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_i)\big)\big\|_2^2\right)^{1/2}$$

for $m,n\in\mathbb{N}$ and $k=1,\dots,d$. Analogously, we define $\overline{A}_{m,n}^k$ with $\|\overline{b}_{k,m}(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_i))\|_2$ replaced by $\|b_k(X(t_i))\|_2$.

For the rest of this subsection, for ease of notation, we write η and S_M instead of η_ε and $S_{M,\varepsilon}$, respectively, for $M\in\mathbb{N}$.

Lemma 6.3.2
Let $m,n,M\in\mathbb{N}$, let $k\in\{1,\dots,d\}$, and let $\gamma^k\in\mathbb{R}^{m\times r}$ such that (5.3.3) as well as (5.3.4) are satisfied. Then there exists a constant $c(s,\varepsilon)\in(0,\infty)$ neither depending on m,n nor on M such that

$$\Big(\mathbb{E}\Big[\big\|\overline{B}^{(\gamma^k)}-\psi_{m,n,M}^{(\gamma^k,q,\varepsilon)}(\overline{B}^{(\gamma^k)})\big\|_\infty^{s\eta}\Big]\Big)^{1/s\eta}\leqslant\delta_M^{(s\eta)}+c(s,\varepsilon)\cdot n^{-1}.$$

Proof:
By (5.3.2), (5.3.12), (5.3.8) as well as (5.3.10), and by Hölder's inequality, we have

$$\begin{aligned}\Big(\mathbb{E}\Big[\big\|\overline{B}^{(\gamma^k)}-\psi_{m,n,M}^{(\gamma^k,q,\varepsilon)}(\overline{B}^{(\gamma^k)})\big\|_\infty^{s\eta}\Big]\Big)^{1/s\eta} &= \Big(\mathbb{E}\Big[\big\|B^{(\gamma^k)}-\psi_{m,n,M}^{(\gamma^k,q,\varepsilon)}(B^{(\gamma^k)})\big\|_\infty^{s\eta}\Big]\Big)^{1/s\eta}\\ &\leqslant\Big(\mathbb{E}\Big[\big\|W^{(\gamma^k)}-S_M(W^{(\gamma^k)})\big\|_\infty^{s\eta}\Big]\Big)^{1/s\eta}\\ &\quad+\left(\mathbb{E}\Big[\max_{l=1,\dots,m}\big|W^{(\gamma^k)}(s_l^k)-\widetilde{W}_{m,n}^{(\gamma^k,q)}(s_l^k)\big|^{s\eta}\Big]\right)^{1/s\eta}\\ &\leqslant\delta_M^{(s\eta)}+\left(\mathbb{E}\Big[\max_{l=1,\dots,m}\big|W^{(\gamma^k)}(s_l^k)-\widetilde{W}_{m,n}^{(\gamma^k,q)}(s_l^k)\big|^{q}\Big]\right)^{1/q}.\end{aligned}\tag{6.3.4}$$

It remains to further estimate the second summand in the last line of (6.3.4). To this end, we put $\overline{U}_0^k:=0$ and

$$\overline{U}_l^k := W^{(\gamma^k)}(s_l^k)-\widetilde{W}_{m,n}^{(\gamma^k,q)}(s_l^k)$$

for $l=1,\dots,m$. In view of Lemma 5.3.2 c) and (5.3.9), it holds

$$\overline{U}_l^k = m^{-1/2}\cdot\sum_{i=1}^{l}\|\gamma_i^k\|_2\cdot\big(Z_i-\widetilde{Z}_{i,n}^{(q)}\big)$$

for all $l=1,\dots,m$. By the choice of the random variables $Z_1,\dots,Z_m$, by the properties of standard normally distributed random variables, and by **(N1)**, we derive that $(\overline{U}_l^k)_{l=0,\dots,m}$ is a martingale with

respect to its natural filtration. Therefore, Doob's maximal inequality implies the existence of a constant $c_1 = c_1(s,\varepsilon) \in (0,\infty)$ such that

$$\mathbb{E}\Big[\max_{l=0,\dots,m} \big|\overline{U}_l^k\big|^q\Big] \leqslant c_1 \cdot \mathbb{E}\left[\big|\overline{U}_m^k\big|^q\right]. \tag{6.3.5}$$

Furthermore, by the definition of $\overline{U}_l^k$, by the choice of the random variables $Z_1,\dots,Z_m$, by **(N1)** and the properties of standard normally distributed random variables, by (5.3.4), and by **(N3)**, we conclude that there exist constants $c_2 = c_2(s,\varepsilon), c_3 = c_3(s,\varepsilon) \in (0,\infty)$ such that

$$\begin{aligned}
\mathbb{E}\left[\big|\overline{U}_{l+1}^k\big|^q\right] &= \mathbb{E}\left[\big|\overline{U}_l^k\big|^q\right] + \sum_{\mu=2}^{q} \binom{q}{\mu} \cdot \mathbb{E}\left[\big|\overline{U}_l^k\big|^{q-\mu}\right] \cdot \big(m^{-1} \cdot \|\gamma_{l+1}^k\|_2^2\big)^{\mu/2} \cdot \mathbb{E}\left[\big(Z_{l+1} - \widetilde{Z}_{l+1,n}^{(q)}\big)^\mu\right] \\
&\leqslant \mathbb{E}\left[\big|\overline{U}_l^k\big|^q\right] + c_2 \cdot m^{-1} \cdot \|\gamma_{l+1}^k\|_2^2 \cdot \sum_{\mu=2}^{q} \binom{q}{\mu} \cdot \mathbb{E}\left[\big|\overline{U}_l^k\big|^{q-\mu}\right] \cdot n^{-\mu} \\
&\leqslant \mathbb{E}\left[\big|\overline{U}_l^k\big|^q\right] + c_2 \cdot m^{-1} \cdot \|\gamma_{l+1}^k\|_2^2 \cdot \mathbb{E}\left[\big(\big|\overline{U}_l^k\big| + n^{-1}\big)^q\right] \\
&\leqslant \mathbb{E}\left[\big|\overline{U}_l^k\big|^q\right] \cdot \big(1 + c_3 \cdot m^{-1} \cdot \|\gamma_{l+1}^k\|_2^2\big) + c_3 \cdot m^{-1} \cdot \|\gamma_{l+1}^k\|_2^2 \cdot n^{-q}
\end{aligned}$$

for all $l = 0,\dots,m-1$. Together with (5.3.4) we have that all assumptions of the discrete version of Gronwall's inequality are satisfied, which then yields

$$\mathbb{E}\left[\big|\overline{U}_m^k\big|^q\right] \leqslant c_3 \cdot n^{-q}. \tag{6.3.6}$$

Finally, (6.3.4)--(6.3.6) imply the existence of a constant $c(s,\varepsilon) \in (0,\infty)$ such that

$$\Big(\mathbb{E}\Big[\big\|\overline{B}^{(\gamma^k)} - \psi_{m,n,M}^{(\gamma^k,q,\varepsilon)}(\overline{B}^{(\gamma^k)})\big\|_\infty^{s\eta}\Big]\Big)^{1/s\eta} \leqslant \delta_M^{(s\eta)} + c(s,\varepsilon) \cdot n^{-1}.$$

□

Lemma 6.3.3
Let $n, M \in \mathbb{N}$, and let $\tilde{s} \in (2,\infty)$ with $\tilde{s} \geqslant s$. Then there exists an $\alpha_{\tilde{s}} \in (3/4,1)$, an index $N(\tilde{s}) \in \mathbb{N}_3$ and a constant $c(s,\varepsilon) \in (0,\infty)$ which neither depends on n nor on M such that, for all $m \geqslant N(\tilde{s})$,

$$e^{(s)}\big(X, \widetilde{X}_{m,n,M}^{(q,\varepsilon)}, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant c \cdot \big(m^{-1} + n^{-1}\big)^{\alpha_{\tilde{s}}} + \left(\mathbb{E}\Big[\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k\big)^s\Big]\right)^{1/s} \cdot (1+\varepsilon) \cdot \big(\delta_M^{s\eta} + c(s,\varepsilon) \cdot n^{-1}\big).$$

Proof:
First, note that Hölder's inequality and Lemma 6.3.1 guarantee the existence of an $\alpha_{1,\tilde{s}} \in (3/4,1)$ such that

$$\Big(\mathbb{E}\Big[\|X - \overline{X}_m\|_\infty^s\Big]\Big)^{1/s} \leqslant \Big(\mathbb{E}\Big[\|X - \overline{X}_m\|_\infty^{\tilde{s}}\Big]\Big)^{1/\tilde{s}} \leqslant c \cdot \ln m \cdot m^{-\alpha_{1,\tilde{s}}}$$

for all $m \in \mathbb{N}_3$. Now, choose $\beta \in (0, \alpha_{1,\tilde{s}} - 3/4)$ and put $\alpha_{2,\tilde{s}} := \alpha_{1,\tilde{s}} - \beta$. Note that the choice of β implies $\alpha_{2,\tilde{s}} \in (3/4,1)$. Moreover, since $\lim_{m\to\infty} \ln m / m^\beta = 0$, there exists an index $N(\tilde{s}) \in \mathbb{N}_3$ such that for all $m \geqslant N(\tilde{s})$ one has $\ln m \leqslant m^\beta$. This leads to

$$\ln m \cdot m^{-\alpha_{1,\tilde{s}}} \leqslant m^{\beta - \alpha_{1,\tilde{s}}} = m^{-\alpha_{2,\tilde{s}}}$$

and therefore

$$\Big(\mathbb{E}\Big[\|X-\overline{X}_m\|_\infty^s\Big]\Big)^{1/s} \leqslant c\cdot\big(m^{-1}+n^{-1}\big)^{\alpha_{2,\tilde{s}}}$$

for all $m \geqslant N(\tilde{s})$. Consequently,

$$\begin{aligned} e^{(s)}\big(X,\widetilde{X}^{(q,\varepsilon)}_{m,n,M},\mathcal{C}([0,1];\mathbb{R}^d)\big) &\leqslant \Big(\mathbb{E}\Big[\|X-\overline{X}_m\|_\infty^s\Big]\Big)^{1/s} + \Big(\mathbb{E}\Big[\|\overline{X}_m-\widetilde{X}^{(q,\varepsilon)}_{m,n,M}\|_\infty^s\Big]\Big)^{1/s} \\ &\leqslant c\cdot\big(m^{-1}+n^{-1}\big)^{\alpha_{2,\tilde{s}}} + \Big(\mathbb{E}\Big[\|\overline{X}_m-\widetilde{X}^{(q,\varepsilon)}_{m,n,M}\|_\infty^s\Big]\Big)^{1/s} \end{aligned} \tag{6.3.7}$$

for all $m \geqslant N(\tilde{s})$.

