



**Asymptotic Expansion
for the Time Evolution of the
Probability Distribution Given by
the Brownian Motion on
Semialgebraic Sets**

Julia Ruppert

Dissertation

Submitted to the Faculty of Computer Science and Mathematics
of the University of Passau in Partial Fulfilment of the Requirements
for the Degree Doctor Rerum Naturalium

May 2017

Supervisor: **Prof. Dr. Tobias Kaiser** Professorship of Mathematics,
Faculty of Computer Science
and Mathematics,
University of Passau

Acknowledgements

I am not one for using big words, but however, I would like to give a special thank to some people.

First of all, I would like to thank my supervisor, Prof. Dr. Tobias Kaiser, for accepting me as a PhD student, for supporting me in all matters, whether scientific or otherwise, and especially for his open-door policy. Furthermore, I would like to express my gratitude to my colleagues, who became friends over the years and made the time at the University and of this PhD unforgettable. A special thank goes to my friend and officemate Dr. Sabrina Lehner. Even in stressful times we laughed ourselves to tears.

Last but not least, I thank my parents, my partner, my whole family, and my friends for supporting me and believing in me all these years.

Thank you for your encouragement in every respect!

Julia Ruppert

Passau, May 2017

Notation

\mathbb{N}_0	$:= \{0, 1, 2, 3, \dots\}$	set of natural numbers including 0
\mathbb{N}	$:= \{1, 2, 3, \dots\}$	set of natural numbers
\mathbb{Z}		set of integers
$\mathbb{Z}_{\leq 0}$	$:= \{x \in \mathbb{Z} \mid x \leq 0\}$	set of negative integers with 0
\mathbb{Q}		field of rational numbers
\mathbb{R}		field of real numbers
$\mathbb{R}_{\geq 0}$	$:= \{x \in \mathbb{R} \mid x \geq 0\}$	set of positive real numbers with 0
$\overline{\mathbb{R}}$	$:= \mathbb{R} \cup \{-\infty, +\infty\}$	
$\mathbb{R}[x]$		polynomial ring in x over \mathbb{R}
\mathbb{R}		o-minimal structure formed by the semialgebraic sets
\mathbb{R}_{an}		o-minimal structure; \mathbb{R} expanded by restricted analytic functions
\mathbb{R}_{exp}		o-minimal structure; \mathbb{R} expanded by the exponential function
$\mathbb{R}_{\text{an,exp}}$		o-minimal structure; \mathbb{R} expanded by restricted analytic functions and the exponential function
$\mathcal{P}(\mathcal{M})$		Pfaffian closure of the o-minimal structure \mathcal{M}
\mathcal{C}^1		class of continuously differentiable functions

e^x		exponential function $\exp(x)$
$\log(x)$		natural logarithm
$\operatorname{erf}(x)$	$:= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$	(Gauss) error function
$\operatorname{erfc}(x)$	$:= 1 - \operatorname{erf}(x)$	complementary error function
$\Gamma(x)$	$:= \int_0^\infty t^{x-1} e^{-t} dt$	gamma function
$\Gamma'(x)$	$:= \int_0^\infty t^{x-1} e^{-t} \log(t) dt$	first derivative of $\Gamma(x)$
$\Gamma(x, y)$	$:= \int_y^\infty t^{x-1} e^{-t} dt$	incomplete gamma function
$\Theta(r, \varphi)$		polar coordinate transformation map
$n!$	$:= 1 \cdot 2 \cdot 3 \cdots n$	factorial function
γ	$\approx 0,5772156649$	Euler-Mascheroni constant
$f \approx g$		g is an asymptotic (series) expansion of f
$f \sim g$		f is asymptotically similar to g
$f = \mathcal{O}(g)$		$ f $ is bounded above by g asymptotically
$f = o(g)$		f is dominated by g asymptotically
(Ω, \mathcal{F}, P)		probability space
μ		mean
σ^2		variance
$\mathcal{N}(\mu, \sigma^2)$		normal distribution
$\delta_x(A)$		Dirac measure
$(X_t)_{t \geq 0}$		stochastic process
B_t		Brownian Motion at time t
\bar{A}		closure of set A
A_x	$:= \{y \in \mathbb{R}^n \mid (x, y) \in A\}$	family of sets
$S _N(x)$		partial sum up to the first summand with order equal or greater than N of series $S(x)$

Contents

Acknowledgements	i
Notation	iii
1 Introduction	1
1.1 Motivation	1
1.2 Outline and Results	5
2 Preliminaries	9
2.1 Basic Definitions and Theorems	9
2.2 Asymptotic Expansions	17
2.3 O-minimal Structures	21
2.4 The Brownian Motion	28
3 Definability and Asymptotic Expansion for the Time Evolution of the Probability Distribution Given by the Brownian Motion on Semialgebraic Sets	33
3.1 Definability in the One-Dimensional Case	35
3.2 The Two-Dimensional Case	40
3.2.1 Preparations	40
3.2.2 Definability and Asymptotic Expansions	51
The Case t at Zero	52
The Case t at Infinity	63
3.3 The Three-Dimensional Case	84
3.3.1 Preparations	84
3.3.2 Definability and Asymptotic Expansions	113
The Case t at Zero	114
The Case t at Infinity	133
4 Conclusion and Outlook	169
Bibliography	171

1

Introduction

1.1 Motivation

The modern theory of *asymptotic expansions* goes back to the research of Thomas Stieltjes [Sti86] and Henri Poincaré [Poi86] in 1886. The asymptotic analysis is that branch of mathematics, more precisely of analysis, devoted to the study of the behaviour and the representation of a given function by simple functions at a limit point of the function's domain. Let us consider the example if the function is analytic. In this special case the asymptotic expansion can be given by its Taylor series. As long ago as 1812, Pierre-Simon Laplace [dL12] showed two methods of establishing an asymptotic expansion of functions defined by an integral. The method in which we are interested is finding a function's asymptotics by integrating by parts. In this way, he represented the complementary error function, which is defined by the integral

$$\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt,$$

by a divergent power series, cf. [Cop04, p.2].

We combine this topic with a field of model theory that comes up about one hundred years later after Poincaré. But first, we take a closer look at the so-called *semialgebraic sets* and go back to a significant theorem, which was established in the middle of the 20th century. Semialgebraic sets can be found in a lot of areas in mathematics. They are a boolean combination of equalities and inequalities of real polynomials, that means, they are a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}.$$

These sets are “truly a special feature in real algebraic geometry”, see [BCR98, p.2]. It is a consequence of the many pleasant properties of this class of sets: for example, they are closed under union, intersection, and complementation. Therefore, the finite union and intersection of semialgebraic sets is still semialgebraic, as well as their complements. Their most significant feature was proven and published by the logician Alfred Tarski [Tar48] in 1948 and popularized by Abraham Seidenberg six years later by using it in his work [Sei54]. The results of both are known as the *Tarski-Seidenberg Theorem* which states that semialgebraic sets are also stable under projection. In addition to this, semialgebraic sets have only finitely many connected components and each of them is also semialgebraic. These fundamental properties give great conveniences. In the early 1980s the model theorist Lou van den Dries realized that few axioms provide many properties of semialgebraic sets, cf. [vdD98, p.vii]. Thereby, he based the theory of the so-called *o-minimal structures* in [vdD84], which has become a recognized area since then.

The theory of o-minimal structures could be located as branch of model theory and mathematical logic with ties to real algebraic geometry and real analysis. In its set-theoretic definition, an o-minimal structure, in our context on the real ordered field, is a structure \mathcal{M} , that means a family of sets $(M_n)_{n \in \mathbb{N}}$ which fulfills specific axioms, in which the sets in M_1 are precisely the finite union of intervals and points. From this point of view, the semialgebraic sets form the simplest o-minimal structure: The expansion of the real ordered field by all semialgebraic sets. O-minimality can be seen as a generalization of semialgebraic sets and real algebraic geometry. By this, it is possible to expand the area of analysis to them and to do, for example, integration. The theory of o-minimal structures offers great potential for research in different areas. In the late 1980s and the 1990s the community had a particular interest in finding structures which are o-minimal. One of these determined structures is \mathbb{R}_{an} , the expansion of the real ordered field by all restricted analytic functions, see [vdD86]. Their definable sets are also known as the *globally subanalytic sets*. Another field of research is studying the effects of o-minimality e.g. to stability properties or to measures, see for example [Kai12]. From this point of view, Jean-Marie Lion, Jean-Philippe Rolin, and Georges Comte

considered parameterized integrals

$$\int f(x, y) dy$$

under the condition that f is a globally subanalytic function, see [LR98] and [CLR00]. These papers state the significant result that $\int f(x, y) dy$ has the form of a polynomial $P(\varphi_1, \dots, \varphi_n, \log \varphi_1, \dots, \log \varphi_n)$ in globally subanalytic functions φ_i and their logarithms. In 2011, Raf Cluckers and Daniel Miller treated the integrals

$$\int f(x, y) (\log(g(x, y)))^n dy,$$

where f and g are globally subanalytic functions, see [CM11]. In their focus of interest are the so-called *constructible* functions, which are represented by a finite sum of finite products of globally subanalytic functions and their logarithms. Cluckers and Miller showed stability under integration for the class of constructible functions. As a consequence, they proved that the constructible functions form the smallest class, “which is stable under integration and contains all the globally subanalytic functions”, see [CM11, p.312]. Hence, this paper is an extension of the research of Lion, Rolin, and Comte, which we mentioned above. In 2013, Tobias Kaiser gave a sharper result in the case of semialgebraic functions in [Kai13]. He showed that parameterized integrals of semialgebraic functions can be completely described if the semialgebraic functions are enlarged by the global logarithm and the iterated antiderivatives of power series, see [Kai13, p.349]. The newest work of Cluckers, Comte, Miller, Rolin, together with Tamara Servi about the integration of oscillatory and globally subanalytic functions continues the research in this area, see [CCM⁺16]¹. Oscillatory integrals have the form

$$\int e^{ig(x, y)} f(x, y) dy$$

and in this context, the functions f and g are globally subanalytic. An example of oscillatory integrals of this kind is given by Fourier transforms, cf. [CCM⁺16, p.4]. The authors “prove[d] the stability under integration and under Fourier transform of [...] globally subanalytic functions and their complex exponentials”, see [CCM⁺16, p.1].

¹This paper is not even published yet at the present moment. It was submitted on 8 January, 2016, see <https://arxiv.org/abs/1601.01850>.

A further aim of research, in this context, is studying integrals of the form

$$\int e^{g(x,y)} f(x,y) dy, \quad (1.1)$$

where f and g are globally subanalytic functions. Issues regarding this integral are far from being solved at present. Our research concentrates on the following kind of integrals:

$$\int e^{-\frac{y^2}{2t}} f(y) dy \quad (1.2)$$

and

$$\int e^{-\frac{y^2}{2t}} f(y) \log(y) dy, \quad (1.3)$$

where f is a globally subanalytic function and t is a positive real number. These special cases of the integrals in (1.1) attract attention in the following way: without regard to the prefactor, the integrals in (1.2) and (1.3) arise when we consider

$$F_n(t) := \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx. \quad (1.4)$$

Thereby A is a semialgebraic set and n the dimension of A . By examining $F_n(t)$, we continue combining the theory of o-minimality with another mathematical field, the stochastics. The reason is that the integrals in (1.4) also represent the probability distribution given by the *Brownian Motion*.

The Brownian Motion is a jittery motion of particles in liquid discovered of the botanist Robert Brown. In the early 20th century, Albert Einstein gave the explanation that this physical phenomenon is caused by the molecular kinetic theory of heat, cf. [Ein05]. In financial mathematics and stochastic processes the Brownian Motion has gained a growing importance among other things due to Louis Bachelier, who used it to study fluctuations in stock prices in [Bac00], and Norbert Wiener, who proved its mathematical existence in [Wie21].

The significant role of the Brownian Motion in the theory of stochastic processes is caused by its many interesting properties. Furthermore, it is a popular example for the so-called *Gauss processes* and it is one of the best known *Lévy processes*. In addition with other Lévy processes, the Brownian motion is located at the interface of martingale and Markov processes, cf. [KW14, p.V]. Since our research puts no value in the Brownian Motion as stochastic process and we do not use its properties in any essential way, we refer for more detailed information to relevant literature, e.g. [KS14] and [KW14]. Our interest in the Brownian Motion pertains

solely to the probability that the Brownian Motion is in a semialgebraic set A at each time t , which is given by $F_n(t)$ in (1.4). This probability distribution does not tell us how the values of the Brownian Motion at different times are related. Due to the fact that the Brownian Motion is a jittery and “wild” motion in its microscopic view as we will see in Figure 2.7 and 2.8 in Chapter 2, the following question arises: Is the Brownian Motion tame in its macroscopic perspective if A is tame?

Against this background, we study $F_n(t)$, the probability distribution given by the Brownian Motion, in two ways: on the one hand, we examine definability in an o-minimal structure; on the other hand, our purpose is to establish asymptotic expansions for the time evolution of $F_n(t)$ at its limit points. In other words, time t approaches zero or tends towards infinity. Establishing asymptotic expansions in connection with stochastics is not a new idea. [GKS61] gives a good overview about asymptotic expansions in probability theory. Asymptotic expansions have been investigated for probability density functions, for example, of the arithmetic mean of identically distributed summands in [Dan54] and are a general method for finding corrections of the limiting distribution if the remainder terms decrease slowly, cf. [GKS61, p.153,155]. There are already works about the asymptotic expansion of the multivariate normal distribution, see [Rub64], and of the stationary distribution for the reflected brownian motion, see [FK16]². There is no need to go further in these publications, because we do not use their results. In contrast to these listed works, we take the approach that we use the properties of o-minimality to achieve our goals.

1.2 Outline and Results

In the next paragraphs, we give a short outline of this thesis and its research results. More detailed information can be found at the beginning of the respective chapter and section. Chapter 2 presents the preliminaries. First, we compile some basic definitions and theorems. Afterwards, we take a closer look at the theory of asymptotic expansions in Section 2.2 and at o-minimal structures in 2.3. A brief excursion in stochastic theory ends this chapter.

In the following main chapter, which is Chapter 3, we start with considering the one-dimensional case. Examining the definability of the probability distribution in (1.4) on a family of semialgebraic sets results in the following proposition.

²This paper is not even published yet, at the present moment. The last revised version is from 5 April, 2017, see <https://hal.archives-ouvertes.fr/hal-01295562>.

Proposition

The probability distribution, which is given by the Brownian Motion on a family of one-dimensional sets A_a definable in an o -minimal structure \mathcal{M} , is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$.

After required preparations, we consider the integrals $F_2(t)$ on two-dimensional semialgebraic sets in Section 3.2, where our main results are stated and proven in Subsection 3.2.2. We show:

Theorem A

The probability distribution $F_2(t)$, which is given by the Brownian Motion on a semialgebraic set $A \subset \mathbb{R}^2$, has the following asymptotic series expansions:

i) As $t \rightarrow 0$, $F_2(t)$ has the asymptotic series expansion

$$\sum_{k=0}^{\infty} \gamma_k t^{\frac{k}{q}},$$

which is not necessarily convergent and where $\gamma_k \in \mathbb{R}$ and $q \in \mathbb{N}$.

ii) As $t \rightarrow \infty$:

a) If A is bounded, $F_2(t)$ is definable in \mathbb{R}_{an} and therefore, $F_2(t)$ is a convergent Puiseux series.

b) If A is unbounded, $F_2(t)$ has the asymptotic expansion

$$\varrho_0(t) + \varrho_1(t) \log(t),$$

where $\varrho_0(t) = \sum_{k=0}^{\infty} \gamma_k t^{-\frac{k}{q}}$ and $\varrho_1(t) = \sum_{k=1}^{\infty} \delta_k t^{-k}$, where at least one of the series is not zero and $\gamma_k, \delta_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and therefore, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

The proof is split into t approaches zero and in t tends towards infinity, where we gather our work in Theorem 3.2.15, respectively in Theorem 3.2.25. For this purpose, we examine integrals of the form $\int e^{-\frac{r^2}{2t}} r \psi(r) dr$, where $\psi(r)$ is a bounded globally subanalytic function and then we investigate asymptotic series expansions of them. The significant point to note here is that we achieve definability in \mathbb{R}_{an} in case A is bounded as t approaches infinity.

Section 3.3 is devoted to the study of $F_3(t)$. For investigating the asymptotic expansions in the three-dimensional case, we need more preparations. These are compiled in Subsection 3.3.1. As in the two-dimensional case we split Subsection 3.3.2 in two parts and examine t approaches zero first, then t tends towards infinity. Each

of these parts ends again in a theorem. By investigating asymptotic expansions of the probability distribution given by the Brownian Motion, we are also interested in finding asymptotic expansions of $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$ and $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ in Subsection 3.3.2. In conclusion, it is remarkable that we get definability of $F_3(t)$ in $\mathbb{R}_{\text{an,exp}}$ in the case t approaches infinity and if A is bounded. Summarizing Theorem 3.3.30 and 3.3.43 we gain

Theorem B

The probability distribution $F_3(t)$, which is given by the Brownian Motion on a semialgebraic set $A \subset \mathbb{R}^3$, has the following asymptotic series expansions:

i) As $t \rightarrow 0$, $F_3(t)$ has the asymptotic expansion

$$\sum_{k=0}^{\infty} (\gamma_k + \delta_k \log(t)) t^{\frac{k}{q}},$$

which is not necessarily convergent and where $\gamma_k, \delta_k \in \mathbb{R}$ and $q \in \mathbb{N}$.

ii) As $t \rightarrow \infty$:

a) If A is bounded, $F_3(t)$ has the form $\varrho_0(t) + \varrho_1(t) \log(t)$, where $\varrho_0(t)$ and $\varrho_1(t)$ are globally subanalytic functions, and thus, $F_3(t)$ is a constructible function and definable in $\mathbb{R}_{\text{an,exp}}$.

b) If A is unbounded, $F_3(t)$ has the asymptotic expansion

$$\varrho_0(t) + \varrho_1(t) \log(t) + \varrho_2(t) \log(t)^2,$$

where $\varrho_0(t) = \sum_{k=0}^{\infty} \gamma_k t^{-\frac{k}{q}}$, $\varrho_1(t) = \sum_{k=1}^{\infty} \delta_k t^{-\frac{k}{q}}$, and $\varrho_2(t) = \sum_{k=1}^{\infty} \lambda_k t^{\frac{1}{2}-k}$, where at least one of the series is not zero and $\gamma_k, \delta_k, \lambda_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series. Thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Finally, in Chapter 4 we give a short conclusion and discuss further research.

2

Preliminaries

In this chapter we would like to take up some necessary preliminaries. First, we give some well-known *basic definitions and theorems*. In the second section we go into the theory of *asymptotic series expansions*. In this connection, the relation between asymptotic series and the *Landau Notation* which denotes order relations is pointed out. Furthermore, the asymptotic series expansions of the gamma function, the incomplete gamma function and the error function defined in Section 2.1 are given. Subsequently, we proceed with so-called *o-minimal structures* and some significant theorems. Finally, we make a brief excursion in stochastic theory, where we dip into stochastic processes and introduce the *Brownian Motion* as special example. This is caused by our study of the definability and the asymptotic behaviour of the probability distribution given by the Brownian Motion on a set in the main chapter.

2.1 Basic Definitions and Theorems

First, we fix a convention for the empty sum and the empty product. Let $m, j \in \mathbb{N}_0$ where $m < j$. We define the empty sum as $\sum_{k=j}^m a_k = 0$ and the empty product as

$\prod_{k=j}^m a_k = 1$. Furthermore, we use the following notation: Let $S(x)$ be an infinite sum in x and let $N \in \mathbb{N}$. Then $S|_N(x)$ denotes the partial sum of $S(x)$ which contains the summands of $S(x)$ up to the first summand with order equal or greater than N . For the following definitions, remarks, and theorems we adopt the notation as given by Abramowitz and Stegun in [AS12], by Graham in [GKP94], by Koenigsberger in [Kön13], by Sibagaki in [Sib52], and by Siegel in [Sie69].

Definition 2.1.1. *The function*

$$\begin{aligned} \pi_n: \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^n, \\ (x_1, \dots, x_{n+1}) &\longmapsto (x_1, \dots, x_n), \end{aligned}$$

denotes the *projection map on the first n coordinates*.

Definition 2.1.2. *Let $e: \mathbb{R} \rightarrow \mathbb{R}$ denote the exponential function. The (non-elementary) (Gauss) error function $\operatorname{erf}: \mathbb{R} \rightarrow]-1, 1[$ is defined as*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

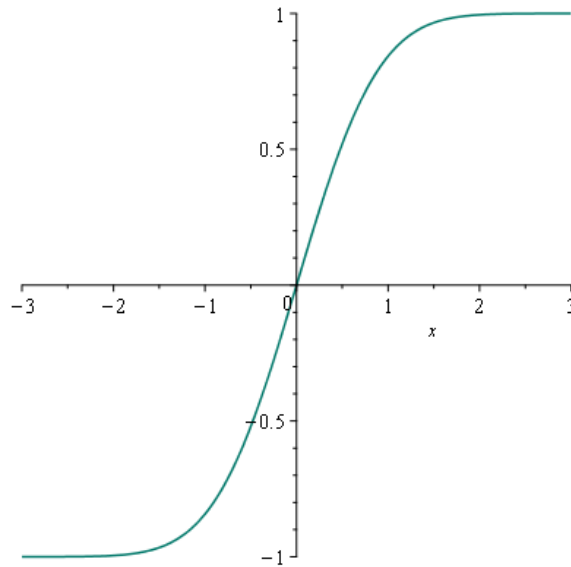


Figure 2.1. The graph of the error function $\operatorname{erf}(x)$ for $x \in [-3, 3]$.

Remark 2.1.3. *The error function has the following properties:*

- *The error function is an odd function, that means $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ for $x \in \mathbb{R}$.*
- *The error function has the following special (limit) values:*
 - $\lim_{x \rightarrow \infty} \operatorname{erf}(x) = 1$.
 - $\lim_{x \rightarrow -\infty} \operatorname{erf}(x) = -1$.
 - $\operatorname{erf}(0) = 0$.
- *Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$ be the cumulative distribution function of the standard normal distribution. The error function $\operatorname{erf}(x)$ is related to $\Phi(x)$ by $\Phi(x) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) \right)$.*

Definition 2.1.4. *The complementary error function $\operatorname{erfc} : \mathbb{R} \rightarrow]0, 2[$ is defined as*

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt.$$

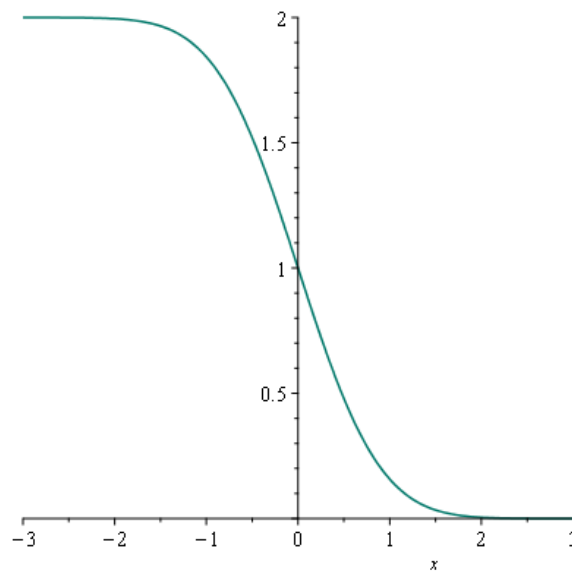


Figure 2.2. The graph of the complementary error function $\operatorname{erfc}(x)$ for $x \in [-3, 3]$.

Remark 2.1.5. *The complementary error function is related to the error function by*

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

The well-known *gamma function* and its first derivative are defined for all complex numbers except the non-positive integers. For positive real numbers, and also for complex numbers with a positive real part, we can define the gamma function via the following convergent improper integral.

Definition 2.1.6. *The **gamma function** $\Gamma : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is defined in Euler's integral representation as*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

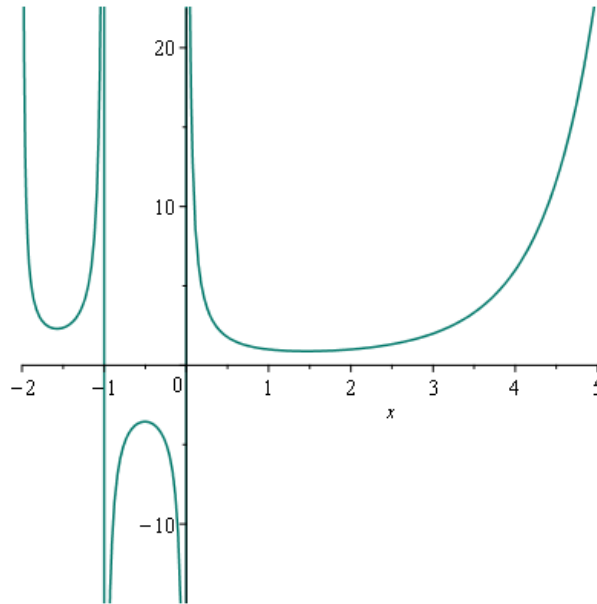


Figure 2.3. The graph of the gamma function $\Gamma(x)$ for $x \in]-2, 5]$.

Remark 2.1.7. *Let $x \in \mathbb{R}_{>0}$. The gamma function has the following properties:*

- $\Gamma(1) = 1$.
- $\Gamma(x + 1) = x\Gamma(x)$.

As well as the gamma function above we can write its first derivative for positive real numbers as the following convergent improper integral.

Definition 2.1.8. *The first derivative of the gamma function $\Gamma' : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is given in its integral representation by*

$$\Gamma'(x) = \int_0^{\infty} t^{x-1} e^{-t} \log(t) dt.$$

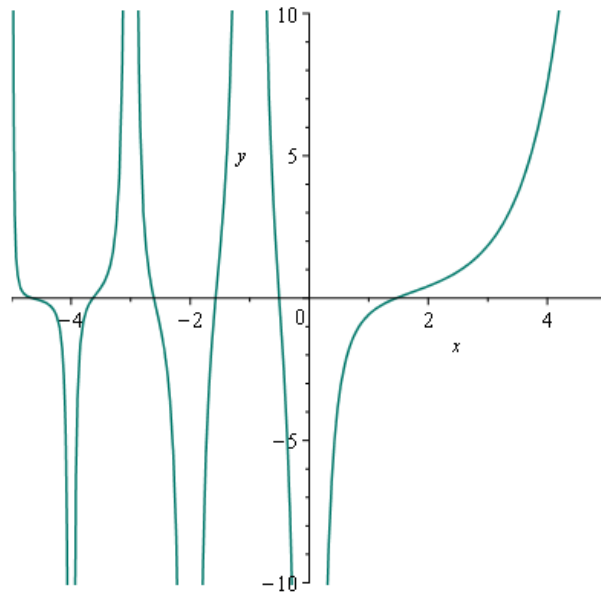


Figure 2.4. The graph of the first derivative of the gamma function $\Gamma'(x)$ for $x \in]-5, 5]$.

Remark 2.1.9. *There is only a single positive zero x_0 of the first derivative of the gamma function. Its first six digits are*

$$x_0 = 1,461632\dots$$

Remark 2.1.10. *Let $x \in \mathbb{R}_{>0}$ and let x_0 be the positive zero of the first derivative of the gamma function. If $x > x_0$, then $\Gamma'(x) > 0$.*

For further information to the first derivative of the gamma function and its zeros we refer to [Sib52].

Definition 2.1.11. *The incomplete gamma function $\Gamma : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ is defined in integral representation as*

$$\Gamma(x, y) = \int_y^{\infty} t^{x-1} e^{-t} dt.$$

Remark 2.1.12. The incomplete gamma function $\Gamma : \mathbb{R}_{\geq 0}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ has the following properties:

- $\Gamma(x + 1, y) = x\Gamma(x, y) + y^x e^{-y}$.
- The incomplete gamma function has the special limit value $\lim_{y \rightarrow \infty} \Gamma(x, y) = 0$.

Remark 2.1.13. For $x \in \mathbb{R}_{> 0}$ the gamma function is related to the incomplete gamma function by

$$\Gamma(x, 0) = \Gamma(x).$$

Remark 2.1.14. For $y \in \mathbb{R}_{\geq 0}$ the incomplete gamma function has the special value

$$\Gamma(1, y^2) = e^{-y^2}.$$

Remark 2.1.15. For $y \in \mathbb{R}_{\geq 0}$ the incomplete gamma function is related to the error function by

$$\Gamma\left(\frac{1}{2}, y^2\right) = \sqrt{\pi}(1 - \operatorname{erf}(y)).$$

Proof. By Definition 2.1.11 of the incomplete gamma function, we get

$$\begin{aligned} \Gamma\left(\frac{1}{2}, y^2\right) &\stackrel{2.1.11}{=} \int_{y^2}^{\infty} t^{\frac{1}{2}-1} e^{-t} dt \\ &= \int_{y^2}^{\infty} t^{-\frac{1}{2}} e^{-t} dt. \end{aligned}$$

We make a change of variables $t = x^2$ and obtain by Definition 2.1.4 of the complementary error function

$$\begin{aligned} \int_y^{\infty} \frac{1}{x} e^{-x^2} 2x dx &= 2 \int_y^{\infty} e^{-x^2} dx \\ &\stackrel{2.1.4}{=} 2 \frac{\sqrt{\pi}}{2} \operatorname{erfc}(y) \\ &\stackrel{2.1.5}{=} \sqrt{\pi}(1 - \operatorname{erf}(y)). \end{aligned}$$

□

Corollary 2.1.16. The gamma function has the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proof. It follows easily by 2.1.15 that

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right) &\stackrel{2.1.13}{=} \Gamma\left(\frac{1}{2}, 0\right) \\ &\stackrel{2.1.15}{=} \sqrt{\pi}(1 - \operatorname{erf}(0)) \\ &\stackrel{2.1.3}{=} \sqrt{\pi}. \end{aligned}$$

□

The two following theorems play a vital part in integration theory. The *Transformation Theorem* describes the relation between two different coordinate systems. *Fubini's Theorem* gives conditions under which, for example, it is possible to calculate a double integral by using one-dimensional integrals.

Theorem 2.1.17 (Transformation Theorem). *Let $U \subset \mathbb{R}^d$ be an open set and let $\Theta : U \rightarrow \Theta(U) \subset \mathbb{R}^d$ be a C^1 -diffeomorphism. The function f is integrable on $\Theta(U)$ if the function $x \mapsto f(\Theta(x))|\det(D\Theta(x))|$ is integrable on U . Then*

$$\int_{\Theta(U)} f(y)dy = \int_U f(\Theta(x))|\det(D\Theta(x))|dx,$$

where $D\Theta(x)$ is the Jacobian matrix and $\det(D\Theta(x))$ is the functional determinant of Θ .

Theorem 2.1.18 (Fubini's Theorem). *Let f be an integrable function on $\mathbb{R}^n \times \mathbb{R}^m$. For every fixed $y \in \mathbb{R}^m$, except the elements of a null set $N \subset \mathbb{R}^m$, the function $x \mapsto f(x, y)$ is integrable over \mathbb{R}^n . Set $F(y) := \int_{\mathbb{R}^n} f(x, y)dx$ for $y \in \mathbb{R}^m \setminus N$ and $F(y) := 0$ for $y \in N$, then F is integrable over \mathbb{R}^m and we can write*

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y)d(x, y) = \int_{\mathbb{R}^m} F(y)dy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x, y)dx \right) dy.$$

This subsection concludes with the definition of *analytic functions* and *Puiseux series*.

Definition 2.1.19. Let $U \subset \mathbb{R}$ be an open set and let $f : U \rightarrow \mathbb{R}$. The function f is called **analytic at $x_0 \in U$** if there exists a power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

and some $r > 0$ such that the series converges absolutely for $x \in]x_0 - r, x_0 + r[$, and such that we have

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for all $x \in]x_0 - r, x_0 + r[$. The function f is called **analytic in U** if f is analytic at every $x \in U$.

We define so-called *Puiseux series* which are a significant generalization of power series. Isaac Newton introduced them in his letter to Henry Oldenburg in 1676, see [Te60, p.1]. They are named after Victor Puiseux, who rediscovered them in 1850. Puiseux series are used to represent an arbitrary function as an infinite sum of simple functions.

Definition 2.1.20. Let $q \in \mathbb{N}$, let $k \in \mathbb{Z}$, and let $a_n \in \mathbb{R}$ for all $n \in \mathbb{Z}$. A series of the form

$$\sum_{n=k}^{\infty} a_n (x - x_0)^{\frac{n}{q}}$$

is called **Puiseux series at (center) x_0** .

In addition, we want to remark that a Puiseux series can appear in three different contexts. In certain cases, we understand $\sum_{n=k}^{\infty} a_n x^{\frac{n}{q}}$ as a formal series by means of a purely algebraic object with no assertion of convergence; so x is simply an indeterminate. We can add and multiply two series as usual and we also get information of the formal series, but possibly no analytic information like asymptotic formulas, see [Wil14, p.27]. In another context, we interpret the Puiseux series as an asymptotic series expansion which we cover in the next section. In this case, it can be convergent or divergent. As opposed to this, a Puiseux series, which is definable in the o-minimal structure \mathbb{R}_{an} (see Section 2.3), is always a convergent series.

2.2 Asymptotic Expansions

As already mentioned in Chapter 1, the theory of *asymptotic series expansions* goes back to Stieltjes and Poincaré in 1886. It can be split into two parts. According to Copson, see [Cop04, p.3], “the first part deals with the summability of asymptotic series and with the validity of such operations as term by term differentiation or integration; the second is concerned with the actual construction of a series which represents a given function asymptotically.” To narrow this down, we have a closer look at the second part. Our aim is to investigate a convergent or divergent infinite series which represents a given function asymptotically in a neighbourhood of a limit point of the function’s domain. That means that any partial sum of that series provides a very good approximation of the function with an error of the order of the first term omitted. This order of the error is noted by the order symbols \mathcal{O} and o , the so-called *Landau notation*. We follow the definitions and remarks as given by Erdélyi in [Erd56], by Copson in [Cop04], and by Murray in [Mur12]; the examples for asymptotic series expansions are given by Abramowitz and Stegun in [AS12].

Definition 2.2.1. *Let $D \subset \mathbb{R}$ be a nonempty set, let \overline{D} be the closure of D in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Let $x_0 \in \overline{D}$ be a limit point and let $f, g : D \rightarrow \mathbb{R}$ be real functions. We write*

$$f(x) = \mathcal{O}(g(x))$$

as $x \rightarrow x_0$ if there exists a constant C and a neighbourhood U of x_0 such that for all $x \in U \cap D$

$$|f(x)| \leq C|g(x)|.$$

We write

$$f(x) = o(g(x))$$

as $x \rightarrow x_0$ if for every $\delta > 0$ there exists a neighbourhood U of x_0 such that for all $x \in U \cap D$

$$|f(x)| \leq \delta|g(x)|.$$

Remark 2.2.2. *Let $g(x) \neq 0$ in a punctured neighbourhood of x_0 . As $x \rightarrow x_0$, then $f(x) = \mathcal{O}(g(x))$ if $\frac{f(x)}{g(x)}$ is bounded on a neighbourhood of x_0 and $f(x) = o(g(x))$ if $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$.*

The \mathcal{O} -notation gives us an asymptotic upper bound on the growth rate of a function. This growth rate is also referred to as an order of this function. Consequently, it describes how fast a function f increases in a neighbourhood of a particular value x_0 or infinity when the argument tends towards it. As mentioned previously, in the case of asymptotic expansions it is used to estimate the error

that occurs when we represent a function asymptotically by a series. Usually the error is described in terms of more simple functions. Therefore, functions with the same growth rate may be represented by the same \mathcal{O} -notation. The \mathcal{O} -notation is a little bit vague in the following sense. Strictly speaking, it should be written as $f \in \mathcal{O}(g)$, but the use of “=” is common. Associated with the \mathcal{O} -notation, the symbol o makes a stronger statement: $f = o(g)$ means that g grows much faster than f . Therefore, every function f that is $o(g)$ is also $\mathcal{O}(g)$, but not generally vice versa.

When going into more detail of asymptotic series expansions it occurs that an asymptotic expansion describes the asymptotic behaviour of a function in terms of an *asymptotic sequence* of functions, which is defined as follows.

Definition 2.2.3. Let $D \subset \mathbb{R}$ be a nonempty set, let \overline{D} be the closure of D in $\overline{\mathbb{R}}$, and let $x_0 \in \overline{D}$ be a limit point of D . For $n \in \mathbb{N}_0$ let $g_n : D \rightarrow \mathbb{R}$ be a function. The sequence of functions $(g_n(x))_{n \in \mathbb{N}_0}$ is called **asymptotic sequence** as $x \rightarrow x_0$ if for each $n \in \mathbb{N}$ there is a punctured neighbourhood of x_0 , in which $g_n(x) \neq 0$ and if for all $n \in \mathbb{N}$

$$g_{n+1}(x) = o(g_n(x)),$$

as $x \rightarrow x_0$.

A series with real coefficients, which is formed by such an asymptotic sequence, is an asymptotic series expansion of a function if the series fulfil the following definition.

Definition 2.2.4. Let $D \subset \mathbb{R}$ be a nonempty set, where $\overline{D} \subseteq \overline{\mathbb{R}}$, and $x_0 \in \overline{D}$ be a limit point of D , let $(g_n(x))_{n \in \mathbb{N}_0}$ be an asymptotic sequence in \mathbb{R} as $x \rightarrow x_0$, let $f : D \rightarrow \mathbb{R}$ be a function, and $(a_n)_{n \in \mathbb{N}_0}$ be a real sequence. The (formal) series $\sum_{k=0}^{\infty} a_k g_k(x)$ is called **asymptotic (series) expansion** of $f(x)$ as $x \rightarrow x_0$, denoted by

$$f(x) \approx \sum_{n=0}^{\infty} a_n g_n(x),$$

if for every value of $N \in \mathbb{N}_0$

$$f(x) - \sum_{n=0}^N a_n g_n(x) = o(g_N(x)), \quad (1)$$

as $x \rightarrow x_0$.

Remark 2.2.5. *Since*

$$f(x) - \sum_{n=0}^{N-1} a_n g_n(x) = a_N g_N(x) + o(g_N(x)),$$

the partial sum

$$\sum_{n=0}^{N-1} a_n g_n(x)$$

is an approximation to $f(x)$ with an error $\mathcal{O}(g_N(x))$ as $x \rightarrow x_0$; this error is of the same order of magnitude as the first term omitted. That means that the equation (1) in Definition 2.2.4 may be written as

$$f(x) - \sum_{n=0}^{N-1} a_n g_n(x) = \mathcal{O}(g_N(x)).$$

Remark 2.2.6. *If an asymptotic expansion of $f(x)$ as in Lemma 2.2.4 exists, it is unique and the coefficients are uniquely given by*

$$a_N = \lim_{x \rightarrow x_0} \frac{f(x) - \sum_{n=0}^{N-1} a_n g_n(x)}{g_N(x)}.$$

Remark 2.2.7. *A function may have several asymptotic series expansions, but every expansion is unique in respect to its asymptotic scale.*

Definition 2.2.8. *The first non-zero term $a_k g_k(x)$ is called **dominant term** and we write*

$$f(x) \sim a_k g_k(x)$$

*as $x \rightarrow x_0$. That means $\lim_{x \rightarrow x_0} \frac{f(x)}{g_k(x)} = a_k \in \mathbb{R} \setminus \{0\}$. We also say f **behaves like** g_k or f is **asymptotically similar** to g_k at x_0 .*

The expression *asymptotically similar* is not only used to characterize a function by the dominant term of its asymptotic series expansion, but also to describe the asymptotic behaviour between two functions f and g .

Definition 2.2.9. *Let $D \subset \mathbb{R}$ be a nonempty set, let $f, g : D \rightarrow \mathbb{R}$ be real functions, where $g(x) \neq 0$ in a punctured neighbourhood of x_0 . We say f is **asymptotically similar** to g at the limit point x_0 if*

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \in \mathbb{R} \setminus \{0\}.$$

Furthermore we say f is **asymptotically equivalent** to g at the limit point x_0 if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1.$$

There are a number of rules for operating with order relations. They are often only shown for the \mathcal{O} -symbol, but also hold for the o -symbol. For these rules we refer to Erdélyi [Erd56, p.5].

Next, we see some examples for asymptotic expansions of well-known functions, which are vital for this thesis. Some of them are familiar by the next remark.

Remark 2.2.10. *Let the function $f(x)$ be analytic at the limit point x_0 . The convergent Taylor series of f in x_0 is an asymptotic series expansion of $f(x)$ as $x \rightarrow x_0$.*

In consequence of that, we can easily specify the asymptotic expansion for the exponential function as x approaches zero.

Lemma 2.2.11. *As $x \rightarrow 0$, the exponential function has the asymptotic series expansion*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

An asymptotic series expansion can be both – a convergent or a divergent series. In most cases it is less useful if it is convergent, because in the case of a divergent series sometimes only a few terms are required to give an accurate approximation of the function. A familiar example is the asymptotic expansion of the error function. If a function is defined by a definite integral, a simple way of finding the asymptotic expansion is by repeated integration by parts. Laplace introduced the idea that the error function can be represented by a convergent power series by integration by parts. Since the series converges slowly for big values, he investigated an asymptotic expansion for the related complementary error function $\operatorname{erfc} = 1 - \operatorname{erf}$ by the same method. This allows a quicker computation of large values. The divergent asymptotic expansion of the error function is shown below.

Lemma 2.2.12. *As $x \rightarrow \infty$, the error function has the asymptotic series expansion*

$$\operatorname{erf}(x) \approx 1 + e^{-x^2} x^{-1} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2x^2)^n} \prod_{l=1}^n (2l - 1)$$

and as $x \rightarrow 0$

$$\operatorname{erf}(x) \approx \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \frac{1}{(2n + 1)n!}.$$

Another example of finding an asymptotic expansion by integration by parts is the incomplete gamma function. Copson shows this in detail in [Cop04, p.13].

Lemma 2.2.13. *Let $x \in \mathbb{R}_{\geq 0}$. As $y \rightarrow \infty$, the incomplete gamma function has the asymptotic series expansion*

$$\Gamma(x, y) \approx \sum_{n=1}^{\infty} e^{-y} y^{x-n} \prod_{l=1}^{n-1} (x-l)$$

and as $y \rightarrow 0$ and $x \neq 0$ the asymptotic series expansion

$$\Gamma(x, y) \approx \Gamma(x) + y^x \sum_{n=0}^{\infty} (-1)^{n+1} y^n \frac{1}{(x+n)n!}.$$

Moreover, in the special case where $x = 0$ we get as $y \rightarrow 0$ the asymptotic series expansion

$$\Gamma(0, y) \approx (-\log(y) - \gamma) + \sum_{n=1}^{\infty} (-1)^{n+1} y^n \frac{1}{n \cdot n!},$$

where γ denotes the Euler-Mascheroni constant.

2.3 O-minimal Structures

The story of the *o-minimal structures* began in the early 1980s. Lou van den Dries looks back and writes in [vdD98, p.vii]: “I had noticed that many properties of semi-algebraic sets and maps could be derived from a few simple axioms”. Stimulated by van den Dries’ paper *Remarks on Tarski’s problem concerning $(R, +, \cdot, \exp)$* , see [vdD84], Pillay and Steinhorn introduced the notation *o-minimality* in their work *Definable sets and ordered structures* [PS84]. The designation *o-minimal* is the abbreviation for order-minimal. This thesis focuses on o-minimal structures that expand the field of reals. In this case the most simple one is the structure formed by the semialgebraic sets. Thus, the study of o-minimal structures and o-minimal theory generalizes real algebraic geometry. First, the major line of research was based on discovering expansions of the real ordered field that are o-minimal. Van den Dries proved in [vdD86] that the structure \mathbb{R}_{an} , the real ordered field expanded by restricted analytic functions, is o-minimal, but it does not contain the full exponential function. In 1991 Wilkie could show that \mathbb{R}_{exp} , the real ordered field expanded by the exponential function, is also an o-minimal structure, see [Wil96]. O-minimality is a class of tame geometry. “The o-minimal structures provide an excellent framework for developing tame topology”, is written in [vdD98, p.vii] by van den Dries.

