Dissertation

Border Basis Schemes

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For my mother, who supported me all the way since the beginning of my studies

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Notation

$\mathbb{N}$	set of natural numbers, $\mathbb{N} = \{0, 1, 2, ...\}$
$\mathbb{N}_+$	set of natural numbers larger than 0
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mid S \mid$	number of elements of the finite set S
$A \subset B$	set A is proper subset of B
$A \subseteq B$	set A is a subset of B
$A \setminus B$	set difference of A and B
$\Psi \circ \varphi$	composition of two maps $\varphi : A \longrightarrow B$ and $\Psi : B \longrightarrow C$
$\mathrm{Im}(\varphi)$	image of the map $\varphi : A \longrightarrow B$
$\mathrm{Ker}(\varphi)$	kernel of the homomorphism $\varphi : A \longrightarrow B$
$\mathrm{K}[x_1, ..., x_n]$	polynomial ring in the indeterminates $x_1, ..., x_n$
$\mathrm{K}[\mathbf{X}]$	short form of $\mathrm{K}[x_1, ..., x_n]$
$\mathbb{A}^n_K$	n-dimensional affine space over a field K
$\mathbb{T}^n$	set of terms in the indeterminates $x_1, .., x_n$
`Lex`	lexicographic term ordering
$\log(x_1^{\alpha_1} \cdots x_n^{\alpha_n})$	exponent vector of a term $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}^n$, $(\alpha_1, ...\alpha_n) \in \mathbb{N}^n$
$\mathcal{O}$	order ideal in $\mathbb{T}^n$
$\partial\mathcal{O}$	border of an order ideal
$\{c_{11}, ..., c_{\mu\nu}\}$	set of indeterminates where $\mu =\mid \mathcal{O} \mid$ and $\nu =\mid \partial\mathcal{O} \mid$
$\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$	polynomial ring in the indeterminates $c_{11}, ..., c_{\mu\nu}$

$\mathcal{Z}(I)$	zero set of the ideal I
$\mathcal{I}(V)$	vanishing ideal of the K-variety V
τ	set of the generators of the vanishing ideal of a border basis scheme
$\mathrm{K}[V]$	coordinate ring of the K-variety
$\mathbb{B}_{\mathcal{O}}$	$\mathcal{O}$-border basis scheme
$\mathcal{I}(\mathbb{B}_{\mathcal{O}})$	vanishing ideal of $\mathcal{O}$-border basis scheme
$\mathrm{B}_{\mathcal{O}}$	coordinate ring of the $\mathcal{O}$-border basis scheme
$\mathbb{C}_{\mathcal{O}}$	principal component of $\mathcal{O}$-border basis scheme
R^n	the set of all n-tuples of elements of R
$(e_1, ..., e_k)$	canonical basis of a finitely generated module
$\mathrm{Jac}(\mathcal{G})$	Jacobian matrix of a tuple of polynomials
$\mathrm{Trace}(\mathcal{A})$	trace of a matrix $\mathcal{A}$
$\mathcal{A}^{tr}$	transposed matrix of $\mathcal{A}$
$\mathrm{Syz}_{\mathrm{R}}(\mathfrak{G})$	syzygy module of the tuple $\mathfrak{G}$
dim	Krull dimension
$\dim_{\mathrm{K}}$	dimension as K-vector space
edim	embedding dimension
$\deg(f)$	degree of a polynomial f
$\deg_W(f)$	degree of f with respect to the grading given by W
$\frac{\partial f}{\partial x_i}$	partial derivative of a f
Spec(R)	spectrum of a ring R

Chapter 1

Introduction

1.1 Overview

The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of terms $\mathcal{O}$ whose residue classes form a K-vector space basis of P/I. The $\mathcal{O}$-border basis scheme is a scheme that parametrizes all zero-dimensional ideals that have an $\mathcal{O}$-border basis. In [Hai98] it is pointed out that, in general, the $\mathcal{O}$-border basis scheme is not an affine space. Subsequently, in [Huib09]) it is proved that if an order ideal $\mathcal{O}$ with μ elements is defined in a two dimensional polynomial ring and it is of some special shapes, then the $\mathcal{O}$-border basis scheme is isomorphic to the affine space $\mathbb{A}^{2\mu}$. This thesis is dedicated to find a more general condition for an $\mathcal{O}$-border basis scheme to be isomorphic to an affine space of dimension $n\mu$ that is independent of the shape of the order ideal and the dimension of the polynomial ring that the order ideal is defined in.

We accomplish this in 6 Chapters. In Chapters 2 and 3 we develop the concepts and properties of border basis schemes. In Chapter 4 we transfer the smoothness criterion for the point $(0, ..., 0)$ in a Hilbert scheme of points in [Huib05] to the monomial point $\mathfrak{o} = (0, ..., 0)$ of the border basis scheme by employing the tools from border basis theory. In Chapter 5 we explain trace and Jacobi identity syzygies of the defining equations of a $\mathcal{O}$-border basis scheme and characterize them by the arrow grading. In Chapter 6 we give a criterion for the isomorphism between $\mathbb{A}^{n\mu}$ and $\mathbb{B}_{\mathcal{O}}$ by using the results from Chapters 4 and Chapter 5. The techniques from other chapters are applied in Chapter 6.4 to segment border basis schemes and in Chapter 6.4 to $\mathcal{O}$-border basis schemes for which $\mathcal{O}$ is of the *sawtooth* form.

1.2 Motivation

Border basis schemes were generally treated as open subsets of Hilbert schemes of points until the pioneering article [KrRo08] which introduced a construction of border basis schemes independent of Hilbert schemes of points and the name border basis scheme, as well. Before explaining border basis schemes, let us take a brief look at border basis theory. Let P be a polynomial ring, and let I be a zero-dimensional ideal in P. The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of terms $\mathcal{O}$ whose residue classes form a K-vector space basis of P/I (for further information we refer to [KrRo05]). Let $\mathcal{O}$ be an order ideal with μ terms. The $\mathcal{O}$-border basis scheme is a scheme that parametrizes all zero-dimensional ideals that have an $\mathcal{O}$-border basis. It is an open sub-scheme of the Hilbert scheme of μ points $H_n^\mu = Hilb^\mu(\mathbb{A}^n)$. The affine varieties mentioned in [Hai98] Proposition 2.1 and [Huib09] are exactly the $\mathcal{O}$-border basis schemes of points on the plane. Let us present which questions motivated this work.

It is first stated in [Hai98] that, in general, border basis schemes are not affine cells, i.e. they are not isomorphic to affine spaces. This note leads us to ask the following question which is the starting point of our research.

Question 1.2.1. *Under which condition (or conditions) is a given $\mathcal{O}$-border basis scheme isomorphic to an affine space?*

In [MilSt05], page 363 an algebraic method is used to show the smoothness of a given Hilbert scheme of points in the plane. This motivated us to ask whether we can use a similar method to answer Question 1.2.1. We show that the given method is insufficient but it directs us to consider the minors of the Jacobian matrix of the set of defining equations of the $\mathcal{O}$-border basis scheme.

Two further valuable sources of our work are [Huib05] and [Huib09]. In the first article [Huib05], a criterion for the smoothness of the monomial point $\mathfrak{o}$ of a Hilbert scheme of points is given. This leads us to reprove this result for the monomial point $\mathfrak{o}$ of a given $\mathcal{O}$-border basis scheme, after which the following question arises:

Question 1.2.2. *Is there a way to generalize this local smoothness criterion to a global smoothness criterion for an $\mathcal{O}$-border basis scheme?*

The second article is [Huib09] which introduces new ways to compute the syzygies of the tuple $\mathcal{T}$ constituting of the defining equations of the $\mathcal{O}$-border basis scheme. This article motivated us to use those syzygies to examine when a given $\mathcal{O}$-border

basis scheme is an ideal-theoretic complete intersection. In general, we find that these syzygies are not enough to reach such a conclusion. By employing further algebraic techniques (see Chapter 6.4), we get results for some specific $\mathcal{O}$-border basis schemes. Let us explain our novel contributions in more detail in the next section.

1.3 Main Results

The first contribution starts in Chapter 3, where we introduce a grading on the polynomial ring, in which the vanishing ideal of a border basis scheme is defined. We shall call this grading *the arrow grading*. Then we show that the vanishing ideal of the $\mathcal{O}$-border basis scheme is homogenous with respect to the corresponding arrow grading. Although this is a known fact (see, for example, Lemma 4.1 [Huib09]), we give an alternative proof by employing the generic multiplication matrices. Moreover, in the same chapter, we show that the torus action (see page 208 of [Hai98] and page 363 of [MilSt05]) on $\mathbb{B}_{\mathcal{O}}$ results in the arrow grading. The arrow grading is neither of positive type nor of non-negative type as defined in ([KrRo05], Definition 4.2.4). Thus it follows that in $\mathrm{B}_{\mathcal{O}}$ there might exist more than one maximal homogenous ideal. This shows that the corresponding claim on page 363 of [MilSt05] is incorrect. Consequently, giving an algebraic proof of any claim about border basis schemes is more difficult because of the peculiarity of the grading which is defined on the coordinate ring of a border basis scheme.

In the subsequent chapter, Chapter 5, we investigate the cotangent space of a border basis scheme at the monomial point $\mathfrak{o} = (0, ..., 0)$. In [Hai98] a new method of constructing the cotangent space of a border basis scheme of μ points in the plane is proposed. We call this method the *arrow method*. The main idea is to consider every indeterminate from the set $\{c_{11}, ..., c_{\mu\nu}\}$ which parametrizes the border basis scheme as an arrow from an order ideal term to a border term. Then one constructs a basis of the cotangent space by using the relations of border terms and order ideal terms. Later, in Chapter 4 and 5 of [Huib05][1], this arrow method is generalized to n dimensions and a way for constructing a basis of the cotangent space of a border basis scheme is given. Although none of those citations contains the name border basis scheme, in [Hai98] the set U_μ given in Equation 2.3, and in [Huib05] the set U_β given in Equation (4) in Chapter 2.2 are both border basis schemes. Our work is inspired by both of those sources, but mainly by [Huib05]. In contrast to these articles we do not use the arrow

[1] [Huib05] is a pre-print that is only published in arXiv.

method. The foundation of our work depends on the generating set of the vanishing ideal of a border basis scheme which encodes all relations between border terms and order ideal terms. Thus every *arrow* relation, as used in [Huib09] and [Hai98] to construct a basis of the cotangent space of Hilbert scheme of μ points at the monomial point $(0, ..., 0)$, is coded by the elements of the defining equations of a border basis scheme. By using these equations one can compute all equivalence classes modulo $\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal generated by the indeterminants of the coordinate ring of the $\mathcal{O}$-border basis scheme. Let us denote this set of equivalence classes by $\mathfrak{E}$. We prove that the arrow degree is well-defined on this set. Then we show that $\mathfrak{E}$ contains at least $n\mu$ elements. Consequently, by using the set $\mathfrak{E}$, we give a K-basis of the cotangent space. From this we deduce the following smoothness criterion for the monomial point of a border basis scheme.

Theorem 1.3.1. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal with μ elements. The monomial point $\mathfrak{o} \in \mathbb{B}_\mathcal{O}$ is smooth if and only if the number of elements of $\mathfrak{E}$ is $n\mu$.*

A similar result can be found in [Huib05], Theorem 5.1.1. We reprove the statements of Chapter 4 and 5 from[Huib05] for border basis schemes by using the tools we have in border basis scheme theory such as border relations and the arrow grading.

The trace and the Jacobi identity syzygies of $\mathcal{T}$, where $\mathcal{T}$ denotes the set of the defining equations of a given $\mathcal{O}$-border basis scheme, were first introduced in [Huib09]. In Section 10 of [Huib09] the trace syzygies of $\mathcal{T}$ were used to show that any border basis scheme of points contained in a plane is an *ideal-theoretic complete intersection*. Moreover, in the same article, for the specific case $\mathcal{O} = \{1, x_1\} \subset \mathrm{K}[x_1, x_2, x_3]$ both trace syzygies and Jacobi identity syzygies of $\mathcal{T}$ were used to prove that $\mathbb{B}_\mathcal{O}$ is an ideal-theoretic complete intersection. Let $\mathcal{B}(2, 2, 2)$ denote the order ideal $\{1, x_3, x_2, x_1, x_2x_3, x_1x_3, x_1x_2, x_1x_2x_3\} \in \mathrm{K}[x_1, x_2, x_3]$. We show that these methods are not sufficient to decide whether $\mathbb{B}_{\mathcal{B}(2,2,2)}$ is a complete intersection or not. We reprove some of the properties of the trace and Jacobi identity syzygies. Our goal is to characterize these syzygy computations by the arrow grading. We show that there are at most 60 redundant polynomials in the set of defining equations of the border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$ that can be found by Jacobi identity and trace syzygies of $\mathcal{T}$ and we illustrate the reasons why these methods are not sufficient. Moreover, we implemented the Jacobi identity and the trace syzygy computation methods in the package *bbsmingen* of the computer algebra system ApCoCoA.

In [Huib02], Proposition 7.3.1 states that if the order ideal $\mathcal{O} \in \mathrm{K}[x_1, x_2]$ is of a

specific shape ("*sawtooth*") then the $\mathcal{O}$-border basis scheme is isomorphic to an affine space. We give a more general condition that applies to an order ideal $\mathcal{O} \subset \mathbb{T}^n$, independent of the shape of $\mathcal{O}$. The matrix $\mathcal{M}_\mathcal{S}$ is a specially chosen sub-matrix of the Jacobian matrix of $\mathcal{T}$. The way how to construct this matrix is explained in 6.2.3 iv). Then we prove the following result.

Theorem 1.3.2. *Let $\mathcal{O}$ denote an order ideal in $\mathrm{K}[x_1, ..., x_n]$, having μ elements. If there exists a set $\bar{\mathcal{S}} \subset \mathrm{B}_\mathcal{O}$ with $n\mu$ elements such that the entries of the corresponding matrix $\mathcal{M}_\mathcal{S}$ are in the polynomial ring $\mathrm{K}[\mathcal{S}]$ and the determinant of this matrix is ± 1, then $\mathbb{B}_\mathcal{O}$ is isomorphic to an affine space of dimension $n\mu$.*

In other words, for a given $\mathcal{O}$-border basis scheme, if there exists a maximal minor of $\mathrm{Jac}(\mathcal{T})$ that is ± 1 and the corresponding sub-matrix is in $\mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$, the $\mathcal{O}$-border basis scheme is isomorphic to an affine space. Moreover, it is an ideal theoretic complete intersection where the vanishing ideal is generated by $\mathcal{T}_\mathcal{S}$. We note that if $\mathbb{B}_\mathcal{O}$ is isomorphic to an affine space of dimension $n\mu$, then it is not trivial that it is an ideal theoretic complete intersection (see [Kum77]).

In Chapter 6.4 we apply Theorem 1.3.2 to prove Proposition 7.3.1 of [Huib02] which is as follows.

Theorem 1.3.3. *Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal in $\mathbb{T}^2$. If the order ideal $\mathcal{O}$ has the sawtooth form then $\mathbb{B}_\mathcal{O}$ is isomorphic to an affine space.*

An order ideal has the *sawtooth* form if the diagram of $\mathcal{O}$ is of the form such that every step width, except possibly the topmost, and every step height, except possibly the rightmost, is of size one.

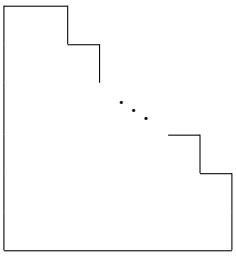

Table 1.1: Diagram of $\mathcal{O}$ has the *sawtooth* form

Our aim is to compare the two methods and to emphasize the fact that Theorem 1.3.2 is more general than the methods applied in the proof of Theorem 7.3.1 of [Huib02].

Let $\mathcal{O} = \{1, x_1, x_1^2, ..., x_1^\mu\} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. We call such an order

ideal as a *segment order ideal* and the $\mathcal{O}$-border basis scheme as a *segment border basis scheme*. We first compute Jacobi identity and trace syzygies of the defining equations of a segment border basis scheme. Next, we show that they are not sufficient to give all redundant generators of the vanishing ideal of this scheme. Then we give a new method to find all redundant generators of the vanishing ideal of a segment border basis scheme. This method leads us to the following result.

Theorem 1.3.4. *A segment border basis scheme can be generated by exactly $\mu\nu - n\mu$ polynomials from $\mathcal{T}$.*

Then, by applying the methods we develop in Chapter 6, we prove the following theorem.

Theorem 1.3.5. *A segment border basis scheme is isomorphic to an affine space. Moreover, it is an ideal-theoretic complete intersection.*

In [Ro09], Corollary 3.13, it is also shown that a segment border basis scheme is isomorphic to an affine space. Our result depends on Theorem 1.3.2 which differs from [Ro09].

For further research we strongly believe that Theorems 1.3.2 can be genralized to the following criterion.

Conjecture 1.3.6. *Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let* K *be a perfect field. The border basis scheme $\mathbb{B}_\mathcal{O}$ is isomorphic to an affine space, if there exists a spanning set $\bar{\mathcal{S}}$ of $\mathfrak{m}/\mathfrak{m}^2$ such that $\mathrm{Det}(\mathcal{M}_\mathcal{S}) = \pm 1$ holds.*

1.4 Outline

Above we presented our contributions and motivation in detail. The organization of every chapter is given at the beginning of the chapters, as well. Therefore in this section we avoid detailed explanations and provided a rough outline of the thesis.

Chapter 2 briefly introduces the border bases theory, the construction of border basis schemes, and the principal component of a border basis scheme. One can find a brief discussion on the basic algebraic geometry, as well. Moreover, the number of non-trivial generators of a box and of a segment border basis is given.

Chapter 3 focuses on the $\mathbb{Z}^m$-gradings. We define a $\mathbb{Z}^m$-grading on the ring $\mathrm{K}[\mathcal{C}]$ and call it the *arrow grading*. First we show that the arrow grading is neither of positive nor of non-negative type. Then we show that the only maximal ideal that is

homogenous in $\mathrm{K}[\mathcal{C}]$ with respect to the arrow grading is the ideal generated by the indeterminates $\{c_{11}, ..., c_{\mu\nu}\}$. We shall denote this ideal by $\mathfrak{M}$. Furthermore, we give a proof that the arrow grading is also defined on the coordinate ring of the border basis scheme $\mathrm{B}_{\mathcal{O}} = \mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})$. This implies the well-known fact that the vanishing ideal of a border basis scheme is generated by homogenous polynomials with respect to the arrow grading. Moreover, we show that the arrow grading is exactly the grading of the torus action on the coordinate ring $\mathrm{B}_{\mathcal{O}}$ which is mentioned in [Hai98]. Since we show that the arrow grading is neither of positive nor of negative type, our results contradict to the claims concerning grading and the homogenous maximal ideal on page 363 of [MilSt05].

Chapter 4 introduces a smoothness criterion for the monomial point $\mathfrak{o} = (0, ..., 0)$ of a given border basis scheme $\mathbb{B}_{\mathcal{O}}$. Therefore we focus on the maximal ideal $\mathfrak{m} = \langle \bar{c}_{11}, ..., \bar{c}_{\mu\nu} \rangle$ and investigate the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ of the border basis schemes $\mathbb{B}_{\mathcal{O}}$ at the point $\mathfrak{o}$. We reprove some results (see Chapter 3, of [Huib05]) for the Hilbert scheme of points in the setting of $\mathcal{O}$-border basis schemes. Although the main ideas of the proofs are the same, the difference in our approach comes from border basis theory. We use tools such as generic multiplication matrices, neighborhood relations of the border elements, properties of order ideals and, most importantly, the arrow grading. Despite the fact that in [Hai98] and [Huib05] a border basis scheme is described (see Equation 2.3 in [Hai98] and Section 2.3 in [Huib05]), the tools we have mentioned are not used. This distinguishes our work on constructing the generating set of the cotangent space of a border basis scheme. Note that the article [Huib05] is an unpublished arXiv article.

Chapter 5 discusses the *trace* and the *Jacobi identity syzygies* of $\mathcal{T}$ which were first introduced in [Huib09]. We prove some properties of those syzygies by using the *arrow grading*. Using Jacobi identity and trace syzygies, we compute the redundant generators of the box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$. We explain the reasons why these syzygies are not conclusive on whether $\mathbb{B}_{\mathcal{B}(2,2,2)}$ is a complete intersection. Section 5.4 is dedicated to finding a way to solve these problems for segment border basis schemes. Our strategy is the following: we start by generalizing the Jacobi identity syzygy and the trace syzygy computations to an arbitrary segment border basis scheme. Then, by using the arrow grading and the construction of the defining equations of the segment border basis scheme, we give a method to compute the redundant generators of the vanishing ideal of an arbitrary segment border basis scheme. Consequently, we prove that an arbitrary segment border basis scheme is an ideal-theoretic complete intersection.

Chapter 6 is dedicated to finding conditions for a general order ideal $\mathcal{O} \subset \mathbb{T}^n$ which make an $\mathcal{O}$-border basis scheme an affine cell, i.e. isomorphic to an affine space. If $\mathfrak{o}$ is a smooth point of the $\mathcal{O}$-border basis scheme, where $\mathcal{O} \subset \mathbb{T}^n$ has μ terms, then $\mathbb{B}_\mathcal{O}$ is locally isomorphic to an affine space of dimension $n\mu$. Our aim is to find a condition which extents this local property to all of $\mathbb{B}_\mathcal{O}$. The main results of this section (see Theorems 6.2.20) imply that this extension is possible. To be precise, we show that the coordinate ring $\mathrm{B}_\mathcal{O}$ is isomorphic to a polynomial ring of dimension $n\mu$ if there exists a set $\mathcal{S} \subset \mathcal{C}$ (see Chapter 4.2) and if there exists a maximal minor of the Jacobian matrix of $\mathcal{T}$ that is ± 1 and the matrix corresponding to this minor is in $\mathrm{Mat}_{\mu\nu-n\mu}(\mathrm{K}[\mathcal{S}])$. Moreover, we show that if a given $\mathcal{O}$-border basis scheme is an affine space and the vanishing ideal is generated by the subset $\mathcal{T}_\mathcal{S}$ (see Notation 6.2.3) of $\mathcal{T}$, then there exists a maximal minor of the Jacobian matrix of $\mathcal{T}$ that is ± 1. Since our results neither depend on the shape nor on n, they are more general than the existing results.

By using Theorem 6.2.20, in Section 6.4, we show that an arbitrary segment border basis scheme is isomorphic to an affine space of dimension $n\mu$. Section 6.4 discusses Theorem 7.3.1 of [Huib02] which states that $\mathcal{O}$-border basis schemes are affine spaces if $\mathcal{O}$ is of the form *sawtooth*. This section is mainly dedicated to prove this result by using Theorem 6.2.20, and to comparing the method given in [Huib02] to the method we use. Moreover, this application shows a way to employ Theorem 6.2.20 in a proof which gives a general condition of a special order ideal. In addition Theorem 6.2.20 is independent of the shape of the order ideal and the dimension of the ring that the order ideal is defined.

Chapter 2

Preliminaries

This chapter starts with known theorems and basic concepts in algebra and algebraic geometry (see Section 2.1) such as varieties, Zariski topology, dimension theory, smoothness and complete intersections without proofs. Our aim is to introduce the algebraic geometric background that is used in this thesis without going deep into the topics. We continue our summary on the basics by giving introductory definitions and theory of border bases (see Section 2.2). This section is important to see the connection between border basis theory and border basis schemes.

Afterwards, we explain how border basis schemes are constructed (see Definition 2.3.1.c and Construction 2.3.17) and recall preliminary results from the theory of border basis schemes focusing on the *monomial point* $\mathfrak{o}$ in these schemes (see Definition 2.3.7 and Lemma 2.3.6). Furthermore, in Section 2.4 we give a brief discussion on the *principal component* of a border basis scheme (see Definition 2.4.1). It should be noted that we give the explicit generating system of the vanishing ideal of a border basis scheme. Therefore we are able to use basic algebraic and algebraic geometric tools we introduce in Section 2.1 instead of scheme theoretic ones.

The last section (see Section 2.5) is dedicated to giving the exact number of non-trivial generators of the vanishing ideals of *box border basis schemes* (see Definition 2.5.1) and *segment border basis schemes* (see Definition 2.5.5). The importance of this section will become clear as we investigate the redundant elements of the set of defining equations of those border basis schemes in Chapters 5 and 6.

2.1 Algebraic Geometry

This section is intended to give an overview of the well- known algebraic and algebraic geometric tools we use throughout this thesis. Unless stated otherwise, throughout this section we let K be a field and $\bar{\mathrm{K}}$ the algebraic closure of K. Let $\mathrm{L} \supseteq \mathrm{K}$ be a field extension. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$.

Definition 2.1.1. Let f be a polynomial in P and let I be an ideal in P.

a) A point $(a_1, ..., a_n) \in \mathrm{L}^n$ is said to be a **zero of the polynomial** f in L^n if $f(a_1, ..., a_n) = 0$ holds.

b) The set of all zeros of f in L^n is denoted by $\mathcal{Z}_{\mathrm{L}}(f)$.

c) The set of simultaneous zeros of all polynomials of an ideal I in L^n is called the **zero-set of I** in L^n and is denoted by $\mathcal{Z}_{\mathrm{L}}(I)$.

For further information on zero-sets we refer to Chapter 2.6.B of [KrRo00]. Note that $\emptyset$ and L^n are zero-sets. Furthermore, arbitrary intersections and finite unions of zero-sets are again zero-sets. This implies that zero-sets form the closed sets of a topology.

Definition 2.1.2. Let L be equal to K. Then the topology formed by the zero-sets are called the **Zariski topology** on K^n. The space K^n with the Zariski topology is called the **n-dimensional affine space** over K^n and it is denoted by $\mathbb{A}^n_{\mathrm{K}}$.

Definition 2.1.3. Let V be a subset of L^n. The set of all polynomials $f \in \mathrm{P}$ such that $f(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in V$ forms an ideal of P. This ideal is called the **vanishing ideal** of V in P and is denoted by $\mathcal{I}(V)$. Moreover, the ring $\mathrm{P}/\mathcal{I}(V)$ is called **the coordinate ring** of V and it is denoted by $\mathrm{K}[\mathbf{V}]$.

Now we recall the well-known Hilbert's Nullstellensatz.

Theorem 2.1.4. *Let I be a proper ideal of $\bar{\mathrm{K}}[x_1, ..., x_n]$. Let $\sqrt{I}$ denote the radical of I. Then the following holds.*

$$\mathcal{I}(\mathcal{Z}(I)) = \sqrt{I}$$

Proof. The proof follows from [KrRo00], Theorem 2.6.16. □

Definition 2.1.5. Let V be a zero-set from L^n.

a) The zero-set V is called **irreducible** if the following holds: If V_1 and V_2 are zero-sets in L^n such that $V = V_1 \cup V_2$, then $V = V_1$ or $V = V_2$.

b) Let W be an irreducible zero-set in L^n where $W \subseteq V$. The zero-set W is called an **irreducible component** of V, if it is a maximal irreducible subset of V

Theorem 2.1.6. *Let V be a zero-set from L^n. The zero-set V has only finitely many irreducible components. Moreover, there exist finitely many irreducible zero-sets $V_1, ..., V_s$ of V such that $V = V_1 \cup ... \cup V_s$.*

Proof. The proof follows from [Ku85], Proposition 2.14 and Corollary 2.15. □

Theorem 2.1.7. *A zero-set $V \subseteq \mathrm{L}^n$ is irreducible if and only if its vanishing ideal $\mathcal{I}(V)$ is prime.*

Proof. The proof follows from [Ku85], Chapter 1, Proposition 1.10. □

Definition 2.1.8. Let $V \subseteq \mathrm{L}^n$ be a non-empty zero-set. The **Krull dimension** of V is the supremum of the lengths of all chains of non-empty distinct irreducible subsets of V. It is denoted by $\dim(V)$.

Note that the Krull dimension of a zero-set $V \subset \mathrm{L}^n$ is equal to the Krull dimension of the coordinate ring $\mathrm{K}[V]$ (see page 40, [Ku85]). That is

$$\dim(V) = \dim(\mathrm{K}[V]) = \dim(\mathrm{P}/\mathcal{I}(V)).$$

Definition 2.1.9. Let V be a zero-set in L^n with $\dim(V) = d$ where $1 \leqslant d < n$. Then $n - d$ is called the **codimension** of V and denoted by $\mathrm{codim}(V)$.

Definition 2.1.10. Let V be a zero-set from L^n with $\mathrm{codim}(V) = c$. The zero-set V is called an **ideal-theoretic complete intersection**, if $\mathcal{I}(V) \subset \mathrm{P}$ can be generated by c polynomials.

For further information on complete intersections we refer to [Ku85], Chapter 3.3.

Lemma 2.1.11. (Nakayama Lemma) *Let R be a ring and let I be an ideal that is contained in the intersection of all maximal ideals in R. Let M be an R-module and let N be a submodule of M and let M/N be finitely generated. If we have $M = N + IM$, then $M = N$ holds.*

Proof. This follows from [Ku85], Chapter IV, Lemma 2.2. □

Definition 2.1.12. A ring that has a unique maximal ideal, is called a **local ring.**

In the following we let V be a zero-set in L^n. Let $\mathfrak{o}$ denote the point $(0, ..., 0) \in \mathrm{L}^n$ and let V contain $\mathfrak{o}$. Let I denote the vanishing ideal $\mathcal{I}(V)$. Let P/I be denoted by R. Let $\mathfrak{m}$ be the maximal ideal generated by $\{\bar{x}_1, ..., \bar{x}_n\} \subset \mathrm{R}$.

Definition 2.1.13. The localization of R at the multiplicative set $S = \mathrm{R} \setminus \mathfrak{m}$ is called the **local ring of V at $\mathfrak{o}$** and is denoted by $\mathrm{R}_\mathfrak{m}$.

The ring $\mathrm{R}_\mathfrak{m}$ contains a unique maximal ideal, namely the ideal $\mathfrak{m}\mathrm{R}_\mathfrak{m}$. Let $\mathrm{L} = \mathrm{R}_\mathfrak{m}/\mathfrak{m}$ be the residue field of $\mathrm{R}_\mathfrak{m}$. Let $\mu(\mathfrak{m})$ denote the minimal number of generators of $\mathfrak{m}$. Then by Lemma 2.1.11 we have the following equality.

$$\mu(\mathfrak{m}\mathrm{R}_\mathfrak{m}) = \dim_\mathrm{L}(\mathfrak{m}/\mathfrak{m}^2) \tag{2.1}$$

Definition 2.1.14. The L-vector space $\mathfrak{m}/\mathfrak{m}^2$ is called the **cotangent space** at $\mathfrak{o}$. The number $\mu(\mathfrak{m}\mathrm{R}_\mathfrak{m})$ is called the **embedding dimension** of $\mathrm{R}_\mathfrak{m}$. It is denoted by $\mathrm{edim}(\mathrm{R}_\mathfrak{m})$.

As a consequence of Definition 2.1.14, Equation (2.1) is as follows:

$$\mathrm{edim}(\mathrm{R}_\mathfrak{m}) = \dim_\mathrm{L}(\mathfrak{m}/\mathfrak{m}^2) \tag{2.2}$$

Proposition 2.1.15. *The dimension of the local ring of V at $\mathfrak{o}$ is smaller or equal to the minimal number of generators of $\mathfrak{m}\mathrm{R}_\mathfrak{m}$. That is*

$$\dim(\mathrm{R}_m) \leqslant \mu(\mathfrak{m}\mathrm{R}_\mathfrak{m}) = \mathrm{edim}(\mathrm{R}_\mathfrak{m}).$$

Proof. This follows from [Ku85], Chapter IV, Theorem 3.4 (Generalized Krull Principal Ideal Theorem). □

The ring $\mathrm{R}_\mathfrak{m}$ is called a **regular local ring** if $\dim(\mathrm{R}_m)$ is equal to $\mathrm{edim}(\mathrm{R}_\mathfrak{m})$. Otherwise it is called a **singular local ring**. If $\mathrm{R}_\mathfrak{m}$ is a singular local ring then the point $\mathfrak{o}$ is said to be a **singular point** of V (see page 473 [KrRo05]).

Definition 2.1.16. A system of elements $\{a_1, ..., a_m\}$ $(m > 0)$ of a ring R is called *independent* if the following conditions hold:

a) $(a_1, ..., a_m) \neq R$

b) If $F \in R[x_1, ..., x_m]$ is a homogenous polynomial with $F(a_1, ..., a_m) = 0$, then all coefficients of F are contained in $\mathrm{Rad}(a_1, ..., a_m)$.

A system of elements $\{a_1, ..., a_m\}$ $(m > 0)$ of a ring R is called *algebraically independent* if the following conditions hold:

a) $(a_1, ..., a_m) \neq R$

b) If $F \in R[x_1, ..., x_m]$ is a homogenous polynomial with $F(a_1, ..., a_m) = 0$, then all coefficients of F are 0.

Definition 2.1.17. Let I be an ideal from P and be generated by $\{f_1, ..., f_s\}$. The **Jacobian matrix** of the tuple $\mathcal{F} = (f_1, ..., f_s)$ is the matrix $\mathrm{Jac}(\mathcal{F}) = (\frac{\partial f_i}{\partial x_j})_{i=1,..,s,j=1,...,n} \in \mathrm{Mat}_{s,n}(\mathrm{P})$. The Jacobian matrix of the tuple $\mathcal{F}$ that is evaluated at the point $\mathfrak{o} = (0, ..., 0) \in \mathrm{K}^n$ is denoted by $\mathrm{Jac}(\mathcal{F})_{\mathfrak{o}}$.

Theorem 2.1.18. *Let $V \subseteq \mathrm{L}^n$ and let $\mathcal{I}(V)$ denote the vanishing ideal from $\mathrm{P} = \mathrm{K}[x_1, ..., x_n]$ and let $\mathrm{char}(\mathrm{K}) = 0$. Then the following holds.*

a) Let $\mathcal{I}(V)$ be generated by a tuple $\mathcal{F} = (f_1, ..., f_s)$. Then $\mathrm{rank}(\mathrm{Jac}(\mathcal{F})_{\mathfrak{o}}) \leqslant n - \dim(\mathrm{R}_{\mathfrak{m}})$ holds.

b) Let $\mathcal{I}(V)$ be generated by a tuple $\mathcal{F} = (f_1, ..., f_s)$. If we have $\mathrm{rank}(\mathrm{Jac}(\mathcal{F})_{\mathfrak{o}}) = n - \dim(\mathrm{R}_{\mathfrak{m}})$, then $\mathrm{R}_{\mathfrak{m}}$ is regular.

c) The local ring $\mathrm{R}_{\mathfrak{m}}$ is regular if and only if there exists a tuple of polynomials $\mathcal{F} = (f_1, ..., f_s)$, which generates $\mathcal{I}(V)$ and we have $\mathrm{rank}(\mathrm{Jac}(\mathcal{F})_{\mathfrak{o}}) = n - \dim(\mathrm{R}_{\mathfrak{m}})$.

Proof. This follows from [Ku85], Chapter VI, Proposition 1.5. □

Theorem 2.1.19. *Let $V \subset \mathrm{L}^n$ be irreducible with dimension d and let $\mathcal{I}(V)$ denote the vanishing ideal in $\mathrm{P} = \mathrm{K}[x_1, ..., x_n]$ that is generated by a tuple $\mathcal{F} = (f_1, ..., f_s)$. Then the singular locus of V is the common zero set in V of the polynomials obtained as the $(n-d)\times(n-d)$ minors of the Jacobian matrix $\mathrm{Jac}(\mathcal{F}) = (\frac{\partial f_i}{\partial x_j})_{i=1,..,s,j=1,...,n} \in \mathrm{Mat}_{s,n}(\mathrm{P})$.*

Proof. This follows from [KaKeSt00], Chapter 6.2, Theorem. □

Definition 2.1.20. Let J be the ideal generated by the minors of size $n-d$ in Theorem 2.1.19. Then the ideal $\bar{J}$ in $\mathrm{P}/\mathcal{I}(V)$ is called the **Jacobian ideal** of $\mathcal{I}(V)$ and denoted by $\mathfrak{J}$.

Corollary 2.1.21. *Let $V \subset \mathrm{L}^n$ be irreducible with dimension d and let $\mathcal{I}(V)$ denote the vanishing ideal in $\mathrm{P} = \mathrm{K}[x_1, ..., x_n]$ that is generated by a tuple $\mathcal{F} = (f_1, ..., f_s)$. The Jacobian ideal $\mathfrak{J}$ is a unit ideal of $\mathrm{P}/\mathcal{I}(V)$ if and only if V is smooth.*

2.2 Border Bases

In order to understand the theory of border basis schemes, it is necessary to discuss border basis theory. Although we shall not go deeply into the theory, we discuss the basic concepts necessary for the rest of this thesis. For further information we refer to [KrRo05].

Definition 2.2.1. Let $I \subset \mathrm{P}$ be a zero-dimensional ideal. A polynomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathrm{P}$ such that $(\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ is called a **term**. Let $\mathbb{T}^n$ denote the set of all terms in P and let $\mathbb{T}_1^n = \{x_1, ..., x_n\}$.

(a) Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ denote a non-empty set of terms from $\mathbb{T}^n$. If $\mathcal{O}$ is closed under forming divisors, then it is called an **order ideal**.

(b) The set of terms defined as $\mathbb{T}_1^n \cdot \mathcal{O} \backslash \mathcal{O} = (x_1\mathcal{O} \cup ... \cup x_n\mathcal{O}) \backslash \mathcal{O}$ is called the **border of** $\mathcal{O}$ and denoted by $\partial\mathcal{O}$.

(c) An $\mathcal{O}$-border **prebasis** $G = \{g_1, ..., g_\nu\}$ is a set that consists of polynomials of the form

$$g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i$$

where $\{\alpha_{ij}\}_{\substack{i \in \{1,..,\mu\} \\ j \in \{1,..\nu\}}} \subset \mathrm{K}$.

(d) Let G be an $\mathcal{O}$-border prebasis. Then G is called an **$\mathcal{O}$-border basis** of I if the residue classes of the elements of $\mathcal{O}$ form a K-basis of P/I.

The basic idea of border basis theory is to describe a zero-dimensional ring P/I by an order ideal of terms $\mathcal{O}$ whose residue classes form a K-vector space basis of P/I.

Example 2.2.2. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2, x_1x_2\} \in \mathrm{K}[x_1, x_2]$. Then the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$. The figure below justifies the name border.

Example 2.2.3. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2, x_1x_2\} \in \mathrm{K}[x_1, x_2]$. Then the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$. Let I be an ideal of $\mathrm{K}[x_1, x_2]$ that is generated

$\mathbf{x_2^2}$	$\mathbf{x_1x_2^2}$	
x_2	x_1x_2	$\mathbf{x_1^2x_2}$
1	x_1	$\mathbf{x_1^2}$

by $G = (g_1, g_2, g_3, g_4)$ where

$$g_1 = x_2^2 + x_1 + 1, \qquad g_2 = x_1^2 + x_1 + x_2,$$
$$g_3 = x_1x_2^2 - x_2, \qquad g_4 = x_1^2x_2 + x_1x_2 - x_1 - 1.$$

Then G is an $\mathcal{O}$-border prebasis of I.

In Example (2.2.3) if $\mathcal{O}$ is the K-basis of P/I, then G is an $\mathcal{O}$-border basis of I. The following definition gives us one tool to verify whether the given $\mathcal{O}$-border prebasis is an $\mathcal{O}$-border basis or not.

Definition 2.2.4. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal. Then $\partial\mathcal{O} = \{b_1, ..., b_\upsilon\}$ be the border of $\mathcal{O}$. For indices $i = 1, ..., \mu$ and $j = 1, .., \nu$, we let α_{ij} be from K. Finally, we let $G = \{g_1, ..., g_\upsilon\}$ be the $\mathcal{O}$-border prebasis with

$$g_j = b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i.$$

For $1 \leqslant r \leqslant n$, define the r^{th} **formal multiplication matrix** $\mathcal{A}_r = (\xi_{kl}^{(r)})$ of G by

$$\xi_{kl}^{(r)} = \begin{cases} \delta_{ki} & \text{if } t_i = x_r t_l \\ \alpha_{kj} & \text{if } b_j = x_r t_l \end{cases}$$

where δ_{ki} is the Kronecker delta.

The next theorem gives a very important property of multiplication matrices. This property will play a crucial role in constructing the defining equations of border basis schemes (see Definition 2.3.1).

Theorem 2.2.5. *Let $\mathcal{O}$ be an order ideal and let G be an $\mathcal{O}$-border prebasis and let I be a zero-dimensional ideal generated by the elements of G. The set G is an $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices of G pairwise commute.*

Proof. This follows from [KrRo05], Theorem 6.4.30. □

Example 2.2.6. (continues) We show that the $\mathcal{O}$-border prebasis in Example (2.2.3)

is an $\mathcal{O}$-border basis of I. With respect to our notation we have $t_1 = 1$, $t_2 = x_1$, $t_3 = x_2$, $t_4 = x_1x_2$, $b_1 = x_2^2$, $b_2 = x_1^2$, $b_3 = x_1x_2^2$ and finally $b_4 = x_1^2x_2$. Let us construct the first and second formal multiplication matrices.

$$\begin{aligned}
g_1 = x_2^2 + x_1 + 1 = b_1 - \alpha_{11}t_1 - \alpha_{21}t_2, &\qquad \alpha_{11} = -1, \quad \alpha_{21} = -1 \\
g_2 = x_1^2 + x_1 + x_2 = b_2 - \alpha_{22}t_2 - \alpha_{32}t_3, &\qquad \alpha_{22} = -1, \quad \alpha_{32} = -1 \\
g_3 = x_1x_2^2 - x_2 = b_3 - \alpha_{33}t_3, &\qquad \alpha_{33} = 1 \\
g_4 = x_1^2x_2 + x_1x_2 - x_1 - 1 = b_4 - \alpha_{44}t_4 - \alpha_{24}t_2 - \alpha_{14}t_1, &\qquad \alpha_{44} = -1, \alpha_{24} = 1, \alpha_{14} = 1
\end{aligned}$$

$$\begin{aligned}
k = 1, \xi_{11}^{(1)} = \delta_{12} \quad \xi_{12}^{(1)} = \alpha_{12} \quad \xi_{13}^{(1)} = \delta_{14} \quad \xi_{14}^{(1)} = \alpha_{14} \\
k = 2, \xi_{21}^{(1)} = \delta_{22} \quad \xi_{22}^{(1)} = \alpha_{22} \quad \xi_{23}^{(1)} = \delta_{24} \quad \xi_{24}^{(1)} = \alpha_{24} \\
k = 3, \xi_{31}^{(1)} = \delta_{32} \quad \xi_{32}^{(1)} = \alpha_{32} \quad \xi_{33}^{(1)} = \delta_{34} \quad \xi_{34}^{(1)} = \alpha_{34} \\
k = 4, \xi_{41}^{(1)} = \delta_{42} \quad \xi_{42}^{(1)} = \alpha_{42} \quad \xi_{43}^{(1)} = \delta_{44} \quad \xi_{44}^{(1)} = \alpha_{44}
\end{aligned}$$

$$\mathcal{A}_{\mathbf{1}} = \begin{pmatrix} \delta_{12} & \alpha_{12} & \delta_{14} & \alpha_{14} \\ \delta_{22} & \alpha_{22} & \delta_{24} & \alpha_{24} \\ \delta_{32} & \alpha_{32} & \delta_{34} & \alpha_{34} \\ \delta_{42} & \alpha_{42} & \delta_{44} & \alpha_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

In the same way, we get the generic multiplication matrix $\mathcal{A}_2$.

$$\mathcal{A}_{\mathbf{2}} = \begin{pmatrix} \delta_{13} & \delta_{14} & \alpha_{11} & \alpha_{13} \\ \delta_{23} & \delta_{24} & \alpha_{21} & \alpha_{23} \\ \delta_{33} & \delta_{34} & \alpha_{31} & \alpha_{33} \\ \delta_{43} & \delta_{44} & \alpha_{41} & \alpha_{43} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ are pairwise commutative i.e., $\mathcal{A}_1\mathcal{A}_2 = \mathcal{A}_2\mathcal{A}_1$. Then by Theorem 2.2.5, $G = \{g_1, g_2, g_3, g_4\}$ is an $\mathcal{O}$-border basis.

2.3 The Border Basis Scheme

An $\mathcal{O}$-border basis scheme is a scheme that parametrizes all zero-dimensional ideals that have an $\mathcal{O}$-border basis. Therefore it is an open sub-scheme of the Hilbert scheme $H_n^\mu = Hilb^\mu(\mathbb{A}^n)$ of μ points. In this section, instead of handling an $\mathcal{O}$-border basis scheme in a scheme theoretic manner, we construct the the generators of the vanishing

ideal of an $\mathcal{O}$-border basis scheme in a concrete way by using the border basis theory.

Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be an order ideal and $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ be its border. Let $I \subset \mathrm{P}$ be a zero-dimensional ideal with an $\mathcal{O}$-border basis. Let $\{c_{11}, ..., c_{\mu\nu}\}$ be set of further indeterminates and let $\mathrm{K}[\mathcal{C}]$ denote the polynomial ring $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$.

Definition 2.3.1. The **generic $\mathcal{O}$-border prebasis** is the set of polynomials $G = \{g_1, ..., g_\nu\}$ in $\mathrm{K}[x_1, .., x_n, c_{11}, ..., c_{\mu\nu}]$ given by

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i. \tag{2.3}$$

Definition 2.3.2. Let $\mathcal{O} \subset \mathrm{P}$ denote an order ideal. Let G be the generic $\mathcal{O}$-border prebasis.

a) Let $\mathcal{A}_k \in \mathrm{Mat}_\mu(\mathrm{K}[c])$ be the k^{th} formal multiplication matrix associated to G (see Definition 2.2.4) where $k \in \{1, ..., n\}$. Then $\mathcal{A}_k$ is called the k^{th} **generic multiplication matrix** with respect to $\mathcal{O}$.

b) The affine scheme $\mathbb{B}_\mathcal{O} \subseteq \mathbb{A}^{\mu\nu}$ defined by the ideal $\mathcal{I}(\mathbb{B}_\mathbb{O})$ generated by the entries of the matrices $\mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k$ with $1 \leq k < l \leq n$ is called the **$\mathcal{O}$-border basis scheme**.

c) The ring $\mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{B}_\mathcal{O})$ is called the **coordinate ring** of the scheme $\mathbb{B}_\mathcal{O}$ and is denoted by $\mathrm{B}_\mathcal{O}$.

Lemma 2.3.3. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, ..., x_n]$ *and let* $\mathcal{O}$ *be an order ideal. Let* t_q *and* t_p *be distinct terms from the order ideal* $\mathcal{O}$ *such that* $x_l t_q = t_p$. *Let* l *be an index from the set* $\{1, ..., n\}$ *and* $\mathcal{A}_l$ *be the generic multiplication matrix with respect to* $\mathcal{O}$. *Then we have* $\mathcal{A}_l e_q^{tr} = e_p^{tr}$.

Proof. This is a result of the definition of the generic multiplication matrices with respect to $\mathcal{O}$. □

In this example we show how to construct an $\mathcal{O}$-border basis scheme with a given order ideal $\mathcal{O}$. We explicitly give the generating system of the vanishing ideal. The main idea is to construct the vanishing ideal of $\mathbb{B}_\mathcal{O}$ by imposing Theorem 2.2.5 to the generic multiplication matrices with respect to the order ideal $\mathcal{O}$.

Example 2.3.4. Let P denote the polynomial ring $\mathrm{K}[x_1, x_2]$. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2, x_1x_2\} \in \mathrm{P}$. Then the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$. With respect to our notation we have $t_1 = 1$, $t_2 = x_1$, $t_3 = x_2$, $t_4 = x_1x_2$, $b_1 = x_2^2$, $b_2 = x_1^2$, $b_3 = x_1x_2^2$ and finally $b_4 = x_1^2x_2$. The generic $\mathcal{O}$-border prebasis is

$$\begin{aligned} G \;=\; \{&x_2^2 - c_{11} - c_{21}x_1 - c_{31}x_2 - c_{41}x_1x_2,\\ &x_1^2 - c_{12} - c_{22}x_1 - c_{32}x_2 - c_{42}x_1x_2,\\ &x_1x_2^2 - c_{13} - c_{23}x_1 - c_{33}x_2 - c_{43}x_1x_2,\\ &x_1^2x_2 - c_{14} - c_{24}x_1 - c_{34}x_2 - c_{44}x_1x_2\}. \end{aligned}$$

Now let us construct the generic multiplication matrices with respect to $\mathcal{O}$ which is basically constructing formal multiplication matrices of the generic $\mathcal{O}$-border prebasis G.

$$A_1 = \begin{pmatrix} 0 & c_{12} & 0 & c_{14} \\ 1 & c_{22} & 0 & c_{24} \\ 0 & c_{32} & 0 & c_{34} \\ 0 & c_{42} & 1 & c_{44} \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & c_{11} & c_{13} \\ 0 & 0 & c_{21} & c_{23} \\ 1 & 0 & c_{31} & c_{33} \\ 0 & 1 & c_{41} & c_{43} \end{pmatrix}$$

The vanishing ideal of the border basis scheme that is generated by the entries of $(\mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_2\mathcal{A}_1)$ is the following ideal.

$$\begin{aligned} \mathcal{I}(\mathbb{B}_{\mathcal{O}}) \;=\; \langle &c_{11}c_{32} + c_{13}c_{42} - c_{14}, c_{12}c_{21} + c_{14}c_{41} - c_{13},\\ &c_{12}c_{23} - c_{11}c_{34} + c_{14}c_{43} - c_{13}c_{44}, c_{21}c_{32} + c_{23}c_{42} - c_{24},\\ &c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23}, c_{22}c_{23} - c_{21}c_{34} + c_{24}c_{43} - c_{23}c_{44} + c_{13},\\ &c_{31}c_{32} + c_{33}c_{42} + c_{12} - c_{34}, c_{21}c_{32} + c_{34}c_{41} - c_{33},\\ &c_{23}c_{32} - c_{31}c_{34} + c_{34}c_{43} - c_{33}c_{44} - c_{14}, c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44},\\ &c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}, c_{34}c_{41} - c_{23}c_{42} + c_{24} - c_{33} \rangle \end{aligned}$$

Definition 2.3.5. A **monomial ideal** is an ideal generated by a set of terms from $\mathbb{T}^n$. The monomial ideal generated by the elements of $\partial\mathcal{O}$ is called the **border term ideal** of $\mathcal{O}$ and is denoted by $\mathrm{BT}_{\mathcal{O}}$.

The next two lemmas can be found in [KrRo08] as notes. In order to make clear reference, we turn them into lemmas and give explicit proofs.

Lemma 2.3.6. *Let $\mathcal{O} \subset \mathrm{P}$ be an order ideal. There is a one-to-one correspondence between the zero-dimensional ideals that have an $\mathcal{O}$-border basis and the* K*-rational points of the border basis scheme. In particular, the point $\mathfrak{o} = (0, ..., 0)$ in the scheme corresponds to the border term ideal.*

Proof. Let σ be the substitution homomorphism defined as follows,

$$\begin{aligned} \sigma : \mathrm{K}[c_{11}, ..., c_{\mu\nu}, x_1, ..., x_n] &\longrightarrow \mathrm{K}[x_1, ..., x_n] \\ c_{ij} &\longmapsto \alpha_{ij} \\ x_k &\longmapsto x_k \end{aligned}$$

where $\alpha_{ij} \in \mathrm{K}$ and $k = 1, ..., n$. Let G be the generic $\mathcal{O}$-border prebasis. By Equation (2.3), the image of G under σ is

$$\sigma(G) = \{b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i \mid j = 1, ..., \nu\} \tag{2.4}$$

which is an $\mathcal{O}$-border prebasis for some zero-dimensional ideal in P, say I. Let $\mathcal{A}_k, \mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be two distinct generic multiplication matrices with respect to $\mathcal{O}$. The images of generic multiplication matrices $\mathcal{A}_k, \mathcal{A}_l$ under the map σ are the k^{th} and l^{th} formal multiplication matrices with respect to the prebasis $\sigma(G)$. By Theorem 2.2.5, $\sigma(G)$ is an $\mathcal{O}$-border basis of I if and only if the formal multiplication matrices commute. Then $\sigma(G)$ is an $\mathcal{O}$-border basis if and only if for each $k, l \in \{1, ..., n\}$ and $k \neq l$

$$\sigma(\mathcal{A}_k\mathcal{A}_l) = \sigma(\mathcal{A}_k)\sigma(\mathcal{A}_l) = \sigma(\mathcal{A}_l)\sigma(\mathcal{A}_k)$$

holds. By Theorem 2.2.5, this holds if and only if the point $(\alpha_{11}, ..., \alpha_{\mu\nu})$ is a point in $\mathbb{B}_\mathcal{O}$. In particular, after plugging the point $(0, ..., 0)$ in G we have the ideal $\langle b_1, ..., b_\nu \rangle$, which is the border term ideal. □

Definition 2.3.7. A point in the scheme $\mathbb{B}_\mathcal{O}$ is called a **monomial point** if it corresponds to a monomial ideal.

Lemma 2.3.8. *Let $\mathcal{O} \subset \mathrm{P}$ be an order ideal. The point $\mathfrak{o} = (0, ..., 0)$ in $\mathbb{B}_\mathcal{O}$ is the only monomial point of this border basis scheme.*

Proof. By Lemma 2.3.6, the point $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ corresponds to the border term ideal $\langle b_1, ..., b_\nu \rangle$. We do the proof by showing that the border term ideal is the only monomial ideal with $\mathcal{O}$-border basis. For a contradiction, suppose there is a monomial zero-dimensional ideal I with an $\mathcal{O}$-border basis that is different than $\mathrm{BT}_{\mathcal{O}}$. We let

$$\{b_j - \sum_{i=1}^{\mu} \alpha_{ij} t_i\}_{\substack{j=1,...,\nu \\ i=1,...,\mu}},$$

where $\alpha_{ij} \in \mathrm{K}$ for $i = 1, ..., \mu$ and $j = 1, ..., \nu$ be the $\mathcal{O}$-border basis of I. Since I is a monomial ideal, for every $i = 1, ..., \mu$ and $j = 1, ..., \nu$ we have $\alpha_{ij} = 0$. But this contradicts to the assumption that I is different than $\mathrm{BT}_{\mathcal{O}}$. □

In Example (2.3.4) we constructed the defining equations for $\mathbb{B}_{\mathcal{O}}$ by imposing the Theorem 2.2.5 to the generic multiplication matrices. Before introducing an alternative construction method which employs the relations of border terms, let us give a closer look what those border relations are and how they can be used to compute polynomials in $\mathrm{K}[\mathcal{C}]$.

Definition 2.3.9. Let $G = \{g_1, ..., g_\nu\}$ be a generic $\mathcal{O}$-border prebasis where $g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i \in \mathrm{K}[c_{11}, ..., c_{\mu\nu}, x_1, ..., x_n]$. Let $\mathcal{A}_1, ..., \mathcal{A}_n \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be the generic multiplication matrices with respect to $\mathcal{O}$. Finally, for each index $j \in \{1, ..., \mu\}$ let c_j denote the column matrix $(c_{1j}, ..., c_{\mu j})^{tr} \in \mathrm{Mat}_{\mu 1}(\mathrm{K}[\mathcal{C}])$. Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms.

a) The border terms b_i, b_j are called **next-door neighbors** if we have $b_i = x_k b_j$ for some $x_k \in \{x_1, ..., x_n\}$. We let $\mathrm{ND}(i, j)$ be the set of polynomial entries of $c_i - \mathcal{A}_k c_j$ where b_i and b_j are next-door neighbors. We shall denote the union of all such sets by ND.

b) The border terms b_i, b_j are called **across-the-street neighbors** if we have $x_k b_i = x_l b_j$. We let $\mathrm{AS}(i, j)$ be the set of polynomial entries of $\mathcal{A}_k c_i - \mathcal{A}_l c_j$ if b_i and b_j are across-the-street-neighbors We shall denote the union of all such sets by AS.

c) Let b_i and b_j be across-the-corner neighbors such that $x_k b_i = x_l b_j$. If there exists $b_m \in \partial\mathcal{O}$ with $b_i = x_l b_m$ and $b_j = x_k b_m$, then b_i and b_j are called the **across-the-corner-neighbors**.

Lemma 2.3.10. *Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms. If $b_i = x_k b_j$ holds, then there exists $t_q \in \mathcal{O}$ for some $q \in \{1, ..., \mu\}$ and $x_l \in \{x_1, ..., x_n\}$ such that $x_k t_q \in \mathcal{O}$ and $x_k x_l t_q = b_i$.*

Proof. This is a result of the Definitions 2.2.1 and 2.3.9.a. □

Lemma 2.3.11. *Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms. If $x_l b_i = x_k b_j$ holds, then there exists $t_q \in \mathcal{O}$ for some $q \in \{1, ..., \mu\}$ such that $x_k t_q = b_i$ and $x_l t_q = b_j$.*

Proof. This is a result of the Definitions 2.2.1 and 2.3.9.b. □

Proposition 2.3.12. *Recall the setting in Definition 2.3.9. The union of all sets in* ND *and all sets in* AS *contains the non-trivial entries of the commutators $\mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k$ with $1 \leqslant k < l \leqslant n$.*

a) If one removes from this union all sets in AS *such that b_i, b_j are across-the-corner neighbors, one gets precisely the set of the non-trivial entries of the commutators $\mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k$ with $1 \leqslant k < l \leqslant n$. In particular, the remaining union generates the vanishing ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$.*

b) The polynomials in the sets of AS *corresponding to across-the-corner neighbors b_i, b_j are contained in $\mathcal{I}(\mathbb{B}_\mathcal{O})$.*

Proof. This follows from [KrRo08], Proposition 4.1. □

Definition 2.3.13. Recall the setting given in Definition 2.3.9. We shall name the polynomials that are from the union of AS and ND as **neighbor polynomials** and denote the set of neighbor polynomials by θ.

Lemma 2.3.14. *Recall the setting given in Definition 2.3.9. Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct terms from $\partial\mathcal{O}$ such that $x_k b_j = b_i$. Let $\theta_\kappa \in \mathrm{ND}(i, j)$ be the polynomial in p^{th} row of the matrix $c_i - \mathcal{A}_k c_j$. Then for every $r \in \{1, ..., \mu\}$ we have*

$$\theta_\kappa = c_{pi} - (\textstyle\sum_{r=1}^{\mu} f_r c_{rj}) \quad \textit{with } f_r = \begin{cases} 0 & \textit{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \textit{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pm} & \textit{if } x_k t_r = b_m \in \partial\mathcal{O}. \end{cases}$$

Proof. The p^{th}-row of $c_i - \mathcal{A}_k c_j$ is equal to $e_p(c_i - \mathcal{A}_k c_j)$. Then we have

$$\theta_\kappa = c_{pi} - (\sum_{r=1}^{\mu} f_r c_{rj}).$$

We focus on the p^{th}-row of the generic multiplication matrix $\mathcal{A}_k$. As a result of Defini-

tion 2.2.4, we have

$$(\mathcal{A}_k)_{pr} = \begin{cases} \delta_{sp} & \text{if } x_k t_r = t_s \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O} \end{cases}$$

where δ_{sp} is the Kronecker delta. Then we have

$$f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O}. \end{cases}$$

□

Lemma 2.3.15. *Recall the setting given in Definition 2.3.9. Let $b_i, b_j \in \partial\mathcal{O}$ be two distinct border terms where $x_k b_j = x_l b_i$. If $\theta_\kappa \in \mathrm{AS}(i,j)$ is the polynomial in the p^{th}-row of the matrix $\mathcal{A}_k c_j - \mathcal{A}_l c_i$, then for every $r \in \{1, ..., \mu\}$ we have*

$$\theta_\kappa = -(\sum_{s=1}^{\mu} h_s c_{si}) + (\sum_{r=1}^{\mu} f_r c_{rj}), \quad \textit{where}$$

$$f_r = \begin{cases} 0 & \textit{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \textit{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pn} & \textit{if } x_k t_r = b_n \in \partial\mathcal{O} \end{cases} \quad \textit{and} \quad h_s = \begin{cases} 0 & \textit{if } x_l t_s \in \mathcal{O} \setminus \{t_p\} \\ 1 & \textit{if } x_l t_s = t_p \in \mathcal{O} \\ c_{pm} & \textit{if } x_l t_s = b_m \in \partial\mathcal{O}. \end{cases}$$

Proof. The p^{th}-row of $\mathcal{A}_k c_j - \mathcal{A}_l c_i$ is equal to $e_p(\mathcal{A}_k c_j - \mathcal{A}_l c_i)$. Then we have

$$\theta_\kappa = -(\sum_{s=1}^{\mu} h_s c_{si}) + (\sum_{r=1}^{\mu} f_r c_{rj}).$$

We focus on the p^{th}-row of the generic multiplication matrices $\mathcal{A}_k$ and $\mathcal{A}_l$ to find h_s and f_r. As a result of Definition 2.2.4, we have

$$(\mathcal{A}_k)_{pr} = \begin{cases} \delta_{up} & \text{if } x_k t_r = t_u \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O}, \end{cases} \quad (\mathcal{A}_l)_{ps} = \begin{cases} \delta_{up} & \text{if } x_l t_s = t_u \in \mathcal{O} \\ c_{pm} & \text{if } x_l t_s = b_n \in \partial\mathcal{O} \end{cases}$$

where δ_{up} is the Kronecker delta. Then we have

$$f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O}. \end{cases} \quad \textit{and} \quad h_s = \begin{cases} 0 & \text{if } x_l t_s \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_l t_s = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_l t_s = b_m \in \partial\mathcal{O}. \end{cases}$$

□

Let k and l be two distinct integers from $\{1, ..., n\}$ where $k < l$. We denote a polynomial entry of the commutator operation $[\mathcal{A}_k, \mathcal{A}_l] = \mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k$ in position (p, q) by τ_{pq}^{kl} where $p, q \in \{1, ..., \mu\}$. Then the result of this commutator operation will be as follows.

$$[\mathcal{A}_k, \mathcal{A}_l] = \mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k = \begin{pmatrix} \tau_{11}^{kl} & \cdots & \tau_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{kl} & \cdots & \tau_{\mu\mu}^{kl} \end{pmatrix} \tag{2.5}$$

Let $\mathcal{T}$ be the set of all polynomial entries of the multiplication matrices, i.e.

$$\mathcal{T} = \{\tau_{pq}^{kl} \mid k \neq l \ \ \textit{and} \ \ k, l \in \{1, ..., n\}, \ \ p, q \in \{1, ..., \mu\}\}. \tag{2.6}$$

Then we have $\mathcal{I}(\mathbb{B}_\mathcal{O}) = \langle \mathcal{T} \rangle$.

Remark 2.3.16. By Proposition 2.3.12 the vanishing ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ can be generated by the lifting neighbor syzygies and by imposing Theorem 2.2.5 to the generic multiplication matrices with respect to order ideal $\mathcal{O}$ (see Example 2.3.4 and Proposition 2.3.12). The above enumeration on the elements of set $\mathcal{T}$ allows us to create a link between these two methods.

This approach is given in Section 2 of [Huib09]. Since we use it frequently, we want to revise the construction with our notation.

Construction 2.3.17. Recall the setting given in Definition 2.3.9 and Equality (2.6). Let t_q, t_p be two distinct terms from $\mathcal{O}$. By Equation (2.5), we have $\tau_{pq}^{kl} = e_p(\mathcal{A}_l\mathcal{A}_k - \mathcal{A}_k\mathcal{A}_l)e_q^{tr}$. We first show explicitly that τ_{pq}^{kl} is 0 if it is contained in neither ND nor AS. If this is not the case, we give the shape of the polynomial τ_{pq}^{kl} by using Lemmas 2.3.14 and 2.3.15.

a) If $x_k x_l t_q = t_j \in \mathcal{O}$ with $x_l t_q = t_p \in \mathcal{O}$ and $x_k t_q = t_r \in \mathcal{O}$ holds, then τ_{pq}^{kl} is neither

in ND nor in AS and the following holds.

$$\begin{aligned} e_p\mathcal{A}_k\mathcal{A}_l e_q^{tr} &= e_p\mathcal{A}_k e_p^{tr} = 0 \\ e_p\mathcal{A}_l\mathcal{A}_k e_q^{tr} &= 0. \end{aligned}$$

Hence we have $\tau_{pq}^{kl} = e_p\mathcal{A}_k\mathcal{A}_l e_q^{tr} - e_p\mathcal{A}_l\mathcal{A}_k e_q^{tr} = 0$.

b) If we have $x_k x_l t_q = b_i \in \partial\mathcal{O}$ with $x_l t_q = t_p \in \mathcal{O}$ and $x_k t_q = t_r \in \mathcal{O}$, then τ_{pq}^{kl} is neither in ND nor in AS. We show that this polynomial is 0 in this case as in the previous case.

$$\begin{aligned} e_p\mathcal{A}_k\mathcal{A}_l e_q^{tr} &= e_p\mathcal{A}_k e_p^{tr} = c_{ri} \\ e_p\mathcal{A}_l\mathcal{A}_k e_q^{tr} &= c_{ri} \end{aligned}$$

Hence we have $\tau_{pq}^{kl} = e_p\mathcal{A}_k\mathcal{A}_l e_q^{tr} - e_p\mathcal{A}_l\mathcal{A}_k e_q^{tr} = 0$.

c) If $x_k t_q = t_r \in \mathcal{O}$ and $x_l t_q = b_j \in \partial\mathcal{O}$ with $x_l(x_k t_q) = x_l t_r = b_i \in \partial\mathcal{O}$ holds, then $x_k b_j = b_i$, i.e. b_i and b_j are next-door neighbors and $\tau_{pq}^{kl} \in \mathrm{ND}(i,j)$. In order to see this, consider the following.

$$\begin{aligned} \tau_{pq}^{kl} &= e_p(\mathcal{A}_l\mathcal{A}_k - \mathcal{A}_k\mathcal{A}_l)e_q^{tr} \\ &= e_p(\mathcal{A}_l\mathcal{A}_k)e_q^{tr} - e_p(\mathcal{A}_k\mathcal{A}_l)e_q^{tr} \\ &= e_p\mathcal{A}_l e_r^{tr} - e_p\mathcal{A}_k\mathcal{A}_l e_q^{tr} \\ &= e_p \begin{pmatrix} c_{1i} \\ \vdots \\ c_{\mu i} \end{pmatrix} - e_p\mathcal{A}_k \begin{pmatrix} c_{1j} \\ \vdots \\ c_{\mu j} \end{pmatrix} \\ &= c_{pi} - e_p\mathcal{A}_k \begin{pmatrix} c_{1j} \\ \vdots \\ c_{\mu j} \end{pmatrix} \end{aligned}$$

This is the p^{th} row of $c_i - \mathcal{A}_k c_j$. Then by Lemma 2.3.14, we have

$$c_{pi} - \sum_{r=1}^{\mu} f_r c_{rj} \quad \text{where} \quad f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O}. \end{cases}$$

d) If we have $x_k t_q = b_i \in \partial\mathcal{O}$ and $x_l t_q = b_j \in \partial\mathcal{O}$, i.e. $x_k b_j = x_l b_i$, then b_i and b_j are across-the-street neighbors and $\tau_{pq}^{kl} \in \mathrm{AS}(i,j)$. To see this consider the following.

$$\begin{aligned} \tau_{pq}^{kl} &= e_p(\mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k)e_q^{tr} \\ &= e_p(\mathcal{A}_k\mathcal{A}_l)e_q^{tr} - e_p(\mathcal{A}_l\mathcal{A}_k)e_q^{tr} \\ &= e_p(\mathcal{A}_k c_j - \mathcal{A}_l c_i) \end{aligned}$$

That is the p^{th}-row of the matrix $\mathcal{A}_k c_j - \mathcal{A}_l c_i$, whose entries give the elements of the set $\mathrm{AS}(i,j)$. Then by Lemma 2.3.15, we have the following

$$\tau_{pq}^{kl} = -\sum_{s=1}^{\mu} h_s c_{si} + \sum_{r=1}^{\mu} f_r c_{rj} \quad \text{where}$$

$$f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pn} & \text{if } x_k t_r = b_n \in \partial\mathcal{O}, \end{cases} \quad \text{and} \quad h_s = \begin{cases} 0 & \text{if } x_l t_s \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_l t_s = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_l t_s = b_m \in \partial\mathcal{O}. \end{cases}$$

Remark 2.3.18. Construction 2.3.17 is in agreement with Proposition 2.3.12.

In order to avoid index problems, in the next lemma we denote an arbitrary polynomial in $\mathcal{T}$ simply by τ_κ instead of τ_{pq}^{kl}. We use this notation when the information on how the polynomial τ_{pq}^{kl} is constructed is not necessary.

Lemma 2.3.19. *The polynomials from the set $\mathcal{T}$ are of degree two with no constant. In particular, a polynomial, say τ_κ, is of the following form:*

$$\tau_\kappa = \tau_\kappa^{(2)} + \tau_\kappa^{(1)} \in \mathcal{T} \tag{2.7}$$

where $\tau_\kappa^{(2)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 2 with respect to the standard grading and $\tau_\kappa^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 1 with respect to the standard grading. Moreover, the set $\mathrm{supp}(\tau_\kappa^{(1)})$ has at most two linear monomials.

Proof. This follows from Construction 2.3.17. □

2.4 The Principal Component

Our aim in this section is to familiarize the reader with the concept and the properties of the principal component of an $\mathcal{O}$-border basis scheme which we will use in Chapter 4. For further information on the principal component we recommend the reader [KrRo10], Section 3. We continue to work over the polynomial ring $\mathrm{P} = \mathrm{K}[x_1, ..., x_n]$ over a field K. Let $\bar{\mathrm{K}}$ denote the algebraic closure of K. Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ denote an order ideal of terms in $\mathbb{T}^n$ and we let $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ be the border of it.

The tuples $\mathbf{y}^{(i)} = (y_1^{(i)}, ..., y_n^{(i)})$ for $i = 1, ..., \mu$ that are given in the following definition constitute of the indeterminates $y_j^{(i)}$, for $j = 1, ..., n$ which represent the coordinates of the i$^{\text{th}}$ point in the set $\mathbb{X} \subseteq \mathbb{A}^n_{\bar{\mathrm{K}}}$, which has μ elements.

Definition 2.4.1. Let Q denote the ring $\mathrm{K}[y^{(1)}, ..., y^{(\mu)}]$. We define the following polynomials in Q.

(a) Let $\triangle_{\mathcal{O}}$ denote the $\det(t_j(y^{(i)}))_{i,j=1,...,\mu}$ where $t_j(y^{(i)})$ denotes the result of the substitutions $x_k \longmapsto y_k^{(i)}$ in t_j.

(b) For $i = 1, ..., \mu$ and $j = 1, ..., \nu$ we let

$$\triangle_{ij} = \det(t_1(y^\bullet) \mid ... \mid b_j(y^\bullet) \mid ... \mid t_\mu(y^\bullet)).$$

Here $t_k(y^\bullet)$ denotes the k$^{\text{th}}$ column of the matrix $\Delta_{\mathcal{O}}$. If we replace the i$^{\text{th}}$ column of $\Delta_{\mathcal{O}}$ by $b_j(\mathbf{y}^\bullet)$, then we have the determinant Δ_{ij}.