Secondly, for the remaining part of this proof we fix $m \geqslant N(\tilde{s})$, and for $k=1,\dots,d$ let U_1^k be the real-valued stochastic process defined in Step 1 in the proof of Lemma 6.2.2, and let $U_2^k=(U_2^k(t))_{t\in[0,1]}$ and $U_3^k=(U_3^k(t))_{t\in[0,1]}$ be real-valued stochastic processes defined by

$$U_2^k(t)=\sum_{j=1}^r\Big(b_{k,j}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big)-\overline{b}_{k,j,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_{l-1})\big)\Big)\cdot B_l^j(t)$$

for $t\in[t_{l-1},t_l]$ and $l=1,\dots,m$, and

$$U_3^k=\widetilde{A}^k_{m,n}\cdot\Big(\overline{B}^{(\Gamma^k)}-\psi^{(\Gamma^k,q,\varepsilon)}_{m,n,M}\big(\overline{B}^{(\Gamma^k)}\big)\Big),$$

respectively. Then,

$$\overline{X}_m-\widetilde{X}^{(q,\varepsilon)}_{m,n,M}=U_1+U_2+U_3$$

where $U_i=(U_i^1,\dots,U_i^d)'$ for $i=1,2,3$, and hence

$$\Big(\mathbb{E}\Big[\|\overline{X}_m-\widetilde{X}^{(q,\varepsilon)}_{m,n,M}\|_\infty^s\Big]\Big)^{1/s} \leqslant \big(\mathbb{E}\big[\|U_1\|_\infty^s\big]\big)^{1/s}+\big(\mathbb{E}\big[\|U_2\|_\infty^s\big]\big)^{1/s}+\big(\mathbb{E}\big[\|U_3\|_\infty^s\big]\big)^{1/s}. \tag{6.3.8}$$

As a next step, we further estimate $\big(\mathbb{E}\big[\|U_1\|_\infty^s\big]\big)^{1/s}$. Since b is globally Lipschitz continuous, we obtain

$$\begin{aligned} \sup_{t\in[0,1]}\big|U_1^k(t)\big|^s \leqslant c\cdot\Big(&\max_{l=0,\dots,m}\|\widehat{X}^{\mathrm{M}}_m(t_l)-\widetilde{X}^{(q)}_{m,n}(t_l)\|_\infty^s \\ &+\sum_{j=1}^r\Big(\max_{l=0,\dots,m}\|\widehat{X}^{\mathrm{M}}_m(t_l)-\widehat{X}^{\mathrm{E}}_m(t_l)\|_\infty^s\Big)\cdot\Big(\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}\big|B_l^j(t)\big|^s\Big) \\ &+\sum_{j=1}^r\Big(\max_{l=0,\dots,m}\|\widehat{X}^{\mathrm{E}}_m(t_l)-\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_l)\|_\infty^s\Big)\cdot\Big(\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}\big|B_l^j(t)\big|^s\Big)\Big) \end{aligned} \tag{6.3.9}$$

for all $k=1,\dots,d$. Additionally, we put $\alpha_{\tilde{s}}:=(1+\alpha_{2,\tilde{s}})/2$. Note that, due to $\alpha_{2,\tilde{s}}\in(3/4,1)$, it holds $\alpha_{\tilde{s}}\in(3/4,1)$. Now, in view of (6.3.9), by the choice of q combined with Lemma 6.1.2 and (5.1.9) in the proof of Proposition 5.1.3, by Lemma 5.2.1 c), by Lemma C.5, by Hölder's inequality, by Corollary 3.3.4, and by applying the same arguments as in the beginning of this proof, we arrive at

$$\begin{aligned}
\Big(\mathbb{E}\big[\|U_1(t)\|^s\big]\Big)^{1/s} &\leqslant c\cdot\Big(\big(m^{-1}+n^{-1}\big)^s+\sum_{j=1}^r\mathbb{E}\Big[\Big(\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{M}}(t_l)-\widehat{X}_m^{\mathrm{E}}(t_l)\|_\infty^s\Big)\cdot\Big(\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}\big|B_l^j(t)\big|^s\Big)\Big]\\
&\qquad+\sum_{j=1}^r\mathbb{E}\Big[\max_{l=0,\dots,m}\|\widehat{X}_m^{\mathrm{E}}(t_l)-\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_l)\|_\infty^s\Big]\cdot\mathbb{E}\Big[\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}\big|B_l^j(t)\big|^s\Big]\Big)^{1/s}\\
&\leqslant c\cdot\Big(\big(m^{-1}+n^{-1}\big)^s+\big(m^{-1}+n^{-1}\big)^{s/2}\cdot\big(\ln m\cdot m^{-1}\big)^{s/2}\Big)^{1/s}\\
&\leqslant c\cdot\Big(\big(m^{-1}+n^{-1}\big)^s+\big(m^{-1}+n^{-1}\big)^{s\cdot(1+\alpha_{2,\bar{s}})/2}\Big)^{1/s}\\
&\leqslant c\cdot\big(m^{-1}+n^{-1}\big)^{\alpha_{\bar{s}}}.
\end{aligned}$$

Next, we derive an upper bound for $\big(\mathbb{E}\big[\|U_2\|_\infty^s\big]\big)^{1/s}$. Clearly, for all $k=1,\dots,d$ and $j=1,\dots,r$, we have

$$\big|\overline{b}_{k,j,m}(x)-b_{k,j}(x)\big|\leqslant\big||b_{k,j}(x)|-\max(|b_{k,j}(x)|,m^{-1})\big|\leqslant m^{-1}$$

for all $x\in\mathbb{R}^d$. Thus, together with Lemma C.5 and by employing the same arguments as in the beginning of this proof, we obtain

$$\big(\mathbb{E}\big[\|U_2\|_\infty^s\big]\big)^{1/s}\leqslant c\cdot m^{-1}\cdot\sum_{j=1}^r\Big(\mathbb{E}\Big[\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}\big|B_l^j(t)\big|^s\Big]\Big)^{1/s}\leqslant c\cdot\big(m^{-1}+n^{-1}\big)^{\alpha_{\bar{s}}}.\tag{6.3.10}$$

It remains to further analyze $\big(\mathbb{E}\big[\|U_3\|_\infty^s\big]\big)^{1/s}$. As in Step 4 in the proof of Lemma 6.2.2 we consider the random matrix

$$\mathcal{V}=\big(\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_0),\dots,\widetilde{X}_{m,n}^{\mathrm{E},(q)}(t_{m-1})\big).\tag{6.3.11}$$

Due to the definition of the random elements $\widetilde{A}_{m,n}^k$ and Γ^k the vector $\big(\widetilde{A}_{m,n}^1,\Gamma^1,\dots,\widetilde{A}_{m,n}^d,\Gamma^d\big)$ is measurable with respect to $\sigma(\mathcal{V})$. Moreover, Lemma 5.2.1 c) implies that $\big(B_1^1,\dots,B_m^1,\dots,B_1^r,\dots,B_m^r\big)$ and $\mathcal{V}$ are independent. Hence Proposition B.1 b), Proposition B.2, Jensen's inequality, Lemma 6.3.2, and the choice of η, imply the existence of a constant $c(s,\varepsilon)\in(0,\infty)$ which neither depends on m,n nor on M such that

$$\begin{aligned}
\mathbb{E}\big[\|U_3\|_\infty^s\,\big|\,\mathcal{V}=v\big]&=\mathbb{E}\Big[\max_{k=1,\dots,d}\big\|\widetilde{A}_{m,n}^k\cdot\big(\overline{B}^{(\Gamma^k)}-\psi_{m,n,M}^{(\Gamma^k,q,\varepsilon)}\big(\overline{B}^{(\Gamma^k)}\big)\big)\big\|_\infty^s\,\Big|\,\mathcal{V}=v\Big]\\
&\leqslant\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot\mathbb{E}\Big[\max_{k=1,\dots,d}\big\|\overline{B}^{(\Gamma^k(v))}-\psi_{m,n,M}^{(\Gamma^k(v),q,\varepsilon)}\big(\overline{B}^{(\Gamma^k(v))}\big)\big\|_\infty^s\Big]\\
&\leqslant\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot\mathbb{E}\left[\left(\sum_{k=1}^d\big\|\overline{B}^{(\Gamma^k(v))}-\psi_{m,n,M}^{(\Gamma^k(v),q,\varepsilon)}\big(\overline{B}^{(\Gamma^k(v))}\big)\big\|_\infty^{s\eta}\right)^{1/\eta}\right]\\
&\leqslant\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot\left(\sum_{k=1}^d\mathbb{E}\left[\big\|\overline{B}^{(\Gamma^k(v))}-\psi_{m,n,M}^{(\Gamma^k(v),q,\varepsilon)}\big(\overline{B}^{(\Gamma^k(v))}\big)\big\|_\infty^{s\eta}\right]\right)^{1/\eta}\\
&\leqslant\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot\left(\sum_{k=1}^d\big(\delta_M^{(s\eta)}+c(s,\varepsilon)\cdot n^{-1}\big)^{s\eta}\right)^{1/\eta}\\
&=\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot\big(d^{1/s\eta}\big)^s\cdot\big(\delta_M^{(s\eta)}+c(s,\varepsilon)\cdot n^{-1}\big)^s\\
&\leqslant\Big(\max_{k=1,\dots,d}\big(\widetilde{A}_{m,n}^k(v)\big)^s\Big)\cdot(1+\varepsilon)^s\cdot\big(\delta_M^{(s\eta)}+c(s,\varepsilon)\cdot n^{-1}\big)^s
\end{aligned}\tag{6.3.12}$$

for $P_{\mathcal{V}}$-a.a. $v \in \mathbb{R}^{d\times m}$ where $\widetilde{A}^k_{m,n}(v)$ and $\Gamma^k(v)$ denote the realizations associated to v of $\widetilde{A}^k_{m,n}$ and Γ^k, respectively. Thus,

$$\big(\mathbb{E}\big[\|U_3\|_\infty^s\big]\big)^{1/s} = \big(\mathbb{E}\big[\mathbb{E}\,[\|U_3\|_\infty^s \mid V]\,\big]\big)^{1/s} \leqslant \left(\mathbb{E}\Big[\max_{k=1,\dots,d}\big(\widetilde{A}^k_{m,n}\big)^s\Big]\right)^{1/s} \cdot (1+\varepsilon)\cdot\big(\delta_M^{(s\eta)} + c(s,\varepsilon)\cdot n^{-1}\big).$$

Combining this with (6.3.7)--(6.3.10) finishes the proof. □

6.3.3. Proof of Theorem 5.3.4

We prove Theorem 5.3.4 by utilizing, among other things, Lemma 6.3.3.

