

Peano-Differentiable Functions in O-Minimal Structures

Dissertation

zur Erlangung des akademischen Grades

des Doktors der Naturwissenschaften

an der Fakultät für Mathematik und Informatik

der Universität Passau

Andreas Fischer

November 2005

Andreas Fischer
UFR Mathématiques
Université de Rennes 1
Campus de Beaulieu
35042 Rennes cedex
France
E-mail: el.fischerandreas@web.de

Danksagung

Zuerst möchte ich mich ganz herzlich bei Prof. Dr. N. Schwartz bedanken. Er hat diese Arbeit souverän und weitsichtig betreut. In der Schlußphase der Erstellung der Arbeit übernahm Prof. Dr. K. Kurdyka diese Aufgabe. Auch ihm ein herzliches Dankeschön. Beide sorgten für eine inspirierende Atmosphäre, die viele Ideen reifen ließ.

Prof. Dr. O. Forster, der mich ermutigte, mich mit reeller Geometrie zu befassen, gilt mein besonderer Dank. Ohne die Unterstützung meiner Kollegen, Eltern und Freunde wäre diese Arbeit ebenfalls nicht zustande gekommen.

Diese Arbeit ist mit finanzieller Unterstützung des EC-IHP-Network RAAG (Contract-No: HPRN-CT-2001-00271) erstellt worden.

Contents

1	Introduction	1
2	Fundamental Definitions and Propositions	7
	Structures and Formulas	7
	Formulas	7
	Real Closed Fields	9
	Semialgebraic Structure	10
	O-minimal Structures	10
	O-minimal Expansions of Real Closed Fields	11
	Topology	12
	Monotonicity-Theorem	12
	Connectedness	13
	Differentiable Functions	14
	Cell Decomposition	14
	Dimension	16
	Approximation	17
	Curve-Selection-Lemma	18
3	Estimates for Definable Derivatives	21
4	Λ^p-regular Stratification	27
	Λ^p -regular Functions and Cells	27
	Bounding the Gradient	30
	Distances on the Grassmannian	30
	ϵ -flat Submanifolds	34
	Finite Open Covers of the Grassmannian	37
	Metric-Combinatorial Properties of Lines	39
	Proof of Theorem 4.5	43
	Preliminary Lemmata	43
	Proof of Theorem 4.5	52
5	Ordinary Differentiable Functions	55
	Examples	56
	One-Dimensional Differentiable Functions	60
	A Characterisation of Ordinary Differentiability	62
	Definable Ordinary Differentiable Functions along Curves	66
	Definable Ordinary Differentiable Functions along \mathcal{C}^1 -Curves	68
	Definable Ordinary Differentiable Functions along Ordinary Differentiable Curves	78

6	Definable Peano-Differentiable Functions	82
	Peano-Differentiability	82
	Examples	83
	Basic Properties of Peano-Differentiable Functions	90
	Definability of Peano-Derivatives	90
	Basic Algebraic Properties of Peano-Differentiable Functions	91
7	Differentiability Properties of Peano-Derivatives	95
	One-Dimensional Peano-Differentiable Functions	95
	Oliver's Theorem in Higher Dimension	96
8	Singularities of Peano-Differentiable Functions	102
	\mathcal{C}^m -Singularities in Higher Dimension	102
	Characterising \mathcal{C}^m -Singularity Sets	108
	Necessary Conditions for Singularity Sets	109
	Preliminary Lemmata	110
	Constructing Singularity Sets	113
9	Extending Definable Peano-Differentiable Functions	121
	Introduction	121
	Preliminary Lemmata	124
	Proof of Theorem 9.5	133
	References	136

1 Introduction

In this thesis we examine m -times Peano-differentiable functions which are definable in \mathcal{M} , an o-minimal expansion of the real closed field R , and compare the results of classical Analysis for these functions with those in o-minimal context.

Let $f : U \rightarrow R$ be a function where $U \subset R^n$ is an open subset. We call f at x_0 m -times Peano-differentiable if there exists a polynomial $p \in R[X_1, \dots, X_n]$ of degree less than or equal to m with $p(0) = 0$ such that

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - p(x - x_0)}{\|x - x_0\|^m} = 0.$$

In 1891, Giuseppe Peano introduced this concept of higher order differentiability for functions of one real variable. Already the title of this work, “Sulla formula di Taylor”, shows that Peano worked on functions which admit an approximation with a Taylor-polynomial. But it was Peano himself who first noted that in general these functions are not continuously differentiable.

Until about 1965, only few results about Peano-differentiable functions were published. Exceptions are a paper of Denjoy in 1935 and one of Oliver in 1954. Oliver’s work is an improvement of Denjoy’s paper.

Since then an ever increasing number of papers were written, and we recognise three main interests of research. One is continuity properties of Peano-derivatives, another is extendibility problems of Peano-differentiable functions defined on closed sets, and the third is generalisations of this differentiability concept.

In this work we do not take generalisations into consideration but we concentrate on the other two research directions, restricting ourselves to definable functions in o-minimal structures. Additionally, we identify 1-times Peano-differentiable functions as the differentiable functions in the usual sense. We will have a special look on these functions which we call ordinary differentiable from now on.

In chapter 5 we begin by giving a couple of examples of definable ordinary differentiable functions of two variables. Besides the cognition that in o-minimal context ordinary differentiable functions are not necessarily continuously differentiable, we get a first impression of the functions which arise as derivatives. It can happen that the derivative is not locally bounded. But even if the derivative is locally bounded we do not obtain continuity of it. Setting additional conditions on the derivative, such as semidefiniteness of the partial derivatives, do not imply their continuity.

Above all we present an example which shows that the Implicit-Function-Theorem is generally not valid for this class of functions.

In contrast to classical Analysis there is no difference between ordinary and continuous differentiability for definable functions of one variable. This fact enables us to analyse differentiability properties of definable ordinary differentiable functions composed with curves. First we can characterise definable ordinary differentiable functions by means of definable continuously differentiable curves and the chain-rule.

But especially the case when the curves are non-definable is of particular interest. For definable ordinary differentiable functions $f : R^n \rightarrow R$ we consider the following property: For all continuously differentiable curves $\varphi : (-1, 1) \rightarrow R^n$ let $f \circ \varphi$ be continuously differentiable.

It is a priori not clear that this is a tame property. But we can give a simple description of these functions: They are locally Lipschitz-continuous and definable ordinary differentiable.

However, we want to check whether f is continuously differentiable or not by studying the differentiability class of f composed with curves. The class of continuously differentiable curves is not big enough to detect all points at which a derivative is not continuous. But ordinary differentiable curves with bounded derivative are a suitable class of curves.

In chapter 6 we generalise to higher order Peano-differentiability. Before we prove that these functions are closed under addition, multiplication and composition, we present a family of semialgebraic m -times Peano-differentiable functions.

When we look at this family of functions we can make the following special observation. There are functions of this family where the $(m - 1)^{th}$ Peano-derivatives are ordinary differentiable, but there is no function of which the $(m - 1)^{th}$ Peano-derivatives are not ordinary differentiable if the m^{th} Peano-derivatives are locally bounded. We go into detail with this phenomena in chapter 7.

Moreover, this family provides us with several kinds of singularities which stimulate our interest in examining them. These are points at which the function is not m -times continuously differentiable but possibly of some weaker differentiability class. We will analyse the sets of such points in chapter 8.

In chapter 7 we consider differentiability properties of Peano-differentiable functions respectively conditions which imply differentiability of certain Peano-derivatives. This has already been studied for real functions of one variable.

Theorem (Oliver) *Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ be an m -times Peano-differentiable function. If, for $x_0 \in I$, the m^{th} Peano-derivative is locally bounded from above or below at x_0 , the $(m - 1)^{\text{th}}$ Peano-derivative is ordinary differentiable at x_0 .*

In o-minimal context we do not need the assumption of boundedness. The first result of this chapter states that definable m -times Peano-differentiable functions are always m -times continuously differentiable.

According to the examples of chapter 6, we know that this does not hold true for definable functions of several variables. However, we give, inspired by Oliver's theorem, a sufficient condition for definable m -times Peano-differentiable functions which implies that the $(m - 1)^{\text{th}}$ Peano-derivatives are ordinary differentiable.

Theorem 7.3 *Let $U \subset \mathbb{R}^n$ be a definable open subset and $f : U \rightarrow \mathbb{R}$ a definable m -times Peano-differentiable function. Let the m^{th} Peano-derivatives all be bounded from above or all be bounded from below. Then the $(m - 1)^{\text{th}}$ Peano-derivatives are ordinary differentiable at each $x_0 \in U$.*

This is also the main result of this chapter.

In chapter 8 we begin with the analysis of the sets of points at which a definable m -times Peano-differentiable function is not m -times continuously differentiable. For continuity properties of Peano-derivatives we find in [25] the following statements.

Theorem (Fejzić, Rinne) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an m -times Peano-differentiable function. Then all Peano-derivatives up to order m are of class Baire-1,*
and

Theorem (Fejzić, Rinne) *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be an m -times Peano-differentiable function. Then all Peano-derivatives up to order $m - 1$ are continuous on some dense open subset of \mathbb{R}^n .*

This paper is of course not the only one on that issue, but it includes many former results about continuity properties of Peano-derivatives.

In o-minimal context we are in a more comfortable situation since definable functions are m -times continuously differentiable outside a definable set of lower dimension; so the second of the theorems by Fejzić and Rinne is trivial in o-minimal context. We are further able to formulate more precise statements about continuity of Peano-derivatives.

We call a point at which a function is not m -times continuously differentiable a \mathcal{C}^m -singularity and denote by $\text{sing}^m(f)$ the set of \mathcal{C}^m -singularities of f . For definable m -times Peano-differentiable functions, the following bound for the dimension of the \mathcal{C}^m -singularity set applies. This is the main result of this chapter.

Theorem 8.5 *Let $U \subset \mathbb{R}^n$ be open and let $f : U \rightarrow \mathbb{R}$ be a definable m -times Peano-differentiable function. Then*

$$\dim(\text{sing}^m(f)) \leq n - 2.$$

Also the converse of this theorem holds true, i.e. for each definable subset $A \subset \mathbb{R}^n$ with $\dim(A) \leq n - 2$, there is a definable m -times Peano-differentiable function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ with $A = \text{sing}^m(F)$.

Furthermore, we can distinguish between several kinds of \mathcal{C}^m -singularities, i.e. points at which a function is k -times ordinary differentiable with some additional properties. We can characterise the sets of points of a definable m -times Peano-differentiable function which belong to such a subclass of \mathcal{C}^m -singularities; moreover, we discuss the relationship between the different singularity sets.

For the constructions we make in chapter 8, we use a Λ^p -regular stratification of the \mathbb{R}^n . We develop this tool in chapter 3 and 4.

In chapter 3 we improve a result about estimates for analytic subanalytic functions.

Proposition (Kurdyka, Pawłucki) *Let $U \subset \mathbb{R}^n$ be an open set and let $\Phi : U \rightarrow \mathbb{R}$ be an analytic subanalytic function. Then there exists a closed nowhere dense subanalytic set $Z \subset U$ such that for each $u \in U \setminus Z$ and each $r > 0$ such that the closed ball $\text{cl}(B_r(u)) \subset U \setminus Z$,*

$$|D_\alpha \Phi(u)| \leq C(n, p) \sup_{x \in B_r(u)} \frac{\Phi(x)}{r^{|\alpha|}}, \quad 1 \leq |\alpha| \leq p,$$

where $C(n, p)$ is a constant only depending on n and p .

In this chapter we transfer this proposition to arbitrary o-minimal expansions of real closed fields, and we improve the constant $C(n, p)$, i.e. we show that

$$C(n, p) := 2^{\binom{p+2}{2}-2}$$

is reasonable. We point out that this constant is independent from the dimension n .

This estimation result is part of the regularity conditions of Λ^p -regular functions.

The Λ^p -regular stratification has its origin in a paper of Kurdyka and Pawłucki. There, this stratification has been developed for subanalytic sets in order to prove a subanalytic version of Whitney's Extension Theorem. The general idea of stratifications is as follows. We partition sets into finitely many sets of simpler or more convenient form.

Here the simple sets are, after an eventual orthogonal change of coordinates, standard Λ^p -regular cells. We obtain the convenient form by the class of Λ^p -regular functions which are used to define these cells. Λ^p -regular functions are a subclass of the \mathcal{C}^p -functions with additional regularity properties. These regularity properties are of technical nature, and we will point out and prove them for the particular applications in chapter 8 and 9.

In chapter 9 we concentrate on the extendibility of definable Peano-differentiable functions defined on closed sets.

Definition *Let $A \subset \mathbb{R}^n$ be a closed set. We call $f : A \rightarrow \mathbb{R}$, together with $(f_{[\alpha]})_{|\alpha| \leq m} : A \rightarrow \mathbb{R}$, m -times Peano-differentiable on A , if*

$$f(x) = \sum_{|\alpha| \leq m} \frac{f_{[\alpha]}(y)}{\alpha!} (x - y)^\alpha + R(x, y), \quad \text{for } x, y \in A,$$

$$\text{where } \lim_{x \rightarrow y} \frac{R(x, y)}{\|x - y\|^m} = 0.$$

Extending Peano-differentiable functions is a very difficult problem in classical Analysis. Even the one-dimensional case is not yet completely solved.

We recognise two strategies in solving this problem.

One of them is to restrict the class of closed sets, and the other is to claim additional properties for the formal derivatives $f_{[\alpha]}$. Especially for functions of several variables, very little is known.

In o-minimal context we make weaker assumptions on the formal derivatives. The main result of this chapter is the following extension-theorem.

Theorem 9.5 *Let $A \subset \mathbb{R}^n$ be a closed definable set, and let $f : A \rightarrow \mathbb{R}$ together with the definable $(f_{[\alpha]})_{|\alpha| \leq m} : A \rightarrow \mathbb{R}$ be a definable m -times Peano-differentiable function. Moreover, there is a definable partition A_1, \dots, A_r of A such that*

(*) *for each $i = 1, \dots, r$ and for all multi-indices β , $1 \leq |\beta| \leq m - 1$, $f_{[\beta]}|_{A_i}$ together with the $f_{[\beta+\gamma]}|_{A_i}$, $|\gamma| \leq m - |\beta|$, is $(m - |\beta|)$ -times Peano-differentiable on A_i .*

Then, there is a definable \mathcal{P}^m -function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

(I) $D_\alpha F(x) = f_{[\alpha]}(x)$ for $x \in A$, $|\alpha| \leq m$.

(II) F is of class \mathcal{C}^m on $\mathbb{R}^n \setminus A$.

2 Fundamental Definitions and Propositions

We examine differentiable functions which are definable in an o-minimal structure expanding a real closed field. The definable sets and functions are described by formulas. We briefly recall the concepts of “structure”, “formula” and “definable” in the first section. Afterwards we will concentrate on o-minimal structures. We will focus on the semialgebraic structure which is an example of an o-minimal expansion of each real closed field. This structure has some special features. In the second section we state fundamental results of o-minimal structures expanding a real closed field to which we will go back in the later chapters.

Structures and Formulas

A language \mathcal{L} is a triple consisting of a set of function symbols \mathcal{F} , a set of relation symbols \mathcal{R} and a set of constant symbols \mathcal{C} . To each $f \in \mathcal{F}$, respectively $\rho \in \mathcal{R}$, a positive natural number n_f , respectively n_ρ , is assigned. The language of ordered fields \mathcal{L}_{or} is of special interest for us. In this case, $\mathcal{F}_{or} = \{+, -, \cdot\}$, $\mathcal{R}_{or} = \{<\}$, and $\mathcal{C}_{or} = \{0, 1\}$. Now we define \mathcal{L} -structures.

Definition 2.1 *An \mathcal{L} -structure \mathcal{M} is given by the following data:*

- (i) *a nonempty set M called domain,*
- (ii) *a function $f^{\mathcal{M}} : M^{n_f} \rightarrow M$ for each $f \in \mathcal{F}$,*
- (iii) *a set $\rho^{\mathcal{M}} \subset M^{n_\rho}$ for each $\rho \in \mathcal{R}$,*
- (iv) *an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$.*

For the symbols $f \in \mathcal{F}$, $\rho \in \mathcal{R}$, and $c \in \mathcal{C}$ we denote by $f^{\mathcal{M}}$, $\rho^{\mathcal{M}}$ and $c^{\mathcal{M}}$ their interpretation in \mathcal{M} . Often we write the structure \mathcal{M} as $\mathcal{M} = (M, f^{\mathcal{M}}, \rho^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, \rho \in \mathcal{R}, \text{ and } c \in \mathcal{C})$. We can write the \mathcal{L}_{or} -structure with domain \mathbb{R} as $\mathcal{M} = (\mathbb{R}, +, -, \cdot, <, 0, 1)$ where the $+$, $-$, \cdot , $<$, 0 , 1 have the obvious interpretations.

Formulas

Our aim is to describe the sets which are *definable* in a structure. This is done with the help of formulas. Formulas are built using symbols of the language \mathcal{L} , variable symbols v_1, v_2, \dots , the equality symbol $=$, the Boolean connectives \wedge, \vee , and \neg , the quantifiers \exists and \forall , and $(,), [, \text{ and }]$.

We begin with defining terms.

Definition 2.2 *The set of \mathcal{L} -terms is the smallest set \mathcal{T} such that $\mathcal{C} \subset \mathcal{T}$, each variable symbol $v_i \in \mathcal{T}$, $i = 1, 2, \dots$, and if $t_1, \dots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$, then $f(t_1, \dots, t_{n_f}) \in \mathcal{T}$.*

We interpret a term t built by using variables from $(v_{i_1}, \dots, v_{i_m})$ in an \mathcal{L} -structure \mathcal{M} as a function $t^{\mathcal{M}} : M^{i_1} \times \dots \times M^{i_m} \rightarrow M$. This is done inductively. Let $a = (a_{i_1}, \dots, a_{i_m}) \in M^{i_1} \times \dots \times M^{i_m}$ and let s be a subterm of t . $s^{\mathcal{M}}(a)$ is defined as follows.

- (i) If s is a constant symbol c , then $s^{\mathcal{M}}(a) = c^{\mathcal{M}}$.
- (ii) If s is the variable v_{i_j} , then $s^{\mathcal{M}}(a) = a_{i_j}$.
- (iii) If s is the term $f(t_1, \dots, t_{n_f})$ where f is a function symbol of \mathcal{L} and t_1, \dots, t_{n_f} are terms, then $s^{\mathcal{M}}(a) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(a), \dots, t_{n_f}^{\mathcal{M}}(a))$.

Now we are able to define formulas.

Definition 2.3 *ϕ is an atomic \mathcal{L} -formula if ϕ is either $t_1 = t_2$ where t_1 and t_2 are terms, or $\rho(t_1, \dots, t_{n_\rho})$ where $\rho \in \mathcal{R}$ and t_1, \dots, t_{n_ρ} are terms. The set of \mathcal{L} -formulas is the smallest set \mathcal{W} containing the atomic formulas such that*

- (i) if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$,
- (ii) if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ are in \mathcal{W} , and
- (iii) if $\phi \in \mathcal{W}$, then $\exists v_i \phi$ and $\forall v_i \phi$ are in \mathcal{W} .

For formulas φ and ψ we use $\varphi \rightarrow \psi$ for $\neg\varphi \vee \psi$. We say that a variable v occurs free in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier.

Definition 2.4 *Let ϕ be a formula with free variables from $v = (v_{i_1}, \dots, v_{i_m})$, and let $a = (a_{i_1}, \dots, a_{i_m}) \in M^{i_1} \times \dots \times M^{i_m}$. We inductively define $\mathcal{M} \models \phi(a)$ as follows.*

- (i) If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(a)$ if $t_1^{\mathcal{M}}(a) = t_2^{\mathcal{M}}(a)$.
- (ii) If ϕ is $\rho(t_1, \dots, t_{n_\rho})$, then $\mathcal{M} \models \phi(a)$ if $(t_1^{\mathcal{M}}, \dots, t_{n_\rho}^{\mathcal{M}}) \in \rho^{\mathcal{M}}$.
- (iii) If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \not\models \psi(a)$.
- (iv) If ϕ is $(\psi \wedge \theta)$, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \models \psi(a)$ and $\mathcal{M} \models \theta(a)$.
- (v) If ϕ is $(\psi \vee \theta)$, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \models \psi(a)$ or $\mathcal{M} \models \theta(a)$.
- (vi) If ϕ is $\exists v_j \psi(v, v_j)$, then $\mathcal{M} \models \phi(a)$ if there is $b \in M$ such that $\mathcal{M} \models \psi(a, b)$.

(vii) If ϕ is $\forall v_j \psi(v, v_j)$, then $\mathcal{M} \models \phi(a)$ if $\mathcal{M} \models \psi(a, b)$ for all $b \in M$.

If $\mathcal{M} \models \phi(a)$ we say $\phi(a)$ is true in \mathcal{M} .

We now give the explanation of definable sets.

Definition 2.5 Let \mathcal{M} be an \mathcal{L} -structure. We say that $X \subset M^n$ is definable if and only if there is an \mathcal{L} -formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $b \in M^m$ such that

$$(1) \quad X = \{a \in M^n : \mathcal{M} \models \phi(a, b)\}.$$

We say that $\phi(v, b)$ defines X . We say that X is A -definable if there is a formula $\psi(v, w_1, \dots, w_l)$ and $b \in A^l$ such that $\psi(v, b)$ defines X .

We denote by $S_n^{\mathcal{M}}$ the set of definable sets $X \subset M^n$ and put $S^{\mathcal{M}} := (S_n^{\mathcal{M}})_{n \in \mathbb{N}}$. Often, if there is no doubt about \mathcal{M} , we say that X is definable if there is an $n \in \mathbb{N}$ such that $X \in S_n^{\mathcal{M}}$. Moreover, if $f : M^n \rightarrow M^m$ is a map, we say that f is definable if the graph $\Gamma(f)$ of f is definable.

Real Closed Fields

We will now specialise on certain structures, namely the *o-minimal* structures expanding a real closed field. We recall that an ordered field $(K, <)$ is a field K equipped with an ordering $<$ such that for $x, y, z \in K$ the following applies.

(i) $x < y \Rightarrow x + z < y + z$, and

(ii) $0 < x, 0 < y \Rightarrow 0 < xy$.

Examples of ordered fields are the field of rational numbers \mathbb{Q} or the field of real numbers \mathbb{R} with the well known orderings. The field of real numbers has an additional property: Its only nontrivial algebraic extension is the algebraic closure of \mathbb{R} , the field of complex numbers, which cannot be ordered. This property of \mathbb{R} is shared by a whole class of fields which we call the real closed fields.

Definition 2.6 A real closed field R is an ordered field which has no non-trivial ordered algebraic extension.

The smallest real closed field is the field of real algebraic numbers $\mathbb{R}_{alg} := \{z \in \mathbb{R} \mid \exists 0 \neq p \in \mathbb{Q}[Z] : p(z) = 0\}$, cf. [5] example 1.2.3. For a detailed introduction to real closed fields see [5] chapter 1.

Semialgebraic Structure

Let R be a real closed field. Then we call the structure $S^{\text{semialg}}(R) = (R, +, -, \cdot, <, 0, 1)$ the semialgebraic structure. The definable sets of this structure are called semialgebraic. We can give an alternative definition of semialgebraic sets.

Definition 2.7 *The set $X \subset R^n$ is semialgebraic if*

$$(2) \quad X = \bigcup_{i=1}^s \bigcap_{j=1}^{r_i} \{x \in R^n \mid p_{i,j} *_{i,j} 0\}$$

where $p_{i,j} \in R[X_1, \dots, X_n]$, and $*_{i,j}$ is either $<$ or $=$, for $i = 1, \dots, s$ and $j = 1, \dots, r_i$.

By [5] chapter 2, the two definitions of semialgebraic sets coincide. Since R is ordered we can speak of intervals. An interval is a set $I \subset R$ of the form

$$(3) \quad I = (a, b) = \{t \in R \mid a < t < b\}$$

where $-\infty \leq a < b \leq \infty$. Moreover we use $[a, b] := \{t \in R \mid a \leq t \leq b\}$ etc. Proposition 2.1.7 in [5] gives the following characterisation of semialgebraic subsets of R .

Proposition 2.8 *Semialgebraic subsets of R are exactly the finite unions of points and open intervals.*

For the semialgebraic structures we have a very powerful transfer-principle, cf. [5] proposition 5.2.3. Note that the language of $S^{\text{semialg}}(R)$ is \mathcal{L}_{or} .

Transfer Principle 2.9 *Let $R \subset K$ both be real closed fields. Let $\Phi(v)$ be an \mathcal{L}_{or} -formula with free variables from $v = (v_{i_1}, \dots, v_{i_n})$. Let $a \in R^n$. Then*

$$(4) \quad S^{\text{semialg}}(R) \models \Phi(a) \Leftrightarrow S^{\text{semialg}}(K) \models \Phi(a).$$

O-minimal Structures

Definition 2.10 *Let R be a real closed field. An o-minimal structure expanding R is a structure \mathcal{M} such that*

- (i) $\mathcal{M} = (R, +, -, \cdot, <, 0, 1, \dots)$ where the dots imply that there may be more interpretations, i.e. functions and relations.
- (ii) If $X \subset R$ is definable, then there are finitely many intervals I_1, \dots, I_m and a finite set X_0 such that $X = X_0 \cup I_1 \cup \dots \cup I_m$.

We also say that \mathcal{M} is an o-minimal expansion of R . A very important example of an o-minimal expansion of R is the semialgebraic structure which is considered as trivial expansion. This provides us with an o-minimal expansion for all real closed fields. In [68] it is shown that, if we add the exponential function to the semialgebraic structure over \mathbb{R} , we get an o-minimal expansion. We now present some elementary facts about definable sets. We state them in o-minimal context although they hold true in a more general situation. The proofs can be read in [62] chapter 1.2, lemma 2.3.

Proposition 2.11 *Let \mathcal{M} be an o-minimal expansion of R . Then the following holds true:*

- (i) *For each $m \in \mathbb{N}$, $S_m^{\mathcal{M}}$ is a boolean algebra of subsets of R^m .*
- (ii) *If A is definable, then $R \times A$ and $A \times R$ are definable.*
- (iii) *The set $\{(x_1, \dots, x_m) \in R^m : x_1 = x_m\}$ is definable.*
- (iv) *If $A \in S_{m+1}^{\mathcal{M}}$, then $\pi(A)$ is definable, where $\pi : R^{m+1} \rightarrow R^m$ is the projection map on the first m coordinates.*

Proposition 2.12 *Let \mathcal{M} be an o-minimal expansion of R . Let $f : S \rightarrow R^n$ be definable. Then:*

- (i) *S is definable.*
- (ii) *If $A \subset S$ is definable, then $f(A)$ is definable and $f|_A$ is definable.*
- (iii) *If $B \subset R^n$ is definable, then $f^{-1}(B)$ is definable.*
- (iv) *If f is injective, its inverse f^{-1} is definable.*
- (v) *The composition of definable maps is a definable map.*

O-minimal Expansions of Real Closed Fields

We now fix a real closed field R and an o-minimal expansion of R . O-minimality restricts our class of definable functions so that we deal with several but not all important analogies of theorems of classical Analysis. Furthermore, several tools have been developed for o-minimal structures so that we are able to sidestep some classical constructions which cannot be applied in o-minimal context.

Topology

An R -norm and an R -metric are defined as for the real numbers except that they take their values in R instead of \mathbb{R} . An R -norm and an R -metric induce a topology in the obvious way.

We equip R^n with the ‘‘Euclidean’’ scalar-product: $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$, $x, y \in R^n$. By $\|x\| := \langle x, x \rangle^{1/2}$ we denote the Euclidean R -norm which induces the Euclidean R -metric in the obvious way.

If a set is supposed to be bounded, then we mean bounded with respect to the Euclidean R -norm.

The product topology on R^n is induced by the maximum- R -norm $|a|_\infty = \max_i |a_i|$.

As a further example of an R -norm we use the operator norm. For $A \in R^{d \times n}$ we set $\|A\|_{op} = \sup_{\|x\|=1} \|Ax\|$. We obtain the following inequalities.

$$(5) \quad \frac{1}{\sqrt{n}} \|x\| = \sqrt{\frac{1}{n} \sum_{i=1}^n x_i^2} \leq \sqrt{\sup_i x_i^2} = |x|_\infty \leq \sqrt{\sum_{i=1}^n x_i^2} = \|x\|, \quad x \in R^n,$$

$$(6) \quad \frac{1}{\sqrt{d}} \|A\| \leq \|A\|_{op} \leq \|A\|, \quad A \in R^{d \times n}.$$

Inequality (6) is follows from $\frac{1}{d} \sum_{i=1}^d \sum_{j=1}^n a_{ij}^2 \leq \sup_{\|x\|=1} \sum_{i=1}^d \left(\sum_{j=1}^n a_{ij} x_j \right)^2$ and $\sup_{\|x\|=1} \sum_{i=1}^d \left(\sum_{j=1}^n a_{ij} x_j \right)^2 \leq \sum_{i=1}^d \sum_{j=1}^n a_{ij}^2$.

We use the Euclidean topology as standard-topology for R^n . Closure $\text{cl}(A)$, interior $\text{int}(A)$, and boundary $\text{bd}(A) = \text{cl}(A) \setminus \text{int}(A)$ of a definable set A are definable, cf. [62] chapter 1 lemma 3.4.

O-minimality preserves some properties of closed and bounded sets of R^n which are in general not compact if $R \neq \mathbb{R}$. So, supremum and infimum of a definable subset of R exist in $R \cup \{\pm\infty\}$. Moreover, if $A \subset R^n$ is closed, bounded and definable, and if $f : A \rightarrow R$ is a definable continuous function, then $f(A)$ possesses maximum and minimum in R , cf. [62] chapter 1 lemma 3.3.

Monotonicity-Theorem

A very powerful tool is given by the Monotonicity-Theorem, [62] chapter 3 theorem 1.2.

Monotonicity-Theorem 2.13 *Let $f : (a, b) \rightarrow R$ be a definable function on the interval (a, b) . Then there are points $a_1 < \dots < a_k$ in (a, b) such that on each subinterval (a_j, a_{j+1}) , with $a_0 = a$, $a_{k+1} = b$, the function is either constant or strictly monotone and continuous.*

We state a very important consequence, cf. [62] chapter 3 corollary 1.6.

Corollary 2.14 *Let $f : (a, b) \rightarrow R$ be definable. Then for each $c \in (a, b)$ the limits $\lim_{x \nearrow c} f(x)$ and $\lim_{x \searrow c} f(x)$ exist in $R \cup \{\pm\infty\}$. Also, the limit $\lim_{x \nearrow b} f(x)$ and $\lim_{x \searrow a} f(x)$ exist in $R \cup \{\pm\infty\}$.*

Connectedness

With the exception of the real numbers, all real closed fields are totally disconnected sets. In order to transform the idea of connected sets to o-minimal structures, one uses the concept of definable connectedness.

Definition 2.15 *A set $X \subset R^n$ is called definably connected if X is definable and X is not the union of two disjoint nonempty definable open subsets of X .*

The class of definable functions is sufficiently small so that the Intermediate-Value-Theorem holds true for definable continuous functions of one variable, cf. [62] chapter 1 lemma 3.6 (2)

Proposition 2.16 *Let $f : [a, b] \rightarrow R$ be a definable and continuous function. Then f assumes all values between $f(a)$ and $f(b)$.*

Let $X \subset R^n$. A definable path in X is a definable continuous map $\gamma : [a, b] \rightarrow R^n$ with $a, b \in R$, $a < b$, taking values in X . Then for definably connected sets the following proposition holds true, see [62] chapter 6 proposition 3.2.

Proposition 2.17 *Suppose the definable set X is definably connected. Then any two points of X can be connected by a definable path in X .*

With the last proposition we can give a more general formulation of the Intermediate-Value-Theorem.

Intermediate-Value-Theorem 2.18 *Let $X \subset R^n$ be definably connected and $f : X \rightarrow R$ be a definable continuous function. Then, for all $x, y \in X$, f assumes all values between $f(x)$ and $f(y)$.*

Proof: We take a path $\gamma : [a, b] \rightarrow R^n$ in X connecting x and y . Then, $f \circ \gamma : [a, b] \rightarrow R$ is definable and continuous so that we can apply proposition 2.16 to it. \square

Differentiable Functions

Let $X \subset R^n$ be open and let $f : X \rightarrow R$ be a function. We define partial differentiability as for $R = \mathbb{R}$. f is called partially differentiable at $x \in X$ with respect to the i^{th} variable if

$$(7) \quad \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h} \in R.$$

Note that for $n = 1$ we only speak of differentiable or ordinary differentiable. For $\alpha \in \mathbb{N}^n$ we denote by D_α the differential-operator $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. If f is definable and partially differentiable in X with respect to the i^{th} variable, then we can describe the graph of the function $g : X \rightarrow R$, $g(x) = \frac{\partial f}{\partial x_i}(x)$, in the following way. Let $x = (x_1, \dots, x_n)$ and $\phi_f(x, y)$ be a formula defining the graph of f . Then let $\psi(x, y)$ be the formula:

$$(8) \quad \forall \epsilon \left[\epsilon > 0 \rightarrow \exists \delta \left[\delta > 0 \rightarrow \forall h \left[h^2 < \delta^2 \rightarrow \right. \right. \right. \\ \left. \left. \left. \forall z_1, z_2 \left[\phi_f(x, z_1) \wedge \phi_f(x + he_i, z_2) \rightarrow (z_2 - z_1 - hy)^2 \leq \epsilon^2 h^2 \right] \right] \right] \right]$$

Then (x, z) belongs to the graph of g if $\mathcal{M} \models \psi(x, z)$ so that g is definable. This induces that the derivative of continuously differentiable definable functions is definable, and, by induction, the derivatives up to order p of definable \mathcal{C}^p -mappings are definable. Moreover, o-minimality provides us with the Mean-Value-Theorem in the following form, cf [62] chapter 7 theorem 2.3.

Mean-Value-Theorem 2.19 *Suppose $a < b$ in R , $f : [a, b] \rightarrow R$ is definable and continuous, and differentiable at each point of (a, b) . Then, for some $c \in (a, b)$, we have $f(b) - f(a) = (b - a) \cdot f'(c)$.*

Cell Decomposition

In order to solve certain constructing problems in o-minimal context it is often very useful to solve the problem on simple sets. Cell-decompositions provide us with a technique to decompose a given set into sets of simple form. Definable \mathcal{C}^0 -cell-decomposition was first established in [32]. The improvement of this decomposition, namely \mathcal{C}^p -cell-decomposition, can be found in [62] 3 and 7. We use this reference for the results presented in this section. We need to extend the notion of \mathcal{C}^p -mappings.

Definition 2.20 *A definable map $f : A \rightarrow R^n$, where $A \subset R^m$ is not necessarily open, is called a \mathcal{C}^p -map if there are a definable open neighbourhood $U \subset R^m$ containing A and a definable \mathcal{C}^p -map $F : U \rightarrow R^n$ such that $F|_A = f$.*

The cell-decomposition requires some terminology. For each $p \in \mathbb{N}$ and each definable set $X \subset R^n$, we put

$$\begin{aligned}\mathcal{C}^p(X) &:= \{f : X \rightarrow R \mid f \text{ is a definable } \mathcal{C}^p\text{-function}\}, \\ \mathcal{C}_\infty^p(X) &:= \mathcal{C}^p(X) \cup \{-\infty, \infty\}\end{aligned}$$

where we regard $-\infty$ and ∞ as constant functions on X . For $f, g \in \mathcal{C}_\infty^p(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X := \{(x, y) \in X \times R \mid f(x) < y < g(x)\}.$$

Let $h \in \mathcal{C}^p(X)$. Then

$$(9) \quad (h)_X := \{(x, y) \in X \times R \mid y = h(x)\}.$$

Having agreed on this terminology we are now able to define \mathcal{C}^p -cells.

Definition 2.21 *Let (i_1, \dots, i_n) be a sequence of zeros and ones of length n . An (i_1, \dots, i_n) - \mathcal{C}^p -cell is a definable subset of R^n obtained by induction on n as follows:*

A (0)- \mathcal{C}^p -cell is a one-element set $\{x\} \subset R$, and a (1)- \mathcal{C}^p -cell is an interval $(a, b) \subset R$.

Suppose (i_1, \dots, i_{n-1}) - \mathcal{C}^p -cells are already defined. Then

(graph) *an $(i_1, \dots, i_{n-1}, 0)$ - \mathcal{C}^p -cell is the graph $(h)_X$ of a function $h \in \mathcal{C}^p(X)$ where X is an (i_1, \dots, i_{n-1}) - \mathcal{C}^p -cell;*

(band) *an $(i_1, \dots, i_{n-1}, 1)$ - \mathcal{C}^p -cell is a set $(f, g)_X$ where X is an (i_1, \dots, i_{n-1}) - \mathcal{C}^p -cell and $f, g \in \mathcal{C}_\infty^p(X)$, $f < g$.*

A \mathcal{C}^p -cell in R^n is an (i_1, \dots, i_n) - \mathcal{C}^p -cell for some sequence (i_1, \dots, i_n) .

A cell-decomposition is a special kind of partition.

Definition 2.22 *A \mathcal{C}^p -cell decomposition of R^n is a partition of R^n into finitely many \mathcal{C}^p -cells. The definition is by induction on n :*

(i) *A \mathcal{C}^p -cell decomposition of $R^1 = R$ is a collection*

$$(10) \quad \{(-\infty, a_1), (a_1, a_2), \dots, (a_l, +\infty), \{a_1\}, \dots, \{a_l\}\}$$

where $a_1 < \dots < a_l$ are points in R .

(ii) *A decomposition of R^{n+1} is a finite partition of R^{n+1} into \mathcal{C}^p -cells A such that the set of projections $\pi_{R^n}(A)$ is a \mathcal{C}^p -cell decomposition of R^n .*

The following theorem, cf. [62] chapter 7.3 especially theorem 3.2, motivates our special interest in \mathcal{C}^p -cells.

Theorem 2.23 *Let $A_1, \dots, A_l \subset R^n$ be definable sets.*

- (i) *For any definable sets $A_1, \dots, A_k \subset R^n$ there is a decomposition of R^n into \mathcal{C}^p -cells partitioning each of the A_i .*
- (ii) *For every definable function $f : A \rightarrow R$, $A \subset R^n$ there is a decomposition of R^n into \mathcal{C}^p -cells partitioning A such that each restriction $f|_C : C \rightarrow R$ is \mathcal{C}^p for each cell $C \subset A$ of the decomposition.*

Dimension

The dimension of a nonempty definable set $X \subset R^n$ is defined by

$$(11) \quad \dim(x) := \max\{i_1 + \dots + i_m : X \text{ contains an } (i_1, \dots, i_n)\text{-cell}\}.$$

Although this is an ad hoc definition, it provides us with the properties we have in mind if we consider $R = \mathbb{R}$, cf. [62] chapter 4 proposition 1.3 and corollary 1.6..

Proposition 2.24 *The following holds true:*

- (i) *If $X \subset Y \subset R^n$ are definable subsets, then $\dim(X) \leq \dim(Y) \leq n$.*
- (ii) *If $X \subset R^n$ and $Y \subset R^m$ are definable and there is a definable bijection between X and Y , then $\dim(X) = \dim(Y)$.*
- (iii) *If $X, Y \subset R^n$ are definable, then $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.*
- (iv) *For definable sets X and Y , $\dim(X \times Y) = \dim(X) + \dim(Y)$.*

A very excellent result on dimension in o-minimal context concerns the frontier of a definable set. We recall that the *frontier* ∂X of a definable set X is defined by $\partial X = \text{cl}(X) \setminus X$. We come to the following result, cf. [62] chapter 4 theorem 1.8:

Theorem 2.25 *Let $X \subset R^n$ be a nonempty definable set. Then*

$$(12) \quad \dim(\partial X) < \dim(X).$$

Approximation

In [17], approximation of definable \mathcal{C}^p -functions by \mathcal{C}^{p+1} -functions in the \mathcal{C}^p -topology is proved. For our purposes, theorem 1.6 of [17] with the remarks made after this theorem, reads as follows.

Theorem 2.26 *Let $U \subset \mathbb{R}^n$ be a definable open set, $f : U \rightarrow \mathbb{R}$ a definable \mathcal{C}^p -function, and let $\epsilon : U \rightarrow \mathbb{R}$ be a definable positive continuous function. Then there exists a definable \mathcal{C}^{p+1} -function $g : U \rightarrow \mathbb{R}$ such that*

$$(13) \quad |D_\alpha(f(x) - g(x))| < \epsilon(x), \quad x \in U, \quad 0 \leq |\alpha| \leq p.$$

Applying theorem 2.26 k times with $\frac{\epsilon}{k}$, we obtain that we may suppose that the function g is of class \mathcal{C}^{p+k} . We can use this approximation theorem to obtain special definable positive semidefinite m -times continuously differentiable functions. In [50], theorem C.9, we find the following generalised Łojasiewicz inequality.

Theorem 2.27 *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous definable functions which are p -times continuously differentiable in $\mathbb{R}^n \setminus g^{-1}(\{0\})$, with $f^{-1}(\{0\}) \subset g^{-1}(\{0\})$. Then there is a definable p -times continuously differentiable bijection $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ with $\Phi^{(l)}(0) = 0$, $l = 0, \dots, p$, and a definable p -times continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$(14) \quad \Phi \circ g = hf.$$

As a consequence of the two previous theorems we receive the following corollary.

Corollary 2.28 *Let $A \subset \mathbb{R}^n$ be a closed definable set. Then there is a positive semidefinite p -times continuously differentiable definable function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F^{-1}(\{0\}) = A$.*

Proof: Let $G : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $G(x) := \text{dist}(x, A)$. According to theorem 2.26 there is a definable function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ such that H restricted to $\mathbb{R}^n \setminus A$ is p -times continuously differentiable with

$$(15) \quad |H(x) - G(x)| \leq \frac{1}{2} \text{dist}(x, A), \quad x \in \mathbb{R}^n \setminus A.$$

If we set $H|_A \equiv 0$, H is a continuous function.

We apply theorem 2.27 to $g = H$ and $f = 1$ and get a definable p -times continuously differentiable function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ which vanishes exactly at the points of A . Hence, $F = h^2$ has the desired properties. \square

Curve-Selection-Lemma

Definable curves play a role similar to that of sequences in \mathbb{R}^n but they have better properties. The following version of the Curve-Selection-Lemma is presented in [62] chapter 6 corollary 1.5.

Proposition 2.29 *If $a \in \partial X$ where X is definable, there is a definable continuous injective map $\gamma : (0, \epsilon) \rightarrow X$ for some $\epsilon > 0$ such that $\lim_{t \rightarrow 0} \gamma(t) = a$.*

We need a variant of this proposition which we will call Curve-Selection-Lemma from now on. In the proof of this variant we use proposition 5.7, i.e. that a definable function $f : I \rightarrow R$ of an open definable subset I of R is continuously differentiable if it is ordinary differentiable. The proof of this proposition uses only the Mean-Value-Theorem and o-minimality.