This example enlightens Definition 2.4.1.

Example 2.4.2. Let $\mathcal{O} = \{1, x_1, x_2, x_1x_2\}$ be the order ideal. Then the border of it is $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$. With respect to our notation we have $t_1 = 1$, $t_2 = x_1$, $t_3 = x_2$, $t_4 = x_1x_2$, $b_1 = x_2^2$, $b_2 = x_1^2$, $b_3 = x_1x_2^2$ and finally $b_4 = x_1^2x_2$. Let $\mathbb{X} = \{(-2, 4), (1, 1), (-3, 9), (4, 16)\}$ be a set of points on the plane. Consider the following equation system,

$$\begin{aligned} b_1(y^{(i)}) &= \gamma_{11}t_1(y^{(i)}) + ... + \gamma_{\mu 1}t_\mu(y^{(i)}), \quad i = 1, ..., \mu \\ b_1(y^{(1)}) = y^2((-2,4)) = 16 &= \gamma_{11} - 2\gamma_{21} + 4\gamma_{31} - 8\gamma_{41} \\ b_1(y^{(2)}) = y^2((1,1)) = 1 &= \gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} \\ b_1(y^{(3)}) = y^2((-2,4)) = 18 &= \gamma_{11} - 3\gamma_{21} + 9\gamma_{31} - 27\gamma_{41} \\ b_1(y^{(4)}) = y^2((-2,4)) = 256 &= \gamma_{11} + 4\gamma_{21} + 16\gamma_{31} + 64\gamma_{41}. \end{aligned}$$

By Cramer's Rule the solution of the system for $\{\gamma_{ij}\}$ is, $\frac{\triangle_{ij}}{\triangle_{\mathcal{O}}}$. Thus we compute the following.

$$\gamma_{11} = \frac{\det \begin{pmatrix} 16 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \\ 81 & -3 & 9 & -27 \\ 256 & 4 & 16 & 64 \end{pmatrix}}{\det \begin{pmatrix} 1 & -2 & 4 & -8 \\ 1 & 1 & 1 & 1 \\ 1 & -3 & 9 & -27 \\ 1 & 4 & 16 & 64 \end{pmatrix}} = -24.$$

By the same method we can calculate $\gamma_{12} = 10, \gamma_{13} = 15, \gamma_{41} = 0, \gamma_{32} = 1$ and the rest is zero.

Notation 2.4.3. Let $\mathrm{K}(y^{(1)}, ..., y^{(\mu)})$ denote the quotient field of Q.

a) Let $C_{\mathcal{O}}$ denote the K-subalgebra of $\mathrm{K}(y^{(1)}, ..., y^{(\mu)})$ generated by the elements $\frac{\triangle_{ij}}{\triangle_{\mathcal{O}}}$ with $i \in \{1, ..., \mu\}$ and $j \in \{1, ..., \nu\}$.

b) Let Φ denote the following surjective K-algebra homomorphism.

$$\begin{aligned} \Phi : \mathrm{K}[\mathcal{C}] &\longrightarrow C_{\mathcal{O}} \\ c_{ij} &\longmapsto \frac{\triangle_{ij}}{\triangle_{\mathcal{O}}} \end{aligned} \tag{2.8}$$

Definition 2.4.4. Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. The **principal component** of $\mathbb{B}_{\mathcal{O}}$ is an irreducible component of $\mathbb{B}_{\mathcal{O}}$ whose vanishing ideal is $\mathrm{Ker}(\Phi)$. The principal component is denoted by $\mathbb{C}_{\mathcal{O}}$.

Let us consider the zero-dimensional radical ideals having an $\mathcal{O}$-border basis. Such ideals correspond to the points on the border basis scheme which are called **radical points**.

Theorem 2.4.5. *The principal component $\mathbb{C}_{\mathcal{O}}$ is smooth of dimension $n\mu$ at its radical points.*

Proof. This follows from [KrRo10] Theorem 4.2. □

Corollary 2.4.6. *If the monomial point $\mathfrak{o}$ in $\mathbb{B}_{\mathcal{O}}$ is smooth, then it is included in $\mathbb{C}_{\mathcal{O}}$, and we have $\dim((\mathbb{B}_{\mathcal{O}})_{\mathfrak{m}}) = n\mu$.*

Proof. This follows from [KrRo10], Theorem 4.2.b and Proposition 4.6. □

2.5 Box and Segment Border Basis Schemes

Recall the Construction 2.3.17 and let $\mathcal{T}$ denote the set of generators of a given border basis. In this section we shall investigate *box* and *segment border basis schemes* with order ideals which have special shapes. Our main interest is to give an exact number of non-trivial polynomials of the set $\mathcal{T}$ for each border basis scheme. We will need the number of non-trivial defining equations of those schemes as we look into the ideal-theoretic complete intersection (see Definition 2.1.10) property of them. Throughout this section we let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$.

Let us start with the formal definition of a box border basis scheme.

Definition 2.5.1. Let $d_1, ..., d_n$ be integers $\geqslant 2$.

(a) The order ideal $\mathcal{B}(d_1, ..., d_n) = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}^n \mid \alpha_i < d_i \quad for \quad i = 1, ..., n\}$ is called the **box** of size $(d_1, ..., d_n)$. If it is clear which size we are dealing with, we simply write $\mathcal{B}$.

(b) Let I be a zero-dimensional ideal of P. A $\mathcal{B}$-border basis of I is called a **box border basis** of I.

(c) The $\mathcal{B}$-border basis scheme is called the **box border basis scheme** and is denoted by $\mathbb{B}_{\mathcal{B}_{(d_1,...,d_n)}}$ or simply by $\mathbb{B}_{\mathcal{B}}$.

Lemma 2.5.2. *Let $\mathcal{B}(d_1, ..., d_n)$ be the box of size $(d_1, ..., d_n)$. The number of terms in the order ideal is*

$$|\mathcal{B}| = d_1 \cdots d_n.$$

The number of terms in the border is

$$|\partial\mathcal{B}| = \nu = \sum_{i=1}^{n}(\prod_{j=1}^{n-1} d_{i_j}),$$

where the set $\{d_{i_1}, ..., d_{i_{n-1}}\}$ *is a subset of* $\{d_1, ..., d_n\}$.

Proof. The first part that $|\mathcal{B}| = d_1 \cdots d_n$ is clear. Now we prove the second part of the claim. Let $t_i \in \mathcal{B}$ with

$$\log(t_i) = (\alpha_1, ..., \alpha_k, ..., \alpha_n).$$

We have $x_k t_i \in \partial\mathcal{B}$ if and only if $\alpha_k + 1 = d_k$ with $\alpha_j \in \{0, 1, ..., d_j - 1\}$ and $j \in \{1, ..., n\}$. By fixing the k^{th}-component of $\log(t_i)$ as $d_k - 1$ and letting the other components of this vector be one of the possible values from the set $\{0, 1, ..., d_j - 1\}$, we can find the number of border elements that are calculated by multiplying the order ideal elements with the indeterminate x_k. Therefore for one indeterminate x_k the number of border elements is $d_1 \cdots d_{k-1} d_{k+1} \cdots d_n$. Clearly, the set $\{d_1, ..., d_{k-1}, d_{k+1}, ..., d_n\}$ has $(n-1)$ elements and is a subset of $\{d_1, ..., d_n\}$. By summing up these multiplications for every indeterminate, we find the desired result. □

Lemma 2.5.3. *Let* $\mathcal{B}(d_1, ..., d_n)$ *be the box of size* $(d_1, ..., d_n)$. *The number of non-trivial polynomials in the generating set* $\mathcal{T}$ *of* $\mathcal{I}(\mathbb{B}_\mathcal{B})$ *is*

$$\mu((n-1)\nu - \nu') \quad \textit{with} \quad \nu' = \sum_{i=1}^{m}(\prod_{j=1}^{n-2} d_{i_j}),$$

where $m = n(n-1)/2$ *and* $\{d_{i_1}, ..., d_{i_{n-2}}\}$ *is a subset of* $\{d_1, ..., d_n\}$.

Proof. Let $\log(t_q) = (\alpha_1, ..., \alpha_k, ..., \alpha_n) \in \mathrm{K}^n$. As a result of the definition of generic multiplication matrices, the entries of the generic multiplication matrices are non-trivial if and only if $x_k t_q$ is in $\partial\mathcal{B}$ or $x_l t_q \in \partial\mathcal{B}$. Our aim is to find the number of order ideal elements with this property. In order to find the non-trivial entries of the matrix $\mathcal{A}_k\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_k$, we fix the k^{th}-component of $\log(t_q)$ as $d_k - 1$ and for $j \neq k, l$ let α_j be from $\{1, ..., d_j - 1\}$. Thus from this product of generic multiplication matrices we have

$$(d_1 \cdots d_{k-1} d_{k+1} \cdots d_l \cdots d_n) + (d_k - 1) d_1 \cdots d_{k-1} d_{k+1} \cdots d_{l-1} d_{l+1} \cdots d_n.$$

It is the number of the order ideal elements which become border elements after the

multiplication with x_k or x_l. We compute this for every pair $x_k, x_l \in \{x_1, ..., x_n\}$ with $l \neq k$. The number of such pairs is $n(n-1)/2$. Therefore we have

$$(d_2 d_3 \cdots d_n + (d_1 - 1)d_3 d_4 \cdots d_n) + \cdots + (d_1 \cdots d_{n-1} + (d_n - 1)d_1 \cdots d_{n-2})).$$

This is equal to the following.

$$\begin{aligned}&((n-1)(d_2 d_3 \cdots d_n + \cdots + d_{k-1}d_{k+1} \cdots d_n)) - \cdots - (d_3 d_4 \cdots d_n + \cdots + d_1 \cdots d_{n-2}) \\ &= (n-1)\nu - \nu'\end{aligned}$$

Since τ_{pq}^{kl} is computed for every $t_p \in \mathcal{B}$, the result is $\mu((n-1)\nu - \nu')$. □

Example 2.5.4. Consider the box order ideal $\mathcal{B}(2,2,2,2)$. By Lemma 2.5.2, we have $\mu = 2^4 = 16$ and by Lemma 2.5.2, we have $\nu = n(2 \cdot 2 \cdot 2) = 32$. By Lemma 2.5.3, the number of non-trivial indeterminates is

$$\mu((n-1)\nu - \nu') = 16((4-1)32 - \nu')$$

where $\nu' = m \cdot (2 \cdot 2)$ with $m = \binom{4}{2} = 6$. Therefore the number of non-trivial elements in the generating set of $\mathcal{I}(\mathbb{B}_\mathcal{B})$ is

$$\mu((n-1)\nu - \nu') = 16(3 \cdot 32 - 12) = 1152.$$

Definition 2.5.5. Let I be a zero-dimensional ideal of P.

(a) The order ideal $\mathcal{O} = \{1, x_1, ..., x_1^{\mu-1}\}$ is called the **segment order ideal**.

(b) An $\mathcal{O}$-border basis of I is called a **segment border basis** of I.

(c) The $\mathcal{O}$-border basis scheme is called the **segment border basis scheme**.

Lemma 2.5.6. *Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\}$ and let $\mathbb{B}_\mathcal{O}$ be a segment border basis scheme. Then the number of terms in $\partial\mathcal{O}$ is $(n-1)\mu + 1$ and the number of non-trivial polynomials in $\mathcal{T}$ is $\mu^2(\frac{n(n-1)}{2})$.*

Proof. The first part is a result of Definition 2.2.1. For the second claim, recall that there are n-different generic multiplication matrices with μ^2 entries and one uses pairs

of them to construct $\mathcal{T}$. Since the shape of the order ideal is a segment, $n-1$ generic multiplication matrices have neither 0 nor 1 as entry. Therefore the matrix $[\mathcal{A}_j, \mathcal{A}_i]$ (see Equation (2.5)) for every pair $\mathcal{A}_i, \mathcal{A}_j \in \{\mathcal{A}_1, ..., \mathcal{A}_n\}$ does not have trivial entries. This gives the desired result. □

Chapter 3

The Arrow Grading

This chapter introduces a grading (see Definition 3.1.1) on the polynomial ring where the vanishing ideal of a border basis scheme is defined. We shall call this grading *the arrow grading* (see Definition 3.2.1). Before looking into the properties of the arrow grading, in Section 3.1 we give a short discussion on $\mathbb{Z}^m$-graded polynomial rings where m is a positive integer. Our focus point in this section is gradings defined by matrices, whose properties we recall in Definition 3.1.6.

The second section (see Section 3.2) is devoted to investigating the arrow grading in detail. First we illustrate that the arrow grading is not like any of the special gradings that we define in the first section (see Example 3.2.12). This shows that it is not a trivial work to deal with the problems we introduce in the rest of the thesis. Then we prove that the vanishing ideal of a border basis scheme is homogenous with respect to the arrow grading. Although it is a known fact (see for example Lemma 4.1 [Huib09]), we give an alternative proof to the known ones. Moreover, we remark that the torus action (see page 208 of [Hai98] and page 363 of [MilSt05]) on $\mathbb{B}_{\mathcal{O}}$ results in the arrow grading (see Remark 3.2.8). Since the arrow grading is not of positive type, we easily show that there might exist more than one maximal homogenous ideal in $\mathrm{B}_{\mathcal{O}}$. This contradicts to the claims on page 363 of [MilSt05] (see Counterexamples 3.2.12 and 3.2.16).

Consequently, the main idea delivered in this chapter is to show that giving an algebraic proof of any claim on a border basis scheme is not straightforward by showing the peculiarity of the grading which is defined on the coordinate ring of a border basis scheme.

3.1 Gradings

In this section we discuss *gradings* and *gradings defined by a matrix*. Unless otherwise stated, throughout this section let K be a field. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$.

Definition 3.1.1. Let R be a ring. Let $(M, +)$ be a monoid. The ring R is called an **M-graded ring** if there exists a family of additive subgroups $\{\mathrm{R}_m\}_{m\in M}$ such that the following properties are satisfied.

a) $\mathrm{R} = \bigoplus_{m\in M} \mathrm{R}_m$

b) $\mathrm{R}_m\mathrm{R}_{m'} \subseteq \mathrm{R}_{m+m'}$ for all $m, m' \in M$

The elements of R_m are called the **homogenous elements** of degree m of R and for every $r \in \mathrm{R}_m$ it is written $\deg(r) = m$. If R is an M-graded ring, then for every element $r \in \mathrm{R}$ we have $r = \sum_{m\in M} r_m$ such that for each $m \in M$ we have $r_m \in M_m$. For every $m \in M$ the element r_m is called the **homogeneous component of degree m**.

Remark 3.1.2. Since 0 is included in every additive subgroup, we say that $0 \in \mathrm{R}$ is homogenous of every degree.

Example 3.1.3. Consider the polynomial ring $\mathrm{P} = \mathrm{K}[x_1, ..., x_n]$. We equip P with an $\mathbb{N}^n$-grading by letting P_α denote an additive subgroup of P, so that for each polynomial $f \in \mathrm{P}_\alpha$, all monomials in the support of f are of degree $\alpha \in \mathbb{N}^n$. This grading is called the **standard grading** of P.

Definition 3.1.4. Let R be a graded ring. An ideal of R is called **a homogeneous ideal** if it can be generated by homogenous elements.

Lemma 3.1.5. *Let I be an ideal of a graded polynomial ring. The ideal I is homogenous if and only if the homogenous components of each polynomial $f \in I$ are included in I, as well.*

Proof. This follows from [KrRo00], Proposition 1.7.10. □

Definition 3.1.6. Let m be a positive integer. Let P be $\mathbb{Z}^m$-graded polynomial ring where $\mathrm{K} \subseteq \mathrm{P}_0$ and $x_1, ..., x_n$ are homogeneous elements. Let W denote a matrix from the set $\mathrm{Mat}_{m,n}(\mathbb{Z})$. Let $w_1, ..., w_m$ denote the rows of W (see Definitions 4.1.6 and 4.2.4 of [KrRo05]).

(a) If for every $i \in \{1, ..., n\}$ the i^{th}-column of W has the degree of the indeterminate x_i, then W is called the **degree matrix** of the grading.

(b) We say P is **graded** by W if P is $\mathbb{Z}^m$-graded and the degrees of the indeterminates are the columns of W with $\mathrm{K} \subseteq \mathrm{P}_0$.

(c) Let P be graded by W. The grading on P is said to be **of non-negative type** if there exists a tuple $(\alpha_1, ..., \alpha_m) \in \mathbb{Z}^m$ such that the entries of $\alpha_1 w_1 + ... + \alpha_m w_m$ which correspond to non-zero columns of W are positive.

(d) Let P be graded by W. The grading on P is called **of positive type** if there exists a tuple $(\alpha_1, ..., \alpha_m) \in \mathbb{Z}^m$ such that all entries of $\alpha_1 w_1 + ... + \alpha_m w_m$ are positive.

(e) Let P be graded by W. The grading on P is called **non-negative** if the first non-zero element in each non-zero column of W is positive.

(f) Let P be graded by W. The grading on P defined by W is called **positive** if no column of W is zero and the first non-zero element in each column is positive.

Remark 3.1.7. If the grading given is positive (resp. non-negative) then it is of positive type (resp. of non-negative type).

Now we give some examples for gradings defined by matrices.

Example 3.1.8. Let P be graded by $W = (1, ..., 1)$. The grading on P is a positive grading and it is also of positive type. Actually, it is equal to the standard grading on P (see Example 3.1.3).

Example 3.1.9. Let P denote $\mathrm{K}[x_1, x_2]$ and be graded by $W = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the grading defined on P is of non-negative type, since for $\alpha = (\alpha_1, \alpha_2) = (-1, -1) \in \mathbb{Z}^2$, we have $\alpha_1 w_1 + \alpha_2 w_2 = (1, 1)$

Example 3.1.10. Let P denote $\mathrm{K}[x_1, x_2]$ and be graded by $W = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Then we have

$$a(-1, 1) + b(1, -1) = (-a + b, -b + a).$$

Since for any $a, b \in \mathbb{Z} \setminus \{0\}$ the first and the second entry of this tuple $(-a+b, -b+a)$ can not be simultaneously larger than 0, this implies that the grading on P given by W is neither of positive type nor of non-negative type.

We want to mention one more grading before ending this section. Let R be a ring and I be an ideal of it. For all $n \in \mathbb{N}$ we let $gr_I^n(R) := I^n/I^{n+1}$ and $gr_I^0 := R/I$.

Lemma 3.1.11. *$gr_I(R) = \bigoplus_{n\in\mathbb{N}} gr_I^n(R)$ is an $\mathbb{N}$-graded ring.*

Proof. Let $x = a + I^{n+1} \in gr_I^n(R)$ $y = b + I^{m+1} \in gr_I^m(R)$. Then we have

$$xy = ab + I^{n+m+1} \in gr_I^{m+n}.$$

□

Definition 3.1.12. The ring $gr_I(R)$ is called graded ring of I.

3.2 The Arrow Grading on $\mathrm{K}[\mathcal{C}]$

In this section we introduce a new grading on $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$, namely the *arrow grading*. We attribute the name *arrow grading* to the proof of [Hai98], Proposition 2.4, which employs arrows to indicate indeterminates from the set $\{c_{11}, ..., c_{\mu\nu}\}$.

Unless otherwise is stated, let K be a field and $\bar{\mathrm{K}}$ be the algebraic closure of K. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $I \subset \mathrm{P}$ denote a zero-dimensional ideal in P. Moreover, we let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ denote an order ideal and $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ denote the border of it. Let $\{c_{11}, ..., c_{\mu\nu}\}$ be a set of further indeterminates and $\mathrm{K}[\mathcal{C}]$ denote the polynomial ring $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$. Let $\mathbb{B}_\mathcal{O}$ denote the $\mathcal{O}$-border basis scheme, let $\mathcal{I}(\mathbb{B}_\mathcal{O})$ denote the vanishing ideal of $\mathbb{B}_\mathcal{O}$ and let $\mathrm{B}_\mathcal{O}$ denote the coordinate ring of $\mathbb{B}_\mathcal{O}$. Finally, let $\mathcal{T}$ denote the set $\{\tau_{pq}^{kl} \mid k < l \,\text{and}\, k, l \in \{1, ..., n\},\, p, q \in \{1, ..., \mu\}\}$, which is the generating set of $\mathcal{I}(\mathbb{B}_\mathcal{O})$ (see Chapter 2.3 for further information).

Definition 3.2.1. Let $\mathcal{O} \subset \mathrm{P}$ be an order ideal and $\partial\mathcal{O}$ be the border of it. Let m denote the number of indeterminates in the ring $\mathrm{K}[\mathcal{C}]$. Let c_{ij} be an indeterminate from the ring $\mathrm{K}[\mathcal{C}]$. We define the degree of the indeterminate c_{ij} by

$$\deg(c_{ij}) = \log(b_j) - \log(t_i) = (u_1, ..., u_n) \in \mathbb{Z}^n$$

where $t_i \in \mathcal{O}$ and $b_j \in \partial\mathcal{O}$. This way we define a $\mathbb{Z}^n$-grading on the ring $\mathrm{K}[\mathcal{C}]$. Let W be a matrix with m columns and n rows. We let $(u_1, ..., u_n)^{tr}$ be the column of the matrix W corresponding to the indeterminate c_{ij}. Then the ring $\mathrm{K}[\mathcal{C}]$ is graded by matrix W. We shall name this grading the **$\mathcal{O}$-arrow grading** and denote it by $\deg_W$. If it is clear which order ideal we are dealing with, then we simply call the **arrow grading**.

Let us write some observations on the generators of a vanishing ideal of an $\mathcal{O}$-border basis scheme with respect to the $\mathcal{O}$-arrow grading. But first let us recall Lemma 2.3.19. The polynomials from the set $\mathcal{T}$ are of degree two with no constant. In particular, a polynomial, say τ_κ, is of the following form:

$$\tau_\kappa = \tau_\kappa^{(2)} + \tau_\kappa^{(1)} \in \mathcal{T} \tag{3.1}$$

where $\tau_\kappa^{(2)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 2 with respect to the standard grading and $\tau_\kappa^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 1 with respect to the standard grading. Moreover, the set $\mathrm{supp}(\tau_\kappa^{(1)})$ has at most two linear monomials.

Corollary 3.2.2. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, ..., x_n]$. *Let* $\mathcal{O}$ *denote an order ideal* $\{t_1, ..., t_\mu\} \in \mathrm{P}$. *Then* $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ *is the border of it. Let* c_{ij} *be an indeterminate from the set* $\{c_{11}, ..., c_{\mu\nu}\}$. *Let* $\tau_{pq}^{kl} \in \mathcal{T}$ *be a non-zero polynomial.*

a) The degree vector $\deg_W(\tau_{pq}^{kl}) = (d_1, ..., d_n) \in \mathbb{Z}^n$ *has at least one positive entry.*

b) For every $c_{ij} \in \{c_{11}, ..., c_{\mu\nu}\}$, *we have* $\deg_W(c_{ij}) \neq (0, ..., 0)$ *and their degree vector has at least one positive entry.*

c) If the indeterminate c_{ij} *is in the support of* τ_{pq}^{kl}, *then for every quadratic monomial* $cd \in \mathrm{supp}(g)$, *we have* $c_{ij} \nmid cd$.

Definition 3.2.3. Let $\mathcal{A}$ be a matrix from $\mathrm{Mat}_{r,s}(\mathrm{P})$. If there exist two tuples $d_0 = (d_{01}, ..., d_{0r}) \in (\mathbb{Z}^m)^r$ and $d_1 = (d_{11}, ..., d_{1s}) \in (\mathbb{Z}^m)^s$ such that for every $i = 1, ..., r$ and $j = 1, ..., s$ the entry (a_{ij}) in position (i, j) of $\mathcal{A}$ has degree $\deg_W(a_{ij}) = d_{1j} - d_{0i}$, then the matrix $\mathcal{A}$ is called a **homogenous matrix**. In this case, the pair $(d_0, d_1) \in (\mathbb{Z}^m)^r \times (\mathbb{Z}^m)^s$ is called a **degree pair** of $\mathcal{A}$.

Recall the generic multiplication matrices (see Definition 2.3.1). Let us give a closer look at them with respect to the arrow grading.

Lemma 3.2.4. *Let* $\mathcal{O}$ *be an order ideal. The generic multiplication matrices with respect to* $\mathcal{O}$ *are homogenous with respect to the* $\mathcal{O}$*-arrow grading.*

Proof. Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let l be an index from $\{1, ..., n\}$. Let $\mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be the l^{th} generic multiplication matrix with respect to $\mathcal{O}$. We show that this matrix is a homogenous matrix with respect to the arrow

grading. We let

$$d_l = (\log(x_l t_1), ..., \log(x_l t_\mu)) \text{ and } d_0 = (\log(t_1), ..., \log(t_\mu)). \tag{3.2}$$

Let a_{ij} be the entry in position (i, j) of $\mathcal{A}_l$. There are three possible values for a_{ij}, namely $0, 1$ or an indeterminate from the set $\{c_{11}, ..., c_{\mu\nu}\}$. If a_{ij} is 0, then it is homogenous of every degree. If a_{ij} is 1, then as a result of the construction of the multiplication matrix $\mathcal{A}_l$, we have $x_l t_j = t_i$ and this implies $\deg_W(a_{ij}) = \log(x_l t_j) - \log(t_i) = d_{lj} - d_{0i} = 0$. If a_{ij} is an indeterminate, say c_{mn}, from $\mathrm{K}[\mathcal{C}]$, then we have $b_n = x_l t_j$ and $m = i$. This implies

$$\deg_W(c_{mn}) = \log(b_n) - \log(t_m) = \log(x_l t_j) - \log(t_i) = d_{lj} - d_{0i}.$$

Therefore the matrix $\mathcal{A}_l$ is a homogenous matrix with the degree pair (d_0, d_l) in Equation 3.2. □

Lemma 3.2.5. *Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathrm{K}[x_1, ..., x_n]$ denote an order ideal. Let $\mathcal{A}_k$ be the k^{th} generic multiplication matrix with respect to $\mathcal{O}$ and $\mathcal{A}_l$ be the l^{th} generic multiplication matrix with respect to $\mathcal{O}$ where $k, l \in \{1, ..., n\}$ and $k \neq l$. Then the results of the matrix multiplications $\mathcal{A}_k \cdot \mathcal{A}_l$ and $\mathcal{A}_l \cdot \mathcal{A}_k$ are both homogenous matrices with the same degree pair $(d_0, d_{(kl)})$ where $d_{(kl)} = (\log(x_k x_l t_1), ..., \log(x_k x_l t_\mu))$ and $d_0 = (\log(t_1), ..., \log(t_\mu))$.*

Proof. In Lemma 3.2.4 we show that for each $l \in \{1, .., n\}$ the generic multiplication matrix $\mathcal{A}_l$ is a homogenous matrix such that for each element a_{ij} in (i, j) position of $\mathcal{A}_l$ has the degree pair (d_0, d_l) where the following holds.

$$\deg_W(a_{ij}) = \log(x_l t_j) - \log(t_i) = d_{l_j} - d_{0_i} \tag{3.3}$$

Recall that $\mathrm{B}_\mathcal{O}$ is the coordinate ring of the $\mathcal{O}$-border basis scheme (see Definition 2.3.1). Consider the graded $\mathrm{B}_\mathcal{O}$ module $\bigoplus_{i=1}^{\mu} \mathrm{B}_\mathcal{O}(-d_{l_i})$. Let $e_1, ..., e_\mu$ be the canonical basis of $\bigoplus_{i=1}^{\mu} \mathrm{B}_\mathcal{O}(-d_{l_i})$ and $(u_1, ..., u_\mu)$ be the canonical basis of $\bigoplus_{j=1}^{\mu} \mathrm{B}_\mathcal{O}(-d_{0_j})$. Let Ψ_l be defined as follows.

$$\begin{aligned} \Psi_l : \bigoplus_{j=1}^{\mu} \mathrm{B}_\mathcal{O}(-d_{l_j}) &\longrightarrow \bigoplus_{i=1}^{\mu} \mathrm{B}_\mathcal{O}(-d_{0_j}) \\ e_j &\longmapsto u\mathcal{A}_l e_j^{tr} \end{aligned}$$

Its representation matrix with respect to the basis $(e_1, ..., e_\mu)$ is the l^{th} generic multiplication matrix $\mathcal{A}_l$. Moreover, by Lemma 3.2.4, $\mathcal{A}_l$ is a homogenous matrix. Thus Ψ_l is a homogeneous $B_\mathcal{O}$-linear map and hence the following holds.

$$\begin{aligned}\deg_W(\Psi_l(e_j)) &= \deg_W(u_i + \deg_W(a_{ij})) \\ &= \log(t_i) + \log(x_l t_j) - \log(t_i) \\ &= \log(x_l t_j)\end{aligned}$$

Let k be an index from $\{1, ..., n\} \setminus \{l\}$. Let Ψ_k be defined as follows.

$$\begin{aligned}\Psi_k : \bigoplus_{j=1}^{\mu} B_\mathcal{O}(-d_{k_j}) &\longrightarrow \bigoplus_{i=1}^{\mu} B_\mathcal{O}(-d_{0_j}) \\ e_j &\longmapsto u\mathcal{A}_k e_j^{tr}\end{aligned}$$

Its representation matrix with respect to the basis $(e_1, ..., e_\mu)$ is the k^{th} generic multiplication matrix $\mathcal{A}_k$. We consider the homogenous maps $\Psi_k \circ \Psi_l$ and $\Psi_l \circ \Psi_k$. The representation matrix of $\Psi_k \circ \Psi_l$ is $\mathcal{A}_k\mathcal{A}_l$ and the representation matrix of $\Psi_l \circ \Psi_k$ is $\mathcal{A}_l\mathcal{A}_k$. Let b_{ij} be the element in position (i, j) of $\mathcal{A}_k$ and a_{ij} be the element in position (i, j) of $\mathcal{A}_l$. Then the following holds.

$$\begin{aligned}\deg_W(\Psi_l \circ \Psi_k(e_j)) &= \deg_W(u_j \sum_{i=1}^{\mu}(a_{ij}b_{ji})) \\ &= \log(t_j) + \log(x_l t_j) - \log(t_i) + \log(x_k t_i) - \log(t_j) \\ &= \log(x_l x_k t_j) \\ &= \log(t_j) + \log(x_k t_j) - \log(t_i) + \log(x_l t_i) - \log(t_j) \\ &= \deg_W(u_j \sum_{i=1}^{\mu}(b_{ij}a_{ji})) \\ &= \deg_W(\Psi_k \circ \Psi_l(e_j))\end{aligned}$$

Let $d_{(kl)_j}$ denote $\log(x_k x_l t_j)$. Then the entry in (i, j) position of the matrix $\mathcal{A}_k\mathcal{A}_l$ and $\mathcal{A}_l\mathcal{A}_k$ has the same arrow degree which is

$$\log(x_k x_l t_j) - \log(t_i) = d_{(kl)_j} - d_{0_i}. \tag{3.4}$$

Thus $\mathcal{A}_k\mathcal{A}_l$ and $\mathcal{A}_l\mathcal{A}_k$ are homogenous matrices with the same degree pair $(d_0, d_{(kl)})$ where $d_{(kl)} = (\log(x_k x_l t_1), ..., \log(x_k x_l t_\mu))$ and $d_0 = (\log(t_1), ..., \log(t_\mu))$. □

Despite it is a well-known fact that the vanishing ideal of a $\mathcal{O}$-border basis scheme is homogenous with respect to the arrow grading, in the following proposition we give a different proof.

Proposition 3.2.6. *Let $\mathcal{O}$ be an order ideal. The vanishing ideal of the $\mathcal{O}$-border basis scheme $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is a homogenous ideal with respect to the $\mathcal{O}$-arrow grading.*

Proof. By Lemma 3.2.5 and Equation 3.4, the matrix $\mathcal{A}_k\mathcal{A}_l$ and the matrix $\mathcal{A}_l\mathcal{A}_k$ are homogenous matrices with the same degree pair $(d_0, d_{(lk)})$ where $(d_{0_1}, ..., d_{0_\mu})$ and $(d_{(lk)_1}, ..., d_{(lk)_\mu})$. Therefore we can deduce that the entry in the position (p, q) of the matrix $\tau^{kl} = \mathcal{A}_l\mathcal{A}_k - \mathcal{A}_kA_l$ which is τ^{kl}_{pq} is homogenous of degree

$$d_{lk_q} - d_{0_p} = \log(x_kx_lt_q) - \log(t_p). \tag{3.5}$$

□

One can find the next corollary as Lemma 4.1 in [Huib09]. The enumeration of the elements of $\mathcal{T}$ (see Equation (2.5)) with help of order ideal elements gives a direct method to determine their arrow degree. The following corollary emphasizes the importance of Costruction 2.3.17 and gives this method.

Corollary 3.2.7. *Let $\tau^{kl}_{pq} \in \tau$ be the non-trivial polynomial entry in the position (p, q) of $[\mathcal{A}_k, \mathcal{A}_l]$. Then we have*

$$\deg_W(\tau^{kl}_{pq}) = \log(x_kx_lt_q) - \log(t_p).$$

Proof. This follows from Lemma 3.2.5 and Proposition 3.2.6. □

Let $\mathcal{O}$ denote an order ideal from $\mathbb{T}^2$. Next remark emphasizes the link between the $\mathcal{O}$-arrow grading and the 2-dimensional torus action given in [Hai98].

Remark 3.2.8. Let us take a closer look at the torus action mentioned in [Hai98] (see page 208) and [MilSt05] (see page 363). Let $\mathbb{C}^*$ denote the group of complex numbers without 0. Let G denote the 2-dimensional torus group $(\mathbb{C}^*)^2$. The group G acts algebraically on $\mathbb{C}^2$ via

$$\begin{array}{rcl} \alpha : G \times \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ ((u, v), (\xi, \zeta)) & \longmapsto & (u\xi, v\zeta) \end{array}$$

The co-action on the polynomial ring $\mathbb{C}[x, y]$ is as follows.

$$\begin{aligned} \alpha' : \mathbb{C}[x,y] &\longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[x,y] \\ f &\longmapsto f((u,v)\cdot(x,y)) = f(ux, vy) \end{aligned}$$

It is a known fact that this action results with a $\mathbb{Z}^2$-grading on the polynomial ring $\mathbb{C}[x, y]$ (see for instance [Bri09], Example 1.7).

On page 208 of [Hai98] it is implied that there exists an induced action on the border basis scheme, given by

$$(u, v)c_{ij} = u^{r-h}v^{s-k}c_{ij} \tag{3.6}$$

where $b_j = x^r y^s$ and $t_i = x^h y^k$. Clearly, the power of u and v, which is $(r - h, s - k)$, is equal to $\log(b_j) - \log(t_i)$ i.e., to $\deg_W(c_{ij})$. This is the first clue that suggests the link between the grading mentioned on page 363 of [MilSt05] and the arrow grading. We claim that they are the same. Let $\mathbb{C}[G]$ act on the coordinate ring $\mathbb{C}[\mathcal{C}]$ as given in Equation 3.6.

$$\begin{aligned} \beta' : \mathbb{C}[\mathcal{C}] &\longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[\mathcal{C}] \\ g &\longmapsto g((u,v)\cdot c_{ij}) = g(u^{r-h}v^{s-k}c_{ij}) \end{aligned}$$

[1] Let us pick an arbitrary element τ_{pq} from the generating set $\mathcal{T}$ of $\mathcal{I}(\mathbb{B}_\mathcal{O})$. Under the given action we have

$$(\tau_{pq})(u^{r-h}v^{s-k}c_{ij}).$$

We showed that for a border element $b_j \in \partial\mathcal{O}$ and an order ideal element $t_i \in \mathcal{O}$ the arrow degree satisfies $\deg_W(c_{ij}) = (r - h, s - k)$. Moreover, Proposition 3.2.6 states that τ_{pq} is homogenous of arrow degree

$$\deg_W(\tau_{pq}) = \log(xyt_q) - \log(t_p) = (d_1, d_2).$$

These imply

$$(\tau_{pq})(u^{r-h}v^{s-k}c_{ij}) = u^{d_1}v^{d_2}(\tau_{pq}).$$

Therefore the ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is invariant under this group action if and only if the grading

[1] Note that there are exactly two generic multiplication matrices with respect to an order ideal $\mathcal{O} \subset \mathrm{K}[x_1, x_2]$. Thus every element from the generating set $\mathcal{T}$ of $\mathcal{I}(\mathbb{B}_\mathcal{O})$ has the same upper index (1 2) which we omit for the sake of simplicity.

is the arrow grading.

Lemma 3.2.9. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. The induced action given in Equation (3.6) on $\mathbb{B}_\mathcal{O}$ results in the $\mathcal{O}$-arrow grading on $\mathrm{B}_\mathcal{O}$.*

Proof. Let c_{ij} be an indeterminate from $\mathbb{C}[\mathcal{C}]$ where the border term b_j is $x_1^{r_1} \cdots x_n^{r_n}$ and the order ideal term t_i is $x_1^{r_1} \cdots x_n^{r_n}$. Consider Equation 3.6 for c_{ij} and $(u_1, ..., u_n)$. Then we have

$$(u_1, ..., u_n)c_{ij} = u_1^{r_1-h_1} \cdots u_n^{r_n-h_n} c_{ij}. \tag{3.7}$$

Let G denote the n-dimensional torus group $(\mathbb{C}^*)^n$. Let $\mathbb{C}[G]$ act on the coordinate ring $\mathbb{C}[\mathcal{C}]$ as given in Equation 3.7.

$$\begin{array}{rcl} \beta' : \mathbb{C}[\mathcal{C}] & \longrightarrow & \mathbb{C}[G] \otimes \mathbb{C}[\mathcal{C}] \\ g & \longmapsto & g((u_1, ..., u_n)) \cdot c_{ij}) = g(u_1^{r_1-h_1} \cdots u_n^{r_n-h_n} c_{ij}) \end{array}$$

Pick an arbitrary generator τ_{pq}^{kl} from the generating set $\mathcal{T}$ of $\mathcal{I}(\mathbb{B}_\mathcal{O})$. By Proposition 3.2.6, the polynomial τ_{pq}^{kl} is a homogenous polynomial of arrow degree

$$\deg_W(\tau_{pq}^{kl}) = \log(x_k x_l t_q) - \log(t_p) = (d_1, ..., d_n).$$

Then under the given action we have

$$(\tau_{pq}^{kl})(u_1^{r_1-h_1} \cdots u_n^{r_n-h_n} c_{ij}) = u^{d_1} \cdots u^{d_n} (\tau_{pq}^{kl}).$$

Therefore, the ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is invariant under this group action if and only if the grading is the $\mathcal{O}$-arrow grading.

□

Now we focus on the maximal ideals that are homogenous with respect to the arrow grading in $\mathrm{B}_\mathcal{O} \subset \bar{\mathrm{K}}[\mathcal{C}]$.

Proposition 3.2.10. *Let $\mathcal{O} \subset \mathbb{T}^n$ denote an order ideal and let $\partial\mathcal{O}$ denote the border of it. The ideal $\mathfrak{M} = \langle c_{11}, ..., c_{\mu\nu} \rangle \subset \mathrm{K}[c_{11}, ..., c_{\mu\nu}]$ is the only maximal ideal that is homogenous with respect to the $\mathcal{O}$-arrow grading.*

Proof. Let $\bar{\mathrm{K}}$ denote the algebraic closure of the field K. Assume that there exists a maximal ideal $\mathfrak{N}$, which is different than $\mathfrak{M}$ and assume that $\mathfrak{N}$ is homogenous with respect to the $\mathcal{O}$-arrow grading. By Hilbert's Nullstellensatz 2.1.4 every maximal ideal

in the polynomial ring $\bar{\mathrm{K}}[\mathcal{C}]$ is of the form $\langle c_{11}-\alpha_{11},...,c_{\mu\nu}-\alpha_{\mu\nu}\rangle$ where $\{\alpha_{11},...,\alpha_{\mu\nu}\} \subset \bar{\mathrm{K}}$. Since $\mathfrak{N}$ is different than $\mathfrak{M}$, there exists a polynomial $f = c_{ij} - \alpha$ with a constant term $\alpha \neq 0$ that is in the generating set of $\mathfrak{N}$ but not in $\mathfrak{M}$. The maximal ideal $\mathfrak{N}$ is a homogenous ideal with respect to the $\mathcal{O}$-arrow grading. Hence the polynomial f is a homogenous polynomial with respect to the arrow grading. As a result of the definition of the arrow grading, the constant term of the polynomial f has the arrow degree $\deg_W(\alpha) = (0,...,0) \subset \mathbb{Z}^n$. Since f is a homogenous polynomial with respect to the arrow grading, we have $\deg(c_{ij}) = (0,...,0)$. This contradicts to Corollary 3.2.2b). Hence $\mathfrak{M} = \langle c_{11},...,c_{\mu\nu}\rangle \subset \bar{\mathrm{K}}[c_{11},...,c_{\mu\nu}]$ is the only maximal ideal that is homogenous with respect to the $\mathcal{O}$-arrow grading.

Now we show that the ideal $\mathfrak{M} = \langle c_{11},...,c_{\mu\nu}\rangle \subset \mathrm{K}[c_{11},...,c_{\mu\nu}]$ is the only maximal ideal that is homogenous with respect to the $\mathcal{O}$-arrow grading. The field $\bar{\mathrm{K}}$ is algebraic closure of K and the base change is a (faithfully) flat extension. Then by going down theorem the only maximal ideal that is homogenous in $\bar{\mathrm{K}}[c_{11},...,c_{\mu\nu}]$ correspond to the only maximal ideal that is homogenous in $\mathrm{K}[c_{11},...,c_{\mu\nu}]$. Hence $\mathfrak{M} = \langle c_{11},...,c_{\mu\nu}\rangle \subset \mathrm{K}[c_{11},...,c_{\mu\nu}]$ is the only maximal ideal that is homogenous with respect to the $\mathcal{O}$-arrow grading.

□

Corollary 3.2.11. *Let $\mathcal{O}$ denote an order ideal from $\mathrm{K}[x_1,...,x_n]$. Let $\mathrm{B}_\mathcal{O} \subset \mathrm{K}[\mathcal{C}]$ denote the coordinate ring of the $\mathcal{O}$-border basis scheme. The ideal $\mathfrak{m} = \langle \bar{c}_{11},...,\bar{c}_{\mu\nu}\rangle \subset \mathrm{B}_\mathcal{O}$ is the only maximal ideal that is homogenous with respect to the arrow grading.*

Let us give an example which shows that the arrow grading is neither positive nor negative. Moreover, it is neither of positive nor of non-negative type (see Definition 3.1.6 from Chapter 3.1).

Example 3.2.12. (counterexample) This example is a counterexample to the claim in page 363 of [MilSt05], that for any $\mathcal{O} \subset \mathbb{T}^2$ the $\mathcal{O}$-arrow grading is positive. Let P denote the polynomial ring $\mathrm{K}[x_1,x_2]$. Let $\mathcal{O}$ denote the order ideal $\{1,x_1,x_2,x_1x_2\}$. Then its border is $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_1\}$, where with respect to our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_2, t_4 = x_1x_2, b_1 = x_2^2, b_2 = x_1^2, b_3 = x_1x_2^2$ and $b_4 = x_1^2x_1$. We construct the grading matrix W whose columns are of the form

$$\deg_W(c_{ij}) = \log(b_j) - \log(t_i)$$

with $c_{ij} \in \{c_{11},...,c_{\mu\nu}\}$. Let us first find the first column of W which corresponds to

$\deg_W(c_{11})$.

$$\deg_W(c_{11}) = \log(b_1) - \log(t_1) = \log(x_2^2) - \log(1) = (0,2) - (0,0) = (0,2).$$

If we proceed in this way, then we get the grading matrix W.

$$W = \begin{array}{c} \begin{matrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{21} & c_{22} & c_{23} & c_{24} & c_{31} & c_{32} & c_{33} & c_{34} & c_{41} & c_{42} & c_{43} & c_{44} \end{matrix} \\ \begin{pmatrix} 0 & 2 & 1 & 2 & -1 & 1 & 0 & 1 & 0 & 2 & 1 & 2 & -1 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix} \end{array}$$

The first non-zero element of the column that corresponds to the degree of c_{21} is $(-1,2)$ i.e., it is negative. By Definition 3.1.6.e) and f) the grading given by W is neither positive nor non-negative. Let w_1 indicate the first row and w_2 indicate the second row of W. If W is of positive or non-negative type, then by Definition 3.1.6.c) and d) for some $a, b \in \mathbb{Z}$ the following must hold.

$$a(w_1) + b(w_2) > (0, ..., 0)$$

Let us look closely to the arrow degree of c_{41} and c_{42}. The degree of c_{41},

$$\deg_W(c_{41}) = \log(b_1) - \log(t_4) = (-1,1),$$

which gives the 13^{th} column. The degree of c_{42}, is

$$\deg_W(c_{42}) = \log(b_2) - \log(t_4) = (1,-1)$$

which gives the 14^{th} column. Moreover we have

$$\begin{aligned} a \cdot w_{1\,13} + b \cdot w_{2\,13} &= -a + b \\ a \cdot w_{1\,14} + b \cdot w_{2\,14} &= a - b. \end{aligned}$$

If $-a + b > 0$ holds, then $b > a$ holds. If we have $a - b > 0$, then we have $a > b$. Therefore W is neither of positive nor of non-negative type.

We want to underline the fact that the peculiarity of the arrow grading does not allow us to give an algebraic proof as easy as mentioned on page 363 of [MilSt05]. For this purpose we show that there is more than one *maximal homogenous ideals* in the ring $\mathrm{B}_{\mathcal{O}}$.

Definition 3.2.13. Let R be a graded ring. A **maximal homogenous ideal** is a homogenous ideal of R which is not contained in any other homogenous ideal of R.

Theorem 3.2.14. (Graded Version of Krull's Theorem) *Let R be a graded ring. Then each proper homogenous ideal is contained in a maximal homogeneous ideal.*

Proof. This follows from [Per01], Theorem 3.4. □

Remark 3.2.15. We should note that by Proposition 3.2.10 there exists only one maximal ideal that is homogenous in $\mathrm{B}_{\mathcal{O}}$. For an arbitrary field K the following example shows that there might be homogenous ideals that are not contained in the maximal ideal $\mathfrak{M} = \langle c_{11}, ..., c_{\nu\mu} \rangle$.

Example 3.2.16. (counterexample) Let us recall Example 3.2.12. We show that this example is a counter example to the consecutive claim in page 363 of [MilSt05] that the maximal homogenous ideal in $\mathrm{B}_{\mathcal{O}}$ is unique, as well. The polynomial $f = c_{41}c_{42} - 1 \in \mathrm{K}[c_{11}, ..., c_{44}]$ is a homogenous irreducible polynomial of degree 0,

$$\deg_W(f) = \deg_W(c_{41}) + \deg_W(c_{42}) + \deg_W(1) = (-1, 1) + (1, -1) + (0, 0) = (0, 0).$$

f is not included in the maximal ideal $\mathfrak{M}$ that is generated by $\{c_{11}, ..., c_{44}\}$, since it has a constant summand. Thus by graded version of Krull's theorem there exists a maximal homogenous ideal $\mathfrak{N}$, different than $\mathfrak{M}$ which contains $\langle f \rangle$. Clearly, the polynomial $\bar{f} = \bar{c}_{41}\bar{c}_{42} - 1 \in \mathrm{B}_{\mathcal{O}}$ is not in the maximal ideal $\mathfrak{m} \in \mathrm{B}_{\mathcal{O}}$, either.

Chapter 4

The Cotangent Space of a Border Bases Scheme

In this chapter we investigate the cotangent space of an $\mathcal{O}$-border basis scheme, $\mathbb{B}_{\mathcal{O}}$ at the monomial point $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$. In [Hai98] M. Haiman proposed a new method of constructing the cotangent space of a border basis scheme of μ points in the plane. We call this method the *arrow method*. The main idea is to consider every indeterminate from the set $\{c_{11}, ..., c_{\mu\nu}\}$ as an arrow from an order ideal term to a border term, then to construct a basis of the cotangent space by using the relations of border terms and order ideal terms. Then in Chapter 4 and 5 of [Huib05][1], M. Huibregste generalizes this "arrow method" to n dimensions and gives a way for constructing a basis of the cotangent space of a border basis scheme. We note that although in [Hai98] the set U_μ that is given in Equation 2.3 and in [Huib05] the set U_β that is given in Equation (4) in Chapter 2.2 are both border basis schemes, none of those citations contains the name border basis scheme.

Our work is inspired by both of those sources, but mainly by [Huib05]. In contrast to these articles, we do not use the arrow method. In order to understand the foundation of our work, one should recall how we construct the generating set $\mathcal{T}$ of the vanishing ideal of a border basis scheme (see Chapter 2.3). This construction depends on all the relations of border terms and order ideal terms. Thus every relation that is used in [Huib05] and [Hai98] to construct a basis set of the cotangent space at the point $\mathfrak{o}$ is coded in elements of $\mathcal{T}$. In Lemmas 4.1.3 and 4.1.4 we show how these relations are coded. In Propositon 4.1.6 we show how to encode it. Therefore if we

[1] [Huib05] is a pre-print that is only published in arXiv.

let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal and $\mathcal{T}$ generate $\mathcal{I}(\mathbb{B}_\mathcal{O})$, then just by observing the elements of $\mathcal{T}$ we have all the equivalence classes modulo the ideal $\mathfrak{m}^2 \subset \mathrm{B}_\mathcal{O}$, where $\mathfrak{m}$ is generated by $\{\bar{c}_{11}, ..., \bar{c}_{\mu\nu}\} \subset \mathrm{K}[c_{11}, ..., c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_\mathcal{O})$. We denote this set of equivalence classes by $\mathfrak{E}$. We show that the arrow degree is well-defined on this set. Then we show that every equivalence class in $\mathfrak{E}$ contains a *minimal indeterminate* (see Definition 4.1.15), and in Theorem 4.2.9 we show that the subset of $\mathfrak{E}$ that consists of equivalence classes of minimal standard indeterminates has at least $n\mu$ elements. The set $\mathcal{S}$ (see Lemma 4.2.12) of indeterminates that is constructed in [Huib05], Chapter 4 is a set where elements are chosen from each equivalence class from the set $\mathfrak{E}$.

Consequently, in Section 4.2 by using the set $\mathfrak{E}$, we give a basis of the cotangent space. Moreover, we give a smoothness criterion for the monomial point of a border basis scheme (see Theorem 4.2.15). A similar result for Hilbert scheme of points can be found in [Huib05], Theorem 5.1.1. Our main inspiration is the paper [Huib05], we reprove the statements in Chapter 4 and 5 of [Huib05] for border basis schemes by using the tools we have in the border basis scheme theory such as border relations and the arrow grading.

4.1 Equivalence Classes of Elements from $\bar{\mathcal{C}}$ modulo $\mathfrak{m}^2$

This section is dedicated to investigating equivalence relations among elements from $\mathrm{B}_\mathcal{O}$ modulo $\mathfrak{m}^2$. Unless otherwise stated, throughout this section we let K be a field of characteristic 0. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O}$ denote an order ideal $\{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ and then the border of it is $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$. Let $\mathcal{C} = \{c_{11}, ..., c_{\mu\nu}\}$ be a set of further indeterminates and $\mathrm{K}[\mathcal{C}] = \mathrm{K}[c_{11}, ..., c_{\mu\nu}]$. Let $\mathcal{T}$ denote the set of generators of the vanishing ideal of an $\mathcal{O}$-border basis scheme given in Equation (2.6). We denote the image of an indeterminate c_{ij} in $\mathrm{B}_\mathcal{O}$ as $\bar{c}_{ij}$ and we denote the set of such elements by $\bar{\mathcal{C}}$. We let $\mathfrak{M}$ denote the maximal ideal of $\mathrm{K}[\mathcal{C}]$ that is generated by $\{c_{11}, ..., c_{\mu\nu}\}$. Moreover, we let $\mathfrak{m}$ denote the maximal ideal of $\mathrm{B}_\mathcal{O}$ that is generated by $\{\bar{c}_{11}, ..., \bar{c}_{\mu\nu}\}$.

Definition 4.1.1. We say the elements $\bar{c}_{ij}$ and $\bar{c}_{kl}$ of $\bar{C}$ are equivalent to each other modulo $\mathfrak{m}^2$ if we have $\bar{c}_{kl} = \bar{c}_{ij} + \mathfrak{m}^2 \in \mathrm{B}_\mathcal{O}$. We let $\widehat{C}$ denote the set $\{\bar{c}_{ij} + \mathfrak{m}^2 \mid \bar{c}_{ij} \in \bar{C}\}$. We let $[\bar{c}_{ij}] \subset \widehat{C}$ denote the equivalence class of $\bar{c}_{ij}$ modulo $\mathfrak{m}^2$.

Remark 4.1.2. By Lemma 2.3.19, an element τ_κ of $\mathcal{T}$ is of the following form,

$$\tau_\kappa = \tau_\kappa^{(2)} + \tau_\kappa^{(1)} \in \mathcal{T} \tag{4.1}$$

where $\tau_\kappa^{(2)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 2 with respect to the standard grading and $\tau_\kappa^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 1 with respect to the standard grading. Moreover, there are at most two monomials in the support of $\tau_\kappa^{(1)}$.

a) If a polynomial τ_κ from $\mathcal{T}$ is of the form $\tau_\kappa^{(2)} \pm c_{ij}$, then we have $\bar{c}_{ij} \in \mathfrak{m}^2$.

b) If a polynomial τ_κ from $\mathcal{T}$ is of the form $\tau_\kappa^{(2)} \pm c_{ij} \pm c_{pq}$, then we have $[\bar{c}_{ij}] = [\bar{c}_{pq}]$.

The next two lemmas show how neighborhood relations of border terms affect the equivalence classes of elements from $\bar{\mathcal{C}}$ modulo $\mathfrak{m}^2$.

Lemma 4.1.3. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let b_i and b_j be two distinct border terms that are next-door neighbors where $b_i = x_k b_j$ and $k \in \{1, ..., n\}$.*

a) If there exists a term $t_r \in \mathcal{O}$ such that $x_k t_r = t_p \in \mathcal{O}$, then there exist indeterminates $c_{pi}, c_{rj} \in \mathcal{C}$ such that $[\bar{c}_{rj}] = [\bar{c}_{pi}]$.

b) If for every $t_r \in \mathcal{O}$ either $x_k t_r \in \partial\mathcal{O}$ or $x_k t_r \in \mathcal{O} \setminus \{t_p\}$ holds, then we have $\bar{c}_{pi} \in \mathfrak{m}^2$.

Proof. Let $\mathcal{A}_k$ denote the k^{th} generic multiplication matrix, where $k \in \{1, ..., n\}$. For each $i \in \{1, ..., \nu\}$ let c_i denote the column matrix $(c_{1i}, ..., c_{\mu i})^{tr}$. From the next-door neighborhood relation $b_i = x_k b_j$ we get $\mathrm{ND}(i, j)$. This set $\mathrm{ND}(i, j)$ contains μ polynomials which are the entries of the $\mu \times 1$ matrix $c_i - \mathcal{A}_k c_j$. For an arbitrary term $t_p \in \mathcal{O}$ the polynomial in the p^{th} row of the matrix $\mathrm{ND}(i, j)$, has the following form:

$$c_{pi} - (\sum_{r=1}^{\mu} f_r c_{rj}) \quad \text{where} \quad f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_k t_r = b_m \in \partial\mathcal{O} \end{cases} \tag{4.2}$$

In Construction 2.3.17.c we show that the elements of $\mathrm{ND}(i, j)$ are also in $\mathcal{T}$. Clearly, for each $t_p \in \mathcal{O}$, a polynomial from $\mathrm{ND}(i, j)$ is in $\mathcal{I}(\mathbb{B}_\mathcal{O})$. Thus for each $p \in \{1, ..., \mu\}$ Polynomial 4.2 satisfies

$$\bar{c}_{pi} = (\sum_{r=1}^{\mu} f_r \bar{c}_{rj}) \in \mathrm{B}_\mathcal{O}.$$

Consider the polynomial f_r in 4.2 and recall that by Lemma 2.3.19, a polynomial from $\mathcal{T}$ has at most two linear monomials in its support. If we have $f_r = 1$, then we have $[\bar{c}_{pi}] = [\bar{c}_{rj}]$. Otherwise, we have $\bar{c}_{pi} \in \mathfrak{m}^2$. Note that in this case we can not have any information about $\bar{c}_{rj}$. □

Lemma 4.1.4. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let b_i and b_j be two distinct border terms that are across-the-street neighbors where $x_l b_i = x_k b_j$, $k \neq l$ and $x_k, x_l \in \{x_1, ..., x_n\}$.*

a) If there exist terms $t_r, t_s, t_p \in \mathcal{O}$ such that $x_k t_r = x_l t_s = t_p \in \mathcal{O}$, then we have

$$[\bar{c}_{si}] = [\bar{c}_{rj}].$$

b) If there exist $t_s, t_r, t_p \in \mathcal{O}$ such that $x_l t_s = t_p \in \mathcal{O}$ with $x_k^{-1} t_p \notin \mathcal{O}$ or $x_k t_r \in \partial\mathcal{O}$, then we have

$$\bar{c}_{si} \in \mathfrak{m}^2.$$

c) If there exist $t_r, t_s, t_p \in \mathcal{O}$ such that $x_k t_r = t_p \in \mathcal{O}$ with $x_l^{-1} t_p \notin \mathcal{O}$ or $x_l t_s \in \partial\mathcal{O}$, then we have

$$\bar{c}_{rj} \in \mathfrak{m}^2.$$

Proof. Let $\mathcal{A}_k$ denote the k^{th} generic multiplication matrix, where $k \in \{1, ..., n\}$. For each $i \in \{1, ..., \nu\}$ let c_i denote the column matrix $(c_{1i}, ..., c_{\mu i})^{tr}$. From the across-the-street neighborhood relation $x_l b_i = x_k b_j$, it follows that $\mathrm{AS}(i, j)$ contains μ many polynomials which are the entries of the $\mu \times 1$ matrix $\mathcal{A}_k c_j - \mathcal{A}_l c_i$. Consider the polynomial in the p$^{\text{th}}$-row of the matrix $\mathcal{A}_k c_j - \mathcal{A}_l c_i$ whose entries are the elements of the set $\mathrm{AS}(i, j)$. Then by Lemma 2.3.15, this polynomial has the following form.

$$-\sum_{s=1}^{\mu} h_s c_{si} + \sum_{r=1}^{\mu} f_r c_{rj} \tag{4.3}$$

$$f_r = \begin{cases} 0 & \text{if } x_k t_r \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_k t_r = t_p \in \mathcal{O} \\ c_{pn} & \text{if } x_k t_r = b_n \in \partial\mathcal{O} \end{cases}, h_s = \begin{cases} 0 & \text{if } x_l t_s \in \mathcal{O} \setminus \{t_p\} \\ 1 & \text{if } x_l t_s = t_p \in \mathcal{O} \\ c_{pm} & \text{if } x_l t_s = b_m \in \partial\mathcal{O} \end{cases}$$

Since this polynomial is in $\mathcal{I}(\mathbb{B}_\mathcal{O})$, for each $p \in \{1, ..., \mu\}$ Polynomial (4.4) satisfies

$$(\sum_{s=1}^{\mu} h_s \bar{c}_{si}) = (\sum_{r=1}^{\mu} f_r \bar{c}_{rj}) \in \mathrm{B}_\mathcal{O}.$$

Consider the polynomials f_r and h_s in 4.4. By Lemma 2.3.19, a polynomial from $\mathcal{T}$ has at most two linear terms in its support. If there exists $r, s \in \{1, ..., \mu\}$ such that $f_r, h_s = 1$, then we have $[\bar{c}_{si}] = [\bar{c}_{rj}]$. If there exists $s \in \{1, .., \mu\}$ such that $h_s = 1$ and for every $r \in \{1, ..., \mu\}$ we have $f_r \neq 1$, then we have $\bar{c}_{si} \in \mathfrak{m}^2$. If for every $s \in \{1, ..., \mu\}$ we have $h_s \neq 1$ and there exists $r \in \{1, .., \mu\}$ such that $f_r = 1$, then we have $\bar{c}_{rj} \in \mathfrak{m}^2$. If for both $r, s \in \{1, ..., \mu\}$ we have $f_r \neq 1$ and $h_s \neq 1$, then we can not have any information neither on $\bar{c}_{si}$ nor on $\bar{c}_{rj}$. □

Now we show that the arrow grading is well-defined on the set of equivalence classes of indeterminates modulo $\mathfrak{m}^2$.

Lemma 4.1.5. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $\bar{c}_{pq}$ and $\bar{c}_{ij}$ be two elements from $\bar{\mathcal{C}}$ but not in $\mathfrak{m}^2$. If $\bar{c}_{ij}$ and $\bar{c}_{pq}$ have the same equivalence class modulo $\mathfrak{m}^2$, then the indeterminates c_{ij} and c_{pq} have the same arrow degree.*

Proof. Assume c_{ij} and c_{pq} are distinct indeterminates, otherwise the proof is trivial. If we have $[\bar{c}_{ij}] = [\bar{c}_{pq}]$, then there exists a polynomial $\bar{\rho} \in \mathfrak{m}^2$, where

$$\pm\bar{c}_{ij} \mp \bar{c}_{pq} + \bar{\rho} = 0 \in \mathrm{B}_{\mathcal{O}}.$$

Let the number of polynomials in $\mathcal{T}$ be denoted by λ. Then by Equation (2.6) the equality above can be re-written as

$$\pm\, c_{ij} \mp c_{pq} + \rho = \sum_{\kappa=1}^{\lambda} f_\kappa \tau_\kappa \tag{4.4}$$

where $\tau_\kappa \in \mathcal{T}$, $f_\kappa \in \mathrm{K}[\mathcal{C}]$ and $\rho \in \mathfrak{M}^2$. Let $\tau_\kappa^{(1)}$ indicate the homogenous component of degree 1 of τ_κ (see Lemma 2.3.19). Thus by equating the linear parts of Equation (4.4), we get

$$\pm\, \bar{c}_{ij} \mp \bar{c}_{pq} = \sum_{\kappa=1}^{\lambda} \beta_\kappa \tau_\kappa^{(1)}, \tag{4.5}$$

where $\beta_\kappa \in \{1, -1, 0\}$. Let T denote the set $\{\tau_{\kappa_1}, ..., \tau_{\kappa_m}\}$ which is a subset of $\mathcal{T}$ of all τ_κ, where $\beta_\kappa \neq 0$. Since c_{ij} and c_{pq} are distinct, we have $m \geqslant 1$. By Lemma 2.3.19, a polynomial from $\mathcal{T}$ has at most two linear monomials in its support. Since we have $\bar{c}_{pq}, \bar{c}_{ij} \notin \mathfrak{m}^2$, neither $\pm c_{pq}$ nor $\pm c_{ij}$ can be the only linear monomial of the support of a polynomial from $\mathcal{T}$. Let τ_{κ_i} be a polynomial in T that has two monomials in its support and one of them is either $\pm c_{ij}$ or $\pm c_{pq}$. In other words, let c be an element in

$\{c_{ij}, c_{pq}\}$ and let

$$\tau_{\kappa_i}^{(1)} = \pm c_{uv} \mp c$$

where $c_{uv} \in \mathcal{C}$. Then there is no polynomial τ_κ in $\mathcal{T}$ such that the monomial $\pm c_{uv}$ is the homogenous component of τ_κ of degree 1. Otherwise, by Lemmas 4.1.3 and 4.1.4, this contradicts to the assumption that $\bar{c}_{ij}, \bar{c}_{pq} \notin \mathfrak{m}^2$. Thus by Lemma 2.3.19, the supports of the elements of T have exactly two linear monomials. Then Equation (4.4) becomes

$$\pm\, c_{ij} \mp c_{pq} = \sum_{u=1}^{m} \beta_{\kappa_u}(c_{s_u r_u} \pm c_{l_u n_u}) \tag{4.6}$$

where $m \in \mathbb{N}^+$ and $\beta_{\kappa_u} \in \{1, -1\}$. There are two cases. The first case is that there exists a polynomial $\tau_{\kappa_i} \in T$ where $\tau_{\kappa_i}^{(1)} = \pm(c_{pq} - c_{ij})$. Proposition 3.2.6 states that polynomials from $\mathcal{T}$ are homogenous with respect to the arrow grading, and hence we have $\deg_W(c_{ij}) = \deg_W(c_{pq})$. In the second case Equation (4.6) becomes

$$\pm\, \mathbf{c_{ij}} \mp \mathbf{c_{pq}} = \pm(\mathbf{c_{ij}} \mp c_{l_1 n_1}) \pm (c_{l_1 n_1} \mp c_{s_2 r_2}) \pm \cdots \pm (c_{s_m r_m} \mp \mathbf{c_{pq}}) \tag{4.7}$$

and we get a telescopic sum of indeterminates where each $(c_{s_u r_u} \pm c_{l_u n_u})$ is $\tau_{\kappa_u}^{(1)}$. Then for each $\kappa_i \in \{\kappa_1, ..., \kappa_m\}$ the pairwise intersection of the supports of τ_{κ_i} and $\tau_{\kappa_{i+1}}$ is nonempty. By Proposition 3.2.6 every polynomial in $\mathcal{T}$ is homogenous with respect to the arrow degree. Hence the arrow degrees of each τ_{κ_i} in the telescopic sum 4.7 are the same. Thus we have

$$\deg_W(c_{ij}) = \deg_W(c_{l_1 n_1}) = ... = \deg_W(c_{s_m r_m}) = \deg_W(c_{pq}),$$

which is the desired result. □

Proposition 4.1.6. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $[\bar{c}_{ij}]$ indicate the equivalence class of $\bar{c}_{ij}$ modulo $\mathfrak{m}^2$. Let $\tau_\kappa^{(1)}$ denote the homogenous component of the polynomial $\tau_\kappa \in \mathcal{T}$ of degree 1 with respect to the standard grading.*

a) *An indeterminate $\bar{c}_{ij} \in \mathrm{B}_\mathcal{O}$ is in $\mathfrak{m}^2$ if and only if there exist polynomials in $\mathcal{T}$ such that $\bar{c}_{ij} = \sum_{\kappa=1}^{\lambda} \alpha_\kappa \tau_\kappa^{(1)}$, where $\alpha_\kappa \in \{0, 1, -1\}$.*

b) *Let c_{sm} and c_{ij} be two distinct indeterminates from $\mathcal{C}$. Then $\bar{c}_{sm}$ is in $[\bar{c}_{ij}]$ if and only if $\pm(\bar{c}_{ij} - \bar{c}_{sm}) = \sum_{\kappa=1}^{\lambda} \alpha_\kappa \tau_\kappa^{(1)}$ where $\alpha_\kappa \in \{0, 1, -1\}$ holds.*

c) *Let $\bar{c}_{ij}$ be a non-zero indeterminate from $\mathrm{B}_\mathcal{O}$. The equivalence class $[\bar{c}_{ij}]$ contains*

only $\bar{c}_{ij}$ if and only if there is no $\tau_\kappa \in \mathcal{T}$ that contains c_{ij} in its support.

Proof. First we prove a). If we have $c_{ij} = \sum_{k=1}^{\lambda} \alpha_k \tau_k^{(1)}$, then there exists at least one index $k \in \{1, .., \lambda\}$ where $\alpha_k \neq 0$ and $\tau_k^{(1)}$ is a monomial of degree 1, say c_{pq}. This implies that $\bar{c}_{pq}$ is in $\mathfrak{m}^2$ and the claim follows from the transitivity. For the converse, if we have $\bar{c}_{ij} \in \mathfrak{m}^2$, then there exists a polynomial $\rho \in \mathfrak{M}^2$ such that $\rho \mp c_{ij} \in \mathcal{I}(\mathbb{B}_\mathcal{O})$. This implies $\rho \mp c_{ij} = \sum_{\kappa=1}^{\lambda} \beta_\kappa \tau_\kappa$, where $\beta_k \in \mathrm{K}[\mathcal{C}]$. By equating the linear polynomials, we get the desired result.

Now we prove b). Let $\bar{c}_{sm}$ be an element of $\bar{\mathcal{C}}$. We have $\bar{c}_{sm} \in [\bar{c}_{ij}]$ if and only if there exists a polynomial $\rho \in \mathfrak{M}^2$ such that $\rho \pm (c_{ij} - c_{sm}) \in \mathcal{I}(\mathbb{B}_\mathcal{O})$. Therefore we have $\rho \pm (c_{ij} - c_{sm}) = \sum_{\kappa=1}^{\lambda} f \tau_\kappa$, where $f \in \mathrm{K}[\mathcal{C}]$. Thus the claim follows by considering each $\tau_\kappa^{(1)}$.

Finally, we prove c) by way of contradiction. Suppose $[\bar{c}_{ij}]$ contains $\bar{c}_{sm}$ and c_{ij} is not an element of any support of τ_κ, for any κ. This contradicts b). Now suppose $\pm c_{ij}$ is in the support of a τ_κ. By a) and b), we have either $\bar{c}_{ij} \in \mathfrak{m}^2$ or $\bar{c}_{ij} \in [\bar{c}_{km}]$, which contradicts to our assumption. □

Remark 4.1.7. Proposition 4.1.6 implies that, by examining the polynomials in $\mathcal{T}$ modulo $\mathfrak{m}^2$, we can find every equivalence class and its elements.

