UNIVERSITÄT PASSAU
Fakultät für Informatik und Mathematik

Dissertation

Generalizations and Applications of Border Bases

Eingereicht an der Fakultät für Informatik und Mathematik
der Universität Passau als Dissertation zur Erlangung des
Grades eines Doktors der Naturwissenschaften

Markus Kriegl

September 2015

Betreuer:	**Prof. Dr. Martin Kreuzer**	Lehrstuhl für Symbolic Computation, Fakultät für Informatik und Mathematik, Universität Passau
externer Gutachter:	**Prof. Dr. Franz Winkler**	Research Institute for Symbolic Computation, Johannes Kepler Universität Linz

Abstract

Border bases of zero-dimensional ideals have turned out to be a very useful generalization of Gröbner bases in recent research in computational commutative algebra. Though border bases share many properties with Gröbner bases, they are still limited to zero-dimensional ideals in polynomial rings. This doctoral thesis is devoted to generalize border bases to the module setting and to apply them in various ways.

In the first part of this thesis, we generalize border bases of zero-dimensional ideals in a polynomial ring $P = K[x_1, \ldots, x_n]$, where K denotes a field, to border bases of submodules of free P-modules of finite rank with finite K-codimension. In particular, we prove their existence and uniqueness, characterize them in various ways, and give an algorithm for their computation that is based on linear algebra techniques. Then we introduce generalized border bases of submodules of arbitrary finitely generated P-modules with finite K-codimension. We characterize these generalized border bases by lifting them to border bases in free modules and show that we can compute them under certain circumstances. As an application of generalized border bases, we are able to characterize subideal border bases in various new ways and give an algorithm for their computation that is based on linear algebra techniques instead of Gröbner bases techniques. Moreover, we prove Schreyer's Theorem for border bases of submodules of free P-modules of finite rank with finite K-codimension, i.e. we prove that the set of all neighbor liftings of such a border basis forms a Gröbner basis of the first syzygy module of the border basis with respect to specific term orderings and we explicitly construct such a term ordering. As a byproduct, we deduce a new, alternative proof of the characterization of border bases via liftings of border syzygies which—in contrast to all previous proofs—does not depend on the characterization of border bases via commuting matrices.

In the second part of this thesis, we study the effect of homogenization to border bases of zero-dimensional ideals in P and applications of border bases in algebraic geometry. This yields the new concept of projective border bases of homogeneous one-dimensional ideals in $P[x_0]$, where x_0 denotes the homogenizing indeterminate. We prove that dehomogenization and homogenization yield a one-to-one correspondence between projective border bases in $P[x_0]$ and border bases in P of a specific shape. Then we explicitly describe the multiplicative structure of both a residue class ring R of $P[x_0]$ modulo a projective border basis and of the canonical module of R by means of formal multiplication matrices that only depend on the projective border basis. After that, we turn our attention to algebraic geometry and show that there is a one-to-one correspondence between projective border bases and zero-dimensional closed subschemes of weighted projective spaces that have no point on the hyperplane at infinity. This correspondence allows us to study schemes that satisfy certain uniformity conditions, e.g. Cayley-Bacharach schemes or schemes in uniform position, by means of the multiplicative structure of their coordinate ring and the corresponding canonical module. In particular, this approach allows us to characterize (i, j)-uniform zero-dimensional closed subschemes of weighted projective spaces that have a K-rational support in various ways without assuming that the base field K is algebraically closed or that the subscheme is reduced. If the base field K is algebraically closed or if the subscheme is reduced, we show that these characterizations immediately yield algorithms that allow us to check whether a given zero-dimensional closed subscheme is (i, j)-uniform or not. Finally, we introduce the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with respect to a given order ideal $\mathcal{O}$ as a specific subscheme of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$. We show that the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ parametrizes all zero-dimensional closed subschemes of a weighted projective space whose defining ideals possess a projective $\mathcal{O}$-border basis. Applying the above methods in this general setting and assuming that the base field K is algebraically closed, we are able to prove that the set of all closed points of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to an (i, j)-uniform subscheme is a constructive set with respect to the Zariski topology.

Zusammenfassung

In der jüngeren Forschung in der berechnenden kommutativen Algebra haben sich Randbasen von null-dimensionalen Idealen als eine nützliche Verallgemeinerung von Gröbnerbasen herausgestellt. Obwohl Randbasen viele Eigenschaften mit Gröbnerbasen gemein haben, sind sie immer noch auf das Studium null-dimensionaler Ideale in Polynomringen limitiert. Die folgende Doktorarbeit dient dazu, eine Theorie der Randbasen in endlich erzeugten Moduln über Polynomringen einzuführen und Randbasen auf verschiedene Arten anzuwenden.

Im ersten Teil dieser Arbeit verallgemeinern wir Randbasen von null-dimensional Idealen in einem Polynomring $P = K[x_1, \ldots, x_n]$, wobei K einen Körper bezeichne, zu Randbasen von Untermoduln von freien P-Moduln von endlichem Rang mit endlicher K-Kodimension. Dabei beweisen wir insbesondere deren Existenz und Eindeutigkeit, charakterisieren sie auf vielfältige Art und Weise und geben einen auf linearer Algebra basierenden Algorithmus zu ihrer Berechnung an. Im Anschluss daran führen wir verallgemeinerte Randbasen von Untermoduln von beliebigen endlich erzeugten P-Moduln mit endlicher K-Kodimension ein. Diese verallgemeinerten Randbasen charakterisieren wir dann, indem wir Sie auf Randbasen in freien P-Moduln zurückführen. Unter bestimmten Voraussetzungen können wir damit verallgemeinerte Randbasen berechnen. Als eine Anwendung von verallgemeinerten Randbasen finden wir einige neue Charakterisierungen für Unterideal-Randbasen und können einen Algorithmus zu deren Berechnung angeben, der auf linearer Algebra statt einer Gröbnerbasis-Berechnung beruht. Des Weiteren beweisen wir den Satz von Schreyer für Randbasen von Untermoduln von freien P-Moduln von endlichem Rang mit endlicher K-Kodimension, d. h. wir zeigen dass die Menge aller Nachbarliftungen einer solchen Randbasis bezüglich spezieller Termordnungen eine Gröbnerbasis des ersten Syzygienmoduls der Randbasis bildet und wir konstruieren eine solche Termordnung explizit. Als Nebenprodukt des Beweises zum Satz von Schreyer erhalten wir einen neuen, alternativen Beweis für die Charakterisierung von Randbasen mittels Liftungen von Randsyzygien, der im Gegensatz zu allen bisherigen Beweisen nicht auf der Charakterisierung von Randbasen mittels kommutierender Matrizen beruht.

Im zweiten Teil dieser Arbeit studieren wir den Effekt von Homogenisierungen auf Randbasen von null-dimensionalen Idealen in P und Anwendungen von Randbasen in der algebraischen Geometrie. Dies führt zur Definition von projektiven Randbasen von homogenen ein-dimensionalen Idealen in $P[x_0]$, wobei hier x_0 die Homogenisierungsunbestimmte bezeichne. Wir beweisen, dass die projektiven Randbasen in $P[x_0]$ durch Dehomogenisierung und Homogenisierung eineindeutig den Randbasen in P einer speziellen Form entsprechen. Weiter beschreiben wir die multiplikative Struktur sowohl eines Restklassenrings R von $P[x_0]$ modulo einer projektiven Randbasis als auch des kanonischen Moduls von R mithilfe von formalen Multiplikationsmatrizen, welche nur von

der projektiven Randbasis abhängen. Anschließend wenden wir uns der algebraischen Geometrie zu und beweisen, dass projektive Randbasen eineindeutig null-dimensionalen abgeschlossenen Unterschemata eines gewichteten projektiven Raumes entsprechen, welche keinen Punkt der unendlich fernen Hyperebene enthalten. Durch diese Beziehung können wir das Studium gewisser uniformer Schemata, z. B. Cayley-Bacharach Schemata oder Schemata in uniformer Lage, auf das Studium der multiplikativen Struktur des zugehörigen Koordinatenrings und dessen kanonischen Moduls zurückführen. Dieser Ansatz ermöglicht es uns, (i, j)-uniforme null-dimensionale abgeschlossene Unterschemata von gewichteten projektiven Räumen mit K-rationalem Träger auf verschiedene Arten zu charakterisieren, ohne dabei anzunehmen, dass der Grundkörper K algebraisch abgeschlossen oder das Unterschema reduziert sei. Falls der Grundkörper algebraisch abgeschlossen oder das Unterschema reduziert ist, liefern diese Charakterisierungen direkt einen Algorithmus zum Test auf (i, j)-Uniformität. Schließlich führen wir noch das projektive $\mathcal{O}$-Randbasisschema $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ bezüglich eines gegebenen Ordnungsideals $\mathcal{O}$ als ein spezielles Unterschema des $\mathcal{O}$-Randbasisschemas $\mathbb{B}_{\mathcal{O}}$ ein. Wir zeigen, dass $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ alle null-dimensionalen abgeschlossenen Unterschemata eines gewichteten projektiven Raumes parametrisiert, deren definierendes Ideal eine projektive $\mathcal{O}$-Randbasis besitzen. Indem wir die obigen Methoden auf diese allgemeine Situation anwenden und annehmen, dass der Grundkörper K algebraisch abgeschlossen ist, können wir beweisen, dass die Menge aller abgeschlossenen Punkte von $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, die einem (i, j)-uniformen Schema entsprechen, bezüglich der Zariski-Topologie eine konstruierbare Menge bildet.

Danksagung

An dieser Stelle möchte ich einigen Wegbegleiterinnen und Wegbegleitern meinen Dank aussprechen.

Allen voran möchte ich mich bei Prof. Martin Kreuzer für die Betreuung, konstruktive Zusammenarbeit und die Möglichkeit, an seinem Lehrstuhl zu promovieren, bedanken. Ebenso gilt mein Dank auch meinem externen Gutachter, Prof. Franz Winkler.

Des Weiteren seien an dieser Stelle alle meine Kolleginnen und Kollegen genannt, die mich auf dem Weg zur Promotion begleitet haben. Meinen besonderen Dank verdient neben Nathalie Vollstädt, die mich in allen organisatorischen Belangen entlastet hat, vor allem Thomas Stadler, mit dem ich über fünf Jahre ein Büro geteilt habe und der immer ein offenes Ohr für mich hatte.

Zu guter Letzt möchte ich noch all meinen Freunden, meiner Familie, und vor allem meiner Freundin Isabell danken. Ohne deren stetige Unterstützung wäre all dies nicht möglich gewesen.

Markus Kriegl
Passau, September 2015

Contents

1 Introduction

Since the introduction of Buchberger's Algorithm in 1965, cf. [Buc65] and [Buc06], Gröbner bases have become a standard tool in computational algebra. Though their computation is quite hard in general, namely it is exponential space hard as it was shown in [MM82], Gröbner bases allow us to constructively solve many problems both theoretically and computationally, cf., for instance, [KR00] and [KR05]. Unfortunately, even if we restrict ourselves to the class of zero-dimensional ideals, Gröbner bases do not behave very well in an approximate setting as described in [Ste04, Subsect. 8.4.4]. This drawback has led to a more general notion of bases, e.g. to the notion of border bases of zero-dimensional ideals. Though border bases behave more nicely in an approximate setting, they can only be applied to zero-dimensional ideals. Nevertheless, border bases turned out to be a good choice and much effort is put in the study of them. The theory of border bases can be divided into two parts. First one is particularly interested in their numerical behaviour. This is due to the fact that we can use them to approximately model a physical system that is described by a finite amount of data and thus yields a zero-dimensional ideal in a suitable polynomial ring. We refer, for instance, to [Ste04], [Lim14], [HKPP09], and [KPR10] for such numerical analyses and applications and draw the attention in this thesis to the second part, namely the algebraic behaviour of border bases and their applications in the exact setting.

The study of border bases of zero-dimensional ideals has brought to light that they share many of the nice properties Gröbner bases have. E.g. there is an explicit division algorithm by [KKR03, Subsext. 4.3.2], they share many characterizations according to [KK05], and they can be computed according to [KK06]. But border bases do not only share many properties of Gröbner bases. Some theorems are true for border bases but have no analogous version in the theory of Gröbner bases. For us, the main advantage of border bases is a characterization which has no analogon in the theory of Gröbner bases, namely the characterization via commuting multiplication matrices, which was introduced in [Mou99, Thm. 3.1]. This theorem states that we only have to check whether the matrices that represent the multiplication by an indeterminate are pairwise commuting. Though there were many advances in the theory of border bases, many well-known results from Gröbner bases lack a border basis version, e.g. the theory of border bases is not applicable to modules and there is no analogon of Schreyer's Theorem for border bases, cf. [Sch80] or [KR00, Prop. 3.1.4] for the Gröbner bases version.

The first part of this thesis, namely the Chapters 2 and 3, solve these disparities between Gröbner and border bases. In the second part, namely in Chapter 4, we study the behaviour of border bases under homogenization. During the last part of this thesis, which consists of the Chapters 5 and 6, we apply the previous results to zero-dimensional closed subschemes of weighted projective spaces in order to study uniformity conditions.

We now describe the content of each chapter in detail. To this end, we let K be an arbitrary field, $P = K[x_1, \dots, x_n]$ be the polynomial ring over K in $n \in \mathbb{N} \setminus \{0\}$ indeterminates, and we denote the set of all terms in P by $\mathbb{T}^n$.

In Chapter 2, we develop a theory of border bases for finitely generated P-modules. To achieve this goal, we introduce border bases of P-submodules of finite K-codimension in the free P-module P^r where we have $r \in \mathbb{N} \setminus \{0\}$. Let $\{e_1, \dots, e_r\}$ denote the canonical P-module basis of P^r and let $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ be the set of terms in P^r. We define order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ to be unions of sets of terms of the form $\mathcal{O} = \mathcal{O}_1 \cdot e_1 \cup \cdots \cup \mathcal{O}_r \cdot e_r$ by Definition 2.1.6. Here $\mathcal{O}_1, \dots, \mathcal{O}_r$ are order ideals in $\mathbb{T}^n$, i.e. sets of terms that are closed under forming divisors. In other words, order ideals $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ consist of n order ideals $\mathcal{O}_i$ in $\mathbb{T}^n$, one for every component of P^r. Note that in contrast to the definition of order ideals of other authors, e.g. the one in [KR05, Defn. 6.4.3], our version in Definition 2.1.1 regards the empty set as an order ideal, too. Then we define the border $\partial\mathcal{O}$ of $\mathcal{O}$ to be the set $\partial\mathcal{O} = \partial\mathcal{O}_1 \cdot e_1 \cup \cdots \cup \partial\mathcal{O}_r \cdot e_r$ by Definition 2.1.7. Here the border of the order ideal $\mathcal{O}_i$ in $\mathbb{T}^n$ is $\partial\mathcal{O}_i = (x_1 \cdot \mathcal{O}_i \cup \cdots \cup x_n \cdot \mathcal{O}_i \cup \{1\}) \setminus \mathcal{O}_i$ according to Definition 2.1.2. As for border bases of zero-dimensional ideals, for a finite order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$, we define an $\mathcal{O}$-border prebasis $G = \{g_b \mid b \in \partial\mathcal{O}\}$ in Definition 2.1.14 to be a set of vectors in P^r of the following specific form: for every $b \in \partial\mathcal{O}$, $g_b = b - \sum_{t \in \mathcal{O}} a_{t,b} t \in P^r$ with $a_{t,b} \in K$ for all $t \in \mathcal{O}$. An $\mathcal{O}$-border prebasis G is called an $\mathcal{O}$-border basis of a P-submodule $U \subseteq P^r$ if $G \subseteq U$ and if the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U. In particular, for $r = 1$, this definition yields nothing but the usual border bases of zero-dimensional ideals in the polynomial ring P.

With this definition, we are able to generalize the Border Division Algorithm in Theorem 2.2.1, prove the existence and uniqueness of border bases in Proposition 2.3.2, and compute border bases in Theorem 2.5.3 using linear algebra techniques in the module setting. The whole Section 2.4 is dedicated to characterizations of border bases. In detail, we characterize border bases via a special generation property in Theorem 2.4.1, via border form modules in Theorem 2.4.5, via rewrite rules in Theorem 2.4.13, via commuting matrices in Theorem 2.4.19, via liftings of border syzygies in Theorem 2.4.26, and we derive a Buchberger Criterion in Theorem 2.4.31. Altogether, we see that most of the concepts of border bases of zero-dimensional ideals in P can be carried over to the module setting if the P-module is free and of finite rank.

The final Section 2.6 of this chapter then establishes a border bases theory in arbitrary finitely generated P-modules. Every finitely generated P-module $M = \langle m_1, \dots, m_r\rangle$ induces a P-module epimorphism $\varphi : P^r \twoheadrightarrow M$, $e_i \mapsto m_i$ for all $i \in \{1, \dots, r\}$. We define a (generalized) order ideal in $\varphi(\mathbb{T}^n\langle e_1, \dots, e_r\rangle)$ to be the image of an order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ under φ in Definition 2.6.1 and a (generalized) $\varphi(\mathcal{O})$-border prebasis to be the image of an $\mathcal{O}$-border prebasis in P^r under φ in Definition 2.6.3. As in the case of free modules, a $\varphi(\mathcal{O})$-border prebasis $G \subseteq M$ is called a $\varphi(\mathcal{O})$-border basis of a P-submodule $U \subseteq M$ if $G \subseteq U$ and if the residue classes of the elements of $\varphi(\mathcal{O})$ in M/U form a K-vector space basis of M/U. Instead of reestablishing all ideas of the previous sections again, we introduce a process of lifting a $\varphi(\mathcal{O})$-border prebasis in M to an $\mathcal{O}$-border prebasis in $\varphi^{-1}(M) = P^r$. If such a lifting exists, we can use it to characterize the

corresponding border prebasis in M according to Corollary 2.6.10. Moreover, if we can compute the kernel of φ, we show that we can even compute $\varphi(\mathcal{O})$-border bases of arbitrary P-submodules $U \subseteq M$ with finite K-codimension in M using Corollary 2.6.12. In particular, since there are algorithms to compute syzygy modules of polynomials in P, we can apply the whole new theory to so-called subideal border bases as defined in [KP11]. This yields a way to compute arbitrary subideal border bases using linear algebra techniques and the computation of a single syzygy module, cf. Example 2.6.13. As indicated in [KP11, Sect. 6], the "standard approach" for the computation of subideal border bases by now needs a Gröbner basis computation instead of the linear algebra techniques.
As indicated at the beginning of the introduction and as described in [KP11, Sect. 7], (subideal) border bases can be used in the modelling of physical systems and this was our starting point of the theory of border bases in the module setting. Note that there is a preprint version of this chapter available, cf. [Kri13].

Chapter 3 is devoted to prove an analogon of Schreyer's Theorem for border bases in free P-modules of finite rank as introduced in Chapter 2. Schreyer's Theorem, as first proven in [Sch80] or stated using our notation in [KR00, Prop. 3.1.4], says that given a Gröbner basis of a P-submodule $U \subseteq P^r$, the set of all liftings corresponding to the S-vectors forms a Gröbner basis of the (first) syzygy module of the given Gröbner basis with respect to a suitably chosen term ordering. For border bases, the analogon of this theorem can be stated as follows: given an $\mathcal{O}$-border basis $G \subseteq P^r$ of a P-submodule $U \subseteq P^r$, the set of neighbor liftings forms a Gröbner basis of the (first) syzygy module of the given border basis with respect to a suitably chosen term ordering. One part of this theorem, namely that the neighbor liftings generate the syzygy module, was proven for border bases in P in [Hui06]. The corresponding proof makes use of two special operations called "degree lowering" and "column clearing". By applying these operations in a specific way, the author of [Hui06] was able to show that every syzygy can be reduced to zero by substracting suitable multiples of neighbor liftings. Based on these proofs, we were able to prove Schreyer's Theorem for border basis in P in [KK14]. We turned the two operations degree lowering and column clearing into explicit algorithms and deduced an algorithmic version of the reduction process used in the proofs of [Hui06]. Moreover, we also showed that the reduction process can actually be interpreted as a set of reduction steps using the rewrite rule defined by the neighbor liftings as defined in [KR00, Defn. 2.2.1] with respect to specific term orderings. Such a specific term ordering was also algorithmically constructed by us. Altogether, we concluded that Schreyer's Theorem holds for border bases in P.
In order to prove that Schreyer's Theorem also holds for border bases in P^r, we generalize the concepts of [KK14] to the module setting. In particular, we have to take empty order ideals into account. The full version of Schreyer's Theorem for border bases in P^r is a direct consequence of the Theorems 3.4.1 and 3.4.5. As a byproduct, we are also able to give an alternative proof of the characterization of border bases in P^r via liftings of border syzygies in Corollary 3.3.9. This alternative proof is quite remarkable since, in stark contrast to the standard proof in the literature, cf. Theorem 2.4.26, it does not depend on the characterization of border bases via commuting matrices in Theorem 2.4.19 but only on the characterizations via the special generation property in Theorem 2.4.1

and via rewrite rules in Theorem 2.4.13. Thus this proof might yield a possible way to characterize border bases in a non-commutative setting, e.g. as defined in [BTBQM00].

As described above, the Chapters 4 to 6 are dedicated to study zero-dimensional closed subschemes of weighted projective spaces with the help of border bases. In Chapter 4, we lay the algebraic foundation of this process, i.e. we find a generalization of border bases that is suitable for homogeneous ideals in $\overline{P} = P[x_0]$ and that are, in particular, one-dimensional. Therefore, we equip P with the grading defined by a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ in the sense of [KR05, Defn. 4.1.6 and 4.2.4] and we let x_0 be the homogenizing indeterminate. Then $\overline{P} = P[x_0] = K[x_0, \dots, x_n]$ is equipped with the induced grading given by the positive matrix $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$. The main idea is then based on [KR05, Thm. 4.3.22] which states that for a proper ideal $I \subset P$, $\overline{P}/I^{\mathrm{hom}}$ is a free $K[x_0]$-module. This led to the definition of projective border bases in Definition 4.1.2. A projective $\mathcal{O}$-border prebases for some finite order ideal $\mathcal{O}$ in $\mathbb{T}^n$ is a set of polynomials $G = \{g_b \mid b \in \partial\mathcal{O}\}$ of the form $g_b = b - \sum_{t \in \mathcal{O}} a_{t,b} t$ with $a_{t,b} \in K[x_0]$ for all $b \in \partial\mathcal{O}$ and $t \in \mathcal{O}$. It is said to be a projective $\mathcal{O}$-border basis of a homogeneous ideal $I \subseteq \overline{P}$ if $G \subseteq I$ and if the residue classes of the elements of the order ideal $\mathcal{O}$ in $\overline{P}/I$ form a $K[x_0]$-module basis of $\overline{P}/I$. Then it turns out that this definition implies many interesting properties of projective border bases in Proposition 4.1.7. The most important ones are that the elements of a projective $\mathcal{O}$-border basis of a homogeneous ideal $I \subseteq \overline{P}$ are homogeneous and of a uniquely determined shape, that dehomogenization yields a border basis of I^{deh} which is also a Macaulay basis (also called H-basis) with respect to the grading given by W, and that it indeed generates the ideal I. In particular, we get the (Projective) Border Basis Correspondence in Corollary 4.1.9 which can be visualized as follow.

$$\begin{tikzcd}
\begin{array}{c}\text{projective } \mathcal{O}\text{-border bases of homogeneous ideals in } \overline{P} \\ \text{where } \overline{P} \text{ is graded by } \overline{W} = (1 \mid W) \text{ with a positive matrix } W \in \mathrm{Mat}_{1,n}(\mathbb{Z})\end{array} \arrow[d, shift right=8ex, "\text{dehomogenization}"'] \\
\begin{array}{c}\mathcal{O}\text{-border bases of ideals in } P \text{ with } b \in \mathrm{DF}_W(g_b) \text{ for all } b \in \partial\mathcal{O} \\ \text{where } g_b \text{ denotes the } \mathcal{O}\text{-border basis element corresponding to } b\end{array} \arrow[u, shift right=8ex, "\text{homogenization}"']
\end{tikzcd}$$

With these properties in mind, we are able to characterize and eventually compute projective border bases in the Corollaries 4.1.10 and 4.1.14 and we can explicitly describe the elements of the residue class ring of $\overline{P}$ modulo a projective $\mathcal{O}$-border basis by means of $\mathcal{O}$ according to Proposition 4.1.15.

In the second section of Chapter 4, we study the multiplicative structure of a residue class ring $\overline{P}/I$ were I is given by a projective $\mathcal{O}$-border basis. It turns out that the characterization of border bases via commuting matrices in Theorem 2.4.19 allows us to explicitly describe the multiplicative structure of $\overline{P}/I$ by means of $\mathcal{O}$ and the multiplication matrices in the Propositions 4.2.5 and 4.2.8. After that, we study the canonical module of $\overline{P}/I$, i.e. the dual module of $\overline{P}/I$, in the third section. After proving some basic facts of the canonical module, we describe its multiplicative structure, again, in terms of $\mathcal{O}$ and the multiplication matrices in the Propositions 4.3.8 and 4.3.10. These descriptions of the multiplicative structure will turn out to be very helpful in the study of geometric properties in the last two chapters of this thesis.

In Chapter 5, we study zero-dimensional closed subschemes of weighted projective spaces $\mathbb{P}_K(\overline{W})$. First we recall basic facts about weighted projective spaces. As before, we equip P with the grading given by a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$, let x_0 be the homogenizing indeterminate, and equip $\overline{P} = P[x_0] = K[x_0, \ldots, x_n]$ with the grading given by the matrix $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$. Then the weighted projective space of type $\overline{W}$ over K is defined to be the projective scheme $\mathbb{P}_K(\overline{W}) = \mathrm{Proj}(\overline{P})$ corresponding to the $\mathbb{Z}$-graded K-algebra $\overline{P}$ in Definition 5.1.1. We are particularly interested in zero-dimensional closed subschemes $\mathbb{X}$ of $\mathbb{P}_K(\overline{W})$ and determine their structure. In particular, in Proposition 5.1.8, it turns out that there is a projective border basis of the defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ of a non-empty zero-dimensional closed subscheme $\mathbb{X}$ if and only if no point of $\mathbb{X}$ lies on the hyperplane at infinity. In Proposition 5.1.10, we prove that in most cases, e.g. if K is infinite and P standard graded, there is a generic homogeneous linear change of coordinates such that the above equivalence relation is satisfied. The remainder of this chapter is then dedicated to the study of the correspondence between geometric properties of the non-empty zero-dimensional closed subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ and algebraic properties of the residue class ring of $\overline{P}$ modulo the defining ideal $\mathcal{I}^+(\mathbb{X})$ of $\mathbb{X}$. We can visualize this correspondence as follows.

zero-dimensional closed subschemes $\mathbb{X}$ of weighted projective spaces $\mathbb{P}_K(\overline{W})$
that have no point on the hyperplane at infinity

projective border basis of the defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ $\downarrow$ $\qquad$ $\uparrow$ projective subscheme defined by the homogeneous ideal

projective $\mathcal{O}$-border bases of homogeneous ideals in $\overline{P}$
where $\overline{P}$ is graded by $\overline{W} = (1 \mid W)$ with a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$

The main geometric properties we are interested in in this thesis are uniformity conditions as defined in Definition 5.2.1. The most general uniformity condition, namely the (i, j)-uniformity, can (in the reduced case) be geometrically interpreted as follows: given a set of projective points $\mathbb{X}$, is there a subset $\widehat{\mathbb{X}}$ of i points of $\mathbb{X}$ and a homogeneous polynomial p of degree j in $\overline{P}$ such that p vanishes in all points of $\mathbb{X} \setminus \widehat{\mathbb{X}}$ but not in all points of $\mathbb{X}$. If no such subset of i points exists, $\mathbb{X}$ is said to be (i, j)-uniform. In the standard graded setting, some special cases of this question, e.g. whether the scheme has the Cayley-Bacharach property or is in uniform position, have already been studied. For instance, under the assumption that the base field is algebraically closed, i-uniform zero-dimensional closed subschemes of the projective n-space have been characterized

in [Kre94] and [Kre98] with the help of the multiplicative structure of the corresponding canonical module. In [Kre01], the same author described algorithms to check this property for reduced subschemes under the same assumptions. Another totally different approach was used by the authors in [MP04]. They have characterized and described an algorithm to check (i, j)-uniformity conditions using Chow forms under the assumptions that the given subscheme is reduced and that the base field is of characteristic zero or its characteristic is large enough.
Our approach in the second section of this chapter is also based on the idea that the multiplicative structure of the canonical module of the projective coordinate ring of the given subscheme $\mathbb{X}$ contains information about the geometry of $\mathbb{X}$. As we have already noted above, we can describe the multiplicative structure of the canonical module by means of the multiplication matrices if the defining ideal of $\mathbb{X}$ possesses a projective border basis, i.e. if no point of $\mathbb{X}$ lies on the hyperplane at infinity. With this assumption and the additional assumption that $\mathbb{X}$ has a K-rational support, i.e. that all the prime ideals of the support of $\mathbb{X}$ are homogeneous vanishing ideals of a K-rational projective point by Definition 5.1.5, we characterize (i, j)-uniform subschemes of weighted projective spaces $\mathbb{P}_K(\overline{W})$ in Theorem 5.2.7. Note that the first condition is often satisfied after a generic homogeneous linear change of coordinates and that the second assumption trivially holds by Hilbert's Nullstellensatz if the base field K is algebraically closed. Therefore, our results also hold in the non-reduced case as well as in the non-standard graded setting. In particular, this affirmatively answers (generalizations of) [Kre01, Questions 1 and 3]. In the third section of that chapter, we turn the methods of Section 5.3 to check uniformity conditions into explicit algorithms.

Finally, in Chapter 6, we combine the results of the Chapters 4 and 5. The following figure shows the intersection of the two previous chapters.

zero-dimensional closed subschemes $\mathbb{X}$ of weighted projective spaces $\mathbb{P}_K(\overline{W})$
that have no point on the hyperplane at infinity

projective border basis of the defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ $\downarrow$ $\uparrow$ projective subscheme defined by the homogeneous ideal

projective $\mathcal{O}$-border bases of homogeneous ideals in $\overline{P}$
where $\overline{P}$ is graded by $\overline{W} = (1 \mid W)$ with a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$

dehomogenization $\downarrow$ $\uparrow$ homogenization

$\mathcal{O}$-border bases of ideals in P with $b \in \mathrm{DF}_W(g_b)$ for all $b \in \partial\mathcal{O}$
where g_b denotes the $\mathcal{O}$-border basis element corresponding to b

We use this intersection to study uniformity conditions for all ideals that possess a projective $\mathcal{O}$-border basis for some given order ideal $\mathcal{O}$ in $\mathbb{T}^n$, at once. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be a finite, non-empty order ideal in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$. In [KR08] and [Rob09], the authors introduced the $\mathcal{O}$-border bases scheme $\mathbb{B}_\mathcal{O}$, cf. Definition 6.1.1, as an affine scheme that parametrizes all zero-dimensional ideals in the polynomial ring P that possess an $\mathcal{O}$-border basis. The construction was done as follows: First they defined the generic $\mathcal{O}$-border prebasis by replacing the scalar $a_{ij} \in K$ of every border basis element $g_j = b_j - \sum_{i=1}^{\mu} a_{ij}t_i$ by a newly introduced indeterminate c_{ij} for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$. Then they considered the corresponding formal multiplication matrices of that generic $\mathcal{O}$-border prebasis and ensured that an affine point $a = (a_{11}, \ldots, a_{\mu\nu})$ is contained in $\mathbb{B}_\mathcal{O}$ if and only if the multiplication matrices are pairwise commuting after applying the substitution homomorphism $c_{ij} \mapsto a_{ij}$. Since border bases can be characterized via commuting matrices by Theorem 2.4.19 and are unique by Proposition 2.3.2, $\mathbb{B}_\mathcal{O}$ parametrizes all zero-dimensional ideals in P that possess an $\mathcal{O}$-border basis. The projective $\mathcal{O}$-border basis scheme is then defined in Definition 6.1.5 to be the subscheme of $\mathbb{B}_\mathcal{O}$ that parametrizes all zero-dimensional ideals in P that possess an $\mathcal{O}$-border basis $\{g_1, \ldots, g_\nu\}$ and additionally satisfy $b_j \in \mathrm{DF}_W(g_j)$ for all $j \in \{1, \ldots, \nu\}$. As indicated by the above figure and defined in Definition 6.4.2, the projective border basis schemes parametrize all zero-dimensional closed subschemes of $\mathbb{P}_K(\overline{W})$ that have no point on the hyperplane at infinity. The main goal of this chapter is to show that the subset of $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ that parametrizes all zero-dimensional closed subschemes of $\mathbb{P}_K(\overline{W})$ that have no point on the hyperplane at infinity and that are (i,j)-uniform is a constructible set with respect to the Zariski topology in Theorem 6.4.4.

More precisely, we do the following: In Definition 6.1.5, we introduce the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$, its coordinate ring $B_\mathcal{O}^{\mathrm{proj}}$, and the corresponding universal projective $\mathcal{O}$-border basis family $B_\mathcal{O}^{\mathrm{proj}} \to U_\mathcal{O}^{\mathrm{proj}}$. Then we prove that the residue classes of the elements of $\mathcal{O}$ in $U_\mathcal{O}^{\mathrm{proj}}$ form a $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-module basis of $U_\mathcal{O}^{\mathrm{proj}}$ in Theorem 6.1.13. In the second section, we then study the multiplicative structure of $U_\mathcal{O}^{\mathrm{proj}}$ by means of the generic (projective) multiplication matrices in the Propositions 6.2.4 and 6.2.7. In the third section, we define the generic canonical module of $U_\mathcal{O}^{\mathrm{proj}}$ in Definition 6.3.2, show that it is also a free $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-module and that the dual objects of the elements in $\mathcal{O}$ form the corresponding basis in Proposition 6.3.4, and also study the multiplicative structure of the generic canonical module of $U_\mathcal{O}^{\mathrm{proj}}$ by means of the generic (projective) multiplication matrices in the Propositions 6.3.5 and 6.3.7. In the final section of this chapter, we additionally assume that the base field K is algebraically closed. This ensures that for all zero-dimensional closed subschemes $\mathbb{X}$ of $\mathbb{P}_K(\overline{W})$, every element of the support of $\mathbb{X}$ is, by Hilbert's Nullstellensatz, the vanishing ideal of a K-rational projective point. Therefore, we can apply the characterizations of (i,j)-uniform subschemes of Section 5.2 to all zero-dimensional closed subschemes $\mathbb{X}$ parametrized by $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$. As a main result, we characterize all points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ that correspond to an (i,j)-uniform zero-dimensional closed subscheme of $\mathbb{P}_K(\overline{W})$ in Theorem 6.4.4. In particular, we show that the set of all points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ with the property that the corresponding zero-dimensional closed subschemes of $\mathbb{P}_K(\overline{W})$ are (i,j)-uniform is constructible with respect to the Zariski topology in Theorem 6.4.4.

For the whole thesis, we assume that the reader is familiar with the basics of algebra as, for instance, taught in a two semester course at university and has basic knowledge about Gröbner bases as, for instance, introduced in [KR00] or [AL94]. If not mentioned otherwise, we use the notation and terminology of [KR00] and [KR05]. In particular, the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ contains zero, we let K be an arbitrary field, and $P = K[x_1, \ldots, x_n]$ be the polynomial ring in $n \in \mathbb{N} \setminus \{0\}$ indeterminates over K. By terms, we denote polynomials in $P = K[x_1, \ldots, x_n]$ of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$. The monoid of all terms in P is denoted by $\mathbb{T}^n$. We let P^r denote the free P-module of rank $r \in \mathbb{N} \setminus \{0\}$ with the P-module basis $\{e_1, \ldots, e_r\}$ and a term in P^r is an element of the form te_i where $t \in \mathbb{T}^n$ and $i \in \{1, \ldots, r\}$. The set of all terms in P^r is denoted by $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Products of a scalar and a term are called monomials. For a subset S of a $\mathbb{Z}$-graded module $M = \bigoplus_{\gamma \in \mathbb{Z}} M_\gamma$ and for all $\gamma \in \mathbb{Z}$, we denote $S_\gamma = S \cap M_\gamma$, $S_{\leq \gamma} = S \cap \bigoplus_{\gamma' = -\infty}^{\gamma} M_{\gamma'}$, and similarly for $S_{\geq \gamma}$, $S_{<\gamma}$, and $S_{>\gamma}$. In algorithms, we often need an ordering on the elements of a given set, e.g. $G = \{g_1, \ldots, g_\nu\}$. If no confusion arises, we always keep that ordering in mind and treat the set as if it was a tuple during algorithms.

2 Border Bases of Finitely Generated Modules

Border bases of zero-dimensional ideals in a polynomial ring have been studied for several years in various ways, cf., for instance, [Mou99], [Ste04], [Hui06], [KR08], [Rob09], and [KPR10]. But despite of the special case of subideal border bases, cf. [KP11], the theory of border bases is restricted to zero-dimensional ideals in a polynomial ring. The goal of this chapter is to overcome this limitation and generalize the concept of border bases to finitely generated P-modules. To this end, we first introduce a border bases theory for submodules of free modules of finite rank over the polynomial ring P in the Sections 2.1 to 2.5. After that, we introduce a border basis theory for submodules of finitely generated P-modules in Section 2.6. More precisely, we do the following.

In Section 2.1, we generalize the basic concepts needed for a border basis theory in P^r with $r \in \mathbb{N} \setminus \{0\}$, namely order ideals in Definition 2.1.6, borders of order ideals in Definition 2.1.7, and border prebases and border bases in Definition 2.1.14. Moreover, we introduce the index with respect to an order ideal in Definition 2.1.11, which allows us to measure the distance between an arbitrary term and the order ideal.

After the introduction of the basic concepts, we prove a division algorithm and direct consequences of it in Section 2.2. The Border Division Algorithm in Theorem 2.2.1. allows us to divide an arbitrary vector $v \in P^r$ by a border prebasis $G \subseteq P^r$ and thus compute a representative of $\overline{v}$ in the residue class module $P^r/\langle G \rangle$, cf. Corollary 2.2.5. Moreover, we show that border bases of submodules can be used to define normal forms in Definition 2.2.9.

Section 2.3 is dedicated to study whether every P-submodule $U \subseteq P^r$ that has a finite K-codimension in P^r possesses a border bases or not. We affirmatively answer this question in Proposition 2.3.2. Moreover, we show that border bases in P^r are unique in Proposition 2.3.2 and we prove that reduced Gröbner bases of submodules $U \subseteq P^r$ with finite K-codimension in P^r are special border bases in Proposition 2.3.5. As a consequence, we give a first naive algorithm based on a Gröbner basis computation that allows us to compute border bases in P^r in Remark 2.3.6.

In Section 2.4, we characterize border bases in various ways. We characterize border bases via a special generation property in Theorem 2.4.1. After that, we characterize them via border form modules, which are the border bases analogon of leading term modules, in Theorem 2.4.5. Then we define rewrites rules associated to border prebases in Definition 2.4.7 and prove that a border prebasis is a border basis if and only if the corresponding rewrite rule is confluent in Theorem 2.4.13. In Theorem 2.4.19, we characterize border bases via commuting multiplication matrices. This characterization will play a key role in the latter part of this thesis since the multiplication matrices allow us to explicitly describe the multiplicative structure of a residue class module P^r/U if U is given by a border basis. In particular, we will use this description in Chapter 5 to study geometric properties of zero-dimensional projective schemes that are given by a border basis. Finally, we characterize border bases via liftings of border syzygies in Theorem 2.4.26 and prove a Buchberger Criterion for border bases in Theorem 2.4.31. As already mentioned above, there is a naive way to compute border bases with Gröbner bases techniques. Section 2.5 is dedicated to find a more efficient algorithm for the computation of border bases which uses linear algebra techniques. This refined Border Bases Algorithm is proven in Theorem 2.5.3.

All the concepts of the Sections 2.1 to 2.5 are generalizations (with minor changes) of the corresponding well-known concepts for border bases in P. The corresponding version of the results in the Sections 2.1 to 2.5 for border bases in polynomial rings are summarized in [KR05, Section 6.4]. In particular, the characterizations in Section 2.4 are due to [Mou99], where commuting matrices were used for the first time to characterize ideal bases, and [KK05], where the notion of border bases was already developed and the above characterizations were proven for the first time. The general framework for an algorithm for the computation of border bases without the need of a Gröbner basis computation was laid in [Mou99] and was turned into an explicit algorithm in [KK06].

Up to that point, we were able to generalize border bases in a straightforward way to free P-modules of finite rank. In Section 2.6, we go another step further and enter unfamiliar territory by developing a border bases theory in arbitrary finitely generated P-modules. To this end, we first generalize order ideals and their borders in Definition 2.6.1 and then define generalized border prebasis and generalized border bases in Definition 2.6.3. Then, under certain assumptions, we lift questions about generalized border bases to questions about border bases in P^r. In this way, we are able to characterize generalized border bases in Theorem 2.6.8 and can reformulate the characterizations of Section 2.4 in the generalized case in Corollary 2.6.10. Moreover, if we can compute certain kernels, we are even able to compute generalized border bases according to Corollary 2.6.12. Finally, we apply the theory of generalized border bases in Example 2.6.13 to subideal border bases as introduced in [KP11]. This allows us to characterize subideal border bases in various ways and to compute arbitrary subideal border bases with one syzygy module computation and linear algebra techniques. This heavily extends the theory of subideal border bases introduced in [KP11] where subideal border bases are only characterized via a special generation property and can only be computed with a naive algorithm that is based on Gröbner bases computations.

If not mentioned otherwise, we equip P with the standard grading and for all $\gamma \in \mathbb{Z}$, we write $\mathbb{T}^n_\gamma\langle e_1, \ldots, e_r\rangle = \{te_k \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle \mid \deg(t) = \gamma\}$ for the set of all terms in P^r of degree γ. Similarly, we define $\mathbb{T}^n_{<\gamma}\langle e_1, \ldots, e_r\rangle$, $\mathbb{T}^n_{\geq\gamma}\langle e_1, \ldots, e_r\rangle$, etc. for all $\gamma \in \mathbb{Z}$. In particular, for all $\gamma \in \mathbb{Z}$, the set $\mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle$ is then a K-vector space basis of $P^r_{\leq\gamma}$.

2.1 Basic Definitions

In this section, we generalize the basic concepts of border bases in the polynomial ring P to free P-modules of finite rank. More precisely, we define order ideals in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ in Definition 2.1.6, their border in Definition 2.1.7, the index with respect to an order ideal in Definition 2.1.11, and border prebases and border bases in P^r in Definition 2.1.14. We refer to [KR05, Section 6.4] for the corresponding definitions and theorems about border bases in P.

The definition of order ideals in $\mathbb{T}^n$ in the literature, for instance in [KR05, Defn. 6.4.3], does not consider the empty set as an order ideal. But it turns out in Remark 2.5.5 that empty order ideals in $\mathbb{T}^n$ might also occur during the computation of border bases in P^r. Therefore, we consider the empty set as an order ideal in $\mathbb{T}^n$, too. For the sake of completeness, we give explicit proofs for the basic properties of order ideals in $\mathbb{T}^n$ and their border in this more general setting.

The following two definitions generalize [KR05, Defn. 6.4.3 and 6.4.4].

Definition 2.1.1. A set $\mathcal{O} \subseteq \mathbb{T}^n$ is called an **order ideal** in $\mathbb{T}^n$ if it is closed under forming divisors.

Definition 2.1.2. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$.

a) We call the set $\partial^1\mathcal{O} = \partial\mathcal{O} = ((\mathbb{T}^n_1 \cdot \mathcal{O}) \cup \{1\}) \setminus \mathcal{O} \subseteq \mathbb{T}^n$ the **(first) border** of $\mathcal{O}$. The **(first) border closure** of $\mathcal{O}$ is the set $\overline{\partial^1\mathcal{O}} = \overline{\partial\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O} \subseteq \mathbb{T}^n$.

b) For every $k \in \mathbb{N} \setminus \{0\}$, we inductively define the $(k+1)^{\mathbf{st}}$ **border** of $\mathcal{O}$ by the rule $\partial^{k+1}\mathcal{O} = \partial(\overline{\partial^k\mathcal{O}}) \subseteq \mathbb{T}^n$ and we define the $(k+1)^{\mathbf{st}}$ **border closure** by the rule $\overline{\partial^{k+1}\mathcal{O}} = \overline{\partial^k\mathcal{O}} \cup \partial^{k+1}\mathcal{O} \subseteq \mathbb{T}^n$. For convenience, we let $\partial^0\mathcal{O} = \overline{\partial^0\mathcal{O}} = \mathcal{O}$.

The following proposition yields the basic properties of borders as in [KR05, Prop. 6.4.6] and also takes the empty set into account.

Proposition 2.1.3. *Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$.*

a) For every $k \in \mathbb{N}$, we have a disjoint union $\overline{\partial^k\mathcal{O}} = \bigcup_{i=0}^k \partial^i\mathcal{O}$.

b) We have a disjoint union $\mathbb{T}^n = \bigcup_{i=0}^\infty \partial^i\mathcal{O}$.

c) For every $k \in \mathbb{N} \setminus \{0\}$, we have

$$\partial^k\mathcal{O} = ((\mathbb{T}^n_k \cdot \mathcal{O}) \cup \mathbb{T}^n_{k-1}) \setminus (\mathbb{T}^n_{<k} \cdot \mathcal{O}) = \begin{cases} (\mathbb{T}^n_k \cdot \mathcal{O}) \setminus (\mathbb{T}^n_{<k} \cdot \mathcal{O}) & \text{if } \mathcal{O} \neq \emptyset, \\ \mathbb{T}^n_{k-1} & \text{if } \mathcal{O} = \emptyset. \end{cases}$$

d) Let $t \in \mathbb{T}^n$ be a term. Then there exists a factorization of the form $t = t'b$ with a term $t' \in \mathbb{T}^n$ and $b \in \partial\mathcal{O}$ if and only if $t \in \mathbb{T}^n \setminus \mathcal{O}$.

Proof. First we prove claim a) by induction on $k \in \mathbb{N}$. For $k = 0$, Definition 2.1.2 yields $\overline{\partial^0 \mathcal{O}} = \partial^0 \mathcal{O} = \bigcup_{i=0}^{0} \partial^i \mathcal{O}$. For $k \in \mathbb{N} \setminus \{0\}$, Definition 2.1.2 and the induction hypothesis imply that $\overline{\partial^{k+1} \mathcal{O}} = \overline{\partial^k \mathcal{O}} \cup \partial^{k+1} \mathcal{O} = \bigcup_{i=0}^{k} \partial^i \mathcal{O} \cup \partial^{k+1} \mathcal{O} = \bigcup_{i=0}^{k+1} \partial^i \mathcal{O}$. Moreover, for all $i, j \in \mathbb{N}$ with $i \neq j$, the borders $\partial^i \mathcal{O}$ and $\partial^j \mathcal{O}$ are disjoint according to Definition 2.1.2 and the first claim follows.

Since every term in $\mathbb{T}^n$ is in $\partial^i \mathcal{O}$ for some $i \in \mathbb{N}$ by Definition 2.1.2, claim b) is a direct consequence of claim a).

Next we prove claim c) by induction on $k \in \mathbb{N} \setminus \{0\}$. For the induction start $k = 1$, Definition 2.1.2 yields $\partial^1 \mathcal{O} = ((\mathbb{T}^n_1 \cdot \mathcal{O}) \cup \{1\}) \setminus \mathcal{O} = ((\mathbb{T}^n_1 \cdot \mathcal{O}) \cup \mathbb{T}^n_0) \setminus (\mathbb{T}^n_{<1} \cdot \mathcal{O})$. For the induction step, let now $k > 1$. In this situation, Definition 2.1.2 and claim a) imply that $\partial^k \mathcal{O} = \partial(\overline{\partial^{k-1} \mathcal{O}}) = \partial(\bigcup_{i=0}^{k-1} \partial^i \mathcal{O})$. If $\mathcal{O} = \emptyset$, the induction hypothesis and Definition 2.1.2 yield $\partial^k \mathcal{O} = \partial(\bigcup_{i=0}^{k-1} \mathbb{T}^n_{i-1}) = \partial(\mathbb{T}^n_{\leq k-2}) = \mathbb{T}^n_{k-1}$. If $\mathcal{O} \neq \emptyset$, we have $1 \in \mathcal{O}$. Thus if $\mathcal{O} \neq \emptyset$, the induction hypothesis and Definition 2.1.2 yield

$$
\begin{aligned}
\partial^k \mathcal{O} &= \partial \left(\bigcup_{i=0}^{k-1} ((\mathbb{T}^n_i \cdot \mathcal{O}) \setminus \mathbb{T}^n_{<i} \cdot \mathcal{O}) \right) \\
&= \partial(\mathbb{T}^n_{\leq k-1} \cdot \mathcal{O}) \\
&= (\mathbb{T}^n_1 \cdot \mathbb{T}^n_{\leq k-1} \cdot \mathcal{O} \cup \{1\}) \setminus (\mathbb{T}^n_{\leq k-1} \cdot \mathcal{O}) \\
&= (\mathbb{T}^n_k \cdot \mathcal{O}) \setminus (\mathbb{T}^n_{\leq k-1} \cdot \mathcal{O})
\end{aligned}
$$

and claim c) follows.

Finally, we prove claim d). We distinguish two cases. For the first case, suppose that $\mathcal{O} = \emptyset$. Then we have $\partial \mathcal{O} = \{1\}$ by Definition 2.1.2 and for every term $t \in \mathbb{T}^n \setminus \mathcal{O} = \mathbb{T}^n$, there is the factorization $t = t \cdot 1$. For the second case, suppose that $\mathcal{O} \neq \emptyset$. Let $t \in \mathbb{T}^n \setminus \mathcal{O}$. Then there exists a $k \in \mathbb{N} \setminus \{0\}$ such that $t \in \partial^k \mathcal{O} = (\mathbb{T}^n_k \cdot \mathcal{O}) \setminus (\mathbb{T}^n_{<k} \cdot \mathcal{O})$ according to b), c), and Definition 2.1.2. In particular, we can write $t = x_\ell t_1 t_2$ with $\ell \in \{1, \ldots, n\}$, $t_1 \in \mathbb{T}^n_{k-1}$, and $t_2 \in \mathcal{O}$. Assume that $x_\ell t_2 \in \mathcal{O}$. Then we get the contradiction $t = t_1(x_\ell t_2) \in \mathbb{T}^n_{<k} \cdot \mathcal{O}$. Thus Definition 2.1.2 yields $x_\ell t_2 \in \partial \mathcal{O}$ and the first implication follows from $t = t_1(x_\ell t_2)$. For the converse implication, let $t' \in \mathbb{T}^n$ and $b \in \partial \mathcal{O}$. Assume that $t'b \in \mathcal{O}$. Then Definition 2.1.1 yields the contradiction $b \in \mathcal{O}$. Thus we have $t'b \in \mathbb{T}^n \setminus \mathcal{O}$ and the claim follows. □

This result enables us to define the index with respect to an order ideal in $\mathbb{T}^n$ just as in [KR05, Defn. 6.4.7]. The index measures the distance from a term to an order ideal.

Definition 2.1.4. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$.

a) For every term $t \in \mathbb{T}^n$, the number $i \in \mathbb{N}$ such that $t \in \partial^i \mathcal{O}$, which is unique according to Proposition 2.1.3, is called the **$\mathcal{O}$-index** of t and denoted by $\operatorname{ind}_{\mathcal{O}}(t)$.

b) For a polynomial $p \in P \setminus \{0\}$, we define $\operatorname{ind}_{\mathcal{O}}(p) = \max\{\operatorname{ind}_{\mathcal{O}}(t) \mid t \in \operatorname{Supp}(p)\}$ to be the **$\mathcal{O}$-index** of p.

The following proposition gathers the basic properties of the index with respect to an order ideal in $\mathbb{T}^n$. It is a generalization of [KR05, Prop. 6.4.8].

Proposition 2.1.5. *Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n$.*

a) For a term $t \in \mathbb{T}^n \setminus \mathcal{O}$, the number $i = \operatorname{ind}_{\mathcal{O}}(t)$ is the smallest natural number such that there is a factorization $t = t'b$ with $t' \in \mathbb{T}^n_{i-1}$ and $b \in \partial\mathcal{O}$.

b) Given a term $t \in \mathbb{T}^n$ and a term $t' \in \mathbb{T}^n$, we have $\operatorname{ind}_{\mathcal{O}}(tt') \leq \deg(t) + \operatorname{ind}_{\mathcal{O}}(t')$.

c) For two polynomials $p, q \in P \setminus \{0\}$ such that $p + q \neq 0$, we have the inequality $\operatorname{ind}_{\mathcal{O}}(p+q) \leq \max\{\operatorname{ind}_{\mathcal{O}}(p), \operatorname{ind}_{\mathcal{O}}(q)\}$.

d) For two polynomials $p, q \in P \setminus \{0\}$, we have $\operatorname{ind}_{\mathcal{O}}(pq) \leq \deg(p) + \operatorname{ind}_{\mathcal{O}}(q)$.

Proof. For the proof of a), let $t \in \mathbb{T}^n \setminus \mathcal{O}$ and $i = \operatorname{ind}_{\mathcal{O}}(t) \in \mathbb{N}$. Then $i > 0$ and $t \in \partial\mathcal{O}^i$ by the Definitions 2.1.4 and 2.1.7. If $\mathcal{O} = \emptyset$, $1 \in \partial\mathcal{O}$ by Definition 2.1.2 and $\partial^i\mathcal{O} = \mathbb{T}^n_{i-1}$ by Proposition 2.1.3. Thus $t = t \cdot 1$ is a factorization with the desired properties in this situation. Suppose now that $\mathcal{O} \neq \emptyset$. According to Proposition 2.1.5 and Definition 2.1.2, there is a factorization $t = x_\ell t_1 t_2$ with $\ell \in \{1, \ldots, n\}$, $t_1 \in \mathbb{T}^n_{i-1}$, and $t_2 \in \mathcal{O}$, and $t \notin \mathbb{T}^n_{<i} \cdot \mathcal{O}$. In particular, $x_\ell t_2 \notin \mathcal{O}$. Thus $x_\ell t_2 \in \partial\mathcal{O}$ by Definition 2.1.2 and $t = t_1(x_\ell t_2)$ is a desired factorization. Assume that there is a factorization $t = t'b$ such that $t' \in \mathbb{T}^n_{<i-1}$ and $b \in \partial\mathcal{O}$. Then $b = x_m t''$ with $m \in \{1, \ldots, n\}$ and $t'' \in \mathcal{O}$ by Definition 2.1.2. In this situation, we get the contradiction $t = (x_m t')t'' \in \mathbb{T}^n_{<i} \cdot \mathcal{O}$ and the claim follows.

Claim b) follows immediately from claim a). Since $\operatorname{Supp}(p + q) \subseteq \operatorname{Supp}(p) \cup \operatorname{Supp}(q)$, claim c) follows immediately from Definition 2.1.4. At last, claim d) follows from claim b) since $\operatorname{Supp}(pq) \subseteq \{tt' \mid t \in \operatorname{Supp}(p), t'' \in \operatorname{Supp}(q)\}$. □

Now we are able generalize order ideals in $\mathbb{T}^n$ to the module setting. The key idea is that for each component of P^r, we consider a separate order ideal in $\mathbb{T}^n$.

Definition 2.1.6. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be order ideals in the monoid of terms $\mathbb{T}^n$. Then we call the set $\mathcal{O} = \mathcal{O}_1 \cdot e_1 \cup \cdots \cup \mathcal{O}_r \cdot e_r \subseteq \mathbb{T}^n\langle e_r, \ldots, e_r\rangle$ an **order ideal** in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.

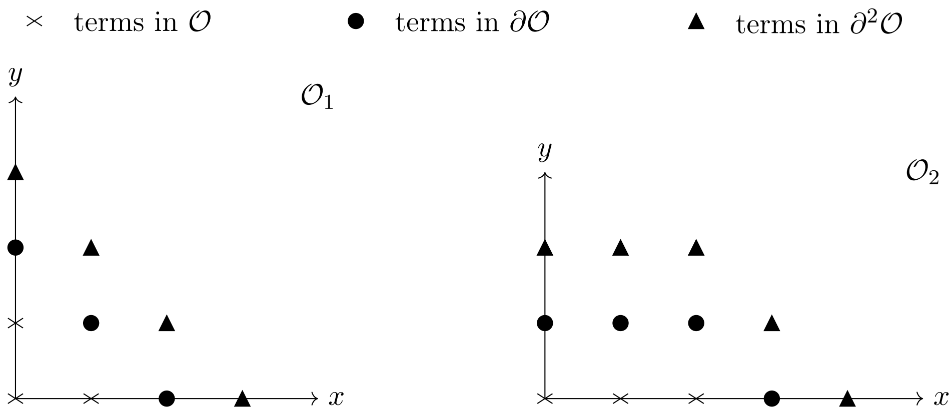
Definition 2.1.7. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ and let $\mathcal{O} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_r e_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.

a) We call the set $\partial^1\mathcal{O} = \partial\mathcal{O} = ((\mathbb{T}^n_1 \cdot \mathcal{O}) \cup \{e_1, \ldots, e_r\}) \setminus \mathcal{O} \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ the **(first) border** of $\mathcal{O}$. The **(first) border closure** of the order ideal $\mathcal{O}$ is defined by $\overline{\partial^1\mathcal{O}} = \overline{\partial\mathcal{O}} = \mathcal{O} \cup \partial\mathcal{O} \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.

b) For every $k \in \mathbb{N} \setminus \{0\}$, we inductively define the $(k+1)^{\text{st}}$ **border** of $\mathcal{O}$ by the rule $\partial^{k+1}\mathcal{O} = \partial(\overline{\partial^k\mathcal{O}}) \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ and the $(k+1)^{\text{st}}$ **border closure** by the rule $\overline{\partial^{k+1}\mathcal{O}} = \overline{\partial^k\mathcal{O}} \cup \partial^{k+1}\mathcal{O} \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. For convenience, we let $\partial^0\mathcal{O} = \overline{\partial^0\mathcal{O}} = \mathcal{O}$.

Remark 2.1.8. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ and let $\mathcal{O} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_r e_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. The P-module structure of P^r and the Definitions 2.1.2 and 2.1.7 yield that for all $k \in \mathbb{N}$, we have $\partial^k\mathcal{O} = \partial^k\mathcal{O}_1 \cdot e_1 \cup \cdots \cup \partial^k\mathcal{O}_r \cdot e_r$ and $\overline{\partial^k\mathcal{O}} = \overline{\partial^k\mathcal{O}_1} \cdot e_1 \cup \cdots \cup \overline{\partial^k\mathcal{O}_r} \cdot e_r$.

Example 2.1.9. Let $P = \mathbb{Q}[x, y]$. The sets $\mathcal{O}_1 = \{1, y, x\}$ and $\mathcal{O}_2 = \{1, x, x^2\}$ are both order ideals in $\mathbb{T}^2$ with first borders $\partial\mathcal{O}_1 = \{y^2, xy, x^2\}$ and $\partial\mathcal{O}_2 = \{y, xy, x^2y, x^3\}$, and second borders $\partial^2\mathcal{O}_1 = \{y^3, xy^2, x^2y, x^3\}$ and $\partial^2\mathcal{O}_2 = \{y^2, xy^2, x^2y^2, x^3y, x^4\}$, respectively, according to the Definitions 2.1.1 and 2.1.2. Let $\{e_1, e_2\}$ be the canonical P-module basis of P^2. Then the set $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\}$ is an order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ with first border $\partial\mathcal{O} = \{y^2e_1, xye_1, x^2e_1, ye_2, xye_2, x^2ye_2, x^3e_2\}$ and second border $\partial^2\mathcal{O} = \{y^3e_1, xy^2e_1, x^2ye_1, x^3e_1, y^2e_2, xy^2e_2, x^2y^2e_2, x^3ye_2, x^4e_2\}$ by the Definitions 2.1.6 and 2.1.7. By identifying terms with their logarithms, i.e. their exponent vectors, we can visualize the order ideal $\mathcal{O}$ in $\mathbb{T}^2\langle e_1, e_2\rangle$ and its first and second borders as follows.



Order ideals and their borders in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ behave similarly as order ideals and their borders in $\mathbb{T}^n$. The following proposition is a module version of Proposition 2.1.3.

Proposition 2.1.10. *Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ and let $\mathcal{O} = \mathcal{O}_1e_1 \cup \dots \cup \mathcal{O}_re_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$.*

a) For every $k \in \mathbb{N}$, we have a disjoint union $\overline{\partial^k\mathcal{O}} = \bigcup_{i=0}^k \partial^i\mathcal{O}$.

b) We have a disjoint union $\mathbb{T}^n\langle e_1, \dots, e_r\rangle = \bigcup_{i=0}^\infty \partial^i\mathcal{O}$.

c) For every $k \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned}\partial^k\mathcal{O} &= ((\mathbb{T}^n_k \cdot \mathcal{O}) \cup \mathbb{T}^n_{k-1}\langle e_1, \dots, e_r\rangle) \setminus (\mathbb{T}^n_{<k} \cdot \mathcal{O}) \\ &= (\mathbb{T}^n_k \cdot \mathcal{O} \setminus \mathbb{T}^n_{<k} \cdot \mathcal{O}) \cup \bigcup_{\substack{i \in \{1,\dots,n\} \\ \mathcal{O}_i = \emptyset}} \mathbb{T}^n_{k-1} \cdot e_i\end{aligned}$$

d) Let $te_k \in \mathbb{T}^n\langle e_1, \dots, e_r\rangle$ be a term. Then there exists a factorization of the form $te_k = t'be_k$ with a term $t' \in \mathbb{T}^n$ and $be_k \in \partial\mathcal{O}$ if and only if $te_k \in \mathbb{T}^n\langle e_1, \dots, e_r\rangle \setminus \mathcal{O}$.

Proof. For all $s \in \{1, \dots, r\}$, we have $\{(p_1, \dots, p_r) \in \mathcal{O} \mid p_s \neq 0\} = \mathcal{O}_se_s$ according to Definition 2.1.6. Thus the claim immediately follows from Proposition 2.1.3 and Definition 2.1.7. □

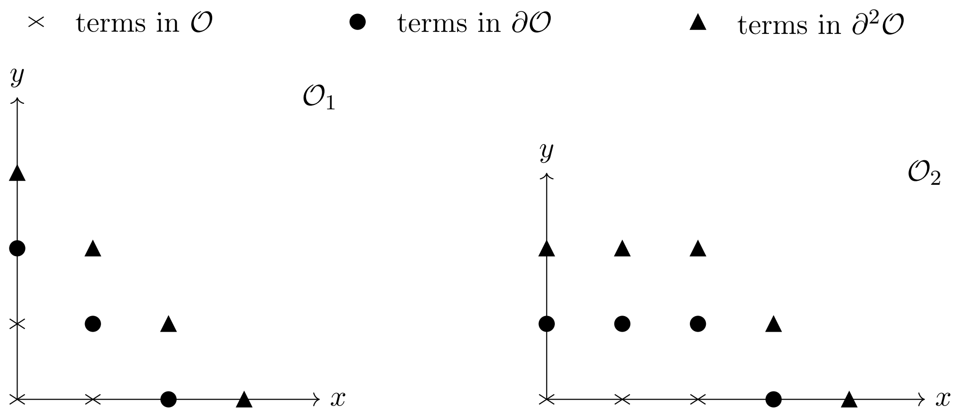
Proposition 2.1.10 gives rise to the definition of the index with respect to an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. This index measures the distance between a term and a given order ideal.

Definition 2.1.11. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.

a) For every term $te_k \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$, the number $i \in \mathbb{N}$ such that $te_k \in \partial^i\mathcal{O}$, which is unique according to Proposition 2.1.10, is called the **$\mathcal{O}$-index** of te_k and is denoted by $\operatorname{ind}_{\mathcal{O}}(te_k)$.

b) For a vector $v \in P^r \setminus \{0\}$, we define $\operatorname{ind}_{\mathcal{O}}(v) = \max\{\operatorname{ind}_{\mathcal{O}}(te_k) \mid te_k \in \operatorname{Supp}(v)\}$ to be the **$\mathcal{O}$-index** of v.

Example 2.1.12. Let $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\}$ be the order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ of Example 2.1.9, again.



Then $\operatorname{ind}_{\mathcal{O}}(xe_1) = 0$, $\operatorname{ind}_{\mathcal{O}}(x^2y^2e_2) = 2$, and hence $\operatorname{ind}_{\mathcal{O}}(xe_1 + x^2y^2e_2) = \max\{0, 2\} = 2$ according to Definition 2.1.11.

This definition allows us to prove a module version of Proposition 2.1.5.

Proposition 2.1.13. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ and $\mathcal{O} = \mathcal{O}_1e_1 \cup \cdots \cup \mathcal{O}_re_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.*

a) For every term $te_k \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$, the number $i = \operatorname{ind}_{\mathcal{O}}(te_k)$ is the smallest natural number such that there is a factorization $t = t'b$ with $t' \in \mathbb{T}^n_{i-1}$ and $b \in \partial\mathcal{O}_k$.

b) For all $t \in \mathbb{T}^n$ and $t'e_k \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$, we have $\operatorname{ind}_{\mathcal{O}}(tt'e_k) \leq \deg(t) + \operatorname{ind}_{\mathcal{O}}(t'e_k)$.

c) For two vectors $v, w \in P^r \setminus \{0\}$ such that $v + w \neq 0$, we have the inequality $\operatorname{ind}_{\mathcal{O}}(v + w) \leq \max\{\operatorname{ind}_{\mathcal{O}}(v), \operatorname{ind}_{\mathcal{O}}(w)\}$.

d) For a vector $v \in P^r \setminus \{0\}$ and a polynomial $p \in P \setminus \{0\}$, we have the inequality $\operatorname{ind}_{\mathcal{O}}(pv) \leq \deg(p) + \operatorname{ind}_{\mathcal{O}}(v)$.

Proof. Let $s \in \{1, \dots, r\}$. Then we have $\{(p_1, \dots, p_r) \in \mathcal{O} \mid p_s \neq 0\} = \mathcal{O}_s e_s$ by Definition 2.1.6. Thus the claim follows immediately from Definition 2.1.11 and Proposition 2.1.5. $\square$

After all the basic concepts concerning order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$, we are now able to define border bases in P^r. If $r = 1$, our notion of border bases exactly yields the usual border bases as, for instance, defined in [KR05, Defn. 6.4.10 and 6.4.13].

Definition 2.1.14. Let $\mathcal{O}$ be a finite order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$. We write the order ideal $\mathcal{O} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$ and $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$ with $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$, and with $\alpha_i, \beta_j \in \{1, \dots, r\}$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$.

a) A set of vectors $G = \{g_1, \dots, g_\nu\} \subseteq P^r$ is called an **$\mathcal{O}$-border prebasis** if the vectors have the form $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$ with $a_{ij} \in K$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$.

b) Let $G = \{g_1, \dots, g_\nu\} \subseteq P^r$ be an $\mathcal{O}$-border prebasis and $U \subseteq P^r$ be a P-submodule. We call G an **$\mathcal{O}$-border basis** of U if $G \subseteq U$ and if the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U.

Example 2.1.15. Consider the order ideal $\mathcal{O}$ in $\mathbb{T}^2\langle e_1, e_2\rangle$ of Example 2.1.12, again. Recall that $P = \mathbb{Q}[x, y]$, that $\mathcal{O} = \{e_1, xe_1, ye_1, e_2, xe_2, x^2e_2\}$, and that the border was of the form $\partial\mathcal{O} = \{y^2e_1, xye_1, x^2e_1, ye_2, xye_2, x^2ye_2, x^3e_2\}$. Let $G = \{g_1, \dots, g_7\} \subseteq P^2$ be with $g_1 = y^2e_1 - xe_2$, $g_2 = xye_1 - e_2$, $g_3 = x^2e_1 - ye_1 + e_2$, $g_4 = ye_2 - e_1 - ye_1 - xe_1 - e_2$, $g_5 = xye_2 + 3e_1$, $g_6 = x^2ye_2 - e_1 - e_2$, and $g_7 = x^3e_2 - e_1$. Then G is an $\mathcal{O}$-border prebasis according to Definition 2.1.14. Since we have $xg_5 - g_6 = 3xe_1 + e_1 + e_2$, we get $0 = 3\overline{xe_1} + \overline{e_1} + \overline{e_2}$ in $P^2/\langle G\rangle$. Thus G is not an $\mathcal{O}$-border basis of $\langle G\rangle$ by Definition 2.1.14.

Remark 2.1.16. If $r = 1$, there is a canonical P-algebra isomorphism between the free P-module P^r and the polynomial ring P. To shorten the notation, we use the correspondence given by this P-module isomorphism without mention.

2.2 The Border Division Algorithm

The following section is dedicated to the introduction of a division algorithm for border prebases. It serves as the basic part of many of the latter proofs in this chapter. In particular, it allows us to compute representatives of elements of residue class modules $P^r/\langle G\rangle$ for any border prebasis G and enables us to define a normal form with respect to a submodule $U \subseteq P^r$ that are given by a border basis.

For the whole section, we let $\mathcal{O} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$ be an order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ with border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$ where we have $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$, and where $\alpha_i, \beta_j \in \{1, \dots, r\}$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$. We let $G = \{g_1, \dots, g_\nu\} \subseteq P^r$ always be an $\mathcal{O}$-border prebasis and for all $j \in \{1, \dots, \nu\}$, we let $a_{1j}, \dots, a_{\mu j} \in K$ be such that $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$.

The following division algorithm allows us to divide any vector $v \in P^r$ by the $\mathcal{O}$-border prebasis G and get a representative of the residue class of v in $P^r/\langle G \rangle$ that is contained in the K-vector space $\langle \mathcal{O} \rangle_K$. It is a generalization of [KR05, Prop. 6.4.11].

Algorithm 1: $\texttt{divAlg}(v, G)$

Input:
$v \in P^r$;
$G = \{g_1, \dots, g_\nu\} \subseteq P^r$ is an $\mathcal{O}$-border prebasis where $\mathcal{O} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$, and $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$ with $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$ and $\alpha_i, \beta_j \in \{1, \dots, r\}$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$;

1 $(p_1, \dots, p_\nu) := 0 \in P^\nu$
2 $(q_1, \dots, q_r) := v$
3 **while** $(q_1, \dots, q_r) \neq 0$ **and** $\operatorname{ind}_{\mathcal{O}}((q_1, \dots, q_r)) > 0$ **do**
4   **choose** $te_k \in \operatorname{Supp}((q_1, \dots, q_r))$ with $\operatorname{ind}_{\mathcal{O}}(te_k) = \operatorname{ind}_{\mathcal{O}}((q_1, \dots, q_r))$.
5   Determine the smallest index $j \in \{1, \dots, \nu\}$ such that there exists a term $t' \in \mathbb{T}^n$ with $\deg(t') = \operatorname{ind}_{\mathcal{O}}((q_1, \dots, q_r)) - 1$ and $te_k = t' b_j e_{\beta_j}$.
6   Let $a \in K$ be the coefficient of $te_k = t' b_j e_{\beta_j}$ in $(q_1, \dots, q_r)$.
7     $p_j := p_j + at'$
8     $(q_1, \dots, q_r) := (q_1, \dots, q_r) - at' g_j$
9 **end**
10 Determine $c_1, \dots, c_\mu \in K$ such that $(q_1, \dots, q_r) = c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu}$.
11 **return** $((p_1, \dots, p_\nu), (c_1, \dots, c_\mu))$

Theorem 2.2.1. (The Border Division Algorithm)
Let $v \in P^r$. Then Algorithm 1 is actually an algorithm and the result

$$((p_1, \dots, p_\nu), (c_1, \dots, c_\mu)) := \texttt{divAlg}(v, G)$$

of Algorithm 1 applied to the input data v and G satisfies the following conditions.

i) The result $((p_1, \dots, p_\nu), (c_1, \dots, c_\mu))$ is a tuple in $P^\nu \times K^\mu$ and it does not depend on the choice of the term te_k in line 4.

ii) We have $v = p_1 g_1 + \dots + p_\nu g_\nu + c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu}$.

iii) For all $j \in \{1, \dots, \nu\}$ with $p_j \neq 0$, we have $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1$.

Proof. First we show that every step of the procedure can be computed. We start to consider an iteration of the while-loop starting in line 3. Let $(q_1, \dots, q_r) \in P^r$ be as in line 4 and let $m = \operatorname{ind}_{\mathcal{O}}((q_1, \dots, q_r))$. Then $(q_1, \dots, q_r) \neq 0$ and $m > 0$. Thus the existence of a term $te_k \in \operatorname{Supp}((q_1, \dots, q_r))$ such that $\operatorname{ind}_{\mathcal{O}}(te_k) = m$ in line 4 follows from Definition 2.1.11. We now take a closer look at line 5. Since the while-loop in line 3 is executed, we have $\operatorname{ind}_{\mathcal{O}}(te_k) > 0$, i.e. $te_k \in \mathbb{T}^n\langle e_1, \dots, e_r \rangle \setminus \mathcal{O}$ by the

Definitions 2.1.2 and 2.1.11. By Proposition 2.1.13, there is a factorization $te_k = t'b_je_{\beta_j}$ with a term $t' \in \mathbb{T}^n$ of degree $\deg(t') = \mathrm{ind}_{\mathcal{O}}(te_k)-1 = m-1$ and an index $j \in \{1,\dots,\nu\}$. At last, in line 10, the while-loop has already been executed, i.e. $(q_1,\dots,q_r) = 0$ or $\mathrm{ind}_{\mathcal{O}}((q_1,\dots,q_r)) = 0$. In this situation, we have $\mathrm{Supp}((q_1,\dots,q_r)) \subseteq \mathcal{O}$ according to Definition 2.1.11 and can hence compute line 10. Altogether, we see that every step of the procedure can actually be computed.
In order to prove the termination, we show that the while-loop starting in line 3 is executed only finitely many times. Let $j \in \{1,\dots,\nu\}$ be defined as in line 5 and let $m = \mathrm{ind}_{\mathcal{O}}((q_1,\dots,q_r))$ in this situation. Taking a closer look at the subtraction in line 8, we see that we subtract the vector $at'g_j = at'b_je_{\beta_j} - at'\sum_{i=1}^{\mu} a_{ij}t_ie_{\alpha_i}$ from $(q_1,\dots,q_r)$. By the choices of $j \in \{1,\dots,\nu\}$ and $t' \in \mathbb{T}^n_{m-1}$ in line 5 and the choice of $a \in K$ in line 6, it follows that the term $te_k = t'b_je_{\beta_j}$ with $\mathcal{O}$-index m is replaced by terms of the form $t't_ie_{\alpha_i} \in \overline{\partial^{m-1}\mathcal{O}}$ with $i \in \{1,\dots,\mu\}$, which have strictly smaller $\mathcal{O}$-index than $te_k = t'b_je_{\beta_j}$ according to Proposition 2.1.13. The procedure hence terminates after finitely many steps because there are only finitely many terms in the support of a given vector whose $\mathcal{O}$-index is smaller than or equal to the $\mathcal{O}$-index of a given term. Altogether, we see that the procedure is actually an algorithm.

Next we prove the correctness. To this end, we start to prove that the equation $v = p_1g_1 + \cdots + p_\nu g_\nu + (q_1,\dots,q_r)$ is an invariant of the while-loop in line 3. Before the first iteration of the while-loop, we have $p_1 = \cdots = p_\nu = 0$ and $(q_1,\dots,q_r) = v$, i.e. the invariant is obviously satisfied. We now consider the changes of $(p_1,\dots,p_\nu) \in P^\nu$ and $(q_1,\dots,q_r) \in P^r$ during one iteration of the while-loop. Let $(p_1,\dots,p_\nu) \in P^\nu$ and $(q_1,\dots,q_r) \in P^r$ be such that the invariant holds, and let $(p_1',\dots,p_\nu') \in P^\nu$ and $(q_1',\dots,q_r') \in P^r$ be the values of $(p_1,\dots,p_\nu)$ and $(q_1,\dots,q_r)$ after one iteration of the while-loop. The values of the vectors $(p_1,\dots,p_\nu) \in P^\nu$ and $(q_1,\dots,q_r) \in P^r$ are only changed in the lines 7 and 8. Let $j \in \{1,\dots,\nu\}$ and $t' \in \mathbb{T}^n$ be as in line 5. Then we have $p_j' = p_j + at'$, $p_i' = p_i$ for all $i \in \{1,\dots,\nu\} \setminus \{j\}$, and $(q_1',\dots,q_r') = (q_1,\dots,q_r) - at'g_j$. This yields

$$\begin{aligned} v &= p_1g_1 + \cdots + p_\nu g_\nu + (q_1,\dots,q_r) \\ &= p_1'g_1 + \cdots + p_{j-1}'g_{j-1} + (p_j' - at')g_j + p_{j+1}'g_{j+1} + \cdots + p_\nu' g_\nu + ((q_1',\dots,q_r') + at'g_j) \\ &= p_1'g_1 + \cdots p_\nu' g_\nu + (q_1',\dots,q_r'), \end{aligned}$$

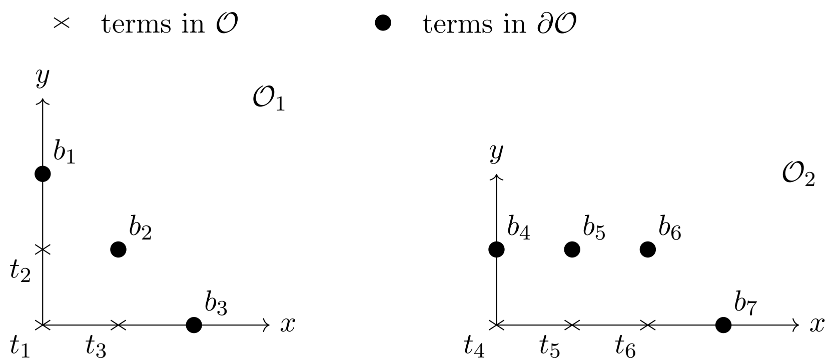
i.e. the invariant is also satisfied after one iteration of the while-loop. By induction on the number of iterations of the while-loop, we see that the invariant is always satisfied.
As we have already seen in the proof of the termination, the term t' in line 5 always has at most the degree $\mathrm{ind}_{\mathcal{O}}(v) - 1$. Since the support of the polynomials $p_1,\dots,p_\nu$ consists precisely of these terms t' by the lines 1 and 7, it follows that $\deg(p_j) \le \mathrm{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1,\dots,\nu\}$ with $p_j \neq 0$ at the end of the algorithm. In line 10, we have $v = p_1g_1 + \cdots + p_\nu g_\nu + (q_1,\dots,q_r) = p_1g_1 + \cdots + p_\nu g_\nu + c_1t_1e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with scalars $c_1,\dots,c_\mu \in K$. Thus the algorithm returns a tuple with the claimed properties in line 11.
Finally, we prove that the result does not depend on the choice of the term te_k in line 4. This fact follows from the observation that te_k in $(q_1,\dots,q_r)$ is replaced by terms of

strictly smaller $\mathcal{O}$-index in line 8 during every iteration of the while-loop in line 3. Thus different choices of the term te_k in line 4 do not interfere with one another. Altogether, we see that the final result, after all terms of maximal $\mathcal{O}$-index have been rewritten, is independent of the ordering in which they are handled. $\square$

For every vector $v \in P^r$, the Border Division Algorithm 2.2.1 can be used to compute a representative of the residue class of v in $P^r/\langle G\rangle$ that is a K-linear combination of the elements of $\mathcal{O}$. This gives rise to the following definition of the normal remainder with respect to the $\mathcal{O}$-border prebasis G.

Definition 2.2.2. Let $v \in P^r$. We apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = p_1 g_1 + \cdots + p_\nu g_\nu + c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with $p_1, \ldots, p_\nu \in P$, $c_1, \ldots, c_\mu \in K$, and such that for all $j \in \{1, \ldots, \nu\}$ where $p_j \neq 0$, $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1$. Then the vector $\mathrm{NR}_G(v) = c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu} \in \langle \mathcal{O}\rangle_K \subseteq P^r$ is called the **normal remainder** of v with respect to G.

Example 2.2.3. In order to illustrate Algorithm 1, we consider the $\mathcal{O}$-border prebasis $G \subseteq P^2$ of Example 2.1.15, again.



Recall that $P = \mathbb{Q}[x, y]$, $\{e_1, e_2\}$ was the canonical P-module basis of P^2, that we had $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\} = \{t_1 e_{\alpha_1} \ldots, t_6 e_{\alpha_6}\}$, that the border of $\mathcal{O}$ was of the form $\partial\mathcal{O} = \{y^2e_1, xye_1, x^2e_1, ye_2, xye_2, x^2ye_2, x^3e_2\} = \{b_1 e_{\beta_1}, \ldots, b_7 e_{\beta_7}\}$, and that we had $G = \{g_1, \ldots, g_7\}$ with $g_1 = y^2e_1 - xe_2$, $g_2 = xye_1 - e_2$, $g_3 = x^2e_1 - ye_1 + e_2$, $g_4 = ye_2 - e_1 - ye_1 - xe_1 - e_2$, $g_5 = xye_2 + 3e_1$, $g_6 = x^2ye_2 - e_1 - e_2$, and $g_7 = x^3e_2 - e_1$. We now consider the steps of the Border Division Algorithm 1 applied to the input data $v = x^3e_1 + xye_1 + x^3ye_2 \in P^2$ and G in detail.

The initialization process in the lines 1 and 2 yields $(p_1, \ldots, p_7) = (0, 0, 0, 0, 0, 0, 0)$ and $(q_1, q_2) = (x^3 + xy, x^3y)$. Since $\operatorname{ind}_{\mathcal{O}}((q_1, q_2)) = \operatorname{ind}_{\mathcal{O}}(x^3e_1) = \operatorname{ind}_{\mathcal{O}}(x^3ye_2) = 2 > 0$, the while-loop in line 3 is executed.
We choose the term x^3ye_2 in line 4. Then we have $j = 6$ and get the factorization $x^3ye_2 = x \cdot x^2ye_2 = x \cdot b_6 e_{\beta_6}$ in line 5. After line 7, we have $(p_1, \ldots, p_7) = (0, 0, 0, 0, 0, x, 0)$ and after line 8, we have $(q_1, q_2) = (x^3 + xy, x^3y) - x \cdot (-1, x^2y - 1) = (x^3 + xy + x, x)$. Now the $\mathcal{O}$-index is $\operatorname{ind}_{\mathcal{O}}((q_1, q_2)) = \operatorname{ind}_{\mathcal{O}}(x^3e_1) = 2 > 0$.
Thus we must choose the term x^3e_1 in line 4. Then we have $j = 3$ and get the factorization $x^3e_1 = x \cdot x^2e_1 = x \cdot b_3 e_{\beta_3}$ in line 5. This yields $(p_1, \ldots, p_7) = (0, 0, x, 0, 0, x, 0)$ after

line 7 and $(q_1, q_2) = (x^3 + xy + x, x) - x \cdot (x^2 - y, 1) = (2xy + x, 0)$ after line 8. Now the $\mathcal{O}$-index has decreased to $\operatorname{ind}_{\mathcal{O}}((q_1, q_2)) = \operatorname{ind}_{\mathcal{O}}(xye_1) = 1 > 0$.
Hence we must choose the term xye_1 in line 4. Then we have $j = 2$ and get the factorization $xye_1 = 1 \cdot xye_1 = 1 \cdot b_2 e_{\beta_2}$ in line 5. This yields $(p_1, \dots, p_7) = (0, 2, x, 0, 0, x, 0)$ after line 7 and $(q_1, q_2) = (2xy + x, 0) - 2 \cdot (xy, -1) = (x, 2)$ after line 8.
After that, $(q_1, q_2) \neq 0$ and $\operatorname{ind}_{\mathcal{O}}((q_1, q_2)) = 0$. As $(q_1, q_2) = (x, 2) = t_3 e_{\alpha_3} + 2t_4 e_{\alpha_4}$, the algorithm returns the tuple $((0, 2, x, 0, 0, x, 0), (0, 0, 1, 2, 0, 0)) \in P^7 \times K^6$ in line 10. Moreover, this yields that $v = x^3 e_1 + xye_1 + x^3 ye_2 = 2g_2 + xg_3 + xg_6 + \mathrm{NR}_G(v)$ with $\mathrm{NR}_G(v) = xe_1 + 2e_2 \in \langle \mathcal{O} \rangle_K$ according to Theorem 2.2.1 and Definition 2.2.2.

As a first consequence, we can prove that an $\mathcal{O}$-border basis of a P-submodule $U \subseteq P^r$ is indeed a basis, i.e. that G generates U. This generalizes [KR05, Prop. 6.4.15] to the module setting.

Corollary 2.2.4. *Assume that G is an O-border basis of a P-submodule $U \subseteq P^r$. Then we have $\langle G \rangle = U$.*

Proof. According to Definition 2.1.14, we have $\langle G \rangle \subseteq U$. For the converse implication, we let $v \in U$. We apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = w + c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with $w \in \langle G \rangle \subseteq U$ and $c_1, \dots, c_\mu \in K$. It follows that $0 = \overline{v} = c_1 \overline{t_1 e_{\alpha_1}} + \cdots + c_\mu \overline{t_\mu e_{\alpha_\mu}}$ in P^r/U. Since G is an $\mathcal{O}$-border basis of U, the residue classes of the elements of $\mathcal{O}$ form a K-vector space basis of P^r/U by Definition 2.1.14. Thus we get $c_1 = \cdots = c_\mu = 0$ and this implies $v = w \in \langle G \rangle$. □

The Border Division Algorithm 2.2.1 also allows us to compute representatives of residue classes modulo $\langle G \rangle$.

Corollary 2.2.5. *The residue classes of the elements of $\mathcal{O}$ in $P^r/\langle G \rangle$ generate the K-vector space $P^r/\langle G \rangle$. In particular, for every vector $v \in P^r$, the normal remainder $\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$ of v with respect to G is a representative of the residue class $\overline{v} \in P^r/\langle G \rangle$.*

Proof. Let $v \in P^r$. We apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = p_1 g_1 + \cdots + p_\nu g_\nu + \mathrm{NR}_G(v)$ with $p_1, \dots, p_\nu \in P$ and $\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$. Thus $\overline{v} = \overline{\mathrm{NR}_G(v)}$ in $P^r/\langle G \rangle$. The converse inclusion follows trivially from $\mathcal{O} \subseteq P^r$. □

Another consequence is this first characterization of border bases.

Corollary 2.2.6. *Let $U \subseteq P^r$ be a P-submodule with $G \subseteq U$. Then the following conditions are equivalent.*

i) The $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis of U.

ii) We have $U \cap \langle \mathcal{O} \rangle_K = \{0\}$.

iii) We have $P^r = U \oplus \langle \mathcal{O} \rangle_K$.

Proof. First we prove that i) implies ii). Let $v \in U \cap \langle \mathcal{O} \rangle_K$. Then there are $c_1, \dots, c_\mu \in K$ with $v = c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu}$. In P^r/U, this yields $0 = \overline{v} = c_1 \overline{t_1 e_{\alpha_1}} + \dots + c_\mu \overline{t_\mu e_{\alpha_\mu}}$. As G is an $\mathcal{O}$-border basis of U, it follows that $c_1 = \dots = c_\mu = 0$ by Definition 2.1.14 and thus $v = 0$. Since $0 \in U \cap \langle \mathcal{O} \rangle_K$ trivially holds, the claim follws.

Next we prove that ii) implies iii). As we have $U \cap \langle \mathcal{O} \rangle_K = \{0\}$, it suffices to prove that $P^r = U + \langle \mathcal{O} \rangle_K$. Obviously, we have $P^r \supseteq U + \langle \mathcal{O} \rangle_K$. In order to prove the converse inclusion, we let $v \in P^r$. According to Corollary 2.2.5, $v = w + \mathrm{NR}_G(v)$ with $w \in \langle G \rangle$ and $\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$. The hypothesis $G \subseteq U$ hence yields the claim.

Finally, we prove that iii) implies i). Let $c_1, \dots, c_\mu \in K$ be coefficients such that $c_1 \overline{t_1 e_{\alpha_1}} + \dots + c_\mu \overline{t_\mu e_{\alpha_\mu}} = 0$ in P^r/U. Then we have $c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu} \in U \cap \langle \mathcal{O} \rangle_K$. Because of $P^r = U \oplus \langle \mathcal{O} \rangle_K$, we see that $c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu} = 0$. As $\mathcal{O}$ is K-linearly independent, it follows that $c_1 = \dots = c_\mu = 0$. Hence the residue classes of the elements of $\mathcal{O}$ in P^r/U are K-linearly independent. Moreover, every vector $v \in P^r$ can be written in the form $v = w + \mathrm{NR}_G(v)$ where $w \in \langle G \rangle$ according to Corollary 2.2.5. As $G \subseteq U$ and as $\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$ by Definition 2.2.2, we see that the residue classes of $\mathcal{O}$ in P^r/U form also a K-generating set of P^r/U. The claim now follows from Definition 2.1.14. □

As we have seen in Corollary 2.2.5, we can use the Border Division Algorithm 2.2.1 to compute normal remainders with respect to the $\mathcal{O}$-border prebasis G of every vector $v \in P^r$ and that $\overline{\mathrm{NR}_G(v)} = \overline{v}$ in $P^r/\langle G \rangle$. But normal remainders are not unique since they depend on the ordering of the elements of G used during the computation. Therefore, we cannot use normal remainders to decide whether two vectors in P^r represent the same residue class modulo $\langle G \rangle$. Fortunately, it turns out that we have unique normal remainders if the $\mathcal{O}$-border prebasis G is an $\mathcal{O}$-border basis of $\langle G \rangle$.