Tameness implies many pleasant properties as example stratification or triangulation. It also guarantees piecewise smoothness of semialgebraic functions. The so-called *Cell Decomposition Theorem* allows to partition a definable set X into finitely many disjoint subsets X_1, \dots, X_n , $n \in \mathbb{N}$, of a special form. These X_i are called *cells*. This partition of X can moreover be chosen such that all restrictions of a semialgebraic function $f : X \rightarrow \mathbb{R}$ to X_i are continuous. This fact is significant for our work. Furthermore, tameness implies that every set has finitely many connected components, there are no functions which oscillate, there are strong properties of finiteness, and there is also held a good notion of dimension. Everything is controlled or to express it with Coste's words: It is "without pathological behaviour" [Cos99, p.6].

In the following, we only refer to a few aspects of the theory mentioned above. If not stated otherwise, we follow the definitions as given by van den Dries in [vdD86] and [vdD98].

Definition 2.3.1. A *semialgebraic set* $A \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\},$$

where $f, g_1, \dots, g_k \in \mathbb{R}[x_1, \dots, x_n]$.

Example 2.3.2. The sets, which are represented by the coloured surfaces in Figure 2.5, are semialgebraic.



Figure 2.5. Examples for a semialgebraic sets in \mathbb{R}^2 .

Definition 2.3.3. A set $A \subset \mathbb{R}^n$ is called *semianalytic* if for each $x_0 \in \mathbb{R}^n$ there is an open neighbourhood U of x_0 and there are real analytic functions f, g_1, \dots, g_k on U , such that $A \cap U$ is a finite union of sets of the form

$$\{x \in U \mid f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}.$$

Definition 2.3.4. Let π_n be the projection map on the first n coordinates. A set $A \subset \mathbb{R}^n$ is called **subanalytic** if for each $x_0 \in \mathbb{R}^n$ there is an open neighbourhood U of x_0 and a bounded semianalytic set $B \subset \mathbb{R}^m$ where $m \geq n$ such that $A \cap U = \pi_n(B)$.

Definition 2.3.5. A set $A \subset \mathbb{R}^n$ is called **globally subanalytic** if it is a subanalytic set after applying the semialgebraic homeomorphism

$$\begin{aligned} \mathbb{R}^n &\longrightarrow]-1, 1[^n, \\ x_i &\longmapsto \frac{x_i}{\sqrt{1+x_i^2}}. \end{aligned}$$

Definition 2.3.6. Let $A \subset \mathbb{R}^n$ be a globally subanalytic set. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **constructible on A** if there are globally subanalytic functions $f_i : A \rightarrow \mathbb{R}$ and $f_{i,j} : A \rightarrow (0, \infty)$ such that

$$f(x) = \sum_{i=1}^k f_i(x) \prod_{j=1}^{\ell_i} \log(f_{i,j}(x)),$$

where $k, \ell_i \in \mathbb{N}$.

Definition 2.3.7. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **restricted analytic function** if there exists a real convergent power series p in n variables which converges on an open neighbourhood of $[-1, 1]^n$ such that

$$f(x) = \begin{cases} p(x), & x \in [-1, 1]^n, \\ 0, & \text{else.} \end{cases}$$

Definition 2.3.8. A **structure** on the ordered field \mathbb{R} is axiomatically defined as a family $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ of sets such that for each n :

- i) M_n is a boolean algebra of subsets of \mathbb{R}^n (i.e. if $A, B \in M_n$ then $A \cup B \in M_n, A \cap B \in M_n, \mathbb{R}^n \setminus A \in M_n$), which contains the semialgebraic subsets of \mathbb{R}^n .
- ii) $A \in M_m$ and $B \in M_n$ implies $A \times B \in M_{m+n}$.
- iii) If $A \in M_{n+1}$ and π_n is the projection map on the first n coordinates then $\pi_n(A) \in M_n$.

A structure \mathcal{M} is called **o-minimal** if it satisfies the extra axiom:

- iv) The sets in M_1 are precisely the finite unions of intervals and points.

Definition 2.3.9. Let $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ be a structure. A set $A \subset \mathbb{R}^n$ is **definable in \mathcal{M}** if $A \in M_n$.

Definition 2.3.10. Let $A \subset \mathbb{R}^n$ and let $f: A \rightarrow \mathbb{R}^m$ be a function, then f is a **function definable in \mathcal{M}** if its graph $\{(x, f(x)) \mid x \in A\}$ is definable in \mathcal{M} .

Example 2.3.11. Examples for o-minimal structures are:

- i) The structure \mathbb{R} is the expansion of the real ordered field by all semialgebraic sets. It is o-minimal.
- ii) The structure \mathbb{R}_{an} is the expansion of the real ordered field by all restricted analytic functions. It is o-minimal. The sets definable in \mathbb{R}_{an} are known as the globally subanalytic sets.
- iii) The structure \mathbb{R}_{exp} is the expansion of the real ordered field by the exponential function $\exp: \mathbb{R} \rightarrow \mathbb{R}$. It is o-minimal.
- iv) $\mathbb{R}_{\text{an,exp}}$ is the structure which is generated by restricted analytic functions and extended by the exponential function. It is o-minimal.

The **Pfaffian closure** is another o-minimal structure. It goes beyond the scope of this thesis to give the exact definition. For more details we refer to Speissegger [Spe99, p.210], [MRS12, p.181ff]. To put it crudely, the Pfaffian closure $\mathcal{P}(\mathcal{M})$ is an o-minimal expansion of an o-minimal structure \mathcal{M} where $\mathcal{P}(\mathcal{M})$ is inductively generated. It is closed under Pfaffian chains with arbitrary definable functions instead of polynomials. The following theorem is listed for the interested reader for the sake of completeness.

Theorem 2.3.12. Let \mathcal{M} be an o-minimal expansion of the real field. Then there is an o-minimal expansion $\mathcal{P}(\mathcal{M})$ of \mathcal{M} which is closed under solutions to Pfaffian equations in the following strong sense. Whenever U is a definable and connected open subset of \mathbb{R}^n , $\omega = a_1 dx_1 + \dots + a_n dx_n$ is a 1-form on U with definable coefficients $a_i: U \rightarrow \mathbb{R}$ of class \mathcal{C}^1 , and $L \subseteq U$ is a Rolle leaf of $\omega = 0$, then L is also definable in $\mathcal{P}(\mathcal{M})$.

Proof. We refer to Speissegger [Spe99, p.189]. □

In other words, every o-minimal expansion $\tilde{\mathbb{R}}$ of the real field has an o-minimal expansion $\mathcal{P}(\mathcal{M})$, the Pfaffian closure of \mathcal{M} , in which the solutions to Pfaffian equations are definable. For more details on the theorem and the definition of **Rolle leafs** we refer to Speissegger [Spe99, p.189] and Moussu and Roche [MR92]. A central point for our research is the following consequence of this theorem.

Corollary 2.3.13. *Suppose that $I \subseteq \mathbb{R}$ is an open interval, $a \in I$ and $g : I \rightarrow \mathbb{R}$ is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$ and continuous. Then its antiderivative $F : I \rightarrow \mathbb{R}$ given by $F(x) := \int_a^x g(t)dt$ is also definable in $\mathcal{P}(\mathcal{M})$.*

Subsequently, we proceed with significant theorems and definitions. We can decompose every definable set in finitely many disjoint subsets of a special form called *cells*. This concept plays an essential role in o-minimal structures. Generally, proofs are more intelligible if we show the claim for every single cell type and finally gather the results.

Definition 2.3.14. *Fix an o-minimal structure \mathcal{M} . Let $B \subset \mathbb{R}^n$ be nonempty and definable. **Cells** are defined inductively as follows:*

- i) $B \subseteq \mathbb{R}$ is a cell if B is either a point or an open interval.
- ii) Let $B \subseteq \mathbb{R}^n$ be a cell and $f : B \rightarrow \mathbb{R}$ be a definable continuous function, then the graph of f is a cell in \mathbb{R}^{n+1} .
- iii) Let $B \subseteq \mathbb{R}^n$ be a cell and $f, g : B \rightarrow \mathbb{R}$ be definable continuous functions such that $f(x) < g(x)$ for all $x \in B$, then
 - $\{(x, y) \in B \times \mathbb{R} : f(x) < y < g(x)\}$,
 - $\{(x, y) \in B \times \mathbb{R} : f(x) < y\}$,
 - $\{(x, y) \in B \times \mathbb{R} : y < f(x)\}$,
 - finally $B \times \mathbb{R}$

are cells in \mathbb{R}^{n+1} .

Then B is also called **base of the cell**.

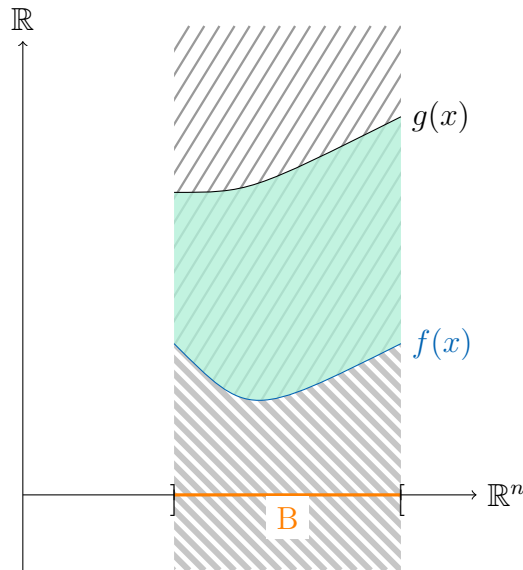


Figure 2.6. Different cell types in \mathbb{R}^{n+1} on the base B

Theorem 2.3.15 (Cell Decomposition Theorem). *Let \mathcal{M} be an o-minimal structure.*

- i) *Each definable set $A \subset \mathbb{R}^m$ has a finite partition $A = C_1 \cup \dots \cup C_k$ into cells C_i for $i \in \{1, \dots, k\}$.*
- ii) *If $f : A \rightarrow \mathbb{R}^n$ is a definable map, this partition of A can be chosen such that all restrictions $f|_{C_i}$ are continuous.*

Proof. We refer to van den Dries [vdD98, p.52]. □

The next theorem is a major result of Comte, Lion and Rolin. It provides us the form of an integral of globally subanalytic functions over a globally subanalytic set after integration.

Theorem 2.3.16. *Let $Y \subset \mathbb{R}^n \times \mathbb{R}^m$ be a globally subanalytic set. Suppose that the dimension of every fiber Y_x is at most k . Let $v(x) = v_k(Y_x)$ denote the k -dimensional volume of Y_x . The set of points, where $v(x)$ is finite, is a globally subanalytic set $B \subset \mathbb{R}^n$. The restriction of v on B is of the form*

$$P(f_1, \dots, f_d, \log f_1, \dots, \log f_d),$$

where $f_1 = f_1(x), \dots, f_d = f_d(x)$ are strictly positive globally subanalytic functions and P is a polynomial.

Proof. We refer to Comte, Lion, and Rolin [CLR00, Theorem 1, p.885] and [LR98, Remark, p.756]. \square

Remark 2.3.17. *Let $a \in \mathbb{R}^m$. If $n = 1$, we get a finite partition of \mathbb{R} in intervals such that the volume $v(x)$ on each interval has the form of a polynomial in $\log|x-a|$ with analytic functions of $|x-a|^{\frac{1}{p}}$ as coefficients, where $p \in \mathbb{N}$.*

Proof. We refer to Lion and Rolin [LR98, Remark, p.756]. \square

The next theorem says broadly speaking: By specific requirements it is possible to choose a cell decomposition of a set X such that a function on X , which is definable in \mathbb{R}_{an} , has the form of a Puiseux series.

Theorem 2.3.18 (Preparation Theorem). *Let $X \subseteq \mathbb{R}^{n+1}$ and let $f : X \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$, be a function definable in \mathbb{R}_{an} . There is a cell decomposition \mathcal{C} of X such that the following holds: Let $C \in \mathcal{C}$ and let B be the base of cell C . Let C be fat with respect to the last variable y , that means, C_x is a nonempty open interval for every $x \in B$. Then the function $f|_C$ can be written as*

$$f|_C(x, y) = g(x)|y - \xi(x)|^\sigma u(x, y - \xi(x))$$

where $\sigma \in \mathbb{Q}$, the functions $g, \xi : B \rightarrow \mathbb{R}$ are definable and real analytic with $y \neq \xi(x)$ on C , and $u(x, y)$ is a so-called special unit on

$$C^\xi := \{(x, y - \xi(x)) \mid (x, y) \in C\},$$

that is of the form

$$u(x, y) = v(b_1(x), \dots, b_M(x), b_{M+1}(x)|y|^{\frac{1}{q}}, b_{M+2}(x)|y|^{-\frac{1}{q}}),$$

where $q \in \mathbb{N}$. The function

$$\begin{aligned} \phi : B \times \mathbb{R} \setminus \{0\} &\longrightarrow \mathbb{R}^{M+2}, \\ (x, y) &\longmapsto (b_1(x), \dots, b_M(x), b_{M+1}(x)|y|^{\frac{1}{q}}, b_{M+2}(x)|y|^{-\frac{1}{q}}), \end{aligned}$$

is a definable and real analytic function with $\phi(C^\xi) \subset [-1, 1]^{M+2}$ and v is a real power series in $M + 2$ variables that converges on an open neighbourhood of $[-1, 1]^{M+2}$ and is away from 0.

Proof. We refer to Lion and Rolin [LR97, p.862]. \square

The constructible functions form the smallest class of functions which contains all the globally subanalytic functions and which is stable under integration.

Theorem 2.3.19. *The class of constructible functions is stable under integration.*

Proof. We refer to Cluckers and Miller [CM11, p.314]. \square

2.4 The Brownian Motion

The *Brownian Motion* is a physical phenomenon which is named after the botanist Robert Brown (* 1773, † 1858). In 1827 Brown studied pollen grains of plants suspended in water under a microscope and he observed that every single particle executes a jittery motion. In 1880, Thorvald N. Thiele developed the first mathematical model of the Brownian Motion and described the mathematics behind it. Notwithstanding the above Louis Bachelier used the Brownian Motion in his PhD thesis “The theory of speculation” to study fluctuations in stock prices. In 1905, Albert Einstein gave the explanation for the jittery motion of the particles in his paper about the molecular kinetic theory of heat, see [Ein05]. His explanation was a milestone for the proof of the existence of molecules and atoms. The mathematical existence of the Brownian Motion was finally proved by Norbert Wiener in 1923. For that reason it is also called Wiener process.

In financial mathematics and stochastics the Brownian Motion is an example for many significant classes of stochastic processes. One of these classes are the so-called Gauss processes, see [MS06, p.341]. Furthermore, the Brownian Motion is the basis of many financial market models. The theory of stochastic processes goes beyond the scope of this thesis, therefore we only want to provide a short insight into the Brownian Motion as a stochastic process in this part. For deeper studies of stochastic processes we refer for example to Meintrup [MS06], which is also the source for the following definitions.

Before we get to the Brownian Motion, we call back to one’s mind the definition of normal distribution and some basics about stochastic processes.

Definition 2.4.1. Let $\sigma > 0$ and $\mu \in \mathbb{R}$. The probability measure $\mathcal{N}(\mu, \sigma^2)$ with density $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

is called **normal** (or **Gaussian**) **distribution** with mean μ and variance σ^2 .

Definition 2.4.2. Let (Ω, \mathcal{F}, P) be a given probability space, (Z, \mathcal{Z}) a measurable space, and let $I \subset \mathbb{R}_{\geq 0}$ be an index set. A family $X = (X_t)_{t \in I}$ of measurable maps

$$X_t : \Omega \longrightarrow Z,$$

where $t \in I$, is called **stochastic process** (with state space Z).

Definition 2.4.3. Let $(X_t)_{t \geq 0}$ be a stochastic process and $\omega \in \Omega$. The map

$$\begin{aligned} X(\omega) : I &\longrightarrow Z, \\ t &\longmapsto X_t(\omega), \end{aligned}$$

is called **path** of ω .

Remark 2.4.4. A stochastic process X is continuous if for almost all $\omega \in \Omega$ the path $X(\omega)$ is continuous.

Definition 2.4.5. Let $z \in \mathbb{R}$. A one-dimensional stochastic process $(B_t)_{t \geq 0}$ with state space \mathbb{R} is called **Brownian Motion in \mathbb{R}** with start value z if it satisfies the following properties:

- i) $B_0 = z$ almost sure.
- ii) Let $0 \leq s < t$. Then $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.
- iii) Let $n \geq 1$, $0 \leq t_0 < t_1 < \dots < t_n$. Then $B_{t_0}, B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent random variables.
- iv) Almost surely, every path, $t \rightarrow B_t$, is continuous.

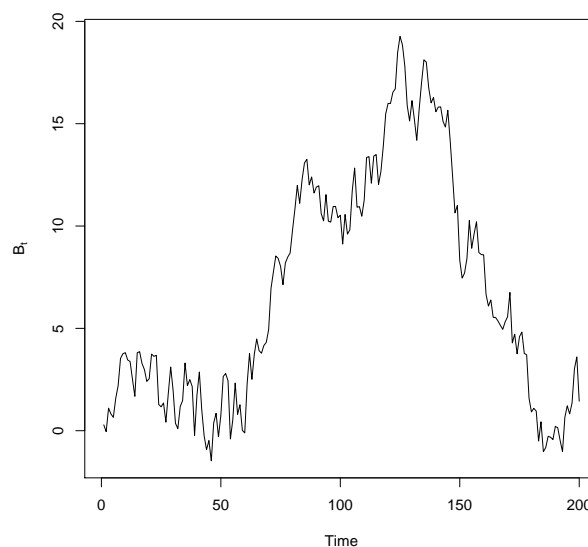


Figure 2.7. Simulation of the one-dimensional Brownian Motion

Remark 2.4.6. *The Brownian Motion with start value zero is called **standard Brownian Motion**.*

In other words, the Brownian Motion is the interpretation of a single particle, which executes a jittery motion caused by the thermal motion of molecules and started at a given position z at time $t = 0$. Therefore, the random variable B_t denotes the position of the particle at time t . The probability of presence of a particle at time t at a position is normally distributed. Moreover, the Brownian Motion has independent increments and the path of the particle is continuous, which means, the particle does not “jump”.

Definition 2.4.7. *An n -dimensional stochastic process $(B_t = (B_t^1, \dots, B_t^n))_{t \geq 0}$ is called **Brownian Motion in \mathbb{R}^n** with start value $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ if every stochastic process $(B_t^i)_{t \geq 0}$ is a Brownian Motion in \mathbb{R} with start value z_i for $i \in \{1, \dots, n\}$ and the stochastic processes B_t^1, \dots, B_t^n are independent for every $t \geq 0$.*

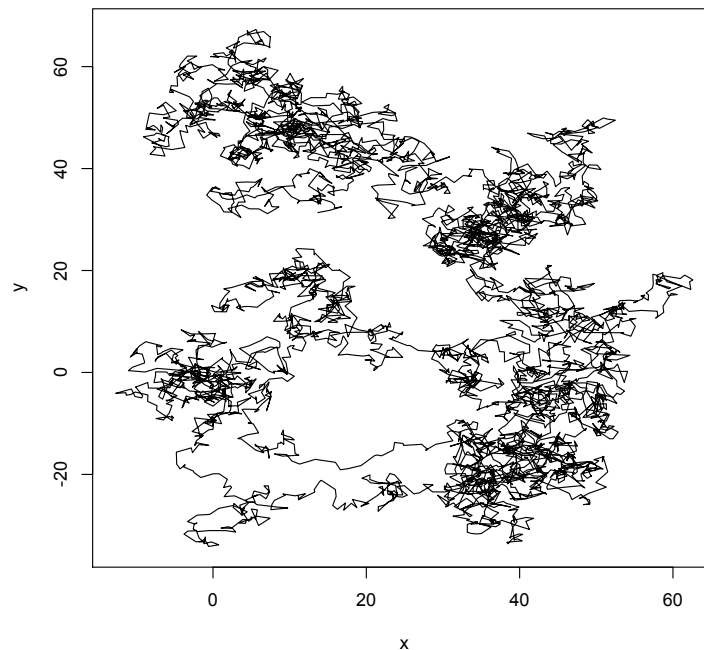


Figure 2.8. Simulation in 5000 time steps of a discrete approximation to a two-dimensional Brownian Motion. It shows the random variable B_t , that means the (x, y) -position of a single particle.

The Brownian Motion has a number of interesting properties. We refer for more detailed information about these to specialist literature, e.g. [KS14] and [KW14]. In this thesis we put no value to the Brownian Motion as a stochastic process and its properties, but rather in the probability that the Brownian Motion is in a given set A at time t . We conclude this chapter by the probability distribution given by the Brownian Motion on a set, which we examine in the next chapter.

Remark 2.4.8. *Let $A \subset \mathbb{R}^n$ be a Borel set and let $z \in \mathbb{R}^n$ be the start value. The probability for $B_t \in A$ at time t is given by*

$$P_z(B_t \in A) = \begin{cases} \delta_z(A), & t = 0, \\ \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_A e^{-\frac{|x-z|^2}{2t}} dx, & t > 0, \end{cases}$$

where $\delta_z(A)$ denotes the Dirac measure which is defined as

$$\delta_z(A) = \begin{cases} 1, & z \in A, \\ 0, & z \notin A. \end{cases}$$

3

Definability and Asymptotic Expansion for the Time Evolution of the Probability Distribution Given by the Brownian Motion on Semialgebraic Sets

In this main chapter we devote our attention to the probability that the Brownian Motion is in a semialgebraic set $A \subset \mathbb{R}^n$, $n \leq 3$, at time t . This probability is given by

$$P_z(B_t \in A) = \begin{cases} \delta_z(A), & t = 0, \\ \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_A e^{-\frac{|x-z|^2}{2t}} dx, & t > 0, \end{cases}$$

where z denotes the start value, see Remark 2.4.8. We are interested in the definability of $t \mapsto P_z(B_t \in A)$ in an o-minimal structure and in finding asymptotic expansions for this time evolution.

In the one-dimensional case in Section 3.1 we do not consider the probability distribution of the Brownian Motion on a single set A , which, in this case, is defin-

able in an arbitrary o-minimal structure \mathcal{M} , but on a definable family of sets A_a . At time zero, $(a, z) \mapsto P_z(B_0 \in A_a)$ is definable in this o-minimal structure \mathcal{M} . As a result for $t > 0$, we show that $(a, z, t) \mapsto P_z(B_t \in A_a)$ is definable in the Pfaffian closure of \mathcal{M} . For this purpose, we prove in Remark 3.1.1 the definability of the error function using a result of Speissegger [Spe99]. In addition, we will use, here and in every dimension, the important Cell Decomposition Theorem. By this, we are able to decompose the integral over the set A into integrals over more simple sets, more precisely over cells, which are intervals in the one-dimensional case, for example.

In higher dimensions we restrict our research to a single arbitrary set A , which is globally subanalytic and not only semialgebraic. The reason is that we would have to enlarge the o-minimal structure \mathbb{R} to \mathbb{R}_{an} in the proof quite early anyway. Furthermore, we assume without loss of generality that the dimension of A is equal or greater than two. We can clearly concentrate on the case $t > 0$. For the sake of convenience, we only study the standard Brownian Motion, where the start value is zero. In consequence of the translation invariance, we may translate the start value of a Brownian Motion from any time back to the origin. In other words, we can transform every Brownian Motion to the standard Brownian Motion without a loss. With respect to our probability distribution, we have only a translation of A by the start value z . Under these conditions, we aggregate our setting for \mathbb{R}^2 and \mathbb{R}^3 in the next remark.

Remark (General setting for higher dimensions). *In higher dimensions than one let $A \subset \mathbb{R}^n$ be a globally subanalytic set with dimension n and the probability distribution, which is given by the standard Brownian Motion, is represented as*

$$f(t) := \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx,$$

where $t > 0$.

Extending our studies to \mathbb{R}^2 and \mathbb{R}^3 we face the problem that we cannot give a general statement about the definability of $f(t)$. Consequently, we focus establishing asymptotic expansions of $f(t)$ in Section 3.2 and 3.3. Each of them starts with some preparations.

In the two-dimensional case we are faced with integrals of the form $\int e^{-s^2} s^{\frac{k}{q}+1} ds$, for which we derive a recursive formula in Lemma 3.2.1, and furthermore some asymptotic expansions. To find an asymptotic expansion of $f(t)$, we will also come across integrals of the form $\int e^{-\frac{r^2}{2t}} r \psi(r) dr$ in the main part of 3.2, where $\psi(r)$ is a bounded globally subanalytic function, which can be represented as a convergent Puiseux series, Their asymptotic expansions will be ascertained in Lemma

3.2.12 as t tends towards zero and in Lemma 3.2.24 as t approaches infinity in several steps. As a conclusion of this section, we obtain that the asymptotic expansion of $f(t)$ is a Puiseux series with positive exponents as t approaches zero, which is stated in Theorem 3.2.15. As $t \rightarrow \infty$, it is remarkable that $f(t)$ is definable in the o-minimal structure \mathbb{R}_{an} by an important theorem of Comte, Lion, and Rolin in [CLR00] if A is bounded. If A is unbounded we investigate an asymptotic expansion of $f(t)$ of the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where φ_i are convergent Puiseux series, and thus, the expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

The three-dimensional case also starts with some preparations. They contain recursive formulas for $\int e^{-s^2} s^{\frac{k}{q}} ds$, see Lemma 3.3.1, and for $\int e^{-s^2} s^{\frac{k}{q}} \log(s) ds$, see Lemma 3.3.13, and some asymptotic expansions. Afterwards, we are confronted with integrals of the form $\int e^{\frac{r^2}{2t}} r^2 \psi(r) dr$ and $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ in the main part. They are achieved by applying the polar transformation once again to $f(t)$ and by using an important result of Come, Lion, and Rolin [CLR00]. As t approaches zero, the asymptotic expansion of $\int e^{\frac{r^2}{2t}} r^2 \psi(r) dr$ will be established in Lemma 3.3.25 and of $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ in Lemma 3.3.27. In the case t tends towards infinity, their asymptotics are stated in Lemma 3.3.42. Our results of this section are resumed in Theorem 3.3.23. More precisely, $f(t)$ has an asymptotic expansion of the form $\sum (c_k + d_k \log(t)) t^{\frac{k}{q}}$, which is not necessarily convergent, as t tends towards zero. This is stated in Theorem 3.3.30. As t approaches infinity and if A is bounded, $f(t)$ is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ and of the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where φ_i are globally subanalytic functions. If A is unbounded, the asymptotic expansion of $f(t)$ is a constructible function of the form $\varphi_0 + \varphi_1 \log(t) + \varphi_2 \log(t)^2$, where φ_i are convergent Puiseux series, and thus, it is definable in $\mathbb{R}_{\text{an,exp}}$. These results are proven in Theorem 3.3.43.

3.1 Definability in the One-Dimensional Case

First, we study the definability of the probability distribution given by the Brownian Motion in \mathbb{R} on a family of sets $A_a := \{x \in \mathbb{R} \mid (a, x) \in A\}$, where $A \subset \mathbb{R}^n \times \mathbb{R}$ is a set definable in an o-minimal structure \mathcal{M} . For $t > 0$ this probability is represented as

$$f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{>0} \longrightarrow [0, 1],$$

$$(a, z, t) \longmapsto P_z(B_t \in A_a) = \frac{1}{\sqrt{2\pi t}} \int_{A_a} e^{-\frac{(x-z)^2}{2t}} dx,$$

by Remark 2.4.8. We show that f is definable in an o-minimal expansion of \mathcal{M} ; in the Pfaffian closure of \mathcal{M} . The trick of the proof is that the exponential function

fits the requirements of Corollary 2.3.13. First, we prove that the error function is definable in the Pfaffian closure in the next remark.

Remark 3.1.1. *Let \mathcal{M} be an o-minimal structure. The error function is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$.*

Proof. Let g be the function defined as

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R}, \\ y &\longmapsto e^{-y^2}. \end{aligned}$$

Then g is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$. By Corollary 2.3.13, the antiderivative of e^{-y^2} , that is

$$\int_0^x e^{-y^2} dy \stackrel{2.1.2}{=} \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$

by Definition 2.1.2, is definable in $\mathcal{P}(\mathcal{P}(\mathcal{M})) = \mathcal{P}(\mathcal{M})$. \square

The first goal of this chapter is to show definability of $f(a, z, t)$ in the Pfaffian closure by the next proposition.

Proposition 3.1.2. *Let \mathcal{M} be an arbitrary, but fixed o-minimal structure. Let $A \subset \mathbb{R}^n \times \mathbb{R}$ be a set definable in \mathcal{M} and $A_a := \{x \in \mathbb{R} \mid (a, x) \in A\}$ for $a \in \mathbb{R}^n$. The function*

$$\begin{aligned} f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_{>0} &\longrightarrow [0, 1], \\ (a, z, t) &\longmapsto \frac{1}{\sqrt{2\pi t}} \int_{A_a} e^{-\frac{(x-z)^2}{2t}} dx, \end{aligned}$$

is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$.

Proof. By Cell Decomposition Theorem 2.3.15, we can partition A into finitely many disjoint cells C_1, \dots, C_p . Let $\pi_n : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be the projection on the first n coordinates. Let $\psi_j, \varphi_j : \pi_n(C_j) \rightarrow \mathbb{R}$ be continuous functions definable in \mathcal{M} such that $\varphi_j < \psi_j$ for all $j \in \{1, \dots, p\}$. Then C_j has either the form

$$\begin{aligned} C_j &= \{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } \varphi_j(a) < x < \psi_j(a)\} \\ &\text{or} \\ C_j &= \{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } \varphi_j(a) < x\} \\ &\text{or} \end{aligned}$$

$$C_j = \{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } x < \varphi_j(a)\}$$

or

$$C_j = \{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } \varphi_j(a) = x\}$$

or

$$C_j = \{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j)\}.$$

By Definition 2.3.14 of a cell, the set $(C_j)_a = \{x \in \mathbb{R} \mid (a, x) \in C_j\}$ is also a cell for every $a \in \pi_n(C_j)$ and uniformly a point $\{\varphi_j(a)\}$ or an open interval. We look at the different cell types one by one.

Let C_j be of the form $\{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } \varphi_j(a) < x < \psi_j(a)\}$. By substitution $y = \frac{x-z}{\sqrt{2t}}$, we get

$$\begin{aligned} \int_{(C_j)_a} e^{\frac{-(x-z)^2}{2t}} dx &= \int_{\varphi_j(a)}^{\psi_j(a)} e^{\frac{-(x-z)^2}{2t}} dx \\ &= \sqrt{2t} \int_{\frac{\varphi_j(a)-z}{\sqrt{2t}}}^{\frac{\psi_j(a)-z}{\sqrt{2t}}} e^{-y^2} dy \\ &\stackrel{2.1.2}{=} \sqrt{2t} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(y) \right]_{\frac{\varphi_j(a)-z}{\sqrt{2t}}}^{\frac{\psi_j(a)-z}{\sqrt{2t}}} \\ &= \sqrt{2t} \frac{\sqrt{\pi}}{2} \left(\operatorname{erf} \left(\frac{\psi_j(a)-z}{\sqrt{2t}} \right) - \operatorname{erf} \left(\frac{\varphi_j(a)-z}{\sqrt{2t}} \right) \right) \end{aligned}$$

and we obtain definability in the Pfaffian closure of \mathcal{M} by Remark 3.1.1.

Let C_j be of the form $\{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } \varphi_j(a) < x\}$. By substitution $y = \frac{x-z}{\sqrt{2t}}$, we get

$$\begin{aligned} \int_{(C_j)_a} e^{\frac{-(x-z)^2}{2t}} dx &= \int_{\varphi_j(a)}^{\infty} e^{\frac{-(x-z)^2}{2t}} dx \\ &= \sqrt{2t} \int_{\frac{\varphi_j(a)-z}{\sqrt{2t}}}^{\infty} e^{-y^2} dy \end{aligned}$$

$$\begin{aligned} &\stackrel{2.1.2}{=} \sqrt{2t} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(y) \right]_{\frac{\varphi_j(a)-z}{\sqrt{2t}}}^{\infty} \\ &\stackrel{2.1.3}{=} \sqrt{\frac{\pi t}{2}} \left(1 - \operatorname{erf} \left(\frac{\varphi_j(a) - z}{\sqrt{2t}} \right) \right) \end{aligned}$$

and we obtain definability in the Pfaffian closure of \mathcal{M} by Remark 3.1.1.

Let C_j be of the form $\{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } x < \varphi_j(a)\}$. By substitution $y = \frac{x-z}{\sqrt{2t}}$, we get

$$\begin{aligned} \int_{(C_j)_a} e^{-\frac{(x-z)^2}{2t}} dx &= \int_{-\infty}^{\varphi_j(a)} e^{-\frac{(x-z)^2}{2t}} dx \\ &= \sqrt{2t} \int_{-\infty}^{\frac{\varphi_j(a)-z}{\sqrt{2t}}} e^{-y^2} dy \\ &\stackrel{2.1.2}{=} \sqrt{2t} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(y) \right]_{-\infty}^{\frac{\varphi_j(a)-z}{\sqrt{2t}}} \\ &\stackrel{2.1.3}{=} \sqrt{\frac{\pi t}{2}} \left(\operatorname{erf} \left(\frac{\varphi_j(a) - z}{\sqrt{2t}} \right) + 1 \right) \end{aligned}$$

and we obtain definability in the Pfaffian closure of \mathcal{M} by Remark 3.1.1.

Let C_j be of the form $\{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j) \text{ and } x = \varphi_j(a)\}$. Then

$$\int_{(C_j)_a} e^{-\frac{(x-z)^2}{2t}} dx = 0$$

for all $a \in \pi_n(C_j)$, and we obtain obviously definability in \mathcal{M} .

Let C_j be of the form $\{(a, x) \in \mathbb{R}^n \times \mathbb{R} \mid a \in \pi_n(C_j)\}$. By substitution $y = \frac{x-z}{\sqrt{2t}}$, we get

$$\begin{aligned} \int_{(C_j)_a} e^{-\frac{(x-z)^2}{2t}} dx &= \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{2t}} dx \\ &= \sqrt{2t} \int_{-\infty}^{\infty} e^{-y^2} dy \end{aligned}$$

$$\begin{aligned}
&\stackrel{2.1.2}{=} \sqrt{2t} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(y) \right]_{-\infty}^{\infty} \\
&\stackrel{2.1.3}{=} \sqrt{\frac{\pi t}{2}} 2 \\
&= \sqrt{2\pi t}.
\end{aligned}$$

This function is semialgebraic and therefore obviously definable in \mathcal{M} .

Hence, we can conclude that

$$\begin{aligned}
f(z, t, a) &= \frac{1}{\sqrt{2\pi t}} \int_{A_a} e^{-\frac{(x-z)^2}{2t}} dx \\
&= \frac{1}{\sqrt{2\pi t}} \sum_{j=1}^p \int_{(C_j)_a} e^{-\frac{(x-z)^2}{2t}} dx
\end{aligned}$$

as a finite sum of definable functions is definable in the Pfaffian closure of \mathcal{M} . \square

Corollary 3.1.3. *The probability distribution given by the Brownian Motion with start value z on a family of sets A_a , which is definable in an o -minimal structure \mathcal{M} , is definable in the Pfaffian closure $\mathcal{P}(\mathcal{M})$.*

3.2 The Two-Dimensional Case

3.2.1 Preparations

As part of this chapter we need the antiderivative of $e^{-s^2} s^{\frac{k}{q}+1}$, where $s \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{Z}$ and $q \in \mathbb{N}$, and some of its asymptotic expansions. The idea is to find a recursive formula for $\int e^{-s^2} s^{\frac{k}{q}+1} ds$ by integrating by parts. The recursion skips $2q$ steps; for this reason we rewrite integrals of the form $\int e^{-s^2} s^{\frac{k}{q}+1} ds$ as

$$F_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k-1} ds,$$

where $j \in \{0, \dots, 2q-1\}$. We denote definite integrals which belong to $F_{q,j,k}$ as

$$G_{q,j,k} := \int_a^b e^{-s^2} s^{\frac{j}{q}+2k-1} ds.$$

Furthermore, we show some asymptotic expansions. For this purpose, we reduce these integrals above to finitely many integrals by recursion. That recursive formula is shown in the next lemma.

Lemma 3.2.1. *Let $k \in \mathbb{Z}$, $q \in \mathbb{N}$, and $j \in \{0, \dots, 2q-1\}$. Up to an additive constant, $F_{q,j,k}$ is given by the recursive formula*

$$F_{q,j,k} = -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k-1)} + \left(\frac{j}{2q} + (k-1) \right) F_{q,j,k-1}.$$

Proof. We proof the recursive formula by integration by parts. We choose $e^{-s^2} s$ and $s^{\frac{j}{q}+2(k-1)}$ as parts and get

$$\begin{aligned} F_{q,j,k} &= \int e^{-s^2} s^{\frac{j}{q}+2k-1} ds \\ &= \int e^{-s^2} s s^{\frac{j}{q}+2(k-1)} ds \\ &= -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k-1)} + \frac{1}{2} \left(\frac{j}{q} + 2(k-1) \right) \int e^{-s^2} s^{\frac{j}{q}+2(k-1)-1} ds \\ &= -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k-1)} + \left(\frac{j}{2q} + (k-1) \right) F_{q,j,k-1}. \end{aligned}$$

□

In this way we can put down all the integrals $\int e^{-s^2} s^{\frac{j}{q}+2k-1} ds$ to finitely many

integrals $\int e^{-s^2} s^{\frac{j}{q}-1} ds$, which means to $k = 0$. For these special integrals, for which $k = 0$, we need the antiderivative of $e^{-s^2} s^{\frac{j}{q}-1}$.

Lemma 3.2.2. *Let $u \in \mathbb{Q}_{\geq 0}$ and let $s \in \mathbb{R}_{>0}$. Up to an additive constant, we get*

$$\int e^{-s^2} s^{u-1} ds = -\frac{1}{2} \Gamma\left(\frac{u}{2}, s^2\right).$$

Proof. We make the change of variables $s = \sqrt{t}$ and obtain

$$\begin{aligned} \int e^{-s^2} s^{u-1} ds &= \int e^{-t} \sqrt{t}^{u-1} \frac{1}{2} \sqrt{t}^{-1} dt \\ &= \frac{1}{2} \int e^{-t} \sqrt{t}^{u-2} dt \\ &= \frac{1}{2} \int e^{-t} t^{\frac{u}{2}-1} dt. \end{aligned} \tag{1}$$

In Definition 2.1.11 we have already seen that the incomplete gamma function is the antiderivative of $e^{-t} t^{\frac{u}{2}-1}$. Particularly, the integration limits should be taken into consideration, because the mentioned incomplete gamma function depends on them. We may assume $a, b \in \mathbb{R}_{>0}$, where $a < b$, are the integration limits and we proceed with the results above:

$$\begin{aligned} \int_a^b e^{-s^2} s^{u-1} ds &\stackrel{(1)}{=} \frac{1}{2} \int_{a^2}^{b^2} e^{-t} t^{\frac{u}{2}-1} dt \\ &= \frac{1}{2} \int_{a^2}^{\infty} e^{-t} t^{\frac{u}{2}-1} dt - \frac{1}{2} \int_{b^2}^{\infty} e^{-t} t^{\frac{u}{2}-1} dt \\ &\stackrel{2.1.11}{=} \frac{1}{2} \Gamma\left(\frac{u}{2}, a^2\right) - \frac{1}{2} \Gamma\left(\frac{u}{2}, b^2\right) \\ &= -\frac{1}{2} \left(\Gamma\left(\frac{u}{2}, b^2\right) - \Gamma\left(\frac{u}{2}, a^2\right) \right). \end{aligned}$$

Hence, we can conclude

$$\int e^{-s^2} s^{u-1} ds = -\frac{1}{2} \Gamma\left(\frac{u}{2}, s^2\right).$$

□

Corollary 3.2.3. *Let $u \in \mathbb{Q}_{>0}$. We get*

$$\int_0^{\infty} e^{-s^2} s^{u-1} ds = \frac{1}{2} \Gamma\left(\frac{u}{2}\right).$$

Proof. By the same substitution as in Lemma 3.2.2 above, we achieve by Definition 2.1.6 of the gamma function

$$\begin{aligned} \int_0^{\infty} e^{-s^2} s^{u-1} ds &= \frac{1}{2} \int_0^{\infty} e^{-t} t^{\frac{u}{2}-1} dt \\ &\stackrel{2.1.6}{=} \frac{1}{2} \Gamma\left(\frac{u}{2}\right). \end{aligned}$$

□

Perhaps the question arises why we investigate a recursive formula when we easily get the antiderivative of $F_{q,j,k}$ by Lemma 3.2.2? We could answer that sometimes it is an advantage to reduce the integrals $F_{q,j,k}$ to a finite number of integrals $F_{q,j,0}$. But the crucial point is, that we cannot use Lemma 3.2.2 for $F_{q,j,k}$ if k is a non-positive integer. Therefore we use recursion to reduce these integrals to integrals $F_{q,j,0}$ as we see later.

The special integrals for $k = 0$ are gathered in the next lemma. In fact, $(j, k) = (0, 1)$ is also a special case which is listed. We want to remark that for $j = q$ the antiderivative of the integral $F_{q,q,0}$ can also be represented by $\frac{\sqrt{\pi}}{2} \operatorname{erf}(s)$ because of Definition 2.1.2 respectively the relation which is stated in Remark 2.1.15.

Lemma 3.2.4. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q-1\}$. Up to an additive constant, we obtain*

$$\begin{aligned} F_{q,j,0} &= \int e^{-s^2} s^{\frac{j}{q}-1} ds \\ &= -\frac{1}{2} \Gamma\left(\frac{j}{2q}, s^2\right) \end{aligned}$$

and for $(j, k) = (0, 1)$

$$\begin{aligned} F_{q,0,1} &= \int e^{-s^2} s ds \\ &= -\frac{1}{2} e^{-s^2}. \end{aligned}$$

Proof. First, we assume $k = 0$. We obtain up to an additive constant by Lemma 3.2.2

$$\begin{aligned} F_{q,j,0} &= \int e^{-s^2} s^{\frac{j}{q}-1} ds \\ &\stackrel{3.2.2}{=} -\frac{1}{2}\Gamma\left(\frac{j}{2q}, s^2\right) \end{aligned}$$

if the conditions $j \neq 0$ or $s^2 \neq 0$ are fulfilled, see Definition 2.1.6.

For the special case $(j, k) = (0, 1)$ we get up to an additive constant

$$\begin{aligned} F_{q,0,1} &= \int e^{-s^2} s ds \\ &= -\frac{1}{2}e^{-s^2}. \end{aligned}$$

□

Now, we give some preparations for the case $k > 0$. First, we calculate the integral $\int_0^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds$ in the next lemma and afterwards, we devote our attention to the asymptotic series expansions of the integrals $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds$, where $k > 0$ and x tends towards infinity.