We call a definable \mathcal{C}^1 -map $\gamma : I \rightarrow R^n$ where $I \subset R$ is an interval a regular definable \mathcal{C}^1 -curve if the derivative φ' does not vanish at each point of I .

Curve-Selection-Lemma 2.30 *If $a \in \partial X$ where $X \subset R^n$ is definable, there is a regular definable \mathcal{C}^1 -curve $\varphi : (-\epsilon, \epsilon) \rightarrow R^n$ for some $\epsilon > 0$ such that $\varphi((0, \epsilon)) \subset X$ and $\varphi(0) = a$.*

Before we begin with the proof we prepend a lemma.

Lemma 2.31 *Let $f_1, \dots, f_n : (0, \delta) \rightarrow R$, $\delta > 0$, be continuous definable functions. Then there is an $\epsilon > 0$ and j such that for each $1 \leq i \leq n$*

$$(16) \quad |f_i(t)| \leq |f_j(t)|, \quad t \in (0, \epsilon).$$

In this case we say that f_j dominates f_1, \dots, f_n at 0.

Proof: We proceed by induction on n .

The case $n = 1$ is evident.

Now we assume that the statement of the lemma holds true for $n-1$ functions f_1, \dots, f_{n-1} , i.e. there is an $\epsilon_{n-1} > 0$ and $j \leq n-1$ such that for each $i \leq n-1$ the following holds true:

$$(17) \quad |f_i(t)| \leq |f_j(t)|, \quad t \in (0, \epsilon_{n-1})$$

Therefore we can reduce our consideration to the functions f_j and f_n restricted to $(0, \epsilon_{n-1})$. For these functions we show the following:

There is an $\epsilon > 0$ such that either

(i) $|f_j(t)| < |f_n(t)|$, $t \in (0, \epsilon)$, or

$$(ii) \quad |f_j(t)| \geq |f_n(t)|, \quad t \in (0, \epsilon),$$

applies.

Let M be the set $\{t \mid t \in (0, \epsilon_{n-1}) \wedge |f_n(t)| - |f_j(t)| > 0\}$. This set is definable, and since the f_i are continuous it is open. According to o-minimality, there are finitely many disjoint intervals I_1, \dots, I_k such that $M = I_1 \cup \dots \cup I_k$. If one of these intervals is of the form $(0, \epsilon)$ for some $\epsilon > 0$, then (i) holds true. Otherwise, the set $(0, \epsilon_{n-1}) \setminus M$ contains an interval of the form $(0, \epsilon)$ for some $\epsilon > 0$ which implies (ii). \square

Proof: of the Curve-Selection-Lemma.

We may assume that $a = 0$ and take the map $\gamma : (0, \epsilon) \rightarrow X$ which is provided by proposition 2.29. By o-minimality, we may further assume that γ is continuously differentiable as long as ϵ is chosen sufficiently small. Hence, we can write γ as $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ where each of the γ_i is continuously differentiable, and $\lim_{t \searrow 0} \gamma_i(t) = 0$. Since all γ_i are continuous, we can apply lemma 2.31 to the γ_i and find an $\epsilon > 0$ and a j such that γ_j dominates $\gamma_1, \dots, \gamma_n$.

According to the Monotonicity-Theorem there is a positive $\epsilon_1 < \epsilon$ such that γ_j is strictly monotone in $(0, \epsilon_1)$, since otherwise, γ would not be injective. Therefore, γ_j has an inverse $\gamma_j^{-1} : (0, \gamma_j(\epsilon_1)) \rightarrow (0, \epsilon_1)$. This inverse is of course continuously differentiable, and, since $\lim_{t \rightarrow 0} \gamma_j(t) = 0$, $\lim_{t \rightarrow 0} \gamma_j^{-1}(t) = 0$.

γ_j dominates the γ_i ; hence, for $1 \leq i \leq n$,

$$(18) \quad |\gamma_i(\gamma_j^{-1}(t))| \leq |\gamma_j(\gamma_j^{-1}(t))| = t$$

so that the function

$$(19) \quad t \mapsto \frac{\gamma_i(\gamma_j^{-1}(t))}{t}, \quad t \in (0, \gamma_j(\epsilon_1)),$$

is bounded. By o-minimality, the limit

$$(20) \quad \lim_{t \searrow 0} \frac{\gamma_i(\gamma_j^{-1}(t))}{t}$$

exists in R and we denote it by c_i . Let $c = (c_1, \dots, c_n)$. We define the curve $\varphi : (-\epsilon_2, \epsilon_2) \rightarrow R^n$ by

$$(21) \quad \varphi(t) = \begin{cases} \gamma(\gamma_j^{-1}(t)), & t > 0 \\ tc, & t \leq 0 \end{cases}$$

for an $0 < \epsilon_2 < \min(\epsilon_1, \gamma_j(\epsilon_1))$. By construction of φ we see that

$$(22) \quad \lim_{t \searrow 0} \frac{\varphi(t)}{t} = \lim_{t \searrow 0} \frac{\gamma(\gamma_j^{-1}(t))}{t} = c = \lim_{t \nearrow 0} \frac{\varphi(t)}{t}$$

applies. Hence, φ is differentiable at 0 with derivative $\varphi'(0) = c$, and $c \neq 0$ since $c_j = 1$. The continuity of φ' is a consequence of proposition 5.7 since each coordinate function is a function of one variable. \square

3 Estimates for Definable Derivatives

In this section we study definable \mathcal{C}^{p+1} -functions with semidefinite derivatives. For a point u of the domain of the function we are able to give bounds for the derivatives up to order p evaluated at this point. We further show that these bounds depend only on the distance of the point u to the boundary of the domain and on the values of the function in a certain neighbourhood of u . We follow an idea of Gromov [29] which is presented in [35], and we generalise and improve the corresponding results in [35] further. We start with the one-dimensional case. For a function of one variable we denote by $f^{(p)}$ the p^{th} derivative of f .

Lemma 3.1 *Let $\lambda : I \rightarrow \mathbb{R}$ be a definable \mathcal{C}^2 -function, and we have either $\lambda^{(2)} \geq 0$ on I or $\lambda^{(2)} \leq 0$ on I . Let $t \in I$ and $r > 0$ such that $[t-r, t+r] \subset I$. Then*

$$(23) \quad |\lambda^{(1)}(t)| \leq \frac{2}{r} \sup_{s \in [t-r, t+r]} |\lambda(s)|.$$

Proof: Without loss of generality we may assume that $\lambda^{(2)} \leq 0$ on I so that $\lambda^{(1)}$ is monotonically decreasing on I . According to the Mean-Value-Theorem we get the two following inequations:

$$(24) \quad \lambda(t) - \lambda(t-r) \geq \inf_{s \in [t-r, t]} \lambda^{(1)}(s)r = \lambda^{(1)}(t)r$$

$$(25) \quad \lambda(t+r) - \lambda(t) \leq \sup_{s \in [t, t+r]} \lambda^{(1)}(s)r = \lambda^{(1)}(t)r.$$

Since $|\lambda(t) - \lambda(t-r)|$ and $|\lambda(t+r) - \lambda(t)|$ are both less than or equal to

$$(26) \quad 2 \sup_{s \in [t-r, t+r]} |\lambda(s)|,$$

we get

$$(27) \quad |\lambda^{(1)}(t)| \leq \frac{2}{r} \sup_{s \in [t-r, t+r]} |\lambda(s)|.$$

□

Lemma 3.2 *Let $\lambda : I \rightarrow \mathbb{R}$ be a definable \mathcal{C}^{p+1} -function such that for $i = 2, \dots, p+1$, we have either $\lambda^{(i)} \geq 0$ on I or $\lambda^{(i)} \leq 0$ on I . Let $t \in I$ and $r > 0$ such that $[t-r, t+r] \subset I$. Then, for $j = 1, \dots, p$, we get the following inequation:*

$$(28) \quad |\lambda^{(j)}(t)| \leq \frac{2^{\frac{j^2+3j-2}{2}}}{r^j} \sup_{s \in [t-r, t+r]} |\lambda(s)|$$

Proof: We proceed by induction on j . The case $j = 1$ was proved in lemma 3.1. So we assume that the statement of the lemma holds true for j . Let $u \in [t - \frac{r}{2}, t + \frac{r}{2}] \subset [t - r, t + r] \subset I$. Then the interval $[u - \frac{r}{2}, u + \frac{r}{2}] \subset I$. Hence,

$$(29) \quad |\lambda^{(j)}(u)| \leq 2^{\frac{j^2+3j-2}{2}} \frac{1}{\left(\frac{r}{2}\right)^j} \sup_{s \in [t-r, t+r]} |\lambda(s)|.$$

Since $\lambda^{(j+2)}$ is semidefinite on I , we can apply lemma (3.1) to $\lambda^{(j)}$ and get

$$(30) \quad \begin{aligned} |\lambda^{(j+1)}(t)| &\leq \frac{2}{\left(\frac{r}{2}\right)} \sup_{u \in [t-\frac{r}{2}, t+\frac{r}{2}]} |\lambda^{(j)}(u)| \\ &\leq \frac{4}{r} 2^{\frac{j^2+3j-2}{2}} \frac{1}{\left(\frac{r}{2}\right)^j} \sup_{s \in [t-r, t+r]} |\lambda(s)| \\ &= \frac{2^{\frac{(j+1)^2+3(j+1)-2}{2}}}{r^{j+1}} \sup_{s \in [t-r, t+r]} |\lambda(s)|. \end{aligned}$$

□

The following lemma is an n -dimensional version of lemma 3.2. In [35] it is shown that there is a constant depending only on p and n . Here, we compute this constant to be

$$(31) \quad 2^{\frac{p^2+3p-2}{2}}.$$

It is surprising that the constant is independent from the dimension n since the proofs in [35] do not give us any idea as to how we can compute the constant without using the dimension.

For the multi-indices $\alpha, \beta \in \mathbb{N}^n$ we put $\alpha! = \prod_{i=1}^n \alpha_i!$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\alpha \prec \beta$ if $\alpha_i \leq \beta_i$ for $i = 1, \dots, n$. Furthermore, if $x \in \mathbb{R}^n$, we set $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.

Lemma 3.3 *Let $\varphi : U \rightarrow \mathbb{R}$ be a definable \mathcal{C}^{p+1} -function on the open set $U \subset \mathbb{R}^n$. For all multi-indices $\beta \in \mathbb{N}^n$ with $2 \leq |\beta| \leq p + 1$ let $D_\beta \varphi$ be semidefinite. Then, for each closed ball $\text{cl}(B_r(u)) \subset U$ and each $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq p$ the following inequality holds true:*

$$(32) \quad |D_\alpha \varphi(u)| \leq \frac{2^{\frac{|\alpha|^2+3|\alpha|-2}{2}}}{r^{|\alpha|}} \sup_{x \in B_r(u)} |\varphi(x)|$$

Proof: For this proof we make the following conventions. For $1 \leq l \leq n$ let $V_l \subset R^n$ be the subspace generated by e_1, \dots, e_l . Furthermore, we set

$$(33) \quad B_r^l(u) = \{u' \in R^n \mid u' - u \in V_l, \|u' - u\| \leq r\},$$

and for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we put

$$(34) \quad i := \max\{l \mid \alpha_l > 0\}.$$

We show by induction on $|\alpha|$ that

$$(35) \quad |D_\alpha \varphi(u)| \leq \frac{2^{\frac{|\alpha|^2 + 3|\alpha| - 2}{2}}}{r^{|\alpha|}} \sup_{x \in B_r^i(u)} |\varphi(x)|, \quad u \in U, \text{cl}(B_r(u)) \subset U.$$

Let $|\alpha| = 1$. Then $D_\alpha = \frac{\partial}{\partial x_i}$. Let $\lambda : [-r, r] \rightarrow R$ be defined by

$$(36) \quad \lambda(s) := \varphi(u + se_i).$$

Then $\lambda^{(2)}(s) = \frac{\partial^2}{\partial x_i^2} \varphi(u + se_i)$, and therefore it is semidefinite. So we can apply lemma 3.2 to λ and get

$$(37) \quad \begin{aligned} |D_\alpha \varphi(u)| &= |\lambda^{(1)}(0)| \\ &\leq \frac{2}{r} \sup_{s \in [-r, r]} |\lambda(s)| \\ &= \frac{2}{r} \sup_{s \in [-r, r]} |\varphi(u + se_i)| \\ &\leq \frac{2}{r} \sup_{x \in B_r^i(u)} |\varphi(x)|. \end{aligned}$$

Let $2 \leq |\alpha| = j + 1 \leq p$. We assume that inequality (35) holds true for $|\alpha| \leq j$. We now have to distinguish between two different cases.

case 1: $\alpha = |\alpha| e_i$.

Then $D_\alpha = \frac{\partial^{j+1}}{\partial x_i^{j+1}}$. Let $\lambda : [-r, r] \rightarrow R$ be defined by

$$(38) \quad \lambda(s) := \varphi(u + se_i).$$

$\lambda^{(k)}(s) = \frac{\partial^k}{\partial x_i^k} \varphi(u + se_i)$ for each $2 \leq k \leq j+2$. Therefore, $\lambda^{(k)}$ is semidefinite. Again, we apply lemma 3.2 to λ and get

$$\begin{aligned}
(39) \quad |D_\alpha \varphi(u)| &= |\lambda^{(j+1)}(0)| \\
&\leq \frac{2^{\frac{(j+1)^2+3(j+1)-2}{2}}}{r^{j+1}} \sup_{s \in [-r, r]} |\lambda(s)| \\
&= \frac{2^{\frac{(j+1)^2+3(j+1)-2}{2}}}{r^{j+1}} \sup_{s \in [-r, r]} |\varphi(u + se_i)| \\
&\leq \frac{2^{\frac{(j+1)^2+3(j+1)-2}{2}}}{r^{j+1}} \sup_{x \in B_r^i(u)} |\varphi(x)|.
\end{aligned}$$

case 2: At least two of the α_l are positive.

Then $\alpha = (\alpha_1, \dots, \alpha_i, 0, \dots, 0)$, and we set $\beta = (\alpha_1, \dots, \alpha_{i-1}, 0, \dots, 0)$ so that $\alpha = \beta + \alpha_i e_i$ and $\beta \neq 0$. We define the function $\lambda : [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}] \rightarrow R$ by

$$(40) \quad \lambda(s) := D_\beta \varphi(u + se_i).$$

Hence, λ is well-defined and each derivative of λ is semidefinite. We note that for each $s \in [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}]$,

$$(41) \quad B_{\frac{r}{\sqrt{2}}}^{i-1}(u + se_i) \subset B_r^i(u) \subset \text{cl}(B_r(u)).$$

So we can apply lemma 3.2 to λ and we get

$$\begin{aligned}
(42) \quad |D_\alpha \varphi(u)| &= |\lambda^{(\alpha_i)}(0)| \\
&\leq 2^{\frac{\alpha_i^2+3\alpha_i-2}{2}} \frac{1}{\left(\frac{r}{\sqrt{2}}\right)^{\alpha_i}} \sup_{s \in [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}]} |\lambda(s)| \\
&= 2^{\frac{\alpha_i^2+3\alpha_i-2}{2}} \frac{1}{\left(\frac{r}{\sqrt{2}}\right)^{\alpha_i}} \sup_{s \in [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}]} |D_\beta(\varphi(u + se_i))| \\
&\leq \frac{2^{\frac{\alpha_i^2+3\alpha_i-2}{2}}}{\left(\frac{r}{\sqrt{2}}\right)^{\alpha_i}} \sup_{s \in [-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}]} \left(\frac{2^{\frac{|\beta|^2+3|\beta|-2}{2}}}{\left(\frac{r}{\sqrt{2}}\right)^{|\beta|}} \sup_{x \in B_{\frac{r}{\sqrt{2}}}^{i-1}(u+se_i)} |\varphi(x)| \right) \\
&\leq 2^{\frac{\alpha_i^2+3\alpha_i-2}{2}} 2^{\frac{\alpha_i+|\beta|}{2}} 2^{\frac{|\beta|^2+3|\beta|-2}{2}} \frac{1}{r^{\alpha_i+|\beta|}} \sup_{u' \in B_r^i(u)} |\varphi(u')|.
\end{aligned}$$

Since $\alpha_i + |\beta| = |\alpha| = j + 1$, it remains to prove that

$$(43) \quad \frac{\alpha_i^2 + 3\alpha_i - 2}{2} + \frac{\alpha_i + |\beta|}{2} + \frac{|\beta|^2 + 3|\beta| - 2}{2} \leq \frac{(j+1)^2 + 3(j+1) - 2}{2}.$$

In order to do that we note that $1 \leq \alpha_i \leq j$ and therefore

$$(44) \quad \begin{aligned} 2\alpha_i^2 - 2\alpha_i(j+1) + j - 1 &\leq 2\alpha_i^2 - (2\alpha_i - 1)(j+1) \\ &\leq 2\alpha_i^2 - (2\alpha_i - 1)(\alpha_i + 1) \\ &= 1 - \alpha_i \leq 0. \end{aligned}$$

So,

$$(45) \quad \begin{aligned} \frac{\alpha_i^2 + 3\alpha_i - 2}{2} + \frac{\alpha_i + |\beta|}{2} + \frac{|\beta|^2 + 3|\beta| - 2}{2} \\ = \frac{(j+1)^2 + 3(j+1) - 2}{2} + \frac{2\alpha_i^2 - 2\alpha_i(j+1) + j - 1}{2} \\ \leq \frac{(j+1)^2 + 3(j+1) - 2}{2} \end{aligned} \quad \square$$

Proposition 3.4 *Let $\varphi : U \rightarrow R$ be a definable \mathcal{C}^{p+1} -function. There is a definable subset $Z \subset U$ of $\dim(Z) \leq n - 1$ which is closed in U such that for each multi-index $\alpha \in \mathbb{N}^n$, $1 \leq |\alpha| \leq p$, and for each $u \in U \setminus Z$ it holds true that*

$$(46) \quad |D_\alpha \varphi(u)| \leq \frac{2^{\frac{|\alpha|^2 + 3|\alpha| - 2}{2}}}{\text{dist}(u, \partial U \cup Z)^{|\alpha|}} \sup\{|\varphi(x)|; x \in U, \|x - u\| < \text{dist}(u, \partial U \cup Z)\}.$$

Proof: For each $\beta \in \mathbb{N}^n$ with $2 \leq |\beta| \leq p + 1$ let

$$(47) \quad A_\beta := \{x \in U \mid D_\beta \varphi(x) > 0\}.$$

We choose a definable \mathcal{C}^1 -cell decomposition of R^n which is compatible with the A_β and U . Let B_1, \dots, B_{r_B} be the open cells which are contained in U . Then

$$(48) \quad Z := U \setminus \cup_{i=1}^{r_B} B_i$$

is definable and closed in U , and $\dim(Z) \leq n - 1$. Furthermore, for each i , $D_\beta \varphi$ is semidefinite on B_i . Let $u \in U \setminus Z$. Then there is an i with $u \in B_i$. We apply lemma 3.3 to φ restricted to B_i . For each $r > 0$ with $\text{cl}(B_r(u)) \subset B_i$ we then get

$$(49) \quad |D_\alpha \varphi(u)| \leq \frac{2^{\frac{|\alpha|^2 + 3|\alpha| - 2}{2}}}{r^{|\alpha|}} \sup\{|\varphi(x)| \mid x \in \text{cl}(B_r(u))\}.$$

Thus,

(50)

$$\begin{aligned}
|D_\alpha \varphi(u)| &\leq \inf_{r>0} \frac{2^{\frac{|\alpha|^2+3|\alpha|-2}{2}}}{r^{|\alpha|}} \sup\{|\varphi(x)| \mid x \in \text{cl}(B_r(u)) \subset B_i\} \\
&\leq \inf_{r>0} \frac{2^{\frac{|\alpha|^2+3|\alpha|-2}{2}}}{r^{|\alpha|}} \sup\{|\varphi(x)| \mid \|x-u\| < \text{dist}(u, \partial B_i)\} \\
&\leq \frac{2^{\frac{|\alpha|^2+3|\alpha|-2}{2}}}{\text{dist}(u, \partial B_i)^{|\alpha|}} \sup\{|\varphi(x)| \mid \|x-u\| < \text{dist}(u, \partial B_i)\} \\
&\leq \frac{2^{\frac{|\alpha|^2+3|\alpha|-2}{2}}}{\text{dist}(u, \partial U \cup Z)^{|\alpha|}} \sup\{|\varphi(x)| \mid x \in U, \|x-u\| < \text{dist}(u, \partial U \cup Z)\}.
\end{aligned}$$

□

We will make use of the following corollary for the proof of theorem 4.5.

Corollary 3.5 *Let $\phi : U \rightarrow R$ be a definable \mathcal{C}^{p+1} -function. Suppose that $\left| \frac{\partial \phi}{\partial x_j} \right| \leq M$ on U , $j = 1, \dots, n$. Then there is a closed definable subset $Z \subset U$ of dimension $\dim(Z) \leq n - 1$ such that*

$$(51) \quad |D^\alpha \phi(u)| \leq 2^{\binom{p+2}{2}-2} \cdot M \cdot \text{dist}(u, \partial U \cup Z)^{1-|\alpha|}$$

for any $u \in U \setminus Z$, $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq p$.

Proof: We apply proposition 3.4 to the derivatives $\frac{\partial \phi}{\partial x_j}$.

□

4 Λ^p -regular Stratification

In this chapter we develop the concept of Λ^p -regular stratification. The strata are Λ^p -regular cells which are a kind of \mathcal{C}^p -cells with additional regularity conditions in a suitably chosen linear coordinate system. In the case that $p = 1$ these cells are closely related to L -regular cells, cf. [34]. Here we use this stratification concept in order to prove extendibility of definable m -times Peano-differentiable functions defined on closed sets and to describe singularity sets of definable m -times Peano-differentiable functions. So, for convenience, we define the Λ^p -regular cells as it is done in [35]. In this paper, Λ_p -regular cell decomposition is developed for proving a subanalytic version of Whitney's Extension Theorem. We generalise the concept of Λ_p -regular cell decomposition to arbitrary o-minimal structures expanding a real closed field. We follow ideas given in [33], [34] and [35].

One important regularity condition of Λ^p -regular cells is that the gradient of the functions with which we define the cells is bounded by a constant¹. In the papers [33], [34] and [35] no explicit estimation for this constant is given, but the proofs induce a constant which increases exponentially with the dimension n of the ambient space. We show that the constant

$$(52) \quad 2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$$

is valid.

We first introduce the concepts of Λ^p -regular functions, cells and stratification, and state the main theorem. After that we prepare the proof of this theorem. A big problem is to get a coordinate system in which all gradients of the functions which define a cell have a bounded gradient. We treat this problem by examining ϵ -flat submanifolds. This requires an intensive study of a certain distance function defined on the Grassmannian. In [33] and [34] the open cover property of compact sets is used to obtain finite open covers of the Grassmannian which are compact sets. This property does not apply for the Grassmannian if we deal with real closed fields different from the real numbers. We circumvent this problem by using model theoretic arguments. We obtain the regularity conditions for the higher derivatives of the defining functions by corollary 3.5.

Λ^p -regular Functions and Cells

We introduce the class of Λ^p -regular functions. These functions are inspired by the Λ_p -regular functions introduced in [35] which are defined on bounded open subanalytic subsets of \mathbb{R}^n . The definitions coincide for bounded sets.

¹The same constant is also valid for L -regular cells

Definition 4.1 Let $X \subset R^n$ be an open and definable set and $\varphi : X \rightarrow R^k$ a definable C^p -mapping. We call φ Λ^p -regular with constant $C > 0$ if for each multi-index $\alpha \in \mathbb{N}^n$, $1 \leq |\alpha| \leq p$, the following holds true: If $X \neq R^n$, then

$$(53) \quad \|D_\alpha \varphi(x)\| \leq \frac{C}{\text{dist}(x, \partial X)^{|\alpha|-1}}, \quad x \in X.$$

If $X = R^n$, then, for each definable open $Y \subsetneq R^n$,

$$(54) \quad \|D_\alpha \varphi(y)\| \leq \frac{C}{\text{dist}(y, \partial Y)^{|\alpha|-1}}, \quad y \in Y.$$

We call φ Λ^p -regular if φ is Λ^p -regular with some constant $C > 0$.

By $\Lambda^p(X, R^k)$ we denote the set of all Λ^p -regular functions $\varphi : X \rightarrow R^k$. If $X \subsetneq R^n$ and $f : X \rightarrow R^k$ is Λ^p -regular with constant $C > 0$, then, for each definable open subset $Y \subset X$, $\varphi|_Y$ is Λ^p -regular with the same constant C . This can be directly concluded from the fact that for all $y \in Y$, $\text{dist}(y, \partial Y) \leq \text{dist}(y, \partial X)$.

One easily gets that definable Λ^2 -regular functions with domain R^n are affine linear functions. We do not make use of this fact.

Our next item is to define Λ^p -regular cells. In order to give a straightforward description of these cells, we agree on the following terminology. For any definable open subset $X \subset R^n$ let

$$\Lambda_\infty^p(X) := \Lambda^p(X, R) \cup \{\pm\infty\}.$$

Here, $\pm\infty$ are regarded as constant functions. For $f, g \in \Lambda_\infty^p(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X := \{(x, y) \in X \times R \mid f(x) < y < g(x)\}.$$

If $h \in \Lambda^p(X, R^k)$ we put

$$(h)_X := \{(x, y) \in X \times R^k \mid y = h(x)\}.$$

If $n = 0$, all functions $f : R^0 \rightarrow R^k$ are constant functions so that we can interpret them as being Λ^p -regular. Moreover, we identify $R^0 \times R^n$ with R^n .

We now define Λ^p -regular cells as a generalisation of the definition given in [35]. There it is assumed that the functions which are used to define the cells are analytic functions. They are able to claim this property because they work with subanalytic functions which are piecewise analytic. In o-minimal

structures we do not have this property in general, and, moreover, it is not yet known whether all definable functions are piecewise C^∞ . Our proofs are of such a nature that we do not need these assumptions. We use the fact that definable functions are piecewise C^p which is a reasonable assumption by theorem 2.23.

Definition 4.2 *The only standard Λ^p -regular cell in R^0 is R^0 itself. If we know all standard Λ^p -regular cells in R^0, \dots, R^{n-1} , we say that $A \subset R^n$ is a standard Λ^p -regular cell if A is either of the mode*

(band) $A = (f, g)_X$ where $X \subset R^{n-1}$ is an open standard Λ^p -regular cell and
 $f, g \in \Lambda_\infty^p(X)$ with $f < g$, or

(graph) $A = (h)_X$ where $X \subset R^k$ is an open standard Λ^p -regular cell and
 $h \in \Lambda^p(X, R^{n-k})$.

The set X is called the base of the cell.

If $A \subset R^n$ is a standard Λ^p -regular cell of dimension d , there is a sequence of sets (A_0, \dots, A_d) such that $A_i \subset R^i$ is a Λ^p -regular cell of dimension i and $A_i = (f_i, g_i)_{A_{i-1}}$, for $i = 1, \dots, d$, and/or $A = (h)_{A_d}$. The f_i, g_i are in $\Lambda_\infty^p(A_{i-1})$ such that $f_i < g_i$, and $h \in \Lambda^p(A_d, R^{n-d})$. We call the f_i, g_i and h the defining functions of A . Note that if $n = 1$ or $d = 0$, the defining functions are constant functions. In the same way we define the defining functions of a C^p -cell.

Definition 4.3 *We call a definable set $Y \subset R^n$ a Λ^p -regular cell with constant C if there is a linear orthogonal endomorphism $\varphi : R^n \rightarrow R^n$ such that $\varphi(Y)$ is a standard Λ^p -regular cell and all defining functions of it are Λ^p -regular with constant C .*

We now go on to Λ^p -regular stratification.

Definition 4.4 *Let p be a positive integer and $C > 0$. A Λ^p -stratification of R^n with constant C is a partition of R^n into finitely many subsets S_1, \dots, S_r , called strata, such that:*

I *Each stratum is a Λ^p -regular cell with constant C .*

II *For each stratum S_i , the frontier ∂S_i is the union of some of the strata.*

We say that the stratification S_1, \dots, S_r is compatible with the sets A_1, \dots, A_k if each A_j is the union of some of the strata.

The aim of this chapter is to prove the following theorem.

Theorem 4.5 *Let $A_1, \dots, A_{r_A} \subset R^n$ be definable sets. There exists a Λ^p -regular stratification of R^n with constant*

$$(55) \quad C := 2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$$

which is compatible with A_1, \dots, A_{r_A} .

One important advantage of stratifications compared to cell decompositions is that each stratum B has a definable open neighbourhood which has empty intersection with all strata apart from B which have at most the dimension of B .

Bounding the Gradient

Distances on the Grassmannian

We recall that the Grassmannian $\mathbb{G}_{k,n}(R)$ is the set of vector subspaces of dimension k in R^n . The standard distance function Δ is defined by means of the Hausdorff-distance.

Let $A, B \subset R^n$ be two nonempty definable closed and bounded sets. The Hausdorff-distance $\text{Hd}(A, B)$ is defined by

$$(56) \quad \text{Hd}(A, B) = \max(\max_{a \in A} d(a, B), \max_{b \in B} d(b, A))$$

where $d(a, B) = \min_{b \in B} |a - b|_\infty$.

We use the Hausdorff-distance to define an R -metric on the Grassmannian.

The standard R -metric Δ on $\mathbb{G}_{k,n}(R)$ is defined in the following way:

$$(57) \quad \Delta(M, N) = \text{Hd}(M \cap S^{n-1}(R), N \cap S^{n-1}(R)), \text{ if } k > 0,$$

where $S^{n-1}(R)$ defines the unit-sphere in R^n . Δ is an R -metric on $\mathbb{G}_{k,n}(R)$, cf. [6] beginning of chapter 2.5.

If $k = 0$ we note that $\mathbb{G}_{0,n}(R)$ consists of a single point, and we denote the unique metric on $\mathbb{G}_{0,n}(R)$ by Δ .

We find the proof of the next proposition in [6], lemma 2.10 and theorem 2.11.

Proposition 4.6 *Let $X \subset R^n$ be a definable \mathcal{C}^1 -submanifold of dimension k . Let $\tau : X \rightarrow \mathbb{G}_{k,n}(R)$ be the tangent-mapping. Then τ is definable and continuous with respect to Δ .*

The distance Δ has not the right geometrical properties for our purposes. In order to deal with a more descriptive distance on the Grassmannian we introduce the distance $\delta(U, V)$ for linear subspaces of R^n . This mapping defines an R -metric on the Grassmannian which is even equivalent to Δ , cf. proposition 4.8. Therefore the tangent-mapping from a C^1 -submanifold of R^n to the Grassmannian is continuous with respect to δ .

Let $P, S \subset R^n$ be one-dimensional linear subspaces. The distance δ between P and S is defined by

$$\delta(P, S) := \sqrt{1 - \langle p, s \rangle^2}$$

where $p \in P, s \in S$ with $\|p\| = \|s\| = 1$. For a linear subspace $U \subset R^n$ the distance between P and U is given by

$$\delta(P, U) := \inf\{\delta(P, S') \mid S' \subset U \text{ one-dimensional subspace}\}.$$

For another linear subspace V of R^n ,

$$\delta(V, U) := \sup\{\delta(P', U) \mid P' \subset V \text{ one-dimensional subspace}\}.$$

If $\dim(U) = 0$, then $\delta(U, V) := 0 =: \delta(V, U)$ for all linear subspaces $V \subset R^n$. Let $V \subset R^n$ be a linear subspace. We denote by π_V the projection onto V and by π_{V^\perp} the projection onto the orthogonal complement of V . Agreeing to this denomination, we can write $\delta(V, W)$ as a closed formula.

Lemma 4.7 *Let V, W be linear subspaces of R^n with positive dimension. Then*

$$(58) \quad \delta(V, W) = \|\pi_{W^\perp} \pi_V\|_{op}.$$

Proof: $\delta(U, V)$ can be written as

$$(59) \quad \sup_{\substack{v \in V \\ \|v\|=1}} \inf_{\substack{w \in W \\ \|w\|=1}} (1 - \langle v, w \rangle^2)^{\frac{1}{2}}$$

so that the validity of equation (58) is easily seen by

$$\begin{aligned}
(60) \quad \delta(V, W)^2 &= \sup_{\substack{v \in V \\ \|v\|=1}} \inf_{\substack{w \in W \\ \|w\|=1}} \{1 - \langle v, w \rangle^2\} \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} (1 - \sup_{\substack{w \in W \\ \|w\|=1}} \langle v, w \rangle^2) \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} (1 - \sup_{\substack{w \in W \\ \|w\|=1}} \langle \pi_W(v), w \rangle^2) \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} (1 - \langle \pi_W(v), \pi_W(v) \rangle) \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} (\langle v, v \rangle - \langle v, \pi_W(v) \rangle) \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} (\langle v, \pi_{W^\perp}(v) \rangle) \\
&= \sup_{\substack{v \in V \\ \|v\|=1}} \|\pi_{W^\perp}(v)\|^2 \\
&= \sup_{\substack{x \in R^n \\ \|x\|=1}} \|\pi_{W^\perp} \pi_V(x)\|^2 \\
&= \|\pi_{W^\perp} \pi_V\|_{op}^2
\end{aligned}$$

□

By the formulation (59) it is evident that the mapping

$$(61) \quad \mathbb{G}_{k,n}(R) \times \mathbb{G}_{l,n}(R) \ni (U, V) \mapsto \delta(U, V)$$

is semialgebraic.

The following results about the distance function are known for the real numbers, see for example [34]. They also apply to any arbitrary real closed field.

Proposition 4.8 *The following holds true.*

(i) $\delta(U, V) = \delta(V, U)$, for $U, V \in \mathbb{G}_{k,n}(R)$.

(ii) For $U, V \in \mathbb{G}_{k,n}(R)$ and $U' := U \times R^m$, $V' := V \times R^m$,

$$\delta(U', V') = \delta(U, V).$$

(iii) Let $U, V, W \subset R^n$ be linear subspaces of positive dimension. Then

$$\delta(U, W) \leq \delta(U, V) + \delta(V, W).$$

(iv) $\delta : \mathbb{G}_{k,n}(R) \times \mathbb{G}_{k,n}(R) \rightarrow R$ is an R -metric which is equivalent to Δ .

Proof:

(i): Let $\varphi : R^n \rightarrow R^n$ be a linear orthogonal mapping with $\varphi(U) = V$ and $\varphi(V) = U$. Then, for each $u \in U$, $v \in V$ with $\|u\| = 1 = \|v\|$, it holds true that $\langle u, v \rangle = \langle \varphi(u), \varphi(v) \rangle = \langle v, u \rangle$. Hence, $\delta(U, V) = \delta(V, U)$.

(ii): Let $V, W \in \mathbb{G}_{k,n}(R)$, and let $V' = V \times R^m$, $W' = W \times R^m$. Since

$$(62) \quad \pi_{(W \times R^m)^\perp} \pi_{V \times R^m}((0, \dots, 0, x_{n+1}, \dots, x_{n+m})) = 0$$

we conclude that $\|\pi_{(W \times R^m)^\perp} \pi_{V \times R^m}\|_{op} = \|\pi_{W^\perp} \pi_V\|_{op}$. Hence,

$$(63) \quad \begin{aligned} \delta(V', W') &= \|\pi_{(W \times R^m)^\perp} \pi_{V \times R^m}\|_{op} \\ &= \|\pi_{W^\perp} \pi_V\|_{op} \\ &= \delta(V, W). \end{aligned}$$

(iii): Let $U, V, W \subset R^n$ be linear subspaces, and let $u \in U$ where $\|u\| = 1$ and $\delta(U, W) = \|u - \pi_W(u)\|$. Then

$$(64) \quad \begin{aligned} \delta(U, W) &= \|u - \pi_W(u)\| \leq \|u - \pi_W(\pi_V(u))\| \\ &\leq \|u - \pi_V(u)\| + \|\pi_V(u) - \pi_W(\pi_V(u))\| \\ &\leq \|\pi_{V^\perp} \pi_U\|_{op} + \|\pi_{W^\perp} \pi_V\|_{op} \\ &= \delta(U, V) + \delta(V, W). \end{aligned}$$

(iv): We prove the following inequality:

$$(65) \quad \frac{1}{\sqrt{n}} \delta(U, V) \leq \Delta(U, V) \leq \sqrt{2} \delta(U, V)$$

Since Δ is an R -metric, it follows from this inequality that $\delta(U, V) = 0$ if and only if $U = V$ so that δ is an R -metric. At the same time we obtain the equivalence of δ and Δ .

In order to prove inequality (65), we first show that for each $v \in V$ with $\|v\| = 1$ the following inequalities hold true:

$$(66) \quad \inf_{u \in U, \|u\|=1} \sqrt{1 - \langle u, v \rangle^2} \leq \inf_{u \in U, \|u\|=1} \|u - v\|$$

$$(67) \quad \sqrt{2} \inf_{u \in U, \|u\|=1} \sqrt{1 - \langle u, v \rangle^2} \geq \inf_{u \in U, \|u\|=1} \|u - v\|$$

Inequality (66) can be understood with the aid of the following estimation.

$$\begin{aligned}
(68) \quad \inf_{u \in U, \|u\|=1} \|u - v\|^2 &= \inf_{u \in U, \|u\|=1} \langle u - v, u - v \rangle \\
&= \inf_{u \in U, \|u\|=1} 2 - 2\langle u, v \rangle \\
&= \inf_{u \in U, \|u\|=1} 1 + (1 - \langle u, v \rangle)^2 - \langle u, v \rangle^2 \\
&\geq \inf_{u \in U, \|u\|=1} 1 - \langle u, v \rangle^2.
\end{aligned}$$

For inequality (67) we look at

$$\begin{aligned}
(69) \quad 2 \inf_{u \in U, \|u\|=1} 1 - \langle u, v \rangle^2 &= \inf_{u \in U, \|u\|=1} 2 - 2\langle u, v \rangle^2 \\
&\geq \inf_{u \in U, \|u\|=1} 2 - 2|\langle u, v \rangle| \\
&= \inf_{u \in U, \|u\|=1} 2 - 2\langle u, v \rangle \\
&= \inf_{u \in U, \|u\|=1} \langle u - v, u - v \rangle \\
&= \inf_{u \in U, \|u\|=1} \|u - v\|^2.
\end{aligned}$$

In order to verify inequality (65), we make use of the equivalence of the Euclidean and the maximum R -norm, i.e. of inequality (5).

According to inequality (5), (66) and (67), we obtain for each $v \in V$ with $\|v\| = 1$

$$(70) \quad \frac{1}{\sqrt{n}} \inf_{u \in U, \|u\|=1} \sqrt{1 - \langle u, v \rangle^2} \leq \inf_{u \in U, \|u\|=1} |u - v|_\infty \leq \sqrt{2} \inf_{u \in U, \|u\|=1} \sqrt{1 - \langle u, v \rangle^2}.$$

In addition, we can express $\Delta(U, V)$ by

$$(71) \quad \Delta(U, V) = \max\left(\sup_{v \in V, \|v\|=1} \inf_{u \in U, \|u\|=1} |u - v|_\infty, \sup_{u \in U, \|u\|=1} \inf_{v \in V, \|v\|=1} |u - v|_\infty \right)$$

so that inequality (65) is now a direct conclusion of inequality (70) and equation (71). \square

ϵ -flat Submanifolds

The boundary of a \mathcal{C}^p -cell is a finite union of of lower dimensional \mathcal{C}^p -cells, and these \mathcal{C}^p -cells are graphs of \mathcal{C}^p -mappings, cf. [6] lemma 2.6, so that they are \mathcal{C}^p -submanifolds. Moreover, the tangential-space exists at each point of

the manifold.

In general the gradient of these functions is not bounded. We want to cover the boundary by ϵ -flat submanifolds. These manifolds can be locally represented as graphs of \mathcal{C}^p -mappings with bounded derivative in some suitable coordinates.

First, we recall the concept of ϵ -flat submanifold as it is presented in [33].

Definition 4.9 *Let $M \subset \mathbb{R}^n$ be a \mathcal{C}^1 -submanifold and $\epsilon > 0$. M is called ϵ -flat if for all $x, y \in M$*

$$(72) \quad \delta(T_x(M), T_y(M)) < \epsilon.$$

where $T_x(M)$ denotes the tangent space of M at x .

The following proposition shows one of our interests in ϵ -flat submanifolds, i.e that, after a suitable change of the coordinate system, we can cover them by a finite disjoint union of some lower dimensional set and of graphs of functions with bounded derivative. Furthermore, this proposition provides us with an estimation for this bound.

Proposition 4.10 *Let $M \subset \mathbb{R}^n$ be an definable \mathcal{C}^p -submanifold of dimension $d < n$ such that the image of the tangent mapping is contained in k $\epsilon/2$ -balls of $\mathbb{G}_{d,n}(\mathbb{R})$, and let $U \subset \mathbb{R}^n$ be a linear subspace of dimension d satisfying*

$$(73) \quad \delta(T_x(M), U) < c < 1$$

whenever $x \in M$ for a suitable $c > 0$. Then the following holds true:

- (i) *There exist finitely many disjoint ϵ -flat \mathcal{C}^p -submanifolds $M_1, \dots, M_r \subset M$ of dimension d such that*

$$(74) \quad \dim(M \setminus \cup_{i=1}^r M_i) < d,$$

and $\pi_U : M_i \rightarrow U$ is \mathcal{C}^p -diffeomorphic onto its image.

- (ii) *Let $F_i : \pi_U(M_i) \rightarrow \pi_{U^\perp}(M_i)$ which sends $u \in \pi_U(M_i)$ to $\pi_{U^\perp}(\pi_U^{-1}(u))$. Then F_i is a definable \mathcal{C}^p -mapping with*

$$(75) \quad \|\nabla F_i\| \leq \sqrt{n-d} \frac{c}{\sqrt{1-c^2}}.$$

Proof:

(i): We consider M in an orthogonal coordinate system induced by $U \times U^\perp$. Let D_1, \dots, D_k be the $\epsilon/2$ -balls in $\mathbb{G}_{d,n}(R)$ and $\tau : M \rightarrow \mathbb{G}_{d,n}(R)$ the tangent mapping. Let $(B_l)_{l \in L}$ be a finite \mathcal{C}^p -cell decomposition compatible with M and $\tau^{-1}(D_1), \dots, \tau^{-1}(D_k)$. Let the M_i be the cells with $M_i \subset M$ such that $\dim(M_i) = d$. By construction, each M_i is ϵ -flat.

Claim: Every cell M_i is the graph of a \mathcal{C}^p -mapping from an open \mathcal{C}^p -cell of U into U^\perp .