Lemma 2.2.7. *Let $U \subseteq P^r$ be a P-submodule and let G and G' be two $\mathcal{O}$-border bases of U. Then we have $\mathrm{NR}_G(v) = \mathrm{NR}_{G'}(v)$ for every vector $v \in P^r$.*

Proof. Let $v \in P^r$. We apply the Border Division Algorithm 2.2.1 to v and G, and to v and G' in order to obtain two representations $w + \mathrm{NR}_G(v) = v = w' + \mathrm{NR}_{G'}(v)$ with vectors $w, w' \in \langle G \rangle$ and $\mathrm{NR}_G(v), \mathrm{NR}_{G'}(v) \in \langle \mathcal{O} \rangle_K$. As G and G' are $\mathcal{O}$-border bases of U, Corollary 2.2.6 implies that $U \cap \langle \mathcal{O} \rangle_K = \{0\}$. Thus the claim follows from $\mathrm{NR}_G(v) - \mathrm{NR}_{G'}(v) = -w + w' \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K \subseteq U \cap \langle \mathcal{O} \rangle_K = \{0\}$. □

Remark 2.2.8. Let $v \in P^r$ be a vector. Similar to the situation of Gröbner bases and border bases in P, the normal remainder of v with respect to the $\mathcal{O}$-border prebasis G is a representative of the residue class $\overline{v} \in P^r/\langle G \rangle$ by Corollary 2.2.5. But the normal remainder of v with respect to G depends on the particularly chosen $\mathcal{O}$-border prebasis G and on the ordering of the elements in G by Definition 2.2.2. Fortunately, if G is even an $\mathcal{O}$-border basis of $\langle G \rangle$, Lemma 2.2.7 shows that the normal remainder of v is independent of the particularly chosen $\mathcal{O}$-border basis G and the ordering of the elements in G. Thus the normal remainder defines a normal form of v with respect to $\langle G \rangle$ in this situation. In particular, we can also compute this normal form with the Border Division Algorithm 2.2.1.

If G is an $\mathcal{O}$-border basis of a P-submodule $U \subseteq P^r$, Corollary 2.2.4 and Remark 2.2.8 give rise to a normal form with respect to U. This normal form generalizes the normal form defined in [KR05, Defn. 6.4.20].

Definition 2.2.9. Let G be an $\mathcal{O}$-border basis of a P-submodule $U \subseteq P^r$ and let $v \in P^r$. Then we call the vector $\mathrm{NF}_{\mathcal{O},U}(v) = \mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K \subseteq P^r$, which is unique according to Remark 2.2.8, the **normal form** of v with respect to $\mathcal{O}$ and U.

Before we end this section, we prove the basic properties of the normal form $\mathrm{NF}_{\mathcal{O},U}$ as defined in Definition 2.2.9 and compare it with the normal form $\mathrm{NF}_{\sigma,U}$ defined by a term ordering σ as in [KR00, Defn. 2.4.8] for a given P-submodule $U \subseteq P^r$. This proposition generalizes [KR05, Prop. 6.4.21] to the module setting. Recall that, for every term ordering σ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ and every submodule $U \subseteq P^r$, the complement $\mathcal{O}_\sigma(U) = \mathbb{T}^n\langle e_1, \ldots, e_r\rangle \setminus \mathrm{LT}_\sigma\{U\}$ of the monomodule of all leading terms of the elements of U is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ as defined in Definition 2.1.6.

Proposition 2.2.10. *Let G be an $\mathcal{O}$-border basis of a P-submodule $U \subseteq P^r$.*

a) Assume that there exists a term ordering σ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ such that $\mathcal{O} = \mathcal{O}_\sigma(U)$. Then we have $\mathrm{NF}_{\mathcal{O},U}(v) = \mathrm{NF}_{\sigma,U}(v)$ for all $v \in P^r$.

b) We have $\mathrm{NF}_{\mathcal{O},U}(cv + c'v') = c\,\mathrm{NF}_{\mathcal{O},U}(v) + c'\,\mathrm{NF}_{\mathcal{O},U}(v')$ for all $c, c' \in K$ and for all $v, v' \in P^r$.

c) For every $v \in P^r$, we have $\mathrm{NF}_{\mathcal{O},U}(\mathrm{NF}_{\mathcal{O},U}(v)) = \mathrm{NF}_{\mathcal{O},U}(v)$.

d) For all $p \in P$ and $v \in P^r$, we have $\mathrm{NF}_{\mathcal{O},U}(pv) = \mathrm{NF}_{\mathcal{O},U}(p\,\mathrm{NF}_{\mathcal{O},U}(v))$.

Proof. Claim a) follows since for all $v \in P^r$, both $\mathrm{NF}_{\mathcal{O},U}(v)$ and $\mathrm{NF}_{\sigma,U}(v)$ are equal to the unique vector in $\overline{v} \in P^r/U$ whose support is contained in $\mathcal{O} = \mathcal{O}_\sigma(U)$ according to Definition 2.2.9 and [KR00, Defn. 2.4.8]. The other claims follow from the same uniqueness. □

2.3 Existence and Uniqueness

In this section, we prove that every submodule $U \subseteq P^r$ with finite K-codimension in P^r possesses an $\mathcal{O}$-border basis if we do not a priori fix the order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Moreover, we show that $\mathcal{O}$-border bases are unique for a predetermined order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ and we give a method to deduce reduced Gröbner bases from border bases if the corresponding order ideal comes from a term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.

Before we start with the proofs, we recall the definition of fields of definitions of [KR00, Defn. 2.4.14].

Definition 2.3.1. Let $U \subseteq P^r$ be a P-submodule.

a) Let $k \subseteq K$ be a subfield. We say that U is **defined over** k if there exist elements in $(k[x_1, \ldots, x_n])^r$ which generate U as a P-module.

b) A subfield $k \subseteq K$ is called a **field of definition** of U if U is defined over k and there exists no proper subfield $k' \subset k$ such that U is defined over k'.

Now we are able prove the existence and uniqueness of border bases in P^r. The following proposition generalizes [KR05, Prop. 6.4.17] to the module setting.

Proposition 2.3.2. (Existence and Uniqueness of Border Bases)
Let $U \subseteq P^r$ be a P-submodule and let $\mathcal{O}$ be a finite order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Moreover, assume that the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U.

a) There exists a unique $\mathcal{O}$-border basis of U.

b) Let G be an $\mathcal{O}$-border prebasis with $G \subseteq U$. Then G is the $\mathcal{O}$-border basis of U.

c) Let K' be the field of definition of U. Then the $\mathcal{O}$-border basis of U is contained in $K'[x_1, \ldots, x_n]$.

Proof. In order to prove a), we write the order ideal $\mathcal{O} = \{t_1e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\}$ and its border $\partial\mathcal{O} = \{b_1e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}\}$ with $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$, and $\alpha_i, \beta_j \in \{1, \ldots, r\}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$. Let $j \in \{1, \ldots, \nu\}$. As $b_je_{\beta_j} \notin \mathcal{O}$ by Definition 2.1.7 and as the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U by assumption, there are $a_{1j}, \ldots, a_{\mu j} \in K$ with $\overline{b_je_{\beta_j}} = a_{1j}\overline{t_1e_{\alpha_1}} + \cdots + a_{\mu j}\overline{t_\mu e_{\alpha_\mu}}$ in P^r/U. In particular, $g_j = b_je_{\beta_j} - \sum_{i=1}^{\mu} a_{ij}t_ie_{\alpha_i} \in U$. Thus $G = \{g_1, \ldots, g_\nu\} \subseteq U$ is an $\mathcal{O}$-border prebasis by Definition 2.1.14. As the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U, G is even an $\mathcal{O}$-border basis of U by Definition 2.1.14.
It remains to prove the uniqueness. Let $G' = \{g_1', \ldots, g_\nu'\} \subseteq U$ be another $\mathcal{O}$-border basis of U where $g_j' = b_je_{\beta_j} - \sum_{i=1}^{\mu} a_{ij}'t_ie_{\alpha_i}$ and $a_{ij}' \in K$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$. Assume that there exist an $i \in \{1, \ldots, \mu\}$ and a $j \in \{1, \ldots, \nu\}$ such that $a_{ij} \neq a_{ij}'$. Then Corollary 2.2.6 yields the contradiction $0 \neq g_j - g_j' \in U \cap \langle\mathcal{O}\rangle_K = \{0\}$. Altogether, claim a) follows.

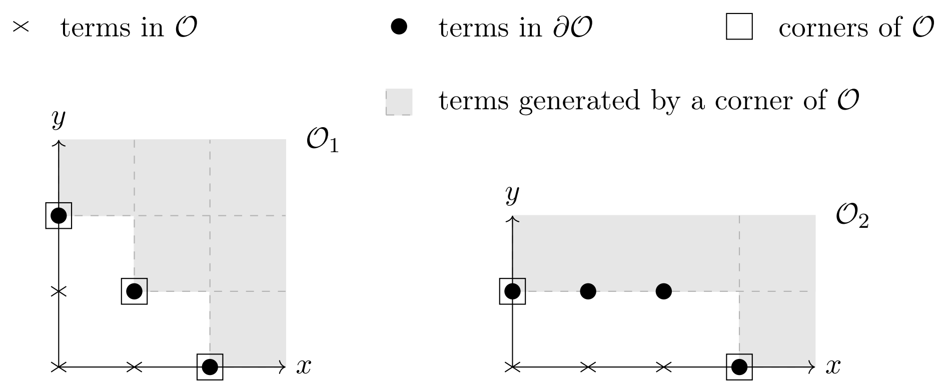
Next we show claim b). Since the residue classes of the elements of $\mathcal{O}$ in P^r/U form a K-vector space basis of P^r/U, we see that G is an $\mathcal{O}$-border basis of U by Definition 2.1.14.

Finally, we prove c). Let $P' = K'[x_1, \ldots, x_n]$ and $U' = U \cap (P')^r$. Moreover, let σ be a term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Then the P-submodules $U \subseteq P^r$ and $U' \subseteq (P')^r$ have the same reduced σ-Gröbner basis and $\mathrm{LT}_\sigma\{U\} = \mathrm{LT}_\sigma\{U'\}$ by [KR00, Lemma 2.4.16]. Hence we see that $\dim_K((P')^r/U') = \dim_K(P^r/U) = \#\mathcal{O}$ according to Macaulay's Basis Theorem [KR00, Thm. 1.5.7]. Moreover, the elements of $\mathcal{O}$ are contained in $(P')^r$ and they are K-linearly independent modulo $U' \subseteq U$. Thus the residue classes of the elements of $\mathcal{O}$ in $(P')^r/U'$ form a K-vector space basis of $(P')^r/U'$. According to a), there exists a unique $\mathcal{O}$-border basis $G' \subseteq P^r$ of U'. Since G' is an $\mathcal{O}$-border prebasis with $G' \subseteq U$, claim c) follows from b). $\square$

Next we show that for all term orderings σ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$, the reduced σ-Gröbner basis of a P-submodule $U \subseteq P^r$ which has a finite K-codimension in P^r is subset of the $\mathcal{O}_\sigma(U)$-border bases of U. Recall that for a term ordering σ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$, we denote the complement of the momonodule $\mathrm{LT}_\sigma\{U\}$ of all leading terms with respect to σ of a P-submodule U by $\mathcal{O}_\sigma(U) = \mathbb{T}^n\langle e_1, \ldots, e_r\rangle \setminus \mathrm{LT}_\sigma\{U\}$ and that $\mathcal{O}_\sigma(U)$ is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ by Definition 2.1.6.

Definition 2.3.3. Let $\mathcal{O}$ be an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. We call the elements of the minimal generating set of the monomial submodule $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$ the **corners** of $\mathcal{O}$.

Example 2.3.4. Let $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\}$ be the order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ of Example 2.1.9, again.



Then y^2e_1, xye_1, x^2e_1, ye_2, and x^3e_2 are precisely the corners of $\mathcal{O}$ by Definition 2.3.3.

Proposition 2.3.5. *Let $U \subseteq P^r$ be a P-submodule and let σ be a term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. Then there exists a unique $\mathcal{O}_\sigma(U)$-border basis G of U and the reduced σ-Gröbner basis of U is the subset of G corresponding to the corners of $\mathcal{O}_\sigma(U)$.*

Proof. By Macaulay's Basis Theorem [KR00, Thm. 1.5.7], the residue classes of the elements of $\mathcal{O}_\sigma(U)$ in P^r/U form a K-vector space basis of P^r/U. Thus Proposition 2.3.2 implies the existence of a unique $\mathcal{O}_\sigma(U)$-border basis G of U.

For the other claim, let $be_k \in \mathrm{LT}_\sigma\{U\}$ with $b \in \mathbb{T}^n$ and $k \in \{1, \ldots, r\}$ be a corner of the order ideal $\mathcal{O}_\sigma(U)$. According to the definition of the reduced σ-Gröbner basis of U, cf. [KR00, Defn. 2.4.12], the corners of $\mathcal{O}_\sigma(U)$ are precisely the leading terms of the elements of the reduced σ-Gröbner basis of U. Moreover, the element of the reduced σ-Gröbner basis of U with leading term be_k has the form $be_k - \mathrm{NF}_{\sigma,U}(be_k) = be_k - \mathrm{NF}_{\mathcal{O}_\sigma(U),U} \in G$ and the claim follows. $\square$

Remark 2.3.6. The proof of Proposition 2.3.5 gives rise to a naive algorithm for the computation of a border basis of a given P-submodule $U \subseteq P^r$ that has a finite K-codimension in P^r. Let σ be any term ordering on $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$. We first have to compute a σ-Gröbner basis H of P^r/U to determine the order ideal $\mathcal{O}_\sigma(U)$ in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ with Macaulay's Basis Theorem [KR00, Thm. 1.5.7]. Then we compute its border $\partial\mathcal{O}_\sigma(U) = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$. Recall that $\mathrm{NF}_{\sigma,U} = \mathrm{NF}_{\mathcal{O}_\sigma(U),U}$ according to Proposition 2.2.10. Using H and the Division Algorithm for Gröbner Bases [KR00, Thm. 1.6.4], we then compute $g_j = b_j e_{\beta_j} - \mathrm{NF}_{\sigma,U}(b_j e_{\beta_j}) = b_j e_{\beta_j} - \mathrm{NR}_{\sigma,H}(b_j e_{\beta_j}) \in U \subseteq P^r$ for all $j \in \{1, \dots, \nu\}$. According to Proposition 2.3.2, the set $G = \{g_1, \dots, g_\nu\} \subseteq P^r$ is then the $\mathcal{O}_\sigma(U)$-border basis of U.

2.4 Characterizations

Now that we know that every submodule of P^r with finite K-codimension in P^r indeed possesses a border basis, we want to characterize border bases in several ways. All the characterizations of this section are generalizations of the corresponding ones in [KK05] and [KR05, Subsection 6.4.B]. More precisely, we want to determine whether a given $\mathcal{O}$-border prebasis $G \subseteq P^r$ is the $\mathcal{O}$-border basis of $\langle G\rangle$.

First we prove that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if G satisfies a certain special generation property in Theorem 2.4.1. After that, we show that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if its border form module is generated by the border forms of the elements of G in Theorem 2.4.5. Here the border form of a vector is a generalization of the concept of leading terms that is also applicable if the order ideal does not come from a term ordering. Then we introduce rewrite rules corresponding to border prebases and show that such a rewrite rule is confluent if and only if we have a border bases in Theorem 2.4.13. In Definition 2.4.24, we introduce the notion of liftings of border syzygies. We prove that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if we are able to lift the neighbor syzygies with respect to $\mathcal{O}$ in Theorem 2.4.26. Here the neighbors with respect to $\mathcal{O}$ are the border basis analogon of the critical pairs of Gröbner bases. Finally, we also prove a Buchberger Criterion for border bases in P^r in Theorem 2.4.31. It allows us to easily check whether we are given a border bases or not by applying the Border Division Algorithm to the S-vectors corresponding to the neighbors with respect to $\mathcal{O}$.

All the previously mentioned characterizations have analogous versions for Gröbner bases, cf., for instance, [KR00, Thm. 2.4.1]. But we will also prove a characterization of border bases that has no analogon in the Gröbner bases theory. The characterization in Theorem 2.4.19 says that we only have to check whether the formal multiplication matrices with respect to G, i.e. certain matrices that describe the multiplication by an indeterminate in the residue class module $P^r/\langle G\rangle$, are pairwise commuting. For border bases in P, this characterization was first proven in [Mou99]. It will play an important role in the latter part of this thesis, namely in the Chapters 4 to 6. For instance, we can study geometric properties of zero-dimensional closed subschemes of weighted projective spaces by taking a careful look at the multiplicative structure of the corresponding projective coordinate rings which, in fact, is fully determined by the corresponding formal multiplication matrices.

For the whole section, we let $\mathcal{O} = \{t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\}$ be an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ with border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}\}$ where we have $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$ and where $\alpha_i, \beta_j \in \{1, \ldots, r\}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$. Let $G = \{g_1, \ldots, g_\nu\}$ be an $\mathcal{O}$-border prebasis and for all $j \in \{1, \ldots, \nu\}$, let $a_{1j}, \ldots, a_{\mu j} \in K$ be such that $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \in P^r$.

2.4.1 Special Generation

The first characterization shows that border bases are precisely the border prebases which satisfy a special generation property. For border bases in P, this characterization was first proven in [KK05, Prop. 9] and can also be found in [KR05, Prop. 6.4.23].

Theorem 2.4.1. (Border Bases and Special Generation)
The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if the following equivalent conditions are satisfied.

A_1) *For every vector $v \in \langle G\rangle \setminus \{0\}$, there exist polynomials $p_1, \ldots, p_\nu \in P$ such that $v = p_1 g_1 + \cdots + p_\nu g_\nu$ and $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1, \ldots, \nu\}$ with $p_j \neq 0$.*

A_2) *For every vector $v \in \langle G\rangle \setminus \{0\}$, there exist polynomials $p_1, \ldots, p_\nu \in P$ such that $v = p_1 g_1 + \cdots + p_\nu g_\nu$ and $\max\{\deg(p_j) \mid j \in \{1, \ldots, \nu\}, p_j \neq 0\} = \operatorname{ind}_{\mathcal{O}}(v) - 1$.*

Proof. In order to show that A_1) holds if G is the $\mathcal{O}$-border basis of $\langle G\rangle$, let $v \in \langle G\rangle \setminus \{0\}$. We apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = p_1 g_1 + \cdots + p_\nu g_\nu + c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with $p_1, \ldots, p_\nu \in P$, $c_1, \ldots, c_\mu \in K$, and $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1, \ldots, \nu\}$ with $p_j \neq 0$. Since $v \in \langle G\rangle$, this yields that $0 = \overline{v} = c_1 \overline{t_1 e_{\alpha_1}} + \cdots + c_\mu \overline{t_\mu e_{\alpha_\mu}}$ in $P^r/\langle G\rangle$. As G is the $\mathcal{O}$-border basis of $\langle G\rangle$, we have $c_1 = \cdots = c_\mu = 0$ by Definition 2.1.14 and the claim follows.

Next we prove that A_1) implies A_2). Let $v \in \langle G\rangle \setminus \{0\}$ and let $v = p_1 g_1 + \cdots + p_\nu g_\nu$ be written with $p_1, \ldots, p_\nu \in P$ as in A_1), i.e. such that $p_j = 0$ or $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1, \ldots, \nu\}$. If $p_j \neq 0$ and $\deg(p_j) < \operatorname{ind}_{\mathcal{O}}(v) - 1$ for some $j \in \{1, \ldots, \nu\}$, Proposition 2.1.13 yields $\operatorname{ind}_{\mathcal{O}}(p_j g_j) \leq \deg(p_j) + \operatorname{ind}_{\mathcal{O}}(g_j) = \deg(p_j) + 1 < \operatorname{ind}_{\mathcal{O}}(v)$. Moreover, as $v \neq 0$, $\operatorname{ind}_{\mathcal{O}}(v) \leq \max\{\operatorname{ind}_{\mathcal{O}}(p_j g_j) \mid j \in \{1, \ldots, \nu\}, p_j \neq 0\} \leq \operatorname{ind}_{\mathcal{O}}(v)$ is also a consequence of Proposition 2.1.13. Altogether, we see that there has to be at least one index $j \in \{1, \ldots, \nu\}$ such that $p_j \neq 0$ and $\deg(p_j) = \operatorname{ind}_{\mathcal{O}}(v) - 1$.

At last, we show that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if A_2) holds. Let $v \in \langle G\rangle \cap \langle\mathcal{O}\rangle_K$. Assume that $v \neq 0$. Then A_2) yields the existence of polynomials $p_1, \ldots, p_\nu \in P$ such that $v = p_1 g_1 + \cdots + p_\nu g_\nu$ and such that $p_j = 0$ or $\deg(p_j) \leq \operatorname{ind}_{\mathcal{O}}(v) - 1 = -1$ for all $j \in \{1, \ldots, \nu\}$. Thus we see that $p_1 = \cdots = p_\nu = 0$ and this contradicts $v \neq 0$. Altogether, we have $\langle G\rangle \cap \langle\mathcal{O}\rangle_K = \{0\}$ and the claim follows from Corollary 2.2.6. $\square$

Example 2.4.2. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.2.3, again. Recall that $P = \mathbb{Q}[x, y]$, that the order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ was $\mathcal{O} = \{e_1, xe_1, ye_1, e_2, xe_2, x^2 e_2\}$, and that $g_5 = xye_2 + 3e_1$ and $g_6 = x^2 ye_2 - e_1 - e_2$.

Then we have $xg_5 - g_6 = 3xe_1 + e_1 + e_2 \in \langle \mathcal{O} \rangle_{\mathbb{Q}}$ and thus $\operatorname{ind}_{\mathcal{O}}(xg_5 - g_6) = 0$ by Definition 2.1.11. In particular, as there are no non-zero polynomials of degree less than or equal to $\operatorname{ind}_{\mathcal{O}}(xg_5 - g_6) - 1 = -1$, Condition A_1) of Theorem 2.4.1 cannot be satisfied for $xg_5 - g_6 \in \langle G \rangle \setminus \{0\}$. Thus G is not the $\mathcal{O}$-border basis of $\langle G \rangle$ according to Theorem 2.4.1. Note, that this result coincides with Example 2.1.15.

2.4.2 Border Form Modules

In this subsection, we introduce the notion of border forms. It serves as a generalization of the concept of leading terms and is also applicable if the order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r \rangle$ does not come from a term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r \rangle$. For border bases in the polynomial ring P, this characterization was first proven in [KK05, Prop. 11] and can also be found in [KR05, Prop. 6.4.25].

Definition 2.4.3. a) Let $v \in P^r \setminus \{0\}$. We write v in the form $v = c_1 u_1 + \cdots + c_s u_s$ where $s \in \mathbb{N} \setminus \{0\}$, $c_1, \ldots, c_s \in K \setminus \{0\}$ are scalars, and $u_1, \ldots, u_s \in \mathbb{T}^n\langle e_1, \ldots, e_r \rangle$ are terms such that $\operatorname{ind}_{\mathcal{O}}(u_1) \geq \operatorname{ind}_{\mathcal{O}}(u_2) \geq \cdots \geq \operatorname{ind}_{\mathcal{O}}(u_s)$. Let $j \in \{1, \ldots, s\}$ be the maximal index such that $\operatorname{ind}_{\mathcal{O}}(u_j) = \operatorname{ind}_{\mathcal{O}}(v)$. Then we call the vector $\operatorname{BF}_{\mathcal{O}}(v) = \sum_{i=1}^{j} c_i u_i \in P^r$ the **border form** of v with respect to $\mathcal{O}$.

b) Let $U \subseteq P^r$ be a P-submodule. Then the **border form module** of U with respect to $\mathcal{O}$ is defined to be the P-submodule $\operatorname{BF}_{\mathcal{O}}(U) = \langle \operatorname{BF}_{\mathcal{O}}(v) \mid v \in U \setminus \{0\} \rangle \subseteq P^r$.

Example 2.4.4. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.2.3, again. Recall that we had $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2, that $\mathcal{O} = \{e_1, xe_1, ye_1, e_2, xe_2, x^2e_2\}$, and that $g_5 = xye_2 + 3e_1$ and $g_6 = x^2ye_2 - e_1 - e_2$. Then we have $xg_5 - 3xg_6 = -3x^3ye_2 + x^2ye_2 + 6xe_1 + 3xe_2$ and $xg_5 - g_6 = 3xe_1 + e_1 + e_2$, i.e. $\operatorname{ind}_{\mathcal{O}}(xg_5 - 3xg_6) = 2$ and $\operatorname{ind}_{\mathcal{O}}(xg_5 - g_6) = 0$ by Definition 2.1.11. Thus we have $\operatorname{BF}_{\mathcal{O}}(xg_5 - 3xg_6) = -3x^3ye_2$ and $\operatorname{BF}_{\mathcal{O}}(xg_5 - g_6) = 3xe_1 + e_1 + e_2$ according to Definition 2.4.3.

Theorem 2.4.5. (Border Bases and Border Form Modules)
The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if and only if the following equivalent conditions are satisfied.

B_1) *For every $v \in \langle G \rangle \setminus \{0\}$, we have $\operatorname{Supp}(\operatorname{BF}_{\mathcal{O}}(v)) \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r \rangle \setminus \mathcal{O}$.*

B_2) *We have $\operatorname{BF}_{\mathcal{O}}(\langle G \rangle) = \langle \operatorname{BF}_{\mathcal{O}}(g_1), \ldots, \operatorname{BF}_{\mathcal{O}}(g_\nu) \rangle = \langle b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu} \rangle$.*

Proof. First we prove that condition B_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G \rangle$. Let $v \in \langle G \rangle \setminus \{0\}$. Assume that $\operatorname{BF}_{\mathcal{O}}(v)$ contains a term of $\mathcal{O}$ in its support. Then we have $\operatorname{ind}_{\mathcal{O}}(v) = 0$ and thus $v = \operatorname{BF}_{\mathcal{O}}(v) \subseteq \langle \mathcal{O} \rangle_K$ by Definition 2.4.3. Now Corollary 2.2.6 yields the contradiction $v \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K = \{0\}$. Thus $\operatorname{BF}_{\mathcal{O}}(v)$ does not contain a term of $\mathcal{O}$ in its support and the claim follows.

Next we show that B_1) implies B_2). For all $j \in \{1, \ldots, \nu\}$, we have $g_j \in \langle G \rangle \setminus \{0\}$ and thus $b_j e_{\beta_j} = \operatorname{BF}_{\mathcal{O}}(g_j) \in \operatorname{BF}_{\mathcal{O}}(\langle G \rangle)$ by Definition 2.4.3. For the converse inclusion, let $v \in \langle G \rangle \setminus \{0\}$. Then B_1) implies that $\operatorname{Supp}(\mathrm{BF}_{\mathcal{O}}(v)) \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r \rangle \setminus \mathcal{O}$. Thus Proposition 2.1.10 yields that every term in the support of $\mathrm{BF}_{\mathcal{O}}(v)$ is divisible by a term in $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}\}$. In other words, $\mathrm{BF}_{\mathcal{O}}(v) \in \langle b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu} \rangle$.

Finally, we prove that G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if B_2) is satisfied. Assume that there exists a vector $v \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K \setminus \{0\}$. Then we have $\mathrm{ind}_{\mathcal{O}}(v) = 0$ and as a consequence we have $\mathrm{BF}_{\mathcal{O}}(v) = v \subseteq \langle \mathcal{O} \rangle_K$ by Definition 2.4.3. Condition B_2) and Proposition 2.1.10 now yield the contradiction $v = 0$. Altogether, the claim follows from Corollary 2.2.6. $\square$

Example 2.4.6. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.4.4, again. Recall that $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2, and that $\mathrm{BF}_{\mathcal{O}}(xg_5 - g_6) = 3xe_1 + e_1 + e_2 \in \langle \mathcal{O} \rangle_{\mathbb{Q}}$. Then Condition B_1) of Theorem 2.4.5 does not hold for G. Therefore, G is not the $\mathcal{O}$-border basis of $\langle G \rangle$ according to Theorem 2.4.5. Note, that this result coincides with Example 2.4.2.

2.4.3 Rewrite Rules

In this subsection, we define rewrite rules defined by border prebases. Though these rewrite rules are, in general, not Noetherian according to Remark 2.4.11, it turns out that they are confluent if and only if the corresponding border prebasis is a border basis. For border bases in P, this characterization was first proven in [KK05, Prop. 14] and can also be found in [KR05, Prop. 6.4.28].

Definition 2.4.7. a) Let $v \in P^r \setminus \{0\}$ and let $te_k \in \operatorname{Supp}(v)$ be such that there exists a factorization $te_k = t' b_j e_{\beta_j}$ with $t' \in \mathbb{T}^n$ and $j \in \{1, \ldots, \nu\}$. Let $c \in K$ be the coefficient of te_k in v. Then the vector $v - ct'g_j \in P^r$ does not contain the term te_k in its support anymore. We say that v **reduces to** $v - ct'g_j$ **in one step** using the **rewrite rule** $\xrightarrow{g_j}$ defined by g_j and write $v \xrightarrow{g_j} v - ct'g_j$. The passage from v to $v - ct'g_j$ is also called a **reduction step** using g_j.

b) The reflexive, transitive closure of the rewrite rules $\xrightarrow{g_j}$ defined by g_j for all indices $j \in \{1, \ldots, \nu\}$ is called the **rewrite relation** associated to G and is denoted by $\xrightarrow{G}$.

c) The equivalence relation generated by $\xrightarrow{G}$ is denoted by $\xleftrightarrow{G}$.

Example 2.4.8. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.2.3, again. Recall that $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2, that the order ideal in $\mathbb{T}^2\langle e_1, e_2 \rangle$ was $\mathcal{O} = \{e_1, xe_1, ye_1, e_2, xe_2, x^2e_2\}$, and that $g_2 = xye_1 - e_2$, $g_6 = x^2ye_2 - e_1 - e_2$, and $g_7 = x^3e_2 - e_1$. Since we have $x^4y^2 = x^3y \cdot xy$ and $x^3y = x \cdot x^2y = y \cdot x^3$, we see that $2x^4y^2e_1 \xrightarrow{g_2} 2x^4y^2e_1 - 2x^3yg_2 = 2x^3ye_2$, $2x^3ye_2 \xrightarrow{g_6} 2x^3ye_2 - 2xg_6 = 2xe_1 + 2xe_2$, and $2x^3ye_2 \xrightarrow{g_7} 2x^3ye_2 - 2yg_7 = 2ye_1$ by Definition 2.4.7. Therefore, it follows that $x^4y^2 \xrightarrow{G} 2ye_1$ and $2xe_1 + 2xe_2 \xleftrightarrow{G} 2ye_1$ according to Definition 2.4.7.

The following proposition gathers the basic properties of rewrite rules and generalizes [KK05, Prop. 13] and [KR05, Rem. 6.4.27] to the module setting. In particular, we give an explicit, complete proof of it.

Proposition 2.4.9. *a) If $v_1, v_2 \in P^r$ satisfy $v_1 \xrightarrow{G} v_2$ and if $c \in K$ and $t \in \mathbb{T}^n$, then we have $ctv_1 \xrightarrow{G} ctv_2$.*

b) If $v_1, v_2 \in P^r$ satisfy $v_1 \xrightarrow{g_j} v_2$ for $j \in \{1, \dots, \nu\}$ and if $v_3 \in P^r$, then there exists a vector $v_4 \in P^r$ such that $v_1 + v_3 \xrightarrow{G} v_4$ and $v_2 + v_3 \xrightarrow{G} v_4$.

c) If $v_1, v_2, v_3, v_4 \in P^r$ satisfy $v_1 \xleftrightarrow{G} v_2$ and $v_3 \xleftrightarrow{G} v_4$, then $v_1 + v_3 \xleftrightarrow{G} v_2 + v_4$.

d) If $v_1, v_2 \in P^r$ satisfy $v_1 \xleftrightarrow{G} v_2$ and if $p \in P$, then we have $pv_1 \xleftrightarrow{G} pv_2$.

e) For a vector $v \in P^r$, we have $v \xleftrightarrow{G} 0$ if and only if $v \in \langle G \rangle$.

f) For vectors $v_1, v_2 \in P^r$, we have $v_1 \xleftrightarrow{G} v_2$ if and only if $v_1 - v_2 \in \langle G \rangle$.

Proof. First we prove a). Let $c \in K$ and $t \in \mathbb{T}^n$. If $c = 0$, nothing needs to be shown. Thus suppose that $c \neq 0$. By induction on the number of reduction steps, it suffices to prove the claim for a single reduction step using g_j where $j \in \{1, \dots, \nu\}$. Let $v_1, v_2 \in P^r$ and $j \in \{1, \dots, \nu\}$ be such that $v_1 \xrightarrow{g_j} v_2$. Then Definition 2.4.7 yields the existence of a term $\hat{t}e_{\beta_j} \in \operatorname{Supp}(v_1)$, a term $t' \in \mathbb{T}^n$, and a coefficient $c' \in K$ such that $v_2 = v_1 - c't'g_j$ does not contain the term $\hat{t}e_{\beta_j} = t'b_je_{\beta_j}$ in its support anymore. Therefore, the vector $ctv_2 = ctv_1 - ctc't'g_j$ does also not contain the term $t\hat{t}e_{\beta_j} = tt'b_je_{\beta_j}$ in its support anymore, i.e. we have $ctv_1 \xrightarrow{g_j} ctv_2$ by Definition 2.4.7.

Next we show b). Let $v_1, v_2, v_3 \in P^r$ be such that $v_1 \xrightarrow{g_j} v_2$ for some $j \in \{1, \dots, \nu\}$. According to Definition 2.4.7, there exist a term $tb_je_{\beta_j} \in \operatorname{Supp}(v_1)$ where $t \in \mathbb{T}^n$ and a coefficient $c \in K \setminus \{0\}$ such that $v_2 = v_1 - ctg_j$ and such that v_2 does not contain the term $tb_je_{\beta_j}$ in its support anymore. Let $c' \in K$ be the coefficient of $tb_je_{\beta_j}$ in v_3. We distinguish two cases. If $c' = -c$, we have $v_1 + v_3 = v_2 + ctg_j + v_3 = v_2 + v_3 - c'tg_j$ and $v_2 + v_3 - c'tg_j$ does not contain the term $tb_je_{\beta_j}$ in its support anymore, i.e. we have $v_1 + v_3 \xrightarrow{g_j} v_2 + v_3$. The claim now follows with $v_4 = v_2 + v_3$. If $c' \neq -c$, we define $v_4 = v_1 + v_3 - (c + c')tg_j = v_2 + ctg_j + v_3 - (c + c')tg_j = v_2 + v_3 - c'tg_j$. Then we see that v_4 does not contain the term $tb_je_{\beta_j}$ in its support anymore and the claim follows.

For the proof of c), we let $v_1, \dots, v_4 \in P^r$ be such that $v_1 \xleftrightarrow{G} v_2$ and $v_3 \xleftrightarrow{G} v_4$. Then by Definition 2.4.7, there are vectors $v'_0, \dots, v'_k \in P^r$ where $k \in \mathbb{N}$ satisfying $v'_0 = v_1$, $v'_k = v_2$, and $v'_{\ell-1} \xrightarrow{g_{i_\ell}} v'_\ell$ or $v'_{\ell-1} \xleftarrow{g_{i_\ell}} v'_\ell$ where $i_\ell \in \{1, \dots, \nu\}$ for all $\ell \in \{1, \dots, k\}$. By b), for all $\ell \in \{1, \dots, k\}$, there is a $\tilde{v}'_\ell \in P^r$ satisfying $v'_{\ell-1} + v_3 \xrightarrow{G} \tilde{v}'_\ell \xleftarrow{G} v'_\ell + v_3$. Therefore, for all indices $\ell \in \{1, \dots, k\}$, we have $v'_{\ell-1} + v_3 \xleftrightarrow{G} v'_\ell + v_3$ and induction on $\ell \in \{1, \dots, k\}$ yields $v_1 + v_3 \xleftrightarrow{G} v_2 + v_3$ by Definition 2.4.7. An analogous construction yields the claim $v_2 + v_4 \xleftrightarrow{G} v_2 + v_3 \xleftrightarrow{G} v_1 + v_3$.

In order to show d), we let $v_1, v_2 \in P^r$ be vectors such that $v_1 \overset{G}{\longleftrightarrow} v_2$ and we let $p = c_1u_1 + \cdots + c_su_s \in P$ be with $s \in \mathbb{N}$, $c_1, \ldots, c_s \in K$, and $u_1, \ldots, u_s \in \mathbb{T}^n$. Then we have $c_iu_iv_1 \overset{G}{\longleftrightarrow} c_iu_iv_2$ for all $i \in \{1, \ldots, s\}$ by a). Induction on $i \in \{1, \ldots, s\}$ and c) now yield the claim $pv_1 = c_1u_1v_1 + \cdots + c_su_sv_1 \overset{G}{\longleftrightarrow} c_1u_1v_2 + \cdots + c_su_sv_2 = pv_2$.

For the proof of e), let $v \in P^r$. If $v \overset{G}{\longleftrightarrow} 0$, we collect the monomials used in the various reduction steps and get polynomials $p_1, \ldots, p_\nu \in P$ such that $v = p_1g_1 + \cdots + p_\nu g_\nu \in \langle G \rangle$. For the converse implication, suppose that $v \in \langle G \rangle$. Then there exist $p_1, \ldots, p_\nu \in P$ such that $v = p_1g_1 + \cdots + p_\nu g_\nu \in \langle G \rangle$. Obviously, we have $g_j \overset{G}{\longleftrightarrow} 0$ by Definition 2.4.7 and thus d) yields $p_jg_j \overset{G}{\longleftrightarrow} 0$ for all $j \in \{1, \ldots, \nu\}$. Therefore, induction on $j \in \{1, \ldots, \nu\}$ together with claim c) shows $v = p_1g_1 + \cdots + p_\nu g_\nu \overset{G}{\longleftrightarrow} 0$.

Finally, we show the equivalence in f). Let $v_1, v_2 \in P^r$. We have $v \overset{G}{\longleftrightarrow} v$ for all $v \in P^r$ by Definition 2.4.7. In particular, we see that $-v_2 \overset{G}{\longleftrightarrow} -v_2$. Thus the condition $v_1 \overset{G}{\longleftrightarrow} v_2$ is equivalent to the condition $v_1 - v_2 \overset{G}{\longleftrightarrow} v_2 - v_2 = 0$ by c). Hence f) is a direct consequence of e). $\square$

After we have introduced rewrite rules and seen basic properties of them concerning specific elements in P^r, we are now particularly interested in the overall properties of rewrite rules.

Definition 2.4.10. a) A vector $v_1 \in P^r$ is called **irreducible** with respect to the rewrite relation $\xrightarrow{G}$ if there are no $j \in \{1, \ldots, \nu\}$ and $v_2 \in P^r$ such that $v_1 \xrightarrow{g_j} v_2$.

b) The rewrite relation $\xrightarrow{G}$ is called **Noetherian** if there is no infinitely descending chain $v_0 \xrightarrow{g_{i_0}} v_1 \xrightarrow{g_{i_1}} v_2 \xrightarrow{g_{i_2}} \cdots$ with $i_j \in \{1, \ldots, \nu\}$ and $v_j \in P^r$ for all $j \in \mathbb{N}$.

c) The rewrite relation $\xrightarrow{G}$ is called **confluent** if for all vectors $v_1, v_2, v_3 \in P^r$ satisfying $v_1 \xrightarrow{G} v_2$ and $v_1 \xrightarrow{G} v_3$, there exists a vector $v_4 \in P^r$ such that $v_2 \xrightarrow{G} v_4$ and $v_3 \xrightarrow{G} v_4$.

$$\begin{tikzcd}
 & v_2 \arrow[dr, dashed, "G"] & \\
v_1 \arrow[ur, "G"] \arrow[dr, "G"'] & & v_4 \\
 & v_3 \arrow[ur, dashed, "G"'] &
\end{tikzcd}$$

Remark 2.4.11. a) For $r = 1$, border bases in P^r coincide with the usual border bases in P by Remark 2.1.16. Thus [KR05, Exmp. 6.4.26] shows that the rewrite relation $\xrightarrow{G}$ is not Noetherian, in general.

b) A vector $v \in P^r$ is irreducible with respect to $\xrightarrow{G}$ if and only if $v \in \langle \mathcal{O} \rangle_K$ according to Proposition 2.1.10 and the Definitions 2.4.7 and 2.4.10.

c) Considering the steps of the Border Division Algorithm 2.2.1 in detail, we see that it performs reduction steps using g_j where $j \in \{1, \ldots, \nu\}$ to compute the normal remainder with respect to G of a given vector. Thus for every $v \in P^r$, we have $v \xrightarrow{G} \mathrm{NR}_G(v)$. In particular, $\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$ is irreducible with respect to $\xrightarrow{G}$ by b).

Example 2.4.12. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.4.8, again. Recall that $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2, $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\}$, and that $2xe_1 + 2xe_2 \xleftarrow{G} 2x^3ye_2 \xrightarrow{G} 2ye_1$. Since $2xe_1 + 2xe_2, 2ye_1 \in \langle \mathcal{O} \rangle_{\mathbb{Q}}$, the vectors $2xe_1 + 2xe_2$ and $2ye_1$ are both irreducible with respect to $\xrightarrow{G}$ by Remark 2.4.11. In particular, it follows that the rewrite relation $\xrightarrow{G}$ is not confluent according to Definition 2.4.10.

Now we have all ingredients to characterize border bases via rewrite rules. The following theorem generalizes [KK05, Prop. 14] and [KR05, Prop. 4.2.28] to the module setting.

Theorem 2.4.13. (Border Bases and Rewrite Rules)
The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if and only if the following equivalent conditions are satisfied.

C_1) *For all $v \in P^r$, we have $v \xrightarrow{G} 0$ if and only if $v \in \langle G \rangle$.*

C_2) *If $v \in \langle G \rangle$ is irreducible with respect to $\xrightarrow{G}$, then we have $v = 0$.*

C_3) *For all $v \in P^r$, there is a unique vector $w \in P^r$ such that $v \xrightarrow{G} w$ and such that w is irreducible with respect to $\xrightarrow{G}$.*

C_4) *The rewrite relation $\xrightarrow{G}$ is confluent.*

Proof. First we show that C_1) implies C_2). Let $v \in \langle G \rangle$ be irreducible with respect to $\xrightarrow{G}$. As we have $v \xrightarrow{G} 0$ by C_1), v must be zero by the Definitions 2.4.7 and 2.4.10.

Next we show that C_2) implies C_3). Let $v \in P^r$. According to Remark 2.4.11, $\mathrm{NR}_G(v)$ is a vector with the claimed properties. In order to show the uniqueness, we let $w \in P^r$ be irreducible with respect to $\xrightarrow{G}$ and satisfying $v \xrightarrow{G} w$. Then we see that $\mathrm{NR}_G(v) \xleftrightarrow{G} w$ by Definition 2.4.7 and it follows $\mathrm{NR}_G(v) - w \in \langle G \rangle$ according to Proposition 2.4.9. Additionally, Remark 2.4.11 yields that $\mathrm{NR}_G(v) - w \in \langle \mathcal{O} \rangle_K$ is irreducible with respect to $\xrightarrow{G}$. Thus the claim follows from C_2).

In order to prove that C_3) implies C_4), we let $v_1, v_2, v_3 \in P^r$ be satisfying $v_1 \xrightarrow{G} v_2$ and $v_1 \xrightarrow{G} v_3$. According to Remark 2.4.11, we see that $v_1 \xrightarrow{G} v_2 \xrightarrow{G} \mathrm{NR}_G(v_2) \in \langle \mathcal{O} \rangle_K$ and $v_1 \xrightarrow{G} v_3 \xrightarrow{G} \mathrm{NR}_G(v_3) \in \langle \mathcal{O} \rangle_K$, and that both $\mathrm{NR}_G(v_2)$ and $\mathrm{NR}_G(v_3)$ are irreducible with respect to the rewrite relation $\xrightarrow{G}$. Thus C_3) implies $\mathrm{NR}_G(v_2) = \mathrm{NR}_G(v_3)$ and the claim follows by Definition 2.4.10.

For the proof that C_4) implies C_1), let $v \in P^r$ be satisfying $v \xleftrightarrow{G} 0$. Then Proposition 2.4.9 yields $v \in \langle G \rangle$. For the converse implication, we let $v \in \langle G \rangle$. Then Proposition 2.4.9 yields $v \xleftrightarrow{G} 0$. Let $v_0, \dots, v_k \in P^r$ with $k \in \mathbb{N}$ be such that $v_0 = v$, $v_k = 0$, and $v_{\ell-1} \xrightarrow{G} v_\ell$ or $v_{\ell-1} \xleftarrow{G} v_\ell$ for all $\ell \in \{1, \dots, k\}$. If there exists no index $\ell \in \{1, \dots, k\}$ such that $v_{\ell-1} \xleftarrow{G} v_\ell$, the claim follows immediately from Definition 2.4.7. Thus suppose that $v_{\ell-1} \xleftarrow{G} v_\ell$ for some $\ell \in \{1, \dots, k\}$. By Definition 2.4.7, we see that $v_{k-1} \xrightarrow{G} v_k = 0$. Let $s \in \{1, \dots, k-1\}$ be maximal such that $v_{s-1} \xleftarrow{G} v_s$. Then we have $v_s \xrightarrow{G} 0$ and $v_s \xrightarrow{G} v_{s-1}$. Moreover, $0 \in \langle \mathcal{O} \rangle_K$ and thus 0 is irreducible with respect to $\xrightarrow{G}$ according to Remark 2.4.11. C_4) and Definition 2.4.10 thus yield $v_{s-1} \xrightarrow{G} 0$. If we replace the sequence $v_0, \dots, v_{k-1}, 0$ with the shorter sequence $v_0, \dots, v_{s-1}, 0$, we see that the claim follows by induction on the number of reduction steps $v_{\ell-1} \xleftarrow{G} v_\ell$ where $\ell \in \{1, \dots, k\}$.

Next we show that condition C_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G \rangle$. If a vector $v \in P^r$ satisfies $v \xrightarrow{G} 0$, we have $v \in \langle G \rangle$ by Proposition 2.4.9. Conversely, let $v \in \langle G \rangle$. Then $v \xrightarrow{G} \mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$ follows from Remark 2.4.11. Since $v \in \langle G \rangle$, we also have $\mathrm{NR}_G(v) \in \langle G \rangle$ according to Definition 2.4.7. By Corollary 2.2.6, we see that $\mathrm{NR}_G(v) \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K = \{0\}$ and the claim follows.

Finally, we prove that G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if C_2) holds. Assume that there exists a $v \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K \setminus \{0\}$. Then v is irreducible with respect to $\xrightarrow{G}$ according to Remark 2.4.11 and C_2) yields the contradiction $v = 0$. The claim hence follows from Corollary 2.2.6. □

Example 2.4.14. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_7\} \subseteq P^2$ of Example 2.4.12, again. Recall that the rewrite relation $\xrightarrow{G}$ was not confluent. Therefore, G is not the $\mathcal{O}$-border basis of $\langle G \rangle$ by Theorem 2.4.13. Note that this result coincides with Example 2.4.6.

2.4.4 Commuting Matrices

Next we characterize border bases via commuting matrices. This characterization is due to [Mou99]. It is outstanding as there is no Gröbner bases analogon of it. The theorem states that it suffices to consider the formal multiplication matrices, i.e. matrices depending only on G that describe the multiplication by an indeterminate in the residue class module $P^r/\langle G \rangle$, and check whether these matrices are pairwise commuting. This is quite remarkable because by knowing that these matrices commute, we can explicitly describe the whole multiplicative structure of the residue class module $P^r/\langle G \rangle$. Based on this explicit description of the multiplicative structure, we study geometric properties of zero-dimensional schemes in the Chapters 4 to 6.

Definition 2.4.15. Given $s \in \{1,\dots,n\}$, we define the s^{th} **formal multiplication matrix** $\mathcal{X}_s = (\xi_{k\ell}^{(s)})_{1\le k,\ell\le\mu} \in \mathrm{Mat}_\mu(K)$ of G by

$$\xi_{k\ell}^{(s)} = \begin{cases} \delta_{ki} & \text{if } x_s t_\ell e_{\alpha_\ell} = t_i e_{\alpha_i} \in \mathcal{O}, \\ a_{kj} & \text{if } x_s t_\ell e_{\alpha_\ell} = b_j e_{\beta_j} \in \partial\mathcal{O}. \end{cases}$$

Example 2.4.16. Consider the $\mathcal{O}$-border prebasis $G \subseteq P^2$ of Example 2.2.3, again. Recall that $P = \mathbb{Q}[x,y]$, that $\{e_1, e_2\}$ was the canonical P-module basis of P^2, that we had $\mathcal{O} = \{e_1, ye_1, xe_1, e_2, xe_2, x^2e_2\} = \{t_1e_{\alpha_1},\dots,t_6e_{\alpha_6}\}$, and that $G = \{g_1,\dots,g_7\}$ was given by $g_1 = y^2e_1 - xe_2$, $g_2 = xye_1 - e_2$, $g_3 = x^2e_1 - ye_1 + e_2$, $g_4 = ye_2 - e_1 - ye_1 - xe_1 - e_2$, $g_5 = xye_2 + 3e_1$, $g_6 = x^2ye_2 - e_1 - e_2$, and $g_7 = x^3e_2 - e_1$. Then the formal multiplication matrices $\mathcal{X}, \mathcal{Y} \in \mathrm{Mat}_6(\mathbb{Q})$ of G are

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 2.4.17. Similar to the interpretation of the formal multiplication matrices of a border prebasis in P in [KR05, p. 434], we can interpret the multiplication matrices of the $\mathcal{O}$-border prebases $G \subseteq P^r$ the following way: Let $s \in \{1,\dots,n\}$ be an index and let $\mathcal{X}_s \in \mathrm{Mat}_\mu(K)$ be the s^{th} formal multiplication matrix of G. We can identify every vector $v = c_1t_1e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu} \in \langle\mathcal{O}\rangle_K$ with the corresponding coordinate vector $(c_1,\dots,c_\mu)^{\mathrm{tr}} \in K^\mu$. Then the vector $(c_1',\dots,c_\mu')^{\mathrm{tr}} = \mathcal{X}_s \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \in K^\mu$ corresponds to the vector $c_1't_1e_{\alpha_1} + \cdots + c_\mu't_\mu e_{\alpha_\mu} = \mathrm{NR}_G(x_sv) \in \langle\mathcal{O}\rangle_K \subseteq P^r$. In particular, we have $c_1'\overline{t_1e_{\alpha_1}} + \cdots + c_\mu'\overline{t_\mu e_{\alpha_\mu}} = \overline{x_sv}$ in $P^r/\langle G\rangle$ in this situation.

Before we actually prove the characterization of border bases via commuting matrices, we show that if the formal multiplication matrices are pairise commuting, then $\langle\mathcal{O}\rangle_K$ can be equipped with a P-module structure.

Lemma 2.4.18. *Assume that $\mu \neq 0$, i.e. $\mathcal{O} \neq \emptyset$. Let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of G and assume that $\mathcal{X}_1,\dots,\mathcal{X}_n$ are pairwise commuting. Then the K-vector subspace $\langle\mathcal{O}\rangle_K \subseteq P^r$ is a P-module with scalar multiplication*

$$p \circ (c_1t_1e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}) = (t_1e_{\alpha_1},\dots,t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1,\dots,\mathcal{X}_n) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}$$

for all $p \in P$ and all $c_1,\dots,c_\mu \in K$. Moreover, the set $\{e_1,\dots,e_r\} \cap \mathcal{O}$ generates the P-module $\langle\mathcal{O}\rangle_K$ and for all $c \in K$ and $i \in \{1,\dots,\mu\}$, we have $(ct_i) \circ e_{\alpha_i} = ct_ie_{\alpha_i}$.

Proof. Since the order ideal $\mathcal{O}$ is K-linearly independent, the map $\circ$ is well-defined. Moreover, the K-vector subspace $\langle\mathcal{O}\rangle_K \subseteq P^r$ is obviously an additive group. For all

$p, q \in P$ and $c_1, \dots, c_\mu, d_1, \dots, d_\mu \in K$, the commutativity of the formal multiplication matrices $\mathcal{X}_1, \dots, \mathcal{X}_n$ with respect to G yields

$$\begin{aligned}
&1 \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot 1(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot \mathcal{I}_\mu \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu},
\end{aligned}$$

and

$$\begin{aligned}
&(pq) \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot (pq)(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot q(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= p \circ ((t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot q(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}})\\
&= p \circ (q \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})),
\end{aligned}$$

and

$$\begin{aligned}
&(p+q) \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot (p+q)(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot (p(\mathcal{X}_1, \dots, \mathcal{X}_n) + q(\mathcal{X}_1, \dots, \mathcal{X}_n)) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&\quad + (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot q(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&= (p \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})) + (q \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})),
\end{aligned}$$

and

$$\begin{aligned}
&p \circ ((c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}) + (d_1 t_1 e_{\alpha_1} + \cdots + d_\mu t_\mu e_{\alpha_\mu}))\\
&= p \circ ((c_1 + d_1) t_1 e_{\alpha_1} + \cdots + (c_\mu + d_\mu) t_\mu e_{\alpha_\mu})\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1 + d_1, \dots, c_\mu + d_\mu)^{\mathrm{tr}}\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot ((c_1, \dots, c_\mu)^{\mathrm{tr}} + (d_1, \dots, d_\mu)^{\mathrm{tr}})\\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}\\
&\quad + (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot p(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (d_1, \dots, d_\mu)^{\mathrm{tr}}\\
&= (p \circ (c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu})) + (p \circ (d_1 t_1 e_{\alpha_1} + \cdots + d_\mu t_\mu e_{\alpha_\mu})).
\end{aligned}$$

Altogether, we see that that $(\langle \mathcal{O} \rangle_K, +, \circ)$ is indeed a P-module.

It remains to prove that the P-module $\langle \mathcal{O} \rangle_K$ is generated by $\{e_1, \dots, e_r\} \cap \mathcal{O}$. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ such that $\mathcal{O} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_r e_r$ as in Definition 2.1.6 and for all $s \in \{1, \dots, n\}$, write $\mathcal{X}_s = (\xi_{k\ell}^{(s)}) \in \mathrm{Mat}_\mu(K)$ as in Definition 2.4.15. Without loss of generality we suppose that $e_1, \dots, e_\ell \in \mathcal{O}$ and $e_{\ell+1}, \dots, e_r \notin \mathcal{O}$ for some $\ell \in \{1, \dots, r\}$ and that $t_k e_{\alpha_k} = e_k$ for all $k \in \{1, \dots, \ell\}$. Let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ and let $k \in \{1, \dots, \ell\}$. We prove by induction on the degree of t that $t \circ e_k = te_k$ for all $t \in \mathcal{O}_k$. For the induction start, we have

$$\begin{aligned}
1 \circ e_k &= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot 1(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot \mathcal{E}_k^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot \mathcal{I}_\mu \cdot \mathcal{E}_k^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot \mathcal{E}_k^{\mathrm{tr}} \\
&= t_k e_{\alpha_k} \\
&= e_k.
\end{aligned}$$

For the induction step, suppose that there is a $t \in \mathcal{O}_k$ with $\deg(t) > 0$. Then we have $te_k = t_i e_{\alpha_i} = x_s t_j e_{\alpha_j}$ for some $i, j \in \{1, \dots, \mu\}$ and $s \in \{1, \dots, n\}$ by Definition 2.1.6. The induction hypothesis yields $t \circ e_k = (x_s t_j) \circ e_{\alpha_j} = x_s \circ (t_j \circ e_{\alpha_j}) = x_s \circ (t_j e_{\alpha_k})$. Thus we see that

$$\begin{aligned}
t \circ e_k &= x_s \circ (t_j e_{\alpha_j}) \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot x_s(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot \mathcal{E}_j^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot \mathcal{X}_s \cdot \mathcal{E}_j^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot (\xi_{1j}^{(s)}, \dots, \xi_{\mu j}^{(s)})^{\mathrm{tr}} \\
&= \delta_{1i} t_1 e_{\alpha_1} + \dots + \delta_{\mu i} t_\mu e_{\alpha_\mu} \\
&= t_i e_{\alpha_i} \\
&= te_k
\end{aligned}$$

by Definition 2.4.15 i.e. the above claim has been proven by induction. For all $c \in K$ and $i \in \{1, \dots, \mu\}$, we also have

$$\begin{aligned}
c \circ t_i e_{\alpha_i} &= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot c(\mathcal{X}_1, \dots, \mathcal{X}_n) \cdot \mathcal{E}_i^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot c\mathcal{I}_\mu \cdot \mathcal{E}_i^{\mathrm{tr}} \\
&= (t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}) \cdot c\mathcal{E}_i^{\mathrm{tr}} \\
&= ct_i e_{\alpha_i}.
\end{aligned}$$

Altogether, since $\ell \in \{1, \dots, r\}$ is chosen such that $\{e_1, \dots, e_r\} \cap \mathcal{O} = \{e_1, \dots, e_\ell\}$, it follows that $(c_1 t_1) \circ e_{\alpha_1} + \dots + (c_\mu t_\mu) \circ e_{\alpha_\mu} = c_1 t_1 e_{\alpha_1} + \dots + c_\mu t_\mu e_{\alpha_\mu}$ for all $c_1, \dots, c_\mu \in K$, i.e. the P-module $\langle \mathcal{O} \rangle_K$ is generated by $\{e_1, \dots, e_r\} \cap \mathcal{O}$. □

After all, we are now able to prove the characterization of border bases via commuting matrices. Note that we do not generalize the original proof in [Mou99, Thm. 3.1] for border bases in P. We rather generalize the proofs of [KK05, Prop. 16] and [KR05, Thm. 6.4.30]. Moreover, we explicitly determine equations which the coefficients of a border prebasis must satisfy to determine a border basis. The corresponding proof of these vanishing conditions for border bases in P can be found in the proof of [KK05, Prop. 16] and in [KR05, Prop. 6.4.32].

Theorem 2.4.19. (Border Bases and Commuting Matrices)
For all $s \in \{1, \dots, n\}$, we let $\mathcal{X}_s = (\xi^{(s)}_{k\ell})_{1 \leq k,\ell \leq \mu} \in \mathrm{Mat}_\mu(K)$ denote the formal multiplication matrix of G as in Definition 2.4.15 and we define the map

$$\varrho_s : \{1, \dots, \mu\} \to \mathbb{N}, \quad i \mapsto \begin{cases} j & \text{if } x_s t_i e_{\alpha_i} = t_j e_{\alpha_j} \in \mathcal{O}, \\ k & \text{if } x_s t_i e_{\alpha_i} = b_k e_{\beta_k} \in \partial\mathcal{O}. \end{cases}$$

Let $\overline{\mathcal{O}} = \{\overline{t_1 e_{\alpha_1}}, \dots, \overline{t_\mu e_{\alpha_\mu}}\}$ denote the image of $\mathcal{O}$ in $P^r/\langle G\rangle$. Then the $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if the following equivalent conditions are satisfied.

D_1) *The formal multiplication matrices $\mathcal{X}_1, \dots, \mathcal{X}_n$ of G are pairwise commuting.*

D_2) *For all $p \in \{1, \dots, \mu\}$ and all $s, u \in \{1, \dots, n\}$ with $s \neq u$, the following equations are satisfied:*

1) *If $x_s t_i e_{\alpha_i} = t_j e_{\alpha_j}$, $x_u t_i e_{\alpha_i} = b_k e_{\beta_k}$, and $x_s b_k e_{\beta_k} = b_\ell e_{\beta_\ell}$ with $i, j \in \{1, \dots, \mu\}$ and $k, \ell \in \{1, \dots, \nu\}$, we have*

$$\sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \mathcal{O}}} \delta_{p\varrho_s(m)} a_{mk} + \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \partial\mathcal{O}}} a_{p\varrho_s(m)} a_{mk} = a_{p\ell}.$$

2) *If we have $x_s t_i e_{\alpha_i} = b_j e_{\beta_j}$ and $x_u t_i e_{\alpha_i} = b_k e_{\beta_k}$ with indices $i \in \{1, \dots, \mu\}$ and $j, k \in \{1, \dots, \nu\}$, we have*

$$\begin{aligned} &\sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \mathcal{O}}} \delta_{p\varrho_s(m)} a_{mk} + \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \partial\mathcal{O}}} a_{p\varrho_s(m)} a_{mk} \\ &= \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_u t_m e_{\alpha_m} \in \mathcal{O}}} \delta_{p\varrho_u(m)} a_{mj} + \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_u t_m e_{\alpha_m} \in \partial\mathcal{O}}} a_{p\varrho_u(m)} a_{mj}. \end{aligned}$$

If these equivalent conditions are satisfied, then for every $s \in \{1, \dots, n\}$, the formal multiplication matrix $\mathcal{X}_s$ represents the multiplication endomorphism of the K-vector space $P^r/\langle G\rangle$ defined by $\overline{v} \mapsto \overline{x_s v}$, where $v \in P^r$, with respect to the K-vector space basis $\overline{\mathcal{O}}$.

Proof. If $\mu = 0$, i.e. if $\mathcal{O} = \emptyset$ and $\partial\mathcal{O} = \{e_1, \dots, e_r\} = G$, the claim is obviously true. Thus suppose that $\mu \neq 0$ and let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ.

First we prove that condition D_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G\rangle$, i.e. that $\overline{\mathcal{O}}$ is a K-vector space basis of $P^r/\langle G\rangle$ by Definition 2.1.14. Let $s \in \{1, \dots, n\}$. The formal multiplication matrix $\mathcal{X}_s$ defines a K-vector space endomorphism ϕ_s of $P^r/\langle G\rangle$

with respect to the K-vector space basis $\overline{\mathcal{O}}$. We show that $\phi_s(\overline{v}) = \overline{x_s v}$ for all $v \in P^r$, i.e. that ϕ_s is the multiplication by x_s. Consider the expansions

$$\begin{aligned}\phi_s(\overline{t_1 e_{\alpha_1}}) &= \xi_{11}^{(s)}\overline{t_1 e_{\alpha_1}} + \cdots + \xi_{\mu 1}^{(s)}\overline{t_\mu e_{\alpha_\mu}},\\ &\vdots\\ \phi_s(\overline{t_\mu e_{\alpha_\mu}}) &= \xi_{1\mu}^{(s)}\overline{t_1 e_{\alpha_1}} + \cdots + \xi_{\mu\mu}^{(s)}\overline{t_\mu e_{\alpha_\mu}}.\end{aligned}$$

Let $u \in \{1, \ldots, \mu\}$. Then we see that $x_s t_u e_{\alpha_u} \in \overline{\partial\mathcal{O}}$ according to Definition 2.1.7. If we have $x_s t_u e_{\alpha_u} = t_i e_{\alpha_i} \in \mathcal{O}$ for some $i \in \{1, \ldots, \mu\}$, Definition 2.4.15 yields that $\phi_s(\overline{t_u e_{\alpha_u}}) = \delta_{1i}\overline{t_1 e_{\alpha_1}} + \cdots + \delta_{\mu i}\overline{t_\mu e_{\alpha_\mu}} = \overline{t_i e_{\alpha_i}} = \overline{x_s t_u e_{\alpha_u}}$. If $x_s t_u e_{\alpha_u} = b_j e_{\beta_j} \in \partial\mathcal{O}$ for some $j \in \{1, \ldots, \nu\}$, we have $\phi_s(\overline{t_u e_{\alpha_u}}) = a_{1j}\overline{t_1 e_{\alpha_1}} + \cdots + a_{\mu j}\overline{t_\mu e_{\alpha_\mu}} = \overline{b_j e_{\beta_j}} = \overline{x_s t_u e_{\alpha_u}}$ in $P^r/\langle G\rangle$ by Definition 2.4.15. Therefore, ϕ_s represents the multiplication by x_s with respect to $\overline{\mathcal{O}}$. Since the multiplication in $P^r/\langle G\rangle$ is commutative and since the formal multiplication matrices $\mathcal{X}_1, \ldots, \mathcal{X}_n$ represent the endomorphisms $\phi_1, \ldots, \phi_n$ of $P^r/\langle G\rangle$ with respect to the K-vector space basis $\overline{\mathcal{O}}$, it follows that the formal multiplication matrices $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are pairwise commuting.

Next we show that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if D_1) holds. As $\{e_1, \ldots, e_r\} \cap \mathcal{O}$ generates the P-module $\langle \mathcal{O}\rangle_K$ according to Lemma 2.4.18, the map

$$\{e_1, \ldots, e_r\} \to \langle \mathcal{O}\rangle_K, \quad e_k \mapsto \begin{cases} e_k & \text{if } e_k \in \mathcal{O},\\ \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} & \text{if } e_k = e_{\beta_j} \in \partial\mathcal{O}\end{cases}$$

induces the P-module epimorphism

$$\varphi : P^r \twoheadrightarrow \langle \mathcal{O}\rangle_K, \quad e_k \mapsto \begin{cases} e_k & \text{if } e_k \in \mathcal{O},\\ \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} & \text{if } e_k = e_{\beta_j} \in \partial\mathcal{O}\end{cases}$$

by the Universal Property of the Free Module P^r. Thus the Isomorphism Theorem induces the P-module isomorphism

$$\overline{\varphi} : P^r/\ker(\varphi) \xrightarrow{\sim} \langle \mathcal{O}\rangle_K, \quad \overline{e}_k \mapsto \begin{cases} e_k & \text{if } e_k \in \mathcal{O},\\ \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} & \text{if } e_k = e_{\beta_j} \in \partial\mathcal{O}.\end{cases}$$

In particular, as $\mathcal{O}$ is a K-vector space basis of $\langle \mathcal{O}\rangle_K$, it follows that $\overline{\varphi}^{-1}(\mathcal{O})$, i.e. the set of residue classes of the elements of $\mathcal{O}$ in $P^r/\ker(\varphi)$, is a K-vector space basis of $P^r/\ker(\varphi)$. We now show that $\langle G\rangle \subseteq \ker(\varphi)$. Without loss of generality we may suppose that $e_1, \ldots, e_\ell \in \mathcal{O}$ and $e_{\ell+1}, \ldots, e_r \notin \mathcal{O}$ for some index $\ell \in \{1, \ldots, r\}$ and that $t_k e_{\alpha_k} = e_k$ for all $k \in \{1, \ldots, \ell\}$. Furthermore, we let $j \in \{1, \ldots, \nu\}$. We have to distinguish two cases. For the first case, suppose that $e_{\beta_j} \in \partial\mathcal{O}$. Then we have $b_j = 1$ by Definition 2.1.7.

Hence Lemma 2.4.18 yields

$$\begin{aligned}
\varphi(g_j) &= \varphi\left(b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}\right) \\
&= b_j \circ \varphi(e_{\beta_i}) - \sum_{i=1}^{\mu} (a_{ij} t_i) \circ \varphi(e_{\alpha_i}) \\
&= 1 \circ \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \\
&= \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \\
&= 0.
\end{aligned}$$

For the second case, suppose now that $\mathcal{O}_{\beta_j} \in \mathcal{O}$. Then $\deg(b_j) \geq 1$ and there hence exist an $s \in \{1, \ldots, n\}$ and a $k \in \{1, \ldots, \mu\}$ such that $b_j e_{\beta_j} = x_s t_k e_{\alpha_k}$ by Definition 2.1.7. In particular, we see that $\beta_j = \alpha_k$ and $b_j = x_s t_k$. Thus Lemma 2.4.18 yields

$$\begin{aligned}
\varphi(g_j) &= \varphi\left(b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}\right) \\
&= b_j \circ \varphi(e_{\beta_j}) - \sum_{i=1}^{\mu} (a_{ij} t_i) \circ \varphi(e_{\alpha_i}) \\
&= b_j \circ e_{\beta_j} - \sum_{i=1}^{\mu} (a_{ij} t_i) \circ e_{\alpha_i} \\
&= x_s \circ (t_k \circ e_{\alpha_k}) - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \\
&= x_s \circ (t_k e_{\alpha_k}) - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \\
&= (t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}) \cdot \left(x_s(\mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot \mathcal{E}_k^{\mathrm{tr}} - \sum_{i=1}^{\mu} a_{ij} \cdot \mathcal{E}_i^{\mathrm{tr}}\right) \\
&= (t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}) \cdot \left(\mathcal{X}_s \cdot \mathcal{E}_k^{\mathrm{tr}} - \sum_{i=1}^{\mu} a_{ij} \cdot \mathcal{E}_i^{\mathrm{tr}}\right) \\
&= (t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}) \cdot \left(\sum_{i=1}^{\mu} \xi_{ik}^{(s)} \cdot \mathcal{E}_i^{\mathrm{tr}} - \sum_{i=1}^{\mu} a_{ij} \cdot \mathcal{E}_i^{\mathrm{tr}}\right) \\
&= (t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}) \cdot \left(\sum_{i=1}^{\mu} a_{ij} \cdot \mathcal{E}_i^{\mathrm{tr}} - \sum_{i=1}^{\mu} a_{ij} \cdot \mathcal{E}_i^{\mathrm{tr}}\right) \\
&= 0.
\end{aligned}$$

Altogether, it follows that $\langle G\rangle \subseteq \ker(\varphi)$.
The Universal Property of the Residue Class Module $P^r/\langle G\rangle$ now induces the P-module epimorphism

$$\psi : P^r/\langle G\rangle \twoheadrightarrow P^r/\ker(\varphi), \quad \bar{e}_k \mapsto \begin{cases} \bar{e}_k & \text{if } e_k \in \mathcal{O}, \\ \sum_{i=1}^{\mu} a_{ij}\overline{t_i e_{\alpha_i}} & \text{if } e_k = e_{\beta_j} \in \partial\mathcal{O}. \end{cases}$$

Moreover, we have $\psi(\overline{\mathcal{O}}) = \overline{\varphi}^{-1}(\mathcal{O})$. Since $\overline{\mathcal{O}}$ generates the K-vector space $P^r/\langle G\rangle$ by Corollary 2.2.5 and since $\overline{\varphi}^{-1}(\mathcal{O})$ is a K-vector space basis of $P^r/\ker(\varphi)$, we see that $\overline{\mathcal{O}}$ is also a K-vector space basis of $P^r/\langle G\rangle$, i.e. that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ by Definition 2.1.14.

Finally, we prove that D_1) and D_2) are equivalent. Let $p, i \in \{1,\dots,\mu\}$ and let $s, u \in \{1,\dots,n\}$ be such that $s \neq u$. In order to show this equivalence, we translate the commutativity condition $\mathcal{E}_p\mathcal{X}_s\mathcal{X}_u\mathcal{E}_i^{\mathrm{tr}} = \mathcal{E}_p\mathcal{X}_u\mathcal{X}_s\mathcal{E}_i^{\mathrm{tr}}$ back into the language of $\langle\mathcal{O}\rangle_K$. As the resulting condition depends on the position of $t_i e_{\alpha_i}$ relative to the border of $\mathcal{O}$, we distinguish four cases.

$t_k e_{\alpha_k}$	$t_\ell e_{\alpha_\ell}$
$t_i e_{\alpha_i}$	$t_j e_{\alpha_j}$

First case: $x_s x_u t_i e_{\alpha_i} \in \mathcal{O}$

Since $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$, it follows that $x_s t_i e_{\alpha_i}, x_u t_i e_{\alpha_i} \in \mathcal{O}$ by Definition 2.1.6. Say, $x_s t_i e_{\alpha_i} = t_j e_{\alpha_j}$, $x_u t_i e_{\alpha_i} = t_k e_{\alpha_k}$, and $x_s x_u t_i e_{\alpha_i} = t_\ell e_{\alpha_\ell}$ with $j, k, \ell \in \{1,\dots,\mu\}$. Then we have

$$\mathcal{E}_p\mathcal{X}_s\mathcal{X}_u\mathcal{E}_i^{\mathrm{tr}} = \mathcal{E}_p\mathcal{X}_s\mathcal{E}_k^{\mathrm{tr}} = \xi_{pk}^{(s)} = \delta_{p\ell} = \xi_{pj}^{(u)} = \mathcal{E}_p\mathcal{X}_u\mathcal{E}_j^{\mathrm{tr}} = \mathcal{E}_p\mathcal{X}_u\mathcal{X}_s\mathcal{E}_i^{\mathrm{tr}},$$

i.e. the commutativity condition holds in this case by Definition 2.4.15.

$t_k e_{\alpha_k}$	$b_\ell e_{\beta_\ell}$
$t_i e_{\alpha_i}$	$t_j e_{\alpha_j}$

Second case: $x_s x_u t_i e_{\alpha_i} \in \partial\mathcal{O}$ and $x_s t_i e_{\alpha_i}, x_u t_i e_{\alpha_i} \in \mathcal{O}$

Say, $x_s t_i e_{\alpha_i} = t_j e_{\alpha_j}$, $x_u t_i e_{\alpha_i} = t_k e_{\alpha_k}$, and $x_s x_u t_i e_{\alpha_i} = b_\ell e_{\beta_\ell}$ with $j, k \in \{1,\dots,\mu\}$ and $\ell \in \{1,\dots,\nu\}$. Then we have

$$\mathcal{E}_p\mathcal{X}_s\mathcal{X}_u\mathcal{E}_i^{\mathrm{tr}} = \mathcal{E}_p\mathcal{X}_s\mathcal{E}_k^{\mathrm{tr}} = \xi_{pk}^{(s)} = a_{p\ell} = \xi_{pj}^{(u)} = \mathcal{E}_p\mathcal{X}_u\mathcal{E}_j^{\mathrm{tr}} = \mathcal{E}_p\mathcal{X}_u\mathcal{X}_s\mathcal{E}_i^{\mathrm{tr}},$$

i.e. the commutativity condition holds in this case by Definition 2.4.15, again.

$b_k e_{\beta_k}$	$b_\ell e_{\beta_\ell}$
$t_i e_{\alpha_i}$	$t_j e_{\alpha_j}$

Third case: $x_s t_i e_{\alpha_i} \in \mathcal{O}$ and $x_u t_i e_{\alpha_i} \in \partial\mathcal{O}$

Since $\overline{\partial\mathcal{O}}$ and $\mathcal{O}$ are both order ideals in $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$, we see that $x_s x_u t_i e_{\alpha_i} \in \partial\mathcal{O}$ by the Definitions 2.1.6 and 2.1.7. Say, $x_s t_i e_{\alpha_i} = t_j e_{\alpha_j}$, $x_u t_i e_{\alpha_i} = b_k e_{\beta_k}$, and $x_s x_u t_i e_{\alpha_i} = b_\ell e_{\beta_\ell}$

with $j \in \{1,\dots,\mu\}$ and $k,\ell \in \{1,\dots,\nu\}$. Then we have

$$\begin{aligned}
\mathcal{E}_p \mathcal{X}_s \mathcal{X}_u \mathcal{E}_i^{\mathrm{tr}} &= \mathcal{E}_p \mathcal{X}_s (a_{1k},\dots,a_{\mu k})^{\mathrm{tr}} \\
&= \sum_{m=1}^{\mu} \xi_{pm}^{(s)} a_{mk} \\
&= \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m}\in\mathcal{O}}} \delta_{p\varrho_s(m)} a_{mk} + \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{p\varrho_s(m)} a_{mk}
\end{aligned}$$

and

$$\mathcal{E}_p \mathcal{X}_u \mathcal{X}_s \mathcal{E}_i^{\mathrm{tr}} = \mathcal{E}_p \mathcal{X}_u \mathcal{E}_j^{\mathrm{tr}} = \xi_{pj}^{(u)} = a_{p\ell}$$

by Definition 2.4.15. Thus the commutativity condition holds in this case if and only if equation 1) is satisfied for s, u and p.

$b_k e_{\beta_k}$	$*$
$t_i e_{\alpha_i}$	$b_j e_{\beta_j}$

Fourth case: $x_s t_i e_{\alpha_i} \in \partial\mathcal{O}$ and $x_u t_i e_{\alpha_i} \in \partial\mathcal{O}$

Say, $x_s t_i e_{\alpha_i} = b_j e_{\beta_j}$ and $x_u t_i e_{\alpha_i} = b_k e_{\beta_k}$ with $j,k \in \{1,\dots,\nu\}$. Then we have

$$\begin{aligned}
\mathcal{E}_p \mathcal{X}_s \mathcal{X}_u \mathcal{E}_i^{\mathrm{tr}} &= \mathcal{E}_p \mathcal{X}_s (a_{1k},\dots,a_{\mu k})^{\mathrm{tr}} \\
&= \sum_{m=1}^{\mu} \xi_{pm}^{(s)} a_{mk} \\
&= \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m}\in\mathcal{O}}} \delta_{p\varrho_s(m)} a_{mk} + \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{p\varrho_s(m)} a_{mk}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_p \mathcal{X}_u \mathcal{X}_s \mathcal{E}_i^{\mathrm{tr}} &= \mathcal{E}_p \mathcal{X}_u (a_{1j},\dots,a_{\mu j})^{\mathrm{tr}} \\
&= \sum_{m=1}^{\mu} \xi_{pm}^{(u)} a_{mj} \\
&= \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_u t_m e_{\alpha_m}\in\mathcal{O}}} \delta_{p\varrho_u(m)} a_{mj} + \sum_{\substack{m\in\{1,\dots,\mu\} \\ x_u t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{p\varrho_u(m)} a_{mj}.
\end{aligned}$$

according to Definition 2.4.15. Thus the commutativity condition holds in this case if and only if equation 2) is satisfied for s, u and p.
Altogether, we have regarded all possible cases and have seen that condition D_1) holds if and only if the equations 1) and 2) are satisfied for all $s,u \in \{1,\dots,n\}$ such that $s \neq u$ and for all $p \in \{1,\dots,\mu\}$, i.e. if and only if D_2) is satisfied. □

Example 2.4.20. Consider Example 2.4.16, again. We have seen that the formal multiplication matrices $\mathcal{X}, \mathcal{Y} \in \mathrm{Mat}_7(\mathbb{Q})$ of G are

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathcal{X} \cdot \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathcal{Y} \cdot \mathcal{X},$$

condition D_1) of Theorem 2.4.19 is not satisfied. Thus G is not the $\mathcal{O}$-border basis of $\langle G \rangle$ according to Theorem 2.4.19. Note that this result coincides with Example 2.4.14.

2.4.5 Liftings of Border Syzygyies

Next we characterize border bases via liftings of border syzygies. Border syzygies are syzygies of $(b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu})$ and can sometimes be lifted to syzygies of $(g_1, \dots, g_\nu)$. The goal of this subsection is to prove that we can lift all syzygies induced by neighbors with respect to $\mathcal{O}$ to syzygies in $(g_1, \dots, g_\nu)$ if and only if G is an $\mathcal{O}$-border basis of $\langle G \rangle$. The proof of this characterization in Theorem 2.4.26 is based on the characterization via commuting matrices. In Corollary 3.3.9, we will give an alternative proof of this theorem that is not based on the characterization via commuting matrices but on a deeper insight into the structure of liftings of border syzygies. All ideas of this subsection follow the corresponding concepts of [KK05, Section 5].

Since we study border syzygyies and their liftings, we must distinguish between the elements of P^r in which G lives and the elements of P^ν in which the syzygies live. Therefore, we denote the canonical basis of the P-module P^ν by $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ in contrast to the canonical basis of the P-module P^r which is, as usual, denoted by $\{e_1, \dots, e_r\}$.

First we introduce neighbors with respect to the order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$ and the corresponding border syzygies. These neighbor syzygies play the same role for border bases as the critical pairs do for Gröbner bases. For the definition of neighbors with respect to an order ideal in $\mathbb{T}^n$, we refer to [KK05, Defn. 17] and [KR05, Defn. 6.4.33].

Definition 2.4.21. Let $\mathcal{O} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_r e_r$ be with order ideals $\mathcal{O}_1, \ldots, \mathcal{O}_r$ in $\mathbb{T}^n$.

a) A syzygy $(q_1, \ldots, q_\nu) \in \operatorname{Syz}_P(b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}) \subseteq P^\nu$ is called a **border syzygy** with respect to $\mathcal{O}$.

b) For all $i, j \in \{1, \ldots, \nu\}$ with $\beta_i = \beta_j$, the **fundamental syzygy** of $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ is defined to be $\sigma_{ij} = \frac{\operatorname{lcm}(b_i, b_j)}{b_i} \varepsilon_i - \frac{\operatorname{lcm}(b_i, b_j)}{b_j} \varepsilon_j$.

c) Let $i, j \in \{1, \ldots, \nu\}$ be with $\beta_i = \beta_j$ and let σ_{ij} be the corresponding fundamental syzygy.

1) The border terms $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are called **next-door neighbors** with respect to $\mathcal{O}$ if $\beta_i = \beta_j$ and if $b_i, b_j \in \partial\mathcal{O}_{\beta_i}$ are next-door neighbors with respect to $\mathcal{O}_{\beta_i}$, i.e. if we have $x_k b_i e_{\beta_i} = b_j e_{\beta_j}$ for some $k \in \{1, \ldots, n\}$. In this case, the border syzygy $\sigma_{ij} = x_k \varepsilon_i - \varepsilon_j$ is called a **next-door neighbor syzygy** with respect to $\mathcal{O}$.

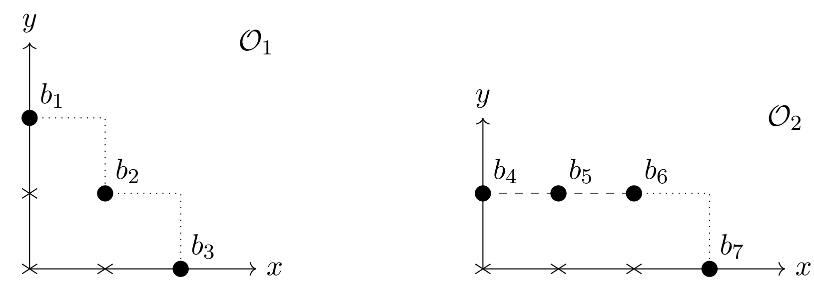
2) The border terms $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are called **across-the-street neighbors** with respect to $\mathcal{O}$ if $\beta_i = \beta_j$ and if $b_i, b_j \in \partial\mathcal{O}_{\beta_i}$ are across-the-street neighbors with respect to $\mathcal{O}_{\beta_i}$, i.e. if we have $x_k b_i e_{\beta_i} = x_\ell b_j e_{\beta_j}$ for some $k, \ell \in \{1, \ldots, n\}$ with $k \neq \ell$. In this case, the border syzygy $\sigma_{ij} = x_k \varepsilon_i - x_\ell \varepsilon_j$ is called an **across-the-street neighbor syzygy** with respect to $\mathcal{O}$.

3) The border terms $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are called **neighbors** with respect to $\mathcal{O}$ if $\beta_i = \beta_j$ and if $b_i, b_j \in \partial\mathcal{O}_{\beta_i}$ are neighbors with respect to $\mathcal{O}_{\beta_i}$, i.e. if they are next-door or across-the-street neighbors with respect to $\mathcal{O}$. In this case, the corresponding border syzygy σ_{ij} is called a **neighbor syzygy** with respect to $\mathcal{O}$.

Example 2.4.22. Consider the $\mathcal{O}$-border prebasis $G \subseteq P^2$ of Example 2.2.3, again. Recall that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2 and that the border of $\mathcal{O}$ was $\partial\mathcal{O} = \{y^2 e_1, xy e_1, x^2 e_1, y e_2, xy e_2, x^2 y e_2, x^3 e_2\} = \{b_1 e_{\beta_1}, \ldots, b_7 e_{\beta_7}\}$.

● terms in $\partial\mathcal{O}$ - - - next-door neighbors with respect to $\mathcal{O}$

× terms in $\mathcal{O}$ across-the-street neighbors with respect to $\mathcal{O}$



We see that $x \cdot b_4 e_{\beta_4} = xy e_2 = b_5 e_{\beta_5}$ and $x \cdot b_5 e_{\beta_5} = x^2 y e_2 = b_6 e_{\beta_6}$, i.e. the border terms $b_4 e_{\beta_4}$ and $b_5 e_{\beta_5}$, and the border terms $b_5 e_{\beta_5}$ and $b_6 e_{\beta_6}$ are next-door neighbors with respect to $\mathcal{O}$. Moreover, $x \cdot b_1 e_{\beta_1} = xy^2 e_1 = y \cdot b_2 e_{\beta_2}$, $x \cdot b_2 e_{\beta_2} = x^2 y e_1 = y \cdot b_3 e_{\beta_3}$, and $x \cdot b_6 e_{\beta_6} = x^3 y e_2 = y \cdot b_7 e_{\beta_7}$, i.e. the border terms $b_1 e_{\beta_1}$ and $b_2 e_{\beta_2}$, the border terms $b_2 e_{\beta_2}$ and $b_3 e_{\beta_3}$, and the border terms $b_6 e_{\beta_6}$ and $b_7 e_{\beta_7}$ are across-the-street neighbors with respect to $\mathcal{O}$. Obviously, there are no further neighbors with respect to $\mathcal{O}$.

The following proposition shows that the set of all neighbor syzygies with respect to the order ideal $\mathcal{O}$ generates $\mathrm{Syz}_P(b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu})$. This is another example of the key role neighbors with respect to $\mathcal{O}$ play for border bases. The proposition is a module version of [KK05, Prop. 21].

Proposition 2.4.23. *The set of all neighbor syzygies with respect to $\mathcal{O}$ generates the P-submodule $\mathrm{Syz}_P(b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}) \subseteq P^\nu$.*

Proof. For all $i, j \in \{1, \ldots, \nu\}$ with $\beta_i = \beta_j$, we let σ_{ij} be the fundamental syzygy of $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ as in Definition 2.4.21. By [KR00, Thm. 2.3.7], the P-submodule $\mathrm{Syz}_P(b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}) \subseteq P^\nu$ is generated by $\Sigma = \{\sigma_{ij} \mid i, j \in \{1, \ldots, \nu\}, i < j, \beta_i = \beta_j\}$. If $\Sigma = \emptyset$, the claim follows trivially. Thus suppose that $\Sigma \neq \emptyset$ and let $i, j \in \{1, \ldots, \nu\}$ be such that $i < j$ and $\beta_i = \beta_j$. We now prove that the fundamental syzygy σ_{ij} is a P-linear combination of neighbor syzygies with respect to $\mathcal{O}$. Let $b_{ij} = \frac{\mathrm{lcm}(b_i, b_j)}{b_i}$ and $b_{ji} = \frac{\mathrm{lcm}(b_i, b_j)}{b_j}$ and let $\mathcal{O} = \mathcal{O}_1 e_1 \cup \cdots \cup \mathcal{O}_r e_r$ be with order ideals $\mathcal{O}_1, \ldots, \mathcal{O}_r$ in $\mathbb{T}^n$. Since $\beta_i = \beta_j$, we see that $b_i, b_j \in \partial \mathcal{O}_{\beta_i}$ by Definition 2.1.7 and thus σ_{ij} is also a syzygy of $(b_1, \ldots, b_\nu)$. Moreover, [KK05, Prop. 21] yields that σ_{ij} is a P-linear combination of fundamental syzygies $\sigma_{k\ell}$ such that $k, \ell \in \{1, \ldots, \nu\}$, $b_k, b_\ell \in \partial \mathcal{O}_{\beta_i}$, i.e. $\beta_k = \beta_\ell = \beta_i$, and such that $b_k, b_\ell \in \partial \mathcal{O}_{\beta_i}$ are neighbors with respect to $\mathcal{O}_{\beta_i}$. Furthermore, we see that two border terms $b_k, b_\ell \in \partial \mathcal{O}_{\beta_i}$ with $k, \ell \in \{1, \ldots, \nu\}$ are neighbors with respect to $\mathcal{O}_{\beta_i}$ if and only if $b_k e_{\beta_i}$ and $b_\ell e_{\beta_i}$ are neighbors with respect to $\mathcal{O}$ by Definition 2.4.21. Altogether, it follows that the fundamental syzygy σ_{ij} is also a P-linear combination of neighbor syzygies $\sigma_{k\ell}$ such that $k, \ell \in \{1, \ldots, \nu\}$ and such that $b_k e_{\beta_k}$ is a neighbor of $b_\ell e_{\beta_\ell}$ with respect to $\mathcal{O}$. □

Next we introduce the notion of liftings of border syzygies. Liftings of border syzygies in P were introduced in [KK05, Defn. 22].

Definition 2.4.24. a) Let $(p_1, \ldots, p_\nu) \in \mathrm{Syz}_P(b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu})$ be a border syzygy with respect to $\mathcal{O}$. Then we call a syzygy $(P_1, \ldots, P_\nu) \in \mathrm{Syz}_P(g_1, \ldots, g_\nu)$ a **lifting** of $(p_1, \ldots, p_\nu)$, if one of the following conditions holds for $v = p_1 g_1 + \cdots + p_\nu g_\nu$.

- We have $v = 0$ and $(P_1, \ldots, P_\nu) = (p_1, \ldots, p_\nu)$.
- We have $v \neq 0$ and $\deg(P_j - p_j) \leq \mathrm{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1, \ldots, \nu\}$ such that $P_j - p_j \neq 0$.

In this situation, we also say that the border syzygy $(p_1, \ldots, p_\nu)$ with respect to $\mathcal{O}$ **lifts** to the syzygy $(P_1, \ldots, P_\nu)$ of $(g_1, \ldots, g_\nu)$.

b) Let $i, j \in \{1, \ldots, \nu\}$ be such that $\beta_i = \beta_j$ and let σ_{ij} be the fundamental syzygy of $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$.

1) If $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are next-door neighbors with respect to $\mathcal{O}$, we call every lifting $\lambda_{ij} \in \mathrm{Syz}_P(g_1, \dots, g_\nu)$ of σ_{ij} a **next-door neighbor lifting** of σ_{ij}.
2) If $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are across-the-street neighbors with respect to $\mathcal{O}$, we call every lifting $\lambda_{ij} \in \mathrm{Syz}_P(g_1, \dots, g_\nu)$ of σ_{ij} an **across-the-street neighbor lifting** of σ_{ij}.
3) If $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are neighbors with respect to $\mathcal{O}$, we call every lifting $\lambda_{ij} \in \mathrm{Syz}_P(g_1, \dots, g_\nu)$ of σ_{ij} a **neighbor lifting** of σ_{ij}.

Example 2.4.25. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_7\} \subseteq P^2$ of the Examples 2.2.3 and 2.4.22, again. Recall that $P = \mathbb{Q}[x, y]$, $\{e_1, e_2\}$ was the canonical P-module basis of P^2, that the border terms $b_5 e_{\beta_5}$ and $b_6 e_{\beta_6}$ are next-door neighbors with respect to $\mathcal{O}$ and that $g_5 = xye_2 + 3e_1$ and $g_6 = x^2ye_2 - e_1 - e_2$. The vector $(0, 0, 0, 0, x, -1, 0) \in P^7$ is a next-door neighbor syzygy with respect to $\mathcal{O}$ according to Definition 2.4.24. As $xg_5 - g_6 = 3xe_1 + e_1 + e_2 \in \langle\mathcal{O}\rangle_{\mathbb{Q}} \setminus \{0\}$, we have $\mathrm{ind}_{\mathcal{O}}(xg_5 - g_6) = 0$ according to Definition 2.1.11. In particular, since there is no polynomial $p \in P \setminus \{0\}$ such that $\deg(p) \leq \mathrm{ind}_{\mathcal{O}}(xg_5 - g_6) - 1 = -1$ and since $xg_5 - g_6 \neq 0$, there exists no next-door neighbor lifting of $(0, 0, 0, 0, x, -1, 0)$ according to Definition 2.4.24.