Example 4.1.8. Let P denote the polynomial ring $\mathrm{K}[x_1, x_2]$. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2, x_1x_2\}$. Then $\partial\mathcal{O} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$ is the border of it, where in our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_2, t_4 = x_1x_2, b_1 = x_2^2, b_2 = x_1^2, b_3 = x_1x_2^2$ and $b_4 = x_1^2x_2$. Let us compute the set $\mathcal{T}$ for $\mathcal{I}(\mathbb{B}_\mathcal{O})$.

$$
\begin{aligned}
\tau_{12} &= c_{11}c_{32} + c_{13}c_{42} - c_{14} \\
\tau_{13} &= c_{12}c_{21} + c_{14}c_{41} - c_{13} \\
\tau_{14} &= c_{12}c_{23} - c_{11}c_{34} - c_{14}c_{43} + c_{13}c_{44} \\
\tau_{22} &= c_{21}c_{32} + c_{23}c_{42} - c_{24} \\
\tau_{23} &= c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23} \\
\tau_{24} &= c_{22}c_{23} - c_{21}c_{34} - c_{24}c_{43} + c_{23}c_{44} + c_{13} \\
\tau_{32} &= c_{13}c_{32} + c_{33}c_{42} + c_{12} - c_{34} \\
\tau_{33} &= c_{21}c_{32} + c_{34}c_{41} - c_{33} \\
\tau_{34} &= c_{23}c_{32} - c_{31}c_{34} - c_{34}c_{43} + c_{33}c_{44} - c_{14} \\
\tau_{42} &= c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}
\end{aligned}
$$

$$\begin{aligned}\tau_{43} &= c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}\\ \tau_{44} &= c_{34}c_{41} - c_{23}c_{42} - c_{24} + c_{33}\end{aligned}$$

By Proposition 4.1.6 a) and b) and by examining $\tau_{12}, \tau_{13}, \tau_{33}, \tau_{24}$, we identify the set of indeterminates in $\mathfrak{m}^2$ and get $\{\bar{c}_{14}, \bar{c}_{24}, \bar{c}_{13}, \bar{c}_{33}\}$. From Proposition 4.1.6 b) and by computing τ_{23} modulo $\mathfrak{m}^2$, we get that $\bar{c}_{11}$ and $\bar{c}_{23}$ are in the same equivalence class. In the same way, $\bar{c}_{31}$ and $\bar{c}_{43}$ are in the same equivalence class, $\bar{c}_{12}$ and $\bar{c}_{24}$ are in the same equivalence class, and finally $\bar{c}_{21}$ and $\bar{c}_{34}$ are in the same equivalence class. By Proposition 4.1.6 c), the remaining equivalence classes contain just one indeterminate which are given by $\{\bar{c}_{41}\}, \{\bar{c}_{42}\}, \{\bar{c}_{32}\}$ and $\{\bar{c}_{21}\}$. All non-zero equivalence classes are then given by

$$\{\bar{c}_{41}\},\ \{\bar{c}_{42}\},\ \{\bar{c}_{21}\},\ \{\bar{c}_{32}\},\ \{\bar{c}_{31}, \bar{c}_{43}\},\ \{\bar{c}_{11}, \bar{c}_{23}\},\ \{\bar{c}_{22}, \bar{c}_{44}\},\ \{\bar{c}_{12}, \bar{c}_{34}\}.$$

We choose one indeterminate from each non-zero equivalence class as a representative and form the set of equivalence classes. The result is

$$\mathfrak{E} = \{[\bar{c}_{11}], [\bar{c}_{21}], [\bar{c}_{31}], [\bar{c}_{41}], [\bar{c}_{12}], [\bar{c}_{22}], [\bar{c}_{32}], [\bar{c}_{42}]\}.$$

Remark 4.1.9. Consider the polynomial

$$\tau_{44} = c_{24}c_{41} - c_{33}c_{42} - c_{23} + c_{34}$$

in Example 4.1.8. Clearly, the elements $\bar{c}_{23}, \bar{c}_{34}$ of $\bar{\mathcal{C}}$ are equivalent modulo $\mathfrak{m}^2$. We denote this relation by

$$\bar{c}_{23} \sim \bar{c}_{34}.$$

The set $\mathcal{T}$ also contains

$$\tau_{33} = c_{22}c_{31} + c_{33}c_{42} - c_{34}.$$

Then Proposition 4.1.6a) and b) imply $\bar{c}_{23}, \bar{c}_{34} \in \mathfrak{m}^2$. Despite the fact that in Example 4.1.8 there is a neighborhood relation between b_4 and b_3 which results in the relation between c_{23} and c_{34}, this information cannot be obtained from the set in (4.1.8).

To keep this information we introduce the following notation.

Notation 4.1.10. We let $\mathfrak{E}$ denote the set of non-zero equivalence classes of elements

from $\bar{\mathcal{C}}$. Let $\{\bar{c}_{i_1j_1}, ..., \bar{c}_{i_mj_m}\}$ be a subset of $\bar{\mathcal{C}}$, where $m > 1$. If we have

$$\bar{c}_{i_1j_1} \sim \bar{c}_{i_2j_2} \sim \cdots \sim \bar{c}_{i_mj_m},$$

then we denote this subset by $C_{i_pj_p}$ where $\bar{c}_{i_pj_p}$ is an arbitrary element of $\{\bar{c}_{i_1j_1}, ..., \bar{c}_{i_mj_m}\}$. If for each $p \in \{1, ..., m\}$ we have $\bar{c}_{i_pj_p} \notin \mathfrak{m}^2$, then we have $C_{i_pj_p} = [\bar{c}_{i_pj_p}] \in \mathfrak{E}$.

Example 4.1.11. (**continues**) Recall Example 4.1.8. The set in (4.1.8) is $\mathfrak{E}$ i.e.,

$$\mathfrak{E} = \{[\bar{c}_{11}], [\bar{c}_{21}], [\bar{c}_{31}], [\bar{c}_{41}], [\bar{c}_{12}], [\bar{c}_{22}], [\bar{c}_{32}], [\bar{c}_{42}]\}.$$

We have $\bar{c}_{23} \in \mathfrak{m}^2$ and $\bar{c}_{13} \in \mathfrak{m}^2$, as well. The set $C_{23} = \{\bar{c}_{23}, \bar{c}_{34}\}$ however does not contain $\bar{c}_{13}$, since there exists no polynomial in $\mathcal{T}$ that gives a relation between c_{23} (or similarly c_{34}) and c_{13}. Moreover, we have $C_{13} = \{\bar{c}_{13}\}$, $C_{14} = \{c_{14}\}$, $C_{11} = [\bar{c_{11}}], C_{21} = [\bar{c_{21}}], C_{31} = [\bar{c_{31}}], C_{41} = [\bar{c_{41}}], C_{12} = [\bar{c_{12}}], C_{22} = [\bar{c_{22}}], C_{32} = [\bar{c_{32}}]$, and finally, $C_{42} = [\bar{c_{42}}]$.

Definition 4.1.12. Let c_{ij} be an element from $\mathcal{C}$. Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal and $\partial\mathcal{O}$ be its border.

a) The indeterminate c_{ij} is called **standard**, if $\deg_W(c_{ij})$ has exactly one positive component. Otherwise it is called **non-standard**.

b) The indeterminate c_{ij} is called $\mathbf{x_k}$**-standard**, if only the k^{th} component of the degree vector $\deg_W(c_{ij})$ is positive.

Definition 4.1.13. Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let b_j be a border term, $x_k \in \{x_1, ..., x_n\}$ and w_k be a positive integer. If we have $b_j = x_k^{\omega_k}$, then b_j is called an $\mathbf{x_k}$**-pure power term**. For any order ideal term t_i, the indeterminate $c_{ij} \in \mathcal{C}$ is called an $\mathbf{x_k}$**-pure power indeterminate**.

Example 4.1.14. Every pure power indeterminate is standard. More precisely, if an indeterminate c_{ij} is an x_k-pure power indeterminate, then the indeterminate c_{ij} is an x_k-standard indeterminate.

Definition 4.1.15. Let $\mathrm{BT}_\mathcal{O}$ denote the border term ideal and $\mathcal{G}$ the minimal generating set of the border term ideal consisting of border terms. Let c_{ij} be an indeterminate from $\mathcal{C}$. If the border term b_j is from the minimal generating set of the border term ideal $\mathcal{G}$, then c_{ij} is called a **minimal indeterminate**. If the indeterminate c_{ij} is a standard indeterminate, then it is called a **minimal standard indeterminate**.

Corollary 4.1.16. *Let c_{ij} be an x_k-standard indeterminate from $\mathcal{C}$. If for every $x_l \in \{x_1, ..., x_n\} \setminus \{x_k\}$ we have $x_l t_i \in \partial\mathcal{O}$, then the element $\bar{c}_{ij} \in \bar{\mathcal{C}}$ is not in $\mathfrak{m}^2$. In particular, if we have $\bar{c}_{ij} \notin \mathfrak{m}^2$, then there exists $\bar{c}_{i_p j_p} \in [\bar{c}_{ij}]$ such that for each $x_l \in \{x_1, ..., x_n\} \setminus \{x_k\}$ we have $x_l t_{i_p} \in \partial\mathcal{O}$.*

Example 4.1.17. (continues) Consider Example 4.1.8 again. The set of equivalence classes is

$$\mathfrak{E} = \{[\bar{c}_{11}], [\bar{c}_{21}], [\bar{c}_{31}], [\bar{c}_{41}], [\bar{c}_{12}], [\bar{c}_{22}], [\bar{c}_{32}], [\bar{c}_{42}]\}.$$

The border term ideal is generated by $\{x_1^2, x_2^2, x_1^2x_2, x_1x_2^2\}$. The minimal generating set of the border term ideal consists of the pure power terms $b_2 = x_1^2$ and $b_1 = x_2^2$. The pure power indeterminates are always standard. In this case they are also minimal.

Now we ask ourselves whether it is possible to form the set of equivalence classes $\mathfrak{E}$ by using only pure power indeterminates. The following example gives a negative answer to this question.

Example 4.1.18. Let P denote the polynomial ring $\mathrm{K}[x_1, x_2]$ and let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2\}$. Then the border of it is $\partial\mathcal{O} = \{x_2^2, x_1x_2, x_1^2\}$ where with respect to our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_2, b_1 = x_2^2, b_2 = x_1x_2$ and $b_3 = x_1^2$. The vanishing ideal of the border basis scheme is generated by the following polynomials.

$$\begin{aligned}
\tau_{12} &= c_{13}c_{22} - c_{12}c_{23} + c_{12}c_{32} - c_{11}c_{33},\\
\tau_{13} &= c_{13}c_{21} - c_{12}c_{22} + c_{12}c_{31} - c_{11}c_{32},\\
\tau_{22} &= c_{22}c_{32} - c_{21}c_{33} + c_{12},\\
\tau_{23} &= -c_{22}^2 + c_{21}c_{23} + c_{2,2}c_{31} - c_{21}c_{3,2} + c_{11},\\
\tau_{32} &= -c_{23}c_{32} + c_{32}^2 + c_{22}c_{33} - c_{31}c_{33} - c_{13},\\
\tau_{33} &= -c_{22}c_{32} + c_{21}c_{33} - c_{12}
\end{aligned}$$

The elements of the set $\{\bar{c}_{13}, \bar{c}_{12}, \bar{c}_{11}\}$ are in $\mathfrak{m}^2$. The equivalences classes are

$$\{\bar{c}_{21}\}, \{\bar{c}_{22}\}, \{\bar{c}_{23}\}, \{\bar{c}_{31}\}, \{\bar{c}_{32}\}, \{\bar{c_{33}}\}.$$

Therefore the set of equivalence classes is $\mathfrak{E} = \{[\bar{c}_{21}], [\bar{c}_{22}], [\bar{c}_{23}], [\bar{c}_{31}], [\bar{c}_{32}], [\bar{c}_{33}]\}$. The minimal generating set of the border term ideal is $\{x_1^2, x_1x_2, x_2^2\}$. This means that every indeterminate here is minimal. Moreover, one should note that the indeterminates c_{13}, c_{11} are pure power indeterminates that are standard and minimal, yet $\bar{c}_{13}$ and $\bar{c}_{11}$ are in $\mathfrak{m}^2$.

Proposition 4.1.19. *Every equivalence class in $\mathfrak{E}$ contains at least one element, say $\bar{c}_{ij}$, from $\bar{\mathcal{C}}$ where c_{ij} is a minimal indeterminate.*

Proof. Let $\bar{c}_{pq}$ be an element of $\bar{\mathcal{C}}$ and let $[\bar{c}_{pq}]$ be an equivalence class in $\mathfrak{E}$. As a consequence of the definition of $\mathfrak{E}$, the element $\bar{c}_{pq}$ is not in $\mathfrak{m}^2$. Let $[\bar{c}_{pq}]$ be given by

$$[\bar{c}_{pq}] = \{\bar{c}_{pq}, \bar{c}_{p_1q_1}, ..., \bar{c}_{p_mq_m}\}$$

where $m \geqslant 0$. Keep the elements of $[\bar{c}_{pq}]$ in mind and consider the set of border elements $B = \{b_q, b_{q_1}, ..., b_{q_m}\}$. Choose the smallest border term from this set with respect to the `Lex` term ordering, say b_{q_k}. We want to show that $c_{p_kq_k}$ is a minimal indeterminate i.e., $b_{q_k} \in \mathcal{G}$. For contradiction suppose that there exists a term $b_j \in \partial\mathcal{O}$ such that $b_j \mid b_{q_k}$ and $b_j <_{\texttt{Lex}} b_{q_k}$. Then there exists a term $x_k \in \{x_1, ..., x_n\}$ such that $x_k b_j = b_{q_k}$. There are two cases to examine. The first case is that there exists a term $t_i \in \mathcal{O}$ such that $x_k t_i = t_{p_k}$. Then from Lemma 4.1.3 follows $\bar{c}_{ij} \in [\bar{c}_{pq}]$. Thus the border term b_j is in B. This contradicts the fact that b_{q_k} is the smallest element of B. The second case is that an order ideal term t_i as in the first case does not exist. That is $x_k^{-1} t_i \notin \mathcal{O}$ and $x_k b_j = b_{q_k}$. Then by Lemma 4.1.3, the element $\bar{c}_{p_kq_k}$ is in $\mathfrak{m}^2$. This contradicts the assumption that $\bar{c}_{pq} \in \mathfrak{E}$. □

4.2 A Smoothness Criterion for the Monomial Point of a Border Basis Scheme

In [Huib05], Theorem 5.1.1 gives a smoothness criterion for the monomial point $(0, ..., 0)$ in a Hilbert scheme. Inspired by this result, in this section we give a smoothness criterion for the monomial point of a border basis scheme based on methods we introduced in the previous section. The smoothness criterion we give can be deduced from Theorem 5.1.1, [Huib05]. Instead of using this deduction, we choose to employ the border basis scheme theory. This approach will play a fundamental role in Chapter 6.

Before starting, let us recall some of our definitions and notation. Let K be a field and L ⊃ K be a field extension. Let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be an order ideal and $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ its border. Let $\mathcal{C} = \{c_{11}, ..., c_{\mu\nu}\}$ be a set of further indeterminates and $\mathrm{K}[\mathcal{C}] = \mathrm{K}[c_{11}, ..., c_{\mu\nu}]$. Let $\bar{\mathcal{C}}$ denote the set $\{\bar{c}_{11}, ..., \bar{c}_{\mu\nu}\}$, where each $\bar{c}_{ij} \in \mathrm{B}_\mathcal{O}$. Let $\mathcal{T}$ denote the set of generators of the vanishing ideal of a border basis scheme as given in Equation (2.6). We let $\mathfrak{m}$ denote the maximal ideal of $\mathrm{B}_\mathcal{O}$ that is generated by the elements of $\bar{\mathcal{C}}$. Moreover, we

assume that $(\mathrm{B}_{\mathcal{O}})_{\mathfrak{m}}/\mathfrak{m}$ is isomorphic to the field L. Then the cotangent space of $\mathbb{B}_{\mathcal{O}}$ at the point $\mathfrak{o} = (0, ..., 0)$, which is $\mathfrak{m}/\mathfrak{m}^2$, is an L-vector space. Let $\mathfrak{E}$ be the set of non-zero equivalence classes of elements from $\bar{\mathcal{C}}$. Let $\mathrm{BT}_{\mathcal{O}}$ denote the border term ideal and $\mathcal{G}$ denote the minimal generating set of the border term ideal.

Example 4.2.1. Let P denote the polynomial ring $\mathrm{K}[x_1, x_2, x_3]$ and let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2, x_3\}$. Then the border of it is $\partial\mathcal{O} = \{x_3^2, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2\}$ where with respect to our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_2, t_4 = x_3 b_1 = x_3^2, b_2 = x_2x_3, b_3 = x_1x_3, b_4 = x_2^2, b_5 = x_1x_2$ and $b_6 = x_1^2$. The vanishing ideal of the border basis scheme is generated by homogenous (with respect to the standard grading) quadratic polynomials plus the following polynomials.

$$
\begin{aligned}
\tau_{22}^{12} &= c_{25}c_{35} - c_{24}c_{36} + c_{23}c_{45} - c_{22}c_{46} + c_{15}\\
\tau_{23}^{12} &= -c_{25}^2 + c_{24}c_{26} + c_{25}c_{34} - c_{24}c_{35} + c_{23}c_{44} - c_{22}c_{45} + c_{14}\\
\tau_{24}^{12} &= -c_{23}c_{25} + c_{22}c_{26} + c_{25}c_{32} - c_{24}c_{33} + c_{23}c_{42} - c_{22}c_{43} + c_{12}\\
\tau_{32}^{12} &= -c_{26}c_{35} + c_{35}^2 + c_{25}c_{36} - c_{34}c_{36} + c_{33}c_{45} - c_{32}c_{46} - c_{16}\\
\tau_{33}^{12} &= -c_{25}c_{35} + c_{24}c_{36} + c_{33}c_{44} - c_{32}c_{45} - c_{15}\\
\tau_{34}^{12} &= -c_{33}c_{34} - c_{23}c_{35} + c_{32}c_{35} + c_{22}c_{36} + c_{33}c_{42} - c_{32}c_{43} - c_{13}\\
\tau_{22}^{13} &= c_{25}c_{33} - c_{22}c_{36} + c_{23}c_{43} - c_{21}c_{46} + c_{13}\\
\tau_{23}^{13} &= -c_{23}c_{25} + c_{22}c_{26} + c_{25}c_{32} - c_{22}c_{35} + c_{23}c_{42} - c_{21}c_{45} + c_{12}\\
\tau_{24}^{13} &= -c_{23}^2 + c_{21}c_{26} + c_{25}c_{31} - c_{22}c_{33} + c_{23}c_{41} - c_{21}c_{43} + c_{11}\\
\tau_{42}^{13} &= -c_{36}c_{42} - c_{26}c_{43} + c_{43}^2 + c_{33}c_{45} + c_{23}c_{46} - c_{41}c_{46} - c_{16}\\
\tau_{43}^{13} &= -c_{35}c_{42} - c_{25}c_{43} + c_{42}c_{43} + c_{32}c_{45} - c_{41}c_{45} + c_{22}c_{46} - c_{15}\\
\tau_{44}^{13} &= -c_{33}c_{42} - c_{23}c_{43} + c_{31}c_{45} + c_{21}c_{46} - c_{13}\\
\tau_{32}^{23} &= -c_{25}c_{33} + c_{33}c_{34} + c_{23}c_{35} - c_{32}c_{35} + c_{32}c_{43} - c_{31}c_{45} + c_{13}\\
\tau_{33}^{23} &= -c_{24}c_{33} + c_{22}c_{35} + c_{32}c_{42} - c_{31}c_{44} + c_{12}\\
\tau_{34}^{23} &= -c_{32}^2 - c_{22}c_{33} + c_{31}c_{34} + c_{21}c_{35} + c_{32}c_{41} - c_{31}c_{42} + c_{11}\\
\tau_{42}^{23} &= -c_{35}c_{42} - c_{25}c_{43} + c_{42}c_{43} + c_{33}c_{44} + c_{23}c_{45} - c_{41}c_{45} - c_{15}\\
\tau_{43}^{23} &= -c_{34}c_{42} + c_{42}^2 - c_{24}c_{43} + c_{32}c_{44} - c_{41}c_{44} + c_{22}c_{45} - c_{14}\\
\tau_{44}^{23} &= -c_{32}c_{42} - c_{22}c_{43} + c_{31}c_{44} + c_{21}c_{45} - c_{12}
\end{aligned}
$$

By Proposition 4.1.6 a), the residue classes of indeterminates contained in $\mathfrak{m}^2$ are

$$\{\bar{c}_{16}, \bar{c}_{15}, \bar{c}_{14}, \bar{c}_{13}, \bar{c}_{12}, \bar{c}_{11}\}.$$

The minimal generating set of the border term ideal is exactly the border. Therefore all the indeterminates in the set $\mathcal{C}$ are minimal. This means that border terms do not have next door neighbors and all polynomials from $\mathcal{T}$ are from AS. By Proposition 4.1.6 c), the set of equivalence classes is $\mathfrak{E} = \{[\bar{c}_{ij}] : c_{ij} \in \mathcal{C} \setminus \{c_{16}, c_{15}, c_{14}, c_{13}, c_{12}, c_{11}\}\}$. Note that c_{14}, c_{16}, c_{11} are pure power indeterminates that are standard and minimal, yet their residue classes are contained in $\mathfrak{m}^2$.

Remark 4.2.2. The indeterminates c_{22}, c_{23} and c_{45} in Example 4.2.1 are non-standard indeterminates, yet their residue classes are not in $\mathfrak{m}^2$. One should also compare the number of elements in $\mathfrak{E}$ in Examples 4.2.1, 4.1.18 and 4.1.8. In Examples 4.1.18 and 4.1.8, the set $\mathfrak{E}$ contains $n\mu$ elements but in the Example 4.2.1 the number of elements of $\mathfrak{E}$ is larger than $n\mu$.

Notation 4.2.3. Let $t = x_1^{\alpha_1} \cdots x_k^{\alpha_k} \cdots x_n^{\alpha_n}$ be a term in $\mathbb{T}^n$. We denote the k^{th}-component of the vector $\log(t)$ by $\deg_{x_k}(t)$. In this case we have

$$\deg_{x_k}(t) = \alpha_k.$$

Remark 4.2.4. Recall that $\mathcal{G}$ is the set of minimal generators of the border term ideal consisting of border terms.

a) If the set $\mathcal{G}$ consists of pure power terms only, then every border term can be divided by exactly one pure power term.

b) If an indeterminate c_{pq} is x_k-standard, then for $l \neq k$ the border term b_q can not be divisible by an x_l-pure power term.

c) For distinct indices k and l, let the indeterminate c_{ij} be an x_k-pure power indeterminate, and let c_{pq} be an x_l-pure power indeterminate. Distinct pure power indeterminates have different arrow degrees and by Lemma 4.1.5, the equivalence classes of $\bar{c}_{ij}$ and $\bar{c}_{pq}$ modulo $\mathfrak{m}^2$ are distinct.

Lemma 4.2.5. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. If the minimal generating set of $\mathrm{BT}_{\mathcal{O}}$ consists of only pure power terms, then $\mathfrak{E}$ has exactly $n\mu$ elements.*

Proof. Clearly, the number of pure power indeterminates is $n\mu$, which is in our case also the number of minimal standard indeterminates. By Proposition 4.1.19, every equivalence class in $\mathfrak{E}$ has at least one element from $\bar{\mathcal{C}}$ that corresponds to a minimal indeterminate in $\mathcal{C}$. Since two distinct elements from $\bar{\mathcal{C}}$, which are mapped to distinct

pure power indeterminates, can not be in the same equivalence class of $\mathfrak{E}$, it follows that every equivalence class in $\mathfrak{E}$ contains exactly an element that corresponds to a pure power indeterminate.

If we show that there exists no pure power indeterminate, say c_{ij}, such that $\bar{c}_{ij} \in \mathfrak{m}^2$, then we have the desired result. For this purpose, let the indeterminate c_{ij} be an x_k-pure power indeterminate. By examining the set $\mathcal{T}$, we construct the set of elements from $\bar{\mathcal{C}}$ that are in relation $\sim$ with c_{ij}, i.e. C_{ij}. Let $B_j = \{b_j, b_1, ..., b_{j_m}\}$ denote the set of border terms which are divisible by b_j and x_l. Now we choose an element $\bar{c}_{i_r j_r} \in C_{ij}$. Let $x_l b_{j_r}$ be a term, say b_{j_l}, in B. Since $c_{i_r j_r}$ is an x_k-standard indeterminate, there are two cases. The first case is that we have a border term $x_l b_{j_r} \in B$, and therefore $x_l t_{i_r}$ is a term in $\mathcal{O}$, say t_{j_l}. Then by Lemma 4.1.3a) and Lemma 4.1.4 b),c), we have $\bar{c}_{i_l j_l} \sim \bar{c}_{i_r j_r}$. The second case is that $x_l b_{j_r} \notin B$. Since $\mathcal{G}$ consists of pure powers only, the term $x_l b_{j_r}$ is divisible by the x_l-pure power term. Keep in mind that $c_{i_r j_r}$ is an x_k-standard indeterminate i.e., $\deg_{x_l}(t_{i_r}) > \deg_{x_l}(b_{j_r})$. Hence $x_l t_i$ is a term in $\partial\mathcal{O}$. Thus we have either an equivalence relation with an element of C_{ij} or $x_l t_{i_r} \in \partial\mathcal{O}$. Since this holds for any $x_l \in \{x_1..., x_n\} \setminus \{x_k\}$, by Corollary 4.1.16 and Proposition 4.1.6 b),c), we have $c_{i_l j_l} \notin \mathfrak{m}^2$. □

Definition 4.2.6. Let c_{ij} be an indeterminate from $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$. Let the arrow degree of c_{ij} be $(\alpha_1, ..., \alpha_k, ..., \alpha_n)$. The **$\mathbf{x_k}$-offset** of c_{ij} is the arrow degree of c_{ij} without the x_k-component. We denote it by

$$\mathbf{offset_{x_k}(c_{ij})} = (\alpha_\mathbf{1}, ..., \alpha_{\mathbf{k-1}}, \alpha_{\mathbf{k+1}}, ..., \alpha_\mathbf{n}).$$

The idea of replacing an x_k-standard pure power indeterminate that is an element of $\mathfrak{m}^2$ with a minimal standard indeterminate which has the same x_k-offset and that is not contained in $\mathfrak{m}^2$, is due to [Huib05]. The process described in [Huib05] is called *Shadow Promotion* and it is defined for *arrows*. Inspired by this technique, we introduce a similar method in our setting for border basis schemes.

There are $n\mu$ distinct pure power indeterminates in $\mathcal{C}$. Recall that pure power indeterminates are standard and distinct pure power indeterminates indicate distinct equivalence classes in $\mathfrak{m}^2$. Let $[\bar{c}_{ij}]$ be the equivalence class of an x_k-pure power indeterminate c_{ij} where $x_l t_i \in \mathcal{O}$ for $x_l \in \{x_1, ..., x_n\} \setminus \{x_k\}$. The following algorithm shows how to find an equivalence class $[\bar{c}_{pq}] \in \mathfrak{E}$ where c_{pq} is a minimal x_k-standard indeterminate which is determined by the term t_i and the x_k-offset of c_{ij}.

Algorithm 4.2.7. Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $b_j = x_k^{\omega_k}$ be an x_k-pure power indeterminate and $t_i \in \mathcal{O}$. Let c_{ij} be an x_k-pure power indeterminate where there exists an indeterminate $x_l \in \{x_1, ..., x_n\} \setminus \{x_k\}$ such that $x_l t_i \in \mathcal{O}$. Consider the following sequence of instructions.

1. Let $l = 0$.

2. Increase l by one. If we have $l = k$, then increase l by one again. If we have $l = n + 1$, then let $t_r := t_{i_l}$ and continue with step 4) otherwise continue with step 3)

3. Let $t_{i_l} := t_i$ and $\beta_l := 0$. Repeat computing $t_{i_l} = x_l \cdot t_{i_l}$ and increasing β_l by one until $t_{i_l} \in \partial\mathcal{O}$. Then let $t_{i_l} := t_{i_l}/x_l$, $\beta_l - 1$ and continue with step 2).

4. Let $x_1^{\beta_1} \cdots x_n^{\beta_n}$ be a term in $\mathbb{T}^n$ that is denoted by b where $\beta_k = 0$ and $b \cdot t_i = t_r \in \mathcal{O}$. Repeat computing $x_k \cdot b := b$ until $b \in \partial\mathcal{O}$ and let $b_s := b$. Then construct the equivalence class $[\bar{c}_{rs}]$ and pick a minimal standard indeterminate $\bar{c}_{pq}$ from $[\bar{c}_{rs}]$.

This is an algorithm which returns a minimal x_k-standard indeterminate, c_{pq} which has the same x_k-offset as the x_k-pure power indeterminate c_{ij} and $[\bar{c}_{pq}] \in \mathfrak{E}$.

Proof. There are finitely many terms in $\mathcal{O}$ and finitely many indeterminates in $\{x_1, ..., x_n\}$. Thus this algorithm stops after finitely many steps. Since in Step 3) after computation t_{l_i} is still an order ideal term, the term b and $b \cdot t_i = t_r$ are order ideal terms where $\deg_{x_k}(b) = 0$. Therefore there exists an integer α such that $x_k^\alpha b \in \partial\mathcal{O}$ which is computed in step 4). From this follows the existence of the border term b_s in step 4). By steps 2) and 3), the order ideal term t_r has the following property. For each $x_l \in \{x_1, ..., x_n\} \setminus \{x_k\}$ we have $x_l t_r \in \partial\mathcal{O}$. Then by Corollary 4.1.16, we have $\bar{c}_{rs} \notin \mathfrak{m}^2$. Therefore by Proposition 4.1.19, there exists a minimal indeterminate c_{pq} such that $\bar{c}_{pq} \in [\bar{c}_{rs}]$. Moreover, from step 4) we have $b_s = x_k^m \cdot b$ and $t_r = b \cdot t_i$ where $m \in \mathbb{N}_+$ and $\beta_k = 0$. Let us determine the arrow degree of c_{rs}.

$$\begin{aligned} \deg_W(c_{rs}) &= \log(b_s) - \log(t_r) \\ &= \log(x_1^{\beta_1} \cdots x_k^m \cdots x_n^{\beta_n}) - \log(x_1^{\beta_1} \cdots x_{k-1}^{\beta_{k-1}} \cdot x_{k+1}^{\beta_{k+1}} \cdots x_n^{\beta_n} \cdot t_i) \\ &= \log(x_k^m) - \log(t_i) \end{aligned}$$

From the above equation follows that c_{rs} is an x_k-standard indeterminate. Since $b_j = x_k^{\omega_j}$ is a pure power term, we have $\mathrm{offset}_{x_k}(c_{rs}) = \mathrm{offset}_{x_k}(t_i) = \mathrm{offset}_{x_k}(c_{ij})$.

Consequently, from $\bar{c}_{pq} \in [\bar{c}_{rs}]$ follows that c_{pq} is a minimal x_k-standard indeterminate and by Lemma 4.1.5 we have $\text{offset}_{x_k}(c_{rs}) = \text{offset}_{x_k}(c_{pq}) = \text{offset}_{x_k}(c_{ij})$. □

Example 4.2.8. (continues) Recall Example 4.1.18 where $\mathcal{O}$ is the order ideal $\{1, x_1, x_2\}$. Then $\partial\mathcal{O} = \{x_2^2, x_1x_2, x_1^2\}$ is the border of it. The set of pure power indeterminates is $\{\mathbf{c_{11}}, c_{21}, c_{31}, \mathbf{c_{13}}, c_{23}, c_{33}\}$. The indeterminates c_{13}, c_{11} are pure power indeterminates that are standard and minimal yet $\bar{c}_{13}$ and $\bar{c}_{11}$ are in $\mathfrak{m}^2$. We replace c_{13} with a minimal standard indeterminate that is an output of Algorithm 4.2.7. We have $b_3 = x_1^2$, $k = 1$, $t_1 = 1$ and $\deg_W(c_{13}) = (2, 0)$ with $\text{offset}_{x_1} = (0)$

2) Set $l = 2$ since $k = 1$.

3) We have $x_2t_1 = t_3 \in \mathcal{O}$ but $x_2^2 \in \partial\mathcal{O}$. Then we let $t_{i_2} = t_3$ and $\beta_l := 1$.

2) We have $l = 3$ then we let $t_r := t_3$.

4) The term b is x_2 and $x_1 \cdot x_2 = b_2 \in \partial\mathcal{O}$ and $t_i = 1$. Then we let $b_s := b_2$. Since b_2 is already minimal, we have $c_{pq} = c_{rs} = c_{32}$

By the same way we can replace c_{11} with c_{22}. Therefore we have

$$\mathfrak{E} = \{[\bar{\mathbf{c}}_{\mathbf{22}}], [\bar{c}_{21}], [\bar{c}_{31}], [\bar{\mathbf{c}}_{\mathbf{32}}], [\bar{c}_{23}], [\bar{c_{33}}]\}.$$

Theorem 4.2.9. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $\mathfrak{E}$ denote the set which contains the equivalence classes of indeterminates modulo $\mathfrak{m}^2$. The set $\mathfrak{E}$ has at least $n\mu$ elements.*

Proof. Let S_{min} denote a subset of $\mathfrak{E}$ that contains the equivalence classes of minimal standard indetermates modulo $\mathfrak{m}^2$.

We divide the proof into two cases. The first case is where the minimal generating set of the border term ideal consists of pure power terms only. The second case is where the minimal generating set of the border term ideal has not only pure power terms. We show that in both cases there are at least $n\mu$ elements in $S_{min} \subseteq \mathfrak{E}$. If the minimal generating set consists of only pure power terms, then by Lemma 4.2.5, we have $\mathfrak{E} = S_{min}$ and the cardinality of $\mathfrak{E}$ is $n\mu$. For the second case, recall Algorithm 4.2.7. Let x_k and x_l be two distinct indeterminates from $\mathrm{K}[x_1, ..., x_n]$. If an x_k-standard pure power indeterminate is in $\mathfrak{m}^2$, then by Algorithm 4.2.7, there exists an x_k-standard minimal indeterminate whose equivalence class is not in $\mathfrak{m}^2$ which has the same x_k-offset as the pure power indeterminate.

We claim that the output of the Algorithm 4.2.7 for distinct pure power indeterminates are distinct indeterminates with distinct equivalence classes in $\mathfrak{E}$. Keep in mind that the output indeterminate of the Algorithm 4.2.7 is a minimal standard indeterminate whose equivalence class is in $\mathfrak{E}$. From Lemma 4.1.5, it follows that the equivalence class of the output of Algorithm 4.2.7 for x_k-standard and x_l-standard pure power indeterminates with $k \neq l$ cannot be equal. Therefore we let c_{ij} and c_{uj} be distinct x_k-pure power indeterminates. Then the terms t_i and t_u in $\mathcal{O}$ are different from each other. Apply Algorithm 4.2.7, and let $c_{p_iq_i}$ be the output for c_{ij} and $c_{p_uq_u}$ be the output for c_{uj}. If the indeterminates $c_{p_iq_i}$ and $c_{p_uq_u}$ have different x_k-offset, then by Lemma 4.1.5, we have $[\bar{c}_{p_iq_i}] \neq [\bar{c}_{p_uq_u}]$. If these two indeterminates c_{ij} and c_{uj} have the same x_k-offset with distinct order ideal terms , then we have

$$\deg_{x_k}(t_i) \neq \deg_{x_k}(t_u). \tag{4.8}$$

Let $c_{r_is_i}$ be the indeterminate in step 4) of Algorithm 4.2.7 where $\bar{c}_{p_iq_i} \in [\bar{c}_{r_is_i}]$. Let $c_{r_us_u}$ be the indeterminate in step 4) of Algorithm 4.2.7 where $\bar{c}_{p_uq_u} \in [\bar{c}_{r_us_u}]$. By steps 2) and 3) of the Algorithm 4.2.7, $\deg_{x_k}(t_i)$ remains the same i.e., $\deg_{x_k}(t_{r_i}) = \deg_{x_k}(t_i)$. Then by (4.8), we have $\deg_{x_k}(t_{r_i}) \neq \deg_{x_k}(t_{r_u})$. From this and step 4) of Algorithm 4.2.7 where b_{s_u} and b_{s_i} are defined, follows $\deg_W(c_{r_us_u}) \neq \deg_W(c_{r_is_i})$. Consequently, from Lemma 4.1.5 follows that $\deg_W(c_{p_uq_u}) \neq \deg_W(c_{p_iq_i})$ and we have $[\bar{c}_{p_iq_i}] \neq [\bar{c}_{p_uq_u}]$. Altogether, we have shown that for distinct pure power indeterminates whose equivalence classes are in $\mathfrak{m}^2$, there exist distinct minimal standard indeterminates with distinct equivalence classes in $\mathfrak{E}$. Since there are $n\mu$ pure power indeterminates, there are at least $n\mu$ equivalence classes of minimal standard indeterminates. □

As the following example shows, there can be non-standard indeterminates whose equivalence classes are in $\mathfrak{E}$

Example 4.2.10. (continues) Recall Example 4.2.1. Clearly, the number of indeterminates in $\mathrm{B}_\mathcal{O}$ is $\mu * \nu = 24$ in this example. There are 6 indeterminates in $\mathfrak{m}^2$ and $\mathfrak{E}$ contains equivalence classes of 3 indeterminates which have a non-standard arrow degree. Thus the set of equivalence classes of minimal standard indeterminates modulo $\mathfrak{m}^2$ has exactly 15 equivalence classes which is larger than $n\mu = 12$.

Lemma 4.2.11. *Let $\mathcal{O} \subset \mathbb{T}^2$ be an order ideal with μ elements. Then the set $\mathfrak{E}$ has exactly 2μ elements.*

Proof. Let $b_w = x_1^{\omega_1}$ and $b_r = x_2^{\omega_2}$ be the two pure power terms in $\partial\mathcal{O}$. Then there are

exactly 2μ pure power indeterminates and by Algorithm 4.2.7, there are exactly 2μ minimal standard indeterminates which are not contained in $\mathfrak{m}^2$. Thus the cardinality of $\mathfrak{E}$ is larger or equal to 2μ. For contradiction suppose $\mid \mathfrak{E} \mid > 2\mu$ i.e., suppose that there exists an indeterminate $c_{ij} \in \mathcal{C}$ such that for each $t_u \in \mathcal{O}$ the element $\bar{c}_{ij}$ is neither in C_{uw} nor in C_{ur} and the indeterminate c_{ij} is not an output of the Algorithm 4.2.7 yet $[\bar{c}_{ij}] \in \mathfrak{E}$ holds. The indeterminate c_{ij} is either a standard or a non-standard indeterminate.

If the indeterminate c_{ij} is a non-standard indeterminate, then we construct the set $C_{ij} = \{\bar{c}_{i_1j_1}, ..., \bar{c}_{i_mj_m}\}$ where for each index $k \in \{1, ..., m-1\}$ we have

$$\deg_{x_1}(t_{i_m}) < \deg_{x_1}(t_{i_k}) \text{ and } \deg_{x_2}(t_{i_m}) > \deg_{x_2}(t_{i_k}). \tag{4.9}$$

Since $\bar{c}_{ij}$ is neither in C_{uw} nor in C_{ur}, we have $\deg_{x_1}(b_{j_m}) < \deg_{x_1}(b_w)$ and $\deg_{x_2}(b_{j_m}) < \deg_{x_2}(b_r)$. Thus there exists a border term b_k such that either $x_2b_{j_m} = b_k$ or $x_2b_{j_m} = x_1b_k$. If there exists an order ideal term t_l such that $x_2t_{i_m} = t_l$ or $x_2t_{i_m} = x_1t_l$, then we have $\bar{c}_{lk} \in C_{ij}$ with $\deg_{x_2}(t_l) > \deg_{x_2}(t_{i_m})$, which contradicts to (4.9). Since there exists no order ideal term t_l with the above property, by Lemmas 4.1.3 and 4.1.4, we have $\bar{c}_{i_mj_m} \in \mathfrak{m}^2$ and consequently $\bar{c}_{ij} \in \mathfrak{m}^2$.

If c_{ij} is a standard indeterminate, then c_{ij} is either an x_1-standard or x_2-standard indeterminate. Without loss of generality we let c_{ij} be an x_1-standard indeterminate. Then by Corollary 4.1.16, there exists an element $\bar{c}_{i_mj_m} \in [\bar{c}_{ij}]$ such that $x_2t_{i_m} \in \partial\mathcal{O}$ and for each $\bar{c}_{i_kj_k} \in [\bar{c}_{ij}]$ we have $\deg_{x_2}(t_{i_m}) \geqslant \deg_{x_2}(t_{i_k})$ and $\deg_{x_1}(t_{i_m}) \leqslant \deg_{x_1}(t_{i_k})$. Moreover,

$$\deg_{x_2}(b_{j_m}) \geqslant \deg_{x_2}(b_{j_k}) \text{ and } \deg_{x_1}(b_{j_m}) \leqslant \deg_{x_1}(b_{j_k}) \tag{4.10}$$

hold. We let the arrow degree of $c_{i_mj_m}$ be $\deg_W(c_{i_mj_m}) = \log(b_{j_m}) - \log(t_{i_m}) = (\alpha_1, \beta_1) - (\alpha_2, \beta_2) = (\alpha, \beta)$ where $\alpha > 0$ and $\beta \leqslant 0$.

Then there exists an order ideal term t_u such that $x_2^{\beta_1}t_u = t_{i_m}$ and $\text{offset}_{x_1}(c_{i_mj_m}) = \text{offset}_{x_1}(c_{uw})$. We apply Algorithm 4.2.7 to c_{uw}. Then by Step 2) of Algorithm 4.2.7 we have $t_r = t_{i_m}$ and we compute the smallest integer γ where

$$b_s = x_2^{\beta_1}x_1^{\gamma} \in \partial\mathcal{O}. \tag{4.11}$$

Since γ is the smallest integer that satisfies Equation (4.11), it cannot be larger than the $\deg_{x_1}(b_{j_m})$ i.e., α_1. If the integer γ smaller than α_1, then b_s has a next-door neighbor and this contradicts (4.10). Thus we have $b_s = x_2^{\beta_1}x_1^{\alpha_1} = b_{j_m}$ and $[\bar{c}_{i_mj_m}] = [\bar{c}_{rs}]$. □

Lemma 4.2.12. *Let $\mathcal{S}$ be a subset of $\mathcal{C}$ which contains d elements, where each element $\bar{c}_{ij} \in \bar{\mathcal{C}}$ and $[\bar{c_{ij}}] \in \mathfrak{E}$. Then the subspace of $\mathfrak{m}/\mathfrak{m}^2$ spanned by the elements of $\bar{\mathcal{S}}$ has the dimension d.*

Proof. Every indeterminate in $\bar{\mathcal{S}}$ is chosen from a non-zero distinct equivalence class modulo $\mathfrak{m}^2$. Therefore they are L-linearly independent in the vector space $\mathfrak{m}/\mathfrak{m}^2$, where $\bar{\mathcal{S}}$ is a subset of $\mathfrak{m}/\mathfrak{m}^2$. The L- subspace of $\mathfrak{m}/\mathfrak{m}^2$ spanned by $\bar{\mathcal{S}}$ has the dimension d, which is exactly the number non zero equivalence classes in $\mathfrak{E}$. □

Remark 4.2.13. The set $\mathcal{S}$ is the same as the one that is constructed in [Huib05], Theorem 4.1.3. Although the set $\mathcal{S}$ is not unique, the set $\mathfrak{E}$ is unique.

Lemma 4.2.14. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. If the monomial point of an $\mathcal{O}$-border basis scheme is smooth, then the dimension of the local ring $(\mathrm{B}_\mathcal{O})_\mathfrak{m}$ is $n\mu$. In particular, if the point $\mathfrak{o}$ is not smooth, then the dimension of $(\mathrm{B}_\mathcal{O})_\mathfrak{m}$ is larger than $n\mu$.*

Proof. This follows from Theorem 2.4.5 and Corollary 2.4.6. □

Theorem 4.2.15. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. The monomial point $\mathfrak{o} \in \mathbb{B}_\mathcal{O}$ is smooth if and only if the number of elements of $\mathfrak{E}$ is $n\mu$.*

Proof. Assume first that the monomial point in a border basis scheme is smooth. Then, by Lemma 4.2.14, we have $\dim((\mathrm{B}_\mathcal{O})_\mathfrak{m}) = n\mu$. By the smoothness, we also have $\dim((\mathrm{B}_\mathcal{O})_\mathfrak{m}) = \mathrm{edim}(\mathbb{B}_\mathcal{O}) = n\mu$. This means the maximum number of the L-linearly independent indeterminates in the vector space $\mathfrak{m}/\mathfrak{m}^2$ is $n\mu$. Then by Theorem 4.2.9, the cardinality of $\mathfrak{E}$ is $n\mu$.

Conversely, let us assume that the cardinality of $\mathfrak{E}$ is $n\mu$. By Lemma 4.2.14, in the general case we have

$$\dim((\mathrm{B}_\mathcal{O})_\mathfrak{m}) \geqslant n\mu. \tag{4.12}$$

By Proposition 2.1.15, we have

$$\dim_\mathrm{L}(\mathfrak{m}/\mathfrak{m}^2) \geqslant \dim((\mathrm{B}_\mathcal{O})_\mathfrak{m}). \tag{4.13}$$

Moreover, the set $\mathcal{S}$, which is constructed by choosing an element from each equivalence class of $\mathfrak{E}$, is a set of maximal L-linear independent indeterminates in the L-vector space $\mathfrak{m}/\mathfrak{m}^2$. Then by Lemma 4.2.12 and by the equality $\mid \mathfrak{E} \mid = \mid \mathcal{S} \mid$, we have

$$n\mu = \mid \mathfrak{E} \mid = \mid \mathcal{S} \mid = \dim_\mathrm{L}(\mathfrak{m}/\mathfrak{m}^2). \tag{4.14}$$

Then by considering Inequalities (4.12) and (4.13) with Equality (4.14), we have the following result.

$$n\mu =\mid \mathcal{S} \mid= \dim_{\mathrm{L}}(\mathfrak{m}/\mathfrak{m}^2) \geqslant \dim((\mathrm{B}_{\mathcal{O}})_{\mathfrak{m}}) \geqslant n\mu$$

Thus we have $\dim((\mathrm{B}_{\mathrm{O}})_{\mathfrak{m}}) = n\mu = \mathrm{edim}(\mathrm{B}_{\mathcal{O}})$ and $\mathfrak{o}$ is a smooth point of $\mathbb{B}_{\mathcal{O}}$. □

Example 4.2.16. (continues) Let us recall Example 4.1.8. The set $\mathfrak{E}$ consists of the equivalence classes of minimal standard indeterminates only. More precisely, it consists of the equivalence classes of pure power indeterminates. Thus the monomial point in this example is smooth. On the other hand, in Example 4.2.1 we have seen that the set $\mathfrak{E}$ has more elements than the set of the equivalence classes of minimal standard indeterminates. Therefore the monomial point of the border basis scheme in that example is not smooth.

Corollary 4.2.17. *Let $\mathcal{O} \subset \mathbb{T}^2$ be an order ideal. The monomial point of an $\mathcal{O}$-border basis scheme is smooth.*

Proof. This follows from Lemma 4.2.11 and Theorem 4.2.15. □

Recall that the order ideal $\mathcal{B}(d_1, ..., d_n) = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}^n \mid \alpha_i < d_i \quad for \quad i = 1, ..., n\}$ is called the **box** order ideal of size $(d_1, ..., d_n)$.

Corollary 4.2.18. *Let $\mathcal{B} \in \mathbb{T}^n$ be a box order ideal. The monomial point a $\mathcal{B}$-border basis scheme is always smooth.*

Proof. This follows from the fact that the minimal generating set of the border term ideal of any box border basis scheme just consists of the pure power terms. □

Chapter 5

Trace and Jacobi Identity Syzygies

In this chapter we shall present two types of syzygies of the tuple whose entries are the elements of $\mathcal{T}$ (see Equation (2.6)) of a given $\mathcal{O}$-border basis scheme, the *trace syzygies of* $\mathcal{T}$ (see Definition 5.1.12 and Remark 5.1.9) and the *Jacobi identity syzygies of* $\mathcal{T}$ (see Definition 5.2.2 and Remark 5.2.3), which are computed by the help of the generic multiplication matrices (see Definitions 2.2.4 and 2.3.1.b). The trace and the Jacobi identity syzygies of $\mathcal{T}$ were first introduced in [Huib09]. In Section 10 of [Huib09] the trace syzygies of $\mathcal{T}$ are used to show that all $\mathcal{O}$-border basis schemes are *ideal-theoretic complete intersections*, where $\mathcal{O} \subset \mathbb{T}^2$ (see Definition 2.1.10). Moreover, in the same article, for the specific case $\mathcal{O} = \{1, x_1\} \subset \mathrm{K}[x_1, x_2, x_3]$, both trace syzygies and Jacobi identity syzygies were used to prove that $\mathbb{B}_{\mathcal{O}}$ is an ideal-theoretic complete intersection. In this chapter we show that for box order ideal $\mathcal{B}(2,2,2) \subset \mathrm{K}[x_1, x_2, x_3]$, these syzygies of $\mathcal{T}$ are not sufficient to decide whether $\mathbb{B}_{\mathcal{B}(2,2,2)}$ is a complete intersection or not.

In the first two sections we reprove some of the properties of trace and Jacobi identity syzygies of $\mathcal{T}$. Our aim is to characterize these syzygies by the arrow grading (see Lemmas 5.1.5, 5.1.6 and 5.2.6). In Section 5.1 we give an example where one can illustrate that the given border basis scheme is a complete intersection (see Example 5.1.22) by employing the trace syzygies of $\mathcal{T}$.

The last section (see Section 5.4) is dedicated to finding redundant elements of the set $\mathcal{T}$ of defining equations of the box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$ by using the Jacobi identity and the trace syzygies of $\mathcal{T}$ together. We show that there are at most 60 redundant polynomials in the generating set $\mathcal{T}$ of $\mathcal{I}(\mathbb{B}_{\mathcal{B}(2,2,2)})$ (see Proposition 5.4.3). We observe that trace and Jacobi identity syzygies of $\mathcal{T}$ do not give conclusive results whether $\mathbb{B}_{\mathcal{B}(2,2,2)}$ has the complete intersection property or not. We provide the reasons

in Remarks 5.4.1 and 5.4.4.

Moreover, we implemented the Jacobi identity and the trace syzygy computation methods in the package *bbsmingensyz* of the computer algebra system ApCoCoA. The vanishing ideal of this box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$ has 144 generators. Therefore it is to hard to perform the syzygy computations by hand. In some cases, where defining equations of border bases schemes have larger cardinalities, even computers might fail to succeed with those computations. In such cases, to find at least the *rewritable elements* with respect to trace or Jacobi identity syzygies of $\mathcal{T}$, we introduce a practical method which we call *simplified* trace and Jacobi identity syzygies of $\mathcal{O}$ (see Section 5.1). In Section 5.4, we first compute the simplified syzygies trace and Jacobi identity syzygies which reduces the number of computations to identify the redundant elements of $\mathcal{T}$.

5.1 Trace Syzygies

Unless stated otherwise, throughout this section we let K be a field and P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be an order ideal and let $\partial\mathcal{O}$ denote its border $\{b_1, ..., b_\nu\}$. Let $\mathcal{C}$ denote a set of indeterminates $\{c_{11}, ..., c_{\mu\nu}\}$. Let $\mathrm{K}[\mathcal{C}]$ be the polynomial ring $K[c_{11}, ..., c_{\mu\nu}]$. Recall that $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is the vanishing ideal of the border basis scheme $\mathbb{B}_\mathcal{O}$ which is generated by the set $\mathcal{T} = \{\tau_{pq}^{kl} \mid p, q \in \{1, ...\mu\}, \quad k < l \quad k, l \in \{1, .., n\}$ (see Section 2.3).

Let $\mathcal{A}, \mathcal{B} \in \mathrm{Mat_m}(\mathrm{K}[\mathcal{C}])$ be square matrices, where $m \in \mathbb{N}_+$. Then $[\mathcal{A}, \mathcal{B}]$ denotes the commutator $\mathcal{A} \cdot \mathcal{B} - \mathcal{B} \cdot \mathcal{A}$.

Lemma 5.1.1. *Let $\mathcal{A}, \mathcal{B} \in \mathrm{Mat_m}(\mathrm{K}[\mathcal{C}])$ be square matrices, where $m \in \mathbb{N}_+$ Then the* $\mathrm{Trace}([\mathcal{A}, \mathcal{B}])$ *is* 0.

Proof. Let $\mathcal{A} = (a_{ij})_{m\times m}$ and $\mathcal{B} = (b_{ij})_{m\times m}$, where $a_{ij}, b_{ij} \in \mathrm{K}[\mathbf{c}]$. Then the $\mathrm{Trace}([\mathcal{A}, \mathcal{B}])$ is as follows.

$$\begin{aligned}
\mathrm{Trace}([\mathcal{A}, \mathcal{B}]) &= \sum_{u=1}^{m} e_u(\mathcal{AB} - \mathcal{BA})e_u^{tr} = \sum_{u=1}^{m} e_u\mathcal{AB}e_u^{tr} - e_u\mathcal{BA}e_u^{tr} \\
&= \sum_{u=1}^{m}(\sum_{j=1}^{m} a_{uj}b_{ju}) - \sum_{u=1}^{m}(\sum_{j=1}^{m} b_{uj}a_{ju}) \\
&= \sum_{u=1}^{m}(\sum_{j=1}^{m} a_{uj}b_{ju}) - \sum_{j=1}^{m}(\sum_{u=1}^{m} a_{uj}b_{ju}) = 0
\end{aligned}$$

Hence we have $\mathrm{Trace}([\mathcal{A}, \mathcal{B}]) = 0$. □

Lemma 5.1.2. *Let $\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}, \mathcal{A}_l \in \mathrm{Mat_m(K[\mathcal{C}])}$ be square matrices where $m \in \mathbb{N}_+$. Then the following equality holds.*

$$[\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}, \mathcal{A}_l] = \sum_{m=1}^{s} \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_m} [\mathcal{A}_l, \mathcal{A}_{k_m}] \mathcal{A}_{k_{m+1}} \cdots \mathcal{A}_{k_s} \tag{5.1}$$

Proof. Let us consider the $\mathrm{u^{th}}$ summand on the right hand side of Equation (5.1).

$$\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{u-1}} \quad [\mathcal{A}_l, \mathcal{A}_{k_u}] \quad \mathcal{A}_{k_{u+1}} \cdots \mathcal{A}_{k_s} = \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{u-1}} (\mathcal{A}_l \mathcal{A}_{k_u} - \mathcal{A}_{k_u} \mathcal{A}_l) \mathcal{A}_{k_{u+1}} \cdots \mathcal{A}_{k_s}$$

$$= \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{u-1}} \mathcal{A}_l \mathcal{A}_{k_u} \mathcal{A}_{k_{u+1}} \cdots \mathcal{A}_{k_{s-1}} - \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{u-1}} \mathcal{A}_{k_u} \mathcal{A}_l \mathcal{A}_{k_{u+1}} \cdots \mathcal{A}_{k_s} \tag{5.2}$$

Now let us check the $(\mathrm{u}+1)^{\mathrm{st}}$ summand in Equation (**??**), which is

$$\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_u} \mathcal{A}_l \mathcal{A}_{k_{u+1}} \mathcal{A}_{k_{u+2}} \cdots \mathcal{A}_{k_s} - \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_u} \mathcal{A}_{k_{u+1}} \mathcal{A}_l \mathcal{A}_{k_{u+2}} \cdots \mathcal{A}_{k_s}. \tag{5.3}$$

If we take the sum of Equations (5.2) and (5.3), then the negative term of Equation (5.2) and the positive term of Equation (5.3) cancel each other out. This implies that the sum in Equation (5.1) is a telescopic sum and the result of it is just the sum of the positive term of the first summand and the negative term of the last summand i.e.,

$$\mathcal{A}_l \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_u} \mathcal{A}_{k_{u+1}} \mathcal{A}_{k_{u+2}} \cdots \mathcal{A}_{k_s} - \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_u} \mathcal{A}_{k_{u+1}} \mathcal{A}_{k_{u+2}} \cdots \mathcal{A}_{k_s} \mathcal{A}_l = [\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}, \mathcal{A}_l].$$

Hence we have

$$[\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}, \mathcal{A}_l] = \sum_{m=1}^{s} \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_m} [\mathcal{A}_l, \mathcal{A}_{k_m}] \mathcal{A}_{k_{m+1}} \cdots \mathcal{A}_{k_s}.$$

□

Example 5.1.3. Let $\mathcal{O} \subset \mathrm{K}[x_1, x_2, x_3]$ be an order ideal. Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ be the generic multiplication matrices with respect to $\mathcal{O}$. We consider the commutator operation $[\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 A_1 \mathcal{A}_2, \mathcal{A}_1]$. By Lemma 5.1.2, the following equality holds.

$$\begin{aligned}
[\mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 A_1 \mathcal{A}_2, \mathcal{A}_1] &= [\mathcal{A}_1, \mathcal{A}_1] \mathcal{A}_2 \mathcal{A}_3 A_1 \mathcal{A}_2 + \mathcal{A}_1 [\mathcal{A}_1, \mathcal{A}_2] \mathcal{A}_3 A_1 \mathcal{A}_2 \\
&\quad + \mathcal{A}_1 \mathcal{A}_2 [\mathcal{A}_1, \mathcal{A}_3] A_1 \mathcal{A}_2 + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 [\mathcal{A}_1, A_1] \mathcal{A}_2 + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 A_1 [\mathcal{A}_1, \mathcal{A}_2] \\
&= \mathcal{A}_1 [\mathcal{A}_1, \mathcal{A}_2] \mathcal{A}_3 A_1 \mathcal{A}_2 + \mathcal{A}_1 \mathcal{A}_2 [\mathcal{A}_1, \mathcal{A}_3] A_1 \mathcal{A}_2 + \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 A_1 [\mathcal{A}_1, \mathcal{A}_2]
\end{aligned}$$

Remark 5.1.4. Recall that the polynomial ring $\mathrm{K}[\mathcal{C}]$ is a $\mathbb{Z}^n$-graded ring such that the indeterminate $c_{ij} \in \mathrm{K}[\mathcal{C}]$ has the arrow degree $\deg_W(c_{ij}) = \log(b_j) - \log(t_i)$. The details on the arrow grading are given in Section 3.2.

Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let $\mathcal{A}_l$ be the l^{th} generic multiplication matrix where $l \in \{1, ..., n\}$. Then by Lemma 3.2.4, A_l is a homogenous matrix with the degree pair (d_0, d_l) where the following holds.

$$d_l = (\log(x_l t_1), ..., \log(x_l t_\mu)) = (d_{l_1}, ..., d_{l_\mu}),\ d_0 = (\log(t_1), ..., \log(t_\mu)) = (d_{0_1}, ..., d_{0_\mu})$$

Lemma 5.1.5. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}, \mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$, where $s \in \mathbb{N}_+$ and $k_1, ..., k_s, l \in \{1, ..., n\}$. The matrix $\mathcal{A}_l\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}$ and the matrix $\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}\mathcal{A}_l$ are both homogenous matrices with the same degree pair $(d_0, d_{(k_1 \cdots k_s l)})$ where*

$$d_{(k_1 \cdots k_s l)} = (\log(x_{k_1} \cdots x_{k_s} x_l t_1), ..., \log(x_{k_1} \cdots x_{k_s} x_l t_\mu)) \text{ and } d_0 = (\log(t_1), ..., \log(t_\mu)).$$

Proof. Let $\mathcal{A}_{k_i}$ be a generic multiplication matrix in $\{\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}\}$ with the degree pair (d_0, d_{k_i}) where

$$d_{k_i} = (\log(x_{k_i} t_1), ..., \log(x_{k_i} t_\mu)) = (d_{(}k_i)_1, ..., d_{(}k_i)_\mu),\ d_0 = (\log(t_1), ..., \log(t_\mu)) = (d_{0_1}, ..., d_{0_\mu}).$$

By Lemma 3.2.5, the degree pairs of the matrices $\mathcal{A}_{k_i}\mathcal{A}_l$ and $\mathcal{A}_l\mathcal{A}_{k_i}$ are the same which is $(d_0, d_{(k_i l)})$, where

$$d_{(k_i l)} = (\log(x_{k_i} x_l t_1), ..., \log(x_{k_i} x_l t_\mu)) \text{ and } d_0 = (\log(t_1), ..., \log(t_\mu)) \tag{5.4}$$

Hence, the Lemma follows. □

Lemma 5.1.6. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal. Let $\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}, \mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$, where $s \in \mathbb{N}_+$ and $k_1, ..., k_s, l \in \{1, ..., n\}$. Then the* $\mathrm{Trace}(\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}\mathcal{A}_l)$ *is a homogenous polynomial in* $\mathrm{K}[\mathcal{C}]$ *with respect to the arrow grading. Moreover, if we let* Π *denote the term $x_{k_1} \cdots x_{k_s} x_l \in \mathbb{T}^n$, then the following holds.*

$$\deg_W(\mathrm{Trace}(\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}\mathcal{A}_l)) = \deg_W(\mathrm{Trace}(\mathcal{A}_l\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s})) = \log(\Pi)$$

Proof. The trace of a product of matrices is invariant under the cyclic permutation of

the positions of the matrices i.e.,

$$\text{Trace}(\mathcal{A}_l\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}) = \text{Trace}(\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}\mathcal{A}_l). \tag{5.5}$$

By Lemma 5.1.5, the matrix $\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}\mathcal{A}_l$ has the degree pair $(d_0, d_{(k_1\cdots k_s l)})$ where

$$d_{(k_1\cdots k_s l)} = (\log(x_{k_1}\cdots x_{k_s}x_l t_1), ..., \log(x_{k_1}\cdots x_{k_s}x_l t_\mu)),\ d_0 = (\log(t_1), ..., \log(t_\mu)).$$

Therefore the entries in (i,i) position of $\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}\mathcal{A}_l$ are of the arrow degrees

$$(\log(x_{k_1}\cdots x_{k_s}x_l t_i) - \log(t_i)) = d_{(lk_1\ldots k_s)_i} - d_{0_i} = \log(x_{k_1}\cdots x_{k_s}x_l).$$

Then the polynomial $\text{Trace}(\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}\mathcal{A}_l)$ has the following arrow degree.

$$\begin{aligned}\deg_W(\text{Trace}(\mathcal{A}_l\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s})) &= \deg_{\text{W}}(\sum_{i=1}^{\mu} a_{ii}) = d_{(lk_1\ldots k_s)_i} - d_{0_i} \\ &= \log(x_{k_1}\cdots x_{k_n}x_l)\end{aligned}$$

By Equality 5.5, follows the proof. □

Let $\mathcal{T}$ denote the set of generators of the ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$, where m denotes the number of polynomials in $\mathcal{T}$. Recall that a polynomial in $\mathcal{T}$ is of the form τ_{pq}^{kl} where $k < l$ and $k, l \in \{1, ..., n\}$ and $p, q \in \{1, ...\mu\}$. We enumerate the polynomials from $\mathcal{T}$. Let

$$\{(k,l) \mid k < l, \text{ and } k, l \in \{1, ..., n\}\} = \{(1,2), (1,3), ..., (1,n), (2,3), ..., (n-1,n)\}$$

denote the set of upper indices of the polynomials from $\mathcal{T}$ and let (k_i, l_i) and (k_j, l_j) be two different tuples from this set. Let

$$\{(p,q) \mid p, q \in \{1, ..., \mu\}\} = \{(1,1), (1,2), ..., (1,\mu), (2,1), ..., (\mu,\mu)\},$$

be the set of lower indices of the polynomials from $\mathcal{T}$ and let (p_i, q_i) and (p_j, q_j) be two different tuples from this set. We order the indices as follows. We have $(k_i, l_i) <_{\texttt{Lex}} (k_j, l_j)$ holds and in the case of equality $(p_i, q_i) <_{\texttt{Lex}} (p_j, q_j)$ holds.

Lemma 5.1.7. *Consider the above setting. If we enumerate the polynomials in $\mathcal{T}$ with respect to the above ordering of the indices, then a polynomial τ_{pq}^{kl} is the α^{th} polynomial where $\alpha = [\frac{n(n-1)}{2} - \frac{(n-k)(n-k+1)}{2} + (l-k-1)]\mu^2 + p \cdot q$.*

Proof. There are $\frac{n(n-1)}{2}$ tuples in the set $\{(k,l) \mid k < l, \text{and}\, k,l \in \{1,...,n\}\}$. With respect to the ordering we defined above there are $\frac{n(n-1)}{2} - \frac{(n-k)(n-k+1)}{2} + (l-k-1)$ many tuples that are smaller than (k,l). The tuple (p,q) is the $(p \cdot q)^{th}$ tuple in the set $\{(p,q) \mid p,q \in \{1,...,\mu\}\}$. Thus with respect to the given ordering the polynomial is the α^{th} polynomial in $\mathcal{T}$ where $\alpha = [\frac{n(n-1)}{2} - \frac{(n-k)(n-k+1)}{2} + (l-k-1)]\mu^2 + p \cdot q$. □

Recall that we let m be the number of polynomials in $\mathcal{T}$. Let $(e_1,...,e_m)$ be the canonical basis of $\mathrm{K}[\mathcal{C}]^m$. We let e_{pq}^{kl} denote the canonical basis element e_α, where by Lemma 5.1.7 $\alpha = [\frac{n(n-1)}{2} - \frac{(n-k)(n-k+1)}{2} + (l-k-1)]\mu^2 + p \cdot q$. Then $(e_{pq}^{kl})_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}}$ is the canonical basis of $\mathrm{K}[\mathcal{C}]^m$.

Example 5.1.8. Let $\mathcal{O} = \{1, x_1\} \subset \mathrm{K}[x_1,x_2,x_3]$. Then the vanishing ideal of the $\mathcal{O}$-border basis scheme $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by the set $\mathcal{T} = \{\tau_{pq}^{kl} \mid p,q \in \{1,...\mu\}$, and $k < l \quad k,l \in \{1,..,n\}\}$. With respect to the ordering above, the polynomial τ_{11}^{12} is the first element and τ_{22}^{23} is the 12^{th} element i.e., the last element. By Lemma 5.1.7, the polynomial τ_{22}^{13} is the 8^{th} element of $\mathcal{T}$ where $((3-1-1)\mu^2 + p \cdot q = 4+4 = 8$. Since the number of elements in the set $\mathcal{T}$ is $m = 12$, we have $\mathrm{K}[\mathcal{C}]^{12}$ with the canonical basis

$$(e_{11}^{12}, e_{12}^{12}, e_{21}^{12}, e_{22}^{12}, e_{11}^{13}, e_{12}^{13}, e_{21}^{13}, e_{22}^{13}, e_{11}^{23}, e_{12}^{23}, e_{21}^{23}, e_{22}^{23}) = (e_1, e_2, ..., e_{12}).$$

Remark 5.1.9. We note that for the sake of simplicity instead of calling the syzygy of the tuple whose elements are the elements of $\mathcal{T}$, we call it the syzygy of $\mathcal{T}$. If it is clear from the context which $\mathcal{O}$-border basis scheme is used, then we simply call it syzygy.

The ring $\mathrm{K}[\mathcal{C}]$ is a graded ring with respect to the arrow grading, where the arrow degree $\deg_W(c_{ij})$ is $\log(b_j) - \log(t_i)$. Moreover, by Proposition 3.2.6, that $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is a homogenous ideal with respect to the arrow grading, where the arrow degree $\deg_W(\tau_{pq}^{kl})$ is $\log(x_k x_l t_q) - \log(t_p)$. By letting $\deg_W(e_{pq}^{kl}) = \deg_W(\tau_{pq}^{kl}) = d_{pq}^{kl} \in \mathbb{Z}^n$, we make the $\mathrm{K}[\mathcal{C}]$-module $\mathrm{K}[\mathcal{C}]^m$ a graded free module,

$$\bigoplus_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \mathrm{K}[\mathcal{C}](-d_{pq}^{kl}).$$

Then we have the following graded $\mathrm{K}[\mathcal{C}]$-module homomorphism with respect to the

arrow grading:

$$\begin{array}{rcl} \varphi : \bigoplus\limits_{\substack{1\leqslant p,q\leqslant \mu \\ 1\leqslant i<l\leqslant n}} \mathrm{K}[\mathcal{C}](-d_{pq}^{kl}) & \longrightarrow & \mathcal{I}(\mathbb{B}_{\mathcal{O}}) \\ e_{pq}^{kl} & \longmapsto & \tau_{pq}^{kl} \end{array} \tag{5.6}$$

The kernel of the map φ is the set $\{\sum_{\substack{1\leqslant p,q\leqslant \mu \\ 1\leqslant i<l\leqslant n}} \rho_{pq}^{kl} e_{pq}^{kl} \mid \sum_{\substack{1\leqslant p,q\leqslant \mu \\ 1\leqslant i<l\leqslant n}} \rho_{pq}^{kl} \tau_{pq}^{kl} = 0\}$. In other words, the kernel $\mathrm{Ker}(\varphi)$ is the syzygy module of $\mathcal{T}$. Furthermore, the syzygy module $\mathrm{Syz}(\mathcal{T})$ is a homogenous sub-module of $\mathrm{K}[\mathcal{C}]^m$ with respect to the arrow grading. Let $\mathcal{H}$ denote a set of further indeterminates $\{h_{11}^{12}, ..., h_{pq}^{kl}, ..., h_{\mu\mu}^{(n-1)n}\}$ where the cardinality of the set $\mathcal{H}$ is m. Let θ denote the following map:

$$\begin{array}{rcl} \theta : \mathrm{K}[\mathcal{C}]^m & \longrightarrow & \mathrm{K}[\mathcal{C}][h_{11}^{12}, ..., h_{pq}^{kl}, ..., h_{\mu\mu}^{(n-1)n}] = \mathrm{K}[\mathcal{C}][\mathcal{H}] \\ e_{pq}^{kl} & \longmapsto & h_{pq}^{kl} \end{array} \tag{5.7}$$

Then θ is a well-defined $\mathrm{K}[\mathcal{C}]$-module morphism that is injective. A polynomial of the form $\sum_{\substack{1\leqslant p,q\leqslant \mu \\ 1\leqslant i<l\leqslant n}} \rho_{pq}^{kl} h_{pq}^{kl}$ has the inverse image $\sum_{\substack{1\leqslant p,q\leqslant \mu \\ 1\leqslant i<l\leqslant n}} \rho_{pq}^{kl} e_{pq}^{kl}$ under the map θ.
Let $\Pi = x_{l_1} \cdots x_{l_s} \cdot x_{l_{s+1}} \in \mathbb{T}^n$ be a term where $s \in \mathbb{N}_+$. Let us choose an index l from the set $\{l_1, ..., l_s, l_{s+1}\}$, and let $x_{k_1}, ..., x_{k_s} = \frac{\Pi}{x_l}$, where $k_1, ..., k_s \in \{1, ..., n\}$. Then with respect to the new indexing we have

$$\Pi = x_{k_1} \cdots x_{k_s} x_l \in \mathbb{T}^n \tag{5.8}$$

and we call x_l as the **distinguished indeterminate**. Let $\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}, \mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$. Now we consider the following polynomial in the ring $\mathrm{K}[\mathcal{C}, \mathcal{H}]$.

$$\sum_{i=1}^{s} \sum_{u=1}^{\mu} e_u (\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{i-1}} \begin{pmatrix} h_{11}^{k_i l} & \dots & h_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{k_i l} & \dots & h_{\mu\mu}^{k_i l} \end{pmatrix} \mathcal{A}_{k_{i+1}} \cdots \mathcal{A}_{k_s}) e_u^{tr} \tag{5.9}$$

We let

$$\begin{pmatrix} f_{11}^{(k_{i-1})} & \dots & f_{1\mu}^{(k_{i-1})} \\ \vdots & \ddots & \vdots \\ f_{\mu 1}^{(k_{i-1})} & \dots & f_{\mu\mu}^{(k_{i-1})} \end{pmatrix} = \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{i-1}} \quad \text{and} \quad \begin{pmatrix} g_{11}^{(k_{i+1})} & \dots & g_{1\mu}^{(k_{i+1})} \\ \vdots & \ddots & \vdots \\ g_{\mu 1}^{(k_{i+1})} & \dots & g_{(\mu\mu}^{k_{i+1})} \end{pmatrix} = \mathcal{A}_{k_{i+1}} \cdots \mathcal{A}_{k_s} \tag{5.10}$$

where $f_{ij}^{(k_i-1)}, g_{ij}^{(k_i+1)} \in \mathrm{K}[\mathcal{C}]$. Note that by Definitions 2.2.4 and 2.3.1, the generic multiplication matrices have the following two properties: First, they are non-zero matrices. Second, for $k \neq l$, we have $\mathcal{A}_k \neq \mathcal{A}_l$. Therefore the products of the generic multiplication matrices $\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{i-1}} \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ and $\mathcal{A}_{k_{i+1}} \cdots \mathcal{A}_{k_s} \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ are non-zero matrices. Then Polynomial (5.9) is equal to the following polynomial.

$$\sum_{i=1}^{s} \sum_{u=1}^{\mu} \begin{pmatrix} f_{u1}^{(k_{i-1})} & \dots & f_{u\mu}^{(k_{i-1})} \end{pmatrix} \begin{pmatrix} h_{11}^{k_i l} & \dots & h_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{k_i l} & \dots & h_{\mu\mu}^{k_i l} \end{pmatrix} \begin{pmatrix} g_{1u}^{(k_{i+1})} \\ \vdots \\ g_{\mu u}^{(k_{i+1})} \end{pmatrix} = \sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} h_{pq}^{k_i l} \tag{5.11}$$

Thus Equation (5.11) is a $\mathrm{K}[\mathcal{C}]$-linear combination of the indeterminates $h_{pq}^{k_i l}$, where each $\kappa_{pq}^{k_i l} \in \mathrm{K}[\mathcal{C}]$ represents the coefficient of $h_{pq}^{k_i l}$ in Equation (5.11). The inverse image of the left hand side of Equation (5.11) under the map θ is

$$\sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}.$$

In other words, under the map φ we have $\varphi(\sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}) = \sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} \tau_{pq}^{k_i l}$.