Proof of Theorem 5.3.4:
Let $\tilde{s} \in (2,\infty)$ with $\tilde{s} \geqslant s$. Moreover, let $(m_N)_{N\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ which satisfies **(Lim1)** and **(Lim3)**. Due to Lemma 6.3.3 and **(Lim3)**, we have

$$\begin{aligned}
&\limsup_{N\to\infty}\Big(\sqrt{\ln N}\cdot e^{(s)}\big(X,\widetilde{X}_N^{(q,\varepsilon)},\mathcal{C}([0,1];\mathbb{R}^d)\big)\Big)\\
&\leqslant \limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\big(\widetilde{A}^k_{m_N,m_N}\big)^s\Big]\right)^{1/s}\cdot(1+\varepsilon)\cdot\big(\delta_{M_N}^{(s\eta)} + c(s,\varepsilon)\cdot\ln m_N\cdot m_N^{-\alpha_{\tilde{s}}}\big)\right)
\end{aligned}$$

with $\alpha_{\tilde{s}} \in (3/4,1)$.

Next, similar to Lemma 6.2.3 one proves that

$$\lim_{N\to\infty}\frac{\ln M_N}{\ln N} = \frac{1}{d}. \tag{6.3.13}$$

Moreover, due to similar arguments as employed in (6.2.22) we conclude that

$$\begin{aligned}
&\left|\left(\mathbb{E}\Big[\max_{k=1,\dots,d}\big(\overline{A}^k_{m_N,m_N}\big)^s\Big]\right)^{1/s} - \left(\mathbb{E}\Big[\max_{k=1,\dots,d}\big(\widetilde{A}^k_{m_N,m_N}\big)^s\Big]\right)^{1/s}\right|\\
&\leqslant\left(\mathbb{E}\left[\max_{k=1,\dots,d}\left(m_N^{-1}\cdot\sum_{i=0}^{m_N-1}\Big|\big\|b_k(X(t_i))\big\|_2 - \big\|\bar{b}_{k,m_N}\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_i)\big)\big\|_2\Big|^2\right)^{s/2}\right]\right)^{1/s}\\
&\leqslant\left(\mathbb{E}\left[\max_{k=1,\dots,d}\left(m_N^{-1}\cdot\sum_{i=0}^{m_N-1}\big\|b_k(X(t_i)) - \bar{b}_{k,m_N}\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_i)\big)\big\|_2^2\right)^{s/2}\right]\right)^{1/s}\\
&=\left(\mathbb{E}\left[\max_{k=1,\dots,d}\left(m_N^{-1}\cdot\sum_{i=0}^{m_N-1}\sum_{j=1}^{r}\big|b_{k,j}\big(X(t_i)\big) - \bar{b}_{k,j,m_N}\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_i)\big)\big|^2\right)^{s/2}\right]\right)^{1/s}\\
&\leqslant\left(\mathbb{E}\left[\max_{k=1,\dots,d}\left(m_N^{-1}\cdot\sum_{i=0}^{m_N-1}\sum_{j=1}^{r}\Big(\big|b_{k,j}\big(X(t_i)\big) - b_{k,j}\big(\widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_i)\big)\big| + m_N^{-1}\Big)^2\right)^{s/2}\right]\right)^{1/s}\\
&\leqslant c\cdot\left(\mathbb{E}\left[\Big(\max_{l=0,\dots,m_N}\big\|X(t_l) - \widetilde{X}^{\mathrm{E},(q)}_{m_N,m_N}(t_l)\big\|_\infty + m_N^{-1}\Big)^s\right]\right)^{1/s}\\
&\leqslant c\cdot m_N^{-1/2}
\end{aligned} \tag{6.3.14}$$

for all $N \in \mathbb{N}$. Now, **(Lim3)** implies

$$\lim_{N\to\infty} m_N^{-1/2} = 0,$$

and hence we obtain

$$\limsup_{N\to\infty} \left| \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \big(\overline{A}^k_{m_N,m_N}\big)^s \Big] \right)^{1/s} - \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \big(\widetilde{A}^k_{m_N,m_N}\big)^s \Big] \right)^{1/s} \right| = 0.$$

Furthermore, by arguing similarly to Step 3 in the proof of Lemma 6.2.4, we get

$$\limsup_{N\to\infty} \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \big(\overline{A}^k_{m_N,m_N}\big)^s \Big] \right)^{1/s} \leqslant \left(\mathbb{E}\left[\max_{k=1,\dots,d} \Big\| \big\| b_k\big(X(\cdot)\big) \big\|_2 \Big\|^s_{L_2[0,1]} \right] \right)^{1/s}.$$

Thus, by the previous two detached formulas we have

$$\limsup_{N\to\infty} \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \big(\widetilde{A}^k_{m_N,m_N}\big)^s \Big] \right)^{1/s} \leqslant \left(\mathbb{E}\left[\max_{k=1,\dots,d} \Big\| \big\| b_k\big(X(\cdot)\big) \big\|_2 \Big\|^s_{L_2[0,1]} \right] \right)^{1/s}. \tag{6.3.15}$$

In particular, note that the properties of b together with Theorem 3.2.3 guarantee

$$\left(\mathbb{E}\left[\max_{k=1,\dots,d} \Big\| \big\| b_k\big(X(\cdot)\big) \big\|_2 \Big\|^s_{L_2[0,1]} \right] \right)^{1/s} < \infty.$$

Since we assumed that $(S_M)_{M\in\mathbb{N}}$ is a strongly asymptotically optimal sequence of order $s\eta$ of M-quantizers $S_M : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$, the same holds true for the sequence $(S_{M_N})_{N\in\mathbb{N}}$. Thus, by Theorem 2.3.11 (ii), by (6.3.13) as well as (6.3.15), and by **(Lim3)**, we conclude that

$$\begin{aligned}
&\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot \left(\mathbb{E}\Big[\max_{k=1,\dots,d} \big(\widetilde{A}^k_{m_N,m_N}\big)^s \Big] \right)^{1/s} \cdot (1+\varepsilon) \cdot \big(\delta^{(s\eta)}_{M_N} + c(s,\varepsilon) \cdot \ln m_N \cdot m_N^{-\alpha_{\tilde{s}}} \big) \right) \\
&\leqslant \left(\mathbb{E}\left[\max_{k=1,\dots,d} \Big\| \big\| b_k\big(X(\cdot)\big) \big\|_2 \Big\|^s_{L_2[0,1]} \right] \right)^{1/s} \cdot (1+\varepsilon) \cdot \left(\limsup_{N\to\infty} \sqrt{\ln N} \cdot \delta^{(s\eta)}_{M_N} \right) \\
&= \left(\mathbb{E}\left[\max_{k=1,\dots,d} \Big\| \big\| b_k\big(X(\cdot)\big) \big\|_2 \Big\|^s_{L_2[0,1]} \right] \right)^{1/s} \cdot (1+\varepsilon) \cdot \left(\lim_{N\to\infty} \frac{\sqrt{\ln N}}{\sqrt{\ln M_N}} \right) \cdot \left(\limsup_{N\to\infty} \sqrt{\ln M_N} \cdot \delta^{(s\eta)}_{M_N} \right) \\
&\leqslant \kappa_\infty \cdot C^{(\infty,s)} \cdot (1+\varepsilon)
\end{aligned}$$

where κ_∞ is the constant in Theorem 2.3.11 (ii), which finishes the proof.

The statement on the size of the range of $\widetilde{X}^{(q,\varepsilon)}_N$ directly follows from (5.3.15) and (5.3.16). □

6.4. Quantization in $\mathcal{C}([0,1];\mathbb{R}^d)$: Proof of Theorem 5.4.1

We consider fixed parameters $s, q \in [1,\infty)$ with $q \geqslant \min\{\widetilde{q} \in 2\mathbb{N} \mid \widetilde{q} \geqslant s\}$. Recall that $(S_K)_{K\in\mathbb{N}}$ is a sequence of K-quantizers $S_K : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ for a Brownian motion on $[0,1]$, and we put

$$\delta_K := e^{(s)}\big(\overline{W}, S_K(\overline{W}), \mathcal{C}[0,1]\big)$$

for $K \in \mathbb{N}$ where $\overline{W}$ denotes a Brownian motion on $[0,1]$.

Moreover, recall that

$$\widetilde{A}^k_{m,n} = \left(m^{-1} \cdot \sum_{i=0}^{m-1} \big\| \overline{b}_{k,m}\big(\widetilde{X}^{\mathrm{E},(q)}_{m,n}(t_i)\big)\big\|_2^2 \right)^{1/2} \qquad \text{and} \qquad \overline{A}^k_{m,n} = \left(m^{-1} \cdot \sum_{i=0}^{m-1} \big\| b_k\big(X(t_i)\big)\big\|_2^2 \right)^{1/2}$$

for $m, n \in \mathbb{N}$ and $k = 1, \ldots, d$.

6.4.1. Auxiliary Lemmas 8 and 9

In analogy to Lemma 6.3.3 we derive the following result.