Finally, we are now able to generalize [KK05, Prop. 25] to the module setting and characterize border bases in P^r via liftings of border syzygies.

Theorem 2.4.26. (Border Bases and Liftings of Border Syzygies) *The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if the following equivalent conditions are satisfied.*

E_1) *Every border syzygy with respect to $\mathcal{O}$ lifts to a syzygy of $(g_1, \dots, g_\nu)$.*

E_2) *Every neighbor syzygy with respect to $\mathcal{O}$ lifts to a syzygy of $(g_1, \dots, g_\nu)$.*

Proof. First we show that condition E_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G\rangle$. Let $(p_1, \dots, p_\nu)$ be a border syzygy with respect to $\mathcal{O}$ and $v = p_1 g_1 + \cdots + p_\nu g_\nu$. If $v = 0$, we see that $(p_1, \dots, p_\nu)$ is a lifting of $(p_1, \dots, p_\nu)$ by Definition 2.4.24. Thus suppose that $v \neq 0$. Since $v \in \langle G\rangle \setminus \{0\}$, condition A_1) of Theorem 2.4.1 yields a representation $v = p_1 g_1 + \cdots + p_\nu g_\nu = q_1 g_1 + \cdots + q_\nu g_\nu$ such that $q_1, \dots, q_\nu \in P$ and such that $q_j = 0$ or $\deg(q_j) \leq \mathrm{ind}_{\mathcal{O}}(v) - 1$ for every index $j \in \{1, \dots, \nu\}$. Let $(P_1, \dots, P_\nu) = (p_1, \dots, p_\nu) - (q_1, \dots, q_\nu)$. Then $(P_1, \dots, P_\nu)$ is a syzygy of $(g_1, \dots, g_\nu)$ by construction. Moreover, we have $\deg(P_j - p_j) = \deg(-q_j) \leq \mathrm{ind}_{\mathcal{O}}(v) - 1$ for all $j \in \{1, \dots, \nu\}$ with $P_j - p_j \neq 0$, i.e. $(p_1, \dots, p_\nu)$ lifts to $(P_1, \dots, P_\nu)$ by Definition 2.4.24.

Since E_1) logically implies E_2) according to Definition 2.4.21, it remains to prove that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if E_2) holds. For all $s \in \{1, \dots, n\}$, we let

$$\varrho_s : \{1, \dots, \mu\} \to \mathbb{N}, \quad i \mapsto \begin{cases} j & \text{if } x_s t_i e_{\alpha_i} = t_j e_{\alpha_j} \in \mathcal{O}, \\ k & \text{if } x_s t_i e_{\alpha_i} = b_k e_{\beta_k} \in \partial\mathcal{O}. \end{cases}$$

be as in Theorem 2.4.19. We have to distinguish two cases.
Given next-door neighbors $b_k e_{\beta_k}, b_\ell e_{\beta_\ell} \in \partial\mathcal{O}$ with respect to $\mathcal{O}$, say $x_s b_k e_{\beta_k} = b_\ell e_{\beta_\ell}$

where $s \in \{1, \dots, n\}$ and $k, \ell \in \{1, \dots, \nu\}$, the corresponding next-door neighbor syzygy with respect to $\mathcal{O}$ is of the form $\sigma_{k\ell} = x_s \varepsilon_k - \varepsilon_\ell$ by Definition 2.4.21. Let $\lambda_{k\ell}$ be a lifting of $\sigma_{k\ell}$. If $v = x_s g_k - g_\ell \neq 0$, Proposition 2.1.13 yields $\operatorname{ind}_{\mathcal{O}}(x_s g_k - g_\ell) \leq 1$. Hence there exist $c_1, \dots, c_\nu \in K$ such that $\lambda_{k\ell} = x_s \varepsilon_k - \varepsilon_\ell - \sum_{w=1}^{\nu} c_w \varepsilon_w$ and $\lambda_{k\ell}$ is also a syzygy of $(g_1, \dots, g_\nu)$ by Definition 2.4.24. Thus we have

$$
\begin{aligned}
0 &= x_s g_k - g_\ell - \sum_{w=1}^{\nu} c_w g_w \\
&= x_s \left(b_k e_{\beta_k} - \sum_{m=1}^{\mu} a_{mk} t_m e_{\alpha_m} \right) - \left(b_\ell e_{\beta_\ell} - \sum_{m=1}^{\mu} a_{m\ell} t_m e_{\alpha_m} \right) \\
&\quad - \sum_{w=1}^{\nu} c_w \left(b_w e_{\beta_w} - \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m} \right) \\
&= - \sum_{m=1}^{\mu} a_{mk} (x_s t_m e_{\alpha_m}) + \sum_{m=1}^{\mu} a_{m\ell} t_m e_{\alpha_m} - \sum_{w=1}^{\nu} c_w b_w e_{\beta_w} + \sum_{w=1}^{\nu} c_w \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m} \\
&= - \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \mathcal{O}}} a_{mk} t_{\varrho_s(m)} e_{\alpha_{\varrho_s(m)}} - \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \partial\mathcal{O}}} a_{mk} b_{\varrho_s(m)} e_{\beta_{\varrho_s(m)}} \\
&\quad + \sum_{m=1}^{\mu} a_{m\ell} t_m e_{\alpha_m} - \sum_{w=1}^{\nu} c_w b_w e_{\beta_w} + \sum_{w=1}^{\nu} c_w \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m}.
\end{aligned}
$$

As $\partial\mathcal{O}$ is K-linearly independent, comparison of the coefficients of the term $b_w e_{\beta_w}$ for all $w \in \{1, \dots, \nu\}$ yields

$$
c_w = \begin{cases} -a_{mk} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} \in x_s \mathcal{O}, \\ 0 & \text{if } b_w e_{\beta_w} \notin x_s \mathcal{O}. \end{cases}
$$

As $\mathcal{O}$ is K-linearly independent, comparison of the coefficients of the term $t_p e_{\alpha_p}$ for all $p \in \{1, \dots, \mu\}$ yields

$$
\begin{aligned}
0 &= - \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \mathcal{O}}} a_{mk} \delta_{p \varrho_s(m)} + a_{p\ell} + \sum_{w=1}^{\nu} c_w a_{pw} \\
&= - \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \mathcal{O}}} a_{mk} \delta_{p \varrho_s(m)} + a_{p\ell} - \sum_{\substack{m \in \{1,\dots,\mu\} \\ x_s t_m e_{\alpha_m} \in \partial\mathcal{O}}} a_{mk} a_{p \varrho_s(m)},
\end{aligned}
$$

i.e. the equations 1) of condition D_2) of Theorem 2.4.19 are satisfied.
Given across-the-street neighbors $b_k e_{\beta_k}, b_j e_{\beta_j} \in \partial\mathcal{O}$ with respect to the order ideal $\mathcal{O}$, say $x_s b_k e_{\beta_k} = x_u b_j e_{\beta_j}$ where $s, u \in \{1, \dots, n\}$ and $k, j \in \{1, \dots, \nu\}$, the corresponding across-the-street neighbor syzygy with respect to $\mathcal{O}$ is of the form $\sigma_{kj} = x_s \varepsilon_k - x_u \varepsilon_j$ according to Definition 2.4.21. Let λ_{kj} be a lifting of σ_{kj}. If $v = x_s g_k - x_u g_j \neq 0$, Proposition 2.1.13 yields $\operatorname{ind}_{\mathcal{O}}(x_s g_k - x_u g_j) \leq 1$. Hence there exist scalars $c_1, \dots, c_\nu \in K$

such that $\lambda_{kj} = x_s\varepsilon_k - x_u\varepsilon_j - \sum_{w=1}^{\nu} c_w\varepsilon_w$ and λ_{kj} is also a syzygy of $(g_1,\dots,g_\nu)$ by Definition 2.4.24. Thus we have

$$\begin{aligned}
0 &= x_s g_k - x_u g_j - \sum_{w=1}^{\nu} c_w g_w \\
&= x_s\left(b_k e_{\beta_k} - \sum_{m=1}^{\mu} a_{mk} t_m e_{\alpha_m}\right) - x_u\left(b_j e_{\beta_j} - \sum_{m=1}^{\mu} a_{mj} t_m e_{\alpha_m}\right) \\
&\quad - \sum_{w=1}^{\nu} c_w\left(b_w e_{\beta_w} - \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m}\right) \\
&= -\sum_{m=1}^{\mu} a_{mk}(x_s t_m e_{\alpha_m}) + \sum_{m=1}^{\mu} a_{mj}(x_u t_m e_{\alpha_m}) \\
&\quad - \sum_{w=1}^{\nu} c_w b_w e_{\beta_w} + \sum_{w=1}^{\nu} c_w \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m} \\
&= -\sum_{\substack{m\in\{1,\dots,\mu\}\\ x_s t_m e_{\alpha_m}\in\mathcal{O}}} a_{mk} t_{\varrho_s(m)} e_{\alpha_{\varrho_s(m)}} - \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_s t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{mk} b_{\varrho_s(m)} e_{\beta_{\varrho_s(m)}} \\
&\quad + \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_u t_m e_{\alpha_m}\in\mathcal{O}}} a_{mj} t_{\varrho_u(m)} e_{\alpha_{\varrho_u(m)}} + \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_u t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{mj} b_{\varrho_u(m)} e_{\beta_{\varrho_u(m)}} \\
&\quad - \sum_{w=1}^{\nu} c_w b_w e_{\beta_w} + \sum_{w=1}^{\nu} c_w \sum_{m=1}^{\mu} a_{mw} t_m e_{\alpha_m}.
\end{aligned}$$

As $\partial\mathcal{O}$ is K-linearly independent, comparison of the coefficients of the term $b_w e_{\beta_w}$ for all $w \in \{1,\dots,\nu\}$ yields

$$c_w = \begin{cases} -a_{mk} + a_{m'j} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} = x_u t_{m'} e_{\alpha_{m'}} \in x_s\mathcal{O}\cap x_u\mathcal{O}, \\ -a_{mk} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} \in x_s\mathcal{O}\setminus x_u\mathcal{O}, \\ a_{m'j} & \text{if } b_w e_{\beta_w} = x_u t_{m'} e_{\alpha_{m'}} \in x_u\mathcal{O}\setminus x_s\mathcal{O}, \\ 0 & \text{if } b_w e_{\beta_w} \notin x_s\mathcal{O}\cup x_u\mathcal{O}. \end{cases}$$

As $\mathcal{O}$ is K-linearly independent, comparison of the coefficients of the term $t_p e_{\alpha_p}$ for all $p \in \{1,\dots,\mu\}$ yields

$$\begin{aligned}
0 &= -\sum_{\substack{m\in\{1,\dots,\mu\}\\ x_s t_m e_{\alpha_m}\in\mathcal{O}}} a_{mk}\delta_{p\varrho_s(m)} + \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_u t_m e_{\alpha_m}\in\mathcal{O}}} a_{mj}\delta_{p\varrho_u(m)} + \sum_{w=1}^{\nu} c_w a_{pw} \\
&= -\sum_{\substack{m\in\{1,\dots,\mu\}\\ x_s t_m e_{\alpha_m}\in\mathcal{O}}} a_{mk}\delta_{p\varrho_s(m)} + \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_u t_m e_{\alpha_m}\in\mathcal{O}}} a_{mj}\delta_{p\varrho_u(m)} \\
&\quad - \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_s t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{mk} a_{p\varrho_s(m)} + \sum_{\substack{m\in\{1,\dots,\mu\}\\ x_u t_m e_{\alpha_m}\in\partial\mathcal{O}}} a_{mj} a_{p\varrho_u(m)},
\end{aligned}$$

i.e. the equations 2) of condition D_2) of Theorem 2.4.19 are satisfied.
Altogether, we see that the condition D_2) of Theorem 2.4.19 is satisfied and thus G is the $\mathcal{O}$-border basis of $\langle G\rangle$. □

Example 2.4.27. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_7\} \subseteq P^2$ of Example 2.4.25, again. Recall that $(0, 0, 0, 0, x, -1, 0) \in \operatorname{Syz}_P(b_1 e_{\beta_1}, \ldots, b_7 e_{\beta_7})$ is a next-door neighbor syzygy with respect to $\mathcal{O}$ that cannot be lifted to a syzygy of $(g_1, \ldots, g_7)$, i.e. condition E_2) of Theorem 2.4.26 is not satisfied. Therefore, G is not the $\mathcal{O}$-border basis of $\langle G\rangle$ according to Theorem 2.4.26. Note that this result coincides with Example 2.4.20.

It is important to remark that the proof of the preceding theorem yields an algorithmic way to compute liftings of border syzygies with respect to $\mathcal{O}$ if G is the $\mathcal{O}$-border basis of $\langle G\rangle$ and that liftings of neighbor syzygies with respect to $\mathcal{O}$ are uniquely determined and only depend on G.

Remark 2.4.28. a) Suppose that G is the $\mathcal{O}$-border basis of $\langle G\rangle$. Given an arbitrary border syzygy $(p_1, \ldots, p_\nu)$, we have seen in the proof of Theorem 2.4.26 that every special generation $p_1 g_1 + \cdots + p_\nu g_\nu = q_1 g_1 + \cdots + q_\nu g_\nu$ with $q_1, \ldots, q_\nu \in P$ as in condition A_1) of Theorem 2.4.1 implies that $(p_1 - q_1, \ldots, p_\nu - q_\nu) \in \operatorname{Syz}_P(g_1, \ldots, g_\nu)$ is a lifting of $(p_1, \ldots, p_\nu)$. In particular, as shown in the proof of Theorem 2.4.1, we can compute such a special generation by applying the Border Division Algorithm 2.2.1 to the input data $p_1 g_1 + \cdots + p_\nu g_\nu$ and G. Note that the Border Division Algorithm 2.2.1 depends on the ordering of the elements in G and thus also the lifting depends on this ordering. Hence the lifting $(p_1 - q_1, \ldots, p_\nu - q_\nu)$ is, in general, not uniquely determined by the $\mathcal{O}$-border basis G.

b) Let $k, \ell \in \{1, \ldots, \nu\}$ be such that $b_k e_{\beta_k}, b_\ell e_{\beta_\ell} \in \partial\mathcal{O}$ are neighbors with respect to $\mathcal{O}$, and let $\sigma_{k\ell}$ denote the corresponding neighbor syzygy with respect to $\mathcal{O}$. Suppose that there exists a neighbor lifting $\lambda_{k\ell}$ of $\sigma_{k\ell}$.

1) Suppose that $b_k e_{\beta_k}, b_\ell e_{\beta_\ell} \in \partial\mathcal{O}$ are next-door neighbors with respect to $\mathcal{O}$, i.e. there exists an $s \in \{1, \ldots, n\}$ such that $x_s b_k e_{\beta_k} = b_\ell e_{\beta_\ell}$ by Definition 2.4.21. As shown in the proof of Theorem 2.4.26, the lifting $\lambda_{k\ell}$ of $\sigma_{k\ell}$ has the form $\lambda_{k\ell} = \sigma_{k\ell} - c_1\varepsilon_1 - \cdots - c_\nu\varepsilon_\nu$ with $c_1, \ldots, c_\nu \in K$ and for every $w \in \{1, \ldots, \nu\}$, we have

$$c_w = \begin{cases} a_{mk} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} \in x_s\mathcal{O}, \\ 0 & \text{if } b_w e_{\beta_w} \notin x_s\mathcal{O}. \end{cases}$$

2) Suppose that $b_k e_{\beta_k}, b_\ell e_{\beta_\ell} \in \partial\mathcal{O}$ are across-the-street neighbors with respect to $\mathcal{O}$, i.e. there exist $s, u \in \{1, \ldots, n\}$ such that $x_s b_k e_{\beta_k} = x_u b_\ell e_{\beta_\ell}$ by Definition 2.4.21. As shown in the proof of Theorem 2.4.26, the lifting $\lambda_{k\ell}$ of $\sigma_{k\ell}$ has the form $\lambda_{k\ell} = \sigma_{k\ell} - c_1\varepsilon_1 - \cdots - c_\nu\varepsilon_\nu$ with $c_1, \ldots, c_\nu \in K$ and for every

$w \in \{1, \dots, \nu\}$, we have

$$c_w = \begin{cases} a_{mk} - a_{m'\ell} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} = x_u t_{m'} e_{\alpha_{m'}} \in x_s\mathcal{O} \cap x_u\mathcal{O}, \\ a_{mk} & \text{if } b_w e_{\beta_w} = x_s t_m e_{\alpha_m} \in x_s\mathcal{O} \setminus x_u\mathcal{O}, \\ -a_{m'\ell} & \text{if } b_w e_{\beta_w} = x_u t_{m'} e_{\alpha_{m'}} \in x_u\mathcal{O} \setminus x_s\mathcal{O}, \\ 0 & \text{if } b_w e_{\beta_w} \notin x_s\mathcal{O} \cup x_u\mathcal{O}. \end{cases}$$

In both cases, the neighbor lifting $\lambda_{k\ell}$ of $\sigma_{k\ell}$ is uniquely determined and solely depends on the $\mathcal{O}$-border basis G.

Remark 2.4.29. The proof of the preceding theorem is based on the characterization of border bases via commuting matrices as shown in Theorem 2.4.19. In Corollary 3.3.9, we give another proof of this theorem that does not depend on Theorem 2.4.19 but on a deeper insight into the structure of the neighbor liftings of a border basis.

2.4.6 Buchberger's Criterion

In the final subsection of this section, we introduce S-vectors and prove a border bases version of Buchberger's Criterion in Theorem 2.4.31. Again, the neighbors with respect to $\mathcal{O}$ play the key role in this theorem. Moreover, this characterizations allows us to easily and algorithmically check whether a given border prebasis is a border basis or not. The corresponding version for border bases in P can be found in [KK05, Prop. 18] and [KR05, Prop. 6.4.34].

As in the previous subsection, we let $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ denote the canonical basis of the free P-module P^ν.

Definition 2.4.30. Let $i, j \in \{1, \dots, \nu\}$. Then the S-**vector** of g_i and g_j is defined by $\mathrm{S}(g_i, g_j) = \frac{\mathrm{lcm}(b_i, b_j)}{b_i} g_i - \frac{\mathrm{lcm}(b_i, b_j)}{b_j} g_j \in \langle G \rangle \subseteq P^r$

Theorem 2.4.31. (Buchberger's Criterion for Border Bases)
The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if and only if the following equivalent conditions are satisfied.

F_1) *We have* $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = 0$ *for all* $i, j \in \{1, \dots, \nu\}$.

F_2) *We have* $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = 0$ *for all* $i, j \in \{1, \dots, \nu\}$ *such that* $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ *are neighbors with respect to* $\mathcal{O}$.

Proof. First we show that condition F_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G \rangle$. Let $i, j \in \{1, \dots, \nu\}$. We apply the Border Division Algorithm 2.2.1 to the S-vector $\mathrm{S}(g_i, g_j)$ and G to obtain a representation $\mathrm{S}(g_i, g_j) = v + \mathrm{NR}_G(\mathrm{S}(g_i, g_j))$ with $v \in \langle G \rangle$ and with $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) \in \langle \mathcal{O} \rangle_K$. As $\mathrm{S}(g_i, g_j) \in \langle G \rangle$ by Definition 2.4.30 and as G is the $\mathcal{O}$-border basis of $\langle G \rangle$, Corollary 2.2.6 implies $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = \mathrm{S}(g_i, g_j) - v \in \langle G \rangle \cap \langle \mathcal{O} \rangle_K = \{0\}$.

Since F_2) follows logically from F_1), it remains to prove that G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if F_2) holds. If there are no neighbors with respect to $\mathcal{O}$, the claim follows trivially

from Theorem 2.4.26. Thus suppose that there are $i, j \in \{1, \dots, \nu\}$ such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$ and let σ_{ij} be the corresponding neighbor syzygy. We distinguish two cases.
First suppose that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are next-door neighbors with respect to $\mathcal{O}$, i.e. there is an $s \in \{1, \dots, n\}$ with $x_s b_i e_{\beta_i} = b_j e_{\beta_j}$ by Definition 2.4.21. Then $\sigma_{ij} = x_s \varepsilon_i - \varepsilon_j$ by Definition 2.4.21 and $\mathrm{S}(g_i, g_j) = x_s g_i - g_j$ by Definition 2.4.30. If $\mathrm{S}(g_i, g_j) = 0$, we see that σ_{ij} is a lifting of σ_{ij} according to Definition 2.4.24. Thus suppose that $\mathrm{S}(g_i, g_j) \neq 0$. Since $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = 0$ according to F_2), the Border Division Algorithm 2.2.1 applied to $\mathrm{S}(g_i, g_j)$ and G yields $\mathrm{S}(g_i, g_j) = p_1 g_1 + \cdots + p_\nu g_\nu + \mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = p_1 g_1 + \cdots + p_\nu g_\nu$ for some polynomials $p_1, \dots, p_\nu \in P$ that satisfy $p_\ell = 0$ or $\deg(p_\ell) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$ for all $\ell \in \{1, \dots, \nu\}$. We now prove that $(P_1, \dots, P_\nu) \in P^\nu$ defined by $P_i = x_s - p_i$, $P_j = 1 - p_j$, and $P_\ell = -p_\ell$ for all $\ell \in \{1, \dots, \nu\} \setminus \{i, j\}$ is a lifting of σ_{ij}. By construction, we see that $(P_1, \dots, P_\nu) \in \mathrm{Syz}_P(g_1, \dots, g_\nu)$. Moreover, we have $P_i - x_s = -p_i = 0$ or $\deg(P_i - x_s) = \deg(-p_i) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$, we have $P_j - 1 = -p_j = 0$ or we have $\deg(P_j - 1) = \deg(-p_j) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$, and for all $\ell \in \{1, \dots, \nu\} \setminus \{i, j\}$, we have $P_\ell - 0 = -p_\ell = 0$ or $\deg(P_\ell - 0) = \deg(-p_\ell) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$. Hence $(P_1, \dots, P_\nu)$ is a lifting of σ_{ij} according to Definition 2.4.24.
Now suppose that $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are across-the-street neighbors with respect to $\mathcal{O}$, i.e. $x_s b_i e_{\beta_i} = x_u b_j e_{\beta_j}$ for some $s, u \in \{1, \dots, n\}$ by Definition 2.4.21. Then $\sigma_{ij} = x_s \varepsilon_i - x_u \varepsilon_j$ by Definition 2.4.21 and $\mathrm{S}(g_i, g_j) = x_s g_i - x_u g_j$ by Definition 2.4.30. If $\mathrm{S}(g_i, g_j) = 0$, σ_{ij} is a lifting of σ_{ij} according to Definition 2.4.24. Thus suppose that $\mathrm{S}(g_i, g_j) \neq 0$. Since we have $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = 0$ by F_2), the Border Division Algorithm 2.2.1 applied to $\mathrm{S}(g_i, g_j)$ and G yields $\mathrm{S}(g_i, g_j) = p_1 g_1 + \cdots + p_\nu g_\nu + \mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = p_1 g_1 + \cdots + p_\nu g_\nu$ for some polynomials $p_1, \dots, p_\nu \in P$ that satisfy $p_\ell = 0$ or $\deg(p_\ell) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$ for all $\ell \in \{1, \dots, \nu\}$. We now prove that $(P_1, \dots, P_\nu) \in P^\nu$ defined by $P_i = x_s - p_i$, $P_j = x_u - p_j$, and $P_\ell = -p_\ell$ for all $\ell \in \{1, \dots, \nu\} \setminus \{i, j\}$ is a lifting of of the border syzygy σ_{ij}. By construction, $(P_1, \dots, P_\nu) \in \mathrm{Syz}_P(g_1, \dots, g_\nu)$. Moreover, we have $P_i - x_s = -p_i = 0$ or $\deg(P_i - x_s) = \deg(-p_i) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$, we have $P_j - x_u = -p_j = 0$ or $\deg(P_j - x_u) = \deg(-p_j) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$, and for all $\ell \in \{1, \dots, \nu\} \setminus \{i, j\}$, we have $P_\ell - 0 = -p_\ell = 0$ or $\deg(P_\ell - 0) = \deg(-p_\ell) \leq \mathrm{ind}_{\mathcal{O}}(\mathrm{S}(g_i, g_j)) - 1$. Hence the vector $(P_1, \dots, P_\nu)$ is a lifting of σ_{ij} according to Definition 2.4.24.
Altogether, we have proven that every neighbor syzygy with respect to $\mathcal{O}$ lifts to a syzygy of $(g_1, \dots, g_\nu)$. Therefore, condition E_2) of Theorem 2.4.26 yields that G is the $\mathcal{O}$-border basis of $\langle G \rangle$. □

Example 2.4.32. Consider the $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_7\} \subseteq P^2$ of Example 2.4.22, again. Recall that $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical basis of the free P-module P^2, that $g_5 = xye_2 + 3e_1$ and $g_6 = x^2ye_2 - e_1 - e_2$, and that $b_5 e_{\beta_5} = xye_2$ and $b_6 e_{\beta_6} = x^2ye_2$ are neighbors with respect to $\mathcal{O}$. Since $\mathrm{NR}_G(\mathrm{S}(g_5, g_6)) = \mathrm{NR}_G(xg_5 - g_6) = 3xe_1 + e_1 + e_2 \neq 0$, condition F_2) of Buchberger's Criterion for border bases 2.4.31 yields that G is not the $\mathcal{O}$-border basis of $\langle G \rangle$. Note that this result coincides with Example 2.4.27.

2.5 Computation

After all the characterizations in the previous section, we now continue Section 2.3. We have proven in Proposition 2.3.2 that every submodule of finite K-codimension in P^r possesses a border basis. Moreover, we have also given a first naive algorithm for the computation of border bases in Remark 2.3.6, which depends on a Gröbner basis computation. In this section, we introduce an algorithm that uses linear algebra techniques instead to compute a border basis. A first generic form of this algorithm for border bases in P was given in [Mou99, Algo. 4.3]. Building upon this generic algorithm, a first explicit form of an algorithm to compute border bases in P was given in [KK06]. We now generalize the latter to the module setting.

For the whole section, we let $W = (w_1, \dots, w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ be a positive matrix in the sense of [KR05, Defn. 4.2.4], i.e. such that $w_i > 0$ for all $i \in \{1, \dots, n\}$. Then the matrix W induces a $\mathbb{Z}$-grading on P which satisfies $\deg_W(x_i) = w_i$ for all indices $i \in \{1, \dots, n\}$ according to [KR05, Defn. 4.1.6]. In particular, $P = \bigoplus_{\gamma \in \mathbb{Z}} P_\gamma$ where we have $P_\gamma = \{p \in P \mid \deg_W(t) = \gamma \text{ for all } t \in \mathrm{Supp}(p)\}$ for all $\gamma \in \mathbb{Z}$. By defining the set of all homogeneous components of the free P-module P^r of degree $\gamma \in \mathbb{Z}$ to be $P^r_\gamma = \{(p_1, \dots, p_r) \in P^r \mid p_i \in P_\gamma \text{ for all } i \in \{1, \dots, r\}\}$, $P^r = \bigoplus_{\gamma \in \mathbb{Z}} P^r_\gamma$ becomes a $\mathbb{Z}$-graded free P-module, cf. [KR00, Defn. 1.7.4 and 1.7.6]. We always consider P and P^r as $\mathbb{Z}$-graded P-modules using the above $\mathbb{Z}$-gradings.

Definition 2.5.1. Let $r, s \in \mathbb{N} \setminus \{0\}$.

a) Let $(c_1, \dots, c_s) \in K^s$, $(d_1, \dots, d_s) \in K^s \setminus \{0\}$, and let $i \in \{1, \dots, s\}$ be minimal such that $d_i \neq 0$. Then the i^{th} component of $(c_1, \dots, c_s) - \frac{c_i}{d_i}(d_1, \dots, d_s)$ is zero. If $c_i \neq 0$ in this situation, we say that $(d_1, \dots, d_s)$ is a **reducer** of $(c_1, \dots, c_s)$.

b) Let $v \in K^s$ and $\mathcal{M} \in \mathrm{Mat}_{r,s}(K)$. We say that v can be **reduced** against $\mathcal{M}$ if there is a row $w \in K^s$ of $\mathcal{M}$ such that w is a reducer of v.

Lemma 2.5.2. *Let $\gamma \in \mathbb{N}$, let $\varrho \in \mathbb{N} \setminus \{0\}$, let $V = \langle v_1, \dots, v_\varrho \rangle \subseteq P^r_{\leq\gamma}$ with $\{v_1, \dots, v_\varrho\} \subseteq P^r_{\leq\gamma} \setminus \{0\}$ be a K-vector subspace such that $(V + x_1 V + \cdots + x_n V) \cap P^r_{\leq\gamma} = V$, and let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$. Then Algorithm 2 is actually an algorithm and the result*

$$\mathcal{O} := \texttt{computeOrderIdeal}(\gamma, \{v_1, \dots, v_\varrho\}, W, \sigma)$$

of Algorithm 2 applied to the input data γ, $\{v_1, \dots, v_\varrho\}$, W, and σ satisfies the following conditions.

1) The set $\mathcal{O} \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1, \dots, e_r \rangle$ is an order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$.

2) The residue classes of the elements of the order ideal $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form a K-vector space basis of $P^r_{\leq\gamma}/V$.

Algorithm 2: `computeOrderIdeal`$(\gamma, \{v_1, \ldots, v_\varrho\}, (w_1, \ldots, w_n), \sigma)$

Input:
$\gamma \in \mathbb{N}$;
$\varrho \in \mathbb{N} \setminus \{0\}$ and $\{v_1, \ldots, v_\varrho\} \subseteq P^r_{\leq\gamma} \setminus \{0\}$;
$V \coloneqq \langle v_1, \ldots, v_\varrho \rangle_K \subseteq P^r_{\leq\gamma}$ is a K-vector subspace with the property that $(V + x_1 V + \cdots + x_n V) \cap P^r_{\leq\gamma} = V$;
$(w_1, \ldots, w_n) \in \operatorname{Mat}_{1,n}(\mathbb{Z})$ such that $w_i > 0$ for all $i \in \{1, \ldots, n\}$;
σ is a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_{(w_1,\ldots,w_n)}$;

1 Let $\ell_1, \ldots, \ell_s \in \mathbb{T}^n$ be terms and $u_1, \ldots, u_s \in \{1, \ldots, r\}$ be indices such that $\mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle = \{\ell_1 e_{u_1}, \ldots, \ell_s e_{u_s}\}$ and $\ell_1 e_{u_1} >_{\sigma\,\mathrm{Pos}} \ell_2 e_{u_2} >_{\sigma\,\mathrm{Pos}} \cdots >_{\sigma\,\mathrm{Pos}} \ell_s e_{u_s}$.
2 Determine a K-vector space basis $\{\widetilde{v}_1, \ldots, \widetilde{v}_k\} \subseteq P^r_{\leq\gamma}$ of V.
3 **for** $i \coloneqq 1$ **to** k **do**
4 | Determine $c_{i1}, \ldots, c_{is} \in K$ such that $\widetilde{v}_i = c_{i1}\ell_1 e_{u_1} + \cdots + c_{is}\ell_s e_{u_s}$.
5 **end**
6 $\mathcal{V} \coloneqq (c_{ij})_{1\leq i\leq k, 1\leq j\leq s} \in \operatorname{Mat}_{k,s}(K)$.
7 Compute a row echolon form $\widetilde{\mathcal{V}} \in \operatorname{Mat}_{k,s}(K)$ of $\mathcal{V}$ using row operations.
8 Let $\mathcal{O} \subseteq L$ be the set of terms in L corresponding to the pivot-free columns of $\widetilde{\mathcal{V}}$, i.e. the columns of $\widetilde{\mathcal{V}}$ in which no row of $\widetilde{\mathcal{V}}$ has its first non-zero entry.
9 **return** $\mathcal{O}$

Proof. First we show that the procedure is actually an algorithm. All the operations in the lines 2, 4, and 7 can be computed with linear algebra techniques. Moreover, the procedure obviously stops after a finite amount of time. Thus the procedure is an algorithm.

Next we show the correctness of the algorithm. We start to prove that the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form a K-vector space basis of $P^r_{\leq\gamma}/V$. We write $\mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle = \{\ell_1 e_{u_1}, \ldots, \ell_s e_{u_s}\}$ with $s \in \mathbb{N}$, $\ell_1, \ldots, \ell_s \in \mathbb{T}^n$, $u_1, \ldots, u_s \in \{1, \ldots, r\}$, and $\ell_1 e_{u_1} >_{\sigma\,\mathrm{Pos}} \ell_2 e_{u_2} >_{\sigma\,\mathrm{Pos}} \cdots >_{\sigma\,\mathrm{Pos}} \ell_s e_{u_s}$ as in line 2, and let $\{\widetilde{v}_1, \ldots, \widetilde{v}_k\} \subseteq P^r_{\leq\gamma}$ with $k \in \mathbb{N}$ be a K-vector space basis of V as in line 2. For all indices $i \in \{1, \ldots, k\}$, we let $c_{i1}, \ldots, c_{is} \in K$ be such that $\widetilde{v}_i = c_{i1}\ell_1 e_{u_1} + \cdots + c_{is}\ell_s e_{u_s}$ as in line 4. Let $\mathcal{V} = (c_{ij})_{1\leq i\leq k, 1\leq j\leq s} \in \operatorname{Mat}_{k,s}(K)$ be as in line 6 and let $\widetilde{\mathcal{V}} \in \operatorname{Mat}_{k,s}(K)$ be a row echolon form of $\mathcal{V}$ as in line 7. Moreover, let $\mathcal{O} = \{\ell_{j_1} e_{u_{j_1}}, \ldots, \ell_{j_\mu} e_{u_{j_\mu}}\}$ with $\mu \in \mathbb{N}$ and with $j_1, \ldots, j_\mu \in \{1, \ldots, r\}$ be as in line 8, and let $c_1, \ldots, c_\mu \in K$ be such that $v = c_1 \ell_{j_1} e_{u_{j_1}} + \cdots + c_\mu \ell_{j_\mu} e_{u_{j_\mu}} \in V$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_s\}$ denote the canonical K-vector space basis of K^s. Then the corresponding vector $c_1\mathcal{E}_{j_1} + \cdots + c_\mu \mathcal{E}_{j_\mu} \in K^s$ of v has all its non-zero entries in the columns corresponding to elements of $\mathcal{O}$ by line 8. As $\widetilde{\mathcal{V}}$ is in row echolon form according to line 7, this vector cannot be further reduced against $\widetilde{\mathcal{V}}$ by Definition 2.5.1. Since the rows of $\mathcal{V}$ correspond to the K-vector space basis $\{\widetilde{v}_1, \ldots, \widetilde{v}_k\}$ of V according to line 4, since $\widetilde{\mathcal{V}}$ corresponds to K-linear combinations of these basis elements by the lines 2 and 7, and since we also have $v \in V$, it follows that $v = 0$. Thus

we get $c_1 = \cdots = c_\mu = 0$, i.e. the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ are K-linearly independent. Let $w = d_1\ell_1 e_{u_1} + \cdots + d_s\ell_s e_{u_s} \in P^r_{\leq\gamma}$ be with $d_1, \ldots, d_s \in K$. Then the corresponding vector $d_1\mathcal{E}_1 + \cdots + d_s\mathcal{E}_s \in K^s$ can be reduced against the matrix $\widetilde{\mathcal{V}}$ to obtain a vector $d_1'\mathcal{E}_{j_1} + \cdots + d_\mu'\mathcal{E}_{j_\mu} \in K^s$ with $d_1', \ldots, d_\mu' \in K$ by Definition 2.5.1 and line 8. Let $w' = d_1'\ell_{j_1} e_{u_{j_1}} + \cdots + d_\mu'\ell_{j_\mu} e_{u_{j_\mu}} \in \langle\mathcal{O}\rangle_K$ be the corresponding element in $\langle\mathcal{O}\rangle_K$. Since the vector w' was constructed from w by a reduction against the matrix $\widetilde{\mathcal{V}}$ and since the rows of $\widetilde{\mathcal{V}}$ correspond to the K-vector space basis $\{\widetilde{v}_1, \ldots, \widetilde{v}_k\}$ of V according to the lines 2, 4, 6, and 7, we get $\overline{w} = \overline{w}'$ in $P^r_{\leq\gamma}/V$. Hence the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form also a generating set of the K-vector space $P^r_{\leq\gamma}/V$. Altogether, we have proven that the residue classes of the elements of $\mathcal{O}$ form K-vector space basis of $P^r_{\leq\gamma}/V$.
Finally, we prove that $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. If $\mathcal{O} = \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle$, then $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ according to Definition 2.1.6. Thus suppose that $\mathcal{O} \subset \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle$. Let $i \in \{1, \ldots, s\}$ be such that $\ell_i e_{u_i} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$ and let $t \in \mathbb{T}^n$ be such that $t\ell_i e_{u_i} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle$. The set $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ by Definition 2.1.6 if we show that $t\ell_i e_{u_i} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$. As $\ell_i e_{u_i} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$, one row of $\widetilde{\mathcal{V}}$ has the form $(0, \ldots, 0, c_i, \ldots, c_s) \in K^s$ where $c_i \neq 0$ according to the construction of $\mathcal{O}$ in line 8. The corresponding vector in $P^r_{\leq\gamma}$ is $v = c_i\ell_i e_{u_i} + \cdots + c_s\ell_s e_{u_s} \in P^r_{\leq\gamma}$. Moreover, line 1 yields that $\ell_i e_{u_i} = \mathrm{LT}_{\sigma\,\mathrm{Pos}}(v)$. Hence we see that $t\ell_i e_{u_i} = \mathrm{LT}_{\sigma\,\mathrm{Pos}}(tv)$. Since the term ordering σ is compatible with $\deg_W$ and since $W = (w_1, \ldots, w_n)$ satisfies $w_k > 0$ for all $k \in \{1, \ldots, n\}$, it follows that $\mathrm{Supp}(tv) \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle$. Since every line of the matrix $\mathcal{V}$ corresponds to a vector in V by line 4 and since $\widetilde{\mathcal{V}}$ is constructed from $\mathcal{V}$ using row operation by line 7, we see that $v \in V$. Hence the hypothesis $(V + x_1V + \cdots + x_nV) \cap P^r_{\leq\gamma} = V$ and induction on the degree of t imply $tv \in V$. Thus we see that the vector in K^s corresponding to tv can be reduced against $\widetilde{\mathcal{V}}$ to zero according to Definition 2.5.1. In particular, we have to reduce the entry of this vector that corresponds to $t\ell_i e_{u_i}$, i.e. there has to be one row in $\widetilde{\mathcal{V}}$ which has its first non-zero entry in the column that corresponds to the term $t\ell_i e_{u_i}$. Altogether, line 8 yields $t\ell_i e_{u_i} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \ldots, e_r\rangle \setminus \mathcal{O}$ and the claim follows. □

Theorem 2.5.3. (The Border Basis Algorithm)
Let $k \in \mathbb{N}$ and $U = \langle v_1, \ldots, v_k\rangle \subseteq P^r$ with vectors $v_1, \ldots, v_k \in P^r \setminus \{0\}$ be a P-submodule such that $\mathrm{codim}_K(U, P^r) < \infty$, and let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$. Then Algorithm 3 is actually an algorithm and the result

$$(\mathcal{O}, G) := \mathtt{moduleBB}(\{v_1, \ldots, v_k\}, W, \sigma)$$

of Algorithm 3 applied to the input data $\{v_1, \ldots, v_k\}$, W, and σ satisfies the following conditions.

1) *The set $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$.*

2) *The set $G \subseteq P^r$ is the $\mathcal{O}$-border basis of U.*

Algorithm 3: $\texttt{BBasis}(\{v_1,\dots,v_k\},(w_1,\dots,w_n),\sigma)$

Input:
$k \in \mathbb{N}$ and $\{v_1,\dots,v_k\} \subseteq P^r \setminus \{0\}$ such that $\operatorname{codim}_K(\langle v_1,\dots,v_k\rangle, P^r) < \infty$;
$(w_1,\dots,w_n) \in \operatorname{Mat}_{1,n}(\mathbb{Z})$ such that $w_i > 0$ for all $i \in \{1,\dots,n\}$;
σ is a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_{(w_1,\dots,w_n)}$;

1 $V := \langle v_1,\dots,v_k\rangle_K$
2 $\gamma := \max\{\deg(t) \mid \ell \in \{1,\dots,k\}, te_u \in \operatorname{Supp}(v_\ell)\}$
3 **repeat**
4 $\quad V' := (V + x_1V + \cdots + x_nV) \cap P^r_{\leq\gamma}$
5 $\quad$**while** $V \neq V'$ **do**
6 $\quad\quad V := V'$
7 $\quad\quad V' := (V + x_1V + \cdots + x_nV) \cap P^r_{\leq\gamma}$
8 $\quad$**end**
9 $\quad$Compute $\varrho \in \mathbb{N}$ and $\{\widetilde{v}_1,\dots,\widetilde{v}_\varrho\} \subseteq P^r_{\leq\gamma} \setminus \{0\}$ with $V = \langle \widetilde{v}_1,\dots,\widetilde{v}_\varrho\rangle_K$.
10 $\quad \mathcal{O} := \texttt{computeOrderIdeal}(\gamma, \{\widetilde{v}_1,\dots,\widetilde{v}_\varrho\}, (w_1,\dots,w_n), \sigma)$
11 $\quad \gamma := \gamma + 1$
12 **until** $\partial\mathcal{O} \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1,\dots,e_r\rangle$
13 Let $\mu \in \mathbb{N}$, let $t_1,\dots,t_\mu \in \mathbb{T}^n$, and let $\alpha_1,\dots,\alpha_\mu \in \{1,\dots,r\}$ be such that $\mathcal{O} = \{t_1e_{\alpha_1},\dots,t_\mu e_{\alpha_\mu}\}$.
14 Let $\nu \in \mathbb{N}$, let $b_1,\dots,b_\nu \in \mathbb{T}^n$, and let $\beta_1,\dots,\beta_\nu \in \{1,\dots,r\}$ be such that $\partial\mathcal{O} = \{b_1e_{\beta_1},\dots,b_\nu e_{\beta_\nu}\}$.
15 $G := \emptyset$
16 **for** $j := 1$ **to** ν **do**
17 $\quad$Determine $a_{1j},\dots,a_{\mu j} \in K$ such that $\overline{b_je_{\beta_j}} = \sum_{i=1}^{\mu} a_{ij}\overline{t_ie_{\alpha_i}}$ in P^r/V.
18 $\quad G := G \cup \{b_je_{\beta_j} - \sum_{i=1}^{\mu} a_{ij}t_ie_{\alpha_i}\}$
19 **end**
20 **return** $(\mathcal{O}, G)$

3) *For all $g \in G$, we have $\operatorname{LT}_{\sigma\operatorname{Pos}}(g) \in \partial\mathcal{O}$. In particular, the degree form of an element of G with respect to W contains the corresponding border term.*

Proof. First we prove that every step of the procedure can be computed. As $r \in \mathbb{N}\setminus\{0\}$ and as $\operatorname{codim}_K(U, P^r) < \infty$, we see that $k \neq 0$ and thus the maximum in line 2 can be computed. In particular, it follows that $\dim_K(V) \geq 1$ in line 1. We can compute the intersection of K-vector spaces for the computation of V' in the lines 4 and 7 with linear algebra techniques. In line 10, the while-loop starting in line 5 has already been finished. In this situation, the construction of V in line 1 and during the while-loop in line 5 yields $\varrho \geq \dim_K(V) \geq 1$ and $V = V' = (V + x_1V + \cdots + x_nV) \cap P^r_{\leq\gamma}$ after the while-loop. In other words, the input data γ, $\{\widetilde{v}_1,\dots,\widetilde{v}_\varrho\}$, W, and σ in line 9 satisfy the requirements of Algorithm 2. Thus we can compute an order ideal $\mathcal{O} \subseteq \mathbb{T}^n_{\leq d}\langle e_1,\dots,e_r\rangle$ in $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ in line 10 such that the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$

form a K-vector space basis of $P^r_{\leq\gamma}/V$ according to Lemma 2.5.2. Moreover, the repeat-until-loop starting in line 3 only stops if $\partial\mathcal{O} \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1,\ldots,e_r\rangle$. Thus we can compute the coefficients $a_{1j},\ldots,a_{\mu j} \in K$ for all $j \in \{1,\ldots,\nu\}$ in line 17 with linear algebra techniques, too. All the other steps of the procedure can be trivially computed.
Second we show that the procedure stops after a finite amount of time. We start to show that the while-loop in line 5 eventually terminates. By the construction of V and V' in the lines 1, 4, 6, and 7, we see that we always have $V \subseteq V' \subseteq P^r_{\leq\gamma}$. Assume that $V \neq V'$ in this situation, i.e. the while-loop in line 5 is executed at least one time. For every $i \in \mathbb{N}$, we let V'_i denote the K-vector subspace $V' \subseteq P^r_{\leq\gamma}$ after the i^{th} iteration of the while-loop. Since we have $\dim_K(P^r_{\leq\gamma}) = \#\mathbb{T}^n_{\leq\gamma}\langle e_1,\ldots,e_r\rangle < \infty$, the chain $V'_0 \subseteq V'_1 \subseteq V'_2 \subseteq \cdots$ eventually gets stationary. In this situation, we have $V'_i = V'_{i-1}$ for some $i \in \mathbb{N}\setminus\{0\}$ and, therefore, $V = V'$ in line 5. Thus the while-loop terminates after the i^{th} iteration.
Third we prove that the repeat-until-loop starting in line 3 stops after a finite amount of time. Let $H = \{h_1,\ldots,h_\eta\} \subseteq P^r$ with $\eta \in \mathbb{N}$ be the reduced σ Pos-Gröbner basis of U. Since $k > 0$, we have $U \neq \{0\}$ and thus see that $\eta > 0$. For every index $j \in \{1,\ldots,\eta\}$, there exist polynomials $p_{j1},\ldots,p_{jk} \in P$ such that $h_j = p_{j1}v_1 + \cdots + p_{jk}v_k$. We denote $\gamma' = \max\{\deg(t) \mid j \in \{1,\ldots,\eta\}, \ell \in \{1,\ldots,k\}, te_u \in \operatorname{Supp}(p_{j\ell}v_\ell)\}$. As we have already seen above, $\eta > 0$ and $k > 0$ and thus this maximum exists. Then we have $H \subseteq P^r_{\leq\gamma}$ after the while-loop in the case that $\gamma = \gamma'$. Now suppose that we are in the situation that $\gamma = \gamma'$ during the repeat-until-loop. Since $V = (V + x_1V + \cdots + x_nV) \cap P^r_{\leq\gamma}$ after the while-loop starting in line 5, we then have $H \subseteq V$. Let $\mathcal{O}$ be the result of Algorithm 2 computed in line 10. Then $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1,\ldots,e_r\rangle$ such that the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form a K-vector space basis of $P^r_{\leq\gamma}/V$ according to Lemma 2.5.2. Moreover, we let $\mathbb{T}^n_{\leq\gamma}\langle e_1,\ldots,e_r\rangle = \{\ell_1e_{u_1},\ldots,\ell_se_{u_s}\}$ with $\ell_1,\ldots,\ell_s \in \mathbb{T}^n$ and $u_1,\ldots,u_s \in \{1,\ldots,r\}$ be such that $\ell_1e_{u_1} >_{\sigma\,\mathrm{Pos}} \cdots >_{\sigma\,\mathrm{Pos}} \ell_se_{u_s}$ as in line 1 of Algorithm 2 during the computation of $\mathcal{O}$ in line 10. Furthermore, we let $\widetilde{\mathcal{V}} \in \operatorname{Mat}_{m,s}(K)$ with $m \in \mathbb{N}$ be the matrix in row echolon form as in line 7 of Algorithm 2 used during the computation of $\mathcal{O}$ in line 10. Let $j \in \{1,\ldots,\eta\}$ and $h_j = c_1\ell_1e_{u_1} + \cdots + c_s\ell_se_{u_s}$ be with $c_1,\ldots,c_s \in K$. Then the vector $(c_1,\ldots,c_s) \in K^s$ corresponds to the Gröbner basis element $h_j \in H$. Let $\mathrm{LT}_{\sigma\,\mathrm{Pos}}(h_j) = \ell_we_{u_w}$ be with $w \in \{1,\ldots,s\}$. Then it follows that $(c_1,\ldots,c_s) = (0,\ldots,0,1,c_{w+1},\ldots,c_s)$ according to [KR00, Defn. 2.4.12]. Since $h_j \in V$, we see that there exists a vector in K^s corresponding to a vector in V which has its first non-zero entry in the w^{th} column of $\widetilde{\mathcal{V}}$, namely the vector $(0,\ldots,0,1,c_{w+1},\ldots,c_s) \in K^s$ corresponding to $h_j \in H$. Therefore, the construction of $\mathcal{O}$ in line 8 of Algorithm 2 yields that $\mathrm{LT}_{\sigma\,\mathrm{Pos}}(h_j) \notin \mathcal{O}$. In particular, this implies $\mathrm{LT}_{\sigma\,\mathrm{Pos}}\{U\} \cap \mathcal{O} = \emptyset$ and thus $\mathcal{O} \subseteq \mathbb{T}^r\langle e_1,\ldots,e_r\rangle \setminus \mathrm{LT}_{\sigma\,\mathrm{Pos}}\{U\} = \mathcal{O}_{\sigma\,\mathrm{Pos}}(U)$. Now the Definitions 2.1.6 and 2.1.7 and $H \subseteq P^r_{\leq\gamma}$ yield $\partial\mathcal{O} \subseteq \overline{\partial\mathcal{O}} \subseteq \overline{\partial\mathcal{O}_{\sigma\,\mathrm{Pos}}(U)} \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1,\ldots,e_r\rangle$. Hence the repeat-until-loop terminates in the case that $\gamma = \gamma'$. Altogether, we see that the procedure is actually an algorithm.

It remains to prove the correctness. As the set $\mathcal{O}$ is computed in line 10 with the use of Algorithm 2, Lemma 2.5.2 yields that $\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1,\ldots,e_r\rangle$, i.e. claim 1) holds. For every index $j \in \{1,\ldots,\nu\}$, we let $g_j = b_je_{\beta_j} - \sum_{i=1}^{\mu} a_{ij}t_ie_{\alpha_i} \in P^r$ with $a_{1j},\ldots,a_{\mu j} \in K$ be as in line 18. Then $G = \{g_1,\ldots,g_\nu\} \subseteq V$ is an $\mathcal{O}$-border prebasis by Definition 2.1.14.

Next we prove 3). Let $j \in \{1, \dots, \nu\}$ and consider the matrix $\widetilde{\mathcal{V}} \in \mathrm{Mat}_{m,s}(K)$ and $\ell_1 e_{u_1}, \dots, \ell_s e_{u_s} \in \mathbb{T}^n_{\leq\gamma}\langle e_1, \dots, e_r\rangle$ as above, again. Write $g_j = c_1\ell_1 e_{u_1} + \cdots + c_s \ell_s e_{u_s}$ with $c_1, \dots, c_s \in K$. Then the vector $(c_1, \dots, c_s) \in K^s$, which corresponds to g_j, represents a K-linear dependency between the terms in the set $\mathbb{T}^n_{\leq\gamma}\langle e_1, \dots, e_r\rangle$. Assume that $\mathrm{LT}_{\sigma\,\mathrm{Pos}}(h_j) = \ell_w e_{u_w} \neq b_j e_{\beta_j}$ for some $w \in \{1, \dots, s\}$. As σ is compatible with $\deg_W$ and as $W = (w_1, \dots, w_n)$ satisfies $w_\ell > 0$ for all $\ell \in \{1, \dots, n\}$, $(c_1, \dots, c_s) \in K^s$ is a vector which has its first non-zero entry in the column corresponding to the term $\ell_w e_{u_w} \in \mathcal{O}$. But this is a contradiction to the construction of the order ideal $\mathcal{O}$ in line 8 of Algorithm 2, i.e. we have $\mathrm{LT}_{\sigma\,\mathrm{Pos}}(g_j) = b_j e_{\beta_j} \in \partial\mathcal{O}$ and claim 3) follows.
Finally, we show claim 2) by proving that the normal remainders of all S-vectors of all neighbors with respect to $\mathcal{O}$ vanish. Let $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ with $i, j \in \{1, \dots, \nu\}$ be neighbors with respect to $\mathcal{O}$. We have already shown that $\overline{\partial\mathcal{O}} \subseteq \mathbb{T}^n_{\leq\gamma}\langle e_1, \dots, e_r\rangle$ at the end of the algorithm. In particular, $G \subseteq \langle \overline{\partial\mathcal{O}}\rangle_K \cap V$. Moreover, by the construction of V in the lines 1, 4, 6, and 7, we have $V \cap x_1 V \cap \cdots \cap x_n V = V$. Since $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are neighbors with respect to $\mathcal{O}$, we hence get $\mathrm{S}(g_i, g_j) \in V$ by Definition 2.4.21. Using the Border Division Algorithm 2.2.1 applied to $\mathrm{S}(g_i, g_j)$ and G, we can compute scalars $c_1, \dots, c_\nu \in K$ such that $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) = \mathrm{S}(g_i, g_j) - \sum_{w=1}^{\nu} c_w g_w \in \langle\mathcal{O}\rangle_K$. Since we have also already seen that $G \subseteq V$, we get $0 = \overline{\mathrm{NR}_G(\mathrm{S}(g_i, g_j))}$ in $P^r_{\leq\gamma}/V$. In particular, as the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form a K-vector space basis of $P^r_{\leq\gamma}/V$, we see that $\mathrm{NR}_G(\mathrm{S}(g_i, g_j)) \in V \cap \langle\mathcal{O}\rangle_K = \{0\}$. Altogether, we see that condition F_2) of Buchberger's Criterion for border bases 2.4.31 holds, i.e. G actually is the $\mathcal{O}$-border basis of $\langle G\rangle$.
Therefore, the claim follows if we show that G generates U. For every $j \in \{1, \dots, \nu\}$, we have already seen that $g_j \in V \subseteq U$, i.e. we have $\langle G\rangle \subseteq U$. For the converse inclusion, we let $w \in \{1, \dots, k\}$. We apply the Border Division Algorithm 2.2.1 to v_w and G to obtain a representation $v_w = v'_w + \mathrm{NR}_G(v_w)$ with $v'_w \in \langle G\rangle$ and $\mathrm{NR}_G(v_w) \in \langle\mathcal{O}\rangle_K$. During the Border Division Algorithm 2.2.1, we always subtract multiples of the form tg_j with $t \in \mathbb{T}^n$ and $j \in \{1, \dots, \nu\}$ to eliminate the term $tb_j e_{\beta_j}$. Since σ is a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$, since $W = (w_1, \dots, w_n)$ satisfies $w_\ell > 0$ for all $\ell \in \{1, \dots, n\}$, and since we have $b_j e_{\beta_j} = \mathrm{LT}_{\sigma\,\mathrm{Pos}}(g_j)$, it follows that all the vectors that are used for these reductions satisfy $tg_j \in P^r_{\leq\gamma}$. Thus we have $v'_w \in V \subseteq P^r_{\leq\gamma}$ because $G \subseteq V$. Altogether, we see that $0 = \overline{v_w} = \overline{\mathrm{NR}_G(v_w)}$ in $P^r_{\leq\gamma}/V$. Since the residue classes of the elements of $\mathcal{O}$ in $P^r_{\leq\gamma}/V$ form a K-vector space basis of $P^r_{\leq\gamma}/V$ by Lemma 2.5.2, it follows that $\mathrm{NR}_G(v_w) \in V \cap \langle\mathcal{O}\rangle_K = \{0\}$. In particular, we get $v_w = v'_w + \mathrm{NR}_G(v_w) = v'_w \in \langle G\rangle$. Therefore, we see that $U = \langle v_1, \dots, v_k\rangle \subseteq \langle G\rangle$. Altogether, we have proven that G is the $\mathcal{O}$-border basis of U. □

Example 2.5.4. Let $P = \mathbb{Q}[x, y]$, $\{e_1, e_2\}$ be the canonical basis of the P-module P^2, $W = (1, 1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$, and $\sigma = \mathtt{DegRevLex}$. Then σ is compatible with $\deg_W$, i.e. compatible with the standard grading. Furthermore, we let $U = \langle v_1, \dots, v_5\rangle \subseteq P^2$ be with the vectors $v_1 = (-2, 3x - 1)$, $v_2 = (3x + 4, 2)$, $v_3 = (0, y - 1)$, $v_4 = (y - 1, 0)$, and $v_5 = (x + y + 1, -x + y)$. Then $xe_2, xe_1, ye_2, ye_1 \in \mathrm{LT}_{\sigma\,\mathrm{Pos}}(U)$. Thus Macaulay's Basis Theorem [KR00, Thm. 1.5.7] yields $\mathrm{codim}_K(U, P^2) = \#\mathcal{O}_{\sigma\,\mathrm{Pos}}(U) \leq \#\{e_1, e_2\} = 2 < \infty$.

In particular, the requirements of the Border Basis Algorithm 3 are satisfied.

In order to illustrate it, we consider the steps of the Border Basis Algorithm 3 applied to the input data $\{v_1, \ldots, v_5\}$, W, and σ in detail. We initialize $V = \langle v_1, \ldots, v_5 \rangle_{\mathbb{Q}}$ in line 1 and thus have $\gamma = 1$ in line 2. Moreover, we compute $V' = (V + xV + yV) \cap P^2_{\leq 1} = V$ with linear algebra techniques in line 4. Thus the while-loop in line 5 does not need to be executed and we have $\{\widetilde{v}_1, \ldots, \widetilde{v}_5\} = \{v_1, \ldots, v_5\}$ in line 9.

Next we consider the computation of $\mathcal{O}$ in line 10 by applying Algorithm 2 to the input data 1, $\{v_1, \ldots, v_5\}$, W, and σ. We order the terms in L according to $\sigma\,\mathrm{Pos}$ decreasingly and compute the matrix

$$\mathcal{V} = \begin{pmatrix} 0 & 3 & 0 & 0 & -2 & -1 \\ 3 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_{5,6}(\mathbb{Q})$$

as in line 6 of Algorithm 2. The (reduced) row echolon form of $\mathcal{V}$ needed in line 7 of Algorithm 2 is then

$$\widetilde{\mathcal{V}} = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{5,6}(\mathbb{Q}),$$

i.e. we get the order ideal $\mathcal{O} = \{e_1, e_2\}$ in $\mathbb{T}^2\langle e_1, e_2\rangle$ after line 10 of Algorithm 3.

As the border of $\mathcal{O}$ satisfies $\partial\mathcal{O} = \{xe_1, xe_2, ye_1, ye_2\} \subseteq \mathbb{T}^2_{\leq 1}\langle e_1, e_2\rangle$, we stop the computation of the repeat-until-loop in line 3. We proceed with the computation of the for-loop in line 16 and get the set $G = \{g_1, \ldots, g_4\} \subseteq P^2$ with $g_1 = xe_1 + \frac{4}{3}e_1 + \frac{2}{3}e_2$, $g_2 = xe_2 - \frac{2}{3}e_1 - \frac{1}{3}e_2$, $g_3 = ye_1 - e_1$, and $g_4 = ye_2 - e_2$. According to Theorem 2.5.3, the set $\mathcal{O}$ is an order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ and G is the $\mathcal{O}$-border basis of U. In particular, since the set of all corners of $\mathcal{O}$ is precisely $\{xe_1, xe_2, ye_1, ye_2\}$ by Definition 2.3.3, the set G is also the reduced $\sigma\,\mathrm{Pos}$-Gröbner basis of U according to Proposition 2.3.5.

Now we are able to give the reason for allowing empty order ideals in Section 2.1.

Remark 2.5.5. In contrast to the theory of border bases as in [KR05, Section 6.4] or [KK05], we have explicitly allowed that order ideals in $\mathbb{T}^n$ may be empty by Definition 2.1.1. The reason for that is as follows: Let $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ be with $w_i > 0$ for all $i \in \{1, \ldots, n\}$, let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$, and let $k \in \mathbb{N} \setminus \{0\}$. Consider vectors $\{v_1, \ldots, v_k\} \subseteq P^r \setminus \{0\}$ such that the P-submodule $U = \langle v_1, \ldots, v_k\rangle \subseteq P^r$ satisfies $\mathrm{codim}_K(U, P^r) < \infty$. Moreover, assume that $r \geq 2$ and $e_1 - e_2 \in U$, i.e. U contains a K-linear dependency between the elements of $\{e_1, \ldots, e_r\}$. Then the resulting order ideal $\mathcal{O}$ in line 10 of the Border Bases Algorithm 2.5.3 applied to $\{v_1, \ldots, v_k\}$, W, and σ does not contain all the elements of $\{e_1, \ldots, e_r\}$, namely the above K-linear dependency yields $e_1 \notin \mathcal{O}$ as $e_1 >_{\sigma\,\mathrm{Pos}} e_2$. By allowing empty order ideals in Definition 2.1.1, this fact does stills occur but the result of the algorithm is still an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ according to Definition 2.1.6.

2.6 Generalized Border Bases

In the Sections 2.1 to 2.5, we have generalized the notion of border bases from the polynomial ring P to free P-modules of finite rank. This was done by generalizing the corresponding concepts of border bases in P. The goal of the final section of this chapter is to establish a border bases theory for arbitrary finitely generated P-modules. A border prebasis in a finitely generated P-module $M = \langle m_1, \ldots, m_r\rangle$ is defined to be the image of a border prebasis in P^r under the P-module epimorphism $P^r \twoheadrightarrow M$, $e_k \mapsto m_k$ for all indices $k \in \{1, \ldots, r\}$ induced by M. Unfortunately, it turns out that we cannot simply refine the theory established in the previous sections but we have to invent a totally different one. The reason is that the image of an order ideal under the above epimorphism does not behave like an order ideal in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ at all. E.g. we will see in Example 2.6.2 that the order ideal and its border can have common elements. Therefore, the basic propositions of Section 2.1 do no longer hold in this general setting. Nevertheless, we can characterize and compute border bases if the corresponding order ideal is not degenerated. More precisely, we do the following.

First we introduce generalized border prebases and border bases in Definition 2.6.3. A generalized border prebasis is the image of a border prebasis in P^r and a generalized border basis has the additional property that the residue classes of the images of the elements of the order ideal form a K-vector space basis of the corresponding residue class module. Then we introduce the notion of characterizing order ideals and characterizing border prebases in Definition 2.6.5. If the generalized order ideal of a given generalized border prebasis is not degenerated, we can use the corresponding characterizing border prebasis to characterize the given generalized border basis in Theorem 2.6.8. This is the key result of this section and yields a characterization of generalized border bases via the characterizations introduced in Section 2.4 in Corollary 2.6.10. Moreover, if we know the kernel of the P-module epimophism induced by the finitely generated P-module in which we want to establish a border bases theory, we can even compute generalized border bases according to Corollary 2.6.12. Finally, we apply the theory of generalized border bases to subideal border bases in Example 2.6.13. Subideal border bases were introduced in [KP11] and, by now only a characterization via a special generation property and an algorithm that uses Gröbner bases techniques for their computation is known. Our new theory allows us to characterize subideal border bases in various other ways and to compute them with linear algebra techniques.

For the whole section, we let $M = \langle m_1, \ldots, m_r\rangle$ be a finitely generated P-module and we let $\varphi : P^r \twoheadrightarrow M$, $e_k \mapsto m_k$ for all $k \in \{1, \ldots, r\}$ be the corresponding P-module epimophism.

First we introduce the notion of generalized order ideals as images of order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$.

Definition 2.6.1. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be order ideals in $\mathbb{T}^n$ and $\mathcal{O} = \mathcal{O}_1 e_1 \cup \dots \cup \mathcal{O}_r e_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$.

a) We call the set $\varphi(\mathcal{O}) = \mathcal{O}_1 \cdot \varphi(e_1) \cup \dots \cup \mathcal{O}_r \cdot \varphi(e_r) \subseteq M$ a **(generalized) order ideal** in $\varphi(\mathbb{T}^n\langle e_1, \dots, e_r\rangle)$.

b) The set $\partial\varphi(\mathcal{O}) = \varphi(\partial\mathcal{O}) = \partial\mathcal{O}_1 \cdot \varphi(e_1) \cup \dots \cup \partial\mathcal{O}_r \cdot \varphi(e_r) \subseteq M$ is called the **(first) border** of $\varphi(\mathcal{O})$.

The following example shows that generalized order ideals do not behave like order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ at all.

Example 2.6.2. Let $P = \mathbb{Q}[x, y]$, let $\{e_1, e_2\}$ be the canonical P-module basis of P^2, and let $S = \langle xye_2 - xe_1\rangle \subseteq P^2$. Consider the P-module $M = P^2/S$ and the corresponding canonical P-module epimorphism $\varphi : P^2 \twoheadrightarrow M$, $e_1 \mapsto \bar{e}_1$, $e_2 \mapsto \bar{e}_2$. Additionally, let $\mathcal{O}_1 = \{1, x\} \subseteq \mathbb{T}^2$, $\mathcal{O}_2 = \{1, y, y^2\} \subseteq \mathbb{T}^2$, and $\mathcal{O} = \mathcal{O}_1 e_1 \cup \mathcal{O}_2 e_2 \subseteq \mathbb{T}^2\langle e_1, e_2\rangle$. Then the set $\varphi(\mathcal{O}) = \mathcal{O}_1 \cdot \bar{e}_1 \cup \mathcal{O}_2 \cdot \bar{e}_2 = \{\bar{e}_1, x\bar{e}_1, \bar{e}_2, y\bar{e}_2, y^2\bar{e}_2\}$ is an order ideal in $\varphi(\mathbb{T}^2\langle e_1, e_2\rangle)$ with the border $\partial\varphi(\mathcal{O}) = \partial\mathcal{O}_1 \cdot \bar{e}_1 \cup \partial\mathcal{O}_2 \cdot \bar{e}_2 = \{y\bar{e}_1, xy\bar{e}_1, x^2\bar{e}_1, x\bar{e}_2, xy\bar{e}_2, xy^2\bar{e}_2, y^3\bar{e}_2\}$ according to Definition 2.6.1.

Since $x\bar{e}_1 = xy\bar{e}_2$ in M, we see that $x\bar{e}_1 = xy\bar{e}_2 \in \varphi(\mathcal{O}) \cap \partial\varphi(\mathcal{O})$, i.e. an analogous version of Proposition 2.1.10 does not hold true. Moreover, generalized order ideals in $\varphi(\mathbb{T}^n\langle e_1, \dots, e_r\rangle)$ are not closed under forming divisors in contrast to order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$, cf. Definition 2.1.6, since $y \cdot x\bar{e}_2 = xy\bar{e}_2 = x\bar{e}_1 \in \varphi(\mathcal{O})$ but $x\bar{e}_2 \notin \varphi(\mathcal{O})$.

Now we are able introduce generalized border basis.

Definition 2.6.3. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be finite order ideals in $\mathbb{T}^n$ and $\mathcal{O} = \mathcal{O}_1 e_1 \cup \dots \cup \mathcal{O}_r e_r$ be the corresponding order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$. We write $\mathcal{O} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$ and the border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$ with $\mu, \nu \in \mathbb{N}$, $t_i, b_j \in \mathbb{T}^n$, and $\alpha_i, \beta_j \in \{1, \dots, r\}$ for all $i \in \{1, \dots, \mu\}$ and for all $j \in \{1, \dots, \nu\}$. Moreover, we let $G = \{g_1, \dots, g_\nu\} \subseteq P^r$ be an $\mathcal{O}$-border prebasis with $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$ where $a_{1j}, \dots, a_{\mu j} \in K$ for all indices $j \in \{1, \dots, \nu\}$.

a) The set $\varphi(G) = \{\varphi(g_1), \dots, \varphi(g_\nu)\} \subseteq M$, where $\varphi(g_j) = b_j m_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i m_{\alpha_i}$ for all $j \in \{1, \dots, \nu\}$, is called a **(generalized) $\varphi(\mathcal{O})$-border prebasis**.

b) Let $U \subseteq M$ be a P-submodule. The $\varphi(\mathcal{O})$-border prebasis $\varphi(G) \subseteq M$ is called an **(generalized) $\varphi(\mathcal{O})$-border basis** of U if $\varphi(G) \subseteq U$ and if the residue classes of the elements of $\varphi(\mathcal{O})$ in M/U form a K-vector space basis of M/U.

One might think that all the definitions and propositions about border bases in free P-modules of the Sections 2.1 to 2.5 can be generalized to generalized border bases in arbitrary finitely generated P-modules in a straightforward way. Unfortunately, the situation is more complicated than expected. We have already seen one big difference concerning border bases and generalized border bases in Example 2.6.2. Namely we have seen that it can happen that a generalized order ideal and its border have some elements in common, i.e. the straightforward, analogous version of Proposition 2.1.10 is wrong, and that generalized order ideals are not closed under forming divisors. Since most of the propositions of the Sections 2.1 to 2.5 are based upon these properties, the theory of generalized border bases needs much more care in the definitions and proofs. Nevertheless, some results can be extended to generalized border bases. The following proposition shows that generalized border bases are unique just as it was the case for border bases in free P-modules in Proposition 2.3.2.

Proposition 2.6.4. (Uniqueness of Generalized Border Bases)
Let $\mathcal{O}^\varphi$ be a finite order ideal in $\varphi(\mathbb{T}^n\langle e_1,\dots,e_r\rangle)$, let $U \subseteq M$ be a P-submodule, and let $G^\varphi, G'^\varphi \subseteq M$ be two $\mathcal{O}^\varphi$-border bases of U. Then we have $G^\varphi = G'^\varphi$.

Proof. Write $\mathcal{O}^\varphi = \{t_1 m_{\alpha_1},\dots,t_\mu m_{\alpha_\mu}\}$ and $\partial\mathcal{O}^\varphi = \{b_1 m_{\beta_1},\dots,b_\nu m_{\beta_\nu}\}$ with $\mu,\nu \in \mathbb{N}$, terms $t_i, b_j \in \mathbb{T}^n$, and $\alpha_i, \beta_j \in \{1,\dots,r\}$ for all $i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$. Then the $\mathcal{O}^\varphi$-border bases G^φ and G'^φ are of the form $G^\varphi = \{g_1^\varphi,\dots,g_\nu^\varphi\} \subseteq M$ with $g_j^\varphi = b_j m_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i m_{\alpha_i}$ where $a_{1j},\dots,a_{\mu j} \in K$ for all $j \in \{1,\dots,\nu\}$, and of the form $G'^\varphi = \{g_1'^\varphi,\dots,g_\nu'^\varphi\} \subseteq M$ with $g_j'^\varphi = b_j m_{\beta_j} - \sum_{i=1}^{\mu} a'_{ij} t_i m_{\alpha_i}$ where $a'_{1j},\dots,a'_{\mu j} \in K$ or all $j \in \{1,\dots,\nu\}$ by Definition 2.6.3. Assume that $a_{ij} \neq a'_{ij}$ for some $i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$. As $G^\varphi, G'^\varphi \subseteq U$ according to Definition 2.6.1, we then see that $0 = \overline{g_j^\varphi} - \overline{g_j'^\varphi} = \sum_{k=1}^{\mu}(-a_{kj} + a'_{kj})\overline{t_k m_{\alpha_k}}$ in M/U. Since G is an $\mathcal{O}^\varphi$-border basis of U, Definition 2.6.3 yields the contradiction $a_{ij} = a'_{ij}$. Thus the claim follows. □

Next we introduce the notion of characterizing order ideals and characterizing border prebases. These exist if a given generalized order ideal is not degenerated and will later turn out to be very useful to characterize generalized border bases.

Definition 2.6.5. Let $G^\varphi \subseteq M$ be an $\mathcal{O}^\varphi$-border prebasis. We write the order ideal $\mathcal{O}^\varphi = \{t_1 m_{\alpha_1},\dots,t_\mu m_{\alpha_\mu}\} \subseteq M$ in $\varphi(\mathbb{T}^n\langle e_1,\dots,e_r\rangle)$ and the corresponding border $\partial\mathcal{O}^\varphi = \{b_1 m_{\beta_1},\dots,b_\nu m_{\beta_\nu}\} \subseteq M$ with $\mu,\nu \in \mathbb{N}$ and $t_i, b_j \in \mathbb{T}^n$ and $\alpha_i, \beta_j \in \{1,\dots,r\}$ for all $i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$. Moreover, we write $G^\varphi = \{g_1^\varphi,\dots,g_\nu^\varphi\}$ with $g_j^\varphi = b_j m_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} m_{\alpha_i}$ where $a_{1j},\dots,a_{\mu j} \in K$ for all $j \in \{1,\dots,\nu\}$.

a) An order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ is said to **characterize** $\mathcal{O}^\varphi$ if $\varphi(\mathcal{O}) = \mathcal{O}^\varphi$ and if the restriction $\varphi|_\mathcal{O}$ of φ to $\mathcal{O}$ is injective.

b) Let $\mathcal{O} \subseteq \mathbb{T}^n\langle e_1,\dots,e_r\rangle$ be an order ideal that characterizes $\mathcal{O}^\varphi$. By choosing suitable preimages and reordering the elements of $\mathcal{O}^\varphi$ and $\partial\mathcal{O}^\varphi$, we can without loss of generality assume that we have $\mathcal{O} = \{t_1 e_{\alpha_1},\dots,t_\mu e_{\alpha_\mu}\}$ and that $\partial\mathcal{O}$ has the form $\partial\mathcal{O} = \{b_1 e_{\beta_1},\dots,b_\nu e_{\beta_\nu}, b_{\nu+1} e_{\beta_{\nu+1}},\dots,b_\omega e_{\beta_\omega}\}$ with $\omega \in \mathbb{N}$, $\omega \geq \nu$, and $b_j \in \mathbb{T}^n$ and $\beta_j \in \{1,\dots,r\}$ for all $j \in \{\nu+1,\dots,\omega\}$. For all $j \in \{1,\dots,\nu\}$, we define $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i} \in P^r$. For all $j \in \{\nu+1,\dots,\omega\}$, there exists a unique index $k \in \{1,\dots,\nu\}$ such that $b_j m_{\beta_j} = \varphi(b_j e_{\beta_j}) = \varphi(b_k e_{\beta_k}) = b_k m_{\beta_k}$ according to Definition 2.6.1 and we define $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\nu} a_{ik} t_i e_{\alpha_i} \in P^r$. We say that the $\mathcal{O}$-border prebasis $G = \{g_1,\dots,g_\omega\} \subseteq P^r$ **characterizes** G^φ.

Example 2.6.6. Consider the generalized order ideal $\mathcal{O}^\varphi = \varphi(\mathcal{O})$ of Example 2.6.2, again. Recall that $P = \mathbb{Q}[x, y]$, that $\{e_1, e_2\}$ denoted the canonical P-module basis of P^2, that we had $\varphi : P^2 \twoheadrightarrow M$, $e_1 \mapsto \bar{e}_1$, $e_2 \mapsto \bar{e}_2$ where we had $M = P^2/S$ with $S = \langle xye_2 - xe_1 \rangle \subseteq P^2$, and that the generalized order ideal in $\varphi(\mathbb{T}^2\langle e_1, e_2\rangle)$ was of the form $\mathcal{O}^\varphi = \{1, x\} \cdot \bar{e}_1 \cup \{1, y, y^2\} \cdot \bar{e}_2$. Moreover, since $xy^2\bar{e}_2 = xy\bar{e}_1$ in M, the border of $\mathcal{O}^\varphi$ is $\partial\mathcal{O}^\varphi = \{y, xy, x^2\} \cdot \bar{e}_1 \cup \{x, xy, xy^2, y^3\} \cdot \bar{e}_2 = \{y\bar{e}_1, xy\bar{e}_1, x^2\bar{e}_1, x\bar{e}_2, xy\bar{e}_2, y^3\bar{e}_2\}$. Then the set $G^\varphi = \{g_1^\varphi, \ldots, g_6^\varphi\} \subseteq M$ with $g_1^\varphi = y\bar{e}_1 - \bar{e}_1 - \bar{e}_2$, $g_2^\varphi = xy\bar{e}_1 - y\bar{e}_2$, $g_3^\varphi = x^2\bar{e}_1 - x\bar{e}_1 + \bar{e}_2$, $g_4^\varphi = x\bar{e}_2 - \bar{e}_2$, $g_5^\varphi = xy\bar{e}_2 + \bar{e}_1$, and $g_6^\varphi = y^3\bar{e}_2 - x\bar{e}_1 + y\bar{e}_2$ is an $\mathcal{O}^\varphi$-border prebasis according to Definition 2.6.3.

Let $\mathcal{O} = \{1, x\} \cdot e_1 \cup \{1, y, y^2\} \cdot e_2$. Then $\mathcal{O}$ is an order ideal in $\mathbb{T}^2\langle e_1, e_2\rangle$ that characterizes $\mathcal{O}^\varphi$ by Definition 2.6.5. Its is $\partial\mathcal{O} = \{ye_1, xye_1, x^2e_1, xe_2, xye_2, xy^2e_2, y^3e_2\}$ Furthermore, as we have $x^2y\bar{e}_2 = xy\bar{e}_1$ in M, the set $G = \{g_1, \ldots, g_7\} \subseteq P^2$ with $g_1 = ye_1 - e_1 - e_2$, $g_2 = xye_1 - ye_2$, $g_3 = x^2e_1 - xe_1 + e_2$, $g_4 = xe_2 - e_2$, $g_5 = xye_2 + e_1$, $g_6 = y^3e_2 - xe_1 + ye_2$, and $g_7 = xy^2e_2 - ye_2$ is the $\mathcal{O}$-border prebasis characterizing G^φ by Definition 2.6.5. Note that $\#G^\varphi = 6 < 7 = \#G$ and that the construction yields $\varphi(g_7) = xy^2\bar{e}_2 - y\bar{e}_2 = xy\bar{e}_1 - y\bar{e}_2 = \varphi(g_2) = g_2^\varphi$.

Before we give the main proof of this section, we need an auxiliary lemma.

Lemma 2.6.7. *Let $v_1, \ldots, v_k \in P^r$ be with $k \in \mathbb{N}$ and let $v_\ell^\varphi = \varphi(v_\ell) \subseteq M$ be for all $\ell \in \{1, \ldots, k\}$. Furthermore, let $U = \langle v_1, \ldots, v_k\rangle \subseteq P^r$ and $U^\varphi = \langle v_1^\varphi, \ldots, v_k^\varphi\rangle \subseteq M$. Then $\varphi(U) = U^\varphi$ and $\varphi^{-1}(U^\varphi) = U + \ker(\varphi)$. In particular, if $G^\varphi \subseteq M$ is an $\mathcal{O}^\varphi$-border prebasis and there exists an $\mathcal{O}$-border prebasis $G \subseteq P^r$ characterizing G^φ, then we have $\varphi(\langle G\rangle) = \langle G^\varphi\rangle$ and $\varphi^{-1}(\langle G^\varphi\rangle) = \langle G\rangle + \ker(\varphi)$.*

Proof. Since the definitions of U and U^φ immediately yield the first equality $\varphi(U) = U^\varphi$, it suffices to prove the second equality $\varphi^{-1}(U^\varphi) = U + \ker(\varphi)$. For the first inclusion, we let $v = p_1v_1 + \cdots + p_kv_k + w$ be with $p_1, \ldots, p_k \in P$ and $w \in \ker(\varphi)$. Then we see that $\varphi(v) = p_1v_1^\varphi + \cdots + p_kv_k^\varphi + \varphi(w) = p_1v_1^\varphi + \cdots + p_kv_k^\varphi \in U^\varphi$. For the converse inclusion, we let $w \in \varphi^{-1}(U^\varphi) \subseteq P^r$. Then there exist polynomials $p_1, \ldots, p_k \in P$ such that $\varphi(w) = p_1v_1^\varphi + \cdots + p_kv_k^\varphi = \varphi(p_1v_1 + \cdots + p_kv_k)$. Thus $w - (p_1v_1 + \cdots + p_kv_k) \in \ker(\varphi)$ and this yields $w \in U + \ker(\varphi)$. The other claims are a direct consequence of this. $\square$

Now we have all ingredients to prove the main result of this section. Under the assumption that the generalized order ideal $\mathcal{O}^\varphi$ is not degenerated, i.e. that there is an order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ that characterizes $\mathcal{O}^\varphi$, we prove that a given generalized $\mathcal{O}^\varphi$-border prebasis is a generalized $\mathcal{O}^\varphi$-border basis if and only if its characterizing $\mathcal{O}$-border prebasis is an $\mathcal{O}$-border basis which generates a submodule that contains the kernel of φ.

Theorem 2.6.8. (Characterization of Generalized Border Bases)
Let $G^\varphi \subseteq M$ be an $\mathcal{O}^\varphi$-border prebasis and assume that there exists an $\mathcal{O}$-border prebasis $G \subseteq P^r$ that characterizes G^φ. Then the following conditions are equivalent.

i) The $\mathcal{O}^\varphi$-border prebasis G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi\rangle$.

ii) The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G \rangle$ and $\ker(\varphi) \subseteq \langle G \rangle$.

Proof. Let $\mathcal{O} = \{t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\} \subseteq \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ and $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_\omega e_{\beta_\omega}\}$, let $\mathcal{O}^\varphi = \{t_1 m_{\alpha_1}, \ldots, t_\mu m_{\alpha_\mu}\} \subseteq M$ and $\partial\mathcal{O}^\varphi = \{b_1 m_{\beta_1}, \ldots, b_\nu m_{\beta_\nu}\}$ be with $\mu, \nu, \omega \in \mathbb{N}$, $\nu \leq \omega$, and $t_i, b_j \in \mathbb{T}^n$ and $\alpha_i, \beta_j \in \{1, \ldots, r\}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \omega\}$, and let $G = \{g_1, \ldots, g_\omega\} \subseteq P^r$ and $G^\varphi = \{g_1^\varphi, \ldots, g_\nu^\varphi\} \subseteq M$ be with $\varphi(g_j) = g_j^\varphi$ for all $j \in \{1, \ldots, \nu\}$ and $\varphi(g_k) \in G^\varphi$ for all $k \in \{\nu + 1, \ldots, \omega\}$ as in Definition 2.6.5.

First we prove that $\ker(\varphi) \subseteq \langle G \rangle$ if G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle$. Suppose that G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle$. Assume that there exists a $v \in \ker(\varphi) \setminus \langle G \rangle$. We apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = p_1 g_1 + \cdots + p_\omega g_\omega + c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with $p_1, \ldots, p_\omega \in P$ and $c_1, \ldots, c_\mu \in K$. Since $v \notin \langle G \rangle$, there exists an $i \in \{1, \ldots, \mu\}$ such that $c_i \neq 0$. Moreover, Definition 2.6.5 and Lemma 2.6.7 yield $0 = \overline{\varphi(v)} = c_1 \overline{t_1 m_{\alpha_1}} + \cdots + c_\mu \overline{t_\mu m_{\alpha_\mu}}$ in $M/\langle G^\varphi \rangle$. As $\varphi(\mathcal{O}) = \mathcal{O}^\varphi$ and $\varphi|_\mathcal{O}$ is injective according to Definition 2.6.5 and as $c_i \neq 0$, the residue classes of the elements of $\mathcal{O}^\varphi$ in $M/\langle G^\varphi \rangle$ are K-linearly dependent in contradiction to Definition 2.6.3. Altogether, we have proven that $\ker(\varphi) \subseteq \langle G \rangle$.

Second we prove the claimed equivalence. Suppose that $\ker(\varphi) \subseteq \langle G \rangle$ holds. Then Lemma 2.6.7 yields that $\varphi^{-1}(\langle G^\varphi \rangle) = \langle G \rangle$ and hence φ induces a P-module isomorphism $P^r/\langle G \rangle = P^r/\varphi^{-1}(\langle G^\varphi \rangle) \cong \varphi(P^r)/\langle G^\varphi \rangle = M/\langle G^\varphi \rangle$ according to the Second Noetherian Isomorphism Theorem. As $\varphi(\mathcal{O}) = \mathcal{O}^\varphi$ and as $\varphi|_\mathcal{O}$ is injective by Definition 2.6.5, the Definitions 2.1.14 and 2.6.3 yield that G is the $\mathcal{O}$-border basis of $\langle G \rangle$ if and only if G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle$. □

Example 2.6.9. Let $P = \mathbb{Q}[x, y]$ and $\{e_1, e_2\}$ be the canonical P-module basis of P^2. Furthermore, let $s = (x + y + 1, -x + y) \in P^2$, $M = P^2/\langle s \rangle$, and $\varphi : P^2 \twoheadrightarrow M$, $e_1 \mapsto \overline{e}_1$, $e_2 \mapsto \overline{e}_2$ be the corresponding canonical P-module epimorphism. Consider the order ideal $\mathcal{O} = \{e_1, e_2\}$ in $\mathbb{T}^2\langle e_1, e_2\rangle$ and the $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_4\} \subseteq P^2$ with $g_1 = xe_1 + \frac{4}{3}e_1 + \frac{2}{3}e_2$, $g_2 = xe_2 - \frac{2}{3}e_1 - \frac{1}{3}e_2$, $g_3 = ye_1 - e_1$, and $g_4 = ye_2 - e_2$. In Example 2.5.4, we have shown that G is the $\mathcal{O}$-border basis of $U = \langle v_1, v_2, v_3, v_4, s\rangle \subseteq P^2$ where $v_1 = (-2, 3x-1)$, $v_2 = (3x+4, 2)$, $v_3 = (0, y-1)$, and $v_4 = (y-1, 0)$. In particular, we have $U = \langle G \rangle$ by Corollary 2.2.4.

Let $\mathcal{O}^\varphi = \varphi(\mathcal{O}) = \{\overline{e}_1, \overline{e}_2\} \subseteq M$ and let $G^\varphi = \{g_1^\varphi, \ldots, g_4^\varphi\} = \varphi(G) \subseteq M$ be with $g_1^\varphi = \varphi(g_1) = x\overline{e}_1 + \frac{4}{3}\overline{e}_1 + \frac{2}{3}\overline{e}_2$, $g_2^\varphi = \varphi(g_2) = x\overline{e}_2 - \frac{2}{3}\overline{e}_1 - \frac{1}{3}\overline{e}_2$, $g_3^\varphi = \varphi(g_3) = y\overline{e}_1 - \overline{e}_1$, $g_4^\varphi = \varphi(g_4) = y\overline{e}_2 - \overline{e}_2$. Then we see that $\mathcal{O}$ is an order ideal characterizing $\mathcal{O}^\varphi$ and G is the $\mathcal{O}$-border prebasis characterizing the $\mathcal{O}^\varphi$-border prebasis G^φ by Definition 2.6.5. As we also have $\ker(\varphi) = \langle s \rangle \subseteq \langle G \rangle$ and as G is the $\mathcal{O}$-border basis of $\langle G \rangle$, Theorem 2.6.8 yields that G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle = \langle \varphi(G) \rangle = \langle G \rangle / \langle s \rangle$.

As a first corollary, we can apply the characterizations of Section 2.4 to generalized border bases.

Corollary 2.6.10. (Characterizations of Generalized Border Bases)
Let $G^\varphi \subseteq M$ be an $\mathcal{O}^\varphi$-border prebasis and assume that there exists an $\mathcal{O}$-border prebasis $G \subseteq P^r$ that characterizes G^φ. Then G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle$ if and only if $\ker(\varphi) \subseteq \langle G \rangle$ *and one of the following conditions is satisfied.*

A) The $\mathcal{O}$-border prebasis G has the special generation property of Theorem 2.4.1.

B) The border form module $\mathrm{BF}_{\mathcal{O}}(\langle G\rangle)$ satisfies the conditions of Theorem 2.4.5.

C) The rewrite rule $\xrightarrow{G}$ defined by G satisfies the conditions of Theorem 2.4.13.

D) The formal multiplication matrices with respect to G are pairwise commuting, cf. Theorem 2.4.19.

E) The border syszygies with respect to $\mathcal{O}$ can be lifted, cf. Theorem 2.4.26.

F) The $\mathcal{O}$-border prebasis G satisfies Buchberger's Criterion for border bases, cf. Theorem 2.4.31.

Proof. These equivalences are a direct consequence of Theorem 2.6.8 and the corresponding characterizations in Section 2.4. □

The assumption of Theorem 2.6.8 that there exists an $\mathcal{O}$-border prebasis G that characterizes the $\mathcal{O}^\varphi$-border prebasis G^φ, i.e. the existence of an order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1,\dots,e_r\rangle$ that characterizes the order ideal $\mathcal{O}^\varphi$ in $\varphi(\mathbb{T}^n\langle e_1,\dots,e_r\rangle)$ according to Definition 2.6.5, cannot be omitted as the following example shows.

Example 2.6.11. Let $P=\mathbb{Q}[x,y]$, $\{e_1,e_2\}$ be the canonical P-module basis of P^2, $M=P^2/S$ be with $S=\langle xe_1-ye_2\rangle\subseteq P^2$, and $\varphi: P^2\twoheadrightarrow M$, $e_1\mapsto\bar{e}_1$, $e_2\mapsto\bar{e}_2$. Consider the order ideal $\mathcal{O}^\varphi=\{1,x,x^2\}\cdot\bar{e}_1\cup\{1,y,y^2\}\cdot\bar{e}_2=\{1\}\cdot\bar{e}_1\cup\{1,y,y^2,xy\}\cdot\bar{e}_2\subseteq M$. Since $x\bar{e}_1=y\bar{e}_2\in\mathcal{O}^\varphi$, we have $\#\mathcal{O}^\varphi=5$. Moreover, we see that $x^2\bar{e}_1=xy\bar{e}_2\in\mathcal{O}^\varphi$.

Assume that there exists an order ideal $\mathcal{O}$ in $\mathbb{T}^2\langle e_1,e_2\rangle$ that characterizes $\mathcal{O}^\varphi$. Since we have $y\bar{e}_1,x^3\bar{e}_1,x\bar{e}_2,y^3\bar{e}_2\notin\mathcal{O}^\varphi$, we get $ye_1,x^3e_1,xe_2,y^3e_2\notin\mathcal{O}$. Therefore, we see that $\mathcal{O}\subseteq\{1,x,x^2\}\cdot e_1\cup\{1,y,y^2\}\cdot e_2$. As $\varphi|_{\mathcal{O}}$ is injective by Definition 2.6.5 and as $\varphi(xe_1)=\varphi(ye_2)$, we have $xe_1\notin\mathcal{O}$ or $ye_2\notin\mathcal{O}$ and thus $\#\mathcal{O}\le 4$ by Definition 2.1.6. In particular, we get the contradiction $5=\#\mathcal{O}^\varphi=\#\mathcal{O}\le 4$.