Lemma 5.1.10. *In the above setting the vector* $\sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}$ *is an element of the syzygy module* $\mathrm{Syz}(\mathcal{T})$.

Proof. We show that $\varphi(\sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l})$ is 0. In Equation (5.11) we substitute the $h_{pq}^{k_i l}$ by $\tau_{pq}^{k_i l}$, and we get the following.

$$\sum_{i=1}^{s} \sum_{u=1}^{\mu} \begin{pmatrix} f_{u1}^{(k_{i-1})} & \dots & f_{u\mu}^{(k_{i-1})} \end{pmatrix} \begin{pmatrix} \tau_{11}^{k_i l} & \dots & \tau_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{k_i l} & \dots & \tau_{\mu\mu}^{k_i l} \end{pmatrix} \begin{pmatrix} g_{1u}^{(k_{i+1})} \\ \vdots \\ g_{\mu u}^{(k_{i+1})} \end{pmatrix} = \sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} \tau_{pq}^{k_i l} \tag{5.12}$$

Recall that, $[\mathcal{A}_l, \mathcal{A}_k]$ denotes the commutator $\mathcal{A}_l \cdot \mathcal{A}_k - \mathcal{A}_k \cdot \mathcal{A}_l$, where $k, l \in \{1, ..., n\}$

and $k \neq l$. Then by Equation (2.5), we have

$$[\mathcal{A}_l, \mathcal{A}_k] = \begin{pmatrix} \tau_{11}^{kl} & \cdots & \tau_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{kl} & \cdots & \tau_{\mu\mu}^{kl} \end{pmatrix}. \tag{5.13}$$

Then by Equalities (5.10) and by Equation (5.13), Polynomial (5.12) becomes the following.

$$\sum_{i=1}^{s}\sum_{u=1}^{\mu} e_u(\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_{i-1}} \begin{pmatrix} \tau_{11}^{k_i l} & \cdots & \tau_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{k_i l} & \cdots & \tau_{\mu\mu}^{k_i l} \end{pmatrix} \mathcal{A}_{k_{i+1}}\cdots\mathcal{A}_{k_s}) e_u^{tr} \tag{5.14}$$

By using the definition of the commutator and (5.14), we get the following.

$$\begin{aligned} &\sum_{u=1}^{\mu} e_u(\sum_{i=1}^{s} \mathcal{A}_{k_1}\cdots\mathcal{A}_{k_{i-1}}[\mathcal{A}_l, \mathcal{A}_{k_i}]\mathcal{A}_{k_{i+1}}\cdots\mathcal{A}_{k_{s-1}}) e_u^{tr} \\ &= \text{Trace}(\sum_{i=1}^{s} \mathcal{A}_{k_1}\cdots\mathcal{A}_{k_{i-1}}[\mathcal{A}_l, \mathcal{A}_{k_i}]\mathcal{A}_{k_{i+1}}\cdots\mathcal{A}_{k_s}) \\ &= \text{Trace}([\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}, \mathcal{A}_l]) \end{aligned}$$

By Lemma 5.1.1, we have $\text{Trace}([\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}, \mathcal{A}_l]) = 0$. Therefore we have

$$\sum_{i=1}^{s}\sum_{u=1}^{\mu} \begin{pmatrix} f_{u1}^{(k_{i-1})} & \cdots & f_{u\mu}^{(k_{i-1})} \end{pmatrix} \begin{pmatrix} \tau_{11}^{k_i l} & \cdots & \tau_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{k_i l} & \cdots & \tau_{\mu\mu}^{k_i l} \end{pmatrix} \begin{pmatrix} g_{1u}^{(k_{i+1})} \\ \vdots \\ g_{\mu u}^{(k_{i+1})} \end{pmatrix} = 0, \tag{5.15}$$

where $\sum_{\substack{1\leqslant p,q\leqslant\mu \\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l}\tau_{pq}^{k_i l} = 0$ and thus $\sum_{\substack{1\leqslant p,q\leqslant\mu \\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}$ is an element of the syzygy module $\text{Syz}(\mathcal{T})$. □

Recall how we get Term (5.8) which is $\Pi = x_{k_1}\cdots x_{k_s}x_l \in \mathbb{T}^n$, where x_l is the distinguished indeterminate. By Lemma 5.1.6, we have

$$\deg_W(\text{Trace}(\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s}\mathcal{A}_l)) = \deg_W(\text{Trace}(\mathcal{A}_l\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s})) = \log(\Pi).$$

Lemma 5.1.11. *In the above setting, if the coefficient of $h_{pq}^{k_i l}$ which is $\kappa_{pq}^{k_i l}$, and the polynomial $\tau_{pq}^{k_i l}$ are non-zero polynomials from $\mathrm{K}[\mathcal{C}]$, then the arrow degree $\deg_W(\kappa_{pq}^{k_i l} e_{pq}^{k_i l})$ is $\log(\Pi)$.*

Proof. Recall that the map φ is a homogenous map and we have $\deg_W(e_{pq}^{kl}) = \deg_W(\tau_{pq}^{kl})$. Moreover, by (5.10), we have $\deg_W(f_{up}^{(k_i-1)}) = \log(x_{k_1} \cdots x_{k_{i-1}} t_p)$ and $\deg_W(g_{qu}^{(k_i+1)}) = \log(x_{k_{i+1}} \cdots x_{k_s} t_u)$. Then the following holds.

$$\begin{aligned}
\deg_W(\kappa_{pq}^{k_i l} e_{pq}^{k_i l}) &= \deg_W(\kappa_{pq}^{k_i l}) + \deg_W(e_{pq}^{k_i l}) \\
&= \deg_W(f_{up}^{(k_i-1)}) + \deg_W(\tau_{pq}^{k_i l}) + \deg_W(g_{qu}^{(k_i+1)}) \\
&= \log(x_{k_1} \cdots x_{k_{i-1}} t_p) - \log(t_u) \\
&\quad + \log(x_{k_i} x_l t_q) - \log(t_p) \\
&\quad + \log(x_{k_{i+1}} \cdots x_{k_s} t_u) - \log(t_q)) \\
&= \log(x_{k_1} \cdots x_{k_s} x_l)
\end{aligned}$$

Thus if the entry $\tau_{pq}^{k_i l}$ is a non-zero polynomial from $\mathcal{T}$ and the indeterminate $h_{pq}^{k_i l}$ has a non-zero coefficient in Equation (5.11), then we get $\deg_W(\kappa_{pq}^{k_i l} e_{pq}^{k_i l}) = \log(\Pi)$. □

Now let us summarize what we have until now. We start with a term

$$\Pi = x_{l_1} \cdots x_{l_s} \cdot x_{s+1} \in \mathbb{T}^n.$$

Then we choose a variable x_l from $\{x_{l_1}, ..., x_{l_s}, x_{s+1}\}$ and we let $\frac{\Pi}{x_l} = x_{k_1} \cdots x_{k_s}$. As a result we get $\Pi = x_{k_1} \cdots x_{k_s} x_l$ (see (5.8)). For each indeterminate x_i from $\{x_{k_1}, ..., x_{k_s}, x_l\}$ there exists a generic multiplication matrix $\mathcal{A}_i$ and vice-versa. Thus there is a bijection between terms in $\mathbb{T}^n$ and the products of generic multiplication matrices. Then the term $\Pi = x_{k_1} \cdots x_{k_s}, x_l$ corresponds to the product of the generic multiplication matrices $\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s} \mathcal{A}_l$. Moreover, by Lemma 5.1.6, we have

$$\deg_W(\text{Trace}(\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s} \mathcal{A}_l)) = \deg_W(\text{Trace}(\mathcal{A}_l \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s})) = \log(\Pi),$$

where $\mathcal{A}_l \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s} - \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s} \mathcal{A}_l$ is the commutator $[\mathcal{A}_{k_1} \cdots \mathcal{A}_{k_s}, \mathcal{A}_l]$. Thus by Lemma 5.2.1, the polynomial

$$\text{Trace}(\sum_{i=1}^{s} \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{i-1}} \begin{pmatrix} h_{11}^{k_i l} & \dots & h_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{k_i l} & \dots & h_{\mu\mu}^{k_i l} \end{pmatrix} \mathcal{A}_{k_{i+1}} \cdots \mathcal{A}_{k_s}) = \sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} h_{pq}^{k_i l} \in \mathrm{K}[\mathcal{C}, \mathcal{H}]$$

leads us to the element $\sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}$ of the syzygy module $\text{Syz}(\mathcal{T})$, which is a homogenous module with respect to the arrow grading that is of arrow degree $\log(\Pi)$.

In the above equation, by substituting $h_{pq}^{k_i l}$ with $\tau_{pq}^{k_i l}$, then we get

$$\sum_{i=1}^{s}\sum_{u=1}^{\mu} e_u(\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_{i-1}}\begin{pmatrix}\tau_{11}^{k_i l} & \dots & \tau_{1\mu}^{k_i l}\\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{k_i l} & \dots & \tau_{\mu\mu}^{k_i l}\end{pmatrix}\mathcal{A}_{k_{i+1}}\cdots\mathcal{A}_{k_s})e_u^{tr} = \mathrm{Trace}([\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s},\mathcal{A}_l]).$$

In other words, for the product Π and the distinguished variable x_l, if we compute $\mathrm{Trace}([\mathcal{A}_{k_1}\cdots\mathcal{A}_{k_s},\mathcal{A}_l])$ by letting

$$[A_{k_i}, Al] = \begin{pmatrix}h_{11}^{k_i l} & \dots & h_{1\mu}^{k_i l}\\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{k_i l} & \dots & h_{\mu\mu}^{k_i l}\end{pmatrix}, \tag{5.16}$$

then we get $\sum_{\substack{1\leqslant p,q\leqslant\mu\\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l} h_{pq}^{k_i l}$ which leads us to the syzygy module element $\sum_{\substack{1\leqslant p,q\leqslant\mu\\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}$. By this way the syzygy module element $\sum_{\substack{1\leqslant p,q\leqslant\mu\\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}$ can be computed which depends on the distinguished variable x_l and the term Π.

Definition 5.1.12. In the above setting, the syzygy of $\mathcal{T}$ $(\sum_{\substack{1\leqslant p,q\leqslant\mu\\ 1\leqslant i<l\leqslant n}} \kappa_{pq}^{k_i l} e_{pq}^{k_i l}) \in \mathrm{K}[\mathcal{C}]^m$ from Lemma (5.2.1) is called the **trace syzygy of** $\mathcal{T}$ with respect to **the distinguished indeterminate** x_l and the term Π. We shall denote this trace syzygy by T_{Π,x_l}.

Corollary 5.1.13. *Let* Π *denote* $x_i^\alpha \in \mathbb{T}^n$ *and* $\alpha \in \mathbb{N}_+$. *Then the trace syzygy* T_{Π,x_i} *is* $(0,...,0) \in \mathrm{K}^n$.

Definition 5.1.14. A polynomial τ_{pq}^{kl} from $\mathcal{T}$ is called a **rewritable element with respect to a trace syzygy** if the component κ_{pq}^{kl} of T_{Π,x_l} is a non-zero constant from the field K. A polynomial τ_{pq}^{kl} from $\mathcal{T}$ is called a **redundant** if the polynomial τ_{pq}^{kl} is in the ideal $\langle(\mathcal{T}\setminus\{\tau_{pq}^{kl}\})\rangle$.

Example 5.1.15. Let $\mathcal{O}$ denote the order ideal $\{1, x_1\} \subset \mathrm{K}[x_1, x_2]$. Its border is $\partial\mathcal{O} = \{x_2, x_1x_2, x_1^2\}$. We compute the vanishing ideal of the $\mathcal{O}$-border basis scheme

$\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ with their arrow degrees by the help of the ApCoCoA package bbsmingensyz.

$$\begin{array}{rr} \text{Arrow degree}(1,1) & \text{Arrow degree}(2,1) \\ \tau_{11}^{12} = c_{13}c_{21} - c_{12} & \tau_{12}^{12} = -c_{11}c_{13} + c_{13}c_{22} - c_{12}c_{23} \\ \text{Arrow degree}(0,1) & \text{Arrow degree}(1,1) \\ \tau_{21}^{12} = c_{21}c_{23} + c_{11} - c_{22} & \tau_{22}^{12} = -c_{13}c_{21} + c_{12} \end{array}$$

We let $\Pi = x_1x_2$ which corresponds to the product of the generic multiplication matrices $\mathcal{A}_1\mathcal{A}_2$. (We will explain later why we choose Π that way.) Since we have $[\mathcal{A}_1, \mathcal{A}_2] = -[\mathcal{A}_2, \mathcal{A}_1]$, both x_1 and x_2 as the distinguished indeterminate will lead to the same trace syzygy. Let x_1 be the distinguished indeterminate and let us compute the trace syzygy $\mathrm{T}_{x_1x_2,x_1}$ as we explained. First we compute the trace of the commutator by using Equation (5.16).

$$\text{Trace}([\mathcal{A}_1, \mathcal{A}_2]) = \text{Trace}(\begin{pmatrix} h_{11}^{12} & h_{12}^{12} \\ h_{21}^{12} & h_{22}^{12} \end{pmatrix}) = h_{11}^{12} + h_2^{12}.$$

Then the trace syzygy is $\mathrm{T}_{x_1x_2,x_1} = e_{11}^{12} + e_{22}^{12}$ From this equality follows that the polynomials τ_{11} and τ_{22} are both rewritable elements. Only one of them is redundant.

In Example 5.1.15, which term from $\mathbb{T}^2$ should we choose to show τ_{12}^{12} is redundant? In other words, how should we choose Π so that we can get all the redundant elements in $\mathcal{T}$? The following proposition gives answers to our questions.

Proposition 5.1.16. *Let $\mathcal{O} \subset \mathbb{T}^n$ denote an order ideal. Let $\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_s}, \mathcal{A}_l \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$, where $s \in \mathbb{N}_+$ and $k_1, ..., k_s, l \in \{1, ..., n\}$. Let Π denote the term $x_{k_1} \cdots x_{k_s}x_l \in \mathbb{T}^n$. Let τ_{pq}^{kl} be a polynomial from $\mathcal{T}$, where $k \in \{k_1, ..., k_s\}$, $l \in \{1, ..., n\}\backslash\{k_1, ..., k_s\}$ and $p, q \in \{1, ..., \mu\}$. Then we have $\deg_W(\tau_{pq}^{kl}) = \log(\Pi)$ if and only if the component κ_{pq}^{kl} of T_{Π,x_l} is a non-zero constant.*

Proof. This follows from [Huib09], Proposition 8.3. □

Definition 5.1.17. Let τ_{pq}^{kl} be a non-trivial polynomial from $\mathcal{T}$ (see Equation 2.6). The polynomial τ_{pq}^{kl} is called a **standard polynomial** if

$$\deg_W(\tau_{pq}^{kl}) = \log(x_kx_lt_q) - \log(t_p)$$

has exactly one positive component. Otherwise, it is called a **non-standard polynomial**. If the arrow degree vector has no negative entries, then it is said to be of **non-negative arrow degree.** Otherwise, it is said to be of **non-positive arrow degree**.

Example 5.1.18. (continues) Recall Example 5.1.15 where the vanishing ideal of the $\mathcal{O}$-border basis scheme $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by the following elements:

$$\begin{array}{ll} \text{Arrow degree}(1,1) & \text{Arrow degree}(2,1) \\ \tau_{11} = c_{13}c_{21} - c_{12} & \tau_{12} = -c_{11}c_{13} + c_{13}c_{22} - c_{12}c_{23} \\ \text{Arrow degree}(0,1) & \text{Arrow degree}(1,1) \\ \tau_{21} = c_{21}c_{23} + c_{11} - c_{22} & \tau_{22} = -c_{13}c_{21} + c_{12} \end{array}$$

Since all the polynomials in $\mathcal{T}$ have the same upper index, we omit the upper index. We want to find all the redundant and rewritable elements in $\mathcal{T}$. Then by Proposition 5.1.16, we need to compute the trace syzygies whose arrow degrees are the same as the arrow degrees of the elements of $\mathcal{T}$. Recall that in Example 5.1.15 the trace syzygy is $\mathrm{T}_{x_1x_2,x_1} = e_{11} + e_{22}$ for arrow degree $(1,1)$. From this equality follows that the polynomials τ_{11} and τ_{22} are both rewritable elements of $\mathcal{T}$.

Next we consider the arrow degree $(0,1)$. Then we have $\log(\Pi) = (0,1)$ i.e., $\Pi = x_2$. By Corollary 5.1.13, we have $\mathrm{T}_{\Pi,x_2} = (0,...,0) \in \mathrm{K}^n$.

The remaining arrow degree is $(2,1)$. Then Π is $x_1^2x_2$ which corresponds to the product of the generic multiplication matrices $\mathcal{A}_1\mathcal{A}_1\mathcal{A}_2$. By Lemma 5.1.2, we have the following equations.

$$\begin{aligned} \mathrm{Trace}([\mathcal{A}_2,\mathcal{A}_1]\mathcal{A}_1 + \mathcal{A}_1[\mathcal{A}_2,\mathcal{A}_1]) &= 2\mathrm{Trace}([\mathcal{A}_1,\mathcal{A}_2]\mathcal{A}_1) \\ &= \mathrm{Trace}([\mathcal{A}_1,\mathcal{A}_2]\mathcal{A}_1 + \mathcal{A}_2[\mathcal{A}_1,\mathcal{A}_1]). \end{aligned}$$

Therefore choosing either x_1 or x_2 won't change the trace syzygy. Let x_1 be the distinguished indeterminate and let us compute the trace syzygy $\mathrm{T}_{x_1^2x_2,x_1}$ as we explained. First we compute the trace of the commutator by using Equation (5.16).

$$\mathrm{Trace}([\mathcal{A}_1^2,\mathcal{A}_2]) = \mathrm{Trace}(\begin{pmatrix} h_{11}^{12} & h_{12}^{12} \\ h_{21}^{12} & h_{22}^{12} \end{pmatrix}\mathcal{A}_1) = c_{13}h_{21}^{12} + c_{23}h_{22}^{12} + h_{12}^{12}.$$

Hence the trace syzygy is $\mathrm{T}_{x_1x_2,x_1} = c_{13}e_{21}^{12} + e_{23}h_{22}^{12} + e_{12}^{12}$. From this equality follows

that the polynomial τ_{12} is rewritable by $\{\tau_{21}, \tau_{22}\}$. From the equality $\tau_{11} = -\tau_{22}$ follows that τ_{11} and τ_{12} are redundant. Then the ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by the set $\{\tau_{21}, \tau_{22}\}$.

Lemma 5.1.19. *Let* Π *denote the term* $x_i^{\alpha} x_j^{\beta} \in \mathbb{T}^n$, *where* $\alpha, \beta \in \mathbb{N}_+$. *Then the trace syzygy* T_{Π,x_i} *is the same as the trace syzygy* T_{Π,x_j}.

Proof. If the exponents of x_i and x_j are both 1 i.e., Π is $x_i x_j$, then the claim follows from the equality $[\mathcal{A}_i, \mathcal{A}_j] = -[\mathcal{A}_j, \mathcal{A}_i]$. Assume we have $\alpha, \beta > 1$. If we choose x_i as the distinguished variable, then by Lemma 5.1.2 we have the following.

$$\begin{aligned}[\mathcal{A}_i \cdots \mathcal{A}_i \mathcal{A}_j \cdots \mathcal{A}_j, \mathcal{A}_i] \;=\; & (\mathcal{A}_i \cdots \mathcal{A}_i [\mathcal{A}_i, \mathcal{A}_j] \mathcal{A}_j \cdots \mathcal{A}_j) \\ & + (\mathcal{A}_i \cdots \mathcal{A}_i \mathcal{A}_j [\mathcal{A}_i, \mathcal{A}_j] \mathcal{A}_j \cdots \mathcal{A}_j) + \cdots \\ & + (\mathcal{A}_i \cdots \mathcal{A}_i \mathcal{A}_j \cdots \mathcal{A}_j [\mathcal{A}_i, \mathcal{A}_j])\end{aligned} \tag{5.17}$$

If we we choose x_j as the distinguished variable, then by Lemma 5.1.2 we have the following.

$$\begin{aligned}[\mathcal{A}_i \cdots \mathcal{A}_i \mathcal{A}_j \cdots \mathcal{A}_j, \mathcal{A}_j] \;=\; & ([\mathcal{A}_j, \mathcal{A}_i] \mathcal{A}_i \cdots \mathcal{A}_i \mathcal{A}_j \cdots \mathcal{A}_j) \\ & + (\mathcal{A}_i [\mathcal{A}_j, \mathcal{A}_i] \mathcal{A}_i \cdots \mathcal{A}_i \cdot \mathcal{A}_j \cdots \mathcal{A}_j) + \cdots \\ & + (\mathcal{A}_i \cdots \mathcal{A}_i [\mathcal{A}_j, \mathcal{A}_i] \mathcal{A}_j \cdots \mathcal{A}_j)\end{aligned} \tag{5.18}$$

Since the trace is invariant under the cyclic permutation of the matrices, the traces of the matrices in the Equation (5.18) and in Equation (5.17) are the same. Thus we have $\mathrm{T}_{\Pi,x_i} = \mathrm{T}_{\Pi,x_j}$. □

Remark 5.1.20. Proposition 5.1.16 helps us to determine the rewritable elements (see Definition 5.1.12) with respect to a trace syzygy. First we recall how one can compute the arrow degree of an element of $\mathcal{T}$. Let τ_{pq}^{kl} be polynomial in $\mathcal{T}$, where $k, l \in \{1, ..., n\}$ and $p, q \in \{1, ..., \mu\}$. Then we have

$$\deg_W(\tau_{pq}^{kl}) = \log(x_k x_l t_q) - \log(t_p).$$

By Proposition 5.1.16, the component κ_{pq}^{kl} of T_{Π,x_l} is a non-zero constant if and only if $\log(\Pi) = (\log(x_k x_l t_q) - \log(t_p)) = \deg_W(\tau_{pq}^{kl})$. Since Π is a product of indeterminates, Proposition 5.1.16 implies that we should choose every $t_p, t_q \in \mathcal{O}$ such that the vector $\log(\Pi)$ is in $\mathbb{N}^n$. Thus the only rewritable polynomials with respect to trace syzygies are the ones that are of non-negative arrow degrees (see Definition 5.1.17). Let $\tau_{pq}^{kl} \in \mathcal{T}$

be a redundant polynomial of non-positive arrow degree. Even if τ_{pq}^{kl} is redundant, the component κ_{pq}^{kl} of a trace syzygy is never a non-zero constant.

Moreover, at least two components of $\log(\Pi)$ must be larger than zero, since the commutator operation needs at least two different generic multiplication matrices for a non-zero result. Thus for a non-standard polynomial from $\tau_{pq}^{kl} \in \mathcal{T}$ with non-negative arrow degree the trace syzygy contains a non-zero constant κ_{pq}^{kl}. Therefore by finding the non-standard, non-negative arrow degrees of the elements of $\mathcal{T}$, one can compute every rewritable element with respect to trace syzygies.

Definition 5.1.21. Let $\mathcal{O} \subset \mathrm{P}$ be an order ideal. Let $\mathbb{B}_{\mathcal{O}}$ denote the $\mathcal{O}$-border basis scheme. Let $\mathfrak{o}$ denote the monomial point $(0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$.

a) We call the generic multiplication matrices we get by evaluating $(c_{11}, ..., c_{\mu\nu})$ at $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ as **simplified generic multiplication matrices**. We denote the simplified generic multiplication matrix we get from a generic multiplication matrix $\mathcal{A}$ by $\mathcal{A}'$.

b) If we evaluate the polynomials $\kappa_{pq}^{kl} \in \mathrm{K}[\mathcal{C}]$ at $\mathfrak{o}$, which is a component of the trace syzygy T_{Π,x_l} in Equation (5.8) and if we let T'_{Π,x_l} denote the result of this process, then the non-zero components of T'_{Π,x_l} will indicate the rewritable elements with respect to T_{Π,x_l}. We call T'_{Π,x_l} as **simplified trace syzygy** of $\mathcal{T}$.

In particular, one can compute T'_{Π,x_l} without computing the whole trace syzygy T_{Π,x_l}. Recall that we started our computation of T_{Π,x_l} with

$$\mathrm{Trace}(\sum_{i=1}^{s} \mathcal{A}_{k_1} \cdots \mathcal{A}_{k_{i-1}} \begin{pmatrix} h_{11}^{k_i l} & \dots & h_{1\mu}^{k_i l} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{k_i l} & \dots & h_{\mu\mu}^{k_i l} \end{pmatrix} \mathcal{A}_{k_{i+1}} \cdots \mathcal{A}_{k_s}) = \sum_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \kappa_{pq}^{k_i l} h_{pq}^{k_i l} \in \mathrm{K}[\mathcal{C}, \mathcal{H}]. \tag{5.19}$$

In Equation (5.19) by replacing the multiplication matrices

$$\mathcal{A}_{k_1}, ..., \mathcal{A}_{k_{i-1}}, \mathcal{A}_{k_{i+1}}, ..., \mathcal{A}_{k_s}$$

with the simplified multiplication matrices without changing the entries from $\mathrm{K}[\mathcal{H}]$, one can easily find T'_{Π,x_l}. The advantage we get by using simplified generic multiplication matrices is as follows. If the computation of the trace syzygy T_{Π,x_l} cannot be performed or takes too long by computer, then computing T'_{Π,x_l} will give us the rewritable elements with respect to T_{Π,x_l} faster. In the next example we compute the trace syzygies of the

defining equations of box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2)}$. The trace syzygies of $\mathcal{T}$ give all the redundant elements from $\mathcal{T}$.

Example 5.1.22. Let P denote the polynomial ring $\mathrm{K}[x_1, x_2]$. Let $\mathcal{B}(2,2)$ denote the order ideal $\{1, x_1, x_2, x_1x_2\}$. The border of $\mathcal{B}(2,2)$ is $\partial\mathcal{B} = \{x_2^2, x_1^2, x_1x_2^2, x_1^2x_2\}$, so that in our terminology we have $t_1 = 1, t_2 = x_1, t_3 = x_2, t_4 = x_1x_2, b_1 = x_2^2, b_2 = x_1^2, b_3 = x_1x_2^2$ and $b_4 = x_1^2x_2$. The arrow degrees of the elements of $\mathcal{T}$ (see Definition 5.1.17) can be calculated as $\mathrm{d} = (d_1, d_2) = \deg_W(xyt_q) - \deg_W(t_p)$. By using the ApCoCoA package bbsmingensyz we compute the elements of $\mathcal{T}$ with their arrow degrees.

$$
\begin{aligned}
&\text{Arrow degree}(2,1)\\
\tau_{12} &= c_{11}c_{32} + c_{13}c_{42} - c_{14}\\
\tau_{34} &= c_{23}c_{32} - c_{31}c_{34} - c_{34}c_{43} + c_{33}c_{44} - c_{14}\\
&\text{Arrow degree}(1,2)\\
\tau_{13} &= c_{12}c_{21} + c_{14}c_{41} - c_{13}\\
\tau_{24} &= c_{22}c_{23} - c_{21}c_{34} - c_{24}c_{43} + c_{23}c_{44} + c_{13}\\
&\text{Arrow degree}(2,2)\\
\tau_{14} &= c_{12}c_{23} - c_{11}c_{34} - c_{14}c_{43} + c_{13}c_{44}\\
&\text{Arrow degree}(1,1)\\
\tau_{22} &= c_{21}c_{32} + c_{23}c_{42} - c_{24}\\
\tau_{33} &= c_{21}c_{32} + c_{34}c_{41} - c_{33}\\
\tau_{44} &= c_{34}c_{41} - c_{23}c_{42} - c_{24} + c_{33}\\
&\text{Arrow degree}(0,2)\\
\tau_{23} &= c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23}\\
&\text{Arrow degree}(2,0)\\
\tau_{32} &= c_{13}c_{32} + c_{33}c_{42} + c_{12} - c_{34}\\
&\text{Arrow degree}(1,0)\\
\tau_{42} &= c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}\\
&\text{Arrow degree}(0,1)\\
\tau_{43} &= c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43}
\end{aligned}
$$

Note that $\mathcal{T}$ has non-standard polynomials which have non-negative arrow degree and there are no polynomials of non-positive arrow degrees.The non-standard arrow degrees

are $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$. For every non-standard arrow degree, say d, we compute the corresponding trace syzygy.

i) d $= (1,1)$: The arrow degree d is equal to $\log(x_1x_2)$. Then let Π denote the term x_1x_2. By Lemma 5.1.1 we have $[\mathcal{A}_1, \mathcal{A}_2] = -[\mathcal{A}_2, \mathcal{A}_1]$. Then we can choose x_1 as the distinguished indeterminate and compute

$$\text{Trace}([\mathcal{A}_1, \mathcal{A}_2]) = \tau_{22} + \tau_{33} + \tau_{44} = 0.$$

Hence the trace syzygy is $\text{T}_{x_1x_2,x_1} = e_{22} + e_{33} + e_{44}$. The polynomial τ_{44} is a rewritable element with respect to the trace syzygy $\text{T}_{x_1x_2,x_1}$. It is rewritable by τ_{33} and τ_{22}, since we have

$$\tau_{44} = \tau_{22} - \tau_{33} \tag{5.20}$$

ii) d $= (1,2)$: The degree d is equal to $\log(x_1x_2^2)$. Let Π denote the term $x_1x_2^2$. Let us choose first x_1 as the distinguished variable.

$$\begin{aligned}
\text{Trace}([\mathcal{A}_1, \mathcal{A}_2]\mathcal{A}_2 + \mathcal{A}_2[\mathcal{A}_1, \mathcal{A}_2]) &= 2\text{Trace}(\mathcal{A}_2[\mathcal{A}_1, \mathcal{A}_2]) \\
&= 2\text{Trace}([\mathcal{A}_1, \mathcal{A}_2]\mathcal{A}_2) \\
&= 2(c_{11}h_{22} + c_{21}h_{32} + c_{41}h_{34} + c_{13}h_{41} \\
&\quad + c_{23}h_{42} + c_{33}h_{43} + h_{13} + h_{24}).
\end{aligned}$$

By using Equation (5.20) we get the following trace syzygy.

$$\begin{aligned}
\text{T}_{x_1x_2^2,x_1} &= c_{11}e_{22} + c_{21}e_{32} + c_{41}e_{34} + c_{13}e_{41} \\
&\quad + c_{23}e_{42} + c_{33}e_{43} + e_{13} + e_{24}.
\end{aligned}$$

Note that if we choose x_2 as the distinguished variable, then we will have the same trace syzygy as $\text{T}_{x_1x_2^2,x_1}$ as the following equation shows.

$$\text{Trace}([\mathcal{A}_2, \mathcal{A}_1]\mathcal{A}_2 + \mathcal{A}_1[\mathcal{A}_2, \mathcal{A}_2]) = \text{Trace}(\mathcal{A}_2[\mathcal{A}_1, \mathcal{A}_2])$$

After the next case, we give the redundant element of $\mathcal{T}$, which we get from this step.

iii) d $= (2,1)$: The arrow degree d is equal to $\log(x_1^2x_2)$. Let Π denote the term $x_1^2x_2$. We choose x_1 as the distinguished variable.

$$\begin{aligned}\text{Trace}([\mathcal{A}_1,\mathcal{A}_2]\mathcal{A}_1+\mathcal{A}_2[\mathcal{A}_1,\mathcal{A}_1]) &= \text{Trace}([\mathcal{A}_1,\mathcal{A}_2]\mathcal{A}_1)\\ &= 2(c_{22}h_{22}+c_{32}h_{23}+c_{42}h_{24}+c_{24}h_{42}\\ &\quad +c_{34}h_{43}+c_{44}h_{44}+h_{12}+h_{34}).\end{aligned}$$

By using Equation (5.20), we get the following trace syzygy.

$$\begin{aligned}\text{T}_{x_1^2x_2,x_1} &= (c_{22}+c_{44})e_{22}+c_{32}e_{23}+c_{42}e_{24}+c_{24}e_{42}\\ &\quad +c_{34}e_{43}-c_{44}e_{44}+e_{12}+e_{34}.\end{aligned}$$

Note that if we choose x_2 as the distinguished variable, then we will have the same trace syzygy as the following equation shows:

$$\text{Trace}([\mathcal{A}_2,\mathcal{A}_1]\mathcal{A}_1+\mathcal{A}_1[\mathcal{A}_2,\mathcal{A}_1]) = 2\text{Trace}([\mathcal{A}_1,\mathcal{A}_2]\mathcal{A}_1).$$

We have the following trace syzygies of $\mathcal{T}$.

$$\text{T}_{x_1x_2,x_1} = e_{22}+e_{33}+e_{44} \tag{5.21}$$

$$\begin{aligned}\text{T}_{x_1x_2^2,x_1} &= c_{11}e_{22}+c_{21}e_{32}+c_{41}e_{34}+c_{13}e_{41}\\ &\quad +c_{23}e_{42}+c_{33}e_{43}+e_{13}+e_{24}.\end{aligned} \tag{5.22}$$

$$\begin{aligned}\text{T}_{x_1^2x_2,x_1} &= (c_{22}+c_{44})e_{22}+c_{32}e_{23}+c_{42}e_{24}+c_{24}e_{42}\\ &\quad +c_{34}e_{43}-c_{44}e_{44}+e_{12}+e_{34}\end{aligned} \tag{5.23}$$

From Syzygy (5.21), it follows that $\tau_{22},\tau_{33},\tau_{44}$ are all rewritable and one of them is redundant. Since κ_{33} is zero in both $\text{T}_{x_1x_2^2,x_1}, \text{T}_{x_1^2x_2,x_1}$, we can say τ_{33} is redundant. From Syzygy (5.22), it follows that τ_{13} and τ_{24} are rewritable. Since κ_{13} is zero in Syzygies (5.21) and (5.23), τ_{13} is redundant. Finally, from Syzygy (5.23), it follows that τ_{12} and τ_{34} are rewritable. Since κ_{12} is zero in the Syzygies (5.21) and (5.22), τ_{12} is redundant. Thus the polynomials τ_{12},τ_{13} and τ_{33} are redundant.

iv) d $= (2,2)$: The arrow degree d is equal to $\log(x_1^2x_2^2)$. Let Π denote the term $x_1^2x_2^2$.

The trace is invariant under cyclic permutations. Then by Lemma 5.1.2, we have

$$\begin{aligned}\text{Trace}([\mathcal{A}_1\mathcal{A}_1\mathcal{A}_2, \mathcal{A}_2] &= \text{Trace}([\mathcal{A}_2, \mathcal{A}_1]\mathcal{A}_1\mathcal{A}_2 + \mathcal{A}_1[\mathcal{A}_1, \mathcal{A}_2] \\ &= \text{Trace}(\mathcal{A}_1[\mathcal{A}_1, \mathcal{A}_2]\mathcal{A}_2 + \mathcal{A}_1\mathcal{A}_2[\mathcal{A}_1, \mathcal{A}_2] \\ &= \text{Trace}([\mathcal{A}_1\mathcal{A}_2\mathcal{A}_2, \mathcal{A}_1]\end{aligned}$$

Thus choosing either x_1 or x_2 will lead to the same trace syzygy. Let x_1 be the distinguished variable. In this case let us compute the trace syzygy by using the simplified multiplication matrices.

$$\text{Trace}(\mathcal{A}'_1[\mathcal{A}_1, \mathcal{A}_2]\mathcal{A}'_2 + \mathcal{A}'_1\mathcal{A}'_2[\mathcal{A}_1, \mathcal{A}_2]) = \tau_{14}.$$

The simplified trace syzygy is $\mathrm{T}'_{\Pi,x_1} = e_{14}$. Thus τ_{14} is the rewritable element with respect to $\mathrm{T}_{x_1^2x_2^2,x_1}$. Since the arrow degree $\deg_W(\tau_{14})$ is larger than $\deg_W(\tau_{44}), \deg_W(\tau_{12})$ and $\deg_W(\tau_{13})$ with respect to the ordering $<_{\texttt{Lex}}$ and since the indeterminates from $\mathrm{K}[\mathcal{C}]$ have the arrow degree vector with at least one positive component, κ_{14} is not a component of Syzygies (5.21), (5.22) and (5.23). Thus without computing the whole syzygy T_{Π,x_1}, we can say that τ_{14} is redundant.

Proposition 5.1.23. *Let $\mathcal{O} \subset \mathbb{T}^2$ be an order ideal. Then $\mathbb{B}_\mathcal{O}$ is an ideal theoretic complete intersection.*

Proof. This follows from [Huib09], Theorem 11.24. □

Example 5.1.24. (continues) We continue with Example 5.1.22. By Corollary 4.2.17 the $(\mathrm{B}_\mathcal{O})_\mathfrak{m}$ is smooth of dimension $n\mu = 8$. From i, ii, iii and iv, it follows that $\mathcal{I}(\mathbb{B}_\mathcal{O})$ can be generated by $8 = \mu\nu - n\mu$ polynomials which is the co-dimension of $\mathbb{B}_\mathcal{O}$. Thus $\mathbb{B}_\mathcal{O}$ is an ideal theoretic complete intersection.

Remark 5.1.25. Let $\mathcal{O} \subset \mathbb{T}^2$ be an order ideal. In [Huib09] an ordering on the set arrow degrees

$$\{(d_1, d_2) \mid \deg_W(\tau_{pq}) = (d_1, d_2)) \;\; d_1, d_2 \in \mathbb{N}_+, \tau_{pq} \in \mathcal{T}\}$$

is defined in order to find the exact redundant generators of $\mathcal{I}(\mathbb{B}_\mathcal{O})$. Let m denote the monomial $x_1x_2t_q$ and let t_p and t_q be terms from the order ideal. The main idea of this ordering process depends on the fact that there exist a power $s \in \mathbb{N}_+$ such that

$t_{r_s} = x_2^s$ is in $\mathcal{O}$ and there exists a term $t_{q'} \in \mathcal{O}$ such that a non-standard arrow degree has the following property.

$$(d_1, d_2) = \log(x_1 x_2 t_q) - \log(t_p) = \log(x_1 x_2 t_{q'}) - \log(t_{r_s})$$

By using this property, in [Huib09] Proposition 5.1.23 is proved. This is a property that only holds in special cases. For instance, in Section 5.4 we show that for the box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$ where $\mathcal{B}(2,2,2) \subset \mathbb{T}^3$, this does not hold.

5.2 Jacobi Identity Syzygies

Let $\mathcal{A}, \mathcal{B}, \mathcal{D} \in \mathrm{Mat_m}(\mathrm{K}[\mathcal{C}])$ be square matrices, where $m \in \mathbb{N}_+$. Then the equality $[\mathcal{D}, [\mathcal{A}, \mathcal{B}]] + [\mathcal{A}, [\mathcal{D}, \mathcal{B}] + [\mathcal{B}, [\mathcal{A}, \mathcal{D}]] = 0$ is called the **Jacobi identity**. Let $\mathbb{B}_{\mathcal{O}}$ denote an $\mathcal{O}$-border basis scheme and let $\mathcal{T}$ be the set of defining equations of $\mathbb{B}_{\mathcal{O}}$. As in the previous subsection, we first reprove some more or less well-known properties of the generic multiplication matrices with respect to the Jacobi identity, then we give a definition of a Jacobi identity syzygy of $\mathcal{T}$. Recall that

$$\begin{array}{rccc} \varphi : & \bigoplus\limits_{\substack{1 \leqslant p,q \leqslant \mu \\ 1 \leqslant i < l \leqslant n}} \mathrm{K}[\mathcal{C}](-d_{pq}^{kl}) & \longrightarrow & \mathcal{I}(\mathbb{B}_{\mathcal{O}}) \\ & e_{pq}^{kl} & \longmapsto & \tau_{pq}^{kl} \end{array} \tag{5.24}$$

is a surjective graded $\mathrm{K}[\mathcal{C}]$-module morphism. The kernel of the map φ is the syzygy module $\mathrm{Syz}(\mathcal{T})$. The syzygy module $\mathrm{Syz}(\mathcal{T})$ is a homogenous module with respect to the arrow grading. Moreover, let the set $\mathcal{H}$ denote a set of further indeterminates $\{h_{11}^{12}, ..., h_{pq}^{kl}, ..., h_{\mu\mu}^{(n-1)n}\}$, where the cardinality of the set $\mathcal{H}$ is m. The map

$$\begin{array}{rcl} \theta : \mathrm{K}[\mathcal{C}]^m & \longrightarrow & \mathrm{K}[\mathcal{C}][h_{11}^{12}, ..., h_{pq}^{kl}, ..., h_{\mu\mu}^{(n-1)n}] = \mathrm{K}[\mathcal{C}][\mathcal{H}] \\ e_{pq}^{kl} & \longmapsto & h_{pq}^{kl} \end{array}$$

is an injective $\mathrm{K}[\mathcal{C}]$-module morphism. Let $\mathcal{O} \subset \mathbb{T}^n$ denote an order ideal. Let $\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_m \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$, where $k, l, m \in \{1, ..., n\}$. Then the matrix

$$[\mathcal{A}_k, \begin{pmatrix} h_{11}^{ml} & \dots & h_{1\mu}^{ml} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{ml} & \dots & h_{\mu\mu}^{ml} \end{pmatrix}] + [\mathcal{A}_l, \begin{pmatrix} h_{11}^{mk} & \dots & h_{1\mu}^{mk} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{mk} & \dots & h_{\mu\mu}^{mk} \end{pmatrix}] + [\mathcal{A}_m, \begin{pmatrix} h_{11}^{kl} & \dots & h_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{kl} & \dots & h_{\mu\mu}^{kl} \end{pmatrix}] \tag{5.25}$$

is an element of $\mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}, \mathcal{H}])$. For $i, j \in \{1, ..., \mu\}$ in position (i, j) of this matrix is a polynomial in $\mathrm{K}[\mathcal{C}, \mathcal{H}]$, which is as follows.

$$e_i[\mathcal{A}_k, \begin{pmatrix} h_{11}^{ml} & \dots & h_{1\mu}^{ml} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{ml} & \dots & h_{\mu\mu}^{ml} \end{pmatrix}]e_j^{tr} + e_i^{tr}[\mathcal{A}_l, \begin{pmatrix} h_{11}^{mk} & \dots & h_{1\mu}^{mk} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{mk} & \dots & h_{\mu\mu}^{mk} \end{pmatrix}]e_j^{tr} + e_i[\mathcal{A}_m, \begin{pmatrix} h_{11}^{kl} & \dots & h_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{kl} & \dots & h_{\mu\mu}^{kl} \end{pmatrix}]e_j^{tr} \tag{5.26}$$

For $s \in \{k,l,m\}$, we let $\mathcal{A}_s = \begin{pmatrix} f_{11}^{(s)} & \cdots & f_{1\mu}^{(s)} \\ \vdots & \ddots & \vdots \\ f_{\mu 1}^{(s)} & \cdots & f_{\mu\mu}^{(s)} \end{pmatrix}$, where $f_{pq}^{(s)}$ is either a constant from $\{0,1\}$ or an indeterminate from $\mathcal{C}$. Then Polynomial (5.26) is as follows.

$$(f_{i1}^{(l)} \dots f_{i\mu}^{(l)}) \begin{pmatrix} h_{1j}^{mk} \\ \vdots \\ h_{\mu j}^{mk} \end{pmatrix} - (h_{i1}^{mk} \dots h_{i\mu}^{mk}) \begin{pmatrix} f_{1j}^{(l)} \\ \vdots \\ f_{\mu j}^{(l)} \end{pmatrix} + (f_{i1}^{(k)} \dots f_{i\mu}^{(k)}) \begin{pmatrix} h_{1j}^{ml} \\ \vdots \\ h_{\mu j}^{ml} \end{pmatrix} -$$
$$(h_{i1}^{ml} \dots h_{i\mu}^{ml}) \begin{pmatrix} f_{1j}^{(k)} \\ \vdots \\ f_{\mu j}^{(k)} \end{pmatrix} + (f_{i1}^{(m)} \dots f_{i\mu}^{(m)}) \begin{pmatrix} h_{1j}^{lk} \\ \vdots \\ h_{\mu j}^{lk} \end{pmatrix} - (h_{i1}^{lk} \dots h_{i\mu}^{lk}) \begin{pmatrix} f_{1j}^{(m)} \\ \vdots \\ f_{\mu j}^{(m)} \end{pmatrix} \quad (5.27)$$

Then Polynomial (5.27) is equal to the following polynomial.

$$\sum_{p=1}^{\mu} (f_{ip}^{(l)} h_{pj}^{mk} - f_{pj}^{(l)} h_{ip}^{mk}) + (f_{ip}^{(k)} h_{pj}^{ml} - f_{pj}^{(k)} h_{ip}^{ml}) + (f_{ip}^{(m)} h_{pj}^{kl} - f_{pj}^{(m)} h_{ip}^{kl}) \quad (5.28)$$

Let us denote the $\mathrm{K}[\mathcal{C}]$-coefficient of an indeterminate $h_{pq}^{mk} \in \mathcal{H}$ as $\rho_{pq}^{mk} \in \mathrm{K}[\mathcal{C}]$. Then Polynomial (5.28) is as follows.

$$\sum_{1 \leqslant p,q \leqslant \mu} \rho_{pq}^{mk} h_{pq}^{mk} + \rho_{pq}^{ml} h_{pq}^{ml} + \rho_{pq}^{kl} h_{pq}^{kl} \quad (5.29)$$

The inverse image of Polynomial (5.29) under the map θ is

$$\sum_{1 \leqslant p,q \leqslant \mu} \rho_{pq}^{mk} e_{pq}^{mk} + \rho_{pq}^{ml} e_{pq}^{ml} + \rho_{pq}^{kl} e_{pq}^{kl}. \quad (5.30)$$

Then under the map φ, we have

$$\varphi(\sum_{1 \leqslant p,q \leqslant \mu} \rho_{pq}^{mk} e_{pq}^{mk} + \rho_{pq}^{ml} e_{pq}^{ml} + \rho_{pq}^{kl} e_{pq}^{kl}) = \sum_{1 \leqslant p,q \leqslant \mu} \rho_{pq}^{mk} \tau_{pq}^{mk} + \rho_{pq}^{ml} \tau_{pq}^{ml} + \rho_{pq}^{kl} \tau_{pq}^{kl}. \quad (5.31)$$

Lemma 5.2.1. *In the above setting, the vector* $\sum_{1 \leqslant p,q \leqslant \mu} \rho_{pq}^{mk} e_{pq}^{mk} + \rho_{pq}^{ml} e_{pq}^{ml} + \rho_{pq}^{kl} e_{pq}^{kl}$ *is an element of the syzygy module* $\mathrm{Syz}(\mathcal{T})$.

Proof. We show that both sides in Equation (5.31) are 0. Recall Equation (5.28). We substitute $h_{pq}^{k_i l}$ with $\tau_{pq}^{k_i l}$ in the polynomial $\sum_{p=1}^{\mu} (f_{ip}^{(l)} h_{pj}^{mk} - f_{pj}^{(l)} h_{ip}^{mk}) + (f_{ip}^{(k)} h_{pj}^{ml} - f_{pj}^{(k)} h_{ip}^{ml}) +$

$(f_{ip}^{(m)}h_{pj}^{kl} - f_{pj}^{(m)}h_{ip}^{kl})$. Then we have

$$\sum_{p=1}^{\mu}(f_{ip}^{(l)}\tau_{pj}^{mk} - f_{pj}^{(l)}\tau_{ip}^{mk}) + (f_{ip}^{(k)}\tau_{pj}^{ml} - f_{pj}^{(k)}\tau_{ip}^{ml}) + (f_{ip}^{(m)}\tau_{pj}^{kl} - f_{pj}^{(m)}\tau_{ip}^{kl}). \tag{5.32}$$

Polynomial (5.32) is in position (i, j) of the following matrix.

$$\begin{aligned} & [\mathcal{A}_k, \begin{pmatrix} \tau_{11}^{ml} & \dots & \tau_{1\mu}^{ml} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{ml} & \dots & \tau_{\mu\mu}^{ml} \end{pmatrix}] + [\mathcal{A}_l, \begin{pmatrix} \tau_{11}^{mk} & \dots & \tau_{1\mu}^{mk} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{mk} & \dots & \tau_{\mu\mu}^{mk} \end{pmatrix}] + [\mathcal{A}_m, \begin{pmatrix} \tau_{11}^{kl} & \dots & \tau_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{kl} & \dots & \tau_{\mu\mu}^{kl} \end{pmatrix}] \\ = \quad & [\mathcal{A}_k, [\mathcal{A}_l, \mathcal{A}_m]] + [\mathcal{A}_l, [\mathcal{A}_k, \mathcal{A}_m]] + [\mathcal{A}_m, [\mathcal{A}_k, \mathcal{A}_l]] \end{aligned} \tag{5.33}$$

The Jacobi identity $[\mathcal{A}_k, [\mathcal{A}_l, \mathcal{A}_m]] + [\mathcal{A}_l, [\mathcal{A}_k, \mathcal{A}_m]] + [\mathcal{A}_m, [\mathcal{A}_k, \mathcal{A}_l]]$ is a zero matrix. Hence the polynomial $\sum_{p=1}^{\mu}(f_{ip}^{(l)}\tau_{pj}^{mk} - f_{pj}^{(l)}\tau_{ip}^{mk}) + (f_{ip}^{(k)}\tau_{pj}^{ml} - f_{pj}^{(k)}\tau_{ip}^{ml}) + (f_{ip}^{(m)}\tau_{pj}^{kl} - f_{pj}^{(m)}\tau_{ip}^{kl})$ which is in the (i, j) position of $[\mathcal{A}_k, [\mathcal{A}_l, \mathcal{A}_m]] + [\mathcal{A}_l, [\mathcal{A}_k, \mathcal{A}_m]] + [\mathcal{A}_m, [\mathcal{A}_k, \mathcal{A}_l]]$ is 0. Then we have $\varphi(\sum_{1\leqslant p,q\leqslant\mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl}) = \sum_{1\leqslant p,q\leqslant\mu} \rho_{pq}^{mk}\tau_{pq}^{mk} + \rho_{pq}^{ml}\tau_{pq}^{ml} + \rho_{pq}^{kl}\tau_{pq}^{kl} = 0$. Thus the vector $\sum_{1\leqslant p,q\leqslant\mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl}$ is an element of the syzygy module $\mathrm{Syz}(\mathcal{T})$. □

Definition 5.2.2. In the setting above, the vector $\sum_{1\leqslant p,q\leqslant\mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl} \in \mathrm{Syz}(\mathcal{T})$ from Lemma (5.2.1) is called a **Jacobi identity syzygy** $\mathcal{T}$ and it is denoted by $\mathcal{J}_{ij}^{klm}$.

Remark 5.2.3. We note that for the sake of simplicity instead of calling the Jacobi identity syzygy of the tuple whose elements are the elements of $\mathcal{T}$, we call it the Jacobi identity syzygy of $\mathcal{T}$. If it is clear from the context which $\mathcal{O}$-border basis scheme is used, then we simply call it Jacobi identity syzygy.

Definition 5.2.4. In the setting above, let $\mathcal{J}_{ij}^{klm}$ be a Jacobi identity syzygy of $\mathcal{T}$, where k, l, m are distinct indices from $\{1, ..., n\}$ and $i, j \in \{1, ..., \mu\}$. A polynomial τ_{pq}^{kl} from $\mathcal{T}$ is called **rewritable** with respect to the Jacobi identity syzygy of $\mathcal{T}$, $\mathcal{J}_{ij}^{klm}$ if ρ_{pq}^{kl} is a non-zero constant component of $\mathcal{J}_{ij}^{klm}$.

Example 5.2.5. Let $\mathcal{O} = \{1, x_1, x_1^2\} \subset \mathrm{K}[x_1, x_2, x_3]$ be the order ideal. The border is $\partial\mathcal{O} = \{x_3, x_2, x_3x_1, x_2x_1, x_3x_1^2, x_2x_1^2, x_1^3\}$ and with respect to our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_1^2, b_1 = x_3, b_2 = x_2, b_3 = x_3x_1, b_4 = x_2x_1, b_5 = x_3x_1^2, b_6 = x_2x_1^2$

and $b_7 = x_1^3$. We compute the Jacobi identity syzygy $\mathcal{J}_{11}^{123}$ with the ApCoCoA package bbsmingensyz and the result is as follows.

$$\mathcal{J}_{11}^{123} = -c_{21}e_{12}^{12} - c_{31}e_{13}^{12} + c_{13}e_{21}^{12} + c_{15}e_{31}^{12} - c_{22}e_{12}^{13} - c_{32}e_{13}^{13} + c_{14}e^{1321} + c_{16}e_{31}^{13} + c_{17}e_{31}^{23} - e_{12}^{23}$$

The polynomial τ_{12}^{23} is a rewritable polynomial. In Section **??** we show that it is redundant, as well.

Lemma 5.2.6. *In the above setting, the vector* $\sum_{1\leqslant p,q\leqslant \mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl}$ *has the arrow degree* $\deg_W(\sum_{1\leqslant p,q\leqslant \mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl}) = \log(x_kx_lx_mt_j) - \log(t_i)$.

Proof. Recall that the map φ in (5.24) is homogenous and we have $\deg_W(e_{pq}^{kl}) = \deg_W(\tau_{pq}^{kl})$. Let $\mathcal{A}_s$ be a generic multiplication matrix from $\{\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_m\}$. By Lemma 3.2.4, the generic multiplication matrix $\mathcal{A}_s$ is a homogenous matrix. For the entry in position (i,p) of $\mathcal{A}_s$ we have $\deg_W(f_{ip}^{(s)}) = \log(x_st_p) - \log(t_i)$. For the entry in position (p,j) of $\mathcal{A}_s$ we have $\deg_W(f_{pj}^{(s)}) = \log(x_st_j) - \log(t_p)$. Then for each $p \in \{1, ..., \mu\}$, the following holds.

$$\begin{aligned}
\deg_W(f_{ip}^{(l)}e_{pj}^{mk}) &= \deg_W(f_{ip}^{(l)}) + \deg_W(e_{pj}^{mk}) \\
&= \log(x_lt_p) - \log(t_i) + \log(x_mx_kt_j) - \log(t_p) \qquad (5.34)\\
&= \log(x_kx_lx_mt_j) - \log(t_i) \\
&= \log(x_lt_j) - \log(t_p) + \log(x_mx_kt_p) - \log(t_i) = \deg_W(f_{pj}^{(l)}e_{ip}^{mk})
\end{aligned}$$

Similarly, we have the following equality.

$$\begin{aligned}
\deg_W(f_{ip}^{(k)}e_{pj}^{ml}) &= \deg_W(f_{pj}^{(k)}e_{ip}^{ml}) = \deg_W(f_{ip}^{(m)}e_{pj}^{kl}) \\
&= \deg_W(f_{pj}^{(m)}e_{ip}^{kl})) = \log(x_kx_lx_mt_j) - \log(t_i).
\end{aligned}$$

Therefore we have the following equations which gives the desired result.

$$\begin{aligned}
&\deg_W(\sum_{1\leqslant p,q\leqslant \mu} \rho_{pq}^{mk}e_{pq}^{mk} + \rho_{pq}^{ml}e_{pq}^{ml} + \rho_{pq}^{kl}e_{pq}^{kl}) \\
&= \deg_W(\sum_{p=1}^{\mu}(f_{ip}^{(l)}e_{pj}^{mk} - f_{pj}^{(l)}e_{ip}^{mk}) + (f_{ip}^{(k)}e_{pj}^{ml} - f_{pj}^{(k)}e_{ip}^{ml}) + (f_{ip}^{(m)}e_{pj}^{kl} - f_{pj}^{(m)}e_{ip}^{kl})) \\
&= \log(x_kx_lx_mt_j) - \log(t_i)
\end{aligned}$$

. □

Definition 5.2.7. Let $\mathcal{O} \subset \mathbb{T}^n$ denote an order ideal. Let $\mathcal{A}_k, \mathcal{A}_l, \mathcal{A}_m \in \mathrm{Mat}_\mu(\mathrm{K}[\mathcal{C}])$ be generic multiplication matrices with respect to $\mathcal{O}$, where $k, l, m \in \{1, ..., n\}$. Let $\mathcal{A}'_k, \mathcal{A}'_l, \mathcal{A}'_m \in \mathrm{Mat}_\mu(\mathrm{K})$ be the corresponding simplified generic multiplication matrices. Then the Jacobi identity syzygy $\mathcal{J}_{ij}^{klm}$ we compute by using simplified generic multiplication matrices is called **simplified Jacobi identity syzygy** and it is denoted by J_{ij}^{klm}.

Simplified Jacobi identity syzygies indicate the rewritable polynomials in $\mathcal{T}$ as the next example shows.

Example 5.2.8. (continues) Recall Example 5.2.5 where $\mathcal{O} = \{1, x_1, x_1^2\} \subset \mathrm{K}[x_1, x_2, x_3]$. We compute the Jacobi identity by using the simplified generic multiplication matrices with the ApCoCoA package bbsmingensyz.

$$\begin{aligned} J^{123} &= [\mathcal{A}'_1, [\mathcal{A}_2, \mathcal{A}_3]] + [\mathcal{A}'_2, [\mathcal{A}_1, \mathcal{A}_3]] + [\mathcal{A}'_3, [\mathcal{A}_1, \mathcal{A}_3]] \\ &= \begin{pmatrix} -h_{12}^{23} & -h_{13}^{23} & 0 \\ h_{11}^{23} - h_{22}^{23} & h_{12}^{23} - h_{23}^{23} & h_{13}^{23} \\ h_{21}^{23} - h_{32}^{23} & h_{22}^{23} - h_{33}^{23} & h_{23}^{23} \end{pmatrix} \end{aligned}$$

Hence the polynomials $\tau_{11}^{23}, \tau_{12}^{23}, \tau_{13}^{23}, \tau_{21}^{23}, \tau_{22}^{23}, \tau_{32}^{23}, \tau_{31}^{23}, \tau_{32}^{23}, \tau_{33}^{23}$ are rewritable polynomials.

5.3 Segment Border Basis Schemes

Let $\mathcal{O}$ denote the set $\{1, x_1, ..., x_1^\mu\} \subset \mathrm{K}[x_1, ..., x_n]$. Then $\mathcal{O}$ is called the *segment order ideal* and $\mathbb{B}_\mathcal{O}$ is called the *segment border basis scheme* (see Section 2.5). We let $\mathcal{T}$ denote the set of defining equations of $\mathbb{B}_\mathcal{O}$. In this chapter we compute Jacobi identity and trace syzygies of the defining equations of a segment border basis scheme. Besides, we introduce a new algebraic method to compute the remaining redundant generators of the vanishing ideal of the segment border basis scheme (see Propositions 5.3.9 and 5.3.14) that cannot be computed by using trace and Jacobi identity syzygies of $\mathcal{T}$.

Our aim in this section is to prove that segment border basis scheme is an ideal-theoretic complete intersection. We proceed in the following way: First we compute redundant polynomials in $\mathcal{T}$ via Jacobi identity syzygies of $\mathcal{T}$ and then we let $\mathcal{T}'$ denote the set of defining equations of $\mathbb{B}_\mathcal{O}$ without the redundant elements that we computed. Then we compute the redundant elements of $\mathcal{T}'$ via the trace syzygies of $\mathcal{T}'$ and we let $\mathcal{T}''$ denote the set of $\mathcal{T}'$ without the redundant elements that we computed. Finally, we use our method to compute the redundant elements of $\mathcal{T}''$ and show that the vanishing ideal of the given segment border basis scheme can be generated by $\mu\nu - n\mu$ polynomials, where $\mu\nu$ is the dimension of the ring $\mathrm{K}[\mathcal{C}]$, n is the dimension of the polynomial ring $\mathrm{K}[x_1, ..., x_n]$, and μ is the number of elements in the order ideal.

In Section 5.3.1 we start our computation with an $\mathcal{O}$-border basis scheme where $\mathcal{O} = \{1, x_1, ..., x_1^{\mu-1}\} \subset \mathrm{K}[x_1, x_2, x_3]$ by following our strategy and we prove that the vanishing ideal of the $\mathcal{O}$-border basis scheme can be generated by $\mu\nu - n\mu$ polynomials.

In Section 5.3.2 we generalize the results of Section 5.3.1 to the $\mathcal{O}$-border basis scheme where $\mathcal{O} = \{1, x_1, ..., x_1^{\mu-1}\} \subset \mathrm{K}[x_1, ..., x_n]$. In other words, we generalize this property for an arbitrary segment border basis scheme and we show that the generating set of the vanishing ideal of an arbitrary segment border basis has exactly $\mu\nu - n\mu$ polynomials (see Proposition 5.3.14 and Corollary 5.3.15).

In [Ro09], Corollary 3.13 it is shown that a segment border basis is isomorphic to an affine space of dimension $n\mu$. Hence we show that an arbitrary segment border basis scheme is an ideal theoretic complete intersection (see Corollary 5.3.17). This is not a known property of segment border basis schemes. In Chapter 6 we give a different method to show that given border basis scheme is an affine space and we verify the result from [Ro09], Corollary 3.13.

5.3.1 Segment Order Ideals in $\mathbb{T}^3$

Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be a segment order ideal. Let the number of the terms in $\mathcal{O}$ be μ and the number terms in $\partial\mathcal{O}$ be ν. Let $\mathcal{T}$ denote the generating set $\langle \tau_{pq}^{kl} \mid q, p = 1, ..., \mu, \quad k, l \in \{1, ..., n\}\rangle$ of the vanishing ideal of the segment border basis scheme $\mathbb{B}_{\mathcal{O}}$. This section is dedicated to show that some of the elements of $\mathcal{T}$ are redundant and in fact the vanishing ideal of an arbitrary segment border basis scheme can be generated by the set $\langle \tau_{pq}^{1k} \mid p = 1, ..., \mu, q = 1, ..., \mu - 1 \quad k \in \{2, ..., n\}\rangle$, which has exactly $\mu\nu - n\mu$ elements. We shall denote this set $\mathcal{T}_{seg}$. We start with the order ideal $\mathcal{O} = \{1, x_1, ..., x_1^n\} \subset \mathrm{K}[x_1, x_2, x_3]$ and generalize to a segment border basis scheme in $\mathrm{K}[x_1, ..., x_n]$.

Let $\mathcal{O}$ denote the order ideal $\{1, x_1, ..., x_1^{\mu-1}\} \subset \mathbb{T}^3$. Then its border is $\partial\mathcal{O} = \{x_3, x_2, x_3x_1,$
$x_2x_1, x_3x_1^2, x_2x_1^2, ..., x_1^\mu\}$ and with respect to our notation we have $t_1 = 1, t_2 = x_1, ..., t_\mu = x_1^{\mu-1}, b_1 = x_3, b_2 = x_2, b_3 = x_3x_1, b_4 = x_2x_1, b_5 = x_3x_1^2, b_6 = x_2x_1^2, ..., b_\nu = x_1^\mu$. Recall that for each border term b_i we have $c_i = (c_{1i}, ..., c_{\mu i})$. The generic multiplication matrices are as follows.

$$\mathcal{A}_1 = (e_2^{tr} \mid, \cdots, \mid e_m^{tr} \mid c_{2\mu+1}^{tr}) \tag{5.35}$$

$$\mathcal{A}_2 = (c_2^{tr} \mid c_4^{tr} \mid, \cdots, \mid c_{2\mu}^{tr}) \tag{5.36}$$

$$\mathcal{A}_3 = (c_1^{tr} \mid c_3^{tr} \mid, \cdots, \mid c_{2\mu-1}^{tr}) \tag{5.37}$$

Remark 5.3.1. The entries of $\mathcal{A}_2$ and $\mathcal{A}_3$ are from $\mathcal{C}\backslash\{c_{1\,2\mu-1}, ..., c_{\mu\,2\mu-1}\}$. In particular, ± 1 and 0 are not among the entries of $\mathcal{A}_2$ and $\mathcal{A}_3$.

Lemma 5.3.2. *The elements of the set $\{\tau_{ij}^{23} \mid i = 1, ..., \mu-1, j = 1, ..., \mu\}$ are redundant generators of the vanishing ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$. That is, if we let $\mathcal{T}' = \{\tau_{pq}^{12}, \tau_{pq}^{13}, \tau_{\mu-1\,q}^{23}, \mid q, p = 1, ..., \mu\}$, then the vanishing ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by $\mathcal{T}'$.*

Proof. We compute the Jacobi identity matrix.

$$\begin{aligned}
\mathbf{J}^{123} &= [\mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_2]] + [\mathcal{A}_2, [\mathcal{A}_1, \mathcal{A}_3]] + [\mathcal{A}_1, [\mathcal{A}_2, \mathcal{A}_3]] \\
&= [\mathcal{A}_3, \begin{pmatrix} h_{11}^{12} & \dots & h_{1\mu}^{12} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{12} & \dots & h_{\mu\mu}^{12} \end{pmatrix}] + [\mathcal{A}_2, \begin{pmatrix} h_{11}^{13} & \dots & h_{1\mu}^{13} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{13} & \dots & h_{\mu\mu}^{13} \end{pmatrix}] + [\mathcal{A}_1, \begin{pmatrix} h_{11}^{23} & \dots & h_{1\mu}^{23} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{23} & \dots & h_{\mu\mu}^{23} \end{pmatrix}]
\end{aligned}$$

By considering the generic multiplication matrix $\mathcal{A}_1$ (see (5.35)), we can deduce that

the Jacobi identity matrix has the following form.

$$\mathbf{J}^{123} = [\mathcal{A}_3, \begin{pmatrix} h_{11}^{12} & \dots & h_{1\mu}^{12} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{12} & \dots & h_{\mu\mu}^{12} \end{pmatrix}] + [\mathcal{A}_2, \begin{pmatrix} h_{11}^{13} & \dots & h_{1\mu}^{13} \\ \vdots & \ddots & \vdots \\ h_{\mu 1}^{13} & \dots & h_{\mu\mu}^{13} \end{pmatrix}] \quad (5.38)$$

$$+ \begin{pmatrix} c_{1\nu}h_{\mu 1}^{23} & \dots & c_{1\nu}h_{\mu\,\mu-1}^{23} & (\sum_{i=1}^{\mu} -c_{i\nu}h_{1\,i}^{23} + c_{1\nu}h_{\mu\mu}^{23}) \\ \vdots & \ddots & \vdots & \vdots \\ c_{\mu\nu}h_{\mu\,1}^{23} & \dots & c_{\mu\nu}h_{\mu\,\mu-1}^{23} & (\sum_{i=1}^{\mu-1} -c_{i\,\nu}h_{\mu\,i}^{23} \end{pmatrix} \quad (5.39)$$

$$+ \begin{pmatrix} h_{12}^{23} & \dots & h_{1p}^{23} & \dots & h_{1\mu}^{23} & 0 \\ h_{11}^{23} - h_{22}^{23} & \dots & h_{1\,p-1}^{23} - h_{2p}^{23} & \dots & h_{1\mu-1}^{23} - h_{2\mu}^{23} & h_{1\mu}^{23} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ h_{\mu-1\,1}^{23} - h_{\mu 2}^{23} & \dots & h_{\mu-1\,p-1}^{23} - h_{\mu p}^{23} & \dots & h_{\mu-1\,\mu-1}^{23} - h_{\mu\mu}^{23} & h_{\mu-1\,\mu}^{23} \end{pmatrix} \quad (5.40)$$

As we showed in Equations (5.36) and (5.37) all entries of the matrices $\mathcal{A}_3$ and $\mathcal{A}_2$ are from the set $\{c_{11}, ..., c_{1\mu-1}, ..., c_{1\nu-1}, ..., c_{\mu\,\nu-1}\}$. Hence the entries of Matrices (5.38) and (5.39) are polynomials of degree larger than 1. Additionally, only the entries of Matrices (5.39) and (5.40) are polynomials with the top index (23). Since we want to show that the set of the polynomials $\{\tau_{ij}^{23} \mid i = 1, ..., \mu-1, j = 1, ..., \mu\}$ is redundant, we focus on the first $\mu - 1$ columns of Matrix (5.40) and examine the following Jacobian identity syzygies of $\mathcal{T}$. We consider the first row of Matrix (5.40). For each $k \in \{1, .., \mu-1\}$ we have the following Jacob identity syzygy of $\mathcal{T}$.

$$\mathcal{J}_{1k}^{123} = \sum_{1\leqslant q\leqslant \mu} \rho_{1q}^{12}e_{1q}^{12} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{12}e_{pk}^{12} + \sum_{1\leqslant q\leqslant \mu} \rho_{1q}^{13}e_{1q}^{13} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{13}e_{pk}^{13} + c_{1\nu}e_{\mu k}^{23} + e_{1k+1}^{23} \quad (5.41)$$

Hence for each $k \in \{1, .., \mu-1\}$ the polynomial $\tau_{1\,(k+1)}^{23}$ is rewritable as follows.

$$\tau_{1k+1}^{23} = -(\sum_{1\leqslant q\leqslant \mu} \rho_{1q}^{12}\tau_{1q}^{12} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{12}\tau_{pk}^{12} + \sum_{1\leqslant q\leqslant \mu} \rho_{1q}^{13}e_{1q}^{13} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{13}\tau_{pk}^{13} + c_{1\nu}\tau_{\mu k}^{23}) \quad (5.42)$$

We focus on the rest of the rows of Matrix (5.40). If we let $l \in \{2, ..., \mu\}$, then from matrix J^{123}, for each k and l we get the following Jacobi identity syzygy.

$$\mathcal{J}_{lk}^{123} = \sum_{1\leqslant q\leqslant \mu} \rho_{lq}^{12}e_{lq}^{12} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{12}e_{pk}^{12} + \sum_{1\leqslant q\leqslant \mu} \rho_{lq}^{13}e_{lq}^{13} - \sum_{1\leqslant p\leqslant \mu} \rho_{pk}^{13}e_{pk}^{13} + c_{l\nu}e_{\mu k}^{23} + e_{l-1k}^{23} - e_{lk+1}^{23} \quad (5.43)$$

Hence for each index $l \in \{2, .., \mu-1\}$ and $k \in \{1, .., \mu-1\}$, the polynomials $\tau_{l\,(k+1)}^{23}$ are

rewritable as follows.

$$\tau^{23}_{l-1\,k}-\tau^{23}_{l\,k+1} = -(\sum_{1\leqslant q\leqslant \mu}\rho^{12}_{lq}\tau^{12}_{lq}-\sum_{1\leqslant p\leqslant \mu}\rho^{12}_{pk}\tau^{12}_{pk}+\sum_{1\leqslant q\leqslant \mu}\rho^{13}_{lq}\tau^{13}_{lq}-\sum_{1\leqslant p\leqslant \mu}\rho^{13}_{pk}\tau^{13}_{pk}+c_{l\nu}\tau^{23}_{\mu k}) \quad (5.44)$$

From Equations (5.42) and (5.44) follows that $\tau^{23}_{11},...,\tau^{23}_{\mu-1\mu}$ are rewritable. Moreover, $\rho^{23}_{11},...,\rho^{23}_{\mu-1\mu}$ are either 0 or constants in the Jacobi identity syzygies. Thus the elements of the set $\{\tau^{23}_{11},...,\tau^{23}_{\mu-1\mu}\}$ are redundant and contained in the ideal generated by the set $\mathcal{T}' = \{\tau^{12}_{ij}, \tau^{13}_{ij,}\tau^{23}_{\mu\, j} \mid i,j = 1,..,\mu\}$. □

Remark 5.3.3. Note that by Lemma 5.3.2, we can not conclude whether $\tau^{23}_{\mu\, k}$ is redundant of not where $k \in \{1,...,\mu\}$.