Lemma 3.2.5. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. We obtain*

$$\int_0^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds = \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell\right) \frac{1}{2}\Gamma\left(\frac{j}{2q} + 1\right).$$

Proof. We set $G_{q,j,k} := \int_0^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds$. Let $k = 1$. By Definition 2.1.6 of the gamma function and Remark 2.1.7, we ascertain

$$\int_0^\infty e^{-s^2} s^{\frac{j}{q}+2-1} ds \stackrel{3.2.2}{\stackrel{2.1.6}{=}} \frac{1}{2}\Gamma\left(\frac{j}{2q} + 1\right).$$

Therefore, we get by the recursive formula

$$\begin{aligned}
 \int_0^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds &\stackrel{3.2.1}{=} \left[-\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k-1)} \right]_0^\infty + \left(\frac{j}{2q} + (k-1) \right) G_{q,j,k-1} \\
 &\stackrel{I.H.}{=} \left(\frac{j}{2q} + (k-1) \right) \prod_{\ell=1}^{k-2} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) \\
 &= \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right).
 \end{aligned}$$

□

Lemma 3.2.6. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q-1\}$. As $x \rightarrow \infty$, we achieve the asymptotic expansion*

$$\begin{aligned}
 \int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds &\approx \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) \\
 &\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j}{q}+2k-2n} \prod_{l=1}^{n-1} \left(\frac{j}{2q} + k - l \right).
 \end{aligned}$$

Proof. By using the asymptotic expansion of the incomplete gamma function in Lemma 2.2.13 and Lemma 3.2.5, we get

$$\begin{aligned}
 \int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds &= \int_0^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds - \int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds \\
 &\stackrel{3.2.5}{=} \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) - \frac{1}{2} \Gamma \left(\frac{j}{2q} + k, x^2 \right) \\
 &\stackrel{2.2.13}{\approx} \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) \\
 &\quad - e^{-x^2} \sum_{n=1}^{\infty} (x^2)^{\frac{j}{2q}+k-n} \prod_{l=1}^{n-1} \left(\frac{j}{2q} + k - l \right)
 \end{aligned}$$

$$\begin{aligned}
&= \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) \\
&\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j}{q} + 2(k-n)} \prod_{l=1}^{n-1} \left(\frac{j}{2q} + k - l \right).
\end{aligned}$$

□

Corollary 3.2.7. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow \infty$, we get the dominant term*

$$\int_0^x e^{-s^2} s^{\frac{j}{q} + 2k - 1} ds \sim \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right).$$

Proof. It is an obvious consequence of Lemma 3.2.6. □

In the next paragraph we give preparations for the case that k is a non-positive integer. For calculating integrals with negative exponents we reorganize the recursive formula of Lemma 3.2.1 in the following way.

Corollary 3.2.8. *Let $k \in \mathbb{Z}_{<0}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. Up to an additive constant, $F_{q,j,k}$ can also recursively be given by*

$$F_{q,j,k} = \frac{1}{\frac{j}{2q} + k} \left(\frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2k} \right) + \frac{1}{\frac{j}{2q} + k} F_{q,j,k+1}.$$

Proof. We use the recursive formula

$$F_{q,j,k} = -\frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2(k-1)} + \left(\frac{j}{2q} + (k-1) \right) F_{q,j,k-1}$$

of Lemma 3.2.1. By reorganizing to the last term, we get for $k < 1$

$$F_{q,j,k-1} = \frac{1}{\frac{j}{2q} + (k-1)} \left(\frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2(k-1)} \right) + \frac{1}{\frac{j}{2q} + (k-1)} F_{q,j,k}$$

and thus

$$F_{q,j,k} = \frac{1}{\frac{j}{2q} + k} \frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2k} + \frac{1}{\frac{j}{2q} + k} F_{q,j,k+1}$$

for $k < 0$. □

In the next Lemma we see an application of the recursive formula in its form for negative exponents and we calculate $\int_x^{\infty} e^{-s^2} s^{\frac{j}{q} + 2k - 1} ds$ where $k < 0$.

Lemma 3.2.9. *Let $k \in \mathbb{Z}_{<0}$, $x \in \mathbb{R}_{>0}$, where $x \geq 1$, $q \in \mathbb{N}$, and $j \in \{0, \dots, 2q-1\}$. We get*

$$\begin{aligned} \int_x^\infty e^{-s^2} s^{\frac{j}{2q}+2k-1} ds &= \frac{-1}{\frac{j}{2q}+k} \frac{1}{2} e^{-x^2} x^{\frac{j}{2q}+2k} - \sum_{m=1}^{-k-1} \left(\prod_{\ell=0}^{-k-m} \frac{1}{\frac{j}{2q}+k+\ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{2q}-2m} \\ &\quad + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q}-\ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right). \end{aligned}$$

Proof. We set $G_{q,j,k}(x) := \int_x^\infty e^{-s^2} s^{\frac{j}{2q}+2k-1} ds$. We start the induction with $k = -1$. We ascertain by the special values for $k = 0$ in Lemma 3.2.4 and the recursive formula in Corollary 3.2.8

$$\begin{aligned} G_{q,j,-1}(x) &\stackrel{3.2.8}{=} \left[\frac{1}{\frac{j}{2q}-1} \frac{1}{2} e^{-s^2} s^{\frac{j}{2q}-2} \right]_x^\infty + \frac{1}{\frac{j}{2q}-1} G_{q,j,0} \\ &\stackrel{3.2.4}{=} \frac{-1}{\frac{j}{2q}-1} \frac{1}{2} e^{x^2} x^{\frac{j}{2q}-2} + \frac{1}{\frac{j}{2q}-1} \left[-\frac{1}{2} \Gamma\left(\frac{j}{2q}, s^2\right) \right]_x^\infty \\ &\stackrel{2.1.12}{=} \frac{-1}{\frac{j}{2q}-1} \frac{1}{2} e^{x^2} x^{\frac{j}{2q}-2} + \frac{1}{\frac{j}{2q}-1} \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right). \end{aligned}$$

For the induction step we get by applying Corollary 3.2.8

$$\begin{aligned} G_{q,j,k}(x) &\stackrel{3.2.8}{=} \left[\frac{1}{\frac{j}{2q}+k} \frac{1}{2} e^{-s^2} s^{\frac{j}{2q}+2k} \right]_x^\infty + \frac{1}{\frac{j}{2q}+k} G_{q,j,k+1}(x) \\ &\stackrel{I.H.}{=} \frac{-1}{\frac{j}{2q}+k} \frac{1}{2} e^{-x^2} x^{\frac{j}{2q}+2k} + \frac{1}{\frac{j}{2q}+k} \left[\frac{-1}{\frac{j}{2q}+(k+1)} \frac{1}{2} e^{-x^2} x^{\frac{j}{2q}+2(k+1)} \right. \\ &\quad \left. - \sum_{m=1}^{-(k+1)-1} \left(\prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j}{2q}+(k+1)+\ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{2q}-2m} \right. \\ &\quad \left. + \left(\prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j}{2q}-\ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right) \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{-1}{\frac{j}{2q} + k} \frac{1}{2} e^{-x^2} x^{\frac{j}{q} + 2k} + \frac{1}{\frac{j}{2q} + k} \left[\frac{-1}{\frac{j}{2q} + (k+1)} \frac{1}{2} e^{-x^2} x^{\frac{j}{q} - 2(-(k+1))} \right. \\
&\quad \left. - \sum_{m=1}^{-(k+2)} \left(\prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j}{2q} + (k+1) + \ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q} - 2m} \right] \\
&\quad + \frac{1}{\frac{j}{2q} + k} \left(\prod_{\ell=1}^{-k-1} \frac{1}{\frac{j}{2q} - \ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right) \\
&= \frac{-1}{\frac{j}{2q} + k} \frac{1}{2} e^{-x^2} x^{\frac{j}{q} + 2k} \\
&\quad + \frac{1}{\frac{j}{2q} + k} \left(- \sum_{m=1}^{-k-1} \left(\prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j}{2q} + k + 1 + \ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q} - 2m} \right) \\
&\quad + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q} - \ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right) \\
&= \frac{-1}{\frac{j}{2q} + k} \frac{1}{2} e^{-x^2} x^{\frac{j}{q} + 2k} \\
&\quad + \frac{1}{\frac{j}{2q} + k} \left(- \sum_{m=1}^{-k-1} \left(\prod_{\ell=1}^{-(k+1)-m+1} \frac{1}{\frac{j}{2q} + k + \ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q} - 2m} \right) \\
&\quad + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q} - \ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right) \\
&= \frac{-1}{\frac{j}{2q} + k} \frac{1}{2} e^{-x^2} x^{\frac{j}{q} + 2k} \\
&\quad - \sum_{m=1}^{-k-1} \left(\prod_{\ell=0}^{-k-m} \frac{1}{\frac{j}{2q} + k + \ell} \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q} - 2m} \\
&\quad + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q} - \ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, x^2\right).
\end{aligned}$$

□

Next, we establish the asymptotic expansion of $\int_x^\infty e^{-s^2} s^{\frac{j}{q} + 2k - 1} ds$, where $k \leq 0$, as x tends towards zero by using the asymptotic series expansion of the exponential function. We have to consider that $\int_x^\infty e^{-s^2} s^{-1} ds$ is special for x approaching zero.

Lemma 3.2.10. *Let $k \in \mathbb{Z}_{\leq 0}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow 0$,*

$$\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds$$

has for $j \neq 0$ the asymptotic expansion

$$d_{j,k} + \sum_{n=0}^{\infty} b_{j,k,n} x^{\frac{j}{q}+2(n+k)}$$

and for $j = 0$ the asymptotic expansion

$$d_{0,k} + \sum_{n=0}^{\infty} b_{0,k,n} x^{2(n+k)} + (-1)^{-k+1} \frac{1}{(-k)!} \log(x),$$

where

$$b_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = -k, \\ (-1)^{n+1} \frac{1}{(\frac{j}{q}+2(n+k))n!}, & \text{else,} \end{cases}$$

and

$$d_{j,k} := \begin{cases} \sum_{n=0}^{\infty} (-b_{j,0,n}) + \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k = 0, \\ \sum_{n=0}^{\infty} (-b_{j,k,n}) - \frac{1}{\frac{j}{2q}+k} \frac{1}{2} e^{-1} - \sum_{m=1}^{-k-1} \left(\prod_{\ell=1}^{-k-m} \frac{1}{\frac{j}{2q}+k+\ell} \right) \frac{1}{2} e^{-1} \\ + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q}-\ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k < 0. \end{cases}$$

Proof. We set $G_{q,j,k}(x) := \int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds$. As $x \rightarrow 0$, we can write

$$G_{q,j,k}(x) = \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k-1} ds + \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k-1} ds.$$

According to Lemma 3.2.4, we set for $k = 0$

$$\begin{aligned} c_{j,0} &:= \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}-1} ds \\ &\stackrel{3.2.4}{=} \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right) \end{aligned}$$

and for $k < 0$ according to Lemma 3.2.9

$$\begin{aligned} c_{j,k} &:= \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k-1} ds \\ &\stackrel{3.2.9}{=} \frac{-1}{\frac{j}{2q} + k} \frac{1}{2} e^{-1} - \sum_{m=1}^{-k-1} \left(\prod_{\ell=1}^{-k-m} \frac{1}{\frac{j}{2q} + k + \ell} \right) \frac{1}{2} e^{-1} + \left(\prod_{\ell=1}^{-k} \frac{1}{\frac{j}{2q} - \ell} \right) \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right). \end{aligned}$$

Hence, we attain with the asymptotic expansion of the exponential function in Lemma 2.2.11 and above

$$\begin{aligned} G_{q,j,k}(x) &= \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k-1} ds + \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k-1} ds \\ &\stackrel{2.2.11}{=} \int_x^1 \sum_{n=0}^{\infty} (-1)^n s^{2n} \frac{1}{n!} s^{\frac{j}{q}+2k-1} ds + c_{j,k} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)-1} ds + c_{j,k}. \end{aligned}$$

It is necessary to distinguish between two cases. First, we may assume $j \neq 0$ and consequently we get $\frac{j}{q} + 2(n+k) \neq 0$ for all $n \in \mathbb{N}_0$. We have

$$\begin{aligned} G_{q,j,k}(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)-1} ds + c_{j,k} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left[\frac{1}{\frac{j}{q} + 2(n+k)} s^{\frac{j}{q}+2(n+k)} \right]_x^1 + c_{j,k} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{\frac{j}{q} + 2(n+k)} x^{\frac{j}{q}+2(n+k)} + d_{j,k}, \end{aligned}$$

where we were able to set

$$d_{j,k} := \sum_{n=0}^{\infty} (-1)^n \frac{1}{\binom{j}{q} + 2(n+k)} n! + c_{j,k},$$

because the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\binom{j}{q} + 2(n+k)} n!$ converges. Next, we may assume $j = 0$. Since $2(n+k) = 0$ for $n = -k$, we split the sum and achieve

$$\begin{aligned} G_{q,0,k}(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \int_x^1 s^{2(n+k)-1} ds + c_{0,k} \\ &= \sum_{\substack{n=0 \\ n \neq -k}}^{\infty} (-1)^n \frac{1}{n!} \int_x^1 s^{2(n+k)-1} ds + (-1)^k \frac{1}{(-k)!} \int_x^1 s^{-1} ds + c_{0,k} \\ &= \sum_{\substack{n=0 \\ n \neq -k}}^{\infty} (-1)^n \frac{1}{n!} \left[\frac{1}{2(n+k)} s^{2(n+k)} \right]_x^1 + (-1)^k \frac{1}{(-k)!} [\log(s)]_x^1 + c_{0,k} \\ &= d_{0,k} + \sum_{\substack{n=0 \\ n \neq -k}}^{\infty} (-1)^{n+1} \frac{1}{n!} \frac{1}{2(n+k)} x^{2(n+k)} + (-1)^{k+1} \frac{1}{(-k)!} \log(x), \end{aligned}$$

where we were able to set

$$d_{0,k} := \sum_{\substack{n=0 \\ n \neq -k}}^{\infty} (-1)^n \frac{1}{2(n+k)n!} + c_{0,k},$$

because the series $\sum_{\substack{n=0 \\ n \neq -k}}^{\infty} (-1)^n \frac{1}{2(n+k)n!}$ converges.

Finally, we set

$$b_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = -k, \\ (-1)^{n+1} \frac{1}{\binom{j}{q} + 2(n+k)} n!, & \text{else,} \end{cases}$$

and

$$d_{j,k} := \sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k}.$$

□

3.2.2 Definability and Asymptotic Expansions

Extending our research to \mathbb{R}^2 we work on the definability and the asymptotic expansion of the probability distribution given by the standard Brownian Motion on a globally subanalytic set $A \subset \mathbb{R}^2$ for time $t > 0$. By Remark 2.4.8, this probability is given by

$$\begin{aligned} f : \mathbb{R}_{>0} &\longrightarrow [0, 1], \\ t &\longmapsto P_0(B_t \in A) = \frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx. \end{aligned}$$

In the following main part, as well in \mathbb{R}^3 below, we consider the probability distribution on a globally subanalytic set and not only on a semialgebraic set. The reason is that we would have to enlarge the o-minimal structure \mathbb{R} to \mathbb{R}_{an} in our proof quite early anyway. As we mentioned above, we assume without loss of generality that the dimension of A is equal two to avoid integration over a null set. In \mathbb{R}^2 we face the problem that we cannot give a general statement about the definability of $f(t)$. Therefore, we devote our attention to the limit points of t . We investigate asymptotic expansions of $f(t)$ as t tends towards zero and towards infinity and we examine the definability of $f(t)$ there. The next theorem gathers our results :

Theorem 3.2.11. *Let $t \in \mathbb{R}_{>0}$ and let $A \subset \mathbb{R}^2$ be a globally subanalytic set. Let $f(t)$ be the probability distribution given by the standard Brownian Motion at time t on A , which is defined as*

$$f(t) := \frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

i) As $t \rightarrow 0$, $f(t)$ has the asymptotic series expansion

$$\sum_{k=0}^{\infty} \gamma_k t^{\frac{k}{q}},$$

which is not necessarily convergent, where $\gamma_k \in \mathbb{R}$ and $q \in \mathbb{N}$.

ii) As $t \rightarrow \infty$:

a) If A is bounded, $f(t)$ is definable in \mathbb{R}_{an} and therefore a convergent Puiseux series.

b) If A is unbounded, $f(t)$ has the asymptotic series expansion

$$f(t) \approx \rho_0(t) + \rho_1(t) \log(t),$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \delta_k t^{-\frac{k}{q}}$ and $\rho_1(t) = \sum_{k=1}^{\infty} \gamma_k t^{-k}$, where at least one of the series is not zero and $\delta_k, \gamma_k \in \mathbb{R}$, are convergent Puiseux series and therefore, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

As t tends towards infinity, we notice that it makes a vital difference if the set A is bounded or not. We prove this theorem in several small steps below and part it into the cases t tends towards zero and t approaches infinity.

The Case t at Zero

Our first aim is to show that $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$ behaves like a Puiseux series as t approaches zero. For this purpose, we need the asymptotic behaviour respectively the order of integrals of the form

$$\int_D e^{-\frac{r^2}{2t}} r \psi(r) dr,$$

where $\psi(r)$ is a globally subanalytic continuous function, which is bounded on an open interval $D \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$. As we will see later on, we can reduce D to three forms of intervals by using the so-called cell decomposition. They are given by: $]0, \beta[$, where $\beta < 1$, $] \alpha, \beta[$, where $0 < \alpha < \beta < \infty$, and $] \alpha, \infty[$, where $\alpha \geq 1$.

First, we may assume $D =]0, \beta[$. The function $\psi(r)$ can be written as a convergent Puiseux series around the center zero by the so-called Preparation Theorem as we will argue later. As part of the next lemma we obtain integrals of the form $\int_0^\beta e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k-1} dr$ which we can convert to $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds$ as x approaches infinity by substitution. These integrals and their asymptotics expansions have already been pointed out in Lemma 3.2.6.

Lemma 3.2.12. *Let $\beta \in \mathbb{R}_{>0}$. Let $\psi(r)$ be a bounded function on $]0, \beta[$, which can be represented as the convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k}$ at 0, where $q \in \mathbb{N}$. As $t \rightarrow 0$, we get the asymptotic series expansion*

$$\int_0^\beta e^{-\frac{r^2}{2t}} r \psi(r) dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+1},$$

where $\hat{a}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k+1} \prod_{\ell=1}^k \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right)$.

Proof. We set $F(t) := \int_0^\beta e^{-\frac{r^2}{2t}} r \psi(r) dr$. We can split $F(t)$ into

$$\begin{aligned} F(t) &= \int_0^\beta e^{-\frac{r^2}{2t}} r \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} dr \\ &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr \\ &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr + \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr. \end{aligned}$$

First, we show as $t \rightarrow 0$

$$F(t) - \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr = \mathcal{O}(t^{N+2})$$

for sufficiently large $N \in \mathbb{N}$, where N is a multiple of q . We make a change of variables $s = \frac{r}{\sqrt{2t}}$ and achieve by the split of $F(t)$ above

$$\begin{aligned} F(t) - \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr \\ &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} dr \\ &= \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \left(s\sqrt{2t} \right)^{\frac{j}{q}+2k+1} \sqrt{2t} ds. \end{aligned}$$

By estimation of the remainder, we have

$$\sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \left(s\sqrt{2t} \right)^{\frac{j}{q}+2k+1} \leq C \left(s\sqrt{2t} \right)^{2(N+1)+1}$$

for $0 \leq s\sqrt{2t} \leq \beta$ and a sufficiently large constant $C \in \mathbb{R}$. Thus, we can continue

$$\begin{aligned}
 & \left| \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \left(s\sqrt{2t} \right)^{\frac{j}{q}+2k+1} \sqrt{2t} \, ds \right| \\
 & \leq C \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \left(s\sqrt{2t} \right)^{2(N+1)+1} \sqrt{2t} \, ds \\
 & = \sqrt{2t}^{2(N+1)+2} C \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{2(N+1)+1} \, ds \\
 & \leq \sqrt{2t}^{2(N+2)} C \int_0^{\infty} e^{-s^2} s^{2(N+1)+2-1} \, ds \\
 & = \sqrt{2t}^{2(N+2)} C \int_0^{\infty} e^{-s^2} s^{2(N+2)-1} \, ds \\
 & \stackrel{3.2.3}{=} \sqrt{2t}^{2(N+2)} C \frac{1}{2} \Gamma(N+2). \tag{1}
 \end{aligned}$$

By Definition 2.2.1, we conclude

$$\begin{aligned}
 F(t) - \int_0^{\beta} e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} \, dr &= \int_0^{\beta} e^{-\frac{r^2}{2t}} \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+1} \, dr \\
 &\stackrel{(1)}{=} \mathcal{O}(t^{N+2}). \tag{2}
 \end{aligned}$$

We proceed by a change of variables $s = \frac{r}{\sqrt{2t}}$ and write

$$\begin{aligned}
 F(t) &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}+2k+2} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k+1} \, ds \\
 &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}+2(k+1)} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2(k+1)-1} \, ds
 \end{aligned}$$

$$= \sum_{k=0}^{\infty} F_{k+1}(t),$$

where $F_k(t) := \sum_{j=0}^{2q-1} a_{j,k-1} \sqrt{2t}^{\frac{j}{q}+2k} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k-1} ds$. The asymptotic series expansion of $\int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k-1} ds$ is

$$\prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) - e^{-\frac{\beta^2}{2t}} \sum_{n=1}^{\infty} \left(\frac{\beta}{\sqrt{2t}} \right)^{\frac{j}{q}+2k-2n} \prod_{l=1}^{n-1} \left(\frac{j}{2q} + k - l \right)$$

and has already been pointed out in Lemma 3.2.6. In addition, we have proven in Corollary 3.2.7

$$\int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k-1} ds \stackrel{3.2.7}{\sim} \prod_{l=1}^{k-1} \left(\frac{j}{2q} + l \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right). \quad (3)$$

Due to the fact that $e^{-\frac{\beta^2}{2t}}$ tends towards zero very fast in comparison to t^N , $N \in \mathbb{N}$, if t approaches zero, only the constant terms are of interest. Hence, we get the asymptotic expansion $\hat{F}_k(t)$ of $F_k(t)$ as $t \rightarrow 0$, that is

$$\begin{aligned} \hat{F}_k(t) &= \sum_{j=0}^{2q-1} a_{j,k-1} \sqrt{2t}^{\frac{j}{q}+2k} \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) \\ &= \sum_{j=0}^{2q-1} a_{j,k-1} 2^{\frac{j}{2q}+k} \prod_{\ell=1}^{k-1} \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right) t^{\frac{j}{2q}+k} \\ &= \sum_{j=0}^{2q-1} \hat{a}_{j,k-1} t^{\frac{j}{2q}+k}, \end{aligned}$$

where $\hat{a}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k+1} \prod_{\ell=1}^k \left(\frac{j}{2q} + \ell \right) \Gamma \left(\frac{j}{2q} + 1 \right)$, and we observe that

$$\hat{F}_k(t) = \mathcal{O}(t^k). \quad (4)$$

We set

$$\begin{aligned}\hat{F}(t) &:= \sum_{k=0}^{\infty} \hat{F}_{k+1}(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+1}.\end{aligned}$$

Due to this, it is evident by

$$\begin{aligned}F(t) - \hat{F}|_N(t) &= \sum_{k=0}^{\infty} F_{k+1}(t) - \left(\sum_{k=0}^{\infty} \hat{F}_{k+1}(t) \right) |_N \\ &\stackrel{(4)}{=} \sum_{k=0}^{\infty} F_{k+1}(t) - \sum_{k=0}^N \hat{F}_{k+1}|_N(t) \\ &= \sum_{k=0}^N F_{k+1}(t) - \sum_{k=0}^N \hat{F}_{k+1}|_N(t) + \sum_{k=N+1}^{\infty} F_{k+1}(t) \\ &\stackrel{(2)}{=} \sum_{k=0}^N \left(F_{k+1}(t) - \hat{F}_{k+1}|_N(t) \right) + \mathcal{O}(t^{N+2}) \\ &\stackrel{(3)}{=} \mathcal{O}(t^{N+2})\end{aligned}$$

that

$$F(t) \approx \hat{F}(t)$$

and as a consequence, $F(t)$ behaves like a Puiseux series. \square

To achieve our aim, it is enough to show that the integrals $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr$ are in the scale $\mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right)$, where $\psi(r)$ is a continuous and bounded function on $] \alpha, \infty[$.

Lemma 3.2.13. *Let $\alpha \in \mathbb{R}$, where $\alpha \geq 1$, and let $\psi(r)$ be a continuous and bounded function on $] \alpha, \infty[$. As $t \rightarrow 0$, we obtain*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right).$$

Proof. It exists a sufficiently large constant C such that $\psi(r) \leq C$ on $] \alpha, \infty[$. By

a change of variables $s = \frac{r}{\sqrt{2t}}$, we attain

$$\begin{aligned}
 \left| \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr \right| &\leq C \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r dr \\
 &= C \sqrt{2t} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s \sqrt{2t} ds \\
 &\stackrel{3.2.4}{=} C 2t \left[-\frac{1}{2} e^{-s^2} \right]_{\frac{\alpha}{\sqrt{2t}}}^{\infty} \\
 &= C t e^{-\frac{\alpha^2}{2t}}
 \end{aligned}$$

and with Definition 2.2.1 we come to the conclusion

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O} \left(e^{-\frac{\alpha^2}{2t}} t \right).$$

□

The next lemma considers $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr$, where $\psi(r)$ is also a continuous and bounded function on $] \alpha, \beta[$ and $0 < \alpha < \beta < \infty$. An argument quite similar to the one used in the last lemma shows that we achieve essentially the same order as above.

Lemma 3.2.14. *Let $] \alpha, \beta[\subset \mathbb{R}_{>0}$, where $0 < \alpha < \beta < \infty$, and let $\psi(r)$ be a continuous and bounded function on $] \alpha, \beta[$. As $t \rightarrow 0$, we have*

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O} \left(e^{-\frac{\alpha^2}{2t}} t \right).$$

Proof. There is a sufficiently large constant $C \in \mathbb{R}$ such that $\psi(r) \leq C$ on $] \alpha, \beta[$.

We make a change of variables $s = \frac{r}{\sqrt{2t}}$ and get

$$\begin{aligned} \left| \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr \right| &\leq \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r C dr \\ &\leq C \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r dr \end{aligned}$$

Thus, we obtain with the same proof as in Lemma 3.2.13 as $t \rightarrow 0$

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr \stackrel{2.2.1}{=} \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right).$$

□

Now, we return to the integrals $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$ and ascertain their asymptotic series expansion as t tends towards zero. The idea of the proof is that we transform $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$ in polar coordinates and apply the Cell Decomposition Theorem.

As a result, we obtain integrals of the form $\int_0^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr$, $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr$, and $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr$, which we have already pointed out in the lemmas above. Afterwards, we show that $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$ behaves like a Puiseux series as t approaches zero by using these preceding lemmas.

Theorem 3.2.15. *Let $A \subset \mathbb{R}^2$ be a globally subanalytic set. Let $f : \mathbb{R}_{>0} \rightarrow [0, 1]$ be the function given by*

$$f(t) = \frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

As $t \rightarrow 0$, we have the asymptotic series expansion

$$f(t) \approx \sum_{k=0}^{\infty} \gamma_k t^{\frac{k}{q}},$$

where $\gamma_k \in \mathbb{R}$.

Proof. Let

$$\begin{aligned} \Theta : \mathbb{R}_{\geq 0} \times]-\pi, \pi] &\longrightarrow \mathbb{R} \times \mathbb{R}, \\ (r, \varphi) &\longmapsto (r \cos \varphi, r \sin \varphi), \end{aligned}$$

be the two dimensional polar coordinate transformation. First, we transform $f(t)$ in polar coordinates by Transformation Theorem 2.1.17 and get

$$\begin{aligned} f(t) &= \frac{1}{2\pi t} \int_A e^{\frac{-|x|^2}{2t}} dx \\ &\stackrel{2.1.17}{=} \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{\frac{-|(r \cos \varphi, r \sin \varphi)|^2}{2t}} r d(r, \varphi) \\ &= \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{\frac{-r^2}{2t}} r d(r, \varphi). \end{aligned}$$

The function Θ is continuous and definable in the o-minimal structure \mathbb{R}_{an} , consequently $B := \Theta^{-1}(A) \subset \mathbb{R}_{\geq 0} \times]-\pi, \pi]$ is also definable in \mathbb{R}_{an} . By Cell Decomposition Theorem 2.3.15, we can partition B into finitely many disjoint cells C_1, \dots, C_p . For the sake of convenience we fix $j \in \{1, \dots, p\}$ and write $C := C_j$. Let $\pi_1 : \mathbb{R}_{\geq 0} \times]-\pi, \pi] \rightarrow \mathbb{R}_{\geq 0}$ be the projection on the first coordinate, let $D := \pi_1(C)$, and let $\psi, \eta : D \rightarrow]-\pi, \pi]$ be definable continuous functions such that $\eta < \psi$ and such that

$$C = \{(r, \varphi) \in \mathbb{R}_{\geq 0} \times]-\pi, \pi] \mid r \in D \text{ and } \eta(r) < \varphi < \psi(r)\}.$$

Due to integration, we can reduce the different cell types of Definition 2.3.14 to this single cell C . As angular part of the cell both $|\eta|$ and $|\psi|$ are bounded by π . Cells are defined inductively, for this reason $D \subset \mathbb{R}$ is also a cell and either a point $\{\alpha\}$ or an open interval $]\alpha, \beta[$ with $\alpha, \beta \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\alpha < \beta$. If D is a point, the integrals over D will be zero.

Using Fubini's Theorem 2.1.18, we get

$$\begin{aligned}
 \int_C e^{-\frac{r^2}{2t}} r d(r, \varphi) &\stackrel{2.1.18}{=} \int_D \int_{\varphi=\eta(r)}^{\psi(r)} e^{-\frac{r^2}{2t}} r d\varphi dr \\
 &= \int_D e^{-\frac{r^2}{2t}} r \int_{\eta(r)}^{\psi(r)} d\varphi dr \\
 &= \int_D e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr \\
 &= \int_D e^{-\frac{r^2}{2t}} r \psi(r) dr - \int_D e^{-\frac{r^2}{2t}} r \eta(r) dr.
 \end{aligned}$$

If B is a bounded set, then the cells C of its cell decomposition are also bounded and D is either a point or a bounded interval. If B is unbounded, so the cells C of its cell decomposition could be bounded, but at least one cell is unbounded. That means D , if an unbounded cell C is based on, could also be an interval of the form $]\alpha, \infty[$.

First, we study $\alpha = 0$ and $\beta < \infty$. By a refined cell decomposition, we may assume $\beta < 1$. By the Preparation Theorem 2.3.18, there is such a cell decomposition of B , so that $\psi(r)$ can be represented as a convergent Puiseux series at zero. Caused by the boundedness of $\psi(r)$, this Puiseux series has only positive exponents. As $t \rightarrow 0$, we have established in Lemma 3.2.12

$$\int_0^\beta e^{-\frac{r^2}{2t}} r \psi(r) dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+1}, \quad (1)$$

where $q \in \mathbb{N}$ and $\hat{a}_{j,k}$ are constants in \mathbb{R} . Analogously, we obtain the asymptotic expansion of $\int_0^\beta e^{-\frac{r^2}{2t}} r \eta(r) dr$. By the requirements above, it is $\eta(r) < \psi(r)$ and thus, $\psi(r) - \eta(r) > 0$. We get that

$$\int_0^\beta e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr > 0 \quad (2)$$

and it has the asymptotic series expansion $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} c_{j,k} t^{\frac{j}{2q}+k+1}$, where $c_{j,k} \in \mathbb{R}$ is the

difference of the coefficients of the asymptotic expansion in (1) and that one of $\eta(r)$. Consequently, it behaves like a Puiseux series.

For $0 < \alpha < \beta < \infty$ we have proven as $t \rightarrow 0$

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right)$$

in Lemma 3.2.14. With the same arguments as above we attain

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr - \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \eta(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right).$$

If C is unbounded, as previously mentioned, at least one D is consequently unbounded. That means D could also be an interval of the form $]\alpha, \infty[$ where $\alpha \in \mathbb{R}_{>0}$. By refined cell decomposition, we can assume $\alpha \geq 1$. In Lemma 3.2.13 we have already seen that

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right),$$

as $t \rightarrow 0$. With the same arguments as above we get

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr - \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \eta(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right).$$

Finally, we gather the results. We have the fact $e^{-\frac{\alpha^2}{2t}} t = \mathcal{O}(t^N)$ for an arbitrary $N \in \mathbb{N}$. Hence, the cells that are based on cells D of the form $]\alpha, \beta[$ or $]\alpha, \infty[$, where $\alpha \neq 0$, do not contribute to the integral over the total set in our scale t^N . Thus, we sum up

$$\begin{aligned} \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r d(r, \varphi) &= \frac{1}{2\pi t} \sum_{j=1}^p \int_{C_j} e^{-\frac{r^2}{2t}} r d(r, \varphi) \\ &\approx \sum_{k=0}^{\infty} \sum_{m=0}^{2q-1} d_{k,m} t^{\frac{m}{2q}+k}, \end{aligned}$$

where $d_{k,m} \in \mathbb{R}$. This is the desired conclusion. \square

In this thesis we focus on asymptotic expansions in a polynomial scale. Therefore, we want to remark the following points about the established asymptotic expansion above.

Remark 3.2.16. *a) If the closure of the globally subanalytic set A does not contain zero or if the dimension of \overline{A} at zero is one, the asymptotic expansion of the function $f(t)$ above vanishes in respect to the polynomial scale.*

b) If zero is in the closure of A and the dimension of \overline{A} at zero is two, the asymptotic expansion of $f(t)$ does not vanish in respect to the polynomial scale.

Proof. First, we give the arguments for the statement in a). If zero is not in the closure of the set A , we will have only integrals of the form $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr$ and

$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr$, $\alpha > 0$, after cell decomposition. We have proven in Lemma 3.2.14

that $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right)$ and in Lemma 3.2.13 that $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr =$

$\mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right)$ as $t \rightarrow 0$. Thus, the asymptotic expansion of $f(t)$ has a scale like $e^{-\frac{\alpha^2}{2t}} t$

and the asymptotic expansion of the function $f(t)$ above vanishes in respect to the polynomial scale. If the dimension of \overline{A} at zero is one, the integral over the cell which contains zero vanishes and by the same arguments as above the asymptotic expansion of the function $f(t)$ above vanishes in respect to the polynomial scale.

For b) we consider $\int_0^{\beta} e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr$ in (2) in the proof of Theorem 3.2.15.

We recall that $\psi(r) - \eta(r) > 0$ and thus, there are constants $\sigma \in \mathbb{Q}_{>0}$ and $C > 0$

such that $\psi(r) - \eta(r) \geq Cr^{\sigma}$ for all $r \in]0, \beta]$. We obtain $\int_0^{\beta} e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr \geq$

$C \int_0^{\beta} e^{-\frac{r^2}{2t}} r^{\sigma+1} dr$, of which the asymptotic expansion has polynomial scale as we

have already seen. We can conclude that the asymptotic expansion of $f(t)$ does not vanish in respect to the polynomial scale. \square

Finally, we would like to point out another property of the asymptotic expansion.

Remark 3.2.17. *The asymptotic expansion of $f(t)$ as t tends towards zero is not necessarily convergent.*

Proof. We show this statement by a short example.

Let B be the globally subanalytic set $\{(r, \varphi) \in [0, \frac{1}{2}] \times [0, \pi] \mid 0 < \varphi < \frac{1+r}{1-r}\}$, which

is given in polar coordinates. We set $\psi(r) := \frac{1+r}{1-r}$ and obviously $|\psi(r)|$ is bounded on $[0, \frac{1}{2}]$ by π . Using Fubini's Theorem 2.1.18, we get

$$\begin{aligned} \int_B e^{-\frac{r^2}{2t}} r d(r, \varphi) &\stackrel{2.1.18}{=} \int_{r=0}^{\frac{1}{2}} \int_{\varphi=0}^{\psi(r)} e^{-\frac{r^2}{2t}} r d\varphi dr \\ &= \int_0^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r \psi(r) dr. \end{aligned}$$

The well-known geometric series $\sum_{n=0}^{\infty} q^n$ has the limit $\frac{1}{1-q}$ if $q < 1$. Using this, $\psi(r)$ can be represented as $(1+r) \sum_{k=0}^{\infty} r^k = \sum_{k=0}^{\infty} \sum_{j=0}^1 r^{j+k}$. In (1) in Theorem 3.2.15 we have shown the asymptotic expansion of $\int_0^{\beta} e^{-\frac{r^2}{2t}} r \psi(r) dr$, $\beta < 1$, which is $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+1}$. The coefficients $\hat{a}_{j,k}$ are given in Lemma 3.2.12 as

$$\hat{a}_{j,k} = a_{j,k} 2^{\frac{j}{2q}+k+1} \prod_{\ell=1}^k \left(\frac{j}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j}{2q} + 1 \right),$$

where $a_{j,k}$ are the coefficients of the series representation of $\psi(r)$. In our example it is $a_{j,k} = 1$ for all j and k . Thus, the asymptotic expansion of $\int_0^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r \psi(r) dr$ does not converge and in consequence, the asymptotic expansion of $\int_B e^{-\frac{r^2}{2t}} r d(r, \varphi)$ does not converge. \square

The Case t at Infinity

Now, we consider the case t tends towards infinity and we study the definability respectively the asymptotic behaviour of $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$. For this purpose, we consider again integrals of the form

$$\int_D e^{-\frac{r^2}{2t}} \psi(r) dr, \tag{1}$$

where $\psi(r)$ is a globally subanalytic function, which is bounded on an open interval $D \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$. More precisely, we show these integrals are definable in \mathbb{R}_{an} if A , consequently D , is bounded. The harder part is if A respectively D is unbounded. In this case, we ascertain asymptotic expansions by reducing the integrals in (1) to $\int_x^\infty e^{-s^2} s^{\frac{j}{q}-2k-1} ds$ as x approaches zero, which is discussed further in this work. First, we may assume D is bounded and we consider the following important remark.

Remark 3.2.18. *Let $D \subset \mathbb{R}_{\geq 0}$ be a bounded interval and let $\psi(r)$ be a bounded function on D definable in \mathbb{R}_{an} . As $t \rightarrow \infty$,*

$$\int_D e^{-\frac{r^2}{2t}} r \psi(r) dr$$

is definable in \mathbb{R}_{an} and strictly positive.

Proof. For large t we get $e^{-\frac{r^2}{2t}}$ is a restricted analytic function on D and subsequently it is definable in \mathbb{R}_{an} . Due to the fact that $e^{-\frac{r^2}{2t}}$ and $\psi(r)$ are definable in \mathbb{R}_{an} , they fulfill the conditions of Theorem 2.3.16. Applying this theorem, we attain that, in our context,

$$\int_D e^{-\frac{r^2}{2t}} r \psi(r) dr$$

is of the form $\varphi(t^{\frac{1}{q}})$, $q \in \mathbb{N}$, where φ is a strictly positive globally subanalytic function and as a result the integral is definable in \mathbb{R}_{an} . In our case, the log-term which is stated in Theorem 2.3.16 does not appear. We could additionally prove this by using the Taylor expansion of the exponential function and by integrating it as usual. \square

Now, we may assume D is unbounded and therefore we consider $\int_\alpha^\infty e^{-\frac{r^2}{2t}} r \psi(r) dr$, where $\alpha > 1$. By arguments which we will bring later, $\psi(r)$ is representable as a convergent Puiseux series. Due to the boundedness of $\psi(r)$, it can be written as $a_0 + \sum_{k=0}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$, where a_0 and $a_{j,k}$ are coefficients in \mathbb{R} . In several lemmas we show the asymptotic expansion of $\int_\alpha^\infty e^{-\frac{r^2}{2t}} r \psi(r) dr$ as t tends towards infinity by establishing expansions of these integrals for parts of $\psi(r)$.

Lemma 3.2.19. *Let $\alpha \in \mathbb{R}$. As $t \rightarrow \infty$, we have the asymptotic series expansion*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \, dr = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{2^n n!} t^{1-n},$$

which is definable in \mathbb{R}_{an} .

Proof. We achieve by a change of variables $s = \frac{r}{\sqrt{2t}}$

$$\begin{aligned} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \, dr &= \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s \sqrt{2t} \sqrt{2t} ds \\ &= 2t \left[-\frac{1}{2} e^{-s^2} \right]_{\frac{\alpha}{\sqrt{2t}}}^{\infty} \\ &= t e^{-\frac{\alpha^2}{2t}}. \end{aligned}$$

For large t we get $e^{-\frac{\alpha^2}{2t}}$ is a restricted analytic function and subsequently $t e^{-\frac{\alpha^2}{2t}}$ is definable in \mathbb{R}_{an} . Applying Lemma 2.2.11, we get as $t \rightarrow \infty$

$$\begin{aligned} t e^{-\frac{\alpha^2}{2t}} &= t \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\alpha^2}{2t}\right)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{2^n n!} t^{1-n}. \end{aligned}$$

□

Subsequently, we ascertain the asymptotic series expansion of the integrals $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)} dr$, where the sum $\sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ is a summand of the convergent Puiseux series which represents $\psi(r)$, as preparation for Lemma 3.2.23.

The idea of the proof is to reduce our problem to $\int_x^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds$ as x approaches zero. We have already ascertained the asymptotics for these integrals in 3.2.10 in Preparations 3.2.1.

Lemma 3.2.20. *Let $q \in \mathbb{N}$, $\alpha \in \mathbb{R}$, where $\alpha > 1$, and let $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ be a convergent Puiseux series, which describes a bounded function on $]\alpha, \infty[$. As $t \rightarrow \infty$, we achieve for $k \geq 0$*

$$\sum_{j=0}^{2q-1} a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}-2(k+1)} dr \approx \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \lambda_k \log(t) t^{-k} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n},$$

where $\lambda_k := \frac{1}{2} a_{0,k} (-1)^k 2^{-k} \frac{1}{k!}$,

$$\gamma_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = k, \\ \frac{(-1)^{n+1}}{2^n n!} a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{\frac{j}{q}+2(n-k)}, & \text{else,} \end{cases}$$

and

$$\delta_{j,k} := \begin{cases} a_{0,k} 2^{-k} [d_{0,k} + (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2))], & j = 0, \\ a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k}, & \text{else,} \end{cases}$$

where

$$d_{j,k} := \begin{cases} \sum_{n=0}^{\infty} (-b_{j,0,n}) + \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k = 0, \\ \sum_{n=0}^{\infty} (-b_{j,k,n}) - \frac{1}{\frac{j}{2q}-k} \frac{1}{2} e^{-1} - \sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \left(\frac{1}{\frac{j}{2q}-k+\ell}\right) \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^k \frac{1}{\frac{j}{2q}-\ell} \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k > 0, \end{cases}$$

and

$$b_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = k, \\ (-1)^{n+1} \frac{1}{\left(\frac{j}{q}+2(n-k)\right)n!}, & \text{else.} \end{cases}$$

Proof. We make a change of variables $s = \frac{r}{\sqrt{2t}}$ and get

$$\begin{aligned} \sum_{j=0}^{2q-1} a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}-2(k+1)} dr &= \sum_{j=0}^{2q-1} a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}-2k-1} dr \\ &= \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds. \end{aligned}$$

Let $b_{j,k,n}$ and $d_{j,k}$ be as in Lemma 3.2.10, where we have also proven that $\sum_{n=0}^{\infty} (-b_{j,k,n})$ is convergent. It should be noted that $b_{j,k,n}$ and $d_{j,k}$ are stated for the integrals $\int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}+2k-1} ds$ there, where $k < 0$. Therefore we have to rewrite the coefficients for $\int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds$, where $k > 0$. Applying Lemma 3.2.10, we achieve

$$\begin{aligned} &\sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds \\ &= a_{0,k} \sqrt{2t}^{-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{-2k-1} ds \\ &\quad + \sum_{j=1}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds \\ &\stackrel{3.2.10}{\approx} a_{0,k} \sqrt{2t}^{-2k} \left(d_{0,k} + \sum_{n=0}^{\infty} b_{0,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{2(n-k)} t^{k-n} + (-1)^{k+1} \frac{1}{k!} \log \left(\frac{\alpha}{\sqrt{2t}} \right) \right) \\ &\quad + \sum_{j=1}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \left(d_{j,k} + \sum_{n=0}^{\infty} b_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)} t^{-\frac{j}{2q}+k-n} \right) \end{aligned}$$

$$\begin{aligned}
&= a_{0,k} \sqrt{2t}^{-2k} \left(d_{0,k} + \sum_{n=0}^{\infty} b_{0,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{2(n-k)} t^{k-n} + (-1)^k \frac{1}{k!} \frac{1}{2} \log(t) \right) \\
&\quad + (-1)^{k+1} \frac{1}{k!} \left(\log(\alpha) - \frac{1}{2} \log(2) \right) \\
&\quad + \sum_{j=1}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \left(d_{j,k} + \sum_{n=0}^{\infty} b_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)} t^{-\frac{j}{2q}+k-n} \right) \\
&= \delta_{0,k} t^{-k} + \sum_{n=0}^{\infty} a_{0,k} b_{0,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{2(n-k)} 2^{-k} t^{-n} + (-1)^k \frac{1}{k!} 2^{-k} a_{0,k} \frac{1}{2} \log(t) t^{-k} \\
&\quad + \sum_{j=1}^{2q-1} a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k} t^{\frac{j}{2q}-k} + \sum_{j=1}^{2q-1} a_{j,k} 2^{\frac{j}{2q}-k} \sum_{n=0}^{\infty} b_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)} t^{-n} \\
&= \delta_{0,k} t^{-k} + (-1)^k \frac{1}{k!} 2^{-k} a_{0,k} \frac{1}{2} \log(t) t^{-k} \\
&\quad + \sum_{j=1}^{2q-1} a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k} t^{\frac{j}{2q}-k} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} 2^{\frac{j}{2q}-k} b_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)} t^{-n},
\end{aligned}$$

where $\delta_{0,k} := a_{0,k} 2^{-k} \left(d_{0,k} + (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2)) \right)$.