Thus there exists no order ideal in $\mathbb{T}^2\langle e_1,e_2\rangle$ that characterizes $\mathcal{O}^\varphi$. In particular, we see that we cannot apply the characterization of Theorem 2.6.8 to arbitrary $\mathcal{O}^\varphi$-border prebases.

Another consequence of Theorem 2.6.8 is that it allows us to compute generalized border bases if we can compute the kernel of φ.

Corollary 2.6.12. (Computation of Generalized Border Bases)
Let $U^\varphi=\langle v_1^\varphi,\dots,v_k^\varphi\rangle\subseteq M$ with $k\in\mathbb{N}$ and $v_1^\varphi,\dots,v_k^\varphi\in M\setminus\{0\}$ be a P-submodule such that $\mathrm{codim}_K(U^\varphi,M)<\infty$ and for all $j\in\{1,\dots,k\}$, let $v_j\in P^r$ be such that $\varphi(v_j)=v_j^\varphi$. Moreover, let $\ker(\varphi)=\langle\kappa_1,\dots,\kappa_\ell\rangle\subseteq P^r$ be with $\ell\in\mathbb{N}$ and $\kappa_1,\dots,\kappa_\ell\in P^r\setminus\{0\}$. Use Algorithm 3 to compute an $\mathcal{O}$-border basis $G\subseteq P^r$ of $\langle v_1,\dots,v_k\rangle+\langle\kappa_1,\dots,\kappa_\ell\rangle$. Then $\varphi(G)\subseteq M$ is the $\varphi(\mathcal{O})$-border basis of U^φ.

Note that we can use any matrix $W=(w_1,\dots,w_n)\in\mathrm{Mat}_{1,n}(\mathbb{Z})$ with the property that $w_i>0$ for all $i\in\{1,\dots,n\}$ and any term ordering σ on $\mathbb{T}^n$ that is compatible with $\deg_W$

for the computation of G, e.g. we can use the standard grading defined by $W = (1, \ldots, 1)$ and $\sigma =$ `DegRevLex`.

Proof. Let $U = \langle v_1, \ldots, v_k \rangle + \langle w_1, \ldots, w_\ell \rangle = \langle v_1, \ldots, v_k \rangle + \ker(\varphi) \subseteq P^r$, let $\mathcal{O}^\varphi = \varphi(\mathcal{O})$, and let $G^\varphi = \varphi(G)$. By the Definitions 2.6.1 and 2.6.3, G^φ is an $\mathcal{O}^\varphi$-border prebasis.

First we prove that we can use Algorithm 3 to compute the $\mathcal{O}$-border basis of U, i.e. that $\operatorname{codim}_K(U, P^r) < \infty$. Since $\varphi^{-1}(U^\varphi) = \langle v_1, \ldots, v_k \rangle + \ker(\varphi) = U$ by Lemma 2.6.7, it follows that $P^r/U = P^r/\varphi^{-1}(U^\varphi) \cong \varphi(P^r)/U^\varphi = M/U^\varphi$ by the Second Noetherian Isomorphism Theorem. In particular, we get $\operatorname{codim}_K(U, P^r) = \operatorname{codim}_K(U^\varphi, M) < \infty$ and the requirements of Algorithm 3 are satisfied.

Next we show that $\mathcal{O}$ characterizes $\mathcal{O}^\varphi$. As U is generated by G according to Corollary 2.2.4, Lemma 2.6.7 shows that $U^\varphi = \varphi(U) = \varphi(\langle G \rangle) = \langle \varphi(G) \rangle = \langle G^\varphi \rangle$. We write $\mathcal{O} = \{t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\}$ with $\mu \in \mathbb{N}$, $t_1, \ldots, t_\mu \in \mathbb{T}^n$, and $\alpha_1, \ldots, \alpha_\mu \in \{1, \ldots, r\}$. Let $i, j \in \{1, \ldots, \mu\}$ be such that $\varphi(t_i e_{\alpha_i}) = \varphi(t_j e_{\alpha_j})$. According to the definition of U, we then get $t_i e_{\alpha_i} - t_j e_{\alpha_j} \in \ker(\varphi) \cap \langle \mathcal{O} \rangle_K \subseteq U \cap \langle \mathcal{O} \rangle_K$. As G is the $\mathcal{O}$-border basis of U, Corollary 2.2.6 yields $t_i e_{\alpha_i} = t_j e_{\alpha_j}$. Hence the restriction $\varphi|_{\mathcal{O}}$ is injective and thus $\mathcal{O}$ characterizes $\mathcal{O}^\varphi = \varphi(\mathcal{O})$ by Definition 2.6.5.

Finally, we show that G characterizes $G^\varphi \subseteq M$. Therefore, let $b_1, \ldots, b_\omega \in \mathbb{T}^n$ and $\beta_1, \ldots, \beta_\omega \in \{1, \ldots, r\}$ with $\omega \in \mathbb{N}$ be such that $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_\omega e_{\beta_\omega}\}$. Additionally, we write $G = \{g_1, \ldots, g_\omega\}$ with $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$ where $a_{1j}, \ldots, a_{\mu j} \in K$ for all $j \in \{1, \ldots, \omega\}$. Let $k, \ell \in \{1, \ldots, \nu\}$ be indices such that $\varphi(b_k e_{\beta_k}) = \varphi(b_\ell e_{\beta_\ell})$. Then $b_k e_{\beta_k} - b_\ell e_{\beta_\ell} \in \ker(\varphi) \subseteq U$ and $\sum_{i=1}^{\mu} (a_{ik} - a_{i\ell}) t_i e_{\alpha_i} = (b_k e_{\beta_k} - b_\ell e_{\beta_\ell}) - (g_k - g_\ell) \in U$. As G is the $\mathcal{O}$-border basis of U, this implies $a_{ik} = a_{i\ell}$ for all $i \in \{1, \ldots, \mu\}$ by Definition 2.1.14. Altogether, we see that G is the $\mathcal{O}$-border prebasis that characterizes G^φ according to Definition 2.6.5. In particular, as G is the $\mathcal{O}$-border basis of U, G^φ is the $\mathcal{O}^\varphi$-border basis of $\langle G^\varphi \rangle = U^\varphi$ by Theorem 2.6.8. $\square$

In [KP11], border bases of zero-dimensional polynomial ideals were generalized to border bases of zero-dimensional polynomial ideals that are contained in another ideal, a so-called subideal. In the following example, we show that these subideal border bases are special cases of generalized border bases. It turns out that our methods allow us to characterize subideal border bases in various ways and that we can compute them with linear algebra techniques. By now the only characterization of subideal border bases was proven in [KP11, Cor. 3.6] and characterized subideal border bases via a special generation property. Moreover, the only general approach to compute subideal border bases was described in [KP11, Section 6] and needs one syzygy computation to determine the kernel of a certain P-linear map and one Gröbner basis computation. Our method is based on the same syzygy computation. But after that, we use linear algebra techniques instead of a Gröbner basis computation.

Example 2.6.13. (Application to Subideal Border Bases)
Let $I = \langle p_1, \ldots, p_k \rangle \subseteq P$ where $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in P \setminus \{0\}$ be a zero-dimensional ideal and let $J = \langle F \rangle \subseteq P$ where $F = \{f_1, \ldots, f_\ell\}$ with $\ell \in \mathbb{N}$ and $f_1, \ldots, f_\ell \in P \setminus \{0\}$ be another ideal. We let $\{e_1, \ldots, e_{k+\ell}\}$ be the canonical P-module basis of $P^{k+\ell}$ and let $\varphi : P^{k+\ell} \twoheadrightarrow I + J$ be the P-module epimorphism defined by $e_i \mapsto p_i$ for every

$i \in \{1, \ldots, k\}$ and by $e_i \mapsto f_{i-k}$ for every $i \in \{k+1, \ldots, k+\ell\}$, which is induced by the Universal Property of the Free Module $P^{k+\ell}$. According to [KP11, Defn. 2.1], the ideal I is said to have an $\mathcal{O}_F$-subideal border basis if there are order ideals $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ in $\mathbb{T}^n$ such that the residue classes of the elements of $\mathcal{O}_F = \mathcal{O}_1 \cdot f_1 \cup \cdots \cup \mathcal{O}_\ell \cdot f_\ell$ in $J/I \cap J$ form a K-vector space basis of $J/I \cap J$. Moreover, in this situation, we see that $\mathcal{O}_F = \mathcal{O}_1 \cdot \varphi(e_{k+1}) \cup \cdots \cup \mathcal{O}_\ell \cdot \varphi(e_{k+\ell})$, i.e. $\mathcal{O}_F$ is an order ideal in $\varphi(\mathbb{T}^n\langle e_1, \ldots, e_{k+\ell}\rangle)$ according to Definition 2.6.1. By the First Noetherian Isomorphism Theorem, there is a canonical K-algebra isomorphism $J/I \cap J \cong I + J/I$ and thus an $\mathcal{O}_F$-subideal border basis of I as defined in [KP11, Defn. 2.1] is nothing but a generalized $\mathcal{O}_F$-border basis of $I \subseteq I + J$ as defined in Definition 2.6.3. In particular, the characterizations of border bases in Corollary 2.6.10 also hold for subideal border bases. By now there was only one characterization for subideal border bases, namely the characterization via the special generation property in [KP11, Coro. 3.6].

Furthermore, we are now able to compute arbitrary subideal border bases using the method of Corollary 2.6.12, as follows. The kernel of the P-module epimorphism φ is $\ker(\varphi) = \mathrm{Syz}_P(p_1, \ldots, p_k, f_1, \ldots, f_\ell) \subseteq P^{k+\ell}$ and we can compute it with standard Gröbner bases techniques, e.g. using the method described in [KR00, Thm. 3.1.8]. Let $\ker(\varphi) = \langle s_1, \ldots, s_m\rangle$ be with $m \in \mathbb{N}$ and with vectors $s_1, \ldots, s_m \in P^{k+\ell} \setminus \{0\}$ and let $U = \langle e_1, \ldots, e_k, s_1, \ldots, s_m\rangle \subseteq P^{k+\ell}$. By Lemma 2.6.7, we have $U = \varphi^{-1}(I)$. According to the Second Noetherian Isomorphism Theorem, φ induces the canonical P-module isomorphism $P^{k+\ell}/U = P^{k+\ell}/\varphi^{-1}(I) \cong \varphi(P^{k+\ell})/I = I + J/I \subseteq P/I$. In particular, we get $\mathrm{codim}_K(U, P^{k+\ell}) = \mathrm{codim}_K(I, I+J) \leq \mathrm{codim}_K(I, P) < \infty$ since $I \subseteq P$ is a zero-dimensional ideal and we can hence use Algorithm 3 to compute the $\mathcal{O}$-border basis $G \subseteq P^{k+\ell}$ of U. By Corollary 2.6.12, $\varphi(G) \subseteq I + J/I$ is the $\varphi(\mathcal{O})$-border basis of $I + J/I$.

Remark 2.6.14. Although we have already seen before that we cannot reuse all the results about border bases in P^r in a straightforward way, we can use Theorem 2.6.8 to identify border prebases in M with their characterizing border prebases in P^r if such exist. This allows us to define many concepts about border bases in P^r for border bases in M. The following example shows such a generalization of the Border Division Algorithm 2.2.1 and of the normal remainder defined in Definition 2.2.2.

Let $G^\varphi = \{g_1^\varphi, \ldots, g_\nu^\varphi\} \subseteq M$ be an $\mathcal{O}^\varphi$-border prebasis and assume that there exists an $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\omega\} \subseteq P^r$ that characterizes G^φ. We write the order ideal $\mathcal{O} = \{t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ with $\mu \in \mathbb{N}$, $t_i \in \mathbb{T}^n$, and $\alpha_i \in \{1, \ldots, r\}$ for all $i \in \{1, \ldots, \mu\}$. Given $v^\varphi \in M$, we first have to determine a preimage $v \in P^r$ of v under φ and after that we have to apply the Border Division Algorithm 2.2.1 to v and G to obtain a representation $v = p_1 g_1 + \cdots + p_\omega g_\omega + c_1 t_1 e_{\alpha_1} + \cdots + c_\mu t_\mu e_{\alpha_\mu}$ with $p_1, \ldots, p_\omega \in P$ and $c_1, \ldots, c_\mu \in K$. Then we must apply φ to this result and get a similar representation of $v^\varphi \in M$, namely

$$\begin{aligned} v^\varphi &= p_1\varphi(g_1) + \cdots + p_\omega\varphi(g_\omega) + c_1 t_1 m_{\alpha_1} + \cdots + c_\mu t_\mu m_{\alpha_\mu} \\ &= q_1 g_1^\varphi + \cdots + q_\nu g_\nu^\varphi + c_1 t_1 m_{\alpha_1} + \cdots + c_\mu t_\mu m_{\alpha_\mu} \end{aligned}$$

with polynomials $q_1, \ldots, q_\nu \in P$, cf. Lemma 2.6.7 and Definition 2.6.5. We can then define the element $c_1 t_1 m_{\alpha_1} + \cdots + c_\mu t_\mu m_{\alpha_\mu} \in M$, which is a representative of the residue class $\overline{v^\varphi} \in M/\langle G^\varphi \rangle$, to be the normal remainder of v^φ with respect to G^φ and v. In particular, using this construction, we are then able to generalize the Corollaries 2.2.4 and 2.2.5 to border bases in M.

Many other concepts could be defined for border bases in M the same way, e.g. an $\mathcal{O}^\varphi$-index as in Definition 2.1.11 or the special generation property in Theorem 2.4.1. But note that the result of the last step, namely applying φ to the result in P^r, can lead to inconsistencies if we do not distinguish between different preimages. The following example shows such a inconsistency: We consider Example 2.6.2, again. Recall that the canonical basis of the free P-module P^2 was $\{e_1, e_2\}$ and that $x^2\bar{e}_1 = xy\bar{e}_2 \in \mathcal{O}^\varphi \cap \partial\mathcal{O}^\varphi$. Then the above construction assigns $x^2\bar{e}_1$ the $\mathcal{O}^\varphi$-index $\mathrm{ind}_{\mathcal{O}}(x^2 e_1) = 0$, whereas the same element $xy\bar{e}_2$ is also assigned the $\mathcal{O}^\varphi$-index $\mathrm{ind}_{\mathcal{O}}(xye_2) = 1$.

Altogether, we see that we can reuse the concepts of Sections 2.1 to 2.5 but we sometimes must not directly define these concepts for a given element in M but only for a specific preimage of it in P^r.

3 Syzygies of Border Bases

In this chapter, we devote our attention to the (first) syzygy module of border bases. As we have already seen in Theorem 2.4.26, we can characterize border bases via liftings of the neighbor syzygyies with respect to the given order ideal. But these liftings have also other nice properties. If we think about Gröbner bases, we see that the liftings of the syzygies corresponding to S-vectors also form a Gröbner basis of the (first) syzygy module with respect to a specific term ordering. This theorem is known as Schreyer's Theorem and was first proved in [Sch80]. A version of Schreyer's Theorem for Gröbner bases using our notation can be found in [KR00, Prop. 3.1.4]. Unfortunately, the methods used for the proof of the Gröbner bases version are not applicable for border bases. The reason is that the structure of border bases does not depend on an underlying term ordering but only on an order ideal. There are even $\mathcal{O}$-border bases of ideals in $\mathbb{Q}[x, y]$ such that $\mathcal{O} \neq \mathcal{O}_\sigma$ for every term ordering σ on $\mathbb{T}^2$. An example of such a border basis can be found in Remark 6.1.15.

The main goal of this chapter is to prove a version of Schreyer's Theorem that is applicable for border bases of submodules of free modules of finite rank in Theorem 3.4.5. More precisely, for a given $\mathcal{O}$-border basis $G = \{g_1, \dots, g_\nu\}$ of a P-submodule $U \subseteq P^r$ as defined in Definition 2.1.14, we prove that the corresponding set of neighbor liftings Λ with respect to $\mathcal{O}$ as defined in Definition 2.4.24 is a τ-Gröbner basis of the (first) syzygy module of $(g_1, \dots, g_\nu)$. Here τ is a term ordering that we can explicitly construct with the help of Algorithm 7. This generalizes the corresponding result in [KK14, Thm. 6.5] to border bases of free modules of finite rank and it generalizes the results of [Hui06, Thm. 22]. In the latter theorem, the author of [Hui06] proved that the set of neighbor liftings Λ with respect to $\mathcal{O}$ of a border basis of an ideal of a polynomial ring generates the (first) syzygy module of the border basis. In order to reach the above goal, the author reduces arbitrary vectors in P^ν in a very special way using the rewrite relation $\xrightarrow{\Lambda}$ defined by Λ as defined in [KR00, Defn. 2.2.1]. The corresponding procedures called "degree lowering" and "column clearing" served as the basis for the proof of Schreyer's Theorem for border bases of polynomial ideals in [KK14, Thm. 6.5]. The authors of that paper turn the methods "degree lowering" and "column clearing" into explicit algorithms and deduce Schreyer's Theorem for border bases of polynomial ideals.

We now go another step further and generalize these methods to border bases of submodules of free modules of finite rank. In Section 3.1, we divide the border terms of order ideals in different parts depending on the structure of the border terms. More precisely, in Definition 3.1.1, we divide the set of border terms into faces, and into non-exposable, extreme and non-extreme border terms. Non-exposable border terms were introduced by us in order to be able to handle order ideals in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ as defined in Definition 2.1.6 with an empty component, i.e. such that $e_k \notin \mathcal{O}$ for some $k \in \{1, \dots, r\}$. Such a strange situation cannot happen in the polynomial case. But we cannot thrust aside such situations in the module case, cf. Remark 2.5.5. After that we formulate the procedures of "degree lowering" and "column clearing" as explicit algorithms in Section 3.2. We then combine these methods in Section 3.3 to get an algorithmic version of the reduction process used in [Hui06, Lemma 33]. Moreover, we prove a generalized version of [Hui06, Thm. 22] and use these results in order show that the set of neighbor liftings Λ with respect to $\mathcal{O}$ generates the (first) syzygy module of an $\mathcal{O}$-border basis. As a byproduct, we give an alternative proof for the characterization of border bases via liftings of border syzygies with respect to $\mathcal{O}$ in Corollary 3.3.9. It is remarkable that, in stark contrast the proof in Theorem 2.4.26 and other proofs of this theorem, this proof does not depend on commuting matrices, at all. After that we have all ingredients to prove Schreyer's Theorem in Section 3.4, First we deduce conditions on a term ordering τ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$ in order for Λ to be a τ-Gröbner basis of the (first) syzygy module of the corresponding border basis in P^r in Theorem 3.4.1. Finally, we algorithmically construct such term orderings in Theorem 3.4.5.

In this chapter, we use the following notation. For every $t = x_1^{\delta_1} \cdots x_n^{\delta_n} \in \mathbb{T}^n$ with $\delta_1, \ldots, \delta_n \in \mathbb{N}$ and some $i \in \{1, \ldots, n\}$, we denote the x_i-degree by $\deg_{x_i}(t) = \delta_i$, and for $p \in P \setminus \{0\}$, we denote the x_i-degree by $\deg_{x_i}(p) = \max\{\deg_{x_i}(t) \mid t \in \operatorname{Supp}(p)\}$. Moreover, we let $G \subseteq P^r$ be an $\mathcal{O}$-border prebasis. As in Definition 2.1.14, we can write the order ideal $\mathcal{O} = \{t_1 e_{\alpha_1}, \ldots, t_\mu e_{\alpha_\mu}\}$ and its border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_\nu e_{\beta_\nu}\}$ with $\mu, \nu \in \mathbb{N}$, and $t_i, b_j \in \mathbb{T}^n$ and $\alpha_i, \beta_j \in \{1, \ldots, r\}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$, and we let G be of the form $G = \{g_1, \ldots, g_\nu\}$ with polynomials $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$, where $a_{ij} \in K$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$.

3.1 Extreme and Non-Extreme Border Terms

In this section, we divide the border terms into different parts. In particular, we generalize the definitions and results about extreme and non-extreme border terms of [Hui06] and [KK14, Section 2] to the module setting.

First of all, we define a generalization of extreme and non-extreme border terms that is suitable for the module setting. In particular, we introduce a new kind of border term that cannot occur in the ideal setting, namely non-exposable border terms. With these non-exposable border terms, we are capable of order ideals that do not contain all the basis vectors $e_1, \ldots, e_r$. The following definition generalizes [KK14, Defn. 2.1] to the module setting and is based on the corresponding definitions in [Hui06].

Definition 3.1.1. Let $j \in \{1, \ldots, \nu\}$ and $k \in \{1, \ldots n\}$.

a) The border term $b_j e_{\beta_j} \in \partial\mathcal{O}$ is called x_k-**exposable** (or simply **exposable**) if there exists a factorization $b_j e_{\beta_j} = x_k t_i e_{\alpha_i}$ for some $i \in \{1, \ldots, \mu\}$. If $b_j = 1$, the border term $b_j e_{\beta_j} = e_{\beta_j} \in \partial\mathcal{O}$ is called **non-exposable**.

b) The border term $b_j e_{\beta_j} \in \partial\mathcal{O}$ is said to be on **face** k of $\mathcal{O}$ if $b_j e_{\beta_j}$ is x_k-exposable, but not x_ℓ-exposable for all $\ell \in \{k+1, \ldots, n\}$. We call the set of all border terms on face k the k^{th} **face set** of $\mathcal{O}$ and denote it by $\mathcal{F}_k(\mathcal{O})$.

c) A border term $b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})$ is called **extreme** if we have $b_j \in K[x_1, \ldots, x_k]$. Otherwise, the border term $b_j e_{\beta_j}$ is called **non-extreme**.

The ideal version of following lemma has been stated in [Hui06, Lemma 17] and has been proved in [KK14, Prop. 2.2].

Proposition 3.1.2. *The set of all face sets of $\mathcal{O}$ is a partition of the set of all exposable border terms in $\partial\mathcal{O}$, i.e. of the set $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\}$.*

Proof. By Definition 3.1.1, a border term $b_j e_{\beta_j} \in \partial\mathcal{O}$ with $j \in \{1, \ldots, \nu\}$ is exposable if and only if $b_j \neq 1$. Thus $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\}$ is exactly the set of all exposable border terms. If $\mathcal{O} = \emptyset$, then $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\} = \emptyset$ by the Definitions 2.1.6 and 2.1.7 and the claim follows trivially. Thus suppose that $\mathcal{O} \neq \emptyset$, i.e. that $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\} \neq \emptyset$. Let $j \in \{1, \ldots, \nu\}$ be such that $b_j e_{\beta_j} \in \partial\mathcal{O} \setminus \{e_1, \ldots, e_r\}$ and such that $b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O}) \cap \mathcal{F}_\ell(\mathcal{O})$ for some $k, \ell \in \{1, \ldots, n\}$. Without loss of generality, we may assume that $k \leq \ell$. Then the border term $b_j e_{\beta_j}$ is not x_m-exposable for all $m \in \{k+1, \ldots, n\}$ but x_k-exposable by Definition 3.1.1. Therefore, we have $\ell \leq k$ and hence $k = \ell$. The claim now follows as every border term in $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\}$ is by the Definitions 2.1.7 and 3.1.1 x_m-exposable for some $m \in \{1, \ldots, n\}$. □

Therefore, we can define a map which associates to each border term contained in the set $\partial\mathcal{O} \setminus \{e_1, \ldots, e_r\}$ the index of the face set containing it. This definition generalizes [KK14, Def. 2.3] to the module setting.

Definition 3.1.3. According to Proposition 3.1.2, there exists a well-defined map $\text{face} : \partial\mathcal{O} \setminus \{e_1, \ldots, e_r\} \to \{1, \ldots, n\}$ which associates to every exposable border term $b_j e_{\beta_j}$ with $j \in \{1, \ldots, \nu\}$ the unique index $k \in \{1, \ldots, n\}$ such that $b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})$. We call this map the **face (index) map**.

The following example will guide us through the remainder of this chapter. It illustrates all the basic concepts used in [Hui06] and in [KK14] as well as our generalizations of these definitions and results.

Example 3.1.4. Let K be a field, $P = K[x, y, z]$, $\{e_1, e_2, e_3\}$ denote the canonical P-module basis of P^3, and

$$\mathcal{O} = \{1, z, y, z^2, y^2, z^3, y^3\} \cdot e_1 \cup \emptyset \cdot e_2 \cup \{1, y\} \cdot e_3.$$

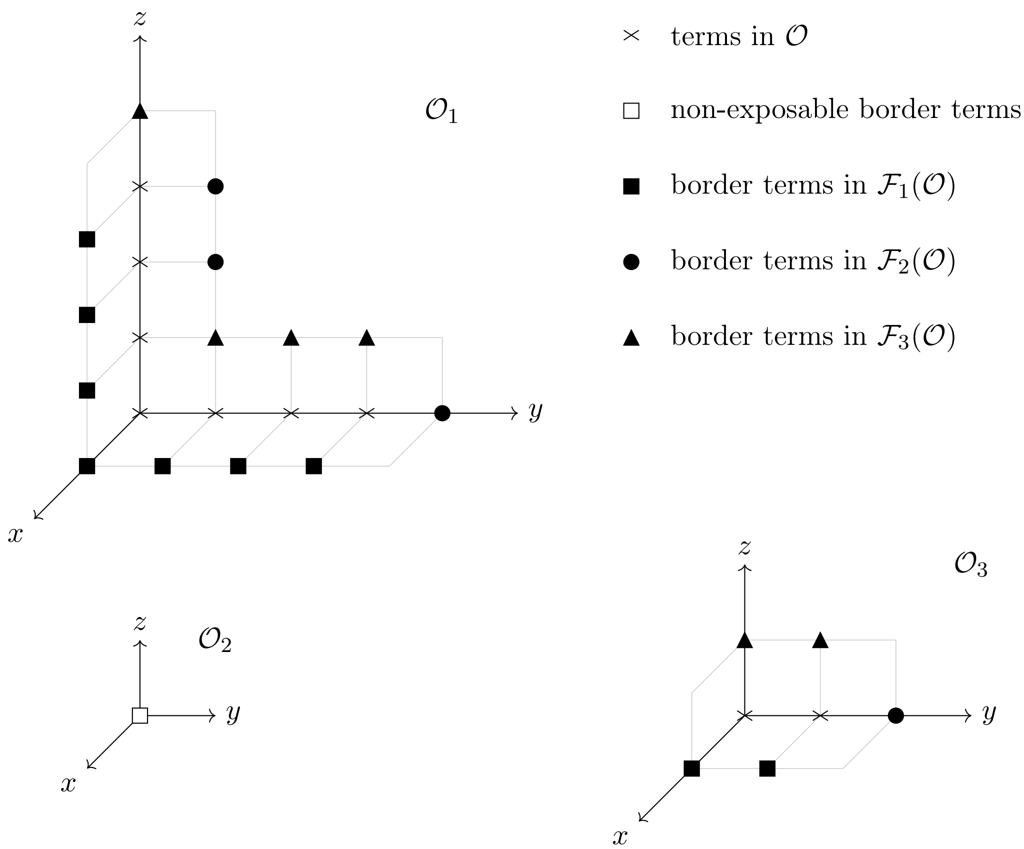
Then $\mathcal{O}$ is an order ideal in $\mathbb{T}^3\langle e_1, \ldots, e_r\rangle$ with border

$$\begin{aligned} \partial\mathcal{O} &= \{b_1, \ldots, b_{14}\} \cdot e_1 \cup \{b_{15}\} \cdot e_2 \cup \{b_{16}, \ldots, b_{20}\} \cdot e_3 \\ &= \{x, yz, xz, xy, yz^2, xz^2, y^2z, xy^2, z^4, yz^3, xz^3, y^3z, y^4, xy^3\} \cdot e_1 \\ &\quad \cup \{1\} \cdot e_2 \cup \{z, x, yz, y^2, xy\} \cdot e_3 \end{aligned}$$

by the Definitions 2.1.6 and 2.1.7. The only non-exposable border term is $b_{15}e_{\beta_{15}} = e_2$ and the face sets of $\mathcal{O}$ are

$$\mathcal{F}_1(\mathcal{O}) = \{x, xz, xy, xz^2, xy^2, xz^3, xy^3\} \cdot e_1 \cup \{x, xy\} \cdot e_3,$$
$$\mathcal{F}_2(\mathcal{O}) = \{yz^2, yz^3, y^4\} \cdot e_1 \cup \{y^2\} \cdot e_3,$$
$$\mathcal{F}_3(\mathcal{O}) = \{yz, y^2z, z^4, y^3z\} \cdot e_1 \cup \{z, yz\} \cdot e_3.$$

according to Definition 3.1.1. By identifying terms with their logarithms, i.e. with their exponent vectors, we can illustrate all these sets with the following pictures.



Moreover, we see that the set of all extreme border terms is

$$\{x, yz, y^2z, z^4, y^3z, y^4\} \cdot e_1 \cup \{x, z, y^2, yz\} \cdot e_3$$

and the set of all non-extreme border terms is

$$\{xz, xy, yz^2, xz^2, xy^2, yz^3, xz^3, xy^3\} \cdot e_1 \cup \{xy\} \cdot e_3.$$

Next we subdivide the set of non-extreme border terms in an even finer way. The corresponding notion in the ideal setting was first introduced in [Hui06, Section 6.5] and can also be found in [KK14, Defn. 2.5].

Definition 3.1.5. Let $i \in \{1,\dots,n-1\}$ and $j \in \{1,\dots,\nu\}$ be such that $b_j e_{\beta_j} \in \mathcal{F}_i(\mathcal{O})$ is a non-extreme border term. Let $\delta_1,\dots,\delta_n \in \mathbb{N}$ be such that $b_j = x_1^{\delta_1}\cdots x_n^{\delta_n}$ and let $k = \min\{\ell \in \{i+1,\dots,n\} \mid \delta_\ell > 0\}$. Then we call the set

$$\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^\ell b_j e_{\beta_j} \in \mathcal{F}_i(\mathcal{O}) \mid \ell \in \mathbb{Z}, x_k^\ell b_j e_{\beta_j} \text{ non-extreme}, x_k \mid x_k^\ell b_j\}$$

the **column** containing $b_j e_{\beta_j}$. We say that it is in the x_k**-direction**.

In the ideal setting, the following description of the column containing a non-extreme border term $b_j e_{\beta_j}$ with $j \in \{1,\dots,\nu\}$ is contained in [Hui06, Lemma 28]. We generalize the corresponding formulation in [KK14, Lemma 2.6] to the module setting.

Lemma 3.1.6. *Let $j \in \{1,\dots,\nu\}$ be such that $b_j e_{\beta_j}$ is non-extreme, let $i = \mathrm{face}(b_j e_{\beta_j})$, and let $k \in \{i+1,\dots,n\}$ be such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. Then there exist natural numbers $s,t \in \mathbb{N}$ such that*

$$\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, x_k^{-s+1} b_j, \dots, x_k^{t-1} b_j, x_k^t b_j\} \cdot e_{\beta_j}.$$

In addition, for every $\ell \in \{1,\dots,\nu\}$ such that $b_\ell e_{\beta_\ell} \in \mathrm{Col}(b_j e_{\beta_j})$, we have $x_m \nmid b_\ell$ for all $m \in \{i+1,\dots,k-1\}$.

Proof. In order to prove the claim, we define $s = \max\{\ell \in \mathbb{N} \mid x_k^{-\ell} b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j})\}$ and $t = \max\{\ell \in \mathbb{N} \mid x_k^\ell b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j})\}$. Note that both of these maxima exist since we have $b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j}) \neq \emptyset$ and since $\#(\mathrm{Col}(b_j e_{\beta_j})) \leq \#\partial\mathcal{O} = \nu < \infty$. Let $\ell \in \{-s,\dots,t\}$. Then the first claim follows if we show that $x_k^\ell b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j})$. As $x_k^{-s} b_j e_{\beta_j}, x_k^t b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j}) \subseteq \partial\mathcal{O}$ by Definition 3.1.5, we get $x_k^\ell b_j e_{\beta_j} \in \partial\mathcal{O}$ according to the Definitions 2.1.6 and 2.1.7. Moreover, we have $x_k^{-s} b_j \neq 1$ by Definition 3.1.1 as $x_k^{-s} b_j e_{\beta_j}$ is non-extreme.
First we prove that $\mathrm{face}(x_k^\ell b_j e_{\beta_j}) = i$. Since $x_k^t b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j})$, $\mathrm{face}(x_k^t b_j e_{\beta_j}) = i$ by Definition 3.1.5. Therefore, it follows $\frac{x_k^t b_j}{x_i} e_{\beta_j} \in \mathcal{O}$ from Definition 3.1.1 and $\ell \leq t$ yields $\frac{x_k^\ell b_j}{x_i} e_{\beta_j} \in \mathcal{O}$ by Definition 2.1.6. In particular, we get $\mathrm{face}(x_k^\ell b_j e_{\beta_j}) \geq i$ according to Definition 3.1.1. Assume that $\mathrm{face}(x_k^\ell b_j e_{\beta_j}) = m$ for some $m \in \{i+1,\dots,n\}$. Then we have $\frac{x_k^\ell b_j}{x_\ell} e_{\beta_j} \in \mathcal{O}$ by Definition 3.1.1 and thus $-s \leq \ell$ yields $\frac{x_k^{-s} b_j}{x_m} e_{\beta_j} \in \mathcal{O}$ by Definition 2.1.6. Therefore, it follows that $\mathrm{face}(x_k^{-s} b_j e_{\beta_j}) \geq m > i$ from Definition 3.1.1. As we also have $\mathrm{face}(x_k^{-s} b_j e_{\beta_j}) = i$ according to Definition 3.1.5, we get a contradiction. Altogether, it follows that $\mathrm{face}(x_k^\ell b_j e_{\beta_j}) = i$.
Next we show that $x_k^\ell b_j e_{\beta_j} \in \partial\mathcal{O}$ is non-extreme. As $x_k^{-s} b_j \neq 1$ by Definition 3.1.5 and as $-s \leq \ell$, it follows $x_k^\ell b_j \neq 1$, i.e. $x_k^\ell b_j e_{\beta_j}$ is exposable by Definition 3.1.1. Assume that $x_k^\ell b_j e_{\beta_j}$ is extreme. Then we have $x_k^\ell b_j \in K[x_1,\dots,x_i]$ by Definition 3.1.1. In particular, it follows that $x_k^{-s} b_j e_{\beta_j} \in \mathcal{F}_i(\mathcal{O})$ is contained in $K[x_1,\dots,x_i]$ and, therefore, extreme according to Definition 3.1.1. As $x_k^{-s} b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j})$ is non-extreme according to Definition 3.1.5, we get a contradiction. Thus we have proven that the border term $x_k^\ell b_j e_{\beta_j}$ is non-extreme.

In order to show the remaining claim, let $m \in \{i+1, \dots, k-1\}$. Assume that $x_m \mid x_k^\ell b_j$. Then, as $m < k$, we get $x_m \mid b_j$ in contradiction to Definition 3.1.5. Thus we see that there exists no index $m \in \{i+1, \dots, k-1\}$ such that $x_m \mid x_k^\ell b_j e_{\beta_j}$. □

Our next lemma shows that the columns form a partition of the non-extreme border terms in $\partial\mathcal{O}$. The corresponding ideal version can be found in [KK14, Lemma 2.7].

Lemma 3.1.7. *The set of all columns is a partition of the set of all non-extreme border terms in $\partial\mathcal{O}$.*

Proof. Let $i, j \in \{1, \dots, \nu\}$ be such that both border terms $b_i e_{\beta_i}, b_j e_{\beta_j} \in \partial\mathcal{O}$ are non-extreme and let $k, \ell \in \{2, \dots, n\}$ be such that the column $\mathrm{Col}(b_i e_{\beta_i})$ is in the x_k-direction and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_ℓ-direction. Suppose that there exists an index $m \in \{1, \dots, \nu\}$ such that $b_m e_{\beta_m} \in \mathrm{Col}(b_i e_{\beta_i}) \cap \mathrm{Col}(b_j e_{\beta_j})$. Then Definition 3.1.5 implies that $s = \mathrm{face}(b_i e_{\beta_i}) = \mathrm{face}(b_m e_{\beta_m}) = \mathrm{face}(b_j e_{\beta_j})$. Without loss of generality, we let $k \leq \ell$. Assume that $k < \ell$. Then we have $s < k < \ell$ according to Definition 3.1.5. Since $b_m e_{\beta_m} \in \mathrm{Col}(b_i e_{\beta_i})$ and since $\mathrm{Col}(b_i e_{\beta_i})$ is in the x_k-direction, we see that $x_k \mid b_m$ by Definition 3.1.5. But, as we also have $b_m e_{\beta_m} \in \mathrm{Col}(b_j e_{\beta_j})$ and as $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_ℓ-direction with $s < k < \ell$, it follows $x_k \nmid b_m$ from Lemma 3.1.6. This is obviously a contradiction. Thus we get $k = \ell$ and hence $\mathrm{Col}(b_i e_{\beta_i}) = \mathrm{Col}(b_j e_{\beta_j})$ according to Definition 3.1.5. The claim now follows from the observation that every non-extreme border term is contained in a column by Definition 3.1.5. □

As in [Hui06, Lemma 28] and [KK14, Def. 2.8], the terms below and above every column receive special names as follows.

Definition 3.1.8. Let $j \in \{1, \dots, \nu\}$ be such that the border term $b_j e_{\beta_j}$ is non-extreme and let $k \in \{2, \dots, n\}$ be such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. Let $s, t \in \mathbb{N}$ be such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \dots, x_k^t b_j\} \cdot e_{\beta_j}$. Then the term $\mathrm{lcb}(b_j e_{\beta_j}) = x_k^{-s-1} b_j e_{\beta_j}$ is called the **lower (column) bound** of $\mathrm{Col}(b_j e_{\beta_j})$ and the term $\mathrm{ucb}(b_j e_{\beta_j}) = x_k^{t+1} b_j e_{\beta_j}$ is called the **upper (column) bound** of $\mathrm{Col}(b_j e_{\beta_j})$.

In [Hui06, Lemma 28], the following properties of the upper and lower bound of a column in $\partial\mathcal{O}$ are proven. We generalize these results to the module setting.

Lemma 3.1.9. *Let $j \in \{1, \dots, \nu\}$ be such that the border term $b_j e_{\beta_j}$ is non-extreme, let $i = \mathrm{face}(b_j e_{\beta_j})$, and let $k \in \{i+1, \dots, n\}$ be such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. Let $s, t \in \mathbb{N}$ be such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \dots, x_k^t b_j\} \cdot e_{\beta_j}$.*

a) Exactly one of the following conditions holds for $\mathrm{ucb}(b_j e_{\beta_j}) = x_k^{t+1} b_j e_{\beta_j}$.

1) $\mathrm{ucb}(b_j e_{\beta_j}) \notin \overline{\partial\mathcal{O}}$.

2) $\mathrm{ucb}(b_j e_{\beta_j}) \in \mathcal{F}_\ell(\mathcal{O})$ for some $\ell \in \{1, \dots, i-1\}$.

b) Exactly one of the following conditions holds for $\mathrm{lcb}(b_j e_{\beta_j}) = x_k^{-s-1} b_j e_{\beta_j}$.

1) $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_i(\mathcal{O})$ is extreme.

2) $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_i(\mathcal{O})$ *is non-extreme and belongs to a column in the* x_ℓ*-direction for some* $\ell \in \{k+1,\dots,n\}$*. In particular, we have* $x_k \nmid x_k^{-s-1} b_j$*.*

3) $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_\ell(\mathcal{O})$ *for some* $\ell \in \{k,\dots,n\}$*.*

Proof. First we prove claim a). According to Definition 3.1.5, we have $x_k^t b_j e_{\beta_j} \in \partial\mathcal{O}$. Thus by the Definitions 2.1.6 and 2.1.7, we have $\mathrm{ucb}(b_j e_{\beta_j}) \notin \mathcal{O}$. In particular, it follows $\mathrm{ucb}(b_j e_{\beta_j}) \in \partial\mathcal{O}$ or $\mathrm{ucb}(b_j e_{\beta_j}) \notin \overline{\partial\mathcal{O}}$. Suppose that $\mathrm{ucb}(b_j e_{\beta_j}) \in \partial\mathcal{O}$. Since $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction, $x_k \mid b_j$ by Definition 3.1.5. As $t \in \mathbb{N}$, this implies $x_k \mid x_k^{t+1} b_j$.
For a contradiction, assume that $\mathrm{face}(\mathrm{ucb}(b_j e_{\beta_j})) = i$. Since $x_k \mid x_k^{t+1} b_j$, it follows that $\mathrm{ucb}(b_j e_{\beta_j}) = x_k^{t+1} b_j \notin K[x_1,\dots,x_i]$, i.e. $\mathrm{ucb}(b_j e_{\beta_j})$ is non-extreme by Definition 3.1.1. Then Definition 3.1.5 yields $\mathrm{ucb}(b_j e_{\beta_j}) \in \mathrm{Col}(b_j e_{\beta_j})$ in contradiction to Definition 3.1.8. Altogether, we see that $\mathrm{face}(\mathrm{ucb}(b_j e_{\beta_j})) \neq i$.
Now assume that $\mathrm{face}(\mathrm{ucb}(b_j e_{\beta_j})) = \ell$ for some $\ell \in \{i+1,\dots,n\}$. Then we have $\frac{x_k^{t+1} b_j}{x_\ell} e_{\beta_j} \in \mathcal{O}$ by Definition 3.1.1. In particular, as $x_k^t b_j e_{\beta_j} \in \mathrm{Col}(b_j e_{\beta_j}) \subseteq \partial\mathcal{O}$ by Definition 3.1.5, it follows $k \neq \ell$ from the Definitions 2.1.6 and 2.1.7. Thus we get $x_\ell \mid b_j$ and this implies $\frac{b_j}{x_\ell} e_{\beta_j} \in \mathcal{O}$ by Definition 2.1.6, i.e. we get $\mathrm{face}(b_j e_{\beta_j}) \geq \ell > i$ by Definition 3.1.1 and this clearly contradicts $\mathrm{face}(b_j e_{\beta_j}) = i$. Altogether, we see that $\mathrm{face}(b_j e_{\beta_j}) < i$ and claim a) follows.

Next we show claim b). According to Definition 3.1.5, we have $x_k \mid x_k^{-s} b_j$. Thus $\mathrm{lcb}(b_j e_{\beta_j}) = x_k^{-s-1} b_j e_{\beta_j} \in \overline{\partial\mathcal{O}}$ by the Definitions 2.1.6 and 2.1.7. Assume that we have $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{O}$. Then $x_k^{-s} b_j e_{\beta_j} = x_k \, \mathrm{lcb}(b_j e_{\beta_j})$ is x_k-exposable by Definition 3.1.1. As we have $\mathrm{face}(x_k^{-s} b_j e_{\beta_j}) = i$ by Lemma 3.1.6 and as $i < k$ by Definition 3.1.5, this is a contradiction. Hence we get $\mathrm{lcb}(b_j e_{\beta_j}) \in \partial\mathcal{O}$.
As $\mathrm{face}(x_k^{-s} b_j e_{\beta_j}) = i$ according to Lemma 3.1.6, it follows $\frac{x_k^{-s} b_j}{x_i} e_{\beta_j} \in \mathcal{O}$. Since we also have $i < k$ by Definition 3.1.5 and since we have already shown that $\mathrm{lcb}(b_j e_{\beta_j}) \in \partial\mathcal{O}$, we see that $\frac{x_k^{-s-1} b_j}{x_i} e_{\beta_j} \in \mathcal{O}$ by the Definitions 2.1.6 and 2.1.7. In particular, it follows $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_\ell(\mathcal{O})$ for some $\ell \in \{i,\dots,n\}$ from Definition 3.1.1.
For every $\ell \in \{i+1,\dots,k-1\}$, since $x_\ell \nmid b_j$ by Definition 3.1.5, it also follows $x_\ell \nmid x_k^{-s-1} b_j$. Therefore, we see that $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_\ell(\mathcal{O})$ with $\ell \in \{i\} \cup \{k,\dots,n\}$. In particular, the only remaining part of the proof is that, if $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathcal{F}_i(\mathcal{O})$ is non-extreme, then $\mathrm{Col}(\mathrm{lcb}(b_j e_{\beta_j}))$ is in the x_ℓ-direction for some $\ell \in \{k+1,\dots,n\}$.
Suppose that $\mathrm{lcb}(b_j e_{\beta_j})$ is non-extreme. Assume that $x_k \mid x_k^{-s-1} b_j$. Then we have $\mathrm{lcb}(b_j e_{\beta_j}) \in \mathrm{Col}(b_j e_{\beta_j})$ by Definition 3.1.5. Since this is clearly a contradiction to Definition 3.1.8, it follows $x_k \nmid x_k^{-s-1} b_j$. Moreover, we have already shown that $x_\ell \nmid x_k^{-s-1} b_j$ for all $\ell \in \{i+1,\dots,k-1\}$. Altogether, $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_ℓ-direction for some $\ell \in \{k+1,\dots,n\}$ according to Definition 3.1.5. □

Example 3.1.10. Consider the order ideal $\mathcal{O}$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ of Example 3.1.4, again. Recall that the set of all extreme border terms was

$$\{x, yz, y^2z, z^4, y^3z, y^4\} \cdot e_1 \cup \{x, z, y^2, yz\} \cdot e_3 \subseteq \partial\mathcal{O}$$

and the set of all non-extreme border terms was

$$\{xz, xy, yz^2, xz^2, xy^2, yz^3, xz^3, xy^3\} \cdot e_1 \cup \{xy\} \cdot e_3 \subseteq \partial\mathcal{O}.$$

We compute all columns and get

$$\begin{aligned}
\mathrm{Col}(b_3 e_{\beta_3}) = \mathrm{Col}(b_6 e_{\beta_6}) = \mathrm{Col}(b_{11} e_{\beta_{11}}) &= \{xz, xz^2, xz^3\} \cdot e_1 \subseteq \mathcal{F}_1(\mathcal{O}),\\
\mathrm{Col}(b_4 e_{\beta_4}) = \mathrm{Col}(b_8 e_{\beta_8}) = \mathrm{Col}(b_{14} e_{\beta_{14}}) &= \{xy, xy^2, xy^3\} \cdot e_1 \subseteq \mathcal{F}_1(\mathcal{O}),\\
\mathrm{Col}(b_5 e_{\beta_5}) = \mathrm{Col}(b_{10} e_{\beta_{10}}) &= \{yz^2, yz^3\} \cdot e_1 \subseteq \mathcal{F}_2(\mathcal{O}),\\
\mathrm{Col}(b_{20} e_{\beta_{20}}) &= \{xy\} \cdot e_3 \subseteq \mathcal{F}_2(\mathcal{O})
\end{aligned}$$

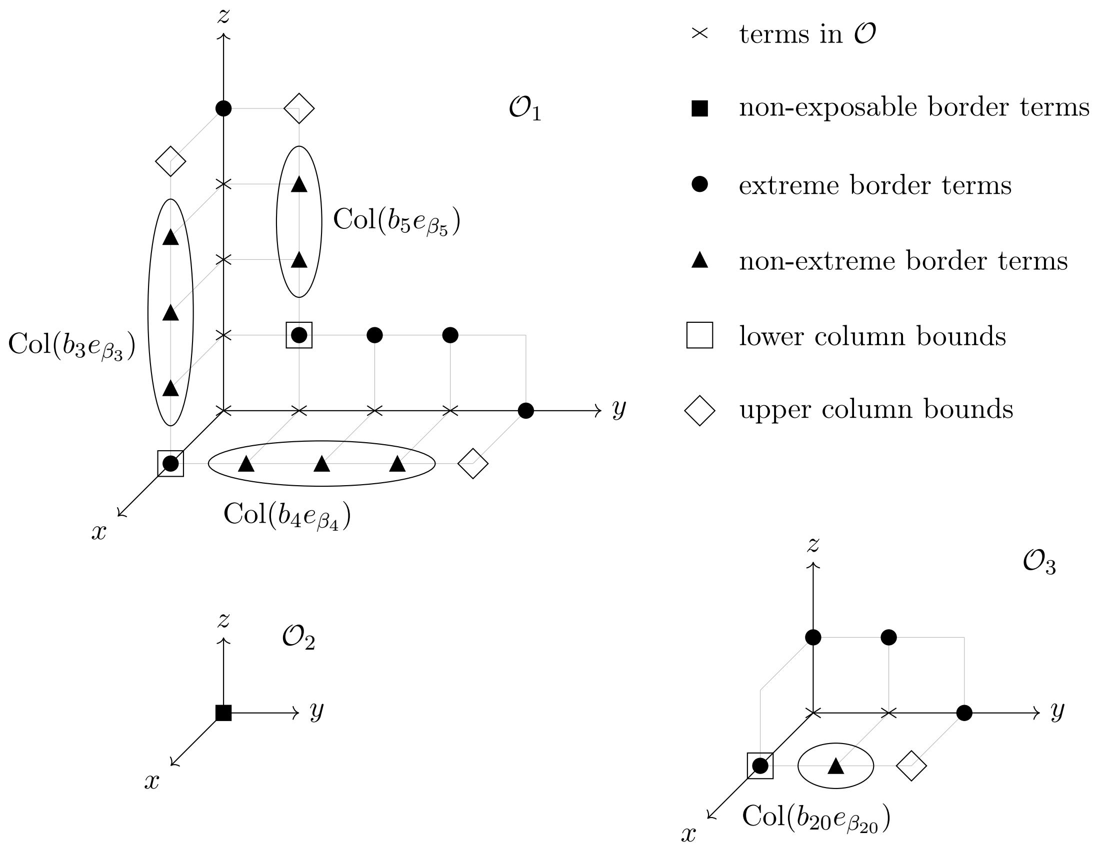
by Definition 3.1.5. Here both $\mathrm{Col}(b_3 e_{\beta_3})$ and $\mathrm{Col}(b_5 e_{\beta_5})$ are in the z-direction, and both $\mathrm{Col}(b_4 e_{\beta_4})$ and $\mathrm{Col}(b_{20} e_{\beta_{20}})$ are in the y-direction. Moreover, all the lower bounds

$$\begin{aligned}
\mathrm{lcb}(b_3 e_{\beta_3}) = \mathrm{lcb}(b_6 e_{\beta_6}) = \mathrm{lcb}(b_{11} e_{\beta_{11}}) &= x e_1 \in \mathcal{F}_1(\mathcal{O}),\\
\mathrm{lcb}(b_4 e_{\beta_4}) = \mathrm{lcb}(b_8 e_{\beta_8}) = \mathrm{lcb}(b_{14} e_{\beta_{14}}) &= x e_1 \in \mathcal{F}_1(\mathcal{O}),\\
\mathrm{lcb}(b_5 e_{\beta_5}) = \mathrm{lcb}(b_{10} e_{\beta_{10}}) &= yz e_1 \in \mathcal{F}_3(\mathcal{O}),\\
\mathrm{lcb}(b_{20} e_{\beta_{20}}) &= x e_3 \in \mathcal{F}_1(\mathcal{O})
\end{aligned}$$

are extreme, and the upper bounds are of the form

$$\begin{aligned}
\mathrm{ucb}(b_3 e_{\beta_3}) = \mathrm{ucb}(b_6 e_{\beta_6}) = \mathrm{ucb}(b_{11} e_{\beta_{11}}) &= xz^4 e_1 \notin \partial\mathcal{O},\\
\mathrm{ucb}(b_4 e_{\beta_4}) = \mathrm{ucb}(b_8 e_{\beta_8}) = \mathrm{ucb}(b_{14} e_{\beta_{14}}) &= xy^4 e_1 \notin \partial\mathcal{O},\\
\mathrm{ucb}(b_5 e_{\beta_5}) = \mathrm{ucb}(b_{10} e_{\beta_{10}}) &= yz^4 e_1 \notin \partial\mathcal{O},\\
\mathrm{ucb}(b_{20} e_{\beta_{20}}) &= xy^2 e_3 \notin \partial\mathcal{O}
\end{aligned}$$

according to Definition 3.1.8. The following sketch illustrates the columns of $\partial\mathcal{O}$:



3.2 Degree Lowering and Column Clearing

In this section, we prepare some material from [Hui06, Sect. 6.4 and 6.5] for its application in the proof of the main theorem of this chapter. In particular, we formulate everything in our notation and transform some proofs into explicit algorithms. A similar work in the ideal setting was already done in [KK14, Section 4]. In this way, we are able to achieve an explicit reduction algorithm, i.e. an effectively implementable version of the proof of [Hui06, Lemma 33]. For the intuitive meaning of the processes of "degree lowering" and "column clearing", we refer to [Hui06, Sect. 6.4 and 6.5].

As in Subsection 2.4.5, we use the following notation. We let $\{e_1, \ldots, e_r\}$ denote the canonical P-module basis of P^r and we let $\{\varepsilon_1, \ldots, \varepsilon_\nu\}$ denote the canonical P-module basis of P^ν. Moreover, we assume that the $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G \rangle$. For every $i, j \in \{1, \ldots, \nu\}$ such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we denote the corresponding neighbor syzygy with respect to $\mathcal{O}$ by σ_{ij} as in Definition 2.4.21 and according to Remark 2.4.28, σ_{ij} can be lifted to a unique neighbor lifting λ_{ij} with respect to $\mathcal{O}$ as defined in Definition 2.4.24. Moreover, we denote the set of all neighbor liftings with respect to $\mathcal{O}$ by Λ.

The special shape of neighbor liftings with respect to $\mathcal{O}$ that was described in Remark 2.4.28 gives us more insight into the structure of neighbor liftings and serves as a basic part of [Hui06] and of this section. In particular, though deduced in a totally different way, it is a reformulated version of [Hui06, Lemma 19]. Because of its importance, we recall it using the definitions of Section 3.1.

Remark 3.2.1. Let $i, j \in \{1, \ldots, \nu\}$.

a) Suppose that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are next-door neighbours with respect to $\mathcal{O}$, i.e. that there exists an index $k \in \{1, \ldots, n\}$ such that $x_k b_i e_{\beta_i} = b_j e_{\beta_j}$. Then we have $\varepsilon_m \notin \operatorname{Supp}(\lambda_{ij} - \sigma_{ij})$ for all $m \in \{1, \ldots, \nu\}$ such that $b_m e_{\beta_m}$ is non-exposable or $\operatorname{face}(b_m e_{\beta_m}) < k$ according to Remark 2.4.28 and Definition 3.1.1.

b) Suppose that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are across-the-street neighbours with respect to $\mathcal{O}$, i.e. that there are $k, \ell \in \{1, \ldots, n\}$ such that $x_k b_i e_{\beta_i} = x_\ell b_j e_{\beta_j}$. Then we have $\varepsilon_m \notin \operatorname{Supp}(\lambda_{ij} - \sigma_{ij})$ for all $m \in \{1, \ldots, \nu\}$ such that $b_m e_{\beta_m}$ is non-exposable or $\operatorname{face}(b_m e_{\beta_m}) < \min\{k, \ell\}$ according to Remark 2.4.28 and Definition 3.1.1.

The following proposition can be shown using a suitably adapted version of the proof of [Hui06, Lemma 27]. For the ideal setting, the corresponding result was proven in [KK14, Prop. 4.1]. We generalize this proposition to the module setting.

Proposition 3.2.2. (Properties of Degree Lowering)
Let $i \in \{1, \ldots, n-1\}$ and let $j \in \{1, \ldots, \nu\}$ be such that $b_j e_{\beta_j}$ is exposable and such that $k = \operatorname{face}(b_j e_{\beta_j}) > i$. Given $(p_1, \ldots, p_\nu) \in P^\nu$ with $p_j \neq 0$, let $d \in \mathbb{N}$ be maximal such that x_i^d divides one of the terms in the support of p_j. Then there exist unique polynomials $p' \in P \setminus \{0\}$ and $p'' \in P$ such that $p_j = p' x_i^d + p''$, and such that no term in the support of p'' is divisible by x_i^d.
In this situation, there exists an exposable border term $b_\ell e_{\beta_\ell}$, where $\ell \in \{1, \ldots, \nu\}$, with the property that $b_\ell e_{\beta_\ell} = x_i b_j e_{\beta_j}$, or that $x_k b_\ell e_{\beta_\ell} = x_i b_j e_{\beta_j}$ and $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq i$. In particular, the border terms $b_j e_{\beta_j}$ and $b_\ell e_{\beta_\ell}$ are neighbors with respect to $\mathcal{O}$.
Assume that $d > 0$. Then the vector

$$(q_1, \ldots, q_\nu) = (p_1, \ldots, p_\nu) - p' x_i^{d-1} \lambda_{j\ell},$$

satisfies the following conditions.

1) *For every $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\operatorname{face}(b_\varrho e_{\beta_\varrho}) < i$, we have $q_\varrho = p_\varrho$.*

2) *Let $\varrho \in \{1, \ldots, \nu\}$ and $s \in \{1, \ldots, i\}$ be any indices such that both p_ϱ and p_j are contained in $K[x_s, \ldots, x_n]$. Then we have $q_\varrho \in K[x_s, \ldots, x_n]$.*

3) *For all $\varrho \in \{1, \ldots, \nu\}$, every term in the support of q_ϱ that is divisible by x_i^d is also contained in the support of p_ϱ.*

4) *We have $q_j = 0$ or $\deg_{x_i}(q_j) \leq d - 1$.*

Proof. The existence and uniqueness of the polynomials $p' \in P \setminus \{0\}$ and $p'' \in P$ such that $p_j = p' x_i^d + p''$ is clear. Hence we start by proving the existence of an exposable border term $b_\ell e_{\beta_\ell}$, where $\ell \in \{1, \dots, \nu\}$, with the property that $b_\ell e_{\beta_\ell} = x_i b_j e_{\beta_j}$, or that $x_k b_\ell e_{\beta_\ell} = x_i b_j e_{\beta_j}$ and $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq i$. As $\operatorname{face}(b_j e_{\beta_j}) = k$, there exists a $u \in \{1, \dots, \mu\}$ such that $x_k t_u e_{\alpha_u} = b_j e_{\beta_j}$ by Definition 3.1.1. Then we have either $x_i t_u e_{\alpha_u} \in \mathcal{O}$ or $x_i t_u e_{\alpha_u} \in \partial\mathcal{O}$ by Definition 2.1.7.
If we have $x_i t_u e_{\alpha_u} \in \partial\mathcal{O}$, there is an $\ell \in \{1, \dots, \nu\}$ such that $x_i t_u e_{\alpha_u} = b_\ell e_{\beta_\ell}$ and this yields $x_k b_\ell e_{\beta_\ell} = x_k x_i t_u e_{\alpha_u} = x_i b_j e_{\beta_j}$. In this first case, $b_\ell e_{\beta_\ell} = x_i t_u e_{\alpha_u}$ is x_i-exposable and thus $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq i$ by Definition 3.1.1. For the second case, $x_i t_u e_{\alpha_u} \in \mathcal{O}$, we note that the term $x_k(x_i t_u e_{\alpha_u}) = x_i b_j e_{\beta_j}$ is contained in $\partial\mathcal{O}$ by the Definitions 2.1.6 and 2.1.7. This shows that there exists an index $\ell \in \{1, \dots, \nu\}$ such that $b_\ell e_{\beta_\ell} = x_i b_j e_{\beta_j}$. In particular, by Definition 3.1.1, we see that $b_\ell e_{\beta_\ell} = x_k(x_i t_u e_{\alpha_u})$ is x_k-exposable and thus $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq k > i$ in this second case.
Now we investigate the shape of the neighbor lifting $\lambda_{j\ell}$. Let $\lambda_{j\ell} = (f_1, \dots, f_\nu)$ be with polynomials $f_1, \dots, f_\nu \in P$. According to Remark 2.4.28, the lifting $\lambda_{j\ell}$ is either of the form $\lambda_{j\ell} = x_i \varepsilon_j - \varepsilon_\ell - c_1 \varepsilon_1 - \cdots - c_\nu \varepsilon_\nu$ or of the form $\lambda_{j\ell} = x_i \varepsilon_j - x_k \varepsilon_\ell - c_1 \varepsilon_1 - \cdots - c_\nu \varepsilon_\nu$ with $c_1, \dots, c_\nu \in K$. Hence for all $\varrho \in \{1, \dots, \nu\} \setminus \{j, \ell\}$, we have $f_\varrho = -c_\varrho$, we have $f_j = x_i - c_j$, and we have either $f_\ell = -1 - c_\ell$ or $f_\ell = -x_k - c_\ell$.
For all indices $\varrho \in \{1, \dots, \nu\}$ such that the border term $b_\varrho e_{\beta_\varrho}$ is non-exposable or such that $\operatorname{face}(b_\varrho e_{\beta_\varrho}) < i < k$, Remark 3.2.1 yields $c_\varrho = 0$. As $\operatorname{face}(b_j e_{\beta_j}) > i$, and as we have already seen that $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq i$, it follows that $f_\varrho = 0$ for all $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\operatorname{face}(b_\varrho e_{\beta_\varrho}) < i$. Claim 1) is now a consequence of $(q_1, \dots, q_\nu) = (p_1, \dots, p_\nu) - p' x_i^{d-1}(f_1, \dots, f_\nu)$.

To prove claim 2), let $\varrho \in \{1, \dots, \nu\}$ and $s \in \{1, \dots, i\}$ be any indices such that both p_ϱ and p_j are contained $K[x_s, \dots, x_n]$. Then the construction of p' yields $p' \in K[x_s, \dots, x_n]$ and hence $q_\varrho = p_\varrho - p' x_i^{d-1} f_\varrho \in K[x_s, \dots, x_n]$.

For the claims 3) and 4), we note that by the construction of the polynomial p', we have $p' \in K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ and that for every index $\varrho \in \{1, \dots, \nu\} \setminus \{j\}$, we have $p' x_i^{d-1} f_\varrho = 0$ or $\deg_{x_i}(p' x_i^{d-1} f_\varrho) \leq d - 1$. Moreover, the descriptions of p' and $\lambda_{j\ell}$ show that all the terms in the support of p_j with x_i-degree greater than or equal to d cancel in $q_j = p_j - p' x_i^{d-1} f_j = p_j - p' x_i^{d-1}(x_i - c_j)$. Thus the claims 3) and 4) follow. $\square$

Based on this proposition, we can formulate an algorithm for performing degree lowering steps that generalizes [KK14, Prop. 4.2] to the module setting. Note that the following algorithm proceeds differently from the method in the proof of [Hui06, Lemma 27]: we do not reduce the x_i-degree of a single component of maximal x_i-degree of a vector, but of all such components.

Algorithm 4: `LowerDegree`$(i, (p_1, \dots, p_\nu), \Lambda)$

Input:
$i \in \{1, \dots, n-1\}$
$(p_1, \dots, p_\nu) \in P^\nu$ such that there exists an index $j \in \{1, \dots, \nu\}$ with the properties that $b_j e_{\beta_j}$ is exposable, $\mathrm{face}(b_j e_{\beta_j}) > i$, and the support of p_j contains a term that is divisible by x_i
$\Lambda = \{\lambda_{ij} \mid i, j \in \{1, \dots, \nu\}, b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neigbhors with respect to $\mathcal{O}\}$

1 Let $d \in \mathbb{N} \setminus \{0\}$ be maximal such that there exists an index $j \in \{1, \dots, \nu\}$ with the properties that $b_j e_{\beta_j}$ is exposable, $\mathrm{face}(b_j e_{\beta_j}) > i$, and the support of p_j contains a term that is divisible by x_i^d.
2 **foreach** $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is exposable, $\mathrm{face}(b_j e_{\beta_j}) > i$, $p_j \neq 0$, and $\deg_{x_i}(p_j) = d$ **do**
3   **if** $x_i b_j e_{\beta_j} \in \partial\mathcal{O}$ **then**
4   Let $\ell \in \{1, \dots, \nu\}$ be such that $x_i b_j e_{\beta_j} = b_\ell e_{\beta_\ell}$.
5   **else**
6         $k := \mathrm{face}(b_j e_{\beta_j})$
7   Let $\ell \in \{1, \dots, \nu\}$ be such that $x_i b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$.
8   **end**
9   **while** there exists a $t \in \mathrm{Supp}(p_j)$ with $\deg_{x_i}(t) = d$ **do**
10   **choose** $t \in \mathrm{Supp}(p_j)$ with $\deg_{x_i}(t) = d$.
11   Let $a \in K$ be the coefficient of t in p_j.
12         $(p_1, \dots, p_\nu) := (p_1, \dots, p_\nu) - a\frac{t}{x_i}\lambda_{j\ell}$
13   **end**
14 **end**
15 **return** $(p_1, \dots, p_\nu)$

Proposition 3.2.3. (Algorithmic Version of Degree Lowering)
Let $i \in \{1, \dots, n-1\}$, let $(p_1, \dots, p_\nu) \in P^\nu$, and let $d \in \mathbb{N}$ be maximal with the following properties: there exists an index $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is exposable, such that $\mathrm{face}(b_j e_{\beta_j}) > i$, and such that the support of p_j contains a term that is divisible by x_i^d. Assume that $d > 0$. Then Algorithm 4 is actually an algorithm and its result

$$(q_1, \dots, q_\nu) := \texttt{LowerDegree}(i, (p_1, \dots, p_\nu), \lambda)$$

applied to the input data i, $(p_1, \dots, p_\nu)$, and Λ is independent of the choice of t in line 10 and satisfies the following conditions.

1) *For every $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < i$, we have $q_\varrho = p_\varrho$.*

2) *Let $\varrho \in \{1, \dots, \nu\}$ and $s \in \{1, \dots, i\}$ be any indices such that p_ϱ and every polynomial in $\{p_u \mid u \in \{1, \dots, \nu\}, b_u e_{\beta_u}$ exposable, $\mathrm{face}(b_u e_{\beta_u}) > i\}$ are contained in $K[x_s, \dots, x_n]$. Then we have $q_\varrho \in K[x_s, \dots, x_n]$.*

3) For all $\varrho \in \{1, \dots, \nu\}$, every term in the support of q_ϱ that is divisible by x_i^d is also contained in the support of p_ϱ.

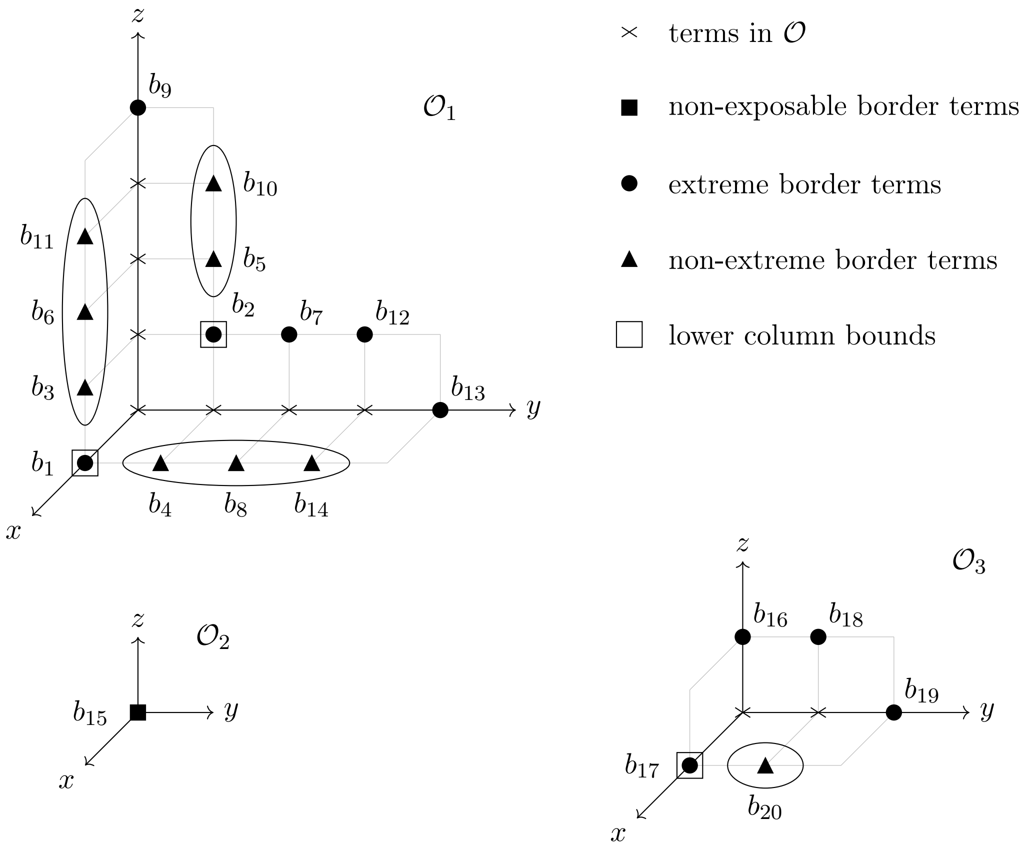
4) For every $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is exposable and such that $\mathrm{face}(b_\varrho e_{\beta_\varrho}) > i$, we have $q_\varrho = 0$ or $\deg_{x_i}(q_\varrho) \le d - 1$.

5) We have $(p_1, \dots, p_\nu) + \langle \Lambda \rangle = (q_1, \dots, q_\nu) + \langle \Lambda \rangle$ in $P^\nu / \langle \Lambda \rangle$.

Proof. First we prove that every step of the procedure can actually be computed and that the procedure terminates after finitely many steps. The existence of an $\ell \in \{1, \dots, \nu\}$ as required in the if-else-clause starting in line 3 follows from Proposition 3.2.2. Moreover, the foreach-loop in line 2 is obviously finite. Therefore, it suffices to prove that the while-loop starting in line 9 is processed only finitely many times. In order to show this, we prove that the number of terms in the support of $(p_1, \dots, p_\nu)$ with maximal x_i-degree decreases at least by one after every loop iteration. Let $t \in \mathrm{Supp}(p_j)$ with $\deg_{x_i}(t) = d$ be chosen as in line 10. We write $t = x_i^d t'$ with a term $t' \in \mathbb{T}^n$ that is not divisible by x_i. By the construction in the if-else-clause starting in line 3, the terms $b_j e_{\beta_j}$ and $b_\ell e_{\beta_\ell}$ are neighbours with respect to $\mathcal{O}$. According to Remark 2.4.28, the corresponding lifting $\lambda_{j\ell}$ has exactly one term with x_i-degree greater than or equal to 1 in its support, namely $x_i \varepsilon_j$. Let $a \in K$ be the coefficient of t in p_j as in line 11. Then the term $t\varepsilon_j$ cancels in the reduction $(p_1, \dots, p_\nu) - a\frac{t}{x_i}\lambda_{j\ell} = (p_1, \dots, p_\nu) - a x_i^{d-1} t' \lambda_{j\ell}$ of line 12. As all terms in $\mathrm{Supp}(\lambda_{j\ell}) \setminus \{x_i \varepsilon_j\}$ have x_i-degree 0, no new term with x_i-degree d is added to the support of $(p_1, \dots, p_\nu)$ in line 12. Hence the number of terms with the maximal x_i-degree in the support of p_j decreases exactly by one. Since the number of terms in the support of $(p_1, \dots, p_\nu)$ with maximal x_i-degree is finite, the while-loop eventually terminates.

Next we prove the correctness of the algorithm. Considering the while-loop in line 9 in detail, we see that we always subtract a vector of the form $a x_i^{d-1} t' \lambda_{j\ell}$ from $(p_1, \dots, p_\nu)$ where $t' \in \mathbb{T}^n$ is a term that is not divisible by x_i. As we have already seen above, exactly one term in the support of $(p_1, \dots, p_\nu)$ with maximal x_i-degree d cancels in line 12, namely the term $t\varepsilon_j = x_i^d t' \varepsilon_j$. If we collect all terms $t\varepsilon_j$ in the iterations of the while-loop, we see that, after the while-loop, we have altogether subtracted a vector of the form $p' x_i^{d-1} \lambda_{j\ell}$ from $(p_1, \dots, p_\nu)$, where $p' \in P \setminus \{0\}$. In particular, we have $p_j = p' x_i^d + p''$ with a polynomial $p'' \in P$ such that no term in the support of p'' is divisible by x_i^d. In other words, the conditions of Proposition 3.2.2 are satisfied during every iteration of the foreach-loop in line 2 and the claims 1) to 3) follow immediately from the corresponding claims in Proposition 3.2.2 and induction on j. In particular, we see that the result of the algorithm does not depend on the choice of t in line 10, since exactly one term with x_i-degree d vanishes in each reduction in line 12. The foreach-loop in line 2 iterates over all exposable border terms that are on faces greater than i. So claim 4) follows by induction on j from claim 4) of Proposition 3.2.2. Since the vector $(p_1, \dots, p_\nu)$ is only changed in line 12 of the algorithm and since we always subtract multiples of $\lambda_{j\ell} \in \Lambda$, claim 5) follows, too. □

Example 3.2.4. Consider the order ideal $\mathcal{O}$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ of Example 3.1.10, again. Recall that $P = \mathbb{Q}[x, y, z]$, that $\{e_1, e_2, e_3\}$ denoted the canonical P-module basis of P^3, and that we could illustrate the order ideal $\mathcal{O} = \{1, z, y, z^2, y^2, z^3, y^3\} \cdot e_1 \cup \{1, y\} \cdot e_3$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ and its border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_{20} e_{\beta_{20}}\}$ with the following figures.



Let $G = \{g_1, \ldots, g_{20}\} \subseteq P^3$ be with $g_1 = xe_1 - e_3$, $g_2 = yze_1$, $g_3 = xze_1$, $g_4 = xye_1 - ye_3$, $g_5 = yz^2e_1$, $g_6 = xz^2e_1$, $g_7 = y^2ze_1$, $g_8 = xy^2e_1$, $g_9 = z^4e_1$, $g_{10} = yz^3e_1$, $g_{11} = xz^3e_1$, $g_{12} = y^3ze_1$, $g_{13} = y^4e_1$, $g_{14} = xy^3e_1$, $g_{15} = e_2$, $g_{16} = ze_3$, $g_{17} = xe_3$, $g_{18} = yze_3$, $g_{19} = y^2e_3$, $g_{20} = xye_3$. Using Theorem 2.4.31, one can easily check that the set G is the $\mathcal{O}$-border basis of $\langle G\rangle$. Let $\{\varepsilon_1, \ldots, \varepsilon_{20}\}$ be the canonical P-module basis of P^{20}. For all $i, j \in \{1, \ldots, 20\}$ such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we denote the corresponding neighbor syzygy with respect to $\mathcal{O}$ by σ_{ij} as in Definition 2.4.21 and the corresponding neighbor lifting with respect to $\mathcal{O}$ by λ_{ij} as in Definition 2.4.24. Using Remark 2.4.28, we can compute the set of all neighbor liftings $\Lambda \subseteq \mathrm{Syz}_P(g_1, \ldots, g_{20})$ with respect to $\mathcal{O}$. In particular, it turns out that for all $i, j \in \{1, \ldots, 20\}$ such that $i < j$, such that $(i, j) \notin \{(1, 3), (2, 4), (3, 4), (4, 8)\}$, and such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we have $\lambda_{ij} = \sigma_{ij}$, and that $\lambda_{1,3} = \sigma_{1,3} + \varepsilon_{16}$, $\lambda_{2,4} = \sigma_{2,4} - y\varepsilon_{16}$, $\lambda_{3,4} = \sigma_{3,4} - y\varepsilon_{16}$, $\lambda_{4,8} = \sigma_{4,8} + \varepsilon_{19}$. Let $v = (p_1, \ldots, p_{20}) = -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16} \in P^{20}$. We now consider the Degree Lowering Algorithm 4 applied to the input data $i = 1$, v, and Λ in detail. Note that this means that we lower the x-degree of all components p_j

with $j \in \{1, \dots, 20\}$ where $b_j e_{\beta_j}$ is expsoable, $\mathrm{face}(b_j e_{\beta_j}) > 1$, and p_j has maximal x-degree.

In our situation, the only non-zero components of v correspond to the exposable border terms $xe_1 = b_1 e_{\beta_1} \in \mathcal{F}_1(\mathcal{O})$, $z^4 e_1 = b_9 e_{\beta_9} \in \mathcal{F}_3(\mathcal{O})$, and $ze_3 = b_{16} e_{\beta_{16}} \in \mathcal{F}_3(\mathcal{O})$ by Definition 3.1.1. As $x \in \mathrm{Supp}(p_9)$ is divisible by x, the requirements of Algorithm 4 are satisfied. Starting with line 1, we see that the maximal x-degree of terms that need to be considered is $d = \deg_x(x) = \deg_x(p_9) = 1$. As $\deg_x(p_1) = \deg_x(p_{16}) = 0$, we only need to consider the index $j = 9$ during the foreach-loop starting in line 2. Since $x \cdot b_9 e_{\beta_9} = xz^4 e_1 \notin \mathcal{O}$, the else-clause starting in line 5 is executed. In line 6, we get $k = \mathrm{face}(z^4 e_1) = 3$. As $x \cdot b_9 e_{\beta_9} = xz^4 e_1 = z \cdot b_{11} e_{\beta_{11}}$, we see that $\ell = 11$ in line 7. Now the while-loop starting in line 9 is executed. The only term of x-degree 1 in the support of p_9 is x. Thus we have to choose $x \in \mathrm{Supp}(p_1)$ in line 10 and get $a = 1$ in line 11. Since the neighbor lifting $\lambda_{9,11}$ is of the form $\lambda_{9,11} = \sigma_{9,11} = x\varepsilon_9 - z\varepsilon_{11}$, we update the value of $(p_1, \dots, p_{20})$ in line 12 to

$$(p_1, \dots, p_{20}) - \lambda_{9,11} = -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16} - \lambda_{9,11} = -z^4\varepsilon_1 + z\varepsilon_{11} - z^3\varepsilon_{16}.$$

Now the while-loop starting in line 9 stops. Since $p_{11} = z$ has x-degree 0, the foreach-loop starting in line 2 also stops. The algorithm finally returns the vector $-z^4\varepsilon_1 + z\varepsilon_{11} - z^3\varepsilon_{16}$ in line 15.

Next we examine the operation of "column clearing" as given in [Hui06, Lemma 29 and Coro. 30]. The following proposition provides particular versions of these results adapted to our setting. For the ideal version of this proposition, we refer to [KK14, Prop. 4.3].

Proposition 3.2.5. (Properties of Column Clearing)
Let $j \in \{1, \dots, \nu\}$ and $k \in \{2, \dots, n\}$ be such that $b_j e_{\beta_j}$ is non-extreme and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. We write $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \dots, x_k^t b_j\} \cdot e_{\beta_j}$ with natural numbers $s, t \in \mathbb{N}$ and we let $u_0, \dots, u_{s+t+1} \in \{1, \dots, \nu\}$ be such that $b_{u_0} e_{\beta_{u_0}} = x_k^{-s-1} b_j e_{\beta_j} = \mathrm{lcb}(b_j e_{\beta_j}), b_{u_1} e_{\beta_{u_1}} = x_k^{-s} b_j e_{\beta_j}, \dots, b_{u_{s+t+1}} e_{\beta_{u_{s+t+1}}} = x_k^t b_j e_{\beta_j}$. Given $(p_1, \dots, p_\nu) \in P^\nu$, we assume that there exists an index $v \in \{1, \dots, s+t+1\}$ such that $p_{u_v} \neq 0$, i.e. such that at least one of the components of $(p_1, \dots, p_\nu)$ corresponding to border terms in $\mathrm{Col}(b_j e_{\beta_j})$ is not zero. Let $\ell \in \{1, \dots, s+t+1\}$ denote the maximal index such that $p_{u_\ell} \neq 0$. Then

$$(q_1, \dots, q_\nu) = (p_1, \dots, p_\nu) - p_{u_\ell} \lambda_{u_\ell u_{\ell-1}}$$

satisfies the following conditions are satisfied.

1) *For all $\varrho \in \{1, \dots, \nu\} \setminus \{u_\ell, u_{\ell-1}\}$ such that the border term $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < k$, we have $q_\varrho = p_\varrho$.*

2) *For every $\varrho \in \{1, \dots, \nu\}$ and every $w \in \{1, \dots, n\} \setminus \{k\}$, we have either $q_\varrho - p_\varrho = 0$ or the x_w-degree of $q_\varrho - p_\varrho$ is equal to the x_w-degree of p_{u_ℓ}.*

3) For every $\varrho \in \{1, \dots, \nu\}$ such that the border term $b_\varrho e_{\beta_\varrho}$ is non-extreme, such that $b_\varrho e_{\beta_\varrho} \notin \mathrm{Col}(b_j e_{\beta_j})$, and such that the column of $b_\varrho e_{\beta_\varrho}$ is in the x_w-direction for some $w \in \{2, \dots, k\}$, we have $q_\varrho = p_\varrho$.

4) For every $v \in \{\ell, \dots, s+t+1\}$, we have $q_{u_v} = 0$.

Proof. Let $\lambda_{u_\ell u_{\ell-1}} = (f_1, \dots, f_\nu)$ be with $f_1, \dots, f_\nu \in P$. By Remark 2.4.28, the neighbor lifting $\lambda_{u_\ell u_{\ell-1}}$ is of the form $\lambda_{u_\ell u_{\ell-1}} = \varepsilon_{u_\ell} - x_k \varepsilon_{u_{\ell-1}} - c_1 \varepsilon_1 - \dots - c_\nu \varepsilon_\nu$ with $c_1, \dots, c_\nu \in K$. Hence we see that $f_\varrho = -c_\varrho$ for all $\varrho \in \{1, \dots, \nu\} \setminus \{u_\ell, u_{\ell-1}\}$. As the column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction, we have $\mathrm{face}(b_{u_v} e_{\beta_{u_v}}) < k$ for every index $v \in \{1, \dots, s+t+1\}$ by Definition 3.1.5. Moreover, for every index $\varrho \in \{1, \dots, \nu\}$ such that the border term $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < k$, Remark 3.2.1 yields $c_\varrho = 0$. Therefore, we see that $f_{u_\ell} = 1 - c_{u_\ell} = 1$ and that $f_{u_{\ell-1}} = -x_k - c_{u_{\ell-1}} = -x_k$ if $\ell > 1$. In particular, for every index $\varrho \in \{1, \dots, \nu\} \setminus \{u_\ell, u_{\ell-1}\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\mathrm{face}(b_\varrho e_{\beta_m}) < k$, we have $q_\varrho = p_\varrho - f_\varrho = p_\varrho$ and claim 1) follows.

For all $\varrho \in \{1, \dots, \nu\} \setminus \{u_\ell, u_{\ell-1}\}$ such that $q_\varrho - p_\varrho \neq 0$ and for all $w \in \{1, \dots, n\} \setminus \{k\}$, the shape of $\lambda_{u_\ell u_{\ell-1}}$ yields $\deg_{x_w}(q_\varrho - p_\varrho) = \deg_{x_w}(-p_{u_\ell} c_\varrho) = \deg_{x_w}(p_{u_\ell})$. Since we have already seen that $f_{u_\ell} = 1$, $q_{u_\ell} = p_{u_\ell} - p_{u_\ell} \cdot 1 = 0$. The equality $f_{u_{\ell-1}} = -x_k - c_{u_{\ell-1}}$ thus yields $\deg_{x_w}(q_{u_{\ell-1}} - p_{u_{\ell-1}}) = \deg_{x_w}(-p_{u_\ell} \cdot (-x_k - c_{u_{\ell-1}})) = \deg_{x_w}(p_{u_\ell})$ for all $w \in \{1, \dots, n\} \setminus \{k\}$. This proves claim 2).

In order to prove 3), suppose that there exists a $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho} \in \partial\mathcal{O}$ is non-extreme, such that $b_\varrho e_{\beta_\varrho} \notin \mathrm{Col}(b_j e_{\beta_j})$, and such that the column $\mathrm{Col}(b_\varrho e_{\beta_\varrho})$ is in the x_w-direction for some $w \in \{2, \dots, k\}$. Then we have $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < w \leq k$. By Lemma 3.1.9, we see that $b_\varrho e_{\beta_\varrho} \neq \mathrm{lcb}(b_j e_{\beta_j})$. Hence we have $f_\varrho = -c_\varrho = 0$. Therefore, we get $q_\varrho = p_\varrho - p_{u_\ell} \cdot 0 = p_\varrho$ and claim 3) follows.

For every index $v \in \{\ell+1, \dots, s+t+1\}$, we have $\mathrm{face}(b_{u_v} e_{\beta_{u_v}}) = \mathrm{face}(b_j e_{\beta_j}) < k$ by Definition 3.1.5 and thus $q_{u_v} = p_{u_v} = 0$ by the choice of ℓ and by claim 1). Since we have already shown that $q_{u_\ell} = 0$, claim 4) follows. □

The following proposition provides an algorithmic version of the process of "column clearing", cf. [Hui06, Lemma 32]. Since it is an essential building block for the main theorem of this chapter, we provide a detailed proof. For the ideal setting, the corresponding result was proven in [KK14, Prop. 4.4]. We generalize this proposition to the module setting.

Algorithm 5: $\texttt{ClearColumns}(i, (p_1, \dots, p_\nu), \Lambda)$

Input:
$i \in \{1, \dots, n-1\}$
$(p_1, \dots, p_\nu) \in P^\nu$
$\Lambda = \{\lambda_{ij} \mid i, j \in \{1, \dots, \nu\}, b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neigbhors with respect to $\mathcal{O}\}$

1 **for** $k = i+1, \dots, n$ **do**
2   **foreach** $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme, such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction, such that $i \leq \mathrm{face}(b_j e_{\beta_j})$, and such that $p_j \neq 0$ **do**
3   Let $s, t \in \mathbb{N}$ be such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \dots, x_k^t b_j\} \cdot e_{\beta_j}$.
4   **for** $\ell := 0$ **to** $s+t+1$ **do**
5   Let $u_\ell \in \{1, \dots, \nu\}$ be such that $b_{u_\ell} e_{\beta_{u_\ell}} = x_k^{\ell-s-1} b_j e_{\beta_j}$.
6   **end**
7   **for** $\ell := s+t+1$ **to** 1 **step** -1 **do**
8   **while** $p_{u_\ell} \neq 0$ **do**
9   **choose** $t' \in \mathrm{Supp}(p_{u_\ell})$
10   Let $a \in K$ be the coefficient of t' in p_{u_ℓ}.
11                 $(p_1, \dots, p_\nu) := (p_1, \dots, p_\nu) - a t' \lambda_{u_\ell u_{\ell-1}}$
12   **end**
13   **end**
14   **end**
15 **end**
16 **return** $(p_1, \dots, p_\nu)$

Proposition 3.2.6. (Algorithmic Version of Column Clearing)
Let $i \in \{1, \dots, n-1\}$ and $(p_1, \dots, p_\nu) \in P^\nu$. Then Algorithm 5 is actually an algorithm and its result

$$(q_1, \dots, q_\nu) := \texttt{ClearColumns}(i, (p_1, \dots, p_\nu), \Lambda)$$

applied to the input data i, $(p_1, \dots, p_\nu)$, and Λ is independent of the choice of t' in line 9 and satisfies the following conditions.