M_i is a $(\sigma_1, \dots, \sigma_n)$ -cell with σ_i either 0 or 1. If one of the $\sigma_{d+1}, \dots, \sigma_n$ is 1, there is at least one $u_0 \in U^\perp \setminus \{0\}$ such that u_0 belongs to the tangent space of M at some x . This contradicts the assumption that $\delta(T_x M, U) < 1$.

Hence π_U restricted to M_i is an injective mapping, and therefore $\pi_U|_{M_i}$ is \mathcal{C}^p -diffeomorphic onto its image.

Furthermore, $\dim(M \setminus \cup_{i=1}^r M_i) < d$ by the choice of the M_i .

(ii): Let V be the tangent space of M_i at x , and denote by ω_U the projection of V onto U . We consider the mapping $u \mapsto \pi_{U^\perp}(\omega_U^{-1}(u))$. The derivative of this mapping and the derivative of F_i at $u = \pi_U(x)$ coincide.

Hence, we have to estimate the Euclidean norm of this linear mapping.

Since $\delta(V, U) < c$, i.e.

$$\sup_{v \in V, \|v\|=1} \inf_{u \in U, \|u\|=1} \sqrt{1 - \langle v, u \rangle}^2 = \sup_{v \in V, \|v\|=1} \sqrt{1 - \|\pi_U(v)\|^2} < c,$$

and $\|\pi_U(v)\|^2 + \|\pi_{U^\perp}(v)\|^2 = 1$ for $\|v\| = 1$, we obtain $\delta(V, U^\perp) > \sqrt{1 - c^2}$. So, $\|\omega_U(v)\| > \sqrt{1 - c^2} \|v\|$ and $\|\pi_{U^\perp}(v)\| < c \|v\|$ such that

$$(76) \quad \|\pi_{U^\perp}(\omega_U^{-1}(u))\| < \frac{c}{\sqrt{1 - c^2}} \|u\|.$$

Therefore the operator R -norm of this linear mapping is bounded by $\frac{c}{\sqrt{1 - c^2}}$, and therefore its Euclidean R -norm is less than $\sqrt{n - d} \frac{c}{\sqrt{1 - c^2}}$. \square

We now transfer the idea of ϵ -flat submanifolds to functions and cells.

Definition 4.11 *We call a function f ϵ -flat if its graph is ϵ -flat. We call Λ^p -regular cells and \mathcal{C}^p -cells ϵ -flat if their defining functions are ϵ -flat.*

The boundary of an ϵ -flat cell is in general no ϵ -flat submanifold. However, except for a sufficiently small set, the boundary is the finite union of ϵ -flat submanifolds.

Lemma 4.12 *Let $n \geq 1$ and let $A \subset R^n$ be a definable ϵ -flat \mathcal{C}^p -cell of dimension n . Then there is a definable subset $B \subset \partial A$ with $\dim(B) < n - 1$ such that $\partial A \setminus B$ is a \mathcal{C}^{p+1} -submanifold, and the image of the tangent-mapping $\tau : \partial A \setminus B \rightarrow \mathbb{G}_{n-1,n}(R)$ is covered by $2n$ ϵ -balls with respect to δ .*

Proof: We proof the lemma by induction on n .

Let $n = 1$.

In this case the defining functions are constant functions, and the boundary consists of two isolated points at most; so no proof is necessary.

Let $n > 1$,

and we suppose that the lemma holds true for $n - 1$.

Then $A = (f, g)_B$ where f, g are ϵ -flat \mathcal{C}^p -mappings, and there is a definable $n-3$ -dimensional set $C \subset \partial B$ such that $\partial B \setminus C$ is a \mathcal{C}^{p+1} -submanifold of R^{n-1} , and the image of the tangent-mapping $\tau|_{\partial B \setminus C}$ in $\mathbb{G}_{n-2, n-1}(R)$ is contained in $2n - 2$ ϵ -balls B_1, \dots, B_{2n-2} of $\mathbb{G}_{n-2, n-1}(R)$ with respect to δ .

Now we set

$$(77) \quad B'_i := \{M \times R \mid M \in B_i\}, \quad i = 1, \dots, 2n - 2.$$

Then, by proposition 4.8 (ii), each B'_i is contained in an ϵ -ball in $\mathbb{G}_{n-1, n}(R)$, and, moreover,

$$(78) \quad \tau((\partial B \setminus C) \times R) \subset \cup_{i=1}^{2n-2} B'_i.$$

Furthermore, we put $B'_{2n-1} := \tau((f)_B)$ and $B'_{2n} := \tau((g)_B)$.

We now choose a \mathcal{C}^{p+1} -cell decomposition of R^n which is compatible with the sets ∂A , $(\partial B \setminus C) \times R$, $(f)_B$, $(g)_B$. Let E_1, \dots, E_r denote the cells of dimension $n - 1$ which are contained in ∂A . Then

$$(79) \quad \dim(\partial A \setminus \cup_{j=1}^r E_j) < n - 1$$

by the choice of the E_j . Each cell E_j is either contained in $(\partial B \setminus C) \times R$, or in $(f)_B$, or in $(g)_B$. In any of the three cases, $\tau(E_j) \subset \cup_{i=1}^{2n} B'_i$. Hence, the lemma is proved. \square

Finite Open Covers of the Grassmannian

By [5] chapter 3.4, the Grassmannian $\mathbb{G}_{k, n}(R)$ is isomorphic to the algebraic set

$$(80) \quad H_{n, k}(R) := \{A \in R^{n \times n} \mid A^t = A, A^2 = A, \text{tr}(A) = k\}.$$

The isomorphism σ maps a vector space V to the matrix π_V .

For $A \in H_{n, k}(R)$, $u \in R^n$ we obtain $\langle Au, (Id - A)u \rangle = \langle u, (A - A^2)u \rangle = 0$, so $\|Au\|^2 + \|(Id - A)u\|^2 = 1$. Hence,

$$(81) \quad \|A\|_{op} \leq 1$$

which induces that the Euclidean R -norm $\|A\|$ of A is bounded by \sqrt{n} . Thus, $H_{n,k}(R)$ is a closed and bounded subset of $R^{n \times n}$. This means that, for the real numbers, $H_{n,k}(\mathbb{R})$ is a compact set, and therefore each open cover of $H_{n,k}(\mathbb{R})$ has a finite subcover. This conclusion does not hold true for $H_{n,k}(R)$ if $R \neq \mathbb{R}$.

Our aim is to show that we can cover $\mathbb{G}_{k,n}(R)$, respectively $H_{n,k}(R)$, by finitely many ϵ -balls with respect to the corresponding R -metric if we choose a positive $\epsilon \in \mathbb{R}_{alg} \subset R$.

We first show an open cover property of $H_{n,k}(R)$, cf. lemma 4.13.

We look at the formula $\Phi_H((x_{ij}))$:

$$(82) \quad \bigwedge_{1 \leq i, j \leq n} \left[x_{ij} = x_{ji} \wedge \sum_{l=1}^n x_{il} x_{lj} = x_{ij} \right] \wedge \sum_{l=1}^n x_{ll} = k$$

So, $A \in R^{n \times n}$ belongs to $H_{n,k}(R)$ iff $S^{semialg}(R) \models \Phi_H(A)$. Moreover we obtain that $H_{n,k}(R)$ is \mathbb{R}_{alg} -definable.

Lemma 4.13 *Let $\epsilon \in \mathbb{R}_{alg} \subset R$ be positive. Then we can cover $H_{n,k}(R)$ with finitely many ϵ -balls*

$$(83) \quad B_\epsilon^R(A_i) := \{A \in H_{n,k}(R) \mid \|A - A_i\| < \epsilon\}$$

where the $A_1, \dots, A_{r_A} \in H_{n,k}(R)$.

Proof: $H_{n,k}(\mathbb{R})$ is compact. Hence, for given $\epsilon \in \mathbb{R}_{alg}$ there are $A_1, \dots, A_r \in H_{n,k}(\mathbb{R})$ such that

$$(84) \quad \bigcup_{i=1}^r B_\epsilon^{\mathbb{R}}(A_i)$$

covers $H_{n,k}(\mathbb{R})$. We formulate for this ϵ and r the following sentence $\Phi(\epsilon, r)$:

$$(85) \quad \exists_{i=1}^r A_i \in H_{n,k}(R) \forall A \in H_{n,k}(R) : \|A_1 - A\| < \epsilon \vee \dots \vee \|A_r - A\| < \epsilon$$

This sentence is \mathbb{R}_{alg} -definable and $S^{semialg}(\mathbb{R}) \models \Phi$. Using the transfer-principle we conclude that $S^{semialg}(\mathbb{R}_{alg}) \models \Phi(\epsilon, r)$. Applying the transfer-principle again we see that $S^{semialg}(R) \models \Phi(\epsilon, r)$ for arbitrary real closed field R . So the lemma is proved. \square

Proposition 4.14 *Let $0 < \epsilon \in \mathbb{R}_{alg} \subset R$. There are finitely many ϵ -balls in $\mathbb{G}_{d,n}(R)$ with respect to δ which cover $\mathbb{G}_{d,n}(R)$.*

Proof: step 1: For $U, V \in \mathbb{G}_{k,n}(R)$,

$$(90) \quad \|\pi_V - \pi_U\| < \epsilon \rightarrow \delta(U, V) < \epsilon$$

applies, where we identify π_V with its corresponding matrix in $H_{n,k}(R)$ if we consider the Euclidean norm.

We consider the following inequality:

$$(91) \quad \begin{aligned} \delta(U, V) &= \|\pi_U \perp \pi_V\|_{op} \\ &= \|\pi_V - \pi_U \pi_V\|_{op} \\ &= \|\pi_V \pi_V - \pi_U \pi_V\|_{op} \\ &\leq \|\pi_U - \pi_V\|_{op} \\ &\leq \|\pi_U - \pi_V\| \end{aligned}$$

Now inequality (90) is evident.

step 2:

Let the sets $B_\epsilon^R(A_i)$ cover $H_{n,k}(R)$, cf. lemma 4.13. Then the ϵ -balls

$$(92) \quad \{U \in \mathbb{G}_{k,n}(R) \mid \delta(U, \sigma^{-1}(A_i)) < \epsilon\} \supset \sigma^{-1}(B_\epsilon^R(A_i))$$

are obviously a cover of $\mathbb{G}_{k,n}(R)$. □

Metric-Combinatorial Properties of Lines

The last lemma of this section plays the key role in the proof of theorem 4.5. Together with proposition 4.10 and by using a suitable coordinate system which is induced by this lemma, we get an estimate for the bounds of the derivatives of finitely many functions whose graphs define ϵ -flat submanifolds.

We denote by τ_n the volume of the unit-ball $B_1(0)$. We will need a suitable lower bound for the fraction $\frac{\tau_n}{\tau_{n-1}}$, for $n \geq 2$. We compute this in the next lemma.

Lemma 4.15 *For $n \geq 2$ applies*

$$(93) \quad \frac{\tau_n}{\tau_{n-1}} > \frac{2}{\sqrt{n}}.$$

Proof: In [28] example 5.7 it is shown that

$$(94) \quad \tau_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}$$

where Γ denotes the Gamma-function. We recall the functional equation of Γ , i.e.

$$(95) \quad \Gamma(x+1) = x\Gamma(x), \quad x > 0$$

and special values of it, i.e. $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1) = 1$. For $n = 1, 2, 3$ we now obtain the values $\tau_1 = 2$, $\tau_2 = \pi$, and $\tau_3 = \frac{4}{3}\pi$, so that

$$(96) \quad \frac{\tau_2}{\tau_1} = \frac{\pi}{2} \geq \sqrt{2} = \frac{2}{\sqrt{2}}, \quad \text{and} \quad \frac{\tau_3}{\tau_2} = \frac{4}{3} \geq \frac{2}{\sqrt{3}}$$

Now we assume the validity of (93) for $n' \leq n$. Then

$$(97) \quad \frac{\tau_{n+1}}{\tau_n} = \frac{\sqrt{\pi}\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2}+1)} = \frac{\sqrt{\pi n}\Gamma(\frac{n-2}{2}+1)}{(n+1)\Gamma(\frac{n-1}{2}+1)} = \frac{n}{n+1} \frac{\tau_{n-1}}{\tau_{n-2}} \geq \frac{n}{n+1} \frac{2}{\sqrt{n-1}}$$

Together with

$$(98) \quad \frac{n^2}{(n+1)^2(n-1)} = \frac{n^2}{n^3+n^2-n-1} \geq \frac{n^2}{n^3+n^2} = \frac{1}{n+1}$$

we obtain the statement of the lemma. \square

In the next lemma we study numerically a metric-combinatorial property of lines in \mathbb{R}^n . This lemma plays the key-role for bounding the gradient. The idea is down to [33], but we improve this result by giving an explicit bound. The proof given in [33] provides us with an exponentially increasing bound whereas we can show that the bound increases only polynomially.

Lemma 4.16 *Let $2 \leq n \leq r \in \mathbb{N}$. For all lines G_1, \dots, G_r of \mathbb{R}^n there exists a line $P \subset \mathbb{R}^n$ with the following property:*

$$(102) \quad \delta(P, G_i) < c, \quad i = 1, \dots, r$$

with $\frac{c}{\sqrt{1-c^2}} = \sqrt{nr^2 - 1}$.

Proof: For each $i = 1, \dots, r$ we look at the sets

$$(103) \quad X_{G_i}(c) := \bigcup_{\substack{H \in \mathbb{G}_{1,n}(\mathbb{R}) \\ \delta(H, G_i) < c}} H \cap B_1(0) = \{x \in B_1(0) \mid \delta(\mathbb{R}x, G_i) < c\}$$

$$Y_{G_i}(c) := \bigcup_{\substack{H \in \mathbb{G}_{1,n}(\mathbb{R}) \\ \delta(H, G_i) \geq c}} H \cap B_1(0) = \{x \in B_1(0) \mid \delta(\mathbb{R}x, G_i) \geq c\}.$$

We find a line P with

$$(104) \quad \delta(P, G_i) < c, \quad i = 1, \dots, r$$

if and only if

$$(105) \quad \bigcap_{i=1}^r X_{G_i}(c) \neq \{0\}.$$

This is equivalent to the following statement:

$$(106) \quad \bigcup_{i=1}^r Y_{G_i}(c) \neq B_1(0) \setminus \{0\}$$

The strategy is to find a c large enough so that the volume of $Y_{G_i}(c)$ is that small that

$$(107) \quad r \text{Vol}(Y_{G_i}(c)) < \text{Vol}(B_1(0) \setminus \{0\}) = \text{Vol}(B_1(0)) = \tau_n.$$

The volume $Y_{G_i}(c)$ does not depend on the choice of the line G_i . It always stays the same for all lines. Therefore we estimate the volume for $G = \mathbb{R}e_n$. We note that

$$(108) \quad Y_G(c) \subset \{(x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n-1} x_j^2 \leq 1, |x_n| \leq \sqrt{1-c^2}\}.$$

Therefore the volume $\text{Vol}(Y_{G_i}(c))$ is bounded by

$$(109) \quad \text{Vol}(Y_{G_i}(c)) \leq 2\sqrt{1-c^2}\tau_{n-1}.$$

From equation (107) and (109) we conclude that c is a suitable constant if it satisfies the following inequality:

$$(110) \quad \sqrt{1-c^2} < \frac{\tau_n}{\tau_{n-1}} \frac{1}{2r}$$

According to lemma 4.15, the estimation

$$(111) \quad \sqrt{1-c^2} = 2 \frac{1}{2\sqrt{nr}} = \frac{1}{\sqrt{nr}}$$

suits our needs, and therefore

$$(112) \quad c = \sqrt{1 - \frac{1}{r^2n}}.$$

Now we can give a suitable estimation for $c/\sqrt{1-c^2}$:

$$(113) \quad \frac{c}{\sqrt{1-c^2}} \leq \sqrt{1 - \frac{1}{r^2n}} \sqrt{nr} = \sqrt{nr^2 - 1}$$

□

In the proof of theorem 4.5 we will refer to the following statement.

Lemma 4.17 *Let $1 \leq n \in \mathbb{N}$ and $0 < \epsilon < \frac{1}{256n^6}$. Let $G_1, \dots, G_{2n} \subset R^n$ be lines. Then there is a line $P \subset R^n$ such that the following holds true: If $H_1, \dots, H_{2n} \subset R^n$ are lines with $\delta(H_i, G_i) < \epsilon$, $i = 1, \dots, 2n$, then*

$$(114) \quad \delta(H_i, P) < c + \epsilon, \quad i = 1, \dots, 2n$$

with $\frac{c+\epsilon}{\sqrt{1-(c+\epsilon)^2}} \leq 2n^{\frac{3}{2}}$.

Proof: step 1:

First we show that we can replace \mathbb{R} in the statement of lemma 4.16 by an arbitrary real closed field R .

Each line $H \subset R^n$ is determined by a unit-vector $y_i \in R^n$. We fix r and n and consider the formula $\Phi(a, b)$ with the variables a and b :

$$(115) \quad \forall_{1 \leq i \leq r, 1 \leq j \leq n} y_{i,j} \left[\left[\bigwedge_{1 \leq i \leq r} \sum_{j=1}^n y_{i,j}^2 = 1 \right] \rightarrow \exists_{1 \leq l \leq n} p_l \left[\sum_{l=1}^n p_l^2 = 1 \rightarrow \left[\bigwedge_{1 \leq i \leq r} \left(\sum_{k=1}^n p_k y_{i,k} \right)^2 > \frac{1}{a^2 b} \right] \right] \right]$$

For $a = r$ and $b = n$ the formula $S^{\text{semialg}}(\mathbb{R}) \models \Phi(r, n)$. Since the numbers $r, n \in \mathbb{R}_{\text{alg}}$, we can apply the transfer principle, proposition 2.9, to $\Phi(r, n)$ so that $S^{\text{semialg}}(\mathbb{R}_{\text{alg}}) \models \Phi(r, n)$. By applying the transfer principle again, $S^{\text{semialg}}(R) \models \Phi(r, n)$ for each real closed field R .

step 2:

In our case $r = 2n$, i.e.

$$(116) \quad 1 - c^2 = \frac{1}{4n^3}, \quad c = \sqrt{1 - \frac{1}{4n^3}}.$$

We make use of the inequality

$$(117) \quad 1 - x \leq \sqrt{1 - x} \leq 1 - \frac{x}{2}, \quad 0 \leq x < \frac{1}{2}.$$

Hence, with $\epsilon_n = \frac{1}{256n^6}$, we get

$$(118) \quad c + \epsilon_n = \sqrt{1 - \frac{1}{4n^3}} + \epsilon_n \leq 1 - \frac{1}{8n^3} + \epsilon_n \leq 1 - \frac{1}{16n^3} \leq \sqrt{1 - \frac{1}{16n^3}}$$

and

$$(119) \quad \begin{aligned} (1 - (c + \epsilon_n)^2)^{-\frac{1}{2}} &= (1 - c^2 - 2c\epsilon_n - \epsilon_n^2)^{-\frac{1}{2}} \\ &\geq \left(\frac{1}{4n^3} - 4\epsilon_n\right)^{-\frac{1}{2}} \\ &= 2n^{\frac{3}{2}} \left(1 - \frac{16n^3}{256n^6}\right)^{-\frac{1}{2}} \\ &= 2n^{\frac{3}{2}} \left(1 - \frac{1}{16n^3}\right)^{-\frac{1}{2}} \end{aligned}$$

which induces

$$(120) \quad \frac{c + \epsilon_n}{\sqrt{1 - (c + \epsilon_n)^2}} \leq 2n^{\frac{3}{2}}.$$

The lemma is now a straight conclusion of the triangle inequality of δ . \square

Proof of Theorem 4.5

Preliminary Lemmata

Before we prove theorem 4.5 we give two technical lemmata.

Lemma 4.18 *Let $U \subset \mathbb{R}^d$ be an open set, $f : U \rightarrow \mathbb{R}^{n-d}$ a \mathcal{C}^p -function, and $u \in U$. For all $\alpha \in \mathbb{N}^d$, $|\alpha| = p$, let*

$$(121) \quad \|D_\alpha f(u)\| \leq M.$$

If $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a linear orthogonal isomorphism, then for $v = \varphi^{-1}(u)$

$$(122) \quad \|D_\alpha(f \circ \varphi)(v)\| \leq Md^{\frac{p}{2}}$$

applies.

Proof: Let $|D_\alpha f_k(u)| \leq M_k$, $k = 1, \dots, n-d$, $|\alpha| = p$. Then we may assume that

$$(123) \quad M = \left(\sum_{k=1}^{n-d} M_k^2 \right)^{\frac{1}{2}}.$$

So we may restrict our examination to the case $n = d + 1$.

Let φ be given by the matrix $(\varphi_{i,j})_{1 \leq i,j \leq d}$. Then for each i , $\sum_{j=1}^d \varphi_{i,j}^2 = 1$, and therefore

$$(124) \quad \sum_{j=1}^d |\varphi_{i,j}| \leq \sqrt{d}.$$

If $D_\alpha = \frac{\partial}{\partial x_{l_1}} \dots \frac{\partial}{\partial x_{l_p}}$, we can express $D_\alpha(f \circ \varphi)(v)$ as

$$(125) \quad D_\alpha(f \circ \varphi)(v) = \sum_{1 \leq k_1, \dots, k_p \leq d} \frac{\partial}{\partial x_{l_1}} \dots \frac{\partial}{\partial x_{l_p}} f(\varphi(v)) \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_p, l_p},$$

so that it remains to show that

$$(126) \quad \left| \sum_{1 \leq k_1, \dots, k_p \leq d} \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_p, l_p} \right| \leq d^{\frac{p}{2}}.$$

We prove this by induction on p .

The case $p = 1$ is evident by inequation (124).

We now assume that

$$(127) \quad \left| \sum_{1 \leq k_1, \dots, k_{p-1} \leq d} \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_{p-1}, l_{p-1}} \right| \leq d^{\frac{p-1}{2}}$$

applies. Then

$$(128) \quad \begin{aligned} \left| \sum_{1 \leq k_1, \dots, k_p \leq d} \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_p, l_p} \right| &= \left| \sum_{1 \leq k_1, \dots, k_{p-1} \leq d} \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_{p-1}, l_{p-1}} \sum_{k_p=1}^d \varphi_{k_p, l_p} \right| \\ &\leq \left| \sum_{1 \leq k_1, \dots, k_{p-1} \leq d} \varphi_{k_1, l_1} \cdot \dots \cdot \varphi_{k_{p-1}, l_{p-1}} \right| \left| \sum_{k_p=1}^d \varphi_{k_p, l_p} \right| \\ &\leq d^{\frac{p-1}{2}} \sqrt{d} \\ &= d^{\frac{p}{2}}. \end{aligned}$$

□

Lemma 4.19 *Let $M \subset R^n$ be a definable connected \mathcal{C}^1 -submanifold of dimension d , and let $X, Y \subset M$ be two disjoint submanifolds of dimension d . Then $\text{cl}(X) \cap Y = \emptyset$.*

Proof: We assume that there is an $x \in \text{cl}(X)$ with $x \in Y$. Since M is a \mathcal{C}^1 -submanifold, there is an open definable neighbourhood U of $x \in M$ and a definable \mathcal{C}^1 -diffeomorphism $\varphi : U \rightarrow V$ where V is an open neighbourhood of 0 such that $\varphi(x) = 0$ and

$$(129) \quad \varphi(U \cap M) = V \cap \{x \in R^n \mid x_{n_d+1} = \dots = x_n = 0\}.$$

Since $\varphi(X \cap U)$ and $\varphi(Y \cap U)$ are open in $V \cap \{x \in R^n \mid x_{n_d+1} = \dots = x_n = 0\}$, there is an open neighbourhood B of 0 such that

$$(130) \quad B \cap V \cap \{x \in R^n \mid x_{n_d+1} = \dots = x_n = 0\} = B \cap \varphi(Y).$$

Hence, since X and Y are disjoint, $x \notin \text{cl}(X)$. □

We now prepare the proof of theorem 4.5 by means of the following lemma.

Lemma 4.20 *Let $d \leq n \in \mathbb{N}$, $n \geq 1$ and $\epsilon_n = \frac{1}{256n^6}$. Let $A \subset R^n$ be a definable set of dimension d . Let $I_{n,d}(A)$ be the following statement:*

For each positive integer p and for all $\epsilon \in \mathbb{R}_{alg}$, $0 < \epsilon \leq \epsilon_n$, we can partition A , each definable subset B of R^n with $\dim(B) < d$, and each definable set $\tilde{B} \subset R^{\tilde{n}}$, $\tilde{n} < n$, into finitely many definable sets X_1, \dots, X_r such that each X_i is an ϵ -flat Λ^p -regular cell with constant $2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$.

Then for all definable $A \subset R^n$ of dimension d , the statement $I_{n,d}(A)$ holds true.

Proof: Throughout the proof we often use the obvious fact that, for disjoint definable sets $X, Y \subset R^n$, $I_{n,d}(X \cup Y)$ holds true if both $I_{n,d}(X)$ and $I_{n,d}(Y)$ hold true.

We now prove $I_{n,d}(A)$ by induction on n and d .

Let $n = 1$.

The cases $I_{1,0}$ and $I_{1,1}$ are evident according to o-minimality.

Let $n > 1$.

We now assume that $I_{n-1,n-1}$ holds true. We show by induction on d that $I_{n,d}$ holds true.

$d = 0$:

$A \subset R^n$ consists of finitely many points $A = \{a_1, \dots, a_r\}$.

Each set $\{a_i\}$ is the graph of the function $h_i : R^0 \rightarrow R^n$, $h_i(0) = a_i$. Since the h_i are constant functions, they are Λ^p -regular with any constant and they

are ϵ -flat.

We now assume that $I_{n,d-1}(B)$ holds true for each definable $B \subset R^n$ of dimension $d-1$. We show that for every definable $A \subset R^n$ of dimension d , $I_{n,d}(A)$ holds true.

case 1: $0 < d < n$

step 1a:

Let $A \subset R^n$ be a definable set of dimension d . According to theorem 2.23, we can decompose R^n into finitely many \mathcal{C}^{p+1} -cells which are compatible with A . We denote by A_1, \dots, A_{r_A} the cells of dimension d which are contained in A so that

$$(131) \quad \dim(A \setminus \cup_{i=1}^{r_A} A_i) < d,$$

and we apply $I_{n,d-1}$ to $A \setminus \cup_{i=1}^{r_A} A_i$.

So it remains to prove the validity of $I_{n,d}(\tilde{A})$ for all d -dimensional \mathcal{C}^{p+1} -cells $\tilde{A} \subset R^n$.

step 1b:

Let $B \subset R^n$ be a \mathcal{C}^{p+1} -cell of dimension d .

According to proposition 4.14, we fix a finite definable open cover

$$(132) \quad \bigcup_{j \in J} B_{\epsilon/2}(G_j) = \mathbb{G}_{d,n}(R)$$

of the Grassmannian with $\epsilon/2$ -balls.

B is a \mathcal{C}^{p+1} -submanifold so that the tangent-mapping $\tau : B \rightarrow \mathbb{G}_{d,n}(R)$ is definable and continuous. Hence, the preimages $B_j = \tau^{-1}(B_{\epsilon/2}(G_j))$ form an open cover of B with respect to the Euclidean topology restricted to B , i.e.

$$(133) \quad B = \bigcup_{j \in J} B_j,$$

and, according to the construction, every B_j is an ϵ -flat submanifold of R^n .

In order to obtain pairwise disjoint ϵ -flat submanifolds, we select a \mathcal{C}^{p+1} -cell decomposition of R^n which is compatible with the B_j , $j \in J$.

For all j we denote by $B_{j,l}$, $l \in L_j$, the cells of dimension d which are contained in B_j . Then

$$(134) \quad \dim(B_j \setminus \cup_{l \in L_j} B_{j,l}) < d.$$

We apply $I_{n,d-1}$ to each $B_j \setminus \cup_{l \in L_j} B_{j,l}$.

Every cell $B_{j,l}$ is an ϵ -flat \mathcal{C}^{p+1} -submanifold so that it remains to show that

$I_{n,d}(\tilde{B})$ holds true for each definable d -dimensional ϵ -flat \mathcal{C}^{p+1} -submanifold \tilde{B} of R^n .

step 1c:

Let $C \subset R^n$ be a definable ϵ -flat \mathcal{C}^{p+1} -submanifold of dimension d . We choose a linear subspace $U \in \mathbb{G}_{d,n}(R)$ such that

$$(135) \quad \delta(V, U) < \epsilon_n < \frac{1}{2}, \quad V \in \tau(C).$$

From now on we consider C in the orthogonal coordinates induced by $U \times U^\perp$. According to proposition 4.10, there are finitely many definable \mathcal{C}^{p+1} -mappings $f_k : C_k \rightarrow U^\perp$, $k = 1, \dots, r_C$, where the C_k are open definable subsets of U such that

$$(136) \quad \|\nabla f_k\| \leq \frac{\frac{1}{2}}{\sqrt{1 - (\frac{1}{2})^2}} \sqrt{n-d} \leq \sqrt{n}$$

and

$$(137) \quad \dim(C \setminus \cup_{k=1}^{r_C} (f_k)_{C_k}) < d.$$

We apply $I_{n,d-1}$ to $C \setminus \cup_{k=1}^{r_C} (f_k)_{C_k}$.

Every f_k is a definable ϵ -flat \mathcal{C}^{p+1} -mapping so that it remains to show the validity of $I_{n,d}(\Gamma(f))$ for all graphs $\Gamma(f)$ of definable ϵ -flat \mathcal{C}^{p+1} -mappings $f : \tilde{C} \rightarrow R^{n-d}$ with $\|\nabla f\| \leq \sqrt{n}$.

step 1d:

Let $D \subset R^d$ be open and $f : D \rightarrow R^{n-d}$ a definable ϵ -flat \mathcal{C}^{p+1} -mapping such that the gradient of f it is bounded by \sqrt{n} .

According to corollary 3.5, there is a closed definable set $Z \subset D$ of dimension less than d such that f restricted to $D \setminus Z$ is a Λ^p -regular function with constant

$$(138) \quad \sqrt{n-d} 2^{\binom{p+2}{2}-2} \sqrt{n}.$$

The factor $\sqrt{n-d}$ is needed because of f being from Z to R^{n-d} . The graph of the restriction remains ϵ -flat.

By means of $I_{d,d}$, we choose a Λ^p -regular ϵ -flat partition of R^d with constant $2^{\binom{p+2}{2}-1} d^{\frac{p+3}{2}}$ which is compatible with the set $D \setminus Z$, and we denote by D_1, \dots, D_{r_D} the ϵ -flat Λ^p -regular cells of dimension d which are contained in $D \setminus Z$. Then

$$(139) \quad \dim(D \setminus \cup_{m=1}^{r_D} D_m) < d,$$

and we apply $I_{n,d-1}$ to $f(D \setminus \cup_{m=1}^{r_D} D_m)$.

Now we consider each D_m in the coordinates in which it is standard cell, i.e. there is a linear orthogonal change of coordinates of R^d given by the matrix φ_m such that $\varphi_m^{-1}(D_m)$ is an ϵ -flat standard Λ^p -regular cell with constant $2^{\binom{p+2}{2}-1} d^{\frac{p+3}{2}}$.

Since ϵ -flatness does not depend on the choice of the coordinate system, the $f \circ \varphi_m$ are ϵ -flat. According to lemma 4.18, they are Λ^p -regular functions with constant $2^{\binom{p+2}{2}-2} \sqrt{n-d} \sqrt{nd}^{\frac{p}{2}}$.

Since $d < n$, both constants are less than $2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$ so that the

$$(140) \quad (f \circ \varphi_m)_{\varphi_m^{-1}(D_m)}, \quad m = 1, \dots, r_D,$$

are ϵ -flat standard Λ^p -regular cells with constant $2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$.

Thus, $I_{n,d}(A)$ holds true for each definable d -dimensional subset $A \subset R^n$.

case 2: $d = n$

step 2a:

Let $\dim A = d = n$.

We choose a \mathcal{C}^{p+1} -cell decomposition of R^n compatible with A and denote by A_1, \dots, A_{r_A} the cells of dimension n which are contained in A . Then

$$(141) \quad \dim(A \setminus \cup_{i=1}^{r_A} A_i) < n,$$

and we can apply $I_{n,n-1}$ to $A \setminus \cup_{i=1}^{r_A} A_i$ so that it remains to show the validity of $I_{n,n}(A)$ for each open \mathcal{C}^{p+1} -cell A of R^n .

step 2b:

Let $B = (f, g)_X$ be an open \mathcal{C}^{p+1} -cell.

According to proposition 4.14 we fix a finite cover of $\mathbb{G}_{n-1,n}(R)$ with $\epsilon/2$ -balls, i.e.

$$(142) \quad \bigcup_{i \in I} B_{\epsilon/2}(G_i) = \mathbb{G}_{n-1,n}(R).$$

Let $\tau_f : (f)_X \rightarrow \mathbb{G}_{n-1,n}(R)$ and $\tau_g : (g)_X \rightarrow \mathbb{G}_{n-1,n}(R)$ denote the tangent-mappings. We set $\tau_{\pm\infty} := R^{n-1} \times \{0\}$ since we treat $\pm\infty$ as constant functions. We denote by $\pi : R^n \rightarrow R^{n-1}$ the projection on the first $n-1$ coordinates.

By $I_{n,n-1}$, we choose an ϵ -flat Λ^{p+1} -regular partition of $\pi(B)$ compatible with

$$(143) \quad \pi(B), \quad \pi(\tau_f^{-1}(B_\epsilon(G_i))), \quad \pi(\tau_g^{-1}(B_\epsilon(G_i))), \quad i \in I.$$

Let B_1, \dots, B_{r_B} be the cells of dimension $n - 1$ which are contained in at least one of the sets of (143).

For each j there is a linear orthogonal coordinate system of R^{n-1} in which B_j is an ϵ -flat Λ^{p+1} -regular standard cell. With respect to this coordinate system, the sets

$$(144) \quad (f, g)_{B_j}, \quad j = 1, \dots, r_B,$$

are ϵ -flat \mathcal{C}^{p+1} -cells since f and g restricted to B_j are both ϵ -flat \mathcal{C}^{p+1} -mappings.

Moreover,

$$(145) \quad \dim(\pi(B) \setminus \cup_{j=1}^{r_B} B_j) < n - 1$$

by the choice of the B_j .

So we conclude that

$$(146) \quad \dim(B \setminus \cup_{j=1}^{r_B} (f, g)_{B_j}) < n.$$

We apply $I_{n,n-1}$ to $B \setminus \cup_{j=1}^{r_B} (f, g)_{B_j}$ so that it remains to show that $I_{n,n}(\tilde{B})$ is valid for each definable open ϵ -flat \mathcal{C}^{p+1} -cell $\tilde{B} \subset R^n$.

step 2c:

Let C be an open ϵ -flat \mathcal{C}^{p+1} -cell in R^n .

According to lemma 4.12 there is a definable subset $Y \subset \partial C$ such that

- (i) $\dim(Y) < n - 1$,
- (ii) $\partial C \setminus Y$ is a \mathcal{C}^{p+1} submanifold, and
- (iii) $\tau : \partial C \setminus Y \rightarrow \mathbb{G}_{n-1,n}(R)$ is covered by $2n$ $\epsilon/2$ -balls $B_{\epsilon/2}(G'_1), \dots, B_{\epsilon/2}(G'_{2n})$.

By lemma 4.17 we find a line P such that for each family of lines H_1, \dots, H_{2n} with $\delta(H_i, G_i^\perp) < \epsilon$, $i = 1, \dots, 2n$,

$$(147) \quad \delta(H_i, P) < c + \epsilon, \quad i = 1, \dots, 2n,$$

applies, where

$$(148) \quad \frac{c + \epsilon}{\sqrt{1 - (c + \epsilon)^2}} \leq 2n^{\frac{3}{2}}.$$

From now on we consider C in an orthogonal coordinate system induced by $P^\perp \times P$. Note that, since P and each G_i^\perp are lines, we have $\delta(P^\perp, G_i) < c + \epsilon$.

Let $\pi : R^n \rightarrow R^{n-1}$ be the projection onto the first $n - 1$ coordinates. According to proposition 4.10 there exist finitely many definable open sets $D_1, \dots, D_{r_D} \subset \pi(\partial C)$ and ϵ -flat \mathcal{C}^{p+1} -functions $f_k : D_k \rightarrow R$ such that

$$(149) \quad (f_k)_{D_k} \subset \partial C \setminus Y, \quad k = 1, \dots, r_D,$$

$$(150) \quad \|\nabla f_k\| \leq \frac{c + \epsilon}{\sqrt{1 - (c + \epsilon)^2}} \leq 2n^{\frac{3}{2}}, \quad k = 1, \dots, r_D, \text{ and}$$

$$(151) \quad \dim((\partial C \setminus Y) \setminus \cup_{k=1}^{r_D} (f_k)_{E_k}) < n - 1.$$

Moreover, the $(f_k)_{D_k}$ are disjoint.

By $\dim(Y) < n - 1$ and inequalities (151) we obtain

$$(152) \quad \dim(\partial C \setminus \cup_{k=1}^{r_D} (f_k)_{D_k}) < n - 1.$$

step 2d:

We choose a \mathcal{C}^{p+1} -cell decomposition of R^n which is compatible with the sets

$$(153) \quad C, \partial C \text{ and } (f_k)_{D_k}, \quad k = 1, \dots, r_D.$$

We further assume that the number K of cells of this decomposition is minimal, i.e. that each \mathcal{C}^{p+1} -cell decomposition of R^n which is compatible with the sets of (153) consists of at least K cells.

By the definition of decompositions, cf. 2.22, the projection of the cells onto the first $n - 1$ coordinates are a decomposition of R^{n-1} .

We denote by E_1, \dots, E_{r_E} the open cells of the decomposition which are contained in C . Then their projections are open cells in R^{n-1} , and we have to distinguish between two cases.

- (i) $\pi(E_l) \subset \pi(C) \setminus \pi(\partial C)$, or
- (ii) $\pi(E_l) \subset \pi(\partial C)$.

subcase (i):

In this case E_l must be of the form

$$(154) \quad E_l = (-\infty, +\infty)_{\pi(E_l)}$$

since by assumption, $(-\infty, +\infty)_{\pi(E_l)} \subset C$, and $(-\infty, +\infty)_{\pi(E_l)} \cap \partial C = \emptyset$, and the number of cells is minimal.

subcase (ii):

Since inequation (152) applies and $\dim(\pi(E_l)) = n - 1$, $\pi(E_l) \subset D_k$, for some k . Since the cell decomposition is compatible with the $(f_k)_{D_k}$ we either have $\pi(E_l) \subset D_k$ or $\pi(E_l) \cap D_k = \emptyset$.

We denote by $f_1^l, \dots, f_{r_l}^l$ those functions of the f_1, \dots, f_{r_k} for which $\pi(E_l) \subset D_k$ holds true. Moreover, we may assume that $f_j^l < f_{j+1}^l$ on $\pi(E_l)$.

By the compatibility of the decomposition with the $(f_k)_{D_k}$, cf. (153), E_l is a subset of one of the following sets.

- (a) $(-\infty, f_1^l)_{\pi(E_l)}$
- (b) $(f_j^l, f_{j+1}^l)_{\pi(E_l)}, j = 1, \dots, r_l - 1$
- (c) $(f_{r_l}^l, \infty)_{\pi(E_l)}$

Moreover, each of the sets (a), (b) and (c) is the union of cells V of the decomposition with $\pi(V) = \pi(E_l)$. Hence, by minimality of the number of cells, E_l equals one of the sets of (a), (b) or (c).

Therefore we have shown that, after a change of the coordinate system, we can partition an open ϵ -flat \mathcal{C}^{p+1} -cell C into some lower dimensional set and finitely many open \mathcal{C}^{p+1} -cells E_1, \dots, E_{r_E} with $E_l = (h_l, g_l)_{\pi(E_l)}$ such that h_l and g_l are ϵ -flat \mathcal{C}^{p+1} -functions with $h_l < g_l$, and both $\|\nabla h_l\|$ and $\|\nabla g_l\|$ are bounded by the constant $2n^{\frac{3}{2}}$.

Moreover,

$$(155) \quad \dim(C \setminus \cup_{l=1}^{r_E} E_l) < n.$$

We apply $I_{n,n-1}$ to $C \setminus \cup_{l=1}^{r_E} E_l$ so that it remains to show the validity of $I_{n,n}(E_1), \dots, I_{n,n}(E_{r_E})$.

step 2e:

Let E be one of the cells E_l with $E = (f, g)_X$.

According to corollary 3.5 there exists a definable subset $Z \subset X$ of dimension less than $n - 1$ such that f and g are both Λ^p -regular on $X \setminus Z$ with constant $2^{\binom{p+2}{2}-2} 2n^{\frac{3}{2}}$; f and g remain ϵ -flat. Since $X \subset R^{n-1}$, we can choose by $I_{n-1,n-1}$ an ϵ -flat Λ^p -regular partition of X compatible with Z with constant $2^{\binom{p+2}{2}-1} (n-1)^{\frac{p+3}{2}}$. We denote by V_1, \dots, V_{r_V} the cells of dimension $n - 1$ and select for each V_m a linear orthogonal coordinate system in which V_m is a standard Λ^p -regular cell. Furthermore, we consider $(f, g)_{V_m}$ in the coordinate system induced by $V_m \times V_m^\perp$.

Hence, by lemma 4.18, each f and g restricted to V_m are ϵ -flat Λ^p -regular functions with constant $2^{\binom{p+2}{2}-2} 2n^{\frac{3}{2}} n^{\frac{p}{2}} = 2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$, and therefore each

$$(156) \quad (f, g)_{V_m}, \quad m = 1, \dots, r_V,$$

is an ϵ -flat Λ^p -regular cell with constant $2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$.

Moreover,

$$(157) \quad \dim(E \setminus \cup_{m=1}^{r_V} (f, g)_{V_m}) < n$$

so that we can apply $I_{n,n-1}$ to $E \setminus \cup_{m=1}^{r_V}(f, g)_{V_m}$.
So the lemma is proved. \square

Proof of Theorem 4.5

Let $C := 2^{\binom{p+2}{2}-1} n^{\frac{p+3}{2}}$.