We set $\lambda_k := (-1)^k \frac{1}{k!} 2^{-k} a_{0,k} \frac{1}{2}$,

$$\begin{aligned}
\gamma_{j,k,n} &:= b_{j,k,n} a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{2^n} \\
&= \begin{cases} 0, & j = 0 \text{ and } n = k, \\ \frac{(-1)^{n+1}}{2^n n!} a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{2^{\frac{j}{q}+2(n-k)}}, & \text{else,} \end{cases}
\end{aligned}$$

and

$$\delta_{j,k} := \begin{cases} a_{0,k} 2^{-k} \left[d_{0,k} + (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2)) \right], & j = 0, \\ a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k}, & \text{else.} \end{cases}$$

□

Before we continue to establish the asymptotic expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr$, we consider the series $\sum_{k=1}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k}$ and show that it is convergent.

Lemma 3.2.21. *Let $j \in \{0, \dots, 2q - 1\}$ and let $\delta_{j,k}$ be as in Lemma 3.2.20. We have that*

$$\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k}$$

converges absolutely for sufficiently large t .

Proof. In Lemma 3.2.20 the coefficients $\delta_{j,k}$ are stated as

$$\delta_{j,k} := \begin{cases} a_{0,k} 2^{-k} [d_{0,k} + (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2))], & j = 0, \\ a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k}, & \text{else,} \end{cases}$$

where $a_{j,k}$ are the coefficients of a convergent Puiseux series and where

$$d_{j,k} := \begin{cases} \sum_{n=0}^{\infty} (-b_{j,0,n}) + \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k = 0, \\ \sum_{n=0}^{\infty} (-b_{j,k,n}) - \frac{1}{\frac{j}{2q}-k} \frac{1}{2} e^{-1} - \sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \left(\frac{1}{\frac{j}{2q}-k+\ell}\right) \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^k \frac{1}{\frac{j}{2q}-\ell} \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right), & k > 0, \end{cases}$$

and

$$b_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = k, \\ (-1)^{n+1} \frac{1}{\left(\frac{j}{2q} + 2(n-k)\right)n!}, & \text{else.} \end{cases}$$

We proof, wether the formal series $\sum_{k=1}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k}$ converges. First, we may assume $j \neq 0$. We get

$$\begin{aligned} \sum_{k=1}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k} &= \sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} d_{j,k} t^{\frac{j}{2q}-k} \\ &= \sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{j}{2q} + 2(n-k)\right)n!} - \frac{1}{\frac{j}{2q}-k} \frac{1}{2} e^{-1} \right. \\ &\quad \left. - \sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \left(\frac{1}{\frac{j}{2q}-k+\ell}\right) \frac{1}{2} e^{-1} + \prod_{\ell=1}^k \frac{1}{\frac{j}{2q}-\ell} \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right) \right] t^{\frac{j}{2q}-k} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} t^{\frac{j}{2q}-k} \\
 &\quad + \frac{1}{2} e^{-1} \left[\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \left(-\frac{1}{\frac{j}{2q}-k} \right) t^{\frac{j}{2q}-k} \right. \\
 &\quad \left. + \sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \left(-\sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \left(\frac{1}{\frac{j}{2q}-k+\ell} \right) \right) t^{\frac{j}{2q}-k} \right] \\
 &\quad + \frac{1}{2} \Gamma\left(\frac{j}{2q}, 1\right) \sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \prod_{\ell=1}^k \frac{1}{\frac{j}{2q}-\ell} t^{\frac{j}{2q}-k}.
 \end{aligned}$$

In the next part we show that each of these series in the sum above is convergent. To determine whether the series $\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} t^{\frac{j}{2q}-k}$ converges, we estimate:

$$\left| \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} \right| \leq \begin{cases} q, & n-k = -1 \text{ and } j > q, \\ 1, & \text{else.} \end{cases} \quad (1)$$

Therefore we get

$$\begin{aligned}
 \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} \right| &\leq \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n!} \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} \right| \\
 &\stackrel{(1)}{\leq} q \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n!} \right| = q e^1.
 \end{aligned}$$

Applying the ratio test to the series $\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} q e^1 t^{\frac{j}{2q}-k}$, we obtain it is absolutely convergent. Thus, we conclude that $\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\left(\frac{j}{q} + 2(n-k)\right)} t^{\frac{j}{2q}-k}$ converges absolutely by the comparison test.

Using the ratio test, we get that

$$\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \left(-\frac{1}{\frac{j}{2q}-k} \right) t^{\frac{j}{2q}-k}$$

also converges absolutely.

Now, we consider the series

$$\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \left(- \sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \frac{1}{\frac{j}{2q} - k + \ell} \right) t^{\frac{j}{2q}-k}. \quad (1)$$

By estimation

$$\left| \sum_{m=1}^{k-1} \prod_{\ell=1}^{k-m} \frac{1}{\frac{j}{2q} - k + \ell} \right| \leq \left| (k-1) \frac{1}{\frac{j}{2q} - k + 1} \right|$$

and by applying the ratio test, that is

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{a_{j,k+1} k 2^{\frac{j}{2q}-k-1} \frac{1}{\frac{j}{2q} - (k+1) + 1} t^{\frac{j}{2q} - (k+1)}}{a_{j,k} (k-1) 2^{\frac{j}{2q}-k} \frac{1}{\frac{j}{2q} - k + 1} t^{\frac{j}{2q} - k}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{a_{j,k+1} k \frac{1}{\frac{j}{2q} - k} t^{-1}}{a_{j,k} (k-1) 2 \frac{1}{\frac{j}{2q} - k + 1}} \right| \\ &= \lim_{k \rightarrow \infty} \left| \frac{a_{j,k+1} (\frac{j}{2q} - k + 1)}{a_{j,k} (k-1) 2 (\frac{j}{2q} - k) t} \right| < 1 \end{aligned}$$

for sufficiently large t , we get that $\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} (k-1) \frac{1}{\frac{j}{2q} - k + 1} t^{\frac{j}{2q}-k}$ converges absolutely. In consequence of that, the series in (1) is absolutely convergent by the comparison test.

For the last series $\sum_{k=1}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k} \prod_{\ell=1}^k \frac{1}{\frac{j}{2q}-\ell} t^{\frac{j}{2q}-k}$ we use the ratio test again and achieve that the series also converges absolutely. Gathering our results, we have proven that the series $\sum_{k=1}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k}$ is absolutely convergent and it follows immediately that $\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k}$ converges absolutely.

Now, we may assume $j = 0$. We have

$$\begin{aligned} \sum_{k=1}^{\infty} \delta_{0,k} t^{\frac{j}{2q}-k} &= \sum_{k=1}^{\infty} a_{0,k} 2^{-k} \left[d_{0,k} + (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2)) \right] t^{\frac{j}{2q}-k} \\ &= \sum_{k=1}^{\infty} a_{0,k} 2^{-k} d_{0,k} t^{\frac{j}{2q}-k} \\ &\quad + \sum_{k=1}^{\infty} a_{0,k} 2^{-k} (-1)^{k+1} \frac{1}{k!} (\log(\alpha) - \frac{1}{2} \log(2)) t^{\frac{j}{2q}-k} \end{aligned}$$

By considering the partial proofs above, it is obvious that $\sum_{k=1}^{\infty} a_{0,k} 2^{-k} d_{0,k} t^{\frac{j}{2q}-k}$ also converges absolutely. We can easily proof the convergence of the second series by the ratio test. In conclusion, we achieve that $\sum_{k=1}^{\infty} \delta_{0,k} t^{\frac{j}{2q}-k}$ is absolutely convergent and therefore $\sum_{k=0}^{\infty} \delta_{0,k} t^{\frac{j}{2q}-k}$. \square

Remark 3.2.22. Let $\sum_{n=0}^{\infty} c_n t^{-n}$ be a convergent series with real coefficients c_n . As $t \rightarrow \infty$, we obtain obviously

$$\sum_{n=N+1}^{\infty} c_n t^{-n} = \mathcal{O}(t^{-N-1}).$$

We would point out that the aim is to ascertain an asymptotic expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr$, where $\psi(r)$ is bounded on $]\alpha, \infty[$ and $\psi(r)$ can be described as the convergent Puiseux series $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$. By using the last lemmas, we can now establish the asymptotic series expansion of integrals of the form $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \left(\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)} \right) dr$.

Lemma 3.2.23. Let $\alpha \in \mathbb{R}$, $\alpha > 1$, and let $\theta(r)$ be a bounded function on $]\alpha, \infty[$, which can be represented as the convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$, $q \in \mathbb{N}$. As $t \rightarrow \infty$, we get the asymptotic series expansion

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \theta(r) dr \approx \varphi_0(t) + \varphi_1(t) \log(t),$$

where $\varphi_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \kappa_{j,k} t^{\frac{j}{2q}-k}$ and $\varphi_1(t) = \sum_{k=0}^{\infty} \lambda_k t^{-k}$, where $\kappa_{j,k}, \lambda_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series. Thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. We set $F(t) := \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \theta(r) dr$. By a change of variables $s = \frac{r}{\sqrt{2t}}$, we get

$$\begin{aligned} F(t) &= \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2k-2} dr \\ &= \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2k-1} dr \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds \\ &= \sum_{k=0}^{\infty} F_k(t), \end{aligned}$$

where $F_k(t) := \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k-1} ds$. In Lemma 3.2.20 we have ascertained the asymptotic expansion $\hat{F}_k(t)$ of $F_k(t)$:

$$\hat{F}_k(t) := \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \lambda_k \log(t) t^{-k} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n},$$

where $\delta_{j,k}, \lambda_k, \gamma_{j,k,n} \in \mathbb{R}$ are as in Lemma 3.2.20 and thus,

$$\gamma_{j,k,n} \stackrel{3.2.20}{=} \begin{cases} 0, & j = 0 \text{ and } n = k, \\ \frac{(-1)^{n+1}}{2^n n!} a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{\frac{j}{q}+2(n-k)}, & \text{else.} \end{cases}$$

We define

$$\hat{\gamma}_{j,k,n} := \begin{cases} 0, & j = 0 \text{ and } n = k, \\ a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{\frac{j}{q}+2(n-k)}, & \text{else,} \end{cases}$$

such that $\gamma_{j,k,n} = \frac{(-1)^{n+1}}{2^n n!} \hat{\gamma}_{j,k,n}$. The series $\sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}$ converges obviously

and we get the m -th remainder for an arbitrary $k \in \mathbb{N}_0$ by

$$\begin{aligned}
 R_{k,m}(t) &:= F_k(t) - \hat{F}_k|_m(t) \\
 &= \begin{cases} F_k(t) - \left(\sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \lambda_k \log(t) t^{-k} + \sum_{n=0}^m \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} \right), & k \leq m, \\ F_k(t) - \sum_{n=0}^m \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}, & k > m, \end{cases} \\
 &= \begin{cases} \sum_{n=m+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}, & k \leq m, \\ \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \lambda_k \log(t) t^{-k} + \sum_{n=m+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}, & k > m. \end{cases} \tag{1}
 \end{aligned}$$

Next, we give the proof that the series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{\gamma}_{j,k,n}$ and $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}$ are convergent. We assume $\hat{\gamma}_{j,k,n} \neq 0$ and estimate

$$\left| \frac{1}{\frac{j}{q} + 2(n-k)} \right| \leq \begin{cases} q, & n-k=1 \text{ and } j > q, \\ 1, & \text{else.} \end{cases}$$

Then we get the estimation of $|\hat{\gamma}_{j,k,n}|$ by

$$\begin{aligned}
 |\hat{\gamma}_{j,k,n}| &= \left| a_{j,k} \frac{\alpha^{\frac{j}{q}+2(n-k)}}{\frac{j}{q} + 2(n-k)} \right| \\
 &\leq a_{j,k} (\alpha^2)^{\frac{j}{2q}+n} q \frac{1}{(\alpha^2)^k}.
 \end{aligned}$$

By the requirements is $\alpha > 1$ and $a_{j,k}$ are the coefficients of a convergent Puiseux series. Subsequently, the series $\sum_{k=0}^{\infty} a_{j,k} \frac{1}{(\alpha^2)^k}$ converges absolutely in consequence of the ratio test. We set

$$C := \sum_{j=0}^{2q-1} \sum_{k=0}^{\infty} (\alpha^2)^{\frac{j}{2q}} a_{j,k} q \frac{1}{(\alpha^2)^k}. \tag{3}$$

Using the comparison test, we can conclude that $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{\gamma}_{j,k,n}$ is convergent and

we set

$$\hat{B}_n := \sum_{j=0}^{2q-1} \sum_{k=0}^{\infty} \hat{\gamma}_{j,k,n}. \quad (4)$$

Moreover, the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n}$$

is also convergent by the comparison test, because we get

$$\begin{aligned} \left| \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n \right| &\stackrel{(4)}{=} \frac{1}{2^n n!} \left| \sum_{j=0}^{2q-1} \sum_{k=0}^{\infty} \hat{\gamma}_{j,k,n} \right| \\ &\leq \frac{1}{2^n n!} \sum_{j=0}^{2q-1} \sum_{k=0}^{\infty} |\hat{\gamma}_{j,k,n}| \\ &\stackrel{(2)}{\leq} \frac{1}{2^n n!} (\alpha^2)^n \sum_{j=0}^{2q-1} (\alpha^2)^{\frac{j}{2q}} q \sum_{k=0}^{\infty} \left| a_{j,k} \frac{1}{(\alpha^2)^k} \right| \\ &\stackrel{(3)}{\leq} \frac{1}{2^n n!} (\alpha^2)^n C \end{aligned}$$

and $\sum_{n=0}^{\infty} \frac{1}{2^n n!} (\alpha^2)^n C$ is absolutely convergent.

We define

$$\hat{F}(t) := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \sum_{k=0}^{\infty} \lambda_k \log(t) t^{-k} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n}.$$

We make a change of the indices to p and set

$$\kappa_{j,p} := \begin{cases} \delta_{0,p} + \frac{(-1)^{p+1}}{2^p p!} \hat{B}_p, & j = 0, \\ \delta_{j,p}, & \text{else.} \end{cases}$$

We gather

$$\hat{F}(t) = \sum_{p=0}^{\infty} \sum_{j=0}^{2q-1} \kappa_{j,p} t^{\frac{j}{2q}-p} + \sum_{p=0}^{\infty} \lambda_p t^{-p} \log(t).$$

Finally, we show that $\hat{F}(t)$ is the asymptotic series expansion of $F(t)$ by the proof

that

$$F(t) - \sum_{p=0}^N \left(\sum_{j=0}^{2q-1} \kappa_{j,p} t^{\frac{j}{2q}} + \lambda_p \log(t) \right) t^{-p} = \mathcal{O} \left(t^{\frac{2q-1}{2q} - (N+1)} \right).$$

We attain by the results above

$$\begin{aligned} & \left| F(t) - \sum_{p=0}^N \left(\sum_{j=0}^{2q-1} \kappa_{j,p} t^{\frac{j}{2q}} + \lambda_p \log(t) \right) t^{-p} \right| \\ &= \left| F(t) - (\hat{F}(t))|_N \right| \\ &= \left| F(t) - \left(\sum_{k=0}^{\infty} \hat{F}_k(t) \right) |_N \right| \\ &= \left| \sum_{k=0}^{\infty} F_k(t) - \sum_{k=0}^{\infty} \hat{F}_k|_N(t) \right| \\ &= \left| \sum_{k=0}^{\infty} \left(F_k(t) - \hat{F}_k|_N(t) \right) \right| \\ &= \left| \sum_{k=0}^N \left(F_k(t) - \hat{F}_k|_N(t) \right) + \sum_{k=N+1}^{\infty} \left(F_k(t) - \hat{F}_k|_N(t) \right) \right| \\ &\stackrel{(1)}{=} \left| \sum_{k=0}^N R_{k,N}(t) + \sum_{k=N+1}^{\infty} R_{k,N}(t) \right| \\ &\stackrel{(1)}{=} \left| \sum_{k=0}^{\infty} \sum_{n=N+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} + \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} + \sum_{k=N+1}^{\infty} \lambda_k \log(t) t^{-k} \right| \\ &\leq \left| \sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n} \right| + \left| \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} \right| + \left| \sum_{k=N+1}^{\infty} \lambda_k \log(t) t^{-k} \right|. \end{aligned}$$

It results from above that $\sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n}$ converges. In addition, we have shown in Lemma 3.2.21 that the series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k}$ is convergent. We consider

$$\sum_{k=0}^{\infty} \lambda_k \log(t) t^{-k} \stackrel{3.2.20}{=} \sum_{k=0}^{\infty} \frac{1}{2} a_{0,k} (-1)^k 2^{-k} \frac{1}{k!} \log(t) t^{-k}.$$

This series also converges by applying the ratio test.

Using Remark 3.2.22 we obtain

$$\sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n} = \mathcal{O}(t^{-(N+1)}),$$

$$\sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k} = \mathcal{O}\left(t^{\frac{2q-1}{2q}-(N+1)}\right),$$

and

$$\sum_{k=N+1}^{\infty} \lambda_k \log(t) t^{-k} = \mathcal{O}(\log(t) t^{-(N+1)}).$$

Being $t^{-(N+1)} = \mathcal{O}\left(t^{\frac{2q-1}{2q}-(N+1)}\right)$ and $\log(t) t^{-(N+1)} = \mathcal{O}\left(t^{\frac{2q-1}{2q}-(N+1)}\right)$ for an arbitrary $N \in \mathbb{N}$, we conclude

$$F(t) - \left(\sum_{p=0}^N \sum_{j=0}^{2q-1} \kappa_{j,p} t^{\frac{j}{2q}-p} + \sum_{p=0}^N \lambda_p t^{-p} \log(t) \right) = \mathcal{O}\left(t^{\frac{2q-1}{2q}-(N+1)}\right).$$

According to Remark 2.2.5 and to Definition 2.2.4 of asymptotic expansions, we have proven

$$F(t) \approx \hat{F}(t).$$

Due to the fact that the series in $\hat{F}(t)$ converge, we can write

$$\hat{F}(t) = \varphi_0(t) + \varphi_1(t) \log(t),$$

where $\varphi_0(t) := \sum_{p=0}^{\infty} \sum_{j=0}^{2q-1} \kappa_{j,p} t^{\frac{j}{2q}-p}$ and $\varphi_1(t) := \sum_{p=0}^{\infty} \lambda_p t^{-p}$ are convergent Puiseux series, and we get that $\hat{F}(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$. \square

To sum it up, we have shown the asymptotic expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \, dr$ in Lemma 3.2.19 and the asymptotic expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \left(\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)} \right) dr$ in Lemma 3.2.23. In addition, we have shown that these asymptotic expansions are definable in $\mathbb{R}_{\text{an,exp}}$. Finally, we can investigate the asymptotic series expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) \, dr$, where $\psi(r)$ is representable as the convergent Puiseux series

$a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2(k+1)}$. In conclusion, we get that the expansion is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Lemma 3.2.24. *Let $\alpha \in \mathbb{R}$, where $\alpha > 1$, let $q \in \mathbb{N}$, and let $\psi(r)$ be a bounded function on $] \alpha, \infty[$, which can be written as a convergent Puiseux series around infinity as $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \tilde{a}_{j,k} r^{\frac{j}{q} - 2(k+1)}$. As $t \rightarrow \infty$, we obtain the asymptotic series expansion*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr \approx a_0 t + \hat{\varphi}_0(t) + \varphi_1(t) \log(t),$$

where $\hat{\varphi}_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q} - k}$ and $\varphi_1(t) = \sum_{k=0}^{\infty} \lambda_k t^{-k}$, where $\mu_{j,k}, \lambda_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. We set $F(t) := \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2k - 1} dr$ and get

$$\begin{aligned} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr &= a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r dr + \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2k - 2} dr \\ &= a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r dr + F(t). \end{aligned}$$

In Lemma 3.2.19 we have shown

$$a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r dr \approx a_0 \sum_{k=0}^{\infty} \gamma_k t^{1-k},$$

where $\gamma_k := \frac{(-1)^k \alpha^{2k}}{2^k k!}$, and furthermore, that the expansion is definable in \mathbb{R}_{an} . In Lemma 3.2.23 we have proven

$$F(t) \approx \varphi_0(t) + \varphi_1(t) \log(t),$$

where $\varphi_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \kappa_{j,k} t^{\frac{j}{2q} - k}$ and $\varphi_1(t) = \sum_{k=0}^{\infty} \lambda_k t^{-k}$ are convergent Puiseux series, where $\kappa_{j,k}, \lambda_k \in \mathbb{R}$, and in consequence of that, this asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

We set

$$\begin{aligned}\hat{\varphi}_0(t) &:= \sum_{k=0}^{\infty} \gamma_{k+1} t^{-k} + \varphi_0(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q}-k},\end{aligned}$$

where

$$\mu_{j,k} := \begin{cases} a_0 \gamma_{k+1} + \kappa_{0,k}, & j = 0, \\ \kappa_{j,k}, & \text{else,} \end{cases}$$

and thus we can conclude

$$\begin{aligned}\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr &\approx a_0 \sum_{k=0}^{\infty} \gamma_k t^{1-k} + \varphi_0(t) + \varphi_1(t) \log(t) \\ &= a_0 \gamma_0 t + a_0 \sum_{k=0}^{\infty} \gamma_{k+1} t^{-k} + \varphi_0(t) + \varphi_1(t) \log(t) \\ &= a_0 t + \hat{\varphi}_0(t) + \varphi_1(t) \log(t)\end{aligned}$$

and $a_0 t + \hat{\varphi}_0(t) + \varphi_1(t) \log(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$. \square

We go back to $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$. Using the lemmas above, we gather our results in the following Theorem. For this purpose, we transform the integrals $\frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx$ in polar coordinates and apply the theorem for cell decomposition, see Theorem 2.3.15. If A is bounded, we show that the integrals are definable in \mathbb{R}_{an} and have the form of a convergent Puiseux series by Remark 3.2.18. If A is unbounded, we get the asymptotic series expansion by Lemma 3.2.24 and furthermore the proof that the expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Theorem 3.2.25. *Let $A \subset \mathbb{R}^2$ be a globally subanalytic set. Let $f : \mathbb{R}_{>0} \rightarrow [0, 1]$ be the function given by*

$$f(t) = \frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

a) *As $t \rightarrow \infty$ and if A is bounded, the function $f(t)$ is definable in \mathbb{R}_{an} and therefore, a convergent Puiseux series.*

b) As $t \rightarrow \infty$ and if A is unbounded, $f(t)$ has the asymptotic series expansion

$$f(t) \approx \rho_0(t) + \rho_1(t) \log(t),$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \kappa_k t^{-\frac{k}{q}}$ and $\rho_1(t) = \sum_{k=1}^{\infty} \lambda_k t^{-k}$, where at least one of the series is not zero and $\kappa_k, \lambda_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and therefore the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. The structure of this proof is quite similar to the proof of Theorem 3.2.15. Let

$$\begin{aligned} \Theta : \mathbb{R}_{\geq 0} \times]-\pi, \pi] &\longrightarrow \mathbb{R} \times \mathbb{R}, \\ (r, \varphi) &\longmapsto (r \cos \varphi, r \sin \varphi), \end{aligned}$$

be the two dimensional polar coordinate transformation. First, we transform $f(t)$ in polar coordinates by Transformation Theorem 2.1.17 and achieve

$$\begin{aligned} f(t) &= \frac{1}{2\pi t} \int_A e^{-\frac{|x|^2}{2t}} dx \\ &\stackrel{2.1.17}{=} \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{-\frac{|(r \cos \varphi, r \sin \varphi)|^2}{2t}} r d(r, \varphi) \\ &= \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r d(r, \varphi). \end{aligned}$$

The function Θ is continuous and definable in the o-minimal structure \mathbb{R}_{an} , hence $B := \Theta^{-1}(A) \subset \mathbb{R}_{\geq 0} \times]-\pi, \pi]$ is definable in \mathbb{R}_{an} . By Cell Decomposition Theorem 2.3.15, we can partition B into finitely many disjoint cells C_1, \dots, C_p . For the sake of convenience, we fix $j \in \{1, \dots, p\}$ and write $C := C_j$ for the next part. Let $\pi_1 : \mathbb{R}_{\geq 0} \times]-\pi, \pi] \rightarrow \mathbb{R}_{\geq 0}$ be the projection on the first coordinate, let $D := \pi_1(C)$, and let $\psi, \eta : D \rightarrow]-\pi, \pi]$ be definable continuous functions such that $\eta < \psi$ and such that

$$C = \{(r, \varphi) \in \mathbb{R}_{\geq 0} \times]-\pi, \pi] \mid r \in D \text{ and } \eta(r) < \varphi < \psi(r)\}.$$

Due to integration, we can reduce the different cell types of Definition 2.3.14 to this single cell C . As angular part of the cell both $|\eta|$ and $|\psi|$ are bounded by π . Cells are defined inductively, for this reason $D \subset \mathbb{R}$ is also a cell and either a point $\{\alpha\}$ or an open interval $]\alpha, \beta[$ where $\alpha, \beta \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $\alpha < \beta$. If D is a point,

the integrals over D will be zero. Using Fubini's Theorem 2.1.18, we get

$$\begin{aligned}
\int_C e^{-\frac{r^2}{2t}} r d(r, \varphi) &\stackrel{2.1.18}{=} \int_D \int_{\varphi=\eta(r)}^{\psi(r)} e^{-\frac{r^2}{2t}} r d\varphi dr \\
&= \int_D e^{-\frac{r^2}{2t}} r \int_{\eta(r)}^{\psi(r)} d\varphi dr \\
&= \int_D e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr \\
&= \int_D e^{-\frac{r^2}{2t}} r \psi(r) dr - \int_D e^{-\frac{r^2}{2t}} r \eta(r) dr.
\end{aligned}$$

If B is a bounded set, the cells C of its cell decomposition are also bounded and D is either a point or a bounded interval. If B is unbounded, the cells C of its cell decomposition could be bounded, but at least one cell is unbounded. That means D , if an unbounded cell C is based on, could also be an interval of the form $] \alpha, \infty[$. First, we give the proof for a). We may assume C is bounded and in consequence also D . By its definition above, $\psi(r)$ is a continuous function definable in \mathbb{R}_{an} . Hence, we get by Remark 3.2.18 that $\int_D e^{-\frac{r^2}{2t}} r \psi(r) dr$ is strictly positive and definable in \mathbb{R}_{an} . By the proof of Remark 3.2.18 we also know that the integral can be represented as a convergent Puiseux series. Moreover, we obtain the same result for the subtrahend $\int_D e^{-\frac{r^2}{2t}} r \eta(r) dr$. By definition of a cell, it is $\eta(r) < \psi(r)$ and thus, $\psi(r) - \eta(r) > 0$. We can conclude that

$$\int_D e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr$$

is strictly positive and definable in \mathbb{R}_{an} and can also be represented by a convergent Puiseux series by the same arguments as above. Therefore, we get for a bounded set $\Theta^{-1}(A)$ that the integral

$$\frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r d(r, \varphi) = \frac{1}{2\pi t} \sum_{j=1}^p \int_{C_j} e^{-\frac{r^2}{2t}} r d(r, \varphi)$$

is definable in \mathbb{R}_{an} and behave like a convergent Puiseux series.

Now, we prove statement b). We may assume C is unbounded. Therefore, at least

one D is unbounded, that means, D could also be an interval of the form $] \alpha, \infty[$ and by a refined cell decomposition, we may assume $\alpha > 1$. By the Preparation Theorem 2.3.18, there is such a cell decomposition of B , so that ψ can be described by a convergent Puiseux series at infinity.

In consequence of the boundedness of $\psi(r)$, the convergent Puiseux series in infinity has only negative exponents and is representable as $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \tilde{a}_{j,k} r^{\frac{j}{q} - 2(k+1)}$, where $q \in \mathbb{N}$. We obtain by Lemma 3.2.24

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r \psi(r) dr \approx a_0 t + \hat{\varphi}_0(t) + \varphi_1(t) \log(t), \quad (1)$$

where $\hat{\varphi}_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q} - k}$, where $\mu_{j,k} \in \mathbb{R}$, and $\varphi_1(t) = \sum_{k=0}^{\infty} \nu_k t^{-k}$, where $\nu_k \in \mathbb{R}$, are convergent Puiseux series and thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$. By the definition of a cell, it is $\eta(r) < \psi(r)$ and therefore, $\psi(r) - \eta(r) > 0$. We gather that the asymptotic expansion of

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr$$

has the same form as in (1). Due to $\psi(r) - \eta(r) > 0$, there are constants $\sigma \in \mathbb{Q}_{<0}$ and $C > 0$ such that $\psi(r) - \eta(r) \geq Cr^{\sigma}$ for all $r \in [\alpha, \infty[$. We get that $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr \geq C \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\sigma+1} dr$, of which the asymptotic expansion has the polynomial scale as we have seen above. We can conclude that at least one coefficient of the expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r (\psi(r) - \eta(r)) dr$ is not zero.

Finally, we bring the achieved facts together by using the results of a). As $t \rightarrow \infty$, we have attained as a result above that these integrals over bounded cells C_j are definable in \mathbb{R}_{an} and thus, they have the form of a convergent Puiseux series if the cells are based on a bounded D . But if $\Theta^{-1}(A)$ is an unbounded set, at least one cell is unbounded and we have

$$\begin{aligned} \frac{1}{2\pi t} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r d(r, \varphi) &= \frac{1}{2\pi t} \sum_{j=1}^p \int_{C_j} e^{-\frac{r^2}{2t}} r d(r, \varphi) \\ &\approx \rho_0(t) + \rho_1(t) \log(t), \end{aligned} \quad (2)$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{2q-1} \gamma_{m,k} t^{\frac{m}{2q}-k}$, where $\gamma_{m,k} \in \mathbb{R}$ and $\gamma_{m,0} = 0$ if $m \neq 0$, and $\rho_1(t) = \sum_{k=1}^{\infty} \lambda_k t^{-k}$, where $\lambda_k \in \mathbb{R}$, are convergent Puiseux series, where at least one coefficient is not zero by the arguments above and because every summand in (2) is positive. Since the Puiseux series ρ_0 and ρ_1 are convergent, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$. \square

3.3 The Three-Dimensional Case

3.3.1 Preparations

As part of this chapter we need the antiderivative of $e^{-s^2} s^{\frac{k}{q}+2}$ and $e^{-s^2} s^{\frac{k}{q}+2} \log(s)$ for a positive real number s , an integer k and a positive integer q . As in Section 3.2.1 the idea is to find a recursive formula for $\int e^{-s^2} s^{\frac{k}{q}+2} ds$ by integrating by parts. The recursion skips $2q$ steps and for this reason we rewrite integrals of the form $\int e^{-s^2} s^{\frac{k}{q}+2} ds$ as

$$\bar{F}_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k} ds,$$

where $k \in \mathbb{Z}$ and $j \in \{0, \dots, 2q-1\}$. We can also establish a recursive formula for $\int e^{-s^2} s^{\frac{k}{q}+2} \log(s) ds$ by integration by parts. Due to the same arguments, we rewrite $\int e^{-s^2} s^{\frac{k}{q}+2} \log(s) ds$ as

$$H_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds,$$

where $k \in \mathbb{Z}$ and $j \in \{0, \dots, 2q-1\}$.

First, we consider $\bar{F}_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k} ds$. We find a recursive formula for $\bar{F}_{q,j,k}$ and we show the relation to

$$F_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k-1} ds,$$

which we have already considered in Section 3.2. Afterwards, we calculate some definite integrals and reorganize the recursive formula for negative integers k . Definite integrals which belong to $\bar{F}_{q,j,k}$ are denoted by

$$\bar{G}_{q,j,k} := \int_a^b e^{-s^2} s^{\frac{j}{q}+2k} ds.$$

Furthermore, we show some asymptotic expansions of $\bar{G}_{q,j,k}$, where a limit of the integral tends towards zero or infinity, e.g. $\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$, where k is a negative integer and x tends towards zero. For this, we reduce $\bar{G}_{q,j,k}$ to finitely many integrals by recursion. We start with the mentioned recursive formula.

Lemma 3.3.1. *Let $k \in \mathbb{Z}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. Up to an additive constant, $\overline{F}_{q,j,k}$ is given by the recursive formula*

$$\overline{F}_{q,j,k} = -\frac{1}{2}e^{-s^2} s^{\frac{j}{q}+2k-1} + \left(\frac{j-q}{2q} + k\right) \overline{F}_{q,j,k-1}.$$

Proof. We proof the recursive formula by integration by parts. We choose $e^{-s^2} s$ and $s^{\frac{j}{q}+2k-1}$ as parts and get

$$\begin{aligned} \overline{F}_{q,j,k} &= \int e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &= \int e^{-s^2} s s^{\frac{j}{q}+2k-1} ds \\ &= -\frac{1}{2}e^{-s^2} s^{\frac{j}{q}+2k-1} + \left(\frac{j}{2q} + k - \frac{1}{2}\right) \int e^{-s^2} s^{\frac{j}{q}+2(k-1)} ds \\ &= -\frac{1}{2}e^{-s^2} s^{\frac{j}{q}+2k-1} + \left(\frac{j-q}{2q} + k\right) \overline{F}_{q,j,k-1}. \end{aligned}$$

□

In our preparations in Section 3.2.1 we have already considered integrals in the quite similar form $F_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k-1} ds$. The subsequent lemma shows how $\overline{F}_{q,j,k}$ belongs to $F_{q,j,k}$.

Lemma 3.3.2. *Let $q \in \mathbb{N}$, $j \in \{0, \dots, 2q - 1\}$, and let $F_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k-1} ds$, where $k \in \mathbb{Z}$. Then $\overline{F}_{q,j,k}$ is related to $F_{q,j,k}$ by*

$$\overline{F}_{q,j,k} = \begin{cases} F_{q,j+q,k}, & j < q, \\ F_{q,j-q,k+1}, & j \geq q. \end{cases}$$

Proof. For $j < q$ we obtain

$$\begin{aligned} \overline{F}_{q,j,k} &= \int e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &= \int e^{-s^2} s^{\frac{j}{q}+2k+1-1} ds \\ &= \int e^{-s^2} s^{\frac{j+q}{q}+2k-1} ds \\ &= F_{q,j+q,k}. \end{aligned}$$

For $j \geq q$ we get

$$\begin{aligned}
 \bar{F}_{q,j,k} &= \int e^{-s^2} s^{\frac{j}{q}+2k} ds \\
 &= \int e^{-s^2} s^{\frac{j-q}{q}+1+2k} ds \\
 &= \int e^{-s^2} s^{\frac{j-q}{q}+1+2k+1-1} ds \\
 &= \int e^{-s^2} s^{\frac{j-q}{q}+2(k+1)-1} ds \\
 &= F_{q,j-q,k+1}.
 \end{aligned}$$

□

By recursion, we can put down each integral $\int e^{-s^2} s^{\frac{j}{q}+2k} ds$ to the finitely many integrals $\int e^{-s^2} s^{\frac{j}{q}} ds$, that means to $k = 0$. For these special integrals, where $k = 0$, we need the antiderivative of $e^{-s^2} s^{\frac{j}{q}}$. To prove the following Lemma we have the choice: we can pick up on Lemma 3.2.4 in Section 3.2 by using the relation of $\bar{F}_{q,j,k}$ to $F_{q,j,k}$, shown in Lemma 3.3.2, or go straight by Definition 2.1.11 of incomplete gamma function. It appears to be less confusing to choose the second way for $j \neq q$.

Lemma 3.3.3. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q - 1\}$. Up to an additive constant, we have*

$$\begin{aligned}
 \bar{F}_{q,q,0} &= \int e^{-s^2} s^{\frac{j}{q}} ds \\
 &= -\frac{1}{2} \Gamma\left(\frac{j+q}{2q}, s^2\right).
 \end{aligned}$$

Proof. By the relation stated in Lemma 3.3.2, by Lemma 3.2.4 in Section 3.2, and by Remark 2.1.14, we attain for $j = q$

$$\begin{aligned}
 \bar{F}_{q,q,0} &\stackrel{3.3.2}{=} F_{q,0,1} \\
 &\stackrel{3.2.4}{=} -\frac{1}{2} e^{-s^2} \\
 &\stackrel{2.1.14}{=} -\frac{1}{2} \Gamma(1, s^2).
 \end{aligned}$$

By Definition 2.1.11 of the incomplete gamma function and Lemma 3.2.2, we attain for $j \neq q$

$$\begin{aligned}\bar{F}_{q,j,0} &= \int e^{-s^2} s^{\frac{j}{q}+1-1} ds \\ &= \int e^{-s^2} s^{\frac{j+q}{q}-1} ds \\ &\stackrel{\substack{2.1.11 \\ 3.2.2}}{=} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, s^2\right).\end{aligned}$$

□

The corollary below and the following remark show the values of special integrals in a short proof by applying the last lemma.

Corollary 3.3.4. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q-1\}$. We get*

$$\int_1^\infty e^{-s^2} s^{\frac{j}{q}} ds = \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right).$$

Proof. It is evident by

$$\begin{aligned}\int_1^\infty e^{-s^2} s^{\frac{j}{q}} ds &\stackrel{3.3.3}{=} \left[-\frac{1}{2} \Gamma\left(\frac{j+q}{2q}, s^2\right) \right]_1^\infty \\ &\stackrel{2.1.12}{=} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right).\end{aligned}$$

□

Remark 3.3.5. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q-1\}$. We have*

$$\int_0^\infty e^{-s^2} s^{\frac{j}{q}} ds = \frac{1}{2} \Gamma\left(\frac{j+q}{2q}\right),$$

where

$$\Gamma\left(\frac{j+q}{2q}\right) = \begin{cases} \sqrt{\pi}, & j=0, \\ 1, & j=q. \end{cases}$$

Proof. By Lemma 3.3.3 and by Lemma 2.1.13, it follows immediately

$$\int_0^\infty e^{-s^2} s^{\frac{j}{q}} ds \stackrel{3.3.3}{\stackrel{2.1.13}{=}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}\right).$$

For $j = 0$ we get the special case

$$\Gamma\left(\frac{q}{2q}\right) = \Gamma\left(\frac{1}{2}\right) \stackrel{2.1.16}{=} \sqrt{\pi}$$

and for $j = q$

$$\Gamma\left(\frac{q+q}{2q}\right) = \Gamma(1) \stackrel{2.1.7}{=} 1.$$

□

In the next paragraph we are interested in the asymptotic series expansion and the asymptotic behaviour of $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k} ds$, where $k > 0$ and x approaches infinity. To investigate these, we use the relation of the integrals $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k} ds$ to $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds$, which is shown in Lemma 3.3.2, and the asymptotic expansion, which has already been pointed out in Lemma 3.2.6 in Section 3.2.1.

Lemma 3.3.6. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q-1\}$. As $x \rightarrow \infty$, we have*

$$\begin{aligned} \int_0^x e^{-s^2} s^{\frac{j}{q}+2k} ds &\approx \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell\right) \frac{1}{2} \Gamma\left(\frac{j+q}{2q}\right) \\ &\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j-q}{q}+2(k+1-n)} \prod_{l=0}^{n-2} \left(\frac{j-q}{2q} + k - l\right). \end{aligned}$$

Proof. We set $\bar{G}_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{q}+2k} ds$ and $G_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{q}+2k-1} ds$. In

Lemma 3.3.2 we have proven that

$$\overline{G}_{q,j,k}(x) \stackrel{3.3.2}{=} \begin{cases} G_{q,j+q,k}(x), & j < q, \\ G_{q,j-q,k+1}(x), & j \geq q. \end{cases}$$

The asymptotic expansion of $G_{q,j,k}(x)$ has already been ascertained in Lemma 3.2.6 as $x \rightarrow \infty$. Thus, we get for $j < q$

$$\begin{aligned} \overline{G}_{q,j,k}(x) &\stackrel{3.3.2}{=} G_{q,j+q,k}(x) \\ &\stackrel{3.2.6}{\approx} \prod_{\ell=1}^{k-1} \left(\frac{j+q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} + 1 \right) \\ &\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j+q}{q} + 2k - 2n} \prod_{l=1}^{n-1} \left(\frac{j+q}{2q} + k - l \right) \\ &\stackrel{2.1.7}{=} \prod_{\ell=1}^{k-1} \left(\frac{j+q}{2q} - 1 + \ell + 1 \right) \frac{1}{2} \left(\frac{j-q}{2q} + 1 \right) \Gamma \left(\frac{j+q}{2q} \right) \\ &\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j-q}{q} + 2(k+1) - 2n} \prod_{l=1}^{n-1} \left(\frac{j+q}{2q} + k - l \right) \\ &= \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \\ &\quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j-q}{q} + 2(k+1-n)} \prod_{l=0}^{n-2} \left(\frac{j-q}{2q} + k - l \right), \end{aligned}$$

and for $j = q$

$$\begin{aligned} \overline{G}_{q,q,k}(x) &\stackrel{3.3.2}{=} G_{q,0,k+1}(x) \\ &\stackrel{3.2.6}{\approx} \prod_{\ell=1}^k \ell \frac{1}{2} \Gamma(1) - e^{-x^2} \sum_{n=1}^{\infty} x^{2(k+1) - 2n} \prod_{l=1}^{n-1} (k+1-l) \\ &= \prod_{\ell=1}^k \ell \frac{1}{2} \Gamma(1) - e^{-x^2} \sum_{n=1}^{\infty} x^{2(k+1-n)} \prod_{l=0}^{n-2} (k-l), \end{aligned}$$

and for $j > q$

$$\overline{G}_{q,j,k}(x) \stackrel{3.3.2}{=} G_{q,j-q,k+1}(x)$$

$$\begin{aligned}
 & \stackrel{3.2.6}{\approx} \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j-q}{2q} + 1 \right) \\
 & \quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j-q}{q} + 2(k+1) - 2n} \prod_{l=1}^{n-1} \left(\frac{j-q}{2q} + k + 1 - l \right) \\
 & = \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \\
 & \quad - e^{-x^2} \sum_{n=1}^{\infty} x^{\frac{j-q}{q} + 2(k+1-n)} \prod_{l=0}^{n-2} \left(\frac{j-q}{2q} + k - l \right).
 \end{aligned}$$

□

Corollary 3.3.7. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow \infty$, we get the dominant term*

$$\int_0^x e^{-s^2} s^{\frac{j}{q} + 2k} ds \sim \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right).$$

Proof. It is an obvious consequence of Lemma 3.3.6. □

Before we come to integrals with non-positive exponents and asymptotic expansions as x tends towards zero, we consider the following lemma. It shows the asymptotic series expansion of $\int_x^{\infty} e^{-s^2} s^2 ds$, that means $k = 1$ and $j = 0$, as x approaches zero by applying the asymptotic series expansion of the exponential function and of the error function.