1) *For every $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ in non-exposable or $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < i$, we have $q_\varrho = p_\varrho$.*

2) *For every $\varrho \in \{1, \dots, \nu\}$ and every $w \in \{1, \dots, i\}$, we have either $q_\varrho - p_\varrho = 0$ or the x_w-degree of $q_\varrho - p_\varrho$ is less than or equal to the maximal x_w-degree of the polynomials in $\{p_u \mid u \in \{1, \dots, \nu\}, p_u \neq 0, b_u e_{\beta_u}$ non-extreme, $\mathrm{face}(b_u e_{\beta_u}) \geq i\}$.*

3) *For every $\varrho \in \{1, \dots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-extreme and $\mathrm{face}(b_\varrho e_{\beta_\varrho}) \geq i$, we have $q_\varrho = 0$.*

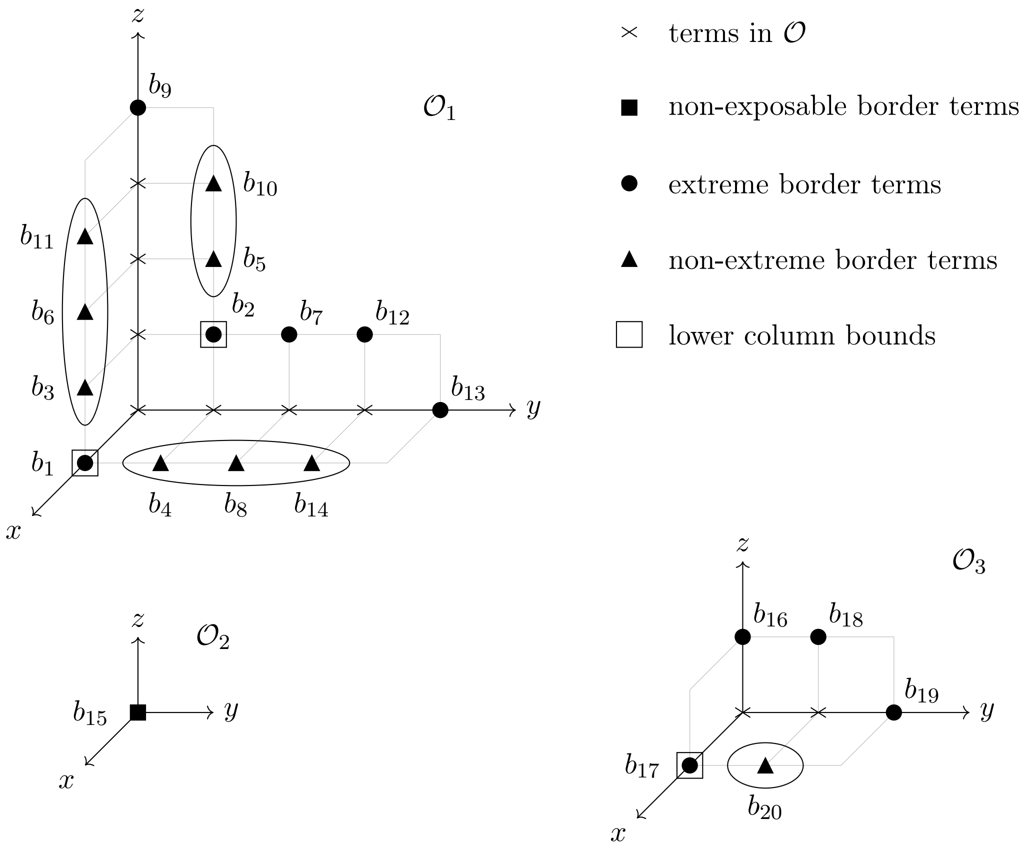
4) *We have $(p_1, \dots, p_\nu) + \langle \Lambda \rangle = (q_1, \dots, q_\nu) + \langle \Lambda \rangle$ in $P^r / \langle \Lambda \rangle$.*

Proof. First we prove that every step of the procedure can actually be executed, that the procedure terminates after finitely many steps, and that the result does not depend on the choice of the term t in line 9. The existence of the natural numbers $s, t \in \mathbb{N}$ as in line 3 follows from Lemma 3.1.6. According to Definition 3.1.5 and Lemma 3.1.9, every element of $\{\mathrm{lcb}(b_j e_{\beta_j})\} \cup \mathrm{Col}(b_j e_{\beta_j})$ is contained in $\partial\mathcal{O}$. Thus there exist indices $u_0, \ldots, u_{s+t+1} \in \{1, \ldots, \nu\}$ as required in line 5 during the for-loop starting in line 4.
Now it only remains to prove termination of the procedure. Since there are only finitely many border terms, the foreach-loop starting in line 2 stops after finitely many steps. Thus it remains to show that the while-loop in line 8 is finite. Let $\ell \in \{1, \ldots, s+t+1\}$ be chosen as in line 7. According to Definition 3.1.5, $b_{u_\ell} e_{\beta_{u_\ell}} \in \mathrm{Col}(b_j e_{\beta_j})$ yields that $i \leq \mathrm{face}(b_{u_\ell} e_{\beta_{u_\ell}}) = \mathrm{face}(b_j e_{\beta_j}) < k$. Let $\lambda_{u_\ell u_{\ell-1}} = (f_1, \ldots, f_\nu)$ be with $f_1, \ldots, f_\nu \in P$. Then Remark 3.2.1 yields $f_{u_\ell} = 1$. Let $t' \in \mathrm{Supp}(p_{u_\ell})$ be as in line 9 and let $a \in K$ be the coefficient of t' in p_{u_ℓ} as in line 10. Since $f_{u_\ell} = 1$, the construction of a and t' implies that the term t' is not contained in the support of $p_{u_\ell} - at' f_{u_\ell} = p_{u_\ell} - at'$ in line 11. Thus the number of terms in the support of p_{u_ℓ} decreases exactly by one during each iteration of the while-loop and hence the while-loop is finite. In particular, as $f_{u_\ell} = 1$, we see that exactly the term t cancels in $p_{u_\ell} - at' f_{u_\ell} = p_{u_\ell} - at'$ in line 11. Since the while-loop starting in line 8 iterates over all terms contained in the support of p_{u_ℓ}, it follows that different choices of t' in line 9 do not interfere with one another. Hence the final result of the while-loop, and hence the whole algorithm, is independent of the order in which the terms in the support of p_{u_ℓ} are handled.
Next we consider the for-loop starting in line 7. Let $k \in \{i+1, \ldots, n\}$ be chosen as in line 1. Moreover, let $j \in \{1, \ldots, \nu\}$ be such that the border term $b_j e_{\beta_j}$ is non-extreme, such that the column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction, and such that $i \leq \mathrm{face}(b_j e_{\beta_j})$ as in line 2. Let $s, t \in \mathbb{N}$ be such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \ldots, x_k^t b_j\} \cdot e_{\beta_j}$ as in line 3 and let $u_0, \ldots, u_{s+t+1} \in \{1, \ldots, \nu\}$ be such that $b_{u_\ell} e_{\beta_{u_\ell}} = x_k^{\ell-s-1} b_j e_{\beta_j}$ for all $\ell \in \{0, \ldots, s+t+1\}$ as in line 5 during the for-loop starting in line 4.
We now show by downward induction on the loop variable $\ell \in \{s+t+1, \ldots, 1\}$ of the for-loop starting in line 7 that the following two properties are satisfied: At the beginning of each iteration of this for-loop that changes the value of $(p_1, \ldots, p_\nu)$ in line 11, the assumptions of Proposition 3.2.5 are satisfied. And at the end of each iteration of this for-loop, we have $p_{u_{s+t+1}} = p_{u_{s+t}} = \cdots = p_{u_\ell} = 0$.
For the induction start, we let $\ell = s+t+1$. If we have $p_{u_\ell} = 0$ here, the while-loop starting in line 8 has no effect. Suppose that $p_{u_\ell} \neq 0$. Collecting all monomials at occurring during the reduction steps in line 11, we see that their sum is exactly p_{u_ℓ}. Thus considering all reductions in line 11 during the while-loop simultaneously, we see that the assumptions of Proposition 3.2.5 are satisfied at the beginning of this iteration of the for-loop starting in line 7. In particular, claim 4) of Proposition 3.2.5 yields $p_{u_{s+t+1}} = p_{u_\ell} = 0$ after the first iteration of the for-loop starting in line 7.
Now let $\ell \in \{s+t+1, \ldots, 1\}$ and assume that $p_{u_{s+t+1}} = p_{u_{s+t}} = \cdots = p_{u_{\ell+1}} = 0$. If we have $p_{u_\ell} = 0$ here, the while-loop starting in line 8 has no effect. Thus suppose that $p_{u_\ell} \neq 0$. Considering all reductions of the inner for-loop simultaneously as in the induction start, we see that the assumptions of Proposition 3.2.5 are satisfied at the

beginning of this iteration of the for-loop. Moreover, claim 4) of Proposition 3.2.5 yields $p_{u_{s+t+1}} = p_{u_{s+t}} = \cdots = p_{u_\ell} = 0$.

Next we show that the four claims hold. As $\mathrm{face}(b_j e_{\beta_j}) \geq i$ by line 2, Lemma 3.1.9 yields $\mathrm{face}(\mathrm{lcb}(b_j e_{\beta_j})) \geq i$. Thus the claims 1) and 2) follow immediately from the claims 1) and 2) of Proposition 3.2.5. Now we show claim 3) by considering the changes of $(p_1, \ldots, p_\nu)$ during the consecutive iterations of the outer for loop starting in line 1. We begin with the first iteration, i.e. with $k = i+1$. Let $j \in \{1, \ldots, \nu\}$ be chosen as in line 2, i.e. such that the border term $b_j e_{\beta_j}$ is non-extreme, such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_{i+1}-direction, and such that $\mathrm{face}(b_j e_{\beta_j}) = i$. As we have already proven in the induction above, we have $p_\varrho = 0$ for all $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho} \in \mathrm{Col}(b_j e_{\beta_j})$ at the end of the inner for-loop starting in line 7. Moreover, we have also shown that Proposition 3.2.5 always holds for the reductions in line 11 during the while-loop starting in line 8. Thus claim 3) of Proposition 3.2.5 implies that for all $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-extreme, such that $\mathrm{Col}(b_\varrho e_{\beta_\varrho}) \neq \mathrm{Col}(b_j e_{\beta_j})$, and such that $\mathrm{Col}(b_\varrho e_{\beta_\varrho})$ is in the x_w-direction with $w \in \{2, \ldots, i+1\}$, the component p_ϱ remains unchanged. In other words, we have cleared the components of $(p_1, \ldots, p_\nu)$ corresponding to the border terms in $\mathrm{Col}(b_j e_{\beta_j})$, whereas all the components corresponding to non-extreme border terms which are contained in a column different from $\mathrm{Col}(b_j e_{\beta_j})$ and whose column is in the x_w-direction for some $w \in \{2, \ldots, i+1\}$ remain unchanged. In particular, we see that two different iterations of the foreach-loop starting in line 2 do not affect one another. Thus induction on the loop variable of the foreach-loop starting in line 2 yields that for every $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-extreme, such that $\mathrm{face}(b_\varrho e_{\beta_\varrho}) \geq i$, and such that $\mathrm{Col}(b_\varrho e_{\beta_\varrho})$ is in the x_{i+1}-direction, we have $p_\varrho = 0$ at the end of the execution of the outer for-loop with $k = i+1$.
Now we consider the next iteration of the outer for-loop starting in line 1, i.e. the iteration with $k = i+2$. As before, this iteration clears all the components of $(p_1, \ldots, p_\nu)$ corresponding to non-extreme border terms which are contained in faces greater than or equal to i and whose columns are in the x_{i+2}-direction, whereas all the components corresponding to non-extreme border terms which are contained in a column that is in the x_w-direction for some $w \in \{2, \ldots, i+1\}$ remain unchanged. In particular, all the components corresponding to non-extreme border terms which are contained in a column that is in the x_{i+1}-direction, i.e. all the components that have been cleared during the previous iteration, remain unchanged. After the second iteration of the outer for-loop starting in line 1, we have then cleared all the components of $(p_1, \ldots, p_\nu)$ corresponding to non-extreme border terms which are on faces greater than or equal to i and whose columns are in the x_w-direction where $w \in \{i+1, i+2\}$.
Continuing in this way, induction on k shows that at the end of the algorithm, we have cleared all the components of $(p_1, \ldots, p_\nu)$ corresponding to non-extreme border terms which are on faces greater than or equal to i and whose columns are in the x_w-direction for some $w \in \{i+1, i+2, \ldots, n\}$. In other words, claim 3) has been proven. Furthermore, claim 4) follows immediately as the vector $(p_1, \ldots, p_\nu)$ is only changed in line 11 during the whole algorithm. $\square$

Example 3.2.7. Consider the situation of Example 3.2.4, again. Recall that we had $P = \mathbb{Q}[x, y, z]$, that $\{e_1, e_2, e_3\}$ denoted the canonical P-module basis of P^3, and that we could illustrate the order ideal $\mathcal{O} = \{1, z, y, z^2, y^2, z^3, y^3\} \cdot e_1 \cup \{1, y\} \cdot e_3$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ and its border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_{20} e_{\beta_{20}}\}$ with the following figures.



Moreover, $G = \{g_1, \ldots, g_{20}\} \subseteq P^3$ denoted the $\mathcal{O}$-border basis of $\langle G\rangle$, $\{\varepsilon_1, \ldots, \varepsilon_{20}\}$ denoted the canonical P-module basis of P^{20}, and $\Lambda \subseteq \mathrm{Syz}_P(g_1, \ldots, g_{20})$ denoted the set of all neighbor liftings with respect to $\mathcal{O}$. For all $i, j \in \{1, \ldots, 20\}$ such that $i < j$, $(i, j) \notin \{(1, 3), (2, 4), (3, 4), (4, 8)\}$, and such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$ the corresponding neighbor liftings with respect to $\mathcal{O}$ were of the form $\lambda_{ij} = \sigma_{ij}$, and we saw that $\lambda_{1,3} = \sigma_{1,3} + \varepsilon_{16}$, $\lambda_{2,4} = \sigma_{2,4} - y\varepsilon_{16}$, $\lambda_{3,4} = \sigma_{3,4} - y\varepsilon_{16}$, $\lambda_{4,8} = \sigma_{4,8} + \varepsilon_{19}$. We now consider the Column Clearing Algorithm 5 applied to $i = 1$, $v = (p_1, \ldots, p_{20}) = (-z^4 + y^3)\varepsilon_1 + x\varepsilon_9 - \varepsilon_{14} - z^3\varepsilon_{16} + y\varepsilon_{19}$, and Λ in detail.

In line 1, we start with $k = i + 1 = 2$. Since $b_1 e_{\beta_1} = xe_1$, $b_9 e_{\beta_9} = z^4 e_1$, $b_{16} e_{\beta_{16}} = ze_3$, and $b_{19} e_{\beta_{19}} = y^2 e_3$ are all extreme border terms by Definition 3.1.1, we only have to consider the non-extreme border term $b_{14} e_{\beta_{14}} = xy^3 e_1 \in \mathcal{F}_1(\mathcal{O})$ during the foreach-loop starting in line 2, i.e. the case $j = 14$. According to Definition 3.1.5 and Lemma 3.1.6, $\mathrm{Col}(b_{14} e_{\beta_{14}}) = \{y^{-2} b_{14}, y^{-1} b_{14}, b_{14}\} \cdot e_{\beta_{14}} = \{xy, xy^2, xy^3\} \cdot e_1 = \{b_4 e_{\beta_4}, b_8 e_{\beta_8}, b_{14} e_{\beta_{14}}\}$. Moreover, $\mathrm{lcb}(b_{14} e_{\beta_{14}}) = xe_1 = b_1 e_{\beta_1}$ by Definition 3.1.8. Thus we have $s = 2$ and $t = 0$ after line 3 and we have $u_0 = 1$, $u_1 = 4$, $u_2 = 8$, and $u_3 = 14$ after the for-loop starting in line 4. During the first iteration of the for-loop starting in line 7, we have $\ell = s+t+1 = 3$. The while-loop starting in line 8 is executed as $p_{u_\ell} = p_{u_3} = p_{14} = -1 \neq 0$. In line 9, we hence have to choose the term $t = 1 \in \mathrm{Supp}(p_{14})$ and, therefore, get $a = -1$ in line 10. Since $\lambda_{u_3,u_2} = \lambda_{14,8} = \sigma_{14,8} = \varepsilon_{14} - y\varepsilon_8$, we update the value of $(p_1, \dots, p_{20})$ to

$$\begin{aligned}(p_1, \dots, p_{20}) + \lambda_{14,8} &= (-z^4 + y^3)\varepsilon_1 + x\varepsilon_9 - \varepsilon_{14} - z^3\varepsilon_{16} + y\varepsilon_{19} + \lambda_{14,8}\\ &= (-z^4 + y^3)\varepsilon_1 - y\varepsilon_8 + x\varepsilon_9 - z^3\varepsilon_{16} + y\varepsilon_{19}.\end{aligned}$$

Since we have $p_{14} = 0$, now the while-loop terminates and the next iteration of the for-loop starting in line 7 with $\ell = 2$ begins. The while-loop starting in line 8 is executed as $p_{u_\ell} = p_{u_2} = p_8 = -y \neq 0$. In line 9, we hence have to choose the term $t = y \in \mathrm{Supp}(p_8)$ and, therefore, get $a = -1$ in line 10. Since $\lambda_{u_2,u_1} = \lambda_{8,4} = \sigma_{8,4} - \varepsilon_{19} = \varepsilon_8 - y\varepsilon_4 - \varepsilon_{19}$, we update the value of $(p_1, \dots, p_{20})$ to

$$\begin{aligned}(p_1, \dots, p_{20}) + y \cdot \lambda_{8,4} &= (-z^4 + y^3)\varepsilon_1 - y\varepsilon_8 + x\varepsilon_9 - z^3\varepsilon_{16} + y\varepsilon_{19} + y \cdot \lambda_{8,4}\\ &= (-z^4 + y^3)\varepsilon_1 - y^2\varepsilon_4 + x\varepsilon_9 - z^3\varepsilon_{16}.\end{aligned}$$

Since we have $p_8 = 0$, now the while-loop terminates and the next iteration of the for-loop starting in line 7 with $\ell = 1$ begins. In the next iteration, we subtract the vector $y^2 \cdot \lambda_{4,1}$ where $\lambda_{4,1} = \sigma_{4,1} = \varepsilon_4 - y\varepsilon_1$ from $(p_1, \dots, p_{20})$ and get

$$\begin{aligned}(p_1, \dots, p_{20}) - y^2 \cdot \lambda_{4,1} &= (-z^4 + y^3)\varepsilon_1 - y^2\varepsilon_4 + x\varepsilon_9 - z^3\varepsilon_{16} + y^2 \cdot \lambda_{4,1}\\ &= -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16}.\end{aligned}$$

Now the for-loop starting in line 7 terminates. Moreover, as all components of the vector $(p_1, \dots, p_{20}) = -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16}$ belonging to non-extreme border terms are zero, the foreach-loop starting in line 2 also terminates.
In the next iteration of the for-loop starting in line 1, nothing happens as there is no component of $(p_1, \dots, p_{20})$ belonging to non-extreme border terms that is not equal to zero and thus the foreach-loop starting in line 2 has no effect. Altogether, the algorithm terminates and returns the vector $(p_1, \dots, p_{20}) = -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16}$ in line 16.

3.3 The Reduction Algorithm

In this Section, we combine the Degree Lowering and the Column Clearing Algorithms in a way which is similar to the method in [Hui06, Section 6.6]. However, we turn the method indicated by the proof of [Hui06, Lemma 33] into an explicit algorithm. This algorithm will be suitable to serve as a basis for the construction of certain term orderings for which the set of neighbor liftings Λ with respect to $\mathcal{O}$ form a Gröbner basis of the syzygy module of the $\mathcal{O}$-border basis G. For the ideal setting, the corresponding result was proven in [KK14, Section 5]. We generalize this proposition to the module setting.

As in the previous section, we let $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ denote the canonical P-module basis of P^ν and we assume that G is the $\mathcal{O}$-border basis of $\langle G \rangle$. For all $i, j \in \{1, \dots, \nu\}$ such that $b_ie_{\beta_i}$ and $b_je_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we let σ_{ij} denote the corresponding neighbor syzygy with respect to $\mathcal{O}$ and we let λ_{ij} denote the corresponding neighbor lifting with respect to $\mathcal{O}$. Moreover, we denote the set of all neighbor liftings with respect to $\mathcal{O}$ by Λ.

As in [Hui06, Section 6.6] and [KK14, Defn. 5.1], we measure the progress of the reduction algorithm using the following notion.

Definition 3.3.1. Let $i \in \{1, \ldots, n\}$. We say that a vector $(p_1, \ldots, p_\nu) \in P^\nu$ is **i-reduced** if the following conditions are satisfied for every $j \in \{1, \ldots, \nu\}$.

a) If the border term $b_je_{\beta_j}$ is non-extreme, we require that $p_j = 0$.

b) If the border term $b_je_{\beta_j}$ is extreme, we let $\ell = \min\{i, \operatorname{face}(b_je_{\beta_j})\}$. Then we require that $p_j \in K[x_\ell, \ldots, x_n]$.

The following lemma provides an extended explicit version of some arguments contained in the proof of [Hui06, Lemma 33]. As before, an ideal version of it can be found in [KK14, Lemma 5.2].

Lemma 3.3.2. *Let $i \in \{1, \ldots, n-1\}$ and let $(p_1, \ldots, p_\nu) \in P^\nu$ be i-reduced. We assume that $d = \max\{\deg_{x_i}(p_j) \mid j \in \{1, \ldots, \nu\}, p_j \neq 0, b_je_{\beta_j}$ is exposable$, \operatorname{face}(b_je_{\beta_j}) > i\}$ exists and that this number is strictly positive. We let*

$$(q_1, \ldots, q_\nu) \coloneqq \texttt{ClearColumns}(i, \texttt{LowerDegree}(i, (p_1, \ldots, p_\nu), \Lambda), \Lambda)$$

be the result of first applying Algorithm 4 and then Algorithm 5 to the input data i, $(p_1, \ldots, p_\nu)$, and Λ. Then the following conditions are satisfied.

1) The vector $(q_1, \ldots, q_\nu)$ is i-reduced.

2) For every $j \in \{1, \ldots, \nu\}$ such that $b_je_{\beta_j}$ is exposable and such that $\operatorname{face}(b_je_{\beta_j}) > i$, we have $q_j = 0$ or $\deg_{x_i}(q_j) \leq d - 1$.

3) We have $(p_1, \ldots, p_\nu) + \langle\Lambda\rangle = (q_1, \ldots, q_\nu) + \langle\Lambda\rangle$ in $P^\nu / \langle\Lambda\rangle$.

Proof. To ease the notation, we let

$$(q'_1, \ldots, q'_\nu) \coloneqq \texttt{LowerDegree}(i, (p_1, \ldots, p_\nu), \Lambda)$$

be the result of applying Algorithm 4 to the input data i, $(p_1, \ldots, p_\nu)$, and Λ.

We start by proving that $(q_1, \ldots, q_\nu)$ is i-reduced. According to Definition 3.3.1, we have to distinguish two cases.

First suppose that there is a $j \in \{1, \ldots, \nu\}$ such that $b_je_{\beta_j}$ is non-extreme. Since the vector $(p_1, \ldots, p_\nu)$ is i-reduced, we have $p_j = 0$ by Definition 3.3.1. If $\operatorname{face}(b_je_{\beta_j}) < i$, the claims 1) of the Propositions 3.2.3 and 3.2.6 yield $q_j = q'_j = p_j = 0$. If $\operatorname{face}(b_je_{\beta_j}) \geq i$, claim 3) of Proposition 3.2.6 yields $q_j = 0$.

Second we let $j \in \{1, \ldots, \nu\}$ be such that $b_je_{\beta_j} \in \partial\mathcal{O}$ is extreme. Let $k = \operatorname{face}(b_je_{\beta_j})$

and $\ell = \min\{i, k\}$. Since $(p_1, \ldots, p_\nu)$ is i-reduced, we have $p_j \in K[x_\ell, \ldots, x_n]$ by Definition 3.3.1. Moreover, for every $u \in \{1, \ldots, \nu\}$ such that $b_u e_{\beta_u}$ is exposable and $\text{face}(b_u e_{\beta_u}) > i$, Definition 3.3.1 yields $p_u \in K[x_i, \ldots, x_n] \subseteq K[x_\ell, \ldots, x_n]$. Thus claim 2) of Proposition 3.2.3 implies $q_j' \in K[x_\ell, \ldots, x_n]$. If $k < i$, claim 1) of Proposition 3.2.6 yields $q_j = q_j' \in K[x_\ell, \ldots, x_n]$. Thus we may suppose that $k \geq i$. Then we have $\ell = i$. Suppose there exists a $u \in \{1, \ldots, \nu\}$ such that $b_u e_{\beta_u}$ is non-extreme and $\text{face}(b_u e_{\beta_u}) \geq i$. Then, similar to above, claim 2) of Proposition 3.2.3 together with Definition 3.3.1 yield $q_u' \in K[x_i, \ldots, x_n]$ as $(p_1, \ldots, p_\nu)$ is i-reduced. In particular, for every $w \in \{1, \ldots, i-1\}$, we have $q_u' = 0$ or the x_w-degree of q_u' equals 0. Since we have $q_j' \in K[x_i, \ldots, x_n]$, it follows now that $q_j - q_j' \in K[x_i, \ldots, x_n]$ by claim 2) of Proposition 3.2.6 and thus $q_j \in K[x_i, \ldots, x_n]$. Altogether, the conditions of Definition 3.3.1 are satisfied, i.e. we have proved that $(q_1, \ldots, q_\nu)$ is i-reduced.

Next we show that claim 2) holds. Let $j \in \{1, \ldots, \nu\}$ be such that $b_j e_{\beta_j}$ is exposable and $\text{face}(b_j e_{\beta_j}) > i$. If $b_j e_{\beta_j}$ is non-extreme, we have $q_j = 0$ by claim 3) of Proposition 3.2.6. Thus we may suppose that $b_j e_{\beta_j}$ is extreme by Definition 3.1.1. Then we have $q_j' = 0$ or $\deg_{x_i}(q_j') \leq d - 1$ by claim 4) of Proposition 3.2.3. Additionally, suppose that there exists a $u \in \{1, \ldots, \nu\}$ such that $b_u e_{\beta_u}$ is non-extreme and such that $\text{face}(b_u e_{\beta_u}) \geq i$. Since $(p_1, \ldots, p_\nu)$ is i-reduced, we have $p_u = 0$. Hence we have $q_u' = 0$ or the x_i-degree of q_u' is less than or equal to $d - 1$ according to claim 3) of Proposition 3.2.3. Altogether, claim 2) of Proposition 3.2.6 now implies that either $q_j = 0$ or the x_i-degree of q_j is less than or equal to $d - 1$. Thus claim 2) follows.

Claim 3) is a direct consequence of claim 5) of Proposition 3.2.3 and claim 4) of Proposition 3.2.6. □

Now we are ready to present the main result of this section: an explicit Reduction Algorithm whose result is a vector which is equivalent to the given vector modulo the submodule $\langle \Lambda \rangle \subseteq P^\nu$ generated by the set of neighbor liftings with respect to $\mathcal{O}$ and which is n-reduced. This proposition generalizes [KK14, Prop. 5.3] to the module setting. The corresponding idea was originally presented in the proof of [Hui06, Lemma 33].

Algorithm 6: `Reduce`$((p_1, \ldots, p_\nu), \Lambda)$

Input:
$(p_1, \ldots, p_\nu) \in P^\nu$
$\Lambda = \{\lambda_{ij} \mid i, j \in \{1, \ldots, \nu\}, b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neigbhors with respect to $\mathcal{O}\}$

$(p_1, \ldots, p_\nu) := $ `ClearColumns`$(1, (p_1, \ldots, p_\nu), \Lambda)$
for $i := 1$ **to** $n - 1$ **do**
  while there exist a $j \in \{1, \ldots, \nu\}$ and a $t \in \text{Supp}(p_j)$ such that $b_j e_{\beta_j}$ is exposable, such that $\text{face}(b_j e_{\beta_j}) > i$, and such that $\deg_{x_i}(t) > 0$ **do**
    $(p_1, \ldots, p_\nu) := $ `ClearColumns`$(i,$ `LowerDegree`$(i, (p_1, \ldots, p_\nu), \Lambda), \Lambda)$
  end
end
return $(p_1, \ldots, p_\nu)$

Theorem 3.3.3. (The Reduction Algorithm)
Let $(p_1, \dots, p_\nu) \in P^\nu$. Then Algorithm 6 is actually an algorithm and its result

$$(q_1, \dots, q_\nu) \coloneqq \mathtt{Reduce}((p_1, \dots, p_\nu), \Lambda)$$

applied to the input data $(p_1, \dots, p_\nu)$ and Λ satisfies the following conditions.

1) The vector $(q_1, \dots, q_\nu)$ is n-reduced.

2) We have $(p_1, \dots, p_\nu) + \langle \Lambda \rangle = (q_1, \dots, q_\nu) + \langle \Lambda \rangle$ in $P^\nu / \langle \Lambda \rangle$.

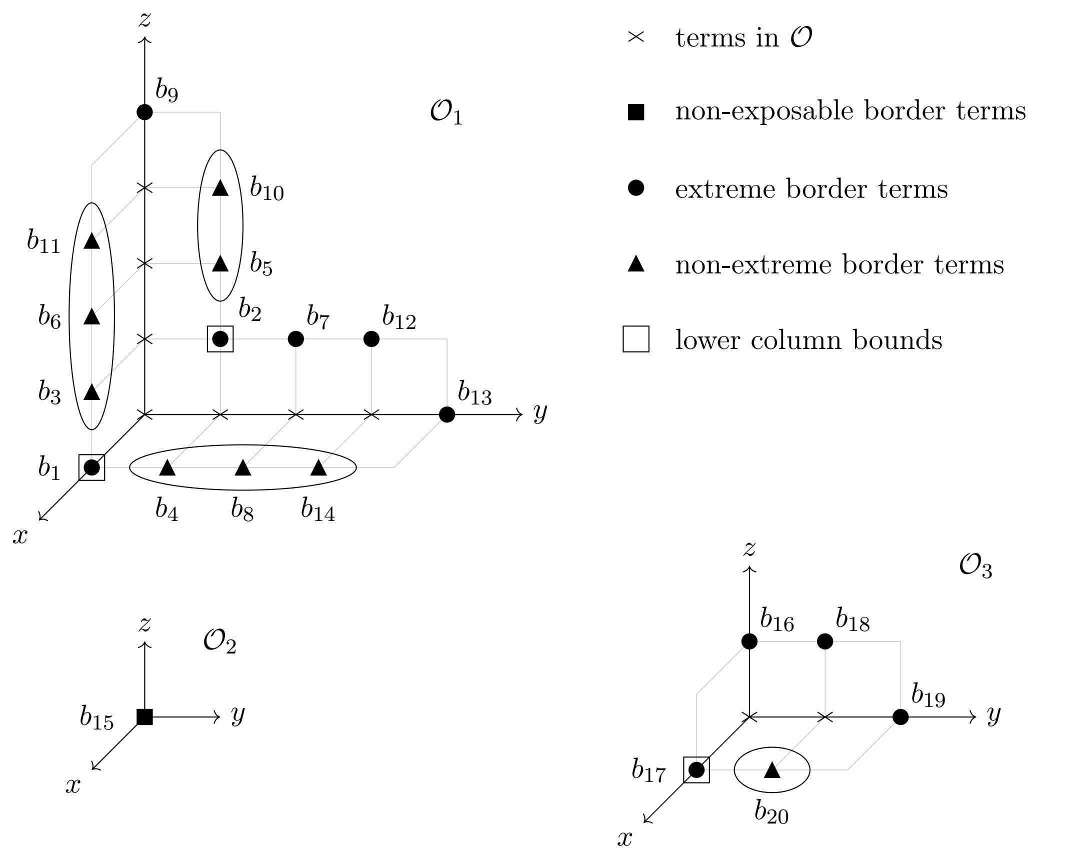
Proof. As the vector $(p_1, \dots, p_\nu)$ in Algorithm 6 is only changed in the lines 1 and 4, claim 2) follows by induction from claim 4) of Proposition 3.2.6 and from claim 3) of Lemma 3.3.2.

Thus it remains to prove claim 1). We let

$$(p_1^{(0)}, \dots, p_\nu^{(0)}) \coloneqq \mathtt{ClearColumns}(1, (p_1, \dots, p_\nu), \Lambda),$$

i.e. we let $(p_1^{(0)}, \dots, p_\nu^{(0)})$ denote the value of $(p_1, \dots, p_\nu)$ before the first iteration of the for-loop starting in line 2. Moreover, for all $i \in \{1, \dots, n-1\}$, we let $(p_1^{(i)}, \dots, p_\nu^{(i)}) \in P^\nu$ be the value of $(p_1, \dots, p_\nu)$ during Algorithm 6 applied to the input data $(p_1, \dots, p_\nu)$ and Λ after the i-th iteration of the for-loop which starts in line 2. We prove by induction on $i \in \{0, \dots, n-1\}$ that $(p_1^{(i)}, \dots, p_\nu^{(i)})$ is $(i+1)$-reduced.
In order to prove the induction start $i = 0$, we note that claim 3) of Proposition 3.2.6 implies that $p_j^{(0)} = 0$ for every index $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme. According to Definition 3.3.1, this means that $(p_1^{(0)}, \dots, p_\nu^{(0)})$ is 1-reduced.
For the induction step, let now $i \in \{1, \dots, n-1\}$. Then the induction hypothesis says that $(p_1^{(i-1)}, \dots, p_\nu^{(i-1)})$ is i-reduced. In other words, for every $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme, we have $p_j^{(i-1)} = 0$, and for every $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is extreme and such that $\ell = \min\{i, \operatorname{face}(b_j e_{\beta_j})\}$, we have $p_j^{(i-1)} \in K[x_\ell, \dots, x_n]$ by Definition 3.3.1. If the while-loop in line 3 is not executed, we have $p_j^{(i-1)} \in K[x_{i+1}, \dots, x_n]$ for every $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is extreme and such that $\operatorname{face}(b_j e_{\beta_j}) > i$. Hence we see that the vector $(p_1^{(i)}, \dots, p_\nu^{(i)}) = (p_1^{(i-1)}, \dots, p_\nu^{(i-1)})$ is $(i+1)$-reduced by Definition 3.3.1 in this case. If the while-loop in line 3 is executed, Lemma 3.3.2 shows that after each iteration of the while-loop starting in line 3, the resulting vector continues to be i-reduced and that for every $j \in \{1, \dots, \nu\}$ such that $p_j \neq 0$, such that $b_j e_{\beta_j}$ is exposable, and such that $\operatorname{face}(b_j e_{\beta_j}) > i$, the maximal x_i-degree of the component p_j has decreased at least by 1. In particular, it follows that the while-loop, and thus the whole procedure, is finite. After finitely many iterations of the while-loop in line 3, for every $j \in \{1, \dots, \nu\}$ such that the border term $b_j e_{\beta_j}$ is exposable and such that $\operatorname{face}(b_j e_{\beta_j}) > i$, we eventually have $p_j = 0$ or $\deg_{x_i}(p_j) = 0$. Hence the resulting vector $(p_1^{(i)}, \dots, p_\nu^{(i)})$ is $(i+1)$-reduced. Altogether, we see that $(q_1, \dots, q_\nu) = (p_1^{(n-1)}, \dots, p_\nu^{(n-1)})$ is n-reduced and claim 1) is proven. □

Example 3.3.4. Consider the situation of Example 3.2.4 again. Recall that we had $P = \mathbb{Q}[x,y,z]$, that $\{e_1, e_2, e_3\}$ denoted the canonical P-module basis of P^3, and that we could illustrate the order ideal $\mathcal{O} = \{1, z, y, z^2, y^2, z^3, y^3\} \cdot e_1 \cup \{1, y\} \cdot e_3$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ and its border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_{20} e_{\beta_{20}}\}$ with the following figures.



Moreover, $G = \{g_1, \ldots, g_{20}\} \subseteq P^3$ denoted the $\mathcal{O}$-border basis of $\langle G\rangle$, we denoted the canonical P-module basis of P^{20} by $\{\varepsilon_1, \ldots, \varepsilon_{20}\}$, and $\Lambda \subseteq \mathrm{Syz}_P(g_1, \ldots, g_{20})$ denoted the set of all neighbor liftings with respect to $\mathcal{O}$. For all $i, j \in \{1, \ldots, 20\}$ such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\prime$, let σ_{ij} denote the corresponding neighbor syzygy with respect to $\prime$ as in Definition 2.4.21 and let λ_{ij} denote the corresponding neighbor lifting with respect to $\prime$ as in Definition 2.4.24. For all indices $i, j \in \{1, \ldots, 20\}$ such that $i < j$, such that $(i,j) \notin \{(1,3), (2,4), (3,4), (4,8)\}$, and such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we had $\lambda_{ij} = \sigma_{ij}$ and we saw that $\lambda_{1,3} = \sigma_{1,3} + \varepsilon_{16}$, $\lambda_{2,4} = \sigma_{2,4} - y\varepsilon_{16}$, $\lambda_{3,4} = \sigma_{3,4} - y\varepsilon_{16}$, $\lambda_{4,8} = \sigma_{4,8} + \varepsilon_{19}$. We now consider the Reduction Algorithm 6 applied to $v = (p_1, \ldots, p_{20}) = (-z^4 + y^3)\varepsilon_1 + x\varepsilon_9 - \varepsilon_{14} - z^3\varepsilon_{16} + y\varepsilon_{19}$ and Λ in detail.

In line 1, we have to apply the Column Clearing Algorithm 5 to the input data 1, v, and Λ. We have already seen in Example 3.2.7 that $(p_1, \ldots, p_{20}) = -z^4\varepsilon_1 + x\varepsilon_9 - z^3\varepsilon_{16}$ is the result of the algorithm. As $b_9 e_{\beta_9}$ is exposable with $\mathrm{face}(b_9 e_{\beta_9}) = 3 > 1$ according to Definition 3.1.1 and as $t \in \mathrm{Supp}(p_9)$ has x-degree $1 > 0$, the while-loop starting in line 3 is executed. First we have to apply the Degree Lowering Proposition 3.2.3 to the input data 1, $(p_1, \dots, p_{20})$, and Λ. Again, we have already computed in Example 3.2.4 that the result is $(q_1, \dots, q_{20}) = -z^4\varepsilon_1 + z\varepsilon_{11} - z^3\varepsilon_{16}$. Second we have to apply the Column Clearing Algorithm 5 to the input data 1, $(q_1, \dots, q_{20})$, and Λ. Similar to the situation in Example 3.2.7, we see that we have to clear the component $q_{11} = z$ as $b_{11}e_{\beta_{11}} = xz^3e_1$ is non-extreme by Definition 3.1.1. Summing up the corresponding reduction steps in line 11 of Algorithm 5, we first update $(q_1, \dots, q_{20})$ to the vector

$$(q_1, \dots, q_{20}) - z \cdot \lambda_{11,6} = -z^4\varepsilon_1 + z\varepsilon_{11} - z^3\varepsilon_{16} - z(\varepsilon_{11} - z\varepsilon_6) = -z^4\varepsilon_1 + z^2\varepsilon_6 - z^3\varepsilon_{16},$$

then we update it to

$$(q_1, \dots, q_{20}) - z^2 \cdot \lambda_{6,3} = -z^4\varepsilon_1 + z^2\varepsilon_6 - z^3\varepsilon_{16} - z^2(\varepsilon_6 - z\varepsilon_3) = -z^4\varepsilon_1 + z^3\varepsilon_3 - z^3\varepsilon_{16},$$

and finally update $(q_1, \dots, q_{20})$ to the vector

$$(q_1, \dots, q_{20}) - z^3 \cdot \lambda_{3,1} = -z^4\varepsilon_1 + z^3\varepsilon_3 - z^3\varepsilon_{16} - z^3 \cdot (\varepsilon_3 - z\varepsilon_1 - \varepsilon_{16}) = 0.$$

In particular, the while-loop starting in line 3 terminates and we have $(p_1, \dots, p_{20}) = 0$ after it. Moreover, this while is not executed anymore, as $(p_1, \dots, p_{20}) = 0$ and the algorithm returns $(p_1, \dots, p_{20}) = 0$ in line 7.

Note that this result is not a coincident. The vector v is a syzygy of $(g_1, \dots, g_{20})$. Thus the result of the Reduction Algorithm 6 applied to v and Λ must be zero as we will prove in Theorem 3.3.7.

The remaining part of this section is devoted to give an alternative proof of the characterization of border bases via liftings of border syzygies in Theorem 2.4.26 that does not use commuting matrices at all. Therefore, we have to generalize [Hui06, Lemma 26] to the module setting. But first we prove the following auxiliary lemma that is a generalized version [Hui06, Lemma 25].

Lemma 3.3.5. *Let $j \in \{1, \dots, \nu\}$ be such that the border term $b_je_{\beta_j}$ is extreme and let $k = \operatorname{face}(b_je_{\beta_j})$. Let $\gamma_1, \dots, \gamma_k \in \mathbb{N}$ be such that $b_j = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$, let $\delta_k, \dots, \delta_n \in \mathbb{N}$, and let $t = x_1^{\gamma_1} \cdots x_{k-1}^{\gamma_{k-1}} x_k^{\delta_k} \cdots x_n^{\delta_n} \in \mathbb{T}^n$. Moreover, assume that either $te_{\beta_j} \in \mathcal{O}$, or that $te_{\beta_j} \in \partial\mathcal{O}$ and x_ℓ-exposable for some $\ell \in \{k+1, \dots, n\}$. Then $\delta_k < \gamma_k$.*

Proof. For a contradiction, assume that $\delta_k \geq \gamma_k$. If $te_{\beta_j} \in \mathcal{O}$, we see that

$$(x_k^{\delta_k - \gamma_k} x_{k+1}^{\delta_{k+1}} \cdots x_n^{\delta_n}) b_je_{\beta_j} = te_{\beta_j} \in \mathcal{O}.$$

If $te_{\beta_j} \in \partial\mathcal{O}$ and x_ℓ-exposable for some $\ell \in \{k+1, \dots, n\}$, we see that $\delta_\ell \geq 1$ and

$$(x_k^{\delta_k - \gamma_k} x_{k+1}^{\delta_{k+1}} \cdots x_{\ell-1}^{\delta_{\ell-1}} x_\ell^{\delta_\ell - 1} x_{\ell+1}^{\delta_{\ell+1}} \cdots x_n^{\delta_n}) b_je_{\beta_j} = \tfrac{t}{x_\ell} e_{\beta_j} \in \mathcal{O}$$

according to Definition 3.1.1. As $\mathcal{O}$ is an order ideal and as $b_je_{\beta_j} \in \partial\mathcal{O}$, both cases contradict the Definitions 2.1.6 and 2.1.7. $\square$

The next lemma encapsulates some arguments used in the induction during the proof of [Hui06, Lemma 26] and generalizes them to the module setting.

Lemma 3.3.6. *Let $(p_1, \ldots, p_\nu) \in P^\nu$ be n-reduced and let $k \in \{1, \ldots, n\}$ be such that for all indices $j \in \{1, \ldots, \nu\}$, we have $p_j = 0$, or $b_j e_{\beta_j}$ is extreme and $\operatorname{face}(b_j e_{\beta_j}) \geq k$. Moreover, we denote $J_k = \{j \in \{1, \ldots, \nu\} \mid p_j \neq 0, b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})\}$, assume that $J_k \neq \emptyset$, and let $j \in J_k$ be such that $\deg_{x_k}(p_j) \geq \deg_{x_k}(p_{j'})$ for every index $j' \in J_k$. Write $p_j = a_s x_k^s + \cdots + a_1 x_k + a_0 \in K[x_{k+1}, \ldots, x_n][x_k]$ with $s \in \mathbb{N}$, $a_0, \ldots, a_s \in K[x_{k+1}, \ldots, x_n]$, and $a_s \neq 0$. Let $\widehat{t} \in \operatorname{Supp}(a_s)$. Then the following conditions hold.*

1) We have $\widehat{t} x_k^s b_j e_{\beta_j} \in \operatorname{Supp}(p_j b_j e_{\beta_j})$.

2) We have $\widehat{t} x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'} b_{j'} e_{\beta_{j'}})$ for all $j' \in J_k \setminus \{j\}$.

3) We have $\widehat{t} x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'}(g_{j'} - b_{j'} e_{\beta_{j'}}))$ for all $j' \in J_k$.

4) We have $\widehat{t} x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'} g_{j'})$ for all $j' \in \{1, \ldots, \nu\} \setminus J_k$.

In other words, the term $\widehat{t} x_k^s b_j e_{\beta_j}$ occurs in the sum $p_1 g_1 + \cdots + p_\nu g_\nu$ exactly once, namely in $p_j b_j e_{\beta_j}$.

Proof. As $b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})$ is extreme, we can write $b_j = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ with $\gamma_1, \ldots, \gamma_k \in \mathbb{N}$ according to Definition 3.1.1. By construction, we have $s = \deg_{x_k}(p_j) \geq \deg_{x_k}(p_{j'})$ for all $j' \in J_k$. Moreover, since $(p_1, \ldots, p_\nu)$ is n-reduced, we have $\deg_{x_\ell}(p_{j'}) = 0$ for all $\ell \in \{1, \ldots, k-1\}$ and $j' \in J_k$ by Definition 3.3.1. We will use these facts during this proof without mention.

Since 1) follows immediately from the definitions of j, a_s, and $\widehat{t}$, we start with the proof of claim 2). If $J_k \setminus \{j\} = \emptyset$ or if $\beta_{j'} \neq \beta_j$ for all $j' \in J_k \setminus \{j\}$, claim 2) follows trivially. Thus assume that there exists a $j' \in J_k \setminus \{j\}$ with $\beta_{j'} = \beta_j$. As $b_{j'} e_{\beta_{j'}} \in \mathcal{F}_k(\mathcal{O})$ is extreme, we can write $b_{j'} = x_1^{\delta_1} \cdots x_k^{\delta_k}$ with $\delta_1, \ldots, \delta_k \in \mathbb{N}$ by Definition 3.1.1. Assume that $\gamma_\ell = \delta_\ell$ for every $\ell \in \{1, \ldots, k-1\}$. Then $b_j \neq b_{j'}$ implies $\gamma_k \neq \delta_k$. If $\gamma_k > \delta_k$, we have $x_k^{\gamma_k - \delta_k - 1} b_{j'} e_{\beta_{j'}} = \frac{b_j}{x_k} e_{\beta_j} \notin \mathcal{O}$ by the Definitions 2.1.6 and 2.1.7. If $\gamma_k < \delta_k$, we have $x_k^{\delta_k - \gamma_k - 1} b_j e_{\beta_j} = \frac{b_{j'}}{x_k} e_{\beta_{j'}} \notin \mathcal{O}$ by the Definitions 2.1.6 and 2.1.7. So both cases yield a contradiction to $b_j e_{\beta_j}, b_{j'} e_{\beta_{j'}} \in \mathcal{F}_k(\mathcal{O})$ and Definition 3.1.1 and we see that there exists an $\ell \in \{1, \ldots, k-1\}$ such that $\gamma_\ell \neq \delta_\ell$. For all $t' \in \operatorname{Supp}(p_{j'})$, we hence get

$$\begin{aligned} \deg_{x_\ell}(t' b_{j'}) &= \deg_{x_\ell}(t') + \deg_{x_\ell}(b_{j'}) \\ &= 0 + \delta_\ell \\ &\neq 0 + 0 + \gamma_\ell \\ &= \deg_{x_\ell}(\widehat{t}) + \deg_{x_\ell}(x_k^s) + \deg_{x_\ell}(b_j) \\ &= \deg_{x_\ell}(\widehat{t} x_k^s b_j), \end{aligned}$$

i.e. $\widehat{t} x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'} b_{j'} e_{\beta_{j'}})$.

Next we prove 3). Let $i \in \{1, \dots, \mu\}$, $\delta_1, \dots, \delta_n \in \mathbb{N}$ be such that $t_i = x_1^{\delta_1} \cdots x_n^{\delta_n}$, $j' \in J_k$, and $t' \in \operatorname{Supp}(p_{j'})$. If $\gamma_\ell \neq \delta_\ell$ for some $\ell \in \{1, \dots, k-1\}$, we get

$$\begin{aligned}\deg_{x_\ell}(t't_i) &= \deg_{x_\ell}(t') + \deg_{x_\ell}(t_i)\\ &= 0 + \delta_\ell\\ &\neq 0 + 0 + \gamma_\ell\\ &= \deg_{x_\ell}(\widehat{t}) + \deg_{x_\ell}(x_k^s) + \deg_{x_\ell}(b_j)\\ &= \deg_{x_\ell}(\widehat{t}x_k^s b_j).\end{aligned}$$

If $\gamma_\ell = \delta_\ell$ for all $\ell \in \{1, \dots, k-1\}$, Lemma 3.3.5 yields $\delta_k < \gamma_k$ and hence we get

$$\begin{aligned}\deg_{x_k}(t't_i) &= \deg_{x_k}(t') + \deg_{x_k}(t_i)\\ &\leq s + \delta_k\\ &< 0 + s + \gamma_k\\ &= \deg_{x_k}(\widehat{t}) + \deg_{x_k}(x_k^s) + \deg_{x_k}(b_j)\\ &= \deg_{x_k}(\widehat{t}x_k^s b_j).\end{aligned}$$

In both cases we get $\widehat{t}x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'} t_i e_{\alpha_i})$. Since $\operatorname{Supp}(g_{j'} - b_{j'} e_{\beta_{j'}}) \subseteq \mathcal{O}$ by Definition 2.1.14, it follows $\widehat{t}x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'}(g_{j'} - b_{j'} e_{\beta_{j'}}))$ and claim 3) follows.

Finally, we prove 4). If $\{1, \dots, \nu\} \setminus J_k = \emptyset$, nothing needs to be shown. Thus suppose that there is a $j' \in \{1, \dots, \nu\} \setminus J_k$. Since claim 4) holds trivially if $p_{j'} = 0$, we suppose that $p_{j'} \neq 0$. The border term $b_{j'} e_{\beta_{j'}} \in \partial\mathcal{O}$ is extreme by Definition 3.3.1 as $(p_1, \dots, p_\nu)$ is n-reduced. Moreover, we have $\operatorname{face}(b_{j'} e_{\beta_{j'}}) > k$ according to the definition of k and J_k. In particular, as $(p_1, \dots, p_\nu)$ is n-reduced, we see that $\deg_{x_\ell}(p_{j'}) = 0$ for all $\ell \in \{1, \dots, k\}$ by Definition 3.3.1. Let $t' \in \operatorname{Supp}(p_{j'})$, $t \in \operatorname{Supp}(g_{j'})$, and $\delta_1, \dots, \delta_n \in \mathbb{N}$ be such that $t = x_1^{\delta_1} \cdots x_n^{\delta_n}$. If $\gamma_\ell \neq \delta_\ell$ for some $\ell \in \{1, \dots, k-1\}$, we get

$$\begin{aligned}\deg_{x_\ell}(t't) &= \deg_{x_\ell}(t') + \deg_{x_\ell}(t)\\ &= 0 + \delta_\ell\\ &\neq 0 + 0 + \gamma_\ell\\ &= \deg_{x_\ell}(\widehat{t}) + \deg_{x_\ell}(x_k^s) + \deg_{x_\ell}(b_j)\\ &= \deg_{x_\ell}(\widehat{t}x_k^s b_j).\end{aligned}$$

If $\gamma_\ell = \delta_\ell$ for all $\ell \in \{1, \dots, k-1\}$, Lemma 3.3.5 yields $\delta_k < \gamma_k$ and hence we get

$$\begin{aligned}\deg_{x_k}(t't) &= \deg_{x_k}(t') + \deg_{x_k}(t)\\ &= 0 + \delta_k\\ &< 0 + s + \gamma_k\\ &= \deg_{x_k}(\widehat{t}) + \deg_{x_k}(x_k^s) + \deg_{x_k}(b_j)\\ &= \deg_{x_k}(\widehat{t}x_k^s b_j).\end{aligned}$$

In both cases we see that $\widehat{t}x_k^s b_j e_{\beta_j} \notin \operatorname{Supp}(p_{j'} g_{j'})$ and claim 4) follows. $\square$

Using these two lemmata, we can now prove a generalization of [Hui06, Lemma 26].

Theorem 3.3.7. *Let* $(p_1, \ldots, p_\nu) \in P^\nu$ *be* n*-reduced with* $p_1 g_1 + \cdots + p_\nu g_\nu \in \langle \mathcal{O} \rangle_K$. *Then* $(p_1, \ldots, p_\nu) = 0$.

Proof. Since $(p_1, \ldots, p_\nu)$ is n-reduced, we have $p_j = 0$ for every $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j} \in \partial\mathcal{O}$ is non-extreme by Definition 3.3.1. Assume that there exists an index $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable and such that $p_j \neq 0$. Let $t \in \operatorname{Supp}(p_j)$. As $b_j e_{\beta_j}$ is non-exposable, it follows $e_{\beta_j} = b_j e_{\beta_j} \in \partial\mathcal{O}$ by Definition 3.1.1. In particular, it follows from the Definitions 2.1.6 and 2.1.7 that $\beta_j \neq \alpha_i$ for every $i \in \{1, \ldots, \mu\}$ and $\beta_j \neq \beta_{j'}$ for every $j' \in \{1, \ldots, \nu\} \setminus \{j\}$. Thus the term $t b_j e_{\beta_j} \in \operatorname{Supp}(p_j g_j)$ cannot cancel in the sum $p_1 g_1 + \cdots + p_\nu g_\nu$ and we get $t b_j e_{\beta_j} \in \operatorname{Supp}(p_1 g_1 + \cdots + p_\nu g_\nu) \subseteq \mathcal{O}$. Therefore, by Definition 2.1.6, we get $b_j e_{\beta_j} \in \mathcal{O}$ and this clearly contradicts Definition 2.1.7. Thus we see that $p_j = 0$ for all $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable. Now we prove by induction on $k \in \{1, \ldots, n\}$ that $p_j = 0$ for every $j \in \{1, \ldots, \nu\}$ with the properties that $b_j e_{\beta_j} \in \partial\mathcal{O}$ is extreme and $\operatorname{face}(b_j e_{\beta_j}) \leq k$.
For the induction start $k = 1$, let $J_1 = \{j \in \{1, \ldots, \nu\} \mid p_j \neq 0, b_j e_{\beta_j} \in \mathcal{F}_1(\mathcal{O})\}$. If we show that $J_1 = \emptyset$, the induction start is proven. For a contradiction, assume that $J_1 \neq \emptyset$. Then there exists an index $j \in J_1$ such that $\deg_{x_1}(p_j) \geq \deg_{x_1}(p_{j'})$ for all indices $j' \in J_1$. We write the polynomial $p_j = a_s x_1^s + \cdots + a_1 x_1 + a_0 \in K[x_2, \ldots, x_n][x_1]$ with $s \in \mathbb{N}$, $a_0, \ldots, a_s \in K[x_2, \ldots, x_n]$, and $a_s \neq 0$. Let $\widehat{t} \in \operatorname{Supp}(a_s)$. As $\mathcal{O}$ is an order ideal and as $b_j e_{\beta_j} \in \partial\mathcal{O}$, it follows $\widehat{t} x_1^s b_j e_{\beta_j} \notin \mathcal{O}$ from the Definitions 2.1.6 and 2.1.7. Moreover, Lemma 3.3.6 shows that the term $\widehat{t} x_1^s b_j e_{\beta_j}$ occurs in the sum $p_1 g_1 + \cdots + p_\nu g_\nu$ exactly once, namely in $p_j b_j e_{\beta_j}$. Since $p_1 g_1 + \cdots + p_\nu g_\nu \in \langle \mathcal{O} \rangle_K$, we hence get the contradiction $\widehat{t} x_1^s b_j e_{\beta_j} \in \mathcal{O}$. Thus it follows $J_1 = \emptyset$ and the induction start is proven. In particular, if $n = 1$, the claim follows from the induction start.
For the induction step, assume that there is a $k \in \{2, \ldots, n\}$ and that the claim holds for $k - 1$. Recall that $p_j = 0$ for all $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable or non-extreme. Let $J_k = \{j \in \{1, \ldots, \nu\} \mid p_j \neq 0, b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})\}$. As in the induction start, it suffices to prove that $J_k = \emptyset$. For a contradiction, assume that $J_k \neq \emptyset$. Then there exists an index $j \in J_k$ such that $\deg_{x_k}(p_j) \geq \deg_{x_k}(p_{j'})$ for all $j' \in J_k$. As $b_j e_{\beta_j} \in \mathcal{F}_k(\mathcal{O})$ is extreme and as $(p_1, \ldots, p_\nu)$ is n-reduced, we have $p_j \in K[x_k, \ldots, x_n]$ by Definition 3.3.1. We write $p_j = a_s x_k^s + \cdots + a_1 x_k + a_0 \in K[x_{k+1}, \ldots, x_n][x_k]$ with $s \in \mathbb{N}$, $a_0, \ldots, a_s \in K[x_{k+1}, \ldots, x_n]$, and $a_s \neq 0$. Let $\widehat{t} \in \operatorname{Supp}(a_s)$. As $\mathcal{O}$ is an order ideal and as $b_j e_{\beta_j} \in \partial\mathcal{O}$, it follows $\widehat{t} x_k^s b_j e_{\beta_j} \notin \mathcal{O}$ according to the Definitions 2.1.6 and 2.1.7. Moreover, the induction hypothesis yields that $p_{j'} = 0$ for all $j' \in \{1, \ldots, \nu\}$ such that $b_{j'} e_{\beta_{j'}} \in \partial\mathcal{O}$ is extreme and such that $\operatorname{face}(b_{j'} e_{\beta_{j'}}) \leq k - 1$. Thus Lemma 3.3.6 implies that $\widehat{t} x_k^s b_j e_{\beta_j}$ occurs in the sum $p_1 g_1 + \cdots + p_\nu g_\nu$ exactly once, namely in $p_j b_j e_{\beta_j}$. Since $p_1 g_1 + \cdots + p_\nu g_\nu \in \langle \mathcal{O} \rangle_K$, we get the contradiction $\widehat{t} x_k^s b_j e_{\beta_j} \in \mathcal{O}$. It follows that $J_k = \emptyset$ and the induction step is proven. □

Now we are able to prove the non-trivial implication of [Hui06, Thm. 22], namely that the set of neighbor liftings is a system of generators of the (first) syzygy module of a border basis, in the module setting. We improve upon this result in the next section.

The following formulation generalizes [KK14, Coro. 5.4].

Corollary 3.3.8. *We have* $\langle\Lambda\rangle = \mathrm{Syz}_P(g_1,\dots,g_\nu)$.

Proof. As the neighbour liftings with respect to $\mathcal{O}$ in Λ are all syzygies of $(g_1,\dots,g_\nu)$ according to Definition 2.4.24, we have $\langle\Lambda\rangle \subseteq \mathrm{Syz}_P(g_1,\dots,g_\nu)$. For the converse inclusion, we let $(p_1,\dots,p_\nu) \in \mathrm{Syz}_P(g_1,\dots,g_\nu)$. Let

$$(q_1,\dots,q_\nu) := \texttt{Reduce}((p_1,\dots,p_\nu),\Lambda)$$

be the result of Algorithm 6 applied to the input data $(p_1,\dots,p_\nu)$ and Λ. Then Theorem 3.3.3 yields that $(p_1,\dots,p_\nu)-(q_1,\dots,q_\nu) \in \langle\Lambda\rangle \subseteq \mathrm{Syz}_P(g_1,\dots,g_\nu)$ and thus $(q_1,\dots,q_\nu) \in \mathrm{Syz}_P(g_1,\dots,g_\nu)$. In particluar, it follows that $q_1g_1+\cdots+q_\nu g_\nu = 0 \in \langle\mathcal{O}\rangle_K$. As $(q_1,\dots,q_\nu)$ is n-reduced by Theorem 3.3.3, Theorem 3.3.7 implies $(q_1,\dots,q_\nu)=0$ and we see that $(p_1,\dots,p_\nu)\in\langle\Lambda\rangle$. □

We end this section with an alternative proof of the characterization of border bases via liftings of border syzygies as in Theorem 2.4.26. This proof is remarkable since, in stark contrast to all previous proofs of this characterization, it does not depend on the characterization of border bases via commuting matrices in Theorem 2.4.19, but only needs the characterizations via the special generation property in Theorem 2.4.1 and via rewrite rules in Theorem 2.4.13. Thus this proof might yield a possible way to characterize border bases in a non-commutative setting, e.g. as defined in [BTBQM00].

Corollary 3.3.9. (Border Bases and Liftings of Border Syzygies)
The $\mathcal{O}$-border prebasis G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if and only if the following equivalent conditions are satisfied.

E_1) *Every border syzygy with respect to $\mathcal{O}$ lifts to a syzygy of $(g_1,\dots,g_\nu)$.*

E_2) *Every neighbor syzygy with respect to $\mathcal{O}$ lifts to a syzygy of $(g_1,\dots,g_\nu)$.*

Proof. First we show that condition E_1) is satisfied if G is the $\mathcal{O}$-border basis of $\langle G\rangle$. Let $(p_1,\dots,p_\nu)$ be a border syzygy with respect to $\mathcal{O}$ and $v = p_1g_1+\cdots+p_\nu g_\nu$. If $v=0$, we see that $(p_1,\dots,p_\nu)$ is a lifting of $(p_1,\dots,p_\nu)$ by Definition 2.4.24. Thus suppose that $v\neq 0$. Since $v\in\langle G\rangle\setminus\{0\}$, condition A_1) of Theorem 2.4.1 yields a representation $v = p_1g_1+\cdots+p_\nu g_\nu = q_1g_1+\cdots+q_\nu g_\nu$ such that $q_1,\dots,q_\nu\in P$ and such that $q_j=0$ or $\deg(q_j)\le \mathrm{ind}_{\mathcal{O}}(v)-1$ for every index $j\in\{1,\dots,\nu\}$. Let $(P_1,\dots,P_\nu) = (p_1,\dots,p_\nu)-(q_1,\dots,q_\nu)$. Then $(P_1,\dots,P_\nu)$ is a syzygy of $(g_1,\dots,g_\nu)$ by construction. Moreover, we have $\deg(P_j-p_j)=\deg(-q_j)\le\mathrm{ind}_{\mathcal{O}}(v)-1$ for all $j\in\{1,\dots,\nu\}$ with $P_j-p_j\neq 0$, i.e. $(p_1,\dots,p_\nu)$ lifts to $(P_1,\dots,P_\nu)$ by Definition 2.4.24.
Since E_1) logically implies E_2) by Definition 2.4.21, it remains to prove that G is the $\mathcal{O}$-border basis of $\langle G\rangle$ if E_2) holds. By E_2), every neighbor syzygy with respect to $\mathcal{O}$ lifts to a neighbor lifting with respect to $\mathcal{O}$. Let $\Lambda\subseteq P^r$ be the set of all neighbor liftings with respect to $\mathcal{O}$ and let $p_1,\dots,p_\nu\in P$ be such that $v=p_1g_1+\cdots+p_\nu g_\nu\in\langle G\rangle$ is irreducible with respect to $\xrightarrow{G}$. Then $v\in\langle\mathcal{O}\rangle_K$ by Remark 2.4.11. By Theorem 3.3.3, there exists

an n-reduced vector $(q_1, \dots, q_\nu) \in P^\nu$ such that $(p_1, \dots, p_\nu) - (q_1, \dots, q_\nu) \in \langle \Lambda \rangle$. As $\Lambda \subseteq \operatorname{Syz}_P(g_1, \dots, g_\nu)$ by Definition 2.4.24, it follows

$$\begin{aligned} v &= p_1 g_1 + \cdots + p_\nu g_\nu \\ &= p_1 g_1 + \cdots + p_\nu g_\nu - (p_1 - q_1) g_1 - \cdots - (p_\nu - q_\nu) g_\nu \\ &= q_1 g_1 + \cdots + q_\nu g_\nu. \end{aligned}$$

In particular, the n-reduced vector $(q_1, \dots, q_\nu)$ satisfies $q_1 g_1 + \cdots + q_\nu g_\nu = v \in \langle \mathcal{O} \rangle$. Hence Theorem 3.3.7 implies $(q_1, \dots, q_\nu) = 0$ and this yields $v = 0$. Altogether, we have proven that condition C_2) of Theorem 2.4.13 is satisfied and thus G is the $\mathcal{O}$-border basis of $\langle G \rangle$. □

3.4 Gröbner Bases for the Syzygy Modules of Border Bases

In the final section of this chapter, we use the above results in order to prove a version of Schreyer's Theorem for border bases of submodules of free modules of finite rank. For polynomial ideals, Schreyer's Theorem was already proven in [KK14, Section 6]. We now generalize the ideas of that paper to the module setting.

Continuing to use the setting of the previous section, we let $\{\varepsilon_1, \dots, \varepsilon_\nu\}$ denote the canonical P-module basis of P^ν and, if not mentioned otherwise, we assume that G is the $\mathcal{O}$-border basis of $\langle G \rangle$. For all $i, j \in \{1, \dots, \nu\}$ such that $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ are neighbors with respect to $\mathcal{O}$, we let σ_{ij} denote the corresponding neighbor syzygy with respect to $\mathcal{O}$ and λ_{ij} denote the corresponding neighbor lifting with respect to $\mathcal{O}$. Moreover, we denote the set of all neighbor liftings with respect to $\mathcal{O}$ by Λ.

As done for the ideal setting in [KK14, Thm. 6.1], we can give explicit conditions that a term ordering τ on $\mathbb{T}^n \langle e_1, \dots, e_r \rangle$ must satisfy to imply that Λ is a τ-Gröbner basis of $\operatorname{Syz}_P(g_1, \dots, g_\nu)$.

Theorem 3.4.1. (Gröbner Bases for the Syzygy Modules of Border Bases) *Let τ be a module term ordering on the set of terms in P^ν which satisfies the following conditions for every $i \in \{1, \dots, n-1\}$ and every $j \in \{1, \dots, \nu\}$.*

1) If $b_j e_{\beta_j}$ is extreme, if $\operatorname{face}(b_j e_{\beta_j}) > i$, and if $k \in \{1, \dots, \nu\}$ is chosen such that $b_k e_{\beta_k}$ is exposable and $\operatorname{face}(b_k e_{\beta_k}) \geq i$, then we have $x_i \varepsilon_j >_\tau \varepsilon_k$.

2) If $b_j e_{\beta_j}$ is extreme, if $x_i b_j e_{\beta_j} \notin \partial \mathcal{O}$, if $k = \operatorname{face}(b_j e_{\beta_j}) > i$, and if $\ell \in \{1, \dots, \nu\}$ is the unique index such that $x_i b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$, then we have $x_i \varepsilon_j >_\tau x_k \varepsilon_\ell$.

3) If $b_j e_{\beta_j}$ is non-extreme, if $k \in \{i+1, \dots, n\}$ is chosen such that $\operatorname{Col}(b_j e_{\beta_j})$ is in the x_k-direction, and if $\ell \in \{1, \dots, \nu\}$ is chosen such that $b_\ell e_{\beta_\ell}$ is exposable and $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq k$, then we have $\varepsilon_j >_\tau \varepsilon_\ell$.

4) *If $b_j e_{\beta_j}$ is non-extreme, if $k \in \{i+1, \dots, n\}$ is chosen such that* $\mathrm{Col}(b_j e_{\beta_j})$ *is in the x_k-direction, and if $\ell \in \{1, \dots, \nu\}$ denotes the unique index with $b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$, then we have $\varepsilon_j >_\tau x_k \varepsilon_\ell$.*

Then the set of all neighbor liftings Λ with respect to $\mathcal{O}$ is a τ-Gröbner basis of the syzygy module $\mathrm{Syz}_P(g_1, \dots, g_\nu)$.

Proof. First we prove that every execution of line 12 of the `LowerDegree` Algorithm 4 can be interpreted as a reduction step with respect to Λ as defined in [KR00, Defn. 2.2.1]. Let $i \in \{1, \dots, n-1\}$, $(p_1, \dots, p_\nu) \in P^\nu$, $j \in \{1, \dots, \nu\}$, and $d \in \mathbb{N} \setminus \{0\}$ be as in the `LowerDegree` Algorithm 4, i.e. such that $b_j e_{\beta_j}$ is extreme, such that $\mathrm{face}(b_j e_{\beta_j}) > i$, and such that the support of p_j contains a term that is divisible by x_i^d but the support of p_j does not contain a term that is divisible by x_i^{d+1}. In order to prove that $\mathrm{LT}_\tau(\lambda_{j\ell}) = x_i \varepsilon_j$ in line 12 of Algorithm 4, we have to distinguish two cases: either $x_i b_j e_{\beta_j} \in \partial\mathcal{O}$ as in line 3 of Algorithm 4 or $x_i b_j e_{\beta_j} \notin \partial\mathcal{O}$ as in line 5 of Algorithm 4.
For the first case, assume that $x_i b_j e_{\beta_j} \in \partial\mathcal{O}$, and let $\ell \in \{1, \dots, \nu\}$ be the index such that $x_i b_j e_{\beta_j} = b_\ell e_{\beta_\ell}$. We have already shown in the proof of Proposition 3.2.2 that $b_\ell e_{\beta_\ell}$ is exposable, that $\mathrm{face}(b_\ell e_{\beta_\ell}) > i$, and that $\lambda_{j\ell} = x_i \varepsilon_j - \varepsilon_\ell - c_1 \varepsilon_1 - \dots - c_\nu \varepsilon_\nu$ with $c_1, \dots, c_\nu \in K$ where we have $c_k = 0$ for every index $k \in \{1, \dots, \nu\}$ such that $b_k e_{\beta_k}$ is non-exposable or $\mathrm{face}(b_k e_{\beta_k}) < i$. According to condition 1), we have $x_i \varepsilon_j >_\tau \varepsilon_k$ for all $k \in \{1, \dots, \nu\}$ such that $b_k e_{\beta_k}$ is exposable and $\mathrm{face}(b_k e_{\beta_k}) \geq i$. Thus it follows that $\mathrm{LT}_\tau(\lambda_{j\ell}) = x_i \varepsilon_j$ in this first case.
For the second case, assume now that $x_i b_j e_{\beta_j} \notin \partial\mathcal{O}$. Let $k = \mathrm{face}(b_j e_{\beta_j})$. Then we have already shown in the proof of Proposition 3.2.2 that there exists a unique index $\ell \in \{1, \dots, \nu\}$ such that $x_i b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$, such that $b_\ell e_{\beta_\ell}$ is exposable, and such that $\mathrm{face}(b_\ell e_{\beta_\ell}) \geq i$. According to condition 2), we have $x_i \varepsilon_j >_\tau x_k \varepsilon_\ell$. Moreover, we have also shown in the proof of Proposition 3.2.2 that $\lambda_{j\ell} = x_i \varepsilon_j - x_k \varepsilon_\ell - c_1 \varepsilon_1 - \dots - c_\nu \varepsilon_\nu$ with $c_1, \dots, c_\nu \in K$, where we have $c_k = 0$ for every index $k \in \{1, \dots, \nu\}$ such that $b_k e_{\beta_k}$ is non-exposable or $\mathrm{face}(b_k e_{\beta_k}) < i$. According to condition 1), we have $x_i \varepsilon_j >_\tau \varepsilon_u$ for all $u \in \{1, \dots, \nu\}$ such that $b_u e_{\beta_u}$ is exposable and $\mathrm{face}(b_u e_{\beta_u}) \geq i$. Thus it follows that $\mathrm{LT}_\tau(\lambda_{j\ell}) = x_i \varepsilon_j$ in this second case.
Let $t \in \mathrm{Supp}(p_j)$ be such that $\deg_{x_i}(t) = d$ as in line 10 of Algorithm 4 and let $a \in K$ be the coefficient of t in p_j as in line 11 of Algorithm 4. Since the term $t\varepsilon_j = \frac{t}{x_j} \cdot \mathrm{LT}_\tau(\lambda_{j\ell})$ cancels in $(p_1, \dots, p_\nu) - a\frac{t}{x_i}\lambda_{j\ell}$, it follows that line 12 of the `LowerDegree` Algorithm 4 actually is a reduction step with respect to Λ according to [KR00, Defn. 2.2.1].

Second we prove that every execution of line 11 of the `ClearColumns` Algorithm 5 can be interpreted as a reduction step with respect to Λ as defined in [KR00, Defn. 2.2.1]. We let $i \in \{1, \dots, n-1\}$, $(p_1, \dots, p_\nu) \in P^\nu$, $k \in \{i+1, \dots, n\}$, and $j \in \{1, \dots, \nu\}$ be as in the `ClearColumns` Algorithm 5, i.e. such that $b_j e_{\beta_j}$ is non-extreme, such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction, and such that $i \leq \mathrm{face}(b_j e_{\beta_j})$. Moreover, let $s, t \in \mathbb{N}$ be such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_k^{-s} b_j, \dots, x_k^t b_j\} \cdot e_{\beta_j}$, let $\ell \in \{1, \dots, s+t+1\}$, let $u_{\ell-1}, u_\ell \in \{1, \dots, \nu\}$ be such that $b_{u_{\ell-1}} e_{\beta_{u_{\ell-1}}} = x_k^{\ell-s-2} b_j e_{\beta_j}$ and $b_{u_\ell} e_{\beta_{u_\ell}} = x_k^{\ell-s-1} b_j e_{\beta_j}$, and assume that $p_{u_\ell} \neq 0$, i.e. as in lines 3 to 9 of Algorithm 5. We now prove that $\mathrm{LT}_\tau(\lambda_{u_\ell u_{\ell-1}}) = \varepsilon_{u_\ell}$.
In the proof of Proposition 3.2.5, we have already shown that the neighbor lifting $\lambda_{u_\ell u_{\ell-1}}$

is of the form $\lambda_{u_\ell u_{\ell-1}} = \varepsilon_{u_\ell} - x_k \varepsilon_{u_{\ell-1}} - c_1\varepsilon_1 - \cdots - c_\nu \varepsilon_\nu$ with $c_1, \ldots, c_\nu \in K$, where we have $c_\varrho = 0$ for all $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-exposable or $\mathrm{face}(b_\varrho e_{\beta_\varrho}) < k$. According to condition 4), we see that $\varepsilon_{u_\ell} >_\tau x_k \varepsilon_{u_{\ell-1}}$. Moreover, for every $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is exposable and $\mathrm{face}(b_\varrho e_{\beta_\varrho}) \geq k$, condition 3) yields $\varepsilon_{u_\ell} >_\tau \varepsilon_\varrho$. Thus it follows that $\mathrm{LT}_\tau(\lambda_{u_\ell u_{\ell-1}}) = \varepsilon_{u_\ell}$.
Furthermore, since $\mathrm{face}(b_{u_\ell} e_{\beta_{u_\ell}}) = \mathrm{face}(b_j e_{\beta_j}) < k$, we have $c_{u_\ell} = 0$, i.e. the u_ℓ^{th} component of $\lambda_{u_\ell u_{\ell-1}}$ is equal to 1. Let $t' \in \mathrm{Supp}(p_{u_\ell})$ be as in line 9 of Algorithm 5 and let $a \in K$ be the coefficient of t' in p_{u_ℓ} as in line 10 of Algorithm 5. Since the term $t'\varepsilon_{u_\ell} = t' \cdot \mathrm{LT}_\tau(\lambda_{u_\ell u_{\ell-1}})$ cancels in $(p_1, \ldots, p_\nu) - at'\lambda_{u_\ell u_{\ell-1}}$, it follows that line 11 of the `ClearColumns` Algorithm 5 is actually a reduction step with respect to Λ as defined in [KR00, Defn. 2.2.1].

Finally, we prove that Λ is a τ-Gröbner basis of the syzygy module $\mathrm{Syz}_P(g_1, \ldots, g_\nu)$. Let $(p_1, \ldots, p_\nu) \in \mathrm{Syz}_P(g_1, \ldots, g_\nu)$ be irreducible with respect to the rewrite rule $\xrightarrow{\Lambda}$ as defined in [KR00, Defn. 2.2.1] and let

$$(q_1, \ldots, q_\nu) := \texttt{Reduce}((p_1, \ldots, p_\nu), \Lambda)$$

be the result of the Reduction Algorithm 6 applied to the input data $(p_1, \ldots, p_\nu)$ and Λ. The Reduction Algorithm 6 calls `LowerDegree` and `ClearColumns` in a certain order to compute the vector $(q_1, \ldots, q_\nu)$. We have proven above that line 12 of the `LowerDegree` Algorithm 4 and line 11 of the `ClearColumns` Algorithm 5 can be interpreted as reduction steps with respect to Λ as defined in [KR00, Defn. 2.2.1]. In particular, we get $(p_1, \ldots, p_\nu) \xrightarrow{\Lambda} (q_1, \ldots, q_\nu)$. According to Theorem 3.3.3, we see that $(q_1, \ldots, q_\nu)$ is n-reduced. As $(p_1, \ldots, p_\nu)$ and thus also $(q_1, \ldots, q_\nu)$ is a syzygy of $(g_1, \ldots, g_\nu)$, Theorem 3.3.7 now yields $(q_1, \ldots, q_\nu) = 0$. In particular, it follows that $(p_1, \ldots, p_\nu) \xrightarrow{\Lambda} 0$. As we assumed $(p_1, \ldots, p_\nu)$ to be irreducible with respect to $\xrightarrow{\Lambda}$, we get $(p_1, \ldots, p_\nu) = 0$. The claim now follows from a standard characterization of Gröbner bases using rewrite rules, cf. [KR00, Thm. 2.4.1]. □

In the remaining part of this section, we show that module term orderings τ satisfying the conditions of this theorem do exist. For the construction of such module term orderings, we use the idealization $\overline{P}$ of P^ν as defined in [KR05, Defn. 4.7.12], i.e. we consider the polynomial ring $\overline{P} = P[\varepsilon_1, \ldots, \varepsilon_\nu]$ where we regard $\varepsilon_1, \ldots, \varepsilon_\nu$ as indeterminates. We identify the elements of P^ν with the corresponding elements of the idealization $\overline{P}$ as described in [KR05, Prop. 4.7.14].

The following algorithm is a generalization of [KK14, Prop. 6.2] and constructs a weight matrix $W \in \mathrm{Mat}_{1,n+\nu}(\mathbb{Z})$ which we later use to explicitly determine module term orderings τ on the terms of P^ν which satisfy the conditions of Theorem 3.4.1. During the algorithm, we assign weights to the indeterminates of $\overline{P}$. We use the symbolic value `null` to state that an indeterminate has not been assigned its final weight yet.

Algorithm 7: `ConstructWeight`($\mathcal{O}$)

Input:
$\mathcal{O}$ is an order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r\rangle$ with border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$

for $i := 1$ **to** $n-1$ **do**
    $w_{x_i} :=$ `null`
end
$w_{x_n} := 1$
foreach $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable or extreme **do**
    $w_{\varepsilon_j} := 1$
end
foreach $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme **do**
    $w_{\varepsilon_j} :=$ `null`
end
$i := n$
while $i > 1$ **do**
    if $w_{\varepsilon_j} \neq$ `null` for all $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction **then**
        $w_{x_{i-1}} := \max\{w_{\varepsilon_j} + w_{x_i} \mid j \in \{1, \dots, \nu\}, w_{\varepsilon_j} \neq \texttt{null}\}$
        $i := i - 1$
    else
        choose any $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme, such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction, and such that $w_{\varepsilon_j} =$ `null`
        Let $\mathrm{Col}(b_j e_{\beta_j}) = \{x_i^{-s} b_j, x_i^{-s+1} b_j, \dots, x_i^t b_j\} \cdot e_{\beta_j}$ be with $s, t \in \mathbb{N}$.
        for $\ell := 0$ **to** $s+t+1$ **do**
            Let $u_\ell \in \{1, \dots, \nu\}$ be such that $b_{u_\ell} e_{\beta_{u_\ell}} = x_i^{\ell-s-1} b_j e_{\beta_j}$.
        end
        for $\ell := 1$ **to** $s+t+1$ **do**
            $w_{\varepsilon_{u_\ell}} := w_{\varepsilon_{u_{\ell-1}}} + w_{x_i} + 1$
        end
    end
end
return $(w_{x_1}, \dots, w_{x_n}, w_{\varepsilon_1}, \dots, w_{\varepsilon_\nu})$

The next proposition shows that this procedure defines indeed an algorithm and that the result of the algorithm is a positive matrix in the sense of [KR05, Defn. 4.2.4].

Proposition 3.4.2. *Algorithm 7 is actually an algorithm and its result*

$$W := \texttt{ConstructWeight}(\mathcal{O})$$

applied to the input data $\mathcal{O}$ is a matrix $W \in \mathrm{Mat}_{1,n+\nu}(\mathbb{Z})$ whose entries are positive integers. In particular, the matrix W is positive and W does not depend on the choice of the index j in line 17.

Proof. First we prove that $W \in \mathrm{Mat}_{1,n+\nu}(\mathbb{Z})$ and that every component of W is a positive integer. Obviously, we see that the result W is a $1 \times (n+\nu)$-matrix. At the beginning of the while-loop starting in line 12, i.e. for $i = n$, we have $w_{\varepsilon_j} = 1 \in \mathbb{N} \setminus \{0\}$ for every $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable or extreme by line 6 and we have $w_{x_n} = 1 \in \mathbb{N} \setminus \{0\}$ by line 4. In particular, there is an $\gamma \in \{1, \ldots, \nu\}$ such that $b_\gamma e_{\beta_\gamma}$ is of the form $b_\gamma e_{\beta_\gamma} = x_1^\delta e_1$ for some $\delta \in \mathbb{N}$. As the border term $b_\gamma e_{\beta_\gamma} = x_1^\delta e_1$ is obviously non-exposable or extreme by Definition 3.1.1, it follows that $w_{\varepsilon_\gamma} = 1$. It suffices now to show that $w_{\varepsilon_j} \in \mathbb{N} \setminus \{0\}$ for all $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme and that $w_{x_i} \in \mathbb{N} \setminus \{0\}$ for all $i \in \{1, \ldots, n-1\}$. We now prove by downward induction on the loop variable $i \in \{n, n-1, \ldots, 2\}$ in line 12 that $w_{x_n}, \ldots, w_{x_{i-1}} \in \mathbb{N} \setminus \{0\}$ and that we have $w_{\varepsilon_j} \in \mathbb{N} \setminus \{0\}$ for all $j \in \{1, \ldots, \nu\}$ and all $k \in \{n, n-1, \ldots, i\}$ such that $b_j e_{\beta_j}$ is non-extreme and $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction.
We start the induction with $i = n$. If there is no index $j \in \{1, \ldots, \nu\}$ such that the border term $b_j e_{\beta_j}$ is non-extreme and its column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_n-direction in line 13, we have $w_{x_{n-1}} = w_{\varepsilon_\gamma} + w_{x_n} = 1 + 1 \in \mathbb{N} \setminus \{0\}$ in line 14. Thus we may assume that there is a $j \in \{1, \ldots, \nu\}$ such that the border term $b_j e_{\beta_j}$ is non-extreme and such that its column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_n-direction. The existence of $s, t \in \mathbb{N}$ as in line 18 is a consequence of Lemma 3.1.6. Let $s, t \in \mathbb{N}$ be such numbers, i.e. that they satisfy $\mathrm{Col}(b_j e_{\beta_j}) = \{x_n^{-s} b_j, \ldots, x_n^t b_j\} \cdot e_{\beta_j}$. The existence of $u_0, \ldots, u_{s+t+1} \in \{1, \ldots, \nu\}$ such that $b_{u_\ell} e_{\beta_{u_\ell}} = x_n^{\ell-s-1} b_j e_{\beta_j}$ for all $\ell \in \{0, \ldots, s+t+1\}$ as in line 20 during the for-loop starting in line 19 follows from Definition 3.1.5 and Lemma 3.1.9. In particular, Definition 3.1.8 and Lemma 3.1.9 also yield that the border term $b_{u_0} e_{\beta_{u_0}} = \mathrm{lcb}(b_j e_{\beta_j})$ is extreme. Therefore, we get $w_{\varepsilon_{u_0}} = 1$ and $w_{\varepsilon_{u_1}} = w_{\varepsilon_{u_0}} + w_{x_n} + 1 = 1+1+1 = 3 \in \mathbb{N} \setminus \{0\}$ in the first two iterations of line 23 during the for-loop starting in line 22. By induction on the natural numbers $\ell \in \{1, \ldots, s+t+1\}$, the definition of $w_{\varepsilon_{u_\ell}}$ in line 23 immediately yields that $w_{\varepsilon_{u_\ell}} = w_{\varepsilon_{u_{\ell-1}}} + w_{x_n} + 1 = w_{\varepsilon_{u_{\ell-1}}} + 2 \in \mathbb{N} \setminus \{0\}$. Recall that the set of all columns forms a partition of the set of all non-extreme border terms according to Lemma 3.1.7. By induction on all columns in the x_n-direction, we get $w_{\varepsilon_\varrho} \in \mathbb{N} \setminus \{0\}$ for all indices $\varrho \in \{1, \ldots, \nu\}$ such that $b_\varrho e_{\beta_\varrho}$ is non-extreme and such that $\mathrm{Col}(b_\varrho e_{\beta_\varrho})$ is in the x_n-direction. Since there is at least one non-exposable or extreme border term, namely $b_\gamma e_{\beta_\gamma} = x_1^\delta e_1$, we get $w_{x_{n-1}} \geq w_{\varepsilon_\gamma} + w_{x_n} = 1 + w_{x_n} = 1 + 1 = 2$ and hence $w_{x_{n-1}} \in \mathbb{N} \setminus \{0\}$ by the definition of $w_{x_{n-1}}$ in line 14.
For the induction step, we let $i \in \{n-1, n-2, \ldots, 2\}$. By the induction hypothesis, we have $w_{\varepsilon_j} \in \mathbb{N} \setminus \{0\}$ for all $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j}$ is non-extreme and $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction for some $k \in \{n, n-1, \ldots, i+1\}$. Furthermore, the induction hypothesis yields $w_{x_n}, \ldots, w_{x_i} \in \mathbb{N} \setminus \{0\}$. If there is no index $j \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j} \in \partial\mathcal{O}$ is non-extreme and such that the column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction in line 13, we get

$w_{x_{i-1}} \in \mathbb{N}\setminus\{0\}$ in line 14 as in the induction start. Thus we may assume that there exists an index $j \in \{1,\dots,\nu\}$ such that $b_j e_{\beta_j}$ is non-extreme and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction. As in the induction start, there exist natural numbers $s,t \in \mathbb{N}$, and indices $u_0,\dots,u_{s+t+1} \in \{1,\dots,\nu\}$ such that $\mathrm{Col}(b_j e_{\beta_j}) = \{x_i^{-s} b_j,\dots,x_i^t b_j\} \cdot e_{\beta_j}$ in line 18, and $b_{u_\ell} e_{\beta_{u_\ell}} = x_i^{\ell-s-1} b_j e_{\beta_j}$ for all $\ell \in \{0,\dots,s+t+1\}$ in line 20. By Definition 3.1.8 and Lemma 3.1.9, the border term $b_{u_0} e_{\beta_0} = \mathrm{lcb}(b_j e_{\beta_j})$ is extreme or $\mathrm{Col}(b_0 e_{\beta_0})$ is in the x_k-direction with $k \in \{i+1,\dots,n\}$. If $b_{u_0} e_{\beta_{u_0}}$ is extreme, we have $w_{\varepsilon_{u_0}} = 1 \in \mathbb{N}\setminus\{0\}$ according to line 6. If $b_{u_0} e_{\beta_{u_0}}$ is non-extreme and $\mathrm{Col}(b_{u_0} e_{\beta_{u_0}})$ is in the x_k-direction with $k \in \{i+1,\dots,n\}$, the induction hypothesis yields $w_{\varepsilon_{u_0}} \in \mathbb{N}\setminus\{0\}$. As the induction hypothesis also yields $w_{x_i} \in \mathbb{N}\setminus\{0\}$, we get $w_{\varepsilon_{u_1}} = w_{\varepsilon_{u_0}} + w_{x_i} + 1 \in \mathbb{N}\setminus\{0\}$ in line 23 in both cases. By induction on $\ell \in \{1,\dots,s+t+1\}$, we immediately get $w_{\varepsilon_{u_\ell}} = w_{\varepsilon_{u_{\ell-1}}} + w_{x_i} + 1 \in \mathbb{N}\setminus\{0\}$ in each iteration of line 23 during the for-loop starting in line 22. By induction on all the columns in the x_i-direction, recalling again Lemma 3.1.7, we get $w_{\varepsilon_\varrho} \in \mathbb{N}\setminus\{0\}$ for all $\varrho \in \{1,\dots,\nu\}$ and $m \in \{n,n-1,\dots,i\}$ such that $\mathrm{Col}(b_\varrho e_{\beta_\varrho})$ is in the x_m-direction. Finally, the induction hypothesis yields $w_{x_i} \in \mathbb{N}\setminus\{0\}$ and we see that $w_{x_{i-1}} \geq w_{x_i} + w_{\varepsilon_\gamma} = w_{x_i} + 1$ and $w_{x_{i-1}} \in \mathbb{N}\setminus\{0\}$ in line 14.
In particular, we have shown that the matrix $W \in \mathrm{Mat}_{1,n+\nu}(\mathbb{Z})$ is positive in the sense of [KR05, Defn. 4.2.4].

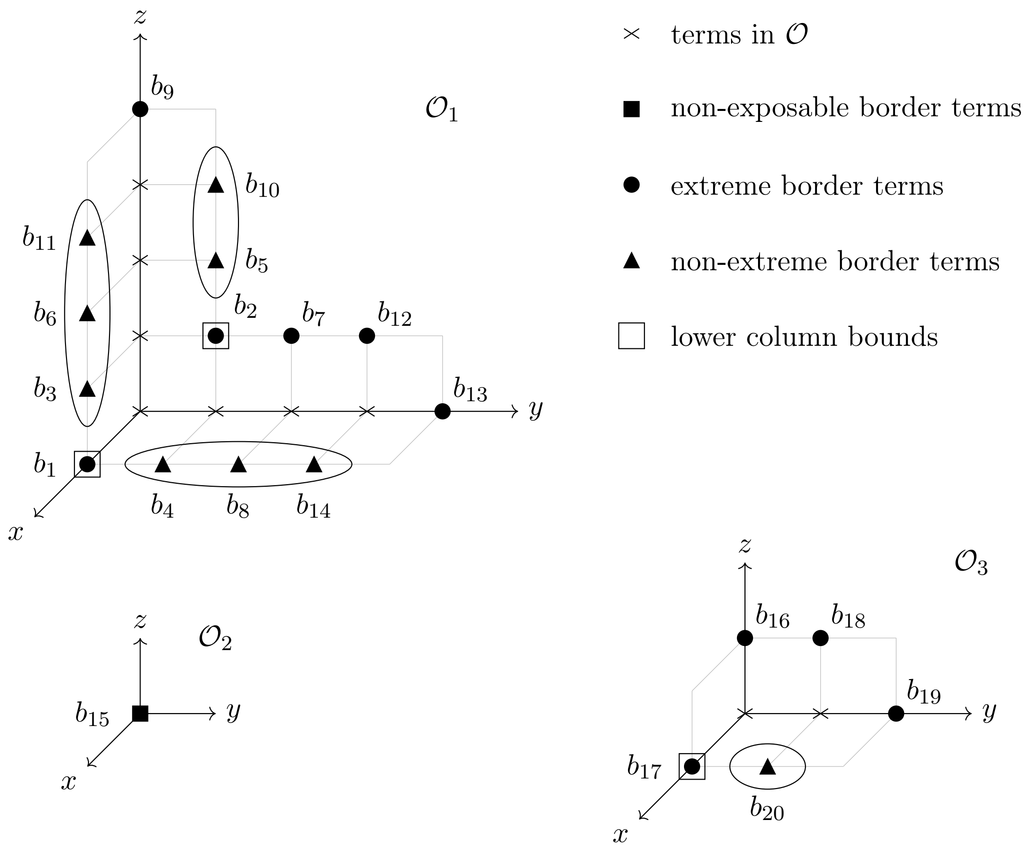
Next we show that the choice of j in line 17 has no effect to the result of the algorithm. The loop variable i of the while-loop starting in line 12 is decreased in line 15 during the if-clause starting in line 13. Hence the else-clause starting in line 16 is executed as long as there exists an index $j \in \{1,\dots,\nu\}$ such that $b_j e_{\beta_j}$ is non-extreme, such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction, and such that $w_{\varepsilon_j} = \texttt{null}$. By Lemma 3.1.7, the set of all columns is a partition of the set of all non-extreme border terms. In each execution of the else-clause during the while-loop starting in line 12, we always assign weights not equal to `null` exactly to w_{ε_ϱ} for all $\varrho \in \{1,\dots,\nu\}$ such that $b_\varrho e_{\beta_\varrho} \in \mathrm{Col}(b_j e_{\beta_j})$. Thus we see that the choice of j in line 17 only has an effect to the ordering in which the algorithm assigns the weights but not to the weights itself. In other words, the result of the algorithm is independent of the choice of j in line 17.