Lemma 6.4.1
Let $\tilde{s} \in (2, \infty)$ with $\tilde{s} \geqslant s$. Then there exists an $\alpha_{\tilde{s}} \in (3/4, 1)$ such that, for all $m \in \mathbb{N}_3$ and $n, M \in \mathbb{N}$,

$$e^{(s)}\big(X, \widetilde{X}^{(q)}_{m,n,M}, \mathcal{C}([0,1];\mathbb{R}^d)\big) \leqslant c \cdot \ln m \cdot \big(m^{-1} + n^{-1}\big)^{\alpha_{\tilde{s}}} + \left(\mathbb{E}\left[\sum_{k=1}^{d} \big(\widetilde{A}^k_{m,n} \cdot \delta_{K_{k,M}} \big)^s \right] \right)^{1/s} + c \cdot n^{-1} \cdot \left(\mathbb{E}\left[\sum_{k=1}^{d} \big(\widetilde{A}^k_{m,n} \big)^s \right] \right)^{1/s}.$$

Proof:
Let $m \in \mathbb{N}_3$, and let $n, M \in \mathbb{N}$. First, Lemma 6.3.1 and Hölder's inequality guarantee the existence of an $\alpha_{\tilde{s}} \in (3/4, 1)$ such that

$$\Big(\mathbb{E}\Big[\,|\!|\!| X - \overline{X}_m |\!|\!|_s^s \,\Big] \Big)^{1/s} \leqslant c \cdot \ln m \cdot \big(m^{-1} + n^{-1}\big)^{\alpha_{\tilde{s}}}. \tag{6.4.1}$$

Secondly, for $k = 1, \ldots, d$ let U^k_1 as well as U^k_2 be as in the proof of Lemma 6.3.3, and let U^k_3 be given by

$$U^k_3 = \widetilde{A}^k_{m,n} \cdot \Big(\overline{B}^{(\Gamma^k)} - \psi^{(\Gamma^k,q)}_{m,n,K_{M,k}}\big(\overline{B}^{(\Gamma^k)}\big) \Big),$$

where $K_{M,k}$ is the random variable in (5.4.3). Then we have

$$\overline{X}_m - \widetilde{X}^{(q)}_{m,n,M} = U_1 + U_2 + U_3$$

where $U_i = (U^1_i, \ldots, U^d_i)'$ for $i = 1, 2, 3$. Hence

$$\Big(\mathbb{E}\Big[|\!|\!| \overline{X}_m - \widetilde{X}^{(q)}_{m,n,M} |\!|\!|_s^s \Big] \Big)^{1/s} \leqslant \sum_{i=1}^{3} \left(\mathbb{E}\left[\sum_{k=1}^{d} \|U^k_i\|_\infty^s \right] \right)^{1/s}. \tag{6.4.2}$$

Due to (**??**) and (6.3.10) in the proof of Lemma 6.3.3 it holds

$$\max\left\{ \left(\mathbb{E}\left[\sum_{k=1}^{d} \|U^k_1\|_\infty^s \right] \right)^{1/s}, \left(\mathbb{E}\left[\sum_{k=1}^{d} \|U^k_2\|_\infty^s \right] \right)^{1/s} \right\} \leqslant c \cdot \ln m \cdot \big(m^{-1} + n^{-1}\big)^{\alpha_{\tilde{s}}}. \tag{6.4.3}$$

Similar to Lemma 6.3.2 one proves that

$$\Big(\mathbb{E}\Big[\big\| \overline{B}^{(\gamma^k)} - \psi^{(\gamma^k,q)}_{m,n,K}(\overline{B}^{(\gamma^k)}) \big\|_\infty^s \Big] \Big)^{1/s} \leqslant \delta_K + c \cdot n^{-1}$$

for all $k = 1, \ldots, d$ and $K \in \mathbb{N}$ where $\gamma^k \in \mathbb{R}^{m \times r}$ satisfies (5.3.3) as well as (5.3.4). Then similar to (6.3.12)

we conclude that

$$\mathbb{E}\left[\|U_3^k\|_\infty^s \,\middle|\, \mathcal{V}\right] \leqslant \left(\widetilde{A}_{m,n}^k\right)^s \cdot \left(\delta_{K_{k,M}} + c \cdot n^{-1}\right)^s, \qquad P\text{a.s.},$$

for all $k = 1, \ldots, d$ where $\mathcal{V}$ is the random matrix in (6.3.11). Therefore,

$$\left(\mathbb{E}\left[\sum_{k=1}^d \|U_3^k\|_\infty^s\right]\right)^{1/s} \leqslant \left(\mathbb{E}\left[\sum_{k=1}^d \left(\widetilde{A}_{m,n}^k \cdot \delta_{K_{k,M}}\right)^s\right]\right)^{1/s} + c \cdot n^{-1} \cdot \left(\mathbb{E}\left[\sum_{k=1}^d \left(\widetilde{A}_{m,n}^k\right)^s\right]\right)^{1/s}. \tag{6.4.4}$$

Now combining (6.4.1)--(6.4.4) finishes the proof. □

For the remaining part of this subsection we consider a sequence $(m_N)_{N\in\mathbb{N}}$ in $\mathbb{N}$ which satisfies **(Lim1)** and **(Lim3)**. Since to a great extend the proof of the following lemma highly resembles the proof of Lemma 6.2.4, we carry out only a sketch of the proof.

Recall that

$$\mathfrak{C}^{(\infty,s)} = \left(\mathbb{E}\left[\left(\sum_{k=1}^d \left\| \|b_k(X(\cdot))\|_2 \right\|_{L_2[0,1]}^{2s/(s+2)}\right)^{(s+2)/2}\right]\right)^{1/s}$$

and

$$\overline{\kappa}_{\infty,s} = \limsup_{K\to\infty} \sqrt{\ln K} \cdot e^{(s)}\left(\overline{W}, S_K(\overline{W}), \mathcal{C}[0,1]\right)$$

where $\overline{W}$ denotes a Brownian motion on $[0,1]$.

Lemma 6.4.2
It holds

$$\limsup_{N\to\infty} \left(\sqrt{\ln N} \cdot \left(\mathbb{E}\left[\sum_{k=1}^d \left(\widetilde{A}_{m_N,m_N}^k \cdot \delta_{K_{k,M_N}}\right)^s\right]\right)^{1/s}\right) \leqslant \overline{\kappa}_{\infty,s} \cdot \mathfrak{C}^{(\infty,s)}.$$

Proof:
We may assume $\overline{\kappa}_{\infty,s} < \infty$. Note that $\overline{\kappa}_{\infty,s} \geqslant \kappa_\infty > 0$ where κ_∞ is the constant in Theorem 2.3.11 (ii). Moreover, for $N \in \mathbb{N}$ we put

$$\kappa_{N,\infty}^{(s)} := \max_{k=1,\ldots,d} \sqrt{\ln K_{k,M_N}} \cdot \delta_{K_{k,M_N}}.$$

From now on, we write $\kappa_{N,\infty}$ instead of $\kappa_{N,\infty}^{(s)}$. Note that $\overline{\kappa}_{\infty,s} < \infty$ implies

$$\sup_{N\in\mathbb{N}} \kappa_{N,\infty} \leqslant c.$$

Additionally, we abbreviate

$$s^* = 2s/(s+2).$$

Now, analogous to (6.2.17), due to the choice of K_{k,M_N}, see (5.4.3), and the definition of μ^k, see (5.4.2), we derive that

$$\begin{aligned}\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^s\cdot\big(\delta_{K_{k,M_N}}\big)^s &\leqslant \big(\kappa_{N,\infty}\big)^s\cdot\big(\ln M_N\big)^{-s/2}\cdot\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^s\cdot\big(\mu^k\big)^{-s/2}\\ &\leqslant \big(\kappa_{N,\infty}\big)^s\cdot\big(\ln M_N\big)^{-s/2}\cdot\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^s\cdot\left(\frac{\big(\widetilde{A}^k_{m_N,m_N}\big)^{s^*}}{\sum_{i=1}^{d}\big(\widetilde{A}^i_{m_N,m_N}\big)^{s^*}}\right)^{-s/2}\\ &= \big(\kappa_{N,\infty}\big)^s\cdot\big(\ln M_N\big)^{-s/2}\cdot\left(\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*},\end{aligned}$$

and therefore

$$\left(\mathbb{E}\left[\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\cdot\delta_{K_{k,M_N}}\big)^s\right]\right)^{1/s}\leqslant\big(\ln M_N\big)^{-1/2}\cdot\left(\mathbb{E}\left[\big(\kappa_{N,\infty}\big)^s\cdot\left(\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*}\right]\right)^{1/s}\tag{6.4.5}$$

for all $N\in\mathbb{N}$ such that $M_N\geqslant 2$.

Similar to Step 2 in the proof of Lemma 6.2.4, one shows

$$\frac{\limsup_{N\to\infty}\kappa_{N,\infty}}{\overline{\kappa}_{\infty,s}}\leqslant 1.\tag{6.4.6}$$

Moreover, since b is continuous and X has continuous paths, it holds

$$\lim_{N\to\infty}\left(\sum_{k=1}^{d}\big(\overline{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*}=\left(\sum_{k=1}^{d}\big\|\,\|b_k(X(\cdot))\|_2\,\big\|^{s^*}_{L_2[0,1]}\right)^{s/s^*}.$$

Now analogous to Step 3 in the proof of Lemma 6.2.4 one argues that all assumptions of Fatou's lemma are satisfied, which in turn, combined with (6.4.6), then leads to

$$\limsup_{N\to\infty}\mathbb{E}\left[\big(\kappa_{N,\infty}\big)^s\cdot\left(\sum_{k=1}^{d}\big(\overline{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*}\right]\leqslant\big(\overline{\kappa}_{\infty,s}\cdot\mathfrak{C}^{(\infty,s)}\big)^s.\tag{6.4.7}$$

As a next step, by employing arguments similar to those employed in Step 4 in the proof of Lemma 6.2.4, as well as by carrying out an analysis analogous to the one in (6.3.14), we conclude that, dependent on s, there exists an $\alpha\in(0,\infty)$ such that

$$\lim_{N\to\infty}\left|\left(\mathbb{E}\left[\big(\kappa_{N,\infty}\big)^s\cdot\left(\sum_{k=1}^{d}\big(\overline{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*}\right]\right)^{\alpha}-\left(\mathbb{E}\left[\big(\kappa_{N,\infty}\big)^s\cdot\left(\sum_{k=1}^{d}\big(\widetilde{A}^k_{m_N,m_N}\big)^{s^*}\right)^{s/s^*}\right]\right)^{\alpha}\right|=0.$$

Consequently, together with (6.4.7), we obtain

$$\limsup_{N\to\infty}\left(\mathbb{E}\left[\left(\kappa_{N,\infty}\right)^s\cdot\left(\sum_{k=1}^{d}\left(\widetilde{A}^k_{m_N,m_N}\right)^{s*}\right)^{s/s*}\right]\right)^{1/s}\leqslant\limsup_{N\to\infty}\left(\mathbb{E}\left[\left(\kappa_{N,\infty}\right)^s\cdot\left(\sum_{k=1}^{d}\left(\overline{A}^k_{m_N,m_N}\right)^{s*}\right)^{s/s*}\right]\right)^{1/s}$$
$$\leqslant\overline{\kappa}_{\infty,s}\cdot\mathfrak{C}^{(\infty,s)}. \tag{6.4.8}$$

As in Lemma 6.2.3 one shows

$$\lim_{N\to\infty}\frac{\ln M_N}{\ln N}=1. \tag{6.4.9}$$

Now by combining (6.4.5)--(6.4.9) we end up with

$$\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\left(\mathbb{E}\left[\sum_{k=1}^{d}\left(\widetilde{A}^k_{m_N,m_N}\cdot\delta_{K_{k,M_N}}\right)^s\right]\right)^{1/s}\right)\leqslant\overline{\kappa}_{\infty,s}\cdot\mathfrak{C}^{(\infty,s)}.$$

□

6.4.2. Proof of Theorem 5.4.1

We prove Theorem 5.4.1 by employing the Lemmas 6.4.1 and 6.4.2.