Lemma 3.3.8. *Let $x \in \mathbb{R}_{>0}$. As $x \rightarrow 0$, we achieve the asymptotic series expansion*

$$\int_x^{\infty} e^{-s^2} s^2 ds \approx \frac{\sqrt{\pi}}{4} + \frac{1}{2} \sum_{n=0}^{\infty} \left(1 - \frac{1}{(2n+1)} \right) \frac{(-1)^n}{n!} x^{2n+1}.$$

Proof. By integration by parts and by the asymptotic expansion of the exponential function, see Lemma 2.2.11, and of the error function, see Lemma 2.2.12, we ascertain as $x \rightarrow 0$

$$\begin{aligned}
 \int_x^{\infty} e^{-s^2} s^2 ds &= \left[-\frac{1}{2} e^{-s^2} s \right]_x^{\infty} + \frac{1}{2} \int_x^{\infty} e^{-s^2} ds \\
 &\stackrel{2.1.2}{=} \frac{1}{2} e^{-x^2} x + \frac{1}{2} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(s) \right]_x^{\infty}
 \end{aligned}$$

$$\begin{aligned}
& \stackrel{2.1.3}{\approx} \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} x + \frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{4} \operatorname{erf}(x) \\
& \stackrel{2.2.12}{\approx} \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!} + \frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{4} \left(\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \frac{1}{(2n+1)n!} \right) \\
& = \frac{\sqrt{\pi}}{4} + \frac{1}{2} \sum_{n=0}^{\infty} \left(1 - \frac{1}{(2n+1)} \right) \frac{(-1)^n}{n!} x^{2n+1}.
\end{aligned}$$

□

Now, we consider integrals $\int e^{-s^2} s^{\frac{j}{q}+2k} ds$, where $k < 0$. Due to the fact that we cannot use the definition of the gamma function in this case, we calculate the integrals by a reorganized form of the recursive formula of Lemma 3.3.1, which we show below. The proof is in the same manner as in Lemma 3.2.8.

Corollary 3.3.9. *Let $k \in \mathbb{Z}_{<0}$, $q \in \mathbb{N}$, $j \in \{0, \dots, 2q-1\}$, and let $(j, k) \neq (q, -1)$. Up to an additive constant, $\bar{F}_{q,j,k}$ can recursively be given by*

$$\bar{F}_{q,j,k} = \frac{1}{\frac{j-q}{2q} + (k+1)} \left(\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k+1)-1} \right) + \frac{1}{\frac{j-q}{2q} + (k+1)} \bar{F}_{q,j,k+1}.$$

Proof. We use the recursive formula

$$\bar{F}_{q,j,k} = -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} + \left(\frac{j-q}{2q} + k \right) \bar{F}_{q,j,k-1}$$

in Lemma 3.3.1. We reorganize to the last term and get for $k < 1$

$$\bar{F}_{q,j,k-1} = \frac{1}{\frac{j-q}{2q} + k} \left(\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} \right) + \frac{1}{\frac{j-q}{2q} + k} \bar{F}_{q,j,k}$$

and in conclusion

$$\bar{F}_{q,j,k} = \frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k+1)-1} + \frac{1}{\frac{j-q}{2q} + (k+1)} \bar{F}_{q,j,k+1}$$

for $k < 0$. □

For $k = 0$ we have already calculated definite integrals $\int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} ds$ in Corollary 3.3.4. We compute the integrals for $k = -1$ among others by applying the recursive formula for negative exponents. We will need this kind of integrals in Lemma 3.3.11.

Lemma 3.3.10. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q - 1\}$. We have*

$$\int_1^\infty e^{-s^2} s^{\frac{j}{q}-2} ds = \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q. \end{cases}$$

Proof. We set $\overline{G}_{q,j,-1} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}-2} ds$. We get for $j < q$ by the recursive formula of Lemma 3.3.9 and Corollary 3.3.4

$$\begin{aligned} \overline{G}_{q,j,-1} &\stackrel{3.3.9}{=} \left[\frac{1}{\frac{j-q}{2q} + (-1+1)} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}-1} \right]_1^\infty + \frac{1}{\frac{j-q}{2q} + (-1+1)} \overline{G}_{q,j,0} \\ &\stackrel{3.3.4}{=} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right). \end{aligned}$$

For $j = q$ we cannot apply the recursive formula, thus we attain by Lemma 3.2.2

$$\begin{aligned} \overline{G}_{q,q,-1} &= \int_1^\infty e^{-s^2} s^{-1} ds \\ &\stackrel{2.1.11}{\stackrel{3.2.2}{=}} \frac{1}{2} \Gamma(0, 1). \end{aligned}$$

For $j > q$ we use the relation to $G_{q,j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k-1}$ of Section 3.2. This relation is stated in Lemma 3.3.2 and we obtain

$$\begin{aligned} \overline{G}_{q,j,-1} &\stackrel{3.3.2}{=} G_{q,j-q,0} \\ &\stackrel{3.2.4}{=} \left[-\frac{1}{2} \Gamma\left(\frac{j-q}{2q}, s^2\right) \right]_1^\infty \\ &\stackrel{2.1.12}{=} \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right). \end{aligned}$$

□

The following Lemma shows the formula for the calculation of $\int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$, where $k < -1$, by applying the recursive formula and the lemma above.

Lemma 3.3.11. *Let $k \in \mathbb{Z}_{<0}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. We achieve for $k < -1$*

$$\int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds = -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q} + (k+1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1},$$

where

$$c_{j,-1} := \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q. \end{cases}$$

Proof. We set $\bar{G}_{q,j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$. According to Lemma 3.3.10, we set

$$c_{j,-1} := \bar{G}_{q,j,-1} \\ \stackrel{3.3.10}{=} \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q. \end{cases}$$

We start the induction with $k = -2$ and get by the recursive formula of Corollary 3.3.9

$$\bar{G}_{q,j,-2} \stackrel{3.3.9}{=} \left[\frac{1}{\frac{j-q}{2q} + (-2+1)} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}-3} \right]_1^\infty + \frac{1}{\frac{j-q}{2q} + (-2+1)} \bar{G}_{q,j,-1} \\ = -\frac{1}{\frac{j-q}{2q} + (-2+1)} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q} - 1} c_{j,-1} \\ = -\frac{1}{\frac{j-q}{2q} + (-2+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^0 \prod_{\ell=0}^{1-m} \frac{1}{\frac{j-q}{2q} + (-2+1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^1 \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1}.$$

By using Corollary 3.3.9 again, we conclude

$$\bar{G}_{q,j,k} \stackrel{3.3.9}{=} -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q} + (k+1)} \bar{G}_{q,j,k+1}$$

$$\begin{aligned}
 & \stackrel{I.H.}{=} -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} \\
 & + \frac{1}{\frac{j-q}{2q} + (k+1)} \left[-\frac{1}{\frac{j-q}{2q} + (k+2)} \frac{1}{2} e^{-1} \right. \\
 & \quad \left. - \sum_{m=1}^{-(k+3)} \prod_{\ell=0}^{-(k+2)-m} \frac{1}{\frac{j-q}{2q} + (k+2) + \ell} \frac{1}{2} e^{-1} + \prod_{\ell=1}^{-(k+2)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1} \right] \\
 & = -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} \\
 & + \frac{1}{\frac{j-q}{2q} + (k+1)} \left[-\sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+2)-m} \frac{1}{\frac{j-q}{2q} + (k+2) + \ell} \frac{1}{2} e^{-1} \right] \\
 & + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1} \\
 & = -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} \\
 & - \sum_{m=1}^{-(k+2)} \frac{1}{\frac{j-q}{2q} + (k+1) + 0} \prod_{\ell=1}^{-(k+2)-m} \frac{1}{\frac{j-q}{2q} + (k+1) + \ell} \frac{1}{2} e^{-1} \\
 & + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1} \\
 & = -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q} + (k+1) + \ell} \frac{1}{2} e^{-1} \\
 & + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1}.
 \end{aligned}$$

□

Next, we establish the asymptotic expansions of $\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$, where k is a non-positive integer and x tends towards zero. We could also prove the next lemma by taking up Lemma 3.2.9 of Section 3.2, but combining the cases of the conclusion seems to be even more complicated than proceeding the same way as in Lemma 3.2.9.

Lemma 3.3.12. *Let $k \in \mathbb{Z}_{\leq 0}$, $q \in \mathbb{N}$, $x \in \mathbb{R}_{>0}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow 0$,*

$$\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$$

has for $j \neq q$ the asymptotic series expansion

$$d_{j,k} + \sum_{n=0}^{\infty} b_{j,k,n} x^{\frac{j}{q}+2(n+k)+1}$$

and for $j = q$

$$d_{q,k} + \sum_{n=0}^{\infty} b_{q,k,n} x^{2(n+k+1)} + \frac{(-1)^k}{(-k-1)!} \log(x),$$

where

$$b_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = -k - 1, \\ (-1)^{n+1} \frac{1}{\left(\frac{j+q}{q}+2(n+k)\right)n!}, & \text{else,} \end{cases}$$

and

$$d_{j,k} := \sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k},$$

where

$$c_{j,k} := \begin{cases} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & k = 0, \\ \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = -1, \\ -\frac{1}{\frac{j-q}{2q}+(k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q}+(k+1)+\ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q}-\ell} c_{j,-1}, & k < -1. \end{cases}$$

Proof. We set $\bar{G}_{q,j,k}(x) := \int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$. As $x \rightarrow 0$, we can split up $\bar{G}_{q,j,k}(x)$ into

$$\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds = \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k} ds + \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds.$$

First, we consider the last summand $\int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$. By using Remark 3.3.4, Lemma 3.3.10, and Lemma 3.3.11, we summarize

$$c_{j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$$

$$\stackrel{\substack{3.3.4 \\ 3.3.10 \\ 3.3.11}}{=} \begin{cases} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & k = 0, \\ \begin{cases} -\frac{1}{-\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = -1, \\ -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q} + (k+1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1}, & k < -1. \end{cases}$$

Hence, we achieve by using the asymptotic expansion of the exponential function, which is stated in Lemma 2.2.11,

$$\begin{aligned} \overline{G}_{q,j,k}(x) &= \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k} ds + \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &\stackrel{2.2.11}{=} \int_x^1 \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^{2n} s^{\frac{j}{q}+2k} ds + c_{j,k} \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)} ds + c_{j,k}. \end{aligned} \quad (1)$$

It is necessary to distinguish between two cases. First, we may assume $j \neq q$. Consequently, it is $\frac{j}{q} + 2(n+k) \neq -1$ and we obtain

$$\begin{aligned} \overline{G}_{q,j,k}(x) &= \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)} ds + c_{j,k} \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{\frac{j}{q} + 2(n+k) + 1} \left[s^{\frac{j}{q}+2(n+k)+1} \right]_x^1 + c_{j,k} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{\frac{j}{q} + 2(n+k) + 1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{1}{\frac{j}{q} + 2(n+k) + 1} x^{\frac{j}{q} + 2(n+k) + 1} + c_{j,k},$$

where $\sum_{n=0}^{\infty} (-1)^n \frac{1}{(\frac{j+q}{q} + 2(n+k))n!}$ converges. We set $b_{j,k,n} := (-1)^{n+1} \frac{1}{(\frac{j+q}{q} + 2(n+k))n!}$ and $d_{j,k} := \sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k}$. Therefore, we can conclude for $j \neq q$

$$\bar{G}_{q,j,k}(x) \approx \sum_{n=0}^{\infty} b_{j,k,n} x^{\frac{j}{q} + 2(n+k) + 1} + d_{j,k}.$$

Now, we may assume $j = q$. Since $1 + 2(n+k) = -1$ if $n = -k - 1$, we split the sum of (1) in

$$\begin{aligned} \bar{G}_{q,j,k}(x) &= \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \int_x^1 s^{\frac{q}{q} + 2(n+k)} ds + \frac{(-1)^{-k-1}}{(-k-1)!} \int_x^1 s^{\frac{q}{q} + 2(-k-1+k)} ds + c_{j,k} \\ &= \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2(n+k) + 2} [s^{2(n+k)+2}]_x^1 + \frac{(-1)^{-k-1}}{(-k-1)!} [\log(s)]_x^1 + c_{j,k} \\ &= \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2(n+k) + 2} + \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{1}{2(n+k) + 2} x^{2(n+k)+2} \\ &\quad + \frac{(-1)^{-k}}{(-k-1)!} \log(x) + c_{j,k}, \end{aligned}$$

where $\sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} (-1)^n \frac{1}{(2(n+k)+2)n!}$ converges obviously. We set

$$b_{q,k,n} = \begin{cases} 0, & n = -k - 1, \\ (-1)^{n+1} \frac{1}{(2(n+k)+2)n!}, & \text{else,} \end{cases}$$

and $d_{q,k} := \sum_{n=0}^{\infty} (-b_{q,k,n}) + c_{j,k}$. Thus, we gather for $j = q$

$$\bar{G}_{q,j,k}(x) \approx \sum_{n=0}^{\infty} b_{q,k,n} x^{2(n+k)+2} + \frac{(-1)^k}{(-k-1)!} \log(x) + d_{q,k}.$$

□

Hereafter, we consider the integrals $H_{q,j,k} := \int e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$ and denote definite integrals which belongs to $H_{q,j,k}$ by

$$I_{q,j,k} := \int_a^b e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds.$$

We present a recursive formula for $H_{q,j,k}$ in Lemma 3.3.13 and then the value of $\int_0^\infty e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds$. Afterwards, we establish some asymptotic expansions. We start with showing the asymptotics of the integral $\int_x^\infty e^{-s^2} s^2 \log(s) ds$ as x approaches zero and after that, we examine $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, where k is a positive integer, and ascertain the dominant term as x approaches infinity. Finally, we analyze $\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, where k is a non-positive integer. First, the next lemma shows the mentioned recursive formula.

Lemma 3.3.13. *Let $q \in \mathbb{N}$, let $k \in \mathbb{Z}$, and let $j \in \{0, \dots, 2q - 1\}$. Up to an additive constant, $H_{q,j,k}$ is given by the recursive formula*

$$H_{q,j,k} = -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} \log(s) + \left(\frac{j-q}{2q} + k \right) H_{q,j,k-1} + \frac{1}{2} \bar{F}_{q,j,k-1},$$

where $\bar{F}_{q,j,k} = \int e^{-s^2} s^{\frac{j}{q}+2k} ds$.

Proof. We proof the recursive formula above by integration by parts and choose $e^{-s^2} s$ and $s^{\frac{j}{q}+2k-1} \log(s)$ as parts. We have

$$\begin{aligned} H_{q,j,k} &= \int e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds \\ &= \int e^{-s^2} s s^{\frac{j}{q}+2k-1} \log(s) ds \\ &= -\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} \log(s) + \frac{1}{2} \left(\frac{j}{q} + 2k - 1 \right) \int e^{-s^2} s^{\frac{j}{q}+2k-2} \log(s) ds \\ &\quad + \frac{1}{2} \int e^{-s^2} s^{\frac{j}{q}+2k-2} ds \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}e^{-s^2}s^{\frac{j}{q}+2k-1}\log(s) + \left(\frac{j}{2q} + k - \frac{1}{2}\right) \int e^{-s^2}s^{\frac{j}{q}+2(k-1)}\log(s)ds \\
&\quad + \frac{1}{2} \int e^{-s^2}s^{\frac{j}{q}+2(k-1)}ds \\
&= -\frac{1}{2}e^{-s^2}s^{\frac{j}{q}+2k-1}\log(s) + \left(\frac{j-q}{2q} + k\right) H_{q,j,k-1} + \frac{1}{2}\overline{F}_{q,j,k-1}.
\end{aligned}$$

□

This recursive formula is based on the integrals $\int e^{-s^2}s^{\frac{j}{q}+2}\log(s)ds$. We do not go further into their antiderivatives, because it contains the so-called *generalized hypergeometric function*¹. For our purposes, it is enough to know that they exist. But we can easily specify the integrals $\int_0^\infty e^{-s^2}s^{\frac{j}{q}+2k}\log(s)ds$, $k > 0$, more precisely, because they are based on the first prime of the gamma function. Therefore, we show these specific integrals, where $k > 0$, first and afterwards that they are not zero.

Lemma 3.3.14. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q - 1\}$. We get*

$$\int_0^\infty e^{-s^2}s^{\frac{j}{q}+2k}\log(s)ds = \frac{1}{4}\Gamma'\left(\frac{j}{2q} + k + \frac{1}{2}\right).$$

Proof. By substitution $s = t^{\frac{1}{2}}$ and Definition 2.1.8 of the first prime of the gamma function, we get

$$\begin{aligned}
\int_0^\infty e^{-s^2}s^{\frac{j}{q}+2k}\log(s)ds &= \int_0^\infty e^{-t}t^{\frac{j}{2q}+k}\log(t^{\frac{1}{2}})\frac{1}{2}t^{-\frac{1}{2}}dt \\
&= \frac{1}{2} \int_0^\infty e^{-t}t^{\frac{j}{2q}+k-\frac{1}{2}}\frac{1}{2}\log(t)dt \\
&= \frac{1}{4} \int_0^\infty e^{-t}t^{(\frac{j}{2q}+k-\frac{1}{2}+1)-1}\log(t)dt \\
&\stackrel{2.1.8}{=} \frac{1}{4}\Gamma'\left(\frac{j}{2q} + k + \frac{1}{2}\right).
\end{aligned}$$

□

¹For more information to the *generalized hypergeometric function* we refer to [AS12] and [Erd81].

Corollary 3.3.15. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q - 1\}$. We have*

$$\int_0^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds > 0.$$

Proof. We get $\int_0^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds = \frac{1}{4} \Gamma' \left(\frac{j}{2q} + k + \frac{1}{2} \right)$ by Lemma 3.3.14. By Remark 2.1.9, we know that the first derivative of the gamma function has only a single positive zero in $x_0 = 1,46\dots$ and in addition $\Gamma'(x) > 0$ for $x > x_0$ by Remark 2.1.10. Due to the fact, that we get for the smallest k , which is $k = 1$: $\frac{j}{2q} + \frac{3}{2} > x_0$ for all $j \in \{0, \dots, 2q - 1\}$ and $q \in \mathbb{N}$, we can conclude $\Gamma' \left(\frac{j}{2q} + k + \frac{1}{2} \right) > 0$. \square

We remain on the integrals $\int e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, where k is a positive integer: our next subgoal is to determine the dominant term of $\int_0^x e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$ as x approaches infinity. We achieve this by reducing the integrals to $k = 1$ and by using the recursive formula of Lemma 3.3.13. First, we show the dominant term for $k = 1$ in the subsequent lemma.

Lemma 3.3.16. *Let $q \in \mathbb{N}$ and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow \infty$, we get*

$$\int_0^x e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds \sim \frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right).$$

Proof. We can split the integral in the following way:

$$\begin{aligned} \int_0^x e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds &= \int_0^{\infty} e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds - \int_x^{\infty} e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds \\ &\stackrel{3.3.14}{=} \frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) - \int_x^{\infty} e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds. \end{aligned}$$

Since $\lim_{x \rightarrow \infty} \int_x^{\infty} e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds = 0$, we conclude that $\frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right)$ is the dominant term. \square

The next step is to prove the formula which reduces $\int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds$ to $\int_0^x e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds$ for an arbitrary $x \in \mathbb{R}_{>0}$.

Lemma 3.3.17. *Let $k \in \mathbb{N}$, where $k > 1$, let $q \in \mathbb{N}$, $j \in \{0, \dots, 2q - 1\}$, and let $x \in \mathbb{R}_{>0}$. We get*

$$\begin{aligned} \int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds &= -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) \\ &\quad - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \\ &\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) \\ &\quad + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \overline{G}_{q,j,k-n}(x), \end{aligned}$$

where $I_{q,j,1}(x) = \int_0^x e^{-s^2} s^{\frac{j}{q}+2} \log(s) ds$ and $\overline{G}_{q,j,k}(x) = \int_0^x e^{-s^2} s^{\frac{j}{q}+2k} ds$.

Proof. We set $I_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds$ and $\overline{G}_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} ds$. By applying the recursive formula of Lemma 3.3.13, we achieve for the initial step $k = 2$

$$\begin{aligned} I_{q,j,2}(x) &= \left[-\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+4-1} \log(s) \right]_0^x + \left(\frac{j-q}{2q} + 2 \right) I_{q,j,1}(x) + \frac{1}{2} \overline{G}_{q,j,2-1}(x) \\ &= -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+4-1} \log(x) \\ &\quad - \sum_{m=2}^1 \prod_{\ell=0}^{2-m-1} \left(\frac{j-q}{2q} + 2 - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \\ &\quad + \prod_{\ell=2}^2 \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) + \frac{1}{2} \sum_{n=1}^1 \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + 2 - \ell \right) \overline{G}_{q,j,2-n}(x). \end{aligned}$$

We obtain with induction

$$\begin{aligned}
 & I_{q,j,k}(x) = \\
 \stackrel{3.3.13}{=} & \left[-\frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} \log(s) \right]_0^x + \left(\frac{j-q}{2q} + k \right) I_{q,j,k-1}(x) + \frac{1}{2} \overline{G}_{q,j,k-1}(x) \\
 \stackrel{I.H.}{=} & -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) + \left(\frac{j-q}{2q} + k \right) \left[-\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2(k-1)-1} \log(x) \right. \\
 & - \sum_{m=2}^{k-2} \prod_{\ell=0}^{k-m-2} \left(\frac{j-q}{2q} + (k-1) - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \\
 & + \prod_{\ell=2}^{k-1} \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) \\
 & \left. + \frac{1}{2} \sum_{n=1}^{k-2} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + (k-1) - \ell \right) \overline{G}_{q,j,k-1-n}(x) \right] + \frac{1}{2} \overline{G}_{q,j,k-1}(x) \\
 = & -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) + \left(\frac{j-q}{2q} + k \right) \left[-\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2(k-1)-1} \log(x) \right. \\
 & \left. - \sum_{m=2}^{k-2} \prod_{\ell=0}^{k-m-2} \left(\frac{j-q}{2q} + k - (\ell+1) \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \right] \\
 & + \left(\frac{j-q}{2q} + k \right) \prod_{\ell=2}^{k-1} \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) \\
 & + \frac{1}{2} \left(\frac{j-q}{2q} + k \right) \sum_{n=1}^{k-2} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - (\ell+1) \right) \overline{G}_{q,j,k-(n+1)}(x) \\
 & + \frac{1}{2} \overline{G}_{q,j,k-1}(x) \\
 = & -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) \\
 & + \left(\frac{j-q}{2q} + k \right) \left[-\sum_{m=2}^{k-1} \prod_{\ell=1}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \right] \\
 & + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) \\
 & + \frac{1}{2} \left(\frac{j-q}{2q} + k \right) \sum_{n=1}^{k-2} \prod_{\ell=1}^{n-1} \left(\frac{j-q}{2q} + k - \ell \right) \overline{G}_{q,j,k-(n+1)}(x) + \frac{1}{2} \overline{G}_{q,j,k-1}(x)
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2k-1}\log(x) - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell\right) \frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2m-1}\log(x) \\
&\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell\right) I_{q,j,1}(x) \\
&\quad + \frac{1}{2} \sum_{n=1}^{k-2} \prod_{\ell=0}^{n-1} \left(\frac{j-q}{2q} + k - \ell\right) \overline{G}_{q,j,k-(n+1)}(x) + \frac{1}{2} \overline{G}_{q,j,k-1}(x) \\
&= -\frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2k-1}\log(x) - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell\right) \frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2m-1}\log(x) \\
&\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell\right) I_{q,j,1}(x) \\
&\quad + \frac{1}{2} \sum_{n=0}^{k-2} \prod_{\ell=0}^{n-1} \left(\frac{j-q}{2q} + k - \ell\right) \overline{G}_{q,j,k-(n+1)}(x) \\
&= -\frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2k-1}\log(x) - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell\right) \frac{1}{2}e^{-x^2}x^{\frac{j}{q}+2m-1}\log(x) \\
&\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell\right) I_{q,j,1}(x) + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell\right) \overline{G}_{q,j,k-n}(x).
\end{aligned}$$

□

Using Lemma 3.3.16 and the last one, Lemma 3.3.17, we attain our subgoal: the ascertainment of the dominant term of $\int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds$ as x tends towards infinity.

Lemma 3.3.18. *Let $k \in \mathbb{N}$, where $k > 1$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow \infty$, we get that*

$$\int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds$$

has the dominant term

$$\begin{aligned}
&\prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell\right) \frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2}\right) \\
&\quad + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell\right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell\right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q}\right).
\end{aligned}$$

Proof. We set $I_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} \log(s) ds$ and $\bar{G}_{q,j,k}(x) := \int_0^x e^{-s^2} s^{\frac{j}{2q}+2k} ds$. By the dominant term of $I_{q,j,1}(x)$, which is stated in Lemma 3.3.16, by the dominant term of $\bar{G}_{q,j,k}$, see Lemma 3.3.7, and by Lemma 3.3.17, we achieve the dominant term of $I_{q,j,k}(x)$ as $x \rightarrow \infty$ in the following way:

$$\begin{aligned}
 I_{q,j,k}(x) &= \\
 &\stackrel{3.3.17}{=} -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \\
 &\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) I_{q,j,1}(x) + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \bar{G}_{q,j,k-n}(x) \\
 &\stackrel{3.3.16}{\approx} -\frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2k-1} \log(x) - \sum_{m=2}^{k-1} \prod_{\ell=0}^{k-m-1} \left(\frac{j-q}{2q} + k - \ell \right) \frac{1}{2} e^{-x^2} x^{\frac{j}{q}+2m-1} \log(x) \\
 &\quad + \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) \\
 &\quad + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \\
 &\sim \prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{4} \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) \\
 &\quad + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right).
 \end{aligned}$$

□

In the subsequent paragraph we consider the integrals $\int e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, where k is a non-positive integer. As above, we reorganize the recursive formula of Lemma 3.3.13 to calculate them.

Remark 3.3.19. Let $k \in \mathbb{Z}_{<0}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q-1\}$, where $(j, k) \neq (q, -1)$. Up to an additive constant, $H_{q,j,k}$ can also recursively be given by

$$H_{q,j,k} = \frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k+1} \log(s) - \frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} \bar{F}_{q,j,k} + \frac{1}{\frac{j+q}{2q} + k} H_{q,j,k+1}.$$

Proof. We use the recursive formula

$$H_{q,j,k} = -\frac{1}{2}e^{-s^2} s^{\frac{j}{q}+2k-1} \log(s) + \left(\frac{j-q}{2q} + k \right) H_{q,j,k-1} + \frac{1}{2} \bar{F}_{q,j,k-1}$$

of Lemma 3.3.13. By reorganizing to $H_{q,j,k-1}$, we get for $k < 1$

$$\begin{aligned} H_{q,j,k-1} &= \frac{1}{\frac{j-q}{2q} + k} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k-1} \log(s) - \frac{1}{\frac{j-q}{2q} + k} \frac{1}{2} \bar{F}_{q,j,k-1} + \frac{1}{\frac{j-q}{2q} + k} H_{q,j,k} \\ &= \frac{1}{\frac{j+q}{2q} + k - 1} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2(k-1)+1} \log(s) - \frac{1}{\frac{j+q}{2q} + k - 1} \frac{1}{2} \bar{F}_{q,j,k-1} \\ &\quad + \frac{1}{\frac{j+q}{2q} + k - 1} H_{q,j,k} \end{aligned}$$

and thus for $k < 0$

$$H_{q,j,k} = \frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+2k+1} \log(s) - \frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} \bar{F}_{q,j,k} + \frac{1}{\frac{j+q}{2q} + k} H_{q,j,k}.$$

□

First, we see an application of the recursive formula in its form for negative exponents stated in the corollary above. We calculate $\int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, $k < 0$. We can put down these integrals to the integrals $\int_1^{\infty} e^{-s^2} s^{\frac{j}{q}} \log(s) ds$ and $\int_1^{\infty} e^{-s^2} s^{-1} \log(s) ds$. As above, we do not go further into their antiderivative, which contains the hypergeometric function respectively the so-called *Meijer G-function*².

Lemma 3.3.20. *Let $k \in \mathbb{Z}_{<0}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. We achieve for $k < -1$*

$$\begin{aligned} \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds &= - \sum_{m=0}^{-(k+2)} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} + (k + \ell)} \right) \frac{1}{2} C_{j,k+m} \\ &\quad + \prod_{\ell=0}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + (k + \ell)} \tilde{c}_{j,-1}, \end{aligned}$$

²For further information to the *Meijer G-function* we refer to [OoSU10] and [Erd81].

where

$$\tilde{c}_{j,-1} := \begin{cases} \int_1^\infty e^{-s^2} s^{-1} \log(s) ds, & j = q, \\ -\frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} c_{j,-1} + \frac{1}{\frac{j+q}{2q}-1} \tilde{c}_{j,0}, & \text{else,} \end{cases}$$

where $\tilde{c}_{j,0} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}} \log(s) ds$ and

$$c_{j,k} := \begin{cases} \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = -1, \\ -\frac{1}{\frac{j-q}{2q}+(k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q}+(k+1)+\ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q}-\ell} c_{j,-1}, & k < -1. \end{cases}$$

Proof. We set $I_{q,j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$ and $\bar{G}_{q,j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} ds$. According to Lemma 3.3.10 and Lemma 3.3.11, we set

$$c_{j,k} := \bar{G}_{q,j,k} \begin{cases} \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = -1, \\ -\frac{1}{\frac{j-q}{2q}+(k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q}+(k+1)+\ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q}-\ell} c_{j,-1}, & k < -1. \end{cases}$$

Furthermore, we set $\tilde{c}_{j,0} := I_{q,j,0}$ and $\tilde{c}_{q,-1} := I_{q,q,-1}$. By the recursive formula of

Corollary 3.3.19 we get for $k = -1$ and $j \neq q$

$$\begin{aligned} I_{q,j,-1} &\stackrel{3.3.19}{=} \left[\frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} e^{-s^2} s^{\frac{j}{q}+-2+1} \log(s) \right]_1^\infty - \frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} \bar{G}_{q,j,-1} + \frac{1}{\frac{j+q}{2q}-1} I_{q,j,0} \\ &= -\frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} c_{j,-1} + \frac{1}{\frac{j+q}{2q}-1} \tilde{c}_{j,0}. \end{aligned}$$

We start the induction with $k = -2$ and obtain by using the recursive formula of Corollary 3.3.19

$$\begin{aligned}
I_{q,j,-2} &\stackrel{3.3.19}{=} \left[\frac{1}{\frac{j+q}{2q} - 2} \frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2(-2)+1} \log(s) \right]_1^\infty - \frac{1}{\frac{j+q}{2q} - 2} \frac{1}{2} \overline{G}_{q,j,-2} \\
&\quad + \frac{1}{\frac{j+q}{2q} - 2} I_{q,j,-1} \\
&= -\frac{1}{\frac{j+q}{2q} - 2} \frac{1}{2} C_{j,-2} + \frac{1}{\frac{j+q}{2q} - 2} \tilde{C}_{j,-1} \\
&= -\sum_{m=0}^0 \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} + (-2 + \ell)} \right) \frac{1}{2} C_{j,-2+m} + \prod_{\ell=0}^0 \frac{1}{\frac{j+q}{2q} + (-2 + \ell)} \tilde{C}_{j,-1}.
\end{aligned}$$

By using Corollary 3.3.19 again, we get for the induction step

$$\begin{aligned}
I_{q,j,k} &\stackrel{3.3.19}{=} \left[\frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} e^{-s^2} s^{\frac{j}{q} + 2k+1} \log(s) \right]_1^\infty - \frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} \overline{F}_{q,j,k} + \frac{1}{\frac{j+q}{2q} + k} I_{q,j,k+1} \\
&\stackrel{I.H.}{=} -\frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} C_{j,k} + \frac{1}{\frac{j+q}{2q} + k} \left[-\sum_{m=0}^{-(k+3)} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} + (k+1 + \ell)} \right) \frac{1}{2} C_{j,k+1+m} \right. \\
&\quad \left. + \prod_{\ell=0}^{-(k+3)} \frac{1}{\frac{j+q}{2q} + (k+1 + \ell)} \tilde{C}_{j,-1} \right] \\
&= -\frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} C_{j,k} - \sum_{m=0}^{-(k+3)} \frac{1}{\frac{j+q}{2q} + k} \left(\prod_{\ell=1}^{m+1} \frac{1}{\frac{j+q}{2q} + k + \ell} \right) \frac{1}{2} C_{j,k+(m+1)} \\
&\quad + \frac{1}{\frac{j+q}{2q} + k} \prod_{\ell=1}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + k + \ell} \tilde{C}_{j,-1} \\
&= -\frac{1}{\frac{j+q}{2q} + k} \frac{1}{2} C_{j,k} - \sum_{m=1}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + k} \left(\prod_{\ell=1}^m \frac{1}{\frac{j+q}{2q} + k + \ell} \right) \frac{1}{2} C_{j,k+m} \\
&\quad + \frac{1}{\frac{j+q}{2q} + k} \prod_{\ell=1}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + k + \ell} \tilde{C}_{j,-1} \\
&= -\sum_{m=0}^{-(k+2)} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} + k + \ell} \right) \frac{1}{2} C_{j,k+m} + \prod_{\ell=0}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + k + \ell} \tilde{C}_{j,-1}.
\end{aligned}$$

□

Remark 3.3.21. *Up to an additive constant, we get*

$$\int s^{-1} \log(s) ds = \frac{1}{2} (\log(s))^2.$$

Proof. Up to an additive constant, we easily achieve

$$\int s^{-1} \log(s) ds = \log(s) \log(s) - \int \log(s) s^{-1} ds$$

by integration by parts. In consequence of reorganization, we get

$$2 \int s^{-1} \log(s) ds = (\log(s))^2$$

and therefore

$$\int s^{-1} \log(s) ds = \frac{1}{2} (\log(s))^2.$$

□

The next Lemma shows the asymptotic series expansion of $\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$, where $k \leq 0$ and x approaches zero. This proof is similarly structured to the proof of Lemma 3.2.10 in Section 3.2 and the proof of Lemma 3.3.12.

Lemma 3.3.22. *Let $k \in \mathbb{Z}_{\leq 0}$, $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q - 1\}$. As $x \rightarrow 0$,*

$$\int_x^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$$

has for $j \neq q$ the asymptotic expansion

$$\sum_{n=0}^{\infty} \tilde{b}_{j,k,n} x^{\frac{j}{q}+2(n+k)+1} + \sum_{n=0}^{\infty} (-\tilde{b}_{j,k,n}) \left(\frac{j+q}{q} + 2(n+k) \right) \log(x) x^{\frac{j}{q}+2(n+k)+1} + \tilde{d}_{j,k}$$

and for $j = q$

$$\begin{aligned} & \sum_{n=0}^{\infty} \tilde{b}_{q,k,n} x^{1+2(n+k)+1} + \sum_{n=0}^{\infty} (-\tilde{b}_{q,k,n}) (2 + 2(n+k)) \log(x) x^{1+2(n+k)+1} \\ & + \frac{(-1)^{-k}}{(-k-1)!} \frac{1}{2} (\log(x))^2 + \tilde{d}_{q,k}, \end{aligned}$$

where

$$\tilde{b}_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = -k - 1, \\ (-1)^n \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right)^2 n!}, & \text{else,} \end{cases}$$

and

$$\tilde{d}_{j,k} := \sum_{n=0}^{\infty} \left(-\tilde{b}_{j,k,n} \right) + \tilde{c}_{j,k},$$

where

$$\tilde{c}_{j,k} := \begin{cases} \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}} \log(s) ds, & k = 0, \\ \int_1^{\infty} e^{-s^2} s^{-1} \log(s) ds, & j = q, \\ -\frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} c_{j,-1} + \frac{1}{\frac{j+q}{2q}-1} \tilde{c}_{j,0}, & \text{else,} \\ -\sum_{m=0}^{-(k+2)} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} + (k+\ell)} \right) \frac{1}{2} c_{j,k+m} + \prod_{\ell=0}^{-(k+2)} \frac{1}{\frac{j+q}{2q} + (k+\ell)} \tilde{c}_{j,-1}, & k < -1, \end{cases}$$

where

$$c_{j,k} := \begin{cases} \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = -1, \\ -\frac{1}{\frac{j-q}{2q} + (k+1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{-(k+2)} \prod_{\ell=0}^{-(k+1)-m} \frac{1}{\frac{j-q}{2q} + (k+1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{-(k+1)} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,-1}, & k < -1. \end{cases}$$

Proof. We set $I_{q,j,k}(x) := \int_x^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$. As $x \rightarrow 0$, we can split up $I_{q,j,k}(x)$ into

$$\int_x^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds = \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds + \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds.$$

The integrals $\int_1^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$ exist and have been calculated in Lemma 3.3.20.

Therefore, we set $\tilde{c}_{j,k} := \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds$. We investigate by using the series expansion of the exponential function in Lemma 2.2.11 that

$$\begin{aligned}
 I_{q,j,k}(x) &= \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds + \int_1^\infty e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds \\
 &= \int_x^1 e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds + \tilde{c}_{j,k} \\
 &\stackrel{2.2.11}{=} \int_x^1 \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^{2n} s^{\frac{j}{q}+2k} \log(s) ds + \tilde{c}_{j,k} \\
 &= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)} \log(s) ds + \tilde{c}_{j,k}. \tag{1}
 \end{aligned}$$

It is necessary to distinguish between two cases. First, we may assume $j \neq q$. Consequently it is $\frac{j}{q} + 2(n+k) \neq -1$ for all $n \in \mathbb{N}_0$ and we ascertain by integration by parts

$$\begin{aligned}
 &I_{q,j,k}(x) \\
 \stackrel{(1)}{=} &\sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_x^1 s^{\frac{j}{q}+2(n+k)} \log(s) ds + \tilde{c}_{j,k} \\
 = &\sum_{n=0}^\infty \frac{(-1)^n}{n!} \left[\left[\frac{1}{\frac{j}{q} + 2(n+k) + 1} s^{\frac{j}{q}+2(n+k)+1} \log(s) \right]_x^1 \right. \\
 &\left. - \int_x^1 \frac{1}{\frac{j}{q} + 2(n+k) + 1} s^{\frac{j}{q}+2(n+k)+1} \frac{1}{s} ds \right] + \tilde{c}_{j,k} \\
 = &\sum_{n=0}^\infty (-1)^n \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right) n!} \left[-x^{\frac{j}{q}+2(n+k)+1} \log(x) \right. \\
 &\left. - \left[\frac{1}{\frac{j}{q} + 2(n+k) + 1} s^{\frac{j}{q}+2(n+k)+1} \right]_x^1 \right] + \tilde{c}_{j,k}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right) n!} \left[-x^{\frac{j}{q} + 2(n+k) + 1} \log(x) \right. \\
&\quad \left. + \frac{1}{\frac{j}{q} + 2(n+k) + 1} x^{\frac{j}{q} + 2(n+k) + 1} - \frac{1}{\frac{j}{q} + 2(n+k) + 1} \right] + \tilde{c}_{j,k} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right)^2 n!} \left[-\left(\frac{j}{q} + 2(n+k) + 1\right) x^{\frac{j}{q} + 2(n+k) + 1} \log(x) \right. \\
&\quad \left. + x^{\frac{j}{q} + 2(n+k) + 1} \right] + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right)^2 n!} + \tilde{c}_{j,k} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\left(\frac{j+q}{q} + 2(n+k)\right)^2 n!} x^{\frac{j}{q} + 2(n+k) + 1} \\
&\quad + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{\left(\frac{j+q}{q} + 2(n+k)\right)^2 n!} \left(\frac{j+q}{q} + 2(n+k)\right) \log(x) x^{\frac{j}{q} + 2(n+k) + 1} \\
&\quad + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{\left(\frac{j}{q} + 2(n+k) + 1\right)^2 n!} + \tilde{c}_{j,k},
\end{aligned}$$

where the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\left(\frac{j+q}{q} + 2(n+k)\right)^2 n!}$ converges. We set $\tilde{b}_{j,k,n} := \frac{(-1)^n}{\left(\frac{j+q}{q} + 2(n+k)\right)^2 n!}$ and furthermore,

$$\tilde{d}_{j,k} := \sum_{n=0}^{\infty} (-\tilde{b}_{j,k,n}) + \tilde{c}_{j,k}.$$

We can conclude

$$\begin{aligned}
I_{q,j,k}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_x^1 s^{\frac{j}{q} + 2(n+k)} \log(s) ds + \tilde{c}_{j,k} \\
&= \tilde{d}_{j,k} + \sum_{n=0}^{\infty} \tilde{b}_{j,k,n} x^{\frac{j}{q} + 2(n+k) + 1} \\
&\quad + \sum_{n=0}^{\infty} (-\tilde{b}_{j,k,n}) \left(\frac{j+q}{q} + 2(n+k)\right) \log(x) x^{\frac{j}{q} + 2(n+k) + 1}. \quad (2)
\end{aligned}$$

We may assume $j = q$. Since $1 + 2(n + k) = -1$ if $n = -k - 1$, we split up the series in (1) in $\sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \int_x^1 s^{1+2(n+k)} \log(s) ds + \frac{(-1)^{-k-1}}{(-k-1)!} \int_x^1 s^{-1} \log(s) ds$. For the first summand we achieve for $j = q$ and $n \neq -k - 1$ the same result as in (2) in an analogous way. For reasons of clarity we use this partial result and set

$$\begin{aligned} \overline{H}_{q,q,k}(x) &:= \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \int_x^1 s^{1+2(n+k)} \log(s) ds + \tilde{c}_{q,k} \\ &= \tilde{d}_{j,k} + \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \tilde{b}_{j,k,n} x^{\frac{j}{q} + 2(n+k) + 1} \\ &\quad + \sum_{n=0}^{\infty} (-\tilde{b}_{j,k,n}) \left(\frac{j+q}{q} + 2(n+k) \right) \log(x) x^{\frac{j}{q} + 2(n+k) + 1}. \end{aligned} \quad (3)$$

Using Remark 3.3.21, we investigate the asymptotic series expansion for $j = q$ by

$$\begin{aligned} I_{q,q,k}(x) &\stackrel{(1)}{=} \sum_{\substack{n=0 \\ n \neq -k-1}}^{\infty} \frac{(-1)^n}{n!} \int_x^1 s^{1+2(n+k)} \log(s) ds + \frac{(-1)^{-k-1}}{(-k-1)!} \int_x^1 s^{-1} \log(s) ds + \tilde{c}_{q,k} \\ &\stackrel{(3)}{=} \overline{H}_{q,q,k}(x) + \frac{(-1)^{k-1}}{(-k-1)!} \int_x^1 s^{-1} \log(s) ds \\ &\stackrel{3.3.21}{=} \overline{H}_{q,q,k}(x) + \frac{(-1)^{k-1}}{(-k-1)!} \left[\frac{1}{2} (\log(s))^2 \right]_x^1 \\ &\stackrel{(3)}{=} \sum_{n=0}^{\infty} \tilde{b}_{q,k,n} x^{2(n+k)+2} + \sum_{n=0}^{\infty} (-\tilde{b}_{q,k,n}) (2(n+k) + 2) \log(x) x^{2(n+k)+2} \\ &\quad + \tilde{d}_{q,k} + \frac{(-1)^k}{(-k-1)!} \frac{1}{2} (\log(x))^2, \end{aligned}$$

where we set $\tilde{b}_{q,k,n} := 0$ if $n = -k - 1$ and else we set $\tilde{b}_{q,k,n}$ and $\tilde{d}_{q,k}$ as above. \square

3.3.2 Definability and Asymptotic Expansions

In this last section of Chapter 3, we come up with the probability distribution given by the standard Brownian Motion on a globally subanalytic set $A \subset \mathbb{R}^3$, which is represented by

$$f : \mathbb{R}_{>0} \longrightarrow [0, 1],$$

$$t \longmapsto P_0(B_t \in A) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

As we mentioned above, we assume without loss of generality that the dimension of A is three to avoid integration over a null set. The three-dimensional case is structural quite similar to \mathbb{R}^2 , but there is a content-related difference. As above, we cannot give a general statement about definability of $f(t)$ and therefore, we devote once again our attention to the limit points of t : zero and infinity. There we establish asymptotic expansions of $f(t)$. It is remarkable that we get definability of $f(t)$ in $\mathbb{R}_{\text{an,exp}}$ if t tends towards infinity and A is bounded. But if A is unbounded the asymptotic expansion of $f(t)$ is also definable in $\mathbb{R}_{\text{an,exp}}$ as t approaches infinity. The main results of this section are gathered in the following theorem.