Finally, we prove that every step of the algorithm can actually be executed and that the algorithm stops after finitely many steps. The existence of $s,t \in \mathbb{N}$ in line 18 has already been shown above. Moreover, we have also seen in the proof of the correctness that we can perform the addition in line 23 and that the set in line 14 is non-empty. Thus every step of the algorithm can be executed.
In order to prove termination, we need to show that the while-loop in line 12 is finite. Taking a closer look at this while-loop, for all $j \in \{1,\dots,\nu\}$ such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_i-direction with $i \in \{2,\dots,n\}$ and for all $\varrho \in \{1,\dots,\nu\}$ such that $b_\varrho e_{\beta_\varrho} \in \mathrm{Col}(b_j e_{\beta_j})$, we see that the for-loop that starts in line 22 assigns values not equal to `null` to w_{ε_ϱ}. Thus in every iteration of the while-loop, the number of non-extreme border terms which have not yet been assigned a weight decreases by at least one. Using Lemma 3.1.7, again, we see that every column and thus every non-extreme border term is considered only once during the algorithm. In particular, it follows that the condition of the if-clause is eventually satisfied and i is decreased by one. Induction on the loop variable i

in line 12 now yields that i decreases to 1 after finitely many steps and the algorithm terminates. $\square$

To get a better understanding of Algorithm 7, we consider the following example.

Example 3.4.3. Let us apply Algorithm 7 to the order ideal $\mathcal{O}$ in $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ of Example 3.3.4. Recall that we can illustrate $\mathcal{O} = \{1, z, y, z^2, y^2, z^3, y^3\} \cdot e_1 \cup \{1, y\} \cdot e_3$ and its border $\partial\mathcal{O} = \{b_1 e_{\beta_1}, \ldots, b_{20} e_{\beta_{20}}\}$ with the following figures.



Let $\{\varepsilon_1, \ldots, \varepsilon_{20}\}$ denote the canonical P-module basis of P^{20}. We consider the intermediate results of the weight matrix $W := \texttt{ConstructWeight}(\mathcal{O})$ during Algorithm 7 in detail. To ease the notation, we record the intermediate value of the matrix

$$W = (w_x, w_y, w_z, w_{\varepsilon_1}, \ldots, w_{\varepsilon_{20}})$$

by a table which looks like

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	`null`	1		1	1	`null`	`null`	`null`	`null`	1	`null`
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	`null`	`null`	1	1	`null`	1	1	1	1	1	`null`

after the initialization steps of the lines 1 to 10.

Then we have $i = 3$ in line 11 and start with the first iteration of the while-loop. There are five non-extreme border terms that have not yet been assigned weights and whose columns are in the z-direction, namely the border terms in

$$\{b_3, b_6, b_{11}\} \cdot e_1 \cup \{b_5, b_{10}\} \cdot e_1 = \{xz, xz^2, xz^3\} \cdot e_1 \cup \{yz^2, yz^3\} \cdot e_1.$$

We choose $b_3 e_{\beta_3} = xze_1$ in line 17 and get $u_0 = 1$, $u_1 = 3$, $u_2 = 6$, and $u_3 = 11$ after the for-loop starting in line 19. As $w_{\varepsilon_1} = 1$, line 23 of the algorithm yields

$$\begin{aligned} w_{\varepsilon_3} &= w_{\varepsilon_1} + w_z + 1 = 1 + 1 + 1 = 3, \\ w_{\varepsilon_6} &= w_{\varepsilon_3} + w_z + 1 = 3 + 1 + 1 = 5, \\ w_{\varepsilon_{11}} &= w_{\varepsilon_6} + w_z + 1 = 5 + 1 + 1 = 7. \end{aligned}$$

The intermediate table then looks like

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	`null`	1		1	1	3	`null`	`null`	5	1	`null`
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	`null`	7	1	1	`null`	1	1	1	1	1	`null`

at the end of the for-loop starting in line 22. As there are still two non-extreme border terms which have not yet been assigned weights and whose columns are in the z-direction, namely the border terms in

$$\{b_5, b_{10}\} \cdot e_1 = \{yz^2, yz^3\} \cdot e_1,$$

the if-clause starting in line 13 is not executed and we proceed with the next iteration of the while-loop without changing i. We now choose $b_5 e_{\beta_5} = yz^2 e_1$ in line 17 and get $w_{\varepsilon_5} = 3$ and $w_{\varepsilon_{10}} = 5$ after the for-loop starting in line 22. The intermediate table then looks like

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	`null`	1		1	1	3	`null`	3	5	1	`null`
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	5	7	1	1	`null`	1	1	1	1	1	`null`

For every $j \in \{1, \dots, 20\}$ such that $b_j e_{\beta_j}$ is non-extreme and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the z-direction, we then have $w_{\varepsilon_j} \neq \texttt{null}$. Thus the if-clause starting in line 13 is executed and we get

$$w_y = \max\{w_{\varepsilon_j} + w_z \mid j \in \{1, \dots, 20\}, w_{\varepsilon_j} \neq \texttt{null}\} = 7 + 1 = 8$$

which yields

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	8	1		1	1	3	`null`	3	5	1	`null`
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	5	7	1	1	`null`	1	1	1	1	1	`null`

and $i = 2$ at the end of the if-clause.

At the beginning of the next iteration of the while-loop, there are four non-extreme border terms which have not yet been assigned weights and whose columns are in the y-direction, namely the border terms in

$$\{b_4, b_8, b_{14}\} \cdot e_1 \cup \{b_{20}\} \cdot e_3 = \{xy, xy^2, xy^3\} \cdot e_1 \cup \{xy\} \cdot e_3.$$

We choose $b_4 e_{\beta_4} = xye_1$ in line 17 and the algorithm yields

$$\begin{aligned} w_{\varepsilon_4} &= w_{\varepsilon_1} + w_y + 1 = 1 + 8 + 1 = 10, \\ w_{\varepsilon_8} &= w_{\varepsilon_4} + w_y + 1 = 10 + 8 + 1 = 19, \\ w_{\varepsilon_{14}} &= w_{\varepsilon_8} + w_y + 1 = 19 + 8 + 1 = 28. \end{aligned}$$

The intermediate table then has the following form

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	8	1		1	1	3	10	3	5	1	19
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	5	7	1	1	28	1	1	1	1	1	`null`

We now choose $b_{20} e_{\beta_{20}} = xye_3$ in line 17 and get $w_{\varepsilon_{20}} = 10$ after the for-loop in line 22. This yields

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
`null`	8	1		1	1	3	10	3	5	1	19
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	5	7	1	1	28	1	1	1	1	1	10

For every $j \in \{1, \ldots, 20\}$ such that $b_j e_{\beta_j}$ is non-extreme and such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the y-direction, we then have $w_{\varepsilon_j} \neq \texttt{null}$. Hence the if-clause starting in line 13 is executed and we get

$$w_x = \max\{w_{\varepsilon_j} + w_y \mid j \in \{1, \ldots, 20\}, w_{\varepsilon_j} \neq \texttt{null}\} = 28 + 8 = 36.$$

Moreover, i is decreased to 1 in line 15, i.e. the algorithm terminates. The final table now has the following form:

w_x	w_y	w_z		w_{ε_1}	w_{ε_2}	w_{ε_3}	w_{ε_4}	w_{ε_5}	w_{ε_6}	w_{ε_7}	w_{ε_8}
36	8	1		1	1	3	10	3	5	1	19
w_{ε_9}	$w_{\varepsilon_{10}}$	$w_{\varepsilon_{11}}$	$w_{\varepsilon_{12}}$	$w_{\varepsilon_{13}}$	$w_{\varepsilon_{14}}$	$w_{\varepsilon_{15}}$	$w_{\varepsilon_{16}}$	$w_{\varepsilon_{17}}$	$w_{\varepsilon_{18}}$	$w_{\varepsilon_{19}}$	$w_{\varepsilon_{20}}$
1	5	7	1	1	28	1	1	1	1	1	10

Thus the algorithm returns the matrix

$$W = (36, 8, 1, 1, 1, 3, 10, 3, 5, 1, 19, 1, 5, 7, 1, 1, 28, 1, 1, 1, 1, 1, 10) \in \mathrm{Mat}_{1,23}(\mathbb{Z}).$$

In Theorem 3.4.5, we will see that for all term orderings τ on $\mathbb{T}^3\langle e_1, e_2, e_3\rangle$ that are compatible with $\deg_W$, the set of all neighbor liftings Λ forms a τ-Gröbner basis of the syzygy module $\mathrm{Syz}_P(g_1, \dots, g_{20})$ where $G = \{g_1, \dots, g_{20}\}$ is a $\mathcal{O}$-border basis.

Before we prove the main result of this chapter, namely that every term ordering on the terms of the idealization $\overline{P} = P[\varepsilon_1, \dots, \varepsilon_\nu]$ that is compatible with the grading $\deg_W$ where the matrix $W \coloneqq \texttt{ConstructWeight}(\mathcal{O})$ is the result of Algorithm 7 applied to the input data $\mathcal{O}$ induces a module term ordering τ on the terms of P^ν that satisfies the conditions of Theorem 3.4.1, we want to show the following auxiliary lemma. It is a generalization of [KK14, Lemma. 6.4].

Lemma 3.4.4. *Let*

$$W = (w_{x_1}, \dots, w_{x_n}, w_{\varepsilon_1}, \dots, w_{\varepsilon_\nu}) \coloneqq \texttt{ConstructWeight}(\mathcal{O})$$

be the result of Algorithm 7 applied to the input data $\mathcal{O}$ and let $i \in \{2, \dots, n\}$. Then

$$\begin{aligned} w_{x_{i-1}} = \max\{w_{\varepsilon_j} + w_{x_i} \mid j \in \{1, \dots, \nu\}, b_j e_{\beta_j} \text{ is non-exposable or extreme} \\ \text{or } \mathrm{Col}(b_j e_{\beta_j}) \text{ is in the } x_k\text{-direction where } k \in \{i, \dots, n\}\}. \end{aligned}$$

In particular, this maximum exists and we have $w_{x_k} > w_{x_\ell}$ for all $k, \ell \in \{1, \dots, n\}$ such that $k < \ell$.

Proof. Taking a closer look at Algorithm 7, we see that during the execution of the while-loop starting in line 12, the algorithm assigns weights to the non-extreme border terms column by column with descending column direction. During the algorithm, w_{x_i} is assigned a weight not equal to `null` in the if-clause starting in line 13. This if-clause is executed exactly if for all $j \in \{1, \dots, \nu\}$ such that the border term $b_j e_{\beta_j}$ is non-extreme and its column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction with $k \in \{i, \dots, n\}$, w_{ε_j} has previously been assigned a weight not equal to `null`. Moreover, for all $j \in \{1, \dots, \nu\}$ such that $b_j e_{\beta_j}$ is non-exposable or extreme, w_{ε_j} is assigned the weight 1 at the beginning of the algorithm. Thus exactly all non-exposable and extreme border terms and the non-extreme border terms on columns in the x_k-direction with $k \in \{i, \dots, n\}$ have been assigned a weight not equal to `null` when line 14 is executed.

As there is at least one non-exposable or extreme border term, namely the border term of the form $x_1^\delta e_1$ for some $\delta \in \mathbb{N}$ is non-exposable or extreme according to Definition 3.1.1, the maximum always exists. In particular, Proposition 3.4.2 implies $w_{x_k} > w_{x_\ell} > 0$ for all $k, \ell \in \{1, \dots, n\}$ such that $k < \ell$. □

Now we are ready to prove the main result of this chapter, namely Schreyer's Theorem for border bases of submodules of free modules of finite rank. The theorem is a generalization of [KK14, Thm. 6.5]. Recall that

$$W = (w_{x_1}, \dots, w_{x_n}, w_{\varepsilon_1}, \dots, w_{\varepsilon_\nu}) := \texttt{ConstructWeight}(\mathcal{O})$$

defines the $\mathbb{Z}$-grading of the idealization $\overline{P} = K[x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_\nu]$ that is given by $\deg_W(x_i) = w_{x_i}$ for all $i \in \{1, \dots, n\}$ and $\deg_W(\varepsilon_j) = w_{\varepsilon_j}$ for all $j \in \{1, \dots, \nu\}$, cf. [KR05, Defn. 4.1.6].

Theorem 3.4.5. (Schreyer's Theorem for Border Bases)
Let

$$W = (w_{x_1}, \dots, w_{x_n}, w_{\varepsilon_1}, \dots, w_{\varepsilon_\nu}) := \texttt{ConstructWeight}(\mathcal{O})$$

be the result of Algorithm 7 applied to the input data $\mathcal{O}$. Then the following conditions are satisfied for every $i \in \{1, \dots, n-1\}$ and every $j \in \{1, \dots, \nu\}$.

1) *Let $b_j e_{\beta_j}$ be extreme, let $\operatorname{face}(b_j e_{\beta_j}) > i$, and let $k \in \{1, \dots, \nu\}$ be such that $b_k e_{\beta_k}$ is exposable and $\operatorname{face}(b_k e_{\beta_k}) \geq i$. Then we have $\deg_W(x_i \varepsilon_j) > \deg_W(\varepsilon_k)$.*

2) *Let $b_j e_{\beta_j}$ be extreme and such that $x_i b_j e_{\beta_j} \notin \partial\mathcal{O}$, let $k = \operatorname{face}(b_j e_{\beta_j}) > i$, and let $\ell \in \{1, \cdots, \nu\}$ be the unique index such that $x_i b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$. Then we have $\deg_W(x_i \varepsilon_j) > \deg_W(x_k \varepsilon_\ell)$.*

3) *Let $b_j e_{\beta_j}$ be non-extreme, let $k \in \{i+1, \dots, n\}$ be such that the column $\operatorname{Col}(b_j e_{\beta_j})$ is in the x_k-direction, and let $\ell \in \{1, \dots, \nu\}$ be such that the border term $b_\ell e_{\beta_\ell}$ is exposable and $\operatorname{face}(b_\ell e_{\beta_\ell}) \geq k$. Then we have $\deg_W(\varepsilon_j) > \deg_W(\varepsilon_\ell)$.*

4) *Let $b_j e_{\beta_j}$ be non-extreme, let $k \in \{i+1, \dots, n\}$ be such that $\operatorname{Col}(b_j e_{\beta_j})$ is in the x_k-direction, and let $\ell \in \{1, \cdots, \nu\}$ be the unique index with $b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$. Then we have $\deg_W(\varepsilon_j) > \deg_W(x_k \varepsilon_\ell)$.*

In particular, every term ordering $\overline{\tau}$ on the terms of $\overline{P}$ that is compatible with $\deg_W$ induces a module term ordering τ on the terms of P^ν such that Λ is a τ-Gröbner basis of $\operatorname{Syz}_P(g_1, \dots, g_\nu)$, and we can explicitly construct such term orderings.

Proof. Let $i \in \{1, \dots, n-1\}$ and let $j \in \{1, \dots, \nu\}$. Since we use Proposition 3.4.2 and Lemma 3.4.4 to show all inequalities of this proof, we will apply them without mention.

First we prove condition 1). Assume that $b_j e_{\beta_j}$ is extreme and $\operatorname{face}(b_j e_{\beta_j}) > i$. Let $k \in \{1, \dots, \nu\}$ be such that the border term $b_k e_{\beta_k}$ is exposable and $\operatorname{face}(b_k e_{\beta_k}) \geq i$.

If $b_k e_{\beta_k}$ is non-extreme, then $\mathrm{Col}(b_k e_{\beta_k})$ is in the x_d-direction for some $d \in \{i+1,\dots,n\}$ by Definition 3.1.5. Thus we have

$$\begin{aligned}
\deg_W(x_i\varepsilon_j) &= w_{x_i} + w_{\varepsilon_j} \\
&> w_{x_i} \\
&= \max\{w_{\varepsilon_\varrho} + w_{x_{i+1}} \mid \varrho \in \{1,\dots,\nu\}, b_\varrho e_{\beta_\varrho} \text{ is non-exposable or extreme} \\
&\qquad\qquad \text{or } \mathrm{Col}(b_\varrho e_{\beta_\varrho}) \text{ is in the } x_s\text{-direction where } s \in \{i+1,\dots,n\}\} \\
&> w_{\varepsilon_k} \\
&= \deg_W(\varepsilon_k).
\end{aligned}$$

Next we prove condition 2). Assume that $b_j e_{\beta_j}$ is extreme, that $x_i b_j e_{\beta_j} \notin \partial\mathcal{O}$, and that $k = \mathrm{face}(b_j e_{\beta_j}) > i$. Let $\ell \in \{1,\dots,\nu\}$ be the unique index with $x_i b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$. The existence and uniqueness of this index ℓ has been shown in Proposition 3.2.2. We have also shown in the proof of Proposition 3.2.2 that $b_\ell e_{\beta_\ell}$ is exposable and that $\mathrm{face}(b_\ell e_{\beta_\ell}) \geq i$. In particular, the border term $b_\ell e_{\beta_\ell}$ is extreme or $\mathrm{Col}(b_\ell e_{\beta_\ell})$ is in the x_d-direction for some $d \in \{i+1,\dots,n\}$ by Definition 3.1.5. Thus we have

$$\begin{aligned}
\deg_W(x_i\varepsilon_j) &= w_{x_i} + w_{\varepsilon_j} \\
&> w_{x_i} \\
&= \max\{w_{\varepsilon_\varrho} + w_{x_{i+1}} \mid \varrho \in \{1,\dots,\nu\}, b_\varrho e_{\beta_\varrho} \text{ is non-exposable or extreme} \\
&\qquad\qquad \text{or } \mathrm{Col}(b_\varrho e_{\beta_\varrho}) \text{ is in the } x_s\text{-direction where } s \in \{i+1,\dots,n\}\} \\
&\geq w_{\varepsilon_\ell} + w_{x_{i+1}} \\
&> w_{\varepsilon_\ell} + w_{x_{i+2}} \\
&\ \ \vdots \\
&> w_{\varepsilon_\ell} + w_{x_k} \\
&= \deg_W(x_k\varepsilon_\ell).
\end{aligned}$$

Now we prove condition 3). Assume that the border term $b_j e_{\beta_j}$ is non-extreme and let $k \in \{i+1,\dots,n\}$ be such that the column $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. According to Definition 3.1.5 and Lemma 3.1.9, there exists an index $m \in \{1,\dots,\nu\}$ such that $b_j e_{\beta_j} = x_k b_m e_{\beta_m}$. Moreover, in view of the construction of w_{ε_j} in line 23 of Algorithm 7, we see that $w_{\varepsilon_j} = w_{\varepsilon_m} + w_{x_k} + 1$. Let $\ell \in \{1,\dots,\nu\}$ be such that $b_\ell e_{\beta_\ell}$ is exposable and $\mathrm{face}(b_\ell e_{\beta_\ell}) \geq k$. If $b_\ell e_{\beta_\ell}$ is extreme, we have

$$\deg_W(\varepsilon_j) = w_{\varepsilon_j} = w_{\varepsilon_m} + w_{x_k} + 1 > 1 = w_{\varepsilon_\ell} = \deg_W(\varepsilon_\ell).$$

If $b_\ell e_{\beta_\ell}$ is non-extreme, we see that $k < n$ and $\mathrm{Col}(b_\ell e_{\beta_\ell})$ is in the x_d-direction for some

$d \in \{k+1, \ldots, n\}$ by Definition 3.1.5. In this situation, we have

$$\begin{aligned}
\deg_W(\varepsilon_j) &= w_{\varepsilon_j} \\
&= w_{\varepsilon_m} + w_{x_k} + 1 \\
&> w_{x_k} \\
&= \max\{w_{\varepsilon_\varrho} + w_{x_{k+1}} \mid \varrho \in \{1, \ldots, \nu\}, b_\varrho e_{\beta_\varrho} \text{ is non-exposable or extreme} \\
&\qquad\qquad \text{or } \mathrm{Col}(b_\varrho e_{\beta_\varrho}) \text{ is in the } x_s\text{-direction where } s \in \{k+1, \ldots, n\}\} \\
&> w_{\varepsilon_\ell} \\
&= \deg_W(\varepsilon_\ell).
\end{aligned}$$

Next we show that condition 4) holds. Assume that $b_j e_{\beta_j}$ is non-extreme and let $k \in \{i+1, \ldots, n\}$ be such that $\mathrm{Col}(b_j e_{\beta_j})$ is in the x_k-direction. By Definition 3.1.5 and Lemma 3.1.9, there exists a unique index $\ell \in \{1, \ldots, \nu\}$ such that $b_j e_{\beta_j} = x_k b_\ell e_{\beta_\ell}$. Moreover, considering the construction of w_{ε_j} in line 23 of Algorithm 7, we see that

$$\deg_W(\varepsilon_j) = w_{\varepsilon_j} = w_{\varepsilon_\ell} + w_{x_k} + 1 > w_{\varepsilon_\ell} + w_{x_k} = \deg_W(x_k \varepsilon_\ell).$$

Finally, we prove all further claims of this theorem. Since we have shown in Proposition 3.4.2 that W is a positive matrix in the sense of [KR05, Defn. 4.2.4], we can easily find a term ordering $\overline{\tau}$ on the terms of $\overline{P}$ that is compatible with $\deg_W$ according to [KR05, Defn. 4.2.1 and Prop. 4.2.3]. Then the term ordering $\overline{\tau}$ on the terms of $\overline{P} = P[\varepsilon_1, \ldots, \varepsilon_\nu]$ induces a module term ordering τ on the terms of P^ν according to [CS99, Thm. 17]. Since the conditions 1) to 4) hold, this module term ordering τ satisfies the conditions 1) to 4) of Theorem 3.4.1 and the claims follow. □

It is important to note that, just as in the ideal setting in [KK14, Rem. 6.6], the construction of the degree matrix W does not depend on the specific border basis G.

Remark 3.4.6. The construction of the degree matrix $W \in \mathrm{Mat}_{1,n+\nu}(\mathbb{Z})$ in the `ConstructWeight` Algorithm 7 only uses the shape of the border $\partial\mathcal{O}$, which, in turn, only depends on the order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. In particular, the constructed matrix W is independent of the specific $\mathcal{O}$-border basis and the corresponding first syzygy module of $(g_1, \ldots, g_\nu)$.

As a first application of the above theorem, we can generalize the method described in [KK14, Rem. 6.7] to quickly compute a free resolution of a submodule of P^r with finite K-codimension in P^r.

Remark 3.4.7. Let $U \subset P^r$ be a P-submodule with $\operatorname{codim}_K(U, P^r) < \infty$. According to Theorem 2.5.3, we can compute an $\mathcal{O}$-border basis $G = \{g_1, \ldots, g_\nu\} \subset P^r$ of U and by Remark 2.4.28, we can then compute the corresponding set of neighbor liftings $\Lambda \subseteq \operatorname{Syz}_P(g_1, \ldots, g_\nu)$ with respect to $\mathcal{O}$. As Theorem 3.4.5 shows, we can compute a term orderings τ on P^ν such that the set of neighbor liftings Λ is a τ-Gröbner basis of the first syzygy module of $(g_1, \ldots, g_\nu)$. Thus we can compute the second syzygy module of $(g_1, \ldots, g_\nu)$ using Schreyer's Theorem for Gröbner bases as in [Sch80] or [KR00, Prop. 3.1.4]. Continuing to use Schreyer's method, we can iteratively compute a free resolution of U. Notice that we are, in general, not in a homogeneous situation. Hence this algorithm yields, in general, neither a graded nor a minimal resolution of U.

4 Border Bases and Homogenization

The remaining chapters of this thesis are dedicated to the study of zero-dimensional closed subschemes of weighted projective spaces by means of border bases. More precisely, we want to represent the defining ideal of the subscheme by a border basis. This enables us to describe the multiplicative structure of the corresponding coordinate very explicitly by only regarding the multiplication matrices of this border basis as follows. If we want to study a geometric property of a given subscheme which is based on specific properties of multiplication maps in the corresponding coordinate ring, we can use the explicit description of the multiplication in the coordinate ring using the multiplication matrices of the corresponding border basis. One example of such a geometric property are uniformity conditions, i.e. is the subscheme in some sense in a uniform position? We will apply these ideas in Chapter 5 to generalizations of the uniformity condition of [Kre98, Defn. 7.12] and [Kre01, Defn. 2.1] and particularly answer [Kre01, Question 1]. Since border bases are a priori not homogeneous, cf. Definition 2.1.14, we must homogenize them in order to use them in a projective setting. The following chapter is dedicated to the study of the effect of homogenization to border bases.

As described above, we are interested in border bases of polynomial rings, i.e. we can restrict ourselves to border bases in $P = K[x_1, \dots, x_n]$ for the remainder of this thesis. We equip the polynomial ring $P = K[x_1, \dots, x_n]$ with the grading defined by a matrix $W = (w_1, \dots, w_n) \in \operatorname{Mat}_{1,n}(\mathbb{Z})$ where $w_i > 0$ for all $i \in \{1, \dots, n\}$. Moreover, we choose a homogenizing indeterminate x_0 and equip $\overline{P} = P[x_0] = K[x_0, \dots, x_n]$ with the grading defined by $\overline{W} = (1 \mid W) \in \operatorname{Mat}_{1,n+1}(\mathbb{Z})$. By [KR05, Defn. 4.1.6 and 4.2.4], both $P = \bigoplus_{\gamma \in \mathbb{Z}} P_\gamma$ and $\overline{P} = \bigoplus_{\gamma \in \mathbb{Z}} \overline{P}_\gamma$ are positively $\mathbb{Z}$-graded K-algebras.

In this setting, we want to use the idea of [KR05, Thm. 4.3.22] which states that for a proper ideal $I \subset P$, $\overline{P}/I^{\mathrm{hom}}$ is a free $K[x_0]$-module. This idea led to the definition of projective border bases of homogeneous ideals in $\overline{P}$ in Definition 4.1.2. A projective $\mathcal{O}$-border prebasis for some order ideal $\mathcal{O} = \{t_1, \dots, t_\mu\}$ in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ is a set of polynomials $G = \{g_1, \dots, g_\nu\}$ of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ with $a_{1j}, \dots, a_{\mu j} \in K[x_0]$ for all $j \in \{1, \dots, \nu\}$. It is said to be a projective $\mathcal{O}$-border basis of a homogeneous ideal $I \subseteq \overline{P}$ if the set of residue classes of the elements of $\mathcal{O}$ in $\overline{P}/I$ form a $K[x_0]$-module basis of $\overline{P}/I$. In Proposition 4.1.7, it then turns out that such a projective border basis $G \subseteq \overline{P}$ of a homogeneous ideal $I \subseteq \overline{P}$ is uniquely determined, its dehomogenization $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ is a border bases in P and a Macaulay basis (also called H-basis) with respect to the grading given by W, and that G indeed generates the ideal I. In particular, we prove the (Projective) Border Bases Correspondence in Corollary 4.1.9 which states that every projective border basis of a homogeneous ideal in $\overline{P}$ induces a border basis of an ideal in P of a specific shape and vice versa. This correspondence can be illustrated as follows.

projective $\mathcal{O}$-border bases of homogeneous ideals in $\overline{P}$
where $\overline{P}$ is graded by $\overline{W} = (1 \mid W)$ with a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$

dehomogenization $\downarrow$ $\quad$ $\uparrow$ homogenization

$\mathcal{O}$-border bases of ideals in P with $b \in \mathrm{DF}_W(g_b)$ for all $b \in \partial\mathcal{O}$
where g_b denotes the $\mathcal{O}$-border basis element corresponding to b

In particular, this correspondence allows us to generalize the concepts and propositions of Chapter 2 to projective border bases. For instance, we can to characterize and eventually compute projective border bases in the Corollaries 4.1.10 and 4.1.14 and we are able to describe the elements of the residue class ring of $\overline{P}$ modulo a projective $\mathcal{O}$-border basis by means of $\mathcal{O}$ in Proposition 4.1.15. Based on these results, we describe the multiplicative structure of the residue class ring $\overline{P}$ modulo a projective $\mathcal{O}$-border basis G of a homogeneous ideal I by means of $\mathcal{O}$ and formal multiplication matrices, which solely depend G, in Section 4.2.

Since it is often useful to consider dual objects of a given object, we study the graded dual module of the residue class ring $R = \overline{P}/\langle G\rangle$ where G is a projective $\mathcal{O}$-border basis in Section 4.3. In this setting, the graded dual $K[x_0]$-module of R turns out to be the canonical module ω_R of R in Definition 4.3.4 and Remark 4.3.5. The remaining part of Section 4.3 is then devoted to describe the $K[x_0]$-module multiplication of ω_R. Again, we show that the order ideal $\mathcal{O}$ and formal multiplication matrices allow us to describe the elements of ω_R in Proposition 4.3.7 and the $K[x_0]$-module multiplication in the Propositions 4.3.8 and 4.3.10, explicitly.

Recall that for a $\mathbb{Z}$-graded module $M = \bigoplus_{\gamma\in\mathbb{Z}} M_\gamma$, a subset $S \subseteq M$, and for an integer $\gamma \in \mathbb{Z}$, we denote $S_\gamma = S \cap M_\gamma$, $S_{<\gamma} = \bigcup_{\gamma'=-\infty}^{\gamma-1} S_{\gamma'}$, $S_{\geq\gamma} = \bigcup_{\gamma'=\gamma}^{\infty} S_{\gamma'}$, etc. Moreover, for every $\mathbb{Z}$-graded module $M = \bigoplus_{\gamma\in\mathbb{Z}} M_\gamma$ and for all $\gamma' \in \mathbb{Z}$, we let $M(\gamma')$ denote the module obtained from M by shifting degrees by γ'. Here the $\mathbb{Z}$-grading of $M(\gamma')$ is given by $(M(\gamma'))_\gamma = M_{\gamma+\gamma'}$ for all $\gamma \in \mathbb{Z}$, cf. [KR05, Defn. 1.7.6].

4.1 The (Projective) Border Bases Correspondence

In this subsection, we introduce projective border bases and we show that there is a one-to-one correspondence between projective border bases of homogeneous ideals in $\overline{P}$ and specific border bases of ideals in P in Corollary 4.1.9. Moreover, we explicitly describe the elements of the residue class ring of $\overline{P}$ modulo a projective $\mathcal{O}$-border basis by means of the residue classes of the elements of $\mathcal{O}$ in Proposition 4.1.15.

The main idea behind the definition of projective border bases is based on [KR05, Thm. 4.3.22]. This theorem states that for a proper ideal $I \subset P$, the residue class ring $\overline{P}/I^{\mathrm{hom}}$ is a free $K[x_0]$-module. In the fashion of this result and starting with a homogeneous ideal $I \subseteq \overline{P}$, we define a projective $\mathcal{O}$-border basis in Definition 4.1.2 as a set of polynomials $G = \{g_1, \dots, g_\nu\}$ of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i \in I$, where $a_{ij} \in K[x_0]$ with the property that the residue classes of the elements of the order ideal $\mathcal{O}$ in $\overline{P}/I$ form a $K[x_0]$-module basis of $\overline{P}/I$. Here $\mathcal{O} = \{t_1, \dots, t_\mu\}$ is a finite order ideal in $\mathbb{T}^n$ with border $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$. In particular, $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\} \subseteq I^{\mathrm{deh}}$ is an $\mathcal{O}$-border prebasis in this situation. Now two questions immediately arise from this definition: Is $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\}$ even the $\mathcal{O}$-border basis of I^{deh}? And the other way round, does every $\mathcal{O}$-border basis of an ideal in P induce a projective $\mathcal{O}$-border basis after homogenization? We will answer these questions in the (Projective) Border Bases Correspondence in Corollary 4.1.9.

But before we can introduce projective border bases, we need to introduce a natural $\mathbb{Z}$-graded $K[x_0]$-module structure for arbitrary residue class rings of $\overline{P}$ modulo a homogeneous ideal $I \subseteq \overline{P}$.

Proposition 4.1.1. *Let $I \subseteq \overline{P}$ be a homogeneous ideal. Then the map*

$$K[x_0] \times \overline{P}/I \to \overline{P}/I, \quad (p, r) \mapsto \overline{p}r$$

turns the ring $\overline{P}/I$ into a $\mathbb{Z}$-graded $K[x_0]$-algebra.

Proof. The canonical K-algebra homomorphism $K[x_0] \twoheadrightarrow K[\overline{x}_0] \hookrightarrow \overline{P}/I$, where we have $\overline{x}_0 \in \overline{P}/I$, equips the ring $\overline{P}/I$ with the given $\mathbb{Z}$-graded $K[x_0]$-module structure. $\square$

The above $K[x_0]$-module structure allows us to define projective border bases.

Definition 4.1.2. Let $\mathcal{O} = \{t_1, \dots, t_\mu\} \subseteq \mathbb{T}^n$ with $\mu \in \mathbb{N}$ be a finite order ideal in $\mathbb{T}^n$ and let $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border.

a) A set of polynomials $G = \{g_1, \dots, g_\nu\} \subseteq \overline{P}$ is called a **projective $\mathcal{O}$-border prebasis** if the polynomials have the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ with $a_{ij} \in K[x_0]$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$.

b) Let $G = \{g_1, \dots, g_\nu\} \subseteq \overline{P}$ be a projective $\mathcal{O}$-border basis and let $I \subseteq \overline{P}$ be a homogeneous ideal. We call G a **projective $\mathcal{O}$-border basis** of I if $G \subseteq I$ and if the residue classes of the elements of $\mathcal{O}$ in $\overline{P}/I$ form a $K[x_0]$-module basis of $\overline{P}/I$.

Example 4.1.3. Let $P = \mathbb{Q}[x, y]$ be standard graded, i.e. graded by the matrix $W = (1, 1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$, let t be a homogenizing indeterminate, and let $\overline{P} = \mathbb{Q}[t, x, y]$ also be standard graded by $\overline{W} = (1, 1, 1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$. Moreover, let $\mathcal{O} = \{1, y, x, y^2, xy\}$ and let $G = \{g_1, \dots, g_4\} \subseteq \overline{P}$ be with $g_1 = x^2 - 1 - (t^2 + t)y$, $g_2 = y^3 - 2t + y$, $g_3 = xy^2 - t - y - xy$, and $g_4 = x^2y + t^4 - xy$. Then G is a projective $\mathcal{O}$-border prebasis by Definition 4.1.2. The decomposition of g_1 into its homogeneous components is $g_1 = (-t^2y) + (x^2 - ty) + (-1)$. Using a Gröbner basis computation as, for instance, described in [KR00, Prop. 2.4.10], we can easily check that $-1 \notin \langle G \rangle$. Therefore, $\langle G \rangle$ is not homogeneous by [KR00, Prop 1.7.10] and G is thus not a projective $\mathcal{O}$-border basis of $\langle G \rangle$ according to Definition 4.1.2.

The specific shape of projective $\mathcal{O}$-border prebases G as defined in Definition 4.1.2 immediately yields that $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ is a border prebasis as defined in Definition 2.1.14. In particular, we can use this fact and the Border Division Algorithm 2.2.1 to deduce a division algorithm.

Proposition 4.1.4. (The Projective Border Division Algorithm)
Let $\mathcal{O} = \{t_1, \dots, t_\mu\}$ with $\mu \in \mathbb{N}$ be a finite order ideal in $\mathbb{T}^n$ and $\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border. Let $G = \{g_1, \dots, g_\nu\} \subseteq \overline{P}$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$, be a projective $\mathcal{O}$-border prebasis and let $p \in \overline{P}$. We identify p and the elements of G with their images under the embedding $\overline{P} = K[x_0, x_1, \dots, x_n] \hookrightarrow K(x_0)[x_1, \dots, x_n]$ and let $p_1, \dots, p_\nu \in K(x_0)[x_1, \dots, x_n]$ and $c_1, \dots, c_\mu \in K(x_0)$ be the result of the Border Division Algorithm 1 applied to the input data $p \in K(x_0)[x_1, \dots, x_n]$ and $G \subseteq K(x_0)[x_1, \dots, x_n]$, i.e. such that $p = p_1 g_1 + \cdots + p_\nu g_\nu + c_1 t_1 + \cdots + c_\mu t_\mu$ by Theorem 2.2.1. Then we have $p_1, \dots, p_\nu \in \overline{P}$ and $c_1, \dots, c_\mu \in K[x_0]$.

Proof. Consider the steps of Algorithm 1 in detail. The coefficient $a \in K(x_0)$ chosen in line 6 is always an element of $K[x_0]$ as $p \in \overline{P}$. Thus the polynomials $p_1, \dots, p_\nu$ in line 7, which are initiated by $0 \in \overline{P}$ in line 1, remain in $\overline{P}$ during the whole algorithm. Therefore, at the end of the algorithm in line 10, we get $c_1 t_1 + \cdots c_\mu t_\mu = p - p_1 g_1 - \cdots - p_\nu g_\nu \in \overline{P}$ and this implies $c_1, \dots, c_\mu \in K[x_0]$. □

Example 4.1.5. Consider the projective $\mathcal{O}$-border prebasis $G \subseteq \overline{P} = \mathbb{Q}[t, x, y]$ of Example 4.1.3, again. Recall that we had $\mathcal{O} = \{1, y, x, y^2, xy\}$ and $G = \{g_1, \dots, g_4\}$ with $g_1 = x^2 - 1 - (t^2 + t)y$, $g_2 = y^3 - 2t + y$, $g_3 = xy^2 - t - y - xy$, and $g_4 = x^2y + t^4 - xy$. Let $p = x^2y^2 + x^3 \in \overline{P}$. Imitating the steps of the Border Division Algorithm 2.2.1 applied to the input data $p \in \mathbb{Q}(t)[x, y]$ and $G \subseteq \mathbb{Q}(t)[x, y]$, we get

$$
\begin{aligned}
p &= x^2y^2 + x^3 \\
&= x \cdot xy^2 + x^3 \\
&= x \cdot g_3 - x \cdot (-xy - y - t) + x^3 \\
&= xg_3 + x^3 + x^2y + xy + tx \\
&= xg_3 + (xg_1 + (t^2 + t)xy + x) + x^2y + xy + tx \\
&= xg_1 + xg_3 + x^2y + (t^2 + t + 1)xy + (t + 1)x \\
&= xg_1 + xg_3 + (g_4 + xy - t^4) + (t^2 + t + 1)xy + (t + 1)x \\
&= xg_1 + xg_3 + g_4 + (t^2 + t + 2)xy + (t + 1)x - t^4.
\end{aligned}
$$

Before we study the basic properties of projective $\mathcal{O}$-border bases, we need an auxiliary lemma.

Lemma 4.1.6. *Let $I \subseteq \overline{P}$ be a homogeneous ideal. Then the following conditions are equivalent.*

i) The element $x_0 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$.

ii) We have $I = I :_{\overline{P}} \langle x_0 \rangle^\infty$.

Proof. First we prove that i) implies ii). The inclusion $I \subseteq I :_{\overline{P}} \langle x_0 \rangle^\infty$ trivially holds. Let $p \in I :_{\overline{P}} \langle x_0 \rangle^\infty$, i.e. $x_0^k p \in I$ for some $k \in \mathbb{N}$. In $\overline{P}/I$, this means that $x_0^k \overline{p} = 0$. By i), we get $\overline{p} = 0$ in $\overline{P}/I$, i.e. $p \in I$.

For the converse implication, let $p \in \overline{P}$ be such that $x_0 p \in I$. Then ii) implies $p \in I$, i.e. $\overline{p} = 0$ in $\overline{P}/I$. In other words, x_0 is a non-zero divisor for $\overline{P}/I$. □

Now we are able to prove the basic properties of projective border bases. In particular, we prove the first part of the (Projective) Border Basis Correspondence 4.1.9, namely that the dehomogenization of a projective $\mathcal{O}$-border basis of a homogeneous ideal I in $\overline{P}$ is indeed an $\mathcal{O}$-border basis of $I^{\mathrm{deh}} \subseteq P$.

Proposition 4.1.7. (Properties of Projective Border Bases)
Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\mu \in \mathbb{N}$ be a finite order ideal in $\mathbb{T}^n$ and let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border. Let $G = \{g_1, \ldots, g_\nu\} \subseteq \overline{P}$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ be a projective $\mathcal{O}$-border basis of a homogeneous ideal $I \subset \overline{P}$.

a) We have $I = \langle G \rangle$.

b) The element $x_0 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$.

c) The element $x_0 - 1 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$.

d) We have $(I^{\mathrm{deh}})^{\mathrm{hom}} = I :_{\overline{P}} \langle x_0 \rangle^\infty = I$.

e) The set $\{g_1^{\mathrm{deh}}, \ldots, g_\nu^{\mathrm{deh}}\} \subseteq P$ is an $\mathcal{O}$-border basis of I^{deh}.

f) For all $j \in \{1, \ldots, \nu\}$, we have $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j^{\mathrm{deh}}))$ and $g_j = (g_j^{\mathrm{deh}})^{\mathrm{hom}}$. In particular, g_j is homogeneous, $a_{ij} = a_{ij}(1) x_0^{\deg_W(b_j) - \deg_W(t_i)}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$, and the $\mathcal{O}$-border basis G of I is unique.

g) The set $\{g_1^{\mathrm{deh}}, \ldots, g_\nu^{\mathrm{deh}}\} \subseteq P$ is a Macaulay basis of I^{deh} with respect to the grading given by W.

Proof. First we prove a). By Definition 4.1.2, we have $\langle G \rangle \subseteq I$. For the converse, let $p \in I$. Then the Projective Border Division Algorithm 4.1.4 yields $p_1, \ldots, p_\nu \in \overline{P}$ and $c_1, \ldots, c_\mu \in K[x_0]$ such that $p = p_1 g_1 + \cdots + p_\nu g_\nu + c_1 t_1 + \cdots + c_\mu t_\mu$. In $\overline{P}/I$, we get $0 = c_1 \overline{t}_1 + \cdots c_\mu \overline{t}_\mu$. As G is the projective $\mathcal{O}$-border basis of I, we have $c_1 = \cdots = c_\mu = 0$ by Definition 4.1.2 and thus $p = p_1 g_1 + \cdots + p_\nu g_\nu \in \langle G \rangle$.

Next we show b). As G is the projective $\mathcal{O}$-border basis of I, the residue classes of the elements of $\mathcal{O}$ in $\overline{P}/I$ form a $K[x_0]$-module basis of $\overline{P}/I$ by Definition 4.1.2. For a contradiction, assume that there exists a $p \in \overline{P} \setminus I$ such that $x_0 p \in I$. Since $p \notin I$ there exist unique polynomials $c_1, \dots, c_\mu \in K[x_0]$ such that $\overline{p} = c_1\bar{t}_1 + \cdots + c_\mu \bar{t}_\mu$ in $\overline{P}/I$ and such that $c_i \neq 0$ for some $i \in \{1, \dots, \mu\}$. As $x_0 p \in I$, it follows $0 = x_0\overline{p} = c_1 x_0 \bar{t}_1 + \cdots + c_\mu x_0 \bar{t}_\mu$ in $\overline{P}/I$ and, therefore, $c_1 x_0 = \cdots = c_\mu x_0 = 0$ by Definition 4.1.2. In particular, we get $c_i x_0 = 0$. Since $K[x_0]$ is an integral domain, this clearly contradicts $c_i \neq 0$. Thus $x_0 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$.

In order to prove c), assume that there exists a $p \in \overline{P} \setminus I$ such that $(x_0 - 1)p \in I$. If $\mathrm{DF}_W(p) \in I$, $p - \mathrm{DF}_W(p) \notin I$ and $(x_0 - 1)(p - \mathrm{DF}_W(p)) = (x_0 - 1)p - (x_0 - 1)\mathrm{DF}_W(p) \in I$. By subtracting $\mathrm{DF}_W(p)$ multiple times, we can thus without loss of generality assume that $\mathrm{DF}_W(p) \notin I$. As I is homogeneous, we then have $\mathrm{DF}_W((x_0 - 1)p) = x_0 \mathrm{DF}_W(p) \in I$ according to [KR05, Defn. 4.2.8 and Rem. 4.2.12] and [KR00, Prop. 1.7.10]. But this is a contradiction to b) and thus $x_0 - 1 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$.

The first equality in d) follows from [KR05, Prop. 4.3.5] and the second one is a direct consequence of b) and Lemma 4.1.6. Thus we go on with the proof of claim e). For all $j \in \{1, \dots, \nu\}$, the dehomogenization g_j^{deh} of the polynomial g_j is of the form $g_j^{\mathrm{deh}} = b_j - \sum_{i=1}^{\mu} a_{ij}^{\mathrm{deh}} t_i \in P$ with $a_{ij}^{\mathrm{deh}} = a_{ij}(1) \in K$ by [KR05, Defn. 4.3.1]. Therefore, $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\} \subseteq P$ is an $\mathcal{O}$-border prebasis by Definition 2.1.14. Let $p \in I^{\mathrm{deh}} \cap \langle \mathcal{O} \rangle_K$. Then there are $c_1, \dots, c_\mu \in K$ with $p = c_1 t_1 + \cdots + c_\mu t_\mu \in I^{\mathrm{deh}}$. Assume that $p \neq 0$ and let $d = \deg_W(p)$. Then $p^{\mathrm{hom}} = c_1 x_0^{d - \deg_W(t_1)} t_1 + \cdots + c_\mu x_0^{d - \deg_W(t_\mu)} t_\mu \in (I^{\mathrm{deh}})^{\mathrm{hom}} = I$ by d). As G is the projective $\mathcal{O}$-border basis of I, it follows $c_i x_0^{d - \deg_W(t_i)} = 0$ for all $i \in \{1, \dots, \mu\}$ by Definition 4.1.2. Since $K[x_0]$ is an integral domain, we see that $c_1 = \cdots = c_\mu = 0$. In particular, we get the contradiction that $p = 0$. Altogether, we have proven that $I^{\mathrm{deh}} \cap \langle \mathcal{O} \rangle_K = \{0\}$ and Corollary 2.2.6 implies that $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\}$ is the $\mathcal{O}$-border basis of I^{deh}.

In order to prove claim f), let $j \in \{1, \dots, \nu\}$. As $g_j^{\mathrm{deh}} = b_j - \sum_{i=1}^{\mu} a_{ij}(1) t_i$, we have $b_j \in \mathrm{Supp}(g_j^{\mathrm{deh}})$ and thus $\deg_W(g_j^{\mathrm{deh}}) \geq \deg_W(b_j)$. Let $d = \deg_W(g_j^{\mathrm{deh}})$. Since $g_j \in G \subseteq I$ according to Definition 4.1.2 and since $I = (I^{\mathrm{deh}})^{\mathrm{hom}}$ by d), it follows that

$$
\begin{aligned}
I &\ni x_0^{d - \deg_W(b_j)} g_j - (g_j^{\mathrm{deh}})^{\mathrm{hom}} \\
&= x_0^{d - \deg_W(b_j)} \left(b_j - \sum_{i=1}^{\mu} a_{ij} t_i \right) - \left(b_j - \sum_{i=1}^{\mu} a_{ij}(1) t_i \right)^{\mathrm{hom}} \\
&= x_0^{d - \deg_W(b_j)} b_j - \sum_{i=1}^{\mu} a_{ij} x_0^{d - \deg_W(b_j)} t_i - x_0^{d - \deg_W(b_j)} b_j + \sum_{i=1}^{\mu} a_{ij}(1) x_0^{d - \deg_W(t_i)} t_i \\
&= \sum_{i=1}^{\mu} \left(-a_{ij} x_0^{d - \deg_W(b_j)} + a_{ij}(1) x_0^{d - \deg_W(t_i)} \right) t_i.
\end{aligned}
$$

As G is a projective $\mathcal{O}$-border basis of I, we get $a_{ij} x_0^{d - \deg_W(b_j)} = a_{ij}(1) x_0^{d - \deg_W(t_i)}$ for all $i \in \{1, \dots, \mu\}$ by Definition 4.1.2. Assume that there is an $i \in \{1, \dots, \mu\}$ such that $\deg_W(b_j) < \deg_W(t_i)$ and such that $a_{ij}(1) \neq 0$. As $a_{ij} \in K[x_0]$ and as $0 \neq a_{ij}(1) \in K$, we see that $d - \deg_W(b_j) \leq d - \deg_W(t_k)$. Clearly, this yields $\deg_W(b_j) \geq \deg_W(t_k)$ in contradiction to our assumption. Therefore, $\deg_W(b_j) \geq \deg_W(t_i)$ for all $i \in \{1, \dots, \mu\}$ such that $a_{ij}(1) \neq 0$ and this implies $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j^{\mathrm{deh}}))$. In particular, we see that

$d = \deg_W(b_j)$ and thus $a_{ij} = a_{ij}(1) x_0^{\deg_W(b_j) - \deg_W(t_i)}$ for all $i \in \{1, \ldots, \mu\}$. Therefore,

$$(g_j^{\mathrm{deh}})^{\mathrm{hom}} = x_0^{d - \deg_W(b_j)} b_j - \sum_{i=1}^{\mu} a_{ij}(1) x_0^{d - \deg_W(t_i)} t_i = b_j - \sum_{i=1}^{\mu} a_{ij} t_i = g_j.$$

Finally, we show g). Claim f) together with [KR08, Thm. 2.4] yield that the set $\{\mathrm{DF}_W(g_1^{\mathrm{deh}}), \ldots, \mathrm{DF}_W(g_\nu^{\mathrm{deh}})\}$ is an $\mathcal{O}$-border basis of $\mathrm{DF}_W(I^{\mathrm{deh}})$. Hence Corollary 2.2.4 implies $\mathrm{DF}_W(I^{\mathrm{deh}}) = \langle \mathrm{DF}_W(g_1^{\mathrm{deh}}), \ldots, \mathrm{DF}_W(g_\nu^{\mathrm{deh}}) \rangle$, i.e. $\{g_1^{\mathrm{deh}}, \ldots, g_\nu^{\mathrm{deh}}\}$ is a Macaulay basis of I with respect to the grading given by W according to [KR05, Defn. 4.2.13]. □

Proposition 4.1.7 yields that given a projective $\mathcal{O}$-border basis $G \subseteq \overline{P}$ of a homogeneous ideal $I \subset \overline{P}$, its dehomogenization $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ is an $\mathcal{O}$-border basis of $I^{\mathrm{deh}} \subseteq P$ with the additional property that the support of the degree form $\mathrm{DF}_W(g^{\mathrm{deh}})$ with respect to W of every polynomial $g \in G$ contains the corresponding border term of g^{deh}. Next we start the other way round and consider the second part of the (Projective) Border Bases Correspondence 4.1.9. Given an $\mathcal{O}$-border basis of an ideal $I \subset P$, we want to decide whether its homogenization is a projective $\mathcal{O}$-border basis of $I^{\mathrm{hom}} \subset \overline{P}$. Obviously, the above result yields the necessary condition that every border term must be contained in the support of the degree form with respect to W of the corresponding border basis element. The next proposition shows that that every border basis with this additional property indeed induces a projective border basis.

Proposition 4.1.8. *Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\mu \in \mathbb{N}$ be a finite order ideal in $\mathbb{T}^n$ and let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border. Let $G = \{g_1, \ldots, g_\nu\} \subseteq P$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ be an $\mathcal{O}$-border basis of an ideal $I \subset P$. Moreover, assume that $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$ for all $j \in \{1, \ldots, \nu\}$. Then $\{g_1^{\mathrm{hom}}, \ldots, g_\nu^{\mathrm{hom}}\} \subseteq \overline{P}$ is the projective $\mathcal{O}$-border basis of I^{hom}.*

Proof. For every index $j \in \{1, \ldots, \nu\}$, the assumption $b_j \in \mathrm{Supp}(\mathrm{DF}_W(g_j))$ implies that $g_j^{\mathrm{hom}} = b_j - \sum_{i=1}^{\mu} a_{ij} x_0^{\deg_W(b_j) - \deg_W(t_i)} t_i$. Thus $\{g_1^{\mathrm{hom}}, \ldots, g_\nu^{\mathrm{hom}}\}$ is a projective $\mathcal{O}$-border prebasis according to Definition 4.1.2. Since $G = \{g_1, \ldots, g_\nu\} \subseteq I$ by Definition 2.1.14, we get $\{g_1^{\mathrm{hom}}, \ldots, g_\nu^{\mathrm{hom}}\} \subseteq I^{\mathrm{hom}}$. Let $c_1, \ldots, c_\mu \in K[x_0]$ be such that $c_1 \bar{t}_1 + \cdots + c_\mu \bar{t}_\mu = 0$ in $\overline{P}/I^{\mathrm{hom}}$. By Definition 4.1.2, the claim follows if we prove that $c_1 = \cdots = c_\mu = 0$. For all $i \in \{1, \ldots, \mu\}$ such that $c_i \neq 0$, we can write $c_i = d_i (x_0 - 1)^{k_i}$ with a unique polynomial $d_i \in K[x_0]$ that satisfies $d_i(1) \neq 0$ and a unique $k_i \in \mathbb{N}$. Assume that $c_r \neq 0$ for some $r \in \{1, \ldots, \mu\}$. Then there is an $s \in \{1, \ldots, \mu\}$ with $k_s = \min\{k_i \mid i \in \{1, \ldots, \mu\}, c_i \neq 0\}$. and we get that $(x_0 - 1)^{k_s} (d_1 (x_0 - 1)^{k_1 - k_s} \bar{t}_1 + \cdots + d_\mu (x_0 - 1)^{k_\mu - k_s} \bar{t}_\mu) = 0$ in $\overline{P}/I^{\mathrm{hom}}$. According to [KR05, Prop. 4.3.5], $x_0 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I^{\mathrm{hom}}$. Just as we did in the proof of Proposition 4.1.7.c, we can deduce that $x_0 - 1 \in \overline{P}$ is also a non-zero divisor for $\overline{P}/I^{\mathrm{hom}}$. Thus we get $(d_1 (x_0 - 1)^{k_1 - k_s} \bar{t}_1 + \cdots + d_\mu (x_0 - 1)^{k_\mu - k_s} \bar{t}_\mu) = 0$ in $\overline{P}/I^{\mathrm{hom}}$. Dehomogenization now yields $d_1(1) \cdot 0^{k_1 - k_s} \cdot \bar{t}_1 + \cdots + d_\mu \cdot 0^{k_\mu - k_s} \cdot \bar{t}_\mu = 0$ in P/I. As G is the $\mathcal{O}$-border basis of I, we get the contradiction $0 = d_s(1) \cdot 0^{k_s - k_s} = d_s(1) \neq 0$ by Definition 2.1.14. Thus $c_1 = \cdots = c_\mu = 0$ and the claim follows. □

As a consequence of the previous two propositions, we are now able to prove the (Projective) Border Basis Correspondence and deduce consequences of it. In particular, we can use this correspondence to characterize projective border bases in Corollary 4.1.10 and to compute them in Corollary 4.1.14.

Corollary 4.1.9. (The (Projective) Border Basis Correspondence)
Let $\mathcal{O} \subseteq \mathbb{T}^n$ be a non-empty order ideal in $\mathbb{T}^n$. Then there is a one-to-one correspondence between the homogeneous ideals in $\overline{P}$ possessing a projective $\mathcal{O}$-border basis and the ideals in P possessing an $\mathcal{O}$-border basis with the property that if $b \in \partial\mathcal{O}$ and g_b denotes the $\mathcal{O}$-border basis element corresponding to b, then we have $b \in \operatorname{Supp}(\mathrm{DF}_W(g_b))$. In particular, this correspondence is given by dehomogenization and homogenization and can be visualized as follows.

$$\begin{tikzcd}
\begin{array}{c}\textit{projective } \mathcal{O}\textit{-border bases of homogeneous ideals in } \overline{P} \\ \textit{where } \overline{P} \textit{ is graded by } \overline{W} = (1 \mid W) \textit{ with a positive matrix } W \in \operatorname{Mat}_{1,n}(\mathbb{Z})\end{array} \arrow[d, shift right=8, "\textit{dehomogenization}"'] \\
\begin{array}{c}\mathcal{O}\textit{-border bases of ideals in } P \textit{ with } b \in \mathrm{DF}_W(g_b) \textit{ for all } b \in \partial\mathcal{O} \\ \textit{where } g_b \textit{ denotes the } \mathcal{O}\textit{-border basis element corresponding to } b\end{array} \arrow[u, shift right=8, "\textit{homogenization}"']
\end{tikzcd}$$

Proof. This is a direct consequence of the Propositions 4.1.7 and 4.1.8. □

Corollary 4.1.10. (Characterization of Projective Border Bases)
Let $\mathcal{O}$ be a non-empty order ideal in $\mathbb{T}^n$ and let $G \subseteq \overline{P}$ be a projective $\mathcal{O}$-border prebasis. Then G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$ if and only if the following conditions are satisfied.

1) *Let $b \in \partial\mathcal{O}$ and let $g_b \in G$ denote the projective $\mathcal{O}$-border basis element corresponding to b. Then we have $b \in \operatorname{Supp}(\mathrm{DF}_W(g_b^{\mathrm{deh}}))$.*

2) *The $\mathcal{O}$-border prebasis $\{g^{\mathrm{deh}} \mid g \in G\}$ is the $\mathcal{O}$-border basis of $\langle g^{\mathrm{deh}} \mid g \in G\rangle \subseteq P$.*

In particular, we can algorithmically check whether G is the projective $\mathcal{O}$-border basis of the ideal $\langle G \rangle$.

Proof. The equivalence is a direct consequence of Corollary 4.1.9. The second claim follows since we can use the characterizations described in Section 2.4 in order to check 2). □

Definition 4.1.11. An order ideal $\mathcal{O}$ of $\mathbb{T}^n$ is said to have a maxdeg_W**-border** if $\deg_W(t) \leq \deg_W(b)$ for all $t \in \mathcal{O}$ and all $b \in \partial\mathcal{O}$.

Corollary 4.1.12. *Let $\mathcal{O} \subseteq \mathbb{T}^n$ be a non-empty order ideal in $\mathbb{T}^n$ which possesses a* maxdeg_W*-border. Then there is a one-to-one correspondence between the homogeneous ideals in $\overline{P}$ possessing a projective $\mathcal{O}$-border basis and the ideals in P possessing an*

$\mathcal{O}$-border basis. In particular, this correspondence is given by dehomogenization and homogenization.

Proof. Let $G \subseteq P$ be an $\mathcal{O}$-border basis of an ideal in P, let $b \in \partial\mathcal{O}$ and let $g_b \in G$ be the border basis element corresponding to b. As $\mathcal{O}$ has a maxdeg_W-border, we have $\deg_W(b) \geq \deg_W(t)$ for all $t \in \mathcal{O}$ by Definition 4.1.11. Thus we get $b \in \mathrm{Supp}(\mathrm{DF}_W(g_b^{\mathrm{deh}}))$ by Definition 2.1.14 and the claim follows from Corollary 4.1.9. $\square$

The following example shows that there are zero-dimensional ideals in P which do not possess an $\mathcal{O}$-border basis with an order ideal $\mathcal{O}$ in $\mathbb{T}^n$ that has a maxdeg_W-border.

Example 4.1.13. Let $P = \mathbb{Q}[x, y]$, let t be a homogenizing indeterminate, and let $\overline{P} = \mathbb{Q}[t, x, y]$, Moreover, we let $w_2 \in \mathbb{N}$ and we equip P respectively $\overline{P}$ with the grading given by $(1, w_2) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ respectively $(1, 1, w_2) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$. Using the method of Corollary 4.1.10, one can easily check that $G = \{y, xy, x^2y, x^3 - 3tx^2 + t^2x\} \subseteq \overline{P}$ is the projective $\{1, x, x^2\}$-border basis of the ideal $I = \langle y, x^3 - 3tx^2 + t^2x\rangle \subseteq \overline{P}$. Moreover, we have $y \in I^{\mathrm{deh}}$ and $x^3 - 3x^2 + x \in I^{\mathrm{deh}}$. According to Corollary 4.1.9 and Definition 2.1.6, there is no projective $\mathcal{O}$-border basis of I for an order ideal $\mathcal{O} \neq \{1, x, x^2\}$ in $\mathbb{T}^2$. Therefore, we see that the ideal I has a projective $\mathcal{O}$-border basis where $\mathcal{O}$ is an order ideal in $\mathbb{T}^2$ with a $\mathrm{maxdeg}_{(1,w_2)}$-border if and only if $w_2 \geq 2$ according to Definition 4.1.11. In particular, this condition is not satisfied in the standard graded case, i.e. for $w_2 = 1$, whereas $\{1, x, x^2\}$ has $\mathrm{maxdeg}_{(1,2)}$-border.

Corollary 4.1.14. (Computation of Projective Border Bases)
Let $I \subseteq \overline{P}$ be a homogeneous ideal such that $\dim(\overline{P}/I) = 1$ and such that the element $x_0 \in \overline{P}$ is a non-zero divisor for $\overline{P}/I$. We use the Border Basis Algorithm 2.5.3 to compute an $\mathcal{O}$-border basis $G \subseteq P$ of $I^{\mathrm{deh}} \subseteq P$. Then the set $\{g^{\mathrm{hom}} \mid g \in G\} \subseteq \overline{P}$ is the projective $\mathcal{O}$-border basis of I.

Proof. By Lemma 4.1.6 and [KR05, Prop. 4.3.5], we have $I = I :_{\overline{P}} \langle x_0\rangle^\infty = (I^{\mathrm{deh}})^{\mathrm{hom}}$. Moreover, as $n \in \mathbb{N} \setminus \{0\}$ and $\dim(\overline{P}/I) = 1$, we have $I \subset \overline{P}$. Thus [KR05, Prop. 5.6.12] implies that $\dim(P/I^{\mathrm{deh}}) = \dim(\overline{P}/I) - 1 = 0$, i.e. $I^{\mathrm{deh}} \subset P$ is a zero-dimensional ideal. Therefore, we can use the Border Basis Algorithm 2.5.3 in order to compute an $\mathcal{O}$-border basis $G \subseteq P$ of the ideal I^{deh}. Moreover, for all $b \in \partial\mathcal{O}$, the corresponding border basis element g_b satisfies $b \in \mathrm{Supp}(\mathrm{DF}_W(g_b))$ by Theorem 2.5.3. Thus $\{g^{\mathrm{hom}} \mid g \in G\} \subseteq \overline{P}$ is a projective $\mathcal{O}$-border basis of $(I^{\mathrm{deh}})^{\mathrm{hom}} = I$ by Corollary 4.1.9. $\square$

Finally, we explicitly describe the homogeneous components of the residue class ring of $\overline{P}$ modulo a projective $\mathcal{O}$-border basis where $\mathcal{O}$ is non-empty order ideal in $\mathbb{T}^n$ by means of $\mathcal{O}$. As a consequence, we prove that such a residue class ring is a one-dimensional Cohen-Macaulay ring and that we can easily read off its (multigraded) Hilbert function.

Proposition 4.1.15. *Let $G \subseteq \overline{P}$ be the projective $\mathcal{O}$-border basis of a homogeneous ideal $I \subset \overline{P}$ and $R = \overline{P}/I$. Let $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1, \ldots, \mu\} \mid \deg_W(t_k) \leq \gamma\}$ and let $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma - 1)$ denote its first difference function. Moreover, we write the order ideal $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\mu \in \mathbb{N}$ and terms $t_1, \ldots, t_\mu \in \mathbb{T}^n$ that* *satisfy* $\deg_W(t_1) \le \cdots \le \deg_W(t_\mu)$ *and for all* $\gamma \in \mathbb{Z}$ *and* $i \in \{1, \ldots, \mu\}$*, we denote* $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$.

a) *There are canonical isomorphisms of* $\mathbb{Z}$*-graded* $K[x_0]$*-algebras*

$$R \cong \bigoplus_{i=1}^{\mu} K[x_0](-\deg_W(t_i)) \cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (K[x_0](-\gamma))^{\Delta H(\gamma)}.$$

In particular, we have $H = \mathrm{HF}_R$.

b) *Let* $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$*. Then the residue classes of the elements of* $\mathcal{O}_0$ *in* R *form a* K*-vector space basis of* R*. In particular, for every* $\gamma \in \mathbb{Z}$*, the residue classes of the elements of* $(\mathcal{O}_0)_\gamma = \{x_0^{d_{\gamma,1}} t_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}\}$ *in* R *form a* K*-vector space basis of* R_γ.

Proof. The first claim is a direct consequence of $\mathcal{O}$ to be a $K[x_0]$-module basis of R according to Definition 4.1.2. Thus it suffices to prove the second claim. Let $\gamma \in \mathbb{Z}$. For all $i \in \{H(\gamma)+1, \ldots, \mu\}$, we have $\deg_W(t_i) > \gamma$ as the elements $t_1, \ldots, t_\mu$ are ordered increasingly with respect to $\deg_W$. Hence a) yields that the residue classes of the elements of $(\mathcal{O}_0)_\gamma = \left(\bigcup_{k=0}^{\infty} x_0^k \mathcal{O}\right) \cap \overline{P}_\gamma = \{x_0^{d_{\gamma,1}} t_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}\}$ in R form a K-vector space basis of R_γ. □

Corollary 4.1.16. *Let* $G \subseteq \overline{P}$ *be the projective* $\mathcal{O}$*-border basis of a homogeneous ideal* $I \subset \overline{P}$ *and let* $R = \overline{P}/I$.

a) *The ring* R *is a one-dimensional Cohen-Macaulay ring.*

b) *Let* $J \subseteq R$ *be a homogeneous ideal and* $S = R/J$ *be such that* $x_0 \in \overline{P}$ *is a non-zero divisor for* S*. Then* S *is a one-dimensional Cohen-Macaulay ring.*

Proof. First we prove a). By Proposition 4.1.7, $x_0 \in \overline{P}$ is a non-zero divisor for R. Thus the R-module homomorphism $\pi_{x_0} : R \to R$, $r \mapsto x_0 \cdot r$, the multiplication by $x_0 \in \overline{P}$, is injective. Let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then the induced $R_\mathfrak{m}$-module homomorphism $(\pi_{x_0})_\mathfrak{m} : R_\mathfrak{m} \to R_\mathfrak{m}$, $\frac{r}{s} \mapsto \frac{x_0 \cdot r}{s} = \frac{\overline{x}_0}{1} \cdot \frac{r}{s}$, the multiplication by $\frac{\overline{x}_0}{1} \in R_\mathfrak{m}$, is also injective according to [Kun85, Rule III.4.7]. Therefore, $\frac{\overline{x}_0}{1} \in R_\mathfrak{m}$ is also a non-zero divisor for $R_\mathfrak{m}$. In particular, the depth of $R_\mathfrak{m}$ satisfies $1 \le \operatorname{depth}(R_\mathfrak{m}) \le \dim(R_\mathfrak{m})$ by [Kun85, Defn. VI.3.3 and Prop. VI.3.9]. Moreover, by Definition 4.1.2, the residue classes of the elements of $\mathcal{O}$ in R form a $K[x_0]$-module basis of R. This yields a Noetherian normalization $K[x_0] \hookrightarrow R$ by [Kun85, Defn. II.3.3] and thus $\dim(R) = 1$ by [Kun85, Prop. II.3.4]. Now [Kun85, Prop. III.4.12] yields $\dim(R_\mathfrak{m}) \le \dim(R)$. Altogether, we get $1 \le \operatorname{depth}(R_\mathfrak{m}) \le \dim(R_\mathfrak{m}) \le 1$. Therefore, $\operatorname{depth}(R_\mathfrak{m}) = 1 = \dim(R_\mathfrak{m})$, i.e. R is a one-dimensional Cohen-Macaulay ring according to [Kun85, Defn. VI.3.10].

Finally, we prove claim b). Let $\varepsilon : \overline{P} \twoheadrightarrow R$ denote the canonical $K[x_0]$-algebra epimorphism. Then the Second Noetherian Isomorphism Theorem yields the existence of the canonical $K[x_0]$-algebra isomorphism $S = R/J = \varepsilon(\overline{P})/J \cong \overline{P}/\varepsilon^{-1}(J)$. Since $x_0 \in \overline{P}$ is a non-zero divisor for $S \cong \overline{P}/\varepsilon^{-1}(J)$ and as $\dim(S) \le \dim(R) = 1$, we can show

$\dim(S) = 1$ just as we did it for R in the proof of a). According to Corollary 4.1.14, there exists a projective $\mathcal{O}'$-border basis of the ideal $\varepsilon^{-1}(J) \subseteq \overline{P}$. Therefore, claim b) follows from the canonical K-algebra isomorphism $S \cong \overline{P}/\varepsilon^{-1}(J)$ and a). $\square$

4.2 The Multiplicative Structure of a Residue Class Ring Modulo a Projective Border Prebasis

Based on the results of the previous section, we now take a closer look at the multiplicative structure of the residue class ring $R = \overline{P}/I$ modulo a homogeneous ideal $I \subset \overline{P}$ that is given by a projective $\mathcal{O}$-border prebasis. In particular, we show that the additional knowledge that I possesses a projective $\mathcal{O}$-border prebasis allows us to describe the multiplication in R only by means of $\mathcal{O}$ and formal multiplication matrices which solely depend on the projective $\mathcal{O}$-border prebasis.

For the whole section, we let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\mu \in \mathbb{N} \setminus \{0\}$ be a finite order ideal in $\mathbb{T}^n$ such that $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$ and we let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border. Moreover, let $G = \{g_1, \ldots, g_\nu\} \subseteq \overline{P}$ be a projective $\mathcal{O}$-border prebasis and let $R = \overline{P}/\langle G\rangle$. According to Proposition 4.1.4, the residue classes of the elements of $\mathcal{O}$ in the ring R form a $K[x_0]$-module generating set of R and thus the residue classes of the elements of $\bigcup_{k=0}^{\infty} x_0^k\mathcal{O}$ in R form a K-vector space generating set of R. The goal of this section is to describe the multiplicative structure of the ring R with respect to the above generating sets by means of matrices as explicitly as possible. To shorten the notation, we define the integer function $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1, \ldots, \mu\} \mid \deg_W(t_k) \leq \gamma\}$ and its first difference function $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma - 1)$.

First we introduce the formal projective multiplication matrices. Similarly to the situation of formal multiplication matrices of border prebases in P, cf. Definition 2.4.15 and Remark 2.4.17, these matrices describe the multiplication in the $K[x_0]$-module R by an indeterminate.

Definition 4.2.1. For all indices $j \in \{1, \ldots, \nu\}$, we let the elements of the projective $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\} \subseteq \overline{P}$ be of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij}t_i$ where we have $a_{1j}, \ldots, a_{\mu j} \in K[x_0]$. Let $\mathcal{X}_0^{\mathrm{proj}} = x_0\mathcal{I}_\mu \in \mathrm{Mat}_\mu(K[x_0])$ and for all $r \in \{1, \ldots, n\}$, let $\mathcal{X}_r^{\mathrm{proj}} = (\xi_{k\ell}^{(r,\mathrm{proj})})_{1\leq k,\ell\leq\mu} \in \mathrm{Mat}_\mu(K[x_0])$ be defined by

$$\xi_{k\ell}^{(r,\mathrm{proj})} = \begin{cases} \delta_{ki} & \text{if } x_r t_\ell = t_i \in \mathcal{O}, \\ a_{kj} & \text{if } x_r t_\ell = b_j \in \partial\mathcal{O}. \end{cases}$$

Then for every $r \in \{0, \ldots, n\}$, we call $\mathcal{X}_r^{\mathrm{proj}}$ the $r^{\mathbf{th}}$ **formal projective multiplication matrix** of G.

Example 4.2.2. Let $P = \mathbb{Q}[x, y]$ be standard graded, i.e. graded by the grading given by the matrix $W = (1, 1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$. Moreover, we let t be a homogenizing indeterminate let $\overline{P} = \mathbb{Q}[t, x, y]$ also be standard graded, i.e. graded by the grading given by the matrix $\overline{W} = (1, 1, 1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$, and we let $G = \{g_1, \dots, g_6\} \subseteq \overline{P}$ be defined by the polynomials $g_1 = y^3 + 2t^2y - 3ty^2$, $g_2 = x^2y - xy^2$, $g_3 = xy^3 + 2t^2xy - 3txy^2$, $g_4 = x^2y^2 + 2t^2xy - 3txy^2$, $g_5 = x^3y + 2t^2xy - 3txy^2$, and $g_6 = x^4 - 6t^3x + 11t^2x^2 - 6tx^3$. Let $\mathcal{O} = \{t_1, \dots, t_8\} = \{1, y, x, y^2, xy, x^2, xy^2, x^3\}$. Applying Corollary 4.1.10, we see that the set G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$. Let $R = \overline{P}/\langle G \rangle$. For all $\tilde{t} \in \mathcal{O}$ and for all $b \in \partial\mathcal{O}$, we have $\deg_W(\tilde{t}) = \deg(\tilde{t}) \leq 3 \leq \deg(b) = \deg_W(b)$, i.e. the order ideal $\mathcal{O}$ has a maxdeg_W-border by Definition 4.1.11. According to Definition 4.2.1, the projective multiplication matrices $\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}} \in \mathrm{Mat}_8(\mathbb{Q}[t])$ of G are

$$\mathcal{T}^{\mathrm{proj}} = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix}, \quad \mathcal{X}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2t^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -11t^2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6t \end{pmatrix},$$

and

$$\mathcal{Y}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2t^2 & -2t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3t & 3t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following definition specializes [KR05, Defn. 4.7.1 and 4.7.5] to our setting.

Definition 4.2.3. Let $\mathcal{M} = (f_{ij})_{1 \leq i \leq k, 1 \leq j \leq \ell} \in \mathrm{Mat}_{k,\ell}(P)$ with $k, \ell \in \mathbb{N}$ be a matrix, let $r = (r_1, \dots, r_k) \in \mathbb{Z}^k$, and let $c = (c_1, \dots, c_\ell) \in \mathbb{Z}^\ell$.

a) The matrix $\mathcal{M}$ is called **homogeneous** with respect to the **degree pair** given by r for the rows and c for the columns if for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \ell\}$, the polynomial f_{ij} is homogeneous of degree $c_j - r_i$.

b) The matrix $\mathcal{M}$ is called **degree-ordered** with respect to degree pair given by r for the rows and c for the columns if $\mathcal{M}$ is homogeneous with respect to the degree pair given by r for the rows and c for the columns and if $r_1 \leq r_2 \leq \cdots \leq r_k$ and $c_1 \leq c_2 \leq \cdots \leq c_\ell$.

The formal multiplication matrices of a border prebasis in P are homogeneous matrices in the sense of Definition 4.2.3, cf. [KR08, Rem. 5.2] for a more general version. The following proposition shows that the formal projective multiplication matrices of a projective border basis are also homogeneous matrices in the sense of Definition 4.2.3.

Proposition 4.2.4. *Let the elements of the projective $\mathcal{O}$-border prebasis $G \subseteq \overline{P}$ be of the form $g_j = b_j - \sum_{i=1}^{H(\deg_W(b_j))} a_{ij} x_0^{\Delta_{ij}} t_i$ where $a_{ij} \in K$ and $\Delta_{ij} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{N}$ for all $j \in \{1, \ldots, \nu\}$ and $i \in \{1, \ldots, H(\deg_W(b_j))\}$. Let $\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0])$ be the formal projective multiplication matrices of G.*

a) For every index $r \in \{0, \ldots, n\}$, the matrix $\mathcal{X}_r^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r t_1), \ldots, \deg_{\overline{W}}(x_r t_\mu))$ for the columns.

b) For all $r, s \in \{0, \ldots, n\}$, the matrices $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}}$ and $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}} - \mathcal{X}_s^{\mathrm{proj}} \mathcal{X}_r^{\mathrm{proj}}$ are both homogeneous degree-ordered matrices with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r x_s t_1), \ldots, \deg_{\overline{W}}(x_r x_s t_\mu))$ for the columns.

In particular, the above assumptions hold if G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$.

Proof. For all $r \in \{0, \ldots, n\}$, we write $\mathcal{X}_r^{\mathrm{proj}} = (\xi_{k\ell}^{(r,\mathrm{proj})})_{1 \le k,\ell \le \mu} \in \mathrm{Mat}_\mu(K[x_0])$. Note that all entries of the matrices $\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}}$ are homogeneous.

For the proof of a), we let $r \in \{0, \ldots, n\}$. If $\mathcal{X}_r^{\mathrm{proj}} = 0$, nothing needs to be shown. Thus assume that there are $k, \ell \in \{1, \ldots, \mu\}$ such that $\xi_{k\ell}^{(r,\mathrm{proj})} \neq 0$. If $r = 0$, $\xi_{k\ell}^{r,\mathrm{proj}} \neq 0$ implies $k = \ell$ and thus

$$\deg_{\overline{W}}(\xi_{k\ell}^{(0,\mathrm{proj})}) = \deg_{\overline{W}}(x_0) = \deg_{\overline{W}}(x_0 t_\ell) - \deg_{\overline{W}}(t_k)$$

by Definition 4.2.1. Now suppose that $r \neq 0$. If $x_r t_\ell = t_i$ for some $i \in \{1, \ldots, \mu\}$, $\delta_{ki} = \xi_{k\ell}^{(r,\mathrm{proj})} \neq 0$ implies $k = i$ by Definition 4.2.1. Thus we get

$$\deg_{\overline{W}}(\xi_{k\ell}^{(r,\mathrm{proj})}) = \deg_{\overline{W}}(\delta_{ki}) = \deg_{\overline{W}}(1) = 0 = \deg_{\overline{W}}(x_r t_\ell) - \deg_{\overline{W}}(t_k)$$

according to Definition 4.2.1. If $x_r t_\ell = b_j$ for some $j \in \{1, \ldots, \nu\}$, we get

$$\deg_{\overline{W}}(\xi_{k\ell}^{(r,\mathrm{proj})}) = \deg_{\overline{W}}(a_{kj} x_0^{\Delta_{kj}}) = \deg_{\overline{W}}(b_j) - \deg_{\overline{W}}(t_k) = \deg_{\overline{W}}(x_r t_\ell) - \deg_{\overline{W}}(t_k)$$

by Definition 4.2.1. In all cases, $\mathcal{X}_r^{\mathrm{proj}}$ is homogeneous with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r t_1), \ldots, \deg_{\overline{W}}(x_r t_\mu))$ for the columns and as $\deg_{\overline{W}}(t_1) \le \cdots \le \deg_{\overline{W}}(t_\mu)$, it is also degree-ordered by Definition 4.2.3.

In order to prove b), we let now $r, s \in \{0, \ldots, n\}$. By a), $\mathcal{X}_r^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r t_1), \ldots, \deg_{\overline{W}}(x_r t_\mu))$ for the columns and $\mathcal{X}_s^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_s t_1), \ldots, \deg_{\overline{W}}(x_s t_\mu))$ for the columns. In particular, $\mathcal{X}_s^{\mathrm{proj}}$ is also a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{W}}(x_r t_1), \ldots, \deg_{\overline{W}}(x_r t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r x_s t_1), \ldots, \deg_{\overline{W}}(x_r x_s t_\mu))$ for the columns according to Definition 4.2.3. If $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}} = 0$, $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}}$ trivially satisfies the claimed properties. Thus assume that there are $k, \ell \in \{1, \ldots, \mu\}$ with

$\sum_{m=1}^{\mu} \xi_{km}^{(r,\mathrm{proj})} \xi_{m\ell}^{(s,\mathrm{proj})} \neq 0$. Let $m \in \{1, \ldots, \mu\}$ be such that $\xi_{km}^{(r,\mathrm{proj})} \neq 0$ and $\xi_{m\ell}^{(s,\mathrm{proj})} \neq 0$. Since the matrices $\mathcal{X}_r^{\mathrm{proj}}$ and $\mathcal{X}_s^{\mathrm{proj}}$ are homogeneous degree-ordered matrices with respect to the above degree pairs, we get

$$\begin{aligned}\deg_{\overline{W}}(\xi_{km}^{(r,\mathrm{proj})} \xi_{m\ell}^{(s,\mathrm{proj})}) &= (\deg_{\overline{W}}(x_r t_m) - \deg_{\overline{W}}(t_k)) + (\deg_{\overline{W}}(x_r x_s t_\ell) - \deg_{\overline{W}}(x_r t_m)) \\ &= \deg_{\overline{W}}(x_r x_s t_\ell) - \deg_{\overline{W}}(t_k)\end{aligned}$$

and $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{W}}(t_1), \ldots, \deg_{\overline{W}}(t_\mu))$ for the rows and $(\deg_{\overline{W}}(x_r x_s t_1), \ldots, \deg_{\overline{W}}(x_r x_s t_\mu))$ for the columns according to Definition 4.2.3. By interchanging the role of r and s, we see that $\mathcal{X}_s^{\mathrm{proj}} \mathcal{X}_r^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the same degree pair, too. In particular, $\mathcal{X}_r^{\mathrm{proj}} \mathcal{X}_s^{\mathrm{proj}} - \mathcal{X}_s^{\mathrm{proj}} \mathcal{X}_r^{\mathrm{proj}}$ is also a homogeneous degree-ordered matrix according to Definition 4.2.3.

The remaining claim is a direct consequence of Proposition 4.1.7.f. □

The formal projective multiplication matrices allow us to explicitly describe the multiplication by homogeneous elements in R by means of $\mathcal{O}$.

Proposition 4.2.5. *For all indices $j \in \{1, \ldots, \nu\}$, we let the elements of the projective $\mathcal{O}$-border prebasis $G = \{g_1, \ldots, g_\nu\} \subseteq \overline{P}$ be of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ where we have $a_{1j}, \ldots, a_{\mu j} \in K[x_0]$. Let $\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0])$ denote the formal projective multiplication matrices of G, let $c_1, \ldots, c_\mu \in K[x_0]$, and let $p = c_1 t_1 + \cdots + c_\mu t_\mu \in \overline{P}$.*

a) For every $k \in \{0, \ldots, n\}$, we have

$$\overline{x}_k \cdot \overline{p} = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \cdot (c_1, \ldots, c_\mu)^{\mathrm{tr}}.$$

in R. In particular, we have

$$(\overline{t}_1, \ldots, \overline{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \mathcal{X}_\ell^{\mathrm{proj}} \cdot (c_1, \ldots, c_\mu)^{\mathrm{tr}} = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot \mathcal{X}_\ell^{\mathrm{proj}} \mathcal{X}_k^{\mathrm{proj}} \cdot (c_1, \ldots, c_\mu)^{\mathrm{tr}}$$

in R for all $k, \ell \in \{0, \ldots, n\}$.

b) For every $p' \in \overline{P}$, we have

$$\overline{p}' \cdot \overline{p} = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot p'(\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1, \ldots, c_\mu)^{\mathrm{tr}}$$

in R.

Proof. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ denote the canonical $K[x_0]$-module basis of $(K[x_0])^\mu$.

In order to prove the equality of a), let $\ell \in \{1, \ldots, \mu\}$. Then we have

$$x_0 \cdot c_\ell \overline{t}_\ell = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot x_0 \mathcal{I}_\mu \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot \mathcal{X}_0^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}$$

in R. Let $k \in \{1, \ldots, n\}$. If $x_k t_\ell = t_i \in \mathcal{O}$ for some $i \in \{1, \ldots, \mu\}$, we have

$$\overline{x}_k \cdot c_\ell \overline{t}_\ell = c_\ell \overline{t}_i = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot (\delta_{1i} c_\ell, \ldots, \delta_{\mu i} c_\ell)^{\mathrm{tr}} = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}$$

in R. If $x_k t_\ell = b_j$ for some $j \in \{1, \dots, \nu\}$, we have

$$\begin{aligned}
\overline{x}_k \cdot c_\ell \bar{t}_\ell &= c_\ell \overline{b}_j \\
&= a_{1j} x_0^{\Delta_{1j}} c_\ell \bar{t}_1 + \cdots + a_{\mu j} x_0^{\Delta_{\mu j}} c_\ell \bar{t}_\mu \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot (a_{1j} x_0^{\Delta_{1j}} c_\ell, \dots, a_{\mu j} x_0^{\Delta_{\mu j}} c_\ell)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}
\end{aligned}$$

in R. Altogether, we see that for all $k \in \{0, \dots, n\}$,

$$\overline{x}_k \cdot \overline{p} = \sum_{\ell=1}^{\mu} \overline{x}_k \cdot c_\ell \bar{t}_\ell = \sum_{\ell=1}^{\mu} (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}$$

in R. In particular, we have proved that for all $k \in \{0, \dots, n\}$, $\mathcal{X}_k^{\mathrm{proj}}$ describes the multiplication by $\overline{x}_k$ in R. Since the multiplication in R is commutative, it follows that

$$\begin{aligned}
(\bar{t}_1, \dots, \bar{t}_\mu) \cdot \mathcal{X}_k^{\mathrm{proj}} \mathcal{X}_\ell^{\mathrm{proj}} \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} &= \overline{x}_k \overline{x}_\ell \overline{p} \\
&= \overline{x}_\ell \overline{x}_k \overline{p} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \mathcal{X}_\ell^{\mathrm{proj}} \mathcal{X}_k^{\mathrm{proj}} \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R for all $k, \ell \in \{0, \dots, n\}$.

Next we prove b). We start to prove the claim for homogeneous polynomials. Let $p' \in \overline{P}_\gamma$ with $\gamma \in \mathbb{Z}$. If $\gamma \leq 0$, we have $p' \in K$ and thus

$$\begin{aligned}
\overline{p}' \cdot \overline{p} &= p' \cdot (c_1 \bar{t}_1 + \cdots + c_\mu \bar{t}_\mu) \\
&= p' c_1 \bar{t}_1 + \cdots + p' c_\mu \bar{t}_\mu \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot p' \mathcal{I}_\mu \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot p'(\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R. For the induction start $\gamma = 1$, there are $d_0, \dots, d_n \in K$ with $p = d_0 x_0 + \cdots d_n x_n$. By a), we get

$$\begin{aligned}
\overline{p}' \cdot \overline{p} &= \sum_{m=1}^{n} d_m \overline{x}_m \cdot \overline{p} \\
&= \sum_{m=1}^{n} (\bar{t}_1, \dots, \bar{t}_\mu) \cdot (d_m x_m)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \left(\sum_{m=1}^{n} d_m x_m \right) (\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot p(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R and the induction start is proven. For the induction step, suppose now that $\gamma > 1$. Then we can write $p' = c_1' t_1' + \cdots + c_s' t_s'$ with $s \in \mathbb{N}$, $c_1', \dots, c_s' \in K$, and $t_1', \dots, t_s' \in \mathbb{T}_\gamma^{n+1}$.

For every $r \in \{1,\dots,s\}$, there exist a $k_r \in \{0,\dots,n\}$ and a term $u_r \in \mathbb{T}^{n+1}_{<\gamma}$ such that $t'_r = x_{k_r} u_r$. Then the induction hypothesis and a) yield

$$\begin{aligned}
c'_r \overline{t}'_r \cdot \overline{p} &= \overline{x}_{k_r} \cdot (c'_r \overline{u}_r \cdot \overline{p}) \\
&= \overline{x}_{k_r} \cdot ((\overline{t}_1,\dots,\overline{t}_\mu) \cdot (c'_r u_r)(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}) \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot \mathcal{X}_{k_r}^{\mathrm{proj}} \cdot (c'_r u_r)(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot (x_{k_r} \cdot c'_r u_r)(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot (c'_r t'_r)(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R for every $r \in \{1,\dots,s\}$. Altogether, we get

$$\begin{aligned}
\overline{p}' \cdot \overline{p} &= \sum_{r=1}^{s} c'_r \overline{t}'_r \cdot \overline{p} \\
&= \sum_{r=1}^{s} (\overline{t}_1,\dots,\overline{t}_\mu) \cdot (c'_r t'_r)(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot \left(\sum_{r=1}^{s} c'_r t'_r p \right) (\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot p'(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R and the claim follows in the homogeneous case.
In order to prove the claim for arbitrary polynomials, we let $p' = \sum_{\gamma \in \mathbb{Z}} p'_\gamma$ be the decomposition of $p' \in \overline{P}$ into its homogeneous components. Then

$$\begin{aligned}
\overline{p}' \cdot \overline{p} &= \sum_{\gamma \in \mathbb{Z}} \overline{p}'_\gamma \cdot \overline{p} \\
&= \sum_{\gamma \in \mathbb{Z}} (\overline{t}_1,\dots,\overline{t}_\mu) \cdot p'_\gamma(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot \left(\sum_{\gamma \in \mathbb{Z}} p'_\gamma \right) (\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1,\dots,\overline{t}_\mu) \cdot p'(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in R from the homogeneous case. □

Corollary 4.2.6. *Assume that G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$ and let $\overline{\mathcal{O}}$ denote the image of the order ideal $\mathcal{O}$ in R. Moreover, let $\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0])$ denote the formal projective multiplication matrices of G, let $\gamma \in \mathbb{Z}$, and let $p \in \overline{P}_\gamma$. Then the matrix $p(\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}) \in \mathrm{Mat}_\mu(K[x_0])$ represents the homogeneous $K[x_0]$-algebra homomorphism $\pi_{\overline{p}} : R(-\gamma) \to R$, $r \mapsto \overline{p} \cdot r$, the multiplication by $\overline{p} \in R_\gamma$, with respect to the $K[x_0]$-module basis $\overline{\mathcal{O}}$ of R.*

In particular, the formal projective multiplication matrices $\mathcal{X}_0^{\mathrm{proj}},\dots,\mathcal{X}_n^{\mathrm{proj}}$ are pairwise commuting in this situation.

Proof. According to Definition 4.1.2, $\overline{\mathcal{O}}$ is a $K[x_0]$-module basis of R. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ denote the canonical $K[x_0]$-module basis of $(K[x_0])^\mu$. Then Proposition 4.2.5 implies that $\pi_{\overline{p}}(\overline{t}_i) = \overline{p} \cdot \overline{t}_i = (\overline{t}_1, \ldots, \overline{t}_\mu) \cdot p(\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}}) \cdot \mathcal{E}_i^{\mathrm{tr}}$ for every $i \in \{1, \ldots, \mu\}$ and the claim follows.