We proof the following statement I_d :

There are finitely many definable subsets $B_{j,l}$, $n-d \leq j \leq n$, $1 \leq l \leq r_j$ of R^n such that the following holds true:

- (i) $B_{j,l}$ is a Λ^p -regular cell with constant C for $n-d \leq j \leq n$, $1 \leq l \leq r_j$,
- (ii) $\dim(B_{j,l}) = j$ for $n-d \leq j \leq n$, $1 \leq l \leq r_j$,
- (iii) $\dim(R^n \setminus \cup_{j=n-d}^n \cup_{l=1}^{r_j} B_{j,l}) < n-d$,
- (iv) for each $n-d \leq j \leq n$, $1 \leq l \leq r_j$ and $k = 1, \dots, r_A$, either $B_{j,l} \cap A_k = \emptyset$ or $B_{j,l} \subset A_k$ applies, and
- (v) for $j, \tilde{j} \geq n-d$, $1 \leq l \leq r_j$ and $1 \leq \tilde{l} \leq r_{\tilde{j}}$ the following holds true:
If $B_{j,l} \subset \text{cl}(B_{\tilde{j},\tilde{l}})$, then $B_{j,l} \subset \partial B_{\tilde{j},\tilde{l}}$ or $B_{j,l} = B_{\tilde{j},\tilde{l}}$.

The statement of theorem 4.5 is I_n .

We proof I_d by induction on d .

Let $d = 0$.

We select a \mathcal{C}^1 -cell decomposition of R^n compatible with the sets A_1, \dots, A_{r_A} , and denote by $X_{n,1}, \dots, X_{n,s_n}$ the \mathcal{C}^1 -cells of dimension n .

According to lemma 4.20 we can partition each of the $X_{n,i}$ into finitely many Λ^p -regular cells with constant C . We denote by $B_{n,1}, \dots, B_{n,r_n}$ the Λ^p -regular cells of dimension n , so

$$(158) \quad \dim(R^n \setminus \cup_{l=1}^{r_n} B_{n,l}) < n.$$

Hence, (i), (ii) and (iii) hold true.

(iv) holds true since the \mathcal{C}^1 -cell decomposition is compatible with the A_k .

(v) holds true since the $B_{n,j}$ are disjoint open sets.

Hence, I_0 is valid.

Let $0 \leq d < n$ We assume that I_d holds true and show the validity of I_{d+1} .

We select a \mathcal{C}^1 -cell decomposition of R^n compatible with the following sets:

$$(159) \quad A_1, \dots, A_{r_A}, B_{j,l}, \partial B_{j,l}, j = n-d, \dots, n, l = 1, \dots, r_j$$

We denote by $X_{n-(d+1),1}, \dots, X_{n-(d+1),r_{d+1}}$ the \mathcal{C}^1 -cells of dimension $n-(d+1)$ which are contained in

$$(160) \quad R^n \setminus \bigcup_{j=n-d}^n \bigcup_{l=1}^{r_j} B_{j,l}$$

so that, by (iii) of I_d ,

$$(161) \quad \dim \left(R^n \setminus \left(\bigcup_{j=n-d}^n \bigcup_{l=1}^{r_j} B_{j,l} \cup \bigcup_{i=1}^{r_{d+1}} X_{d+1,i} \right) \right) < n - (d+1).$$

We now choose a further \mathcal{C}^p -cell decomposition of R^n which is compatible with the sets

$$(162) \quad X_{n-(d+1),1}, \dots, X_{n-(d+1),r_{d+1}}, \partial X_{n-(d+1),1}, \dots, \partial X_{n-(d+1),r_X}.$$

We denote by Y_1, \dots, Y_{r_Y} the cells of dimension $n-(d+1)$ which are contained in some $X_{n-(d+1),i}$.

Since $\dim(\partial X_{n-(d+1),i}) < n-(d+1)$, we conclude by the compatibility of the decomposition that for all Y_k

$$(163) \quad Y_k \cap \partial X_{n-(d+1),i} = \emptyset$$

applies. Moreover, in connection with equation (161) we conclude that

$$(164) \quad \dim \left(R^n \setminus \left(\bigcup_{j=n-d}^n \bigcup_{l=1}^{r_j} B_{j,l} \cup \bigcup_{k=1}^{r_Y} Y_k \right) \right) < n - (d+1).$$

According to lemma 4.20 we can partition each of the sets Y_k into finitely many Λ^p -regular cells with constant C , and by $B_{n-(d+1),1}, \dots, B_{n-(d+1),r_{d+1}}$ we denote the Λ^p -regular cells of dimension $n-(d+1)$.

By the choice of the $B_{n-(d+1),i}$ in connection with I_d , (i) and (ii) are valid.

By the choice of the $B_{n-(d+1),i}$ and inequation (164), (iii) holds true.

(iv) is evident since the $X_{n-(d+1),l}$ either have empty intersection with A_k or they are a subset of it, and each $B_{n-(d+1),i}$ is a subset of one of the $X_{n-(d+1),l}$.

We now show property (v).

By the compatibility of the $X_{n-(d+1),l}$ with the sets of (159), and since each

$B_{n-(d+1),i}$ is a subset of some $X_{n-(d+1),s}$ it is evident that for $j \geq n-d$ we either have $B_{n-(d+1),i} \subset \partial B_{j,l}$ or $B_{n-(d+1),i} \cap \partial B_{j,l} = \emptyset$.

It remains to show that for $B_{n-(d+1),i}$ and $B_{n-(d+1),j}$ it holds true that

$$(165) \quad \text{cl}(B_{n-(d+1),i}) \cap B_{n-(d+1),j} = \emptyset, \quad i \neq j.$$

We have to distinguish between two cases:

Either $B_{n-(d+1),i}$ and $B_{n-(d+1),j}$ are both contained in the same $X_{n-(d+1),l}$, and therefore, by lemma 4.19, $\text{cl}(B_{n-(d+1),i}) \cap B_{n-(d+1),j} = \emptyset$, or they are contained in two different $X_{n-(d+1),l_i} \neq X_{n-(d+1),l_j}$.

In this case

$$(166) \quad \text{cl}(B_{n-(d+1),i}) \subset \text{cl}(X_{n-(d+1),l_i}),$$

and $B_{n-(d+1),j}$ is contained in some $Y_k \subset X_{n-(d+1),l_j}$.

By (163), $Y_k \cap \partial X_{n-(d+1),l_i} = \emptyset$. Therefore,

$$(167) \quad \begin{aligned} \text{cl}(B_{n-(d+1),i}) \cap B_{n-(d+1),j} &\subset \text{cl}(X_{n-(d+1),l_i}) \cap Y_k \\ &= X_{n-(d+1),l_i} \cap Y_k \\ &\subset X_{n-(d+1),l_i} \cap X_{n-(d+1),l_j} \\ &= \emptyset \end{aligned}$$

since different cells of a cell decomposition have empty intersection. \square

5 Ordinary Differentiable Functions

We begin the intrinsic study of differentiable functions. Before we look at higher order Peano differentiability we consider the more widely known ordinary differentiable functions. These are by definition the 1-times Peano-differentiable functions.

For the whole chapter we fix a real closed field R and an o-minimal expansion $\mathcal{M} = (R, +, -, \cdot, 1, 0, <, \dots)$ of R . If we speak of definable sets or functions we always mean definable in the structure \mathcal{M} .

First we recall the definition of ordinary differentiability.

Definition 5.1 *Let $U \subset R^n$ be an open set and let $f : U \rightarrow R$ be a function. f is called ordinary differentiable at $x_0 \in U$ with derivative $Df(x_0) \in R^n$ if the following equation holds true:*

$$(168) \quad \lim_{U \ni x \rightarrow x_0} \frac{f(x) - (f(x_0) + \langle Df(x_0), x - x_0 \rangle)}{\|x - x_0\|} = 0$$

The ordinary differentiable functions form a composition closed ring. It is important to note that the chain-rule is valid. Although this has already been proved in [62] chapter 7, we will prove it again in the next chapter for the more general case of Peano-differentiable functions.

In literature we find the concepts of totally differentiable or Frechet-differentiable functions. Often they are just called differentiable functions. These concepts coincide for the \mathbb{R}^n . In our case we speak of ordinary differentiable in order to underline that the derivative is not necessarily continuous.

We begin the discussion by presenting several examples. These examples show that continuous and ordinary differentiability differ for definable functions of two or more variables. The non-continuous derivatives can be locally bounded or unbounded functions. Semidefinite derivatives are not necessarily continuous either.

Moreover, we give an example which shows that the Implicit-Function-Theorem is not valid for ordinary differentiable definable functions.

All these examples are semialgebraic functions of two variables, and the zero-point is the only point at which the derivatives are not continuous. This is no coincidence. We will see in later chapters that the set of points at which ordinary differentiable functions are not continuously differentiable is of codimension greater than or equal to 2.

After the discussion of the examples we prove that for definable functions of one variable, ordinary and continuous differentiability coincide.

This property enables us to characterise ordinary differentiability of a function at a certain point by means of definable \mathcal{C}^1 -curves. Furthermore, we study ordinary differentiable definable functions composed with non-definable curves. More precisely, the curves are continuously differentiable, respectively ordinary differentiable with bounded derivative.

The differentiability class of these compositions at certain points allows us to decide whether the derivative of the function is locally bounded or continuous at those points.

Examples

We now give four examples of definable ordinary differentiable functions. The examples are semialgebraic functions so that they are definable in \mathcal{M} .

For the following three examples let A be the semialgebraic set

$$(169) \quad A := \{(x, y) \in \mathbb{R}^2 \mid y^2 \leq x \leq 3y^2, y > 0\}.$$

The derivative of the first function is not continuous but it is locally bounded.

Example 5.2

The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$(170) \quad (x, y) \mapsto \begin{cases} y^2 \left(1 - \left(\frac{x}{y^2} - 2\right)^2\right)^2, & (x, y) \in A \\ 0, & (x, y) \notin A \end{cases}$$

is semialgebraic and ordinary differentiable on \mathbb{R}^2 but not continuously differentiable at $(0, 0)$, and the partial derivatives are locally bounded.

Proof: Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be the function $t \mapsto (t - 1)^2(t - 3)^2\chi_{[1,3]}(t)$. h is obviously a semialgebraic continuously differentiable function. Moreover, $h'(1) = h'(3) = 0$ and $h'(3/2) = 3/2$. With the help of h we can write f as

$$(171) \quad f(x, y) = \begin{cases} y^2 h\left(\frac{x}{y^2}\right), & y > 0 \\ 0, & \text{else.} \end{cases}$$

If $(x, y) \notin \text{cl}(A)$, then $f(x, y) = 0$. If $y > 0$, then f is a continuously differentiable function.

Hence, f is continuously differentiable in

$$(R^2 \setminus \text{cl}(A)) \cup (R \times R^+) = R^2 \setminus \{(0, 0)\}.$$

Since h is a bounded function,

$$(172) \quad \lim_{(x,y) \rightarrow (0,0)} \left| \frac{f(x,y)}{\|(x,y)\|} \right| \leq \lim_{(x,y) \rightarrow (0,0)} \left| \frac{y^2}{\|(x,y)\|} \right| \sup_{t \in R} h(t) = 0$$

so that it is evident that f is ordinary differentiable at $(0, 0)$ with derivative $(0, 0)$.

The gradient of f is

$$(173) \quad \nabla f(x, y) = \left(h' \left(\frac{x}{y^2} \right), 2yh \left(\frac{x}{y^2} \right) - \frac{2x}{y} h' \left(\frac{x}{y^2} \right) \right), \quad (x, y) \in A,$$

and vanishes outside A .

Moreover, h' and h are both bounded functions so that $\frac{\partial f}{\partial x}$ is a bounded function. Since for $(x, y) \in A$,

$$(174) \quad 0 \leq \frac{2x}{y} \leq \frac{6y^2}{y} = 6y$$

applies, $\frac{\partial f}{\partial y}$ is locally bounded as well.

$h'(3/2) = 3/2$. Therefore,

$$(175) \quad \frac{\partial f}{\partial x} \left(\frac{3}{2}t^2, t \right) = \begin{cases} \frac{3}{2}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Hence, f is not continuously differentiable at $(0, 0)$. □

In general we do not get locally bounded derivatives as we will point out in the next example.

Example 5.3 Let $g : R^2 \rightarrow R$ be the semialgebraic function defined by

$$(176) \quad g(x, y) := \begin{cases} y^{3/2} \left(1 - (x/y^2 - 2)^2 \right)^2, & (x, y) \in A \\ 0, & (x, y) \notin A. \end{cases}$$

g is ordinary differentiable and the derivative is not locally bounded at $(0, 0)$.

Proof: Since $g(x, y) = f(x, y)/\sqrt{|y|}$ for $y \neq 0$, and 0 elsewhere, we see that g is continuously differentiable in $R^2 \setminus \{(0, 0)\}$ and ordinary differentiable

at $(0, 0)$.

For $t > 0$,

$$(177) \quad \frac{\partial g}{\partial x} \left(\frac{3}{2}t^2, t \right) = \frac{1}{\sqrt{t}} \frac{\partial f}{\partial x} \left(\frac{3}{2}t^2, t \right) = \frac{h' \left(\frac{3}{2} \right)}{\sqrt{t}} = \frac{3}{2\sqrt{t}}.$$

Hence, the partial derivative of g with respect to x is not locally bounded at 0 . \square

The next example gives us a negative answer to the question whether the Implicit-Function-Theorem is valid for definable ordinary differentiable functions or not.

Example 5.4 Let $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the semialgebraic function

$$(178) \quad h(x, y) := \begin{cases} x - x \left(1 - \left(\frac{x}{y^2} - 2 \right)^2 \right)^2, & (x, y) \in A \\ x, & (x, y) \notin A. \end{cases}$$

h is ordinary differentiable, $\frac{\partial h}{\partial x}(0, 0) = 1$ and $h(0, 0) = 0$, but for each $\delta > 0$ there is more than one \mathcal{C}^1 -function $\varphi : (-\delta, \delta) \rightarrow \mathbb{R}$ with $\varphi(0) = 0$ and $h(\varphi(t), t) = 0$.

Proof: We know by the previous examples that h is continuously differentiable in $\mathbb{R}^2 \setminus \{(0, 0)\}$.

We check the differentiability of h at $(0, 0)$. If $(x, y) \in A$,

$$(179) \quad |h(x, y) - x| = \left| -x \left(1 - \left(\frac{x}{y^2} - 2 \right)^2 \right)^2 \right|$$

$$(180) \quad \begin{aligned} &\leq |x| \\ &\leq 3y^2. \end{aligned}$$

Since $3y^2$ is $o(\|(x, y)\|)$ we conclude that $h(x, y) - x$ is $o(\|(x, y)\|)$ in A . Otherwise, if $(x, y) \notin A$, $h(x, y) - x = 0$ so that it is evident that h is ordinary differentiable at $(0, 0)$. Furthermore, the partial derivative of h with respect to x equals 1 at $(0, 0)$ and h vanishes at $(0, 0)$.

We now consider the two functions $\varphi_1 : (-\delta, \delta) \rightarrow \mathbb{R}$ and $\varphi_2 : (-\delta, \delta) \rightarrow \mathbb{R}$ which are defined by

$$(181) \quad \varphi_1(t) = 0,$$

and

$$(182) \quad \varphi_2(t) = \begin{cases} 0, & \text{if } 0 \leq t \\ 2t^2, & \text{else.} \end{cases}$$

Then $h(\varphi_1(t), t) = 0$ and $h(\varphi_2(t), t) = 0$, but φ_1 and φ_2 are different functions in each neighbourhood of 0. \square

The derivative of the next example vanishes at $(0, 0)$, and both partial derivatives are positive semidefinite functions. In spite of this, the function is not continuously differentiable.

Example 5.5 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$(183) \quad h(t) := (t^4 - 4t^3 + 4t^2)\chi_{[0,1]}(t) + \chi_{(1,\infty)}(t).$$

We define $f : B_{\frac{1}{2}}(0) \rightarrow \mathbb{R}$ by

$$(184) \quad f(x, y) := \begin{cases} y^4 h\left(\frac{x}{y^4} + 1\right), & y > 0 \\ 0, & y \leq 0. \end{cases}$$

f is semialgebraic and ordinary differentiable with positive semidefinite partial derivatives. Moreover, the derivative vanishes at $(0, 0)$ and is not continuous at $(0, 0)$.

Proof: step 1: h is continuously differentiable with positive semidefinite derivative, and $h'(\frac{1}{2}) = \frac{3}{2}$.

Let p be the polynomial $p(t) = t^4 - 4t^3 + 4t^2 = t^2(t - 2)^2$. Then the derivative of p is $p'(t) = 4t^3 - 12t^2 + 8t = 4t(t - 1)(t - 2)$. We obtain that $p(0) = p'(0) = p'(1) = 0$ and $p(1) = 1$. Hence, h is continuously differentiable and its derivative is positive semidefinite. Moreover, $h'(\frac{1}{2}) = p'(\frac{1}{2}) = \frac{3}{2}$.

step 2: f restricted to the set $B = B_{\frac{1}{2}}(0) \cap \{(x, y) \mid y < |x|\}$ is the function

$$(185) \quad f|_B(x, y) = \begin{cases} y^4, & x > 0, 0 \leq y < x \\ 0, & \text{else.} \end{cases}$$

Hence, $f|_B$ is obviously continuously differentiable. Moreover, for $y > 0$, f is the composition of continuously differentiable functions and is therefore continuously differentiable itself. Since h is a bounded function, f is ordinary

differentiable with derivative $(0, 0)$.

step 3: The derivative of f vanishes for $y \leq 0$, and for $y > 0$ we can write it as

$$(186) \quad Df(x, y) = \left(h' \left(\frac{x}{y^4} + 1 \right), 4y^3 h' \left(\frac{x}{y^4} + 1 \right) + h' \left(\frac{x}{y^4} + 1 \right) \frac{-4x}{y} \right).$$

Since h' is positive semidefinite, the partial derivative of f with respect to x is positive semidefinite. If $h' \left(\frac{x}{y^4} + 1 \right)$ is positive, then $x/y^4 + 1 \leq 1$, and therefore $x \leq 0$. Hence, the partial derivative with respect to y is also positive semidefinite.

step 4: $\frac{\partial f}{\partial x}$ is not continuous at $(0, 0)$.

We check this by

$$(187) \quad \lim_{t \searrow 0} \frac{\partial f}{\partial x} \left(\frac{-1}{2} t^4, t \right) = \lim_{t \searrow 0} h' \left(\frac{-1}{2} \frac{t^4}{t^4} - 1 \right) = h' \left(\frac{1}{2} \right) = \frac{3}{2} \neq 0 = \frac{\partial f}{\partial x}(0, 0)$$

□

One-Dimensional Differentiable Functions

In this section we will show that in o-minimal context, ordinary differentiability induces continuous differentiability for univariate functions. We begin with an example taken from classical Analysis.

Example 5.6 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be

$$(188) \quad f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We know that this function is ordinary differentiable in \mathbb{R} but the derivative is not continuous at 0. This case does not appear in o-minimal structures, as the following proposition indicates.

Proposition 5.7 Let $I \subset \mathbb{R}$ be an open interval and let $f : I \rightarrow \mathbb{R}$ be a definable ordinary differentiable function. Then f is continuously differentiable.

Proof: Without loss of generality we consider the point $0 =: x_0 \in I$. Moreover, we may assume that $f(0) = 0$ and $f'(0) = 0$. For $\epsilon > 0$ let M_ϵ^+ and M_ϵ^- be the definable sets

$$(189) \quad M_\epsilon^+ := \{x \in I \mid f'(x) > \epsilon\}, \quad M_\epsilon^- := \{x \in I \mid f'(x) < -\epsilon\}.$$

According to o-minimality, both sets are the disjoint union of finitely many open intervals and single points, i.e.

$$(190) \quad M_\epsilon^\pm = \{a_1\} \cup \dots \cup \{a_m\} \cup I_1 \cup \dots \cup I_r.$$

Since $0 \notin M_\epsilon^\pm$, none of the points a_1, \dots, a_m equals 0 and no interval contains 0. We show that the intervals are not of the form $(0, b)$ or $(a, 0)$, with $a < 0$ and $b > 0$.

We assume that $(0, b) \subset M_\epsilon^+$. According to the Mean-Value-Theorem,

$$(191) \quad f(t) - f(0) > \epsilon(t - 0), \quad t \in (0, b),$$

so that

$$(192) \quad f'(0) = \lim_{t \searrow 0} \frac{f(t)}{t} \geq \epsilon$$

which contradicts the assumption that $f'(0) = 0$.

By symmetry, we conclude that $(a, 0) \not\subset M_\epsilon^+$, and neither $(0, b) \subset M_\epsilon^-$ nor $(a, 0) \subset M_\epsilon^-$. Hence, 0 does not belong to the closure of M_ϵ^\pm for any $\epsilon > 0$.

If f' was not continuous at 0 then there would be an $\epsilon > 0$ such that $0 \in \text{cl}(M_\epsilon^\pm)$. Since this does not occur, f' is continuous at 0. \square

The next lemma will be used later in this chapter and in chapter 8.

Lemma 5.8 *Let $f(-1, 1) \rightarrow R$ be a continuous definable function which is ordinary differentiable outside 0. If $\lim_{t \rightarrow 0} f'(t)$ exists in R , f is continuously differentiable at 0.*

Proof: Without loss of generality we may assume that $f(0) = 0$ and $\lim_{t \rightarrow 0} f'(t) = 0$. According to the Mean-Value-Theorem, it holds true that

$$(193) \quad f(t) - f(0) = f'(\xi_t)(t - 0)$$

for some $\xi_t \in (0, t)$, respectively $\xi_t \in (t, 0)$.

Hence,

$$(194) \quad |f(t)| \leq \sup_{\xi \neq 0, |\xi| < |t|} |f'(\xi)| |t|.$$

Therefore,

$$(195) \quad \lim_{t \rightarrow 0} \left| \frac{f(t)}{t} \right| \leq \lim_{t \rightarrow 0} \sup_{\xi \neq 0, |\xi| < |t|} |f'(\xi)| = 0$$

\square

A Characterisation of Ordinary Differentiability

By proposition 5.7, ordinary and continuous differentiability coincide for univariate functions in o-minimal structures.

In order to generalise the one-dimensional differentiability concepts to functions of several variables there exist a couple of concepts beside the one of ordinary differentiability.

We know the partial differentiability which provides us with a unique derivative at a point. The answer to the question whether or not a function is partially differentiable at a certain point depends on the choice of the coordinate system in which we consider the function. This problem is avoided by the concept of Gateaux-differentiability.

We recall that a function $f : R^n \rightarrow R$ is called *Gateaux-differentiable* at $x_0 \in U$ if there is an $a \in R^n$ such that for each $\nu \in R^n$, $\|\nu\| = 1$, the following holds true:

$$(196) \quad \lim_{t \rightarrow 0} \frac{f(x_0 + t\nu) - f(x_0)}{t} = \langle \nu, a \rangle.$$

The next example shows that in o-minimal context as well as in classical Analysis, a Gateaux-differentiable function is not necessarily ordinary differentiable.

Let $A := \{(x, y) \mid y > 0, y^2 < x < 3y^2\}$.

Example 5.9 Let $g : R^2 \rightarrow R$ be the semialgebraic function

$$(197) \quad g(x, y) := \begin{cases} \left(1 - (x/y^2 - 2)^2\right)^2, & (x, y) \in A \\ 0, & (x, y) \notin A. \end{cases}$$

g is Gateaux-differentiable but not ordinary differentiable at $(0, 0)$.

Proof: The same argument as for example 5.2 here induces the continuity of the partial derivatives outside $(0, 0)$, i.e. g is continuously differentiable in $R^2 \setminus \{(0, 0)\}$.

Let $\nu := (a, b)$ be a unit-vector. If $g(at, bt) \neq 0$, the following inequality holds true:

$$(198) \quad b^2 |t|^2 < |a| |t| < 3b^2 |t|^2.$$

So, if $a = 0$ or $b = 0$, then $g(at, bt) = 0$ for all $t \in R$. Otherwise, $g(at, bt) = 0$ for $|t| \leq \frac{|a|}{3b^2}$. Thus, for all directions ν , the derivative at $(0, 0)$ is

$$(199) \quad \lim_{t \rightarrow 0} \frac{g(\nu t) - g((0, 0))}{t} = 0.$$

Hence, g is Gateaux-differentiable at $(0, 0)$.

Since g is obviously not continuous at $(0, 0)$ we conclude that g is not ordinary differentiable at $(0, 0)$. \square

One of the insufficient properties of the Gateaux-differentiable functions is that they are not closed under composition. If we select the function g of example 5.9 and $\varphi : (-1, 1) \rightarrow \mathbb{R}^2$ with $\varphi(t) = (\frac{3}{2}t^2, t)$, we see that $g \circ \varphi(t) = \frac{9}{16}\chi_{(0,1)}(t)$ is not ordinary differentiable at 0, and it is a function of one variable.

An alternative concept is to claim composition closedness a priori. The next example belongs to the differentiability class which consists of all definable functions which are continuously differentiable along all definable continuously differentiable curves.

Example 5.10 *The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by*

$$(200) \quad f(x, y) := \begin{cases} \frac{yx^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{else} \end{cases}$$

is semialgebraic and continuously differentiable along all definable \mathcal{C}^1 -curves but it is not ordinary differentiable at $(0, 0)$.

Proof: The partial derivatives of f vanish at $(0, 0)$; otherwise they are

$$(201) \quad \frac{\partial f}{\partial x}(x, y) = \frac{2xy^3}{(x^2 + y^2)^2}$$

$$(202) \quad \frac{\partial f}{\partial y}(x, y) = \frac{x^4}{(x^2 + y^2)^2} - \frac{2x^2y^2}{(x^2 + y^2)^2}.$$

Hence, the gradient of f is a bounded function.

Claim:

Let $p(x, y) = x^k y^l$ where k and l are non-negative integers with $k + l = 4$.

Let $\varphi : (-\delta, \delta) \rightarrow \mathbb{R}^2$ be a definable \mathcal{C}^1 -curve with $\varphi'(0) = (a, b) \neq (0, 0)$.

Then

$$(203) \quad \lim_{t \rightarrow 0} \frac{p(\varphi_1(t), \varphi_2(t))}{(\varphi_1^2(t) + \varphi_2^2(t))^2}$$

exists in \mathbb{R} .

Proof of the claim:

Let $a \neq 0$. Then

$$(204) \quad \lim_{t \rightarrow 0, t \neq 0} \frac{\varphi_2(t)}{\varphi_1(t)} = \lim_{t \rightarrow 0, t \neq 0} \frac{\frac{\varphi_2(t)}{t}}{\frac{\varphi_1(t)}{t}} = \frac{b}{a}.$$

Thus,

$$(205) \quad \begin{aligned} \lim_{t \rightarrow 0, t \neq 0} \frac{p(\varphi_1(t), \varphi_2(t))}{(\varphi_1^2(t) + \varphi_2^2(t))^2} &= \lim_{t \rightarrow 0, t \neq 0} \frac{\varphi_1^{4-l}(t) \varphi_2^l(t)}{(\varphi_1^2(t) + \varphi_2^2(t))^2} \\ &= \lim_{t \rightarrow 0, t \neq 0} \frac{\varphi_1^4(t) \left(\frac{\varphi_2(t)}{\varphi_1(t)}\right)^l}{\varphi_1^4(t) \left(1 + 2\left(\frac{\varphi_2(t)}{\varphi_1(t)}\right)^2 + \left(\frac{\varphi_2(t)}{\varphi_1(t)}\right)^4\right)} \\ &= \frac{\left(\frac{b}{a}\right)^l}{\left(1 + \frac{b^2}{a^2}\right)^2}. \end{aligned}$$

The case $b \neq 0$ is analogous to $a \neq 0$. Hence, the claim is proved.

Let $\varphi : (-\delta, \delta) \rightarrow R^2$ be a definable \mathcal{C}^1 -curve with $\varphi(0) = 0$. If $\varphi'(0) = 0$ then

$$(206) \quad \lim_{t \rightarrow 0, t \neq 0} \frac{d}{dt}(f \circ \varphi(t)) = \lim_{t \rightarrow 0, t \neq 0} \left(\frac{\partial f}{\partial x}(\varphi(t)) \varphi_1'(t) + \frac{\partial f}{\partial y}(\varphi(t)) \varphi_2'(t) \right) = 0$$

since the partial derivatives of f are bounded.

If $\varphi'(0) = (a, b) \neq 0$ then, for $t \neq 0$,

$$(207) \quad \begin{aligned} \frac{d}{dt}(f \circ \varphi(t)) &= \frac{2\varphi_1(t)\varphi_2^3(t)}{(\varphi_1^2(t) + \varphi_2^2(t))^2} \varphi_1'(t) + \frac{\varphi_1^4(t)}{(\varphi_1^2(t) + \varphi_2^2(t))^2} \varphi_2'(t) \\ &\quad - \frac{2\varphi_1^2(t)\varphi_2^2(t)}{(\varphi_1^2(t) + \varphi_2^2(t))^2} \varphi_2'(t). \end{aligned}$$

By the continuity of φ' and the claim, the limit

$$(208) \quad \lim_{t \rightarrow 0, t \neq 0} \frac{d}{dt}(f \circ \varphi(t))$$

exists in R . Hence, according to lemma 5.8, $f \circ \varphi$ is continuously differentiable at 0. \square

Again, we do not get ordinary differentiability in general.

We now give a characterisation of ordinary differentiability by means of definable regular \mathcal{C}^1 -curves and the chain-rule. This characterisation shows that by using definable \mathcal{C}^1 -curves we are only able to decide whether a definable function is ordinary differentiable or not. In order to get more precise information about the gradient such as local boundedness or continuity we have to use more general classes of curves. This will be discussed later in this chapter.

Proposition 5.11 *Let $U \subset R^n$ be a definable open subset and $f : U \rightarrow R$ a definable function. f is ordinary differentiable at $x_0 \in U$ if and only if there is an $a \in R^n$ such that for all definable regular \mathcal{C}^1 -curves $\varphi : (-1, 1) \rightarrow U$ with $\varphi(0) = x_0$ the following holds true:*

$$(209) \quad \lim_{t \rightarrow 0} \frac{f \circ \varphi(t) - f(x_0)}{t} = \langle \varphi'(0), a \rangle$$

Proof: If f is differentiable at x_0 , then equation (209) is valid for every \mathcal{C}^1 -curve.

Now we show that condition (209) is sufficient.

In this case we assume that f is not differentiable at x_0 . Thus, for every $a \in R^n$ there is an $\epsilon > 0$ and a definable set $A \subset U$ with $x_0 \in \overline{A}$ such that

$$(210) \quad \left| \frac{f(x) - f(x_0) - \langle x - x_0, a \rangle}{\|x - x_0\|} \right| > \epsilon, x \in A.$$

Since f is differentiable along all lines, it is also partially differentiable. So, without loss of generality we may assume that $a = (0, \dots, 0)$ and $f(x_0) = 0$.

According to the Curve-Selection-Lemma, there is a definable regular \mathcal{C}^1 -curve $\varphi : (-\delta, \delta) \rightarrow U$ with $\varphi(0) = x_0$ and $\varphi((\delta, 0)) \subset A$. By regularity of φ , $\|\varphi'(0)\| = c > 0$. Hence, the continuity of φ' implies that there is a $\delta' > 0$ such that $\|\varphi(t) - x_0\| \geq \frac{c}{2} |t|$ for $|t| < \delta'$.

Therefore

$$(211) \quad \epsilon \leq \left| \frac{f \circ \varphi(t)}{\|\varphi(t) - x_0\|} \right| \leq \left| \frac{f \circ \varphi(t)}{\frac{c}{2}t} \right|.$$

But then, $f \circ \varphi$ is not ordinary differentiable at 0 with derivative 0.

Hence, f has to be ordinary differentiable at x_0 . □

As a consequence we can express ordinary differentiable with Gateaux-differentiability in connection with composition closedness.

Corollary 5.12 *Let $U \subset R^n$ be open and $f : U \rightarrow R$ a definable function. Let f be Gateaux-differentiable at $x_0 \in U$, and for each definable regular \mathcal{C}^1 -curve $\varphi : (-1, 1) \rightarrow U$ with $\varphi(0) = x_0$ let $f \circ \varphi$ be continuously differentiable at 0. Then f is ordinary differentiable at x_0 .*

Proof: Let $\varphi : (-1, 1) \rightarrow U$ be a definable regular \mathcal{C}^1 -curve such that $\varphi(0) = x_0$. We have to show that the chain-rule is valid for $f \circ \varphi$ at $t = 0$. We define the curve $\psi : (-\delta, \delta) \rightarrow U$ for some $0 < \delta$ small enough by

$$(212) \quad \psi(t) := \begin{cases} \varphi(t), & t > 0 \\ x_0 + t\varphi'(0), & t \leq 0. \end{cases}$$

We note that $\psi|_{(-\delta, 0]}$ is a line and ψ is a regular \mathcal{C}^1 -curve. Hence,

$$(213) \quad \begin{aligned} \lim_{t \rightarrow 0} \frac{f \circ \varphi(t) - f(x_0)}{t} &= \lim_{t \searrow 0} \frac{f \circ \psi(t) - f(x_0)}{t} \\ &= \lim_{t \nearrow 0} \frac{f \circ \psi(t) - f(x_0)}{t} = \lim_{t \nearrow 0} \frac{f(x_0 + t\varphi'(0)) - f(x_0)}{t} \\ &= \langle \varphi'(0), \nabla f(x_0) \rangle. \end{aligned}$$

This is the chain-rule for $f \circ \varphi$ at $t = 0$. □

Definable Ordinary Differentiable Functions along Curves

We know the differentiability properties of definable ordinary functions along definable \mathcal{C}^1 -curves. As the following example points out, a definable ordinary differentiable function does not implicate continuous differentiability along non-definable \mathcal{C}^1 -curves.

Example 5.13 For $R = \mathbb{R}$ we consider the function g of example 5.3 along the curve $\varphi : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by

$$(214) \quad t \mapsto \begin{cases} \left(2t^2 + t^2 \sin\left(1/\sqrt{|t|}\right), t \right), & \text{if } t \neq 0 \\ 0, & \text{else.} \end{cases}$$

Then φ is a continuously differentiable curve since the derivative of φ ,

$$(215) \quad \varphi'(t) = \left(4t + 2t \sin\left(1/\sqrt{|t|}\right) - \frac{t}{2\sqrt{|t|}} \cos\left(1/\sqrt{|t|}\right), 1 \right), \quad t \neq 0$$

and $\varphi'(0) = (0, 0)$ is a continuous function.

For each $t \in \mathbb{R}^+$ the composition $g \circ \varphi(t)$ reads as follows:

$$(216) \quad t \mapsto t^{3/2} \left(\cos\left(1/\sqrt{t}\right) \right)^4$$

Thus, the derivative of the composition $g \circ \varphi$ is

$$(217) \quad \frac{d}{dt}(g \circ \varphi)(t) = \frac{3}{2}t^{1/2} \cos^4\left(1/\sqrt{t}\right) + 2 \cos^3\left(1/\sqrt{t}\right) \sin\left(1/\sqrt{t}\right).$$

We now have a look at the two zero-sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ which are given by

$$(218) \quad x_n := \frac{1}{(\pi/4 + 2n\pi)^2}, \text{ and } y_n := \frac{1}{4n^2\pi^2}.$$

We evaluate $(g \circ \varphi)'$ at these points:

$$(219) \quad \begin{aligned} (g \circ \varphi)'(x_n) &= \frac{3}{8} \frac{1}{\pi/4 + 2n\pi} + \frac{1}{2} \rightarrow \frac{1}{2} \\ (g \circ \varphi)'(y_n) &= \frac{3}{2} \frac{1}{2n\pi} \rightarrow 0 \end{aligned}$$

This shows that $g \circ \varphi$ is not continuously differentiable at 0. \square

The example motivates the following question:

Which information about the derivative of f do we get by checking the differentiability properties of the composition of $f \circ \varphi$ where the φ are curves of a certain differentiability class?

We know by proposition 5.11 that we can check ordinary differentiability of a definable function f at a point x_0 by using definable continuously differentiable curves.

Now we consider curves which are either continuously differentiable or ordinary differentiable with bounded derivative. Note that in both cases we do not restrict our examinations to definable curves.

If we deal with not necessarily definable continuous or continuously differentiable functions defined on a real closed field $R \neq \mathbb{R}$, there appear some technical problems since in this case, R is a totally disconnected set. For example the function $f : \mathbb{R}_{alg} \rightarrow \mathbb{R}_{alg}$ with

$$(220) \quad f(x) = \begin{cases} 1, & x < \pi \\ 0, & x > \pi \end{cases}$$

is analytic at each point $x \in \mathbb{R}_{alg}$. But neither the Mean-Value-Theorem nor the Intermediate-Value-Theorem apply to this function. So we will have to use very elementary techniques for our proofs when we deal with non-definable functions.

Definable Ordinary Differentiable Functions along C^1 -Curves

The first result which we prove is that if the derivative of a definable ordinary differentiable function f is locally bounded, then the composition of f with a continuously differentiable curve φ is always continuously differentiable.

The idea is the following: If $\varphi : (-1, 1) \rightarrow R^n$ is an ordinary differentiable curve with $\varphi'(0) \neq 0$ then, for $\delta > 0$ small enough, $\varphi((0, \delta))$ is contained in some half-cone. Using some further arguments, we can reduce the problem to sets of the form

$$(221) \quad A_{\epsilon, \alpha} := \left\{ (x_1, \dots, x_n) \in R^n \mid 0 < x_1 < \epsilon, \sum_{i=2}^n x_i^2 \leq \alpha^2 x_1^2 \right\}$$

where ϵ and α are positive.

Lemma 5.14 *Let $\varphi : (-1, 1) \rightarrow R^n$ be an ordinary differentiable curve with*

- (i) $\varphi(0) = 0$ and
- (ii) $\varphi'(0) = (1, 0, \dots, 0)$.

Then, for all $\alpha > 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that

$$(222) \quad \varphi((0, \delta)) \subset A_{\epsilon, \alpha}.$$

Proof: Since all φ_i , $i = 2, \dots, n$, are ordinary differentiable with $\varphi'(0) = 0$, there is a $\delta_1 > 0$ such that

$$(223) \quad |\varphi_i(t)| \leq \frac{\alpha}{2\sqrt{n-1}} |t|, \quad i = 2, \dots, n, \quad 0 < t < \delta_1.$$

Moreover,

$$(224) \quad \lim_{t \searrow 0} \frac{\varphi_1(t)}{t} = 1.$$

Hence, there is a $0 < \delta_2$ such that

$$(225) \quad \frac{1}{2}t < \varphi_1(t) < \frac{3}{2}t, \quad 0 < t < \delta_2.$$

If we choose $\delta < \min(\frac{\epsilon}{2}, \frac{\alpha}{2n}, \delta_1, \delta_2)$ then, for $t \in (0, \delta)$,

$$(226) \quad \sum_{i=2}^n \varphi_i^2(t) \leq \sum_{i=2}^n \frac{\alpha^2}{4(n-1)} t^2 = \frac{1}{4} \alpha^2 t^2 \leq \alpha^2 \varphi_1^2(t).$$

Now,

$$(227) \quad \varphi(t) \in A_{\epsilon, \alpha} \text{ for } 0 < t < \delta.$$

□

So we first study the size of the partial derivative with respect to the first variable in the sets $A_{\epsilon, \alpha}$. In order to do that we need some reparametrisation of definable curves.

Lemma 5.15 *Let ϵ, α, δ be positive. If $\varphi : (-\delta, \delta) \rightarrow R^n$ is a definable regular C^1 -curve with $\varphi(0) = 0$ and $\varphi((0, \delta)) \subset A_{\epsilon, \alpha}$, there is a definable regular C^1 -curve $\psi : (-\delta_1, \delta_1) \rightarrow R^n$ such that*

- (i) $\psi((-\delta_1, \delta_1)) \subset \varphi((-\delta, \delta))$,
- (ii) $\psi(0) = 0$, and
- (iii) $\psi(t) = (t, \psi_2(t), \dots, \psi_n(t))$ with $|\psi'_i(0)| \leq \alpha$, $i = 2, \dots, n$.

Proof: step 1: We show that $\varphi'_1(0) > 0$.

We assume that $\varphi'_1(0) = 0$.

Since φ is continuously differentiable with $\varphi(0) = 0$, for each $0 < \epsilon_2 < \epsilon$ there is a $\delta_2 > 0$ such that

$$(228) \quad |\varphi_1(t)| \leq \epsilon_2 |t|, \quad 0 < t < \delta_2.$$

Since $\varphi(t) \in A_{\epsilon, \alpha}$, $t > 0$, we conclude that for $i = 2, \dots, n$

$$(229) \quad |\varphi_i(t)| \leq \epsilon_2 \alpha t, \quad 0 < t < \delta_2.$$

Therefore, $\varphi'_i(t)$ vanishes at $t = 0$ which contradicts the assumption that φ is regular.

Hence, $\varphi'_1(0) \neq 0$. Since $\varphi(t) \in A_{\epsilon, \alpha}$ for $t > 0$, it is evident that $\varphi'_1(0) > 0$.

step 2:

Since $\varphi'_1(0) > 0$, there is an open neighbourhood $(-\delta', \delta')$ of 0 such that φ_1 has an inverse φ_1^{-1} on this set. This inverse is clearly continuously differentiable and satisfies $\varphi_1^{-1}(0) = 0$. We define $\psi : (-\delta_1, \delta_1) \rightarrow R^n$ by

$$(230) \quad \psi(t) = \varphi(\varphi_1^{-1}(t)), \quad t \in (-\delta_1, \delta_1),$$

where we choose δ_1 that small that $|\varphi_1^{-1}(t)| < \delta'$, for $|t| < \delta_1$. Obviously, ψ is continuously differentiable and (i) and (ii) hold true. Moreover, since

$\varphi_1(\varphi_1^{-1}(t)) = t$, it holds true that $\psi(t) = (t, \psi_2(t), \dots, \psi_n(t))$.
Since $\psi(t) \in A_{\epsilon, \alpha}$,

$$(231) \quad \sum_{i=2}^n \psi_i^2(t) \leq \alpha^2 t^2, \quad t > 0.$$

Hence,

$$(232) \quad \lim_{t \rightarrow 0} \left| \frac{\psi_i(t)}{t} \right| \leq \alpha, \quad i = 2, \dots, n.$$

□

The next lemma provides us with a suitable bound for $\frac{\partial f}{\partial x_1}$ restricted to $A_{\epsilon, \alpha}$ if we choose ϵ sufficiently small.

Lemma 5.16 *Let $f : B_1(0) \rightarrow R$ be a definable ordinary differentiable function with f and ∇f vanishing at 0, and*

$$(233) \quad \|\nabla f|_{B_1(0)}\| \leq M < \infty.$$

Then, for $\alpha > 0$ there is an $\epsilon > 0$ such that

$$(234) \quad \sup_{x \in A_{\epsilon, \alpha}} \left| \frac{\partial f}{\partial x_1}(x) \right| < 2M\alpha n.$$

Proof: We assume that the zero-point belongs to the closure of the set

$$(235) \quad B := \left\{ x \mid x \in A_{1, \alpha} \wedge \left| \frac{\partial f}{\partial x_1}(x) \right| \geq 2M\alpha n \right\}.$$

Obviously, 0 does not belong to B . Hence, it belongs to the frontier of B . B is definable so that we can apply the Curve-Selection-Lemma to B and 0. In combination with lemma 5.15 we obtain a continuously differentiable definable curve $\varphi : (-\delta, \delta) \rightarrow R^n$ such that

(i) $\varphi((0, \delta)) \subset B$,

(ii) $\varphi(0) = 0$, and

(iii) $\varphi(t) = (t, \varphi_2(t), \dots, \varphi_n(t))$ with $|\varphi'_i(0)| \leq \alpha$ for $i = 2, \dots, n$.