Theorem 3.3.23. *Let $t \in \mathbb{R}_{>0}$ and let $A \subset \mathbb{R}^3$ be a globally subanalytic set. Let $f(t)$ be the probability distribution given by the standard Brownian Motion on A , which is defined as*

$$f(t) := \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

i) *As $t \rightarrow 0$, $f(t)$ has the asymptotic series expansion*

$$\sum_{k=0}^{\infty} (\gamma_k + \delta_k \log(t)) t^{\frac{k}{q}},$$

which is not necessarily convergent and where $\gamma_k, \delta_k \in \mathbb{R}$.

ii) *As $t \rightarrow \infty$:*

a) *If A is bounded, $f(t)$ has the form $\varrho_0(t) + \varrho_1(t) \log(t)$, where $\varrho_0(t)$ and $\varrho_1(t)$ are globally subanalytic functions, and thereby $f(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$.*

b) *If A is unbounded, $f(t)$ has the asymptotic expansion*

$$\rho_0(t) + \rho_1(t) \log(t) + \rho_2(t) \log(t)^2,$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \mu_k t^{-\frac{k}{q}}$, $\rho_1(t) = \sum_{k=1}^{\infty} \nu_k t^{-\frac{k}{q}}$, and $\rho_2(t) = \sum_{k=1}^{\infty} \omega_k t^{\frac{1}{2}-k}$, $\mu_k, \nu_k, \omega_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and at least one of the series is not zero. The asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

The proof of this theorem is split up in two parts. For each part we resume the results in a theorem; as t tends towards zero in Theorem 3.3.30 and in Theorem 3.3.43 as t approaches infinity. First, we consider the case t tends to zero in the subsequent paragraph.

The Case t at Zero

Our aim is to show an asymptotic series expansion of $\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx$ as t tends towards zero. In the course of this we will consider integrals of the form

$$\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr,$$

where $\psi_i(r)$ are globally subanalytic functions and $\psi_0(r) + \psi_1(r) \log(r)$ is bounded on an open interval $D \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$. By the Cell Decomposition Theorem 2.3.15, we are able to reduce D to three forms of intervals: $]0, \beta[$, where $\beta < 1$, $] \alpha, \beta[$, where $0 < \alpha < \beta < \infty$, and $] \alpha, \infty[$, where $\alpha > 1$.

First, we assume $D =]0, \beta[$. The functions $\psi_0(r)$ and $\psi_1(r)$ can be represented as a convergent Puiseux series around the center zero by arguments, which we will present later. Before we start with establishing asymptotic expansions, we give a short proof to show the argument that if $\psi_0(r) + \psi_1(r) \log(r)$ is bounded on $]0, \beta[$, $\beta < 1$, it implies $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ is bounded on $]0, \beta[$.

Lemma 3.3.24. *Let $\Psi(r) = \psi_0(r) + \psi_1(r) \log(r)$ be a bounded function on $D :=]0, \beta[$, $\beta < 1$, where $\psi_0(r)$ and $\psi_1(r)$ are globally subanalytic functions, which can be represented as Puiseux series around zero. We have that $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ is bounded on D .*

Proof. Let $\psi_0(r)$ be represented as $\sum_{n=k}^{\infty} a_n r^n$ and $\psi_1(r)$ as $\sum_{m=\ell}^{\infty} b_m r^m$. We prove the claim by contradiction. Without loss of generality we assume $k < 0$. If $k < \ell$

then $\Psi(r) \sim r^k$ is a contradiction to $\Psi(r)$ is bounded on $]0, \beta[$. If $k \geq \ell$ then $\Psi(r) \sim r^\ell \log(r)$, which is also a contradiction. Thus, it is $k, \ell \geq 0$. If $\ell = 0$ then we obtain $\Psi(r) \sim \log(r)$, which is also a contradiction. Therefore, we get that $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ have to be bounded on D and in consequence of this, $k \geq 0$ and $\ell > 0$. \square

Now, we show the asymptotic series expansions of the integrals $\int_D e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$ and $\int_D e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$.

We attract our attention to $\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$. In Section 3.2 we have already studied a form of integrals which is quite similar to the form above. In the next lemma, we show the asymptotic series expansion of $\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$ by applying results of Preparations 3.3.1.

Lemma 3.3.25. *Let $q \in \mathbb{N}$, let $\beta \in \mathbb{R}_{>0}$, where $\beta < 1$, and let $\psi(r)$ be a bounded function on $]0, \beta[$ which can be represented as the convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k}$. As $t \rightarrow 0$, we get the asymptotic series expansion*

$$\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+\frac{3}{2}},$$

where $\hat{a}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k+\frac{3}{2}} \prod_{\ell=1}^{k+1} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right)$.

Proof. As this proof is similar in spirit to the proof for $\int_0^\beta e^{-\frac{r^2}{2t}} r \psi(r) dr$, see Lemma 3.2.12 in Section 3.2, we shorten it to the essentials.

We set $\bar{F}(t) := \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$. First, we show as $t \rightarrow 0$

$$\bar{F}(t) - \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+2} dr = \mathcal{O}(t^{N+2})$$

for a sufficiently large $N \in \mathbb{N}$, where N is a multiple of q . By changing variables

$s = \frac{r}{\sqrt{2t}}$, we get

$$\begin{aligned}
 \bar{F}(t) &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=1}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+2} dr \\
 &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+2} dr \\
 &= \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} \left(s\sqrt{2t}\right)^{\frac{j}{q}+2k+2} \sqrt{2t} ds.
 \end{aligned}$$

By estimation of the remainder, we obtain for $0 \leq s\sqrt{2t} \leq \beta$ and a sufficiently large constant $C \in \mathbb{R}$

$$\begin{aligned}
 &\left| \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} \left(s\sqrt{2t}\right)^{\frac{j}{q}+2k+2} \sqrt{2t} ds \right| \\
 &\leq \sqrt{2t} \int_0^\infty e^{-s^2} C \sqrt{2t}^{2(N+1)+2} s^{2(N+1)+2} ds \\
 &= \sqrt{2t}^{2(N+2)+1} C \int_0^\infty e^{-s^2} s^{2(N+2)+1-1} ds \\
 &\stackrel{3.2.3}{=} \sqrt{2t}^{2(N+2)+1} C \frac{1}{2} \Gamma\left(N + \frac{5}{2}\right). \tag{1}
 \end{aligned}$$

We conclude

$$\begin{aligned}
 \bar{F}(t) - \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+2} dr &= \int_0^\beta e^{-\frac{r^2}{2t}} \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k+2} dr \\
 &\stackrel{(1)}{=} \mathcal{O}\left(t^{N+\frac{5}{2}}\right). \tag{2}
 \end{aligned}$$

Establishing the asymptotic series expansion, we change the variable $s = \frac{r}{\sqrt{2t}}$ as

above and write

$$\begin{aligned}\bar{F}(t) &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q}+2(k+1)+1} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2(k+1)} ds \\ &= \sum_{k=0}^{\infty} \bar{F}_{k+1}(t),\end{aligned}$$

where $\bar{F}_k(t) := \sum_{j=0}^{2q-1} a_{j,k-1} \sqrt{2t}^{\frac{j}{q}+2k+1} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds$. In Lemma 3.3.6 we have al-

ready seen the asymptotic series expansion of $\int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds$. Due to the fact that $e^{-\frac{\beta^2}{2t}}$, which is contained in one summand of the asymptotic expansion, tends towards zero very fast in comparison to t^N , $N \in \mathbb{N}$, if t approaches zero, only the constant terms of the expansion are of interest. In Corollary 3.3.7 we have already seen that

$$\int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \sim \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right). \quad (3)$$

Hence, we get the asymptotic expansion $\hat{F}_k(t)$ of $\bar{F}_k(t)$ as $t \rightarrow 0$, which is

$$\hat{F}_k(t) = \sum_{j=0}^{2q-1} \hat{a}_{j,k-1} t^{\frac{j}{2q}+k+\frac{1}{2}},$$

where $\hat{a}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k+\frac{3}{2}} \prod_{\ell=1}^{k+1} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right)$ and we observe that

$$\hat{F}_k(t) = \mathcal{O} \left(t^{k+\frac{1}{2}} \right). \quad (4)$$

We set

$$\begin{aligned}\hat{F}(t) &= \sum_{k=0}^{\infty} \hat{F}_{k+1}(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} t^{\frac{j}{2q}+k+\frac{3}{2}}.\end{aligned}$$

Alike Lemma 3.2.12 in Section 3.2, we achieve

$$\overline{F}(t) \approx \widehat{F}(t)$$

by

$$\begin{aligned} \overline{F}(t) - \widehat{F}|_N(t) &\stackrel{(4)}{=} \sum_{k=0}^{\infty} \overline{F}_{k+1}(t) - \sum_{k=0}^N \widehat{F}_{k+1}|_N(t) \\ &\stackrel{(2)}{=} \sum_{k=0}^N \left(\overline{F}_{k+1}(t) - \widehat{F}_{k+1}|_N(t) \right) + \mathcal{O}\left(t^{N+\frac{5}{2}}\right) \\ &\stackrel{(3)}{=} \mathcal{O}\left(t^{N+\frac{5}{2}}\right) \end{aligned}$$

and thus $\overline{F}(t)$ behaves like a Puiseux series. \square

Now, we study the integrals $\int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$, where $\beta < 1$ and $\psi(r) \log(r)$ is bounded on $]0, \beta[$ and $\psi(r)$ can be written as a convergent Puiseux series around the center zero. Our next aim is to find the asymptotic expansion of $\int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$. For this purpose, we show the dominant terms of the integrals $\int_0^{\beta} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k} \log(r) dr$ by using several lemmas, which have already been proven in Preparations 3.3.1.

Lemma 3.3.26. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$, $j \in \{0, \dots, 2q - 1\}$, and let $\beta \in \mathbb{R}_{>0}$, where $\beta < 1$. As $t \rightarrow 0$, we have the dominant term*

$$\int_0^{\beta} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k} \log(r) dr \sim (b_{j,k} + c_{j,k} \log(2)) t^{\frac{j}{2q}+k+\frac{1}{2}} + c_{j,k} \log(t) t^{\frac{j}{2q}+k+\frac{1}{2}},$$

where

$$\begin{aligned} b_{j,k} := & 2^{\frac{j}{2q}+k+\frac{1}{2}} \left[\prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) \right. \\ & \left. + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \right] \end{aligned}$$

and

$$c_{j,k} := 2^{\frac{j}{2q}+k-\frac{1}{2}} \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right).$$

Proof. We make a change of variable $s = \frac{r}{\sqrt{2t}}$ and obtain

$$\begin{aligned} \int_0^{\beta} e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k} \log(r) dr &= \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \left(s\sqrt{2t} \right)^{\frac{j}{q}+2k} \log \left(s\sqrt{2t} \right) \sqrt{2t} ds \\ &= \sqrt{2t} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} \left(s\sqrt{2t} \right)^{\frac{j}{q}+2k} \left(\log(s) + \frac{1}{2} \log(2t) \right) ds \\ &= \sqrt{2t}^{\frac{j}{q}+2k+1} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} \left(\log(s) + \frac{1}{2} (\log(2) + \log(t)) \right) ds \\ &= \sqrt{2t}^{\frac{j}{q}+2k+1} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds \\ &\quad + \sqrt{2t}^{\frac{j}{q}+2k+1} \frac{1}{2} \log(2) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &\quad + \sqrt{2t}^{\frac{j}{q}+2k+1} \frac{1}{2} \log(t) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds. \end{aligned}$$

We have already seen the asymptotic expansion of these integrals in Lemma 3.3.6 and 3.3.18. Due to the fact that $e^{-\frac{\beta^2}{2t}}$, which is a factor in a part of each of the asymptotic expansions, tends towards zero very fast in comparison to t^N , $N \in \mathbb{N}$, if t approaches zero, only the constant terms are of interest. For the first summand we have ascertained the dominant term

$$\sqrt{2t}^{\frac{j}{q}+2k+1} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds \sim b_{j,k} t^{\frac{j}{2q}+k+\frac{1}{2}}$$

in Lemma 3.3.18, where

$$b_{j,k} := 2^{\frac{j}{2q}+k+\frac{1}{2}} \left[\prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \right].$$

Using Lemma 3.3.7, the second summand has the dominant term

$$\sqrt{2t^{\frac{j}{q}+2k+1}} \frac{1}{2} \log(2) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \sim c_{j,k} \log(2) t^{\frac{j}{2q}+k+\frac{1}{2}},$$

where

$$c_{j,k} := 2^{\frac{j}{2q}+k-\frac{1}{2}} \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right),$$

and obviously we obtain by the same lemma as above

$$\sqrt{2t^{\frac{j}{q}+2k+1}} \frac{1}{2} \log(t) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \stackrel{3.3.7}{\sim} c_{j,k} t^{\frac{j}{2q}+k+\frac{1}{2}} \log(t)$$

for the third term. In conclusion, we get

$$\begin{aligned} \int_0^{\beta} e^{\frac{-r^2}{2t}} r^{\frac{j}{q}+2k} \log(r) dr &= \sqrt{2t^{\frac{j}{q}+2k+1}} \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} \log(s) ds \\ &+ \sqrt{2t^{\frac{j}{q}+2k+1}} \frac{1}{2} \log(2) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &+ \sqrt{2t^{\frac{j}{q}+2k+1}} \frac{1}{2} \log(t) \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{\frac{j}{q}+2k} ds \\ &\sim b_{j,k} t^{\frac{j}{2q}+k+\frac{1}{2}} + c_{j,k} \log(2) t^{\frac{j}{2q}+k+\frac{1}{2}} + c_{j,k} \log(t) t^{\frac{j}{2q}+k+\frac{1}{2}} \\ &= (b_{j,k} + c_{j,k} \log(2)) t^{\frac{j}{2q}+k+\frac{1}{2}} + c_{j,k} \log(t) t^{\frac{j}{2q}+k+\frac{1}{2}}. \end{aligned}$$

□

Now, we are able to establish the asymptotic series expansion of integrals $\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ in the next lemma. In consequence of the boundedness of $\psi(r) \log(r)$ on $]0, \beta[$, the representation of $\psi(r)$ as a convergent Puiseux series in zero has only positive exponents. Therefore, we can reduce the integrals $\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ to integrals of the form $\int_0^\beta e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k} \log(r) dr$ for which we have already seen the dominant term in Lemma 3.3.26.

Lemma 3.3.27. *Let $\beta \in \mathbb{R}_{>0}$, where $\beta < 1$, let $\psi(r)$ be a function such that $\psi(r) \log(r)$ be bounded on $]0, \beta[$ and $\psi(r)$ can be described by the convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k}$, where $q \in \mathbb{N}$. As $t \rightarrow 0$, we achieve the asymptotic series expansion*

$$\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} (\hat{a}_{j,k} + \hat{c}_{j,k} \log(t)) t^{\frac{j}{2q}+k+\frac{3}{2}},$$

where

$$\begin{aligned} \hat{a}_{j,k} := & a_{j,k} 2^{\frac{j}{2q}+k+\frac{1}{2}} \left[\prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \Gamma \left(\frac{j}{2q} + \frac{3}{2} \right) \right. \\ & + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \\ & \left. + \frac{1}{2} \log(2) \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \right] \end{aligned}$$

and

$$\hat{c}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k-\frac{1}{2}} \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right).$$

Proof. We set $H(t) := \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ and split $H(t)$ into

$$H(t) = \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr$$

$$\begin{aligned}
 &= \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr \\
 &\quad + \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr.
 \end{aligned}$$

First, we show

$$H(t) - \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr = \mathcal{O}\left(\log(t) t^{N+\frac{5}{2}}\right)$$

for a sufficiently large $N \in \mathbb{N}$, where N is a multiple of q , and as $t \rightarrow 0$. By substitution $s = \frac{r}{\sqrt{2t}}$, we get

$$\begin{aligned}
 &H(t) - \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr \\
 &= \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr \\
 &= \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} (s\sqrt{2t})^2 \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} (s\sqrt{2t})^{\frac{j}{q}+2k} \log(s\sqrt{2t}) \sqrt{2t} ds.
 \end{aligned}$$

We estimate the remainder by

$$\sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} (s\sqrt{2t})^{\frac{j}{q}+2k} \leq C (s\sqrt{2t})^{2(N+1)}$$

for $0 \leq s\sqrt{2t} \leq \beta$ and a sufficiently large constant $C \in \mathbb{R}$ and thereby we can continue

$$\left| \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} (s\sqrt{2t})^2 \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} (s\sqrt{2t})^{\frac{j}{q}+2k} \log(s\sqrt{2t}) \sqrt{2t} ds \right| \leq$$

$$\begin{aligned}
&\leq \left| C \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} (s\sqrt{2t})^{2(N+1)+2} \log(s\sqrt{2t}) \sqrt{2t} ds \right| \\
&= \sqrt{2t}^{2(N+2)+1} C \left| \int_0^{\frac{\beta}{\sqrt{2t}}} e^{-s^2} s^{2(N+2)} \log(s\sqrt{2t}) ds \right| \\
&\leq \sqrt{2t}^{2(N+2)+1} C \left| \int_0^{\infty} e^{-s^2} s^{2(N+2)} \log(s\sqrt{2t}) ds \right| \\
&= \sqrt{2t}^{2(N+2)+1} C \left| \int_0^{\infty} e^{-s^2} s^{2(N+2)} \left(\log(s) + \frac{1}{2} \log(2) + \frac{1}{2} \log(t) \right) ds \right| \\
&\leq \sqrt{2t}^{2(N+2)+1} C \left(\left| \int_0^{\infty} e^{-s^2} s^{2(N+2)+1-1} \log(s) ds \right| + \frac{1}{2} \log(2) \int_0^{\infty} e^{-s^2} s^{2(N+2)+1-1} ds \right. \\
&\quad \left. + \frac{1}{2} |\log(t)| \int_0^{\infty} e^{-s^2} s^{2(N+2)+1-1} ds \right) \\
&\stackrel{3.2.3}{=} \stackrel{3.3.14}{=} (2t)^{N+2+\frac{1}{2}} C \left[\frac{1}{4} \left| \Gamma' \left(\frac{2(N+2)+1}{2} \right) \right| + \frac{1}{4} \log(2) \Gamma \left(\frac{2(N+2)+1}{2} \right) \right. \\
&\quad \left. + \frac{1}{4} |\log(t)| \Gamma \left(\frac{2(N+2)+1}{2} \right) \right] \\
&\stackrel{3.3.15}{=} (2t)^{N+2+\frac{1}{2}} C \left[\frac{1}{4} \Gamma' \left(N + \frac{5}{2} \right) + \frac{1}{4} \log(2) \Gamma \left(N + \frac{5}{2} \right) + \frac{1}{4} |\log(t)| \Gamma \left(N + \frac{5}{2} \right) \right] \\
&= \left[2^{N+2+\frac{1}{2}} C \left(\frac{1}{4} \Gamma' \left(N + \frac{5}{2} \right) + \frac{1}{4} \log(2) \Gamma \left(N + \frac{5}{2} \right) \right) \right] t^{N+\frac{5}{2}} \\
&\quad + \left[2^{N+2+\frac{1}{2}} C \frac{1}{4} \Gamma \left(N + \frac{5}{2} \right) \right] |\log(t)| t^{N+\frac{5}{2}}. \tag{1}
\end{aligned}$$

Being $t^{N+\frac{5}{2}} = \mathcal{O} \left(t^{N+\frac{5}{2}} \log(t) \right)$ for an arbitrary $N \in \mathbb{N}$ as $t \rightarrow 0$, we conclude by Definition 2.2.1

$$H(t) - \int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^N \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr =$$

$$\begin{aligned}
 &= \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=N+1}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k-1} \log(r) dr \\
 &\stackrel{(1)}{=} \mathcal{O}\left(t^{N+\frac{5}{2}} \log(t)\right). \tag{2}
 \end{aligned}$$

In the next step we establish the asymptotic expansion of $H(t)$ and for this we rewrite $H(t)$ in

$$\begin{aligned}
 H(t) &= \int_0^\beta e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^\infty \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k} \log(r) dr \\
 &= \sum_{k=0}^\infty \sum_{j=0}^{2q-1} a_{j,k} \int_0^\beta e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k+2} \log(r) dr \\
 &= \sum_{k=0}^\infty H_k(t),
 \end{aligned}$$

where $H_k(t) := \sum_{j=0}^{2q-1} a_{j,k} \int_0^\beta e^{-\frac{r^2}{2t}} r^{\frac{j}{q}+2k+2} \log(r) dr$. By Lemma 3.3.26, we ascertain the asymptotic expansion $\hat{H}_k(t)$ of $H_k(t)$ as $t \rightarrow 0$, which is given by

$$\begin{aligned}
 \hat{H}_k(t) &= \sum_{j=0}^{2q-1} \left(\hat{a}_{j,k} t^{\frac{j}{2q}+(k+1)+\frac{1}{2}} + \hat{c}_{j,k} t^{\frac{j}{2q}+(k+1)+\frac{1}{2}} \log(t) \right) \\
 &= \sum_{j=0}^{2q-1} (\hat{a}_{j,k} + \hat{c}_{j,k} \log(t)) t^{\frac{j}{2q}+k+\frac{3}{2}},
 \end{aligned}$$

where

$$\begin{aligned}
 \hat{a}_{j,k} &:= a_{j,k} 2^{\frac{j}{2q}+k+\frac{1}{2}} \left[\prod_{\ell=2}^k \left(\frac{j-q}{2q} + \ell \right) \Gamma' \left(\frac{j}{2q} + \frac{3}{2} \right) \right. \\
 &\quad \left. + \frac{1}{2} \sum_{n=1}^{k-1} \prod_{\ell=0}^{n-2} \left(\frac{j-q}{2q} + k - \ell \right) \prod_{\ell=1}^{k-n} \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \right. \\
 &\quad \left. + \frac{1}{2} \log(2) \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right) \right]
 \end{aligned}$$

and

$$\hat{c}_{j,k} := a_{j,k} 2^{\frac{j}{2q}+k-\frac{1}{2}} \prod_{\ell=1}^k \left(\frac{j-q}{2q} + \ell \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right).$$

We observe that

$$\hat{H}_k(t) = \mathcal{O} \left(\log(t) t^{k+\frac{3}{2}} \right). \tag{3}$$

Consequently, we set

$$\begin{aligned} \hat{H}(t) &= \sum_{k=0}^{\infty} \hat{H}_k(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} (\hat{a}_{j,k} + \hat{c}_{j,k} \log(t)) t^{\frac{j}{2q}+k+\frac{3}{2}} \end{aligned}$$

and due to this it is evident by

$$\begin{aligned} H(t) - \hat{H}|_N(t) &\stackrel{(3)}{=} \sum_{k=0}^N H_k(t) - \sum_{k=0}^N \hat{H}_k|_N(t) + \sum_{k=N+1}^{\infty} H_k(t) \\ &\stackrel{(2)}{=} \sum_{k=0}^N \left(H_k(t) - \hat{H}_k(t) \right) + \mathcal{O} \left(t^{N+\frac{5}{2}} \log(t) \right) \\ &= \mathcal{O} \left(t^{N+\frac{5}{2}} \log(t) \right) \end{aligned}$$

that

$$H(t) \approx \hat{H}(t).$$

□

For the next part we may assume $D =]\alpha, \beta[$ respectively $D =]\alpha, \infty[$, where $\alpha \neq 0$. We only show the scale of the integrals $\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, where $\psi_0(r)$ and $\psi_1(r)$ are continuous functions. For this, we use the argument that $\psi_0(r) + \psi_1(r) \log(r)$ is bounded on D as we mentioned above.

Lemma 3.3.28. *Let $\alpha \in \mathbb{R}$, where $\alpha \geq 1$. Let $\psi_0(r)$ and $\psi_1(r)$ be continuous functions and let $\psi_0(r) + \psi_1(r) \log(r)$ be bounded on $] \alpha, \infty[$. As $t \rightarrow 0$, we achieve*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O} \left(e^{-\frac{\alpha^2}{2t}} t \right).$$

Proof. In Lemma 2.2.12 we have introduced the asymptotic expansion

$$\operatorname{erf}\left(\frac{\alpha}{\sqrt{2t}}\right) \approx 1 + e^{-\frac{\alpha^2}{2t}} \left(\frac{\alpha}{\sqrt{2t}}\right)^{-1} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{\left(\frac{\alpha^2}{t}\right)^n} \prod_{\ell=1}^n (2\ell - 1)$$

as $t \rightarrow 0$ and by Remark 2.2.5 we obviously have

$$\operatorname{erf}\left(\frac{\alpha}{\sqrt{2t}}\right) = 1 + \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t^{\frac{1}{2}}\right). \quad (1)$$

In consequence of the boundedness of $\psi_0(r) + \psi_1(r) \log(r)$, it exists a sufficiently large constant $C \in \mathbb{R}$ such that $|\psi_0(r) + \psi_1(r) \log(r)| \leq C$ on $] \alpha, \infty[$. Therefore, we get by a change of the variable $s = \frac{r}{\sqrt{2t}}$ and the recursive formula in Lemma 3.3.1

$$\begin{aligned} & \left| \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr \right| \leq \\ & \leq \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 C dr \\ & = \sqrt{2t} C \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} (s\sqrt{2t})^2 ds \\ & = \sqrt{2t}^3 C \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^2 ds \\ & \stackrel{3.3.1}{=} \sqrt{2t}^3 C \left(\left[-\frac{1}{2} e^{-s^2} s \right]_{\frac{\alpha}{\sqrt{2t}}}^{\infty} + \frac{1}{2} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} ds \right) \\ & \stackrel{2.1.2}{=} \sqrt{2t}^3 C \left(\frac{1}{2} e^{-\frac{\alpha^2}{2t}} \frac{\alpha}{\sqrt{2t}} + \frac{1}{2} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}(s) \right]_{\frac{\alpha}{\sqrt{2t}}}^{\infty} \right) \\ & \stackrel{2.1.3}{=} \sqrt{2t}^3 C \left(\frac{1}{2} e^{-\frac{\alpha^2}{2t}} \frac{\alpha}{\sqrt{2t}} + \frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{4} \operatorname{erf}\left(\frac{\alpha}{\sqrt{2t}}\right) \right) \\ & \stackrel{(1)}{=} \sqrt{2t}^3 C \left(\frac{1}{2} e^{-\frac{\alpha^2}{2t}} \frac{\alpha}{\sqrt{2t}} + \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t^{\frac{1}{2}}\right) \right). \end{aligned}$$

Being $e^{-\frac{\alpha^2}{2t}t^{\frac{1}{2}}} = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}t^{-\frac{1}{2}}}\right)$ as $t \rightarrow 0$, we can gather

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 C dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}t^{-\frac{1}{2}}t^{\frac{3}{2}}}\right) \quad (1)$$

and in conclusion

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}t}\right).$$

□

Finally, we show the scale of $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$.

Lemma 3.3.29. *Let $]\alpha, \beta[\subset \mathbb{R}_{>0}$, where $0 < \alpha < \beta < \infty$. Let $\psi_0(r)$ and $\psi_1(r)$ be continuous functions and let $(\psi_0(r) + \psi_1(r) \log(r))$ be bounded on $]\alpha, \infty[$. As $t \rightarrow 0$, we obtain*

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}t}\right).$$

Proof. It exists a sufficiently large constant C such that $|\psi_0(r) + \psi_1(r) \log(r)| \leq C$ on $]\alpha, \infty[$. We get by substitution $s = \frac{r}{\sqrt{2t}}$ and the recursive formula in Lemma 3.3.1

$$\begin{aligned} \left| \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr \right| &\leq \int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 C dr \\ &\leq \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 C dr \end{aligned}$$

In equation (1) in Lemma 3.3.28 we have already pointed out that

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 C dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}t^{-\frac{1}{2}}t^{\frac{3}{2}}}\right)$$

and consequently,

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}} t\right).$$

□

Now, we return to the integrals $\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx$ and ascertain their asymptotic expansion as t tends towards infinity. For this purpose, we transform the integrals in polar coordinates by Transformation Theorem. By Remark 2.3.17 of Theorem 2.3.16, which gives the form of the angular part, and by the preceding lemmas, we show the integrals above have an asymptotic series expansion of the form $\sum_{k=0}^{\infty} (\delta_k + \gamma_k \log(t)) t^{\frac{k}{q}}$ as t approaches zero.

Theorem 3.3.30. *Let $A \subset \mathbb{R}^3$ be a globally subanalytic set. Let $f : \mathbb{R}_{>0} \rightarrow [0, 1]$ be the function given by*

$$f(t) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

As $t \rightarrow 0$, we have the asymptotic series expansion

$$f(t) \approx \sum_{k=0}^{\infty} (\delta_k + \gamma_k \log(t)) t^{\frac{k}{q}},$$

where $\delta_k, \gamma_k \in \mathbb{R}$ and $q \in \mathbb{N}$.

Proof. Let

$$\begin{aligned} \Theta : \mathbb{R}_{\geq 0} \times [0, \pi] \times]-\pi, \pi[&\longrightarrow \mathbb{R}^3, \\ (r, \vartheta, \varphi) &\longmapsto (r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta), \end{aligned}$$

be the three-dimensional polar coordinate transformation with functional determinant $\det(D\Theta(r, \vartheta, \varphi)) = r^2 \sin \vartheta$. By Transformation Theorem 2.1.17, we transform the given integral in polar coordinates and get

$$\begin{aligned} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx &\stackrel{2.1.17}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{|(r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta)|^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi). \end{aligned}$$

The function Θ is continuous and definable in the o-minimal structure \mathbb{R}_{an} , therefore $B := \Theta^{-1}(A) \subset \mathbb{R}_{\geq 0} \times]-\pi, \pi]^2$ is definable in \mathbb{R}_{an} . Let B_r be the set $\{(\vartheta, \varphi) \mid (r, \vartheta, \varphi) \in B\}$. By Fubini's Theorem 2.1.18, we achieve

$$\begin{aligned} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{\frac{-r^2}{2t}} r^2 \sin \vartheta \, d(r, \vartheta, \varphi) &\stackrel{2.1.18}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} \int_{B_r} e^{\frac{-r^2}{2t}} r^2 \sin \vartheta \, d(\vartheta, \varphi) dr \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr. \end{aligned}$$

Applying the Cell Decomposition Theorem 2.3.15, we get a finite partition of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ into finitely many disjoint cells, that means in points and open intervals D_1, \dots, D_p , where $D_j :=]\alpha, \beta[$ with $0 \leq \alpha < \beta \leq \infty$ and we can write

$$\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr = \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{j=1}^p \int_{D_j} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr.$$

For the sake of convenience we fix $j \in \{1, \dots, p\}$ and write $D := D_j$. Since the integrals over D will be zero if D is a point, we ignore this case. If B is a bounded set, consequently D is a bounded interval. If B is unbounded, at least one interval D is unbounded and consequently $\beta = \infty$. Due to the fact that $\sin \vartheta$ is globally subanalytic by its restriction to B_r , we get by using Remark 2.3.17 that

$$\int_{B_r} \sin \vartheta \, d(\vartheta, \varphi), \tag{1}$$

which is the angular part, has the form of a constructible function. In our case, the log-term of the constructible function has at most power one. The reason is, if we apply Fubini's Theorem 2.1.18 and integrate $\sin \vartheta$, there is no log-term in the antiderivative of $\sin \vartheta$ as is well known. Therefore, only the second integration can bring a log-term in the antiderivative of the integral in (1). Thus, the angular part, which is shown in (1), has the form $\psi_0(r) + \psi_1(r) \log(r)$, where $\psi_0(r)$ and $\psi_1(r)$ are strictly positive globally subanalytic functions on D . Since $\psi_0(r) + \psi_1(r) \log(r)$ is the angular part, it is additionally bounded by π .

We obtain

$$\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_D e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) \, dr = \tag{2}$$

$$\begin{aligned} &\stackrel{2.3.17}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) \, dr \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \left(\int_D e^{-\frac{r^2}{2t}} r^2 \psi_0(r) \, dr + \int_D e^{-\frac{r^2}{2t}} r^2 \psi_1(r) \log(r) \, dr \right). \end{aligned}$$

First, we may assume $D =]0, \beta[$ and by a refined cell decomposition we can additionally assume $\beta < 1$. Due to the fact that $\psi_i(r)$ are globally subanalytic functions, they can be represented as convergent Puiseux series around zero by the Preparation Theorem 2.3.18. We get that $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ is bounded on $]0, \beta[$ by Lemma 3.3.24. In consequence of this, $\psi_0(r)$ can be represented as $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}+2k}$ and $\psi_1(r)$ as $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}+2k}$, where $\hat{a}_{0,0} = 0$ and $q \in \mathbb{N}$. As $t \rightarrow 0$, we get immediately by Lemma 3.3.25

$$\int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 \psi_0(r) \, dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} b_{j,k} t^{\frac{j}{2q}+k+\frac{3}{2}}, \quad (3)$$

where $b_{j,k} \in \mathbb{R}$, and by Lemma 3.3.27

$$\int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 \psi_1(r) \log(r) \, dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} (c_{j,k} + d_{j,k} \log(t)) t^{\frac{j}{2q}+k+\frac{3}{2}},$$

where $c_{j,k}, d_{j,k} \in \mathbb{R}$. Subsequently, we gather that we obtain the asymptotic expansion

$$\int_0^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) \, dr \approx \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} ((b_{j,k} + c_{j,k}) + d_{j,k} \log(t)) t^{\frac{j}{2q}+k+\frac{3}{2}}.$$

Next, we may assume $D =]\alpha, \beta[$, where $0 < \alpha < \beta < \infty$. In Lemma 3.3.29 we have shown

$$\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) \, dr = \mathcal{O} \left(e^{-\frac{\alpha^2}{2t}} t \right).$$

Finally, we may assume $D =]\alpha, \infty[$. By a refined cell decomposition we can

assume $\alpha \geq 1$. We have already proven in Lemma 3.3.28 that

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right).$$

At the end, we combine the results. As $t \rightarrow 0$, we get that $e^{-\frac{\alpha^2}{2t}} = \mathcal{O}(\log(t)t^N)$ for an arbitrary $N \in \mathbb{N}$. Hence, the cells, that are based on cells D of the form $]\alpha, \beta[$ or $]\alpha, \infty[$, where $\alpha \neq 0$, do not contribute to the integral over the total set in our scale. Consequently, we sum up

$$\begin{aligned} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{j=1}^p \int_{D_j} e^{-\frac{r^2}{2t}} r^2 \sum_{i=0}^1 \psi_i(r) \log(r)^i dr \\ &\approx \sum_{k=0}^{\infty} (\delta_k + \gamma_k \log(t)) t^{\frac{k}{q}}, \end{aligned}$$

where $\delta_k, \gamma_k \in \mathbb{R}$. This is the desired conclusion. \square

In this thesis we focus on asymptotic expansions in a polynomial scale. Finally, we close this paragraph with two remarks which give us some properties of the asymptotic expansion above.

Remark 3.3.31. *a) If the closure of the globally subanalytic set A does not contain zero or if the dimension of \overline{A} at zero is less than three, the asymptotic expansion of the function $f(t)$ above vanishes in respect to the polynomial scale.*

b) If zero is in the closure of A and the dimension of \overline{A} at zero is three, the asymptotic expansion of $f(t)$ does not vanish in respect to the polynomial scale.

Proof. First, we give the arguments for the statement in *a)*. If zero is not in the closure of A , we will have integrals of the form $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$ and $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, $\alpha > 0$, after cell decomposition. We have shown that $\int_{\alpha}^{\beta} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right)$ in Lemma 3.3.29 and that $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr = \mathcal{O}\left(e^{-\frac{\alpha^2}{2t}}\right)$ in Lemma 3.3.28. Thus, the asymptotic expansion of $f(t)$ have a scale like $e^{-\frac{\alpha^2}{2t}}$. Therefore, all coefficients of

the asymptotic expansion of $f(t)$ in respect to the polynomial scale are zero. If the dimension of \bar{A} at zero is less than three, the integral over the cell which contains zero vanishes and by the same arguments as above the asymptotic expansion of the function $f(t)$ above vanishes in respect to the polynomial scale.

Now, we argue *b*). The integral $\int_{B_r} \sin \vartheta d(\vartheta, \varphi)$ in equation (1) is strictly positive for $(\vartheta, \varphi) \in [0, \pi] \times]-\pi, \pi]$. Thus, $\int_{B_r} \sin \vartheta d(\vartheta, \varphi) = \psi_0(r) + \psi_1(r) \log(r)$ is strictly positive. There are constants $\sigma \in \mathbb{Q}_{>0}$ and $C > 0$ such that $\psi_0(r) + \psi_1(r) \log(r) \geq Cr^\sigma$ for all $r \in]0, \beta[$. We obtain that $\int_0^\beta e^{-\frac{r^2}{2t}} r^2 \int_{B_r} \sin \vartheta d(\vartheta, \varphi) dr \geq C \int_0^\beta e^{-\frac{r^2}{2t}} r^{\sigma+2} dr$, of which the asymptotic expansion has polynomial scale as we have already seen. We can conclude that the asymptotic expansion of $f(t)$ does not vanish in respect to the polynomial scale. \square

Remark 3.3.32. *The asymptotic expansion of $f(t)$ as t tends towards zero is not necessarily convergent.*

Proof. We show the statement by a short example. Let B be the globally subanalytic set $\{(r, \vartheta, \varphi) \in [0, \frac{1}{2}] \times [0, \pi] \times [0, \pi] \mid 0 < \varphi < \frac{1+r}{1-r}\}$, which is given in polar coordinates. We set $\psi(r) := \frac{1+r}{1-r}$ and obviously $|\psi(r)|$ is bounded on $[0, \frac{1}{2}]$ by π . Using Fubini's Theorem 2.1.18, we get

$$\begin{aligned} \int_B e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) &\stackrel{2.1.18}{=} \int_{r=0}^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r^2 \int_{\varphi=0}^{\psi(r)} \int_{\vartheta=0}^{\pi} \sin \vartheta d\vartheta d\varphi dr \\ &= \int_0^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r^2 \int_0^{\psi(r)} [-\cos \vartheta]_0^\pi d\varphi dr \\ &= 2 \int_0^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r^2 \psi(r) dr. \end{aligned}$$

The well-known geometric series $\sum_{n=0}^{\infty} q^n$ has the limit $\frac{1}{1-q}$ if $q < 1$. Using this,

$\psi(r)$ can be represented as $(1+r) \sum_{k=0}^{\infty} r^k = \sum_{k=0}^{\infty} \sum_{j=0}^1 r^{j+k}$. In (3) in Theorem 3.3.30

we have shown the asymptotic expansion of $\int_0^\beta e^{-\frac{r^2}{2t}} r \psi(r) dr$, $\beta < 1$, which is

$\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} b_{j,k} t^{\frac{j}{q} + k + \frac{3}{2}}$. The coefficients $b_{j,k}$ are given in Lemma 3.3.25 as

$$a_{j,k} 2^{\frac{j}{2q} + k + \frac{3}{2}} \prod_{l=1}^{k+1} \left(\frac{j-q}{2q} + l \right) \frac{1}{2} \Gamma \left(\frac{j+q}{2q} \right)$$

where $a_{j,k}$ are the coefficients of the series representation of $\psi(r)$. In our example it is $a_{j,k} = 1$ for all j and k . Thus, the asymptotic expansion of $\int_0^{\frac{1}{2}} e^{-\frac{r^2}{2t}} r \psi(r) dr$ does obviously not converge and in consequence, the asymptotic expansion of $\int_B e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi)$ does not converge.

□

The Case t at Infinity

In this section we study $\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx$ as t tends towards infinity. For this purpose, we consider integrals of the form

$$\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr,$$

where $\psi_0(r)$ and $\psi_1(r)$ are globally subanalytic functions and $\psi_0(r) + \psi_1(r) \log(r)$ is bounded on $D \subset \mathbb{R}_{\geq 0} \cup \{\infty\}$.

If not stated otherwise, we set the following conventions for the next part: $a_{j,k}$ or $\hat{a}_{j,k}$ always denote the coefficients of convergent Puiseux series; coefficients which are in relation to integrals which contains the logarithm are marked with tilde e.g. $\tilde{b}_{j,k}$; an “extension” of coefficients are noted by capital letters e.g. $\tilde{B}_{j,k} = (\log(2) - 1) \tilde{b}_{j,k}$; for limit values of series we use capital letters with hat-symbol for example \hat{C} .

If D is bounded, we can gather that $\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$ has the form $(\varphi_0(t) + \varphi_1(t) \log(t))$ by the following Lemma.

Lemma 3.3.33. *Let $D \subset \mathbb{R}_{\geq 0}$ be a bounded open interval and $\psi_0(r)$ and $\psi_1(r)$ be bounded functions on D and definable in \mathbb{R}_{an} . As $t \rightarrow \infty$,*

$$\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$$

has the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where $\varphi_0(r)$ and $\varphi_1(r)$ are globally subanalytic functions and consequently, the integral is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. By arguments equal to those which we used in proof of Remark 3.2.18 in Section 3.2, we attain $e^{-\frac{r^2}{2t}} r^2 \psi_i(r)$ are definable in \mathbb{R}_{an} for large t and subsequently $e^{-\frac{r^2}{2t}} r^2 \psi_i(r)$ are globally subanalytic functions. Therefore $e^{-\frac{r^2}{2t}} r^2 \psi_0(r) + e^{-\frac{r^2}{2t}} r^2 \psi_1(r) \log(r)$ is a constructible function, see Definition 2.3.6. Due to the fact that constructible functions are stable under integration, see Theorem 2.3.19, we obtain that

$$\int_D \left(e^{-\frac{r^2}{2t}} r^2 \psi_0(r) + e^{-\frac{r^2}{2t}} r^2 \psi_1(r) \log(r) \right) dr \quad (1)$$

has the form of a constructible function. We can argue that, in our context, only a log-term with power one appears, because if we use the series expansion of the exponential function and integrate the integrals as usual by parts, the power of the log-term in (1) decreases in the first step. Hence, the integral in (1) has the form

$$\varphi_0(t) + \varphi_1(t) \log(t), \quad (2)$$

where φ_0 and φ_1 are globally subanalytic functions. Consequently, the integral in (1) is definable in $\mathbb{R}_{\text{an,exp}}$. \square

It is more elaborate to study $\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$ if D is unbounded, that means $D =]\alpha, \infty[$. Our aim is to establish the asymptotic series expansion of the integral $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, where $\alpha < 1$ and $\psi_i(r)$ are globally subanalytic functions and $\psi_0(r) + \psi_1(r) \log(r)$ is bounded on $] \alpha, \infty [$. Furthermore $\psi_0(r)$ and $\psi_1(r)$ can be represented as a convergent Puiseux series around infinity by arguments we show later. We ascertain the total asymptotic expansion in several lemmas step by step.