In particular, for all $k \in \{0, \ldots, n\}$, $\mathcal{X}_k^{\mathrm{proj}}$ represents the $K[x_0]$-algebra endomorphism of R defined by $r \mapsto \overline{x}_k \cdot r$ for all $r \in R$, i.e. the multiplication by $\overline{x}_k$ in R. Since the multiplication in the ring R is commutative, it follows that the matrices $\mathcal{X}_0^{\mathrm{proj}}, \ldots, \mathcal{X}_n^{\mathrm{proj}}$ are pairwise commuting. $\square$

Example 4.2.7. Consider Example 4.2.2 in the standard graded ring $\overline{P} = \mathbb{Q}[t, x, y]$, again. Recall that $G \subseteq \overline{P}$ was the projective $\mathcal{O}$-border basis of $\langle G \rangle$ with the order ideal $\mathcal{O} = \{1, y, x, y^2, xy, x^2, xy^2, x^3\}$, which has a maxdeg_W-border, that $R = \overline{P}/\langle G \rangle$, and that the projective multiplication matrices $\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}} \in \mathrm{Mat}_8(\mathbb{Q}[t])$ of G were

$$\mathcal{T}^{\mathrm{proj}} = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix}, \quad \mathcal{X}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2t^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -11t^2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6t \end{pmatrix},$$

and

$$\mathcal{Y}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2t^2 & -2t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3t & 3t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\overline{\mathcal{O}}$ denote the image of $\mathcal{O}$ in R, let $p = x^2 + ty \in \overline{P}_2$, and let $\pi_{\overline{p}} : R(-2) \to R$,

$r \mapsto \overline{p} \cdot r$ be the multiplication by $\overline{p} \in R_2$. Then the matrix

$$
\begin{aligned}
p(\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}}) &= (\mathcal{X}^{\mathrm{proj}})^2 + \mathcal{T}^{\mathrm{proj}} \cdot \mathcal{Y}^{\mathrm{proj}} \\
&= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t & 0 & 0 & -2t^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6t^3 & 0 & 36t^4 \\
0 & t & 0 & 3t^2 & 0 & 0 & 0 & 0 \\
0 & 0 & t & -2t^2 & -2t^2 & 0 & -8t^3 & -2t^3 \\
1 & 0 & 0 & 0 & 0 & -11t^2 & 0 & -60t^3 \\
0 & 1 & 0 & 3t & 4t & t & 10t^2 & 3t^2 \\
0 & 0 & 1 & 0 & 0 & 6t & 0 & 25t^2
\end{pmatrix} \\
&\in \mathrm{Mat}_8(\mathbb{Q}[t])
\end{aligned}
$$

represents the $\mathbb{Q}[t]$-algebra homomorphism $\pi_{\overline{p}}$ with respect to the $\mathbb{Q}[t]$-module basis $\overline{\mathcal{O}}$ according to Corollary 4.2.6.

In the remaining part of this section, we do not consider the multiplication by a homogeneous element in the whole ring R but we restrict the multiplication to one homogeneous component of R. In this setting, the formal multiplication matrices of the $\mathcal{O}$-border prebasis $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ can be used to explicitly describe the multiplication map only by means of $\mathcal{O}$.

Proposition 4.2.8. *Assume that the elements of $G = \{g_1, \dots, g_\nu\} \subseteq \overline{P}$ are of the form $g_j = b_j - \sum_{i=1}^{H(\deg_W(b_j))} a_{ij} x_0^{\Delta_{ij}} t_i$ where $a_{ij} \in K$ and $\Delta_{ij} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{N}$ for all $j \in \{1, \dots, \nu\}$ and $i \in \{1, \dots, H(\deg_W(b_j))\}$. Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and $i \in \{1, \dots, \mu\}$, let $\mathcal{X}_1, \dots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ denote the multiplication matrices of the $\mathcal{O}$-border prebasis $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\} \subseteq P$, let $c_1, \dots, c_{H(\gamma)} \in K$, and let $p = c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)} \in \overline{P}_\gamma$.*

a) We have

$$
\begin{aligned}
x_0 \cdot \overline{p} = (&x_0^{d_{\gamma+1,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+1,H(\gamma+1)}} \overline{t}_{H(\gamma+1)}, 0, \dots, 0) \\
&\cdot \mathcal{I}_\mu \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in R and for every $k \in \{1, \dots, n\}$, we have

$$
\begin{aligned}
\overline{x}_k \cdot \overline{p} = (&x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \dots, 0) \\
&\cdot \mathcal{X}_k \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in R. In particular, if we denote $w_0 = 1$ and $\mathcal{X}_0 = \mathcal{I}_\mu$, we have

$$
\begin{aligned}
&(x_0^{d_{\gamma+w_k+w_\ell,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+w_k+w_\ell,H(\gamma+w_k+w_\ell)}} \overline{t}_{H(\gamma+w_k+w_\ell)}, 0, \dots, 0) \\
&\quad \cdot \mathcal{X}_k \mathcal{X}_\ell \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+w_k+w_\ell,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+w_k+w_\ell,H(\gamma+w_k+w_\ell)}} \overline{t}_{H(\gamma+w_k+w_\ell)}, 0, \dots, 0) \\
&\quad \cdot \mathcal{X}_\ell \mathcal{X}_k \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in R for all $k, \ell \in \{0, \dots, n\}$.

b) For every $\gamma' \in \mathbb{Z}$ and every $p' \in \overline{P}_{\gamma'}$, we have

$$\begin{aligned}\overline{p}' \cdot \overline{p} = (&x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma)}, 0, \dots, 0) \\ &\cdot p'(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}\end{aligned}$$

in R.

In particular, the above assumptions hold if G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$.

Proof. For every $s \in \{1, \dots, n\}$, we denote $\mathcal{X}_s = (\xi_{k\ell}^{(s)})_{1 \le k, \ell \le \mu} \in \mathrm{Mat}_\mu(K)$ as in Definition 2.4.15 and we let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ. Moreover, we denote $w_0 = 1$ and $\mathcal{X}_0 = \mathcal{I}_\mu$. If $\gamma < 0$, we have $\overline{P}_\gamma = \{0\}$ and the claims follow trivially. Thus suppose that $\gamma \ge 0$.

The first equality of a) follows as we have

$$\begin{aligned}x_0 \cdot \overline{p} &= x_0 \cdot (c_1 x_0^{d_{\gamma,1}} \overline{t}_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}) \\ &= c_1 x_0^{d_{\gamma,1}+1} \overline{t}_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}+1} \overline{t}_{H(\gamma)} \\ &= c_1 x_0^{d_{\gamma+1,1}} \overline{t}_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma+1,H(\gamma)}} \overline{t}_{H(\gamma)} \\ &= (x_0^{d_{\gamma+1,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+1,H(\gamma+1)}} \overline{t}_{H(\gamma+1)}, 0, \dots, 0) \cdot \mathcal{I}_\mu \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}\end{aligned}$$

in R. In order to prove the second equality, let $k \in \{1, \dots, n\}$ and $\ell \in \{1, \dots, H(\gamma)\}$. If $x_k t_\ell = t_i \in \mathcal{O}$ for some $i \in \{1, \dots, \mu\}$, we have $\deg_W(t_i) = \deg_W(t_\ell) + w_k$ and hence

$$\begin{aligned}\overline{x}_k \cdot x_0^{d_{\gamma,\ell}} \overline{t}_\ell &= x_0^{d_{\gamma+w_k,i}} \overline{t}_i \\ &= \sum_{m=1}^{H(\gamma+w_k)} \delta_{mi} x_0^{d_{\gamma+w_k,m}} \overline{t}_m \\ &= \sum_{m=1}^{H(\gamma+w_k)} \xi_{m\ell}^{(k)} x_0^{d_{\gamma+w_k,m}} \overline{t}_m \\ &= (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \dots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \dots, 0) \cdot \mathcal{X}_k \cdot \mathcal{E}_\ell^{\mathrm{tr}}\end{aligned}$$

in R. If $x_k t_\ell = b_j \in \partial\mathcal{O}$ for some $j \in \{1, \dots, \nu\}$, we get $\deg_W(b_j) = \deg_W(t_\ell) + w_k$ and for every index $m \in \{1, \dots, H(\deg_W(b_j))\}$, it follows that

$$\begin{aligned}d_{\gamma,\ell} + \Delta_{mj} &= \gamma - \deg_W(t_\ell) + \deg_W(b_j) - \deg_W(t_m) \\ &= \gamma - \deg_W(t_\ell) + \deg_W(t_\ell) + w_k - \deg_W(t_m) \\ &= \gamma + w_k - \deg_W(t_m) \\ &= d_{\gamma+w_k,m}.\end{aligned}$$

Altogether, we get

$$\begin{aligned}
\overline{x}_k \cdot x_0^{d_{\gamma,\ell}} \overline{t}_\ell &= x_0^{d_{\gamma,\ell}} \overline{b}_j \\
&= \sum_{m=1}^{H(\deg_W(b_j))} a_{mj} x_0^{d_{\gamma,\ell}+\Delta_{mj}} \overline{t}_m \\
&= \sum_{m=1}^{H(\gamma+w_k)} \xi_{m\ell}^{(k)} x_0^{d_{\gamma+w_k,m}} \overline{t}_m \\
&= (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \mathcal{X}_k \cdot \mathcal{E}_\ell^{\mathrm{tr}}
\end{aligned}$$

in R. The first part of a) now follows as

$$\begin{aligned}
\overline{x}_k \cdot \overline{p} &= \sum_{\ell=1}^{H(\gamma)} \overline{x}_k \cdot c_\ell x_0^{d_{\gamma,\ell}} \overline{t}_\ell \\
&= \sum_{\ell=1}^{H(\gamma)} (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \mathcal{X}_k \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \mathcal{X}_k \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}
\end{aligned}$$

in R. In particular, we have proven that for all $k \in \{0, \ldots, n\}$, $\mathcal{X}_k$ can be used to describe the multiplication by $\overline{x}_k$ in R_γ. Since the multiplication in R is commutative, it follows that

$$\begin{aligned}
&(x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot \mathcal{X}_k \mathcal{X}_\ell \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= \overline{x}_k \overline{x}_\ell \overline{p} \\
&= \overline{x}_\ell \overline{x}_k \overline{p} \\
&= (x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot \mathcal{X}_\ell \mathcal{X}_k \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}
\end{aligned}$$

in R for all $k, \ell \in \{0, \ldots, n\}$.

Next we prove by induction on $\gamma' \in \mathbb{N}$ that b) holds in general. For $\gamma' = 0$, it follows that $p' \in K$ and thus

$$\begin{aligned}
\overline{p}' \cdot \overline{p} &= p' c_1 x_0^{d_{\gamma,1}} \overline{t}_1 + \cdots + p' c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)} \\
&= (x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot p' \mathcal{I}_\mu \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot p'(\mathcal{I}_\mu, \mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}
\end{aligned}$$

in R. This shows the claim for $\gamma' = 0$. For the induction step, let now $\gamma' > 0$. Then there exist an $s \in \mathbb{N} \setminus \{0\}$, $c'_1, \ldots, c'_s \in K$, and $t'_1, \ldots, t'_s \in \mathbb{T}^{n+1}_{\gamma'}$ such that $p' = c'_1 t'_1 + \cdots + c'_s t'_s$. For all $r \in \{1, \ldots, s\}$, there are $k_r \in \{0, \ldots, n\}$ and $u_r \in \mathbb{T}^{n+1}_{<\gamma'}$ such that $t'_r = x_{k_r} u_r$.

Hence the induction hypothesis together with a) yield

$$
\begin{aligned}
c'_r\overline{t}'_r \cdot \overline{p} &= \overline{x}_{k_r} \cdot (c'_r\overline{u}_r \cdot \overline{p}) \\
&= \overline{x}_{k_r} \cdot ((x_0^{d_{\gamma+\gamma'-w_{k_r},1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma'-w_{k_r},H(\gamma+\gamma'-w_{k_r})}}\overline{t}_{H(\gamma+\gamma'-w_{k_r})}, 0, \dots, 0) \\
&\quad \cdot (c'_r u_r)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}) \\
&= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot (\mathcal{X}_{k_r} \cdot (c'_r u_r)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n)) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot (c'_r x_{k_r} u_r)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot (c'_r t'_r)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in R for all $r \in \{1, \dots, s\}$. The claim for $\gamma' > 0$ now follows as

$$
\begin{aligned}
\overline{p}' \cdot \overline{p} &= \sum_{r=1}^{s} c'_r\overline{t}'_r \cdot \overline{p} \\
&= \sum_{r=1}^{s} (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\qquad \cdot (c'_r t'_r)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot \left(\sum_{r=1}^{s} c'_r t'_r\right)(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot p'(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in R.

The remaining claim is a direct consequence of Proposition 4.1.7.f. □

Corollary 4.2.9. *Assume that G is the projective $\mathcal{O}$-border basis of the ideal $\langle G \rangle$. We let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and for all indices $i \in \{1, \dots, \mu\}$, let $\mathcal{X}_1, \dots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ denote the multiplication matrices of $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\} \subseteq P$, and let $\overline{\mathcal{O}}_0$ denote the image of $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$ in R. Moreover, we let $\gamma, \gamma' \in \mathbb{Z}$ and $p \in \overline{P}_{\gamma'}$. Then the submatrix of $p(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \in \mathrm{Mat}_\mu(K)$ consisting of the first $H(\gamma+\gamma')$ rows and the first $H(\gamma)$ columns represents the K-algebra homomorphism $\widehat{\pi}_{\overline{p}} : R_\gamma \to R_{\gamma+\gamma'}$, $r \mapsto \overline{p} \cdot r$, the multiplication by $\overline{p} \in R_{\gamma'}$, with respect to the K-vector space bases $(\overline{\mathcal{O}}_0)_\gamma$ of R_γ and $(\overline{\mathcal{O}}_0)_{\gamma+\gamma'}$ of $R_{\gamma+\gamma'}$.*

Proof. Let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ. According to Proposition 4.1.15, the sets $(\overline{\mathcal{O}}_0)_\gamma$ and $(\overline{\mathcal{O}}_0)_{\gamma+\gamma'}$ are K-vector space bases of R_γ and $R_{\gamma+\gamma'}$, respectively. If $\gamma < 0$, $R_\gamma = \{0\}$ by Proposition 4.1.15 and the claim follows trivially. Thus suppose that $\gamma \geq 0$. For all $k \in \{1, \dots, H(\gamma)\}$, Proposition 4.2.8 yields

$$\begin{aligned}\widehat{\pi}_{\overline{p}}(x_0^{d_{\gamma,k}} \overline{t}_k) &= \overline{p} \cdot x_0^{d_{\gamma,k}} \overline{t}_k \\ &= (x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \cdot p(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \cdot \mathcal{E}_k^{\mathrm{tr}}\end{aligned}$$

and the claim is a direct consequence of this. $\square$

Example 4.2.10. Consider the projective $\mathcal{O}$-border basis G in the standard graded ring $\overline{P} = \mathbb{Q}[t, x, y]$ of Example 4.2.7, again. Recall that $W = (1, 1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ defines the standard grading, that $G = \{g_1, \dots, g_6\}$ where $g_1 = y^3 + 2t^2y - 3ty^2$, $g_2 = x^2y - xy^2$, $g_3 = xy^3 + 2t^2xy - 3txy^2$, $g_4 = x^2y^2 + 2t^2xy - 3txy^2$, $g_5 = x^3y + 2t^2xy - 3txy^2$, and $g_6 = x^4 - 6t^3x + 11t^2x^2 - 6tx^3$, and that $\mathcal{O} = \{t_1, \dots, t_8\} = \{1, y, x, y^2, xy, x^2, xy^2, x^3\}$ had a maxdeg_W-border. Moreover, we denote $R = \overline{P}/\langle G\rangle$ and we denote $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1, \dots, 8\} \mid \deg_W(t_k) \leq \gamma\}$. By Proposition 4.1.15, the Hilbert function of R is $\mathrm{HF}_R = H : 1, 3, 6, 8, 8, \dots$ and by Corollary 4.1.12, $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P = \mathbb{Q}[x, y]$ is the $\mathcal{O}$-border basis of the ideal $\langle g^{\mathrm{deh}} \mid g \in G\rangle \subseteq P$. According to Definition 2.4.15, the formal multiplication matrices $\mathcal{X}, \mathcal{Y} \in \mathrm{Mat}_8(\mathbb{Q})$ of $\{g^{\mathrm{deh}} \mid g \in G\}$ are

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $p = x^2 + ty \in \overline{P}_2$. Then

$$p(\mathcal{I}_8, \mathcal{X}, \mathcal{Y}) = \mathcal{X}^2 + \mathcal{I}_8 \cdot \mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 36 \\ 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 0 & -8 & -2 \\ 1 & 0 & 0 & 0 & 0 & -11 & 0 & -60 \\ 0 & 1 & 0 & 3 & 4 & 1 & 10 & 3 \\ 0 & 0 & 1 & 0 & 0 & 6 & 0 & 25 \end{pmatrix} \in \mathrm{Mat}_8(\mathbb{Q}).$$

By Corollary 4.2.9, the submatrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{8,3}(\mathbb{Q})$$

of $p(\mathcal{I}_8, \mathcal{X}, \mathcal{Y})$ consisting of the first $H(3) = 8$ rows and the first $H(1) = 3$ columns represents the homogeneous $\mathbb{Q}[t]$-algebra homomorphism $\widehat{\pi}_{\overline{p}} : R_1 \to R_3$, $r \mapsto \overline{p} \cdot r$, the multiplication by $\overline{p} \in R_2$, with respect to the $\mathbb{Q}$-vector spaces $\{t, \overline{y}, \overline{x}\} \subseteq R$ of R_1 and $\{t^3, t^2\overline{y}, t^2\overline{x}, t\overline{y^2}, t\overline{xy}, t\overline{x^2}, \overline{xy^2}, \overline{x^3}\} \subseteq R$ of R_3. Note that this result can also be deduced from Example 4.2.7 by dehomogenization.

4.3 Projective Border Bases and Dualization

In the final section of this chapter, we study the graded dual of a residue class ring of $\overline{P}$ modulo a homogeneous ideal in $\overline{P}$ that is given by a projective border basis.

Before we actually start with that, we recall that for two $\mathbb{Z}$-graded R-modules M and N, the homogeneous R-module homomorphisms $M \to N$ induce a $\mathbb{Z}$-graded submodule of the set of all (not necessarily homogeneous) R-module homomorphisms $M \to N$. For a more general introduction to gradings and graded dual modules, we refer to [KR00, Sect. 1.7] and [Bou89, Sect. II.§11].

Definition 4.3.1. Let R be a $\mathbb{Z}$-graded ring and let M and N be $\mathbb{Z}$-graded R-modules. An R-module homomorphism $\varphi : M \to N$ is called a **homomorphism of $\mathbb{Z}$-graded R-modules** or a **homogeneous R-module homomorphism** if $\varphi(M_\gamma) \subseteq N_\gamma$ for all $\gamma \in \mathbb{Z}$. The set of all homogeneous R-module homomorphisms $M \to N$ is denoted by $\mathrm{Hom}_R(M, N)$.

Proposition 4.3.2. *Let R be a $\mathbb{Z}$-graded ring and M and N be $\mathbb{Z}$-graded R-modules. Then $\underline{\mathrm{Hom}}_R(M, N) = \bigoplus_{\gamma \in \mathbb{Z}} \mathrm{Hom}_R(M, N(\gamma))$ is a $\mathbb{Z}$-graded R-module. If, in addition, M is finitely generated as an R-module, then $\underline{\mathrm{Hom}}_R(M, N)$ coincides with the set of all R-module homomorphism $M \to N$.*

Proof. This follows immediately from [Bou89, Subsect. II.§11.6]. □

Now we can start with the study of the graded dual module of the residue class ring of $\overline{P}$ modulo a projective border basis. Given a projective $\mathcal{O}$-border basis G of a homogeneous ideal $I \subseteq \overline{P}$, we know that $R = \overline{P}/I$ is a free $K[x_0]$-module and that the residue classes of the elements of the order ideal $\mathcal{O}$ in R form a $K[x_0]$-module basis of R.

Thus the dual $K[x_0]$-module of R is precisely the graded dual $K[x_0]$-module of R, i.e. the $\mathbb{Z}$-graded $K[x_0]$-module $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) = \bigoplus_{\gamma\in\mathbb{Z}} \mathrm{Hom}_{K[x_0]}(R, K[x_0](\gamma))$ by Proposition 4.3.2. Moreover, the dual objects of the residue classes of the elements of $\mathcal{O}$ in R form a $K[x_0]$-module basis of $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$ according to [Bou89, II.§2.6, Prop. 11 and Defn. 7]. In Lemma 4.3.3, we will see that $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$ has also the structure of a $\mathbb{Z}$-graded R-module in this setting. The main goal of this section is to describe this R-module multiplication by means of $\mathcal{O}$ and the (projective) formal multiplication matrices of G as explicitly as possible.

For the remainder of this section, we let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\mu \in \mathbb{N} \setminus \{0\}$ be a finite order ideal in $\mathbb{T}^n$ such that $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$ and we let $\partial\mathcal{O} = \{b_1, \ldots, b_\nu\}$ with $\nu \in \mathbb{N}$ be its border. Moreover, let $G \subseteq \overline{P}$ be the projective $\mathcal{O}$-border basis of the ideal $\langle G\rangle$ and let $R = \overline{P}/\langle G\rangle$. Again, to shorten the notation, we denote the integer function $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1, \ldots, \mu\} \mid \deg_W(t_k) \leq \gamma\}$ and its first difference function by $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma - 1)$. Recall that $H = \mathrm{HF}_R$ by Proposition 4.1.15. Then for all $j \in \{1, \ldots, \nu\}$, the element $g_j \in G$ is of the form $g_j = b_j - \sum_{i=1}^{H(\deg_W(b_j))} a_{ij} x_0^{\Delta_{ij}} t_i$ with $a_{ij} \in K$ and $\Delta_{ij} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{N}$ for all $i \in \{1, \ldots, H(\deg_W(b_j))\}$ according to Proposition 4.1.7.

Lemma 4.3.3. *Let $J \subseteq R$ be a homogeneous ideal and let $S = R/J$. Moreover, let $M = \underline{\mathrm{Hom}}_{K[x_0]}(S, K[x_0])$.*

a) The map

$$R \times S \to S, \quad (r, r' + J) \mapsto rr' + J \quad (r, r' \in R)$$

equips the $\mathbb{Z}$-graded ring S with the structure of a $\mathbb{Z}$-graded R-algebra.

b) The map

$$R \times M \to M, \quad (r, \varphi) \mapsto (r' + J \mapsto \varphi(r \cdot (r' + J))) \quad (r, r' \in R)$$

equips the $\mathbb{Z}$-graded $K[x_0]$-module M with the structure of a $\mathbb{Z}$-graded R-module. In particular, the $K[x_0]$-module structure of M as in Proposition 4.3.2 is compatible with this R-module structure.

Proof. Claim a) follows from [KR00, Rem. 1.7.9].
In order to prove b), let $r, r', r'' \in R$ and $\varphi, \varphi' \in M$. Using a), we see that

$$(1 \cdot \varphi)(\overline{r}'') = \varphi(1 \cdot \overline{r}'') = \varphi(\overline{r}''),$$

i.e. $1 \cdot \varphi = \varphi$,

$$((rr') \cdot \varphi)(\overline{r}'') = \varphi((rr') \cdot \overline{r}'') = \varphi(r' \cdot (r \cdot \overline{r}'')) = (r' \cdot \varphi)(r \cdot \overline{r}'') = (r \cdot (r' \cdot \varphi))(\overline{r}''),$$

i.e. $(rr') \cdot \varphi = r \cdot (r' \cdot \varphi)$,

$$((r + r') \cdot \varphi)(\overline{r}'') = \varphi((r + r') \cdot \overline{r}'') = \varphi(r \cdot \overline{r}'') + \varphi(r' \cdot \overline{r}'') = (r \cdot \varphi)(\overline{r}'') + (r' \cdot \varphi)(\overline{r}'')$$

i.e. $(r + r') \cdot \varphi = r \cdot \varphi + r' \cdot \varphi$, and

$$\begin{aligned}(r \cdot (\varphi + \varphi'))(\overline{r}') &= (\varphi + \varphi')(r \cdot \overline{r}') \\ &= \varphi(r \cdot \overline{r}') + \varphi'(r \cdot \overline{r}') \\ &= (r \cdot \varphi)(\overline{r}') + (r \cdot \varphi')(\overline{r}') \\ &= (r \cdot \varphi + r \cdot \varphi')(\overline{r}'),\end{aligned}$$

i.e. $r \cdot (\varphi + \varphi') = r \cdot \varphi + r \cdot \varphi'$. Thus the given map turns M into an R-module which is obviously compatible with the $K[x_0]$-module structure of M of Proposition 4.3.2. □

Next we prove that the dual $K[x_0]$-module of R with its degrees shifted by 1 is precisely the canonical module of R.

Definition 4.3.4. Let $J \subseteq R$ be a homogeneous ideal and $S = R/J$ be such that $x_0 \in \overline{P}$ is a non-zero divisor for S. Then

$$\omega_S = \underline{\mathrm{Hom}}_{K[x_0]}(S, K[x_0])(-1)$$

is called the **canonical module** of S. If $J = \{0\}$, we identify R with R/J and also write ω_R instead of $\omega_{R/J}$.

Remark 4.3.5. Let $J \subseteq R$ be a homogeneous ideal and $S = R/J$ be such that the polynomial $x_0 \in \overline{P}$ is a non-zero divisor for S. According to Corollary 4.1.16, the $\mathbb{Z}$-graded R-algebras R and S are one-dimensional Cohen-Macaulay rings, and according to [Kun85, Prop. II.3.4], the $\mathbb{Z}$-graded $K[x_0]$-algebra $K[x_0]$ is a one-dimensional ring. Let $\mathfrak{m}_R = \bigoplus_{\gamma=1}^{\infty} R_\gamma = \langle \overline{x}_0, \ldots, \overline{x}_n \rangle$, let $\mathfrak{m}_S = \bigoplus_{\gamma=1}^{\infty} S_\gamma = \langle \overline{x}_0, \ldots, \overline{x}_n \rangle$, and let $\mathfrak{m}_{K[x_0]} = \bigoplus_{\gamma=1}^{\infty} K[x_0] = \langle x_0 \rangle$ denote the maximal homogeneous ideals of R, S, and $K[x_0]$, respectively. As introduced in [HK71, Defn. 5.6] and as also defined in [GW78, Defn. 2.1.2], the R-modules $\omega_R = \underline{\mathrm{Hom}}_K(\underline{H}^1_{\mathfrak{m}_R}(R), K)$ and $\omega_S = \underline{\mathrm{Hom}}_K(\underline{H}^1_{\mathfrak{m}_S}(S), K)$, and the $K[x_0]$-module $\omega_{K[x_0]} = \underline{\mathrm{Hom}}_K(\underline{H}^1_{\mathfrak{m}_{K[x_0]}}(K[x_0]), K)$ are called the canonical modules of R, S, and $K[x_0]$, respectively. Here we denote by $\underline{H}^1_{\mathfrak{m}_R}(R)$, $\underline{H}^1_{\mathfrak{m}_S}(S)$, and $\underline{H}^1_{\mathfrak{m}_{K[x_0]}}(K[x_0])$ the first local cohomology groups of R, S, and $K[x_0]$, respectively. As R and S are one-dimensional Cohen-Macaulay rings, the canonical R-algebra epimorphism $R \twoheadrightarrow R/I = S$ induces a canonical homogeneous R-module isomorphism $\omega_S \cong \underline{\mathrm{Hom}}_K(\underline{H}^1_{\mathfrak{m}_S}(S), K)$ according to [GW78, Prop. 2.2.9 and Prop. 2.1.6]. As the residue classes of the elements of $\mathcal{O}$ in R form a $K[x_0]$-module basis of R according to Proposition 4.1.15, [GW78, Prop. 2.1.5] yields the existence of a canonical homogeneous $K[x_0]$-module isomorphism $\underline{\mathrm{Hom}}_K(\underline{H}^1_{\mathfrak{m}_S}(S), K) \cong \underline{\mathrm{Hom}}_{K[x_0]}(S, \omega_{K[x_0]})$. Moreover, there is a canonical homogeneous $K[x_0]$-algebra isomorphism $\omega_{K[x_0]} \cong K[x_0](-1)$ according to [GW78, Prop. 2.2.8]. Altogether, we get a canonical homogeneous $K[x_0]$-algebra isomorphism $\omega_S \cong \underline{\mathrm{Hom}}_{K[x_0]}(S, K[x_0](-1)) = \underline{\mathrm{Hom}}_{K[x_0]}(S, K[x_0])(-1)$ and with the R-module structure defined in Lemma 4.3.3, we see that our definition of the canonical module in Definition 4.3.4 coincides with the one introduced in [HK71, Defn. 5.6].

The next two propositions describe the structure of such canonical modules. In particular, we determine the canonical module of the residue class ring of R modulo a homogeneous ideal $J \subseteq R$ in Proposition 4.3.6 and describe ω_R and its homogeneous components by means of $\mathcal{O}$ in Proposition 4.3.7.

Proposition 4.3.6. *a) The element $x_0 \in K[x_0]$ is a non-zero divisor for ω_R.*

b) Let $J \subseteq R$ be a homogeneous ideal and let $S = R/J$. Then there exists a canonical homogeneous R-module isomorphism

$$\omega_S \cong \{\varphi \in \omega_R \mid J \cdot \varphi = \{0\}\}.$$

Proof. First we prove a). Assume that there exists a homomorphism $\varphi \in \omega_R \setminus \{0\}$ such that $x_0\varphi = 0$. Then there exists an element $r \in R$ such that $\varphi(r) \neq 0$ and Lemma 4.3.3 yields $0 = 0(r) = (x_0\varphi)(r) = \varphi(x_0 \cdot r) = x_0 \cdot \varphi(r)$. Since $\varphi(r) \in K[x_0] \setminus \{0\}$ and G is the $\mathcal{O}$-border basis of $\langle G\rangle \subset \overline{P}$, this is a contradiction to Proposition 4.1.7. Thus x_0 is a non-zero divisor for ω_R.

Next we prove b). Let $M = \{\varphi \in \omega_R \mid J \cdot \varphi = \{0\}\}$. For all $\varphi_1, \varphi_2 \in M$, $r_1, r_2 \in R$, and $s \in J$, we have

$$\begin{aligned}(s \cdot (\varphi_1 - \varphi_2))(r_1) &= (\varphi_1 - \varphi_2)(sr_1)\\ &= \varphi_1(sr) - \varphi_2(sr_1)\\ &= (s \cdot \varphi_1)(r) - (s \cdot \varphi_2)(r_1)\\ &= 0(r_1) + 0(r_1)\\ &= 0,\end{aligned}$$

i.e. $J \cdot (\varphi_1 - \varphi_2) = \{0\}$ and thus $\varphi_1 - \varphi_2 \in M$, and since $r_1 s \in J$, we have

$$(s \cdot (r_1 \cdot \varphi_1))(r_2) = (r_1 \cdot \varphi_1)(sr_2) = \varphi_1(r_1 s r_2) = ((r_1 s) \cdot \varphi_1)(r_2) = 0(r_2) = 0,$$

i.e. $J \cdot (r_1 \cdot \varphi_1) = \{0\}$ and thus $r_1 \cdot \varphi_1 \in M$. Altogether, it follows that the set M is an R-submodule of ω_R.
For every $\varphi \in M$ and all $s \in J$, we have

$$\varphi(s) = \varphi(s \cdot 1) = (s \cdot \varphi)(1) = 0(1) = 0$$

and hence $J \subseteq \ker(\varphi)$. Let $\varepsilon : R \twoheadrightarrow S = R/J$ be the canonical $K[x_0]$-module epimorphism. Then for every $\varphi \in M$, the Universal Property of the Residue Class Module S induces a unique $K[x_0]$-module homomorphism $\overline{\varphi} : S \to K[x_0]$ which satisfies $\varphi = \overline{\varphi} \circ \varepsilon$ and $\overline{\varphi} \in \omega_S$. In other words, every element $\varphi \in M$ induces a unique element $\overline{\varphi} \in \omega_S$. Thus the map

$$\phi : M \to \omega_S, \quad \varphi \mapsto \overline{\varphi} \quad \text{such that} \quad \varphi = \overline{\varphi} \circ \varepsilon$$

is well-defined. For all $\varphi_1, \varphi_2 \in M$ and $r_1, r_2 \in R$, we have

$$\begin{aligned}
\phi(r_1 \cdot (\varphi_1 - \varphi_2))(r_2 + J) &= \overline{r_1 \cdot (\varphi_1 - \varphi_2)}(r_2 + J) \\
&= (r_1 \cdot (\varphi_1 - \varphi_2))(r_2) \\
&= (\varphi_1 - \varphi_2)(r_1 r_2) \\
&= \varphi_1(r_1 r_2) - \varphi_2(r_1 r_2) \\
&= \overline{\varphi}_1(r_1 r_2 + J) - \overline{\varphi}_2(r_1 r_2 + J) \\
&= (\overline{\varphi}_1 - \overline{\varphi}_2)(r_1 r_2 + J) \\
&= (\phi(\varphi_1) - \phi(\varphi_2))(r_1 r_2 + J) \\
&= (r_1 \cdot (\phi(\varphi_1) - \phi(\varphi_2)))(r_2 + J),
\end{aligned}$$

i.e. $\phi(r_1 \cdot (\varphi_1 - \varphi_2)) = r_1 \cdot (\phi(\varphi_1) - \phi(\varphi_2))$. Altogether, it follows that ϕ is an R-module homomorphism. For all $\gamma, \gamma' \in \mathbb{Z}$, $r \in R_\gamma$, and $\varphi \in M_{\gamma'}$, we also have

$$\phi(\varphi)(r + J) = \overline{\varphi}(r + J) = \varphi(r) \in ((K[x_0])(\gamma' - 1))_\gamma = (K[x_0](-1))_{\gamma+\gamma'},$$

i.e. $\phi(\varphi) \in (\omega_S)_{\gamma'}$. Thus the R-module homomorphism ϕ is homogeneous.
Next we show that ϕ is an R-module isomorphism. For all $\overline{\varphi} \in \omega_S$ and for all $s \in J$, $\varphi = \overline{\varphi} \circ \varepsilon \in \omega_R$ satisfies $(s \cdot \varphi)(1) = \varphi(s) = \overline{\varphi}(s + J) = \overline{\varphi}(0) = 0$, i.e. $s \cdot \varphi = 0$ and thus $J \cdot \varphi = \{0\}$. In particular, for all $\overline{\varphi} \in \omega_S$, $\overline{\varphi} \circ \varepsilon \in M$ and $\phi(\overline{\varphi} \circ \varepsilon) = \overline{\varphi}$. Altogether, it follows that ϕ is surjective. For every $\varphi \in \ker(\phi)$, we have $\varphi = \phi(\varphi) \circ \varepsilon = 0 \circ \varepsilon = 0$, i.e. $\ker(\phi) = \{0\}$ and ϕ is hence injective. □

For the remainder of this thesis, we identify the elements of ω_S for a residue class ring $S = R/J$ with the elements of the corresponding submodule of ω_R as proven in Proposition 4.3.6 without mention.

Next we determine the Hilbert function of ω_R and give K-vector space bases of the homogeneous components of ω_R by means of $\mathcal{O}$. The ideas of the following proposition follow [GKR93, Prop. 3.1].

Proposition 4.3.7. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and all $i \in \{1, \dots, \mu\}$, let $\overline{\mathcal{O}} = \{\bar{t}_1, \dots, \bar{t}_\mu\} \subseteq R$ denote the image of $\mathcal{O}$ in R, and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*, \dots, \bar{t}_\mu^*\} \subseteq \omega_R$ be the dual $K[x_0]$-module basis of $\overline{\mathcal{O}}$, i.e. we have*

$$\bar{t}_i^* : R \to K[x_0], \quad \bar{t}_j \mapsto \delta_{ij} \quad (j \in \{1, \dots, \mu\})$$

for every $i \in \{1, \dots, \mu\}$.

a) The set $\overline{\mathcal{O}}^*$ is a $K[x_0]$-module basis of ω_R and there are canonical isomorphisms of $\mathbb{Z}$-graded $K[x_0]$-modules

$$\omega_R \cong \bigoplus_{i=1}^{\mu} K[x_0](\deg_W(t_i) - 1) \cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (K[x_0](\gamma - 1))^{\Delta H(\gamma)}.$$

In particular, $\mathrm{HF}_{\omega_R}(-\gamma) = \mu - H(\gamma) = \mu - \mathrm{HF}_R(\gamma)$ for all $\gamma \in \mathbb{Z}$.

b) Let $\overline{\mathcal{O}}_0$ denote the image of $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$ in R and let $\overline{\mathcal{O}}_0^* \subseteq \omega_R$ be the set of dual objects of the elements in $\overline{\mathcal{O}}_0$. Then for every integer $\gamma \in \mathbb{Z}$, the set $(\overline{\mathcal{O}}_0^*)_{-\gamma} = \{x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}^*_{H(\gamma)+1}, \ldots, x_0^{-d_{\gamma+1,\mu}} \bar{t}^*_{\mu}\} \subseteq (\omega_R)_{-\gamma}$ is a K-vector space basis of $(\omega_R)_{-\gamma}$.

Proof. First we prove a). Since G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$, $\overline{\mathcal{O}} \subseteq R$ is a $K[x_0]$-module basis of R according to Definition 4.1.2. Thus the dual basis $\overline{\mathcal{O}}^*$ of $\overline{\mathcal{O}}$ is a $K[x_0]$-module basis of ω_R by [Bou89, II.§2.6, Prop. 11 and Defn. 7]. According to [Bou89, II.§1.6, Coro. 1 and II.§11.6, Rem.]), there exists a canonical homogeneous $K[x_0]$-module isomorphism

$$\underline{\mathrm{Hom}}_{K[x_0]}\left(\bigoplus_{i=1}^{\mu} K[x_0](-\deg_W(t_i)), K[x_0]\right) \cong \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{K[x_0]}(K[x_0](-\deg_W(t_i)), K[x_0]).$$

Moreover, there are a canonical homogeneous $K[x_0]$-module isomorphism

$$R = \bigoplus_{i=1}^{\mu} K[x_0] \cdot \bar{t}_i \cong \bigoplus_{i=1}^{\mu} K[x_0](-\deg_{\overline{W}}(t_i))$$

by Proposition 4.1.15 and a canonical homogeneous $K[x_0]$-module isomorphism

$$\underline{\mathrm{Hom}}_{K[x_0]}(K[x_0], K[x_0]) \cong K[x_0]$$

by [Bou89, II.§11.6, p. 376]. Altogether, we get the canonical isomorphism

$$\begin{aligned}
\omega_R &= \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-1) \\
&\cong \underline{\mathrm{Hom}}_{K[x_0]}\left(\bigoplus_{i=1}^{\mu} K[x_0](-\deg_W(t_i)), K[x_0]\right)(-1) \\
&\cong \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{K[x_0]}(K[x_0](-\deg_W(t_i)), K[x_0])(-1) \\
&= \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{K[x_0]}(K[x_0], K[x_0])(\deg_W(t_i) - 1) \\
&\cong \bigoplus_{i=1}^{\mu} K[x_0](\deg_W(t_i) - 1) \\
&\cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (K[x_0](\gamma - 1))^{\Delta H(\gamma)}
\end{aligned}$$

of $\mathbb{Z}$-graded $K[x_0]$-modules and the claim follows. In particular, for all $\gamma \in \mathbb{Z}$, we have

$$\begin{aligned}(\omega_R)_{-\gamma} &\cong \left(\bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} (K[x_0](\tilde{\gamma}-1))^{\Delta H(\tilde{\gamma})} \right)_{-\gamma} \\ &= \bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} ((K[x_0](\tilde{\gamma}-1))_{-\gamma})^{\Delta H(\tilde{\gamma})} \\ &= \bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} (K[x_0]_{\tilde{\gamma}-1-\gamma})^{\Delta H(\tilde{\gamma})}\end{aligned}$$

and as $\mathcal{O}$ is ordered increasingly with respect to $\deg_W$, this implies

$$\mathrm{HF}_{\omega_R}(-\gamma) = \bigoplus_{\tilde{\gamma}=\gamma+1}^{\deg_W(t_\mu)} \Delta H(\tilde{\gamma}) = H(\deg_W(t_\mu)) - H(\gamma) = \mu - H(\gamma).$$

As $\mathrm{HF}_R = H$ by Proposition 4.1.15, the remaining equality follows.

In order to prove b), let $\gamma \in \mathbb{Z}$. For all $i \in \{1, \dots, \mu\}$, we have

$$\bar{t}_i^*(\bar{t}_i) = 1 \in K[x_0]_0 = (K[x_0](-\deg_W(t_i)))_{\deg_W(t_i)},$$

and $\bar{t}_i^*(\bar{t}_j) = 0 \in (K[x_0](-\deg_W(t_i)))_{\deg_W(t_i)}$ for all indices $j \in \{1, \dots, \mu\} \setminus \{i\}$, i.e. we have $\bar{t}_i^* \in (\omega_R)_{-\deg_W(t_i)+1}$. Moreover, for every index $i \in \{H(\gamma)+1, \dots, \mu\}$, we get $-d_{\gamma+1,i} = -\gamma - 1 + \deg_W(t_i) \in \mathbb{N}$ and thus $x_0^{-d_{\gamma+1,i}} \bar{t}_i^* \in (\omega_R)_{-\gamma}$. Since $\bar{t}_i^*(\bar{t}_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, \mu\}$ and since $\mathrm{HF}_{\omega_R}(-\gamma) = \mu - H(\gamma)$ according to a), it follows that $(\overline{\mathcal{O}}_0^*)_{-\gamma} = \{x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}_{H(\gamma)+1}^*, \dots, x_0^{-d_{\gamma+1,\mu}} \bar{t}_\mu^*\} \subseteq (\omega_R)_{-\gamma}$ is a K-vector space basis of $(\omega_R)_{-\gamma}$. $\square$

Just as we did for the multiplicative structure of R in Proposition 4.2.5 and Corollary 4.2.6, we can use the formal projective multiplication matrices of G to explicitly describe the R-module multiplication of ω_R.

Proposition 4.3.8. *Let $\overline{\mathcal{O}} = \{\bar{t}_1, \dots, \bar{t}_\mu\} \subseteq R$ denote the image of the order ideal $\mathcal{O}$ in R and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*, \dots, \bar{t}_\mu^*\} \subseteq \omega_R$ be the dual $K[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, let $\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0])$ be the formal projective multiplication matrices of G, let $c_1, \dots, c_\mu \in K[x_0]$, and let $\varphi = c_1\bar{t}_1^* + \dots + c_\mu \bar{t}_\mu^* \in \omega_R$. Then for all $\gamma \in \mathbb{Z}$ and $p \in \overline{P}_\gamma$, we have*

$$\bar{p} \cdot \varphi = (\bar{t}_1^*, \dots, \bar{t}_\mu^*) \cdot p((\mathcal{X}_0^{\mathrm{proj}})^{\mathrm{tr}}, \dots, (\mathcal{X}_n^{\mathrm{proj}})^{\mathrm{tr}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}},$$

in ω_R, i.e. the matrix $p((\mathcal{X}_0^{\mathrm{proj}})^{\mathrm{tr}}, \dots, (\mathcal{X}_n^{\mathrm{proj}})^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K[x_0])$ represents the homogeneous $K[x_0]$-module homomorphism $\pi_{\bar{p}}^* : \omega_R(-\gamma) \to \omega_R$, $\varphi' \mapsto \bar{p} \cdot \varphi'$, the multiplication by $\bar{p} \in R_\gamma$, with respect to the $K[x_0]$-module basis $\overline{\mathcal{O}}^*$ of ω_R.

Proof. Let $\gamma \in \mathbb{Z}$ and $p \in \overline{P}_\gamma$. By Corollary 4.2.6, $p(\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}}) \in \mathrm{Mat}_\mu(K[x_0])$ represents the homogeneous $K[x_0]$-module homomorphism $\pi_{\overline{p}} : R(-\gamma) \mapsto R$, $r \mapsto \overline{p} \cdot r$, the multiplication by $\overline{p}$, with respect to the $K[x_0]$-module basis $\overline{\mathcal{O}}$. Since the dual $K[x_0]$-module of R is $R^* = \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) = \omega_R(1)$, it follows that the induced homogeneous $K[x_0]$-module homomorphism $\pi_{\overline{p}}^* : \omega_R(-\gamma) \to \omega_R$, $\varphi' \mapsto \varphi' \circ \pi_{\overline{p}} = \overline{p} \cdot \varphi'$, cf. Lemma 4.3.3, with respect to the dual $K[x_0]$-module basis $\overline{\mathcal{O}}^*$ is represented by the matrix $p(\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}})^{\mathrm{tr}} \in \mathrm{Mat}_\mu(K[x_0])$ according to [Bou89, II.§10.4, Prop. 3] and Corollary 4.2.6. Moreover, as G is the projective $\mathcal{O}$-border basis of $\langle G \rangle$ the matrices $\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}}$ are pairwise commuting according to Corollary 4.2.6. The claim now follows from $p(\mathcal{X}_0^{\mathrm{proj}}, \dots, \mathcal{X}_n^{\mathrm{proj}})^{\mathrm{tr}} = p((\mathcal{X}_0^{\mathrm{proj}})^{\mathrm{tr}}, \dots, (\mathcal{X}_n^{\mathrm{proj}})^{\mathrm{tr}})$. $\square$

Example 4.3.9. Consider the projective $\mathcal{O}$-border basis $G \subseteq \overline{P}$ of the standard graded $\mathbb{Q}$-algebra $\overline{P} = \mathbb{Q}[t, x, y]$ of Example 4.2.7, again. Recall that the order ideal $\mathcal{O} = \{1, y, x, y^2, xy, x^2, xy^2, x^3\}$ has a maxdeg_W-border, that $R = \overline{P}/\langle G \rangle$, and that the projective multiplication matrices $\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}} \in \mathrm{Mat}_8(\mathbb{Q}[t])$ of G are

$$\mathcal{T}^{\mathrm{proj}} = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix}, \quad \mathcal{X}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2t^2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -11t^2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6t \end{pmatrix},$$

and

$$\mathcal{Y}^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2t^2 & -2t^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3t & 3t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, we had $p = x^2 + ty \in \overline{P}_2$. Let $\pi_{\overline{p}}^* : \omega_R(-2) \to \omega_R$, $\varphi \mapsto \overline{p} \cdot \varphi$ be the multiplication by $\overline{p} \in R_2$, let $\overline{\mathcal{O}} \subseteq R$ be the image of $\mathcal{O}$ in R, and let $\overline{\mathcal{O}}^* \subseteq \omega_R$ be its dual $\mathbb{Q}[t]$-module basis. Then the matrix

$$
\begin{aligned}
p((\mathcal{T}^{\mathrm{proj}})^{\mathrm{tr}}, (\mathcal{X}^{\mathrm{proj}})^{\mathrm{tr}}, (\mathcal{Y}^{\mathrm{proj}})^{\mathrm{tr}}) &= ((\mathcal{X}^{\mathrm{proj}})^{\mathrm{tr}})^2 + (\mathcal{T}^{\mathrm{proj}})^{\mathrm{tr}} \cdot (\mathcal{X}^{\mathrm{proj}})^{\mathrm{tr}} \\
&= \begin{pmatrix}
0 & t & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & t & 0 & 0 & 1 \\
0 & -2t^3 & 0 & 3t^2 & -2t^2 & 0 & 3t & 0 \\
0 & 0 & 0 & 0 & -2t^2 & 0 & 4t & 0 \\
0 & 0 & 6t^3 & 0 & 0 & -11t^2 & t & 6t \\
0 & 0 & 0 & 0 & -8t^3 & 0 & 10t^2 & 0 \\
0 & 0 & 36t^4 & 0 & -2t^3 & -60t^3 & 3t^2 & 25t^2
\end{pmatrix} \\
&\in \mathrm{Mat}_8(\mathbb{Q}[t])
\end{aligned}
$$

represents the homogeneous $\mathbb{Q}[t]$-module homomorphism $\pi_{\overline{p}}^*$ with respect to the dual $\mathbb{Q}[t]$-module basis $\overline{\mathcal{O}}^*$ of ω_R according to Proposition 4.3.8. Note that the matrices $\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}} \in \mathrm{Mat}_8(\mathbb{Q}[t])$ are pairwise commuting and, therefore, we see that $p((\mathcal{T}^{\mathrm{proj}})^{\mathrm{tr}}, (\mathcal{X}^{\mathrm{proj}})^{\mathrm{tr}}, (\mathcal{Y}^{\mathrm{proj}})^{\mathrm{tr}}) = (p(\mathcal{T}^{\mathrm{proj}}, \mathcal{X}^{\mathrm{proj}}, \mathcal{Y}^{\mathrm{proj}})^{\mathrm{tr}}$. In particular, the above result coincides with the result of Example 4.2.7.

Finally, we restrict the R-module multiplication of ω_R to a homogeneous component R_γ with $\gamma \in \mathbb{Z}$. As done in Proposition 4.2.5 for the multiplication in R, we can use the formal multiplication matrices of $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ to explicitly describe the R-module multiplication by a homogeneous element.

Proposition 4.3.10. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and for all indices $i \in \{1, \ldots, \mu\}$, let $\overline{\mathcal{O}} = \{\bar{t}_1, \ldots, \bar{t}_\mu\} \subseteq R$ denote the image of the order ideal $\mathcal{O}$ in R, and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*, \ldots, \bar{t}_\mu^*\} \subseteq \omega_R$ be the dual $K[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, let $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of $\{g_1^{\mathrm{deh}}, \ldots, g_\nu^{\mathrm{deh}}\}$, $\gamma \in \mathbb{Z}$, $c_{H(\gamma)+1}, \ldots, c_\mu \in K$, and $\varphi = c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}_{H(\gamma)+1}^* + \cdots + c_\mu x_0^{-d_{\gamma+1,\mu}} \bar{t}_\mu^* \in (\omega_R)_{-\gamma}$.*

a) *We have*

$$
\begin{aligned}
x_0 \cdot \varphi = (0, \ldots, 0, x_0^{-d_{\gamma,H(\gamma-1)+1}} \bar{t}_{H(\gamma-1)+1}^*, \ldots, x_0^{-d_{\gamma,\mu}} \bar{t}_\mu^*) \\
\cdot \mathcal{I}_\mu \cdot (0, \ldots, 0, c_{H(\gamma)+1}, \ldots, c_\mu)^{\mathrm{tr}}
\end{aligned}
$$

in ω_R and for every $k \in \{1, \ldots, n\}$, we have

$$
\begin{aligned}
\overline{x}_k \cdot \varphi = (0, \ldots, 0, x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \bar{t}_{H(\gamma-w_k)+1}^*, \ldots, x_0^{-d_{\gamma-w_k+1,\mu}} \bar{t}_\mu^*) \\
\cdot \mathcal{X}_k^{\mathrm{tr}} \cdot (0, \ldots, 0, c_{H(\gamma)+1}, \ldots, c_\mu)^{\mathrm{tr}}
\end{aligned}
$$

in ω_R.

b) *For every $\gamma' \in \mathbb{Z}$ and every $p' \in \overline{P}_{\gamma'}$, we have*

$$
\begin{aligned}
\overline{p}' \cdot \varphi = (0, \ldots, 0, \overline{x}_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}} \bar{t}_{H(\gamma-\gamma')+1}^*, \ldots, \overline{x}_0^{-d_{\gamma-\gamma'+1,\mu}} \bar{t}_\mu^*) \\
\cdot p'(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}}, \ldots, \mathcal{X}_n^{\mathrm{tr}}) \cdot (0, \ldots, 0, c_{H(\gamma)+1}, \ldots, c_\mu)^{\mathrm{tr}}
\end{aligned}
$$

in ω_R.

Proof. For every $s \in \{1,\dots,n\}$, we denote $\mathcal{X}_s = (\xi_{k\ell}^{(s)})_{1\le k,\ell\le\mu} \in \mathrm{Mat}_\mu(K)$ as in Definition 2.4.15 and we let $\{\mathcal{E}_1,\dots,\mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ. If $\gamma \ge \deg_W(t_\mu)$, we have $(\omega_R)_{-\gamma} = \{0\}$ and the claims follow trivially. Thus we suppose that $\gamma < \deg_W(t_\mu)$.

First we prove a). For every $i \in \{1,\dots,\mu\}$, we have

$$-d_{\gamma+1,i}+1 = -(\gamma+1-\deg_W(t_i))+1 = -(\gamma-\deg_W(t_i)) = -d_{\gamma,i}$$

and get

$$\begin{aligned} x_0\cdot\varphi &= x_0\cdot(c_{H(\gamma)+1}x_0^{-d_{\gamma+1,H(\gamma)+1}}\bar{t}^*_{H(\gamma)+1}+\cdots+c_\mu x_0^{-d_{\gamma+1,\mu}}\bar{t}^*_\mu)\\ &= c_{H(\gamma)+1}x_0^{-d_{\gamma+1,H(\gamma)+1}+1}\bar{t}^*_{H(\gamma)+1}+\cdots+c_\mu x_0^{-d_{\gamma+1,\mu}+1}\bar{t}^*_\mu\\ &= c_{H(\gamma)+1}x_0^{-d_{\gamma,H(\gamma)+1}}\bar{t}^*_{H(\gamma)+1}+\cdots+c_\mu x_0^{-d_{\gamma,\mu}}\bar{t}^*_\mu\\ &= (0,\dots,0,x_0^{-d_{\gamma,H(\gamma-1)+1}}\bar{t}^*_{H(\gamma-1)+1},\dots,x_0^{d_{\gamma,\mu}}\bar{t}^*_\mu)\\ &\quad\cdot\mathcal{I}_\mu\cdot(0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}} \end{aligned}$$

in ω_R. For the second part, let $k \in \{1,\dots,n\}$. Moreover, let $i \in \{H(\gamma)+1,\dots,\mu\}$, $\ell \in \{1,\dots,\mu\}$, and $\tilde\gamma = \deg_W(t_\ell)+w_k$. Then Proposition 6.2.7 yields

$$\begin{aligned} (\bar{x}_k\cdot\bar{t}^*_i)(\bar{t}_\ell) &= \bar{t}^*_i(\bar{x}_k\bar{t}_\ell)\\ &= \bar{t}^*_i((x_0^{d_{\tilde\gamma,1}}\bar{t}_1,\dots,x_0^{d_{\tilde\gamma,H(\tilde\gamma)}}\bar{t}_{H(\tilde\gamma)},0,\dots,0)\cdot\mathcal{X}_k\cdot\mathcal{E}_\ell^{\mathrm{tr}})\\ &= \begin{cases} 0 & \text{if } \deg_W(t_i) > \tilde\gamma\\ \xi_{i\ell}^{(k)}x_0^{d_{\tilde\gamma,i}} & \text{if } \deg_W(t_i)\le\tilde\gamma \end{cases}\\ &= \begin{cases} 0 & \text{if } \deg_W(t_\ell) < \deg_W(t_i)-w_k\\ \xi_{i\ell}^{(k)}x_0^{d_{\tilde\gamma,i}} & \text{if } \deg_W(t_\ell)\ge\deg_W(t_i)-w_k \end{cases}. \end{aligned}$$

Assume that there is an $\tilde\ell \in \{1,\dots,\mu\}$ such that $\gamma+1-w_k \le \deg_W(t_{\tilde\ell}) < \deg_W(t_i)-w_k$ and such that $\xi_{i\tilde\ell}^{(k)} \ne 0$. Then $\gamma+1 \le \deg_W(t_{\tilde\ell})+w_k < \deg_W(t_i)$. We distinguish two cases. If $x_kt_{\tilde\ell} = t_r \in \mathcal{O}$ for some $r \in \{1,\dots,\mu\}$, we have $\deg_W(t_{\tilde\ell})+w_k = \deg_W(t_r)$. In particular, we see that $r \ne i$ and thus $\xi_{i\tilde\ell}^{(k)} = \delta_{ir} = 0$ by Definition 2.4.15 in this situation. If $x_kt_{\tilde\ell} = b_s \in \partial\mathcal{O}$ for some $s \in \{1,\dots,\nu\}$, $\deg_W(t_i) > \deg_W(t_{\tilde\ell})+w_k = \deg_W(b_s)$. As G is the projective $\mathcal{O}$-border basis of the ideal $\langle G\rangle$, Proposition 4.1.7 implies that $g_s = b_s - \sum_{m=1}^{H(\deg_W(b_s))} a_{ms}x_0^{\Delta_{ms}}t_m$ and thus we have $\xi_{i\tilde\ell}^{(k)} = 0$ by Definition 2.4.15 in this situation. Since both cases yield a contradiction, there is no such index $\tilde\ell$. In particular, this implies that

$$(\bar{x}_k\cdot\bar{t}^*_i)(\bar{t}_\ell) = \begin{cases} 0 & \text{if } \deg_W(t_\ell) < \gamma-w_k+1\\ \xi_{i\ell}^{(k)}x_0^{d_{\tilde\gamma,i}} & \text{if } \deg_W(t_\ell)\ge\gamma-w_k+1 \end{cases}.$$

Moreover,

$$\begin{aligned}-d_{\gamma+1,i} + d_{\tilde{\gamma},i} &= -(\gamma + 1 - \deg_W(t_i)) + \tilde{\gamma} - \deg_W(t_i)\\ &= -\gamma - 1 + \deg_W(t_\ell) + w_k\\ &= -d_{\gamma-w_k+1,\ell}.\end{aligned}$$

Altogether, we get

$$\begin{aligned}&\overline{x}_k \cdot c_i x_0^{-d_{\gamma+1,i}} \overline{t}_i^*\\ &= \sum_{\ell=H(\gamma-w_k)+1}^{\mu} \xi_{i\ell}^{(k)} c_i x_0^{-d_{\gamma-w_k+1,\ell}} \overline{t}_\ell^*\\ &= (0,\dots,0,x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*) \cdot \mathcal{X}_k^{\mathrm{tr}} \cdot c_i \mathcal{E}_i^{\mathrm{tr}}\end{aligned}$$

in ω_R and thus

$$\begin{aligned}\overline{x}_k \cdot \varphi &= \sum_{i=H(\gamma)+1}^{\mu} \overline{x}_k \cdot c_i x_0^{-d_{\gamma+1,i}} \overline{t}_i^*\\ &= \sum_{i=H(\gamma)+1}^{\mu} (0,\dots,0,x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*) \cdot \mathcal{X}_k^{\mathrm{tr}} \cdot c_i \mathcal{E}_i^{\mathrm{tr}}\\ &= (0,\dots,0,x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*)\\ &\quad \cdot \mathcal{X}_k^{\mathrm{tr}} \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}\end{aligned}$$

in ω_R.

Next we prove b). For all $\gamma' \in \mathbb{Z}$ with $\gamma' < 0$, we have $\overline{P}_{\gamma'} = \{0\}$ and b) holds trivially. We prove by induction on $\gamma' \in \mathbb{N}$ that b) holds in general. For $\gamma' = 0$, it follows that $p' \in K$ and thus

$$\begin{aligned}\overline{p}' \cdot \varphi &= p' c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^* + \cdots + p' c_\mu x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^*\\ &= (0,\dots,0,x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^*, \dots, x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^*)\\ &\quad \cdot p' \mathcal{I}_\mu \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}\\ &= (0,\dots,0,x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^*, \dots, x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^*)\\ &\quad \cdot p'(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}}, \dots, \mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}\end{aligned}$$

in ω_R. For the induction step, let now $\gamma' > 0$. Then there exist an $s \in \mathbb{N}$, $c_1', \dots, c_s' \in K$, and $t_1', \dots, t_s' \in \mathbb{T}_{\gamma'}^{n+1}$ such that $p' = c_1' t_1' + \cdots + c_s' t_s'$. For every $r \in \{1, \dots, s\}$, there is a $k_r \in \{0, \dots, n\}$ and a $u_r \in \mathbb{T}_{<\gamma}^{n+1}$ such that $t_r' = x_{k_r} u_r$. Let $w_0 = 1$, $\mathcal{X}_0 = \mathcal{I}_\mu$, and for all indices $r \in \{1, \dots, s\}$, let $\hat{\gamma}_r = \gamma - \gamma' + w_{k_r}$. Then the induction hypothesis, the commutativity of the formal multiplication matrices of $\{g_1^{\mathrm{deh}}, \dots, g_\nu^{\mathrm{deh}}\} \subseteq P$, which follows from Proposition 4.1.7 and Theorem 2.4.19, together with a) yield

$$\begin{aligned}
c'_r\overline{t}'_r \cdot \varphi &= \overline{x}_{k_r} \cdot (c'_r\overline{u}_r \cdot \varphi) \\
&= \overline{x}_{k_r} \cdot ((0,\dots,0,x_0^{-d_{\hat\gamma_r+1,H(\hat\gamma_r)+1}}\overline{t}^*_{H(\hat\gamma_r)+1},\dots,x_0^{-d_{\hat\gamma_r+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c'_r u_r)(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\hat\gamma_r-w_{k_r}+1,H(\hat\gamma_r-w_{k_r})+1}}\overline{t}^*_{H(\hat\gamma_r-w_{k_r})+1},\dots,x_0^{-d_{\hat\gamma_r-w_{k_r}+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (\mathcal{X}_{k_r}^{\mathrm{tr}} \cdot (c'_r u_r)(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}})) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c'_r x_{k_r} u_r)(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c'_r t'_r)(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}})
\end{aligned}$$

in ω_R. Thus it follows that

$$\begin{aligned}
\overline{p}' \cdot \varphi &= \sum_{r=1}^{s} c'_r\overline{t}_r \cdot \varphi \\
&= \sum_{r=1}^{s}((0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\qquad \cdot (c'_r t'_r)(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot \left(\sum_{r=1}^{s} c'_r\overline{t}'_r\right)(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}} \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot p'(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in ω_R and the claim follows for $\gamma' > 0$. □

Corollary 4.3.11. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and all $i \in \{1,\dots,\mu\}$, let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ denote the multiplication matrices of $\{g_1^{\mathrm{deh}},\dots,g_\nu^{\mathrm{deh}}\} \subseteq P$, let $\overline{\mathcal{O}} = \{\overline{t}_1,\dots,\overline{t}_\mu\} \subseteq R$ denote the image of $\mathcal{O}$ in R and let $\overline{\mathcal{O}}^* = \{\overline{t}_1^*,\dots,\overline{t}_\mu^*\} \subseteq \omega_R$ be the dual $K[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, let $\overline{\mathcal{O}}_0$ be the image of $\mathcal{O}_0 = \bigcup_{k=0}^\infty x_0^k\mathcal{O}$ in R and $\overline{\mathcal{O}}_0^* \subseteq \omega_R$ be the set of dual objects of the elements in $\overline{\mathcal{O}}_0$. Let $\gamma,\gamma' \in \mathbb{Z}$ be integers and $p \in \overline{P}_{\gamma'}$. Then the submatrix of $p(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K)$ obtained by deleting the first $H(\gamma-\gamma')$ rows and the first $H(\gamma)$ columns represents the K-vector space homomorphism $\widehat{\pi}^*_{\overline{p}} : (\omega_R)_{-\gamma} \to (\omega_R)_{-\gamma+\gamma'}$, $\varphi \mapsto \overline{p}\cdot\varphi$, the multiplication by the element $\overline{p} \in R_{\gamma'}$, with respect to the K-vector space bases $(\overline{\mathcal{O}}_0^*)_{-\gamma}$ of $(\omega_R)_{-\gamma}$ and $(\overline{\mathcal{O}}_0^*)_{-\gamma+\gamma'}$ of $(\omega_R)_{-\gamma+\gamma'}$.*

Proof. Let $\{\mathcal{E}_1,\dots,\mathcal{E}_\mu\}$ denote the canonical K-vector space basis of K^μ. According to Proposition 4.3.7, the sets $(\overline{\mathcal{O}}_0^*)_{-\gamma}$ and $(\overline{\mathcal{O}}_0^*)_{-\gamma+\gamma'}$ are K-vector space bases of $(\omega_R)_{-\gamma}$ and $(\omega_R)_{-\gamma+\gamma'}$, respectively. If $\gamma \geq \deg_W(t_\mu)$, we have $(\omega_R)_{-\gamma} = \{0\}$ according to Proposition 4.3.7 and the claim follows trivially. Thus suppose that $\gamma < \deg_W(t_\mu)$. For all indices $k \in \{H(\gamma)+1,\dots,\mu\}$, Proposition 4.3.10 yields

$$\begin{aligned}
&\widehat{\pi}_{\overline{p}}^*(x_0^{-d_{\gamma+1,k}}\overline{t}_k^*)\\
&= \overline{p}\cdot x_0^{-d_{\gamma+1,k}}\overline{t}_k^*\\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}_{H(\gamma-\gamma')+1}^*,\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}_\mu^*)\cdot p(\mathcal{I}_\mu,\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}})\cdot\mathcal{E}_k^{\mathrm{tr}}
\end{aligned}$$

and the claim is a direct consequence of this. $\square$

Example 4.3.12. Consider the $\mathbb{Q}$-algebra $R = \overline{P}/\langle G\rangle$ which is defined by the projective $\mathcal{O}$-border basis $G \subseteq \overline{P} = \mathbb{Q}[t,x,y]$ of the Examples 4.2.10 and 4.3.9, again. Recall that $\overline{P}$ was standard graded and that $\mathcal{O} = \{t_1,\dots,t_8\} = \{1,y,x,y^2,xy,x^2,xy^2,x^3\}$ has a maxdeg_W-border. Moreover, we had $H:\mathbb{Z}\to\mathbb{Z}$, $\gamma\mapsto \#\{k\in\{1,\dots,8\}\mid \deg_W(t_k)\leq\gamma\}$, the Hilbert function of R was $\mathrm{HF}_R = H : 1,3,6,8,8,\dots$, the multiplication matrices $\mathcal{X},\mathcal{Y}\in\mathrm{Mat}_8(\mathbb{Q})$ of $\{g^{\mathrm{deh}}\mid g\in G\}$ were

$$\mathcal{X} = \begin{pmatrix}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&6\\
0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&-2&0\\
0&0&1&0&0&0&0&-11\\
0&0&0&1&1&0&3&0\\
0&0&0&0&0&1&0&6
\end{pmatrix},\qquad
\mathcal{Y} = \begin{pmatrix}
0&0&0&0&0&0&0&0\\
1&0&0&-2&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&1&0&3&0&0&0&0\\
0&0&1&0&0&0&-2&-2\\
0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&3&3\\
0&0&0&0&0&0&0&0
\end{pmatrix},$$

and we had $p = x^2+ty\in\overline{P}_2$. Let $\widehat{\pi}_{\overline{p}}^* : (\omega_R)_{-2}\to(\omega_R)_0$, $\varphi\mapsto\overline{p}\cdot\varphi$ be the multiplication by $\overline{p}\in R_2$. Then

$$\begin{aligned}
p(\mathcal{I}_8,\mathcal{X}^{\mathrm{tr}},\mathcal{Y}^{\mathrm{tr}}) &= (\mathcal{X}^{\mathrm{tr}})^2+\mathcal{I}_8\cdot\mathcal{Y}^{\mathrm{tr}}\\
&= \begin{pmatrix}
0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&1&0\\
0&0&0&0&1&0&0&1\\
0&-2&0&3&-2&0&3&0\\
0&0&0&0&-2&0&4&0\\
0&0&6&0&0&-11&1&6\\
0&0&0&0&-8&0&10&0\\
0&0&36&0&-2&-60&3&25
\end{pmatrix}\\
&\in\mathrm{Mat}_8(\mathbb{Q}).
\end{aligned}$$

By Corollary 4.3.11, the submatrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 0 \\ 4 & 0 \\ 1 & 6 \\ 10 & 0 \\ 3 & 25 \end{pmatrix} \in \mathrm{Mat}_{7,2}(\mathbb{Q})$$

of the matrix $p(\mathcal{I}_8, \mathcal{X}^{\mathrm{tr}}, \mathcal{Y}^{\mathrm{tr}})$ obtained by deleting the first $H(0) = 1$ rows and the first $H(2) = 6$ columns represents the homogeneous $\mathbb{Q}$-vector space homomorphism $\widehat{\pi}_p^*$ with respect to the $\mathbb{Q}$-vector space basis $\{\overline{xy^2}^*, \overline{x^3}^*\} \subseteq \omega_R$ of $(\omega_R)_{-2}$ and the $\mathbb{Q}$-vector space basis $\{\overline{y}^*, \overline{x}^*, t\overline{y^2}^*, t\overline{xy}^*, t\overline{x^2}^*, t^2\overline{xy^2}^*, t^2\overline{x^3}^*\} \subseteq \omega_R$ of $(\omega_R)_0$.

5 Projective Border Bases and Algebraic Geometry

After the algebraic foundations in Chapter 4, we now use projective border bases for the study of zero-dimensional closed subschemes of weighted projective spaces. The main goal of this chapter is to show that we can translate geometric properties of zero-dimensional closed subschemes of weighted projective spaces to algebraic properties of residue class rings of the $\mathbb{Z}$-graded K-algebra $\overline{P} = P[x_0] = K[x_0, \ldots, x_n]$. In this thesis, we are particularly interested in uniformity conditions of zero-dimensional closed subschemes and try to characterize them. Uniformity conditions, e.g. the Cayley-Bacharach property or the uniform position property, have frequently been studied, for instance in [Har80], [Rat87], [Kre98], [Kre94], [GKR93], and [MP04]. The most general notion of uniformity we consider, namely the concept of (i, j)-uniformity as defined in Definition 5.2.1, is due to [Kre98, Defn. 7.12] and [Kre01, Defn. 2.1]. This uniformity condition generalizes both the Cayley-Bacharach property as well as the uniform position property and we characterize (i, j)-uniform subschemes by means of algebraic properties of the corresponding projective coordinate ring and its canonical module in Theorem 5.2.7, Corollary 5.2.9, and Theorem 5.2.14. In the final section, we turn these characterizations into algorithms that allow us to check whether a given zero-dimensional closed subscheme is (i, j)-uniform or not. More precisely, we do the following.

In the first section of this chapter, we recall the basic facts about zero-dimensional closed subschemes of weighted projective spaces. The weighted projective space $\mathbb{P}_K(\overline{W})$ with respect to a positive matrix $\overline{W} \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$ is the projective scheme which is defined by the $\mathbb{Z}$-graded K-algebra $\overline{P} = P[x_0] = K[x_0, \ldots, x_n]$ according to Definition 5.1.1. As in the previous chapter, x_0 denotes a homogenizing indetermine and the $\mathbb{Z}$-gradings of P respectively $\overline{P}$ are defined by the matrices $W = (w_1, \ldots, w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ with the property that $w_i > 0$ for all $i \in \{1, \ldots, n\}$ respectively $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$. Then in Proposition 5.1.8, we show that a zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ has no point on the hyperplane at infinity if and only if there exists a projective border basis of its defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$. This condition can often be satisfied after a generic linear change coordinates, cf. Proposition 5.1.10. Thus we get a one-to-one correspondence between the zero-dimensional subschemes $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ that have no point on the hyperplane at infinity and projective border bases of homogeneous ideals in $\overline{P}$ that can be illustrated as follows:

zero-dimensional closed subschemes $\mathbb{X}$ of weighted projective spaces $\mathbb{P}_K(\overline{W})$
that have no point on the hyperplane at infinity

$$\begin{array}{r|c|l} \text{projective border basis} & \downarrow \quad\quad\quad\quad \uparrow & \text{projective subscheme} \\ \text{of the defining ideal} & & \text{defined by the} \\ \mathcal{I}^+(\mathbb{X}) \subseteq \overline{P} & & \text{homogeneous ideal} \end{array}$$

projective $\mathcal{O}$-border bases of homogeneous ideals in $\overline{P}$
where $\overline{P}$ is graded by $\overline{W} = (1 \mid W)$ with a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$

Moreover, we additionally assume that $\mathbb{X}$ has a K-rational support, i.e. that the common zeroes of the defining ideal of $\mathbb{X}$ are K-rational by Definition 5.1.5. Note that this condition is trivially satisfied if K is algebraically closed. Under the assumptions that $\mathbb{X}$ has a K-rational support and that no point of $\mathbb{X}$ lies on the hyperplane at infinity, we show in Proposition 5.1.19 that for all subschemes $\mathbb{Y} \subseteq \mathbb{X}$ and for all $i \in \{\deg(\mathbb{Y}), \ldots, \deg(\mathbb{X})\}$, there is a subscheme $\mathbb{Y}' \subseteq \mathbb{X}$ of degree i that satisfies $\mathbb{Y} \subseteq \mathbb{Y}'$. This proposition enables us to study uniformity conditions of $\mathbb{X}$ by means of the projective coordinate ring of $\mathbb{X}$ and its canonical module in the remaining part of this chapter.

In Section 5.2, we use the properties of zero-dimensional closed subschemes $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ shown in Section 5.1 and the additional knowledge of the existence of a projective border basis of the defining ideal of $\mathbb{X}$ in order to determine whether $\mathbb{X}$ is (i,j)-uniform or not. After the definition of various uniformity conditions in Definition 5.2.1, we show that we can use (i,j)-uniformity conditions to characterize all the other uniformity conditions in Proposition 5.2.6. Then we characterize (i,j)-uniform subschemes by means of the multiplicative structure of the canonical module of the corresponding projective coordinate ring in Theorem 5.2.7. Since we assume that there is a projective border basis of the defining ideal of $\mathbb{X}$, we are then able to use the explicit description of the multiplicative structure of the corresponding canonical module as developed in Section 4.3. This additional piece of information about the multiplicative structure yields a characterization of (i,j)-uniform schemes by means of zero sets of a specific ideal in Corollary 5.2.9. In particular, if K is algebraically closed, we only have to apply several radical membership tests in order to check whether a given zero-dimensional closed subscheme is (i,j)-uniform or not. In the second subsection, we then restrict ourselves to the reduced case and show that a given reduced zero-dimensional closed subscheme is (i,j)-uniform if and only if certain sets are K-linear independent in Theorem 5.2.14.

In the final section of this chapter, we sum up various ways to check uniformity conditions and turn all the results of Section 5.2 into explicit algorithms.

As in the previous chapter, we let the polynomial ring P be $\mathbb{Z}$-graded by the matrix $W = (w_1, \ldots, w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ with $w_i > 0$ for all $i \in \{1, \ldots, n\}$. Moreover, x_0 denotes a homogenizing indeterminate and $\overline{P} = P[x_0] = K[x_0, \ldots, x_n]$ is $\mathbb{Z}$-graded by the positive matrix $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$. The algebraic closure of the base field K is denoted by $\overline{K}$.

Note that we always consider closed subschemes and closed points and thus simply say subscheme and point instead of closed subscheme and closed point for the whole thesis.

5.1 Zero-Dimensional Schemes in Weighted Projective Spaces

In the first section of this chapter, we recall some of the basic properties of weighted projective spaces and their subschemes and show that projective border bases are suitable to study these subschemes under certain conditions. For a more general introduction to weighted projective spaces, we refer to [IF00], [Dol82], and [Har92] and for a general introduction to algebraic geometry, we refer to [Kun85], [Har77], and [EH00].

First we introduce weighted projective spaces. By [Kun85, Defn. I.5.15], the homogeneous spectrum $\operatorname{Proj}(\overline{P})$ of the $\mathbb{Z}$-graded K-algebra $\overline{P} = \bigoplus_{\gamma=0}^{\infty} \overline{P}_\gamma$ denotes the set of all homogeneous prime ideals of $\overline{P}$ that do not contain the homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{\gamma=1}^{\infty} \overline{P}_\gamma = \langle x_0, \dots, x_n \rangle$.

Definition 5.1.1. Let $\mathfrak{m} = \bigoplus_{\gamma=1}^{\infty} \overline{P}_\gamma = \langle x_0, \dots, x_n \rangle$ denote the homogenous maximal ideal of the $\mathbb{Z}$-graded K-algebra $\overline{P} = \bigoplus_{\gamma=0}^{\infty} \overline{P}_\gamma$.

a) The projective scheme $\mathbb{P}_K(\overline{W}) = \operatorname{Proj}(\overline{P})$ is called the **weighted projective space of type $\overline{W}$** over K. In the standard graded case, i.e. for $\overline{W} = (1, \dots, 1)$, we also write $\mathbb{P}^n_K$ instead of $\mathbb{P}_K(\overline{W})$ and call $\mathbb{P}^n_K$ the **n-dimensional projective space** over K.

b) Let the multiplicative group $K \setminus \{0\}$ act on $\mathbb{A}^{n+1}_K = \operatorname{Spec}(\overline{P})$ via

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n).$$

Then $\mathbb{P}_K(\overline{W})$ is the quotient of $\mathbb{A}^{n+1}_K \setminus \{0\}$ by $K \setminus \{0\}$. The equivalence class of a point $(c_0, \dots, c_n) \in \mathbb{A}^{n+1}_K$ under the above action is called a **(projective) point** in $\mathbb{P}_K(\overline{W})$ and is denoted by $(c_0 : \dots : c_n) \in \mathbb{P}_K(\overline{W})$.

c) Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a subscheme, let R be the coordinate ring of $\mathbb{X}$, and let $J \subseteq R$ be a homogeneous ideal. Then we call $J^{\mathrm{sat}} = \{r \in R \mid \mathfrak{m}^i r \subseteq J \text{ for some } i \in \mathbb{N}\}$ the **saturation** of J. If $J = J^{\mathrm{sat}}$, we say that J is **saturated**.

d) Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a subscheme. The largest ideal in $\overline{P}$ that defines $\mathbb{X}$ scheme-theoretically is called the **defining ideal** of $\mathbb{X}$ and is denoted by $\mathcal{I}^+(\mathbb{X})$. In this situation, the coordinate ring of $\mathbb{X}$ is $\overline{P}/\mathcal{I}^+(\mathbb{X})$.

e) Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ and $\mathbb{Y} \subseteq \mathbb{X}$ be subschemes. The largest ideal in the coordinate ring $\overline{P}/\mathcal{I}^+(\mathbb{X})$ of $\mathbb{X}$ that defines $\mathbb{Y}$ scheme-theoretically is called the **defining ideal** of $\mathbb{Y}$ and is denoted by $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$.

Proposition 5.1.2. *Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a non-empty subscheme, we let $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$, and let $\mathfrak{m} = \bigoplus_{\gamma\in\mathbb{Z}} \overline{P}_\gamma = \langle x_0, \dots, x_n\rangle$ be the homogeneous maximal ideal of the $\mathbb{Z}$-graded K-algebra $\overline{P}$.*

a) *There is a one-to-one correspondence between the subschemes of $\mathbb{X}$ and the saturated homogeneous ideals of R. In particular, the defining ideal $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$ of a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ is saturated and two ideal $I, J \subseteq R$ define the same subscheme of $\mathbb{X}$ if and only if $I^{\mathrm{sat}} = J^{\mathrm{sat}}$.*

b) *Let $J \subseteq R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ be a homogeneous ideal and let $\varepsilon : \overline{P} \twoheadrightarrow R$ denote the canonical $\overline{P}$-algebra epimorphism. The saturation J^{sat} of J is a homogeneous ideal that contains J and we have the equality $J^{\mathrm{sat}} = \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty)$. In particular, we can compute J^{sat}.*

Proof. According to the Second Noetherian Isomorphism Theorem, the ideals of R are in one-to-one correspondence to the ideals in $\overline{P}$ that contain $\mathcal{I}^+(\mathbb{X})$. Claim a) follows from this correspondence and [Har77, Exer. II.5.10].

Next we prove claim b). By Definition 5.1.1, J^{sat} trivially contains J. According to Definition 5.1.1 and [KR00, Defn. 3.5.7],

$$\begin{aligned} J^{\mathrm{sat}} &= \{r \in R \mid \mathfrak{m}^i r \in J \text{ for some } i \in \mathbb{N}\} \\ &= \{\varepsilon(p) \mid p \in \overline{P},\, \mathfrak{m}^i p \in \varepsilon^{-1}(J) \text{ for some } i \in \mathbb{N}\} \\ &= \{\varepsilon(p) \mid p \in \varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty\} \\ &= \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty). \end{aligned}$$

Since $J \subseteq R$ is homogeneous, $J = \langle \overline{f}_1, \dots, \overline{f}_s\rangle$ where $s \in \mathbb{N}$ and $f_1, \dots, f_s \in \overline{P}$ are homogeneous by [KR00, Prop. 1.7.10]. By Lemma 2.6.7, $\varepsilon^{-1}(J) = \langle f_1, \dots, f_s\rangle + \mathcal{I}^+(\mathbb{X})$. Again, [KR00, Prop. 1.7.10] yields that $\langle f_1, \dots, f_s\rangle$ is homogeneous. Since $\mathcal{I}^+(\mathbb{X})$ is also homogeneous by Definition 5.1.1 and since $\mathfrak{m}$ is homogeneous, [KR05, Prop. 4.1.11] implies that both $\varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty$ and thus $J^{\mathrm{sat}} = \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty)$ are homogeneous, too. Moreover, we can compute $J^{\mathrm{sat}} = \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \mathfrak{m}^\infty)$ using [KR00, Thm. 3.5.13] and Lemma 2.6.7. □

Definition 5.1.3. Let $K \subseteq L$ be a field extension and $\overline{K}$ be the algebraic closure of K.

a) According to [IF00, Prop. 5.3], the map $\iota_0 : \mathrm{Spec}(P) = \mathbb{A}^n_K \to \mathbb{P}_K(\overline{W}) = \mathrm{Proj}(\overline{P})$, $(c_1, \dots, c_n) \mapsto (1 : c_1 : \dots : c_n)$ is injective. For a subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$, we call $\mathbb{X} \cap \iota_0(\mathbb{A}^n_K) \subseteq \mathbb{P}_K(\overline{W})$ the **affine part** of $\mathbb{X}$ and we identify it with its preimage $\mathbb{X}^a = \iota_0^{-1}(\mathbb{X} \cap \iota_0(\mathbb{A}^n_K)) \subseteq \mathbb{A}^n_K$.

b) Let $f \in \overline{P}$ be a homogeneous polynomial. A projective point $(c_0 : \dots : c_n) \in \mathbb{P}_L(\overline{W})$ is said to be a **zero** of f in $\mathbb{P}_L(\overline{W})$ if $f(c_0, \dots, c_n) = 0$. The set of all zeros of f in $\mathbb{P}_L(\overline{W})$ is denoted by $\mathcal{Z}^+_L(f)$. For a zero $(c_0 : \dots : c_n)$ of f in $\mathbb{P}_{\overline{K}}(\overline{W})$, we simply call $(c_0 : \dots : c_n)$ a **zero** of f and for the set of all zeros of f in $\mathbb{P}_{\overline{K}}(\overline{W})$ we also write $\mathcal{Z}^+(f)$ instead of $\mathcal{Z}^+_{\overline{K}}(f)$.

c) The set $H^{\mathrm{inf}} = \mathcal{Z}_K^+(x_0) \subseteq \mathbb{P}_K(\overline{W})$ is called the **hyperplane at infinity** of $\mathbb{P}_K(\overline{W})$.

d) For a subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$, the points on $\mathbb{X} \cap H^{\mathrm{inf}}$ are called the **points at infinity** of $\mathbb{X}$.

e) For a homogeneous ideal $I \subseteq \overline{P}$, the **projective zero-set** of I in $\mathbb{P}_L(\overline{W})$ is defined as the set

$$\mathcal{Z}_L^+(I) = \{(c_0 : \cdots : c_n) \in \mathbb{P}_L(\overline{W}) \mid f(c_0, \ldots, c_n) = 0 \text{ for all homogeneous } f \in I\}.$$

Again, we simply call the zero-set of I in $\mathbb{P}_{\overline{K}}(\overline{W})$ the **projective zero-set** of I and denote it by $\mathcal{Z}^+(I)$.

f) Let S be a set of projective points in $\mathbb{P}_L(\overline{W})$. Then the homogeneous ideal

$$\mathcal{I}^+(S) = \langle f \in \overline{P} \mid f \text{ homogeneous, } f(p) = 0 \text{ for all } p \in S\rangle$$

is called the **homogeneous vanishing ideal** of S.

Lemma 5.1.4. *Let S be a set of projective points in $\mathbb{P}_K(\overline{W})$. Then $\mathcal{I}^+(S) \subseteq \overline{P}$ is a homogeneous radical ideal.*

Proof. If $S = \emptyset$, Definition 5.1.3 yields $\mathcal{I}^+(S) = \overline{P}$ and the claim follows trivially. Thus suppose that $S \neq \emptyset$. By Definition 5.1.3, the homogeneous vanishing ideal $\mathcal{I}^+(S)$ is an ideal in $\overline{P}$ and it is generated by homogeneous polynomials. Thus $\mathcal{I}^+(S)$ is also homogeneous according to [KR00, Prop. 1.7.10]. Moreover, $\sqrt{\mathcal{I}^+(S)} \subseteq \overline{P}$ is a homogeneous ideal by [KR05, Prop. 4.1.11]. Let $f \in \sqrt{\mathcal{I}^+(S)}$ and let $f = \sum_{\gamma=0}^{\infty} f_\gamma$ be the decomposition of f into its homogeneous components. Then $f_\gamma \in \sqrt{\mathcal{I}^+(S)}$ for all $\gamma \in \mathbb{N}$ by [KR00, Prop. 1.7.10]. Let $\gamma \in \mathbb{N}$. Then there exists a $k \in \mathbb{N}$ such that $f_\gamma^k \in \mathcal{I}^+(S)$. Let $(c_0 : \cdots : c_n) \in S$. Since K is a field and since $0 = f_\gamma^k(c_0, \ldots, c_n) = (f_\gamma(c_0, \ldots, c_n))^k$ by Definition 5.1.3, it follows $f_\gamma(c_0, \ldots, c_n) = 0$. Thus $f_\gamma \in \mathcal{I}^+(S)$. In particular, we see that $f = \sum_{\gamma=0}^{\infty} f_\gamma \in \mathcal{I}^+(S)$ and $\mathcal{I}^+(S) = \sqrt{\mathcal{I}^+(S)}$ is a homogeneous radical ideal. □

Next we give a necessary and sufficient condition for the defining ideal of a subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ to possess a projective border basis. Later we study such subschemes and explicitly use the additional structure given by the projective border bases, e.g. to characterize (i,j)-uniform subschemes in Corollary 5.2.9.

Definition 5.1.5. a) A projective point $(c_0 : \cdots : c_n) \in \mathbb{P}_{\overline{K}}(\overline{W})$ with $c_0, \ldots, c_n \in K$ is called **K-rational**

b) A subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is said to have a **K-rational support** if all points of the zero set $\mathcal{Z}^+(\mathcal{I}^+(\mathbb{X})) \subseteq \mathbb{P}_{\overline{K}}(\overline{W})$ are K-rational.

Proposition 5.1.6. *Let $p = (1 : c_1 : \cdots : c_n) \in \mathbb{P}_K(\overline{W})$ be a K-rational point. Then we have $\mathcal{I}^+(\{p\}) = \langle x_1 - c_1 x_0^{w_1}, \ldots, x_n - c_n x_0^{w_n}\rangle$ and $\mathcal{Z}^+(\mathcal{I}^+(\{p\})) = \{p\}$.*

Proof. Since $\mathcal{I}^+(\{p\}) \supseteq \langle x_1 - c_1 x_0^{w_1}, \dots, x_n - c_n x_0^{w_n} \rangle$ obviously holds by Definition 5.1.3, it suffices to prove the converse inclusion. Let $f \in \mathcal{I}^+(\{p\}) \setminus \{0\}$. Then Definition 5.1.3 yields $0 = f(1, c_1, \dots, c_n) = f^{\mathrm{deh}}(c_1, \dots, c_n)$. Thus [KR05, Exmp. 6.3.2] implies that $f^{\mathrm{deh}} \in \mathcal{I}(\{(c_1, \dots, c_n)\}) = \langle x_1 - c_1, \dots, x_n - c_n \rangle$ Moreover, [KR05, Prop. 4.3.2] yields that $f = x_0^s (f^{\mathrm{deh}})^{\mathrm{hom}} \in \mathcal{I}(\{(c_1, \dots, c_n)\})^{\mathrm{hom}}$ for some $s \in \mathbb{N}$. Let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$. Then $\{x_1 - c_1, \dots, x_n - c_n\}$ is a σ-Gröbner basis of $\mathcal{I}(\{(c_1, \dots, c_n)\})$ according to [KR00, Coro. 2.5.10]. Now [KR05, Prop. 4.2.15] shows that $\{x_1 - c_1, \dots, x_n - c_n\}$ is also a Macaulay basis of $\mathcal{I}(\{(c_1, \dots, c_n)\})$ with respect to W and thus $f \in \mathcal{I}(\{(c_1, \dots, c_n)\})^{\mathrm{hom}} = \langle x_1 - c_1 x_0^{w_1}, \dots, x_n - c_n x_0^{w_n} \rangle$ by [KR05, Thm. 4.3.19]. The other claim is a direct consequence of this and Definition 5.1.3. □

If not mentioned otherwise, we let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a non-empty, zero-dimensional subscheme that has a K-rational support, let $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ be the coordinate ring of $\mathbb{X}$, and let $\mathfrak{m} = \bigoplus_{\gamma=1}^{\infty} \overline{P}_\gamma = \langle x_0, \dots, x_n \rangle$ denote the homogeneous maximal ideal of the $\mathbb{Z}$-graded K-algebra $\overline{P} = \bigoplus_{\gamma=0}^{\infty} \overline{P}_\gamma$ for the remainder of this section.