Lemma 5.3.4. *Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\} \in K[x_1,...,x_n]$. Let* K[$\mathcal{C}$] *be the polynomial ring where $\mathcal{C} = \{c_{11},...,c_{\mu\nu}\}$. The arrow degree vector of each $c_{ij} \in \mathcal{C}$ has at least one positive component.*

Proof. The border of the order ideal $\mathcal{O}$ is $\partial\mathcal{O} = \{x_3, x_2, x_3x_1, x_2x_1, x_3x_1^2, x_2x_1^2, ..., x_1^\mu\}$. Let c_{ij} be an indeterminate in $\mathcal{C}$ where $\deg_W(c_{ij}) = \log(b_j) - \log(t_i) = (d_1,...,d_n) \in \mathbb{Z}^n$. As a result of the shape of the order ideal and its border, for each $d_i \in \{d_2,...,d_n\}$ we have $d_i \geqslant 0$ and there exists at least one index $\in \{1,...,n\}$ such that $d_k > 0$. Let $g \in$ K[$\mathcal{C}$] be a homogenous polynomial of the arrow degree $(u_1,...,u_n)$. Then for each $u_k \in \{u_2,...,un\}$, we have $u_k \geqslant 0$. If we have $u_1 < 0$, then there exists at least one indeterminate, say $c_{ij} \in \mathcal{C}$ with the arrow degree $(d_1,..,d_k,...,d_n)$, such that $c_{ij} \mid c$, where $c \in \text{supp}(g)$, where $d_1 < 0$ and $d_k > 0$. Since none of the indeterminates in $\mathcal{C}$ has a negative entry in the k^{th} component of their arrow degree, $u_k > 0$ follows. □

Remark 5.3.5. We use the term ordering `Lex` with $x_1 >_{\texttt{Lex}} ... >_{\texttt{Lex}} x_n$.

Lemma 5.3.6. *Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\} \in K[x_1,...,x_n]$. Let $\tau^{1k}_{\omega\, q}$ and $\tau^{1k}_{p\, q}$ be non-standard polynomials from $\mathcal{T}'$ with non-negative arrow degrees where $\tau^{1k}_{p\, q} \neq \tau^{1k}_{\omega\mu}$ and $\deg_W(\tau^{1k}_{p\, q}) >_{\texttt{Lex}} \deg_W(\tau^{1k}_{\omega\mu})$. If there exists a term* Π *such that* $\log(\Pi) = \deg_W(\tau^{1k}_{\omega\mu})$*, then the entry $\kappa^{1k}_{p\, q}$ of the trace syzygy T_{Π,x_1} is* 0.

Proof. We have $\deg_W(\tau^{1k}_{p\, q}) >_{\texttt{Lex}} \deg_W(\tau^{1k}_{\omega\mu})$. Then by Proposition 5.1.16, the entry $\kappa^{1k}_{p\, q}$ of the trace syzygy T_{Π,x_1} is not a constant i.e., it is either 0 or a polynomial from K[$\mathcal{C}$]. Suppose for contradiction, it is a non-zero polynomial from K[$\mathcal{C}$]. By Lemma 5.1.6, the polynomial $\kappa^{1k}_{pq}\tau^{1k}_{pq} + \tau^{1k}_{\omega\mu}$ is homogenous with respect to the arrow grading and has the same arrow degree as $\log(\Pi) = \log(x_1^{\mu-\omega+1}x_k)$. By Corollary 3.2.7, the arrow degree of

$\tau_{p\,q}^{1k}$ is $\log(x_1 x_k t_q) - \log(t_q)$. As a consequence of the shape of the order ideal we get

$$\deg(\tau_{p\,q}^{1k}) = \log(x_1 x_k t_q) - \log(t_q) = \log(x_1^q x_k) - \log(x_1^{q-1}) = \log(x_1^{q-p+1} x_k).$$

Since the polynomial $\tau_{p\,q}^{1k}$ is a non-standard polynomial with the non-negative arrow degree, we have $0 < (q - p + 1) < \mu$. Therefore we have

$$\begin{aligned} \deg_W(\kappa_{p\,q}^{1k}) &= \log(\Pi) - \deg_W(\tau_{p\,q}^{1k}) \\ &= \log(x_1^{\mu-\omega+1} x_k) - \log(x_1^{q-p+1} x_k) \\ &= (\mu - \omega - q + p, 0, ..., 0) \end{aligned}$$

By ordering the following arrow degrees with respect to `Lex` where $x_1 > x_2 > ... > x_n$, we get

$$\deg_W(\tau_{p\,q}^{1k}) >_{\texttt{Lex}} \deg_W(\tau_{\omega\mu}^{1k}),$$

we get the inequality $q - p + 1 > \mu - \omega + 1$ i.e., $\mu - \omega - q + p < 0$, where $\kappa_{p\,q}^{1k} \in \mathrm{K}[\mathcal{C}]$. This contradicts to Lemma 5.3.4, which states that in every arrow degree vector of a homogenous polynomial from $\mathrm{K}[\mathcal{C}]$ there exists at least one entry in the arrow degree vector that is larger than 0. Thus the entry $\kappa_{p\,q}^{1k}$ is 0. □

Proposition 5.3.7. *Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\} \in K[x_1, ..., x_n]$. The elements of the set $\{\tau_{p\mu}^{1k} \mid p = 1, ..., \mu, k = 2, .., n\}$ are redundant generators of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$.*

Proof. We will prove that for each order ideal term $t_\omega \in \{t_1, ..., t_\mu\}$ the polynomial $\tau_{\omega\mu}^{1k}$ is redundant. From Corollary 3.2.7 and from the fact that each order ideal term t_q is x_1^{q-1} follows that the arrow degree of $\tau_{\omega\mu}^{1k}$ is

$$\begin{aligned} \deg_W(\tau_{\omega\mu}^{1k}) &= \log(x_1^{\mu} x_k) - \log(x_1^{(\omega-1)}) \\ &= \log(x_1^{\mu-\omega+1} x_k). \end{aligned} \tag{5.45}$$

The rewritable polynomials that are computed via trace syzygies of $\mathcal{T}$ have non-standard positive arrow degrees. Since the index $\omega - 1$ is smaller than μ, the integer $(\mu - \omega + 1)$ is larger than 0 and the polynomial $\tau_{\omega\mu}^{1k}$ is non-standard with non-negative arrow degree. Then by Proposition 5.1.16, there exists a trace syzygy T_{Π,x_1}, where Π denotes the term $x_1^{\mu-\omega+1} x_k$ such that $\kappa_{\omega\mu}^{1k}$ is a non-zero constant in T_{Π,x_1}. That is, the polynomial $\tau_{\omega\mu}^{1k}$ is rewritable. Note that by Lemma 5.1.19, choosing either x_1 or x_k as

the distinguished indeterminate will give the same trace syzygy.

From the fact that $\Pi = x_1^{\mu-\omega+1}x_k$, follows that $\mathcal{A}_1$ and $\mathcal{A}_k$ are the only generic multiplication matrices used to compute the Trace$[\mathcal{A}_1 \cdots \mathcal{A}_1\mathcal{A}_k \cdots \mathcal{A}_k, \mathcal{A}_1]$, which leads to the trace syzygy T_{Π,x_1}. Then for each $l \in \{2,...,n\} \setminus \{k\}$ the entry κ_{pq}^{1l} of the trace syzygy T_{Π,x_1} is 0. Thus the polynomial $\tau_{\omega\mu}^{1k}$ is rewritable by $\{\tau_{11}^{1k},...,\tau_{\mu\mu}^{1k}\} \setminus \{\tau_{\omega\mu}^{1k}\}$. By using Equation (5.45), for each $\omega \in \{1,...,\mu\}$ we can order the arrow degrees of the polynomial in the set $\{\tau_{11}^{1k},...,\tau_{\mu\mu}^{1k}\}$ as follows.

$$\deg_W(\tau_{\mu\mu}^{1k}) <_{\mathtt{Lex}} \cdots <_{\mathtt{Lex}} \deg_W(\tau_{1\mu}^{1k}). \tag{5.46}$$

We compute the trace syzygy $\mathrm{T}_{x_1x_k,x_1} = \sum_{1\leqslant p\leqslant \mu-1} \kappa_{pp}^{1k}e_{pp}^{1k}$, where $\deg_W(\tau_{\mu\mu}^{1k}) = \log(x_1x_k)$ to get

$$\tau_{\mu\mu}^{1k} \in \langle \tau_{11}^{1k},...,\tau_{\mu\mu-1}^{1k}\rangle.$$

From Equation (5.46) we get $\deg_W(\tau_{\mu\mu}^{1k}) <_{\mathtt{Lex}} \deg_W(\tau_{\mu-1\mu}^{1k}) = \log(x_1^2x_k)$. We compute the trace syzygy $\mathrm{T}_{x_1^2x_k,x_k} = \sum_{1\leqslant p\,q\leqslant \mu-1} \kappa_{pq}^{1k}e_{pq}^{1k}$. Then by Proposition 5.1.16, the component $\kappa_{\mu-1\mu}^{1k}$ is constant in the trace syzygy $\mathrm{T}_{x_1^2x_k,x_k}$ and by Lemma 5.3.6, for each index $p \in \{1,...,\mu-1\}$, the component $\kappa_{p\mu}^{1k}$ is 0 in the trace syzygy $\mathrm{T}_{x_1^2x_k,x_k}$. Since $\tau_{\mu\mu}^{1k} \in \langle \tau_{11}^{1k},...,\tau_{\mu\mu-1}^{1k}\rangle$ holds, we have $\tau_{\mu-1\mu}^{1k} \in \langle \tau_{11}^{1k},...,\tau_{\mu\mu-1}^{1k}\rangle$. As we proceed this way for each $\omega \in \{1,...,\mu\}$ we get $\tau_{\omega\mu}^{1k} \in \langle \tau_{11}^{1k},...,\tau_{\mu\mu-1}^{1k}\rangle$. Thus the polynomials $\tau_{1\mu}^{1k},...,\tau_{\mu\mu}^{1k}$ are redundant. □

Corollary 5.3.8. *The elements of the set $\{\tau_{p\mu}^{12}, \tau_{p\mu}^{13} \mid p = 1,...,\mu\}$ are redundant generators of $\mathcal{I}(\mathbb{B}_\mathcal{O})$. That is, if we let $\mathcal{T}'' = \{\tau_{pq}^{12}, \tau_{pq}^{13}, \tau_{\mu p}^{23} \mid p = 1,...,\mu, q = 1,...,\mu-1\}$, then the vanishing ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by $\mathcal{T}''$.*

Proposition 5.3.9. *The set $\{\tau_{\mu q}^{23} \mid \ q = 1,...,\mu\}$ is contained in the ideal $\langle\{\tau_{pq}^{12}, \tau_{pq}^{13} \mid p = 1,...,\mu, q = 1,...,\mu-1\}\rangle = \mathcal{I}(\mathbb{B}_\mathcal{O})$. In particular, if the index $p > 0$ holds, we have the following.*

$$\begin{aligned}\tau_{\mu q}^{23} \;=\; & \sum_{i=1}^{q-1}\left(-c_{\mu\,(2i-1)}[\sum_{j=0}^{i-1}\tau_{(i-j)\,(q-j-1)}^{12}] + c_{\mu\,2i}[\sum_{j=0}^{i-1}\tau_{(i-j)\,(q-j-1)}^{13}]\right) \\ & + \sum_{i=q}^{\mu-1}\left(-c_{\mu\,(2i-1)}[\sum_{j=1}^{\mu-i}\tau_{(i+j)\,(q+j-1)}^{12}] + c_{\mu\,2i}[\sum_{j=0}^{\mu-i}\tau_{(i+j)\,(q+j-1)}^{13}]\right)\end{aligned} \tag{5.47}$$

If the index $q = 1$ holds, Equation (5.47) has the following form.

$$\tau^{23}_{\mu 1} = \sum_{i=1}^{\mu-1} \left(-c_{\mu\,(2i-1)}[\sum_{j=1}^{\mu} \tau^{12}_{(i+j)\,(q+j-1)}] + c_{\mu\,2i}[\sum_{j=0}^{\mu-u} \tau^{13}_{(i+j)\,(q+j-1)}] \right) \tag{5.48}$$

Proof. Given $\mu - 1$ as the largest value for the indices $q - j - 1$ and $q + j + 1$. To begin, let us consider the first part of Equation (5.47), whereas the index $q > 1$ and the indices i, j are fixed values.

$$\begin{aligned} \tau^{12}_{(i-j)\,(q-j-1)} &= e_{i-j}(\mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_2\mathcal{A}_1)e^{tr}_{q-j-1} \\ &= e_{i-j}\mathcal{A}_1\mathcal{A}_2e^{tr}_{q-j-1} - e_{i-j}\mathcal{A}_2\mathcal{A}_1e^{tr}_{q-j-1} \end{aligned}$$

For each $i \in \{1, .., q-1\}$, the polynomial $\tau^{12}_{1\,(q-i)}$ is the $(\mathrm{i}-1)^{\mathrm{th}}$ summand of the sum $\sum_{j=0}^{i-1} \tau^{12}_{(i-j)\,(q-j-1)}$. By using Equation (5.35), we determine the polynomial $\tau^{12}_{1\,(q-i)}$.

$$\begin{aligned} \tau^{12}_{1\,(q-i)} &= e_{i-j}\mathcal{A}_1\mathcal{A}_2e^{tr}_{q-i} - e_{i-j}\mathcal{A}_2\mathcal{A}_1e^{tr}_{q-i} \\ &= c_{1\,\nu}c_{\mu\,2(q-i)} - c_{1\,2(q-i+1)} \end{aligned}$$

For $j \neq (i-1)$ the summands of $\sum_{j=0}^{i-1} \tau^{12}_{(i-j)\,(q-j-1)}$ have the following shape.

$$\begin{aligned} \tau^{12}_{(i-j)\,(q-j-1)} &= e_{i-j}\mathcal{A}_1\mathcal{A}_2e^{tr}_{q-j-1} - e_{i-j}\mathcal{A}_2\mathcal{A}_1e^{tr}_{q-j-1} \\ &= c_{i-j\,\nu}c_{\mu\,2(q-j-1)} + c_{(i-j-1)\,2(q-j-1)} - c_{i\,2q} \end{aligned}$$

Therefore the sum $\sum_{j=0}^{i-1} \tau^{12}_{(i-j)\,(q-j-1)}$ is equal to the following.

$$\sum_{j=0}^{i-2} c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)} + c_{1\,\nu}c_{\mu\,2(q-i)} + c_{(i-j-1)\,2(q-j-1)} - c_{(i-j)\,2(q-j)} - c_{1\,2(q-i+1)} \tag{5.49}$$

We want to show that Equation 5.49 has the following shape.

$$\sum_{j=0}^{i-1} \tau^{12}_{(i-j)\,(q-j-1)} = \sum_{j=0}^{i-1} c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)} - c_{i\,2q}$$

Let us consider the j^{th} summand of Equation (5.49) which is homogenous of degree one

with respect to the standard grading where $j < i-1$.

$$c_{(i-j-1)\,2(q-j-1)} - c_{(i-j)\,2(q-j)} \tag{5.50}$$

Now let us check the $(\mathrm{j}+1)^{\text{th}}$ summand of Equation (5.49) which is homogenous of degree one with respect to the standard grading where $j+1 < i-1$, as well.

$$c_{(i-j-2)\,2(q-j)} - c_{(i-j-1)\,2(q-j-1)} \tag{5.51}$$

If we take the sum of Equations (5.50) and (5.51), then the positive term of Equation (5.50) and the negative term of Equation (5.51) cancel each other out. Therefore the sum $\sum_{j=0}^{i-2} \tau^{12}_{(i-j)\,(q-j-1)}$ has the following form.

$$\sum_{j=0}^{i-2} c_{(i-j)\,\nu} c_{\mu\,2(q-i+1)} + c_{1\,2(q-i+1)} - c_{i\,2q}.$$

Moreover, we showed that for $j = i-1$ the summand of the Equation (5.49), which is homogenous of degree one with respect to the standard grading, is the following.

$$-c_{1\,2(q-i+1)}.$$

This proves our claim that Equation (5.49) has only one linear summand. Thus the sum

$$\sum_{j=0}^{i-1} \tau^{12}_{(i-j)\,(q-j-1)} = \sum_{j=0}^{i-1} c_{(i-j)\,\nu} c_{\mu\,2(q-i+1)} - c_{i\,2q} \tag{5.52}$$

gives the desired result. Similarly, we identify the summands of $\sum_{j=0}^{i-1} \tau^{13}_{(i-j)\,(q-j-1)}$.

$$\tau^{13}_{(i-j)\,(q-j-1)} = \begin{cases} c_{1\,\nu} c_{\mu\,2(q-i)-1} - c_{1\,2(q-i+1)-1} & \text{if } j = i-1 \\ c_{i-j\,\nu} c_{\mu\,2(q-j-1)-1} - c_{(i-j-1)\,2(q-j-1)-1} + c_{(i-j)\,2(q-j)-1} & \text{otherwise} \end{cases}$$

One can also show similarly that the equality

$$\sum_{j=0}^{i-1} \tau^{13}_{(i-j)\,(q-j-1)} = \sum_{j=0}^{i-1} c_{(i-j)\,\nu} c_{\mu\,2(q-i+1)-1} - c_{i\,2q-1}. \tag{5.53}$$

holds. Now we compute the first part of Equtaion (5.47) i.e.,

$$\sum_{i=1}^{q-1}\left(-c_{\mu\,(2i-1)}[\sum_{j=0}^{i-1}\tau^{12}_{(i-j)\,(q-j-1)}]+c_{\mu\,2i}[\sum_{j=0}^{i-1}\tau^{13}_{(i-j)\,(q-j-1)}]\right). \tag{5.54}$$

We plug Equations (5.52) and (5.53) in Equation (5.54). Then we have

$$\sum_{i=1}^{q-1}\left(\sum_{j=0}^{i-1}-c_{\mu\,(2i-1)}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)}+c_{\mu\,2i}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)-1}+c_{\mu\,(2i-1)}c_{i\,2q}-c_{i\,2q-1}c_{\mu\,2i}\right).$$

For every index $i=1,...,q-1$, if the index j is equal to $i-q+1$, then the following holds.

$$-c_{\mu\,(2i-1)}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)}+c_{\mu\,2i}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)-1}=0$$

Since q is larger than 1 and $j\leqslant i-1$, for every term $-c_{\mu\,(2i-1)}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)}$ there is $c_{\mu\,2i}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)-1}$ which cancels that term. Therefore the following equation holds.

$$\sum_{i=1}^{q-1}\left(\sum_{j=0}^{i-1}-c_{\mu\,(2i-1)}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)}+c_{\mu\,2i}c_{(i-j)\,\nu}c_{\mu\,2(q-i+1)-1}\right)=0$$

Consequently, the sum $\sum_{i=1}^{q-1}\left(-c_{\mu\,(2i-1)}[\sum_{j=0}^{i-1}\tau^{12}_{(i-j)\,(q-j-1)}]+c_{\mu\,2i}[\sum_{j=0}^{i-1}\tau^{13}_{(i-j)\,(q-j-1)}]\right)$ is equal to the following.

$$\sum_{i=1}^{q-1}c_{\mu\,(2i-1)}c_{i\,2q}-c_{i\,2q-1}c_{\mu\,2i} \tag{5.55}$$

Now we compute the second part of Equation (5.47) i.e.,

$$\sum_{i=q}^{\mu-1}\left(-c_{\mu\,(2i-1)}[\sum_{j=1}^{\mu-i}\tau^{12}_{(i+j)\,(q+j-1)}]+c_{\mu\,2i}[\sum_{j=0}^{\mu-i}\tau^{13}_{(i+j)\,(q+j-1)}]\right). \tag{5.56}$$

We follow the same route as we did for computing the first part of Equation (5.47). Let us fix i and j and compute $\sum_{j=1}^{\mu-i}\tau^{12}_{(i+j)\,(q+j-1)}$ and $\sum_{j=1}^{\mu-i}\tau^{13}_{(i+j)\,(q+j-1)}$.

$$\begin{aligned}\tau^{12}_{(i-j)\,(q-j-1)} &= e_{i+j}(\mathcal{A}_1\mathcal{A}_2-\mathcal{A}_2\mathcal{A}_1)e^{tr}_{q+j-1}\\ &= e_{i+j}\mathcal{A}_1\mathcal{A}_2e^{tr}_{q+j-1}-e_{i+j}\mathcal{A}_2\mathcal{A}_1e^{tr}_{q+j-1}\\ &= c_{i+j\,\nu}c_{\mu\,2(q+j-1)}+c_{(i+j-1)\,2(q+j-1)}-c_{i+j\,2(q+j)}\end{aligned}$$

As in the previous summand, the degree 1 homogenous components (with respect to the

standard grading) of the sum $\sum_{j=1}^{\mu-i} \tau^{12}_{(i+j)\,(q+j-1)}$ form a telescopic sum (see Equations (5.50) and (5.51)). Therefore this sum is equal to the following.

$$\sum_{j=1}^{\mu-i} c_{i+j\,\nu} c_{\mu\,2(q+j-1)} + c_{i\,2q} - c_{\mu\,2(q+\mu-i)}. \tag{5.57}$$

Similarly, the sum $\sum_{j=1}^{\mu-i} \tau^{13}_{(i+j)\,(q+j-1)}$ is equal to the following.

$$\sum_{j=1}^{\mu-i} c_{i+j\,\nu} c_{\mu\,2(q+j-1)-1} + c_{i\,2q-1} - c_{\mu\,2(q+\mu-i)-1}. \tag{5.58}$$

We plug Equations (5.57) and (5.58) in Equation (5.56). Then we have

$$\begin{aligned}&\sum_{i=q}^{\mu-1} -c_{\mu\,(2i-1)} c_{i+j\,\nu} c_{\mu\,2(q+j-1)} + c_{\mu\,2i} c_{i+j\,\nu} c_{\mu\,2(q+j-1)-1} \\ &-c_{\mu\,(2i-1)} c_{i\,2q} + c_{\mu\,(2i-1)} c_{\mu\,2(q+\mu-i)} + c_{\mu\,2i} c_{i\,2q-1} - c_{\mu\,2i} c_{\mu\,2(q+\mu-i)-1}.\end{aligned}$$

As in the first part, we have

$$\sum_{i=q}^{\mu-1} -c_{\mu\,(2i-1)} c_{i+j\,\nu} c_{\mu\,2(q+j-1)} + c_{\mu\,2i} c_{i+j\,\nu} c_{\mu\,2(q+j-1)-1} = 0.$$

Moreover, for every $i = q+1, ..., \mu-1$ the summand

$$-c_{\mu\,2i} c_{\mu\,2(q+\mu-i)-1} + c_{\mu\,(2i-1)} c_{\mu\,2(q+\mu-i)}$$

is equal to 0. Therefore Sum (5.56) is equal to the following.

$$\sum_{i=q}^{\mu-1} -c_{\mu\,(2i-1)} c_{i\,2q} + c_{\mu\,2i} c_{i\,2q-1} - c_{\mu\,2q} c_{\mu\,2(\mu)-1} + c_{\mu\,(2q-1)} c_{\mu\,2(\mu)} \tag{5.59}$$

By summing up the results (5.55) and (5.59), we conclude that Equation (5.47) is equal

to the following.

$$\begin{aligned}
& \sum_{i=q+1}^{\mu-1} -c_{\mu\,(2i-1)}c_{i\,2q} + c_{\mu\,2i}c_{i\,2q-1} - c_{\mu\,2q}c_{\mu\,2(\mu)-1} + c_{\mu\,(2q-1)}c_{\mu\,2(\mu)} \\
& + \sum_{i=1}^{q-1} c_{\mu\,(2i-1)}c_{i\,2q} - c_{i\,2q-1}c_{\mu\,2i} \\
= & \sum_{i=1}^{\mu-1} -c_{\mu\,(2i-1)}c_{i\,2q} + c_{\mu\,2i}c_{i\,2q-1} - c_{\mu\,2q}c_{\mu\,2(\mu)-1} + c_{\mu\,(2q-1)}c_{\mu\,2(\mu)} \\
= & \; e_\mu \mathcal{A}_2\mathcal{A}_3 e_q^{tr} - e_\mu \mathcal{A}_3\mathcal{A}_2 e_q^{tr} = \tau^{23}_{\mu\,q}
\end{aligned}$$

□

Corollary 5.3.10. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, x_2, x_3]$. *Let* $\mathcal{O}$ *denote the order ideal* $\{1, x_1, ..., x_1^{\mu-1}\}$ *where with respect to our notation we have* $t_1 = 1, ..., t_\mu = x_1^{\mu-1}$. *Then the vanishing ideal of the* $\mathcal{O}$*-border basis scheme is generated by the set* $\{\tau^{12}_{pq}, \tau^{13}_{pq} \mid p = 1, ..., \mu, q = 1, ..., \mu-1\}$ *which has exactly* $\mu\nu - 3\mu$ *polynomials.*

Proof. It follows from Corollary 5.3.8, Lemma 5.3.2 and Propostion 5.3.9 that $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by $\langle\{\tau^{12}_{pq}, \tau^{13}_{pq} \mid p = 1, ..., \mu, q = 1, ..., \mu-1\}\rangle$. This generating set has $2 \cdot \mu \cdot (\mu - 1)$ polynomials. By Lemma 2.5.6, the number of border elements ν is equal to $2\mu + 1$. Therefore $\mu\nu - 3\mu = (2\mu + 1)\mu - 3\mu = 2 \cdot \mu \cdot (\mu - 1)$. □

Example 5.3.11. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_1^2\} \subset \mathrm{K}[x_1, x_2, x_3]$. Then the border of it is $\partial\mathcal{O} = \{x_3, x_2, x_3x_1, x_2x_1, x_3x_1^2, x_2x_1^2, x_1^3\}$ and with respect to our notation we have $t_1 = 1, t_2 = x_1, t_3 = x_1^2, b_1 = x_3, b_2 = x_2, b_3 = x_3x_1, b_4 = x_2x_1, b_5 = x_3x_1^2, b_6 = x_2x_1^2$ and $b_7 = x_1^3$. The dimension of the ring $\mathrm{K}[\mathcal{C}]$ is 27 and the dimension of $\mathrm{K}[x_1, x_2, x_3]$ is 3. Then we have $\mu\nu - n\mu = 21 - 9 = 12$. Our aim is to show that the ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ can be generated by $\mu\nu - n\mu = (3 \cdot 7) - (3 \cdot 3) = 12$ polynomials. The defining equations of $\mathbb{B}_\mathcal{O}$ are as follows:

$$\begin{aligned}
\tau^{23}_{32} &= -c_{14}c_{31} + c_{13}c_{32} - c_{24}c_{33} + c_{23}c_{34} - c_{34}c_{35} + c_{33}c_{36} \quad , \quad \tau^{12}_{11} = c_{17}c_{32} - c_{14}, \\
\tau^{23}_{13} &= c_{12}c_{15} - c_{11}c_{16} + c_{14}c_{25} - c_{13}c_{26} + c_{16}c_{35} - c_{15}c_{36} \quad , \quad \tau^{12}_{33} = -c_{17}c_{32} - c_{27}c_{34} + c_{26}, \\
\tau^{12}_{23} &= -c_{17}c_{22} - c_{24}c_{27} + c_{27}c_{36} - c_{26}c_{37} + c_{16} \quad , \quad \tau^{13}_{22} = c_{27}c_{33} + c_{13} - c_{25}, \\
\tau^{23}_{22} &= -c_{14}c_{21} + c_{13}c_{22} + c_{26}c_{33} - c_{25}c_{34} \quad , \quad \tau^{12}_{21} = c_{27}c_{32} + c_{12} - c_{24},
\end{aligned}$$

$$\begin{aligned}
\tau_{12}^{23} &= c_{12}c_{13} - c_{11}c_{14} + c_{14}c_{23} - c_{13}c_{24} + c_{16}c_{33} - c_{15}c_{34} \quad , \quad \tau_{31}^{12} = c_{32}c_{37} + c_{22} - c_{34},\\
\tau_{13}^{12} &= -c_{12}c_{17} - c_{14}c_{27} + c_{17}c_{36} - c_{16}c_{37} \quad , \quad \tau_{33}^{13} = -c_{17}c_{31} - c_{27}c_{33} + c_{25},\\
\tau_{23}^{13} &= -c_{17}c_{21} - c_{23}c_{27} + c_{27}c_{35} - c_{25}c_{37} + c_{15} \quad , \quad \tau_{12}^{13} = c_{17}c_{33} - c_{15},\\
\tau_{13}^{13} &= -c_{11}c_{17} - c_{13}c_{27} + c_{17}c_{35} - c_{15}c_{37} \quad , \quad \tau_{21}^{13} = c_{27}c_{31} + c_{11} - c_{23},\\
\tau_{11}^{23} &= c_{14}c_{21} - c_{13}c_{22} + c_{16}c_{31} - c_{15}c_{32} \quad , \quad \tau_{32}^{12} = c_{34}c_{37} + c_{24} - c_{36},\\
\tau_{33}^{23} &= -c_{16}c_{31} + c_{15}c_{32} - c_{26}c_{33} + c_{25}c_{34} \quad , \quad \tau_{11}^{13} = c_{17}c_{31} - c_{13},\\
\tau_{23}^{23} &= -c_{16}c_{21} + c_{15}c_{22} + c_{24}c_{25} - c_{23}c_{26} + c_{26}c_{35} - c_{25}c_{36} \quad , \quad \tau_{12}^{12} = c_{17}c_{34} - c_{16},\\
\tau_{21}^{23} &= -c_{12}c_{21} + c_{11}c_{22} - c_{22}c_{23} + c_{21}c_{24} + c_{26}c_{31} - c_{25}c_{32} \quad , \quad \tau_{22}^{12} = c_{27}c_{34} + c_{14} - c_{26},\\
\tau_{31}^{23} &= -c_{12}c_{31} + c_{11}c_{32} - c_{22}c_{33} + c_{21}c_{34} - c_{32}c_{35} + c_{31}c_{36}
\end{aligned}$$

By Lemma 5.3.2, for $p = 1, 2$ and $q = 1, 2, 3$ the polynomials τ_{pq}^{23}, are redundant. By Lemma 5.3.8, for and $q = 1, 2, 3$ the polynomials $\tau_{p3}^{13}, \tau_{3q}^{12}$ are redundant as well. By using Proposition 5.3.9, we show that the polynomials $\tau_{33}^{23}, \tau_{32}^{23}, \tau_{31}^{23}$ are in $\langle\{\tau_{ij}^{12}, \tau_{ij,}^{13} \mid i, j = 1, 2, 3\}\rangle$.

$$\begin{aligned}
\tau_{31}^{23} &= c_{31}(\tau_{32}^{12} + \tau_{21}^{12}) - c_{32}(\tau_{32}^{13} + \tau_{21}^{13}) + c_{33}\tau_{31}^{12} - c_{34}\tau_{31}^{13}\\
\tau_{32}^{23} &= c_{31}\tau_{11}^{12} - c_{32}\tau_{11}^{13} + c_{33}\tau_{32}^{12} - c_{34}\tau_{32}^{13}\\
\tau_{33}^{23} &= c_{31}\tau_{12}^{12} - c_{32}\tau_{12}^{13} + c_{33}(\tau_{11}^{12} + \tau_{22}^{12}) - c_{34}(\tau_{11}^{13} + \tau_{22}^{13})
\end{aligned}$$

Therefore we conclude that $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) = \langle\{\tau_{pq}^{12}, \tau_{pq}^{13} \mid p = 1, 2, 3,\ q = 1, 2\}\rangle$ and the number of generators are $2{\cdot}\mu{\cdot}(\mu{-}1) = 12$. Now let us emphasize that the number of elements of the border $\partial\mathcal{O}$ is $2\mu{+}1 = 7$ and $n = 3$. The polynomial ring $\mathrm{K}[\mathcal{C}]$ has $\mu\nu$ indeterminates. In our case where $\mathcal{O} = \{1, x_1, x_1^2\}$ we have $n\mu = 3 \cdot 3 = 9$. Then we have $\mu\nu - n\mu = 12$ and it is equal to the number of generators of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$.

5.3.2 Segment Order Ideals in $\mathbb{T}^n$

In this section we generalize Lemma 5.3.2 and Proposition 5.3.9 for the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Note that Lemma 5.3.6 and Proposition 5.3.7 are given for an arbitrary segment order ideal in the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\}$. Then the border of the order ideal $\mathcal{O}$ is $\partial\mathcal{O} = \{x_n, ..., x_2, x_n, b_2x_1, ..., x_1^{\mu-1}x_2, x_1^{\mu}\}$ and with respect to our notation we have $t_1 = 1, t_2 = x_1, ..., t_\mu = x_1^{\mu-1}, b_1 = x_n, ..., b_{n-2} = x_2, b_{n-1} = x_n, b_n = b_2x_1, ..., b_{\nu-1} = x_1^{\mu-1}x_2$ and $b_\nu = x_1^{\mu}$. Recall that for each border term b_i we have $c_i = (c_{1i}, ..., c_{\mu i})$ and that for the

polynomial τ_{pq}^{kl} from $\mathcal{T}$, we have $k < l$.

Lemma 5.3.12. *The generic multiplication matrix $\mathcal{A}_1$ is as follows.*

$$\mathcal{A}_1 = (e_2^{tr} \mid , .. , \mid e_\mu^{tr} \mid c_\nu^{tr}) \tag{5.60}$$

For $i > 1$ the generic multiplication matrix $\mathcal{A}_i$ has the following form

$$\mathcal{A}_i = (c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid , ... , \mid c_{k+(\mu-1)(n-1)}^{tr}) \tag{5.61}$$

where the index $k = (n+1) - i$.

Proof. Since for each $t_i \in \mathcal{O} \setminus \{t_\mu\}$ we have $x_1 t_i \in \mathcal{O}$, the generic multiplication matrix $\mathcal{A}_1$ has the given form. Let t_j be an arbitrary term from $\mathcal{O} \setminus \{t_1\}$. For an arbitrary indeterminate $x_i \in \{x_2, ..., x_n\}$ we have $x_i t_1 = b_{n-i+1}$ and $x_i t_j = b_{(n-i+1)(j-1)(n-1)}$. Thus the generic multiplication matrix $\mathcal{A}_i$ has the form given in (5.61). □

Lemma 5.3.13. *The elements of the set $\{\tau_{ij}^{kl} \mid i = 1, ..., \mu - 1, j = 1, ..., \mu,$ and $k, l \in \{2, ..., n\}\}$ are redundant generators of $\mathcal{I}(\mathbb{B}_\mathcal{O})$.*

Proof. We compute the Jacobi identity matrix.

$$\begin{aligned}
\mathbf{J}^{mkl} &= [\mathcal{A}_k, [\mathcal{A}_l, \mathcal{A}_m]] + [\mathcal{A}_l, [\mathcal{A}_k, \mathcal{A}_m]] + [\mathcal{A}_m, [\mathcal{A}_k, \mathcal{A}_l]] \\
&= [\mathcal{A}_k, \begin{pmatrix} \tau_{11}^{lm} & \dots & \tau_{1\mu}^{lm} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{lm} & \dots & \tau_{\mu\mu}^{lm} \end{pmatrix}] + [\mathcal{A}_l, \begin{pmatrix} \tau_{11}^{km} & \dots & \tau_{1\mu}^{km} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{km} & \dots & \tau_{\mu\mu}^{km} \end{pmatrix}] + [\mathcal{A}_m, \begin{pmatrix} \tau_{11}^{kl} & \dots & \tau_{1\mu}^{kl} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{kl} & \dots & \tau_{\mu\mu}^{kl} \end{pmatrix}]
\end{aligned}$$

Since the order of the indices m, k, l does not change the Jacobi identity syzygies of $\mathcal{T}_{Seg}$, for convenience, we always choose $m < k < l$. From Lemma 5.3.12 follows that if $m \neq 1$, then there is no entry of the Jacobi identity syzygy, which is a non-zero constant. Thus the Jacobi identity syzygy $\mathcal{J}^{mkl}$ is useful for determining redundant generators of the vanishing ideal if and only if $m = 1$. By computing the Jacobian identity matrix with the generic multiplication matrix $\mathcal{A}_1$ (see (5.60)), we have the following.

$$\mathbf{J}^{1kl} = [\mathcal{A}_l, \begin{pmatrix} \tau_{11}^{1k} & \dots & \tau_{1\mu}^{1k} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{1k} & \dots & \tau_{\mu\mu}^{1k} \end{pmatrix}] + [\mathcal{A}_k, \begin{pmatrix} \tau_{11}^{1l} & \dots & \tau_{1\mu}^{1l} \\ \vdots & \ddots & \vdots \\ \tau_{\mu 1}^{1l} & \dots & \tau_{\mu\mu}^{1l} \end{pmatrix}] \tag{5.62}$$

$$+\begin{pmatrix} c_{1\nu}\tau^{kl}_{\mu 1} & \dots & c_{1\nu}\tau^{kl}_{\mu\,\mu-1} & (\sum_{i=1}^{\mu} -c_{i\nu}\tau^{kl}_{1\,i} + c_{1\nu}\tau^{kl}_{\mu\mu}) \\ \vdots & \ddots & \vdots & \vdots \\ c_{\mu\nu}\tau^{kl}_{\mu\,1} & \dots & c_{\mu\nu}\tau^{kl}_{\mu\,\mu-1} & (\sum_{i=1}^{\mu-1} -c_{i\,\nu}\tau^{kl}_{\mu\,i}) \end{pmatrix} \tag{5.63}$$

$$+\begin{pmatrix} \tau^{kl}_{12} & \dots & \tau^{kl}_{1p} & \dots & \tau^{kl}_{1\mu} & 0 \\ \tau^{kl}_{11} - \tau^{kl}_{22} & \dots & \tau^{kl}_{1\,p-1} - \tau^{kl}_{2\,p} & \dots & \tau^{kl}_{1\mu-1} - \tau^{kl}_{2\mu} & \tau^{kl}_{1\mu} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \tau^{kl}_{\mu-1\,1} - \tau^{kl}_{\mu 2} & \dots & \tau^{kl}_{\mu-1\,p-1} - \tau^{kl}_{\mu\,p} & \dots & \tau^{kl}_{\mu-1\,\mu-1} - \tau^{kl}_{\mu\mu} & \tau^{kl}_{\mu-1\,\mu} \end{pmatrix} \tag{5.64}$$

As in Lemma 5.3.2, we focus on the the first $\mu - 1$ columns of Matrix (5.64) and examine the following Jacobi identity syzygies. First we consider the first row of the matrix (5.64). If we let the index $i \in \{1, .., \mu - 1\}$, then we have the following.

$$\mathcal{J}^{1kl}_{1i} = \sum_{1\leqslant q\leqslant \mu} \rho^{1k}_{1q} e^{1k}_{1q} - \sum_{1\leqslant p\leqslant \mu} \rho^{1k}_{pi} e^{1k}_{pi} + \sum_{1\leqslant q\leqslant \mu} \rho^{1l}_{1q} e^{1l}_{1q} - \sum_{1\leqslant p\leqslant \mu} \rho^{1l}_{pi} e^{1l}_{pi} + c_{1\nu} e^{kl}_{\mu k} + e^{kl}_{1i+1} \tag{5.65}$$

Hence for each $i \in \{1, .., \mu - 1\}$ the polynomial $\tau^{23}_{1\,(i+1)}$ is rewritable as follows.

$$\tau^{kl}_{1i+1} = -(\sum_{1\leqslant q\leqslant \mu} \rho^{1k}_{1q} \tau^{1k}_{1q} - \sum_{1\leqslant p\leqslant \mu} \rho^{1k}_{pi} \tau^{1k}_{pi} + \sum_{1\leqslant q\leqslant \mu} \rho^{1l}_{1q} e^{1l}_{1q} - \sum_{1\leqslant p\leqslant \mu} \rho^{1l}_{pi} \tau^{1l}_{pi} + c_{1\nu}\tau^{23}_{\mu i}) \tag{5.66}$$

We focus on the rest of the rows of Matrix (5.64). We let $j \in \{2, ..., \mu\}$ then from the matrix J^{1kl}, for each i and j we get the following Jacobi identity syzygy of $\mathcal{T}$.

$$\mathcal{J}^{1kl}_{ji} = \sum_{1\leqslant q\leqslant \mu} \rho^{1k}_{jq} e^{1k}_{jq} - \sum_{1\leqslant p\leqslant \mu} \rho^{1k}_{pi} e^{1k}_{pi} + \sum_{1\leqslant q\leqslant \mu} \rho^{1l}_{jq} e^{1l}_{jq} - \sum_{1\leqslant p\leqslant \mu} \rho^{1l}_{pi} e^{1l}_{pi} + c_{l\nu} e^{kl}_{\mu i} + e^{kl}_{l-1\,i} - e^{kl}_{li+1} \tag{5.67}$$

Hence for each $l \in \{2, .., \mu - 1\}$ and $i \in \{1, .., \mu - 1\}$, the polynomials $\tau^{23}_{l\,(i+1)}$ are rewritable as follows.

$$\tau^{kl}_{l-1\,k} - \tau^{kl}_{j\,i+1} = -(\sum_{1\leqslant q\leqslant \mu} \rho^{1k}_{jq} \tau^{1k}_{jq} - \sum_{1\leqslant p\leqslant \mu} \rho^{1k}_{pi} \tau^{1k}_{pi} + \sum_{1\leqslant q\leqslant \mu} \rho^{1l}_{jq} \tau^{1l}_{jq} - \sum_{1\leqslant p\leqslant \mu} \rho^{1l}_{pi} \tau^{1l}_{pi} + c_{j\nu}\tau^{kl}_{\mu i}) \tag{5.68}$$

From Equations (5.66) and (5.68) follows that $\tau^{kl}_{11}, ..., \tau^{kl}_{\mu-1\mu}$ are rewritable. Moreover, $\rho^{kl}_{11}, ..., \rho^{kl}_{\mu-1\mu}$ are either 0 or constants in the Jacobi identity syzygies of $\mathcal{T}$. Thus the elements of the set $\{\tau^{kl}_{11}, ..., \tau^{kl}_{\mu-1\mu}\}$ are redundant and contained in the ideal generated by the set $\{\tau^{1k}_{ij}, \tau^{1l}_{ij,}\tau^{2l}_{\mu\,j} \mid i, j = 1, .., \mu\}$. Since this holds for arbitrary $k, l \in \{2, .., n\}$ with $k \neq l$, the claim $\mathcal{T}' = \{\tau^{1k}_{ij}, \tau^{1l}_{ij,}\tau^{kl}_{\mu\,j} \mid i, j = 1, .., \mu, k < l, \quad k, l \in \{2, .., n\}\}$ follows. □

Proposition 5.3.14. *Let $k < l$. Let $B_{kl} = \{b_{m_1}, ..., b_{m_{2\mu}}\}$ be the ordered subset of $\partial\mathcal{O}$ such that each border term $b_{m_i} \in B_{kl}$ is divisible by either x_k or x_l. That is*

$$B_{kl} = \{x_l, x_k, x_1x_l, x_1x_k, ..., x_lx_1^{\mu-1}, x_kx_1^{\mu-1}\}$$

and with respect to our notation we have

$$b_{m_1} = x_l, b_{m_2} = x_k, b_{m_3} = x_1x_l, b_{m_4} = x_1x_k, ..., b_{m_{2\mu-1}} = x_lx_1^{\mu-1}, b_{m_{2\mu}} = x_kx_1^{\mu-1}.$$

Then the set $\{\tau_{\mu p}^{kl} \mid \ p = 1, ..., \mu \, and \, k, l \in \{2, .., n\}\}$ is the set of redundant generators of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$. In particular, for each $q \in \{1, ..., \mu\}$ the polynomial $\tau_{\mu q}^{kl}$ is redundant as the following equalities show.

$$\tau_{\mu q}^{kl} = \sum_{i=1}^{q-1} \left(-c_{\mu\, m_{2i-1}} [\sum_{j=0}^{i-1} \tau_{(i-j)\,(q-j-1)}^{1k}] + c_{\mu\, m_{2i+1}} [\sum_{j=0}^{i-1} \tau_{(i-j)\,(q-j-1)}^{1l}] \right) \tag{5.69}$$

$$+ \sum_{i=q}^{\mu-1} \left(-c_{\mu\,(m_{2i-1})} [\sum_{j=1}^{\mu-i} \tau_{(i+j)\,(q+j-1)}^{1k}] + c_{\mu\,(m_{2i}+1)} [\sum_{j=0}^{\mu-i} \tau_{(i+j)\,(q+j-1)}^{1l}] \right)$$

If q is equal to 1, then the equality has the following form.

$$\tau_{\mu 1}^{kl} = \sum_{i=1}^{\mu-1} \left(-c_{\mu\, m_{2i-1}} [\sum_{j=1}^{\mu} \tau_{(i+j)\,(q+j-1)}^{1k}] + c_{\mu\,(m_{2i}+1)} [\sum_{j=0}^{\mu-u} \tau_{(i+j)\,(q+j-1)}^{1l}] \right) \tag{5.70}$$

Proof. We focus on the set B_{kl}. Let the integers k, l be from $\{1, ..., n\}$ and $k \neq l$. If we consider the map

$$\begin{aligned} h : K[x_1, .., x_n] &\longrightarrow K[x_1, x_k, x_l] \\ x_j &\longmapsto x_j \\ x_i &\longmapsto 0 \end{aligned}$$

where $j \in \{1, k, l\}$, and $i \in \{1, ..., n\} \setminus \{1, k, l\}$, then the set

$$B_{kl} \cup \{x_1^{\mu+1}\} = \{b_{m_1}, ..., b_{m_{2\mu}}, x_1^{\mu+1}\}$$

can be seen as the border of the order ideal $\mathcal{O} = \{x_1, ..., x_1^{\mu-1}\}$ in $K[x_1, x_k, x_l]$ under h.

Thus we let $b_{2\mu+1}$ denote the term $x_1^{\mu+1}$ instead of b_ν. Now let us recall the construction of the generic multiplication matrices $\mathcal{A}_k$ and $\mathcal{A}_l$ (see Definitions 2.2.4 and 2.3.1.) We reconstruct $\mathcal{A}_k$ and $\mathcal{A}_l$ by using B_{kl}, instead of $\partial\mathcal{O}$. Thus we have the following generic multiplication matrices.

$$\mathcal{A}_1 = (e_2^{tr} \mid , .. , \mid e_m^{tr} \mid c_{2\mu+1}^{tr}) \tag{5.71}$$

$$\mathcal{A}_k = (c_{m_2}^{tr} \mid c_{m_4}^{tr} \mid , ... , \mid c_{m_{2\mu}}^{tr}) \tag{5.72}$$

$$\mathcal{A}_l = (c_{m_1}^{tr} \mid c_{m_3}^{tr} \mid , ... , \mid c_{m_{2\mu-1}}^{tr}) \tag{5.73}$$

If we apply Proposition 5.3.9 with these generic multiplication matrices, then we get Equations (5.69) and (5.70). Therefore for arbitrarily chosen $k, l \in \{2, ..., n\}$ the proposition follows. □

Corollary 5.3.15. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, ..., x_n]$. *Let* $\mathcal{O}$ *be a segment order ideal* $\{1, x_1, ..., x_1^{\mu-1}\}$ *and let the number of terms in its border* $\partial\mathcal{O}$ *be* ν. *Then the vanishing ideal of the* $\mathcal{O}$*-border basis scheme is generated by* $\mu\nu - n\mu$ *polynomials.*

Proof. It follows from Lemmas 5.3.7 and 5.3.13 and Propostion 5.3.14 that $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by

$$\langle \tau_{pq}^{1k} \mid p = 1, ..., \mu, q = 1, ..., \mu - 1 \;\; k \in \{2, ..., n\} \rangle.$$

This generating set has

$$(n-1) \cdot \mu \cdot (\mu - 1)$$

polynomials. By Lemma 2.5.6 the number of border elements is equal to $(n-1)\mu + 1$.

$$\nu = (n-1)\mu + 1$$

We plug this in the equation $\mu\nu - n\nu$ and the following holds.

$$\mu\nu - n\mu = ((n-1)\mu + 1)\mu - n\mu = (n-1) \cdot \mu \cdot (\mu - 1).$$

Hence we get the desired result. □

Example 5.3.16. Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_1^2\} \subset \mathrm{K}[x_1, x_2, x_3, x_4]$. Then the border of the order ideal $\mathcal{O}$ is $\partial\mathcal{O} = \{x_4, x_3, x_2, x_1x_4, x_1x_3, x_1x_2, x_1^2x_4, x_1^2x_3, x_1^2x_2, x_1^3\}$ and with respect to our notation we have $b_1 = x_4, b_2 = x_3, b_3 = x_2, b_4 = x_1x_4, b_5 = x_1x_3, b_6 = x_1x_2, b_7 = x_1^2x_4, b_8 = x_1^2x_3, b_9 = x_1^2x_2, b_{10} = x_1^3$. The dimension of the ring $\mathrm{K}[\mathcal{C}]$ is equal to 30 and the dimension of $\mathrm{K}[x_1, x_2, x_3, x_4]$ is 4. Then we have

$\mu\nu - n\mu = 30 - 12 = 18$. We show that the ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by the following polynomials.

$$\begin{aligned}
\tau_{11}^{12} &= c_{110}c_{33} - c_{16}, & \tau_{21}^{13} &= c_{210}c_{32} + c_{12} - c_{25} \\
\tau_{12}^{12} &= c_{110}c_{36} - c_{19}, & \tau_{22}^{13} &= c_{210}c_{35} + c_{15} - c_{28} \\
\tau_{32}^{12} &= c_{36}c_{310} + c_{26} - c_{39}, & \tau_{31}^{13} &= c_{32}c_{310} + c_{22} - c_{35} \\
\tau_{21}^{12} &= c_{210}c_{33} + c_{13} - c_{26}, & \tau_{11}^{13} &= c_{110}c_{32} - c_{15} \\
\tau_{22}^{12} &= c_{210}c_{36} + c_{16} - c_{29}, & \tau_{12}^{13} &= c_{110}c_{35} - c_{18} \\
\tau_{31}^{12} &= c_{33}c_{310} + c_{23} - c_{36}, & \tau_{32}^{13} &= c_{35}c_{310} + c_{25} - c_{38} \\
\tau_{11}^{14} &= c_{110}c_{31} - c_{14}, & \tau_{12}^{14} &= c_{110}c_{34} - c_{17} \\
\tau_{21}^{14} &= c_{210}c_{31} + c_{11} - c_{24}, & \tau_{22}^{14} &= c_{210}c_{34} + c_{14} - c_{27} \\
\tau_{31}^{14} &= c_{31}c_{310} + c_{21} - c_{34}, & \tau_{32}^{14} &= c_{34}c_{310} + c_{24} - c_{37}
\end{aligned} \tag{5.74}$$

By Lemma 5.3.13, the Jacobi identity syzygies of $\mathcal{T}$ indicate that for $p = 1, 2$ and $q = 1, 2, 3$ the polynomials $\tau_{pq}^{23}, \tau_{pq}^{24}, \tau_{pq}^{34}$ are redundant i.e., the following polynomials are redundant.

$$\begin{aligned}
\tau_{11}^{23} &= c_{16}c_{22} - c_{15}c_{23} + c_{19}c_{32} - c_{18}c_{33} \\
\tau_{12}^{23} &= c_{13}c_{15} - c_{12}c_{16} + c_{16}c_{25} - c_{15}c_{26} + c_{19}c_{35} - c_{18}c_{36} \\
\tau_{13}^{23} &= c_{13}c_{18} - c_{12}c_{19} + c_{16}c_{28} - c_{15}c_{29} + c_{19}c_{38} - c_{18}c_{39} \\
\tau_{21}^{23} &= -c_{13}c_{22} + c_{12}c_{23} - c_{23}c_{25} + c_{22}c_{26} + c_{29}c_{32} - c_{28}c_{33} \\
\tau_{22}^{23} &= -c_{16}c_{22} + c_{15}c_{23} + c_{29}c_{35} - c_{28}c_{36} \\
\tau_{23}^{23} &= -c_{19}c_{22} + c_{18}c_{23} + c_{26}c_{28} - c_{25}c_{29} + c_{29}c_{38} - c_{28}c_{39} \\
\tau_{11}^{24} &= c_{16}c_{21} - c_{14}c_{23} + c_{19}c_{31} - c_{17}c_{33} \\
\tau_{12}^{24} &= c_{13}c_{14} - c_{11}c_{16} + c_{16}c_{24} - c_{14}c_{26} + c_{19}c_{34} - c_{17}c_{36} \\
\tau_{13}^{24} &= c_{13}c_{17} - c_{11}c_{19} + c_{16}c_{27} - c_{14}c_{29} + c_{19}c_{37} - c_{17}c_{39} \\
\tau_{21}^{24} &= -c_{13}c_{21} + c_{11}c_{23} - c_{23}c_{24} + c_{21}c_{26} + c_{29}c_{31} - c_{27}c_{33} \\
\tau_{22}^{24} &= -c_{16}c_{21} + c_{14}c_{23} + c_{29}c_{34} - c_{27}c_{36} \\
\tau_{23}^{24} &= -c_{19}c_{21} + c_{17}c_{23} + c_{26}c_{27} - c_{24}c_{29} + c_{29}c_{37} - c_{27}c_{39}
\end{aligned}$$

$$\begin{aligned}
\tau_{11}^{34} &= c_{15}c_{21} - c_{14}c_{22} + c_{18}c_{31} - c_{17}c_{32} \\
\tau_{12}^{34} &= c_{12}c_{14} - c_{11}c_{15} + c_{15}c_{24} - c_{14}c_{25} + c_{18}c_{34} - c_{17}c_{35} \\
\tau_{13}^{34} &= c_{12}c_{17} - c_{11}c_{18} + c_{15}c_{27} - c_{14}c_{28} + c_{18}c_{37} - c_{17}c_{38} \\
\tau_{21}^{34} &= -c_{12}c_{21} + c_{11}c_{22} - c_{22}c_{24} + c_{21}c_{25} + c_{28}c_{31} - c_{27}c_{32} \\
\tau_{22}^{34} &= -c_{15}c_{21} + c_{14}c_{22} + c_{28}c_{34} - c_{27}c_{35} \\
\tau_{23}^{34} &= -c_{18}c_{21} + c_{17}c_{22} + c_{25}c_{27} - c_{24}c_{28} + c_{28}c_{37} - c_{27}c_{38}
\end{aligned}$$

By Lemma 5.3.8, the trace syzygies of $\mathcal{T}$ indicate that for and $q = 1, 2, 3$ the polynomials $\tau_{3q}^{13}, \tau_{3q}^{14}, \tau_{3q}^{12}$ are redundant, as well. That is, the following polynomials are redundant, as well.

$$\begin{aligned}
\tau_{13}^{12} &= -c_{13}c_{110} - c_{16}c_{210} + c_{110}c_{39} - c_{19}c_{310} \\
\tau_{23}^{12} &= -c_{110}c_{23} - c_{26}c_{210} + c_{210}c_{39} - c_{29}c_{310} + c_{19} \\
\tau_{33}^{12} &= -c_{110}c_{33} - c_{210}c_{36} + c_{29} \\
\tau_{13}^{13} &= -c_{12}c_{110} - c_{15}c_{210} + c_{110}c_{38} - c_{18}c_{310} \\
\tau_{23}^{13} &= -c_{110}c_{22} - c_{25}c_{210} + c_{210}c_{38} - c_{28}c_{310} + c_{18} \\
\tau_{33}^{13} &= -c_{110}c_{32} - c_{210}c_{35} + c_{28} \\
\tau_{13}^{14} &= -c_{11}c_{110} - c_{14}c_{210} + c_{110}c_{37} - c_{17}c_{310} \\
\tau_{23}^{14} &= -c_{110}c_{21} - c_{24}c_{210} + c_{210}c_{37} - c_{27}c_{310} + c_{17} \\
\tau_{33}^{14} &= -c_{110}c_{31} - c_{210}c_{34} + c_{27}
\end{aligned}$$

Let B_{23} denote the set $\{x_3, x_2, x_1x_3, x_1x_2, x_1^2x_3, x_1^2x_2, x_1^3\}$ where we have $b_2 = x_3, b_3 = x_2, b_5 = x_1x_3, b_6 = x_1x_2, b_8 = x_1^2x_3, b_9 = x_1^2x_2, b_{10} = x_1^3$. By Proposition 5.3.14, the following holds.

$$\begin{aligned}
\tau_{31}^{23} &= c_{32}(\tau_{32}^{12} + \tau_{21}^{12}) - c_{33}(\tau_{32}^{13} + \tau_{21}^{13}) + c_{35}\tau_{31}^{12} - c_{36}\tau_{31}^{13} \\
&= -c_{13}c_{32} + c_{12}c_{33} - c_{23}c_{35} + c_{22}c_{36} - c_{33}c_{38} + c_{32}c_{39} \\
\tau_{32}^{23} &= c_{32}\tau_{11}^{12} - c_{33}\tau_{11}^{13} + c_{35}\tau_{32}^{12} - c_{36}\tau_{32}^{13} \\
&= -c_{16}c_{32} + c_{15}c_{33} - c_{26}c_{35} + c_{25}c_{36} - c_{36}c_{38} + c_{35}c_{39} \\
\tau_{33}^{23} &= c_{32}\tau_{12}^{12} - c_{33}\tau_{12}^{13} + c_{35}(\tau_{11}^{12} + \tau_{22}^{12}) - c_{36}(\tau_{11}^{13} + \tau_{22}^{13}) \\
&= -c_{19}c_{32} + c_{18}c_{33} - c_{29}c_{35} + c_{28}c_{36}
\end{aligned}$$

Let B_{34} denote the set $\{x_4, x_3, x_1x_4, x_1x_3, x_1^2x_4, x_1^2x_3, x_1^3\}$ where $b_1 = x_4, b_2 = x_3, b_4 = x_1x_4, b_5 = x_1x_3, b_7 = x_1^2x_4, b_8 = x_1^2x_3, b_10 = x_1^3$. By Proposition 5.3.14, the following holds.

$$\begin{aligned}
\tau_{31}^{34} &= c_{31}(\tau_{32}^{12} + \tau_{21}^{12}) - c_{32}(\tau_{32}^{13} + \tau_{21}^{13}) + c_{34}\tau_{31}^{12} - c_{35}\tau_{31}^{13} \\
&= -c_{12}c_{31} + c_{11}c_{32} - c_{22}c_{34} + c_{21}c_{35} - c_{32}c_{37} + c_{31}c_{38} \\
\tau_{32}^{34} &= c_{31}\tau_{11}^{12} - c_{32}\tau_{11}^{13} + c_{34}\tau_{32}^{12} - c_{35}\tau_{32}^{13} \\
&= -c_{15}c_{31} + c_{14}c_{32} - c_{25}c_{34} + c_{24}c_{35} - c_{35}c_{37} + c_{34}c_{38} \\
\tau_{33}^{34} &= c_{31}\tau_{12}^{12} - c_{32}\tau_{12}^{13} + c_{34}(\tau_{11}^{12} + \tau_{22}^{12}) - c_{35}(\tau_{11}^{13} + \tau_{22}^{13}) \\
&= -c_{18}c_{31} + c_{17}c_{32} - c_{28}c_{34} + c_{27}c_{35}
\end{aligned}$$

Let B_{24} denote the set $\{x_4, x_2, x_1x_4, x_1x_2, x_1^2x_4, x_1^2x_2, x_1^3\}$ where we have $b_1x_4, b_3 = x_2, b_4 = x_1x_4, b_6 = x_1x_2, b_7 = x_1^2x_4, b_9 = x_1^2x_2, b_10 = x_1^3$. By Proposition 5.3.14, the following holds.

$$\begin{aligned}
\tau_{31}^{24} &= c_{31}(\tau_{32}^{12} + \tau_{21}^{12}) - c_{33}(\tau_{32}^{13} + \tau_{21}^{13}) + c_{34}\tau_{31}^{12} - c_{36}\tau_{31}^{13} \\
&= -c_{13}c_{31} + c_{11}c_{33} - c_{23}c_{34} + c_{21}c_{36} - c_{33}c_{37} + c_{31}c_{39} \\
\tau_{32}^{24} &= c_{31}\tau_{11}^{12} - c_{33}\tau_{11}^{13} + c_{34}\tau_{32}^{12} - c_{36}\tau_{32}^{13} \\
&= -c_{16}c_{31} + c_{14}c_{33} - c_{26}c_{34} + c_{24}c_{36} - c_{36}c_{37} + c_{34}c_{39} \\
\tau_{33}^{24} &= c_{31}\tau_{12}^{12} - c_{33}\tau_{12}^{13} + c_{34}(\tau_{11}^{12} + \tau_{22}^{12}) - c_{36}(\tau_{11}^{13} + \tau_{22}^{13}) \\
&= -c_{19}c_{31} + c_{17}c_{33} - c_{29}c_{34} + c_{27}c_{36}
\end{aligned}$$

Then the $\mathcal{O}$-border basis scheme is generated by the set $\{\tau_{pq}^{12}, \tau_{pq}^{13}, \tau_{pq}^{14} \mid p = 1, 2, 3\, q = 1, 2\}$, whose elements are the 18 polynomials given in (5.74).

Corollary 5.3.17. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, ..., x_n]$. *Let* $\mathcal{O} = \{1, x_1, ..., x_1^{\mu-1}\}$ *denote a segment order ideal with* μ *terms. Then the border basis scheme* $\mathbb{B}_\mathcal{O}$ *is an ideal-theoretic complete intersection.*

Proof. By Propostion 5.3.14 the ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by $\mathcal{T}_{seg}$, which by Corollary 5.3.15, has $\mu\nu - n\mu$ polynomials. Since the segment border basis scheme is isomorphic to an affine space of dimension $n\mu$, the number of the generators of the vanishing ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is equal to the codimension of the $\mathcal{O}$-border basis scheme. □

5.4 Some Redundant Generators of $\mathcal{I}(\mathbb{B}_{\mathcal{B}(2,2,2)})$

Unless otherwise stated, throughout this section we let K be a field, and let P denote the polynomial ring $\mathrm{K}[x_1, x_2, x_3]$. Let $\mathcal{B}(2,2,2)$ denote the order ideal

$$\{1, x_3, x_2, x_1, x_2x_3, x_1x_3, x_1x_2, x_1x_2x_3\}.$$

The border of $\mathcal{B}(2,2,2)$ is

$$\partial\mathcal{B} = \{x_3^2, x_2^2, x_1^2, x_1^2, x_2x_3^2, x_2^2x_3, x_1^2x_3, x_1x_2^2, x_1^2x_2, x_1x_2x_3^2, x_1x_2^2x_3, x_1^2x_2x_3\}.$$

Let $\mathcal{T}$ be the set of generators of $\mathcal{I}(\mathbb{B}_{\mathcal{B}})$ defined in (2.6). Then by Lemma 2.5.2, the number of elements of $\mathcal{T}$ is 144.

This section is dedicated to finding the maximal number of redundant generators of the vanishing ideal of the box border basis scheme $\mathbb{B}_{\mathcal{B}(2,2,2)}$ by computing the Jacobi identity and trace syzygies of $\mathcal{T}$. As we mentioned in Sections 6.1 and 6.2, the simplified Jacobi identity and trace syzygies indicate the rewritable polynomials (see Definitions 5.2.7 and 5.1.21). Thus we start our computation by finding the simplified generic multiplication matrices.

$$\mathcal{A}'_1 = (e_4^{tr} \mid e_6^{tr} \mid e_8^{tr} \mid 0 \mid 0 \mid 0 \mid 0) \qquad \mathcal{A}'_2 = (e_3^{tr} \mid e_5^{tr} \mid 0 \mid e_7^{tr} \mid 0 \mid e_8^{tr} \mid 0 \mid 0)$$
$$\mathcal{A}'_3 = (e_2^{tr} \mid 0 \mid e_5^{tr} \mid e_6^{tr} \mid 0 \mid 0 \mid e_8^{tr} \mid 0)$$

The non-standard distinct positive degree vectors are

$$(1,2,0), (2,1,0), (1,2,1), (2,1,1), (2,2,0), (2,2,1), (0,1,2), (0,2,1),$$
$$(1,2,1), (0,2,2), (1,2,2), (1,0,2), (2,0,1), (1,1,2), (2,0,2), (2,1,2).$$

We compute the simplified trace syzygies T'_{Π,x_k} by using the ApCoCoA package bbsmingensyz, in order to find the rewritable elements. The non-zero simple trace syzygies are given as follows:

$$\mathrm{T}_{x_1x_2,x_1} = \mathrm{T}'_{x_1x_2,x_1} = e_{11}^{12} + e_{22}^{12} + e_{33}^{12} + e_{44}^{12} + ... + e_{88}^{12},$$
$$\mathrm{T}_{x_1x_3,x_1} = \mathrm{T}'_{x_1x_3,x_1} = e_{11}^{13} + e_{22}^{13} + e_{33}^{13} + e_{44}^{13} + ... + e_{88}^{13},$$
$$\mathrm{T}_{x_2x_3,x_2} = \mathrm{T}'_{x_2x_3,x_2} = e_{11}^{23} + e_{22}^{23} + e_{33}^{23} + e_{44}^{23} + ... + e_{88}^{23},$$

$$
\begin{aligned}
\mathrm{T}'_{x_1^2x_2,x_1} &= e_{14}^{12} + e_{26}^{12} + e_{37}^{12} + e_{58}^{12} \\
\mathrm{T}'_{x_1x_2^2,x_1} &= e_{13}^{12} + e_{25}^{12} + e_{47}^{12} + e_{68}^{12}, \\
\mathrm{T}'_{x_1^2x_2^2,x_1} &= e_{17}^{12} + e_{28}^{12}, \\
\mathrm{T}'_{x_1x_3^2,x_1} &= e_{12}^{13} + e_{35}^{13} + e_{46}^{13} + e_{78}^{13}, \\
\mathrm{T}'_{x_1^2x_3,x_1} &= e_{14}^{13} + e_{26}^{13} + e_{37}^{13} + e_{58}^{13} \\
\mathrm{T}'_{x_1^2x_3^2,x_1} &= e_{16}^{13} + e_{38}^{13}, \\
\mathrm{T}'_{x_2^2x_3,x_2} &= e_{13}^{23} + e_{25}^{23} + e_{47}^{23} + e_{68}^{23}, \\
\mathrm{T}'_{x_2x_3^2,x_2} &= e_{12}^{23} + e_{35}^{23} + e_{46}^{23} + e_{78}^{23}, \\
\mathrm{T}'_{x_2^2x_3^2,x_2} &= e_{15}^{23} + e_{48}^{23}, \\
\mathrm{T}'_{x_1x_2x_3,x_1} &= e_{12}^{12} + e_{35}^{12} + e_{46}^{12} + e_{78}^{12} + e_{13}^{13} + e_{25}^{13} + e_{47}^{13} + e_{68}^{13}, \\
\mathrm{T}'_{x_1x_2x_3,x_2} &= e_{12}^{12} + e_{35}^{12} + e_{46}^{12} + e_{78}^{12} + e_{14}^{23} + e_{26}^{23} + e_{37}^{23} + e_{58}^{23}, \\
\mathrm{T}'_{x_1x_2x_3,x_3} &= e_{13}^{13} + e_{25}^{13} + e_{47}^{13} + e_{68}^{13} + e_{14}^{23} + e_{26}^{23} + e_{37}^{23} + e_{58}^{23}, \\
\mathrm{T}'_{x_1^2x_2x_3,x_1} &= e_{16}^{12} + e_{38}^{12} + e_{17}^{13} + e_{28}^{13}, \\
\mathrm{T}'_{x_1^2x_2x_3,x_2} &= e_{16}^{12} + e_{38}^{12} \\
\mathrm{T}'_{x_1^2x_2x_3,x_3} &= e_{17}^{13} + e_{28}^{13}, \\
\mathrm{T}'_{x_1x_2x_3^2,x_1} &= e_{15}^{13} + e_{48}^{13}, \\
\mathrm{T}'_{x_1x_2x_3^2,x_2} &= e_{16}^{23} + e_{38}^{23}, \\
\mathrm{T}'_{x_1x_2x_3^2,x_3} &= e_{15}^{13} + e_{48}^{13} + e_{16}^{23} + e_{38}^{23}, \\
\mathrm{T}'_{x_1x_2^2x_3,x_1} &= e_{15}^{12} + e_{48}^{12}, \\
\mathrm{T}'_{x_1x_2^2x_3,x_2} &= e_{15}^{12} + e_{48}^{12} + e_{17}^{23} + e_{18}^{23}, \\
\mathrm{T}'_{x_1x_2^2x_3,x_3} &= e_{17}^{23} + e_{28}^{23}, \\
\mathrm{T}'_{x_1^2x_2^2x_3,x_1} &= e_{18}^{12}, \\
\mathrm{T}'_{x_1^2x_2x_3^2,x_1} &= e_{18}^{13}, \\
\mathrm{T}'_{x_1x_2^2x_3^2,x_2} &= e_{18}^{23}.
\end{aligned}
$$

Note that 4 of the above simple trace syzygies, $\mathrm{T}'_{x_1x_2^2x_3,x_2}, \mathrm{T}'_{x_1x_2x_3^2,x_3}, \mathrm{T}'_{x_1^2x_2x_3,x_1}, \mathrm{T}'_{x_1x_2x_3,x_3}$ don't give new rewritable elements, so there are 23 different trace syzygies. Next we compute the Jacobi identity syzygies. Similarly, we use the simplified generic multiplication matrices as the multiple of commutators to find the rewritable elements with respect to Jacobi identity syzygies.

$$
\mathrm{J}^{123} = [\mathcal{A}'_1, [\mathcal{A}_2, \mathcal{A}_3]] + [\mathcal{A}'_2, [\mathcal{A}_3, \mathcal{A}_1]] + [\mathcal{A}'_3, [\mathcal{A}_1, \mathcal{A}_2]] \tag{5.75}
$$

The entries of the matrix in Equation (5.75) are the following.