Proof of Theorem 5.4.1:
Similar to the proof of Theorem 5.3.4, and by Lemma 6.4.1 as well as **(Lim3)**, we obtain

$$\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot e^{(s)}\big(X,\widetilde{X}^{(q)}_N,\mathcal{C}([0,1];\mathbb{R}^d)\big)\right)\leqslant\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\left(\mathbb{E}\left[\sum_{k=1}^{d}\left(\widetilde{A}^k_{m_N,m_N}\cdot\delta_{K_{k,M_N}}\right)^s\right]\right)^{1/s}\right).$$

Now Lemma 6.4.2 leads to

$$\limsup_{N\to\infty}\left(\sqrt{\ln N}\cdot\left(\mathbb{E}\left[\sum_{k=1}^{d}\left(\widetilde{A}^k_{m_N,m_N}\cdot\delta_{K_{k,M_N}}\right)^s\right]\right)^{1/s}\right)\leqslant\overline{\kappa}_{\infty,s}\cdot\mathfrak{C}^{(\infty,s)}.$$

The statement on the size of the range of $\widetilde{X}^{(q)}_N$ directly follows from (5.4.4) and (5.4.5).

□

7 | Final Remarks and Open Problems

We close this thesis by summarizing our main results and by presenting a selection of open problems.

Ad Chapter 2

In section 2.3.1 we applied a quantization procedure for such real-valued random variables which have finite moments of any order, see [DV11] and [MGRY15], for quantization of $Z = \int_0^1 W_1(s)\,dW_2(s)$ where (W_1, W_2) is a two-dimensional Brownian motion. By the construction of the mappings $T_N^{(\gamma)}$, $N \in \mathbb{N}$, see (2.3.8), we know the support points of the quantization $\widetilde{Z}_N^{(\gamma)} := T_N^{(\gamma)}(Z)$ for all $N \in \mathbb{N}$. But so far the construction has only been of a semi-constructive type since, to us, the distribution P_Z is unknown, see also Remark 2.3.8. Hence, to us the determination (or at least approximation) of P_Z remains an open problem, which leads to the following question:

Question 1: How to compute (or at least) approximate the probability weights corresponding to the support points of the quantizations $\widetilde{Z}_N^{(\gamma)}$ for $N \in \mathbb{N}$?

Ad Chapter 4

In Chapter 4 we derived new results on the sharp asymptotics of the Nth minimal quantization error of one-dimensional Itô processes, see Theorem 4.1.2. Subsequently we applied the results obtained in the aforementioned theorem to derive the sharp asymptotics of the Nth minimal quantization error of solutions of such one-dimensional SDEs which are driven by a multidimensional Brownian motion, see Proposition 4.2.1. In case that the dimension r of the driving Brownian motion of the SDE (4.2.1) satisfies $r \geqslant 2$, Proposition 4.2.1 generalizes the results on the sharp asymptotics of the Nth minimal quantization error of solutions of one-dimensional SDEs driven by a one-dimensional Brownian motion presented in [Der08a, Theorem 1.1.] and [Der08b, Theorem 1.1.]. We closed the chapter by providing a lower bound for the Nth minimal quantization error w.r.t. N-uniform-product-quantizations of multidimensional Itô processes in the space $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$, see Theorem 4.3.2.

Although the main results of this section, namely Theorem 4.1.2 and Theorem 4.3.2, basically served as auxiliary results to derive Theorem 5.2.6, Proposition 5.3.5, and Theorem 5.4.2, they give rise to the following research question:

Question 2: The results in Theorem 4.1.2 yield the sharp rate of convergence of $(\ln N)^{-1/2}$ of the Nth minimal quantization error of multidimensional Itô processes in the spaces $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$, $p \in [1,\infty)$ and $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$. What are the corresponding sharp asymptotic constants?

Ad Chapter 5

In Chapter 5 we presented a semi-constructive method which led to sequences of (strongly) asymptotically optimal quantizations of the solution X of the SDE (3.1.1) in the spaces $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$, $p \in [1,\infty)$, $(\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)$, $s \in [1,\infty)$, and $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$. Together with the results obtained in Chapter 4 we were able to derive the sharp asymptotics of the Nth minimal quantization error w.r.t. general N-product-quantizations in the spaces $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$ and $(\mathcal{C}([0,1];\mathbb{R}^d), |||\cdot|||_s)$ as well as of the Nth minimal quantization error w.r.t. N-uniform-product-quantizations in the space $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$. Considering that the following further research question arises:

Question 3: As mentioned in the Introduction, the sharp rate of convergence of $(\ln N)^{-1/2}$ of the Nth minimal quantization error of solutions of multidimensional SDEs in the spaces $(L_p^d[0,1], \|\cdot\|_{L_p^d[0,1]})$, $p \in [1,\infty)$ and $(\mathcal{C}([0,1];\mathbb{R}^d), \|\cdot\|_\infty)$ is already known. What are the corresponding sharp asymptotic constants?

Moreover, the fact that our method is semi-constructive leads to the following open problems:

Question 4: To be able to compute the probability weights corresponding to the Milstein quantization of X we need to know the joint distribution of the random variables $Y_1^j, j = 1,\dots,r$, and $I^1_{(j_1,j_2)}$, $j_1, j_2 = 1,\dots,r$ with $j_1 < j_2$. How to determine (or at least approximate) this distribution?

Question 5: How to compute the probability weights corresponding to the overall quantizations of X constructed in the Sections 5.2.2, 5.3.2, and 5.4.2?

As there can be all kind of interdependencies between the components of X, we conjecture that one can hardly find a general answer to Question 5.

A | Selected Inequalities

Gronwall's Lemma

Lemma A.1
Let $\alpha, \beta \in \mathbb{R}$ with $\beta \geqslant 0$, let $T \in (0, \infty)$, and let $f : [0, T] \to [0, \infty)$ be a bounded Borel measurable function such that

$$f(t) \leqslant \alpha + \beta \cdot \int_0^t f(s)\, ds$$

for all $t \in [0, T]$. Then,

$$\sup_{t \in [0,T]} f(t) \leqslant \alpha \cdot \exp(\beta \cdot T).$$

Proof:
See, for example, [RY99, Appendix, §1] or [Kuo06, Lemma 10.2.2.]. □

The following corollary contains a discrete version of Gronwall's lemma.

Corollary A.2
Let $\alpha, \beta \in [0, \infty)$, let $m \in \mathbb{N}$, and let $T \in (0, \infty)$. Moreover, let $f_0, \ldots, f_m$ and $\Delta_0, \ldots, \Delta_{m-1}$ be non-negative real numbers such that

$$\sum_{l=0}^{m-1} \Delta_l = T$$

and

$$f_{l+1} \leqslant \alpha \cdot \Delta_l + f_l \cdot \big(1 + \beta \cdot \Delta_l\big)$$

for all $l = 0, \ldots, m-1$. Then,

$$\max_{l=0,\ldots,m} f_l \leqslant (f_0 + \alpha \cdot T) \cdot \exp(\beta \cdot T).$$

Proof:
The proof is carried out analogously to the proof of Corollary 1 in [MG02b, Appendix, §1]. □

Minkowski's Integral Inequality

The following result is a special case of Minkowski's integral inequality.

Proposition A.3
Let $T \in [0, \infty)$, let $q \in [1, \infty)$, let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $Y : [0, T] \times \Omega \to [0, \infty)$ be a $\mathcal{B}([0, T]) \otimes \mathcal{A}$-$\mathcal{B}([0, \infty))$-measurable stochastic process. Then,

$$\left(\mathbb{E}\left[\left| \int_0^T Y(s)\, ds \right|^q \right] \right)^{1/q} \leqslant \int_0^T \left(\mathbb{E}\big[|Y(s)|^q \big] \right)^{1/q} ds.$$

Proof:
See, for example, [HLP52, Theorem 202] or [Ste70, Appendix, §A.1]. □

B | Properties of Conditional Expected Values

As a reference for the presented statements and their proofs one might use [GS77, Chapter 5] or [Kle14, Chapter 8].

Part I

First, we collect basic properties of conditional expected values, which are employed in this thesis. All (in)equalities concerning conditional expected values are understood to hold almost surely.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $X, Y : \Omega \to \mathbb{R}$ be $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable with

$$\max\left\{\mathbb{E}\left[|X|\right], \mathbb{E}\left[|Y|\right]\right\} < \infty.$$

Proposition B.1
Let $\mathcal{G} \subseteq \mathcal{A}$ be a σ-algebra.

a) For all $\lambda \in \mathbb{R}$ it holds $\mathbb{E}\left[\lambda \cdot X + Y \mid \mathcal{G}\right] = \lambda \cdot \mathbb{E}\left[X \mid \mathcal{G}\right] + \mathbb{E}\left[Y \mid \mathcal{G}\right]$. (Linearity)

b) If $Y \leqslant X$ P-a.s., then $\mathbb{E}\left[Y \mid \mathcal{G}\right] \leqslant \mathbb{E}\left[X \mid \mathcal{G}\right]$. (Monotonicity)

c) If $\mathbb{E}\left[|X \cdot Y|\right] < \infty$ and Y is $\mathcal{G}$-$\mathcal{B}(\mathbb{R})$-measurable, then

$$\mathbb{E}\left[X \cdot Y \mid \mathcal{G}\right] = Y \cdot \mathbb{E}\left[X \mid \mathcal{G}\right] \qquad \text{and} \qquad \mathbb{E}\left[Y \mid \mathcal{G}\right] = Y.$$

d) If $\sigma(X)$ and $\mathcal{G}$ are independent, then $\mathbb{E}\left[X \mid \mathcal{G}\right] = \mathbb{E}\left[X\right]$. (Independence)

Part II

Secondly, we employ the following fact on conditional expected values at several places within this thesis.

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let $(D, \mathcal{D})$, $(E, \mathcal{E})$ be measurable spaces, let $X : \Omega \to D$ be $\mathcal{A}$-$\mathcal{D}$-measurable, and let $Y : \Omega \to E$ be $\mathcal{A}$-$\mathcal{E}$-measurable.

Proposition B.2
Assume that X and Y are independent, and let $T : D \times E \to \mathbb{R}$ be a $\mathcal{D}\otimes\mathcal{E}$-$\mathcal{B}(\mathbb{R})$-measurable mapping with

$$\mathbb{E}\big[|T(X,Y)|\big] < \infty.$$

Then, for P_X-a.a. $x \in D$,

$$\mathbb{E}\left[T(X,Y) \mid X = x\right] = \mathbb{E}\left[T(x,Y)\right].$$

C | Further Auxiliary Statements

Part I

In this section we provide auxiliary statements which are employed in Chapter 2.