According to proposition 5.7, the composition $f \circ \varphi$ is continuously differentiable because it is a univariate function. The chain-rule implies $(f \circ \varphi)'(0) = 0$ since $\nabla f((0, \dots, 0)) = 0$. Thus, there is a $\delta_1 > 0$ such that

$$(236) \quad |(f \circ \varphi)'(t)| \leq \frac{4}{3}M\alpha, \quad t \in (-\delta_1, \delta_1).$$

Moreover, since the φ'_i are continuous there is a $\delta_2 > 0$ such that

$$(237) \quad |\varphi'_i(t) - \varphi'_i(0)| < \frac{\alpha}{3}, \quad t \in (0, \delta_2), \quad i = 2, \dots, n.$$

Let $\epsilon_1 := \min(\delta_1, \delta_2)$. We apply the chain-rule to $f \circ \varphi$ and get for $0 < t < \epsilon_1$

$$(238) \quad \begin{aligned} \frac{4}{3}M\alpha &\geq |(f \circ \varphi)'(t)| \\ &= \left| \frac{\partial f}{\partial x_1}(\varphi(t))\varphi'_1(t) + \sum_{i=2}^n \frac{\partial f}{\partial x_i}(\varphi(t))\varphi'_i(t) \right| \\ &\geq \left| \frac{\partial f}{\partial x_1}(\varphi(t)) \right| - \left| \sum_{i=2}^n \frac{\partial f}{\partial x_i}(\varphi(t))\varphi'_i(t) \right|. \end{aligned}$$

By assumption, all partial derivatives are bounded by M , and, by (237), the $|\varphi'_i(t)|$ are bounded by $\frac{4}{3}\alpha$ for $i = 2, \dots, n$. Hence,

$$(239) \quad \left| \frac{\partial f}{\partial x_1}(\varphi(t)) \right| \leq \frac{4}{3}M\alpha + \frac{4}{3}M\alpha(n-1) = \frac{4}{3}M\alpha n.$$

This contradicts the assumption that 0 belongs to the closure of B . Hence, there is an $\epsilon > 0$ such that $B_\epsilon(0) \cap B = \emptyset$, i.e.

$$(240) \quad \sup_{x \in A_{\epsilon, \alpha}} \left| \frac{\partial f}{\partial x_1}(x) \right| < 2M\alpha n.$$

□

Proposition 5.17 *Let $f : R^n \rightarrow R$ be a definable ordinary differentiable function. If ∇f is locally bounded, for all C^1 -curves $\psi : (-1, 1) \rightarrow R^n$ the composition $f \circ \psi$ is continuously differentiable.*

Proof: Without loss of generality we may assume that $x_0 = 0$, $f(0) = 0$, $\|\nabla f\|$ is bounded in $B_1(0)$ by the constant $M > 0$, and $\|\nabla f(0)\| = 0$. Let $\psi : (-1, 1) \rightarrow R^n$ be a C^1 -curve with $\psi(0) = 0$. If $\psi'(0) = 0$, the statement is evident. Therefore let $\psi'(0) \neq 0$.

step 1:

We use a suitable linear orthogonal coordinate system of the R^n in which the first coordinate has $\psi'(0)$ as its direction. Thus, in a small neighbourhood of 0 and after a suitable change of the coordinate system, we may assume that $\psi'_i(0) = 0$ for $i = 2, \dots, n$, and $\psi'_1(0) \neq 0$. The limit

$$(241) \quad \lim_{t \rightarrow 0} \frac{d}{dt}(f \circ \psi)(t) = 0 \text{ iff } \lim_{t \rightarrow 0} \frac{d}{dt}(f \circ \psi)(t/\psi'_1(0)) = 0.$$

Hence we may assume that $\psi'_1(0) = 1$.

step 2:

We show that

$$(242) \quad \lim_{t \searrow 0} \left| \frac{\partial f}{\partial x_1}(\psi(t))\psi'_1(t) \right| = 0.$$

Let $\epsilon > 0$. For $\alpha := \frac{\epsilon}{4Mn}$ we choose $\delta_1 > 0$ that small that $|\psi'_1(t)| \leq 2$ for $0 < t < \delta_1$. According to lemma 5.16, we may choose $\delta_2 > 0$ that small that

$$(243) \quad \left| \frac{\partial f}{\partial x_1}(x) \right| \leq 2M\alpha n, \text{ for } x \in A_{\delta_2, \alpha}.$$

Applying lemma 5.14, there is a $\delta_3 > 0$ such that $\psi((0, \delta_3)) \subset A_{\delta_2, \alpha}$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then, for $0 < t < \delta$ it holds true that

$$(244) \quad \left| \frac{\partial f}{\partial x_1}(\psi(t))\psi'_1(t) \right| \leq 4M\alpha n = \epsilon$$

Hence, $\lim_{t \searrow 0} \frac{\partial f}{\partial x_1}(\psi(t))\psi'_1(t) = 0$.

step 3:

For $t > 0$ it holds true that

$$\begin{aligned}
 (245) \quad |(f \circ \psi)'(t)| &= \left| \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\psi(t)) \psi'_i(t) \right| \\
 &\leq \sum_{i=1}^n \left| \frac{\partial f}{\partial x_i}(\psi(t)) \psi'_i(t) \right| \\
 &\leq \left| \frac{\partial f}{\partial x_1}(\psi(t)) \psi'_1(t) \right| + \sum_{i=2}^n \left| \frac{\partial f}{\partial x_i}(\psi(t)) \psi'_i(t) \right| \\
 &\leq \left| \frac{\partial f}{\partial x_1}(\psi(t)) \psi'_1(t) \right| + \sum_{i=2}^n |M \psi'_i(t)|.
 \end{aligned}$$

ψ' is continuous so that $\lim_{t \searrow 0} \psi'_i(t) = 0$ for $i = 2, \dots, n$. This in connection with step 2 implies

$$(246) \quad \lim_{t \searrow 0} (f \circ \psi)'(t) = 0.$$

By replacing t with $-t$ we obtain

$$(247) \quad \lim_{t \searrow 0} (f \circ \psi(-t))' = 0.$$

□

Now we want to show that if the gradient of f is not locally bounded at x_0 , we obtain a continuously differentiable curve $\varphi : (-1, 1) \rightarrow R^n$ with $\varphi(0) = x_0$ such that $f \circ \varphi$ is not continuously differentiable at 0.

In example (5.13) we used the sine function to construct a \mathcal{C}^1 -curve along which the function is not continuously differentiable.

In arbitrary real closed fields we cannot make use of the sine function. We modulate the sine oscillation by a piecewise semialgebraic function. This is done in the following lemmata.

The idea behind creating such a \mathcal{C}^1 -curve is that we disturb a definable \mathcal{C}^1 -curve by adding a quickly oscillating \mathcal{C}^1 -function with certain properties which will be pointed out. For this purpose we construct a discrete subset A_R of R with some further properties in the next lemma. Afterwards we define the disturbing function piecewise between two consecutive points of this set.

Lemma 5.18 *There is a discrete subset $A_R \subset (0, 1]$ including 1 such that*

- (i) $\text{cl}(A_R) = A_R \cup \{0\}$.
- (ii) *If $a \in A_R$ and $A_R \ni b < a$, then $b \leq a/2$.*
- (iii) *If $a \in A_R$, then $\max\{b \in A_R \mid b < a\}$ exists.*
- (iv) *If $a \in A_R$ and $a < 1/2$, then $\min\{b \in A_R \mid b > a\}$ exists.*
- (v) *Each $x \in (0, 1] \setminus A_R$ is contained in an interval $(a/2, a)$ for an $a \in A_R$.*

Proof: If R is Archimedean, let A_R be

$$(248) \quad A_R := \left\{ \frac{1}{2^l} \mid l \in \mathbb{N} \right\}.$$

For a non-Archimedean real closed field R let $B \subset R$ be the convex envelope of \mathbb{Z} in R . Then B is a non-trivial valuation ring. We denote by ν the natural evaluation map $\nu : B \rightarrow (\Gamma, \leq, +)$. For $0 < \gamma \in \Gamma$ let $0 < \rho_\gamma \in \nu^{-1}(\{\gamma\})$ and $\rho_0 = 1$.

We define A_R by

$$(249) \quad A_R := \{2^l \rho_\gamma, \gamma \geq 0, l \in \mathbb{Z}\} \cap (0, 1].$$

Obviously $1 \in A_R$.

We now check the properties.

- (iii) If $a \in A_R$, there is a $\gamma \in \Gamma$, $l \in \mathbb{Z}$ such that $a = \rho_\gamma 2^l$. Then

$$(250) \quad \max\{b \in A_R \mid b < a\} = \rho_\gamma 2^{l-1}.$$

- (iv) is analogous to (iii).

$$(251) \quad \min\{b \in A_R \mid b > a\} = \rho_\gamma 2^{l+1}$$

From (iii) and (iv) we conclude that A_R is discrete, and moreover, (ii) is a direct consequence of the proof of (iii).

Now we look at the properties (i) and (v):

Let $x \in (0, 1] \setminus A_R$. Then $x \in B$. So, there is a $\gamma \in \Gamma$ such that $\nu(x) = \gamma$, and therefore $\nu(x) = \nu(\rho_\gamma)$. Hence, there is a non-infinitesimal $r \in B$ with $x = r\rho_\gamma$, and we find an $l \in \mathbb{Z}$ with $2^l < r < 2^{l+1}$. Now (v) is evident.

We put $\epsilon := \frac{1}{2} \min(r - 2^l, 2^{l+1} - r)\rho_\gamma$ so that $B_\epsilon(x) \cap A_R = \emptyset$. Thus, 0 is the only likely cluster point of A_R .

Since for all $\epsilon > 0$ the inequality $\rho_{2\nu(\epsilon)} < \epsilon$ applies, we see that $0 \in \text{cl}(A)$. \square

We now construct functions defined on an interval of R which are inspired by some properties of the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$ and $f(t) = t^2 \sin(1/t)$ for $t \neq 0$. The idea is that the derivative of these functions in each neighbourhood of 0 have a point at which they are greater than a given positive lower bound. We construct these functions with the help of the semialgebraic function $h : [0, 1] \rightarrow R$,

$$(252) \quad h(s) = s^2(1-s)^2(s-1/2).$$

The derivative of this function is

$$(253) \quad h'(s) = 2s(1-s)^2(s-1/2) + 2s^2(1-s)(s-1/2) + s^2(1-s)^2.$$

Since $0 \leq s \leq 1$,

$$(254) \quad |h'(s)| \leq 4$$

applies. Furthermore,

$$(255) \quad h' \left(\frac{1}{2} \right) = \frac{1}{16}$$

and

$$(256) \quad h(0) = h(1) = h(1/2) = 0 = h'(0) = h'(1).$$

Since the suitable disturbing functions depend very much on the size of the gradient of the examined function near the considered point x_0 , we always have to make an ad hoc construction. In the next lemma we construct a function which reduces these efforts.

Lemma 5.19 *There is a map $H : [0, c] \rightarrow R$ with the following properties:*

(i) $H \in \mathcal{C}^1(0, c]$

(ii) $|H(t)| \leq t$

(iii) $|H'(t)| \leq 9$

(iv) $\forall \delta > 0 \exists 0 < t < \delta : H'(t) = 1/32$

Proof: Let A_R be a set with the properties of lemma 5.18 and $c := \max(A_R)$. For all $a \in A_R$ we set $b_a := \{b \in A_R \mid b < a\}$. Now we construct, with the help of the function h , cf. equation (252), $H : [0, c] \rightarrow R$. For $t \in [b_a, a]$ we set

$$(257) \quad H|_{[b_a, a]}(t) = th \left(\frac{t - b_a}{a - b_a} \right)$$

and $H(0) = 0$.

We see that $H(b_a) = H(a) = 0$ and that $|H(t)| \leq t$ because of the properties of h . Hence, (ii) is proved.

We can write the derivative of H restricted to $[b_a, a]$ as

$$(258) \quad H'(t) = h \left(\frac{t - b_a}{a - b_a} \right) + \frac{t}{a - b_a} h' \left(\frac{t - b_a}{a - b_a} \right).$$

Now we show the boundedness of the derivative of H .

$$(259) \quad |H'|_{[b_a, a]}(t) \leq 1 + \frac{a}{a - b_a} \cdot 4 \leq 1 + 2 \cdot 4 = 9$$

This illustrates property (iii).

Since $H'(a) = H'(b_a) = 0$, we conclude that H' is continuous in $(0, c]$. This proves property (i).

If $t_0 = (a + b_a)/2$,

$$(260) \quad H' \left(\frac{a + b_a}{2} \right) = h \left(\frac{1}{2} \right) + \frac{1}{2} h' \left(\frac{1}{2} \right) = 0 + \frac{1}{2} \frac{1}{16} + 0 = \frac{1}{32}.$$

Hence, property (iv) is evident. \square

We use the function H to prove the following proposition.

Proposition 5.20 *Let $f : R^n \rightarrow R$ be a definable ordinary differentiable function. The gradient ∇f shall not be locally bounded at x_0 . Then there is a \mathcal{C}^1 -curve $\varphi : (-\delta, \delta) \rightarrow R^n$ with $\varphi(0) = x_0$ such that the composition $f \circ \varphi$ is not continuously differentiable at $t = 0$.*

Proof: Without loss of generality we may assume that $x_0 = 0$ and that the partial derivative of f with respect to the first variable is not locally bounded from above at 0, otherwise consider $-f$. Then, by the Curve-Selection-Lemma, there is a regular definable \mathcal{C}^1 -curve $\psi : (-1, 1) \rightarrow R^n$ such that

$$(261) \quad \lim_{t \searrow 0} \frac{\partial f}{\partial x_1} \circ \psi(t) = +\infty.$$

Since ψ is definable, we have $f \circ \psi \in \mathcal{C}^1$.

We consider the function $l : (0, \delta_1) \rightarrow R$,

$$(262) \quad l(t) := \frac{\partial f}{\partial x_1} \circ \psi(t),$$

where we choose δ_1 that small that l is strictly monotone, continuously differentiable and positive. We can do this by the Monotonicity-Theorem. Note that $l(t)$ tends to ∞ at 0, and therefore l is strictly decreasing.

We define the disturbing function $\Delta : (-1, \min(c, \delta_1)) \rightarrow R$ by

$$(263) \quad \Delta(t) := \begin{cases} 0, & \text{for } t \leq 0 \\ \frac{H(t)}{\sqrt{l(t)}}, & \text{for } 0 < t < \min(c, \delta_1) \end{cases}$$

where H and c are taken from lemma 5.19.

Claim: $\Delta \in \mathcal{C}^1(-1, \min(c, \delta_1))$.

Since $H(t)/\sqrt{l(t)}$ is continuously differentiable for $t \neq 0$, we only need to check the point $t_0 = 0$.

First, we observe that $t/\sqrt{l(t)}$ is $o(t)$ and definable. According to proposition 5.7,

$$(264) \quad 0 = \lim_{t \searrow 0} \left(\frac{t}{\sqrt{l(t)}} \right)' = \lim_{t \searrow 0} \left(\frac{\sqrt{l(t)} - \frac{tl'(t)}{2\sqrt{l(t)}}}{l(t)} \right) = \lim_{t \searrow 0} \left(\frac{1}{\sqrt{l(t)}} - \frac{tl'(t)}{2l(t)^{3/2}} \right).$$

We can conclude that

$$(265) \quad \lim_{t \searrow 0} \left| \frac{tl'(t)}{l(t)^{3/2}} \right| = 0.$$

Now we estimate the derivative of Δ at 0.

$$(266) \quad \lim_{t \searrow 0} \Delta'(t) = \lim_{t \searrow 0} \left(\frac{H'(t)}{\sqrt{l(t)}} - \frac{H(t)l'(t)}{2l(t)^{3/2}} \right) = 0,$$

since $|H(t)| \leq t$ and H' are bounded. Hence, Δ is continuously differentiable.

With $\delta = \min(c, \delta_1)$ we now define our curve $\psi : (-\delta, \delta) \rightarrow R^n$ by

$$(267) \quad \psi(t) := \varphi(t) + (\Delta(t), 0, \dots, 0).$$

Thus, ψ is a \mathcal{C}^1 -curve.

We consider the derivative of $f \circ \psi$. Note that for all $\delta_2 > 0$ there is a

$0 < t_{\delta_2} < \delta_2$ such that $\Delta(t_{\delta_2}) = 0$ and $\Delta'(t_{\delta_2}) = \frac{1}{32\sqrt{l(t_{\delta_2})}}$. For such t_{δ_2} we obtain

(268)

$$\begin{aligned}
 (f \circ \psi)'(t_{\delta_2}) &= (Df \circ \psi(t_{\delta_2}))\psi'(t_{\delta_2}) \\
 &= (Df \circ (\varphi(t_{\delta_2}) + (\Delta(t_{\delta_2}), 0, \dots, 0))) (\varphi'(t_{\delta_2}) + (\Delta'(t_{\delta_2}), 0, \dots, 0)) \\
 &= (Df \circ \varphi(t_{\delta_2}))\varphi'(t_{\delta_2}) + (Df \circ \varphi(t_{\delta_2}))(\Delta'(t_{\delta_2}), 0, \dots, 0) \\
 &= (Df \circ \varphi(t_{\delta_2}))\varphi'(t_{\delta_2}) + l(t_{\delta_2})\Delta'(t_{\delta_2}) \\
 &= (Df \circ \varphi(t_{\delta_2}))\varphi'(t_{\delta_2}) + l(t_{\delta_2})\frac{1}{32\sqrt{l(t_{\delta_2})}} \\
 &= (f \circ \varphi)'(t_{\delta_2}) + \frac{\sqrt{l(t_{\delta_2})}}{32}
 \end{aligned}$$

$(f \circ \varphi)'(t_{\delta_2})$ tends towards 0 as δ_2 tends towards 0. Since l is strictly decreasing and unbounded near 0, \sqrt{l} behaves in the same way. Hence, the derivative of $f \circ \psi$ is not locally bounded at 0 and therefore cannot be continuous. \square

As a first summary we remark that in general we cannot decide whether a definable ordinary differentiable function is continuously differentiable if we only test with continuously differentiable functions.

Definable Ordinary Differentiable Functions along Ordinary Differentiable Curves

Now we want to check continuous differentiability of a function at a certain point using curves. The same idea which we used to detect not locally bounded points of the derivative can be applied to detect non-continuous points of it. But of course we need more curves besides the continuously differentiable curves. We use the class of ordinary differentiable curves with bounded derivatives.

Lemma 5.21 *There is a map $H : [0, c] \rightarrow \mathbb{R}$ with the following properties:*

- (i) $H \in \mathcal{C}^1(0, c]$
- (ii) $|H(t)| \leq t^2$
- (iii) $|H'(t)| \leq 6$
- (iv) $\forall \delta > 0 \exists 0 < t < \delta : 1/64 \leq H'(t) \leq 1/16$ and $H(t) = 0$

Proof: Let A_R be a set of lemma 5.18 and let $c := \max(A_R)$. Without loss of generality we may assume that $c \leq \frac{1}{2}$. Moreover, let h be the function of equation (252).

For all $a \in A_R$ we set $b_a := \{b \in A \mid b < a\}$.

Now we construct $H : [0, c] \rightarrow R$. For $t \in [b_a, a]$ we set, with h of equation (252)

$$(269) \quad H|_{[b_a, a]}(t) := \begin{cases} 0, & b_a \leq t \leq a - a^2 \\ t^2 h\left(\frac{t - (a - a^2)}{a^2}\right), & a - a^2 < t \leq a. \end{cases}$$

The derivative of H in $[b_a, a]$ is

$$(270) \quad H'|_{[b_a, a]}(t) := \begin{cases} 0, & b_a \leq t \leq a - a^2 \\ 2th\left(\frac{t - (a - a^2)}{a^2}\right) + \frac{t^2}{a^2} h'\left(\frac{t - (a - a^2)}{a^2}\right), & a - a^2 < t \leq a. \end{cases}$$

Hence, by equation (256), we get $H(b_a) = 0$, $H(a) = a^2 h(1) = 0$, $H'(b_a) = 0$ and $H'(a) = 2ah(1) + h'(1) = 0$. Moreover, $H(a - a^2) = (a - a^2)^2 h(0) = 0$ and $H'(a - a^2) = 2(a - a^2)h(0) + (1 - a)^2 h'(0) = 0$. Hence, H is continuously differentiable in $(0, 1]$, and (i) is proved.

Since h restricted to $[0, 1]$ is bounded by 1, (ii) is a direct consequence of the definition of H .

Since h' is bounded by 4, we conclude that for $H'(t)$ restricted to $[b_a, a]$ it holds true that

$$(271) \quad \begin{aligned} |H'(t)| &\leq \sup_{t \in [b_a, a]} \left| 2th\left(\frac{t - (a - a^2)}{a^2}\right) + \frac{t^2}{a^2} h'\left(\frac{t - (a - a^2)}{a^2}\right) \right| \\ &\leq 2 \sup_{t \in [0, 1]} |h(t)| + \sup_{t \in [0, 1]} |h'(t)| \\ &\leq 6. \end{aligned}$$

Hence, (iii) holds true.

In order to show (iv), we evaluate H and H' at the points $a - a^2/2$, $a \in A_R$. We note that $0 < a \leq 1$. Thus,

$$(272) \quad \begin{aligned} H'\left(a - \frac{a^2}{2}\right) &= (2a - a^2)h\left(\frac{1}{2}\right) + \left(1 - \frac{a}{2}\right)^2 h'\left(\frac{1}{2}\right) \\ &= \left(1 - \frac{a}{2}\right)^2 \frac{1}{16}. \end{aligned}$$

Since

$$(273) \quad \frac{1}{4} \leq \left(1 - \frac{a}{2}\right)^2 \leq 1,$$

(iv) holds true. Moreover,

$$(274) \quad H\left(a - \frac{a^2}{2}\right) = \left(a - \frac{a^2}{2}\right)^2 h\left(\frac{1}{2}\right) = 0.$$

Hence, (iv) holds true. □

Proposition 5.22 *Let $f : R^n \rightarrow R$ be a definable ordinary differentiable function. Then f is continuously differentiable at x_0 if and only if for all ordinary differentiable curves $\varphi : (-1, 1) \rightarrow R^n$ with bounded derivative and $\varphi(0) = x_0$,*

$$(275) \quad t \mapsto f \circ \varphi(t) - \langle \nabla f(x_0), \varphi(t) \rangle$$

is continuously differentiable at $t = 0$.

Proof: We may reduce our considerations to definable ordinary differentiable functions with locally bounded derivative since otherwise, proposition 5.20 verifies non-continuity of the derivative at x_0 .

Without loss of generality we may assume that $x_0 = 0$ and $f(0) = 0$. Since we consider

$$(276) \quad x \mapsto f(x) - \langle \nabla f(0), x \rangle$$

we may further assume that $\nabla f(0) = 0$.

Let the derivative of f be continuous at 0.

If $\varphi : (-1, 1) \rightarrow R^n$ is an ordinary differentiable curve with $\varphi(0) = 0$ such that φ' is bounded by M , we obtain by the chain-rule

$$(277) \quad \left| \frac{d}{dt}(f \circ \varphi)(t) \right| = \left| \sum_{i=1}^n \frac{\partial}{\partial x_i}(\varphi(t)) \varphi'_i(t) \right| \leq M \|\nabla f(\varphi(t))\|.$$

Since $\lim_{x \rightarrow 0} \nabla f(x) = 0$, we get continuity of $\frac{d}{dt}(f \circ \varphi)(t)$ at $t = 0$.

Now let the derivative of f be not continuous at 0.

Without loss of generality we may assume that $\frac{\partial f}{\partial x_1}$ is not continuous at 0. Then there is a $C > 0$ and a definable regular \mathcal{C}^1 -curve $\psi : (-1, 1) \rightarrow R^n$ with $\psi(0) = 0$ such that

$$(278) \quad \lim_{t \searrow 0} \left| \frac{\partial f}{\partial x_1}(\psi(t)) \right| \geq C.$$

Let $\varphi : (-c, c) \rightarrow R^n$ be defined by

$$(279) \quad \varphi(t) = \psi(t) + H(t)e_1$$

where H is the function of lemma 5.21.

By property (iv) of lemma 5.21, for each $\delta > 0$ there exists a $0 < t_\delta < \delta$ with $H(t_\delta) = 0$ and $H'(t_\delta) \geq 1/64$. Thus, we evaluate $(f \circ \varphi)'$ at t_δ .

(280)

$$\begin{aligned} |(f \circ \varphi)'(t_\delta)| &= |(Df \circ \varphi(t_\delta))\varphi'(t_\delta)| \\ &= |(Df \circ (\psi(t_\delta) + (H(t_\delta), 0, \dots, 0)))(\psi'(t_\delta) + (H'(t_\delta), 0, \dots, 0))| \\ &= |(Df \circ \psi(t_\delta))\psi'(t_\delta) + (Df \circ \psi(t_\delta))(H'(t_\delta), 0, \dots, 0)| \\ (281) \quad &\geq \left| \frac{1}{64} \left(\frac{\partial f}{\partial x_1}(\psi(t_\delta)) \right) \right| - |(Df \circ \psi(t_\delta))\psi'(t_\delta)| \\ &\geq \frac{C}{64} - |(Df \circ \psi(t_\delta))\psi'(t_\delta)| \end{aligned}$$

Since $f \circ \psi$ is continuously differentiable with vanishing derivative at 0,

$$(282) \quad \limsup_{t \searrow 0} (f \circ \varphi)'(t) \geq \frac{C}{64} \neq 0 = (f \circ \varphi)'(0)$$

□

6 Definable Peano-Differentiable Functions

In this section we introduce the concept of Peano-differentiable functions and Peano-derivatives. We show some basic algebraic properties of Peano-differentiable functions as well as the behaviour of the Peano-derivatives under basic algebraic operations. Moreover, we discuss a family of definable Peano-differentiable functions which will give us an impression of this class of functions and will motivate questions about them.

Peano-Differentiability

Definition 6.1 Let $U \subset \mathbb{R}^n$ and let $f : U \rightarrow \mathbb{R}$ be a function.

We call f m -times Peano-differentiable at $x_0 \in U$ if there is a polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ with $\deg(p) \leq m$ and $p(0) = 0$ such that

$$(283) \quad f(x) = f(x_0) + p(x - x_0) + \varphi(x - x_0)$$

$$(284) \quad \text{and} \quad \lim_{x \rightarrow x_0} \frac{\varphi(x - x_0)}{\|x - x_0\|^m} = 0.$$

p is called an approximation polynomial of f at x_0 .

We call f m -times Peano-differentiable in U if f is m -times Peano-differentiable at all $x_0 \in U$. We call a function $f : U \rightarrow \mathbb{R}^p$ m -times Peano-differentiable in U if each coordinate function is m -times Peano-differentiable.

By $\mathcal{P}^m(U, W)$ we denote the set of functions $f : U \rightarrow W$ which are m -times Peano-differentiable in U .

Before giving some examples, we note some basic facts.

Lemma 6.2 Let $U \subset \mathbb{R}^n$ be an open set. If $f : U \rightarrow \mathbb{R}$ is a function which is m -times Peano-differentiable at $x_0 \in U$ with approximation polynomial p , then p is uniquely determined by f and x_0 .

Proof: Let p and q be two approximation polynomials of f at x_0 . Then $p(x - x_0) + \Phi(x - x_0) = f(x) - f(x_0) = q(x - x_0) + \Psi(x - x_0)$ where Φ and Ψ are $o(\|x - x_0\|^m)$.

Thus, the polynomial $Q(x - x_0) := p(x - x_0) - q(x - x_0) = \Phi(x - x_0) - \Psi(x - x_0)$ is $o(\|x - x_0\|^m)$. We conclude that for the polynomial Q , the coefficients must be zero up to order m . $\deg(Q) \leq m$ since p and q are of degree less than or equal to m so that Q must be the zero-polynomial, i.e. $p = q$. \square

If $U \subset R^n$ is open and f is m -times Peano-differentiable at x_0 , we denote the unique approximation polynomial of f at x_0 by p_{f,x_0} . We may write this as

$$(285) \quad p_{f,x_0}(X) = \sum_{1 \leq |\alpha| \leq m} \frac{f_{[\alpha]}(x_0)}{\alpha!} X^\alpha$$

where we call the $f_{[\alpha]}(x_0)$ the α^{th} Peano-derivative of f at x_0 . We often use the denomination $D_\alpha f(x_0) := f_{[\alpha]}(x_0)$.

Examples

We fix an o-minimal expansion \mathcal{M} of R .

By Taylor's Theorem it is evident that definable m -times continuously differentiable functions are m -times Peano-differentiable. But as we know from definable ordinary differentiable functions which are the 1-times Peano-differentiable functions, these functions have points at which the derivative is not continuous.

Next we discuss a family of semialgebraic m -times Peano-differentiable functions. These examples point out several interesting types of \mathcal{C}^m -singularities of Peano-derivatives which can appear.

In chapter 8 we will use these functions in order to construct functions which have certain \mathcal{C}^m -singularities on given sets.

Example 6.3

Let $1 \leq k \leq m$ be an integer. Let $F = F_{k,a,b} : R \times R^{n-1} \rightarrow R$ be the semialgebraic function defined by

$$(286) \quad F(x, y) := \begin{cases} \|y\|^a \varphi_k\left(\frac{x}{\|y\|^b} - 2\right), & y \neq 0 \\ 0, & y = 0 \end{cases}$$

where a, b are positive rational numbers and $\varphi_k : R \rightarrow R$ is defined by

$$(287) \quad \varphi_k(t) = \frac{t^k}{k!} (1 - t^2)^{m+1} \chi_{[-1,1]}(t).$$

If $a > m + 2$ and $\frac{a}{k} \leq b < \frac{a}{k-1}$, the following holds true:

- (i) F is m -times Peano-differentiable in R^n .
- (ii) F is $(k - 1)$ -times continuously differentiable in R^n .

- (iii) F is m -times continuously differentiable in $R^n \setminus \{0\}$.
- (iv) F is not k -times continuously differentiable at 0.
- (v) If $b = \frac{a}{k}$, the $(k-1)^{\text{th}}$ derivatives of F are ordinary differentiable with locally bounded derivatives.
- (vi) If $k \geq 2$ and $\frac{a}{k} < b < \frac{a-1}{k-1}$, the $(k-1)^{\text{th}}$ derivatives of F are ordinary differentiable but not all k^{th} derivatives are locally bounded. For $k = 1$ this holds for $a < b$.
- (vii) If $k \geq 2$ and if $\frac{a-1}{k-1} \leq b < \frac{a}{k-1}$, the $(k-1)^{\text{th}}$ derivatives of F are continuous but not all $(k-1)^{\text{th}}$ derivatives are ordinary differentiable.

We will use the next two technical lemmata in order to show the properties (ii) and (iv)-(vii) of the examples above.

Lemma 6.4 *Let P be a homogeneous polynomial of degree l in n variables. Then, for each multi-index $\alpha \in \mathbb{N}^n$ and $0 < k \in \mathbb{N}$*

$$(288) \quad D_\alpha \frac{P(x)}{\|x\|^k} = \frac{Q_\alpha(x)}{\|x\|^{k+2|\alpha|}}, \quad x \neq 0,$$

where Q_α is a homogeneous polynomial of degree $l+|\alpha|$ or the zero-polynomial.

Proof: Let S be a homogeneous polynomial and m a positive integer. Then,

$$(289) \quad \begin{aligned} \frac{\partial}{\partial x_i} \frac{S(x)}{\|x\|^m} &= \frac{\partial S}{\partial x_i}(x) (x_1^2 + \dots + x_n^2) \frac{1}{\|x\|^{m+2}} + S(x) \frac{-m}{\|x\|^{m+1}} \frac{x_i}{\|x\|} \\ &= \frac{Q(x)}{\|x\|^{m+2}} \end{aligned}$$

where Q is the homogeneous polynomial

$$(290) \quad Q(x) = \left(\frac{\partial}{\partial x_i} S(x) \right) (x_1^2 + \dots + x_n^2) - mS(x)x_i.$$

Obviously, $\deg(Q) = \deg(S) + 1$ or it is the zero-polynomial. The statement of the lemma is now evident. \square

Lemma 6.5 *Let $f : R \setminus \{0\} \rightarrow R$ be an m -times continuously differentiable function. Let $g : R^n \setminus \{0\} \rightarrow R$ be defined by $g(x) := f(\|x\|)$. Then, for each multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$ there are homogeneous polynomials $Q_{\alpha,0}, \dots, Q_{\alpha,|\alpha|}$ of degree $|\alpha|$ such that*

$$(291) \quad D_\alpha g(\|x\|) = \sum_{l \leq |\alpha|} \frac{d^l f}{dt^l}(\|x\|) \frac{Q_{\alpha,l}(x)}{\|x\|^{2|\alpha|-l}}.$$

Proof: We proceed by induction on $|\alpha|$.

If $|\alpha| = 0$, the statement is evident.

Let $|\alpha| > 0$ and

$$(292) \quad D_\alpha f(\|x\|) = \sum_{l \leq |\alpha|} f^{(l)}(\|x\|) \frac{Q_{\alpha,l}(x)}{\|x\|^{2|\alpha|-l}}$$

with homogeneous polynomials $Q_{\alpha,l}$ of degree $|\alpha|$.

Then,

$$(293) \quad \begin{aligned} \frac{\partial}{\partial x_j} D_\alpha f(\|x\|) &= \sum_{l \leq |\alpha|} \frac{\partial}{\partial x_j} \left(f^{(l)}(\|x\|) \frac{Q_{\alpha,l}(x)}{\|x\|^{2|\alpha|-l}} \right) \\ &= \sum_{l \leq |\alpha|} f^{(l+1)}(\|x\|) \frac{x_j}{\|x\|} \frac{Q_{\alpha,l}(x)}{\|x\|^{2|\alpha|-l}} + f^{(l)}(\|x\|) \frac{S_{\alpha,l}(x)}{\|x\|^{2|\alpha|-l+2}} \end{aligned}$$

where the $S_{\alpha,l}$ are homogeneous polynomials of degree $\deg(Q_{\alpha,l}) + 1$ or the zero-polynomial, cf. lemma 6.4.

If we put

$$(294) \quad \begin{aligned} Q_{\alpha+e_j,0}(x) &= S_{\alpha,0}(x), \\ Q_{\alpha+e_j,l}(x) &= x_j Q_{\alpha,l}(x) + S_{\alpha,l}(x), \quad 1 \leq l \leq |\alpha|, \\ Q_{\alpha+e_j,|\alpha|+1}(x) &= x_j Q_{\alpha,|\alpha|}, \end{aligned}$$

each $Q_{\alpha+e_j,l}$ is a homogeneous polynomial of degree $|\alpha| + 1$ and we get

$$(295) \quad D_{\alpha+e_j} f(\|x\|) = \sum_{l \leq |\alpha|+1} f^{(l)}(\|x\|) \frac{Q_{\alpha+e_j,l}(x)}{\|x\|^{2|\alpha+e_j|-l}}$$

□

We now show the properties of the examples of 6.3.

Proof: step 1:

Since the zero-order of the polynomial $\frac{t^k}{k!}(1-t^2)^{m+1}$ at $t = 1$ and $t = -1$ is $m+1$, it is evident that φ_k is an m -times continuously differentiable function. Moreover, the functions

$$(296) \quad (x, y) \mapsto \frac{x}{\|y\|^b}, \quad \text{and } (x, y) \mapsto \|y\|^a, \quad y \neq 0,$$

are \mathcal{C}^∞ -functions. Hence, F is an m -times continuously differentiable function on the open set $R^n \setminus (R \times \{0\}^{n-1})$.

The support of F is

$$(297) \quad \text{supp}(F) = \{(x, y) \in R \times R^{n-1} \mid \|y\|^b \leq x \leq 3\|y\|^b\}.$$

Evidently, F is m -times continuously differentiable outside its support. Hence, we can conclude that F is m -times continuously differentiable in $(R^n \setminus (R \times \{0\}^{n-1})) \cup (R^n \setminus \text{supp}(F)) = R^n \setminus \{0\}$ and we have shown (iii).

φ_k is bounded by 1 and $a > m$. This implies that

$$(298) \quad \lim_{(x,y) \rightarrow 0} \left| \frac{F(x, y)}{\|(x, y)\|^m} \right| \leq \lim_{(x,y) \rightarrow 0} \left| \frac{\|y\|^a}{\|(x, y)\|^m} \right| = 0,$$

i.e. F is m -times Peano-differentiable at 0 with the zero-polynomial as approximation polynomial. This in connection with (iii) implies (i).

(iv) is a consequence of (v), (vi) and (vii).

Since F is m -times continuously differentiable outside 0, we have to discuss the Peano-derivatives in a neighbourhood of 0.

step 2:

We have a closer look at the two-dimensional case.

Let the function $f_{k,a,b} : R^2 \rightarrow R$ be defined by

$$(299) \quad f_{k,a,b}(x, y) := \begin{cases} F(x, y) = y^a \varphi_k(xy^{-b} - 2), & y > 0, \\ 0, & y \leq 0 \end{cases}$$

With the analogue argument as for F , $f_{k,a,b}$ is m -times Peano differentiable on R^2 and \mathcal{C}^m on $R^2 \setminus \{(0, 0)\}$. We restrict our examination to the support of $F \cap \{y > 0\} = \text{supp} f_{k,a,b} \setminus \{(0, 0)\}$. Note that $F(x, y) = f_{k,a,b}(x, |y|)$.

For $r + s \leq m$ we can write the $(s, r)^{\text{th}}$ Peano-derivative of $f_{k,a,b}$ in the following form:

$$(300) \quad D_{(s,r)} f_{k,a,b}(x, y) = \sum_{i=0}^r c_{r,s,i} y^{a-bs-r-bi} (-bx)^i \varphi_k^{(i+s)}(xy^{-b} - 2)$$

where $c_{r,s,i} \in R$.

We prove equation (300) by induction on $q := r + s$:

The case $q = 0$ where $r = s = 0$ and $c_{0,0,0} = 1$ is evident.

Now we assume that equation (300) holds true for $s + r \leq q - 1$.

Let the constants $c_{r,s,i}$ for $i = 0, \dots, r$ be given.

We define

$$(301) \quad \begin{aligned} c_{r+1,s,0} &:= c_{r,s,0}(a - bs - r), \\ c_{r+1,s,r+1} &:= c_{r,s,r}, \\ c_{r+1,s,i} &:= c_{r,s,i}(a - bs - r - bi) + c_{r,s,i-1}, \quad 1 \leq i \leq r, \\ c_{r,s+1,i} &:= c_{r,s,i} - b(i+1)c_{r,s,i+1}, \quad 0 \leq i \leq r-1, \\ c_{r,s+1,r} &:= c_{r,s,r}. \end{aligned}$$

Then,

$$(302) \quad \begin{aligned} &\frac{\partial}{\partial y} D_{(s,r)} f_{k,a,b}(x, y) = \\ &= \sum_{i=0}^r (c_{r,s,i}(a - bs - r - bi)y^{a-bs-r-bi-1}(-bx)^i \varphi_k^{(i+s)}(xy^{-b} - 2) + \\ &\quad + c_{r,s,i}y^{a-bs-r-bi}(-bx)^i \varphi_k^{(i+s+1)}(xy^{-b} - 2) \frac{-bx}{y^{b+1}}) \\ &= \sum_{i=0}^{r+1} c_{r+1,s,i}y^{a-bs-(r+1)-bi}(-bx)^i \varphi_k^{(i+s)}(xy^{-b} - 2) \end{aligned}$$

and

$$(303) \quad \begin{aligned} &\frac{\partial}{\partial x} D_{(s,r)} f_{k,a,b}(x, y) = \\ &= \sum_{i=0}^r c_{r,s,i}y^{a-bs-r-bi}(-bx)^i \varphi_k^{(i+s+1)}(xy^{-b} - 2) \frac{1}{y^b} + \\ &\quad + \sum_{i=1}^r c_{r,s,i}(-bi)y^{a-bs-r-bi}(-bx)^{i-1} \varphi_k^{(i+s)}(xy^{-b} - 2) \\ &= \sum_{i=0}^r c_{r,s+1,i}y^{a-b(s+1)-r-bi}(-bx)^i \varphi_k^{(i+s+1)}(xy^{-b} - 2). \end{aligned}$$

step 3: We now present an estimate for the zero-order of the derivatives of F in a pointed neighbourhood of 0.

Note that now $n \geq 2$. We restrict ourselves to $\text{supp}(F) \setminus \{0\}$ since elsewhere, F vanishes together with its Peano-derivatives. We show that

$$(304) \quad D_{(s,r)} F(x, y) \text{ is } O(\|y\|^{a-bs-|r|})$$

where $(s, r) \in \mathbb{N} \times \mathbb{N}^{n-1}$ is a multi-index with $|(s, r)| \leq m$.

For $(x, z) \in \text{supp}(f_{k,a,b}) \setminus \{(0, 0)\}$ applies $1 \leq x/z^b \leq 3$. Thus, by equation (300), for $s + l \leq m$

$$(305) \quad (D_{(s,l)}f_{k,a,b})(x, z) \text{ is } O(|z|^{a-bs-l}).$$

We generalise this estimation to higher dimension.

Therefore we apply lemma 6.5 to $y \mapsto F(x, y) = f_{k,a,b}(x, \|y\|)$.

Let $(x, y) \in \text{supp}(F) \setminus \{0\} \subset R \times R^{n-1}$.

Then, for $|(s, r)| \leq m$

$$(306) \quad \begin{aligned} D_{(s,r)}F(x, y) &= D_{(s,r)}f_{k,a,b}(x, \|y\|) \\ &= \sum_{l \leq |r|} (D_{(s,l)}f_{k,a,b})(x, \|y\|) \frac{Q_{r,l}(y)}{\|y\|^{2|r|-l}}. \end{aligned}$$

By (305), $(D_{(s,l)}f_{k,a,b})(x, \|y\|)$ is $O(\|y\|^{a-bs-l})$.

Moreover,

$$(307) \quad \frac{Q_{r,l}(y)}{\|y\|^{2|r|-l}} \text{ is } O(\|y\|^{l-|r|})$$

since a homogeneous polynomial $Q(Y)$ is obviously $O(\|Y\|^{\deg Q})$. We conclude that $D_{(s,r)}F(x, y)$ is $O(\|y\|^{a-bs-|r|})$.

step 4: We show that the Peano-derivatives up to order $k - 2$ of F are continuously differentiable. Note that this is only interesting for $k \geq 2$.