0	$-h_{15}^{13}-$ h_{16}^{23}	$-h_{15}^{12}-$ h_{17}^{23}	$-h_{16}^{12}-$ h_{17}^{13}	$-h_{18}^{23}$	$-h_{18}^{13}$	$-h_{18}^{12}$	0
0	$h_{12}^{12}-$ $h_{25}^{13}-$ h_{26}^{23}	$h_{13}^{12}-$ $h_{25}^{12}-$ h_{27}^{23}	$h_{14}^{12}-$ $h_{26}^{12}-$ h_{27}^{13}	$h_{15}^{12}-$ h_{28}^{23}	$h_{16}^{12}-$ h_{28}^{13}	$h_{17}^{12}-$ h_{28}^{12}	h_{18}^{12}
0	$h_{12}^{13}-$ $h_{35}^{13}-$ h_{36}^{23}	$-h_{35}^{12}+$ $h_{13}^{13}-$ h_{37}^{23}	$-h_{36}^{12}+$ $h_{14}^{13}-$ h_{37}^{13}	$h_{15}^{13}-$ h_{38}^{23}	$-h_{16}^{13}$ $+h_{38}^{13}$	$-h_{38}^{12}$ $+h_{17}^{13}$	h_{18}^{13}
0	$-h_{45}^{13}+$ $h_{12}^{23}-$ h_{46}^{23}	$-h_{45}^{12}+$ $h_{13}^{23}-$ h_{47}^{23}	$-h_{46}^{12}-$ $h_{47}^{13}+$ h_{14}^{23}	$h_{15}^{23}-$ h_{48}^{23}	$-h_{48}^{13}+$ h_{16}^{23}	$-h_{48}^{12}+$ h_{17}^{23}	h_{18}^{23}
0	$h_{32}^{12}+$ $h_{22}^{13}-$ $h_{55}^{13}-$ h_{56}^{23}	$h_{33}^{12}-$ $h_{55}^{12}+$ $h_{23}^{13}-$ h_{57}^{23}	$h_{34}^{12}-$ $h_{56}^{12}+$ $h_{24}^{13}-$ h_{57}^{13}	$h_{35}^{12}+$ $h_{25}^{13}-$ h_{58}^{23}	$h_{36}^{12}+$ $h_{26}^{13}-$ h_{58}^{13}	$h_{37}^{12}-$ $h_{58}^{12}+$ h_{27}^{13}	$h_{38}^{12}+$ h_{28}^{13}
0	$h_{42}^{12}-$ $h_{65}^{13}+$ $h_{22}^{23}-$ h_{66}^{23}	$h_{43}^{12}-$ $h_{65}^{12}+$ $h_{23}^{23}-$ h_{67}^{23}	$h_{44}^{12}-$ $h_{66}^{12}-$ $h_{67}^{13}+$ h_{24}^{23}	h_{45}^{12} $+h_{25}^{23}-$ h_{68}^{23}	$h_{46}^{12}-$ $h_{68}^{13}+$ h_{26}^{23}	$h_{47}^{12}-$ $h_{68}^{12}+$ h_{27}^{23}	$h_{48}^{12}+$ h_{28}^{23}
0	$h_{42}^{13}-$ $h_{75}^{13}+$ $h_{32}^{23}-$ h_{76}^{23}	$-h_{75}^{12}+$ $h_{43}^{13}+$ $h_{33}^{23}-$ h_{77}^{23}	$-h_{76}^{12}+$ $h_{44}^{13}-$ $h_{77}^{13}+$ h_{34}^{23}	$h_{45}^{13}+$ $h_{35}^{23}-$ h_{78}^{23}	$h_{46}^{13}-$ $h_{78}^{13}+$ h_{36}^{23}	$-h_{78}^{12}+$ $h_{47}^{13}+$ h_{37}^{23}	$h_{48}^{13}+$ h_{38}^{23}
0	$h_{72}^{12}+$ $h_{62}^{13}-$ $h_{85}^{13}+$ $h_{52}^{23}-$ h_{86}^{23}	$h_{73}^{12}-$ $h_{85}^{12}+$ $h_{63}^{13}+$ $h_{53}^{23}-$ h_{87}^{23}	$h_{74}^{12}-$ $h_{86}^{12}+$ $h_{64}^{13}-$ $h_{87}^{13}+$ h_{54}^{23}	$h_{75}^{12}+$ $h_{65}^{13}+$ $h_{55}^{23}-$ h_{88}^{23}	h_{76}^{12} $h_{66}^{13}-$ $h_{88}^{13}+$ h_{56}^{23}	$h_{77}^{12}-$ $h_{88}^{12}+$ $h_{67}^{13}+$ h_{57}^{23}	$h_{78}^{12}+$ $h_{68}^{13}+$ h_{58}^{23}

When we compare this result with the simplified trace syzygies, we observe that some of the rewritable elements occur more than once. For example both trace syzygy and Jacobi identity matrix computations show that τ_{12}^{18} is rewritable. Thus after carefully examining T_{Π,x_k} and J^{123}, we can simplify the repeating rewritable elements and the trivial polynomials. We claim that there are at most 60 redundant elements in the set $\mathcal{T}$ that can be determined using trace and Jacobi identity syzygies. Before proving our claim let us summarize some of our observations that we get from our computations.

Remark 5.4.1. Keep in mind that simplified Jacobi identity matrix is not a zero matrix. By Lemma 5.2.6, a non-zero entry in position (i, j) of the J^{123} has the arrow degree $\log(x_1x_2x_3t_i) - \log(t_j)$. By considering the order ideal,

$$\mathcal{B}(2,2,2) = \{1, x_3, x_2, x_1, x_2x_3, x_1x_3, x_1x_2, x_1x_2x_3\}$$

one can easily deduce that all entries of J^{123} have non-negative arrow degrees (see Definition 5.1.17). In Remark 5.1.20 we explain that by computing the trace syzygies one can only find the rewritable polynomials from $\mathcal{T}$ with a non-standard and non-negative arrow degree. Let us compute the generators of $\mathcal{I}(\mathbb{B}_{\mathcal{B}(2,2,2)})$ and their arrow degrees by using the ApCoCoA package bbsmingensyz There are 12 different non-standard and non-positive arrow degrees.

$$(-1,1,2), (1,-1,2), (-1,2,1), (2,1,-1), (2,-1,1), (1,2,-1),$$

$$(-1,2,2), (2,-1,2), (2,2,-1), (1,1,-1), (1,-1,1), (-1,1,1)$$

If there are redundant polynomials of the above arrow degrees, then neither Jacobi identity nor trace syzygies of $\mathcal{T}$ will give them.

The order ideal $\mathcal{B}(2,2,2)$ is symmetric with respect to swapping x_i and x_j where $i \neq j$ and $i, j \in \{1,2,3\}$. This is a quite useful property, as next lemma shows.

Lemma 5.4.2. *Let σ be a permutation defined on the set $\{1,2,3\}$. Let λ denote the number of rewritable elements (with respect to either Jacobi identity or trace syzygy) from $\mathcal{T}$ with arrow degree $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$. Then there are λ rewritable elements (with respect to either Jacobi identity or trace syzygy) with arrow degree $\alpha_\sigma = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)})$.*

Proof. We start with Jacobi identity syzygies. There are λ rewritable elements from $\mathcal{T}$ with respect to Jacobi identity syzygies with degree α. Let i be an index from the set $\{1,2,3\}$ and let $t_p, t_q \in \mathcal{B}_{(2,2,2)}$ be two order ideal terms such that

$$t_p = x_1^{p_1}x_2^{p_2}x_3^{p_3} \text{ and } t_q = x_1^{q_1}x_2^{q_2}x_3^{q_3}.$$

Let the polynomial $\tau_{pq}^{kl} \in \mathcal{T}$ be one of those λ rewritable elements. Since τ_{pq}^{kl} is rewritable with respect to Jacobi identity syzygies, we have

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) = (1,1,1) + \log(t_p) - \log(t_q).$$

Thus for each i in $\{1,2,3\}$, the α_i is equal to p_i+1-q_i i.e., $\alpha_i = p_i+1-q_i$. Then we have

$$\alpha_{\sigma(i)} = p_{\sigma(i)} - q_{\sigma(i)} + 1$$

where $t_{p_\sigma} = x_1^{p_{\sigma(1)}} x_2^{p_{\sigma(2)}} x_3^{p_{\sigma(3)}}$ and $t_{q_\sigma} = x_1^{q_{\sigma(1)}} x_2^{q_{\sigma(2)}} x_3^{q_{\sigma(3)}}$. The order ideal $\mathcal{B}(2,2,2)$ is a symmetric order ideal. Since the terms t_p, t_q are in $\mathcal{B}(2,2,2)$, we have $t_{p_\sigma}, t_{q_\sigma} \in \mathcal{B}(2,2,2)$. Thus for each order ideal term t_p and t_q there exits a rewritable polynomial from $\mathcal{T}$ in (p,q) position of the matrix J^{klm}, there exists a rewritable polynomial from $\mathcal{T}$ in (p_σ, q_σ) position of the matrix J^{klm}.

Now we focus on the rewritable elements from $\mathcal{T}$ with respect to the trace syzygies. Let $\tau_{pq}^{kl} \in \tau$ be a rewritable element with respect to trace syzygies, where $p,q \in \{1,..,8\}$ and $k,l \in \{1,2,3\}$. There are three cases to consider.

case 1) If two components of the arrow degree vector α are zero, then there are only zero trace syzygies.

case 2) Next assume one component of α, say $\alpha_i = 0$ and the others $\alpha_j, \alpha_r \neq 0$. By Lemma 5.1.19 we have $T_{x_j^{\alpha_j} x_r^{\alpha_r}, x_j} = -T_{x_j^{\alpha_j} x_r^{\alpha_r}, x_r}$. Therefore considering just one trace syzygy $T_{x_j^{\alpha_j} x_r^{\alpha_r}, x_j}$ will be enough to find rewritable elements. By the same argument for some σ, there is only one trace syzygy $T_{x_{\sigma(j)}^{\alpha_{\sigma(j)}} x_{\sigma(r)}^{\alpha_{\sigma(r)}}, x_{\sigma(j)}}$.

case 3) We consider the last case $\alpha_1, \alpha_2, \alpha_3 \neq 0$. Let Π denote term $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \in \mathbb{T}^3$ and Π' denote the term $x_{\sigma(1)}^{\alpha_{\sigma(1)}} x_{\sigma(2)}^{\alpha_{\sigma(2)}} x_{\sigma(3)}^{\alpha_{\sigma(3)}} \in \mathbb{T}^3$. For the term Π, there are exactly three different trace syzygies, since we have $T_\Pi, x_i \neq T_\Pi, x_j$ where $i \neq j$. Then there are also three different trace syzygies $T_{\Pi', x_{\sigma(i)}}$, $i \in \{1,..,3\}$. Moreover, the trace syzygy $T_{\Pi', x_{\sigma(i)}}$ has the same shape as T_{Π, x_i}.

Therefore the number of rewritable elements with respect to trace syzygies are the same for any permutation of the components of given arrow degree vector. □

Before proving our main proposition of this section, let us give an overview of the proof method we will follow. The main aim of the proof is to give the maximal number of the redundant polynomials in $\mathcal{T}$ that can be computed by trace and Jacobi identity syzygies.

1) We fix an arrow degree $d = (d_1, d_2, d_3) \in \mathbb{N}^3$ and compute the Jacobi identity and trace syzygies by using the simple generic multiplication matrices. This will give us the rewritable polynomials with respect to Jacobi identity and trace syzygies.

2) If the both syzygy computation method indicates the same rewritable polynomials, then we compute the trace syzygy and the Jacobi identity syzygy only for this arrow degree. Then we give the maximal number of the redundant polynomials, say λ.

3) Using Lemma 5.4.2, we deduce the maximal number of redundant polynomials for arrow degrees $(d_1, d_3, d_2), (d_2, d_1, d_3), (d_2, d_3, d_1), (d_3, d_1, d_2)$ and (d_3, d_2, d_1) are also λ.

Thus the number of redundant polynomials for the arrow degree (d_1, d_2, d_3) and its re-arrangements are at most $6 \cdot \lambda$. The reason why we cannot give the exact number of redundant polynomials, is given in Remark 5.4.4.

Proposition 5.4.3. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, x_2, x_3]$. *Let* $\mathcal{B}(2,2,2)$ *be the order ideal. Let* $\mathcal{T}$ *denote the set of generators of the vanishing ideal of the box border basis scheme* $\mathbb{B}_{\mathcal{B}(2,2,2)}$. *Then there are at most 60 redundant polynomials from* $\mathcal{T}$ *that can be computed by trace and Jacobi identity syzygies.*

Proof. As we have seen while computing the trace and Jacobi identity syzygies of $\mathcal{T}$, one can only compute the rewritable polynomials from $\mathcal{T}$ that have non-negative arrow degrees (for details see Remark 5.4.1). Therefore it suffices to consider the elements from $\mathcal{T}$ with arrow degree vector from $\mathbb{N}^3$. There are seven different sorts of arrow degree vectors we will consider.

i) We start our computation with degrees, $(0,0,1), (1,0,0), (0,1,0), (0,0,2), (0,2,0)$ and $(2,0,0)$. There is no trace syzygy with respect to those degrees. Thus it is enough to consider only Jacobi identity syzygies. By Lemma 5.2.6, a non-zero entry of Matrix (5.75) in position (i,j) has the degree

$$(1,1,1) + \log(t_j) - \log(t_i) \tag{5.76}$$

where $t_i, t_j \in \mathcal{B}(2,2,2)$. Therefore the only possible position for an entry of the arrow degree $(1,0,0)$ in Matrix (5.75) is $(8,2)$. The entry J_{82}^{123} is non-zero. Since there is no other possible position in the Matrix (5.75) for the given degree, choosing one of the rewritable elements that is indicated by the simplified Jacobi identity syzygy J_{82}^{123} will give us the desired rewritable element from $\mathcal{T}$ of arrow degree $(1,0,0)$. This is the first possible redundant element. By the same argument, we have one rewritable element by the simplified Jacobi identity syzygy

$\mathcal{J}_{83}^{123}$ with degree $(0,0,1)$ and one rewritable by $\mathcal{J}_{84}^{123}$ with degree $(0,1,0)$. For $(0,2,0),(2,0,0),(0,0,2)$ by the same way, we can get one rewritable element by each simplified Jacobi identity syzygies $\mathrm{J}_{63}^{123}, \mathrm{J}_{54}^{123}, \mathrm{J}_{72}^{123}$ respectively. Therefore from this step we have 6 different rewritable polynomials and at most 6 redundant polynomials.

ii) As a result of Equation (5.76), the entries of arrow degrees $(0,2,2)$, $(2,0,2)$ or $(2,2,0)$ are located only in positions $(4,5),(3,6)$ and $(2,7)$ of Matrix (5.75). By examining the simplified Jacobi identity syzygies in these positions, we have $\mathrm{J}_{45}^{123} = e_{15}^{23} - e_{48}^{23}$, $\mathrm{J}_{36}^{123} = e_{16}^{13} - e_{38}^{13}$ and $\mathrm{J}_{27}^{123} = e_{17}^{12} - e_{28}^{12}$. Now let us give a closer look at the Jacobi identity syzygy J_{27}^{123}.

$$\begin{aligned}\mathcal{J}_{45}^{123} \;=\; & \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{23} e_{4q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p5}^{23} e_{p5}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{13} e_{4q}^{13} - \sum_{1\leqslant p\leqslant 8} \rho_{p5}^{13} e_{p5}^{13} \\ & + \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{12} e_{4q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p5}^{12} e_{p5}^{12} + e_{15}^{23} + e_{48}^{23}\end{aligned}$$

Therefore $\tau_{15}^{23} - \tau_{48}^{23}$ can be rewritten as $\mathrm{K}[\mathcal{C}]$-linear combination of tuples of polynomials from the set $\{\tau_{4q}^{kl}, \tau_{p5}^{kl} \mid k<l,\ \ k,l\in\{1,2,3\},\ \ p,q\in\{1,...,8\}\}\backslash\{\tau_{15}^{23},\tau_{48}^{23}\}$. Hence, the polynomial τ_{15}^{23} can be a redundant element from $\mathcal{T}$. Then we examine the simplified trace syzygy $\mathrm{T}'_{x_2^2x_3^2,x_2} = e_{15}^{23} + e_{48}^{23}$ such that $\log(x_2^2x_3^2) = (0,2,2)$. If we show the trace syzygy $\mathrm{T}_{x_2^2x_3^2,x_2}$ is different from the Jacobi identity syzygy $\mathcal{J}_{45}^{123}$, then by substituting e_{15}^{23} with $\mathcal{J}_{45}^{123} - e_{15}^{23}$ in $\mathrm{T}_{x_2^2x_3^2,x_2}$, we can deduce that τ_{48}^{23} can also be a redundant element of $\mathcal{T}$. Recall that we obtain the trace syzygy $\mathrm{T}_{x_2^2x_3^2,x_2}$ from $\mathrm{Trace}(\mathcal{A}_2[\mathcal{A}_2,\mathcal{A}_3]\mathcal{A}_3 + \mathcal{A}_2\mathcal{A}_3[\mathcal{A}_2,\mathcal{A}_3])$ and let us focus on that trace syzygy.

$$\mathrm{T}_{x_2^2x_3^2,x_2} = \sum_{p,q\in\{1,\ldots,8\}} (\kappa_{pq}^{23} e_{pq}^{23}) \tag{5.77}$$

That implies this trace syzygy is different than the Jacobi identity syzygy. Hence, the polynomial τ_{48}^{23} can be a redundant element of $\mathcal{T}$. Then both τ_{15}^{23} and τ_{48}^{23} can be redundant elements of $\mathcal{T}$. By the same method one can show τ_{13}^{16} and $\tau_{38}^{13}, \tau_{17}^{12}, \tau_{28}^{12}$ can be redundant. There are altogether at most 6 redundant elements coming from this step.

iii) Let us consider the arrow degrees $(0,1,1),(1,0,1)$ and $(1,1,0)$. We will first find the rewritable element with degree $(1,1,0)$. By Lemma 5.4.2, the redundant

polynomials of degrees $(1,0,1)$ and $(0,1,1)$ can be found similarly. The trace syzygy and the simplified trace syzygy are equal for degree $(1,1,0)$.

$$\mathrm{T}_{x_1x_2,x_1} = e_{11}^{12} + e_{22}^{12} + e_{33}^{12} + e_{44}^{12} + \cdots + e_{88}^{12} \tag{5.78}$$

There are exactly three different positions in Matrix (5.75), where the simplified Jacobi identity syzygies have degree $(1,1,0)$.

$$\begin{aligned}
\mathcal{J}_{53}^{123} &= \sum_{1\leqslant q\leqslant 8} \rho_{5q}^{23} e_{5q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p3}^{23} e_{p3}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{5q}^{13} e_{5q}^{13} - \sum_{1\leqslant p\leqslant 8} \rho_{p3}^{13} e_{p3}^{13} \qquad (5.79)\\
&\quad + \sum_{1\leqslant q\leqslant 8} \rho_{5q}^{12} e_{5q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p3}^{12} e_{p3}^{12} + e_{55}^{12} + e_{57}^{23}\\
\mathcal{J}_{64}^{123} &= \sum_{1\leqslant q\leqslant 8} \rho_{6q}^{23} e_{6q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p4}^{23} e_{p4}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{6q}^{13} e_{6q}^{13} - \sum_{1\leqslant p\leqslant 8} \rho_{p4}^{13} e_{p4}^{13} \qquad (5.80)\\
&\quad + \sum_{1\leqslant q\leqslant 8} \rho_{6q}^{12} e_{6q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p4}^{12} e_{p4}^{12} + e_{66}^{12} + e_{67}^{13}\\
\mathcal{J}_{87}^{123} &= \sum_{1\leqslant q\leqslant 8} \rho_{8q}^{23} e_{8q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p7}^{23} e_{p7}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{8q}^{13} e_{8q}^{13} - \sum_{1\leqslant p\leqslant 8} \rho_{p7}^{13} e_{p7}^{13} \qquad (5.81)\\
&\quad + \sum_{1\leqslant q\leqslant 8} \rho_{8q}^{12} e_{8q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p7}^{12} e_{p7}^{12} + e_{77}^{12} + e_{88}^{12} + e_{67}^{13} + e_{57}^{23}
\end{aligned}$$

The entries of the tuple $(\rho_{pq}^{k,m})_{\substack{1\leqslant p,q\leqslant 8\\ r,s\in\{k,l,m\}}}$ are either a non-constant polynomial from $\mathrm{K}[\mathcal{C}]$ or 0. If the polynomial τ_{88}^{12} is a redundant polynomial which is rewritable by Equation (5.78), then the polynomials $\tau_{57}^{23}, \tau_{67}^{13}$ can also be redundant polynomials from $\mathcal{T}$, which are rewritable polynomials in $\mathcal{T}$ by the Jacobi identity syzygies (5.79) ,(5.80) and (5.81). Hence the polynomial τ_{77}^{13} can be a redundant polynomial from $\mathcal{T}$, as well. Thus there are at most 4 redundant polynomials of degree $(1,1,0)$. By Lemma 5.4.2, there are also at most 4 redundant elements of degree $(0,1,1)$ and at most 4 more from degree $(1,0,1)$. Altogether, there are at most 12 redundant polynomials of such sort of arrow degrees.

iv) Next we consider the arrow degrees

$$(0,2,1), (2,1,0), (2,0,1), (0,1,2), (1,0,2), \text{ and } (1,2,0).$$

As in iii) we focus on one of those arrow degrees, say $(1,2,0)$, and find the maximal number of redundant polynomials. Then similarly to the previous steps, by using

Lemma 5.4.2 we deduce, that there is the same number of redundant polynomials of the other given arrow degrees. There are two positions in Matrix (5.75) such that the Jacobi identity syzygies have degree $(1,2,0)$.

$$\begin{aligned}\mathcal{J}_{23}^{123} &= \sum_{1\leqslant q\leqslant 8}\rho_{2q}^{23}e_{2q}^{23} - \sum_{1\leqslant p\leqslant 8}\rho_{p3}^{23}e_{p3}^{23} + \sum_{1\leqslant q\leqslant 8}\rho_{2q}^{13}e_{2q}^{13} - \sum_{1\leqslant p\leqslant 8}\rho_{p3}^{13}e_{p3}^{13} \\ &\quad + \sum_{1\leqslant q\leqslant 8}\rho_{2q}^{12}e_{2q}^{12} - \sum_{1\leqslant p\leqslant 8}\rho_{p3}^{12}e_{p3}^{12} + e_{13}^{12} + e_{25}^{12} + e_{27}^{23} \qquad (5.82)\\ \mathcal{J}_{67}^{123} &= \sum_{1\leqslant q\leqslant 8}\rho_{6q}^{23}e_{6q}^{23} - \sum_{1\leqslant p\leqslant 8}\rho_{p7}^{23}e_{p7}^{23} + \sum_{1\leqslant q\leqslant 8}\rho_{6q}^{13}e_{6q}^{13} - \sum_{1\leqslant p\leqslant 8}\rho_{p7}^{13}e_{p7}^{13} \\ &\quad + \sum_{1\leqslant q\leqslant 8}\rho_{6q}^{12}e_{6q}^{12} - \sum_{1\leqslant p\leqslant 8}\rho_{p7}^{12}e_{p7}^{12} + e_{68}^{12} + e_{47}^{12} + e_{27}^{23} \qquad (5.83)\end{aligned}$$

The entries of the tuple $(\rho_{pq}^{k,m})_{\substack{1\leqslant p,q\leqslant 8\\ r,s\in\{k,l,m\}}}$ are either a non-constant polynomial from $\mathrm{K}[\mathcal{C}]$ or 0. Therefore the polynomials τ_{27}^{23} and τ_{68}^{12} can be the redundant polynomials which are rewritable elements by both Jacobi identity syzygies (5.82) and (5.83). Moreover, there is exactly one trace syzygy $\mathrm{T}_{x_1x_2^2,x_1}$ that corresponds to arrow degree $(1,2,0)$.

$$\mathrm{T}_{x_1x_2^2,x_1} = \sum_{1\leqslant p,q\leqslant 8}\kappa_{pq}^{12}e_{pq}^{12} + 2e_{13}^{12} + 2e_{25}^{12} + 2e_{47}^{12} + 2e_{68}^{12} \qquad (5.84)$$

The components of the tuple $(\kappa_{pq}^{12})_{1\leqslant p,q\leqslant 8}$ are either a non-constant polynomial from $\mathrm{K}[\mathcal{C}]$ or 0. Hence, τ_{47}^{12} can be redundant. Therefore there are at most 3 redundant polynomials of arrow degree $(1,2,0)$. By Lemma 5.4.2, there are at most 3 redundant polynomials of each given arrow degree. Consequently, there are at most 18 redundant polynomials of the above arrow degrees.

v) In this case we consider the arrow degrees $(1,2,1)$, $(1,1,2)$ and $(2,1,1)$. As in the previous cases we choose one arrow degree , say $(1,1,2)$. By Equation (5.76), in positions $(3,5)$, $(4,6)$, $(7,8)$ of Matrix (5.75) the simplified Jacobi identity syzygies with degree $(1,1,2)$ are located . Let us give a closer look at the Jacobi identity syzygies.

$$\begin{aligned}\mathcal{J}_{35}^{123} &= \sum_{1\leqslant q\leqslant 8}\rho_{3q}^{23}e_{3q}^{23} - \sum_{1\leqslant p\leqslant 8}\rho_{p5}^{23}e_{p}^{23} + \sum_{1\leqslant q\leqslant 8}\rho_{3q}^{13}e_{3q}^{13} \qquad (5.85)\\ &\quad - \sum_{1\leqslant p\leqslant 8}\rho_{p5}^{13}e_{p5}^{13} + \sum_{1\leqslant q\leqslant 8}\rho_{3q}^{12}e_{3q}^{12} - \sum_{1\leqslant p\leqslant 8}\rho_{p5}^{12}e_{p5}^{12} + e_{13}^{15} + e_{38}^{23}\end{aligned}$$

$$\mathcal{J}_{46}^{123} = \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{23} e_{4q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p6}^{23} e_{p6}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{13} e_{4q}^{13} \tag{5.86}$$
$$- \sum_{1\leqslant p\leqslant 8} \rho_{p6}^{13} e_{p6}^{13} + \sum_{1\leqslant q\leqslant 8} \rho_{4q}^{12} e_{4q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p6}^{12} e_{p6}^{12}$$
$$-e_{48}^{13} + e_{16}^{23} \tag{5.87}$$
$$\mathcal{J}_{78}^{123} = \sum_{1\leqslant q\leqslant 8} \rho_{7q}^{23} e_{7q}^{23} - \sum_{1\leqslant p\leqslant 8} \rho_{p8}^{23} e_{p8}^{23} + \sum_{1\leqslant q\leqslant 8} \rho_{7q}^{13} e_{7q}^{13}$$
$$- \sum_{1\leqslant p\leqslant 8} \rho_{p8}^{13} e_{p8}^{13} + \sum_{1\leqslant q\leqslant 8} \rho_{7q}^{12} e_{7q}^{12} - \sum_{1\leqslant p\leqslant 8} \rho_{p8}^{12} e_{p8}^{12}$$
$$-e_{48}^{13} + e_{38}^{23}$$

The components of the tuple $(\rho_{pq}^{k,m})_{\substack{1\leqslant p,q\leqslant 8 \\ r,s\in\{k,l,m\}}}$ are either a non-constant polynomial from $\mathrm{K}[\mathcal{C}]$ or 0. Hence the polynomials $\tau_{38}^{23}, \tau_{15}^{13}$ and τ_{16}^{23} can be redundant, which are rewritable polynomials by the Jacobi identity syzygies (5.85), (5.86) and (5.88). There are also trace syzygies that correspond to that degree but they are just equal to the $\mathrm{K}[\mathcal{C}]$-combination of the above Jacobi identity syzygies. Therefore by Lemma 5.4.2 , we deduce there are at most 9 redundant elements of the above arrow degrees.

vi) The diagonal positions of Matrix (5.75) contain the polynomials of degree $(1,1,1)$. By examining the simplified Jacobi polynomials in the diagonal position i.e., J_{pp}^{123} for every $p \in \{1,...,8\}$, we deduce there are exactly 7 different Jacobi identity syzygies and therefore at most 7 different redundant polynomials. The trace syzygy of that degree does not give a new rewritable polynomial.

vii) The only remaining degrees are $(2,2,1), (1,2,2)$ and $(2,1,2)$. By Equation (5.76) there is exactly one Jacobi identity syzygy for each given arrow degree. These are $\mathcal{J}_{28}^{123}, \mathcal{J}_{38}^{123}$ and $\mathcal{J}_{48}^{123}$ respectively, where simplified Jacobi identity syzygies have one non-zero component. Therefore there is at most one redundant polynomial for each given degree. It is not necessary to check the trace syzygies, since they indicate the same rewritable elements from $\mathcal{T}$.

As a result, there are at most 60 rewritable elements with respect to Jacobi identity and trace syzygies. □

Remark 5.4.4. Note that in Proposition 5.4.3 we did not claim that there are exactly 60 redundant polynomials from $\mathcal{T}$, instead we claimed that there are at most 60 redundant polynomials that can be computed by Jacobi identity and trace syzygies of

$\mathcal{T}$. There are two reasons: First reason is that there are standard polynomials that are rewritable with respect to the Jacobi identity syzygies. The Second reason is that we can not order the arrow degrees of the polynomials from $\mathcal{T}$ such that it enables us to decide whether a rewritable polynomial is also a redundant one or not, since arrow grading is neither of positive type nor positive. In other words, we cannot use any arguments that depend on the graded version of the Nakayama Lemma 2.1.11. There are two reasons why we don't face the same problem in the case where the polynomial ring is $\mathrm{K}[x_1, x_2]$. First, in that case there are no non-standard non-positive polynomials from $\mathcal{T}$ (see Remark 5.1.20) and therefore the trace syzygies of $\mathcal{T}$ are enough to compute all the redundant polynomials. Second, there are no indeterminates from $\mathrm{K}[\mathcal{C}]$ that have arrow degree vectors with more than one negative entry, such as $\deg_W(c_{82}) = (-1, 1, -1)$.

Chapter 6

Border Basis Schemes and Affine Spaces

This Chapter is dedicated to finding conditions which make an $\mathcal{O}$-border basis scheme an affine cell, i.e. isomorphic to an affine space. For the special case $n = 2$, in [Hai98] $\mathcal{O}$-border basis schemes are described as open coverings of the punctual Hilbert schemes $\mathrm{Hilb}^{\mu}(\mathbb{A}^2)$. Moreover, it is stated as a remark (see Proposition 2.1 [Hai98]) that they are not affine cells in general. In [Huib02], Chapter 7, a necessary and sufficient condition on the order ideal $\mathcal{O} \subset \mathbb{T}^2$ is given which makes $\mathcal{O}$-border basis scheme an affine space. Since our results depend neither on the shape of $\mathcal{O}$ nor on n, they are more general than the existing results. If the monomial point $\mathfrak{o}$ is a smooth point of the $\mathcal{O}$-border basis scheme and μ denotes the number of terms in $\mathcal{O} \subset \mathbb{T}^n$, then $\mathbb{B}_{\mathcal{O}}$ is locally isomorphic to an affine space of dimension $n\mu$. Our aim is to find a condition which extents this local property to $\mathbb{B}_{\mathcal{O}}$. The main results of this section (see Theorems 6.2.20) imply that this extension is possible. To be precise, we show that the coordinate ring $\mathrm{B}_{\mathcal{O}}$ is isomorphic to a polynomial ring of dimension $n\mu$ if there exist a set $\mathcal{S} \subset \mathcal{C}$ (see Chapter 4.2) and a maximal minor of the Jacobian matrix of $\mathcal{T}$ that is ± 1 and the corresponding matrix of this minor is in $\mathrm{Mat}_{\mu\nu-n\mu}(\mathrm{K}[\mathcal{S}])$. Let us have a closer look at the sections of this chapter.

In Section 6.1 we investigate the properties of the Jacobian matrix of the defining equations $\mathcal{T}$ of an $\mathcal{O}$-border basis scheme where $\mathcal{O} \subset \mathbb{T}^n$ and K is a perfect field. Then we show that the Jacobian ideal is a homogenous ideal with respect to the arrow grading. On page 366 of [MilSt05], this property is also mentioned for an order ideal $\mathcal{O} \subset \mathbb{T}^2$.

Before stating our main theorem, in Section 6.2 we give a condition for checking whether the injective homomorphism

$$\begin{array}{rcl} \varphi : \mathrm{K}[\mathcal{S}] & \longrightarrow & \mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) \\ c_{ij} & \longmapsto & \bar{c}_{ij} \end{array}$$

in (6.15) is surjective or not, where $\mathcal{S}$ is a set defined in Lemma 4.2.12. This condition holds under the assumptions that the given border basis scheme is an ideal-theoretic complete intersection. In particular, our claim holds if the vanishing ideal of the $\mathcal{O}$-border basis scheme is generated by the subset $\mathcal{T}_{\mathcal{S}}$ of $\mathcal{T}$. Then we generalize this result to Theorem 6.2.20 in which we give a condition without that assumption. Then, by using our theorem (see Theorem 6.2.20), we show that the coordinate ring of the border basis scheme $\mathrm{B}_{\mathcal{O}}$, where $\mathcal{O} = \{1, x_1, x_2\} \subset \mathbb{T}^3$ is isomorphic to an affine space of dimension 9. The proof of Theorem 6.2.20 is an algorithmic proof. By using the idea of this proof we give an algorithm for checking whether given border basis scheme is isomorphic to an affine space (see Algorithm 6.2.26).

Additionally, we have two sections which we apply Theorem 6.2.20 to prove that $\mathcal{O}$-border basis scheme is isomorphic to an affine space where the order ideal $\mathcal{O}$ is a *segment* order ideal in Section 6.3 and the order ideal $\mathcal{O}$ has the *sawtooth* form in Section 6.4.

6.1 The Jacobian Matrix of $\mathcal{T}$

Unless stated otherwise, throughout this section we let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$, where K is a field. Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be an order ideal and let $\partial\mathcal{O}$ denote its border $\{b_1, ..., b_\nu\}$. Let $\mathcal{C}$ denote a set of indeterminates $\{c_{11}, ..., c_{\mu\nu}\}$. Let $\mathrm{K}[\mathcal{C}]$ be the polynomial ring $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$.

Let $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ be the vanishing ideal of the border basis scheme $\mathbb{B}_{\mathcal{O}}$ and it is generated by the set $\mathcal{T} = \{\tau_{pq}^{kl} \mid p, q \in \{1, ...\mu\}, \quad k < l \quad k, l \in \{1, .., n\}\}$ defined in Section 2.3. By Lemma 2.3.19, the polynomials from the set $\mathcal{T}$ are of degree two with no constant term.

Let τ_κ be a polynomial in $\mathcal{T}$. Recall that no matter which order ideal is used, by Lemma 2.3.19, the polynomial τ_κ is of the following shape:

$$\tau_\kappa = \tau_\kappa^{(2)} \pm \tau_\kappa^{(1)} \tag{6.1}$$

where $\tau_\kappa^{(2)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 2 with respect to the standard grading and $\tau_\kappa^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 1 with respect to the standard grading. For the polynomial $\tau_\kappa^{(1)}$, there are three possibilities: The first is that $\tau_\kappa^{(1)} = (c_{i_1j_1} \pm c_{i_2j_2})$, where $c_{i_1j_1}, c_{i_2j_2} \in \mathrm{K}[\mathcal{C}]$. The second is that $\tau_\kappa^{(2)} = c_{ij}$ and finally, we have the possibility $\tau_\kappa^{(1)} = 0$.

Since the polynomials in $\mathcal{T}$ have a specific shape, we can describe the entries of the Jacobian matrix. This section is dedicated to collecting information on how the shape of the elements of $\mathcal{T}$ effects the Jacobian matrix of $\mathcal{T}$ and the Jacobian ideal. Recall that we denote the Jacobian matrix of $\mathcal{T}$ by $\mathrm{Jac}(\mathcal{T})$.

Corollary 6.1.1. *The non-zero entries of the* $\mathrm{Jac}(\mathcal{T})$ *are either linear polynomials from* $\mathrm{K}[\mathcal{C}]$ *with no constant, or* ± 1. *In particular, if we let* $\tau_{pq}^{kl} \in \mathcal{T}$ *and let* $c_{rs} \in \mathcal{C}$, *then the following holds.*

$$\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}} = \begin{cases} \pm 1 & \text{if } c_{rs} \in \mathrm{Supp}(\tau_{pq}^{kl}) \\ \text{linear polynomial in } \mathrm{K}[\mathcal{C}] & \text{if } c_{rs} \mid c,\ c \text{ is a quadratic term in } \mathrm{Supp}(\tau_{pq}^{kl}) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let c_{rs} be an indeterminate from $\mathcal{C}$. There are three cases we will consider.

case 1) If the term c_{rs} is in the support of the polynomial τ_{pq}^{kl} then, by Proposition 3.2.6, we have $\deg_W(\tau_{pq}^{kl}) = \deg_W(c_{rs})$. By Corollary 3.2.2, none of the indeterminates in $\mathcal{C}$ has the arrow degree $(0, ..., 0) \in \mathrm{K}^n$. Then for each index α, the indeterminate c_{rs} can not divide the monomial $c_{u_\alpha v_\alpha} c_{i_\alpha j_\alpha} \in \mathrm{Supp}(\tau_{pq}^{kl})$ and as a result, we have $\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}} = \pm 1$.

case 2) If there exists an index α such that c_{rs} divides the monomial $c_{u_\alpha v_\alpha} c_{i_\alpha j_\alpha} \in \mathrm{Supp}(\tau_{pq}^{kl})$ then, by Proposition 3.2.6, the indeterminate c_{rs} is not an element of $\mathrm{Supp}(\tau_{pq}^{kl})$, and hence $\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}}$ is a linear polynomial with no constant term.

case 3) If c_{rs} satisfies neither case 1 nor case 2, then we have $\frac{\partial \tau_{pq}^{kl}}{\partial c_{ij}} = 0$.

□

Example 6.1.2. Let $\mathcal{O}$ denote the order ideal $\{1, x_1\} \subset \mathbb{T}^2$. Then $\partial\mathcal{O} = \{x_2, x_1x_2, x_1^2\}$ is its border. By considering the arrow degrees of the elements of the set $\mathcal{C} = \{c_{11}, ..., c_{23}\}$, we construct the degree matrix. The columns of this matrix are the

arrow degree vectors of each $c_{ij} \in \mathcal{C}$ i.e., $\deg_W(c_{ij}) = \log(b_j) - \log(t_i)$.

$$W = \begin{array}{c} \begin{matrix} c_{11} & c_{12} & c_{13} & c_{21} & c_{22} & c_{23} \end{matrix} \\ \begin{pmatrix} 0 & 1 & 2 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

The coordinate ring of the border basis scheme is $\mathrm{B}_{\mathcal{O}} = K[\mathcal{C}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ where the vanishing ideal of the border basis scheme is generated by the set $\mathcal{T} = \{\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}\}$ with

$$\begin{aligned} \tau_{11} &= c_{13}c_{21} - c_{12} & \tau_{12} &= c_{11}c_{13} - c_{13}c_{22} + c_{12}c_{23} \\ \tau_{21} &= c_{21}c_{23} - c_{22} & \tau_{22} &= c_{13}c_{21} - c_{12}. \end{aligned}$$

For details on how we construct these polynomials we refer to Chapter 2. By Proposition 3.2.6, the polynomials from $\mathcal{T}$ are homogeneous with respect to the arrow grading. Let us give their arrow degrees:

$$\deg_W(\tau_{11}) = (1,1), \quad \deg_W(\tau_{12}) = (2,1), \quad \deg_W(\tau_{21}) = (0,1), \quad \deg_W(\tau_{22}) = (1,1)$$

Next we compute the Jacobian matrix and get the following matrix as a result.

$$\mathrm{Jac}(\mathcal{T}) = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{11}}{\partial c_{23}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_{22}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{22}}{\partial c_{23}} \end{pmatrix} = \begin{pmatrix} 0 & -1 & c_{21} & c_{13} & 0 & 0 \\ c_{13} & c_{23} & c_{11} - c_{22} & 0 & -c_{13} & c_{12} \\ 1 & 0 & 0 & c_{23} & -1 & c_{21} \\ 0 & -1 & c_{21} & c_{13} & 0 & 0 \end{pmatrix}.$$

Let us check whether $\mathrm{Jac}(\mathcal{T})$ is a homogeneous matrix (see Definition 3.2.3) or not. First we find suitable vectors e and d such that $e = (e_1, e_2, e_3, e_4)$ determines the degree of the rows and $d = (d_1, ..., d_6)$ determines the degree of the columns. Consider

$$e_j - d_i = \deg_W(\frac{\partial \tau_{pq}}{\partial c_{kl}}),$$

where c_{kl} is the i^{th} element of $\mathcal{C}$ and τ_{pq} is the j^{th} element of $\mathcal{T}$. Choosing $e_j = \deg_W(\tau_{pq})$ and $d_i = \deg_W(c_{kl})$ gives us the desired vectors. Thus the Jacobian matrix $\mathrm{Jac}(\mathcal{T})$ is a homogenous matrix with the degree pair (d, e) where $e = ((1,1),(2,1),(0,1),(1,1))$ and $d = ((0,1),(1,0),(2,0),(0,1),(1,0))$.

Lemma 6.1.3. *Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal with μ elements. Let m be the number of polynomials in $\mathcal{T}$. Then the Jacobian matrix* $\mathrm{Jac}(\mathcal{T})$ *is a homogeneous matrix with respect to the arrow grading, with the degree pair $(e,d) \in (\mathbb{Z}^n)^{n\mu} \times (\mathbb{Z}^n)^m$, where $e_i = \deg_W(\tau_{pq}^{kl}) \in \mathbb{Z}^n$ and $d_j = \deg_W(c_{rs}) \in \mathbb{Z}^n$, for the i^{th} element τ_{pq}^{kl} of $\mathcal{T}$ and the j^{th} element c_{rs} of $\mathcal{C}$. Consequently, the minors of* $\mathrm{Jac}(\mathcal{T})$ *are homogeneous polynomials and the Jacobian ideal is a homogeneous ideal.*

Let τ_{pq}^{kl} be the i^{th} element of $\mathcal{T}$ and let c_{rs} be the j^{th} element of $\mathcal{C}$.

Proof. By Proposition 3.2.6, every polynomial in $\mathcal{T}$ is homogeneous with respect to the arrow grading. Then the polynomial $\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}}$ is homogeneous with respect to the arrow grading, as well. By Corollary 6.1.1, we have

$$\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}} = \begin{cases} \pm 1 & \text{if } c_{rs} \in \mathrm{Supp}(\tau_{pq}^{kl}), \\ \text{linear polynomial in K}[\mathcal{C}] & \text{if } c_{rs} \mid c, \text{ c is a quadratic monomial in Supp}(\tau_{pq}^{kl}) \\ 0 & \text{otherwise.} \end{cases}$$

Note that 0 is homogenous of every degree. Then the arrow degree of the polynomial above is

$$\deg_W(\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}}) = \deg_W(\tau_{pq}^{kl}) - \deg_W(c_{rs}). \tag{6.2}$$

Since τ_{pq}^{kl} is the i^{th} polynomial in $\mathcal{T}$ and c_{rs} is the j^{th} indeterminate in $\mathcal{C}$, we let $\deg_W(\tau_{pq}^{kl}) = e_i$ and $\deg_W(c_{rs}) = d_j$. Then by Corollary 6.1.1, the degree of the homogeneous element $\frac{\partial \tau_{pq}^{kl}}{\partial c_{rs}}$ in the matrix is $(e_i - d_j)$. Thus the Jacobian matrix is a homogeneous matrix with respect to the degree pair (e,d) (see Definition 3.2.3).

Furthermore, every sub-matrix of the Jacobian matrix is also a homogeneous matrix. Therefore r-rowed minors are homogeneous polynomials of degree $\sum_{i=1}^{r}(e_i' - d_i')$ where $d_i' \in \{d_1, ..., d_{\mu\cdot\nu}\}$ and $e_i' \in \{e_1, ..., e_m\}$. In particular, the Jacobian ideal is a homogeneous ideal with respect to the arrow grading. □

Remark 6.1.4. In Chapter 4 we showed that the arrow grading is in general neither of positive nor of non-negative type. Although the Jacobian ideal is homogenous with respect to the arrow grading, it is not necessarily included in the maximal ideal $\langle c_{11}, ..., c_{\mu\nu} \rangle$ (see Example 3.2.12).

6.2 An Affine Space Criterion

Unless stated otherwise, throughout this section we use the following notation.

- We let P denote the polynomial ring $\mathrm{K}[x_1, ..., x_n]$. We assume that K is a perfect field.
- Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be an order ideal and let $\partial\mathcal{O}$ denote its border $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$.
- Let $\mathcal{C}$ denote the set of indeterminates $\{c_{11}, ..., c_{\mu\nu}\}$. Let $\mathrm{K}[\mathcal{C}]$ be the polynomial ring $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$.
- The ideal $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is the vanishing ideal of the border basis scheme $\mathbb{B}_\mathcal{O}$. It is generated by the set $\mathcal{T} = \{\tau_{pq}^{kl} \mid p, q \in \{1, ...\mu\}, \quad k < l \quad k, l \in \{1, .., n\}\}$, which is defined in Chapter 2.3.
- According to Lemma 2.3.19, the polynomials from the set $\mathcal{T}$ are of degree two with no constant. Let us pick a polynomial from $\mathcal{T}$, say τ_κ. Then we write the polynomial τ_κ as
$$\tau_\kappa = \tau_\kappa^{(2)} + \tau_\kappa^{(1)} \in \mathcal{T} \tag{6.3}$$
where $\tau_\kappa^{(2)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 2 with respect to the standard grading and $\tau_\kappa^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a homogenous polynomial of degree 1 with respect to the standard grading. Recall that the set $\mathrm{Supp}(\tau_\kappa^{(1)})$ has at most two linear monomials.

Remark 6.2.1. The field K is isomorphic to $\mathrm{B}_\mathcal{O}/\mathfrak{m}(\mathrm{B}_\mathcal{O})_\mathfrak{m}$ and the point $\mathfrak{o} = (0, ..., 0) \in \mathrm{K}$.

6.2.1 Border Bases Scheme with the Smooth Monomial Point

Throughout this subsection we make the following assumptions.

- The field K is a perfect field.
- The point $\mathfrak{o} = (0, .., 0)$ is a smooth point on the scheme $\mathbb{B}_\mathcal{O}$.

Now let us recall the necessary notations for this subsection.

- Let $\mathfrak{m}$ denote the maximal ideal of $\mathrm{B}_\mathcal{O}$ generated by the elements of $\mathcal{C} = \{\bar{c}_{11}, ..., \bar{c}_{n\mu\mu\nu}\} \subset \mathrm{B}_\mathcal{O}$. Then $(\mathrm{B}_\mathcal{O})_\mathfrak{m}$ is a regular local ring.

- The set $\mathfrak{E}$ denotes the set of non-zero equivalence classes modulo $\mathfrak{m}^2$.
- Since $\mathfrak{o}$ is a smooth point of $\mathbb{B}_{\mathcal{O}}$, according to Theorem 4.2.15, the set $\mathfrak{E}$ has exactly $n\mu$ different equivalence classes.
- The set $\mathcal{S} = \{s_1, ..., s_{n\mu}\}$ is constructed in a way that each $[\bar{s}_i]$ represents a distinct equivalence class in $\mathfrak{E}$. The set $\mathcal{S}$ is not unique.
- We let $\mathrm{K}[\mathcal{S}]$ denote the polynomial ring $\mathrm{K}[s_1, ..., s_{n\mu}]$. We let $\bar{S} = \{\bar{s}_i \in \mathrm{B}_{\mathcal{O}} \mid s_i \in \mathcal{S}\} \subset \bar{\mathcal{C}}$.
- We let $\widehat{\mathcal{S}} = \{\widehat{s}_i \in \mathfrak{m}/\mathfrak{m}^2 \mid s_i \in \mathcal{S}\}$ denote a basis set of the vector space $\mathfrak{m}/\mathfrak{m}^2$.
- We denote the elements of $\mathcal{C} \setminus \mathcal{S}$ as $d_1, ..., d_m$, where $m = \mu\nu - n\mu$.

Remark 6.2.2. Let $\mathcal{O} \subset \mathbb{T}^n$ denote an order ideal. In Chapter 7 of [Huib02] it is proved that the $\mathcal{O}$-border basis scheme is an affine space (see Theorem 7.4.1, [Huib02]), if n is 2 and the order ideal $\mathcal{O}$ has a special shape. For further information on this topic we refer to Chapter 7 of [Huib02]. Note that in this section we present a condition that is valid for $n \geqslant 2$ and independent of the shape of $\mathcal{O}$

Let us introduce the necessary notation we use frequently throughout this section and next section.

Notation 6.2.3. Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal that has μ terms. If we let the number of elements in the i^{th} equivalence class in $\mathfrak{E}$ to be k_i, then by picking exactly one element from each equivalence class, we can construct $k_1 \times ... \times k_{n\mu}$ many different sets $\mathcal{S}$. We denote the number of such sets by $\mathfrak{n}_{\mathcal{S}}$.

$$\mathfrak{n}_{\mathcal{S}} := k_1 \times ... \times k_{n\mu} \tag{6.4}$$

For the rest of the section let us fix one such set $\mathcal{S}$.

Lemma 6.2.4. *For each element d_i of $\mathcal{C} \setminus \mathcal{S}$ there exists at least one polynomial τ_i in $\mathcal{T}$ that contains the term d_i in its support. If the polynomial τ_i has the term d_j that is distinct from d_i in its support, then there exists a polynomial $\tau_j \in \mathcal{T}$ such that $\tau_i \neq \tau_j$ and τ_j contains the term d_j in its support.*

Proof. Let s_i be an indeterminate from $\mathcal{S}$. We write elements in the equivalence class $[\bar{s}_i]$ as follows,

$$[\bar{s}_i] = \{\bar{s}_i, \bar{d}_{i_1}, ..., \bar{d}_{i_k}\} \tag{6.5}$$

and hence this equivalence class has $k+1$ elements. If we have $k=0$, then by Proposition 4.1.6c), there exists no polynomial in $\mathcal{T}$ that contains the term s_i in its support. If we have $k>1$ in (6.5), then by Proposition 4.1.6b), for each element d_{i_j} of $\{\bar{d}_{i_1},...,\bar{d}_{i_k}\} \in \mathcal{C}\setminus\mathcal{S}$ there exists a polynomial in $\mathcal{T}$ that contains the term d_{i_j} in its support. Additionally, if we have $\bar{d}_j \in \mathfrak{m}^2$, then $d_j \in \mathcal{C}\setminus\mathcal{S}$ and by Proposition 4.1.6a), there exist polynomials in $\mathcal{T}$ that contain d_j in their support. To prove our next claim, we let τ_k be a polynomial in $\mathcal{T}$ that contains the distinct terms d_k and d_j in its support. Keep in mind that d_i and d_j are both in $\mathcal{C}\setminus\mathcal{S}$. To get a contradiction, we suppose neither the term d_j nor the term d_i is in the support of a polynomial in $\mathcal{T}$ except for τ_k. Using this assumption with Proposition 4.1.6 b) gives us that $[\bar{d}_j]=\{\bar{d}_i,\bar{d}_j\}$. By Proposition 4.1.6 a), neither $\bar{d}_i$ nor $\bar{d}_j$ is in $\mathfrak{m}^2$. Then we have $[\bar{d}_j]\in\mathfrak{E}$. As a result of the construction of $\mathcal{S}$, either d_i or d_j must be in $\mathcal{S}$. This contradicts with the assumption that d_i and d_j are both in $\mathcal{C}\setminus\mathcal{S}$. □

By Lemma 6.2.4 for each $d_i \in \mathcal{C}\setminus\mathcal{S}$ there exists at least one polynomial in $\mathcal{T}$, say τ_i, such that $d_i \in \mathrm{Supp}(\tau_i)$. By Corollary 6.1.1, for the polynomial $\tau_i \in \mathcal{T}$ we have

$$\frac{\partial \tau_i}{\partial d_i} = \pm 1. \tag{6.6}$$

By using these two results we construct a subset of $\mathcal{T}$ that depends on $\mathcal{S}$.

Construction 6.2.5. According to the Lemma 6.2.4 we form a subset of $\mathcal{T}$, where for each $d_i \in \mathcal{C}\setminus\mathcal{S}$, we pick exactly one polynomial $\tau_i \in \mathcal{T}$ that satisfies (6.6). We let $\mathcal{T}_\mathcal{S}$ denote this set. Recall that $m=\mu\nu-n\mu$ is the number of elements of $\mathcal{C}\setminus\mathcal{S}$. Then there are exactly m polynomials in $\mathcal{T}_\mathcal{S}$. By using the elements of $\mathcal{T}_\mathcal{S}$ and $\mathcal{C}\setminus\mathcal{S}$, we construct the following $m\times m$ matrix.

$$\begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & \dots & d_m \end{array} \\ \begin{pmatrix} \pm 1 & \star & \star \\ \vdots & \ddots & \vdots \\ \star & \star & \pm 1 \end{pmatrix} \end{array} = \begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \cdots & \frac{\partial \tau_1}{\partial d_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \dots & \frac{\partial \tau_m}{\partial d_m} \end{pmatrix} \tag{6.7}$$

We denote the matrix in (6.7) by $\mathcal{M}_\mathcal{S}$.

Remark 6.2.6. Note that there can be more than one τ_i for each $d_i \in M$. Hence the set $\mathcal{T}_\mathcal{S}$ is not uniquely determined.

Let us compute the Jacobian matrix $\mathrm{Jac}(\mathcal{T})$.

$$\mathrm{Jac}(\mathcal{T}) = \begin{array}{c} \\ \tau_{11}^{12} \\ \vdots \\ \tau_{\mu\mu}^{n-1\,n} \end{array} \begin{array}{c} \begin{array}{ccc} c_{11} & \cdots & c_{\mu\,\nu} \end{array} \\ \begin{pmatrix} \frac{\partial \tau_{11}^{12}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{11}^{12}}{\partial c_{\mu\,\nu}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_{\mu\mu}^{n-1\,n}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{\mu\mu}^{n-1\,n}}{\partial c_{\mu\,\nu}} \end{pmatrix} \end{array} \tag{6.8}$$

The matrix $\mathcal{M}_{\mathcal{S}}$ is a sub-matrix of $\mathrm{Jac}(\mathcal{T})$. We rearrange the rows and the columns of (6.8) as follows.

$$\mathrm{Jac}(\mathcal{T}) = \left(\begin{array}{c|c} \mathcal{M}_{\mathcal{S}} & \star \\ \hline \star & \star \end{array}\right) \tag{6.9}$$

Recall that we denote the Jacobian matrix evaluated at the point $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ as $\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}$.

Lemma 6.2.7. *By applying row and column operations on the matrix* $\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}$ *in (6.9) without exchanging rows and columns, we get a matrix of rank* $\mu\nu - n\mu$ *of the following shape.*

$$\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}} \sim \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0 \end{array}\right) \tag{6.10}$$

The matrix I_m *is the unit matrix of rank* m *and it is the matrix* $\mathcal{M}_{\mathcal{S}}$ *evaluated at the point* $\mathfrak{o} = (0, ..., 0)$.

Proof. By Lemma 6.1.1, the entries of the matrix $\mathrm{Jac}(\mathcal{T})$ are 0 or ± 1 or a homogenous (with respect to the standard degree) linear polynomial in $\mathrm{K}[\mathcal{C}]$. If we evaluate the matrix $\mathrm{Jac}(\mathcal{T})$ at the point $\mathfrak{o} = (0, ..., 0)$, we substitute 0 instead of each indeterminate c_{ij} in the matrix $\mathrm{Jac}(\mathcal{T})$, then the homogenous linear polynomials become 0 and the entries of the matrix $\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}$ are 0 or ± 1. Moreover the monomial point $\mathfrak{o} = (0, ..., 0)$ is a smooth point of the scheme $\mathbb{B}_{\mathcal{O}}$. Then by Theorem 2.1.18 the matrix $\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}$ has rank m. Hence by row and column operations we get 6.10. □

6.2.2 Border Basis Schemes with an Arbitrary Monomial Point

Throughout this section we make no assumptions on the monomial point $\mathfrak{o}$ of $\mathbb{B}_{\mathcal{O}}$. Let us recall some of the necessary notations and claims from Chapter 4 that we will use in this subsection.

i) The set $\mathfrak{E}$ denotes the set of non-zero equivalence classes modulo $\mathfrak{m}^2$.

ii) According to Theorem 4.2.9, the set $\mathfrak{E}$ has at least $n\mu$ different equivalence classes.

iii) The set $\mathcal{S} = \{s_1, ..., s_{n\mu}\} \subseteq \mathcal{C}$ is constructed in a way that each $[\bar{s}_i]$ represents a distinct equivalence class in $\mathfrak{E}$. The set $\mathcal{S}$ is not unique. For further information we refer to Chapter 4.

iv) According to Lemma 4.2.12, the elements of $\widehat{\mathcal{S}} = \{\widehat{s}_i \in \mathfrak{m}/\mathfrak{m}^2 \mid s_i \in \mathcal{S}\}$ are linearly independent and the K-vector space $\langle\widehat{\mathcal{S}}\rangle_{\mathrm{K}}$ is a subspace of $\mathfrak{m}/\mathfrak{m}^2$ of dimension at least $n\mu$.

v) We denote the elements of the set $\mathcal{C} \setminus \mathcal{S}$ as $\{d_1, ..., d_m\}$, where $m = \mu\nu - n\mu$.

Our next lemma shows that if we can find a maximal minor of the Jacobian matrix, which is ± 1, then the point $\mathfrak{o}$ is a smooth point of the $\mathcal{O}$-border basis scheme.

Lemma 6.2.8. *If there exists a set $\mathcal{S}$ as given in Notation 6.2.3 with $n\mu$ elements such that the determinant of the matrix $\mathcal{M}_{\mathcal{S}}$ in (6.7) is ± 1, then the monomial point $\mathfrak{o} \in \mathbb{B}_{\mathcal{O}}$ is a smooth point.*

Proof. The matrix $\mathcal{M}_{\mathcal{S}}$ is an $m \times m$ matrix. Since the determinant of $\mathcal{M}_{\mathcal{S}}$ is ± 1, the rank of the matrix $\mathcal{M}_{\mathcal{S}}$ evaluated at any point including $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ is $m = \nu\mu - n\mu$. The matrix $\mathcal{M}_{\mathcal{S}}$ is a submatrix of the matrix $\mathrm{Jac}(\mathcal{T})$. Hence we have the following inequality.

$$\mathrm{Rank}(\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}) \geqslant \mathrm{Rank}(\mathcal{M}_{\mathcal{S}_{\mathfrak{o}}}) = \nu\mu - n\mu = m$$

By Theorem 2.1.18a), the rank of the matrix $\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}=(0,...,0)}$ is less than or equal to $\mu\nu - n\mu$.

$$\mathrm{Rank}(\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}) \leqslant \mu\nu - n\mu.$$

Hence we have $\mathrm{Rank}(\mathrm{Jac}(\mathcal{T})_{\mathfrak{o}}) = \mu\nu - n\mu$. Then by Theorem 2.1.18c), $\mathfrak{o}$ is a smooth point of $\mathbb{B}_{\mathcal{O}}$. □

Lemma 6.2.9. *Let us pick a polynomial $\tau_k \in \mathcal{T}$. If the entries of the matrix $\mathcal{M}_{\mathcal{S}}$ given in (6.7) are in $\mathrm{K}[\mathcal{S}]$, then each monomial in the support of $\tau_k^{(2)}$ is either of the form $s_i d_j \in (\mathrm{K}[\mathcal{S}])[d_1, ..., d_m]$ or a quadratic monomial $s_p s_q \in \mathrm{K}[\mathcal{S}]$.*

Proof. Suppose the matrix $\mathcal{M}_{\mathcal{S}} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$ and there exists a monomial $d_j d_l \in \tau_k^{(2)}$. Then in position (k, j) of $\mathcal{M}_{\mathcal{S}}$ we have $\frac{\partial \tau_k}{\partial d_j} = d_l$ and in (k, l) position of $\mathcal{M}_{\mathcal{S}}$ we have

$\frac{\partial \tau_k}{\partial d_l} = d_j$ This contradicts to the assumption that the entries of the matrix $\mathcal{M}_\mathcal{S}$ are in $\mathrm{K}[\mathcal{S}]$. □

Notation 6.2.10. Suppose that the entries of the matrix $\mathcal{M}_\mathcal{S}$ in 6.7 are from $\mathrm{K}[\mathcal{S}]$. Then, by Lemma 6.2.9, the polynomial τ_k is of the form.

$$\tau_k = \pm \sum_{i=1}^{\alpha} s_{k_i} d_{l_i} \pm \sum_{j}^{\beta} s_{p_j} s_{q_j} \pm d_k \pm t, \tag{6.11}$$

where $t = 0$ or $t \in \mathcal{C}$. Notice that $\pm \sum_j^\beta s_{p_j} s_{q_j} \in \mathrm{K}[\mathcal{S}]$, and $\pm \sum_{i=1}^\alpha s_{k_i} d_{l_i} \in (\mathrm{K}[\mathcal{S}])[d_{1,\ldots,d_m}]$. Let us give a closer look into all the possibilities for t.

case 1) If t is 0, then we let $g_k(\mathcal{S}) = \pm \sum_j^\beta s_{p_j} s_{q_j} \in \mathrm{K}[\mathcal{S}]$.

case 2) If we have $t \in \mathrm{K}[\mathcal{S}]$, then we let $g_k(\mathcal{S}) = \pm \sum_{j=1}^\beta s_{p_j} s_{q_j} \pm t \in \mathrm{K}[\mathcal{S}]$.

case 3 If we have $t \in \mathcal{C} \setminus \mathcal{S}$, then we let $g_k(\mathcal{S}) = \pm \sum_{j=1}^\beta s_{p_j} s_{q_j} \in \mathrm{K}[\mathcal{S}]$ and we have $\frac{\partial \tau_k}{\partial d_l} = \pm 1$ is in $\mathrm{K}[\mathcal{S}]$.

Then we rewrite the polynomial τ_k as follows.

$$\begin{aligned} \tau_k &= \pm \sum_{i=1}^{\alpha} \frac{\partial \tau_k}{d_{l_i}} d_{l_i} \pm \sum_{j}^{\beta} s_{p_j} s_{q_j} \pm \frac{\partial \tau_k}{d_k} d_k \pm t \\ &= \pm \sum_{i=1}^{\alpha} \frac{\partial \tau_k}{d_{l_i}} d_{l_i} \pm \frac{\partial \tau_k}{d_k} d_k + g_k(\mathcal{S}) \end{aligned}$$

Notice that for $j \notin \{l_1, ..., l_\alpha, k\}$ we have $\frac{\partial \tau_k}{d_j} = 0$. This leads us to the following equality.

$$\begin{pmatrix} \tau_1(\mathcal{S}) \\ \vdots \\ \tau_m(\mathcal{S}) \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \cdots & \frac{\partial \tau_1}{\partial d_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \cdots & \frac{\partial \tau_m}{\partial d_m} \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix} + \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix}$$

Recall that we construct the set $\mathcal{T}_\mathcal{S}$ by picking exactly one polynomial, τ_i from $\mathcal{T}$ for each element d_i such that

$$\frac{\partial \tau_i}{\partial d_i} = \pm 1.$$

Notation 6.2.11. If we let $\mathcal{S} = \{s_1, ..., s_{n\mu}\}$, then by Lemma 4.2.12, the vector space $\langle \bar{S} \rangle_\mathrm{K}$ is a subspace of $\mathfrak{m}/\mathfrak{m}^2$ of dimension $n\mu$. Let $\mathcal{J} \subset \mathrm{K}[\mathcal{C}]$ denote the ideal generated by the set $\mathcal{T}_\mathcal{S} = \{\tau_1, ..., \tau_m\} \subset \mathcal{T}$ and we let $\mathcal{Z}(\mathcal{J}) = \mathcal{V}$. Let the coordinate ring of $\mathcal{V}$

be denoted by $\mathrm{K}[\mathcal{V}]$. Let us denote the image of an indeterminate c_{ij} in $\mathrm{K}[\mathcal{V}]$ as $\tilde{c}_{ij}$ and let us denote the set of such elements by $\tilde{\mathcal{C}}$. We denote the maximal ideal that is generated by the elements $\{\tilde{c}_{11}, ..., \tilde{c}_{\mu\nu}\} \in \mathrm{K}[\mathcal{V}]$ by $\mathfrak{n}$. Moreover, the Jacobian matrix of $\mathcal{T}_{\mathcal{S}}$ is as follows.

$$\mathrm{Jac}(\mathcal{T}_{\mathcal{S}}) = \begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{cccccc} d_1 & \dots & d_m & s_1 & \dots & s_{n\mu} \end{array} \\ \begin{pmatrix} \pm 1 & \star & \star & \star & \star & \star \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \pm 1 & \star & \star & \star \end{pmatrix} \end{array} = \begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \cdots & \frac{\partial \tau_1}{\partial d_m} & \frac{\partial \tau_1}{\partial s_1} & \cdots & \frac{\partial \tau_1}{\partial s_{n\mu}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \cdots & \frac{\partial \tau_m}{\partial d_m} & \frac{\partial \tau_m}{\partial s_1} & \cdots & \frac{\partial \tau_m}{\partial s_{n\mu}} \end{pmatrix}.$$

Since the ideal $\mathcal{J}$ is a subset of $\langle c_{ij} \rangle$, the monomial point $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ is also contained in $\mathcal{V}$. Since $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}$ holds where $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) = \langle \mathcal{T} \rangle$, we have $\mathbb{B}_{\mathcal{O}} \subset \mathcal{V}$. Moreover, the shape of the Jacobian matrix of $\mathcal{T}_{\mathcal{S}}$ is as follows;

$$\mathrm{Jac}(\mathcal{T}_{\mathcal{S}}) = \left(\begin{array}{c|c} \mathcal{M}_{\mathcal{S}} & \star \end{array} \right) \tag{6.12}$$

Let us consider the following map.

$$\begin{array}{rcl} \mathfrak{h} : \mathrm{K}[c_{ij}]/\mathcal{J} & \longrightarrow & \mathrm{K}[c_{ij}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) \\ \mathfrak{n} & \longmapsto & \mathfrak{m} \\ \mathfrak{n}^2 & \longmapsto & \mathfrak{m}^2 \end{array}$$

Since $\mathcal{J} \subset \mathcal{I}(\mathbb{B}_{\mathcal{O}})$ holds, the map $\mathfrak{h}$ is a canonical surjective K-algebra homomorphism.

Lemma 6.2.12. *In Setting (6.2.11), suppose $\mathcal{M}_{\mathcal{S}} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$ and the determinant of this matrix is ± 1. Then the point $\mathfrak{o}$ is a smooth point of $\mathcal{V}$. Moreover, the dimension of the local ring $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$ is $n\mu$.*

Proof. Since the determinant of the matrix $\mathcal{M}_{\mathcal{S}}$ is ± 1, the rank of it is $\mu\nu - n\mu$ at any point of $\mathcal{V}$ including the point $\mathfrak{o}$. Then the rank of $(\mathrm{Jac}(\mathcal{T}_{\mathcal{S}}))_{\mathfrak{o}}$ is $\mu\nu - n\mu$. Let $\mathrm{K}[\mathcal{V}]$ denote the coordinate ring of $\mathcal{V}$. Let $\mathfrak{n}$ be the maximal ideal generated by the elements in the set $\{\tilde{c}_{11}, ..., \tilde{c}_{n\mu}\} \subset \mathrm{K}[\mathcal{V}]$. By Theorem 2.1.18a), we have $\mathrm{Rank}(\mathrm{Jac}(\mathcal{T}_{\mathcal{S}}))_{\mathfrak{o}}) \leqslant \mu\nu - \dim(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}})$. Then the following holds.

$$\dim(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}) \leqslant n\mu \tag{6.13}$$

On the other hand, by Lemma 6.2.8 the point $\mathfrak{o}$ is a smooth point of $\mathbb{B}_{\mathcal{O}} \subseteq \mathcal{V}$ and the $\dim((\mathbb{B}_{\mathcal{O}})_{\mathfrak{m}})$ is $n\mu$. Then the following holds.

$$\dim(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}) \geqslant n\mu \tag{6.14}$$

By Inequalities (6.13) and (6.14), we have $\dim(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}) = n\mu$. Then by Theorem 2.1.18c), the point $\mathfrak{o}$ is a smooth point of $\mathcal{V}$. □

Lemma 6.2.13. *In Setting (6.2.11), if the corresponding matrix* $\mathcal{M}_{\mathcal{S}} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$ *has determinant* ± 1, *then the elements of the set* $\tilde{S} = \{\tilde{s}_i \mid s_i \in \mathcal{S}\}$ *form a* K*-vector space basis of* $\mathfrak{n}/\mathfrak{n}^2$.

Proof. By Lemma 6.2.8 the point $\mathfrak{o} = (0, ..., 0)$ of $\mathbb{B}_{\mathcal{O}}$ is smooth of dimension $n\mu$ and by Lemma 6.2.12 the point $(0, ..., 0)$ is a smooth point of $\mathcal{V}$ of dimension $n\mu$. Hence we have $\dim_{\mathrm{K}}(\mathfrak{m}/\mathfrak{m}^2) = \dim_{\mathrm{K}}(\mathfrak{n}/\mathfrak{n}^2) = n\mu$. Moreover, by Lemma 4.2.12 the elements of the set $\widehat{\mathcal{S}} = \{\bar{s}_i + \mathfrak{m}^2 \in \mathrm{B}_{\mathcal{O}} \mid s_i \in \mathcal{S}\}$ form a K-vector space basis of $\mathfrak{m}/\mathfrak{m}^2$. As a result of the canonical surjective K-algebra homomorphism $\mathfrak{h}$, the canonical K-algebra homomorphism

$$\begin{array}{rcl} \epsilon : \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & \mathfrak{n}/\mathfrak{n}^2 \\ \bar{s}_i + \mathfrak{m}^2 & \longmapsto & \tilde{s}_i + \mathfrak{n}^2. \end{array}$$

is surjective. Since both $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{n}/\mathfrak{n}^2$ have the same dimension, the K-algebra homomorphism is an isomorphism. Hence the elements of $\tilde{S}$ are K-linearly independent with dimension $n\mu$. Then they form a K-vector space basis of $\mathfrak{n}/\mathfrak{n}^2$. □

Recall that, as given in Definition 3.1.12, the ring $gr_{\mathfrak{n}}(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}})$ is called the graded ring of $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$.