Lemma C.1
Let X, Y, X_n, Y_n, $n \in \mathbb{N}$, be real-valued random variables such that

(i) $X_n \xrightarrow{\mathcal{L}} X$,

(ii) $Y_n \xrightarrow{\mathcal{L}} Y$, *and*

(iii) for all $n \in \mathbb{N}$ *it holds* $X_n \overset{\mathcal{L}}{=} Y_n$.

Then, $X \overset{\mathcal{L}}{=} Y$.

Proof:
We show that $F_X(t) = F_Y(t)$ for all $t \in \mathbb{R}$. For $Z \in \{X, Y\}$ we put

$$\mathcal{C}(F_Z) := \{t \in \mathbb{R} \mid F_Z \text{ is continuous in } t\}.$$

First, let $t \in \mathcal{C}(F_X) \cap \mathcal{C}(F_Y)$. Then by (i)--(iii) we have

$$F_X(t) = \lim_{n\to\infty} F_{X_n}(t) = \lim_{n\to\infty} F_{Y_n}(t) = F_Y(t). \tag{C.1}$$

Secondly, let $t \in \big(\mathcal{C}(F_X) \cap \mathcal{C}(F_Y)\big)^C$. It is well known that $\big(\mathcal{C}(F_X)\big)^C$ and $\big(\mathcal{C}(F_Y)\big)^C$ are countable. Thus, $\big(\mathcal{C}(F_X) \cap \mathcal{C}(F_Y)\big)^C$ is countable, and hence $\mathcal{C}(F_X) \cap \mathcal{C}(F_Y)$ is dense in $\mathbb{R}$. Consequently, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(F_X) \cap \mathcal{C}(F_Y)$ with $t_n \downarrow t$. Together with the right-continuity property of distribution functions and (C.1) we arrive at

$$F_X(t) = F_X\big(\lim_{n\to\infty} t_n\big) = \lim_{n\to\infty} F_X(t_n) = \lim_{n\to\infty} F_Y(t_n) = F_Y\big(\lim_{n\to\infty} t_n\big) = F_Y(t),$$

which finishes the proof. □

The next result is sometimes also referred to as *Kac's theorem*. As a reference one might use, for example, [App05, Theorem 2.1].

Notation
Let $d \in \mathbb{N}$, and let Z be an $\mathbb{R}^d$-valued random vector. By φ_Z we denote the characteristic function of Z, which is given by $\varphi_Z(t) = \mathbb{E}\big[\exp\big(i \cdot \langle t, Z\rangle\big)\big]$ for $t \in \mathbb{R}^d$ where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^d$.

Proposition C.2
Let $d \in \mathbb{N}$, and let $X^1, \dots, X^d$ be real-valued random variables. Then the following statements are equivalent:

(i) The random variables $X^1, \dots, X^d$ are independent.

(ii) For all $(x^1, \dots, x^d)' \in \mathbb{R}^d$ it holds

$$\varphi_{(X^1,\dots,X^d)}(x^1, \dots, x^d) = \prod_{i=1}^{d} \varphi_{X^i}(x^i).$$

Part II

In this section we provide an auxiliary statement which is employed in the proof of Theorem 4.3.2. More precisely, we present a multivariate extension of the well-known theorems of Dambis, Dubins and Schwarz, see [Dam65] and [DS65].

Throughout this section let $d \in \mathbb{N}$, and we use $(\Omega, \mathcal{A}, P)$ to denote a probability space equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$ which satisfies the usual conditions.

The following result is often referred to as *Knight's theorem*. As a reference one might use, for example, [KS88, Theorem 3.4.13].

Proposition C.3 *(F. B. Knight (1971))*
Let $M = (M(t))_{t\in[0,\infty)} = (M^1(t), \dots, M^d(t))'_{t\in[0,\infty)}$ be a continuous stochastic process defined on $(\Omega, \mathcal{A}, P)$ such that

i) M is adapted to $\mathcal{F}$,

ii) for every $k = 1, \dots, d$ the process $(M^k(t))_{t\in[0,\infty)}$ is a local $\mathcal{F}$-martingale,

iii) for every $k = 1, \dots, d$ it holds $\lim_{t\to\infty}\langle M^k\rangle(t) = \infty$ P-a.s. where $\langle M^k\rangle$ denotes the quadratic variation process of M^k, and

iv) for every $i, k \in \{1, \dots, d\}$ with $i \neq k$ and every $t \in [0, \infty)$ it holds $\langle M^i, M^k\rangle(t) = 0$ where $\langle M^i, M^k\rangle$ denotes the cross-variation process of M^i and M^k.

Furthermore, for $s \in [0, \infty)$ and $k = 1, \dots, d$ consider the stopping time

$$T^k(s) := \inf\big\{t \in [0, \infty) \,\big|\, \langle M^k\rangle(t) \geqslant s\big\}.$$

Then the stochastic process $B = (B(s))_{s\in[0,\infty)} = (B^1(s), \dots, B^d(s))'_{s\in[0,\infty)}$ with $B^k(s) := M^k(T^k(s))$ for $s \in [0, \infty)$ and $k = 1, \dots, d$ is a d-dimensional Brownian motion.

Part III

In this section we assume the setting in Chapter 5, and we provide two auxiliary results which are employed in the proofs of the main results of Chapter 5.

For $m \in \mathbb{N}$ let $0 = t_0 < \cdots < t_m = 1$ be the discretization of $[0,1]$ as defined in (5.0.1), and recall that the independent Brownian bridges B_l^j considered in Chapter 5 are given by

$$B_l^j(t) = W_j(t) - W_j(t_{l-1}) - (t - t_{l-1}) \cdot m \cdot \big(W_j(t_l) - W_j(t_{l-1})\big)$$

for $t \in [t_{l-1}, t_l]$, $l = 1, \ldots, m$ and $j = 1, \ldots, r$.

The following auxiliary result will be used in the proof of Lemma C.5.

Lemma C.4
Let $(B, \|\cdot\|_B)$ be a Banach space, and let X be a B-valued Gaussian random element. Then for all $p, q \in (0,\infty)$ there exists a constant $c(p,q) \in (0,\infty)$ depending only on p and q such that

$$\big(\mathbb{E}\left[\|X\|_B^p\right]\big)^{1/p} \leqslant c(p,q) \cdot \big(\mathbb{E}\left[\|X\|_B^q\right]\big)^{1/q}.$$

Proof:
For a proof we refer the reader to [LT91, Corollary 3.2.]. □

Lemma C.5
Let $q \in [1,\infty)$, and let $j \in \{1, \ldots, r\}$. Then there exists a constant $c \in (0,\infty)$ such that, for all $m \in \mathbb{N}_2$,

$$\left(\mathbb{E}\Big[\max_{l=1,\ldots,m} \sup_{t \in [t_{l-1}, t_l]} |B_l^j(t)|^q\Big]\right)^{1/q} \leqslant c \cdot \left(\frac{\ln m}{m}\right)^{1/2}.$$

Proof:
Let $m \in \mathbb{N}_2$. Throughout the proof c denotes a not further specified positive real constant which may vary from line to line and which does not depend on m.

The definition of the processes B_l^j leads to

$$\begin{aligned}
&\left(\mathbb{E}\Big[\max_{l=1,\ldots,m} \sup_{t \in [t_{l-1}, t_l]} |B_l^j(t)|^q\Big]\right)^{1/q} \\
&\leqslant \left(\mathbb{E}\Big[\max_{l=1,\ldots,m} \sup_{t \in [t_{l-1}, t_l]} |W_j(t) - W_j(t_{l-1})|^q\Big]\right)^{1/q} + \left(\mathbb{E}\Big[\max_{l=1,\ldots,m} |W_j(t_l) - W_j(t_{l-1})|^q\Big]\right)^{1/q}.
\end{aligned}$$

First, we further estimate the second summand in the above inequality. Recall that $Y_1^j, \ldots, Y_m^j$ are independent standard normally distributed random variables where $Y_l^j = m^{1/2} \cdot W_j(t_l) - W_j(t_{l-1})$ for $l = 1, \ldots, m$. By Lemma C.4 and [LT91, Formula (3.6)], we obtain

$$\begin{aligned}
\left(\mathbb{E}\Big[\max_{l=1,\ldots,m} |W_j(t_l) - W_j(t_{l-1})|^q\Big]\right)^{1/q} &= m^{-1/2} \cdot \left(\mathbb{E}\Big[\max_{l=1,\ldots,m} |Y_l^j|^q\Big]\right)^{1/q} \\
&\leqslant c \cdot m^{-1/2} \cdot \mathbb{E}\Big[\max_{l=1,\ldots,m} |Y_l^j|\Big] \\
&\leqslant c \cdot \left(\frac{\ln m}{m}\right)^{1/2}.
\end{aligned}$$

Secondly, the processes $(W_j(t) - W_j(t_{l-1}))_{t \in [t_{l-1}, t_l]}$, $l = 1, \ldots, m$, are independent Brownian motions on the respective subintervals $[t_{l-1}, t_l]$. Then, by the scaling properties of Brownian motions and by

Lemma C.4, we obtain

$$\left(\mathbb{E}\Big[\max_{l=1,\dots,m}\sup_{t\in[t_{l-1},t_l]}|W_j(t)-W_j(t_{l-1})|^q\Big]\right)^{1/q} = m^{-1/2}\cdot\left(\mathbb{E}\Big[\max_{l=1,\dots,m}\|\mathcal{W}_l\|_\infty^q\Big]\right)^{1/q} \leqslant c\cdot m^{-1/2}\cdot\mathbb{E}\Big[\max_{l=1,\dots,m}\|\mathcal{W}_l\|_\infty\Big] \tag{C.2}$$

where $\mathcal{W}_1,\dots,\mathcal{W}_m$ denote independent Brownian motions on $[0,1]$. Moreover, similar to [LT91, Formula (3.6)] and by utilizing [LT91, Lemma 3.1.], one derives that

$$\mathbb{E}\Big[\max_{l=1,\dots,m}\|\mathcal{W}_l\|_\infty\Big] \leqslant c\cdot\big(\ln m\big)^{1/2}, \tag{C.3}$$

which finishes the proof. □

For the next auxiliary lemma recall that

$$|f|_\alpha = \sup_{0\leqslant s<t\leqslant 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha}$$

for $f:[0,1]\to\mathbb{R}$ and $\alpha\in(0,1/2)$, and recall that $X=(X^1,\dots,X^d)'$ is the strong solution of the SDE (3.1.1).