First, we pick up some lemmas of Preparations 3.3.1, because we can reduce the integrals $\int_D e^{-\frac{r^2}{2t}} r^{\frac{i}{q}-2k} dr$ to $\sqrt{2t}^3 \int_D e^{-s^2} s^{\frac{i}{q}-2k} ds$ by the substitution $s = \frac{r}{\sqrt{2t}}$ for example. We include immediately the factor $\sqrt{2t}^3$, which arises in consequence of the mentioned substitution, because this makes the proof in which we use the following lemmas more clearly. We start with a short proof of the asymptotic series expansion of $\sqrt{2t}^3 \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^2 ds$.

Lemma 3.3.34. *Let $\alpha \in \mathbb{R}$, where $\alpha > 1$. As $t \rightarrow \infty$, we achieve*

$$\sqrt{2t}^3 \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^2 ds \approx \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \sum_{n=0}^{\infty} \left(1 - \frac{1}{(2n+1)}\right) (-1)^n \frac{\alpha^{2n+1}}{2^n n!} t^{1-n}.$$

Proof. As $t \rightarrow \infty$, we get by Lemma 3.3.8

$$\begin{aligned} \sqrt{2t}^3 \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^2 ds &\stackrel{3.3.8}{\approx} \sqrt{2t}^3 \frac{\sqrt{\pi}}{4} + \sqrt{2t}^3 \frac{1}{2} \sum_{n=0}^{\infty} \left(1 - \frac{1}{(2n+1)}\right) \frac{(-1)^n}{n!} \left(\frac{\alpha}{\sqrt{2t}}\right)^{2n+1} \\ &= \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \sum_{n=0}^{\infty} \left(1 - \frac{1}{(2n+1)}\right) (-1)^n \frac{\alpha^{2n+1}}{2^n n!} t^{1-n}. \end{aligned}$$

□

Applying Lemma 3.3.12 of Section 3.3.1, we ascertain the asymptotic series expansion of $\sqrt{2t}^{\frac{j}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds$.

Lemma 3.3.35. *Let $\alpha \in \mathbb{R}$, where $\alpha > 1$, let $q \in \mathbb{N}$, let $k \in \mathbb{N}_0$, and let $j \in \{0, \dots, 2q-1\}$. As $t \rightarrow \infty$,*

$$\sqrt{2t}^{\frac{j}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds$$

has for $j \neq q$ the asymptotic series expansion

$$2^{\frac{j}{2q}-k+\frac{1}{2}} d_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} b_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n} t^{-n}$$

and for $j = q$ the asymptotic series expansion

$$\begin{aligned} &2^{1-k} \left(d_{q,k} + \frac{(-1)^k}{(k-1)!} \log\left(\frac{\alpha}{\sqrt{2}}\right) \right) t^{1-k} + \sum_{n=0}^{\infty} b_{q,k,n} \frac{\alpha^{2+2(n-k)}}{2^n} t^{-n} \\ &+ 2^{-k} \frac{(-1)^{k+1}}{(k-1)!} \log(t) t^{1-k}, \end{aligned}$$

where

$$b_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ (-1)^{n+1} \frac{1}{\left(\frac{j+q}{q} + 2(n-k)\right)n!}, & \text{else,} \end{cases}$$

and

$$d_{j,k} := \sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k},$$

where

$$c_{j,k} := \begin{cases} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & k = 0, \\ \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = 1, \\ -\frac{1}{\frac{j-q}{2q} - (k-1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{k-2} \prod_{\ell=0}^{k-1-m} \frac{1}{\frac{j-q}{2q} - (k-1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{k-1} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,1}, & k > 1. \end{cases}$$

Proof. Let $b_{j,k,n}$ and $d_{j,k}$ be as in Lemma 3.3.12. It should be noted that they are stated for the integrals $\int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}+2k} ds$ there, where $k < 0$. First, we may assume

$j \neq q$ and attain by the asymptotic expansion of $\int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds$ in Lemma 3.3.12

$$\begin{aligned} & \sqrt{2t}^{\frac{j}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \\ \stackrel{3.3.12}{\approx} & \sqrt{2t}^{\frac{j}{q}-2k+1} \left(d_{j,k} + \sum_{n=0}^{\infty} b_{j,k,n} \left(\frac{\alpha}{\sqrt{2t}} \right)^{\frac{j}{q}+2(n-k)+1} \right) \\ = & 2^{\frac{j}{2q}-k+\frac{1}{2}} d_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \\ & + 2^{\frac{j}{2q}-k+\frac{1}{2}} t^{\frac{j}{2q}-k+\frac{1}{2}} \sum_{n=0}^{\infty} b_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^{\frac{j}{2q}+(n-k)+\frac{1}{2}}} t^{-\frac{j}{2q}-(n-k)-\frac{1}{2}} \\ = & 2^{\frac{j}{2q}-k+\frac{1}{2}} d_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} b_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n} t^{-n}. \end{aligned}$$

Now, we may assume $j = q$ and we get by Lemma 3.3.12

$$\begin{aligned}
& \sqrt{2t}^{\frac{q}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{q}{q}-2k} ds \\
& \stackrel{3.3.12}{\approx} \sqrt{2t}^{2-2k} \left(d_{q,k} + \sum_{n=0}^{\infty} b_{q,k,n} \left(\frac{\alpha}{\sqrt{2t}} \right)^{2(n-k+1)} + \frac{(-1)^k}{(k-1)!} \log \left(\frac{\alpha}{\sqrt{2t}} \right) \right) \\
& = 2^{1-k} d_{q,k} t^{1-k} + \sum_{n=0}^{\infty} b_{q,k,n} \frac{\alpha^{2+2(n-k)}}{2^n} t^{-n} \\
& \quad + 2^{1-k} t^{1-k} \frac{(-1)^k}{(k-1)!} \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) - \frac{1}{2} \log(t) \right) \\
& = 2^{1-k} \left(d_{q,k} + \frac{(-1)^k}{(k-1)!} \log \left(\frac{\alpha}{\sqrt{2}} \right) \right) t^{1-k} + \sum_{n=0}^{\infty} b_{q,k,n} \frac{\alpha^{2+2(n-k)}}{2^n} t^{-n} \\
& \quad + 2^{-k} \frac{(-1)^{k+1}}{(k-1)!} \log(t) t^{1-k}.
\end{aligned}$$

□

In the same manner we proceed with $\sqrt{2t}^{\frac{j}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds$ and investigate the asymptotic series expansion.

Lemma 3.3.36. *Let $\alpha \in \mathbb{R}$, where $\alpha > 1$, let $k \in \mathbb{N}_0$, let $q \in \mathbb{N}$, and let $j \in \{0, \dots, 2q-1\}$. As $t \rightarrow \infty$, we achieve that*

$$\sqrt{2t}^{\frac{j}{q}-2k+1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds$$

has for $j \neq q$ the asymptotic expansion

$$\begin{aligned}
& \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right) t^{-n} \\
& + \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(t) t^{-n} + \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}
\end{aligned}$$

and has for $j = q$ the asymptotic expansion

$$\begin{aligned} & \sum_{n=0}^{\infty} \tilde{B}_{q,k,n} \left(1 - \left(2(n-k) + 2 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right) t^{-n} \\ & + \sum_{n=0}^{\infty} \tilde{B}_{q,k,n} (2(n-k) + 2) \frac{1}{2} \log(t) t^{-n} + \left(\tilde{D}_{q,k} + u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 \right) t^{1-k} \\ & + (-u_k) \log \left(\frac{\alpha}{\sqrt{2}} \right) \log(t) t^{1-k} + u_k \frac{1}{4} (\log(t))^2 t^{1-k}, \end{aligned}$$

where $u_k := \frac{(-1)^k}{(k-1)!} \frac{1}{2^k}$ and

$$\tilde{B}_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ (-1)^n \frac{\alpha^{\frac{j}{q} + 2(n-k) + 1}}{\left(\frac{j}{q} + 2(n-k) + 1 \right)^2 n! 2^n}, & \text{else,} \end{cases}$$

and

$$\tilde{D}_{j,k} := \begin{cases} 2^{1-k} \left(\sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1 \right)^2 n!} + \tilde{c}_{q,k} \right), & j = q, \\ 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1 \right)^2 n!} + \tilde{c}_{j,k} \right), & \text{else,} \end{cases}$$

where $\tilde{c}_{j,k} := \int_1^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} \log(s) ds$ is as in Lemma 3.3.20.

Proof. Let $\tilde{b}_{j,k,n}$ and $\tilde{d}_{j,k}$ be as in Lemma 3.3.22. It should be noted that they are stated for the integrals $\int_{\frac{\alpha}{\sqrt{2}t}}^{\infty} e^{-s^2} s^{\frac{j}{q} + 2k} \log(s) ds$ there, where $k < 0$. First, we may assume $j \neq q$. By the asymptotic expansions of $\int_{\frac{\alpha}{\sqrt{2}t}}^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} \log(s) ds$, which we have shown in Lemma 3.3.22, we get

$$\sqrt{2} t^{\frac{j}{q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2}t}}^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} \log(s) ds$$

$$\begin{aligned}
& \stackrel{3.3.22}{\approx} \sqrt{2t^{\frac{j}{q}-2k+1}} \sum_{n=0}^{\infty} \tilde{b}_{j,k,n} \left(\frac{\alpha}{\sqrt{2t}} \right)^{\frac{j}{q}+2(n-k)+1} \\
& + \sqrt{2t^{\frac{j}{q}-2k+1}} \sum_{n=0}^{\infty} (-\tilde{b}_{j,k,n}) \left(\frac{j}{2q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2t}} \right) \left(\frac{\alpha}{\sqrt{2t}} \right)^{\frac{j}{q}+2(n-k)+1} \\
& + \sqrt{2t^{\frac{j}{q}-2k+1}} \tilde{d}_{j,k} \\
& = (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \sum_{n=0}^{\infty} \tilde{b}_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)+1} t^{-\frac{j}{2q}-(n-k)-\frac{1}{2}} \\
& + (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \sum_{n=0}^{\infty} \left[(-\tilde{b}_{j,k,n}) \left(\frac{j}{q} + 2(n-k) + 1 \right) \right. \\
& \quad \left. \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) - \frac{1}{2} \log(t) \right) \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)+1} t^{-\frac{j}{2q}-(n-k)-\frac{1}{2}} \right] + (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{d}_{j,k} \\
& = (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \sum_{n=0}^{\infty} \left[\tilde{b}_{j,k,n} \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)+1} \right. \\
& \quad \left. \left(1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right) \right] t^{-\frac{j}{2q}-(n-k)-\frac{1}{2}} \\
& + (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \sum_{n=0}^{\infty} \left[\tilde{b}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \right. \\
& \quad \left. \left(\frac{\alpha}{\sqrt{2}} \right)^{\frac{j}{q}+2(n-k)+1} \frac{1}{2} \log(t) t^{-\frac{j}{2q}-(n-k)-\frac{1}{2}} \right] + (2t)^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{d}_{j,k} \\
& = \sum_{n=0}^{\infty} \tilde{b}_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{-n} \\
& + \sum_{n=0}^{\infty} \tilde{b}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n} \frac{1}{2} \log(t) t^{-n} \\
& + 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{d}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \\
& = \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{-n} \\
& + \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(t) t^{-n} + \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}},
\end{aligned}$$

where $\tilde{B}_{j,k,n} := \tilde{b}_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n}$ and $\tilde{D}_{j,k} := 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{d}_{j,k}$. We also set $\tilde{B}_{j,k,n}$ and $\tilde{D}_{j,k}$ in this way for the subsequent proof for $j = q$. We get in a similar manner as above that

$$\begin{aligned}
 G_{q,k,n}(t) &:= \sqrt{2t}^{2-2k} \sum_{n=0}^{\infty} \tilde{b}_{q,k,n} \left(\frac{\alpha}{\sqrt{2t}} \right)^{2(n-k)+2} + \sqrt{2t}^{2-2k} d_{q,k} \\
 &\quad + \sqrt{2t}^{2-2k} \sum_{n=0}^{\infty} (-\tilde{b}_{q,k,n}) (2(n-k)+2) \log \left(\frac{\alpha}{\sqrt{2t}} \right) \left(\frac{\alpha}{\sqrt{2t}} \right)^{2(n-k)+2} \\
 &= \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2t}} \right) \right] t^{-n} \\
 &\quad + \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(t) t^{-n} + \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}. \quad (1)
 \end{aligned}$$

We have

$$\begin{aligned}
 &\sqrt{2t}^{2-2k} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{q}{q}-2k} \log(s) ds \\
 &\stackrel{3.3,22}{\approx} \sqrt{2t}^{2-2k} \sum_{n=0}^{\infty} \tilde{b}_{q,k,n} \left(\frac{\alpha}{\sqrt{2t}} \right)^{2(n-k)+2} \\
 &\quad + \sqrt{2t}^{2-2k} \sum_{n=0}^{\infty} (-\tilde{b}_{q,k,n}) (2(n-k)+2) \log \left(\frac{\alpha}{\sqrt{2t}} \right) \left(\frac{\alpha}{\sqrt{2t}} \right)^{2(n-k)+2} \\
 &\quad + \sqrt{2t}^{2-2k} d_{q,k} + \sqrt{2t}^{2-2k} \frac{(-1)^k}{(k-1)!} \frac{1}{2} \left(\log \left(\frac{\alpha}{\sqrt{2t}} \right) \right)^2 \\
 &\stackrel{(1)}{=} G_{q,k,n}(t) + (2t)^{1-k} \frac{(-1)^k}{(k-1)!} \frac{1}{2} \left(\log \left(\frac{\alpha}{\sqrt{2t}} \right) - \frac{1}{2} \log(t) \right)^2 \\
 &= G_{q,k,n}(t) + \frac{(-1)^k}{(k-1)!} \frac{1}{2^k} \left(\log \left(\frac{\alpha}{\sqrt{2t}} \right) \right)^2 t^{1-k} \\
 &\quad + \frac{(-1)^{k+1}}{(k-1)!} \frac{1}{2^k} \log(t) \log \left(\frac{\alpha}{\sqrt{2t}} \right) t^{1-k} + \frac{(-1)^k}{(k-1)!} \frac{1}{2^k} \frac{1}{4} (\log(t))^2 t^{1-k} \\
 &\stackrel{(1)}{=} \sum_{n=0}^{\infty} \tilde{B}_{q,k,n} \left[1 - (2(n-k)+2) \log \left(\frac{\alpha}{\sqrt{2t}} \right) \right] t^{-n} \\
 &\quad + \sum_{n=0}^{\infty} \tilde{B}_{q,k,n} (2(n-k)+2) \frac{1}{2} \log(t) t^{-n}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\tilde{D}_{q,k} + \frac{(-1)^k}{(k-1)!} \frac{1}{2^k} \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 \right) t^{1-k} \\
 & + \frac{(-1)^{k+1}}{(k-1)!} \frac{1}{2^k} \log \left(\frac{\alpha}{\sqrt{2}} \right) \log(t) t^{1-k} + \frac{(-1)^k}{(k-1)!} \frac{1}{2^k} \frac{1}{4} (\log(t))^2 t^{1-k}.
 \end{aligned}$$

□

Our subgoal for the next part is establishing the asymptotic series expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$ step by step. Due to the boundedness of $\psi_0(r) + \psi_1(r) \log(r)$ on $]\alpha, \infty[$, we will argue that $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ are bounded on $]\alpha, \infty[$ at a later time. By this and another subsequent argument, $\psi_0(r)$ can be expressed by a convergent Puiseux series $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ and $\psi_1(r)$ by a convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)}$. Before we can ascertain the asymptotic expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, we make some preparations in the following lemmas.

Lemma 3.3.37. *Let $q \in \mathbb{N}$, let $\alpha \in \mathbb{R}$, where $\alpha > 1$, and let $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ and $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)}$ be convergent Puiseux series, which describe bounded functions on $]\alpha, \infty[$. As $t \rightarrow \infty$, we achieve that*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \left(\sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)} + \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)} \log(r) \right) dr$$

has the asymptotic series expansion

$$\sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} + \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} + \kappa_k (\log(t))^2 t^{1-k},$$

where

$$\gamma_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ \frac{\alpha^{\frac{j}{q}+2(n-k)+1} (-1)^{n+1}}{2^n \binom{\frac{j}{q}+2(n-k)+1}{n}} \left(a_{j,k} + \hat{a}_{j,k} \left(\log(\alpha) - \frac{1}{\frac{j}{q}+2(n-k)+1} \right) \right), & \text{else,} \end{cases}$$

$$\begin{aligned} \kappa_k &:= \hat{a}_{q,k} \frac{(-1)^{k+1}}{(k-1)! 2^k 4}, \\ \delta_{j,k} &:= \begin{cases} (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)) D_{q,k} + \hat{a}_{q,k} \tilde{D}_{q,k} \\ + \frac{(-1)^k}{(k-1)! 2^k} \log\left(\frac{\alpha}{\sqrt{2}}\right) (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2})), & j = q, \\ (a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2)) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k}, & \text{else,} \end{cases} \\ \text{and } \lambda_{j,k} &:= \begin{cases} \hat{a}_{q,k} \frac{1}{2} D_{q,k} + [a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)] \frac{(-1)^{k+1}}{(k-1)! 2^k}, & j = q, \\ \hat{a}_{j,k} \frac{1}{2} D_{j,k}, & \text{else,} \end{cases} \end{aligned}$$

where $\tilde{D}_{j,k}$ is as in Lemma 3.3.36 and $D_{j,k} := 2^{\frac{j}{2q} - k + \frac{1}{2}} d_{j,k}$, where $d_{j,k}$ is as in Lemma 3.3.35.

Proof. We set

$$F_k(t) := \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \left(\sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2(k+1)} + \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q} - 2(k+1)} \log(r) \right) dr$$

and obtain by substitution $s = \frac{r}{\sqrt{2t}}$

$$\begin{aligned} F_k(t) &= \sum_{j=0}^{2q-1} a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 r^{\frac{j}{q} - 2(k+1)} dr + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 r^{\frac{j}{q} - 2(k+1)} \log(r) dr \\ &= \sum_{j=0}^{2q-1} a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\frac{j}{q} - 2k} dr + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\frac{j}{q} - 2k} \log(r) dr \\ &= \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} ds \\ &\quad + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \sqrt{2t}^{\frac{j}{q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} \log(s\sqrt{2t}) ds \\ &= \sum_{j=0}^{2q-1} a_{j,k} \sqrt{2t}^{\frac{j}{q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} ds \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds \\
& + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \frac{1}{2} \log(2) \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \\
& + \sum_{j=0}^{2q-1} \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \frac{1}{2} \log(t) \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \\
& = \sum_{j=0}^{2q-1} \left[\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) + \hat{a}_{j,k} \frac{1}{2} \log(t) \right) \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \right. \\
& \quad \left. + \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds \right] \\
& = \sum_{j=0}^{2q-1} F_{j,k}(t),
\end{aligned}$$

where

$$\begin{aligned}
F_{j,k}(t) & := \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) + \hat{a}_{j,k} \frac{1}{2} \log(t) \right) \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \\
& \quad + \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds.
\end{aligned}$$

Next, we investigate the asymptotic expansion of $F_{j,k}(t)$. For this, we set $\tilde{B}_{j,k,n}$ and $\tilde{D}_{j,k}$ as in Lemma 3.3.36 and $b_{j,k,n}$ as in Lemma 3.3.35. The relation

$$B_{j,k,n} := b_{j,k,n} \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{2^n} = -\tilde{B}_{j,k,n} \left(\frac{j+q}{q} + 2(n-k) \right)$$

is obvious. Moreover, let $d_{j,k}$ be as in Lemma 3.3.35 and we set $D_{j,k} := 2^{\frac{j}{2q}-k+\frac{1}{2}} d_{j,k}$ and $u_k := \frac{(-1)^k}{(k-1)! 2^k}$. Applying Lemma 3.3.36 and Lemma 3.3.35, for $j \neq q$ we

establish the asymptotic expansion $\hat{F}_{j,k}$ of $F_{j,k}$ as $t \rightarrow \infty$ by

$$\begin{aligned}
 F_{j,k}(t) &= \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) + \hat{a}_{j,k} \frac{1}{2} \log(t) \right) \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} ds \\
 &\quad + \hat{a}_{j,k} \sqrt{2t^{\frac{j}{q}-2k+1}} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{j}{q}-2k} \log(s) ds \\
 &\stackrel{3.3.35}{\approx} \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) + \hat{a}_{j,k} \frac{1}{2} \log(t) \right) \left(D_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \right) \\
 &\quad + \hat{a}_{j,k} \left(\sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{-n} \right. \\
 &\quad \left. + \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(t) t^{-n} + \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \right) \quad (1) \\
 &= \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) \left(D_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \right) \\
 &\quad + \hat{a}_{j,k} \frac{1}{2} \log(t) D_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \hat{a}_{j,k} \sum_{n=0}^{\infty} B_{j,k,n} \frac{1}{2} \log(t) t^{-n} \\
 &\quad + \hat{a}_{j,k} \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{-n} \\
 &\quad - \hat{a}_{j,k} \sum_{n=0}^{\infty} B_{j,k,n} \frac{1}{2} \log(t) t^{-n} + \hat{a}_{j,k} \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \\
 &= \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \\
 &\quad + \hat{a}_{j,k} D_{j,k} \frac{1}{2} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} \\
 &\quad + \hat{a}_{j,k} \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right] t^{-n} \\
 &\quad + \hat{a}_{j,k} \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(2) t^{-n} \\
 &\quad + \hat{a}_{j,k} \tilde{D}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
&= \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + a_{j,k} \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \\
&\quad + \hat{a}_{j,k} \frac{1}{2} \log(2) \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} + \hat{a}_{j,k} \frac{1}{2} D_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}} \\
&\quad + \hat{a}_{j,k} \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right] t^{-n} \\
&\quad - \hat{a}_{j,k} \sum_{n=0}^{\infty} B_{j,k,n} \frac{1}{2} \log(2) t^{-n} + \hat{a}_{j,k} \tilde{D}_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \\
&= \left[\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k} \right] t^{\frac{j}{2q} - k + \frac{1}{2}} + a_{j,k} \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \\
&\quad + \hat{a}_{j,k} \frac{1}{2} D_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}} \\
&\quad + \hat{a}_{j,k} \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right] t^{-n} \\
&= \left[\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k} \right] t^{\frac{j}{2q} - k + \frac{1}{2}} \\
&\quad + \sum_{n=0}^{\infty} \left[a_{j,k} B_{j,k,n} + \hat{a}_{j,k} \tilde{B}_{j,k,n} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right) \right] t^{-n} \\
&\quad + \hat{a}_{j,k} \frac{1}{2} D_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}}. \tag{2}
\end{aligned}$$

One part of the calculation of $j = q$ is in an analogous way to the calculation above. For reasons of clarity we excerpt the equation in (1) and (2) and we set

$$\begin{aligned}
G_{j,k}(t) &\stackrel{(1)}{:=} \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) + \hat{a}_{j,k} \frac{1}{2} \log(t) \right) \left(D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + \sum_{n=0}^{\infty} B_{j,k,n} t^{-n} \right) \\
&\quad + \hat{a}_{j,k} \left(\sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left[1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{-n} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \tilde{B}_{j,k,n} \left(\frac{j}{q} + 2(n-k) + 1 \right) \frac{1}{2} \log(t) t^{-n} + \tilde{D}_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \right) \\
&\stackrel{(2)}{=} \left[\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k} \right] t^{\frac{j}{2q} - k + \frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{n=0}^{\infty} \left[a_{j,k} B_{j,k,n} + \hat{a}_{j,k} \tilde{B}_{j,k,n} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right) \right] t^{-n} \\
& + \hat{a}_{j,k} \frac{1}{2} D_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}}. \tag{3}
\end{aligned}$$

Using Lemma 3.3.35 and Lemma 3.3.36, we obtain for $j = q$ the asymptotic expansion $\hat{F}_{q,k}$ of $F_{q,k}$ as $t \rightarrow \infty$ by

$$\begin{aligned}
& \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) \sqrt{2} t^{\frac{q}{2q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{q}{2q} - 2k} ds \\
& + \hat{a}_{q,k} \sqrt{2} t^{\frac{q}{2q} - 2k + 1} \int_{\frac{\alpha}{\sqrt{2t}}}^{\infty} e^{-s^2} s^{\frac{q}{2q} - 2k} \log(s) ds \\
& \stackrel{3.3.35}{\approx} \stackrel{3.3.36}{\approx} \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) \left[D_{q,k} t^{1-k} + 2u_k \log\left(\frac{\alpha}{\sqrt{2}}\right) t^{1-k} \right. \\
& \left. + \sum_{n=0}^{\infty} B_{q,k,n} t^{-n} + (-u_k) \log(t) t^{1-k} \right] \\
& + \hat{a}_{q,k} \left[\sum_{n=0}^{\infty} \tilde{B}_{q,k,n} \left(1 - (2(n-k) + 2) \log\left(\frac{\alpha}{\sqrt{2}}\right) \right) t^{-n} \right. \\
& \left. + \sum_{n=0}^{\infty} \tilde{B}_{q,k,n} (2(n-k) + 2) \frac{1}{2} \log(t) t^{-n} \right. \\
& \left. + \left(\tilde{D}_{q,k} + u_k \left(\log\left(\frac{\alpha}{\sqrt{2}}\right) \right)^2 \right) t^{1-k} + (-u_k) \log\left(\frac{\alpha}{\sqrt{2}}\right) \log(t) t^{1-k} \right. \\
& \left. + u_k \frac{1}{4} (\log(t))^2 t^{1-k} \right] \\
& \stackrel{(3)}{=} G_{q,k}(t) + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) \left[2u_k \log\left(\frac{\alpha}{\sqrt{2}}\right) t^{1-k} \right. \\
& \left. + (-u_k) \log(t) t^{1-k} \right] + \hat{a}_{q,k} \left[u_k \left(\log\left(\frac{\alpha}{\sqrt{2}}\right) \right)^2 t^{1-k} \right. \\
& \left. + (-u_k) \log\left(\frac{\alpha}{\sqrt{2}}\right) \log(t) t^{1-k} + u_k \frac{1}{4} (\log(t))^2 t^{1-k} \right]
\end{aligned}$$

$$\begin{aligned}
&= G_{q,k}(t) + \hat{a}_{q,k} \left[u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 t^{1-k} + \right. \\
&\quad \left. (-u_k) \log \left(\frac{\alpha}{\sqrt{2}} \right) \log(t) t^{1-k} + u_k \frac{1}{4} (\log(t))^2 t^{1-k} \right] \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) \left[2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) t^{1-k} \right. \\
&\quad \left. + (-u_k) \log(t) t^{1-k} \right] \\
&= G_{q,k}(t) + \hat{a}_{q,k} u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 t^{1-k} \\
&\quad + \hat{a}_{q,k} (-u_k) \log \left(\frac{\alpha}{\sqrt{2}} \right) \log(t) t^{1-k} + \hat{a}_{q,k} u_k \frac{1}{4} (\log(t))^2 t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) 2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) + \hat{a}_{q,k} \frac{1}{2} \log(t) \right) (-u_k) \log(t) t^{1-k} \\
&= G_{q,k}(t) + \hat{a}_{q,k} u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 t^{1-k} \\
&\quad - \hat{a}_{q,k} u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) \log(t) t^{1-k} + \hat{a}_{q,k} u_k \frac{1}{4} (\log(t))^2 t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) 2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) t^{1-k} + \hat{a}_{q,k} \frac{1}{2} \log(t) 2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k} + \hat{a}_{q,k} \frac{1}{2} \log(t) (-u_k) \log(t) t^{1-k} \\
&= G_{q,k}(t) + \hat{a}_{q,k} u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 t^{1-k} + \hat{a}_{q,k} u_k \left(\frac{1}{4} - \frac{1}{2} \right) (\log(t))^2 t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) 2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) t^{1-k} \\
&\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k} \\
&= G_{q,k}(t) \\
&\quad + \left[\hat{a}_{q,k} u_k \left(\log \left(\frac{\alpha}{\sqrt{2}} \right) \right)^2 + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) 2u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) \right] t^{1-k} \\
&\quad + \hat{a}_{q,k} (-u_k) \frac{1}{4} (\log(t))^2 t^{1-k} + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k}
\end{aligned}$$

$$\begin{aligned}
 &= G_{q,k}(t) + \left[u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2}) \right) \right] t^{1-k} \\
 &\quad + \hat{a}_{q,k}(-u_k) \frac{1}{4} (\log(t))^2 t^{1-k} + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k}.
 \end{aligned}$$

Therefore, we get the asymptotic expansion $\hat{F}_k(t) := \sum_{j=0}^{2q-1} \hat{F}_{j,k}(t)$ of $F_k(t)$ by

$$\begin{aligned}
 \hat{F}_k(t) &= \sum_{\substack{j=0 \\ j \neq q}}^{2q-1} \hat{F}_{j,k}(t) + \hat{F}_{q,k}(t) \\
 &= \sum_{\substack{j=0 \\ j \neq q}}^{2q-1} G_{j,k}(t) + G_{q,k}(t) + \hat{a}_{q,k}(-u_k) \frac{1}{4} (\log(t))^2 t^{1-k} \\
 &\quad + \left[u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2}) \right) \right] t^{1-k} \\
 &\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k} \\
 &= \sum_{j=0}^{2q-1} G_{j,k}(t) + \hat{a}_{q,k}(-u_k) \frac{1}{4} (\log(t))^2 t^{1-k} \\
 &\quad + \left[u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2}) \right) \right] t^{1-k} \\
 &\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k} \\
 &\stackrel{(3)}{=} \sum_{j=0}^{2q-1} \left[\left[\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k} \right] t^{\frac{j}{2q}-k+\frac{1}{2}} \right. \\
 &\quad \left. + \sum_{n=0}^{\infty} \left[a_{j,k} B_{j,k,n} + \hat{a}_{j,k} \tilde{B}_{j,k,n} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1 \right) \log(\alpha) \right) \right] t^{-n} \right. \\
 &\quad \left. + \hat{a}_{j,k} \frac{1}{2} D_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} \right] + \hat{a}_{q,k}(-u_k) \frac{1}{4} (\log(t))^2 t^{1-k} \\
 &\quad + \left[u_k \log \left(\frac{\alpha}{\sqrt{2}} \right) + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2}) \right) \right] t^{1-k} \\
 &\quad + \left(a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2) \right) (-u_k) \log(t) t^{1-k}.
 \end{aligned}$$

We set $\kappa_k := \hat{a}_{q,k}(-u_k)^{\frac{1}{4}}$,

$$\delta_{j,k} := \begin{cases} (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)) D_{q,k} + \hat{a}_{q,k} \tilde{D}_{q,k} \\ + u_k \log\left(\frac{\alpha}{\sqrt{2}}\right) (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2})), & j = q, \\ (a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2)) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k}, & \text{else,} \end{cases}$$

and

$$\begin{aligned} \gamma_{j,k,n} &:= \begin{cases} \hat{a}_{q,k} \tilde{B}_{q,k,n} \left(1 - \left(\frac{q}{q} + 2(n-k) + 1\right) \log(\alpha)\right), & j = q, \\ a_{j,k} B_{j,k,n} + \hat{a}_{j,k} \tilde{B}_{j,k,n} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1\right) \log(\alpha)\right), & \text{else,} \end{cases} \\ &= \begin{cases} 0, & j = q \text{ and} \\ & n = k - 1, \\ \frac{\alpha^{\frac{j}{2q} + 2(n-k) + 1}}{2^n} \left(a_{j,k} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right) n!} \right. \\ \left. + \hat{a}_{j,k} \frac{(-1)^n}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} \left(1 - \left(\frac{j}{q} + 2(n-k) + 1\right) \log(\alpha)\right) \right), & \text{else,} \end{cases} \\ &= \begin{cases} 0, & j = q \text{ and} \\ & n = k - 1, \\ \frac{\alpha^{\frac{j}{2q} + 2(n-k) + 1} (-1)^{n+1}}{2^n \left(\frac{j}{q} + 2(n-k) + 1\right) n!} \left(a_{j,k} + \hat{a}_{j,k} \left(\log(\alpha) - \frac{1}{\frac{j}{q} + 2(n-k) + 1} \right) \right), & \text{else,} \end{cases} \end{aligned}$$

and

$$\lambda_{j,k} := \begin{cases} \hat{a}_{q,k} \frac{1}{2} D_{q,k} + [a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)] (-u_k), & j = q, \\ \hat{a}_{j,k} \frac{1}{2} D_{j,k}, & \text{else.} \end{cases}$$

In conclusion we gather

$$\begin{aligned} \hat{F}_k(t) &= \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} \\ &\quad + \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}} + \kappa_{q,k} (\log(t))^2 t^{1-k}. \end{aligned}$$

□

Before we continue to establish the asymptotic series expansion of the integrals $\int_0^\infty e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, we give the proof that the series $\sum_{k=0}^{\infty} \lambda_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$, $\sum_{k=0}^{\infty} 2^{\frac{j+q}{2q} - k} \tilde{c}_{j,k} t^{\frac{1}{2q} - k + \frac{1}{2}}$, and $\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ are convergent.

Lemma 3.3.38. *Let $j \in \{0, \dots, 2q - 1\}$ and let $\lambda_{j,k}$ be as in Lemma 3.3.37. The series*

$$\sum_{k=0}^{\infty} \lambda_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$$

converges absolutely for sufficiently large t .

Proof. In Lemma 3.3.37 the coefficients $\lambda_{j,k}$ are stated as

$$\lambda_{j,k} := \begin{cases} \hat{a}_{q,k} \frac{1}{2} D_{q,k} + [a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)] \frac{(-1)^{k+1}}{(k-1)! 2^k}, & j = q, \\ \hat{a}_{j,k} \frac{1}{2} D_{j,k}, & \text{else,} \end{cases}$$

where $a_{j,k}$ and $\hat{a}_{j,k}$ are the real coefficients of a convergent Puiseux series, and

$$D_{j,k} := 2^{\frac{j+q}{2q} - k} \left(\sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k} \right),$$

where

$$b_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ (-1)^{n+1} \frac{1}{\left(\frac{j+q}{q} + 2(n-k)\right)n!}, & \text{else,} \end{cases}$$

and

$$c_{j,k} := \begin{cases} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & k = 0, \\ \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} c_{j,0}, & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = 1, \\ -\frac{1}{\frac{j-q}{2q} - (k-1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{k-2} \prod_{\ell=0}^{k-1-m} \frac{1}{\frac{j-q}{2q} - (k-1) + \ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{k-1} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,1}, & k > 1. \end{cases}$$

First, we may assume $j \neq q$. We get

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} &= \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \\ &= \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} 2^{\frac{j+q}{2q} - k} \left(\sum_{n=0}^{\infty} (-b_{j,k,n}) + c_{j,k} \right) t^{\frac{j}{2q} - k + \frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} 2^{\frac{j+q}{2q}-k} \left(\sum_{n=0}^{\infty} (-b_{j,k,n}) \right) t^{\frac{j}{2q}-k+\frac{1}{2}} \\
&\quad + \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} 2^{\frac{j+q}{2q}-k} c_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \\
&= \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} 2^{\frac{j+q}{2q}-k} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(\frac{j+q}{q} + 2(n-k)\right) n!} \right) t^{\frac{j}{2q}-k+\frac{1}{2}} \\
&\quad + \sum_{k=0}^{\infty} \hat{a}_{j,k} \frac{1}{2} 2^{\frac{j+q}{2q}-k} \left(-\frac{1}{\frac{j-q}{2q} - (k-1)} \frac{1}{2} e^{-1} \right. \\
&\quad \left. - \sum_{m=1}^{k-2} \prod_{\ell=0}^{k-1-m} \frac{1}{\frac{j-q}{2q} - (k-1) + \ell} \frac{1}{2} e^{-1} + \prod_{\ell=1}^{k-1} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,1} \right) t^{\frac{j}{2q}-k+\frac{1}{2}}.
\end{aligned}$$

We have already considered quite similar series in Lemma 3.2.21 and with the same arguments as in Lemma 3.2.21 we obtain that the series $\sum_{k=0}^{\infty} \lambda_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ is absolutely convergent.

The case $j = q$ results from Lemma 3.2.21 as well. \square

As a preparation for the following lemma, we determine whether the series $\sum_{k=0}^{\infty} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ is convergent or divergent.

Lemma 3.3.39. *Let $j \in \{0, \dots, 2q-1\}$, let $a_{j,k}$ be the coefficients of a convergent Puiseux series, and let $\tilde{c}_{j,k}$ be as in Lemma 3.3.20. We have that*

$$\sum_{k=0}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$$

converges absolutely for sufficiently large t .

Proof. According to Lemma 3.3.20, it is $\tilde{c}_{j,0} = \int_1^{\infty} e^{-s^2} s^{\frac{j}{q}} \log(s) ds$ for $k = 0$. It should be noted that $\tilde{c}_{j,k}$ and $c_{j,k}$ are stated for $k < -1$ in Lemma 3.3.20. For $k > 1$ the coefficients $\tilde{c}_{j,k}$ are given by

$$\tilde{c}_{j,k} \stackrel{3.3.20}{=} - \sum_{m=0}^{k-2} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} - (k-\ell)} \right) \frac{1}{2} c_{j,k-m} + \prod_{\ell=0}^{k-2} \frac{1}{\frac{j+q}{2q} - (k-\ell)} \tilde{c}_{j,1},$$

where

$$\tilde{c}_{j,1} := \begin{cases} \int_1^{\infty} e^{-s^2} s^{-1} \log(s) ds, & j = q, \\ -\frac{1}{\frac{j+q}{2q}-1} \frac{1}{2} c_{j,1} + \frac{1}{\frac{j+q}{2q}-1} \tilde{c}_{j,0}, & \text{else,} \end{cases}$$

where $\int_1^{\infty} e^{-s^2} s^{-1} \log(s) ds$ exists and

$$c_{j,k} := \begin{cases} \frac{1}{2} \Gamma\left(\frac{j+q}{2q}, 1\right), & k = 0, \\ \begin{cases} -\frac{1}{\frac{j-q}{2q}} \frac{1}{2} e^{-1} + \frac{1}{\frac{j-q}{2q}} c_{j,0}, & j < q, \\ \frac{1}{2} \Gamma\left(\frac{j-q}{2q}, 1\right), & j \geq q, \end{cases} & k = 1, \\ -\frac{1}{\frac{j-q}{2q}-(k-1)} \frac{1}{2} e^{-1} - \sum_{m=1}^{(k-2)} \prod_{\ell=0}^{k-1-m} \frac{1}{\frac{j-q}{2q}-(k-1)+\ell} \frac{1}{2} e^{-1} \\ + \prod_{\ell=1}^{k-1} \frac{1}{\frac{j-q}{2q}-\ell} c_{j,1}, & k > 1. \end{cases}$$

By Lemma 3.3.20, we get for $k > 1$

$$\begin{aligned} & \sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \\ \stackrel{3.3.20}{=} & \sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \left(- \sum_{m=0}^{k-2} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} - (k-\ell)} \right) \frac{1}{2} c_{j,k-m} \right. \\ & \left. + \prod_{\ell=0}^{k-2} \frac{1}{\frac{j+q}{2q} - (k-\ell)} \tilde{c}_{j,1} \right) t^{\frac{j}{2q}-k+\frac{1}{2}} \\ = & \sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \left(- \sum_{m=0}^{k-2} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} - (k-\ell)} \right) \frac{1}{2} c_{j,k-m} \right) t^{\frac{j}{2q}-k+\frac{1}{2}} \\ & + \tilde{c}_{j,1} \sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \prod_{\ell=0}^{k-1} \frac{1}{\frac{j+q}{2q} - (k-\ell)} t^{\frac{j}{2q}-k+\frac{1}{2}}. \end{aligned} \quad (1)$$

First, we consider the series $\sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \prod_{\ell=0}^{k-1} \frac{1}{\frac{j+q}{2q}-(k-\ell)} t^{\frac{j}{2q}-k+\frac{1}{2}}$. We can estimate

$$\left| \frac{1}{\frac{j+q}{2q} - (k+1)} \right| < 1,$$

because $k > 1$ and we apply the ratio test and obtain

$$\begin{aligned} \left| \frac{a_{j,k+1} 2^{\frac{j}{2q} - (k+1) + \frac{1}{2}} \prod_{\ell=0}^{k+1-1} \frac{1}{\frac{j+q}{2q} - (k+1-\ell)} t^{\frac{j}{2q} - (k+1) + \frac{1}{2}}}{a_{j,k} 2^{\frac{j}{2q} - k + \frac{1}{2}} \prod_{\ell=0}^{k-1} \frac{1}{\frac{j+q}{2q} - (k-\ell)} t^{\frac{j}{2q} - k + \frac{1}{2}}} \right| &= \left| \frac{a_{j,k+1} 2^{-1} \prod_{\ell=0}^k \frac{1}{\frac{j+q}{2q} - (k-(\ell-1))} t^{-1}}{a_{j,k} \prod_{\ell=0}^{k-1} \frac{1}{\frac{j+q}{2q} - (k-\ell)}} \right| \\ &= \left| \frac{a_{j,k+1} \frac{1}{\frac{j+q}{2q} - (k+1)}}{a_{j,k} 2t} \right| \end{aligned}$$

Thus the series $\sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q} - k + \frac{1}{2}} \prod_{\ell=0}^{k-1} \frac{1}{\frac{j+q}{2q} - (k-\ell)} t^{\frac{j}{2q} - k + \frac{1}{2}}$ converges absolutely for a sufficiently large t .