Lemma 6.2.14. *In the setting 6.2.11, the ring* $gr_{\mathfrak{n}}(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}})$ *is isomorphic to the polynomial ring* $\mathrm{K}[\mathcal{S}]$. *In particular, the system* $\{\tilde{s}_1, ..., \tilde{s}_{n\mu}\}$ *is independent.*

Proof. By Lemma 6.2.13 the set of residue classes of the elements of $\tilde{\mathcal{S}}$ form a basis of the K-vector space $\mathfrak{n}/\mathfrak{n}^2$. Then by Nakayama's Lemma 2.1.11, it follows that $\mathfrak{n}(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}})$ is minimally generated by the elements of $\tilde{\mathcal{S}}$. Since the monomial point $\mathfrak{o}$ is smooth, any minimal system of generators are regular system of parameters of $(\mathrm{K}[\mathcal{V}])_{\mathfrak{n}}$. Hence the elements of $\tilde{\mathcal{S}}$ form an $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$-regular sequence. Then from the implication from a) to

b) of [Ku85] Chapter V, Corollary 5.13 , it follows that the K-algebra homomorphism

$$\begin{array}{rcl} \theta : \mathrm{K}[\mathcal{S}] & \longrightarrow & gr_{\mathfrak{n}}(\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}) \\ s_i & \longmapsto & \tilde{s}_i + \mathfrak{n}^2 \end{array}$$

is an isomorphism.

□

Lemma 6.2.15. *Let φ denote the* K*-algebra homomorphism*

$$\begin{array}{rcl} \varphi : \mathrm{K}[\mathcal{S}] & \longrightarrow & \mathrm{K}[\mathcal{V}] \\ s_i & \longmapsto & \bar{s}_i. \end{array} \tag{6.15}$$

Then the K*-algebra homomorphism φ is injective.*

Proof. Our aim is first to show that the K-algebra homomorphism

$$\begin{array}{rcl} \Gamma : \mathrm{K}[\mathcal{S}] & \longrightarrow & \mathrm{K}[\mathcal{V}]_{\mathfrak{n}} \\ s_i & \longmapsto & \tilde{s}_i. \end{array}$$

is injective. Our strategy is to show that the elements of $\{\tilde{s}_1, ..., \tilde{s}_{n\mu}\}$ are algebraically independent in $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$. Let F be a polynomial in $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}[x_1, ..., x_{n\mu}]$ such that $F(\tilde{s}_1, ..., \tilde{s}_{n\mu}) = 0$. By Lemma 6.2.14 the system of parameters $\{\tilde{s}_1, ..., \tilde{s}_{n\mu}\}$ are independent. As a result of Definition 2.1.16 the set of coefficients of F are contained in $\mathrm{Rad}(\tilde{s}_1, ..., \tilde{s}_{n\mu})$. For $F \in \mathrm{K}[\mathcal{V}]_{\mathfrak{n}}[x_1, ..., x_{n\mu}]$ the set of coefficients of F are contained in $\mathrm{Rad}(\tilde{s}_1, ..., \tilde{s}_{n\mu}) \cap \mathrm{K}[\mathcal{V}]_{\mathfrak{n}} = \mathfrak{n} \cap \mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$. Then the coefficients of F are 0, and the elements of $\{\tilde{s}_1, ..., \tilde{s}_{n\mu}\}$ are algebraically independent. Hence the map Γ is injective.

Now we prove that the set $\mathrm{K}[\mathcal{V}] \setminus \mathfrak{n}$ has no zero divisors. Let $a + \mathcal{J}$ and $b + \mathcal{J}$ be two elements in $\mathrm{K}[\mathcal{V}] \setminus \mathfrak{n}$. The maximal ideal $\mathfrak{n}$ is generated by the elements of $\tilde{\mathcal{C}}$. Then both a and b polynomials with constant terms and ab has a constant term. Recall that the ideal $\mathcal{J}$ is generated by the polynomials $\tau_\kappa \in \mathcal{T}_{\mathcal{S}}$ that have no constants in their support. Hence the polynomial ab is not in $\mathcal{J}$. Thus the set $\mathrm{K}[\mathcal{V}] \setminus \mathfrak{n}$ has no zero divisors. Hence the K-algebra homomorphism from $\mathrm{K}[\mathcal{V}]$ to the localization $\mathrm{K}[\mathcal{V}]_{\mathfrak{n}}$ is injective. Consequently, the K-algebra homomorphism φ is injective. □

Lemma 6.2.16. *In Setting (6.2.11), if the corresponding matrix $\mathcal{M}_{\mathcal{S}} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$ has determinant ± 1, then the coordinate ring of $\mathcal{V}$ is isomorphic to the polynomial ring*

$\mathrm{K}[\mathcal{S}]$. *In particular,* $\mathcal{V}$ *is a smooth ideal-theoretic complete intersection.*

Proof. By Lemma 6.2.15, the K-algebra homomorphism φ given in (6.15) is injective.

We prove our claim by showing that φ is surjective i.e., for each $d_i \in \mathcal{C} \setminus \mathcal{S}$ there exists a polynomial $f_i \in \mathrm{K}[\mathcal{S}]$ such that $\varphi(f_i) = \bar{d_i} \in \mathrm{K}[\mathcal{V}]$. We assumed that the entries of $\mathcal{M}_\mathcal{S}$ are in $\mathrm{K}[\mathcal{S}]$.

$$\mathcal{M}_\mathcal{S} = \begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & \dots & d_m \end{array} \\ \begin{pmatrix} \pm 1 & \star & \star \\ \vdots & \ddots & \vdots \\ \star & \star & \pm 1 \end{pmatrix} \end{array} = \begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \cdots & \frac{\partial \tau_1}{\partial d_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \cdots & \frac{\partial \tau_m}{\partial d_m} \end{pmatrix} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$$

By Lemma 6.2.9, the polynomial $\tau_k \in \mathcal{T}_\mathcal{S}$ can be written as follows.

$$\begin{aligned} \tau_1 &= \frac{\partial \tau_1}{\partial d_2} d_2 \pm \cdots \pm \frac{\partial \tau_1}{\partial d_k} d_k \pm \cdots \pm \frac{\partial \tau_1}{\partial d_m} d_m + g_1(\mathcal{S}) \mp d_1 \\ &\vdots \\ \tau_m &= \frac{\partial \tau_m}{\partial d_1} d_1 \pm \cdots \pm \frac{\partial \tau_m}{\partial d_k} d_k \pm \cdots \pm \frac{\partial \tau_m}{\partial d_{m-1}} d_{m-1} + g_m(\mathcal{S}) \mp d_m. \end{aligned} \tag{6.16}$$

where for each $k \in \{1, ..., m\}$ the polynomial $g_k(\mathcal{S})$ is in $\mathrm{K}[\mathcal{S}]$ and for each $i \in \{1, ..., m\}$ the polynomial $\frac{\partial \tau_k}{\partial d_i}$ is in $\mathrm{K}[\mathcal{S}]$, as well. By keeping Equation (6.16) in mind, we consider the following system of linear equations over the ring $\mathrm{K}[\mathcal{C}]$.

$$\begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \cdots & \frac{\partial \tau_1}{\partial d_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \cdots & \frac{\partial \tau_m}{\partial d_m} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = - \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix}. \tag{6.17}$$

By Cramer's rule for rings, if $(a_1, ..., a_m) \in \mathrm{K}[\mathcal{C}]$ is a solution of (6.17), then for $i = 1, ..., m$ we have

$$\mathrm{Det}(\mathcal{M}_\mathcal{S}) a_i = \mathrm{Det}(\mathcal{M}_i),$$

where $\mathcal{M}_i$ is the matrix $\mathcal{M}_\mathcal{S}$ in which the i^{th} column has been replaced by

$$\begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix}.$$

Since $\mathcal{M}_\mathcal{S} \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$ holds, for each $i \in \{1, ..., m\}$ the matrix $\mathcal{M}_i \in \mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$.

Then $\mathrm{Det}(\mathcal{M}_i)$ is in $\mathrm{K}[\mathcal{S}]$. Since we assumed that $\mathrm{Det}(\mathcal{M}_\mathcal{S})$ is ± 1, for each index $i \in \{1, ..., m\}$, the element a_i is $\pm\mathrm{Det}(\mathcal{M}_i)$. We claim that $f_i = a_i = \pm\mathrm{Det}(\mathcal{M}_i)$ satisfies $\varphi(f_i) = \bar{d}_i$. In the ring $\mathrm{K}[\mathcal{V}]$ the equation system (6.16) becomes

$$\bar{\mathcal{M}}_\mathcal{S} \begin{pmatrix} \bar{d}_1 \\ \vdots \\ \bar{d}_m \end{pmatrix} = - \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix}$$

Under the map φ the equation system (6.17) becomes,

$$\bar{\mathcal{M}}_\mathcal{S} \begin{pmatrix} \varphi(f_1) \\ \vdots \\ \varphi(f_m) \end{pmatrix} = - \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix} = \bar{\mathcal{M}}_\mathcal{S} \begin{pmatrix} \bar{d}_1 \\ \vdots \\ \bar{d}_m \end{pmatrix}.$$

Since the matrix $\mathcal{M}_\mathcal{S}$ has determinant ± 1 it is invertible in $\mathrm{K}[\mathcal{S}]$, we have the following equality.

$$\begin{pmatrix} \varphi(f_1) \\ \vdots \\ \varphi(f_m) \end{pmatrix} = \begin{pmatrix} \bar{d}_1 \\ \vdots \\ \bar{d}_m \end{pmatrix}$$

Then for each $d_i \in \mathcal{C} \setminus \mathcal{S}$ we have a polynomial f_i, $\varphi(f_i) = \bar{d}_i \in \mathrm{B}_\mathcal{O}$. Moreover, by the Cramer's Rule we have for each $d_i \in \mathcal{C} \setminus \mathcal{S}$ have a polynomial $f_i = \mathrm{Det}(\mathcal{M}_i) \in \mathrm{K}[\mathcal{S}]$. This shows that the injective K-algebra homomorphism φ is surjective, as well. □

Remark 6.2.17. The proof of Lemma 6.2.16 does not only introduce a condition that makes the map φ an isomorphism, it also illustrates a method to find the pre-images of the elements of the set $\mathcal{C} \setminus \mathcal{S}$ concretely.

Let us apply the preceding result to an example.

Example 6.2.18. Let us recall Example 4.1.8, where the order ideal is $\mathcal{O} = \{1, x_1, x_2, x_1x_2\}$.

In Example 3.2.12 we show that $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by the following 12 polynomials.

$$\begin{aligned}
\tau_{\mathbf{22}} &= c_{21}c_{32} + c_{23}c_{42} - c_{24} & \tau_{\mathbf{33}} &= c_{21}c_{32} + c_{34}c_{41} - c_{33}\\
\tau_{\mathbf{44}} &= c_{34}c_{41} - c_{23}c_{42} - c_{24} + c_{33} & \tau_{\mathbf{32}} &= c_{31}c_{32} + c_{33}c_{42} + c_{12} - c_{34}\\
\tau_{\mathbf{12}} &= c_{11}c_{32} + c_{13}c_{42} - c_{14} & \tau_{\mathbf{34}} &= c_{23}c_{32} - c_{31}c_{34} - c_{34}c_{43} + c_{33}c_{44} - c_{14}\\
\tau_{\mathbf{13}} &= c_{12}c_{21} + c_{14}c_{41} - c_{13} & \tau_{\mathbf{24}} &= c_{22}c_{23} - c_{21}c_{34} - c_{24}c_{43} + c_{23}c_{44} + c_{13}\\
\tau_{\mathbf{23}} &= c_{21}c_{22} + c_{24}c_{41} + c_{11} - c_{23} & \tau_{\mathbf{42}} &= c_{32}c_{41} + c_{42}c_{43} + c_{22} - c_{44}\\
\tau_{\mathbf{43}} &= c_{21}c_{42} + c_{41}c_{44} + c_{31} - c_{43} & \tau_{\mathbf{14}} &= c_{12}c_{23} - c_{11}c_{34} - c_{14}c_{43} + c_{13}c_{44}
\end{aligned}$$

In Example 4.1.8, we have computed the set $\mathfrak{E} = \{[\bar{c}_{11}], [\bar{c}_{21}], [\bar{c}_{31}], [\bar{c}_{41}], [\bar{c}_{12}], [\bar{c}_{22}], [\bar{c}_{32}], [\bar{c}_{42}]\}$, where $[\bar{c}_{31}] = \{c_{31}, c_{43}\}$, $[\bar{c}_{11}] = \{c_{11}, c_{23}\}$, $[\bar{c}_{22}] = \{c_{22}, c_{44}\}$, $[\bar{c}_{12}] = \{c_{12}, c_{34}\}$. By choosing one element from each equivalence class, we get a set $\mathcal{S}$ (see Lemma 4.2.12). Let us construct this set as $\mathcal{S} = \{c_{21}, c_{23}, c_{32}, c_{34}, c_{41}, c_{42}, c_{43}, c_{44}\}$. Let M be the set $\mathcal{C} \setminus \mathcal{S}$. Then the set $\mathcal{C} \setminus \mathcal{S}$ is

$$\{c_{33}, c_{24}, c_{22}, c_{31}, c_{12}, c_{11}, c_{13}, c_{14}\}.$$

Note that by Proposition 4.1.6a), no matter which elements from an equivalence class of $\mathfrak{E}$ we choose, the elements $c_{14}, c_{13}, c_{24}, c_{33}$ in $\mathcal{C} \setminus \mathcal{S}$ cannot be replaced. As a result of $\mathcal{O} \subset \mathbb{T}^2$, Corollary 4.2.17 implies that the monomial point $\mathfrak{o}$ is the smooth point of the $\mathcal{O}$-border basis scheme where the dimension of the regular local ring $(\mathrm{B}_{\mathcal{O}})_{\mathfrak{m}}$ is $2\mu = 8$. Then as given in (6.15), there exists an injective homomorphism

$$\begin{aligned}
\varphi : \mathrm{K}[\mathcal{S}] &\longrightarrow \mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})\\
s_i &\longmapsto \bar{s}_i.
\end{aligned}$$

We show that the map φ is surjective, as well. By Proposition 5.1.23, the scheme $\mathbb{B}_{\mathcal{O}}$ is an ideal-theoretic complete intersection where the vanishing ideal of it is generated by a subset $\mathcal{T}_{\mathcal{S}}$ of $\mathcal{T}$. We compute the generating set $\mathcal{T}_{\mathcal{S}} = \{\tau_{22}, \tau_{33}, \tau_{42}, \tau_{43}, \tau_{23}, \tau_{24}, \tau_{32}, \tau_{34}\}$ by using our ApCoCoA package bbsmingensyz. For further information on how to do the computations we refer to Chapter 5.

$$\begin{aligned}
\varphi^{-1}(\bar{c}_{33}) &= \overline{\tau_{33} + c_{33}} = \overline{c_{21}c_{32} + c_{34}c_{41}} = g_{33}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{24}) &= \overline{\tau_{22} + c_{24}} = \overline{c_{21}c_{32} + c_{23}c_{42}} = g_{24}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{22}) &= \overline{\tau_{42} - c_{22}} = \overline{c_{32}c_{41} + c_{42}c_{43} - c_{44}} = g_{22}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},
\end{aligned}$$

$$
\begin{aligned}
\varphi^{-1}(\bar{c}_{22}) &= \overline{\tau_{42} - c_{22}} = \overline{c_{32}c_{41} + c_{42}c_{43} - c_{44}} = g_{22}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{31}) &= \overline{\tau_{43} - c_{31}} = \overline{c_{21}c_{42} + c_{41}c_{44} - c_{43}} = g_{31}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{12}) &= \varphi^{-1}(\bar{c}_{31})\bar{c}_{32} + \varphi^{-1}(\bar{c}_{33})\bar{c}_{42} - \bar{c}_{34} = \varphi^{-1}(\bar{c}_{31})\bar{c}_{32} + \varphi^{-1}(\bar{c}_{33})\bar{c}_{42} - g_{12}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{11}) &= \overline{c_{21}c_{22}} + \varphi^{-1}(\bar{c}_{24})\bar{c}_{41} - \bar{c}_{23} = \varphi^{-1}(\bar{c}_{24})\bar{c}_{41} + g_{11}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{13}) &= \overline{c_{22}c_{23} - c_{21}c_{34}} - \varphi^{-1}(\bar{c}_{24})c_{43} + \overline{c_{23}c_{44}} = -\varphi^{-1}(\bar{c}_{24})c_{43} + g_{13}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}},\\
\varphi^{-1}(\bar{c}_{14}) &= c_{23}c_{32} - \varphi^{-1}(\bar{c}_{31})c_{34} - c_{34}c_{43} + c_{33}c_{44} = -\varphi^{-1}(\bar{c}_{31})c_{34} + g_{14}(\mathcal{S}) \in \mathrm{B}_{\mathcal{O}}.
\end{aligned}
$$

For each $c_{ij} \in M$ if we let $f_{ij} := \varphi^{-1}(\bar{c}_{ij})$, then we have

$$
\begin{aligned}
f_{24} &:= \tau_{22} + c_{24} = c_{21}c_{32} + c_{23}c_{42},\\
f_{11} &:= \frac{\partial \tau_{23}}{\partial c_{24}} f_{24} + \frac{\partial \tau_{23}}{\partial c_{23}} f_{22} - c_{23},\\
f_{13} &:= -\frac{\partial \tau_{24}}{\partial c_{24}} f_{24} + \frac{\partial \tau_{24}}{\partial c_{22}} f_{22} - c_{21}c_{34} + c_{23}c_{44},\\
f_{12} &:= \frac{\partial \tau_{32}}{\partial c_{31}} f_{31} + \frac{\partial \tau_{32}}{\partial c_{33}} f_{33} - c_{34},\\
f_{33} &:= \tau_{33} + c_{33} = c_{21}c_{32} + c_{34}c_{41},\\
f_{14} &:= -\frac{\partial \tau_{34}}{\partial c_{31}} f_{31} + \frac{\partial \tau_{34}}{\partial c_{33}} f_{33} + c_{23}c_{32} - c_{34}c_{43},\\
f_{22} &:= \tau_{42} - c_{22} = c_{32}c_{41} + c_{42}c_{43} - c_{44},\\
f_{31} &:= \tau_{43} - c_{31} = c_{21}c_{42} + c_{41}c_{44} - c_{43}.
\end{aligned}
$$

Let us recall the following equality

$$
\mathcal{M}_{\mathcal{S}} = \begin{pmatrix} \frac{\partial \tau_{22}}{\partial c_{24}} & \frac{\partial \tau_{22}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{22}}{\partial c_{31}} \\ \vdots & \ddots & \vdots & \\ \frac{\partial \tau_{43}}{\partial c_{24}} & \frac{\partial \tau_{43}}{\partial c_{11}} & \cdots & \frac{\partial \tau_{43}}{\partial c_{31}} \end{pmatrix} \times \begin{pmatrix} f_{24} \\ f_{11} \\ \vdots \\ f_{31} \end{pmatrix} = \begin{pmatrix} g_{24}(\mathcal{S}) \\ g_{11}(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix},
$$

which in this example leads us to the following equality.

$$
\mathcal{M}_{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{41} & 1 & 0 & 0 & 0 & 0 & c_{21} & 0 \\ c_{43} & 0 & 1 & 0 & 0 & 0 & c_{23} & 0 \\ 0 & 0 & 0 & 1 & c_{42} & 0 & 0 & c_{32} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_{44} & -1 & 0 & -c_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} f_{24} \\ f_{11} \\ f_{13} \\ f_{12} \\ f_{33} \\ f_{14} \\ f_{22} \\ f_{31} \end{pmatrix} = \begin{pmatrix} c_{21}c_{32} + c_{23}c_{42} \\ c_{23} \\ -c_{21}c_{34} + c_{23}c_{44} \\ c_{34} \\ c_{21}c_{32} + c_{34}c_{41} \\ c_{23}c_{32} - c_{34}c_{43} \\ c_{32}c_{41} + c_{42}c_{43} - c_{44} \\ c_{21}c_{42} + c_{41}c_{44} - c_{43} \end{pmatrix}.
$$

The matrix $\mathcal{M}_{\mathcal{S}}$ is in $\mathrm{Mat}_8(\mathrm{K}[\mathcal{S}])$ and the determinant of it is 1. Then $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space $\mathbb{A}^8$. Let us find explicitly the pre-images of the elements $\bar{c}_{ij} \in \mathrm{B}_{\mathcal{O}}$,

where $c_{ij} \in M$. Let us start with c_{24}.

$$\mathcal{M}_{24} = \begin{pmatrix} c_{21}c_{32} + c_{23}c_{42} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{23} & 1 & 0 & 0 & 0 & 0 & c_{21} & 0 \\ c_{21}c_{34} + c_{23}c_{44} & 0 & 1 & 0 & 0 & 0 & c_{23} & 0 \\ c_{34} & 0 & 0 & 1 & c_{42} & 0 & 0 & c_{32} \\ c_{21}c_{32} + c_{34}c_{41} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ c_{23}c_{32} - c_{34}c_{43} & 0 & 0 & 0 & -c_{44} & -1 & 0 & -c_{34} \\ c_{32}c_{41} + c_{42}c_{43} - c_{44} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ c_{21}c_{42} + c_{41}c_{44} - c_{43} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $f_{24} = \mathrm{Det}(M_{24}) = c_{21}c_{32} + c_{23}c_{42} \in \mathrm{K}[\mathcal{S}]$. By using the same method we have

$$\begin{aligned} f_{11} &= \mathrm{Det}(M_{11}) = -c_{21}c_{32}c_{41} - c_{23}c_{41}c_{42} \in \mathrm{K}[\mathcal{S}] \\ f_{13} &= \mathrm{Det}(M_{13}) = c_{23}c_{32}c_{41} - c_{21}c_{32}c_{43} + c_{21}c_{34} - 2c_{23}c_{44} \in \mathrm{K}[\mathcal{S}] \\ f_{12} &= \mathrm{Det}(M_{12}) = -c_{34}c_{41}c_{42} + c_{32}c_{41}c_{44} - c_{32}c_{43} - c_{34} \in \mathrm{K}[\mathcal{S}] \\ f_{33} &= \mathrm{Det}(M_{33}) = c_{21}c_{32} + c_{34}c_{41} \in \mathrm{K}[\mathcal{S}] \\ f_{14} &= \mathrm{Det}(M_{14}) = c_{21}c_{34}c_{42} - c_{21}c_{32}c_{44} + c_{23}c_{32} - 2c_{34}c_{43} \in \mathrm{K}[\mathcal{S}] \\ f_{22} &= \mathrm{Det}(M_{22}) = c_{32}c_{41} + c_{42}c_{43} - c_{44} \in \mathrm{K}[\mathcal{S}] \\ f_{31} &= \mathrm{Det}(M_{31}) = c_{21}c_{42} + c_{41}c_{44} - c_{43}. \end{aligned}$$

The image of f_{33} under the map φ is as follows.

$$\varphi(f_{33}) = \overline{c_{21}c_{32} \pm c_{34}c_{41}} = \bar{c}_{21}\bar{c}_{32} \pm \bar{c}_{34}\bar{c}_{41} = \bar{c}_{33}$$

In the same way, we have $\varphi(f_{24}) = \bar{c}_{24}, \varphi(f_{22}) = \bar{c}_{22}, \varphi(f_{31}) = \bar{c}_{31}, \varphi(f_{12}) = \bar{c}_{12}, \varphi(f_{11}) = \bar{c}_{11}, \varphi(f_{13}) = \bar{c}_{13}$, and finally $\varphi(f_{24}) = \bar{c}_{24}$.

Recall how we constructed the Matrix (6.7) where $\{d_1, ..., d_m\} = \mathcal{C} \setminus \mathcal{S}$ and $\mathcal{T}_{\mathcal{S}} = \{\tau_1, ..., \tau_m\}$, which is a subset of $\mathcal{T}$ that consists of the polynomials, whose supports contain d_i. Our goal is to get rid of the condition that the vanishing ideal of a given $\mathcal{O}$-border basis scheme must be generated by the set $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}$. In order to reach this goal let us recall some notation and introduce new ones.

In Chapter 2, Proposition 2.1.19 states that if $I \subset \mathrm{K}[x_1, ..., x_n]$ is a prime ideal and K is a perfect field, then the Jacobian ideal defines the singular locus of $\mathcal{Z}(I)$. This is

a crucial part of the proof of our next claim. Before stating the lemma let us recall Definition 2.4.4, i.e. the principal component of $\mathbb{B}_{\mathcal{O}}$ is an irreducible component of $\mathbb{B}_{\mathcal{O}}$ whose vanishing ideal is $\mathrm{Ker}(\Phi)$ in 2.8. The principal component is denoted by $\mathbb{C}_{\mathcal{O}}$.

Lemma 6.2.19. *Let $\mathcal{O}$ denote an order ideal with μ terms from $\mathrm{K}[x_1, ..., x_n]$. If there exists a set $\bar{\mathcal{S}} \subset \bar{\mathcal{C}}$ with $n\mu$ elements such that the entries of the corresponding matrix $\mathcal{M}_{\mathcal{S}}$ are in the polynomial ring $\mathrm{K}[\mathcal{S}]$ and the determinant of this matrix is ± 1, then $\mathbb{C}_{\mathcal{O}}$ is smooth of dimension $n\mu$.*

Proof. Recall from Chapter 2.5 that the principal component $\mathbb{C}_{\mathcal{O}}$ is irreducible and contained in $\mathbb{B}_{\mathcal{O}}$. Let $\mathcal{J}$ denote the ideal that is generated by a set $\mathcal{T}_{\mathcal{S}}$. Then we have $\mathcal{I}(\mathbb{C}_{\mathcal{O}}) \supseteq \mathcal{I}(\mathbb{B}_{\mathcal{O}}) \supseteq \mathcal{J}$ where $\mathcal{J}$ is generated by $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}$. Thus the matrix $\mathcal{M}_{\mathcal{S}}$ in Notation (6.2.11) is a submatrix of the Jacobian matrix of the generators of $\mathcal{I}(\mathbb{C}_{\mathcal{O}})$.

By Theorem 2.4.5, the principal component has the dimension $n\mu$ at its smooth points. Then the submatrices of the Jacobian matrix that have the rank $\mu\nu - n\mu$ give the maximal minors of it, which generate the Jacobian ideal. Since $\mathcal{M}_{\mathcal{S}}$ is a submatrix of the Jacobian matrix of the generators of $\mathcal{I}(\mathbb{C}_{\mathcal{O}})$ with rank $\mu\nu - n\mu$, the $\mathrm{Det}(\mathcal{M}_{\mathcal{S}}) = \pm 1$ is one of the maximal minors of the Jacobian matrix. Then the Jacobian ideal is a unit ideal. The ideal $\mathcal{I}(\mathbb{C}_{\mathcal{O}})$ is a prime ideal and K is a perfect field. Then by Theorem 2.1.19 the Jacobian ideal defines the singular locus of $\mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{C}_{\mathcal{O}})$. As a result of the Corollary 2.1.21, $\mathbb{C}_{\mathcal{O}}$ is smooth of dimension $n\mu$.

□

Theorem 6.2.20. *Let $\mathcal{O}$ denote an order ideal with μ terms from $\mathrm{K}[x_1, ..., x_n]$. If there exists a set $\bar{\mathcal{S}} \subset \bar{\mathcal{C}}$ with $n\mu$ elements such that the entries of the corresponding matrix $\mathcal{M}_{\mathcal{S}}$ are in the polynomial ring $\mathrm{K}[\mathcal{S}]$ and the determinant of this matrix is ± 1, then $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space of dimension $n\mu$.*

Proof. W e have $\mathcal{I}(\mathbb{C}_{\mathcal{O}}) \supseteq \mathcal{I}(\mathbb{B}_{\mathcal{O}}) \supseteq \mathcal{J}$ where $\mathcal{J}$ is generated by $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}$. Then we have

$$\mathcal{V} \supseteq \mathbb{B}_{\mathcal{O}} \supseteq \mathbb{C}_{\mathcal{O}}. \tag{6.18}$$

By Lemma 6.2.16, the zero set $\mathcal{V}$ is an affine space of dimension $n\mu$ and it contains the principal component $\mathbb{C}_{\mathcal{O}}$. By Lemma 6.2.19 the principal component is smooth of dimension $n\mu$. From those results follows that $\mathbb{C}_{\mathcal{O}}$ is an affine space of $n\mu$. Since we have (6.18), the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ is an affine space of dimension $n\mu$, as well. □

Remark 6.2.21. Note that for a given $\mathcal{O}$-border basis scheme Theorem 6.2.20 implies

that if there exists a maximal minor of $\mathrm{Jac}(\mathcal{T})$ that is ± 1 and the corresponding submatrix is in $\mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$, then the $\mathcal{O}$-border basis scheme is isomorphic to an affine space. It is also worth mentioning that it is not necessary to check all the minors of the Jacobian matrix, since it is enough to construct the set $\mathcal{S}$ and pick polynomials $\tau_i \in \mathcal{T}$, for each $d_i \in M$ with the property that $\frac{\partial \tau_i}{\partial d_i} = \pm 1$. This way of constructing the candidate minors of the Jacobian matrix of $\mathcal{T}$ is more efficient than blindly checking all the minors of it.

Remark 6.2.22. If an $\mathcal{O}$-border basis scheme satisfies Theorem 6.2.20, then it is an ideal theoretic complete intersection where the vanishing ideal is generated by $\mathcal{T}_{\mathcal{S}}$. In general, affine spaces of dimension smaller than 5 need not be an ideal theoretic complete intersection (see [Kum77]).

Corollary 6.2.23. *Let $\mathcal{O}$ denote the order ideal $\{1, x_1, x_2\} \subset \mathrm{K}[x_1, x_2, x_3]$. Then the $\mathcal{O}$-border basis scheme is isomorphic to an affine space $\mathbb{A}^9$.*

Proof. The border of $\mathcal{O}$ is $\partial\mathcal{O} = \{x_3, x_2x_3, x_1x_3, x_2^2, x_1x_2, x_1^2\}$. The generators of the vanishing ideal of the $\mathcal{O}$-border basis scheme are as follows.

$$
\begin{aligned}
& \text{Polynomials with arrow degree } (2,1,0) \\
\tau_{12}^{12} &= c_{16}c_{25} - c_{15}c_{26} + c_{15}c_{35} - c_{14}c_{36} \\
& \text{Polynomials with arrow degree } (1,2,0) \\
\tau_{13}^{12} &= c_{16}c_{24} - c_{15}c_{25} + c_{15}c_{34} - c_{14}c_{35} \\
& \text{Polynomials with arrow degree } (1,1,0) \\
\tau_{22}^{12} &= c_{25}c_{35} - c_{24}c_{36} + c_{15} \\
\tau_{33}^{12} &= -c_{25}c_{35} + c_{24}c_{36} - c_{15} \\
& \text{Polynomials with arrow degree } (0,2,0) \\
\tau_{23}^{12} &= -c_{25}^2 + c_{24}c_{26} + c_{25}c_{34} - c_{24}c_{35} + c_{14} \\
& \text{Polynomials with arrow degree } (2,0,1) \\
\tau_{12}^{13} &= -c_{11}c_{16} + c_{16}c_{23} - c_{13}c_{26} + c_{15}c_{33} - c_{12}c_{36}, \\
& \text{Polynomials with arrow degree } (0,2,1) \\
\tau_{13}^{23} &= -c_{11}c_{14} + c_{15}c_{22} - c_{13}c_{24} + c_{14}c_{32} - c_{12}c_{34} \\
& \text{Polynomials with arrow degree } (1,-1,1) \\
\tau_{31}^{13} &= c_{31}c_{35} + c_{21}c_{36} - c_{33}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Polynomials with arrow degree } (2,0,0)\\
\tau_{32}^{12} &= -c_{26}c_{35} + c_{35}^2 + c_{25}c_{36} - c_{34}c_{36} - c_{16}\\
&\text{Polynomials with arrow degree } (1,0,1)\\
\tau_{11}^{13} &= c_{16}c_{21} + c_{15}c_{31} - c_{13}\\
\tau_{22}^{13} &= -c_{16}c_{21} + c_{25}c_{33} - c_{22}c_{36} + c_{13}\\
\tau_{33}^{13} &= -c_{15}c_{31} - c_{25}c_{33} + c_{22}c_{36}\\
\tau_{32}^{23} &= -c_{15}c_{31} - c_{25}c_{33} + c_{33}c_{34} + c_{23}c_{35} - c_{32}c_{35} + c_{13}\\
&\text{Polynomials with arrow degree } (1,1,1)\\
\tau_{13}^{13} &= -c_{11}c_{15} + c_{16}c_{22} - c_{13}c_{25} + c_{15}c_{32} - c_{12}c_{35}\\
\tau_{12}^{23} &= -c_{11}c_{15} + c_{15}c_{23} - c_{13}c_{25} + c_{14}c_{33} - c_{12}c_{35}\\
&\text{Polynomials with arrow degree } (0,0,1)\\
\tau_{21}^{13} &= c_{21}c_{26} + c_{25}c_{31} + c_{11} - c_{23}\\
\tau_{31}^{23} &= c_{31}c_{34} + c_{21}c_{35} + c_{11} - c_{32}\\
&\text{Polynomials with arrow degree } (0,1,1)\\
\tau_{23}^{13} &= -c_{15}c_{21} - c_{23}c_{25} + c_{22}c_{26} + c_{25}c_{32} - c_{22}c_{35} + c_{12}\\
\tau_{11}^{23} &= c_{15}c_{21} + c_{14}c_{31} - c_{12}\\
\tau_{22}^{23} &= -c_{15}c_{21} + c_{24}c_{33} - c_{22}c_{35}\\
\tau_{33}^{23} &= -c_{14}c_{31} - c_{24}c_{33} + c_{22}c_{35} + c_{12}\\
&\text{Polynomials with arrow degree } (2,-1,1)\\
\tau_{32}^{13} &= -c_{16}c_{31} - c_{26}c_{33} + c_{33}c_{35} + c_{23}c_{36} - c_{32}c_{36}\\
&\text{Polynomials with arrow degree } (-1,1,1)\\
\tau_{21}^{23} &= c_{21}c_{25} + c_{24}c_{31} - c_{22}\\
&\text{Polynomials with arrow degree}(-1,2,1)\\
\tau_{23}^{23} &= -c_{14}c_{21} - c_{23}c_{24} + c_{22}c_{25} + c_{24}c_{32} - c_{22}c_{34}
\end{aligned}
$$

As we did in Chapter 4.1, by the help of Proposition 4.1.6, we compute the set of equivalence classes by using the polynomials in $\mathcal{T}$.

$$\mathfrak{E} = \{[\bar{c}_{23}], [\bar{c}_{24}], [\bar{c}_{25}], [\bar{c}_{26}], [\bar{c}_{31}], [\bar{c}_{21}], [\bar{c}_{34}], [\bar{c}_{35}], [\bar{c}_{36}]\}$$

The cardinality of $\mathfrak{E}$ is 9 which is $n\mu$. Then by Theorem 4.2.15, $\mathfrak{o}$ is a smooth point of $\mathbb{B}_{\mathcal{O}}$. Since $\mathfrak{o}$ is the smooth point of the $\mathcal{O}$-border basis scheme, the dimension of

the regular local ring $(\mathrm{B}_{\mathcal{O}})_{\mathfrak{m}}$ is $3\mu = 9$. By observing the equivalence classes in $\mathfrak{E}$, we deduce that except for

$$[\bar{c}_{23}] = \{\bar{c}_{23}, \bar{c}_{11}, \bar{c}_{32}\},$$

all the other equivalence classes contain exactly one element. Then there are exactly 3 different bases of the K-vector space $\mathfrak{m}/\mathfrak{m}^2$. Then there are three different possibilities for the set $\mathcal{S}$. Let us choose the elements of $\mathcal{S}$ as follows.

$$\mathcal{S} = \{c_{23}, c_{24}, c_{25}, c_{26}, c_{31}, c_{21}, c_{34}, c_{35}, c_{36}\}$$

The set $\mathcal{T}_{\mathcal{S}}$ is not unique, either. Since for each element $d_i \in M$ a polynomial $\tau_i \in \mathcal{T}_{\mathcal{S}}$ satisfies

$$\frac{\partial \tau_i}{\partial d_i} = \pm 1, \tag{6.19}$$

the polynomials that are homogenous of standard degree 2 are not in $\mathcal{T}_{\mathcal{S}}$. Moreover, the polynomials $\tau_{21}^{23}, \tau_{31}^{13}, \tau_{32}^{12}, \tau_{23}^{12}, \tau_{31}^{23}, \tau_{21}^{13}$ are the only polynomials that satisfy Equality (6.19) for the elements $c_{22}, c_{33}, c_{11}, c_{32}, c_{16}, c_{14}$ of M and hence they must be $\mathcal{T}_{\mathcal{S}}$. The rest of the elements of $\mathcal{T}_{\mathcal{S}}$ depend on our choice. For example, for the indeterminate c_{15} both of the polynomials $\tau_{22}^{12}, \tau_{33}^{12}$ satisfy Equality 6.19 and we choose τ_{22}^{12}. By choosing exactly one polynomial for each remaining element of M we get

$$\mathcal{T}_{\mathcal{S}} = \{\tau_{22}^{12}, \tau_{23}^{12}, \tau_{32}^{12}, \tau_{32}^{23}, \tau_{21}^{13}, \tau_{31}^{23}, \tau_{33}^{23}, \tau_{31}^{13}, \tau_{21}^{23}\}.$$

Keep in mind that if by using this choice of $\mathcal{T}_{\mathcal{S}}$ we get a non-constant polynomial as determinant, then we check other possibilities. Now we have all we need to construct the matrix $\mathcal{M}_{\mathcal{S}}$.

$$\mathcal{M}_{\mathcal{S}} = \begin{array}{c} \\ \tau_{22}^{12} \\ \tau_{23}^{12} \\ \tau_{32}^{12} \\ \tau_{32}^{23} \\ \tau_{21}^{13} \\ \tau_{31}^{23} \\ \tau_{33}^{23} \\ \tau_{31}^{13} \\ \tau_{21}^{23} \end{array} \begin{array}{c} \begin{array}{ccccccccc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{22} & c_{32} & c_{33} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -c_{31} & 0 & 0 & -c_{35} & -c_{25}+c_{34} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \\ 0 & 1 & 0 & -c_{31} & 0 & 0 & c_{35} & 0 & -c_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \end{array} \in \mathrm{Mat}_9(\mathrm{K}[\mathcal{S}]).$$

We observe that entries of the matrix $\mathcal{M}_{\mathcal{S}}$ are in $\mathrm{K}[\mathcal{S}]$, and the determinant of this matrix is -1. By Theorem 6.2.20, $\mathbb{B}_{\mathcal{O}}$ is isomorphic to the affine space $\mathbb{A}^9$. □

Lemma 6.2.24. *Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let $\mathrm{B}_{\mathcal{O}}$ be the coordinate ring of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$. Let $\mathbb{B}_{\mathcal{O}}$ be a smooth scheme. If the vanishing ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is generated by $\mathcal{T}_{\mathcal{S}}$, then the elements of $\mathcal{T}_{\mathcal{S}}$ forms an $\mathrm{K}[\mathcal{C}]$-regular sequence.*

Proof. The $\mathcal{O}$-border basis scheme is smooth. By Theorem 2.4.5 the dimension of the Principal component at smooth points is $n\mu$. Hence the dimension of $\mathcal{O}$-border basis scheme has dimension $n\mu$. The set $\mathcal{T}_{\mathcal{S}}$ has $\mu\nu - n\mu$ elements, where $\mu\nu - n\mu$ is the codimension. Then the border basis scheme is an ideal theoretic complete intersection. Thus the elements of $\mathcal{T}_{\mathcal{S}}$ form a regular sequence. □

For further research we strongly believe that Theorem 6.2.20 can be generalized to give the following criterion.

Conjecture 6.2.25. *Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let K be a perfect field. The border basis scheme $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space, if there exists a spanning set $\bar{\mathcal{S}}$ of $\mathfrak{m}/\mathfrak{m}^2$ such that $\mathrm{Det}(\mathcal{M}_{\mathcal{S}}) = \pm 1$ holds.*

Algorithm 6.2.26. Let $\mathcal{O} \subset \mathrm{K}[x_1, ..., x_n]$ be an order ideal. Let $\mathfrak{E}$ represent the set of equivalence classes modulo $\mathfrak{m}^2$. Let $\mathcal{S}_i$ represents a set, where each element is chosen from different equivalence classes of $\mathfrak{E}$. Let $\mathcal{M}_i$ denote the set $\mathcal{C} \setminus \mathcal{S}_i$. Let $\mathcal{T}_{\mathcal{S}}$ denote the set of polynomials that for each $d_k \in \mathcal{M}_i$ there exists exactly one τ_k in $\mathcal{T}_{\mathcal{S}}$

Step 1) If the cardinality of $\mathfrak{E}$ is larger than $n\mu$, return "the $\mathcal{O}$-border basis scheme is not isomorphic to an affine space". Else proceed with the next step.

Step 2) Compute $\mathfrak{n}_{\mathcal{S}}$ as in (6.4). Let $E := \{[\bar{c}_{ij}] \mid$ the number of elements of $[\bar{c}_{ij}]$ is more than $1\}$. Let $\mathcal{S}_0 := \{c_{ij} \in \mathcal{C} \mid [\bar{c}_{ij}] \in E_0\}$. We set $i := 0$.

Step 3) $i := i + 1$ and construct the following set

$$\mathcal{S}_i = S_{i-1} \cup \{\text{choose exactly one } c_{ij} \text{ from each of the equivalence classes } [\bar{c}_{ij}] \in E\},$$

where for $j < i$, $\mathcal{S}_i \neq \mathcal{S}_j$ holds.

Step 4) Construct $M_i := \mathcal{C} \setminus \mathcal{S}_i$ and the matrix $\mathcal{M}_{\mathcal{S}i}$ and compute the determinant of $\mathcal{M}_{\mathcal{S}i}$.

Step 5) If the determinant of $\mathcal{M}_{\mathcal{S}i}$ is ± 1 and $\mathcal{M}_{\mathcal{S}i} \in \mathrm{K}[\mathcal{S}_i]$, then return "$\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space of dimension $n\mu$ and it is isomorphic to $\mathrm{K}[\mathcal{S}_i]$ ". If the determinant of $\mathcal{M}_{\mathcal{S}i}$ is a non-constant polynomial and $i \leqslant \mathfrak{n}_{\mathcal{S}}$, then return to step 3) and construct a different $\mathcal{S}_i$. Otherwise, the algorithm is inconclusive.

Step 6) If the determinant of $\mathcal{M}_{\mathcal{S}i}$ is ± 1 and $i \leqslant \mathfrak{n}_{\mathcal{S}}$ then return to step 3) and construct a different $\mathcal{S}_i$. If the index i is equal to the number of all possible sets $\mathcal{S}$, $\mathfrak{n}_{\mathcal{S}}$, then the algorithm is inconclusive.

Under the above assumptions, this algorithm finds out whether $\mathcal{O}$-border basis scheme is isomorphic to an affine space.

Proof. This follows from Theorems 4.2.15 and 6.2.20. □

For further research, our calculations using Algorithm 6.2.26 which is derived from Theorem 1.3.2 and Theorem **??** also suggest the following result.

Conjecture 6.2.27. *Let $\mathcal{O} = \{t_1, ..., t_\mu\} \subset \mathbb{T}^2$ be an order ideal. Let $\mathcal{O}' = \{t_1, ..., t_\mu\} \subset \mathbb{T}^n$ be the same order ideal whose border is in $\mathbb{T}^n$. If the $\mathcal{O}$-border basis scheme is isomorphic to an affine space of dimension 2μ, then the $\mathcal{O}'$-border basis scheme is isomorphic to an affine space of dimension $n\mu$.*

6.3 Segment Border Basis Scheme is an Affine Space

We start Section 6.3 by proving that the monomial point of a segment border basis scheme is smooth of dimension $n\mu$. (see Proposition 6.3.4). Then by using Theorem 6.2.20, we prove that a segment border basis is isomorphic to an affine space of dimension $n\mu$ (see Proposition 6.3.6). This result was also shown in [Ro09],Corollary 3.13 by a different method. In this section we verify this with our method.

Let $\mathcal{C}$ denote a set of indeterminates $\{c_{11}, ..., c_{\mu\nu}\}$. Let $\mathrm{K}[\mathcal{C}]$ be the polynomial ring $\mathrm{K}[c_{11}, ..., c_{\mu\nu}]$. Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, ..., x_1^{\mu-1}\}$. Then the border of the order ideal $\mathcal{O}$ is $\partial\mathcal{O} = \{x_n, ..., x_2, x_n, b_2x_1, ..., x_1^{\mu-1}x_2, x_1^{\mu}\}$ and with respect to our notation we have $t_1 = 1, t_2 = x_1, ..., t_\mu = x_1^{\mu-1}, b_1 = x_n, ..., b_{n-2} = x_2, b_{n-1} = x_n, b_n = b_2x_1, ..., b_{\nu-1} = x_1^{\mu-1}x_2$ and $b_\nu = x_1^{\mu}$. By Corollary 5.3.15, the vanishing ideal of a segment border basis scheme $\mathbb{B}_{\mathcal{O}}$ is generated by the set $\{\tau_{pq}^{1l} \mid p = 1, ..., \mu\, q = 1, ..., \mu - 1\, l \in \{2, ..., n\}\}$. We shall name this set as $\mathcal{T}_{seg}$. Let $\mathfrak{o}$ denote the point $(0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$. We let $\mathfrak{m}$ denote the maximal ideal of $\mathrm{B}_{\mathcal{O}}$ that is generated by the elements of $\bar{\mathcal{C}} = \{\bar{c_{ij}} \in \mathrm{B}_{\mathcal{O}} \mid c_{ij} \in \mathcal{C}\}$. Moreover, for the sake of simplicity we assume

that $(\mathrm{B}_{\mathcal{O}})_{\mathfrak{m}}/\mathfrak{m}$ is isomorphic to the field K. Then the space $\mathfrak{m}/\mathfrak{m}^2$ is an K-vector space. Let $\mathfrak{E}$ denote the set of the equivalence classes modulo $\mathfrak{m}^2$. Let us take a closer look at the elements of $\mathcal{T}_{Seg}$.

Lemma 6.3.1. *The quadratic monomials in the support of the polynomial $\tau_{pq}^{1l} \in \mathcal{T}_{seg}$ are of the form $c_{p\nu}c_{\mu j}$, where $1 \leqslant j < \nu$, $1 < p < \mu$ and $1 < q < \mu - 1$.*

Proof. Let τ_{pq}^{1l} be a polynomial from $\mathcal{T}_{seg}$. Let t_q be an order ideal term such that $x_l t_q = b_m \in \partial\mathcal{O}$ and $x_l t_1 = b_{(n+1)-l} = b_k$. The polynomial τ_{pq}^{1l} is the entry in position (p, q) of the matrix

$$\mathcal{A}_1\mathcal{A}_l - \mathcal{A}_l\mathcal{A}_1.$$

By Lemma 5.3.12, for $i > 1$ the generic multiplication matrix $\mathcal{A}_i$ has the following form

$$\mathcal{A}_1 = (e_2^{tr} \mid ... \mid e_\mu^{tr} \mid c_\nu^{tr}) \tag{6.20}$$

For the indices $l > 1$ the generic multiplication matrix $\mathcal{A}_l$ has the following form

$$\mathcal{A}_l = (c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid ... \mid c_{k+(\mu-1)(n-1)}^{tr}) \tag{6.21}$$

where the index $k = (n+1) - l$. Then by using the generic multiplication matrices above, we have

$$\begin{aligned} \tau_{pq}^{1l} &= e_p[(e_2^{tr} \mid ... \mid e_\mu^{tr} \mid c_\nu^{tr})(c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid ... \mid c_{k+(\mu-1)(n-1)}^{tr})]e_q^{tr} \\ &\quad -e_p[(c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid ... \mid c_{k+(\mu-1)(n-1)}^{tr})(e_2^{tr} \mid ... \mid e_\mu^{tr} \mid c_\nu^{tr})]e_q^{tr} \\ &= c_{p\nu}c_{\mu j} \pm g \end{aligned}$$

where g is a linear polynomial from $\mathrm{K}[\mathcal{C}]$ and $j \in \{1, .., \mu\}$. □

Recall that the vanishing ideal of the $\mathcal{O}$-border basis scheme is generated by the set $\mathcal{T}_{seg} = \{\tau_{pq}^{1l} \mid p = 1, ..., \mu\, q = 1, ..., \mu - 1, \quad l \in \{2, ..., n\}\}$ that has exactly $\mu\nu - n\mu$.

Lemma 6.3.2. *Let $\mathcal{O} = \{1, x_1, ..., x_1^{\mu-1}\}$ denote a segment order ideal in $\mathrm{K}[x_1, ..., x_n]$.*

a) Let τ_{pq}^{1l} be a polynomial from $\mathcal{T}_{seg}$. Then we have $\deg_W(\tau_{pq}^{1l}) = \alpha e_1 + e_l$, where $-\mu + 1 < \alpha < \mu$.

b) A polynomial $\tau_{pq}^{1l} \in \mathcal{T}_{seg}$ has only one indeterminate from $\mathcal{C}$ in its support if and only if we have $p = 1$.

c) *A polynomial* $\tau_{rs}^{1l} \in \mathcal{T}_{seg}$ *has a non-standard arrow degree if and only if there exists an order ideal term* t_q *such that* $\deg_W(\tau_{1q}^{1l}) = qe_1 + e_l = \deg_W(\tau_{rs}^{1l})$.

d) *The equivalence classes* $[\bar{c}_{1\nu}], [\bar{c}_{2\nu}], ..., [\bar{c}_{\mu\nu}], [\bar{c}_{\mu(\nu-1)}] \in \mathfrak{E}$ *have only one element in* $\bar{\mathcal{C}}$.

Proof. a) A polynomial $\tau_{pq}^{1l} \in \mathcal{T}_{seg}$ has the following arrow degree

$$\deg_W(\tau_{pq}^{1l}) = \log(x_1 x_l t_q) - \log(t_p) = \log(x_l x_1^q) - \log(x_1^{p-1}) = (q-p+1)e_1 + e_l.$$

For each order ideal term t_p and t_q we have $-\mu + 1 < (q-p+1) < \mu$.

b) For each index $l \in \{2, ..., \mu\}$, the generic multiplication matrix $\mathcal{A}_l$ does not have any constant entry and only the first row of the generic multiplication matrix $\mathcal{A}_1$ does not have the entry 1. If a polynomial $\tau_{pq}^{1l} \in \mathcal{T}_{seg}$ has only one indeterminate in its support, then the polynomial τ_{pq}^{kl} is located in the first row of the matrix $[\mathcal{A}_1, \mathcal{A}_l]$ i.e., $p = 1$. For the converse let $b_j = x_l t_q$. Then we have the following.

$$\begin{aligned} \tau_{1q}^{1l} &= e_1[(e_2^{tr} \mid ... \mid e_\mu^{tr} \mid c_\nu^{tr})(c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid ... \mid c_{k+(\mu-1)(n-1)}^{tr})]e_q^{tr} \\ &\quad -e_1[(c_k^{tr} \mid ... \mid c_{k+(j-1)(n-1)}^{tr} \mid ... \mid c_{k+(\mu-1)(n-1)}^{tr})(e_2^{tr} \mid ... \mid e_\mu^{tr} \mid c_\nu^{tr})]e_q^{tr} \\ &= g_{1p} \pm c_{1j} \end{aligned}$$

c) Let t_q be an order ideal term and x_l be from $\{x_2, ..., x_n\}$. Consider the polynomial $\tau_{1q}^{1l} \in \mathcal{T}_{seg}$. The arrow degree of τ_{1q}^{1l} is non-standard as follows.

$$\deg_W(\tau_{1q}^{1l}) = qe_1 + e_l$$

For the converse let $\tau_{rs}^{1l} \in \mathcal{T}_{seg}$ with a non-standard arrow degree. Then by part a) we have $\mu > s - r + 1 > 0$ and there exists t_q where $0 < q = s - r + 1 < \mu$ such that $\deg_W(\tau_{rs}^{1l}) = \deg_W(\tau_{1q}^{1l})$.

d) For each index $1 < k < \mu$ the indeterminate $c_{k\nu}$ has the following arrow degree

$$\deg_W(c_{k\nu}) = \log(b_\nu) - \log(t_k) = \log(x_1^\mu) - \log(x_1^k) = (\mu - k)e_1.$$

By part a), there is no polynomial in $\mathcal{T}_{seg}$ with this given arrow degree and by Proposition 4.1.6 c), the claim follows.

□

Corollary 6.3.3. *Let $\mathcal{O}$ denote a segment order ideal in $\mathrm{K}[x_1, ..., x_n]$ with μ terms. For $\bar{c}_{ij} \in \mathrm{B}\mathcal{O}$ we have $\bar{c}_{ij} \in \mathfrak{m}^2$ if and only if the arrow degree of the indeterminate c_{ij} is non-standard.*

Proof. This follows from Lemma 6.3.2 b), c) and Proposition 4.1.6 a). □

Proposition 6.3.4. *Let $\mathcal{O} \subset \mathbb{T}^n$ be a segment order ideal. Then the monomial point $\mathfrak{o} = (0, ..., 0) \in \mathbb{B}_{\mathcal{O}}$ is smooth. Moreover, let $\mathcal{S}_{seg}$ denote the set*

$$\{c_{1\nu}, c_{2\nu}, ..., c_{\mu\nu}, c_{\mu(\nu-1)}, ..., c_{\mu 1}\}.$$

Then the set $\bar{\mathcal{S}}_{seg} = \{\bar{c}_{ij} \in \mathrm{B}_{\mathcal{O}} \mid c_{ij} \in \mathcal{S}_{seg}\}$ is a basis of the K*-vector space $\mathfrak{m}/\mathfrak{m}^2$.*

Proof. Let us examine the border of the segment order ideal. The minimal generating set of the border term ideal is $\{b_1, b_2, ..., b_{n-1}, b_\nu\}$, where $b_1 = x_2, b_2 = x_3, ..., b_{n-1} = x_n$ and $b_\nu = x_1^{\mu-1}$ which are pure power terms. Recall Lemma 4.2.5 which states that if the border term ideal is generated by pure power terms. Hence the set $\mathfrak{E}$ has exactly $n\mu$. Then by Theorem 4.2.15, the monomial point $\mathfrak{o}$ is smooth.

The set $\mathcal{S}_{seg}$ has exactly $\mu + \nu - 1 = \mu + [(n-1)\mu + 1] - 1 = n\mu$ distinct elements. We show that those elements are K-linearly independent in the vector space $\mathfrak{m}/\mathfrak{m}^2$. By Lemma 6.3.2 d), $[\bar{c}_{1\nu}], [\bar{c}_{2\nu}], ..., [\bar{c}_{\mu\nu}], [\bar{c}_{\mu(\nu-1)}] \in \mathfrak{E}$ have only one element in $\bar{\mathcal{C}}$. Then by Proposition 4.1.6c), we have $[\bar{c}_{1\nu}] \neq [\bar{c}_{2\nu}] \neq, ..., \neq [\bar{c}_{\mu\nu}] \neq [\bar{c}_{\mu(\nu-1)}]$. We now focus on the rest of the elements in the set $\mathcal{S}_{seg}$. Since for each border term $b_i \in \{b_1, ..., b_{\nu-2}\} \subset \partial\mathcal{O}$, the index i is smaller than ν, for $\alpha < 0$ the arrow degree of $c_{\mu i}$ is as follows.

$$\deg_W(c_{\mu i}) = \log(b_i) - \log(t_\mu) = \log(x_l x_1^s) - \log(x_1^{\mu-1}) = \alpha e_1 + e_l$$

Consequently, we have the inequality $\mu - 1 > s$. Then each $c_{ij} \in \{c_{\mu(\nu-2)}, ..., c_{\mu 1}\}$ is standard and by Corollary 6.3.3, the equivalence class $\bar{c}_{ij}$ is not in $\mathfrak{m}^2$. Moreover, for distinct border terms b_j and b_k we have $\deg_W(c_{\mu j}) \neq \deg_W(c_{\mu k})$. Then by Lemma 4.1.5, $[\bar{c}_{\mu j}] \neq [\bar{c}_{\mu k}]$. Thus the set $\bar{\mathcal{S}}_{seg}$ is a set of K-linear independent indeterminates in $\mathfrak{m}/\mathfrak{m}^2$ with $n\mu$ elements, which is by the smoothness of $\mathfrak{o}$ the exact number of elements in $\mathfrak{E}$. Hence the proposition follows. □

Corollary 6.3.5. *Let $\mathcal{O} \subset \mathbb{T}^n$ be a segment order ideal. Let τ_{pq}^{1l} be a polynomial in $\mathcal{T}_{Seg}$. The polynomial τ_{pq}^{1l} is of the following form $(\tau_{pq}^{1l})^{(2)} + (\tau_{pq}^{1l})^{(1)}$, where $(\tau_{pq}^{1l})^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a linear polynomial and $(\tau_{pq}^{1l})^{(2)}$ is homogenous polynomial of standard degree 2 and it is in* K$[Seg]$.

Proof. This follows from Lemma 2.3.19, Proposition 6.3.4 and Lemma 6.3.1. □

Proposition 6.3.6. *Let* P *denote the polynomial ring* $\mathrm{K}[x_1, ..., x_n]$. *Let*

$$\mathcal{O} = \{1, x_1, .., x_1^{\mu-1}\} \subset \mathrm{K}[x_1, ..., x_n]$$

be a segment order ideal. Then $\mathbb{B}_\mathcal{O}$ *is isomorphic to an affine space of dimension* $n\mu$.

Proof. By Proposition 6.3.4, the set

$$\bar{\mathcal{S}}_{seg} = \{\bar{c}_{1\nu}, \bar{c}_{2\nu}, ..., \bar{c}_{\mu\nu}, \bar{c}_{\mu(\nu-1)}, ..., \bar{c}_{\mu 1}\}$$

is a K-vector basis of $\mathfrak{m}/\mathfrak{m}^2$. Let $\mathcal{M}$ denote the set of indeterminates $\mathcal{C} \setminus \mathcal{S}_{seg}$. The set $\mathcal{M}$ has exactly $\mu\nu - n\mu$ elements. By the Corollary 5.3.15, the vanishing ideal of the segment border basis scheme $\mathbb{B}_\mathcal{O}$ is generated by the set

$$\mathcal{T}_{seg} = \{\tau_{pq}^{1l} \mid p = 1, ..., \mu\, q = 1, ..., \mu - 1\, l \in \{2, ..., n\}\},$$

which has $\mu\nu - n\mu$ elements, as well. Note that by Proposition 4.1.6a) and b), for each $d_i \in \mathcal{M}$ there exists a polynomial $\tau_i \in \mathcal{T}_{seg}$ such that $\frac{\partial \tau_i}{\partial d_i} = \pm 1$. We construct the following matrix.

$$\mathcal{M}_{\mathcal{S}_{seg}} = \begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & \dots & d_m \end{array} \\ \begin{pmatrix} \frac{\partial \tau_1}{\partial d_1} & \dots & \frac{\partial \tau_1}{\partial d_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tau_m}{\partial d_1} & \dots & \frac{\partial \tau_m}{\partial d_m} \end{pmatrix} \end{array} = \begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & \dots & d_m \end{array} \\ \begin{pmatrix} \pm 1 & \star & \star \\ \vdots & \ddots & \vdots \\ \star & \star & \pm 1 \end{pmatrix} \end{array}$$

Since the set $\mathcal{T}_{seg}$ is a set of generators of $\mathcal{I}(\mathbb{B}_\mathcal{O})$, the Jacobian matrix is as follows:

$$\mathrm{Jac}(\mathcal{T}_{seg}) = \left(\begin{array}{c|c} \mathcal{M}_{\mathcal{S}_{seg}} & * \end{array} \right) \tag{6.22}$$

By Corollary 6.3.5, a polynomial $\tau_{pq}^{1l} \in \mathcal{T}_{seg}$, is of the following shape

$$\tau_{pq}^{1l} = (\tau_{pq}^{1l})^{(2)} \pm (\tau_{pq}^{1l})^{(1)},$$

where $(\tau_{pq}^{1l})^{(2)} \in \mathrm{K}[\mathcal{S}_{seg}]$ and $(\tau_{pq}^{1l})^{(1)} \in \mathrm{K}[\mathcal{C}]$ is a linear polynomial that has at most two terms in its support. Then by Corollary 6.1.1, the entries of $\mathcal{M}_{\mathcal{S}_{seg}}$ are only ± 1 and 0. Since the monomial point $\mathfrak{o} = (0, .., 0)$ is a smooth point of $\mathbb{B}_\mathcal{O}$, by Theorem 2.1.18c, the rank of the matrix $\mathrm{Jac}(\mathcal{T}_{seg})_\mathfrak{o}$ is $\mu\nu - n\mu$. Since the entries of $\mathcal{M}_{\mathcal{S}_{seg}}$ are either ± 1

or 0, it is a sub-matrix of $\mathrm{Jac}(\mathcal{T}_{seg})_{\mathfrak{o}}$, as well. Then the matrix $\mathcal{M}_{\mathcal{S}_{seg}}$ has the rank $\mu\nu - n\mu$. By applying row operations, we have

$$\mathcal{M}_{\mathcal{S}_{seg}} = \begin{array}{c} \\ \tau_1 \\ \vdots \\ \tau_m \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & \dots & d_m \end{array} \\ \begin{pmatrix} \pm 1 & \star & \star \\ \vdots & \ddots & \vdots \\ \star & \star & \pm 1 \end{pmatrix} \end{array} \sim \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \pm 1 \end{pmatrix} = E_m.$$

Hence, the determinant of the matrix $\mathcal{M}_{\mathcal{S}_{seg}}$ is ± 1. Thus by Theorem 6.2.20, the segment border basis scheme is isomorphic to an affine space of dimension $n\mu$. □

Example 6.3.7. Let $\mathcal{O}$ denote the segment order ideal $\{1, x_1, x_1^2\} \subset \mathrm{K}[x_1, x_2, x_3]$. Then the border of $\mathcal{O}$ is

$$\partial\mathcal{O} = \{x_3, x_2, x_1x_3, x_2x_3, x_1^2x_3, x_2x_1^2, x_1^3\},$$

so that in our terminology we have

$$t_1 = 1, t_2 = x_1, t_3 = x_1^2, b_1 = x_3, b_2 = x_2, b_3 = x_1x_3, b_4 = x_2x_3, b_5 = x_1^2x_3, b_6 = x_2x_1^2, b_7 = x_1^3.$$

By Propositions 5.3.14, the defining equations are as follows.

$$\begin{aligned} \tau_{11}^{12} &= c_{17}c_{32} - c_{14} & \tau_{11}^{13} &= c_{17}c_{31} - c_{13} \\ \tau_{22}^{12} &= c_{27}c_{34} + c_{14} - c_{26} & \tau_{22}^{13} &= c_{27}c_{33} + c_{13} - c_{25} \\ \tau_{12}^{12} &= c_{17}c_{34} - c_{16} & \tau_{12}^{13} &= c_{17}c_{33} - c_{15} \\ \tau_{21}^{12} &= c_{27}c_{32} + c_{12} - c_{24} & \tau_{21}^{13} &= c_{27}c_{31} + c_{11} - c_{23} \\ \tau_{32}^{12} &= c_{34}c_{37} + c_{24} - c_{36} & \tau_{32}^{13} &= c_{33}c_{37} + c_{23} - c_{35} \\ \tau_{31}^{12} &= c_{32}c_{37} + c_{22} - c_{34} & \tau_{31}^{13} &= c_{31}c_{37} + c_{21} - c_{33} \end{aligned}$$

By Lemma 6.3.4, the set $\bar{\mathcal{S}}_{seg} = \{\bar{c}_{17}, \bar{c}_{27}, \bar{c}_{37}, \bar{c}_{31}, \bar{c}_{32}, \bar{c}_{33}, \bar{c}_{34}, \bar{c}_{35}, \bar{c}_{37}\}$ is a spanning set of $\mathfrak{m}/\mathfrak{m}^2$. By Proposition 6.3.6, the scheme $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space of dimension $n\mu$.

Let us illustrate this result. Recall that the homomorphism

$$\begin{aligned} \varphi : \mathrm{K}[\mathcal{S}_{seg}] &\longrightarrow \mathrm{K}[\mathcal{C}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}), \\ c_{ij} &\longmapsto \bar{c}_{ij} \end{aligned}$$

is injective (see Chapter 6.2, Map 6.15).

To show that φ is surjective, we compute the determinant of the matrix $\mathcal{M}_{\mathcal{S}_{seg}}$. Clearly, this matrix is in $\mathrm{Mat}_{12}(\mathrm{K}[\mathcal{S}_{Seg}])$ and the determinant of this matrix is 1. By Theorem 6.2.20, the scheme $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space of dimension $n\mu$. Now we illustrate the way to get $f_i \in \mathrm{K}[\mathcal{S}]$ for each $d_i \in M$, concretely. Recall from Chapter 6 that we have the following equation.

$$\mathcal{M}_{\mathcal{S}_{Seg}} \times \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix},$$

For each $c_{ij} \in M$, we define f_{ij}, which are the pre-images of each $c_{ij} \in M$ as follows.

$$\begin{aligned}
f_{14} &:= \tau_{11}^{12} - c_{14} = c_{17}c_{32} & f_{13} &:= \tau_{11}^{13} + c_{13} = c_{17}c_{31} \\
f_{26} &:= c_{27}c_{34} - f_{14} & f_{25} &:= c_{27}c_{33} + f_{13} \\
f_{16} &:= \tau_{12}^{12} + c_{16} = c_{17}c_{34} & f_{15} &:= \tau_{12}^{13} + c_{15} = c_{17}c_{33} \\
f_{24} &:= \tau_{32}^{12} - c_{24} = c_{34}c_{37} - c_{36} & f_{23} &:= \tau_{32}^{13} - c_{23} = c_{33}c_{37} - c_{35} \\
f_{12} &:= c_{27}c_{32} + f_{24} & f_{11} &:= c_{27}c_{31} + f_{23} \\
f_{22} &:= \tau_{31}^{12} - c_{22} = c_{32}c_{37} - c_{34} & f_{21} &:= \tau_{31}^{13} - c_{21} = c_{31}c_{37} - c_{33}.
\end{aligned}$$

Then the matrix $\mathcal{M}_{\mathcal{S}_{Seg}}$ is as follows.

$$\mathcal{M}_{\mathcal{S}_{Seg}} = \begin{array}{c} \\ \tau_{21}^{13} \\ \tau_{21}^{12} \\ \tau_{11}^{13} \\ \tau_{11}^{12} \\ \tau_{12}^{13} \\ \tau_{12}^{12} \\ \tau_{31}^{13} \\ \tau_{31}^{12} \\ \tau_{32}^{13} \\ \tau_{32}^{12} \\ \tau_{22}^{13} \\ \tau_{22}^{12} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Now let us verify the entries of the column matrix

$$G = \begin{pmatrix} g_1(\mathcal{S}) \\ \vdots \\ g_m(\mathcal{S}) \end{pmatrix},$$

which are by Corollary 6.3.5 all in the polynomial ring $\mathrm{K}[\mathcal{S}_{Seg}]$ as follows.

$$\begin{aligned} &g_{14} = c_{17}c_{32},\ g_{13} = c_{17}c_{31}\ g_{26} = c_{27}c_{34},\ g_{25} = c_{27}c_{33} \\ &g_{16} = c_{17}c_{34},\ g_{15} = c_{17}c_{33},\ g_{24} = c_{34}c_{37} - c_{36},\ g_{23} = c_{33}c_{37} - c_{35} \\ &g_{12} = c_{27}c_{32},\ g_{11} = c_{27}c_{31},\ g_{22} = c_{32}c_{37} - c_{34},\ g_{21} = c_{31}c_{37} - c_{33} \end{aligned}$$

By Proposition 6.3.6, all the polynomials that we defined above $f_{11}, f_{12}, ..., f_{26}$ are from the polynomial ring $\mathrm{K}[c_{17}, c_{27}, c_{37}, c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{37}] = \mathrm{K}[\mathcal{S}_{seg}]$. We let $c_{ij} \in M$ and $d_k = c_{ij}$. Hence, for each $d_k \in M$ there exists an $f_k \in \mathrm{K}[\mathcal{S}_{seg}]$ such that $f_k = \mathrm{Det}(M_k) \in \mathrm{K}[\mathcal{S}_{seg}]$, where $\mathrm{Det}(M_k)$ is the determinant of the matrix $\mathcal{M}_{\mathcal{S}_{Seg}}$ that has the the column G in the k^{th} column. In this example, it is straightforward to verify the polynomials f_k by simple substitution. For example, $f_{14} = c_{17}c_{32} \in \mathrm{K}[\mathcal{S}_{Seg}]$ and f_{26} can be found by substituting f_{14} by $c_{17}c_{32}$ and hence we get $f_{26} = c_{27}c_{34} - c_{17}c_{32}$. By proceeding this way, we get

$$\begin{aligned} f_{14} &:= c_{17}c_{32} & f_{13} &:= c_{17}c_{31} \\ f_{26} &:= c_{27}c_{34} - c_{17}c_{32} & f_{25} &:= c_{27}c_{33} + c_{17}c_{31} \\ f_{16} &:= c_{17}c_{34} & f_{15} &:= c_{17}c_{33} \\ f_{24} &:= c_{34}c_{37} - c_{36} & f_{23} &:= c_{33}c_{37} - c_{35} \\ f_{12} &:= c_{27}c_{32} + c_{34}c_{37} - c_{36} & f_{11} &:= c_{27}c_{31} + c_{33}c_{37} - c_{35} \\ f_{22} &:= c_{32}c_{37} - c_{34} & f_{21} &:= c_{31}c_{37} - c_{33}. \end{aligned}$$

6.4 Special Order Ideals from $\mathbb{T}^2$

Before starting the main part of this section, let us present the necessary notation and definitions. Unless stated otherwise, throughout this section the polynomial ring in use is $P = \mathrm{K}[x, y]$ and $\mathcal{O}$ is an order ideal in $\mathbb{T}^2$.

1) Let an element $x^r y^l$ of the order ideal $\mathcal{O}$ be denoted by t_{rl}.

2) The set of indeterminates of $\mathcal{C}$ are of the form $c_{rl\,j}$, where j is the index of b_j and rl are such that t_{rl} is an element of the order ideal $\mathcal{O}$.

3) For $i \geqslant 0$ and $j \geqslant 0$ we place the term $x^i y^j$ at position (i, j) of a diagram in order to visualize $\mathcal{O}$ and $\partial\mathcal{O}$. Let us illustrate how we place the terms and how the diagram of $\mathcal{O}$ is constructed with an example.