For all choices of b and a we have $1 < b < \frac{a}{k-1}$. So, for $|r| + s \leq k - 2$

$$(308) \quad a - bs - |r| \geq a - (k - 2)b > \frac{a}{k - 1} > 1$$

applies. Hence, according to (304), the Peano-derivatives up to order $k - 2$ are $o(|y|^1)$, i.e. they are differentiable at 0.

If $|r| + s = k - 1$,

$$(309) \quad a - bs - |r| \geq a - (k - 1)b > 0.$$

Hence, the $(k - 1)^{th}$ Peano-derivatives of F are continuous at 0. This implies (ii).

step 5: We show that $\varphi_k^{(k)}(0) = 1$, $k \geq 1$:

We use a derivation formula for the product of two \mathcal{C}^k -functions p and q ,

namely

$$(310) \quad (pq)^{(k)} = \sum_{l=0}^k \binom{k}{l} p^{(l)} q^{(k-l)}.$$

In a small neighbourhood of 0 we can write $\varphi_k(t) = \left(\frac{t^k}{k!}\right) \cdot ((1-t^2)^{m+1})$. Hence,

$$(311) \quad \begin{aligned} \varphi_k^{(k)}(0) &= \sum_{l=0}^k \binom{k}{l} \left(\frac{t^k}{k!}\right)^{(l)} \Big|_{t=0} \cdot ((1-t^2)^{m+1})^{(k-l)} \Big|_{t=0} \\ &= \sum_{l=0}^k \binom{k}{l} \left(\frac{t^{k-l}}{(k-l)!}\right) \Big|_{t=0} \cdot ((1-t^2)^{m+1})^{(k-l)} \Big|_{t=0} \\ &= \sum_{l=0}^k \delta_{k,l} ((1-t^2)^{m+1})^{(k-l)} \Big|_{t=0} \\ &= ((1-t^2)^{m+1}) \Big|_{t=0} \\ &= 1. \end{aligned}$$

step 6: We show (v), (vi) and (vii).

(v) $b = \frac{a}{k} > 1$:

Then,

$$(312) \quad \begin{aligned} a - b(k-1) &= a - \frac{a(k-1)}{k} = \frac{a}{k} = b > 1 \text{ and} \\ a - bk &= 0. \end{aligned}$$

Hence, by (304), the $(k-1)^{th}$ derivatives of F are $O(\|y\|^b)$, and therefore they are ordinary differentiable at 0. The k^{th} derivatives are $O(1)$, i.e. they are bounded.

It remains to show that at least one of the k^{th} derivatives is not continuous at 0. We consider $D_{(k,0)}F(x, z, 0, \dots, 0)$ for $z > 0$:

$$(313) \quad \begin{aligned} D_{(k,0)}F(x, z, 0, \dots, 0) &= D_{(k,0)}f_{k,a,b}(x, z) \\ &= c_{0,k,0} z^{a-bk} \varphi_k^{(k)}(xz^{-b} - 2) = \varphi_k^{(k)}(xz^{-b} - 2) \end{aligned}$$

For $x = 2z^b$ we get $\varphi_k^{(k)}(xz^{-b} - 2) = 1$. Hence,

$$(314) \quad \limsup_{(x,y) \rightarrow 0} D_{(k,0)}F(x, y) \geq \limsup_{(x,z) \rightarrow 0, z > 0, x = 2z^b} \varphi_k^{(k)}(xz^{-b} - 2) = 1 \neq 0.$$

We conclude that $D_{(k,0)}F$ is not continuous at $(0, 0)$.

$$(vi) \frac{a}{k} < b < \frac{a-1}{k-1}:$$

Then, $a - (k-1)b > 1$ and $a - kb < 0$. Thus, analogous to (v), the $(k-1)^{th}$ derivatives are ordinary differentiable.

Moreover, for $x = 2z^b$, $z > 0$

$$(315) \quad D_{(k,0)}F(x, z, 0, \dots, 0) = z^{a-kb}$$

which is locally unbounded at $(0, 0)$. Hence, we have proved (vi).

$$(vii) \frac{a-1}{k-1} \leq b < \frac{a}{k-1}:$$

Here we have $0 < a - (k-1)b \leq 1$. The derivative of $D_{(k-1,0)}F$ at 0 should vanish if it exists. But

$$(316) \quad \begin{aligned} & \limsup_{(x,z,0,\dots,0) \rightarrow 0, z > 0, x=2z^b} \frac{D_{(k-1,0)}F(x, z, 0, \dots, 0)}{\|(x, z, 0, \dots, 0)\|} \\ &= \limsup_{(x,z) \rightarrow 0, z > 0, x=2z^b} \frac{D_{(k-1,0)}f_{k,a,b}(x, z)}{\|(x, z)\|} \\ &\geq \limsup_{(x,z) \rightarrow 0, z > 0, x=2z^b} \frac{z^{a-(k-1)b}}{(4z^{2b} + z^2)^{\frac{1}{2}}} \\ &\geq \limsup_{(x,z) \rightarrow 0, z > 0, x=2z^b} \frac{z^1}{(9z^2)^{\frac{1}{2}}} \\ &\geq \frac{1}{3} \end{aligned}$$

which inhibits ordinary differentiability of $D_{(k-1,0)}F$ at 0. □

Basic Properties of Peano-Differentiable Functions

Definability of Peano-Derivatives

Let $U \subset R^n$ be a definable open subset, and let $f : U \rightarrow R$ be a definable function.

Let $\Phi_f(x, y)$ be a formula defining the graph of f , i.e.

$(x, y) \in \Gamma(f) \Leftrightarrow \mathcal{M} \models \Phi_f(x, y)$. We consider the formula $\Psi_f(x, (d_\alpha)_{1 \leq |\alpha| \leq m})$:

$$(317) \quad \forall \epsilon \left[\epsilon > 0 \rightarrow \exists \delta \left[\delta > 0 \rightarrow \forall_{i=1}^n z_i \left[\sum_{i=1}^n (z_i - x_i)^2 < \delta^2 \rightarrow \right. \right. \right. \\ \left. \left. \left. \forall y \forall y' \left[\Phi_f(x, y) \wedge \Phi_f(z, y') \rightarrow \right. \right. \right. \right. \\ \left. \left. \left. \left(y' - \left(y + \sum_{1 \leq |\alpha| \leq m} \frac{d_\alpha}{\alpha!} (z - x)^\alpha \right) \right)^2 \leq \epsilon^2 \left(\sum_{i=1}^n (z_i - x_i)^2 \right)^m \right] \right] \right] \right]$$

So we see that the set X of points at which f is m -times Peano-differentiable is definable since

$$(318) \quad x \in X \quad \Leftrightarrow \quad \mathcal{M} \models \exists_{1 \leq |\alpha| \leq m} d_\alpha \Psi_f(x, (d_\alpha)_{1 \leq |\alpha| \leq m}).$$

If f is m -times Peano-differentiable in U , we denote by $F : U \rightarrow (R_\alpha)_{1 \leq |\alpha| \leq m}$ the function which maps x to $(f_{[\alpha]}(x))_{1 \leq |\alpha| \leq m}$. Then

$$(319) \quad (x, y) \in \Gamma(F) \quad \Leftrightarrow \quad \mathcal{M} \models \Psi_f(x, y).$$

Hence, the Peano-derivatives are definable.

Basic Algebraic Properties of Peano-Differentiable Functions

In this section, definability of the functions is not required.

We will show that the Peano-differentiable functions form a composition-closed ring.

First we investigate compositions of Peano-differentiable functions and generalise the chain-rule for ordinary differentiable functions in the following way:

Proposition 6.6 Chain-Rule

Let $U \subset R^n$, $V \subset R^{\tilde{n}}$ be sets, and let $g : U \rightarrow R^{\tilde{n}}$, $f : V \rightarrow R^k$ be functions with $g(U) \subset V$. If g is m -times Peano-differentiable at x and f at $y = g(x)$, the composition $f \circ g : U \rightarrow R^k$ is m -times Peano-differentiable at x .

Then, an approximation polynomial $p_{f \circ g}$ of $f \circ g$ at the point x is given by

$$(320) \quad p_{f \circ g} = (p_f \circ p_g)_{(m)}.$$

Here, for a polynomial $r(X) = \sum_\alpha r_\alpha X^\alpha$ we put $r_{(m)}(X) = \sum_{|\alpha| \leq m} r_\alpha X^\alpha$.

Proof: With the conventions

$$(321) \quad \begin{aligned} f &= \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} & p_f =: p &= \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix} & \varphi_f =: \psi &= \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_k \end{pmatrix} \\ g &= \begin{pmatrix} g_1 \\ \vdots \\ g_{\bar{n}} \end{pmatrix} & p_g =: q &= \begin{pmatrix} q_1 \\ \vdots \\ q_{\bar{n}} \end{pmatrix} & \varphi_g =: \varphi &= \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{\bar{n}} \end{pmatrix} \end{aligned}$$

we can write f and g in the form

$$(322) \quad \begin{aligned} f(y + \eta) &= f(y) + p(\eta) + \psi(\eta) \\ g(x + \xi) &= g(x) + q(\xi) + \varphi(\xi) \end{aligned}$$

where the functions φ and ψ possess the following property:

$$(323) \quad \lim_{\xi \rightarrow 0} \frac{\varphi(\xi)}{\|\xi\|^m} = 0, \quad \lim_{\eta \rightarrow 0} \frac{\psi(\eta)}{\|\eta\|^m} = 0$$

We put $\eta := g(x + \xi) - g(x) = q(\xi) + \varphi(\xi)$ and get

$$(324) \quad \begin{aligned} f \circ g(x + \xi) &= f(g(x) + \eta) \\ &= f(g(x)) + p(q(\xi) + \varphi(\xi)) + \psi(q(\xi) + \varphi(\xi)) \\ &= f \circ g(x) + \underbrace{(p \circ q)_{(m)}(\xi) + p(q(\xi) + \varphi(\xi)) - (p \circ q)_{(m)}(\xi) + \psi(q(\xi) + \varphi(\xi))}_{=: \phi(\xi)} \\ &= f \circ g(x) + p_f(p_g)_{(m)}(\xi) + \phi(\xi). \end{aligned}$$

So it remains to show that $\lim_{\xi \rightarrow 0} \frac{\phi(\xi)}{\|\xi\|^m} = 0$.

By the assumptions we made about f , g , p , q , φ and ψ , we conclude that there is a $K > 0$ such that for sufficiently small ξ and η it holds true that

$$(325) \quad \begin{aligned} \|p(\eta)\| &\leq K \|\eta\| \\ \|q(\xi)\| &\leq K \|\xi\| \\ \|\psi(\eta)\| &\leq K \|\eta\|^m \quad \text{and} \quad \lim_{\eta \rightarrow 0} \frac{\psi(\eta)}{\|\eta\|^m} = 0 \\ \|\varphi(\xi)\| &\leq K \|\xi\|^m \quad \text{and} \quad \lim_{\xi \rightarrow 0} \frac{\varphi(\xi)}{\|\xi\|^m} = 0. \end{aligned}$$

First we show that for $i = 1, \dots, k$

$$(326) \quad \lim_{\xi \rightarrow 0} \frac{p_i(q(\xi) + \varphi(\xi)) - p_i(q(\xi))_{(m)}}{\|\xi\|^m} = 0.$$

For $x = (x_1, \dots, x_n)$, $Y = (Y_1, \dots, Y_n)$, and the multi-index $\alpha \in \mathbb{N}^n$,

$$(327) \quad (x_1 + Y_1, \dots, x_n + Y_n)^\alpha = \prod_{s=1}^n \left(\sum_{l=0}^{\alpha_s} \binom{\alpha_s}{l} x_s^l Y_s^{\alpha_s-l} \right)$$

so that

$$(328) \quad (x_1 + Y_1, \dots, x_n + Y_n)^\alpha \equiv x^\alpha \pmod{I}$$

where $I \subset R[x_1, \dots, x_n][Y_1, \dots, Y_n]$ is the ideal generated by Y_1, \dots, Y_n . Hence, for all polynomials $r \in R[X_1, \dots, X_n]$,

$$(329) \quad r(x + Y) \equiv r(x) \pmod{I}.$$

So, for the polynomial

$$(330) \quad R(\xi, Y) := \sum_{(\alpha, \beta) \in \mathbb{N}^{2n}} r_{\alpha, \beta} \xi^\alpha Y^\beta := p_i(q(\xi) + Y) - p_i(q(\xi))_{(m)}$$

the coefficients $r_{\alpha, 0}$ with $|\alpha| \leq m$ vanish.

Thus,

$$(331) \quad \begin{aligned} \lim_{\xi \rightarrow 0} \frac{p_i(q(\xi) + \varphi(\xi)) - p_i(q(\xi))_{(m)}}{\|\xi\|^m} &= \lim_{\xi \rightarrow 0} \frac{R(\xi, \varphi(\xi))}{\|\xi\|^m} \\ &= \sum_{|\alpha, 0| > m} r_{(\alpha, 0)} \lim_{\xi \rightarrow 0} \frac{\xi^\alpha}{\|\xi\|^m} + \sum_{(\alpha, \beta), \beta \neq 0} r_{(\alpha, \beta)} \lim_{\xi \rightarrow 0} \xi^\alpha \frac{\varphi(\xi)^\beta}{\|\xi\|^m} \\ &= 0 \end{aligned}$$

since all sums are finite.

We now consider the function $\psi_i(q(-) + \varphi(-))$. For a sufficiently small ξ , $\|q(\xi) + \varphi(\xi)\| \leq C \|\xi\|$ with a constant $C > 0$. Hence,

$$(332) \quad \begin{aligned} \lim_{\xi \rightarrow 0} \left| \frac{\psi_i(q(\xi) + \varphi(\xi))}{\|\xi\|^m} \right| &\leq \lim_{\xi \rightarrow 0} \sup_{\|\xi'\| \leq C \|\xi\|} \left| \frac{\psi_i(\xi')}{\|\xi\|^m} \right| \\ &\leq \lim_{\xi \rightarrow 0} \sup_{\|\xi'\| \leq C \|\xi\|} \left| C^m \frac{\psi_i(\xi')}{\|\xi'\|^m} \right| \\ &= 0 \end{aligned}$$

which induces

$$(333) \quad \begin{aligned} \lim_{\xi \rightarrow 0} \frac{\phi(\xi)}{\|\xi\|^m} &= \lim_{\xi \rightarrow 0} \frac{p(q(\xi) + \varphi(\xi)) - p(q(\xi))_{(m)} + \psi(q(\xi) + \varphi(\xi))}{\|\xi\|^m} \\ &= \lim_{\xi \rightarrow 0} \frac{p(q(\xi) + \varphi(\xi)) - p(q(\xi))_{(m)}}{\|\xi\|^m} + \lim_{\xi \rightarrow 0} \frac{\psi(q(\xi) + \varphi(\xi))}{\|\xi\|^m} \\ &= 0 \end{aligned}$$

□

Now we show the ring-property, i.e. that sum and product of m -times Peano-differentiable functions are again m -times Peano-differentiable.

Proposition 6.7 *Sum and product of m -times Peano-differentiable functions are again m -times Peano-differentiable with*

$$(334) \quad p_{f+g,x_0} = p_{f,x_0} + p_{g,x_0} \text{ and}$$

$$(335) \quad p_{f \cdot g,x_0} = f(x_0)p_{g,x_0} + g(x_0)p_{f,x_0} + (p_{f,x_0}p_{g,x_0})_{(m)}.$$

Proof: The approximation polynomial of the addition $+ : R^2 \rightarrow R$ at (x_0, y_0) is

$$(336) \quad p_{+,(x_0,y_0)}(x, y) = x + y,$$

and the one of the multiplication $\cdot : R^2 \rightarrow R$ at (x_0, y_0) is

$$(337) \quad p_{\cdot,(x_0,y_0)}(x, y) = xy + y_0x + x_0y.$$

Since both $+$ and \cdot are polynomial functions they are \mathcal{P}^m for each m . The chain-rule now says that

$$(338) \quad p_{f+g,x_0} = p_{f,x_0} + p_{g,x_0}$$

and

$$(339) \quad p_{f \cdot g,x_0} = f(x_0)p_{g,x_0} + g(x_0)p_{f,x_0} + (p_{f,x_0}p_{g,x_0})_{(m)}.$$

□

7 Differentiability Properties of Peano-Derivatives

In [54] we find the following statement:

Theorem 7.1 (Oliver)

Let I be an interval in \mathbb{R} and let $f : I \rightarrow \mathbb{R}$ be an m -times Peano-differentiable function. If the m^{th} Peano-derivative is locally bounded from above or below at $x_0 \in I$, the $(m - 1)^{\text{th}}$ Peano-derivative is ordinary differentiable at x_0 .

In this section we will prove a stronger statement for m -times Peano-differentiable functions in o-minimal structures.

Moreover, we will generalise Oliver's result to definable functions of several variables.

One-Dimensional Peano-Differentiable Functions

Oliver's theorem in o-minimal structures can also be regarded as a generalisation of proposition 5.7 to Peano-differentiable functions. Actually, we make use of the fact that ordinary differentiability implies continuous differentiability for definable functions of one variable.

Proposition 7.2 *Let $I \subset R$ be an open interval and $f : I \rightarrow R$ a definable m -times Peano-differentiable function. Then, f is m -times continuously differentiable in I .*

Proof: We prove this by induction on m .

The statement holds true for $m = 1$ because of proposition 5.7.

We now assume that the proposition holds true for m .

Let $f : I \rightarrow R$ be $(m + 1)$ -times Peano-differentiable at 0. Without loss of generality we may assume the $p_{f,0}$ is the zero-polynomial. Hence,

$$(340) \quad \lim_{x \rightarrow 0} \frac{f(x)}{x^{m+1}} = 0.$$

We conclude that the function $h : I \rightarrow R$ with

$$(341) \quad h(x) = \begin{cases} \frac{f(x)}{x^m}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is ordinary differentiable at 0 with $h'(0) = 0$. According to proposition 5.7, h' is continuous at 0.

h' can be expressed by

$$(342) \quad h'(x) = \begin{cases} \frac{f'(x)}{x^m} - m \frac{f(x)}{x^{m+1}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Hence,

$$(343) \quad \lim_{x \rightarrow 0} \frac{f'(x)}{x^m} = \lim_{x \rightarrow 0} \left(h'(x) + m \frac{f(x)}{x^{m+1}} \right) = 0.$$

Therefore, f' is m -times Peano-differentiable at 0 and thus m -times continuously differentiable at 0. This induces that f is $(m + 1)$ -times continuously differentiable at 0. \square

Oliver's Theorem in Higher Dimension

Now we generalise Oliver's theorem to definable functions of several variables.

Theorem 7.3 *Let $U \subset \mathbb{R}^n$ be a definable open subset and $f : U \rightarrow \mathbb{R}$ a definable m -times Peano-differentiable function. Let the m^{th} Peano-derivatives all be bounded from above or all be bounded from below. Then the $(m - 1)^{\text{th}}$ Peano-derivatives are ordinary differentiable at each $x_0 \in U$.*

We now prepare the proof of the theorem. The next lemma gives us a bound for the number of zeros of the derivatives under certain conditions. More precisely, it says that we always find a sufficiently large interval on which the derivatives of a univariate definable \mathcal{C}^{p+1} -function do not change their sign.

Lemma 7.4 *Let $h : (a, b) \rightarrow \mathbb{R}$ be a definable \mathcal{C}^{p+1} -function and*

$$(344) \quad h^{(p+1)}(t) > 0, \quad t \in (a, b).$$

Then there is a subinterval $[c, d] \subset (a, b)$ of length

$$(345) \quad |d - c| \geq \frac{|b - a|}{(p + 1)^2}$$

such that all derivatives of order 2 to $p + 1$ are either positive or negative on (c, d) .

Proof: We show by induction on j that for $j = 0, \dots, p$ the number of zero-points of $h^{(p+1-j)}$ is bounded by j :

$j = 0$:

By assumption, $h^{(p+1-0)}$ is positive. Therefore it has no zero-point.

$j \rightsquigarrow j + 1$:

By the assumption of induction, there are at most j zero-points $x_1 < x_2 < \dots < x_j$ of $h^{(p+1-j)}$ in (a, b) .

We put $x_0 = a$ and $x_{j+1} = b$.

On each of the $j + 1$ intervals (x_l, x_{l+1}) where $l = 0, \dots, j$, the $(p + 1 - j)^{th}$ derivative of h is strictly positive or negative.

Hence, $h^{(p-j)}|_{(x_l, x_{l+1})}$ is a strictly monotone function and may therefore have at most one zero-point.

So there are at most $\sum_{l=0}^{p-1} l = \frac{p(p-1)}{2} \leq p^2$ points at which at least one of the $h^{(i)}$ equals 0 where $i = 2, \dots, p + 1$.

Hence, we find an interval $[c, d] \subset (a, b)$ of length

$$(346) \quad \frac{|b - a|}{(p + 1)^2}$$

such that for $i = 2, \dots, p + 1$, the i^{th} derivative of h is either positive or negative. \square

As an immediate consequence of this lemma and lemma 3.2 we get:

Corollary 7.5 *Let $h : (a, b) \rightarrow R$ be a definable \mathcal{C}^{p+1} -function and*

$$(347) \quad h^{(p+1)}(t) > 0, \quad t \in (a, b).$$

Then there is a subinterval $[c, d] \subset (a, b)$ of length

$$(348) \quad |d - c| \geq \frac{|b - a|}{(p + 1)^2}$$

such that for $j = 1, \dots, p$:

$$(349) \quad |f^{(j)}(t)| \leq 2^{\frac{j^2+5j-2}{2}} \sup_{s \in [c, d]} \frac{|f(s)|}{|d - c|^j}, \quad t \in \left(c + \frac{d - c}{4}, d - \frac{d - c}{4} \right)$$

Proof: According to lemma 7.4 we get a subinterval $[c, d] \subset (a, b)$ of the desired length where each of the derivatives of h is definite. We apply lemma 3.2 to h restricted to $[c, d]$ with $r = (d - c)/4$ and $t \in [c + \frac{d-c}{4}, d - \frac{d-c}{4}]$. \square

The next lemma sets a sufficient condition on the m^{th} partial derivative with respect to the first variable of a definable m -times Peano-differentiable function such that the corresponding $(m-1)^{\text{th}}$ partial derivative is ordinary differentiable.

Lemma 7.6 *Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$ a definable m -times Peano-differentiable function. Let the m^{th} partial derivative of f with respect to the first variable be locally bounded from below (above) at $x_0 \in U$. Then the corresponding $(m-1)^{\text{th}}$ partial derivative is ordinary differentiable at x_0 .*

Proof: Without loss of generality we may assume that $U = B_1(0)$, $x_0 = 0$, and the Peano derivatives up to order m vanish at 0. Furthermore, we may assume that the m^{th} partial derivative with respect to the first variable is bounded from below on $B_1(0)$ by $K < 0$.

We show that the mapping

$$(350) \quad x \mapsto \frac{\partial^{m-1}}{\partial x_1^{m-1}} f(x)$$

is $o(\|x\|)$.

In order to show this we assume that the mapping above is not $o(\|x\|)$, e.g. there is an $L > 0$ such that

$$(351) \quad \forall 0 < \epsilon < \frac{1}{2} \exists x_\epsilon \in B_\epsilon(0) : \frac{\partial^{m-1}}{\partial x_1^{m-1}} f(x_\epsilon) \geq L \|x_\epsilon\|.$$

step 1:

We consider the following family of functions. $g_x : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{R}$, $x \in B_{\frac{1}{2}}(0)$ with

$$(352) \quad g_x(t) := f(x + te_1).$$

Since g_x is a definable m -times Peano-differentiable function of the variable t , we know that g_x is m -times continuously differentiable in $[-\frac{1}{2}, \frac{1}{2}]$, cf. proposition 7.2.

We may apply the Mean-Value-Theorem to the $(m-1)^{\text{th}}$ derivative of g_x . If $s \geq 0$,

$$(353) \quad g_x^{(m-1)}(s) - g_x^{(m-1)}(0) \geq K(s-0)$$

since K is a lower bound for the m^{th} derivative of g_x which is in fact the function $t \mapsto \frac{\partial^m f}{\partial x_1^m}(x + te_1)$.

Hence, if $0 \leq s \leq \frac{L\|x_\epsilon\|}{-2K}$,

$$\begin{aligned}
 (354) \quad g_{x_\epsilon}^{(m-1)}(s) &= g_{x_\epsilon}^{m-1}(0) + Ks \\
 &\geq g_{x_\epsilon}^{(m-1)} - \frac{L\|x_\epsilon\|}{2} \\
 &= \frac{\partial^{m-1}}{\partial x_1^{m-1}} f(x_\epsilon) - \frac{L\|x_\epsilon\|}{2} \\
 &\geq \frac{L\|x_\epsilon\|}{2}.
 \end{aligned}$$

Especially, $g_{x_\epsilon}^{(m-1)}$ is positive on $(0, \frac{L\|x_\epsilon\|}{-2K})$.

step 2:

We apply corollary 7.5 to g_{x_ϵ} restricted to the interval $(0, \frac{L\|x_\epsilon\|}{-2K})$ with $p = m - 2$.

Thus, there is a subinterval $[c_\epsilon, d_\epsilon] \subset (0, \frac{L\|x_\epsilon\|}{-2K})$ such that

$$(355) \quad |g_{x_\epsilon}^{(m-2)}(s)| \leq 2^{\frac{(m-2)^2+5(m-2)+2}{2}} \sup_{s' \in [c_\epsilon, d_\epsilon]} \frac{|g_{x_\epsilon}(s')|}{|d_\epsilon - c_\epsilon|^{m-2}}$$

whenever $s \in [c_\epsilon + (d_\epsilon - c_\epsilon)/4, d_\epsilon - (d_\epsilon - c_\epsilon)/4]$. Moreover, we may assume that for c_ϵ and d_ϵ applies

$$(356) \quad |d_\epsilon - c_\epsilon| = \frac{L\|x_\epsilon\|}{-2K(m-1)^2}.$$

So, for $s \in [c_\epsilon + (d_\epsilon - c_\epsilon)/4, d_\epsilon - (d_\epsilon - c_\epsilon)/4]$ we obtain the inequation

$$\begin{aligned}
 (357) \quad |g_{x_\epsilon}^{(m-2)}(s)| &\leq N_1 \sup_{s' \in [c_\epsilon, d_\epsilon]} \frac{|g_{x_\epsilon}(s')|}{\|2x_\epsilon\|^{m-2}} \\
 &\leq N \sup_{y \in B_{2\|x_\epsilon\|}(0)} \frac{f(y)}{\|y\|^{m-2}}
 \end{aligned}$$

where N_1 and N are positive constants depending only on m, L, K .

step 3a:

Since $g_{x_\epsilon}^{(m-2)}$ is continuous on the closed interval $[\frac{-1}{2}, \frac{1}{2}]$, we conclude that for $s_\epsilon := \frac{c_\epsilon + d_\epsilon}{2} - \frac{d_\epsilon - c_\epsilon}{4}$ and $t_\epsilon := \frac{c_\epsilon + d_\epsilon}{2} + \frac{d_\epsilon - c_\epsilon}{4}$,

$$(358) \quad |g_{x_\epsilon}^{(m-2)}(t_\epsilon) - g_{x_\epsilon}^{(m-2)}(s_\epsilon)| \leq 2N \sup_{y \in B_{2\|x_\epsilon\|}(0)} \frac{f(y)}{\|y\|^{m-2}}.$$

step 3b:

According to the Mean-Value-Theorem, we can give an alternative estimate for $\left|g_{x_\epsilon}^{(m-2)}(t_\epsilon) - g_{x_\epsilon}^{(m-2)}(s_\epsilon)\right|$.

$$\begin{aligned}
 (359) \quad \left|g_{x_\epsilon}^{(m-2)}(t_\epsilon) - g_{x_\epsilon}^{(m-2)}(s_\epsilon)\right| &\geq (t_\epsilon - s_\epsilon) \inf_{u \in (s_\epsilon, t_\epsilon)} g_{x_\epsilon}^{(m-1)}(u) \\
 &\geq (t_\epsilon - s_\epsilon) \frac{L \|x_\epsilon\|}{2} \\
 &= \frac{L \|x_\epsilon\|}{-4K(m-1)^2} \frac{L \|x_\epsilon\|}{2} \\
 &\geq \tilde{N} \|x_\epsilon\|^2
 \end{aligned}$$

with a positive constant \tilde{N} depending only on m, L, K .

step 4:

Since f is $o(\|x\|^m)$ there is an $\epsilon > 0$ such that

$$(360) \quad \sup_{y \in B_{2\|x_\epsilon\|}(0)} \frac{f(y)}{\|y\|^{m-2}} \leq \frac{\tilde{N}}{4N} \|x_\epsilon\|^2.$$

Step 3a and 3b now imply

$$(361) \quad \frac{\tilde{N}}{2} \|x_\epsilon\|^2 \geq 2N \sup_{y \in B_{2\|x_\epsilon\|}(0)} \frac{f(y)}{\|y\|^{m-2}} \geq \left|g_{x_\epsilon}^{(m-2)}(t_\epsilon) - g_{x_\epsilon}^{(m-2)}(s_\epsilon)\right| \geq \tilde{N} \|x_\epsilon\|^2$$

which is a contradiction. So, the assumption of (351) is false.

step 5:

We can show analogously that the assumption

$$(362) \quad \forall 0 < \epsilon < \frac{1}{2} \exists x_\epsilon \in B_\epsilon(0) : \frac{\partial^{m-1}}{\partial x_1^{m-1}} f(x_\epsilon) \leq -L \|x_\epsilon\|$$

is false.

Hence, $\frac{\partial^{m-1}}{\partial x_1^{m-1}} f(x)$ is $o(\|x\|)$, i.e. it is ordinary differentiable at 0. \square

We note an immediate consequence of this lemma.

Corollary 7.7 *Let $U \subset \mathbb{R}^n$ be open, let $f : U \rightarrow \mathbb{R}$ be a definable m -times Peano-differentiable function and $\nu \in \mathbb{R}^n$ unit-vector. If the m^{th} directional derivative of f with respect to ν is locally bounded from below (above) at $x_0 \in U$, the corresponding $(m-1)^{\text{th}}$ directional derivative is ordinary differentiable at x_0 .*

This leads us directly to the following corollary.

Corollary 7.8 *Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}$ a definable 2-times Peano-differentiable function. If every second partial derivative of f is locally bounded from above or below at $x_0 \in U$, then f is 2-times ordinary differentiable at x_0 .*

Corollary 7.8 cannot directly be generalised to higher order Peano-differentiable functions. This is due to the fact that for $m \geq 3$, the $(m-1)^{\text{th}}$ Peano-derivatives can be mixed derivatives, i.e. they are no longer directional derivatives. But, by some stronger assumption as given in theorem 7.3, we can obtain an analogous statement for $m \geq 3$.

Proof of theorem 7.3 Let $\nu \in (R_0^+)^n$ be a unit-vector. Then, the m^{th} directional derivative with respect to ν is bounded from below (above) since all m^{th} Peano-derivatives are bounded from below (above). Hence, by corollary 7.7, the $(m-1)^{\text{th}}$ directional derivative with respect to ν is an ordinary differentiable function.

All unit-vectors of \mathbb{R}^n can be written as a linear combination of unit-vectors with non-negative entries, and a linear combination of ordinary differentiable functions is again ordinary differentiable. Thus, all $(m-1)^{\text{th}}$ directional derivatives are ordinary differentiable functions.

Since for $\alpha \in \mathbb{N}^n$ with $|\alpha| = m-1$ the differential operator D_α can be written as a linear combination of $(m-1)^{\text{th}}$ directional derivatives, we have finished the proof. \square

We note a consequence of theorem 7.3.

Corollary 7.9 *Let $U \subset \mathbb{R}^n$ be a definable open subset and $f : U \rightarrow \mathbb{R}$ a definable m -times Peano-differentiable function. Let the m^{th} Peano-derivatives all be locally bounded at $x_0 \in U$. Then, the $(m-1)^{\text{th}}$ Peano-derivatives are ordinary differentiable at x_0 .*

8 Singularities of Peano-Differentiable Functions

In this chapter we study the set of points at which a definable m -times Peano-differentiable function is not m -times continuously differentiable. By proposition 7.2, we know that definable m -times Peano-differentiable functions of one variable are m -times continuously differentiable, while we obtain by the examples of 6.3 that we can distinguish at least three different types of \mathcal{C}^k -singularities for functions of several variables.

We will first show that the codimension of the \mathcal{C}^m -singularity set of a definable m -times Peano-differentiable function is at least 2.

In the second section we give a full description of the sets which can appear as singularity sets. Moreover, we construct for each possible set a definable m -times Peano-differentiable function which has exactly the given set as set of \mathcal{C}^k -singularities of the corresponding type.

\mathcal{C}^m -Singularities in Higher Dimension

We begin with the definition of a \mathcal{C}^m -singularity of a function.

Definition 8.1 *Let $f : U \rightarrow R^p$ be a definable mapping. If f is not m -times continuously differentiable at $x_0 \in U$, we call x_0 a \mathcal{C}^m -singularity of f . By $\text{sing}^m(f)$ we denote the set of \mathcal{C}^m -singularities of f .*

Since we can describe continuous differentiability by a formula, it is evident that $\text{sing}^m(f)$ of a definable function f is definable.

Later in this chapter we will distinguish between several subclasses of \mathcal{C}^m -singularities.

As our first task we will prove that the codimension of the set of \mathcal{C}^m -singularities of a definable m -times Peano-differentiable function is at least 2.

In order to analyse functions of several variables we agree on the following terminology:

For $\delta > 0$, $B_\delta(x) := \{y \in R^n \mid \|y - x\| < \delta\}$, $U_\delta := B_\delta(0)$.

$$U_\delta^\ddagger := \{(x_1, \dots, x_n) \mid \|(x_1, \dots, x_n)\| < \delta, x_1 \ddagger 0\}$$

where \ddagger is one of the symbols $=, <, >, \leq, \geq, \neq$.

Definable functions are a priori \mathcal{C}^m -functions up to a definable set of lower dimension.

Our idea is to assume that the set of \mathcal{C}^m -singularities is of codimension 1. With the help of the \mathcal{C}^m -cell decomposition we can reduce our examination locally to the cases where this set is of the form U_δ^\neq which leads to a contradiction.

The codimension-2-statement is induced by the following lemma.

Lemma 8.2 *Let $f : U_\delta^\neq \rightarrow R$ be a definable continuous function with*

$$(363) \quad \forall x \in U_\delta^\neq \quad \forall \epsilon, \epsilon' > 0 \quad \exists y \in B_\epsilon(x) \cap U_\delta^\neq \quad \exists y' \in B_\epsilon(x) \cap U_\delta^\neq : \\ |f(y)| < \epsilon' \wedge |f(y')| < \epsilon'.$$

Then, the set of points at which $\bar{f} : U_\delta \rightarrow R$,

$$(364) \quad \bar{f}(x) = \begin{cases} f(x), & x \in U_\delta^\neq \\ 0, & x \in U_\delta^\neq, \end{cases}$$

is not continuous, is of codimension greater than or equal to 2.

Proof: Without loss of generality let f be positive semidefinite. We first look at f restricted to U_δ^\neq .
Let

$$(365) \quad A = \{x \in U_\delta^\neq \mid f \text{ is locally constant at } x\}.$$

Property (363) does not change if we add to f the definable continuous function $x \mapsto \text{dist}(x, R^n \setminus A)$ which vanishes on U_δ^\neq .

So, we may assume that f is nowhere locally constant.

We define the function $g : U_\delta^\neq \rightarrow R$ as

$$(366) \quad g(y) := \inf_{\epsilon > 0} \sup_{z \in B_\epsilon(y) \cap U_\delta^\neq} \{f(z)\}.$$

This function is definable by construction.

The lemma is proved if g is positive only on a set of dimension less than or equal to $n - 2$.

We assume that this is not true, i.e. that the set of positive points of g is of dimension $n - 1$.

According to o-minimality, g is a continuous function up to a definable set of

dimension less than $n - 1$. Hence, there is an $M > 0$, a $\delta_1 > 0$ and a $y \in U_\delta^-$ such that

$$(367) \quad g(x) > M, \quad x \in U_{\delta_1}^- \cap B_{\delta_1}(y).$$

By property (363) we have

$$(368) \quad \forall x \in B_{\delta_1}(y) \cap U_\delta^- \quad \forall \epsilon > 0 \quad \exists z \in B_\epsilon(x) \cap U_\delta^+ : \quad f(z) \leq M.$$

Since $B_\epsilon(x) \cap U_\delta^+$ is definably connected and $f|_{B_\epsilon(x) \cap U_\delta^+}$ is continuous, the Intermediate-Value-Theorem says that there is a $z' \in B_\epsilon(x) \cap U_\delta^+$ with $f(z') = M$.

So, $B_{\delta_1}(y) \cap U_\delta^-$ is a subset of $\partial B(M)$ where $B(M) = f^{-1}(\{M\})$. Therefore we get

$$(369) \quad n - 1 = \dim(B_{\delta_1}(y) \cap U_\delta^-) \leq \dim(\partial B(M)).$$

At the same time, f is nowhere locally constant. Hence, $\dim(B(M)) \leq n - 1$, and theorem 2.25 implies

$$(370) \quad \dim(\partial B(M)) \leq n - 2$$

which contradicts inequation (369).

For $U_\delta^<$ we get the corresponding statement.

Hence, the set of points at which we cannot extend f continuously to 0 in U_δ^- is of dimension less than or equal to $n - 2$ since it is the union of two definable sets of dimension less than or equal to $n - 2$. \square

In order to apply lemma 8.2 we need vanishing derivatives on certain sets. This is provided by the following lemma.

Lemma 8.3 *Let $\delta > 0$ and $f \in \mathcal{C}^{m-1}(U_\delta, R)$ be a definable m -times Peano-differentiable function such that for each $k = 0, \dots, m$,*

$$(371) \quad (x_2, \dots, x_n) \mapsto f_{[(k,0,\dots,0)]}(0, x_2, \dots, x_n)$$

is an m -times continuously differentiable function. There exists a definable m -times continuously differentiable function $F : U_\delta \rightarrow R$ such that

$$(372) \quad D_\alpha F = f_{[\alpha]}, \quad |\alpha| \leq m - 1, \quad \text{and } D_{(m,0,\dots,0)} F = f_{[(m,0,\dots,0)]} \text{ on } U_\delta^-.$$

Proof: We define

$$(373) \quad F(x_1, \dots, x_n) := \sum_{k=0}^m \frac{f_{[(k,0,\dots,0)]}(0, x_2, \dots, x_n)}{k!} x_1^k, \quad (x_1, \dots, x_n) \in U_\delta.$$

Since the $f_{[(k,0,\dots,0)]}$ restricted to U_δ^- are \mathcal{C}^m -functions we see that F is also a \mathcal{C}^m -function. For $|\alpha| \leq m - 1$ we have

$$(374) \quad \begin{aligned} D_\alpha F(x_1, \dots, x_n) &= \sum_{k=0}^m D_\alpha \left(\frac{f_{[(k,0,\dots,0)]}(0, x_2, \dots, x_n)}{k!} x_1^k \right) \\ &= \sum_{k=\alpha_1}^m \frac{D_{(0,\alpha_2,\dots,\alpha_n)} f_{[(k,0,\dots,0)]}(0, x_2, \dots, x_n)}{(k - \alpha_1)!} x_1^{k-\alpha_1} \end{aligned}$$

For $|\alpha| \leq m - 1$ we get

$$(375) \quad D_\alpha F(0, x_2, \dots, x_n) = D_{(0,\alpha_2,\dots,\alpha_n)} f_{[(\alpha_1,0,\dots,0)]}(0, x_2, \dots, x_n) = f_{[\alpha]}(0, x_2, \dots, x_n),$$

since f is \mathcal{C}^{m-1} . Furthermore $D_{(m,0,\dots,0)} F = f_{[m,0,\dots,0]}$ on U_δ^- . \square

We now show that with some further assumptions, U_δ^- cannot be the set of \mathcal{C}^m -singularities of a definable m -times Peano-differentiable function.

Lemma 8.4 *Let $f : U_\delta \rightarrow R$ be a definable \mathcal{C}^{m-1} -function which is m -times Peano-differentiable on U_δ . Further, let $f|_{U_\delta^\neq}$ be of class \mathcal{C}^m and let each of the Peano-derivatives $f_{[\alpha]}$, $|\alpha| \leq m$, be a \mathcal{C}^m -function on U_δ^- . Then, f is of class \mathcal{C}^m except for a set of codimension greater than or equal to 2.*

Proof: According to lemma 8.3, we make the general assumption that $f|_{U_\delta^-} \equiv 0$ together with all Peano-derivatives up to order $m - 1$, and $f_{[(m,0,\dots,0)]}(0, x_2, \dots, x_n) = 0$.

step 1: We show that all m^{th} Peano-derivatives satisfy property (363). Let $|\alpha| = m$.

We assume that property (363) does not hold true for $D_\alpha f$, i.e. that there is an $x_0 \in U_\delta^-$, a $\delta_1 > 0$ and an $M > 0$ such that for all $y \in B_{\delta_1}(x_0) \cap U_\delta^<$ or all $y \in B_{\delta_1}(x_0) \cap U_\delta^>$

$$(376) \quad |D_\alpha f(y)| > M.$$

Without loss of generality we may assume that $x_0 = 0$. We assume that inequality (376) is valid for all $y \in B_{\delta_1}(x_0) \cap U_\delta^>$, the other case is analog.

By the continuity of $D_\alpha f$ in $U_{\delta_1}^>$ and inequality (376), $D_\alpha f$ does not change its sign in $U_{\delta_1}^>$. So we may assume that $D_\alpha f$ is positive in $U_{\delta_1}^>$.

We distinguish between the two possible cases.

case 1: $\alpha_1 < m$.

In this case there is an $i \geq 2$ with $\alpha_i > 0$.

$D_{\alpha-e_i} f$ is continuous in U_δ and, by the general assumption, it vanishes at all points of U_δ^- .

For $x_1 > 0$ we fix all coordinates excepting the i^{th} one and apply the Mean-Value-Theorem to $D_{\alpha-e_i} f$. Then

$$(377) \quad D_{\alpha-e_i} f(x) - D_{\alpha-e_i} f(x - x_i e_i) \geq M x_i.$$

This is not possible since, also for $x_i > 0$, according to the continuity of $D_{\alpha-e_i} f$, the limit

$$(378) \quad \lim_{x_1 \rightarrow 0} (D_{\alpha-e_i} f(x) - D_{\alpha-e_i} f(x - x_i e_i))$$

vanishes for all choices of x_2, \dots, x_n .