Next, we examine the coefficients of the series in the first summand of the sum in equation (1). We have

$$\begin{aligned} & \left| 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(- \sum_{m=0}^{k-2} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q} - (k-\ell)} \right) \frac{1}{2} c_{j,k-m} \right) \right| \leq \\ & \leq \left| 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(-(k-1) \frac{1}{\frac{j+q}{2q} - k} \frac{1}{2} c_{j,k} \right) \right| \\ & \leq \left| 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(-(k-1) \frac{1}{\frac{j+q}{2q} - k} \frac{1}{2} \left[- \frac{1}{\frac{j-q}{2q} - (k-1)} \frac{1}{2} e^{-1} \right. \right. \right. \\ & \quad \left. \left. - \sum_{m=1}^{(k-2)} \prod_{\ell=0}^{k-1-m} \frac{1}{\frac{j-q}{2q} - (k-1) + \ell} \frac{1}{2} e^{-1} + \prod_{\ell=1}^{k-1} \frac{1}{\frac{j-q}{2q} - \ell} c_{j,1} \right] \right) \right| \\ & \leq \left| 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(-(k-1) \frac{1}{\frac{j+q}{2q} - k} \frac{1}{2} \left[- \frac{1}{\frac{j+q}{2q} - k} \frac{1}{2} e^{-1} \right. \right. \right. \\ & \quad \left. \left. - (k-2) \prod_{\ell=0}^1 \frac{1}{\frac{j+q}{2q} - k + \ell} \frac{1}{2} e^{-1} + \prod_{\ell=2}^k \frac{1}{\frac{j+q}{2q} - \ell} c_{j,1} \right] \right) \right| \\ & \leq 2^{\frac{j}{2q} - k + \frac{1}{2}} \frac{1}{4} e^{-1} (k-1) \left(\frac{1}{\frac{j+q}{2q} - k} \right)^2 \\ & \quad + 2^{\frac{j}{2q} - k + \frac{1}{2}} \frac{1}{4} e^{-1} (k-1) \left(\frac{1}{\frac{j+q}{2q} - k} \right)^2 (k-2) \left| \frac{1}{\frac{j+q}{2q} - k + 1} \right| \end{aligned}$$

$$+ 2^{\frac{j}{2q}-k+\frac{1}{2}} \frac{1}{2} \left(\frac{1}{\frac{j+q}{2q}-k} \right)^2 (k-1) \left| c_{j,1} \prod_{\ell=2}^{k-1} \frac{1}{\frac{j+q}{2q}-\ell} \right|.$$

We can also estimate $\left(\frac{1}{\frac{j+q}{2q}-k} \right)^2 \leq 1$ for $k \geq 2$. Therefore, the series

$$\begin{aligned} & \sum_{k=2}^{\infty} a_{j,k} \left[2^{\frac{j}{2q}-k+\frac{1}{2}} \frac{1}{4} e^{-1} (k-1) + 2^{\frac{j}{2q}-k+\frac{1}{2}} \frac{1}{4} e^{-1} (k-1)(k-2) \left| \frac{1}{\frac{j+q}{2q}-k+1} \right| \right. \\ & \left. + 2^{\frac{j}{2q}-k+\frac{1}{2}} \frac{1}{2} (k-1) \left| c_{j,1} \prod_{\ell=2}^{k-1} \frac{1}{\frac{j+q}{2q}-\ell} \right| \right] t^{\frac{j}{2q}-k+\frac{1}{2}} \end{aligned}$$

are convergent, which can be easily proven by the ratio test. Using the comparison test, we obtain that $\sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \left(-\sum_{m=0}^{k-2} \left(\prod_{\ell=0}^m \frac{1}{\frac{j+q}{2q}-(k-\ell)} \right) \frac{1}{2} c_{j,k-m} \right) t^{\frac{j}{2q}-k+\frac{1}{2}}$ converges absolutely. In conclusion of that and the results above, we gather that the series $\sum_{k=2}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ is absolutely convergent.

Finally we get that $\sum_{k=0}^{\infty} a_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ converges absolutely. □

Lemma 3.3.40. *Let $j \in \{0, \dots, 2q-1\}$ and let $\delta_{j,k}$ be as in Lemma 3.3.37. We have that*

$$\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$$

converges absolutely for sufficiently large t .

Proof. In Lemma 3.3.37 the coefficients $\delta_{j,k}$ are indicated as

$$\delta_{j,k} := \begin{cases} (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(2)) D_{q,k} + \hat{a}_{q,k} \tilde{D}_{q,k} \\ + \frac{(-1)^k}{(k-1)! 2^k} \log\left(\frac{\alpha}{\sqrt{2}}\right) (a_{q,k} + \hat{a}_{q,k} \frac{1}{2} \log(\alpha\sqrt{2})), & j = q, \\ (a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2)) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k}, & \text{else,} \end{cases}$$

where $a_{j,k}, \hat{a}_{j,k}$ are the coefficients of a convergent Puiseux series, $D_{j,k}$ is given as

in the lemma before, Lemma 3.3.38, and

$$\tilde{D}_{j,k} := \begin{cases} 2^{1-k} \left(\sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} + \tilde{c}_{q,k} \right), & j = q, \\ 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} + \tilde{c}_{j,k} \right), & \text{else,} \end{cases}$$

where $\tilde{c}_{j,k} := \int_1^{\infty} e^{-s^2} s^{\frac{j}{q} - 2k} \log(s) ds$.

First, we assume $j \neq q$. We get

$$\begin{aligned} & \sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \\ &= \sum_{k=0}^{\infty} \left(\left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} + \hat{a}_{j,k} \tilde{D}_{j,k} \right) t^{\frac{j}{2q} - k + \frac{1}{2}} \\ &= \sum_{k=0}^{\infty} \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + \sum_{k=0}^{\infty} \hat{a}_{j,k} \tilde{D}_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \\ &= \sum_{k=0}^{\infty} \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} \\ & \quad + \sum_{k=0}^{\infty} \hat{a}_{j,k} 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} \right) t^{\frac{j}{2q} - k + \frac{1}{2}} \\ & \quad + \sum_{k=0}^{\infty} \hat{a}_{j,k} 2^{\frac{j}{2q} - k + \frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}. \end{aligned}$$

Due to the fact that the arguments are quite the same and the first series is very similar to the series in Lemma 3.2.21, we refer to this lemma to show that the series $\sum_{k=0}^{\infty} \left(a_{j,k} + \hat{a}_{j,k} \frac{1}{2} \log(2) \right) D_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ is absolutely convergent.

We consider

$$\sum_{k=0}^{\infty} \hat{a}_{j,k} 2^{\frac{j}{2q} - k + \frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} \right) t^{\frac{j}{2q} - k + \frac{1}{2}}$$

and we estimate

$$\left| \frac{1}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2} \right| \leq \begin{cases} q^2, & j < q \text{ and } n - k = 1, \\ 1, & \text{else.} \end{cases}$$

We get

$$\left| \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} \right| \leq \sum_{n=0}^{\infty} q^2 \left| \frac{(-1)^{n+1}}{n!} \right| = q^2 e.$$

By the ratio test we easily prove that $q^2 e \sum_{k=0}^{\infty} \hat{a}_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} t^{\frac{j}{2q}-k+\frac{1}{2}}$ converges. Therefore the series $\sum_{k=0}^{\infty} \hat{a}_{j,k} 2^{\frac{j}{2q}-k+\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\left(\frac{j}{q} + 2(n-k) + 1\right)^2 n!} \right) t^{\frac{j}{2q}-k+\frac{1}{2}}$ is absolutely convergent by the comparison test.

In Lemma 3.3.39 we have already seen that the series $\sum_{k=0}^{\infty} 2^{\frac{j}{2q}-k+\frac{1}{2}} \tilde{c}_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ is convergent.

Gathering the results, we have proven that $\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ converges absolutely. □

We would remind you that the aim is to ascertain an asymptotic series expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, where $\psi_0(r)$ and $\psi_1(r) \log(r)$ is bounded on $] \alpha, \infty[$ and $\psi_0(r)$ can be described by the convergent Puiseux series $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ and $\psi_1(r)$ by $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)}$. By the lemmas above, we can investigate the asymptotic series expansion for the following integrals:

Lemma 3.3.41. *Let $\alpha \in \mathbb{R}_{>0}$, where $\alpha > 1$, and let $\theta_0(r)$ and $\theta_1(r) \log(r)$ be bounded functions on $] \alpha, \infty[$, where $\theta_0(r)$ can be represented as the convergent Puiseux series $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)}$ and $\theta_1(r)$ as $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)}$, $q \in \mathbb{N}$. As $t \rightarrow \infty$, we obtain that*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\theta_0(r) + \theta_1(r) \log(r)) dr$$

has the asymptotic series expansion

$$\varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2,$$

where $\varphi_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$, $\varphi_1(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$, $\varphi_2(t) = \sum_{k=0}^{\infty} \kappa_k t^{1-k}$ are convergent Puiseux series, where $\mu_{j,k}$, $\lambda_{j,k}$, $\kappa_k \in \mathbb{R}$ and $q \in \mathbb{N}$. Thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. We set $F(t) := \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\theta_0(r) + \theta_1(r) \log(r)) dr$ and achieve

$$\begin{aligned} F(t) &= \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\theta_0(r) + \theta_1(r) \log(r)) dr \\ &= \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \left(\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q}-2(k+1)} + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q}-2(k+1)} \log(r) \right) dr \\ &= \sum_{k=0}^{\infty} F_k(t), \end{aligned}$$

where $F_k(t) := \sum_{j=0}^{2q-1} \left(a_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 r^{\frac{j}{q}-2(k+1)} dr + \hat{a}_{j,k} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 r^{\frac{j}{q}-2(k+1)} \log(r) dr \right)$.

In Lemma 3.3.37 we have established the asymptotic series expansion $\hat{F}_k(t)$ of $F_k(t)$, which is given by

$$\begin{aligned} \hat{F}_k(t) &:= \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} + \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} \\ &\quad + \kappa_k (\log(t))^2 t^{1-k}, \end{aligned}$$

where $\delta_{j,k}$, $\gamma_{j,k,n}$, $\lambda_{j,k}$, and κ_k are as in Lemma 3.3.37. Consequently, $\gamma_{j,k,n}$ is set as

$$\gamma_{j,k,n} \stackrel{3.3.37}{=} \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ \frac{(-1)^{n+1} \alpha^{\frac{j}{q}+2(n-k)+1}}{2^n n! \frac{j}{q}+2(n-k)+1} \left(a_{j,k} + \hat{a}_{j,k} \left(\log(\alpha) - \frac{1}{\frac{j}{q}+2(n-k)+1} \right) \right), & \text{else,} \end{cases}$$

and we define

$$\hat{\gamma}_{j,k,n} := \begin{cases} 0, & j = q \text{ and } n = k - 1, \\ \frac{\alpha^{\frac{j}{q} + 2(n-k) + 1}}{\frac{j}{q} + 2(n-k) + 1} \left(a_{j,k} + \hat{a}_{j,k} \left(\log(\alpha) - \frac{1}{\frac{j}{q} + 2(n-k) + 1} \right) \right), & \text{else,} \end{cases}$$

such that $\gamma_{j,k,n} = \frac{(-1)^{n+1}}{2^n n!} \hat{\gamma}_{j,k,n}$. The series $\sum_{n=0}^{\infty} \gamma_{j,k,n} t^{-n}$ converges obviously and we get the m -th remainder for an arbitrary $k \in \mathbb{N}_0$ by

$$\begin{aligned} R_{k,m}(t) &:= F_k(t) - \hat{F}_k|_m(t) \\ &= \begin{cases} F_k(t) - \left(\sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + \sum_{n=0}^m \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} \right. \\ \quad \left. + \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}} + \kappa_k (\log(t))^2 t^{1-k} \right), & k \leq m, \\ F_k(t) - \sum_{n=0}^m \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}, & k > m, \end{cases} \\ &= \begin{cases} \sum_{n=m+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}, & k \leq m, \\ \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}} + \sum_{n=m+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} \\ \quad + \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q} - k + \frac{1}{2}} + \kappa_k (\log(t))^2 t^{1-k}, & k > m. \end{cases} \end{aligned} \quad (1)$$

Next, we show that the series $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \hat{\gamma}_{j,k,n}$ and $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n}$ are convergent.

We assume $\hat{\gamma}_{j,k,n} \neq 0$. We estimate

$$\left| \frac{1}{\frac{j}{q} + 2(n-k) + 1} \right| \leq \begin{cases} q, & j < q \text{ and } n - k = 1, \\ 1, & \text{else,} \end{cases}$$

and consequently

$$\left| \frac{1}{\left(\frac{j}{q} + 2(n-k) + 1 \right)^2} \right| \leq \begin{cases} q^2, & j < q \text{ and } n - k = 1, \\ 1, & \text{else.} \end{cases}$$

By these, we get for $|\hat{\gamma}_{j,k,n}|$ the estimation

$$\begin{aligned}
|\hat{\gamma}_{j,k,n}| &= \left| \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{\frac{j}{q}+2(n-k)+1} \left(a_{j,k} + \hat{a}_{j,k} \left(\log(\alpha) - \frac{1}{\frac{j}{q}+2(n-k)+1} \right) \right) \right| \\
&\leq \left| \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{\frac{j}{q}+2(n-k)+1} a_{j,k} \right| + \left| \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{\frac{j}{q}+2(n-k)+1} \hat{a}_{j,k} \log(\alpha) \right| \\
&\quad + \left| \frac{\alpha^{\frac{j}{q}+2(n-k)+1}}{\left(\frac{j}{q}+2(n-k)+1 \right)^2} \hat{a}_{j,k} \right| \\
&\leq q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |a_{j,k}| + q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |\hat{a}_{j,k}| \log(\alpha) \\
&\quad + q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |\hat{a}_{j,k}|. \tag{2}
\end{aligned}$$

Due to the fact that $\alpha^2 > 1$ and that $a_{j,k}$ and $\hat{a}_{j,k}$ are real coefficients of a convergent Puiseux series, the series $\sum_{k=0}^{\infty} (\alpha^2)^{-k} a_{j,k}$ and $\sum_{k=0}^{\infty} (\alpha^2)^{-k} \hat{a}_{j,k}$ converge absolutely by the ratio test and thus, we can conclude that $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{\gamma}_{j,k,n}$ converges by the comparison test. We set

$$\hat{B}_n := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{\gamma}_{j,k,n} \tag{3}$$

and furthermore,

$$\hat{C} := q^2 \sum_{j=0}^{2q-1} \sum_{k=0}^{\infty} (\alpha^2)^{\frac{j+q}{2q}-k} (|a_{j,k}| + |\hat{a}_{j,k}| (\log(\alpha) + 1)). \tag{4}$$

Moreover, the series $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n}$ converges by the com-

parison test in consequence of

$$\left| \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n \right| \stackrel{(3)}{\leq} \frac{1}{2^n n!} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} |\hat{\gamma}_{j,k,n}|$$

$$\begin{aligned}
 &\stackrel{(2)}{\leq} \frac{1}{2^n n!} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \left[q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |a_{j,k}| \right. \\
 &\quad \left. + q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |\hat{a}_{j,k}| \log(\alpha) + q^2 (\alpha^2)^{\frac{j+q}{2q}+n} (\alpha^2)^{-k} |\hat{a}_{j,k}| \right] \\
 &\stackrel{(4)}{\leq} \frac{1}{2^n n!} (\alpha^2)^n \hat{C}
 \end{aligned}$$

and the fact that $\sum_{n=0}^{\infty} \frac{1}{2^n n!} (\alpha^2)^n \hat{C} t^{-n}$ is absolutely convergent.

We define

$$\begin{aligned}
 \hat{F}(t) &:= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n} \\
 &\quad + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{k=0}^{\infty} \kappa_k (\log(t))^2 t^{1-k}.
 \end{aligned}$$

We change the indices k respectively n to p and we set

$$\mu_{j,p} := \begin{cases} \delta_{q,p} + \frac{(-1)^p}{2^{p-1}(p-1)!} \hat{B}_{p-1}, & j = q \text{ and } p \geq 1, \\ \delta_{j,p}, & \text{else.} \end{cases}$$

Thus, we can gather

$$\hat{F}(t) = \sum_{p=0}^{\infty} \left(\sum_{j=0}^{2q-1} \left(\mu_{j,p} + \lambda_{j,p} \log(t) \right) t^{\frac{j}{2q}} + \kappa_p \log(t)^2 t^{\frac{1}{2}} \right) t^{-p+\frac{1}{2}}.$$

Finally, we give the proof that $\hat{F}(t)$ is the asymptotic expansion of $F(t)$ by showing that

$$F(t) - \sum_{p=0}^N \left(\sum_{j=0}^{2q-1} \left(\mu_{j,p} + \lambda_{j,p} \log(t) \right) t^{\frac{j}{2q}} + \kappa_p \log(t)^2 t^{\frac{1}{2}} \right) t^{-p+\frac{1}{2}}$$

is in $\mathcal{O} \left(\log(t) t^{\frac{2q-1}{2q}-(N+1)+\frac{1}{2}} \right)$ for a sufficiently large $N \in \mathbb{N}$. We attain by our results above

$$\left| F(t) - \sum_{p=0}^N \left(\sum_{j=0}^{2q-1} \left(\mu_{j,p} + \lambda_{j,p} \log(t) \right) t^{\frac{j}{2q}} + \kappa_p \log(t)^2 t^{\frac{1}{2}} \right) t^{-p+\frac{1}{2}} \right| =$$

$$\begin{aligned}
&= \left| F(t) - (\hat{F}(t))|_N \right| \\
&= \left| F(t) - \left(\sum_{k=0}^{\infty} \hat{F}_k(t) \right) \Big|_N \right| \\
&= \left| \sum_{k=0}^{\infty} F_k(t) - \sum_{k=0}^{\infty} \hat{F}_k|_N(t) \right| \\
&= \left| \sum_{k=0}^N \left(F_k(t) - \hat{F}_k|_N(t) \right) + \sum_{k=N+1}^{\infty} \left(F_k(t) - \hat{F}_k|_N(t) \right) \right| \\
&\stackrel{(1)}{=} \left| \sum_{k=0}^N R_{k,N}(t) + \sum_{k=N+1}^{\infty} R_{k,N}(t) \right| \\
&\stackrel{(1)}{=} \left| \sum_{k=0}^{\infty} \sum_{n=N+1}^{\infty} \sum_{j=0}^{2q-1} \gamma_{j,k,n} t^{-n} + \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \right. \\
&\quad \left. + \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} + \sum_{k=N+1}^{\infty} \kappa_k (\log(t))^2 t^{1-k} \right| \\
&\leq \left| \sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n} \right| + \left| \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}} \right| \\
&\quad + \left| \sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} \log(t) t^{\frac{j}{2q}-k+\frac{1}{2}} \right| + \left| \sum_{k=N+1}^{\infty} \kappa_k (\log(t))^2 t^{1-k} \right|
\end{aligned}$$

As we mentioned above the series $\sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n}$ converges and we get

$$\sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} \hat{B}_n t^{-n} = \mathcal{O}(t^{-(N+1)})$$

by Remark 3.2.22. In lemmas above, we have proven that the series $\sum_{k=0}^{\infty} \delta_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$, see Lemma 3.3.40, and $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$, see Lemma 3.3.38, are convergent. By Remark 3.2.22, we obtain

$$\sum_{k=N+1}^{\infty} \sum_{j=0}^{2q-1} (\delta_{j,k} + \lambda_{j,k} \log(t)) t^{\frac{j}{2q}-k+\frac{1}{2}} = \mathcal{O}\left(\log(t) t^{\frac{2q-1}{2q}-(N+1)+\frac{1}{2}}\right).$$

Finally, we consider

$$\sum_{k=0}^{\infty} \kappa_k (\log(t))^2 t^{1-k} \stackrel{3.3.37}{=} \sum_{k=0}^{\infty} \hat{a}_{q,k} \frac{(-1)^{k+1}}{(k-1)! 2^k} \frac{1}{4} (\log(t))^2 t^{1-k}.$$

This series is convergent by the ratio test and by Remark 3.2.22, we get

$$\sum_{k=N+1}^{\infty} \kappa_k (\log(t))^2 t^{1-k} = \mathcal{O}((\log(t))^2 t^{1-(N+1)}).$$

We gather our results. Due to the fact that $t^{-(N+1)} = \mathcal{O}\left(\log(t) t^{\frac{2q-1}{2q}-(N+1)+\frac{1}{2}}\right)$ and the fact that $(\log(t))^2 t^{1-(N+1)} = \mathcal{O}\left(\log(t) t^{\frac{2q-1}{2q}-(N+1)+\frac{1}{2}}\right)$ for an arbitrary $N \in \mathbb{N}$, we conclude that

$$F(t) - \sum_{p=0}^N \left(\sum_{j=0}^{2q-1} \left(\mu_{j,p} + \lambda_{j,p} \log(t) \right) t^{\frac{j}{2q}} + \kappa_p \log(t)^2 t^{\frac{1}{2}} \right) t^{-p+\frac{1}{2}}$$

is in $\mathcal{O}\left(\log(t) t^{\frac{2q-1}{2q}-(N+1)+\frac{1}{2}}\right)$ for a sufficiently large $N \in \mathbb{N}$. According to Definition 2.2.4 and to Remark 2.2.5 we have shown

$$F(t) \approx \hat{F}(t).$$

What is more, each series which the asymptotic expansion $\hat{F}(t)$ contains is convergent. Thereby, we can write $\hat{F}(t)$ as the constructible function

$$\hat{F}(t) = \varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2,$$

where $\varphi_0(t) := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ and $\varphi_1(t) := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ and $\varphi_2(t) := \sum_{k=0}^{\infty} \kappa_k t^{1-k}$ are convergent Puiseux series. Thus, $\hat{F}(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$. \square

After all, we gather our results and establish the asymptotic series expansion of $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2i}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$, where $\psi_0(r)$ and $\psi_1(r)$ are bounded functions on $] \alpha, \infty[$, which can be expressed by convergent Puiseux series in infinity which are very similar to the representations of $\theta_0(r)$ and $\theta_1(r)$ in the last lemma.

Lemma 3.3.42. *Let $\alpha \in \mathbb{R}$, where $\alpha > 1$, let $q \in \mathbb{N}$, and let $\psi_0(r)$ and $\psi_1(r) \log(r)$ be bounded functions on $] \alpha, \infty[$ and $\psi_0(r)$ can be represented as the convergent Puiseux series $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2(k+1)}$ and $\psi_1(r)$ as $\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q} - 2(k+1)}$. As $t \rightarrow \infty$, we obtain that*

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$$

has the asymptotic expansion

$$a_0 \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \hat{\varphi}_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2,$$

where $\hat{\varphi}_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \nu_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ and $\varphi_1(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ and $\varphi_2(t) = \sum_{k=0}^{\infty} \kappa_k t^{1-k}$, where $\nu_{j,k}, \lambda_{j,k}, \kappa_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. We have

$$\begin{aligned} \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr &= \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \psi_0(r) dr + \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \psi_1(r) \log(r) dr \\ &= a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 dr + \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2k} dr \\ &\quad + \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q} - 2k} \log(r) dr \\ &= a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 dr + F(t), \end{aligned}$$

where

$$F(t) := \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} a_{j,k} r^{\frac{j}{q} - 2k - 2} dr + \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q} - 2k - 2} \log(r) dr.$$

In Lemma 3.3.34 we have proven

$$a_0 \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 dr \stackrel{3.3.34}{\approx} a_0 \left(\sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \sum_{k=0}^{\infty} \gamma_k t^{1-k} \right),$$

where $\gamma_k = \left(1 - \frac{1}{(2k+1)}\right) (-1)^k \frac{\alpha^{2k+1}}{2^k k!}$. By applying the ratio test, we obtain that the series converges. Thus, it is definable in \mathbb{R}_{an} . Furthermore, in Lemma 3.3.41 we have shown the asymptotic expansion

$$F(t) \stackrel{3.3.41}{\approx} \varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2,$$

where $\varphi_0(t) := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \mu_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ and $\varphi_1(t) := \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}$ and $\varphi_2(t) := \sum_{k=0}^{\infty} \kappa_k t^{1-k}$ are convergent Puiseux series, where $\mu_{j,k}, \lambda_{j,k}, \kappa_k \in \mathbb{R}$, and thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

We set

$$\begin{aligned} \hat{\varphi}_0(t) &:= a_0 \sum_{k=0}^{\infty} \gamma_{k+1} t^{-k} + \varphi_0(t) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \nu_{j,k} t^{\frac{j}{2q} - k + \frac{1}{2}}, \end{aligned}$$

where

$$\nu_{j,k} := \begin{cases} \mu_{q,k} + a_0 \gamma_{k+1}, & j = q, \\ \mu_{j,k}, & \text{else.} \end{cases}$$

Hence, we obtain

$$\begin{aligned} & \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr \\ & \approx a_0 \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + a_0 \sum_{k=0}^{\infty} \gamma_k t^{1-k} + \varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2 \\ & = a_0 \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + a_0 \gamma_0 t + a_0 \sum_{k=0}^{\infty} \gamma_{k+1} t^{-k} + \varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2 \\ & = a_0 \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \hat{\varphi}_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2 \end{aligned}$$

and $a_0\sqrt{\frac{\pi}{2}}t^{\frac{3}{2}} + \hat{\varphi}_0(t) + \varphi_1(t)\log(t) + \varphi_2(t)\log(t)^2$ is definable in $\mathbb{R}_{\text{an,exp}}$. \square

We conclude the chapter by resuming the results in the following theorem. We go back to $\frac{1}{(2\pi t)^{\frac{3}{2}}}\int_A e^{-\frac{|x|^2}{2t}} dx$ and transform it in polar coordinates by Transformation Theorem. By Remark 2.3.17 of Theorem 2.3.16, which gives us the form of the angular part, we get integrals of the form $\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r)\log(r)) dr$, for which we use our results above.

Theorem 3.3.43. *Let $A \subset \mathbb{R}^3$ be a globally subanalytic set. Let $f : \mathbb{R}_{>0} \rightarrow [0, 1]$ be the function given by*

$$f(t) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

- a) *As $t \rightarrow \infty$ and if A is bounded, $f(t)$ is of the form $\rho_0(t) + \rho_1(t)\log(t)$, where ρ_n are globally subanalytic functions, and hence, $f(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$.*
- b) *As $t \rightarrow \infty$ and if A is unbounded, $f(t)$ has the asymptotic series expansion*

$$\rho_0(t) + \rho_1(t)\log(t) + \rho_2(t)\log(t)^2,$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \mu_k t^{-\frac{k}{q}}$, $\rho_1(t) = \sum_{k=1}^{\infty} \nu_k t^{-\frac{k}{q}}$, and $\rho_2(t) = \sum_{k=1}^{\infty} \omega_k t^{\frac{1}{2}-k}$, where at least one of the series is not zero and $\mu_k, \nu_k, \omega_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series. Thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Proof. The structure of this proof is quite similar to the proof of Theorem 3.3.30. Let

$$\begin{aligned} \Theta : \mathbb{R}_{\geq 0} \times [0, \pi] \times]-\pi, \pi] &\longrightarrow \mathbb{R}^3, \\ (r, \vartheta, \varphi) &\longmapsto (r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta), \end{aligned}$$

be the three-dimensional polar coordinate transformation with functional determinant $\det(D\Theta(r, \vartheta, \varphi)) = r^2 \sin \vartheta$. By Transformation Theorem 2.1.17, we transform $f(t)$ in polar coordinates and get

$$\begin{aligned} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx &\stackrel{2.1.17}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{|(r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta)|^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi). \end{aligned}$$

The function Θ is continuous and definable in the o-minimal structure \mathbb{R}_{an} , therefore $B := \Theta^{-1}(A) \subset \mathbb{R}_{\geq 0} \times]-\pi, \pi]^2$ is definable in \mathbb{R}_{an} . Let B_r be the set $\{(\vartheta, \varphi) \mid (r, \vartheta, \varphi) \in B\}$. By Fubini's Theorem 2.1.18, we get

$$\begin{aligned} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{\frac{-r^2}{2t}} r^2 \sin \vartheta \, d(r, \vartheta, \varphi) &\stackrel{2.1.18}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} \int_{B_r} e^{\frac{-r^2}{2t}} r^2 \sin \vartheta \, d(\vartheta, \varphi) dr \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr. \end{aligned}$$

By applying the Cell Decomposition Theorem 2.3.15 to $\mathbb{R}_{\geq 0} \cup \{\infty\}$, we get a finite partition of $\mathbb{R}_{\geq 0} \cup \{\infty\}$ into finitely many disjoint cells, that means in points and open intervals D_1, \dots, D_p , where $D_j :=]\alpha, \beta[$ with $0 \leq \alpha < \beta \leq \infty$ and we can write

$$\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr = \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{j=1}^p \int_{D_j} e^{\frac{-r^2}{2t}} r^2 \int_{B_r} \sin \vartheta \, d(\vartheta, \varphi) dr.$$

For the sake of convenience we fix $j \in \{1, \dots, p\}$ and write $D := D_j$. Due to the fact that integrals over D will be zero if D is a point, we ignore this case. If B is a bounded set, D is a bounded interval. If B is unbounded, at least one interval D is unbounded and consequently $\beta = \infty$.

In consequence of the restriction to B_r , $\sin \vartheta$ is globally subanalytic and we can apply Remark 2.3.17 and get that

$$\int_{B_r} \sin \vartheta \, d(\vartheta, \varphi), \tag{1}$$

which is the angular part, has the form of a constructible function. In our case, the log-term in the constructible function has at most power one. The reason is, if we apply Fubini's Theorem 2.1.18 to the integral in (1) and integrate $\sin \vartheta$, there is no log-term in the antiderivative of $\sin \vartheta$ as is well known. Therefore, only the second integration can bring a log-term in the antiderivative of $\int_{B_r} \sin \vartheta \, d(\vartheta, \varphi)$. Thus,

the integral has the form $\psi_0(r) + \psi_1(r) \log(r)$, where $\psi_0(r)$ and $\psi_1(r)$ are strictly positive globally subanalytic functions. Since $\psi_0(r) + \psi_1(r) \log(r)$ represents the angular part, it is bounded by π . In a similar way to Lemma 3.3.24, we can argue that $\psi_0(r)$ as well as $\psi_1(r) \log(r)$ is also bounded.

We obtain

$$\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_D e^{-\frac{r^2}{2t}} r^2 \int_{B_r} \sin \vartheta d(\vartheta, \varphi) dr \stackrel{2.3.17}{=} \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr.$$

First, we give the proof for a). We may assume B is bounded, consequently also D . Due to $\psi_0(r)$ and $\psi_1(r)$ are globally subanalytic functions on D , and consequently definable in \mathbb{R}_{an} , we get by Lemma 3.3.33 that

$$\int_D e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$$

has the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where φ_i are globally subanalytic functions. We can conclude about the total set if the set $\Theta^{-1}(A)$ is bounded that

$$\frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{j=1}^p \int_{D_j} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$$

has the form $\rho_0(t) + \rho_1(t) \log(t)$ as $t \rightarrow \infty$, where ρ_n are globally subanalytic functions and $q \in \mathbb{N}$. Therefore $f(t)$ is definable in $\mathbb{R}_{\text{an,exp}}$.

Now, we prove statement b) and we may assume that B is unbounded. In this case, at least one D is unbounded, that means, D could also be an open interval of the form $] \alpha, \infty[$ and by a refined cell decomposition we can assume $\alpha > 1$. In consequence of the definability of $\psi_i(r)$ in \mathbb{R}_{an} , the Preparation Theorem 2.3.18, and the boundedness of $\psi_0(r)$ and $\psi_1(r) \log(r)$ on $] \alpha, \infty[$, which can be shown in the same way as in Lemma 3.3.24, $\psi_0(r)$ is representable as a convergent Puiseux series $a_0 + \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \tilde{a}_{j,k} r^{\frac{j}{q} - 2(k+1)}$ and $\psi_1(r)$ as a convergent Puiseux series

$\sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \hat{a}_{j,k} r^{\frac{j}{q} - 2(k+1)}$, where $q \in \mathbb{N}$. In Lemma 3.3.42 we have already pointed out that

$$\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr$$

has the asymptotic expansion

$$a_0 \sqrt{\frac{\pi}{2}} t^{\frac{3}{2}} + \hat{\varphi}_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2, \quad (2)$$

where $\hat{\varphi}_0(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \nu_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ and $\varphi_1(t) = \sum_{k=0}^{\infty} \sum_{j=0}^{2q-1} \lambda_{j,k} t^{\frac{j}{2q}-k+\frac{1}{2}}$ and $\varphi_2(t) = \sum_{k=0}^{\infty} \kappa_k t^{1-k}$, where $\nu_{j,k}, \lambda_{j,k}, \kappa_k \in \mathbb{R}$ and $q \in \mathbb{N}$, are convergent Puiseux series and thus, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Due to the fact that $\int_{B_r} \sin \vartheta d(\vartheta, \varphi) > 0$ for $(\vartheta, \varphi) \in [0, \pi] \times]-\pi, \pi]$, we get that the angular part $\psi_0(r) + \psi_1(r) \log(r)$ is also strictly positive. Thus, there are constants $\sigma \in \mathbb{Q}_{<0}$ and $C > 0$ such that $\psi_0(r) + \psi_1(r) \log(r) \geq Cr^\sigma$ for all $r \in [\alpha, \infty[$. We obtain that $\int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr \geq C \int_{\alpha}^{\infty} e^{-\frac{r^2}{2t}} r^{\sigma+2} dr$, of which the asymptotic expansion has polynomial scale as we have already seen. Thus, at least one of the coefficients of the series $\hat{\varphi}_0(t), \varphi_1(t), \varphi_2(t)$ or a_0 in (2) is not zero.

Finally, we get the following conclusion: If $\Theta^{-1}(A)$ is an unbounded set, at least one D is unbounded and we achieve, taking also the results of a) in account, that

$$\begin{aligned} & \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\Theta^{-1}(A)} e^{-\frac{r^2}{2t}} r^2 \sin \vartheta d(r, \vartheta, \varphi) = \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}_{\geq 0}} e^{-\frac{r^2}{2t}} r^2 \int_{B_r} \sin \vartheta d(\vartheta, \varphi) dr \\ &= \frac{1}{(2\pi t)^{\frac{3}{2}}} \sum_{j=1}^p \int_{D_j} e^{-\frac{r^2}{2t}} r^2 (\psi_0(r) + \psi_1(r) \log(r)) dr \tag{3} \\ &\approx \rho_0(t) + \rho_1(t) \log(t) + \rho_2(t) \log(t)^2, \end{aligned}$$

where $\rho_0(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{2q-1} \gamma_{m,k} t^{\frac{m}{2q}-k}$, where $\gamma_{m,k} \in \mathbb{R}$ and $\gamma_{m,0} = 0$ if $m \neq 0$, and $\rho_1(t) = \sum_{k=1}^{\infty} \sum_{m=0}^{2q-1} \delta_{m,k} t^{\frac{m}{2q}-k}$, $\delta_{m,k} \in \mathbb{R}$, and $\rho_2(t) = \sum_{k=1}^{\infty} \omega_k t^{\frac{1}{2}-k}$, $\omega_k \in \mathbb{R}$, are convergent Puiseux series, where at least one coefficient of the series is not zero by the arguments above and because every summand in (3) is positive.

Since the Puiseux series ρ_0, ρ_1 and ρ_2 are convergent, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$. \square

4

Conclusion and Outlook

We conclude this thesis with a short summary of the shown results. Motivated by the research of Comte, Lion and Rolin, by Cluckers and Miller, and also by the research of Kaiser, we studied the probability distribution

$$\frac{1}{(2\pi t)^{\frac{n}{2}}} \int_A e^{-\frac{|x|^2}{2t}} dx, \quad (4.1)$$

which is given by the Brownian Motion on a semialgebraic set, as example for the occurrence of integrals of the form $\int e^{-\frac{r^2}{2t}} \psi(r) dr$ and $\int e^{-\frac{r^2}{2t}} \psi(r) \log(r) dr$, where ψ is a globally subanalytic function. In the one-dimensional case we showed that these integrals in (4.1) on a family of sets, which are definable in an arbitrary o-minimal structure \mathcal{M} , are definable in the Pfaffian closure of \mathcal{M} . In the two-dimensional case we examined $\int e^{-\frac{r^2}{2t}} r \psi(r) dr$ at the limit points of t and investigated asymptotic series expansions by applying a recursive formula. We proved that the integrals behave like a Puiseux series as t approaches zero. As t tends towards infinity, we gave the proof that the probability distribution is definable in the o-minimal structure \mathbb{R}_{an} and it is a convergent Puiseux series if A is bounded. For this purpose, we applied the results of Lion and Rolin, see [LR98], to the integrals $\int e^{-\frac{r^2}{2t}} r \psi(r) dr$ over a bounded set. Then, we established the asymptotic

expansion of the integrals over an unbounded set. We were able to demonstrate that the asymptotic expansion has the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where φ_i are convergent Puiseux series with negative exponents, and subsequently, we got that the expansion is definable in $\mathbb{R}_{\text{an,exp}}$. Afterwards, we considered the probability distribution in \mathbb{R}^3 and started with the case t tends towards zero. We proved that the integrals in (4.1) have the expansion $\sum_{k=0}^{\infty} (\gamma_k + \delta_k \log(t)) t^{\frac{k}{q}}$. Then, we distinguished A is bounded and unbounded another time as t approaches infinity. In the first case, it could be ascertained by the stability property of constructible functions that the probability distribution has the form $\varphi_0(t) + \varphi_1(t) \log(t)$, where φ_i are convergent Puiseux series with negative exponents and thereby the probability distribution is definable in $\mathbb{R}_{\text{an,exp}}$. In the other case, we proceeded as in the two-dimensional case and investigated the asymptotic expansions of $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) dr$ and $\int e^{-\frac{r^2}{2t}} r^2 \psi(r) \log(r) dr$ in several single steps by means of recursive formulas. Finally, the last proof showed that the probability distribution given by the Brownian Motion on unbounded sets has an asymptotic expansion of the form $\varphi_0(t) + \varphi_1(t) \log(t) + \varphi_2(t) \log(t)^2$ as t approaches infinity, where φ_i are convergent Puiseux series with negative exponents. In consequence of that, the asymptotic expansion is definable in $\mathbb{R}_{\text{an,exp}}$.

Now, we briefly want to discuss open questions for further research. An obvious way is the enlargement to the n -dimensional case and the investigation of asymptotic expansions of the probability distribution given by the Brownian Motion on a n -dimensional semialgebraic set. We have the vague assumption that there may arise integrals similar to the form $\int e^{-\frac{r^2}{2t}} r^{n-1} \sum_{i=0}^{n-2} \psi_i(r) \log(r)^i dr$. It is another open question, whether these integrals can be reduced to our discussed integrals and their asymptotic expansions by using recursion. Another path could be the study of our probability distribution on families of semialgebraic sets in higher dimensions.

Bibliography

- [AS12] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*. Dover Books on Mathematics. Dover Publications, 2012.
- [Bac00] L. Bachelier. Théorie de la spéculation. *Ann. sci. éc. norm. sup.*, 17:21–86, 1900.
- [BCR98] J. Bochnak, M. Coste, and M.-F. Roy. *Real Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics. Springer, 1998.
- [CCM⁺16] R. Cluckers, G. Comte, D.J. Miller, J.-P. Rolin, and T. Servi. Integration of oscillatory and subanalytic functions. <https://arxiv.org/abs/1601.01850>, preprint, January 2016.
- [CLR00] G. Comte, J.-M. Lion, and J.-P. Rolin. Nature log-analytique du volume des sous-analytiques. *Illinois J. Math.*, 44(4):884–888, 2000.
- [CM11] R. Cluckers and D.J. Miller. Stability under integration of sums of products of real globally subanalytic functions and their logarithms. *Duke Math. J.*, 156(2):311–348, 2011.
- [Cop04] E.T. Copson. *Asymptotic Expansions*. Cambridge Tracts in Mathematics. Cambridge University Press, 2004.
- [Cos99] M. Coste. *An Introduction to o-minimal Geometry*. Institut de Recherche Mathématique de Rennes, 1999.
- [Dan54] H. E. Daniels. Saddlepoint approximations in statistics. *Ann. Math. Statist.*, 25(4):631–650, 12 1954.
- [dL12] P.S. de Laplace. *Théorie analytique des probabilités*. Mme Ve Courcier, 1812.

- [Ein05] A. Einstein. *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen*. Annalen der Physik. Johann Ambrosius Barth, 1905.
- [Erd56] A. Erdélyi. *Asymptotic Expansions*. Dover Books on Mathematics. Dover Publications, 1956.
- [Erd81] A. Erdélyi. *Higher Transcendental Functions*. Krieger Publishing Company, 1981.
- [FK16] S. Franceschi and I. Kourkova. Asymptotic expansion of stationary distribution for reflected Brownian motion in the quarter plane via analytic approach. <https://hal.archives-ouvertes.fr/hal-01295562>, preprint, March 2016.
- [GKP94] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. A foundation for computer science. Addison-Wesley, 1994.
- [GKS61] B. V. Gnedenko, V. S. Koroluk, and A. V. Skorokhod. Asymptotic expansions in probability theory. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Contributions to Probability Theory*, pages 153–170, Berkeley, Calif., 1961. University of California Press.
- [Kai12] T. Kaiser. First order tameness of measures. *Ann. Pure Appl. Logic*, 163(12):1903–1927, 2012.
- [Kai13] T. Kaiser. Integration of semialgebraic functions and integrated Nash functions. *Math. Z.*, 275(1-2):349–366, 2013.
- [Kön13] K. Königsberger. *Analysis 2*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2013.
- [KS14] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer New York, 2014.
- [KW14] G. Kersting and A. Wakolbinger. *Stochastische Prozesse*. Mathematik Kompakt. Springer Basel, 2014.
- [LR97] J.-M. Lion and J.-P. Rolin. Théorème de préparation pour les fonctions logarithmico-exponentielles. *Ann. Inst. Fourier (Grenoble)*, 47(3):859–884, 1997.

- [LR98] J.-M. Lion and J.-P. Rolin. Intégration des fonctions sous-analytiques et volumes des sous-ensembles sous-analytiques. *Ann. Inst. Fourier (Grenoble)*, 48(3):755–767, 1998.
- [MR92] R. Moussu and C. Roche. Théorèmes de finitude pour les variétés pfaffiennes. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):393–420, 1992.
- [MRS12] C. Miller, J.-P. Rolin, and P. Speissegger. *Lecture Notes on O-Minimal Structures and Real Analytic Geometry*. Fields Institute Communications. Springer New York, 2012.
- [MS06] D. Meintrup and S. Schäffler. *Stochastik: Theorie und Anwendungen*. Statistik und ihre Anwendungen. Springer Berlin Heidelberg, 2006.
- [Mur12] J.D. Murray. *Asymptotic Analysis*. Applied Mathematical Sciences. Springer New York, 2012.
- [OoSU10] F.W.J. Olver, National Institute of Standards, and Technology (U.S.). *NIST Handbook of Mathematical Functions Hardback and CD-ROM*. Cambridge University Press, 2010.
- [Poi86] H. Poincaré. Sur les intégrales irrégulières des équations linéaires. *Acta Math.*, 8:295–344, 1886.
- [PS84] A. Pillay and C. Steinhorn. Definable sets in ordered structures. *Bull. Amer. Math. Soc. (N.S.)*, 11(1):159–162, 1984.
- [Rub64] H. Ruben. An asymptotic expansion for the multivariate normal distribution and Mills' ratio. *J. Res. Nat. Bur. Standards Sect. B*, 68B:3–11, 1964.
- [Sei54] A. Seidenberg. A new decision method for elementary algebra. *Ann. of Math. (2)*, 60:365–374, 1954.
- [Sib52] W. Sibagaki. *Theory and Applications of the Gamma Function with a Table of the Gamma Function for Complex Arguments Significant to the Sixth Decimal Place*. Iwanami Syoten, 1952.
- [Sie69] C.L. Siegel. *Topics in Complex Function Theory: Elliptic functions and uniformization theory*. Interscience tracts in pure and applied mathematics. Wiley-Interscience, 1969.
- [Spe99] P. Speissegger. The Pfaffian closure of an o-minimal structure. *J. Reine Angew. Math.*, 508:189–211, 1999.

- [Sti86] T.-J. Stieltjes. Recherches sur quelques séries semi-convergentes. *Ann. sci. èc. norm. sup.*, 3(3):201–258, 1886.
- [Tar48] A. Tarski. *A Decision Method for Elementary Algebra and Geometry*. RAND Corporation, Santa Monica, Calif., 1948.
- [Te60] H.W. Turnbull (ed). *The Correspondence of Isaac Newton*, volume 2 (1676-1687). Cambridge University Press, 1960.
- [vdD84] L. van den Dries. Remarks on Tarski’s problem concerning $(\mathbf{R}, +, \cdot, \exp)$. In *Logic colloquium '82 (Florence, 1982)*, volume 112 of *Stud. Logic Found. Math.*, pages 97–121. North-Holland, Amsterdam, 1984.
- [vdD86] L. van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bull. Amer. Math. Soc. (N.S.)*, 15(2):189–193, 1986.
- [vdD98] L. van den Dries. *Tame Topology and O-minimal Structures*. 150–184. Cambridge University Press, 1998.
- [Wie21] Norbert Wiener. The average of an analytic functional and the brownian movement. *Proceedings of the National Academy of Sciences of the United States of America*, 7(10):294–298, 1921.
- [Wil96] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996.
- [Wil14] H.S. Wilf. *Generatingfunctionology*. Elsevier Science, 2014.