Example 6.4.1. Let $\mathcal{O}$ denote the order ideal

$$\{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3, x^3y, y^4\} \subset \mathbb{T}^2.$$

Then the border of $\mathcal{O}$ is $\partial\mathcal{O} = \{xy^3, x^2y^2, x^4, y^5, xy^4, x^3y^2, x^4y\}$.

$\mathbf{y^5} = b_7$	$\mathbf{xy^5} = b_6$				
y^4	xy^4	$\mathbf{x^2y^4} = b_5$			
y^3	xy^3	x^2y^3	$\mathbf{x^3y^3} = b_4$		
y^2	xy^2	x^2y^2	x^3y^2	$\mathbf{x^4y^2} = b_3$	
y	xy	x^2y	x^3y	x^4y	$\mathbf{x^5y} = b_2$
1	x	x^2	x^3	x^4	$\mathbf{x^5} = b_1$

Table 6.1: Diagram of $\mathcal{O}$

As given above we have $x^r y^l = t_{rl}$. In the diagram of $\mathcal{O}$, for $l = 0$, we have

$$t_{0\,0} = 1,\ t_{1\,0} = x, ..., t_{4\,0} = x^4, b_1 = x^5.$$

4) Let L denote the number of rows and R denote the number of columns. In Diagram 6.1 we have 6 rows, i.e. $R = 6$ and 6 columns, i.e. $L = 6$.

5) Let R_l denote the number of terms placed in the row l and L_r denote the number of terms placed in the column r. For example, in the third row we have $l = 2$, Diagram 6.1 has 5 terms, i.e. $R_2 = 5$ and in the fourth column it has 3 terms, i.e. $L_4 = 3$.

Definition 6.4.2. An order ideal has the *sawtooth* form if the diagram of $\mathcal{O}$ is of the form such that every step width, except possibly the topmost, and every step height, except possibly the rightmost, is of size one.

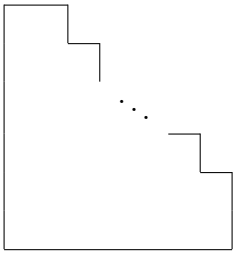

Table 6.2: Diagram of $\mathcal{O}$ has the *sawtooth* form

Let $\mathcal{O}$ be an order ideal that has the *sawtooth* form. Let $\partial\mathcal{O} = \{b_1, ..., b_\nu\}$ denote the border of $\mathcal{O}$, with ν elements. Observe that, except for the top row which consists of only border terms, each row has exactly one border term. We enumerate the border term on the j^{th} row as $b_{j+1} = x^{R_j}y^j$, where $0 \leqslant j < L-1$. For example, in Diagram 6.1 border elements are indexed beginning from b_1 to b_6 in an ascending order between the top row and the bottom row. There are ν border terms and if the border term is on the top row then we enumerate the border terms as $b_j = x^{\nu-j}y^{L-1}$ where $\nu \geqslant j \geqslant L-1$.

In [Huib02] by using Proposition 7.1.2 and Corollary 7.2.2 the following theorem was proven.

Theorem 6.4.3. *Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal in $\mathbb{T}^2$. If the order ideal $\mathcal{O}$ has the sawtooth form then $\mathbb{B}_\mathcal{O}$ is isomorphic to an affine space.*

In this section our aim is to prove the above theorem by applying Theorem 6.2.20 instead of using Proposition 7.1.2 [Huib02]. The main reason is to show that Theorem 6.2.20 is not only a good tool to check whether a given $\mathcal{O}$-border basis scheme is an affine space, but it can be used to give genral results for a specific shape of order ideals like sawtooth. Moreover, it holds for larger dimensional polynomial rings, not just for two dimensional cases.

Next we translate very useful notions introduced in [Huib02] to our notation, that are called the *exposable term*, *exposed term*.

Definition 6.4.4. Let $\mathcal{O} \in \mathrm{K}[x, y]$ be an order ideal. Let t_{rl} be a term in $\mathcal{O}$ and b_j be a term from $\partial\mathcal{O}$.

1. If we have $xt_{rl} = b_j$ then we call b_j as an **x-exposed** border term and t_{rl} is called

an **x-exposable** order ideal term.

2. If we have $yt_{rl} = b_j$ then we call b_j an **y-exposed** border term and t_{rl} is called a **y-exposable** term.

3. If b_j is either x-exposed or y-exposed, then $c_{rl\,j}$ is called an **exposed indeterminate**.

4. Let b_j be an y-exposed border term and t_{fk} is an x-exposable term in the order ideal. Then the indeterminate $c_{fk\,j}$ is called **yx-exposed indeterminate**.

5. Let b_j be an x-exposed border term and t_{fk} is an y-exposable term in the order ideal. Then the indeterminate $c_{fk\,j}$ is called **xy-exposed indeterminate**.

Construction 6.4.5. Let $\mathcal{O}$ be an order ideal that has the sawtooth form. Let $\mathcal{S}$ be a subset of $\mathcal{C}$ that contains xy-exposed indeterminates and yx-exposed indeterminates.

Corollary 6.4.6. *Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal in $\mathbb{T}^2$. Let $\mathcal{S}$ be constructed as in Construction 6.4. Then the set $\bar{\mathcal{S}}$ is a basis of the* K*-vector space $\mathfrak{m}/\mathfrak{m}^2$.*

Proof. By Proposition 2.4 of [Hai98] the set $\bar{\mathcal{S}}$ is a spanning set of $\mathfrak{m}/\mathfrak{m}^2$ and there are exactly 2μ elements in $\bar{\mathcal{S}}$. By Corollary 4.2.17, the monomial point $\mathfrak{o}$ is a smooth point of $\mathbb{B}_\mathcal{O}$ and the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is 2μ. Hence the set $\bar{\mathcal{S}}$ is a basis set of $\mathfrak{m}/\mathfrak{m}^2$. □

We let $m = \mu\nu - 2\mu$. The elements of $\mathcal{C} \setminus \mathcal{S}$ are denoted by d_p for $p \in \{1, ..., m\}$. Recall that for each d_p we choose exactly one polynomial τ in $\mathcal{T}$ with the property $\frac{\partial \tau}{\partial d_p} = \pm 1$. We index this polynomial τ as τ_p and we call the set of these polynomials as $\mathcal{T}_\mathcal{S}$. We note that $\mathcal{T}_\mathcal{S}$ is not unique.

We write each polynomial in $\mathcal{T}$ with respect to our new notation. Recall that two distinct border terms b_i, b_j satisfying $x_k b_i = b_j$ are called x_k-next door neighbors where in our setting we have $x_k \in \{x, y\}$.

For the order ideal that has *sawtooth* form, y-next door neighbors are located only on the rightmost column, i.e. $yb_1 = b_2, ..., yb_{L_R - 1} = b_{L_R}$. Let b_j and b_i be distinct border terms in $\{b_1, ..., b_{L_R}\}$ that are y-next door neighbors, i.e. $yb_j = b_i$ By the enumeration on the border elements, we have $b_j = b_{i-1}$. Recall Construction 2.3.17c) of the polynomial $\tau_{pq} \in \mathrm{ND}(i, j)$ which is a result of y-next door neighbors b_i and b_j. In order to rewrite the polynomial τ_{pq} in the notation of this section, we use t_{rl} and $t_{r'l'}$ for t_p and t_r, respectively. Note that $t_{r'l'}$ is y-exposable and $b_m = yt_{r'l'}$ is a y-exposed

border term. Hence the polynomial in Construction 2.3.17c) that is constructed by lifting y-next door neighbors.

$$\mathcal{Y}_{rl-1\,i-1} = c_{rl\,i} - c_{rl-1\,i-1} - \sum_{\substack{t_{r'l'}\in\mathcal{O}\\ yt_{r'l'}=b_m\in\partial\mathcal{O}}} c_{r'l'\,i-1}c_{rl\,m} \tag{6.23}$$

Next recall Construction 2.3.17c) for the x-next door neighbors. We use t_{rl} and $t_{r'l'}$ instead of t_p and t_r, respectively. Note that $t_{r'l'}$ is y-exposable and $b_n = xt_{r'l'}$ is an x-exposed border term. Moreover, as a result of the enumeration of the border terms, for x-next door neighbor border terms b_i and b_j, we have $j = i-1$ (see Example 6.4.1). The border elements with x-next door neighbor relation are located only on the top row of Diagram 6.2, we have $b_L = xb_{L+1}, b_{L+1} = xb_{L+2}..., b_\nu - 1 = xb_\nu$. Let b_i and b_{i-1} be two distinct border terms in $\{b_L, ..., b_\nu\}$. Then we get the following polynomial by lifting x-next door neighbors.

$$\mathcal{X}_{r-1l\,i} = c_{rl\,i-1} - c_{r-1l\,i} - \sum_{\substack{t_{r'l'}\in\mathcal{O}\\ xt_{r'l'}=b_n\in\partial\mathcal{O}}} c_{r'l'\,i}c_{rl\,n} \tag{6.24}$$

The neighbor relation that we focus on is the across the street neighbors. Since the order ideal in use has the sawtooth form, the border terms we consider are $\{b_{L_R}, ..., b_L\}$ and $b_j = b_{i+1}$. We use t_{rl} and $t_{r'l'}$ and $t_{r"l"}$ instead of t_p, t_r and t_s respectively. The term $t_{r'l'}$ is x-exposable and $t_{r"l"}$ is y-exposable. We let $yt_{r"l"} = b_p$ and $xt_{r'l'} = b_q$. The border term b_q is an x-exposed term and the border term b_p is an y-exposed term. Then the polynomial that we get by lifting across the street neighbors b_i and b_{i-1} with $xb_i = yb_{i-1}$, is as follows.

$$\mathfrak{A}_{r-1l+1\,i} = c_{rl\,i-1} - c_{r-1l+1\,i} - \sum_{\substack{t_{r'l'}\in\mathcal{O}\\ xt_{r'l'}=b_q\in\partial\mathcal{O}}} c_{r'l'\,i-1}c_{r+1l\,q} + \sum_{\substack{t_{r"l"}\in\mathcal{O}\\ yt_{r"l"}=b_m\in\partial\mathcal{O}}} c_{r"l"\,i}c_{r+1l\,m} \tag{6.25}$$

Lemma 6.4.7. *Let $\mathcal{O} \subset \mathbb{T}^2$ be an order ideal that has the sawtooth form. Let $\mathcal{S}$ be as given in Construction 6.4. Then the following statements are true for polynomials $\mathcal{X}_{rl-1\,i-1}, \mathcal{Y}_{r-1l\,i}$ and $\mathfrak{A}_{r-1l+1\,i}$.*

i) Consider the polynomial $\mathcal{Y}_{rl\,i}$. For each $t_{r'l'} \in \mathcal{O}$ with $yt_{r'l'} \in \partial\mathcal{O}$ the indeterminate $c_{r'l'\,i-1}$ in $\mathcal{Y}_{r-1l\,i}$ is in $\mathcal{S}$.

ii) Consider the polynomial $\mathcal{X}_{r-1l\,i-1}$. For each $t_{r'l'} \in \mathcal{O}$ with $xt_{r'l'} \in \partial\mathcal{O}$ the inde-

terminate $c_{r'l'\,i-1}$ *in* $\mathcal{X}_{r-1l\,i-1}$ *is in* $\mathcal{S}$.

iii) Consider the polynomial $\mathfrak{A}_{r-1l+1\,i}$. *For each* $t_{r'l'} \in \mathcal{O}$ *with* $xt_{r'l'} \in \partial\mathcal{O}$ *the indeterminate* $c_{r'l'\,i}$ *and for each* $t_{r"l"} \in \mathcal{O}$ *with* $yt_{r"l"} \in \partial\mathcal{O}$ *the indeterminate* $c_{r"l"\,i-1}$ *are both in* $\mathcal{S}$.

Proof. The first statement i) is a result of the definition of xy-exposed indeterminate, since the border terms on the last column (see, for example, Table 6.1) are x-exposed and $t_{r'l'}$ is y-exposable. The second statement is a result of the definition of yx-exposed indeterminate, since the border terms on the top row are y-exposed and both $t_{r"l"}$ x-exposable. The last statement follows similarly to i) and ii). □

Lemma 6.4.8. *Let* $\mathcal{O} \in \mathbb{T}^2$ *be an order ideal of the shape sawtooth. Then the matrix* $\mathcal{M}_\mathcal{S}$ *is an element of* $\mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$.

Proof. The set $\mathcal{T}_\mathcal{S}$ consists of polynomials of the form $\mathcal{X}, \mathcal{Y}$ and $\mathfrak{A}$. Let τ_k be an element from $\mathcal{T}_\mathcal{S}$ with $\tau_k = \tau_k^{(1)} + \tau_k^{(2)}$ where $\tau_k^{(2)}$ represents the homogenous (with res. standard grading) part of τ of degree 2. Recall that the element in position (k, j) of the matrix $\mathcal{M}_\mathcal{S}$ is

$$\frac{\partial \tau_k}{\partial d_j}, \tag{6.26}$$

where $d_i \in \mathcal{C} \setminus \mathcal{S}$. By Lemma 6.4.7, the monomials in $\mathrm{Supp}(\tau_k^{(2)})$ are of the form $\pm sp$, where $p \in \mathcal{C}$ and $s \in \mathcal{S}$. Then if $p = d_u$ is in $\mathcal{C} \setminus \mathcal{S}$ then we have

$$\frac{\partial \tau_k}{\partial d_u} = s.$$

If p is in $\mathcal{S}$, then 6.26 is 0. Moreover, by Lemma 2.3.19 $\mathrm{Supp}(\tau_k^{(1)})$ has maximum 2 elements and as a result of the choice of $\mathcal{T}_k$ one of them is d_k. Hence the entries of the matrix $\mathcal{M}_\mathcal{S}$ is either $s, 0$ or ± 1. □

Recall that the set $\mathcal{T}_\mathcal{S}$ is not a uniquely chosen set.

Lemma 6.4.9. *Let* $\mathcal{O} \subset \mathbb{T}^2$ *be an order ideal. Let* $\mathcal{T}_\mathcal{S}$ *be a subset of* $\mathcal{T}$ *which is defined in Construction 6.2.5. Then the* $\mathcal{O}$*-border basis scheme* $\mathbb{B}_\mathcal{O}$ *is a complete intersection and the vanishing ideal* $\mathcal{I}(\mathbb{B}_\mathcal{O})$ *is generated by* $\mathcal{T}_\mathcal{S}$.

Proof. By Proposition 5.1.23 the scheme $\mathbb{B}_\mathcal{O}$ is a complete intersection and $\mathcal{I}(\mathbb{B}_\mathcal{O})$ is generated by a subset $\mathcal{T}'$ of $\mathcal{T}$ with $\mu\nu - 2\mu$ polynomials. Our aim is to show that for

each $d_i \in \mathcal{C} \setminus \mathcal{S}$ there exists a polynomial $\tau_i \in \mathcal{T}'$ such that $\frac{\partial \tau_i}{\partial d_i} = \pm 1$. This will show that $\mathcal{T}'$ is $\mathcal{T}_\mathcal{S}$.

By Corollary 4.2.17 the monomial point $\mathfrak{o}$ of $\mathbb{B}_\mathcal{O}$ is smooth and by Lemma the elements of $\mathcal{S}$ form a K-vector space basis of the space $\mathfrak{m}/\mathfrak{m}^2$, then for each $d_i \in \mathcal{C} \setminus \mathcal{S}$ $d_i \in \mathfrak{m}^2$ holds. Then by Proposition 4.1.6 we have $d_i = \sum_{k=1}^{m} \alpha_k \tau_k$ where $\alpha_k = \pm 1$ and $\tau_k \in \mathcal{T}'$. Hence for each $d_i \in \mathcal{C} \setminus \mathcal{S}$ there exists at least one polynomial, τ_i in the set $\mathcal{T}'$ such that $\frac{\partial \tau_i}{\partial d_i} = \pm 1$. Since there are exactly m elements in both sets $\mathcal{T}'$ and $\mathcal{C} \setminus \mathcal{S}$, the set $\mathcal{T}'$ is one possible $\mathcal{T}_\mathcal{S}$. □

Let $d_{i_1} = c_{r_1 l_1 j_1}$ and $d_{i_2} = c_{r_2 l_2 j_2}$ be two distinct elements in $\mathcal{C} \setminus \mathcal{S}$, where $t_{l_1 r_1}$ and $t_{l_2 r_2}$ are order ideal terms and b_{j_1} and b_{j_2} are border terms. Let τ_{i_1} and τ_{i_2} be polynomials in $\mathcal{T}_\mathcal{S}$ which correspond to d_{i_1} and d_{i_2}, respectively. We say

$$d_{i_1} \prec d_{i_2} \tag{6.27}$$

if and only if one of the following statements is true.

i) $r_1 > r_2$

ii) For the case $r_1 = r_2$, we have $l_1 > l_2$

iii) For the case $r_1 = r_2$ and $l_1 = l_2$, the polynomial τ_{i_1} is of the form $\mathfrak{A}_{r_1 l_1 j_1}$ and τ_{i_2} is either of the form $\mathcal{Y}_{r_2 l_2 j_2}$ or $\mathcal{X}_{r_2 l_2 j_2}$.

iv) For case $r_1 = r_2$ and $l_1 = l_2$, the polynomial τ_{i_1} is of the form $\mathcal{Y}_{r_1 l_1 j_1}$ and τ_{i_2} is of the form $\mathcal{X}_{r_2 l_2 j_2}$.

Lemma 6.4.10. *Let b_{j_1} and b_{j_2} be two distinct border terms and neighbors. Let τ_p be the polynomial which satisfies $\tau_p^{(1)} = c_{i_1 k_1 j_1} - c_{i_2 k_2 j_2}$ and $\tau_p^{(2)} = \sum_\lambda c_{r_\lambda q_\lambda l_\lambda} s_\lambda$. If we let s_λ be an element in $\mathcal{S}$, then we have $c_{i_1 k_1 j_1} \prec c_{i_2 k_2 j_2}$ and for each λ, $c_{r_\lambda q_\lambda l_\lambda} \prec c_{i_2 k_2 j_2}$.*

Proof. This is a result of " $\prec$ " with polynomials $\mathcal{Y}_{i_2 k_2 j_2}, \mathcal{X}_{i_2 k_2 j_2}$ and $\mathfrak{A}_{i_2 k_2 j_2}$. □

Lemma 6.4.11. *Let d_p and d_s be two distinct elements in $\mathcal{C} \setminus \mathcal{S}$. Let a_{pu} be the entry in position (p, u) and a_{sv} be the entry in position (s, v) of the matrix $\mathcal{M}_\mathcal{S}$, where $u, v \in \{1, ..., m\}$. If $d_p \prec d_s$ holds, then for each $u > v$ we have $a_{pu} = 0$ and for each $v > s$ we have $a_{sv} = 0$. Moreover, we have $a_{pp} = \pm 1$ and $a_{ss} = \pm 1$. Then the p^{th} and the s^{th} row and the column of the matrix $\mathcal{M}_\mathcal{S}$ have the following shape.*

$$\begin{array}{c} \\ \tau_p \\ \tau_s \end{array} \begin{array}{c} \begin{array}{cccccccc} d_1 & \dots & d_p & d_{p+1} & \dots & d_s & d_{s+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & \pm 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{s1} & \dots & a_{sr} & a_{s\,p+1} & \dots & \pm 1 & 0 & \dots & 0 \end{pmatrix} \end{array} \tag{6.28}$$

Proof. We prove the claim by descending induction on $0 \leqslant l \leqslant L-1$ and on $0 \leqslant r \leqslant R_l$. In order to visualize the process, reducing l represents moving through the rows starting from the top row that contains order ideal terms and reducing r represents moving through the elements of the row starting from the x-exposable ones (See Diagram 6.1).

step 1) Our first step is $l = L - 1$. By Constructions 6.4 and 2.3.17 the only neighbor relation we have to consider is the x-next door neighbor relations and the polynomials in $\mathcal{T}_{\mathcal{S}}$ that are calculated as a result of this relation. We focus on the polynomials $\mathcal{X}_{rl\,i-1}$ and we claim that for each $0 \leqslant r \leqslant R_l$ the polynomial

$$\sum_{\substack{t_{r'l'} \in \mathcal{O} \\ x t_{r'l'} = b_q \in \partial\mathcal{O}}} c_{r'l'\,j} c_{rl\,q},$$

is in $\mathrm{K}[\mathcal{S}]$. To prove this claim we proceed by descending induction on $1 \leqslant r \leqslant R_l$.

sub-step 1) Let $l = R - 1$ and $r = R_l$. Recall that R_l represents the number of order ideal elements in the row l. By Lemma 6.4.7 i) for each $j \in \{1, ..., R-1\}$ the indeterminates $c_{r'l'\,j}$ is in $\mathcal{S}$. Moreover, t_{rl} is on the top row and y-exposable and b_q is an x-exposed term. Therefore for each x-exposed b_q the indeterminate $c_{rl\,q}$, that is in $\mathcal{X}_{rl\,i-1}$, is in $\mathcal{S}$. Then the entries of $\mathcal{M}_{\mathcal{S}}$ in the row p are 0 or ± 1.

sub-step 2) Now we assume that $l = R - 1$ and $r < R_l$. By Lemma 6.4.7 i), for each $j \in \{1, ..., L_r - 1\}$ the indeterminate $c_{r'l'\,j}$ is in $\mathcal{S}$. By Lemma 6.4 ii), for $i \leqslant R - 1$ the indeterminate $c_{rl\,i-1}$ is in $\mathcal{S}$. This holds, since each b_{i-1} is x-exposed and t_{rl} is y-exposable. If we let $d_k = c_{rl\,i-1}$ and $d_s = c_{r-1l\,i}$, then as a result of 6.27 we have $d_k \prec d_s$. By using the induction hypothesis for each $u > r$, the element a_{ru} in position (s, u) of the matrix $\mathcal{M}_{\mathcal{S}}$ is 0, and for each $v < v$, we have $a_{ru} \in \mathrm{K}[\mathcal{S}]$. Hence the k^{th} and the s^{th} rows of the matrix $\mathcal{M}_{\mathcal{S}}$ have the following shape.

$$\begin{array}{c} \\ \tau_k \\ \tau_s \end{array} \begin{array}{c} \begin{array}{cccccccc} d_1 & \dots & d_k & d_{k+1} & \dots & d_s & d_{s+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{ru} & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & -1 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \end{array} \tag{6.29}$$

step 2) We assume $l = l-n$, where $1 \leqslant n \leqslant L-1$. We proceed with descending induction on $0 \leqslant r \leqslant R_{l-n}$. (Note that the number of indeterminates of the polynomial ring $\mathrm{K}[x, y]$ is fixed in this section and it is 2.)

sub-step 1)) For $l = l - n$ and $r = R_{l-n}$, the term in the order ideal that is indexed as $t_{l-nR_{l-n}}$ is x-exposable. In this case only y-next door border elements will result with non-trivial results. These next door relations are between the border terms located on the rightmost column of the diagram of $\mathcal{O}$. Hence it suffices to consider $1 \leqslant i \leqslant L_R$. Let us focus on the polynomial, say τ_s, that is constructed as a result of lifting y-neighbors, i.e. the polynomial $\mathcal{Y}_{rl-n+1\,i}$.

$$c_{rl-n+1\,i} - c_{rl-n\,i-1} - \sum_{\substack{t_{r'l'} \in \mathcal{O} \\ yt_{r'l'} = b_m \in \partial\mathcal{O}}} c_{r'l'\,i-1} c_{rl-n+1\,m}$$

We claim that the polynomial $\sum_{\substack{t_{r'l'} \in \mathcal{O} \\ yt_{r'l'} = b_m \in \partial\mathcal{O}}} c_{r'l'\,i-1} c_{rl\,m}$ is in $\mathrm{K}[\mathcal{S}]$. By Lemma 6.4.7 ii) the indeterminate $c_{r'l'\,i-1}$ is in $\mathcal{S}$. Moreover, for each $m \in \{L_R, ..., \nu\}$, the indeterminate $c_{rl\,m}$ is in $\mathcal{S}$, since each border term b_m is y-exposed and the term $t_{(l-n+1)(R_{l-n})}$ is x-exposable. This proves the claim that the degree two homogenous part of $\mathcal{Y}_{rl-n+1\,i}$ is in $\mathrm{K}[\mathcal{S}]$. If we let $d_k = c_{rl-n+1\,i}$ and $d_s = c_{rl-n\,i-1}$ with $l - n + 1 > l - n$, then by 6.27 we have $d_k \prec d_s$ and the r^{th} and the s^{th} rows of the matrix $\mathcal{M}_{\mathcal{S}}$ are the same as 6.29.

sub-step 2) For the last induction step, we assume $0 \leqslant r < R_{l-n}$.

1) As a result of the property iii) of " $\prec$ " given in 6.27, we first focus on $\mathcal{X}_{r-1l-n+1\,i}$, which is constructed as a result of the lifting of across the street neighbors.

$$c_{rl-n\,i-1} - c_{r-1l-n+1\,i} - \sum_{\substack{t_{r'l'} \in \mathcal{O} \\ xt_{r'l'} = b_q \in \partial\mathcal{O}}} c_{r'l'\,i-1} c_{r+1l-n\,q} + \sum_{\substack{t_{r"l"} \in \mathcal{O} \\ yt_{r"l"} = b_m \in \partial\mathcal{O}}} c_{r"l"\,i} c_{r+1l-n\,m}$$

If we let $d_k = c_{r-1l-n+1\,i}$ and $d_s = c_{rl-n\,i-1}$, then we have $d_k \prec d_s$. By Lemma 6.4.8 we can write the degree two homogenous part $\tau_s^{(2)}$ as a product of an element from $\mathcal{C} \setminus \mathcal{S}$, say d_p, and an element from $\mathcal{S}$. By Lemma 6.4.7 iii), the terms $c_{r'l'\,i-1}$ and $c_{r"l"\,i}$ in the above equation are in $\mathcal{S}$. Then $d_p \in \mathcal{C} \setminus \mathcal{S}$ can be $c_{rl-n+1\,q}$ or $c_{rl-n+1\,m}$. In each case, as a result of Property i) of " $\prec$ ", for $l - n + 1 > l - n$ we have $d_p \prec d_s$.

Moreover, by Property ii) of " $\prec$ ", for each $1 \leqslant i \leqslant \nu$ indeterminate $d_p \prec c_{r-1l-n+1\,i} = d_k$, since we have $r-1 < r$. Then by the induction hypothesis for each index $u > k$ the element in position (k,u) of $\mathcal{M}_{\mathcal{S}}$ is 0 and for each index $v > p$ the element in position (p,v) of $\mathcal{M}_{\mathcal{S}}$ is 0. Hence the k^{th}, p^{th} and s^{th} rows of $\mathcal{M}_{\mathcal{S}}$ have the following form.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_s \end{array} \begin{array}{c} \begin{array}{cccccccccccc} d_1 & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_s & d_{s+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & \pm 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{k1} & \dots & a_{kp} & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{s1} & \dots & a_{sp} & \dots & -1 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \end{array} \tag{6.30}$$

Thus the rows of the matrix $\mathcal{M}_{\mathcal{S}}$ have the desired form. By Lemma 6.4.8 the entries of the matrix $\mathcal{M}_{\mathcal{S}}$ are in $\mathrm{K}[\mathcal{S}]$. The entries which are represented by a_{ij} are elements of $\mathrm{K}[\mathcal{S}]$.

2) Now we consider the polynomials in $\mathcal{T}_{\mathcal{S}}$ that are constructed as a result of lifting y-next door neighbors. We focus on polynomial $\mathcal{Y}_{rl-n\,i-1}$ which has the degree 2 homogenous part

$$c_{rl-n+1\,i} - c_{rl-n\,i-1} - \sum_{\substack{t_{r'l'} \in \mathcal{O} \\ yt_{r'l'} = b_m \in \partial\mathcal{O}}} c_{r'l'\,i-1} c_{rl-n+1\,m}.$$

By Lemma 6.4.7ii), the term $c_{r'l'\,i-1}$ in the above equation is in $\mathcal{S}$. Similarly, the border term b_m is y-exposed, but the indeterminate $c_{rl-n\,m}$ is not necessarily in $\mathcal{S}$. If we let $c_{rl-n+1\,m} = d_p$, $d_k = c_{rl-n+1\,i}$ and $d_s = c_{rl-n\,i-1}$, then we have $d_k \prec d_s$ and $d_p \prec d_s$. By the induction hypothesis, the k^{th} and p^{th} rows of $\mathcal{M}_{\mathcal{S}}$ satisfy the claim that in the k^{th} row, after the k^{th} column there are only zeros and in the p^{th} row, after the p^{th} column there are only zeros. There are cases which τ_p is a result of across the street relation. Then by the relation $\prec$, $d_p \prec d_k$ holds. And there are cases which $\tau_{p'}$ is a result of x-next door relations. Then by the relation $\prec$, $d_k \prec d_{p'}$ holds. Then for $u \in \{p, k, p'\}$ we have

$$\frac{\partial \tau_u}{\partial d_v} = 0$$

if $d_u \prec d_v$ holds and

$$\frac{\partial \tau_u}{\partial d_u} = \pm 1.$$

Then for the row and columns p, k, p' the matrix $\mathcal{M}_{\mathcal{S}}$ has the following form.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_{p'} \end{array} \begin{array}{c} \begin{array}{ccccccccccc} d_1 & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_{p'} & d_{p'+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & \pm 1 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{k1} & \dots & a_{kp} & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{p'1} & \dots & a_{p'p} & \dots & a_{p'k} & a_{p'k+1} & \dots & \pm 1 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

Since we have

$$\frac{\partial \tau_s}{\partial d_v} = 0 \quad \text{for} \quad d_v \notin \{d_p, d_k, d_{p'}, d_s\}$$

and

$$\frac{\partial \tau_s}{\partial d_s} = \pm 1$$

Consequently with

$$d_p \prec d_k \prec d_{p'} \prec d_s,$$

the rows and the columns p, k, p', s of the matrix $\mathcal{M}_{\mathcal{S}}$ have the following form.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_{p'} \\ \tau_s \end{array} \begin{array}{c} \begin{array}{ccccccccccc} d_1 & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_{p'} & \dots & d_s & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & \pm 1 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ a_{k1} & \dots & a_{kp} & \dots & 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ a_{p'1} & \dots & a_{p'p} & \dots & a_{p'k} & a_{p'k+1} & \dots & \pm 1 & \dots & 0 & \dots & 0 \\ a_{s1} & \dots & a_{sp} & \dots & -1 & a_{s\,k+1} & \dots & a_{sp'} & \dots & 1 & \dots & 0 \end{pmatrix} \end{array} \tag{6.31}$$

Note that by Lemma 6.4.8 the entries of the matrix $\mathcal{M}_{\mathcal{S}}$ are in $\mathrm{K}[\mathcal{S}]$.

3) By the property iv) of 6.27, the last step is to consider the polynomials in $\mathcal{T}_{\mathcal{S}}$, that are constructed as a result of the x-next door neighborhood relation. Their form is given in Polynomial 6.24.

$$c_{rl-n\,i-1} - c_{r-1l-n\,i} - \sum_{\substack{t_{r'l'} \in \mathcal{O} \\ x t_{r'l'} = b_q \in \partial\mathcal{O}}} c_{r'l'\,i} c_{rl-n\,q}$$

By Lemma 6.4.7 i) for each $i \in \{1, ..., \nu\}$ the indeterminate $c_{r'l'\,i}$ is in $\mathcal{S}$. If we let $d_s = c_{r-1l-n\,i}$ and $d_k = c_{rl-n\,i-1}$ and for $i \in \{1, .., \nu\}$ we let $c_{rl-n\,q} = d_p$, then we have $d_k \prec d_s$ and $d_p \prec d_s$. In this case such

d_s does not exist. Since the border term b_i is y-exposed, for each τ_p that corresponds to d_p is a result of either an y-next door neighborhood relation or an across the street neighborhood relation, where order ideal elements were located on the row $l-n$ and column r. By considering the properties iii) and iv) of 6.27 , we have $d_p \prec d_k$. Hence by applying the induction hypotheses, the k^{th} and s^{th} rows of $\mathcal{M}_{\mathcal{S}}$ are as follows.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_s \end{array} \begin{array}{c} \begin{array}{ccccccccccc} d_1 & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_s & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & \pm 1 & \dots & 0 & 0 & \dots & 0 & \dots & 0 \\ a_{k1} & \dots & a_{kp} & \dots & 1 & 0 & \dots & 0 & \dots & 0 \\ a_{s1} & \dots & a_{sp} & \dots & -1 & 0 & \dots & 1 & \dots & 0 \end{pmatrix} \end{array} \tag{6.32}$$

Then by the induction hypothesis the rows p and s of the matrix $\mathcal{M}_{\mathcal{S}}$ where $d_p \prec d_s$ is of the form 6.28.

□

Recall that an indeterminate c_{ij} is called standard, if the arrow degree $\deg_W(c_{ij})$ has exactly one positive component. Otherwise it is called non-standard. We let $x_k \in \{x, y\}$. An indeterminate c_{ij} is called x_k-standard, if only the kth component of the degree vector $\deg_W(c_{ij})$ is positive.

Remark 6.4.12. Let us summarize what we showed until now. Recall that $\mathcal{T}_{\mathcal{S}}$ is a system of the generators of the vanishing ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$. As a result of the construction of $\mathcal{T}_{\mathcal{S}}$ for each element τ_p of $\mathcal{T}_{\mathcal{S}}$, there is an element, d_p in $\mathcal{C} \setminus \mathcal{S}$ with $\frac{\partial \tau_p}{\partial d_p} = \pm 1$. For two distinct neighbors b_{j_1} and b_{j_2} we denote the homogenous (with respect to the standard degree) of degree 1 part of τ_p by $\tau_p^{(1)} = c_{i_1 k_1 j_1} - c_{i_2, k_2 j_2}$ and homogenous of degree 2 part of τ_p by $\tau_p^{(2)} = \sum_\lambda c_{r_\lambda q_\lambda l_\lambda} s_\lambda$. As seen, there are two possibilities for d_p. As a result of $\prec$, we have $c_{i_1 j_1} \prec c_{i_2 j_2}$ and for each λ, we have $c_{r_\lambda q_\lambda l_\lambda} \prec c_{i_2 k_2 j_2}$. By Lemma 6.4.10 and Lemma 6.4.11, the choice of $d_p = c_{i_2 k_2 j_2}$ is made with respect to the " $\prec$ ". If $the indeterminate d_p$ is a non-standard indeterminate or an x-standard indeterminate, then by constructing the polynomial τ_p by neighbor relations of b_{j_2} results with the following two possible conclusions. First the indeterminate $c_{i_1 k_1 j_1}$ does not exist and the second one is that the indeterminate $c_{i_1 k_1 j_1}$ is in $\mathcal{S}$. Moreover, by choosing $d_p = c_{i_2 k_2 j_2}$ for τ_p and proceeding this way will place $\frac{\partial \tau_p}{\partial d_p} = \pm 1$ in the diagonal. If τ_p is an y-standard polynomial and we choose $d_p = c_{i_2 k_2, j_2}$, then there are two cases to consider. The first case is that $t_{i_1 k_1}$ is x-exposable or the border terms b_{j_1} and b_{j_2} have only x-next door neighbors. Then $c_{i_1 k_1 j_1}$ is in $\mathcal{S}$ and this case is similar to the previous argument. The second case is that one of the border terms b_{j_1} or b_{j_2} has an across the street neighbor.

We investigate this case in the next proof in detail and show that the matrix is lower triangular.

Lemma 6.4.13. *Let $\mathcal{O} \in \mathbb{T}^2$ be an order ideal. If the order ideal $\mathcal{O}$ has the sawtooth form then the matrix $\mathcal{M}_{\mathcal{S}}$ is similar to a lower triangular matrix that have ± 1 on the diagonal.*

Proof. It suffices to show that row operations on the rows, say the p^{th} row, where the the arrow degree of τ_p is y-standard, will result with the lower diagonal matrix that has ± 1 on the diagonal, since as stated in Remark 6.4.12 the other rows are already in the desired shape. Let us illustrate why the case which y-standard polynomial τ_p is different than the other cases. Let the homogenous component of τ_p of degree 1 be

$$\tau_p^{(1)} = c_{rl\,i-1} - c_{r-1l\,i}.$$

We assume τ_p is constructed as a result of x-next door neighbor relation of b_{i-1} and b_i. By Lemma 6.4.10 and Lemma 6.4.11, we have $d_p = c_{r-1l\,i}$. Then let τ_k and τ_s be polynomials constructed as result of across the street neighbors b_{i-1}, b_{i-2} and b_{i-2}, b_{i-3} respectively. We let $\tau_k^{(1)} = c_{rl\,i-1} - c_{r+1l-1\,i-2}$ and $\tau_s^{(1)} = c_{r+1l-1\,i-2}$. As above we have $d_k = c_{r+1l-1\,i-2}$ and $d_s = c_{r+1l-1\,i-2}$. Moreover, we have $d_p \prec d_k = d_s$ and we let $d = c_{rl\,i-1}$. Hence the corresponding rows and the columns of the matrix $\mathcal{M}_{\mathcal{S}}$ are as follows where we let a_{ij} denote the element of $\mathrm{K}[\mathcal{S}]$ in position (i, j) of $\mathcal{M}_{\mathcal{S}}$.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_s \end{array} \begin{array}{c} \begin{array}{cccccccccc} d_1 & \dots & d_q & d_{q+1} & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & 1 & a_{pq+1} & \dots & -1 & \dots & 0 & 0 & \dots & 0 \\ a_{k1} & \dots & 1 & a_{kq+1} & \dots & a_{kp} & \dots & -1 & 0 & \dots & 0 \\ a_{s1} & \dots & a_{sq} & a_{sq+1} & \dots & a_{sp} & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \end{array} \tag{6.33}$$

Clearly this matrix is not a lower triangular matrix, since $d_k = d_s$ and there are two different polynomials are indexed the same. Since this is only an enumeration problem, we fix it as follows. First we claim that $a_{ki} = 0$ for $q < i < p$. This follows from the fact that in Polynomial 6.25 we have $c_{r+1l\,n} \prec c_{rl\,i-1} \prec c_{r-1l+1\,i}$ and $c_{r+1l\,m} \prec c_{rl\,i-1} \prec c_{r-1l+1\,i}$. Then we claim that $a_{pi} = 0$ for $q \leqslant i < k$. This follows from the fact that in Polynomial 6.24 we have $c_{rl\,i-1} \prec c_{r-1l\,i}$ and $c_{rl\,n} \prec c_{r-1l\,i}$. Since the border term b_n is x-exposed, either $c_{rl\,n} \in \mathcal{S}$ or $c_{rl\,n} = d_r$ where τ_r is a result of either across the street or y-next door relation. As a result of the property (iv) of $\prec$, we have $c_{rl\,n} \prec c_{rl\,i-1} \prec c_{r-1l\,i}$. Our last claim is $a_{si} = 0$ for $q \leqslant i < k$. This follows from the

fact that in Polynomial 6.25. For $\tau_s^{(1)} = c_{r+1l-1\,i-2}$ we have

$$\tau_s^{(2)} = \sum_{\substack{t_{r'l'}\in\mathcal{O}\\ xt_{r'l'}=b_q\in\partial\mathcal{O}}} c_{r'l'\,i-3}c_{r+2l-1\,q} + \sum_{\substack{t_{r''l''}\in\mathcal{O}\\ yt_{r''l''}=b_m\in\partial\mathcal{O}}} c_{r''l''\,i-2}c_{r+2l-1\,m}.$$

As a result of the first property of $\prec$, $c_{r+2l-1\,m}$ and $c_{r+2l-1\,q}$ come before d_k, d_p and d_s. Hence the claim follows. Then the above matrix has the following shape.

$$\begin{array}{c} \\ \tau_p \\ \tau_k \\ \tau_s \end{array} \begin{array}{c} \begin{array}{cccccccccc} d_1 & \dots & d_q & d_{q+1} & \dots & d_p & \dots & d_k & d_{k+1} & \dots & d_m \end{array} \\ \begin{pmatrix} a_{p1} & \dots & 1 & 0 & \dots & -1 & \dots & 0 & 0 & \dots & 0 \\ a_{k1} & \dots & 1 & 0 & \dots & 0 & \dots & -1 & 0 & \dots & 0 \\ a_{s1} & \dots & 0 & 0 & \dots & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix} \end{array} \tag{6.34}$$

There are row operations that can be applied on the above matrix which make it a lower triangular matrix having ± 1 on the diagonal. Matrix 6.34 represents the p^{th}, k^{th} and the s^{th} rows of the matrix $\mathcal{M}_{\mathcal{S}}$ for polynomials τ_p, τ_k and τ_s with y-standard arrow degrees. If the polynomials are not of the y-standard arrow degree, they already satisfy the property of Lemma 6.4.11 and the above problem is avoided as a result of the properties (i) and (ii) of $\prec$. □

Proposition 6.4.14. *Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal in $\mathbb{T}^2$. If the order ideal $\mathcal{O}$ is of the form of a sawtooth then the $\mathcal{O}$-border basis scheme, $\mathbb{B}_{\mathcal{O}}$ is isomorphic to an affine space.*

Proof. By Lemma 6.4.13 the determinant of the matrix $\mathcal{M}_{\mathcal{S}}$ is ± 1 and by Lemma 6.4.8 the entries of this matrix are in K[$\mathcal{S}$]. Then by Theorem 6.2.20 the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ is an affine space.

□

Definition 6.4.15. An order ideal has the *deformed sawtooth* form if the diagram of $\mathcal{O}$ is of the form such that every step, width is decreasing, except possibly the topmost, and every step height, except possibly the rightmost, is of size one (see, for example, Diagram 6.3).

Let $\mathcal{O} = \{t_1, ..., t_\mu\}$ be an order ideal in $\mathbb{T}^2$. If $\mathcal{O}$ has the sawtooth form then the only y-next door neighbors are on the top row. if the $\partial\mathcal{O}$ has deformed sawtooth form, then there are border elements which are not located on the top row but are y-next door neighbors. The next example shows that Theorem 6.2.20 is not useful to verify

whether $\mathcal{O}$-border basis scheme is isomorphic to an affine space, if the order ideal $\mathcal{O}$ has the deformed sawtooth shape.

Example 6.4.16. Let $\mathcal{O} = \{1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3, x^2y^2, x^3y, xy^3, x^3y^2, x^4, y^4, x^4y, x^5, x^5y, x^6, x^6y, x^4y^2, x^5y^2, x^2y^3, x^3y^3\}$. Then the border of it is

$$\partial\mathcal{O} = \{y^5, xy^4, x^2y^4, x^3y^4, x^4y^3, x^7, x^5y^3, x^6y^2, x^7y\}.$$

Let us draw the diagram of $\mathcal{O}$.

$\mathbf{y^5}$							
y^4	$\mathbf{xy^4}$	$\mathbf{x^2y^4}$	$\mathbf{x^3y^4}$				
y^3	xy^3	x^2y^3	x^3y^3	$\mathbf{x^4y^3}$	$\mathbf{x^5y^4}$		
y^2	x^2y	x^3y^2	x^3y^2	x^4y^2	x^5y^2	$\mathbf{x^6y^2}$	
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	$\mathbf{x^7y}$
1	x	x^2	x^3	x^4	x^5	x^6	$\mathbf{x^7}$

Table 6.3: Diagram of $\mathcal{O}$

We compute the generating set $\mathcal{T}$ of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$, by using ApCoCoA and focus on the following polynomials.

$$\begin{aligned}
\tau_{416} &= -c_{42}c_{132} + c_{42}c_{161} - c_{41}c_{162} + c_{46}c_{201} + c_{49}c_{211} - c_{48}c_{212} \\
&\quad -c_{45}c_{222} + c_{48}c_{231} - c_{47}c_{232} - c_{43}c_{242} + c_{45}c_{251} - c_{44}c_{252} - c_{22} + c_{31}, \\
\tau_{513} &= c_{52}c_{162} + c_{56}c_{202} + c_{59}c_{212} + c_{58}c_{232} + c_{55}c_{252} + c_{22} - c_{53}, \\
s\tau_{924} &= c_{92}c_{163} + c_{96}c_{203} + c_{99}c_{213} + c_{98}c_{233} + c_{95}c_{253} + c_{53} - c_{94}, \\
\tau_{515} &= -c_{135}c_{152} + c_{152}c_{164} - c_{151}c_{165} + c_{156}c_{204} + c_{159}c_{214} - c_{158}c_{215} - c_{155}c_{225} \\
&\quad +c_{158}c_{234} - c_{157}c_{235} - c_{153}c_{245} + c_{155}c_{254} - c_{154}c_{255} + c_{94}, \\
\tau_{1216} &= -c_{122}c_{132} + c_{122}c_{161} - c_{121}c_{162} + c_{126}c_{201} + c_{129}c_{211} - c_{128}c_{212} - c_{125}c_{222} \\
&\quad +c_{128}c_{231} - c_{127}c_{232} - c_{123}c_{242} + c_{125}c_{251} - c_{124}c_{252} + c_{71} - c_{92}, \\
\tau_{1513} &= c_{152}c_{162} + c_{156}c_{202} + c_{159}c_{212} + c_{158}c_{232} + c_{155}c_{252} + c_{92} - c_{153}, \\
\tau_{1824} &= c_{163}c_{182} + c_{186}c_{203} + c_{189}c_{213} + c_{188}c_{233} + c_{185}c_{253} + c_{153} - c_{184}, \\
\tau_{2025} &= -c_{165}c_{201} - c_{135}c_{202} + c_{164}c_{202} + c_{204}c_{206} + c_{209}c_{214} - c_{208}c_{215} - c_{205}c_{225} \\
&\quad +c_{208}c_{234} - c_{207}c_{235} - c_{203}c_{245} + c_{205}c_{254} - c_{204}c_{255} + c_{184}, \\
\tau_{1624} &= c_{162}c_{163} + c_{166}c_{203} + c_{169}c_{213} + c_{168}c_{233} + c_{165}c_{253} - c_{164},
\end{aligned}$$

By Proposition 4.1.6 a) $\bar{c}_{16\,4}$ and $\bar{c}_{18\,4}$ are both in $\mathfrak{m}^2$ and hence they are in $\mathcal{C} \setminus \mathcal{S}$. By Proposition 4.1.6 a) and b)

$$c_{2\,2}, c_{31}, c_{5\,3}, c_{9\,4}, c_{9\,2}, c_{7\,1}, c_{15\,3}, c_{18\,4}, c_{20\,5}, c_{22\,5}$$

are in $\mathfrak{m}^2$ and hence in $\mathcal{C} \setminus \mathcal{S}$, as well. Therefore for every possible $\mathcal{S}$, the set $\mathcal{T}_{\mathcal{S}}$ contains the polynomial $\tau_{9\,24} = c_{9\,2}c_{16\,3} + c_{9\,6}c_{20\,3} + c_{9\,9}c_{21\,3} + c_{9\,8}c_{23\,3} + c_{9\,5}c_{25\,3} + c_{5\,3} - c_{9\,4}$ where $\frac{\partial \tau_{9\,24}}{\partial c_{16\,3}} = c_{92} \in M$ and $\frac{\partial \tau_{16\,3}}{\partial c_{9\,2}} = c_{16\,3} \in \mathcal{C} \setminus \mathcal{S}$. Then for any $\mathcal{S}$ the matrix $\mathcal{M}_{\mathcal{S}}$ is not in $\mathrm{Mat}_m(\mathrm{K}[\mathcal{S}])$. But we note that for $\mathcal{S}$ that is given in Construction 6.4, the determinant $\mathcal{M}_{\mathcal{S}}$ is 1.

The Example indicates that Theorem 6.2.20 can't be applied directly to the case which the order ideal has deformed sawtooth form, since the condition $\mathcal{M}_{\mathcal{S}} \in \mathrm{Mat}_{\mu\nu-2\mu}(\mathrm{K}[\mathcal{S}])$ is not satisfied.

Using Theorem 6.2.20 to prove that $\mathbb{B}_{\mathcal{O}}$ is an affine space, where $\mathcal{O}$ is of the shape *sawtooth*, is not simpler than the proof [Huib02]. But we note that, Proposition 7.1.2 of [Huib02] only holds for two dimensional cases whereas Theorem 6.2.20 holds for every number of dimensions. We conjecture that our theorem can be generalized further in Conjecture 6.2.25. It asks whether any given $\mathcal{O}$-border basis scheme is an affine space if the determinant of the matrix $\mathcal{M}_{\mathcal{S}}$, which is not necessarily in $\mathrm{K}[\mathcal{S}]$, is ± 1.

Chapter 7

Appendix

Function 7.0.17. (WMat) *This function computes the Weight Matrix with respect to the arrow grading.*

$$\text{WMat}(\text{OO}, \text{BO}, \text{N}) :$$
$$\text{WMat}(\text{OO} : \text{LIST}, \text{BO} : \text{LIST}, \text{N} : \text{INTEGER}) : \text{MATRIX}$$

InPut*:The order ideal OO, the border BO and the number of Indeterminates of the Polynomial Ring.*
OutPut*: Weight Matrix.*

Example 7.0.18.
```
Use R::=QQ[x[1..2]];
OO:=BB.Box([1,1]);
BO:=BB.Border(OO);
N:=Len(Indets());
W:=BBSGen.Wmat(OO,BO,N);

W;

Mat([
  [0, 2, 1, 2, 0, 2, 1, 2, -1, 1, 0, 1, -1, 1, 0, 1],
  [2, 0, 2, 1, 1, -1, 1, 0, 2, 0, 2, 1, 1, -1, 1, 0]
])
-----------------------------
```

Function 7.0.19. (NonStand) *: This function computes the non-standard Indeterminates in the Coordinate Ring of Border Basis Scheme, $\mathbb{B}_{\mathcal{O}}$.*

NonStand(OO, BO, N, W);

NonStand(OO : LIST, BO : LIST, N : INTEGER, W : MATRIX) : LIST

InPut*:The order ideal OO, border of OO, N the number of Indeterminates of the Polynomial Ring and the Weight Matrix.*
OutPut*: List of Indeterminates and their degree with respect to the arrow grading.*

Example 7.0.20.
```
Use R::=QQ[x[1..2]];
OO:=BB.Box([1,1]);
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2;----Number of indeterminates in the ring R.
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];

BBSGen.NonStand(OO,BO,N,W);
[[c[1,3], [R :: 1, R :: 2]], [c[1,4], [R :: 2, R :: 1]],
[c[2,3], [R :: 1, R :: 1]], [c[3,4], [R :: 1, R :: 1]]]
------------------------------
```

Function 7.0.21. (TraceSyzFull) *: Let $\tau_{ij}^{kl} := t[k,l,i,j]$ be the$(i,j)^{th}$ entry of matrix the operation $[A_k, A_l]$. The result of the Trace Syzygy computation is $K[\mathcal{C}]$-linear combination of τ_{ij}^{kl} that is equal to 0. This function computes such Trace Syzygy polynomials. This function, because of the growth of the polynomials during the computation, may not give result for the ring with three indeterminates.*

TraceSyzFull(OO, BO, N);

TraceSyzFull(OO : LIST, BO : LIST, N : INTEGER) : LIST

InPut*:The order ideal OO, border BO, the number of Indeterminates of the Polynomial.*
OutPut*: The list of Trace Syzygy polynomial.*

Example 7.0.22.
```
Use R::=QQ[x[1..2]];
OO:=BB.Box([1,1]);
BO:=BB.Border(OO);
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
N:=2; -----Number of indeterminates in the ring R.
BBSGen.TraceSyzFull(OO,BO,N);

[
  c[1,2]t[1,2,3,1] + c[2,2]t[1,2,3,2] + c[3,2]t[1,2,3,3] +
  c[4,2]t[1,2,3,4] + c[1,4]t[1,2,4,1] + c[2,4]t[1,2,4,2] +
   c[3,4]t[1,2,4,3] + c[4,4]t[1,2,4,4] + t[1,2,1,3] + t[1,2,2,4],
  2c[1,1]t[1,2,2,1] + 2c[2,1]t[1,2,2,2] + 2c[3,1]t[1,2,2,3] +
   2c[4,1]t[1,2,2,4] + 2c[1,3]t[1,2,4,1] + 2c[2,3]t[1,2,4,2] +
   2c[3,3]t[1,2,4,3] + 2c[4,3]t[1,2,4,4] + 2t[1,2,1,2] + 2t[1,2,3,4],
  c[1,2]t[1,2,3,1] + c[2,2]t[1,2,3,2] + c[3,2]t[1,2,3,3] + c[4,2]t[1,2,3,4] +
  c[1,4]t[1,2,4,1] + c[2,4]t[1,2,4,2] + c[3,4]t[1,2,4,3] + c[4,4]t[1,2,4,4] +
  t[1,2,1,3] + t[1,2,2,4],
  2c[1,2]c[3,1]t[1,2,2,1] + 2c[1,4]c[4,1]t[1,2,2,1] + 2c[2,2]c[3,1]t[1,2,2,2] +
   2c[2,4]c[4,1]t[1,2,2,2] + 2c[3,1]c[3,2]t[1,2,2,3] + 2c[3,4]c[4,1]t[1,2,2,3] +
   2c[3,1]c[4,2]t[1,2,2,4] + 2c[4,1]c[4,4]t[1,2,2,4] + 2c[1,2]c[3,3]t[1,2,4,1] +
   2c[1,4]c[4,3]t[1,2,4,1] + 2c[2,2]c[3,3]t[1,2,4,2] + 2c[2,4]c[4,3]t[1,2,4,2] +
    2c[3,2]c[3,3]t[1,2,4,3] + 2c[3,4]c[4,3]t[1,2,4,3] + 2c[3,3]c[4,2]t[1,2,4,4] +
    2c[4,3]c[4,4]t[1,2,4,4] + 2c[1,1]t[1,2,2,3] + 2c[2,1]t[1,2,2,4] +
   2c[1,4]t[1,2,3,1] + 2c[2,4]t[1,2,3,2] + 2c[3,4]t[1,2,3,3] +
    2c[4,4]t[1,2,3,4] + 2c[1,3]t[1,2,4,3] + 2c[2,3]t[1,2,4,4] + 2t[1,2,1,4]]
------------------------------
-- Done.
------------------------------
```

NOTE: As one can see in the example some syzygies might appear more than once.

Function 7.0.23. (TraceSyzLin) *Let* $\tau_{ij}^{kl} := t[k,l,i,j]$ *be the*$(i,j)^{th}$ *entry of matrix the operation* $[A_k, A_l]$. *The result of the Trace Syzygy computation is* $K[\mathcal{C}]$*-linear combination of* τ_{ij}^{kl} *that is equal to* 0. *This function only computes the summands of Trace*

Syzygy, which has constant and non-zero coefficient.

TraceSyzLin(OO, BO, N);
TraceSyzLin(OO : LIST, BO : LIST, N : INTEGER) : LIST

***InPut**: The order ideal OO, border BO, the number of Indeterminates of the Polynomial.*
***OutPut**: List of polynomials τ_i^{kl}, which have constant coefficients in the Trace Syzygy polynomial.*

Example 7.0.24.

```
Use R::=QQ[x[1..2]];
OO:=BB.Box([1,1]);
BO:=BB.Border(OO);
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
N:=2; ------Number of indeterminates in the ring R.
BBSGen.TraceSyzLin(OO,BO,N);

[
  t[1,2,1,3] + t[1,2,2,4],
  2t[1,2,1,2] + 2t[1,2,3,4],
  t[1,2,1,3] + t[1,2,2,4],
  2t[1,2,1,4]]
------------------------------
-- Done.
---------------------
```

Function 7.0.25. (TraceSyzStep) *Let $\tau_{ij}^{kl} := t[k, l, i, j]$ be the$(i, j)^{th}$ entry of matrix the operation $[A_k, A_l]$. The result of the Trace Syzygy computation is $K[\mathcal{C}]$-linear combination of τ_{ij}^{kl} that is equal to 0. This function only computes the trace syzygy for the degree of the given monomial(not necessarily a target monomial).*

TraceSyzStep(Mon, X, OO, BO, N);
TraceSyzStep(Mon : MON, X : INDET, OO : LIST, BO : LIST, N : INT) : POLY

InPut:*The Monomial Mon, the distinguished indterminate of choice, order ideal OO, border BO, the number of Indeterminates of the Polynomial.*
OutPut: *Trace syzygy of the degree of the given monomial.*

Example 7.0.26. `Use R::=QQ[x[1..2]];`

```
OO:=BB.Box([1,1]);
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2; ----Number of indeterminates in the ring R.
Mon:=x[1]^2x[2];----------Target Monomial
X:=x[1];   ---------------Choice of the Indeterminate
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.TraceSyzStep(Mon,X,OO,BO,N);

  c[1,2]t[1,2,3,1] + c[2,2]t[1,2,3,2] +
   c[3,2]t[1,2,3,3] + c[4,2]t[1,2,3,4] +
   c[1,4]t[1,2,4,1] +
   c[2,4]t[1,2,4,2] + c[3,4]t[1,2,4,3] +
   c[4,4]t[1,2,4,4] + t[1,2,1,3] + t[1,2,2,4]
 ------------------------------
-- Done.
------------------------------
```

Function 7.0.27. (JacobiFull) *Let $R = K[x_1, ..., x_n]$ and A_i is the generic multiplication matrix for x_i. Let $\tau_{ij}^{kl} := t[k, l, i, j]$ be the entry in position (i, j) of matrix the operation $[A_k, A_l]$. This function computes the entries of the Jacobi identity*

$$[A_i[A_k, A_l]] + [A_k[A_l, A_i]] + [A_l[A_i, A_k]] = 0,$$

where i, k, l is from $\{1...n\}$. Note that this function does not work for the case, where $n = 2$.

JacobiFull(OO, BO, N);
JacobiFull(OO : LIST, BO : LIST, N : INTEGER) : MATRIX

***InPut**:Order ideal OO, border BO, the number of Indeterminates of the Polynomial.*
***OutPut**: The entries of the Jacobi Identity.*

Example 7.0.28.
```
Use R::=QQ[x[1..3]];
OO:=[1,x[1]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=3;------Number of indeterminates in the ring R.
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.JacobiFull(OO,BO,N);

[[   [c[1,1]t[1,2,1,1] + c[1,3]t[1,2,2,1] +
     c[1,2]t[1,3,1,1] + c[1,4]t[1,3,2,1] + c[1,5]t[2,3,2,1],
      c[1,1]t[1,2,1,2] + c[1,3]t[1,2,2,2] + c[1,2]t[1,3,1,2] +
       c[1,4]t[1,3,2,2] + c[1,5]t[2,3,2,2]],
    [ c[2,1]t[1,2,1,1] + c[2,3]t[1,2,2,1] + c[2,2]t[1,3,1,1] +
     c[2,4]t[1,3,2,1] + c[2,5]t[2,3,2,1] + t[2,3,1,1],
      c[2,1]t[1,2,1,2] + c[2,3]t[1,2,2,2] +
      c[2,2]t[1,3,1,2] + c[2,4]t[1,3,2,2] + c[2,5]t[2,3,2,2] + t[2,3,1,2]]]]
------------------------------
-- Done.
------------------------------
```

Function 7.0.29. (JacobiLin) *Let $R = K[x_1, ..., x_n]$ and A_i is the generic multiplication matrix for x_i. Let $\tau_{ij}^{kl} := t[k, l, i, j]$ be the entry in position (i, j) of the matrix the operation $[A_k, A_l]$. This function computes the entries of the Jacobi identity*

$$[A_i[A_k, A_l]] + [A_k[A_l, A_i]] + [A_l[A_i, A_k]] = 0,$$

where i, k, l is from $\{1...n\}$, which has constant coefficients. Note that this function does not work for the case, where $n = 2$.

JacobiLin(OO, BO, N);
JacobiLin(OO : LIST, BO : LIST, N : INT) : MATRIX

***InPut**:Order ideal OO, border BO, the number of Indeterminates of the Polynomial.*

OutPut: *The entries of the Jacobi Identity.*

Example 7.0.30.

```
Use R::=QQ[x[1..3]];
OO:=[1,x[1]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=3;-------Number of indeterminates in the ring R.
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.JacobiLin(OO,BO,N);

[  [  [
      -t[2,3,1,2],
      0],
    [
      t[2,3,1,1] - t[2,3,2,2],
      t[2,3,1,2]]]]
------------------------------
-- Done.
------------------------------
```

Function 7.0.31. Class *This function computes the equivalent indeterminates in the cotangent space* $mathfrakm/\mathfrak{m}^2$ *of border basis scheme* $\mathbb{B}_{\mathcal{O}}$ *(BBS) and gives these equivalent indeterminates in the first list and the elements of* $\mathfrak{m}^2$ *in the second list that starts with 0. The base ring is* $K[x_1, .., x_n]$.

Class(OO, BO, W, N);
Class(OO : LIST, BO : LIST, N : INTEGER, W : MATRIX) : LIST

InPut:*The order ideal OO.*
OutPut:*The list of* K*-linearly independent indeterminates in the cotangent space.*

Example 7.0.32.

```
Use R::=QQ[x[1..2]];
OO:=[1,x[1],x[2],x[1]x[2]];
BO:=BB.Border(OO);
Mu:=Len(OO);
```

```
Nu:=Len(BO);
W:=BBSGen.Wmat(OO,BO,N);
N:=2;------Number of indeterminates in the ring R.
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.Class(OO,BO,W,N);
[
  [
    [
      c[2,3],
      c[1,1]],
    [
      c[3,4],
      c[1,2]],
    [
      c[4,4],
      c[2,2]],
    [
      c[4,3],
      c[3,1]]],
  [
    0,
    c[1,4],
    c[1,3],
    c[2,4],
    c[3,3]]]
-------------------------------
-- Done.
-------------------------------
```

Function 7.0.33. EqClass *Computes the equivalence classes of the elements in the coordinate ring of border basis scheme modulo the ideal* $\mathfrak{m}^2$ *by using the defining equations of the given OO-border basis scheme. The ideal* $\mathfrak{m}$ *is the maximal which is homogenous with respect to the arrow grading.*

EqClass(OO, BO, W, N);
EqClass(OO : LIST, BO : LIST, W : MAT, N : INT) : LIST

InPut:*The order ideal OO, Border of the order ideal BO, the weight matrix W, the number of indeterminates of the ground ring.*
OutPut:*List of the list of equivalence classes modulo* $\mathfrak{m}^2$.

Example 7.0.34.

```
Use R::=QQ[x[1..2]];
OO:=[1,x[1],x[2],x[1]x[2]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2;---- Number of indeterminates in the ring R.
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.EqClass(OO,BO,W,N);
[
  [
    c[2,3],
    c[1,1]],
  [
    c[3,4],
    c[1,2]],
  [
    c[4,4],
    c[2,2]],
  [
    c[4,3],
    c[3,1]]]
```

Function 7.0.35. IsMonSmooth *Compares the number of bases elements in* $\mathfrak{m}/\mathfrak{m}^2$ *and the dimension of the pronciple component of the OO-border basis scheme which is* $N * Mu$, *where* N *is the number of indeterminates of the ring which OO is defined and* Mu *is the number of terms in OO.*

IsMonSmooth(OO, BO, W, N);
IsMonSmooth(OO : LIST, BO : LIST, W : MAT, N : INT) : BOOL

InPut:*The order ideal OO, border of the order ideal BO, the weight matrix W, the number of indeterminates of the ground ring.*
OutPut:*BOOL (True or False)*

Example 7.0.36.

```
Use R::=QQ[x[1..2]];
OO:=[1,x[1],x[2],x[1]x[2]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2;---- Number of indeterminates in the ring R.
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.IsMonSmooth(OO,BO,W,N);
The monomial point of the BB scheme is smooth.True
------------------------------
-- Done.
------------------------------
```

Function 7.0.37. IsAffine *This function checks possible minors of the jacocian matrix of the defining equations of border basis scheme. Instead of checking the minors directly, we construct possible sub-matrices by computing the possible generators of the vanishing ideal of the OO-border basis scheme and possible basis sets for the vector space* $\mathfrak{m}/\mathfrak{m}^2$. *Because of the rapid growth of the elements in the Cartesian product, for border basis schemes with slightly large number of generators, one should not always expect result. We recommend the function* IsAffineEff *in that case .*

IsAffine(OO, BO, N, W, M, S, PolDeg, Pol);
IsAffine(OO : LIST, BO : LIST, W : MAT, N : INT, M : LIST, PolDeg : LIST, Pol : LIST)
: BOOL, LIST

InPut:*The order ideal OO, Border of the order ideal BO, the weight matrix W, the number of indeterminates of the ground ring N, the list possible basis elements S, the list of the sets of elements that S does not contain M, List of defining equations of border basis scheme and PolDeg.*
OutPut: *BOOL (True or False)*

Example 7.0.38.

```
Use R::=QQ[x[1..2]];
OO:=[1,x[1],x[2],x[1]x[2]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2;---- Number of indeterminates in the ring R.
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
BBSGen.IsAffine(OO,BO,N,W);
The monomial point of the BB scheme is smooth.
The Coordinate Ring of the given BB scheme is
  isomorphic to a polynomial ring K[S].
K[S]=K[
  c[2,1],
  c[3,2],
  c[4,1],
  c[4,2],
  c[2,3],
  c[3,4],
  c[4,4],
  c[4,3]]
The scheme can be generated by the polynomials[
  c[1,2]c[2,1] + c[1,4]c[4,1] - c[1,3],
  c[2,3]c[3,2] - c[3,1]c[3,4] + c[3,4]c[4,3] - c[3,3]c[4,4] - c[1,4],
  -c[2,1]c[3,2] - c[2,3]c[4,2] + c[2,4],
  c[2,1]c[3,2] + c[3,4]c[4,1] - c[3,3],
  c[2,1]c[2,2] + c[2,4]c[4,1] + c[1,1] - c[2,3],
  -c[3,1]c[3,2] - c[3,3]c[4,2] - c[1,2] + c[3,4],
  -c[3,2]c[4,1] - c[4,2]c[4,3] - c[2,2] + c[4,4],
  c[2,1]c[4,2] + c[4,1]c[4,4] + c[3,1] - c[4,3]]
True
-------------------------------
-- Done.
-------------------------------
```

Function 7.0.39. IsAffineEff *This function checks possible minors of the Jacobian matrix of the defining equations of border basis scheme. Instead of checking the minors directly, we construct possible sub-matrices by computing the possible generators of the vanishing ideal of the OO-border basis scheme and possible basis sets for the vector space* $\mathfrak{m}/\mathfrak{m}^2$. *This function is more efficient version of the function* IsAffineEff(OO, BO, N, W, M, S, PolDeg, Pol). *Instead of computing all the possible redundant polynomial candidates in* $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$, *we compute them step by step. After each step we check the determinant of the matrix* M_S, *and proceed to the next possible candidates if the determinant is not* ± 1. *Note that not all the possibilities in the generating set are computed.*

IsAffineEff(OO, BO, N, W, M, S, PolDeg, Pol).

IsAffineEff(OO : LIST, BO : LIST, W : MAT, N : INT, M : LIST, PolDeg : LIST, Pol : LIST) : BOOL, LIST

InPut*:The order ideal OO, Border of the order ideal BO, the weight matrix W, the number of indeterminates of the ground ring N, the list possible basis elements S, the list of the sets of elements that S does not contain M, List of defining equations of border basis scheme and PolDeg.*
OutPut*: BOOL (True or False)*

Example 7.0.40.

```
Use R::=QQ[x[1..2]];
OO:=[1,x[1],x[2],x[1]x[2]];
BO:=BB.Border(OO);
Mu:=Len(OO);
Nu:=Len(BO);
N:=2;---- Number of indeterminates in the ring R.
W:=BBSGen.Wmat(OO,BO,N);
Use XX::=QQ[c[1..Mu,1..Nu],t[1..N,1..N,1..Mu,1..Mu]];
 M:=BBSGen.FindM(OO,BO,W,N);   ---The elements in m^2 wrt the given S.
 PolDeg:=$apcocoa/bbsmingensys.OrganizePolDeg(OO,BO,W,N);
  ----Gives the tau's  with degrees and their position on the mult matrix
  Pol:= $apcocoa/bbsmingensys.OrganizePoly(OO,BO,W,N);
 ---Gives the defining equations of border basis scheme
 -- in K[C] that excludes the homogenous of degree 2 polynomials.
```

```
The monomial point of the BB scheme is smooth.
The number of possible subsets is 8
Is the matrix in K[S]?
 The matrix M_s is in K[S]
The matrix isMat([
  [-1, -c[4,2], 0, 0, 0, 0, 0, 0],
  [c[4,1], 1, 0, 0, 0, 0, 0, 0],
  [0, 0, 1, c[4,1], 0, 0, 0, 0],
  [0, -c[3,2], 0, 1, 0, 0, 0, 0],
  [c[2,1], 0, 0, 0, -1, 0, 0, 0],
  [0, 0, 0, c[2,1], 0, -1, 0, 0],
  [0, 0, 0, 0, -c[3,2], 0, 1, 0],
  [0, 0, 0, 0, -c[4,2], 0, 0, -1]
])
Computing the Determinant for the evaluated matrix
Determinant is -c[4,1]c[4,2] + 1
The evaluated Determinant is-c[4,1]c[4,2] + 1
Is the matrix in K[S]?
 The matrix M_s is in K[S]
The matrix isMat([
  [-1, 0, 0, 0, 0, 0, 0, 0],
  [c[4,1], -1, 0, 0, 0, 0, 0, 0],
  [0, 0, 1, c[4,1], 0, 0, 0, 0],
  [0, 0, 0, 1, 0, 0, 0, 0],
  [c[2,1], 0, 0, 0, -1, 0, 0, 0],
  [0, 0, 0, c[2,1], 0, -1, 0, 0],
  [0, -c[3,4], 0, 0, -c[3,2], 0, 1, 0],
  [0, -c[4,4], 0, 0, -c[4,2], 0, 0, -1]
])
Computing the Determinant for the evaluated matrix
Determinant is -1
Computing the Full determinant
The evaluated Determinant is-1
The Coordinate Ring of the given BB scheme is  isomorphic to a polynomial ring K[S]
```

```
K[S]=K[[
  c[2,1],
  c[2,3],
  c[3,2],
  c[3,4],
  c[4,1],
  c[4,2],
  c[4,3],
  c[4,4]]]
The scheme can be generated by the polynomials[
  c[1,3],
  c[1,2]c[2,1] + c[1,4]c[4,1] - c[1,3],
  c[1,4],
  c[2,3]c[3,2] - c[3,1]c[3,4] + c[3,4]c[4,3] - c[3,3]c[4,4] - c[1,4],
  c[2,4],
  -c[2,1]c[3,2] - c[2,3]c[4,2] + c[2,4],
  c[1,1],
  c[2,1]c[2,2] + c[2,4]c[4,1] + c[1,1] - c[2,3],
  c[1,2],
  -c[3,1]c[3,2] - c[3,3]c[4,2] - c[1,2] + c[3,4],
  c[2,2],
  -c[3,2]c[4,1] - c[4,2]c[4,3] - c[2,2] + c[4,4],
  c[3,1],
  c[2,1]c[4,2] + c[4,1]c[4,4] + c[3,1] - c[4,3],
  c[3,3],
  c[2,1]c[3,2] + c[3,4]c[4,1] - c[3,3]]
True
-------------------------------
-- Done.
-------------------------------
```

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