Lemma C.6
Let $k\in\{1,\dots,d\}$. Then, for all $q\in[1,\infty)$ and $\alpha\in(0,1/2)$, it holds

$$\mathbb{E}\big[|X^k|_\alpha^q\big]<\infty.$$

Proof:
Let $q\in[1,\infty)$, and let $\alpha\in(0,1/2)$. Throughout this proof c denotes a not further specified positive real constant which may be different at every occurrence and which might only depend on the moment parameter q, on α, and on the constant in **(C2)**.

Recall that P-a.s.

$$X^k(t) = x_0^k + \int_0^t a_k\big(X(s)\big)\,ds + \sum_{j=1}^r\int_0^t b_{k,j}\big(X(s)\big)\,dW_j(s)$$

for all $t\in[0,1]$. We put

$$A(t) := x_0^k + \int_0^t a_k\big(X(s)\big)\,ds \quad\text{and}\quad M_j(t) := \int_0^t b_{k,j}\big(X(s)\big)\,dW_j(s),\ j=1,\dots,r,$$

for $t\in[0,1]$. Then by the triangle inequality we have

$$\big(\mathbb{E}\big[|X^k|_\alpha^q\big]\big)^{1/q} \leqslant \big(\mathbb{E}\big[|A|_\alpha^q\big]\big)^{1/q} + \sum_{j=1}^r\big(\mathbb{E}\big[|M_j|_\alpha^q\big]\big)^{1/q}. \tag{C.2}$$

Moreover, due to the fact that a is of at most linear growth, and due to Theorem 3.2.3, we obtain

$$\begin{aligned}
\big(\mathbb{E}\big[|A|_\alpha^q\big]\big)^{1/q} &= \left(\mathbb{E}\left[\left(\sup_{0\leqslant s<t\leqslant 1}\frac{|A(t)-A(s)|}{|t-s|^\alpha}\right)^q\right]\right)^{1/q}\\
&\leqslant \left(\mathbb{E}\left[\left(\sup_{0\leqslant s<t\leqslant 1}\frac{|A(t)-A(s)|}{|t-s|}\right)^q\right]\right)^{1/q}\\
&= \left(\mathbb{E}\left[\left(\sup_{0\leqslant s<t\leqslant 1}\frac{\left|\int_s^t a_k(X(u))\,du\right|}{|t-s|}\right)^q\right]\right)^{1/q}\\
&\leqslant \left(\mathbb{E}\left[\left(\sup_{0\leqslant s<t\leqslant 1}\frac{\big(\sup_{u\in[0,1]}|a_k(X(u))|\big)\cdot|t-s|}{|t-s|}\right)^q\right]\right)^{1/q}\\
&= \left(\mathbb{E}\Big[\sup_{u\in[0,1]}\big|a_k\big(X(u)\big)\big|^q\Big]\right)^{1/q}\\
&\leqslant c\cdot\Big(1+\mathbb{E}\big[\|X\|_\infty^q\big]\Big)^{1/q}\\
&< \infty.
\end{aligned} \tag{C.3}$$

Furthermore, the properties of b, Fubini's theorem and Theorem 3.2.3 yield

$$\mathbb{E}\left[\int_0^1\big(b_{k,j}\big(X(u)\big)\big)^2\,du\right]\leqslant c\cdot\big(1+\mathbb{E}\big[\|X\|_\infty^2\big]\big)<\infty.$$

Additionally, the processes $\big(b_{k,j}(X(u))\big)_{u\in[0,1]}$, $j=1,\dots,r$, are measurable and adapted to $\mathcal{F}$, where $\mathcal{F}$ is the filtration constructed in Section 3.1.1. Hence for all $j=1,\dots,r$ the process M_j satisfies the assumptions of Theorem 3.1. in [Der08b], which combined with the properties of b, Fubini's theorem, and Theorem 3.2.3 then yields that for every $\kappa\in\big(\max\{q,2/(1-2\alpha)\},\infty\big)$ we have

$$\big(\mathbb{E}\big[|M_j|_\alpha^q\big]\big)^{1/q}\leqslant\big(\mathbb{E}\big[|M_j|_\alpha^\kappa\big]\big)^{1/\kappa}\leqslant c\cdot\left(\int_0^1\mathbb{E}\left[\big|b_{k,j}\big(X(u)\big)\big|^\kappa\right]du\right)^{1/\kappa}\leqslant c\cdot\Big(1+\big(\mathbb{E}\big[\|X\|_\infty^\kappa\big]\big)^{1/\kappa}\Big)<\infty. \tag{C.4}$$

Now combining (C.2)--(C.4) finishes the proof.

□

Notation Index

$\mathbb{N}$	natural numbers without 0
$\mathbb{Z}$	integers
$\mathbb{N}_z$	non-negative integers greater or equal than $z \in \mathbb{Z}$
$\mathbb{R}$	real numbers

Sets

$\varnothing$	empty set
A^c	the complement of a set A
$\lvert A \rvert$	the cardinality of a set A

Real Numbers

$\lfloor x \rfloor$	the largest integer $z \leqslant x$
$\lceil x \rceil$	the smallest integer $z \geqslant x$
$\lvert x \rvert$	the absolute value of x
$x \wedge y$	the minimum of x and y
$\langle \cdot, \cdot \rangle$	the standard scalar product on $\mathbb{R}^d$
$a_n = o(b_n)$	small O-notation for real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, i.e., $\lim_{n\to\infty} a_n/b_n = 0$ if $b_n \neq 0$ for all $n \geqslant N$ for an $N \in \mathbb{N}$

Vectors and Matrices

$\lVert v \rVert_p$	the ℓ_p-norm of a vector $v \in \mathbb{R}^d$ for $p \in [1, \infty)$

$\|v\|_\infty$	the max-norm of a vector $v \in \mathbb{R}^d$
V_i	the ith row of a Matrix $V = (v_{i,j})$
$V^{(j)}$	the jth column of a Matrix $V = (v_{i,j})$
$\|V\|_p$	the p-norm of a Matrix $V = (v_{i,j})$ for $p \in [1, \infty)$

Mappings and Functional Analysis

$\operatorname{ran}(T)$	the image set of a mapping $T : A \to B$
$\frac{\partial g}{\partial x_k}$	the partial derivative of a partially differentiable function $g : \mathbb{R}^d \to \mathbb{R}$ w.r.t. to the kth variable for $k = 1, \ldots, d$
∇f	the Jacobian matrix of a partially differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$
$L(D, E)$	the space of continuous linear operators $T : D \to E$ between normed vector spaces $(D, \|\cdot\|_D)$ and $(E, \|\cdot\|_E)$
$\|\cdot\|_{\mathrm{op}}$	operator norm
D'	the dual space of a normed vector space $(D, \|\cdot\|_D)$
D''	the bidual space of a normed vector space $(D, \|\cdot\|_D)$

Function Spaces

$\mathcal{C}([a, b]; \mathbb{R}^d)$	the space of continuous functions $f : [a, b] \to \mathbb{R}^d$
$\mathcal{C}[a, b]$	shorter notation for $\mathcal{C}([a, b]; \mathbb{R})$
$\|\cdot\|_\infty$	the supremum norm
$\|f\|_{L_p([a,b];\mathbb{R}^d)}$	the $L_p([a, b]; \mathbb{R}^d)$-(quasi-)norm of a function $f : [a, b] \to \mathbb{R}^d$, see p. 8
$\|f\|_{L_p^d[a,b]}$, $\|f\|_{L_p[a,b]}$	shorter notation for $\|f\|_{L_p([a,b];\mathbb{R}^d)}$ and $\|f\|_{L_p([a,b];\mathbb{R})}$
$L_p([a, b], \mathbb{R}^d)$	the space of all (equivalence classes) of functions $f : [a, b] \to \mathbb{R}^d$ with finite $L_p([a, b]; \mathbb{R}^d)$-norm
$L_p^d[a, b]$, $L_p[a, b]$	shorter notation for $L_p([a, b], \mathbb{R}^d)$ and $L_p([a, b], \mathbb{R})$
$\mathcal{C}^\alpha([a, b]; \mathbb{R}^d)$	space of α-Hölder continuous functions $f : [a, b] \to \mathbb{R}^d$ for $\alpha \in (0, 1]$
$\|f\|_{\alpha,a,b}$	norm on space $\mathcal{C}^\alpha([a, b]; \mathbb{R}^d)$, see p. 8

Probability and Measure

$\mathcal{B}(D)$	the Borel σ-algebra on a normed vector space $(D, \|\cdot\|_D)$
λ_d	d-dimensional Lebesgue measure
random variable	term used to refer to a $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable mapping $X : \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{A}, P)$ is a probability space
random vector	term used to refer to a $\mathcal{A}$-$\mathcal{B}(\mathbb{R}^d)$-measurable mapping $X : \Omega \to \mathbb{R}^d$ where $(\Omega, \mathcal{A}, P)$ is a probability space
random matrix	term used to refer to a $\mathcal{A}$-$\mathcal{B}(\mathbb{R}^{d\times m})$-measurable mapping $X : \Omega \to \mathbb{R}^{d\times m}$ where $(\Omega, \mathcal{A}, P)$ is a probability space
random element	term used to refer to a $\mathcal{A}$-$\mathcal{B}(B)$-measurable mapping $X : \Omega \to B$ where $(\Omega, \mathcal{A}, P)$ is a probability space and $(B, \|\cdot\|_B)$ is a Banach space
P_X	distribution (law) of a random element X
$\mathbb{E}[X]$	expected value of a random element X
$\mathbb{E}[X \mid \sigma(Y)]$, $\mathbb{E}[X \mid Y]$	conditional expected value of the random variable X given the σ-algebra $\sigma(Y)$ generated by the random element Y
$\mathrm{Cov}[X, Y]$	covariance of two random variables X and Y
$N(\mu, \sigma^2)$	the normal distribution with mean μ and variance σ^2
$N(0, 1)$	the standard normal distribution
F_X	the distribution function of a random variable X
$X \overset{\mathcal{L}}{=} Y$	equal in law
$X_n \xrightarrow{\mathcal{L}} X$	convergence in law for random elements X and X_n, $n \in \mathbb{N}$
$X_n \xrightarrow{L_p} X$	L_p-convergence, $p \in [1, \infty)$, i.e., $\lim_{n\to\infty} \mathbb{E}\left[\left\|X - X_n\right\|^p\right] = 0$, for random variables X and X_n, $n \in \mathbb{N}$
φ_X	the characteristic function of a random vector X

Quantization

$e^{(s)}(Z, \widetilde{Z}, B)$	the quantization error of order s of a B-valued random element Z corresponding to the quantization $\widetilde{Z}$	p. 12
$e_N^{(s)}(Z, B)$	the Nth minimal quantization error of order s of a B-valued random element Z	p. 12

$C_N^{(s)}(X, B)$	the set of all N-optimal quantizations of order s of a B-valued random element Z	p. 12

Abbreviations

a.a.	almost all
a.e.	almost everywhere
a.s.	almost surely
corresp.	corresponding (to)
w.r.t.	with respect to
SDE	stochastic differential equation
ODE	ordinary differential equation

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Bibliography

[ADI07] S. Ankirchner, S. Dereich, and P. Imkeller. Enlargement of Filtrations and Continuous Girsanov-Type Embeddings. In C. Donati-Martin, M. Émery, A. Rouault, and C. Stricker, editors, *Séminaire de Probabilités XL*, Lect. Notes in Math., vol 1899, pages 389–410. Springer, Berlin, Heidelberg, 2007.