So, property (363) holds true for $D_\alpha f$.

case 2: $\alpha_1 = m$:

For an $x_0 \in U_\delta^-$ and $t \in R$ sufficiently small we look at

$$(379) \quad \frac{d^m}{dt^m} f(x_0 + t e_1) = f_{[(m,0,\dots,0)]}(x_0 + t e_1).$$

By the general assumption,

$$(380) \quad \lim_{t \searrow 0} \frac{f(t, x_2, \dots, x_n)}{|t|^m} = 0$$

for all choices of x_2, \dots, x_n .

According to proposition 7.2, we obtain

$$(381) \quad \lim_{t \rightarrow 0, t \neq 0} f_{[(m,0,\dots,0)]}(x_0 + t e_i) = \lim_{t \rightarrow 0, t \neq 0} \frac{d^m}{dt^m} f(x_0 + t e_1) = 0.$$

Thus, for all $|\alpha| = m$, $D_\alpha f$ has property (363).

This, in connection with lemma 8.2, implies that all m^{th} Peano-derivatives are continuous up to a set of codimension greater than or equal to 2.

step 2:

In U_δ^\neq , $D_\alpha f$ is continuously differentiable for $|\alpha| = m - 1$ by assumption.

Let $E \subset U_\delta^-$ be the closure of the points at which at least one m^{th} Peano-derivative is not continuous. Note that

$$(382) \quad \dim(E) \leq n - 2.$$

For $x \in U_\delta^- \setminus E$, all m^{th} Peano derivatives are continuous at x and so locally bounded. We apply corollary 7.9 to these x and obtain continuous differentiability for all $(m - 1)^{\text{th}}$ Peano derivatives at all points of $U_\delta^- \setminus E$. \square

We are now able to prove the first important result of this chapter.

Theorem 8.5 *Let $U \subset R^n$ be open, $m \in \mathbb{N}$, and let $f : U \rightarrow R$ be a definable m -times Peano-differentiable function. Then*

$$(383) \quad \dim(\text{sing}^m(f)) \leq n - 2.$$

Proof: We prove this by induction on m :

The case $m = 0$ is evident.

$m - 1 \rightsquigarrow m$:

Since f is m -times Peano-differentiable, it is also $(m - 1)$ -times Peano-differentiable so that the assumption of the induction implies that

$$(384) \quad \dim(\text{sing}^{m-1}(f)) \leq n - 2.$$

With the help of theorem 2.25 we obtain

$$(385) \quad \dim(\text{cl}(\text{sing}^{m-1}(f))) \leq n - 2.$$

Let $V := U \setminus \text{cl}(\text{sing}^{m-1}(f))$. Then, V is open, and f restricted to V is an $(m - 1)$ -times continuously differentiable function.

We now assume that

$$(386) \quad \dim(\text{sing}^m(f|_V)) = n - 1.$$

We select by theorem 4.5 a Λ^{2m} -regular stratification of R^n which is compatible with V , $\text{sing}^m(f|_V)$, and also the sets on which all Peano-derivatives of f up to order m are $2m$ -times continuously differentiable functions.

If equation (386) applies, there is at least one Λ^m -regular cell B of dimension $n - 1$ with $B \subset \text{sing}^m(f|_V)$.

Since B is a C^{2m} -submanifold of V , there is an $x_0 \in B$, a neighbourhood W of x_0 in V , and for some $\delta > 0$ there is a C^{2m} -diffeomorphism $\varphi : W \rightarrow B_\delta(0)$ such that $\varphi(W \cap B) = U_\delta^-$.

So, the function $f \circ \varphi^{-1}$ satisfies the conditions of lemma 8.4.

Hence, $f = f \circ \varphi^{-1} \circ \varphi$ is of class \mathcal{C}^m except for a set of codimension ≥ 2 , which contradicts equation (386).

Hence,

$$(387) \quad \dim(\text{sing}^m(f)) \leq \dim(\text{sing}^m(f|_V) \cup \text{cl}(\text{sing}^{m-1}(f))) \leq n - 2$$

□

Characterising \mathcal{C}^m -Singularity Sets

We have seen in the section before that the dimension of the \mathcal{C}^m -singularity set of a definable m -times Peano-differentiable function of n variables is bounded by $n - 2$, cf. theorem 8.5.

So, it is reasonable to ask which sets can be \mathcal{C}^m -singularity sets.

The answer is: For each definable set $A \subset R^n$ with $\dim(A) \leq n - 2$ there is a definable m -times Peano-differentiable function f with $A = \text{sing}^m(f)$.

We will prove even more. Inspired by the examples of 6.3, we distinguish between several types of \mathcal{C}^m -singularities.

Definition 8.6 *Let $f : U \rightarrow R$ and let k be a positive integer. We call $x_0 \in U$ a*

(k, \natural) -singularity of f if f is $(k - 1)$ -times continuously but not k -times ordinary differentiable at x_0 ,

(k, ∞) -singularity of f if f is k -times ordinary differentiable at x_0 but not all k^{th} derivatives are locally bounded at x_0 ,

$(k, <)$ -singularity of f if f is k -times ordinary differentiable at x_0 and all k^{th} derivatives are locally bounded at x_0 .

We denote by $\text{sing}_{k,*}(f)$ the set of $(k, *)$ -singularities of f where $*$ is one of the symbols $\infty, <$ or \natural .

If $k \leq m$, all these singularities are obviously \mathcal{C}^m -singularities.

We will give a full description of the singularity sets $\text{sing}_{k,*}(f)$ which can appear for definable m -times Peano-differentiable functions.

Before we begin to prove several technical lemmata, we discuss several conditions which the singularity sets have to satisfy. Of course these sets are definable and of codimension at least 2.

In this section we will make much use of the concepts of Λ^m -regular stratifications, functions and cells, cf. chapter 4.

Necessary Conditions for Singularity Sets

Let $f : U \rightarrow R$ be a definable m -times Peano-differentiable function and $1 \leq k \leq m$.

If $x_0 \in \text{sing}_{k,\natural}(f)$, at least one of the k^{th} Peano-derivatives is not locally bounded at x_0 , cf. corollary 7.9.

Hence,

$$(388) \quad \partial \text{sing}_{k,\natural}(f) \subset \text{sing}_{k,\infty}(f).$$

Since the set of points at which at least one of the k^{th} Peano-derivatives of f is not locally bounded is closed in U ,

$$(389) \quad \text{sing}_{k,\natural}(f) \cup \text{sing}_{k,\infty}(f)$$

is closed in U . Moreover, this union is disjoint by definition.

We obtain that if $\text{sing}^m(f) = \text{sing}_{k,\natural}(f) \cup \text{sing}_{k,\infty}(f)$, then $\text{sing}^m(f)$ is closed in U .

We now look at $\text{sing}_{k,<}(f)$ for $k < m$.

If $x_0 \in \text{sing}_{k,<}(f)$, at least one of the $(k+1)^{\text{th}}$ Peano-derivatives is not locally bounded at x_0 , cf. corollary 7.9.

Hence,

$$(390) \quad \text{cl}(\text{sing}_{k,<}(f)) \subset \text{sing}^m(f).$$

Let $x_0 \in \partial \text{sing}_{k,<}(f)$. Then, $x_0 \notin \text{sing}_{k+1,<}(f)$ since the $(k+1)^{\text{th}}$ Peano-derivatives are not locally bounded at x_0 .

We will show that it is possible for x_0 to be a $(k+1, \infty)$ -singularity.

Furthermore, the set $\text{sing}_{m,<}(f)$ can be any definable subset of U of codimension greater than or equal to 2. We show this by constructing definable m -times Peano-differentiable functions $f : R^n \rightarrow R$ with $\text{sing}^m(f) = \text{sing}_{m,<}(f)$.

Besides the fact that all singularity sets are definable and of codimension greater than or equal to 2 they possess the following properties:

$k \leq m$	$\text{sing}_{k,<}(f)$ is arbitrary $\text{sing}_{k,\natural}(f)$ is arbitrary $\text{sing}_{k,\infty}(f)$ is arbitrary $\partial \text{sing}_{k,\natural}(f) \subset \text{sing}_{k,\infty}$ $\partial \text{sing}_{k,\infty}(f) \subset \text{sing}_{k,\natural}$ $\text{sing}_{k,\infty}(f) \cup \text{sing}_{k,\natural}$ is closed
$k < m$	$\text{cl}(\text{sing}_{k,<}(f)) \subset \text{sing}^m(f)$

Preliminary Lemmata

We need several technical lemmata.

Lemma 8.7 *Let \mathcal{A} be either \mathcal{O} or \mathcal{O} . Let $X \subsetneq \mathbb{R}^n$ be open and let $f, g : X \rightarrow \mathbb{R}$ be k -times ordinary differentiable functions such that for each multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$,*

$$(391) \quad D_\alpha f(x) \text{ is } \mathcal{A}(\text{dist}(x, \partial X)^{s-|\alpha|}), \quad D_\alpha g(x) \text{ is } \mathcal{O}(\text{dist}(x, \partial X)^{r-|\alpha|})$$

as x tends to ∂X . Then, for $|\alpha| \leq k$,

$$(392) \quad D_\alpha(f \cdot g)(x) \text{ is } \mathcal{A}(\text{dist}(x, \partial X)^{r+s-|\alpha|})$$

as x tends to ∂X .

Proof: By the Leibniz-formula for derivatives, there are $a_\beta \in \mathbb{R}$, $\beta \prec \alpha$ (i.e. $\beta_i \leq \alpha_i$, $i = 1, \dots, n$), such that

$$(393) \quad D_\alpha(f \cdot g)(x) = \sum_{\beta \prec \alpha} a_\beta (D_\beta f)(D_{\alpha-\beta} g)(x).$$

According to property (391), we obtain that

$$(394) \quad (D_\beta f(x))D_{\alpha-\beta} g(x) \text{ is } \mathcal{A}(\text{dist}(x, \partial X)^{s+r-|\alpha|})$$

as x tends to ∂X . Hence, the statement of the lemma is evident. \square

In the following we often write " $f(x)$ is $\mathcal{A}(\text{dist}(x, Y)^k)$ " and omit "as x tends to Y ".

Lemma 8.8 *Let $X, Y \subset \mathbb{R}^n$ be open definable sets and $\varphi : X \rightarrow Y$ a definable bijective Λ^p -regular mapping with Λ^p -regular inverse. Then there is an $L > 0$ such that*

$$(395) \quad \frac{1}{L} \text{dist}(\varphi(x), \partial Y) \leq \text{dist}(x, \partial X) \leq L \text{dist}(\varphi(x), \partial Y).$$

Proof: We choose $L > 0$ that way that both φ and φ^{-1} are Λ^p -regular with L .

Without loss of generality we may assume that X is definably connected.

If $x \in X$, there is an $x_0 \in \partial X$ such that $\|x - x_0\| = \text{dist}(x, \partial X)$.

So, the function $h : [0, 1) \rightarrow X$, $t \mapsto h(t) = x + t(x_0 - x)$ is well defined and

obviously differentiable. We apply the Mean-Value-Theorem to $\varphi \circ h$ and obtain

$$(396) \quad \|\varphi \circ h(t) - \varphi \circ h(0)\| \leq \sup_{0 < s < t} |\nabla \varphi(h(s))h'(s)| (t - 0) \leq L \|x_0 - x\| t.$$

Hence, $\varphi \circ h$ is a bounded Lipschitz-continuous definable function. Thus, the limit

$$(397) \quad \lim_{t \nearrow 1} \varphi \circ h(t) = z$$

exists in R^n .

z belongs to the boundary of Y , otherwise $z \in Y$ so that $x_0 = \varphi^{-1}(z) \in X$ which is not possible. Moreover,

$$(398) \quad \|z - \varphi(h(0))\| \leq L \|x - x_0\|.$$

Therefore we can estimate the distance of $\varphi(x)$ to ∂Y by

$$(399) \quad \text{dist}(\varphi(x), \partial Y) \leq \|\varphi(h(0)) - z\| \leq L \|x - x_0\| = L \text{dist}(x, \partial X).$$

By substituting φ with φ^{-1} we get inequality (395). \square

The next lemma shows the usefulness of Λ^p -regular functions.

Lemma 8.9 *Let q be either $p-1$ or p and let \mathcal{A} be either o or O . If $X \subset R^n$ is open and if $f : X \rightarrow R$ is a definable p -times ordinary differentiable function such that for each multi-index α with $|\alpha| \leq q$*

$$(400) \quad D_\alpha f(x) \text{ is } \mathcal{A}(\text{dist}(x, \partial X)^{p-|\alpha|}),$$

then for each definable open subset $Y \subset R^n$ and each definable bijective Λ^p -regular mapping $\varphi : Y \rightarrow X$,

$$(401) \quad D_\beta(f \circ \varphi) \in \mathcal{A}(\text{dist}(y, \partial Y)^{p-|\beta|}), \quad |\beta| \leq q,$$

applies.

Proof: We have to show the following claim.

Claim: For $|\beta| \leq q$,

$$(402) \quad D_\beta(f \circ \varphi)(y) = \sum_{1 \leq |\alpha| \leq |\beta|} (D_\alpha f)(\varphi(y)) \psi_{\beta, \alpha}(y), \quad y \in Y,$$

where $\psi_{\alpha,\beta}$ is a $\mathcal{C}^{p-|\beta|}$ -function such that

$$(403) \quad D_\gamma \psi_{\alpha,\beta}(y) \text{ is } O(\text{dist}(y, \partial Y)^{|\alpha|-|\beta|-|\gamma|}), \quad |\gamma| \leq q - |\beta|.$$

We will show this claim by induction on $|\alpha|$.

We assume that φ is Λ^p -regular with constant C .

$|\beta| = 1$, i.e. $D_\beta = \frac{\partial}{\partial y_i}$ for a $1 \leq i \leq n$.

$$(404) \quad \frac{\partial}{\partial y_i}(f \circ \varphi)(y) = \sum_{j=1}^n (D_{e_j} f)(\varphi(y)) \cdot \frac{\partial \varphi_j}{\partial y_i}(y) = \sum_{1 \leq |\alpha| \leq |\beta|=1} (D_\alpha f)(\varphi(y)) \psi_{\alpha,\beta}(y)$$

where we set $\psi_{e_j,\beta} := \frac{\partial \varphi_j}{\partial y_i}$. Since φ is Λ^p -regular with constant C ,

$$(405) \quad |D_\gamma \psi_{e_j,\beta}(y)| = |D_{\gamma+e_i} \varphi_j(y)| \leq \frac{C}{\text{dist}(y, \partial Y)^{|\gamma|}}$$

applies for $|\gamma| \leq q - 1 = q - |\beta|$.

Hence, the $\psi_{\alpha,\beta}$ satisfy property (403).

$|\beta| \rightsquigarrow |\beta| + 1$:

We assume that for $1 \leq |\alpha| \leq |\beta|$ the function $\psi_{\alpha,\beta}$ is $(p - |\beta|)$ -times continuously differentiable having property (403).

Then, for $1 \leq i \leq n$,

$$(406) \quad \begin{aligned} D_{\beta+e_i}(f \circ \varphi)(y) &= \\ &= \sum_{1 \leq |\alpha| \leq |\beta|} \frac{\partial}{\partial y_i} ((D_\alpha f)(\varphi(y)) \cdot \psi_{\alpha,\beta}(y)) \\ &= \sum_{1 \leq |\alpha| \leq |\beta|} \sum_{j=1}^n (D_{\alpha+e_j} f)(\varphi(y)) \frac{\partial \varphi_j}{\partial y_i}(y) \psi_{\alpha,\beta}(y) + (D_\alpha f)(\varphi(y)) \frac{\partial \psi_{\alpha,\beta}}{\partial y_i}(y) \\ &= \sum_{1 \leq |\alpha| \leq |\beta+e_i|} (D_\alpha f)(\varphi(y)) \psi_{\alpha,\beta}(y) \end{aligned}$$

where we have put

$$(407) \quad \psi_{\alpha,\beta+e_i}(y) := \begin{cases} \frac{\partial \psi_{\alpha,\beta}}{\partial y_i}(y), & \text{if } |\alpha| = 1 \\ \sum_{j=1}^n \frac{\partial \varphi_j}{\partial y_i}(y) \psi_{\alpha-e_j,\beta}(y) + \frac{\partial \psi_{\alpha,\beta}}{\partial y_i}(y), & \text{if } 1 < |\alpha| \leq |\beta| \\ \sum_{j=1}^n \frac{\partial \varphi_j}{\partial y_i}(y) \psi_{\alpha-e_j,\beta}(y), & \text{if } |\alpha| = |\beta| + 1. \end{cases}$$

The ' at the summation sign \sum means that we only sum over those j where $\alpha_j > 0$.

We now check property (403) for $\psi_{\alpha, \beta + e_i}$.

By assumption, for $|\gamma| \leq q - |\beta| - 1$,

$$(408) \quad D_\gamma \frac{\partial}{\partial y_i} \psi_{\alpha, \beta}(y) = D_{\gamma + e_i} \psi_{\alpha, \beta}(y) \text{ is } O(\text{dist}(y, \partial Y)^{|\alpha| - |\beta| - |\gamma| - |e_i|}),$$

and since φ is Λ^p -regular,

$$(409) \quad D_\gamma \frac{\partial \varphi_j}{\partial y_i}(y) \text{ is } O(\text{dist}(y, \partial Y)^{-|\gamma|}).$$

By induction hypothesis $\psi_{\alpha - e_i, \beta}$ is a $\mathcal{C}^{p - |\beta|}$ -function with $D_\gamma \psi_{\alpha - e_i, \beta}(y)$ being $O(\text{dist}(y, \partial Y)^{1 + |\alpha| - |\beta| - |\gamma|})$ for $|\gamma| \leq p - |\beta|$, c.f. (403). We apply lemma 8.7 to $\psi_{\alpha - e_i, \beta}$ and $\frac{\partial \varphi_j}{\partial y_i}$, hence

$$(410) \quad D_\gamma \left(\frac{\partial \varphi_j}{\partial y_i}(y) \psi_{\alpha - e_j, \beta}(y) \right) \text{ is } O(\text{dist}(y, \partial Y)^{1 + |\alpha| - |\beta| - |\gamma|}).$$

Since $\psi_{\alpha, \beta + e_i}$ is a linear combination of functions of the forms (408) or (410), property (403) applies to it.

Since $\text{dist}(\varphi(y), \partial X)$ is $O(\text{dist}(y, \partial Y))$, the lemma follows immediately from the claim. \square

Constructing Singularity Sets

Now we are able to construct functions with special singularity sets.

The examples of 6.3 provide us for $2 \leq d \leq n$, $1 \leq k \leq m$ and each symbol $*$ $\in \{\natural, \infty, <\}$ with a semialgebraic function $F_{d, k, *}: R^{n-d} \rightarrow R$ such that

- (i) $F_{d, k, *}$ is m -times Peano-differentiable in R^{n-d} ,
- (ii) $F_{d, k, *}$ is m -times continuously differentiable in $R^{n-d} \setminus \{0\}$, and
- (iii) $\text{sing}_{k, *}(F_{d, k, *}) = \{0\}$.

We use these functions for the proof of the following proposition as well as the regularity conditions of Λ^m -regular functions and cells.

Proposition 8.10 *Let $Y \subset R^n$ be a Λ^m -regular cell of dimension $d \leq n - 2$. Then, for each $1 \leq k \leq m$ there are definable m -times Peano-differentiable functions $F_{<}, F_\infty, F_\natural: R^n \rightarrow R$ such that*

- (\langle) if $k < m$, then $\text{sing}^m(F_{\langle}) = \text{cl}(Y)$, $\text{sing}_{k,\langle}(F_{\langle}) = Y$ and
 $\text{sing}_{k+1,\infty}(F_{\langle}) = \partial Y$;
 if $k = m$, then $\text{sing}^m(F_{\langle}) = \text{sing}_{(m,\langle)}(F_{\langle}) = Y$,
- (∞) $\text{sing}^m(F_{\infty}) = \text{sing}_{k,\infty}(F_{\infty}) = \text{cl}(Y)$,
- (\natural) $\text{sing}^m(F_{\natural}) = \text{cl}(Y)$, $\text{sing}_{k,\natural}(F_{\natural}) = Y$ and $\text{sing}_{k,\infty}(F_{\natural}) = \partial Y$.

Proof: Let $Y = (h)_X$ be a standard Λ^m -regular cell in R^n of dimension d , i.e. X is an open Λ^m -regular cell in R^d and $h : X \rightarrow R^{n-d}$ a definable Λ^m -regular function.

step 1:

For the definable set

$$(411) \quad Z := \{(x, y) \in X \times R^{n-d} \mid \|y\| < \min(1, \text{dist}(x, \partial X)^{m+1})\}$$

let $\psi : R^n \rightarrow R$ be a definable \mathcal{C}^{m+1} -function where

- (i) ψ is positive in Z and
 (ii) $\text{supp}(\psi) = \text{cl}(Z)$.

Since ψ is \mathcal{C}^{m+1} with $\text{supp}(\psi) = \text{cl}(Z)$, for all multi-indices $|\beta| \leq k$,

$$(412) \quad D_{\beta}\psi(x, y) \text{ is } o(\text{dist}(x, \partial X)^{m+1-|\beta|}).$$

step 2:

If $* = \langle$, for $|\alpha| \leq k$,

$$(413) \quad D_{\alpha}F_{d,k,\langle}(y) \text{ is } O(\|y\|^{k-|\alpha|}).$$

If $* = \infty$, for $|\alpha| \leq k - 1$,

$$(414) \quad D_{\alpha}F_{d,k,\infty}(y) \text{ is } O(\|y\|^{k-1-|\alpha|}).$$

If $* = \natural$, for $1 \leq |\alpha| \leq k - 1$,

$$(415) \quad D_{\alpha}F_{d,k,\natural}(y) \text{ is } O(\|y\|^{k-1-|\alpha|}).$$

step 3:

Let $H_* : R^d \times R^{n-d} \rightarrow R$ be the function defined by

$$(416) \quad H_*(x, y) := \begin{cases} \psi(x, y)F_{d,k,*}(y), & x \in X \\ 0, & \text{else.} \end{cases}$$

H_* is m -times continuously differentiable on the open sets $R^n \setminus \text{supp}(H_*)$ and $(X \times R^{n-d}) \setminus (X \times \{0\})$. Hence, it is m -times continuously differentiable in the union of these sets, i.e. $H_* \in \mathcal{C}^m(R^n \setminus (\text{cl}(X) \times \{0\}))$.

step 3a: $* = <$:

For $(x, y) \in Z$ and $|\alpha| \leq k$,

$$(417) \quad D_\alpha H_{<}(x, y) = \sum_{\beta \prec \alpha} a_\beta (D_{\alpha-\beta} \psi(x, y)) D_\beta F_{d,k,<}(y)$$

applies, where $a_\beta \in R$ with $a_\alpha = 1$.

By (412) in connection with (413) and (417), we obtain that for $|\alpha| \leq k$,

$$(418) \quad D_\alpha H_{<}(x, y) \text{ is } o(\text{dist}(x, \partial X)^{m+1-|\alpha|}).$$

This implies that $H_{<}$ is $(k+1)$ -times ordinary differentiable in $\partial X \times \{0\}$.

We show that $\text{sing}_{k,<}(H_{<}) = X \times \{0\}$.

If $|\beta| \leq k-1$, then $D_\beta F_{d,k,<}$ is a continuous function. Hence, for $|\alpha| = k$ we can write $D_\alpha H_{<}$ with the help of (417) and (412) as

$$(419) \quad D_\alpha H_{<}(x, y) = f(x, y) + (D_\alpha F_{d,k,<}(y)) \psi(x, y)$$

with a continuous function f .

So, the $(k, <)$ -singularities of $H_{<}|_Z$ correspond to the $(k, <)$ -singularities of $F_{d,k,<}$. Hence, by the properties of $F_{d,k,<}$, $\text{sing}_{k,<}(H_{<}) = X \times \{0\}$, and $H_{<}$ has no other \mathcal{C}^m -singularities in Z .

step 3b: $* = \infty$:

For $(x, y) \in Z$ and $|\alpha| \leq k-1$,

$$(420) \quad D_\alpha H_\infty(x, y) = \sum_{\beta \prec \alpha} A_\beta (D_{\alpha-\beta} \psi(x, y)) D_\beta F_{d,k,\infty}(y)$$

applies, where $a_\beta \in R$ with $a_\alpha = 1$.

By (412) in connection with (414) and (420), we obtain that for $|\alpha| \leq k-1$,

$$(421) \quad D_\alpha H_\infty(x, y) \text{ is } o(\text{dist}(x, \partial X)^{m-|\alpha|}).$$

This implies that H_∞ is k -times ordinary differentiable in $\partial X \times \{0\}$.

We show that $\text{sing}_{k,\infty}(H_\infty) = \text{cl}(X) \times \{0\}$.

If $|\beta| \leq k-1$, then $D_\beta F_{d,k,\infty}$ is a continuous function. Hence, by (420), for $|\alpha| = k-1$ we can write $D_\alpha H_\infty$ as

$$(422) \quad D_\alpha H_\infty(x, y) = f(x, y) + (D_\alpha F_{d,k,\infty}(y)) \psi(x, y)$$

with a continuous function f .

So, the (k, ∞) -singularities of $H_\infty|_Z$ correspond to the (k, ∞) -singularities of $F_{d,k,\infty}$. Hence, by the properties of $F_{d,k,\infty}$, $\text{sing}_{k,\infty}(H_\infty|_Z) = X \times \{0\}$, and H_∞ has no other \mathcal{C}^m -singularities in Z . Since at no point of $\text{cl}(X \times \{0\})$ all k^{th} Peano-derivatives are locally bounded and H_∞ is k -times ordinary differentiable in $\text{cl}(X \times \{0\})$, $\text{sing}_{k,\infty}(H_\infty) = \text{cl}(X \times \{0\})$.

step 3c: $* = \natural$:

For $(x, y) \in Z$ and $|\alpha| \leq k - 1$ it holds true that

$$(423) \quad D_\alpha H_\natural(x, y) = \sum_{\beta \prec \alpha} a_\beta (D_{\alpha-\beta} \psi(x, y)) D_\beta F_{d,k,\natural}(y),$$

where $a_\beta \in R$ with $a_\alpha = 1$. By (412) in connection with (415) and (423), we obtain that for $|\alpha| \leq k - 1$,

$$(424) \quad D_\alpha H_\natural(x, y) \text{ is } o(\text{dist}(x, \partial X)^{m-|\alpha|}).$$

This implies that H_\natural is k -times ordinary differentiable in $\partial X \times \{0\}$.

We show that $\text{sing}_{k,\natural}(H_\natural|_Z) = X \times \{0\}$.

If $|\beta| \leq k - 1$, then $D_\beta F_{d,k,\natural}$ is a continuous function. Hence, by (423), for $|\alpha| = k$ we can write $D_\alpha H_\natural$ as

$$(425) \quad D_\alpha H_\natural(x, y) = f(x, y) + (D_\alpha F_{d,k,\natural}(y)) \psi(x, y)$$

with a continuous function f .

So, the (k, \natural) -singularities of $H_\natural|_Z$ correspond to the (k, \natural) -singularities of $F_{d,k,\natural}$. Hence, by the properties of $F_{d,k,\natural}$, $\text{sing}_{k,\natural}(H_\natural|_Z) = X \times \{0\}$, and H_\natural has no other \mathcal{C}^m -singularities in Z . Since at no point of $\text{cl}(X \times \{0\})$ all k^{th} Peano-derivatives are locally bounded and H_\natural is k -times ordinary differentiable in $\partial X \times \{0\}$, $\text{sing}_{k,\infty}(H_\natural) = \partial X \times \{0\}$.

step 4:

Let $g : X \times R^{n-d} \rightarrow X \times R^{n-d}$ be the function defined by

$$(426) \quad g(x, y) := (x, y - h(x)).$$

g is bijective, and both g and g^{-1} are Λ^m -regular since h is a Λ^m -regular function. Moreover,

$$(427) \quad \text{dist}((x, y), \partial X \times R^{n-d}) = \text{dist}(x, \partial X), \quad (x, y) \in X \times R^{n-d}.$$

We are now able to define $F_* : R^d \times R^{n-d} \rightarrow R$. We put

$$(428) \quad F_*(x, y) := \begin{cases} H_*(g(x, y)), & x \in X \\ 0, & \text{else.} \end{cases}$$

By the same argument as in step 3, we conclude that F_* is m -times continuously differentiable on the set $R^n \setminus \text{cl}(Y)$.

Since g is m -times continuously differentiable,

$$(429) \quad Y = g(X \times \{0\}) \subset \text{sing}_{k,*}(F_*).$$

So, it remains to look at the points of ∂Y .

step 4a: $* = <$:

$F_<$ is m -times Peano- and $(k+1)$ -times ordinary differentiable in ∂Y and in no point of ∂y , all $(k+1)^{\text{th}}$ Peano-derivatives are locally bounded.

In order to show that we obtain by lemma 8.9 and (418) in connection with the remarks of (426) that

$$(430) \quad D_\alpha F_<(x, y) = D_\alpha H_<(g(x, y)) \text{ is } o(\text{dist}(x, \partial X)^{m-|\alpha|}), \quad |\alpha| \leq k, \quad x \in X.$$

So, it is evident that $F_<$ is m -times Peano-differentiable at all points of $\partial Y \subset \partial X \times R^{n-d}$ and that $F_<$ is $(k+1)$ -times ordinary differentiable in ∂Y . In no point of ∂y , the $(k+1)^{\text{th}}$ cannot be all locally bounded since that would imply continuity of the k^{th} derivatives in a neighbourhood of such a point, cf. corollary 7.9, which contradicts (429).

step 4b: $* = \infty$:

F_∞ is m -times Peano- and k -times ordinary differentiable in ∂Y and the k^{th} Peano-derivatives are nowhere locally bounded in ∂Y . In order to show that we obtain by lemma 8.9 and (421) in connection with the remarks of (426) that

$$(431) \quad D_\alpha F_\infty(x, y) = D_\alpha H_\infty(g(x, y)) \text{ is } o(\text{dist}(x, \partial X)^{m-|\alpha|}), \quad |\alpha| \leq k-1, \quad x \in X$$

So, it is evident that F_∞ is m -times Peano-differentiable at all points of $\partial Y \subset \partial X \times R^{n-d}$ and that F_∞ is k -times ordinary differentiable in ∂Y . $\partial Y \subset \text{cl}(Y)$ and the k^{th} Peano-derivatives are not locally bounded at each point of Y . Since being not locally bounded is a closed property, the k^{th} Peano-derivatives are not locally bounded at each point of $\text{cl}(Y)$.

step 4c: $* = \natural$:

F_\natural is m -times Peano- and k -times ordinary differentiable in ∂Y .

In order to show that we obtain by lemma 8.9 and (424) in connection with the remarks of (426) that

$$(432) \quad D_\alpha F_{\mathfrak{h}}(x, y) = D_\alpha H_{\mathfrak{h}}(g(x, y)) \text{ is } o(\text{dist}(x, \partial X)^{m-|\alpha|}), \quad 1 \leq |\alpha| \leq k-1, \quad x \in X.$$

So, it is evident that $F_{\mathfrak{h}}$ is m -times Peano-differentiable in $\partial Y \subset \partial X \times R^{n-d}$ and that $F_{\mathfrak{h}}$ is k -times ordinary differentiable in ∂Y . $\partial Y \subset \text{cl}(Y)$ and the k^{th} Peano-derivatives are not locally bounded at each point of Y . Since being not locally bounded is a closed property, the k^{th} Peano-derivatives are not locally bounded at each point of $\text{cl}(Y)$.

Since a linear change of coordinates is a \mathcal{C}^m operation, we can assume that Y is not necessarily given in coordinates in which it is standard. \square

Applying theorem 4.5 and using proposition 8.10, we can construct definable m -times Peano-differentiable functions with arbitrary and reasonable singularity sets. This shows the usefulness of the concept of Λ^m -regular stratifications.

Proposition 8.11 *Let $\emptyset = A_0 \subset A_1 \subset \dots \subset A_m \subset R^n$ be a sequence of definable sets with $\dim(A_m) \leq n - 2$ and*

$$(433) \quad \text{cl}(A_k) \subset A_{k+1}, \quad k = 1, \dots, m-1.$$

There exists a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ such that

$$(434) \quad \text{sing}^k(F) = A_k, \quad k = 1, \dots, m.$$

Proof: We choose a Λ^m -regular stratification of R^n compatible with the A_1, \dots, A_m and denote by $X_{k,1}, \dots, X_{k,r_k}$ the cells which are contained in A_k , $k = 1, \dots, m$.

By proposition 8.10 (<), for each $X_{k,i}$ there exists a definable m -times Peano-differentiable function $F_{k,i} : R^n \rightarrow R$ such that for $1 \leq k < m$,

$$\begin{aligned} \text{sing}^k(F_{k,i}) &= X_{k,i}, \\ \text{sing}^{k+1}(F_{k,i}) &= \text{cl}(X_{k,i}), \text{ and} \\ \text{sing}^m(F_{k,i}) &= \text{cl}(X_{k,i}). \end{aligned}$$

For $k = m$, $\text{sing}^m(F_{m,i}) = X_{m,i}$.

Hence, the function

$$(435) \quad F = \sum_{k=1}^m \sum_{i=1}^{r_k} F_{k,i}$$

has the desired property. \square

We obtain a consequence of this proposition.

Corollary 8.12 *For each definable subset $A \subset R^n$ with $\dim(A) \leq n - 2$ there exists a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ with*

$$(436) \quad \text{sing}^m(F) = A.$$

Let $1 \leq k \leq m$. With the help of proposition 8.10 we can describe the sets of $(k, *)$ -singularities of definable m -times Peano-differentiable functions.

Corollary 8.13 *Let $A \subset R^n$ be a definable set with $\dim A \leq n - 2$. There is a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ such that*

$$(437) \quad \begin{aligned} \text{sing}_{k, <}(F) = A, \quad \text{sing}_{k+1, \infty}(F) = \partial A, \text{ and} \\ \text{sing}^m(F) = \begin{cases} \text{cl}(A), & k < m \\ A, & k = m. \end{cases} \end{aligned}$$

We can also describe the (k, ∞) - and (k, \natural) -singularity sets.

Corollary 8.14 *Let $A, B \subset R^n$ be disjoint definable subsets with $\dim(A) \leq n - 2$ and $\dim(B) \leq n - 2$. If $A \cup B$ is closed, there is a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ such that*

$$(440) \quad \begin{aligned} \text{sing}_{k, \infty}(F) = A, \quad \text{sing}_{k, \natural}(F) = B \text{ and} \\ \text{sing}^m(F) = A \cup B. \end{aligned}$$

Proof: We choose a Λ^m -regular stratification of R^n compatible with A and B and denote by X_1, \dots, X_r the cells which are contained in A and by X_{r+1}, \dots, X_s those contained in B .

By proposition 8.10 (\natural) and (∞), for each $i = 1, \dots, r$ there is a definable m -times Peano-differentiable function $F_i : R^n \rightarrow R$ such that

$$(441) \quad \text{sing}_{k, \infty}(F_i) = \text{cl}(X_i), \text{ and } \text{sing}^m(F_i) = \text{cl}(X_i);$$

for $j = r + 1, \dots, s$ there is a definable m -times Peano-differentiable function $G_j : R^n \rightarrow R$ such that

$$(442) \quad \text{sing}_{k, \natural}(G_j) = X_j, \text{ sing}_{k, \infty}(G_j) = \partial X_j, \text{ and } \text{sing}^m(G_j) = \text{cl}(X_j).$$

Hence, the function

$$(443) \quad F = \sum_{i=1}^r F_i + \sum_{j=r+1}^s G_j$$

has the desired property. □

As conclusion we see that the necessary conditions for singularity sets are also sufficient. So we have given a characterisation of the singularity sets.

9 Extending Definable Peano-Differentiable Functions

Introduction

In this chapter we prove an extension theorem for definable Peano-differentiable functions defined on a closed set.

As in the chapters before, R denotes a real closed field and we fix an o-minimal expansion \mathcal{M} of R .

We re-define Peano-differentiability in the way it is used in many papers about extending Peano-differentiable functions. Here we are interested in closed sets.

Definition 9.1 *Let $A \subset R^n$ be a set. We call $f : A \rightarrow R$ together with $(f_{[\alpha]})_{|\alpha| \leq m} : A \rightarrow R$ m -times Peano-differentiable on A if*

$$(444) \quad f(x) = \sum_{|\alpha| \leq m} \frac{f_{[\alpha]}(y)}{\alpha!} (x - y)^\alpha + R(x, y) \quad \text{for } x, y \in A$$

where

$$(445) \quad \lim_{x \rightarrow y} \frac{R(x, y)}{\|x - y\|^m} = 0.$$

Using this notion, the definition of the approximation polynomial of f at x_0 remains the same for open sets. In the proof of the chain-rule, c.f. proposition 6.6, we do not use uniqueness of the approximation polynomials. Whenever we compose Peano-differentiable functions with given approximation polynomials we use the approximation polynomial constructed by means of the chain-rule as an approximation polynomial for the composed function.

In classical Analysis, extending Peano-differentiable functions turns out to be very difficult even for the real line.

In [8] there is an example of a 2-times Peano-differentiable function defined on a perfect subset of \mathbb{R} which is not the restriction of a 2-times Peano-differentiable function on \mathbb{R} .

If we assume stronger conditions on a closed subset of \mathbb{R} , we can get extendibility. For example if the subset $A \subset \mathbb{R}$ has finite Denjoy-index, we can extend each m -times Peano-differentiable function f to an m -times Peano-differentiable function F on \mathbb{R} such that f and F coincide on A together with their derivatives, cf.[24]. In this case the class of closed sets is restricted.

Another strategy is to restrict the class of functions which we want to extend as Peano-differentiable functions. The following and very interesting theorem is proved in [26].

Theorem 9.2 *Let $A \subset \mathbb{R}^n$ be a closed set and let f together with the $f_{[\alpha]} : A \rightarrow \mathbb{R}$, $|\alpha| \leq m$, be an m -times Peano-differentiable function.*

If

- (i) *for all multi-indices β , $|\beta| \leq m - 1$, $f_{[\beta]}$ together with the $f_{[\beta+\gamma]}$, $|\gamma| \leq m - |\beta|$, is $(m - |\beta|)$ -times Peano-differentiable, and if*
- (ii) *for all multi-indices β , $|\beta| \leq m$, $f_{[\beta]}$ is a Baire-1-function (i.e. is the pointwise limit of a sequence of continuous function),*

then there is an m -times Peano-differentiable function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $D_\alpha F(x) = f_{[\alpha]}(x)$ for all $x \in A$, $|\alpha| \leq m$.

We should mention that in theorem 9.2, assumption (ii) is not restrictive. In [25], theorem 3 says that the Peano-derivatives of an m -times Peano-differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are necessarily Baire-1-functions.

Theorem 9.2 is optimal in the sense that we do not get more than m -times Peano-differentiability for the extended function, even if the function is definable in an o-minimal structure or the formal derivatives are continuous functions.

Example 9.3 *In \mathbb{R}^2 we consider the sets $A_1 := \{(x, y) \mid y \leq 0 \vee x \geq 0\}$ and $A_2 := \{(x, y) \mid x \geq 0 \wedge y \geq x^4\}$. Then $A = A_1 \cup A_2$ is a closed set. Let*

$$(446) \quad \begin{aligned} f_{[0,0]}|_{A_2}(x, y) &= x^3, \\ f_{[1,0]}|_{A_2}(x, y) &= 3x^2, \text{ and} \\ f_{[2,0]}|_{A_2}(x, y) &= 6x. \end{aligned}$$

Let $f_{[0,1]}|_{A_2}(x, y) = f_{[0,2]}|_{A_2}(x, y) = f_{[1,1]}|_{A_2}(x, y) = 0$. Moreover, we claim that all these functions vanish in A_1 .

Then, $f = f_{[0,0]}$ together with the $f_{[\alpha]}$, $|\alpha| \leq m$, is a 2-times continuously differentiable semialgebraic function.

But there is even no continuously differentiable $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $f|_A \equiv F|_A$.

Proof: We assume that there exists a \mathcal{C}^1 -function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$(447) \quad F(x, y) = f(x, y), \quad (x, y) \in A.$$

For $x \geq 0$ we consider

$$(448) \quad F(x, x^4) - F(x, 0) = f(x, x^4) - f(x, 0) = x^3.$$

Since F is continuously differentiable, we may apply the Mean-Value-Theorem to

$$(449) \quad t \mapsto F(x, t).$$

Hence, there is a $0 < t_x < x^4$ such that

$$(450) \quad F(x, x^4) - F(x, 0) = \frac{\partial F}{\partial y}(x, t_x)(x^4 - 0).$$

Thus,

$$(451) \quad \frac{\partial F}{\partial y}(x, t_x) = \frac{1}{x}.$$

This implies that $\frac{\partial F}{\partial y}$ is not locally bounded at $(0, 0)$, and therefore it is not continuous. \square

This example shows in particular that the concept of k -times ordinary differentiability is not the correct one for closed sets even in o-minimal context. By the example in [8] it is evident that Peano-differentiability also needs several further assumptions to obtain extendibility.

Definition 9.4 *Let $A \subset R^n$ be definable. A definable partition of A is a finite sequence of disjoint definable sets A_1, \dots, A_r such that $A = A_1 \cup \dots \cup A_r$.*

The main goal of this chapter is to prove the following extension theorem for definable m -times Peano-differentiable functions.

Theorem 9.5 *Let $A \subset R^n$ be a closed definable set, and let $f : A \rightarrow R$ together with the definable $(f_{[\alpha]})_{|\alpha| \leq m} : A \rightarrow R$ be a definable m -times Peano-differentiable function. Moreover, there is a definable partition A_1, \dots, A_r of A such that*

(*) *for each $i = 1, \dots, r$ and for all multi-indices β , $1 \leq |\beta| \leq m - 1$, $f_{[\beta]}|_{A_i}$ together with the $f_{[\beta+\gamma]}|_{A_i}$, $|\gamma| \leq m - |\beta|$, is $(m - |\beta|)$ -times Peano-differentiable on A_i .*

Then, there is a definable \mathcal{P}^m -function $F : R^n \rightarrow R$ such that

(I) $D_\alpha F(x) = f_{[\alpha]}(x)$ for $x \in A$, $|\alpha| \leq m$.

(II) F is of class \mathcal{C}^m on $R^n \setminus A$.

The next example shows that we cannot do without the assumption (*).

Example 9.6 Let $A := \{0\} \times R$. Let $f : A \rightarrow R$ and $f_{[\alpha]} : A \rightarrow R$, $|\alpha| \leq 2$ be given by the following data.

$f \equiv f_{[(0,0)]} \equiv f_{[(1,0)]} \equiv f_{[(2,0)]} \equiv f_{[(0,1)]} \equiv f_{[(0,2)]} \equiv 0$ and $f_{[(1,1)]} \equiv 1$. Then f is 2-times Peano-differentiable on A , but it is not the restriction of a 2-times Peano-differentiable function $F : R^2 \rightarrow R$ such that

$$(452) \quad D_\alpha F|_A \equiv f_{[\alpha]}, \quad |\alpha| \leq 2.$$

Proof: For all $(0, a) \neq (0, y) \in A$ we obtain

$$(453) \quad f(0, y) - \sum_{|\alpha| \leq 2} \frac{f_{[\alpha]}(0, a)}{\alpha!} ((0, y) - (0, a))^\alpha = \frac{f_{[(1,1)]}(0, a)}{1!1!} (y - a)^1 0^1 = 0$$

Hence f together with $f_{[\alpha]}$ is 2-times Peano-differentiable on A . We assume that f is the restriction of a definable 2-times Peano-differentiable function $F : R^2 \rightarrow R$. In this case, $D_{(0,1)} D_{(1,0)} F = F_{[(1,1)]}$ on R^2 with the exception of a finite set, cf. theorem 8.5. This implies $D_{(0,1)} f_{[(1,0)]}(0, y) = f_{[(1,1)]}(0, y)$ except for finitely many $y \in R$. Obviously this is not true for any y . \square

On the other side, condition (*) is less restrictive than it seems to be. So for $n = 2$ we obtain by theorem 8.5 for a definable m -times Peano-differentiable function $f : R^2 \rightarrow R$ that this function is \mathcal{C}^m except for a finite set X . So (*) is satisfied on $R^2 \setminus X$, and on each one-element subset $Y \subset X$, $f|_Y$ is trivially \mathcal{C}^2 . So we see that for $n = 2$ (*) is also necessary.

Preliminary Lemmata

We prepare the proof of theorem 9.5 by several lemmata.

The strategy of proving theorem 9.5 is to stratify the set A into finitely many suitable sets such that the $f_{[\alpha]}$ satisfy stronger conditions on each of these sets. These sets are Λ^{3m} -regular cells. We begin with a simple case.

Lemma 9.7 Let $d < n$ be a positive integer, $X \subset R^d$ an open Λ^{3m} -regular cell and let $f : \text{cl}(X) \times \{0\} \rightarrow R$ together with $(f_{[\alpha]})_{|\alpha| \leq m} : \text{cl}(X) \times \{0\} \rightarrow R$ be a definable m -times Peano-differentiable function. If

(i) $X \ni (x_1, \dots, x_d) \mapsto f_{[\alpha]}(x_1, \dots, x_d, 0, \dots, 0)$ is \mathcal{C}^m for $|\alpha| \leq m$,

(ii) for $1 \leq |\beta| \leq m - 1$, $f_{[\beta]}$ together with $f_{[\beta+\gamma]}$, $|\gamma| \leq m - |\beta|$, is $m - |\beta|$ -times Peano-differentiable on $X \times \{0\}$, and

(iii) $f_{[\alpha]}|_{\partial X} \equiv 0$, $|\alpha| \leq m$,

there is for every definable $3m$ -times continuously differentiable function $\rho : R^d \rightarrow R$ with $\text{supp}(\rho) = \text{cl}(X)$ and $\rho|_X > 0$ a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ with the following properties:

$$(454) \quad D_\alpha F(x) = f_{[\alpha]}(x), \quad x \in \text{cl}(X) \times \{0\}, \quad |\alpha| \leq m,$$

$$(455) \quad F \in \mathcal{C}^{3m}(R^n \setminus (\text{cl}(X) \times \{0\})), \text{ and}$$

$$(456) \quad \text{supp}(F) \subset \text{cl}(\{(x, y) \in R^d \times R^{n-d} \mid x \in X \wedge \|y\| < \rho(x)\}).$$

Proof: step 1:

Let $F_1 : X \times R^{n-d} \rightarrow R$ be defined by

$$(457) \quad F_1(x, y) := \sum_{\substack{|\alpha| \leq m \\ \alpha_1 + \dots + \alpha_d = 0}} \frac{f_{[\alpha]}(x, 0)}{\alpha!} y^\alpha.$$

By property (i), we deduce that F_1 is m -times continuously differentiable in $X \times R^{n-d}$, and F_1 is of course definable. Moreover, by property (ii), $D_\alpha F_1(x, 0) = f_{[\alpha]}(x, 0)$ for $|\alpha| \leq m$.

step 2:

Put $\epsilon : X \times (R^{n-d} \setminus \{0\})$, $\epsilon(x, y) = \min(1, \|(0, y)\|^{m+1})$. According to theorem 2.26, we can approximate F_1 on $X \times (R^{n-d} \setminus \{0\})$ with a definable $3m$ -times continuously differentiable function $F_2 : X \times (R^{n-d} \setminus \{0\}) \rightarrow R$, i.e.

$$(458) \quad |D_\alpha(F_1 - F_2)(x, y)| \leq \epsilon(x, y), |\alpha| \leq m.$$

step 3:

Let $\rho_1 : R^d \rightarrow R$ be a definable $3m$ -times continuously differentiable function such that

$$(459) \quad \rho_1|_X > 0,$$

$$(460) \quad \text{supp}(\rho_1) = \text{cl}(X), \text{ and}$$

$$(461) \quad \rho_1(x) < \min\left(\text{dist}(x, \partial X)^{m+1} \left(1 + \sum_{|\alpha| \leq m} |D_\alpha F_1(x)|\right)^{-1}, \rho(x), 1\right), \quad x \in X.$$

Further let $\varphi : [0, \infty) \rightarrow [0, 1]$ be a definable $3m$ -times continuously differentiable function with

$$(462) \quad \varphi|_{[0, \frac{1}{2}]} \equiv 1 \quad \text{and} \quad \varphi|_{[1, \infty)} \equiv 0.$$

step 4:

We now define $F : R^n \rightarrow R$ by

$$(463) \quad F(x, y) := \begin{cases} F_2(x, y) \cdot \varphi\left(\frac{\|y\|^2}{\rho_1(x)^2}\right), & x \in X, y \neq 0 \\ F_1(x, 0), & x \in X, y = 0 \\ 0, & \text{else.} \end{cases}$$

F is evidently definable.

step 5:

We check properties (454), (455) and (456).

Property (456) is evident by the choice of the functions φ and ρ_1 .

Since the function

$$(464) \quad (x, y) \mapsto \varphi\left(\frac{\|y\|^2}{\rho_1(x)^2}\right), \quad (x, y) \in X \times R^{n-d},$$

is $3m$ -times continuously differentiable and F_2 is $3m$ -times continuously differentiable in $X \times (R^{n-d} \setminus \{0\})$, we conclude that $F \in \mathcal{C}^{3m}(R^n \setminus \text{cl}(X \times \{0\}))$. So, property (455) is evident.

For $(x, y) \in X \times R^{n-d}$ with $0 < \|y\| < \frac{1}{\sqrt{2}}\rho_1(x)$, the difference

$$(465) \quad |F(x, y) - F_1(x, y)| = |F_2(x, y) - F_1(x, y)| \leq \|y\|^{m+1}.$$

Hence, $F - F_1$ is m -times Peano-differentiable at all points of $X \times \{0\}$ such that all Peano-derivatives up to order m vanish. Since F_1 is m -times continuously differentiable, F must be m -times Peano-differentiable at each point of $X \times \{0\}$ such that

$$(466) \quad D_\alpha F(x, 0) = D_\alpha F_1(x, 0) = f_{[\alpha]}(x, 0), \quad |\alpha| \leq m.$$

So, it remains to show that F is m -times Peano-differentiable in $\partial X \times \{0\}^{n-d}$ with the zero-polynomial as approximation polynomial.

Let (x, y) be of the interior of $\text{supp}(F)$ with $y \neq 0$. Then $\|y\| < \rho_1(x)$.

$$\begin{aligned}
 (467) \quad |F(x, y)| &= |F_2(x, y)| \varphi\left(\frac{\|y\|^2}{\rho_1^2(x)}\right) \\
 &\leq |F_2(x, y)| \\
 &\leq |F_1(x, y)| + \|y\|^{m+1} \\
 &\leq \sum_{\substack{1 \leq |\alpha| \leq m \\ \alpha_1 + \dots + \alpha_d = 0}} \frac{|f_{[\alpha]}((x, 0))|}{\alpha!} |y^\alpha| + |f((x, 0))| + \|y\|^{m+1} \\
 &\leq \sum_{\substack{1 \leq |\alpha| \leq m \\ \alpha_1 + \dots + \alpha_d = 0}} \frac{|f_{[\alpha]}((x, 0))|}{\alpha!} \rho_1^{|\alpha|}(x) + |f((x, 0))| + \rho_1^{m+1}(x) \\
 &\leq \text{dist}(x, \partial X)^{m+1} + f((x, 0)) + \text{dist}(x, \partial X)^{(m+1)^2}
 \end{aligned}$$

For each $\xi \in \partial X \times \{0\}$,

$$(468) \quad \lim_{(x,y) \rightarrow \xi} \frac{\text{dist}(x, \partial X)^{m+1}}{\|(x, y) - \xi\|^m} = 0 = \lim_{(x,y) \rightarrow \xi} \frac{\text{dist}(x, \partial X)^{(m+1)^2}}{\|(x, y) - \xi\|^m}.$$

By property (iii),

$$(469) \quad \lim_{(x,y) \rightarrow \xi} \frac{f(x, 0)}{\|(x, y) - \xi\|^m} = 0$$

Hence,

$$(470) \quad \lim_{(x,y) \rightarrow \xi} \frac{F(x, y)}{\|(x, y) - \xi\|^m} = 0.$$

□

In order to generalise lemma 9.7 to arbitrary Λ^{3m} -regular cells, we need more information about Λ^1 -regular functions defined on Λ^1 -regular cells.

Lemma 9.8 *Let $X \subset R^n$ be a definable Λ^1 -regular cell of positive dimension with constant C . We can join each pair of points $x \neq y \in X$ by a definable Lipschitz-continuous path $\varphi : [0, \|x - y\|] \rightarrow X$ with constant $(C + 1)^n$.*

Proof: We prove this by induction on n .

The case $n = 1$ is evident.

$n - 1 \rightsquigarrow n$:

Without loss of generality we consider Y in a coordinate system in which it is standard.

We notice that a definable function f of one variable is Lipschitz-continuous with constant L if it is a continuous function and if the derivative of f is bounded by L at the points at which it exists.

case 1:

Let $\dim(Y) = d < n$. Since $x \neq y$ we may assume that $d > 0$. We write $Y = (h)_X$ where $X \subset R^d$ is a definable open Λ^1 -regular cell with constant C and $h : X \rightarrow R^{n-d}$ is a Λ^1 -regular function with constant C .

If $\pi_X(x) = \pi_X(y)$ then

$$(471) \quad x = (\pi_X(x), h(\pi_X(x))) = (\pi_X(y), h(\pi_X(y))) = y$$

which contradicts the assumption that $x \neq y$.

Let $x_0 = \pi_X(x)$ and $y_0 = \pi_X(y)$.

By the assumption of the induction, there is a definable Lipschitz-continuous path $\gamma : [0, \|x_0 - y_0\|] \rightarrow X$ with constant $(C + 1)^d$ joining x_0 and y_0 .

Let $a := \|x_0 - y_0\| / \|x - y\|$. We define $\psi : [0, \|x - y\|] \rightarrow Y$ by

$$(472) \quad \psi(t) = (\gamma(at), h(\gamma(at))).$$

This path obviously joins x and y and it is continuous. For all points at which the derivative of $t \mapsto \gamma(at)$ exists, we can estimate the norm of the derivative by

$$(473) \quad \begin{aligned} \|\psi'(t)\| &= \|a\gamma'(at), a\nabla h(\gamma(at)) \cdot \gamma'(at)\| \\ &\leq a(C + 1)^d + C(C + 1)^d \\ &\leq (C + 1)^n. \end{aligned}$$

case 2:

Let $\dim(Y) = n$. Then $Y = (f, g)_X$ where $X \subset R^{n-1}$ is a definable open Λ^1 -regular cell and f, g are definable Λ^1 -regular functions $f, g : X \rightarrow R$ with constant C .

Let x_0, y_0 and a be defined as in case 1.

By the assumption of the induction there is a definable Lipschitz-continuous path $\gamma : [0, \|x_0 - y_0\|] \rightarrow X$ with constant $(C + 1)^{n-1}$ joining x_0 and y_0 .

Moreover, we assume that for the n^{th} coordinate of y and x , $y_n \geq x_n$ applies.

Let $b := (y_n - x_n) / \|x_0 - y_0\|$, $a := \|x_0 - y_0\| / \|x - y\|$ and let the function $\varphi : [0, \|x - y\|] \rightarrow Y$ be defined by

$$(474) \quad \varphi(t) = (\gamma(at), x_n + abt).$$

We define $\psi : [0, \|x - y\|] \rightarrow Y$ by

$$(475) \quad \psi(t) := \begin{cases} (\gamma(at), f(\gamma(at))), & \text{if } x_n + abt \leq f(\gamma(at)), \text{ (I)} \\ (\gamma(at), g(\gamma(at))), & \text{if } x_n + abt \geq g(\gamma(at)), \text{ (II)} \\ \varphi(t) = (\gamma(at), x_n + abt), & \text{else.} \end{cases}$$

If $b > C(C + 1)^{n-1}$, then

$$(476) \quad g(\gamma(at)) > y_n - b(t - \|x - y\|) \text{ and } f(\gamma(at)) < x_n + tab.$$

Hence, $\varphi(t) \in Y$ for all t . Moreover, at the points where $t \mapsto \gamma(at)$ is differentiable,

$$(477) \quad \|\varphi'(t)\| \leq a \|\gamma'(at)\| + ab \leq (C + 1)^{n-1} + C(C + 1)^{n-1} \leq (C + 1)^n.$$

If $b \leq C(C + 1)^{n-1}$, it can happen that (I) or (II) applies. This situation is analogous to (472) and (473).

Hence, for all points t at which the derivative of ψ exists it holds true that

$$(478) \quad \|\psi'(t)\| \leq (C + 1)^n.$$

□

Corollary 9.9 *Let $X, Y \subset R^n$ be open Λ^1 -regular cells and $\psi : X \rightarrow Y$ be a Λ^1 -regular mapping. Then ψ is Lipschitz-continuous.*

Proof: According to lemma 9.8, there is a constant C such that we can join each pair of points $x_1, x_2 \in X$ by a Lipschitz-continuous curve $\varphi_{x_1, x_2} : [0, \|x_1 - x_2\|] \rightarrow X$ with constant C . If ψ is Λ^1 -regular with constant D ,

$$(479) \quad \|\psi(x_2) - \psi(x_1)\| = \|\psi(\varphi_{x_1, x_2}(\|x_1 - x_2\|)) - \psi(\varphi_{x_1, x_2}(0))\| \leq CD \|x_2 - x_1\|$$

□

We generalise lemma (9.7) to arbitrary Λ^{3m} -regular cells of dimension $d < n$.

Lemma 9.10 *Let Y be a Λ^{3m} -regular cell of dimension $d < n$ and U a definable open neighbourhood of Y . Let $f : \text{cl}(Y) \rightarrow R$ together with $(f_{[\alpha]})_{|\alpha| \leq m}$ be a definable m -times Peano-differentiable function with the following properties:*

(i') $f_{[\alpha]} \in \mathcal{C}^{2m}(Y)$, $|\alpha| \leq m$,

(ii') for $1 \leq |\beta| \leq m - 1$, $f_{[\beta]}|_Y$ together with $f_{[\beta+\gamma]}|_Y$, $|\gamma| \leq m - |\beta|$, is $m - |\beta|$ -times Peano-differentiable on Y , and

(iii') $f_{[\alpha]}(\eta) = 0$, $\eta \in \partial Y$.

Then there is a definable m -times Peano-differentiable function $F : R^n \rightarrow R$ such that

$$(480) \quad D_\alpha F(\eta) = f_{[\alpha]}(\eta), \quad \eta \in \text{cl}(Y), \quad |\alpha| \leq m$$

$$(481) \quad \text{supp}(F) \subset \text{cl}(U),$$

$$(482) \quad F|_{R^n \setminus \text{cl}(Y)} \text{ is } 3m\text{-times continuously differentiable.}$$

Proof: We examine the validity for the special case $d = 0$. In this case, $\partial Y = \emptyset$ and $Y = \{y_0\}$. We choose a definable \mathcal{C}^{3m} -map $\varphi : R \rightarrow R$ which vanishes outside $(-1, 1)$ and equals 1 in $(-1/2, 1/2)$. Moreover, let $\epsilon > 0$ and let it be that small that $B_\epsilon(y_0) \subset U$. Then $F : R^n \rightarrow R$, defined by

$$(483) \quad F(x) := \varphi\left(1 - \frac{1}{\epsilon} \|x - y_0\|^{3m+1}\right) \sum_{|\alpha| \leq m} \frac{f_{[\alpha]}(y_0)}{\alpha!} (x - y_0)^\alpha,$$

satisfies the conditions (480), (481) and (482).

Let $d > 0$.

step 1:

Let $Y = (h)_X$ where $X \subset R^d$ is an open definable Λ^{3m} -regular cell and $h : X \rightarrow R^{n-d}$ a Λ^{3m} -regular mapping.

We define the function $\psi : X \times R^{n-d} \rightarrow X \times R^{n-d}$ by

$$(484) \quad \psi(x, y) := (x, y + h(x)).$$

This function is Λ^{3m} -regular on $X \times R^{n-d}$ and bijective with Λ^{3m} -regular inverse. We use this function to reduce our problem to the situation assumed in lemma 9.7. As a suitable function we choose $G : X \times \{0\} \rightarrow R$ together with the functions $(G_{[\alpha]})_{|\alpha| \leq m} : X \times \{0\} \rightarrow R$ given in (485) and (486).

For $\xi_0 \in X \times \{0\}$ let

$$(485) \quad G(\xi_0) = f(\psi(\xi_0)) \text{ and } p_{G, \xi_0} = p_{f \circ \psi, \xi_0}$$

where the $G_{[\alpha]}$ are defined through the identity

$$(486) \quad \sum_{|\alpha| \leq m} \frac{G_{[\alpha]}(\xi_0)}{\alpha!} (\xi - \xi_0)^\alpha = p_{G, \xi_0}(\xi - \xi_0).$$

step 2:

We extend the $G_{[\alpha]}$ to the closure of X .

$$(487) \quad G_{[\alpha]}(x, 0) := \begin{cases} G_{[\alpha]}(x, 0), & \text{if } x \in X \\ 0, & \text{if } x \in \partial X \end{cases}$$

We now show that G together with the $G_{[\alpha]}$ satisfies the conditions of lemma 9.7.

According to the chain-rule, cf. proposition 6.6, $G|_{X \times \{0\}}$ is an m -times Peano-differentiable function. So, for $\xi_0 \in X \times \{0\}$,

$$(488) \quad p_{f \circ \psi, \xi_0}(\xi) = \left(\sum_{|\alpha| \leq m} \frac{f_{[\alpha]}(\psi(\xi_0))}{\alpha!} \left(\sum_{|\beta| \leq m} \frac{\psi_{[\beta]}(\xi_0)}{\beta!} (\xi - \xi_0)^\beta \right)^\alpha \right)_{(m)}$$

applies (the index m signifies that we set all coefficients of degree greater than m zero). The $f_{[\alpha]}$ and $\psi_{[\beta]}$, $|\alpha| \leq m$ and $|\beta| \leq m$, are m -times continuously differentiable functions. Hence, the coefficients of $p_{f \circ \psi, \xi_0}$ are m -times continuously differentiable functions and therefore, by (485) and (486), the $G_{[\alpha]}$ are m -times continuously differentiable functions in $X \times \{0\}$. This is condition (i) of lemma 9.7. Condition (ii) of lemma 9.7 is a consequence of (ii')

and the fact that ψ is a (definable) \mathcal{C}^m -function on an open set. It remains to show that G is Peano-differentiable at the points $\xi_0 \in \partial X \times \{0\}$, i.e. we must show that

$$(489) \quad \lim_{X \times \{0\} \ni \xi \rightarrow \xi_0} \frac{G(\xi)}{\|\xi - \xi_0\|^m}, \quad \xi_0 \in \partial X \times \{0\}.$$

By equation (485), we obtain $G(x, 0) = f \circ \psi(x, 0)$ for $x \in X$. According to corollary 9.9, ψ is Lipschitz-continuous with some constant $L > 0$ so that ψ extends continuously to $\text{cl}(X) \times R^{n-d}$. Hence, $Y \ni \psi(\xi) \rightarrow \psi(\xi_0) \in \partial Y$ when $X \times \{0\} \ni \xi \rightarrow \xi_0 \in \partial X \times \{0\}$. Now,

$$(490) \quad \begin{aligned} \lim_{\xi \rightarrow \xi_0} \left| \frac{G(\xi)}{\|\xi - \xi_0\|^m} \right| &= \lim_{\xi \rightarrow \xi_0} \left| \frac{f \circ \psi(\xi)}{\|\xi - \xi_0\|^m} \right| \\ &\leq \lim_{\xi \rightarrow \xi_0} \left| \frac{f \circ \psi(\xi) L^m}{\|\psi(\xi) - \psi(\xi_0)\|^m} \right| \\ &\leq \lim_{\psi(\xi) \rightarrow \psi(\xi_0)} \left| \frac{f \circ \psi(\xi) L^m}{\|\psi(\xi) - \psi(\xi_0)\|^m} \right| \\ &= 0. \end{aligned}$$

step 3:

G together with the $G_{[\alpha]}$ satisfies the conditions of lemma 9.7 for $\text{cl}(X) \times \{0\}$.

Let $V := \psi^{-1}(U)$. There is a definable $3m$ -times continuously differentiable function $\rho : R^d \rightarrow R$ with $\text{supp}(\rho) = \text{cl}(X)$ and $\rho > 0$ on X such that $W := \{(x, y) \in R^n \mid \|y\| < \rho(x), x \in X\} \subset V$.

Lemma 9.7 provides us with a definable m -times Peano-differentiable function $H : R^n \rightarrow R$ such that

$$(491) \quad \text{supp}(H) \subset \text{cl}(W),$$

$$(492) \quad H \text{ is } 3m\text{-times continuously differentiable outside } \text{cl}(X) \times \{0\}, \text{ and}$$

$$(493) \quad D_\alpha H(x, 0) = G_{[\alpha]}(x, 0), \quad x \in \text{cl}(X), \quad |\alpha| \leq m.$$

step 4

The function $F : R^n \rightarrow R$ defined by

$$(494) \quad F(x, y) = \begin{cases} H \circ \psi^{-1}(x, y), & (x, y) \in \psi(W) \\ 0, & \text{else} \end{cases}$$

is a function with the desired properties.

step 4a: $\text{supp}(F) \subset \text{cl}(U)$:

This is easily seen by

$$(495) \quad \text{supp}(F) \subset \text{cl}(\psi(W)) \subset \text{cl}(\psi(\psi^{-1}(U))) = \text{cl}(U).$$

step 4b: F is $3m$ -times continuously differentiable outside $\text{cl}(Y)$.

Since $F \equiv 0$ on $R^n \setminus \text{cl}(U)$, it is $3m$ -times continuously differentiable in $R^n \setminus \text{cl}(U)$. Furthermore, by (492), F is $3m$ -times continuously differentiable in $X \times R^{n-d} \setminus Y$. Hence, F is $3m$ -times continuously differentiable in $R^n \setminus \text{cl}(Y)$.

step 4c: F is m -times Peano-differentiable in $\text{cl}(Y)$.

It is obvious that F is m -times Peano-differentiable in Y by step 3.

We now concentrate on ∂Y . Let $\eta \in \partial Y$. We show that F is m -times Peano-differentiable at η with vanishing approximation polynomial.

Since ψ^{-1} is Lipschitz-continuous with a constant L , we can extend it continuously to $\text{cl}(Y)$. Let $\xi := \psi^{-1}(\eta)$ and $\epsilon > 0$. Since H is m -times Peano-differentiable at ξ with the zero-polynomial as approximation polynomial, there is a $\delta > 0$ such that

$$(496) \quad |H(w) - H(\xi)| < \frac{\epsilon}{L^m} \|w - \xi\|^m, \quad w \in B_{\delta \cdot L}(\xi) \cap W.$$

Let $y \in B_\delta(\eta)$.

If $y \in R^n \setminus \psi(W)$, then $F(y) - F(\eta) = 0$ and there is nothing to prove.

Otherwise, $\psi^{-1}(y) \in W \cap B_{\delta,L}(\xi)$ and we obtain the following inequation:

$$\begin{aligned}
 (497) \quad |F(y) - F(\eta)| &= |H \circ \psi^{-1}(y) - H \circ \psi^{-1}(\eta)| \\
 &\leq \frac{\epsilon}{L^m} \|\psi^{-1}(y) - \psi^{-1}(\eta)\|^m \\
 &\leq \frac{\epsilon}{L^m} L^m \|y - \eta\|^m \\
 (498) \quad &= \epsilon \|y - \eta\|^m
 \end{aligned}$$

step 4d: $D_\alpha F(y) = f_{[\alpha]}(y)$, $y \in \text{cl}(Y)$, $|\alpha| \leq m$.

This follows immediately if we consider the corresponding approximation polynomials.

$$(499) \quad p_{F,y} = p_{H \circ \psi^{-1},y} = p_{(f \circ \psi) \circ \psi^{-1},y} = p_{f,y}$$

□

Proof of Theorem 9.5

Suppose that the statement of theorem 9.5 holds true for definable sets of dimension less than n . If the closed definable set $A \subset R^n$ has nonempty interior $\text{int}(A)$, we choose a definable m -times Peano-differentiable function $\tilde{f} : R^n \rightarrow R$ extending f together with the $f_{[\alpha]}$ restricted to $A \setminus \text{int}(A)$. Then $F : R^n \rightarrow R$ defined by $F := f$ on A and $F := \tilde{f}$ on $R^n \setminus A$ is definable and m -times Peano-differentiable everywhere such that the Peano-derivatives of F coincide with the $f_{[\alpha]}$ on A .

So we have to show the validity of theorem 9.5 for definable sets $A \subset R^n$ with $\dim(A) \leq n - 1$.

step 1:

Using theorem 4.5 we choose a Λ^{3m} -regular stratification S_1, \dots, S_q of R^n compatible with A , A_1, \dots, A_r , and compatible with the sets on which each Peano-derivative up to order m is an $2m$ -times continuously differentiable function. Furthermore we assume that $\dim(S_j) \leq \dim(S_{j+1})$.

For $j = 1, \dots, q$ put $T_j := \bigcup_{i \leq j} S_i$.

Since the S_l , $l < j$ are a strata and since all strata of dimension less than the dimension of T_j are contained in T_j , T_j is always a closed set.

We equip each S_j with a suitable neighbourhood

$$(500) \quad U_j := \{x \in R^n \mid \text{dist}(x, T_{j-1}) > \text{dist}(x, S_j)\}.$$

The U_j and T_{j-1} have the following obvious relationship:

$$(501) \quad \text{cl}(U_j) \cap \text{cl}(T)_{j-1} \subset \partial U_j$$

step 2:

We now construct the function $F : R^n \rightarrow R$ inductively using the sequences of helping functions $(H_{[\alpha]}^i)_{i=0, \dots, q-1} : A \rightarrow R$ and $(F^i)_{i=1, \dots, q} : R^n \rightarrow R$.

For $|\alpha| \leq m$ we put $H_{[\alpha]}^0 = f_{[\alpha]}$ so that each $H_{[\alpha]}^0|_{S_1}$ is $2m$ -times continuously differentiable.

According to lemma 9.10 applied to $H^0|_{S_1}$ together with its Peano-derivatives and with R^n as open neighbourhood of S_1 , we obtain a definable m -times Peano-differentiable function $F^1 : R^n \rightarrow R$ which is $3m$ -times continuously differentiable in $R^n \setminus T_1$ and satisfies

$$(502) \quad D_\alpha F^1 = H_{[\alpha]}^0 \text{ on } S_1, \quad |\alpha| \leq m.$$

We put $H_{[\alpha]}^1 := H_{[\alpha]}^0 - D_\alpha F^1|_A$, $|\alpha| \leq m$, so that the $H_{[\alpha]}^1 \equiv 0$ on T_1 and that the $H_{[\alpha]}^1$ are $2m$ -times continuously differentiable on each of the S_i .

We now construct the $H_{[\alpha]}^j$ and F^j , provided that the $H_{[\alpha]}^{j-1}$ is given with the following data:

$$(503) \quad H_{[\alpha]}^{j-1} \equiv 0 \text{ on } T_{j-1},$$

$$(504) \quad H_{[\alpha]}^{j-1} \text{ is } 2m\text{-times continuously differentiable in } S_i, \quad i = 1, \dots, q.$$

$H_{[\alpha]}^{j-1}$ satisfies the conditions of lemma 9.10 on S_j . As neighbourhood of S_j we choose U_j .

Now, by lemma 9.10, we obtain a definable m -times Peano-differentiable $F^j : R^n \rightarrow R$ such that

$$(505) \quad \text{supp}(F^j) \subset \text{cl}(U_j),$$

$$(506) \quad D_\alpha F^j \equiv H_{[\alpha]}^{j-1} \text{ on } S_j, \text{ and}$$

$$(507) \quad F^j|_{R^n \setminus T_j} \text{ is } 3m\text{-times continuously differentiable.}$$

Because of (501) and (507), it holds true that $D_\beta F^j \equiv 0$ on T_{j-1} for $|\beta| \leq 3m$. For every $|\alpha| \leq m$ we put

$$(508) \quad H_{[\alpha]}^j := H_{[\alpha]}^{j-1} - D_\alpha F^j \text{ on } A$$

and receive

$$(509) \quad H_{[\alpha]}^j \equiv 0 \text{ on } T_j,$$

$$(510) \quad H_{[\alpha]}^j \text{ is } 2m\text{-times continuously differentiable in } S_i, \quad i = 1, \dots, q.$$

step 3: Now it is easy to see that

$$(511) \quad F := \sum_{i=1}^q F^i$$

is a possible function we have been looking for.

Since each F^i is m -times continuously differentiable outside T_q , F is m -times continuously differentiable outside A . Moreover, for $x \in A$

$$(512) \quad f_{[\alpha]}(x) - \sum_{i=1}^q D_{\alpha} F^i(x) = H_{[\alpha]}^{j-1}(x) - \sum_{i=j}^q D_{\alpha} F^j(x) = H_{[\alpha]}^{q-1}(x) - D_{\alpha} F^q(x) = 0,$$

by the definition of the $H_{[\alpha]}^i$. Hence, theorem 9.5 is evident. \square

References

- [1] J. M. Ash *A characterization of the Peano derivative* Trans. Amer. Math. Soc. 149 (1970) 489–501
- [2] V. Aversa, M. Laczkovich, D. Preiss *Extension of differentiable functions* Comment. Math. Univ. Carolin. 26 (1985), no. 3, 597-609.
- [3] B. S. Babcock *On properties of the approximate Peano derivatives* Trans. Amer. Math. Soc. 212 (1975), 279–294
- [4] M. Barner, F. Flohr *Analysis I* de Gruyter Lehrbuch Walter de Gruyter & Co. Berlin - New York 1991
- [5] J. Bochnak, M. Coste, M.-F. Roy *Real Algebraic Geometry* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) Springer Verlag Berlin - Heidelberg 1998
- [6] A. Berarducci, M. Otero *Intersection theory for o-minimal manifolds* Annals of Pure and Applied Logic 107 (2001) 87-119
- [7] Z. Buczolich *An existence theorem for higher Peano derivatives in R^m* Real Anal. Exchange 13 (1987/88), no. 1, 245–252
- [8] Buczolich, Z. *Second Peano derivatives are not extendable* Real Anal. Exchange 14 (1988/89), no. 2, 423-428
- [9] Z. Buczolich, C. E. Weil *The non-coincidence of ordinary and Peano derivatives* Math. Bohem. 124 (1999), no. 4, 381-399
- [10] P. S. Bullen, S. N. Mukhopadhyay *On the Peano derivatives* Canad. J. Math. 25 (1973), 127-140
- [11] P. S. Bullen, S. N. Mukhopadhyay *Peano derivatives and general integrals* Pacific J. Math. 47 (1973), 43-58
- [12] P. S. Bullen, S. N. Mukhopadhyay *The Peano derivative and the M_2 property of Zahorski* Indian J. Math. 28 (1986), no. 3, 219-228 (1987)
- [13] P. S. Bullen, S. N. Mukhopadhyay *Properties of Baire*-1 Darboux functions and some mean value theorems for Peano derivatives* Math. Japon. 36 (1991), no. 2, 309-316

- [14] S. Chanillo, R. L. Wheeden *Relations between Peano derivatives and Marcinkiewicz integrals* Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 508-525, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983
- [15] G. E. Cross *The integration of exact Peano derivatives* Canad. Math. Bull. 29 (1986), no. 3, 334-340
- [16] A. Denjoy *Sur l'integration des coefficients differentiels d'ordre suprieur* Fundamentae Mathematicae 25 (1935), p. 273-326
- [17] J. Escribano *Approximation Theorems in o-minimal structures* Illinois Journal of Mathematics 46(1), 2002, 111-128
- [18] M. J. Evans *L_p derivatives and approximate Peano derivatives* Trans. Amer. Math. Soc. 165 (1972), 381-388
- [19] M. J. Evans, C. E. Weil *Peano derivatives: a survey* Real Anal. Exchange 7 (1981/82), no. 1, 5-23
- [20] M. J. Evans *Approximate Peano derivatives and the Baire* one property* The ninth summer real analysis symposium (Louisville, Ky., 1985). Real Anal. Exchange 11 (1985/86), no. 1, 283-289
- [21] H. Fejzić *Decomposition of Peano derivatives* Proc. Amer. Math. Soc. 119 (1993), no. 2, 599-609
- [22] H. Fejzić *On generalized Peano and Peano derivatives* Fund. Math. 143 (1993), no. 1, 55-74
- [23] H. Fejzić *On approximate Peano derivatives* Acta Math. Hungar. 65 (1994), no. 4, 319-332
- [24] H. Fejzić, J. Mařík, C. Weil *Extending Peano derivatives* Math. Bohem. 119 (1994), no. 4, 387-406
- [25] H. Fejzić, D. Rinne *Continuity Properties of Peano Derivaives in Several Variables* Real Analysis Exchange 21(1) (1995/6), p. 292-298
- [26] H. Fejzić, D. Rinne, C. Weil *Extending n times differentiable functions of several variables* Czechoslovak Math. J. 49(124) (1999), no. 4, 825-830
- [27] H. Fejzić *Infinite approximate Peano derivatives* Proc. Amer. Math. Soc. 131 (2003), no. 8, 2527-2536

- [28] O. Forster *Analysis 3* Vieweg Studium 52, Vieweg Braunschweig 1984
- [29] M. Gromov *Entropy, homology and semialgebraic geometry* Seminaire Bourbaki, 38eme annee, 1985-86, no. 663; Asterisque 145-146 (1987), 225-240
- [30] J. Hejduk *On the Peano derivatives of functions having the Baire property* Demonstratio Math. 31 (1998), no. 3, 663-668
- [31] M. Knebusch, C. Scheiderer *Einführung in die reelle Algebra* Vieweg Studium 63, Vieweg Braunschweig - Wiesbaden, 1989
- [32] J. Knight, A. Pillay, C. Steinhorn *Definable sets in ordered structures. II.* Trans. Amer. Math. Soc. 295 (1986), no. 2, 593-605
- [33] K. Kurdyka *On a subanalytic stratification satisfying a Whitney-Property with exponent 1* Proceeding Conference Real Algebraic Geometry - Rennes 1991, Springer LNM 1524 (1992), 316-322.
- [34] K. Kurdyka, A. Parusinski *Quasikonvex Decomposition in o-minimal structures. Application to the Gradient Conjecture*, preprint, available at <http://math.univ-angers.fr/preprint/138.html>
- [35] K. Kurdyka, W. Pawlucki *Subanalytic version of Whitney's extension theorem* Studia Mathematica 124 (3), (1997)
- [36] M. Laczkovich *On the absolute Peano derivatives* Ann. Univ. Sci. Budapest. Etsv Sect. Math. 21 (1978), 83-97 (1979)
- [37] M. Laczkovich, D. Preiss, C. E. Weil *Infinite Peano derivatives* The ninth summer real analysis symposium (Louisville, Ky., 1985). Real Anal. Exchange 11 (1985/86), no. 1, 72-74
- [38] M. Laczkovich, D. Preiss, C. E. Weil *On unilateral and bilateral n th Peano derivatives* Proc. Amer. Math. Soc. 99 (1987), no. 1, 129-134
- [39] M. Laczkovich *Infinite Peano derivatives* Real Anal. Exchange 26 (2000/01), no. 2, 811-825
- [40] S. Lasher *On Peano derivatives in $L^p(E_n)$* Studia Math. 29 1967/1968 195-201
- [41] C. M. Lee, R. J. O'Malley *The second approximate derivative and the second approximate Peano derivative* Bull. Inst. Math. Acad. Sinica 3 (1975), no. 2, 193-197

- [42] C. M. Lee *On the approximate Peano derivatives* J. London Math. Soc. (2) 12 (1975/76), no. 4, 475-478
- [43] C. M. Lee *On functions with summable approximate Peano derivative* Proc. Amer. Math. Soc. 57 (1976), no. 1, 53-57
- [44] C. M. Lee *Regular approximate Peano derivatives are the ordinary ones* Bull. Inst. Math. Acad. Sinica 10 (1982), no. 4, 401-404
- [45] C. M. Lee *On absolute Peano derivatives* Real Anal. Exchange 8 (1982/83), no. 1, 228-243
- [46] C. M. Lee *On generalized Peano derivatives* Trans. Amer. Math. Soc. 275 (1983), no. 1, 381-396
- [47] C. M. Lee *On generalizations of exact Peano derivatives and integrals via the coefficient problems of convergent trigonometric series* Seventh symposium on real analysis (Santa Barbara, Calif., 1984). Real Anal. Exchange 9 (1983/84), no. 2, 369-378
- [48] J. Marik *Derivatives and closed sets* Acta Math. Hungar. 43 (1984), no. 1-2, 25-29
- [49] D. Marker *Model Theory: An Introduction* Graduate Texts in Mathematics, 217, Springer New York, 2002
- [50] C. Miller, L. van den Dries *Geometric categories and o-minimal structures* Duke Math. J. 84, 497-540 (1996)
- [51] S. N. Mukhopadhyay *On the approximate Peano derivatives* Fund. Math. 88 (1975), no. 2, 133-143
- [52] S. N. Mukhopadhyay, S. Mitra *Measurability of Peano derivatives and approximate Peano derivatives* Real Anal. Exchange 20 (1994/95), no. 2, 768-775
- [53] S. N. Mukhopadhyay, S. Ray *On extending Peano derivatives* Acta Math. Hungar. 89 (2000), no. 4, 327-346
- [54] H. W. Oliver *The exact Peano derivative* Trans. AMS 76, (1954), p. 444-456
- [55] R. J. O'Malley *The second Peano derivative as a composite derivative* Eighth symposium on real analysis (Northfield, Minn., 1984). Real Anal. Exchange 10 (1984/85), no. 1, 17-20

- [56] R. J. O'Malley *The second Peano derivative as a composite derivative* Fund. Math. 132 (1989), no. 2, 99–103
- [57] A. Parusiński *Lipschitz stratification of subanalytic sets* Ann. Sci. cole Norm. Sup. (4) 27 (1994), no. 6, 661–696
- [58] G. Peano *Sulla formula di Taylor* Atti Accad. Sci. Torino 27 (1891/92), p. 4046
- [59] H. W. Pu, H. H. Pu *On iterated Peano derivatives* Bull. Inst. Math. Acad. Sinica 7 (1979), no. 3, 323–328
- [60] R. E. Svetic, H. Volkmer *On the ultimate Peano derivative* J. Math. Anal. Appl. 218 (1998), no. 2, 439–452
- [61] L. van den Dries *o -minimal structures* Logic: from foundations to applications (Staffordshire, 1993), 137–185, Oxford Sci. Publ., Oxford Univ. Press, New York, 1996
- [62] L. van den Dries *Tame Topology and O -minimal Structures* LMS Lecture Notes 248, Cambridge University Press 1998
- [63] S. Verblunsky *On the Peano derivatives* Proc. London Math. Soc. (3) 22 1971 313–324
- [64] H. Volkmer *Extending Peano derivatives: necessary and sufficient conditions* Fund. Math. 159 (1999), no. 3, 219–229
- [65] C. E. Weil *On approximate and Peano derivatives* Proc. Amer. Math. Soc. 20 1969 487–490
- [66] C. E. Weil *The Peano derivative: what's known and what isn't* Seventh symposium on real analysis (Santa Barbara, Calif., 1984). Real Anal. Exchange 9 (1983/84), no. 2, 354–365
- [67] H. Whitney *Analytic extensions of differentiable functions defined in closed sets* Trans AMS 36, (1934), 63–89
- [68] A. J. Wilkie *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function* J. Amer. Math. Soc. 9 (1996), no. 4, 1051–1094.
- [69] I. B. Zibman *Some characterizations of the n -dimensional Peano derivative* Studia Math. 63 (1978), no. 1, 89–110

Index

- approximation
 - theorem of Escibano, 17
- approximation polynomial, 83
- cell
 - C^p -cell, 15
 - ϵ -flat, 36
 - Λ^p -regular, 29
- cell decomposition, 15
- Chain-Rule, 92
- connected
 - definably, 13
 - definably path-, 13
- Curve-Selection-Lemma, 18
- curves
 - continuously differentiable, 69
 - ordinary differentiable, 79
- definable, 9
- derivative
 - directional, 101
 - estimates for definable, 22
 - locally bounded, 72
 - Peano-, 84
 - locally bounded, 97
- differentiable
 - Gateaux, 63
 - ordinary, 56
 - partially, 14
 - Peano-, 83
- dimension, 16
- field
 - ordered, 9
 - real closed, 9
- formula, 8
- function
 - Baire-1, 123
 - Λ^p -regular, 28
 - ordinary differentiable of one variable, 61
 - Peano-differentiable
 - Extension of, 124
- Grassmannian, 30
- Implicit-Function-Theorem, 59
- Intermediate-Value-Theorem, 13
- Łojasiewicz inequality, 17
- Mean-Value-Theorem, 14
- Monotonicity-Theorem, 12
- o-minimal expansion, 11
- R -metric, 12
- semialgebraic, 10
- singularity
 - C^m -, 103
 - (k, \natural) -, 109
 - (k, ∞) -, 109
 - $(k, <)$ -, 109
- stratification
 - Λ^p -regular, 29
- structure, 7
 - o-minimal, 10
- submanifold
 - ϵ -flat, 35
- term, 8
- transfer-principle, 10