[App05] D. Applebaum. Lévy Processes in Euclidean Spaces and Groups. In M. Schürmann and U. Franz, editors, *Quantum Independent Increment Processes I*, Lect. Notes in Math., vol 1865, pages 1–98. Springer, Berlin, Heidelberg, 2005.

[Arn74] L. Arnold. *Stochastic Differential Equations.* Wiley, New York, 1974.

[Ben48] W. R. Bennett. Spectra of Quantized Signals. *Bell Syst. Tech. J.*, 27(3):446–472, 1948.

[BW82] J. A. Bucklew and G. L. Wise. Multidimensional Asymptotic Quantization Theory with rth Power Distortion Measures. *IEEE Trans. Inform. Theory*, 28(2):239–247, 1982.

[CDMGR09] J. Creutzig, S. Dereich, T. Müller-Gronbach, and K. Ritter. Infinite-dimensional Quadrature and Approximation of Distributions. *Found. Comput. Math.*, 9(4):391–429, 2009.

[Dam65] K. E. Dambis. On the decomposition of continuous submartingales. *Theory Probab. Appl.*, 10(3):401–410, 1965.

[Der03] S. Dereich. *High Resolution Coding of Stochastic Processes and Small Ball Probabilities.* PhD thesis, Technische Universität Berlin, 2003.

[Der08a] S. Dereich. The coding complexity of diffusion processes under $L^p[0,1]$-norm distortion. *Stoch. Process. Appl.*, 118(6):938–951, 2008.

[Der08b] S. Dereich. The coding complexity of diffusion processes under supremum norm distortion. *Stoch. Process. Appl.*, 118(6):917–937, 2008.

[Der09] S. Dereich. Asymptotic formulae for coding problems and intermediate optimization problems: a review. In J. Blath, P. Mörters, and M. Scheutzow, editors, *Trends in Stochastic Analysis*, London Math.Soc.Lecture Note Ser., vol 353, pages 187–232. Cambridge Univ. Press, Cambridge, 2009.

[DFMS03] S. Dereich, F. Fehringer, A. Matoussi, and M. Scheutzow. On the Link Between Small Ball Probabilities and the Quantization Problem for Gaussian Measures on Banach Spaces. *J. Theoret. Probab.*, 16(1):249–265, 2003.

[DS65] L. E. Dubins and G. Schwarz. On continuous martingales. *Proc. Natl. Acad. Sci. USA*, 53(5):913–916, 1965.

[DS06] S. Dereich and M. Scheutzow. High-resolution quantization and entropy coding for fractional Brownian motion. *Electron. J. Probab.*, 11:700–722, 2006.

[DV11] S. Dereich and C. Vormoor. The High Resolution Vector Quantization Problem with Orlicz Norm Distortion. *J. Theoret. Probab.*, 24(2):517–544, 2011.

[Fau92] O. Faure. *Simulation du mouvement brownien et des diffusions*. PhD thesis, Ecole Nationale des Ponts et Chaussées, 1992.

[Feh01] F. Fehringer. *Kodierung von Gaußmaßen*. PhD thesis, Technische Universität Berlin, 2001.

[GL00] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*. Springer, Berlin, Heidelberg, 2000.

[GLP07] S. Graf, H. Luschgy, and G. Pagès. Optimal quantizers for Radon random vectors in a Banach space. *J. Approx. Theory*, 144(1):27–53, 2007.

[GN98] R. M. Gray and D. L. Neuhoff. Quantization. *IEEE Trans. Inform. Theory*, 44(6):2325–2383, 1998.

[GS77] P. Gänssler and W. Stute. *Wahrscheinlichkeitstheorie*. Springer, Berlin, Heidelberg, 1977.

[HLP52] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Univ. Press, New York, 1952.

[HMGR01] N. Hofmann, T. Müller-Gronbach, and K. Ritter. The Optimal Discretization of Stochastic Differential Equations. *J. Complexity*, 17(1):117–153, 2001.

[IR90] K. Ireland and M. Rosen. *A Classical Introduction to Modern Number Theory*. Springer, New York, 2nd edition, 1990.

[JY85] Th. Jeulin and M. Yor, editors. *Grossissements de filtrations: exemples et applications*. Lect. Notes in Math. vol 1118. Spinger, Berlin, 1985.

[Kle14] A. Klenke. *Probability Theory*. Springer, London, 2nd edition, 2014.

[KP95] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin, 1995.

[KS88] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, New York, 1988.

[Kuo06] H.-H. Kuo. *Introduction to Stochastic Integration*. Springer, New York, 2006.

[LP02] H. Luschgy and G. Pagès. Functional quantization of Gaussian processes. *J. Funct. Anal.*, 196(2):486–531, 2002.

[LP04] H. Luschgy and G. Pagès. Sharp asymptotics of the functional quantization problem for Gaussian processes. *Ann. Probab.*, 32(2):1574–1599, 2004.

[LP06] H. Luschgy and G. Pagès. Functional quantization of a class of Brownian diffusions: A constructive approach. *Stoch. Process. Appl.*, 116(2):310–336, 2006.

[LP08] H. Luschgy and G. Pagès. Functional quantization rate and mean regularity of processes with an application to Lévy processes. *Ann. Appl. Probab.*, 18(2):427–469, 2008.

[LT91] M. Ledoux and M. Talagrand. *Probability in Banach Spaces*. Springer, Berlin, Heidelberg, 1991.

[Mao07] X. Mao. *Stochastic Differential Equations and Applications*. Woodhead Publishing, Cambridge, 2nd edition, 2007.

[Mar55] G. Maruyama. Continuous Markov processes and stochastic equations. *Rend. Circ. Mat. Palermo*, 4:48–90, 1955.

[MG02a] T. Müller-Gronbach. The optimal uniform approximation of systems of stochastic differential equations. *Ann. Appl. Probab.*, 12(2):664–690, 2002.

[MG02b] T. Müller-Gronbach. Strong Approximation of Systems of Stochastic Differential Equations. Habilitation thesis, Technische Universität Darmstadt, 2002.

[MGR10] T. Müller-Gronbach and K. Ritter. A Local Refinement Strategy for Constructive Quantization of Scalar SDEs. *DFG Priority Programme 1324*, Preprint 72, 2010.

[MGR13] T. Müller-Gronbach and K. Ritter. A Local Refinement Strategy for Constructive Quantization of Scalar SDEs. *Found. Comput. Math.*, 13(6):1005–1033, 2013.

[MGRY15] T. Müller-Gronbach, K. Ritter, and L. Yaroslavtseva. On the complexity of computing quadrature formulas for marginal distributions of SDEs. *J. Complexity*, 31(1):110–145, 2015.

[Mil75] G. N. Milstein. Approximate Integration of Stochastic Differential Equations. *Theory Probab. Appl.*, 19(3):557–562, 1975.

[Nov95] E. Novak. The Real Number Model in Numerical Analysis. *J. Complexity*, 11(1):57–73, 1995.

[OPS48] B. M. Oliver, J. R. Pierce, and C. E. Shannon. The Philosophy of PCM. *Proceedings of the IRE*, 36(11):1324–1331, 1948.

[PJ03] G. Pagès and J.Printems. Optimal quadratic quantization for numerics: the Gaussian case. *Monte Carlo Methods Appl.*, 9(2):135–165, 2003.

[PP05] G. Pagès and J. Printems. Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods Appl.*, 11(4):407–446, 2005.

[PP09] G. Pagès and J. Printems. Optimal Quantization for Finance: From Random Vectors to Stochastic Processes. In A. Bensoussan and Q. Zhang, editors, *Mathematical Modeling and Numerical Methods in Finance*, Handbook of Numerical Analysis, vol XV. North-Holland, Amsterdam, 2009.

[PPP04] G. Pagès, H. Pham, and J. Printems. Optimal Quantization Methods and Applications to Numerical Problems in Finance. In S. T. Rachev and G. A. Anastassiou, editors, *Handbook of Computational and Numerical Methods in Finance*, pages 253–297. Birkhäuser, Boston, MA, 2004.

[Pro05] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin, Heidelberg, 2nd edition, 2005.

[PS11] G. Pagès and A. Sellami. Convergence of Multi-Dimensional Quantized SDE's. In C. Donati-Martin, A. Lejay, and A. Rouault, editors, *Séminaire de Probabilités XLIII*, Lect. Notes in Math., vol 2006, pages 269–307. Springer, Berlin, Heidelberg, 2011.

[PZ14] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge Univ. Press, Cambridge, 2nd edition, 2014.

[RW01] T. Rydén and M. Wiktorsson. On the simulation of iterated Itô integrals. *Stoch. Process. Appl.*, 91(1):151–168, 2001.

[RY99] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, Berlin, 3rd edition, 1999.

[Sha48] C. E. Shannon. A mathematical theory of communication. *Bell System Technical*, 27(3):379–423, 1948.

[Ste70] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30. Princeton Univ. Press, Princeton, N.J., 1970.

[Tou08] S. Toussaint. Konstruktive Quantisierung skalarer Diffusionsprozesse. Diploma thesis, Technische Universität Darmstadt, 2008.

[TWW88] J. F. Traub, G. W. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity*. Academic Press, San Diego, 1988.

[Wil08] B. Wilbertz. *Construction of optimal quantizers for Gaussian measures on Banach spaces*. PhD thesis, Universität Trier, 2008.

[WP78] W. Wagner and E. Platen. Approximation of Itô integral equations. *Preprint ZIMM, Akad. der Wiss., DDR, Berlin*, 1978.

[Zad63] P. L. Zador. *Development and Evaluation of Procedures for Quantizing Multivariate Distributions*. PhD thesis, Stanford University, 1963.