Proposition 5.1.7. *Let $\mathcal{I}^+(\mathbb{X}) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ with $s \in \mathbb{N} \setminus \{0\}$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s \subseteq \overline{P}$ be the reduced homogeneous primary decomposition of $\mathcal{I}^+(\mathbb{X})$ as in [KR05, Prop. 5.6.21]. For every $i \in \{1, \dots, s\}$, let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Then the support of $\mathbb{X}$ is $\mathrm{Supp}(\mathbb{X}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ and for every $i \in \{1, \dots, s\}$, there is a K-rational point $p_i \in \mathbb{P}_K(\overline{W})$ such that $\mathfrak{p}_i = \mathcal{I}^+(\{p_i\})$.*

Proof. Let $i \in \{1, \dots, s\}$. The radical ideal $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ of the homogeneous primary ideal $\mathfrak{q}_i \subseteq \overline{P}$ is a homogeneous prime ideal according to [KR05, Prop. 4.1.11] and [Kun85, Rem. V.4.3]. Since $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is a non-empty zero-dimensional subscheme, we get $\mathcal{I}^+(\mathbb{X}) \subset \mathfrak{m}$. As the primary decomposition is reduced, we get $\mathfrak{p}_i \subset \mathfrak{m}$. Altogether, we have the following chain of homogeneous ideals $\mathcal{I}^+(\mathbb{X}) \subseteq \mathfrak{p}_i \subset \mathfrak{m}$. Consider the corresponding vanishing ideals in the affine space $\mathbb{A}_{\overline{K}}^{n+1} = \mathrm{Spec}(\overline{K}[x_0, \dots, x_n])$. Since both $\mathfrak{p}_i$ and $\mathfrak{m}$ are prime ideals, they are also radical ideals according to [Kun85, Defn. I.1.6]. As $\mathfrak{p}_i \subset \mathfrak{m}$, the Ideal-Variety Correspondence [CLO07, Thm. 4.2.7] yields the chain $\mathbb{A}_{\overline{K}}^{n+1} \supseteq \mathcal{Z}(\mathcal{I}^+(\mathbb{X})) \supseteq \mathcal{Z}(\mathfrak{p}_i) \supset \mathcal{Z}(\mathfrak{m}) = \{0\}$. Thus there is an affine point $(c_0, \dots, c_n) \in \mathcal{Z}(\mathfrak{p}_i) \setminus \{0\} \subseteq \mathcal{Z}(\mathcal{I}^+(\mathbb{X}))$. In particular, the corresponding projective point $(c_0 : \dots : c_n) \in \mathbb{P}_{\overline{K}}(\overline{W})$ is also a zero of the homogeneous ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \mathfrak{p}_i$. Since $\mathbb{X}$ has a K-rational support, it follows $(c_0 : \dots : c_n) \in \mathbb{P}_K(\overline{W})$ from Definition 5.1.5. Let $\mathfrak{p} = \mathcal{I}^+(\{c_0 : \dots : c_n\})$. Then $\mathfrak{p}_i \subseteq \mathfrak{p}$ by Definition 5.1.3. Let $f, g \in \overline{P}$ be homogeneous polynomials with $fg \in \mathfrak{p}$. Then we have $f(c_0, \dots, c_n) g(c_0, \dots, c_n) = (fg)(c_0, \dots, c_n) = 0$ by Definition 5.1.3. As K is a field, we thus see that $f(c_0, \dots, c_n) = 0$ or $g(c_0, \dots, c_n) = 0$, i.e. $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$ by Definition 5.1.3. Now [KR00, Prop. 1.7.12] implies that $\mathfrak{p}$ is a homogeneous prime ideal. In particular, we get the chain of homogeneous ideals $\mathcal{I}^+(\mathbb{X}) \subseteq \mathfrak{p}_i \subseteq \mathfrak{p} \subset \mathfrak{m}$. As $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is zero-dimensional and as $\mathfrak{p}_i$ and $\mathfrak{p}$ are both homogeneous prime ideals, it follows that $\mathfrak{p}_i = \mathfrak{p}$. Thus we see that $\mathrm{Supp}(\mathbb{X}) \supseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ and that for all $i \in \{1, \dots, s\}$, $\mathfrak{p}_i = \mathcal{I}^+(\{p_i\})$ for some point $p_i \in \mathbb{P}_K(\overline{W})$.

For the converse inclusion, let $\mathfrak{p} \subset \mathfrak{m}$ be a homogeneous ideal with the property that $\mathcal{I}^+(\mathbb{X}) \subseteq \mathfrak{p}$. As above, there is a zero $p \in \mathbb{P}_K(\overline{W})$ of $\mathfrak{p}$ and since the subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is zero-dimensional, it follows $\mathfrak{p} = \mathcal{I}^+(\{p\})$. Thus $\mathcal{I}^+(\mathbb{X}) \subseteq \mathfrak{p} = \mathcal{I}^+(\{p\})$,

$\mathcal{Z}^+(\mathfrak{p}_i) = \mathcal{Z}^+(\mathfrak{q}_i)$ for all $i \in \{1,\dots,s\}$, together with Definition 5.1.3 and Proposition 5.1.6 yield

$$\begin{aligned} p \in \mathcal{Z}^+(\mathcal{I}^+(\{p\})) \\ &\subseteq \mathcal{Z}^+(\mathcal{I}^+(\mathbb{X})) \\ &= \mathcal{Z}^+(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s) \\ &= \mathcal{Z}^+(\mathfrak{p}_1) \cup \cdots \cup \mathcal{Z}^+(\mathfrak{p}_s) \\ &= \mathcal{Z}^+(\mathcal{I}^+(\{p_1\})) \cup \cdots \cup \mathcal{Z}^+(\mathcal{I}^+(\{p_s\})) \\ &= \{p_1,\dots,p_s\} \end{aligned}$$

Thus there is an index $i \in \{1,\dots,s\}$ such that $p = p_i$ and we get $\mathfrak{p} = \mathfrak{p}_i$. Altogether, we see that $\operatorname{Supp}(\mathbb{X}) \subseteq \{\mathfrak{p}_1,\dots,\mathfrak{p}_s\}$ and the claim follows. □

The following proposition generalizes [KR05, Prop. 6.3.21.e] to our setting.

Proposition 5.1.8. *The following conditions are equivalent.*

i) No point of $\mathbb{X}$ lies on the hyperplane at infinity of $\mathbb{P}_K(\overline{W})$, i.e. $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$.

ii) The element $x_0 \in \overline{P}$ is a non-zero divisor for $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$.

iii) There isx a projective border basis of the homogeneous vanishing ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$.

Proof. Let $\mathcal{I}^+(\mathbb{X}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be the reduced homogeneous primary composition of $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ as in [KR05, Prop. 5.6.21] and for all $i \in \{1,\dots,s\}$, let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. As we have seen in Proposition 5.1.7, we have $\mathfrak{p}_i = \mathcal{I}^+(\{p_i\})$ where $p_i \in \mathbb{P}_K(\overline{W})$ for all $i \in \{1,\dots,s\}$ and $\operatorname{Supp}(\mathbb{X}) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_s\}$.

Now we prove that i) implies ii). Since we assume that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, it follows that $x_0 \notin \mathcal{I}^+(\{p_1\}) \cup \cdots \cup \mathcal{I}^+(\{p_s\}) = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s = \sqrt{\mathfrak{q}_1} \cup \cdots \cup \sqrt{\mathfrak{q}_s}$. Thus x_0 is a non-zero divisor for $\overline{P}/(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s) = R$ according to [KR05, Prop. 5.6.17.c].

Next we prove that ii) implies i). By the Second Noetherian Isomorphism Theorem, we have $R/\langle \overline{x}_0 \rangle \cong \overline{P}/(\mathcal{I}^+(\mathbb{X}) + \langle x_0 \rangle)$ and the prime ideals of $R/\langle \overline{x}_0 \rangle$ are in one-to-one correspondence to the prime ideals of R that contain $\overline{x}_0$. Assume that $x_0 \in \mathfrak{p}_i$ for some $i \in \{1,\dots,s\}$. Let $\overline{\mathfrak{p}}_i$ and $\overline{\mathfrak{m}}$ denote the images of the prime ideals $\mathfrak{p}_i$ and $\mathfrak{m}$ in the ring R, respectively. Then we get the chain of prime ideals $\langle \overline{x}_0 \rangle \subseteq \overline{\mathfrak{p}}_i \subset \overline{\mathfrak{m}} \subset R$. Thus the Krull dimension of $R/\langle \overline{x}_0 \rangle$ is greater than or equal to 1. According to [KR05, Thm. 5.6.36], this implies $\dim(R/\langle \overline{x}_0 \rangle) \geq 1$. Since x_0 is a non-zero divisor for R, it follows from [KR05, Prop. 5.6.33] that $\dim(R/\langle \overline{x}_0 \rangle) = \dim(\overline{P}/(\mathcal{I}^+(\mathbb{X}) + \langle \overline{x}_0 \rangle)) \leq \dim(R) - 1 = 0$. This is clearly a contradiction and we get $x_0 \notin \mathfrak{p}_i = \mathcal{I}^+(\{p_i\})$ for all $i \in \{1,\dots,s\}$ and i) follows.

The equivalence of ii) and iii) is a direct consequence of Proposition 4.1.7 and Corollary 4.1.14 □

Next we show that we can often find a suitable linear change of coordinates such that no point of $\mathbb{X}$ is on the hyperplane at infinity in this new coordinate system.

Definition 5.1.9. A homogeneous isomorphism $\overline{P} \to \overline{P}$ is also called a **homogeneous linear change of coordinates**.

The following proposition follows the ideas of [MR93, beginning of Section 2.1] respectively [KR05, Lemma 6.3.20 and Prop. 5.5.23.a] and generalizes the result to our setting.

Proposition 5.1.10. *Assume that K is an infinite field and $\mathbb{X} \cap \mathcal{Z}^+(\langle \mathbb{T}_1^{n+1} \rangle) = \emptyset$. Then there is a generic homogeneous linear change of coordinates such that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$.*

Proof. According to Proposition 5.1.7, there are K-rational points $p_1, \ldots, p_s \in \mathbb{P}_K(\overline{W})$ where $s \in \mathbb{N} \setminus \{0\}$ such that $\operatorname{Supp}(\mathbb{X}) = \{\mathcal{I}^+(\{p_1\}), \ldots, \mathcal{I}^+(\{p_s\})\}$. For all $i \in \{1, \ldots, s\}$, we write $p_i = (c_{i0} : \cdots : c_{in})$ with $c_{i0}, \ldots, c_{in} \in K$. Without loss of generality, suppose that the indeterminates $x_1, \ldots, x_n$ are ordered increasingly with respect to $\deg_W$. Then $\mathbb{T}_1^{n+1} = \{x_0, \ldots, x_j\}$ for some $j \in \{0, \ldots, n\}$. Let $f = \prod_{i=1}^{s}(c_{i0}x_0 + c_{i1}x_1 + \cdots + c_{ij}x_j)$. Since we assume that $\mathbb{X} \cap \mathcal{Z}^+(\langle x_0, \ldots, x_j \rangle) = \emptyset$, it follows $f \neq 0$. Let $\mathbb{A}_K^{n+1} = \operatorname{Spec}(\overline{P})$. The set $U = \mathbb{A}_K^{n+1} \setminus \mathcal{Z}_K(x_0 f)$ is open with respect to the Zariski topology and it is not empty by [KR05, Prop. 5.5.21]. Let $a = (a_0, \ldots, a_j) \in U$. Then we have $a_0 \neq 0$. Let $\ell_a = \frac{1}{a_0}(x_0 - a_1x_1 - \cdots - a_jx_j) \in \overline{P}_1$ and let $\Phi_a : \overline{P} \to \overline{P}$, $x_0 \mapsto \ell_a$, $x_i \mapsto x_i$ for all $i \in \{1, \ldots, n\}$ be the K-algebra homomorphism induced by the Universal Property of the Polynomial Ring $\overline{P}$.

We show that Φ_a is a homogeneous linear change of coordinates. As $\Phi_a(x_0) = \ell_a \in \overline{P}_1$, it follows that Φ_a is a homogeneous K-algebra homomorphism. Moreover, we have $x_0 = a_0\ell_a + a_1x_1 + \cdots a_jx_j = \Phi_a(a_0x_0 + \cdots + a_jx_j) \in \operatorname{im}(\Phi_a)$ and $x_i = \Phi_a(x_i) \in \operatorname{im}(\Phi_a)$ for all $i \in \{1, \ldots, n\}$, i.e. Φ_a is surjective. Let $p \in \ker(\Phi_a)$. For every $q \in \overline{P} \setminus \{0\}$, the x_0-degree of $\Phi_a(q)$ is exactly the x_0-degree of q. Hence we get $q \in P$. Since $\Phi_a|_P = \mathrm{id}$, it follows that $0 = \Phi_a(q) = q$. Therefore, $\ker(\Phi_a) = \{0\}$ and Φ_a is injective. Altogether, Φ_a is a homogeneous linear of change of coordinates by Definition 5.1.9.

Finally, we show that after the homogeneous linear change of coordinates Φ_a, no point of the support $\operatorname{Supp}(\mathbb{X})$ lies on the hyperplane at infinity H^{inf}. Therefore, let $i \in \{1, \ldots, s\}$. After applying Φ_a, the point $p_i = (c_{i0} : \cdots : c_{in}) \in \mathbb{X}$ has the new coordinates $(\Phi_a^{-1}(x_0)(p_i), \ldots, \Phi_a^{-1}(x_n)(p_i)) = (a_0c_{i0} + \cdots + a_jc_{ij} : c_{i1} : \cdots : c_{in})$. As $a \in U$, we have $a_0c_{i0} + \cdots + a_jc_{ij} \neq 0$, i.e. $p_i \notin H^{\mathrm{inf}}$.

Altogether, we have proven that every point of U defines a homogeneous linear change of coordinates with the property that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. By [KR05, Defn. 5.5.19], this means that there exists a generic homogeneous linear change of coordinates. □

The assumptions of the above proposition are necessary as the following example shows.

Example 5.1.11. a) Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{F}_2}$ be the reduced zero-dimensional subscheme concentrated at the points of $\{(1:0:0), (0:1:0), (0:0:1), (0:1:1), (1:1:1)\}$. Then there does obviously not exist a homogeneous linear change of coordinates such that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. The reason for this is that the projective space $\mathbb{P}^2_{\mathbb{F}_2}$ does not contain enough points.

b) Let $\overline{P} = \mathbb{Q}[x_0, x_1, x_2]$ be $\mathbb{Z}$-graded by the matrix $\overline{W} = (1, 1, 2) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$ and we let $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}(\overline{W})$ be the reduced zero-dimensional subscheme concentrated at the point $(0:0:1)$. Let $\Phi : \overline{P} \to \overline{P}$ be a homogeneous linear change of coordinates. In the new coordinate system, the projective point $(0:0:1)$ then has the coordinates $(\Phi^{-1}(x_0)(0,0,1) : \Phi^{-1}(x_1)(0,0,1) : \Phi^{-1}(x_2)(0,0,1))$. As Φ is homogeneous according to Definition 5.1.9 and as $x_0 \in \overline{P}_1$, we have $\Phi^{-1}(x_0) \in \overline{P}_1$. Since the set $\mathbb{T}_1^3 = \{x_0, x_1\}$ is a $\mathbb{Q}$-vector space basis of $\overline{P}_1$, we see that there exist $a_0, a_1 \in \mathbb{Q}$ such that $\Phi^{-1}(x_0) = a_0 x_0 + a_1 x_1$. In particular, in the new coordinate system, the point $(0:0:1)$ has the x_0-coordinate $\Phi^{-1}(x_0)(0,0,1) = (a_0 x_0 + a_1 x_1)(0,0,1) = 0$. Thus there is no homogeneous linear change of coordinates such that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$ in the new coordinates.

Remark 5.1.12. Assume that K is infinite.

a) Assume that $\mathbb{X} \cap \mathcal{Z}^+(\langle \mathbb{T}_1^{n+1} \rangle) = \emptyset$. Then Proposition 5.1.10 implies the existence of a generic linear change of coordinates such that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. Thus there is a high chance that a random choice of a linear change of coordinates yields $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. For practical purposes, repeatedly choosing random linear changes of coordinates until $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$ holds is enough.

b) Assume that $\overline{P}$ is standard graded, i.e. graded by $\overline{W} = (1, \dots, 1) \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$. Then the condition $\mathbb{X} \cap \mathcal{Z}^+(\langle \mathbb{T}_1^{n+1} \rangle) = \mathbb{X} \cap \mathcal{Z}^+(\langle x_0, \dots, x_n \rangle) = \emptyset$ is trivially holds. As described in a), one can guess a suitable linear change of coordinates such that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$ holds. Moreover, there are also deterministic methods to choose such a linear change of coordinates. An explicit description of such a deterministic method, which is based on a result in [GH91, Subsection 2.3.3], can be found in [MR93, Section 2.1].

The final part of this section is devoted to prove a version of [Kre94, Lemma 2.2] in our setting. This lemma states that for a given subscheme $\mathbb{Y} \subseteq \mathbb{X}$, we can find specific subschemes $\mathbb{Y}' \subseteq \mathbb{X}$ with $\mathbb{Y} \subseteq \mathbb{Y}'$.

The following lemma generalizes [Kre94, Lemma 1.2] and gives an easier description of the saturation of a homogeneous ideal in our setting.

Lemma 5.1.13. *Let $J \subseteq R$ be a homogeneous ideal and let $\varepsilon : \overline{P} \twoheadrightarrow R$ denote the canonical $\overline{P}$-algebra epimorphism. Assume that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. Then the saturation J^{sat} is given by $J^{\mathrm{sat}} = \{r' \in R \mid x_0^i r' \in J$ for some $i \in \mathbb{N}\}$ and $J^{\mathrm{sat}} = \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \langle x_0 \rangle^\infty)$.*

Proof. As $x_0 \in \mathfrak{m}$, the inclusion $J^{\mathrm{sat}} \subseteq \{r \in R \mid x_0^i r \in J \text{ for some } i \in \mathbb{N}\}$ trivially holds according to Definition 5.1.1. For the converse inclusion, let $r \in R$ and $i \in \mathbb{N}$ be such that $x_0^i r \in J$. Since $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, there is a projective $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})$ by Proposition 5.1.8. Since $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is a non-empty subscheme, $\mathcal{I}^+(\mathbb{X}) \neq \overline{P}$ and hence $\mathcal{O} \neq \emptyset$. Let $\sigma_{\mathbb{X}} = \max(\{\deg_W(t) \mid t \in \mathcal{O}\}) - 1$. Then Proposition 4.1.15 implies $R_{\sigma_{\mathbb{X}}+1+m} = x_0^m R_{\sigma_{\mathbb{X}}+1}$ for all $m \in \mathbb{N}$. In particular, $\mathfrak{m}^{\sigma_{\mathbb{X}}+1+i} r = \mathfrak{m}^{\sigma_{\mathbb{X}}+1} x_0^i r \subseteq J$, i.e.

$r \in J^{\mathrm{sat}}$ according to Definition 5.1.1. In particular,

$$\begin{aligned} J^{\mathrm{sat}} &= \{r \in R \mid x_0^i r \in J \text{ for some } i \in \mathbb{N}\} \\ &= \{\varepsilon(p) \mid p \in \overline{P},\, x_0^i p \in \varepsilon^{-1}(J) \text{ for some } i \in \mathbb{N}\} \\ &= \{\varepsilon(p) \mid p \in \varepsilon^{-1}(J) :_{\overline{P}} \langle x_0\rangle^\infty\} \\ &= \varepsilon(\varepsilon^{-1}(J) :_{\overline{P}} \langle x_0\rangle^\infty). \end{aligned}$$

according to Definition 5.1.1 and [KR00, Defn. 3.5.7]. □

Next we consider the local structure of the subscheme $\mathbb{X}$ at the points of its support. The following two lemmata generalize [CLO05, Thm. 4.§2.2.2] to arbitrary fields.

Lemma 5.1.14. *Let $\mathbb{A}^n_K = \mathrm{Spec}(P)$ and let $I \subset P$ be a zero-dimensional ideal such that $\mathcal{Z}(I) = \{p_1, \dots, p_s\} \subseteq \mathbb{A}^n_K$, i.e. such that all zeros of I are K-rational. For all $i \in \{1, \dots, s\}$, we let $\mathfrak{m}_i = \mathcal{I}(\{p_i\}) \subseteq P$ be the maximal ideal corresponding to p_i.*

a) There exists an exponent $d \in \mathbb{N} \setminus \{0\}$ such that $(\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s)^d \subseteq I$.

b) There exist $e_1, \dots, e_s \in P$ such that the following properties hold.

1) We have $e_i \notin \mathfrak{m}_i$ for all $i \in \{1, \dots, s\}$.

2) We have $\overline{e}_1 + \dots + \overline{e}_s = 1$ in P/I.

3) We have $\overline{e}_i^2 = \overline{e}_i$ for all $i \in \{1, \dots, s\}$ in P/I.

4) We have $\overline{e}_i\overline{e}_j = 0$ for all $i, j \in \{1, \dots, s\}$ with $i \neq j$ in P/I.

5) For all $i \in \{1, \dots, s\}$ and for all $f_i \in P \setminus \mathfrak{m}_i$, there exists a polynomial $h_i \in P$ such that $\overline{f}_i\overline{h}_i = \overline{e}_i$ in P/I.

Proof. First we prove a). Let $\mathfrak{M} = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$ and $q_1, \dots, q_r \in P$ with $r \in \mathbb{N} \setminus \{0\}$ be such that $\mathfrak{M} = \langle q_1, \dots, q_r\rangle$ and let $Q = \overline{K}[x_1, \dots, x_n]$. Since the points in $\mathcal{Z}(I) = \{p_1, \dots, p_s\}$ are K-rational, $\mathfrak{m}_1 Q, \dots, \mathfrak{m}_s Q$ are precisely the maximal ideals in Q containing IQ by [KR00, Prop. 2.6.11]. For all $i \in \{1, \dots, r\}$, we have $q_i \in \mathfrak{M}$ and thus [KR00, Coro. 2.6.17] implies that $q_i \in \sqrt{I}$. In particular, for every $i \in \{1, \dots, r\}$, there exists an exponent $d_i \in \mathbb{N} \setminus \{0\}$ such that $q_i^{d_i} \in I$. Let $d = r \cdot \max\{d_1, \dots, d_r\} \in \mathbb{N} \setminus \{0\}$. According to [KR00, Rem. 3.2.1], we have $\mathfrak{M}^d = \langle q_{j_1} \cdots q_{j_d} \mid j_1, \dots, j_d \in \{1, \dots, r\}\rangle$ and thus we get $\mathfrak{M}^d \subseteq I$.

Next we prove b). For all $i \in \{1, \dots, s\}$, [KR05, Prop. 6.3.6] yields the existence of a separator $g_i \in P$ of p_i from the affine point set $\{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_s\}$, i.e. an element with the property that $g_i(p_j) = \delta_{ij}$ for all $j \in \{1, \dots, s\}$. In particular, $g_i \in \mathfrak{m}_j$ and $g_i - 1 \in \mathfrak{m}_i$ for all $i, j \in \{1, \dots, s\}$ with $i \neq j$. For every index $i \in \{1, \dots, s\}$, we define $e_i = 1 - (1 - g_i^d)^d = -\sum_{k=1}^d \binom{d}{k}(-g_i^d)^k$. Then for all $i \in \{1, \dots, s\}$, we have $e_i(p_i) = 1$ and thus $e_i \notin \mathfrak{m}_i$. Moreover, let $i, j \in \{1, \dots, s\}$ be with $i \neq j$. Then $g_i^d \in \mathfrak{m}_j^d$ and thus we see that $e_i \in \mathfrak{m}_j^d$. Furthermore, it follows that $e_i - 1 = -(1 - g_i^d)^d \in \mathfrak{m}_i^d$ and hence we get $e_1 + \dots + e_s - 1 = e_1 + \dots + e_{i-1} + (e_i - 1) + e_{i+1} + \dots + e_s \in \mathfrak{m}_1^d \cap \dots \cap \mathfrak{m}_s^d$. Since the maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s \subseteq P$ are radical ideals by [Kun85, Defn. I.1.6] and pairwise comaximal, [AM69, Prop. 1.16 and 1.10] yield that $\mathfrak{m}_1^d \cap \cdots \cap \mathfrak{m}_s^d = \mathfrak{m}_1^d \cdots \mathfrak{m}_s^d = \mathfrak{M}^d \subseteq I$. Thus $\overline{e}_1 + \cdots + \overline{e}_s = 1$ in P/I. Moreover, we see that $e_i e_j \in \mathfrak{m}_1^d \cap \cdots \cap \mathfrak{m}_s^d \subseteq I$ and hence $\overline{e}_i \overline{e}_j = 0$ in P/I. Combining these two results, we get $\overline{e}_i^2 = \overline{e}_i \overline{e}_1 + \cdots + \overline{e}_i \overline{e}_s = \overline{e}_i \cdot (\overline{e}_1 + \cdots + \overline{e}_s) = \overline{e}_i$ in P/I. For the proof of the last condition, let $f_i \in P \setminus \mathfrak{m}_i$. Then $c = f_i(p_i) \in K \setminus \{0\}$. Let $f_i' = \frac{1}{c} f_i \in P$. Then we have $1 - f_i' \in \mathfrak{m}_i$. We define $h_i' = (1 + (1-f_i') + \cdots + (1-f_i')^{d-1}) e_i$. Then $f_i' h_i' = (1 - (1 - f_i'))h_i' = (1 - (1 - f_i')^d) e_i = e_i - (1 - f_i')^d e_i$. We have already seen above that $e_i \in \mathfrak{m}_j^d$ for all $j \in \{1, \dots, s\} \setminus \{i\}$. Moreover, $(1 - f_i')^d \in \mathfrak{m}_i^d$. Hence we get $f_i' h_i' - e_i \in \mathfrak{m}_1^d \cap \cdots \cap \mathfrak{m}_s^d = \mathfrak{M}^d \subseteq I$. Therefore, if we define $h_i = c h_i'$, it follows that $\overline{f}_i \overline{h}_i = \frac{1}{c} \overline{f}_i' c \overline{h}_i = \overline{f}_i' \overline{h}_i' = \overline{e}_i$ in P/I. □

Lemma 5.1.15. *Let $\mathbb{A}^n_K = \operatorname{Spec}(P)$ and let $I \subset P$ be a zero-dimensional ideal such that $\mathcal{Z}(I) = \{p_1, \dots, p_s\} \subseteq \mathbb{A}^n_K$, i.e. such that all zeros of I are K-rational. For all $i \in \{1, \dots, s\}$, we let $\mathfrak{m}_i = \mathcal{I}(\{p_i\}) \subseteq P$ be the maximal ideal corresponding to p_i and we let $\mathcal{O}_i = P_{\mathfrak{m}_i}$ be the localization of P at $\mathfrak{m}_i$. Then there exists a canonical K-algebra isomorphism $P/I \cong \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$.*

Proof. For every index $i \in \{1, \dots, s\}$, there is a canonical K-algebra homomorphisms $\varphi_i : P \hookrightarrow \mathcal{O}_i \twoheadrightarrow \mathcal{O}_i/I\mathcal{O}_i$. According to the Universal Property of the Direct Product $\mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$, these canonical K-algebra homomorphisms induce the canonical K-algebra homomorphism $\varphi = \varphi_1 \times \cdots \times \varphi_s : P \to \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$. For all $i \in \{1, \dots, s\}$ and for all $q \in P$, we denote the coset of q in $\mathcal{O}_i/I\mathcal{O}_i$ by $[q]_i$.

First we prove that $\ker(\varphi) = I$. For every $q \in I$ and all $i \in \{1, \dots, s\}$, we have $\varphi_i(q) = [q]_i = [0]_i$ and thus $I \subseteq \ker(\varphi)$. For the converse inclusion, let $f \in \ker(\varphi)$. Then for all $i \in \{1, \dots, s\}$, we have $f \in I\mathcal{O}_i$, i.e. there exists an $f_i \in P \setminus \mathfrak{m}_i$ such that $f f_i \in I$. For all $i \in \{1, \dots, s\}$, let $e_i, h_i \in P$ be as in Lemma 5.1.14, i.e. such that $\overline{f}_i \overline{h}_i = \overline{e}_i$ in P/I. Then we have $f(f_1 h_1 + \cdots + f_s h_s) = (f f_1) h_1 + \cdots + (f f_s) h_s \in I$ and thus it follows that $0 = \overline{f}(\overline{f}_1 \overline{h}_1 + \cdots + \overline{f}_s \overline{h}_s) = \overline{f}(\overline{e}_1 + \cdots + \overline{e}_s) = \overline{f}$ in P/I. Altogether, we get $f \in I$ and hence $I = \ker(\varphi)$.

According to the Homomorphism Theorem, the claim follows if φ is surjective. Let $r \in \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$. For all $i \in \{1, \dots, s\}$, there are $r_i \in P$ and $q_i \in P \setminus \mathfrak{m}_i$ such that $r = ([\frac{r_1}{q_1}]_1, \dots, [\frac{r_s}{q_s}]_s)$. By Lemma 5.1.14.b, there exists a $u_i \in P$ such that $\overline{q}_i \overline{u}_i = \overline{e}_i$ in P/I for all $i \in \{1, \dots, s\}$. Let $F = \sum_{i=1}^s r_i u_i e_i \in P$ and $j \in \{1, \dots, s\}$. According to Lemma 5.1.14, we have $e_j \notin \mathfrak{m}_j$ and thus $\frac{1}{e_j} \in \mathcal{O}_j$. In particular, $e_j - 1 = \frac{e_j^2 - e_j}{e_j} \in I\mathcal{O}_j$ and $e_i = \frac{e_i e_j}{e_j} \in I\mathcal{O}_j$ for all $i \in \{1, \dots, s\} \setminus \{j\}$ by Lemma 5.1.14. Altogether, it follows that $\varphi_j(F) = [\sum_{i=1}^s r_i u_i e_i]_j = [\sum_{i=1}^s \frac{r_i}{q_i} e_i^2]_j = [\sum_{i=1}^s \frac{r_i}{q_i} e_i]_j = [\frac{r_j}{q_j}]_j$, i.e. $\varphi(F) = r$. Thus φ is surjective and the claim follows. □

Next we generalize [Kre94, Lemma 1.1] to our setting.

Lemma 5.1.16. *Assume that $\mathbb{X} \cap H^{\inf} = \emptyset$. Then $\overline{\operatorname{deh}} : R \twoheadrightarrow P/\mathcal{I}^+(\mathbb{X})^{\operatorname{deh}}$, $\overline{p} \mapsto \overline{p^{\operatorname{deh}}}$ for all $p \in \overline{P}$ is a K-algebra epimorphism with $\ker(\overline{\operatorname{deh}}) = \langle \overline{x}_0 - 1 \rangle \subseteq R$. In particular, if we denote $\sigma_{\mathbb{X}} = \max\{\gamma \in \mathbb{Z} \mid \Delta \operatorname{HF}_{\mathbb{X}}(\gamma) \neq 0\} - 1$, the restriction $\overline{\operatorname{deh}}|_{R_{\sigma_{\mathbb{X}}+1}}$ of $\overline{\operatorname{deh}}$ to $R_{\sigma_{\mathbb{X}}+1}$ is a K-vector space isomorphism.*

Proof. Consider the K-algebra epimorphism $\varphi : \overline{P} \xrightarrow{\text{deh}} P \twoheadrightarrow P/\mathcal{I}^+(\mathbb{X})^{\text{deh}}$ induced by dehomogenization and the canonical K-algebra epimorphism. By Lemma 2.6.7, we have $\ker(\varphi) = \text{deh}^{-1}(\mathcal{I}^+(\mathbb{X})^{\text{deh}}) = \mathcal{I}^+(\mathbb{X}) + \ker(\text{deh}) = \mathcal{I}^+(\mathbb{X}) + \langle x_0 - 1\rangle$. Thus we see that $\mathcal{I}^+(\mathbb{X}) \subseteq \ker(\varphi)$ and the Universal Property of the Residue Class K-Algebra $P/\mathcal{I}^+(\mathbb{X})^{\text{deh}}$ induces the K-algebra epimorphism $\overline{\text{deh}} : \overline{P}/\mathcal{I}^+(\mathbb{X}) = R \twoheadrightarrow P/\mathcal{I}^+(\mathbb{X})^{\text{deh}}$ defined by $\overline{p} \mapsto \varphi(p) = \overline{p^{\text{deh}}}$ for all $p \in \overline{P}$. In particular, $\ker(\overline{\text{deh}}) = \ker(\varphi)/\mathcal{I}^+(\mathbb{X}) = \langle \overline{x}_0 - 1\rangle \subseteq R$. As $\mathbb{X} \cap H^{\text{inf}} = \emptyset$, Proposition 5.1.8 implies the existence of a projective $\mathcal{O}$-border basis G of $\mathcal{I}^+(\mathbb{X})$. In particular, it follows that $\sigma_{\mathbb{X}} + 1 = \max\{\deg_W(t) \mid t \in \mathcal{O}\}$ and thus $\dim_K(R_{\sigma_{\mathbb{X}}+1}) = \#\mathcal{O}$ according to Proposition 4.1.15. Moreover, Proposition 4.1.7 yields that the set $\{g^{\text{deh}} \mid g \in G\} \subseteq P$ is an $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})^{\text{deh}}$ and thus we have $\dim_K(P/\mathcal{I}^+(\mathbb{X})^{\text{deh}}) = \#\mathcal{O} = \dim_K(R_{\sigma_{\mathbb{X}}+1}) < \infty$ by Definition 2.1.14. In particular, the elements of $P/\mathcal{I}^+(\mathbb{X})$ are K-linear combinations of the residue classes of the elements of $\mathcal{O}$, i.e. of the form $\overline{p}$ with $p \in P_{\leq \sigma_{\mathbb{X}}+1}$. Since we have $\overline{\text{deh}}(x_0^{\sigma_{\mathbb{X}}+1-\deg_W(p)}p) = \overline{p}$ and $x_0^{\sigma_{\mathbb{X}}+1-\deg_W(p)}p \in \overline{P}_{\sigma_{\mathbb{X}}+1}$ for all $p \in P_{\leq \sigma_{\mathbb{X}}+1}$, $\overline{\text{deh}}|_{R_{\sigma_{\mathbb{X}}+1}}$ is a K-algebra epimorphism. Since $\dim_K(R_{\sigma_{\mathbb{X}}+1}) = \dim_K(P/\mathcal{I}^+(\mathbb{X})^{\text{deh}}) < \infty$, it is even a K-vector space isomorphism and the claim follows. □

Before we are able to generalize [Kre94, Lemma 2.2] to our setting, we need to define the concept of separators.

Definition 5.1.17. Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be an arbitrary non-empty zero-dimensional subscheme and let $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ be the corresponding coordinate ring.

a) Since $\mathbb{X}$ is a zero-dimensional subscheme, the Hilbert function $\text{HF}_{\mathbb{X}} = \text{HF}_R$ gets eventually stationary. Its maximum value $\deg(\mathbb{X}) = \max\{\text{HF}_{\mathbb{X}}(\gamma) \mid \gamma \in \mathbb{Z}\} \in \mathbb{N}$ is called the **degree** of $\mathbb{X}$.

b) Let $\mathbb{Y} \subseteq \mathbb{X}$ be a non-empty subscheme of degree $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$. The natural number $\alpha_{\mathbb{X}}(\mathbb{Y}) = \min\{\gamma \in \mathbb{Z} \mid (\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}))_\gamma \neq 0\}$ is called the **initial degree** of $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$.

c) Let $\mathbb{Y} \subseteq \mathbb{X}$ be a non-empty subscheme of degree $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$. We call every element of $(\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}))_{\alpha_{\mathbb{X}}(\mathbb{Y})} \setminus \{0\}$ a **minimal separator** of $\mathbb{Y}$ and every element of $(\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}))_{\sigma_{\mathbb{X}}+1} \setminus \{0\}$ is called a **separator** of $\mathbb{Y}$.

Proposition 5.1.18. *Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be an arbitrary non-empty zero-dimensional subscheme with coordinate ring $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$, let $\mathbb{Y} \subseteq \mathbb{X}$ be a non-empty subscheme of degree $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$, and let $\alpha_{\mathbb{X}}(\mathbb{Y}) = \min\{\gamma \in \mathbb{Z} \mid (\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}))_\gamma \neq 0\}$ be the initial degree of $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$. Moreover, let $\sigma_{\mathbb{X}} = \max\{\gamma \in \mathbb{Z} \mid \text{HF}_{\mathbb{X}}(\gamma) < \deg(\mathbb{X})\}$.*

a) The Hilbert function of $\mathbb{Y}$ satisfies

$$\text{HF}_{\mathbb{Y}}(\gamma) = \begin{cases} \text{HF}_{\mathbb{X}}(\gamma) & \text{if } \gamma < \alpha_{\mathbb{X}}(\mathbb{Y}) \\ \text{HF}_{\mathbb{X}}(\gamma) - 1 & \text{if } \gamma \geq \alpha_{\mathbb{X}}(\mathbb{Y}) \end{cases}$$

for all $\gamma \in \mathbb{Z}$. In particular, we get $\alpha_{\mathbb{Y}}(\mathbb{X}) \leq \sigma_{\mathbb{X}} + 1$.

b) Let $f_{\mathbb{Y}}^* \in (\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_{\alpha_{\mathbb{X}}(\mathbb{Y})} \setminus \{0\}$. Then we have $(\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_\gamma = K \cdot x_0^{\gamma-\alpha_{\mathbb{X}}(\mathbb{Y})} f_{\mathbb{Y}}^*$ for all $\gamma \in \mathbb{Z}$ with $\gamma \geq \alpha_{\mathbb{X}}(\mathbb{Y})$.

Proof. This is proven in [Kre94, Section 1] for K being algebraically closed and $\overline{P}$ being standard graded. In our setting, the proof stays just the same. □

Finally, we are now able to generalize [Kre94, Lemma 2.2] to our setting.

Proposition 5.1.19. *Assume that $\mathbb{X} \cap H^{\inf} = \emptyset$. Moreover, we let $\mathbb{A}_K^n = \operatorname{Spec}(P)$ and denote $\sigma_{\mathbb{X}} = \max(\{\gamma \in \mathbb{Z} \mid \operatorname{HF}_{\mathbb{X}}(\gamma) < \deg(\mathbb{X})\})$.*

a) The ideal $\mathcal{I}^+(\mathbb{X})^{\mathrm{deh}} \subseteq P$ is a zero-dimensional ideal and $\mathcal{Z}(\mathcal{I}^+(\mathbb{X})^{\mathrm{deh}}) \subseteq \mathbb{A}_K^n$, i.e. all the zeros of $\mathcal{I}^+(\mathbb{X})^{\mathrm{deh}}$ are K-rational.

b) Let $k \in \mathbb{N}$ and $r \in R_k \setminus \{0\}$. Then there exists an $r' \in R_{\sigma_{\mathbb{X}}+1}$, a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$, and a separator $f_{\mathbb{Y}} \in R_{\sigma_{\mathbb{X}}+1}$ of $\mathbb{Y}$ such that $rr' = x_0^k f_{\mathbb{Y}}$.

c) If $\mathbb{Y} \subseteq \mathbb{X}$ is a subscheme and $k \in \mathbb{N}$ satisfies $\deg(\mathbb{Y}) \leq k \leq \deg(\mathbb{X})$, then there exists a subscheme $\mathbb{Y}' \subseteq \mathbb{X}$ such that $\deg(\mathbb{Y}') = k$ and such that $\mathbb{Y} \subseteq \mathbb{Y}'$.

Proof. First we prove a). We denote $I = \mathcal{I}^+(\mathbb{X})^{\mathrm{deh}}$. According to Lemma 5.1.16 and Definition 5.1.17, we have $\dim_K(P/I) = \dim_K(R_{\sigma_{\mathbb{X}}+1}) = \deg(\mathbb{X}) < \infty$, i.e. $I \subsetneq P$ is a zero-dimensional ideal according to [KR00, Defn. 3.7.2]. By the Finiteness Criterion [KR00, Prop. 3.7.1], the zero set $\mathcal{Z}(I) \subseteq \mathbb{A}_{\overline{K}}^n = \operatorname{Spec}(\overline{K}[x_1, \dots, x_n])$ is finite. Let $\mathcal{Z}(I) = \{p_1, \dots, p_s\} \subseteq \mathbb{A}_{\overline{K}}^n$ be where $s \in \mathbb{N}$. Since $\mathbb{X}$ is assumed to be non-empty, we also have $\dim_K(R_{\sigma_{\mathbb{X}}+1}) = \deg(\mathbb{X}) > 0$. According to Lemma 5.1.16, it follows that $\dim_K(P/I) = \dim_K(R_{\sigma_{\mathbb{X}}+1}) > 0$, i.e. $I \subset P$. By the Weak Nullstellensatz [KR00, Thm. 2.6.13], we hence get $\mathcal{Z}(I) \neq \emptyset$, i.e. $s \geq 1$. Let $i \in \{1, \dots, s\}$ and write $p_i = (p_{i1}, \dots, p_{in}) \in \mathbb{A}_{\overline{K}}^n$ with $p_{i1}, \dots, p_{in} \in \overline{K}$. As $\mathbb{X} \cap H^{\inf} = \emptyset$ by assumption, we have $\mathcal{I}^+(\mathbb{X}) = I^{\mathrm{hom}}$ according to Proposition 5.1.8 and [KR05, Prop. 4.3.5]. Let $F \in \mathcal{I}^+(\mathbb{X})$ be a homogeneous polynomial and let $d \in \mathbb{N}$ be maximal such that x_0^d divides F. By [KR05, Prop. 4.3.2.h], we have $F = x_0^d (F^{\mathrm{deh}})^{\mathrm{hom}}$. In particular, we see that $F(1, p_{i1}, \dots, p_{in}) = (x_0^d (F^{\mathrm{deh}})^{\mathrm{hom}})(1, p_{i1}, \dots, p_{in}) = F^{\mathrm{deh}}(p_{i1}, \dots, p_{in}) = 0$ since $F^{\mathrm{deh}} \in I$ and since $p_i \in \mathcal{Z}(I)$. In other words, the projective point $(1 : p_{i1} : \cdots : p_{in})$ is a common zero of the ideal $\mathcal{I}^+(\mathbb{X})$. Since the zeros of $\mathcal{I}^+(\mathbb{X})$ are assumed to be K-rational, we see that $p_i = (p_{i1}, \dots, p_{in})$ is also K-rational and hence $\mathcal{Z}(I) \subseteq \mathbb{A}_K^n$ follows.

Next we prove the second claim. For all $i \in \{1, \dots, s\}$, let $\mathfrak{m}_i = \mathcal{I}(\{p_i\}) \subseteq P$ be the maximal ideal corresponding to the point p_i and let $\mathcal{O}_i = P_{\mathfrak{m}_i}$ be the localization of P at $\mathfrak{m}_i$. Let $\varphi : P/I \xrightarrow{\sim} \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$ be the K-algebra isomorphism of Lemma 5.1.15, let $\overline{\mathrm{deh}} : R \twoheadrightarrow P/I$ be the K-algebra epimorphism of Lemma 5.1.16, and let $\iota = \varphi \circ \overline{\mathrm{deh}}$. Note that since the restriction $\iota|_{R_{\sigma_{\mathbb{X}}+1}}$ of ι to $R_{\sigma_{\mathbb{X}}+1}$ is a K-vector space isomorphism according to Lemma 5.1.16, we have $\iota^{-1}(\{f\}) \cap R_{\sigma_{\mathbb{X}}+1} \neq \emptyset$ for all $f \in \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$. We distinguish two cases.

For the first case, suppose that $k \leq \sigma_{\mathbb{X}}+1$. Let $(r_1, \dots, r_s) := \iota(x_0^{\sigma_{\mathbb{X}}+1-k} r)$. Since we have $r \neq 0$, there exists an $i \in \{1, \dots, s\}$ such that $r_i \neq 0$. The K-algebra $\mathcal{O}_i$ is Noetherian by [Kun85, Coro. III.4.11] and it is a local ring with maximal ideal $\mathfrak{m}_i\mathcal{O}_i$ according to

[Kun85, Exmp. d in III.3]. Let $\mathfrak{p} \subseteq \mathcal{O}_i$ be a prime ideal with $I\mathcal{O}_i \subseteq \mathfrak{p}$. Assume that $\mathfrak{p} \neq \mathfrak{m}_i\mathcal{O}_i$. Since $\mathfrak{m}_i\mathcal{O}_i$ is the maximal ideal of the local ring $\mathcal{O}_i$, we get the chain of ideals $I\mathcal{O}_i \subseteq \mathfrak{p} \subset \mathfrak{m}_i\mathcal{O}_i$ in $\mathcal{O}_i$. i.e. $\dim(\mathcal{O}_i/I\mathcal{O}_i) > 0$. As $P/I \cong \mathcal{O}_1/I\mathcal{O}_1 \times \cdots \times \mathcal{O}_s/I\mathcal{O}_s$ according to Lemma 5.1.15 and as I is zero-dimensional by a), $0 = \dim(P/I) \geq \dim(\mathcal{O}_i/I\mathcal{O}_i) > 0$. Hence it follows that $\mathfrak{p} = \mathfrak{m}_i\mathcal{O}_i$. By [Kun85, Defn. I.1.6], this means that $\mathfrak{m}_i\mathcal{O}_i$ is the only minimal prime divisor of the ideal $I\mathcal{O}_i$ and $I\mathcal{O}_i \subseteq \mathcal{O}_i$ is thus an $\mathfrak{m}_i\mathcal{O}_i$-primary ideal by [Kun85, Lemma V.4.4]. Now we consider the socle $\mathfrak{S}(\mathcal{O}_i/I\mathcal{O}_i)$ of the ring $\mathcal{O}_i/I\mathcal{O}_i$, i.e. the set $\mathfrak{S}(\mathcal{O}_i/I\mathcal{O}_i) = \{\overline{f} \in \mathcal{O}_i/I\mathcal{O}_i \mid f \in \mathcal{O}_i, \mathfrak{m}_i\mathcal{O}_i \cdot \overline{f} = \{0\}\} \subseteq \mathcal{O}_i/I\mathcal{O}_i$. As $r_i \neq 0$, [Kun85, p. 189] implies the existence of an element $r_i' \in \mathcal{O}_i/I\mathcal{O}_i$ and a socle element $s_i \in \mathfrak{S}(\mathcal{O}_i/I\mathcal{O}_i)$ such that $r_i r_i' = s_i$. Let $f_{\mathbb{Y}} \in \iota^{-1}((0,\dots,0,s_i,0,\dots,0)) \cap R_{\sigma_{\mathbb{X}}+1}$ and let $r' \in \iota^{-1}((0,\dots,0,r_i',0,\dots,0)) \cap R_{\sigma_{\mathbb{X}}+1}$. Then we have

$$\begin{aligned}
&\iota(x_0^{\sigma_{\mathbb{X}}+1-k} r r' - x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}}) \\
&= \iota(x_0^{\sigma_{\mathbb{X}}+1-k} r)\iota(r') - \iota(x_0^{\sigma_{\mathbb{X}}+1})\iota(f_{\mathbb{Y}}) \\
&= (r_1,\dots,r_s)(0,\dots,0,r_i',0,\dots,0) - (1,\dots,1)(0,\dots,0,s_i,0,\dots,0) \\
&= (0,\dots,0,r_i r_i',0,\dots,0) - (0,\dots,0,s_i,0,\dots,0) \\
&= 0,
\end{aligned}$$

i.e. $x_0^{\sigma_{\mathbb{X}}+1-k} r r' - x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}} \in \ker(\iota)_{2(\sigma_{\mathbb{X}}+1)}$. We have $\ker(\iota) = \ker(\overline{\mathrm{deh}}) = \langle \overline{x}_0 - 1 \rangle \subseteq R$ by Lemma 5.1.16. Let $p \in R$ be such that $x_0^{\sigma_{\mathbb{X}}+1-k} r r' - x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}} = p(x_0 - 1)$. In particular, $p(x_0-1)$ is homogeneous of degree $2(\sigma_{\mathbb{X}}+1)$. Let $p = \sum_{\gamma \in \mathbb{Z}} p_\gamma$ be the decomposition of p in its homogeneous components. Assume that $p \neq 0$. Then the elements $m = \min\{\gamma \in \mathbb{Z} \mid p_\gamma \neq 0\}$ and $d = \max\{\gamma \in \mathbb{Z} \mid p_\gamma \neq 0\}$ exist and we get $p(\overline{x}_0 - 1) = p_d x_0 + \widetilde{p} - p_m \in R_{2(\sigma_{\mathbb{X}}+1)}$ with an element $\widetilde{p} \in \bigoplus_{\gamma=m+1}^{d-1} R_\gamma$. As x_0 is a non-zero divisor for R by Proposition 5.1.8 and as 1 is obviously a non-zero divisor for R, it follows that $d < d+1 = m$ in contradiction to the choice of m and d. Thus we get $p = 0$ and this implies $x_0^{\sigma_{\mathbb{X}}+1-k}(rr' - x_0^k f_{\mathbb{Y}}) = x_0^{\sigma_{\mathbb{X}}+1-k} r r' - x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}} = 0$. Since the element x_0 is a non-zero divisor for R by Proposition 5.1.8, we get the equality $rr' - x_0^k f_{\mathbb{Y}} = 0$. It remains to prove that $f_{\mathbb{Y}}$ is a separator of a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$. Let $\mathbb{Y} \subseteq \mathbb{X}$ be the subscheme of $\mathbb{X}$ defined by the saturated ideal $\langle f_{\mathbb{Y}} \rangle^{\mathrm{sat}} \subseteq R$. For all $k \in \{1,\dots,n\}$, we have $\iota((x_k - p_{ik} x_0^{w_k}) f_{\mathbb{Y}}) = (0,\dots,0,(x_k - p_{ik})s_i,0,\dots,0) = 0$ as $x_k - p_{ik} \in \mathfrak{m}_i$ and $s_i \in \mathfrak{S}(\mathcal{O}_i/I\mathcal{O}_i)$. Thus $(x_k - p_{ik} x_0^{w_k}) f_{\mathbb{Y}} \in \ker(\iota)_{\sigma_{\mathbb{X}}+1+w_k}$ and in an analogous fashion as above, we get $(x_k - p_{ik} x_0^{w_k}) f_{\mathbb{Y}} = 0$ and thus $x_k f_{\mathbb{Y}} = p_{ik} x_0^{w_k} f_{\mathbb{Y}}$ for all $k \in \{1,\dots,n\}$. In particular, we see that $(\langle f_{\mathbb{Y}} \rangle^{\mathrm{sat}})_\gamma = K \cdot x_0^{\gamma - \sigma_{\mathbb{X}} - 1} f_{\mathbb{Y}}$ for all $\gamma \in \mathbb{Z}$ with $\gamma \geq \sigma_{\mathbb{X}} + 1$ and thus $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \mathrm{HF}_{\langle f_{\mathbb{Y}} \rangle^{\mathrm{sat}}}(\sigma_{\mathbb{X}}+1) = \deg(\mathbb{X}) - 1$. By Definition 5.1.17, $f_{\mathbb{Y}}$ is a separator of $\mathbb{Y}$ and the claim follows in the case $k \leq \sigma_{\mathbb{X}} + 1$.
For the second case, suppose now that $k > \sigma_{\mathbb{X}} + 1$. As $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, Proposition 5.1.8 yields the existence of a projective $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})$. In particular, it follows that $\sigma_{\mathbb{X}} + 1 = \max\{\deg_W(t) \mid t \in \mathcal{O}\}$. Now Proposition 4.1.15 implies that r is of the form $r = x_0^{k - \sigma_{\mathbb{X}} - 1} \widetilde{r}$ for some $\widetilde{r} \in R_{\sigma_{\mathbb{X}}+1}$. As we have already shown in the first case, there exists a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ of degree $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$, a separator $f_{\mathbb{Y}} \in R_{\sigma_{\mathbb{X}}+1}$ of $\mathbb{Y}$, and an element $r' \in R_{\sigma_{\mathbb{X}}+1}$ such that $\widetilde{r} r' = x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}}$. Altogether, we see that $rr' = x_0^{k-\sigma_{\mathbb{X}}-1} \widetilde{r} r' = x_0^{k - \sigma_{\mathbb{X}} - 1} x_0^{\sigma_{\mathbb{X}}+1} f_{\mathbb{Y}} = x_0^k f_{\mathbb{Y}}$ and b) follows.

Finally, we prove claim c). Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme and let $k \in \mathbb{N}$ be a natural number such that $\deg(\mathbb{Y}) \leq k \leq \deg(\mathbb{X})$. We prove the claim by downward induction on $k \in \{\deg(\mathbb{X}), \dots, \deg(\mathbb{Y})\}$. Since the induction start $k = \deg(\mathbb{X})$ follows trivially, we only have to prove the claim for $k < \deg(\mathbb{X})$. If $\deg(\mathbb{Y}) = k$, the claim follows trivially. Thus suppose that $\deg(\mathbb{Y}) < k$. By the induction hypothesis, there is a subscheme $\mathbb{Y}'' \subseteq \mathbb{X}$ such that $\deg(\mathbb{Y}'') = k+1$ and $\mathbb{Y} \subseteq \mathbb{Y}''$. We let $S = \overline{P}/\mathcal{I}^+(\mathbb{Y}'')$. As $\mathbb{Y}'' \subseteq \mathbb{X}$ is a non-emtpy subscheme, $\mathcal{I}^+(\mathbb{Y}'') \supseteq \mathcal{I}^+(\mathbb{X})$ and, therefore, $\mathcal{Z}^+(\mathcal{I}^+(\mathbb{Y}'')) \subseteq \mathcal{Z}^+(\mathcal{I}^+(\mathbb{X}))$, i.e. the subscheme $\mathbb{Y}'' \subseteq \mathbb{X}$ has K-rational support, and we get $\mathbb{Y}'' \cap H^{\inf} \subseteq \mathbb{X} \cap H^{\inf} = \emptyset$. As $\deg(\mathbb{Y}) < \deg(\mathbb{Y}'')$, there exist a natural number $m \in \mathbb{N}$ with $(\mathcal{I}^+_{\mathbb{Y}''}(\mathbb{Y}))_m \neq \{0\}$. Let $s \in (\mathcal{I}^+_{\mathbb{Y}''}(\mathbb{Y}))_m \setminus \{0\} \in S_m \setminus \{0\}$. Now b) implies the existence of an element $s' \in S_{\sigma_{\mathbb{Y}''}+1}$, of a subscheme $\mathbb{Y}' \subseteq \mathbb{Y}''$ such that $\deg(\mathbb{Y}') = \deg(\mathbb{Y}'') - 1 = k$, and of a separator $f_{\mathbb{Y}} \in S_{\sigma_{\mathbb{Y}''}+1}$ of $\mathbb{Y}$, such that $ss' = x_0^m f_{\mathbb{Y}}$. Hence Proposition 5.1.18 and Lemma 5.1.13 yield $\mathcal{I}^+_{\mathbb{Y}''}(\mathbb{Y}') = \langle f_{\mathbb{Y}'} \rangle^{\mathrm{sat}} \subseteq \langle s \rangle^{\mathrm{sat}} \subseteq \mathcal{I}^+_{\mathbb{Y}''}(\mathbb{Y})$. Thus we have $\mathbb{Y} \subseteq \mathbb{Y}'$ and the claim follows. $\square$

5.2 Projective Border Bases and Uniformity Conditions

In this section, we study zero-dimensional closed subschemes of weighted projective spaces as introduced in Section 5.1. We restrict ourselves to the study of zero-dimensional subschemes with K-rational support, cf. Definition 5.1.5, and with the property that no point of the subschemes lies on the hyperplane $H^{\inf} = \mathcal{Z}^+(x_0)$ at infinity. By Proposition 5.1.10, there often exists a generic linear change of coordinates such that the second property holds. More precisely, we do the following.

In the first subsection, we introduce the notion of certain uniformity conditions in Definition 5.2.1. We are particularly interested in the study of (i,j)-uniform zero-dimensional subschemes since all the other uniformity condition of Definition 5.2.1 are special cases of them, cf. Proposition 5.2.6. Since we assume that no point of the support of the given subscheme $\mathbb{X}$ lies on the hyperplane at infinity $H^{\inf}$, we know that there exists a projective $\mathcal{O}$-border basis G of the corresponding defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$. Moreover, the formal (projective) multiplication matrices of G fully determine the multplicative structure of the coordinate ring $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ and the corresponding canonical module ω_R, cf. Chapter 4. Those properties provide additional information that allow us to characterize (i,j)-uniform subschemes by means of the multiplicative structure of the corresponding canonical module in Theorem 5.2.7. As a direct consequence, we can characterize (i,j)-uniform schemes via certain vanishing conditions in Corollary 5.2.9, which allow us reduce the question whether $\mathbb{X}$ is (i,j)-uniform to certain radical membership tests if the base field K is algebraically closed.

In the second subsection, we additionally assume that the subscheme $\mathbb{X}$ is reduced. In this case, we can characterize (i,j)-uniform subschemes using separators as introduced in Definition 5.1.17. In particular, we prove that the scheme $\mathbb{X}$ is (i,j)-uniform if and only if certain sets are K-linearly independent in Theorem 5.2.14.

For the whole section, we let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a non-empty zero-dimensional projective subscheme with K-rational support and assume that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. We want to study uniformity conditions with the help of the canonical module ω_R of the projective coordinate ring $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$. Let $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \dim_K(R_\gamma)$ be the multigraded Hilbert function of $\mathbb{X}$ and let $\Delta\,\mathrm{HF}_{\mathbb{X}} : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \mathrm{HF}_{\mathbb{X}}(\gamma) - \mathrm{HF}_{\mathbb{X}}(\gamma - 1)$ denote its Castelnuovo function. Moreover, the invariant

$$\sigma_{\mathbb{X}} = \max\{i \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{X}}(i) < \deg(\mathbb{X})\}$$

will play an important role throughout this section. According to [IK99, Thm. 1.69], $\sigma_{\mathbb{X}} + 2$ is exactly the Castelnuovo-Mumford regularity based on [Mum66, Lect. 14] in the standard graded case. In order to achieve our goals, we study subschemes $\mathbb{Y} \subseteq \mathbb{X}$. We define the Hilbert function $\mathrm{HF}_{\mathbb{Y}}$ and the invariant $\sigma_{\mathbb{Y}}$ similarly to the definitions above. Recall that we always consider closed subschemes and thus simply say subscheme instead of closed subscheme for the whole thesis.

5.2.1 The General Case

The study of projective point sets in uniform position over an algebraically closed field of characteristic zero started with the Uniform Position Lemma in [Har80, Section 2]. Over algebraically closed fields of positive characteristic, projective point sets in uniform position were studied in [Rat87]. In [Kre94, Section 3], the uniform position property was refined to the notion of i-uniformity and applied to non-reduced zero-dimensional subschemes. In [Kre98, Section 7], [Kre00, Section 5], and [Kre01, Section 2] the author further generalized this notion to (i, j)-uniform zero-dimensional subschemes. The authors of [MP04] also studied (i, j)-uniform reduced zero-dimensional subschemes. Note that, as described in [MP04, Rem. 2], their notion slightly differs from ours in Definition 5.2.1, which coincides with the one in [Kre01, Defn. 2.1] in the standard graded case. Similarly, the Cayley-Bacharach Theorem, which is named after theorems in [Cay43] and [Bac86], was first generalized in [Kre94, Section 2] to non-reduced zero-dimensional subschemes over algebraically closed fields.

Our notion of (i, j)-uniformity in Definition 5.2.1, unifies and generalizes all these concepts and allows us to apply them to non-reduced zero-dimensional subschemes of weighted projective spaces over arbitrary fields.

Definition 5.2.1. Let $i \in \{1, \ldots, \deg(\mathbb{X}) - 1\}$ and $j \in \{1, \ldots, \sigma_{\mathbb{X}}\}$.

a) We say that $\mathbb{X}$ is an ***i*-uniform** scheme if for every subscheme $\mathbb{Y} \subseteq \mathbb{X}$ that satisfies $\deg(\mathbb{X}) - i \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$, we have $\mathrm{HF}_{\mathbb{Y}} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$. In particular, we say that $\mathbb{X}$ is **in uniform position** if $\mathbb{X}$ is $(\deg(\mathbb{X}) - 1)$-uniform.

b) We say that $\mathbb{X}$ is an **(i, j)-uniform** scheme if for all subschemes $\mathbb{Y} \subseteq \mathbb{X}$ that satisfy $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$, we have $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$.

c) We say that the scheme $\mathbb{X}$ has the **Cayley-Bacharach property of degree** j, if $\mathbb{X}$ is $(1,j)$-uniform. In particular, if $\mathbb{X}$ has the Cayley-Bacharach propery of degree $\sigma_{\mathbb{X}}$, we call $\mathbb{X}$ a **Cayley-Bacharach scheme**.

Example 5.2.2. Let $\overline{P} = \mathbb{Q}[x_0, x_1, x_2]$ be standard graded, i.e. graded by the matrix $\overline{W} = (1,1,1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$, and consider the ideal $I = \mathfrak{p}_0^2 \cap \mathfrak{p}_1^2 \cap \mathfrak{p}_2^2 \subseteq \overline{P}$ with the homogeneous vanishing ideals $\mathfrak{p}_0 = \mathcal{I}^+(\{(1:0:0)\})$, $\mathfrak{p}_1 = \mathcal{I}^+(\{(1:0:1)\})$, and with the ideal $\mathfrak{p}_2 = \mathcal{I}^+(\{(1:1:0)\})$ of $\overline{P}$ as defined in Definition 5.1.3. According to Proposition 5.1.6, we have $\mathfrak{p}_0 = \langle x_1, x_2 \rangle$, $\mathfrak{p}_1 = \langle x_1, x_2 - x_0 \rangle$, and $\mathfrak{p}_2 = \langle x_1 - x_0, x_2 \rangle$ and thus we can easily compute I according to [KR00, Rem. 3.2.1 and Prop. 3.2.7]. By [KR00, Prop. 1.7.10] and [KR05, Prop. 4.1.11], I is homogeneous. Let $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the subscheme that is scheme-theoretically defined by I. By construction, $\mathbb{X}$ is zero-dimensional and concentrated at the "fat points" $(1:0:0)$, $(1:0:1)$, and $(1:1:0)$. The Hilbert function of the scheme $\mathbb{X}$, which can be computed using Proposition 4.1.15 and Corollary 4.1.14, is of the form $\mathrm{HF}_{\mathbb{X}} : 1, 3, 6, 9, 9, \ldots$. Let $J = \mathfrak{p}_0^2 \cap \langle x_1, (x_2 - x_0)^2 \rangle \cap \langle (x_1 - x_0)^2, x_2 \rangle \subseteq \overline{P}$. Then J is homogeneous by [KR00, Prop. 1.7.10] and [KR05, Prop. 4.1.11]. Since $I \subseteq J$, the homogeneous ideal J scheme-theoretically defines a zero-dimensional subscheme $\mathbb{Y} \subseteq \mathbb{X}$. Using Proposition 4.1.15 and Corollary 4.1.14, again, we compute $\mathrm{HF}_{\mathbb{Y}} : 1, 3, 5, 7, 7, \ldots$ and conclude that the scheme $\mathbb{X}$ is not $(2,2)$-uniform by Definition 5.2.1.

As a first consequence of Definition 5.2.1, we show that higher (i,j)-uniformity conditions imply lower ones.

Proposition 5.2.3. *Let* $i \in \{1, \ldots, \deg(\mathbb{X}) - 1\}$, $j \in \{1, \ldots, \sigma_{\mathbb{X}}\}$, *and assume that the scheme* $\mathbb{X}$ *is* (i,j)*-uniform.*

a) We have $i \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$.

b) If $j > 1$, *then* $\mathbb{X}$ *is* $(i, j-1)$*-uniform.*

c) If $i > 1$, *then* $\mathbb{X}$ *is* $(i-1, j)$*-uniform.*

Proof. First we prove a). For a contradiction, assume that $i > \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$. By Proposition 5.1.19, there is a subscheme $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i < \mathrm{HF}_{\mathbb{X}}(j)$. Since $\mathbb{X}$ is (i,j)-uniform, it follows $\mathrm{HF}_{\mathbb{X}}(j) = \mathrm{HF}_{\mathbb{Y}}(j) \leq \deg(\mathbb{Y}) < \mathrm{HF}_{\mathbb{X}}(j)$ from Definitions 5.1.17 and 5.2.1. This is clearly a contradiction and thus $i \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$.

Next we prove b). Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$ and let $j > 1$. As $\mathbb{X}$ is (i,j)-uniform, we have $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ according to Definition 5.2.1. This yields $\mathrm{HF}_{\mathbb{Y}}(j-1) = \mathrm{HF}_{\mathbb{X}}(j-1)$, i.e. $\mathbb{Y}$ is $(i, j-1)$-uniform by Definition 5.2.1.

In order to prove claim c), we let $\mathbb{Y} \subseteq \mathbb{X}$ now be a subscheme of the zero-dimensional scheme $\mathbb{X}$ such that $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - (i-1)$ and let $i > 1$. According to Proposition 5.1.19, there exists a subscheme $\mathbb{Y}' \subseteq \mathbb{Y}$ with $\deg(\mathbb{Y}') = \deg(\mathbb{Y}) - 1 = \deg(\mathbb{X}) - i$. Since $\mathbb{X}$ is (i,j)-uniform, we have $\mathrm{HF}_{\mathbb{Y}'}(j) = \mathrm{HF}_{\mathbb{X}}(j)$. As $\mathbb{Y}' \subseteq \mathbb{Y} \subseteq \mathbb{X}$, we also have $\mathrm{HF}_{\mathbb{Y}'}(j) \leq \mathrm{HF}_{\mathbb{Y}}(j) \leq \mathrm{HF}_{\mathbb{X}}(j)$. Altogether, we see that $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$, i.e. $\mathbb{X}$ is $(i-1, j)$-uniform according to Definition 5.2.1 □

The Cayley-Bacharach property is obviously a special kind of (i, j)-uniformity according to Definition 5.2.1. Next we show that it suffices to consider (i, j)-uniformity conditions by proving that a subscheme is i-uniform if and only if it satifies certain (i, j)-uniformity conditions in Proposition 5.2.6.

Lemma 5.2.4. *Let $i \in \{1, \dots, \Delta \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1)\}$. Then the following conditions are equivalent.*

i) The zero-dimensional scheme $\mathbb{X}$ is i-uniform.

ii) The zero-dimensional scheme $\mathbb{X}$ is $(i, \sigma_{\mathbb{X}})$-uniform.

Proof. First we prove that i) implies ii). Suppose that the zero-dimensional scheme $\mathbb{X}$ is i-uniform. Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme such that $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$. Since $\mathbb{X}$ is an i-uniform scheme, we have $\operatorname{HF}_{\mathbb{Y}} = \min\{\operatorname{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$ according to Definition 5.2.1. Moreover, as we also have $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i \geq \deg(\mathbb{X}) - \Delta \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1) = \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}})$, it follows that $\operatorname{HF}_{\mathbb{Y}}(\sigma_{\mathbb{X}}) = \min\{\operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}}), \deg(\mathbb{Y})\} = \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}})$, i.e. $\mathbb{X}$ is $(i, \sigma_{\mathbb{X}})$-uniform by Definition 5.2.1.

In order to prove that ii) implies i), suppose now that $\mathbb{X}$ is $(i, \sigma_{\mathbb{X}})$-uniform. Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme with $\deg(\mathbb{X}) - i \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$. Let $j = \deg(\mathbb{X}) - \deg(\mathbb{Y})$. Then $j \leq i$ and Proposition 5.2.3 implies that $\mathbb{X}$ is $(j, \sigma_{\mathbb{X}})$-uniform. By Definition 5.2.1, this means that $\operatorname{HF}_{\mathbb{Y}}(\sigma_{\mathbb{X}}) = \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}})$. Hence we get $\operatorname{HF}_{\mathbb{Y}}(m) = \operatorname{HF}_{\mathbb{X}}(m)$ for all $m \in \mathbb{Z}$ with $m \leq \sigma_{\mathbb{X}}$. As $\sigma_{\mathbb{Y}} \leq \sigma_{\mathbb{X}}$, it follows that $\operatorname{HF}_{\mathbb{Y}}(m) = \deg(\mathbb{Y})$ for all $m \in \mathbb{Z}$ with $m \geq \sigma_{\mathbb{X}} + 1$. Altogether, we see that $\operatorname{HF}_{\mathbb{Y}} = \min\{\operatorname{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$, i.e. $\mathbb{X}$ is i-uniform according to Definition 5.2.1. □

Lemma 5.2.5. *Let $j \in \{1, \dots, \sigma_{\mathbb{X}}\}$. Then the following conditions are equivalent.*

i) The zero-dimensional scheme $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(j))$-uniform.

ii) The zero-dimensional scheme $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(k), k)$-uniform for every element $k \in \{j, \dots, \sigma_{\mathbb{X}}\}$.

Proof. First we prove that i) implies ii). Let $k \in \{j, \dots, \sigma_{\mathbb{X}}\}$ and $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - (\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(k))$. As $j \leq k$, we get $\operatorname{HF}_{\mathbb{X}}(j) \leq \operatorname{HF}_{\mathbb{X}}(k)$ and thus $\deg(\mathbb{X}) - (\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(j)) \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$. Since the zero-dimensional scheme $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(j))$-uniform, we have $\operatorname{HF}_{\mathbb{Y}} = \min\{\operatorname{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$ by Definition 5.2.1. In particular, as $\deg(\mathbb{Y}) = \operatorname{HF}_{\mathbb{X}}(k)$, this implies that $\operatorname{HF}_{\mathbb{Y}}(k) = \operatorname{HF}_{\mathbb{X}}(k)$. Thus $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(k), k)$-uniform by Definition 5.2.1.

Next we prove that ii) implies i) by downward induction on $j \in \{\sigma_{\mathbb{X}}, \dots, 1\}$. Since $\Delta \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}} + 1) = \deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}})$, the induction start $j = \sigma_{\mathbb{X}}$ is a consequence of Lemma 5.2.4. For the induction step, let now $j < \sigma_{\mathbb{X}}$ and suppose that the claim holds for all elements greater than j. Assume that $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(k), k)$-uniform for all $k \in \{j, \dots, \sigma_{\mathbb{X}}\}$. Then $\mathbb{X}$ is $(\deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(j + 1))$-uniform according to the induction hypothesis. For all subschemes $\mathbb{Y} \subseteq \mathbb{X}$ with $\operatorname{HF}_{\mathbb{X}}(j + 1) \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$, we thus get $\operatorname{HF}_{\mathbb{Y}} = \min\{\operatorname{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$ by Definition 5.2.1. Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme such that $\mathrm{HF}_{\mathbb{X}}(j) \leq \deg(\mathbb{Y}) < \mathrm{HF}_{\mathbb{X}}(j+1)$. By Proposition 5.1.19, there exists a subscheme $\mathbb{Y}' \subseteq \mathbb{Y}$ with the property that $\deg(\mathbb{Y}') = \mathrm{HF}_{\mathbb{X}}(j) = \deg(\mathbb{X}) - (\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j))$. As $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j), j)$-uniform, we get $\mathrm{HF}_{\mathbb{X}}(j) = \mathrm{HF}_{\mathbb{Y}'}(j) \leq \mathrm{HF}_{\mathbb{Y}}(j) \leq \mathrm{HF}_{\mathbb{X}}(j)$, i.e. $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ and thus $\mathrm{HF}_{\mathbb{Y}}(m) = \mathrm{HF}_{\mathbb{X}}(m)$ for all $m \in \mathbb{Z}$ with $m \leq j$. According to Proposition 5.1.19, there also exists a subscheme $\mathbb{Y}'' \subseteq \mathbb{X}$ with $\mathbb{Y} \subseteq \mathbb{Y}''$ and $\deg(\mathbb{Y}'') = \mathrm{HF}_{\mathbb{X}}(j+1)$. Again, since $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j+1))$-uniform according to the induction hypothesis, we get $\mathrm{HF}_{\mathbb{Y}''} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y}'')\}$ by Definition 5.2.1. In particular, $\mathrm{HF}_{\mathbb{Y}''}(j+1) = \deg(\mathbb{Y}'')$ and this implies $\sigma_{\mathbb{Y}} \leq \sigma_{\mathbb{Y}''} \leq j+1$. Hence it follows $\mathrm{HF}_{\mathbb{Y}}(m) = \deg(\mathbb{Y})$ for all $m \in \mathbb{Z}$ with $m \geq j+1$ from Definition 5.1.17. Altogether, $\mathrm{HF}_{\mathbb{Y}} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$, i.e. $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j))$-uniform by Definition 5.2.1. □

Finally, we are now able to prove that i-uniform subschemes can be characterized by certain (i, j)-uniformity conditions, in general. Similar ideas have been used to give an algorithm to check arbitrary i-uniformity conditions in [BK96, after Rem. 8.4] if K is algebraically closed and $\overline{P}$ standard graded.

Proposition 5.2.6. *Let $i \in \{1, \ldots, \deg(\mathbb{X}) - 1\}$ and let $j \in \{1, \ldots, \sigma_{\mathbb{X}}\}$ be the unique natural number with $\mathrm{HF}_{\mathbb{X}}(j) < \deg(\mathbb{X}) - i \leq \mathrm{HF}_{\mathbb{X}}(j+1)$. Then the following conditions are equivalent.*

i) The zero-dimensional scheme $\mathbb{X}$ is i-uniform.

ii) The zero-dimensional scheme $\mathbb{X}$ is (i, j)-uniform and $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k), k)$-uniform for all $k \in \{j+1, \ldots, \sigma_{\mathbb{X}}\}$.

Proof. First we prove that i) implies ii), As we have $i \geq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j+1)$ and as $\mathbb{X}$ is i-uniform, $\mathbb{X}$ is also $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j+1))$-uniform by Definition 5.2.1. Thus Lemma 5.2.5 implies that $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k), k)$-uniform for all $k \in \{j+1, \ldots, \sigma_{\mathbb{X}}\}$. Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$. Since $\mathbb{X}$ is i-uniform, we have $\mathrm{HF}_{\mathbb{Y}}(j) = \min\{\mathrm{HF}_{\mathbb{X}}(j), \deg(\mathbb{Y})\}$ by Definition 5.2.1. As $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i > \mathrm{HF}_{\mathbb{X}}(j)$, it follows that $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$. Hence $\mathbb{X}$ is (i, j)-uniform by Definition 5.2.1.

Next we prove that ii) implies i). Since $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k), k)$-uniform for all $k \in \{j+1, \ldots, \sigma_{\mathbb{X}}\}$, Lemma 5.2.5 yields that $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j+1))$-uniform. Hence for every subscheme $\mathbb{Y} \subseteq \mathbb{X}$ with $\mathrm{HF}_{\mathbb{X}}(j+1) \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$, we have $\mathrm{HF}_{\mathbb{Y}} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$ by Definition 5.2.1. Let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme such that $\deg(\mathbb{X}) - i \leq \deg(\mathbb{Y}) < \mathrm{HF}_{\mathbb{X}}(j+1)$. According to Proposition 5.1.19, there exists a subscheme $\mathbb{Y}' \subseteq \mathbb{Y}$ with the property that $\deg(\mathbb{Y}') = \deg(\mathbb{X}) - i$. Since the scheme $\mathbb{X}$ is (i, j)-uniform, we get $\mathrm{HF}_{\mathbb{X}}(j) = \mathrm{HF}_{\mathbb{Y}'}(j) \leq \mathrm{HF}_{\mathbb{Y}}(j) \leq \mathrm{HF}_{\mathbb{X}}(j)$, i.e. $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ and thus $\mathrm{HF}_{\mathbb{Y}}(m) = \mathrm{HF}_{\mathbb{X}}(m)$ for all $m \in \mathbb{Z}$ with $m \leq j$. According to Proposition 5.1.19, there exists a subscheme $\mathbb{Y}'' \subseteq \mathbb{X}$ with $\mathbb{Y} \subseteq \mathbb{Y}''$ and $\deg(\mathbb{Y}'') = \mathrm{HF}_{\mathbb{X}}(j+1)$. As we have already shown above, $\mathrm{HF}_{\mathbb{Y}''} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y}'')\}$. In particular, $\mathrm{HF}_{\mathbb{X}}(j+1) = \deg(\mathbb{Y}'')$ yields $\mathrm{HF}_{\mathbb{Y}''}(j+1) = \deg(\mathbb{Y}'')$. This implies $\sigma_{\mathbb{Y}} \leq \sigma_{\mathbb{Y}''} \leq j$ and thus $\mathrm{HF}_{\mathbb{Y}}(m) = \deg(\mathbb{Y})$ for every $m \in \mathbb{Z}$ with $m \geq j+1$. Altogether, we get $\mathrm{HF}_{\mathbb{Y}} = \min\{\mathrm{HF}_{\mathbb{X}}, \deg(\mathbb{Y})\}$, i.e. $\mathbb{X}$ is i-uniform by Definition 5.2.1. □

The next theorem is the most important result of this chapter. It allows us to characterize (i,j)-uniform subschemes by means of linear algebra. Note that we do not need to assume that K is algebraically closed or $\mathbb{X}$ is reduced and that it does not only hold in $\mathbb{P}^n_K$ but in the weighted projective space $\mathbb{P}_K(\overline{W})$ for every positive matrix $\overline{W} \in \mathrm{Mat}_{1,n+1}(\mathbb{Z})$.

Theorem 5.2.7. (Characterization of (i,j)-Uniform Schemes)
Let ω_R be the canonical module of R, let $i \in \{1,\dots,\deg(\mathbb{X})-1\}$, let $j \in \{1,\dots,\sigma_{\mathbb{X}}\}$, and for all $r \in R_j$, let $\widehat{\pi}^*_r : (\omega_R)_{-j} \to (\omega_R)_0$, $\varphi \mapsto r \cdot \varphi$ be the multiplication by r. Then the following conditions are equivalent.

i) The zero-dimensional scheme $\mathbb{X}$ is (i,j)-uniform.

ii) For every $r \in R_j \setminus \{0\}$, we have $\dim_K(\ker(\widehat{\pi}^*_r)) \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$.

Proof. First we prove that i) implies ii). Suppose that $\mathbb{X}$ is (i,j)-uniform. If $R_j = \{0\}$, nothing has to be shown. Thus suppose that $R_j \neq \{0\}$ and let $r \in R_j \setminus \{0\}$. Assume that $d = \dim_K(\ker(\widehat{\pi}^*_r)) > \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$. Let $\mathbb{Y} \subseteq \mathbb{X}$ be the subscheme that is scheme-theoretically defined by $\langle r \rangle \subseteq R$. Write $r = \overline{p} \in R$ with $p \in \overline{P}_j$. Since $\overline{p} \in \mathcal{I}^+(\mathbb{Y}) \setminus \mathcal{I}^+(\mathbb{X})$, we have $\mathrm{HF}_{\mathbb{Y}}(j) < \mathrm{HF}_{\mathbb{X}}(j)$. As $\mathbb{X}$ is (i,j)-uniform, Definition 5.2.1 yields $\deg(\mathbb{Y}) \neq \deg(\mathbb{X}) - i$. For a contradiction, assume that $\deg(\mathbb{Y}) > \deg(\mathbb{X}) - i$. Then Proposition 5.1.19 yields the existence of a subscheme $\widetilde{\mathbb{Y}} \subseteq \mathbb{Y}$ such that such that $\deg(\widetilde{\mathbb{Y}}) = \deg(\mathbb{X}) - i$. As the scheme $\mathbb{X}$ is (i,j)-uniform, we get the contradiction $\mathrm{HF}_{\mathbb{X}}(j) = \mathrm{HF}_{\widetilde{\mathbb{Y}}}(j) \leq \mathrm{HF}_{\mathbb{Y}}(j) < \mathrm{HF}_{\mathbb{X}}(j)$. Thus we have $\deg(\mathbb{Y}) < \deg(\mathbb{X})-i$. According to Proposition 5.1.19, there is a subscheme $\mathbb{Y}' \subseteq \mathbb{X}$ with $\mathbb{Y} \subseteq \mathbb{Y}'$ and $\deg(\mathbb{Y}') = \deg(\mathbb{X}) - i$. Let $S = R/\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}')$ be the projective coordinate ring of $\mathbb{Y}'$ and ω_S be its canonical module. As $\mathbb{X}$ is (i,j)-uniform, $\mathrm{HF}_{\omega_S}(-j) = \deg(\mathbb{Y}') - \mathrm{HF}_{\mathbb{Y}'}(j) = \deg(\mathbb{X}) - i - \mathrm{HF}_{\mathbb{X}}(j)$ according to Proposition 4.3.7 and the Definitions 5.1.17 and 5.2.1. Moreover, we let $\{\varphi_1,\dots,\varphi_d\} \subseteq (\omega_R)_{-j}$ be a K-vector space basis of $\ker(\widehat{\pi}^*_r)$. For every $k \in \{1,\dots,d\}$, we then have $r \cdot \varphi_k = \widehat{\pi}^*_r(\varphi_k) = 0$, i.e. $\langle r \rangle \cdot \varphi_k = \{0\}$. As x_0 is a non-zero divisor for the canonical module ω_R by Proposition 4.3.6, Proposition 5.1.2 and Lemma 5.1.13 imply that $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) \cdot \varphi_k = \langle r \rangle^{\mathrm{sat}} \cdot \varphi_k = \{0\}$ for every $k \in \{1,\dots,d\}$. Since $\mathbb{Y} \subseteq \mathbb{Y}'$, we also have $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) \supseteq \mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}')$ and thus $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}') \cdot \varphi_k = \{0\}$ for every $k \in \{1,\dots,d\}$. Hence Proposition 4.3.6 yields that $\{\varphi_1,\dots,\varphi_d\} \subseteq (\omega_S)_{-j}$. Altogether, we get the contradiction $d \leq \dim_K((\omega_S)_{-j}) = \mathrm{HF}_{\omega_S}(-j) = \deg(\mathbb{X}) - i - \mathrm{HF}_{\mathbb{X}}(j) < d$. Thus it follows that $d \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$.

Next we prove that ii) implies i). Therefore, let now $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme such that $\deg(\mathbb{Y}) = \deg(\mathbb{X})-i$ holds. Then we have $\mathrm{HF}_{\mathbb{Y}}(j) \leq \mathrm{HF}_{\mathbb{X}}(j)$. For a contradiction, assume that $\mathrm{HF}_{\mathbb{Y}}(j) < \mathrm{HF}_{\mathbb{X}}(j)$. Then there is an $r \in (\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}))_j \setminus \{0\}$. Let $d = \dim_K(\ker(\widehat{\pi}^*_r))$. According to claim ii), we have $d \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$. Let $S = R/\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$ be the projective coordinate ring of $\mathbb{Y}$ and ω_S be its canonical module. Then Proposition 4.3.7 yields the inequality $\mathrm{HF}_{\omega_S}(-j) = \deg(\mathbb{Y}) - \mathrm{HF}_{\mathbb{Y}}(j) > \deg(\mathbb{X}) - i - \mathrm{HF}_{\mathbb{X}}(j) \geq d$. Let $\{\varphi_1,\dots,\varphi_\ell\} \subseteq (\omega_S)_{-j}$ be a K-vector space basis of $(\omega_S)_{-j}$. In particular, $\ell > d$ and for every $k \in \{1,\dots,\ell\}$, Proposition 4.3.6 implies that $\widehat{\pi}^*_r(\varphi_k) = r \cdot \varphi_k \in \mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) \cdot \varphi_k = \{0\}$. Thus we see that $\{\varphi_1,\dots,\varphi_\ell\} \subseteq \ker(\widehat{\pi}^*_r)$. As $\{\varphi_1,\dots,\varphi_\ell\}$ is K-linearly independent, we get the contradiction $\dim_K(\ker(\widehat{\pi}^*_r)) = d < \ell \leq \dim_K(\ker(\widehat{\pi}^*_r))$. Thus we have

$\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ and $\mathbb{X}$ is (i,j)-uniform by Definition 5.2.1. $\square$

Example 5.2.8. Consider the zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ of Example 5.2.2, again. Recall that the defining ideal of the scheme $\mathbb{X}$ in the standard graded polynomial ring $\overline{P} = \mathbb{Q}[x_0, x_1, x_2]$, was $I = \mathcal{I}^+(\{(1:0:0)\})^2 \cap \mathcal{I}^+(\{(1:0:1)\})^2 \cap \mathcal{I}^+(\{(1:1:0)\})^2$. Using the method described in Example 5.2.2 to compute I and Corollary 4.1.14, we can compute the projective $\mathcal{O}$-border basis $G = \{g_1, \dots, g_6\} \subseteq \overline{P}$ of I with the order ideal $\mathcal{O} = \{t_1, \dots, t_9\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_2^3, x_1x_2^2, x_1^3\}$ and $g_1 = x_1^2x_2 + x_1x_2^2 - x_0x_1x_2$, $g_2 = x_2^4 - 2x_0x_2^3 + x_0^2x_2^2$, $g_3 = x_1x_2^3 - x_0x_1x_2^2$, $g_4 = x_1^2x_2^2$, $g_5 = x_1^3x_2 + x_0x_1x_2^2 - x_0^2x_1x_2$, and $g_6 = x_1^4 - 2x_0x_1^3 + x_0^2x_1^2$. By Definition 2.4.15, the formal multiplication matrices $\mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Mat}_9(\mathbb{Q})$ of the $\mathcal{O}$-border basis $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P = \mathbb{Q}[x_1, x_2]$ are

$$\mathcal{X}_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&-1&0&1&0&0\\ 0&0&0&0&0&1&0&0&2 \end{pmatrix}, \quad \mathcal{X}_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&-1&0&0\\ 0&0&1&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&2&0&0\\ 0&0&0&0&1&-1&0&1&-1\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Let $R = \overline{P}/I$, let ω_R be the canonical module of R, let $p = -x_1^2 - x_1x_2 + x_0x_1 \in \overline{P}_2$, and let $\widehat{\pi}^*_{\overline{p}} : (\omega_R)_{-2} \to (\omega_R)_0$, $\varphi \mapsto \overline{p} \cdot \varphi$ be the multiplication by $\overline{p} \in R_2$. We have

$$p(\mathcal{I}_9, \mathcal{X}_1^{\mathrm{tr}}, \mathcal{X}_2^{\mathrm{tr}}) = -(\mathcal{X}_1^{\mathrm{tr}})^2 - \mathcal{X}_1^{\mathrm{tr}} \cdot \mathcal{X}_2^{\mathrm{tr}} + \mathcal{I}_9 \cdot \mathcal{X}_1^{\mathrm{tr}} = \begin{pmatrix} 0&0&1&0&-1&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&1&0&1&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&1&0&1&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&1&0&1&-1 \end{pmatrix}.$$

Recall that $\mathcal{O}$ determines both $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R$ by Proposition 4.1.15 and HF_{ω_R} by Proposition 4.3.7. In particular, $(\overline{\mathcal{O}}_0^*)_{-2} = \{\overline{x_2^3}^*, \overline{x_1x_2^2}^*, \overline{x_1^3}^*\}$ is a $\mathbb{Q}$-vector space basis of $(\omega_R)_{-2}$ and $(\overline{\mathcal{O}}_0^*)_0 = \{\overline{x_2}^*, \overline{x_1}^*, x_0\overline{x_2^2}^*, x_0\overline{x_1x_2}^*, x_0\overline{x_1^2}^*, x_0^2\overline{x_2^3}^*, x_0^2\overline{x_1x_2^2}^*, x_0^2\overline{x_1^3}^*\}$ is a $\mathbb{Q}$-vector space basis of $(\omega_R)_0$ by Proposition 4.3.7. With respect to the $\mathbb{Q}$-vector space bases $(\overline{\mathcal{O}_0}^*)_{-2}$ of $(\omega_R)_{-2}$ and $(\overline{\mathcal{O}_0}^*)_0$ of $(\omega_R)_0$, the $\mathbb{Q}$-vector space homomorphism $\widehat{\pi}^*_{\overline{p}}$ is represented by the submatrix $\mathcal{C} \in \mathrm{Mat}_{8,3}(\mathbb{Q})$ of $p(\mathcal{I}_9, \mathcal{X}_1^{\mathrm{tr}}, \mathcal{X}_2^{\mathrm{tr}}) \in \mathrm{Mat}_9(\mathbb{Q})$ obtained by deleting the first

$\mathrm{HF}_{\mathbb{X}}(0) = 1$ rows and the first $\mathrm{HF}_{\mathbb{X}}(2) = 6$ columns according to Corollary 4.3.11, i.e.

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \in \mathrm{Mat}_{8,3}(\mathbb{Q})$$

In particular, $\dim_{\mathbb{Q}}(\ker(\widehat{\pi}^*_{\overline{p}})) = \dim_{\mathbb{Q}}(\ker(\mathcal{C})) = 2 > 1 = 9-6-2 = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(2) - 2$. Therefore, Theorem 5.2.7 implies that the zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ is not $(2,2)$-uniform.

Corollary 5.2.9. *Let $G \subseteq \overline{P}$ be a projective $\mathcal{O}$-border of $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ where we have $\mathcal{O} = \{t_1, \dots, t_\mu\}$ with $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$ and let $\mathcal{X}_1, \dots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$. Let $j \in \{1, \dots, \sigma_{\mathbb{X}}\}$, let $i \in \{1, \dots, \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)\}$, and let $\{y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)}\}$ be a set of further indeterminates. For every $k \in \{1, \dots, \mathrm{HF}_{\mathbb{X}}(j)\}$, we let $\mathcal{T}_k \in \mathrm{Mat}_{\mu-1,\mu-\mathrm{HF}_{\mathbb{X}}(j)}(K)$ denote the submatrix of $t_k(\mathcal{X}_1^{\mathrm{tr}}, \dots, \mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K)$ obtained by deleting the first row and the first $\mathrm{HF}_{\mathbb{X}}(j)$ columns. Furthermore, let $J \subseteq K[y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)}]$ be the ideal generated by the set of all i-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_{\mathrm{HF}_{\mathbb{X}}(j)}\mathcal{T}_{\mathrm{HF}_{\mathbb{X}}(j)} \in \mathrm{Mat}_{\mu-1,\mu-\mathrm{HF}_{\mathbb{X}}(j)}(K[y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)}])$. Consider the affine zero set $\mathcal{Z}_K(J) \subseteq \mathbb{A}_K^{\mathrm{HF}_{\mathbb{X}}(j)} = \mathrm{Spec}(K[y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)}])$. Then the following conditions are equivalent.*

i) The zero-dimensional scheme $\mathbb{X}$ is (i,j)-uniform.

ii) We have $\mathcal{Z}_K(J) \subseteq \{0\}$.

iii) We have $\mathcal{Z}_K(J) = \{0\}$.

If, in addition, K is algebraically closed, then the above conditions are equivalent to the following ones.

iv) We have $\sqrt{J} \supseteq \langle y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)} \rangle$.

v) We have $\sqrt{J} = \langle y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)} \rangle$.

Proof. Let ω_R denote the canonical module of the coordinate ring $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ and for all $\gamma \in \mathbb{Z}$, let $(\overline{\mathcal{O}}^*)_{-\gamma} = \{x_0^{-d_{\gamma+1,\mathrm{HF}_{\mathbb{X}}(\gamma)+1}} \bar{t}^*_{\mathrm{HF}_{\mathbb{X}}(\gamma)+1}, \dots, x_0^{-d_{\gamma+1,\mu}} \bar{t}^*_\mu\}$. Then $(\overline{\mathcal{O}}^*)_{-\gamma}$ is a K-vector space basis of $(\omega_R)_{-\gamma}$ for every $\gamma \in \mathbb{Z}$ according to Proposition 4.3.7. For every $k \in \{1, \dots, \mathrm{HF}_{\mathbb{X}}(j)\}$, we have $(x_0^{d_{j,k}} t_k)(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}}, \dots, \mathcal{X}_n^{\mathrm{tr}}) = t_k(\mathcal{X}_1^{\mathrm{tr}}, \dots, \mathcal{X}_n^{\mathrm{tr}})$ and $\deg_W(x_0^{d_{j,k}} t_k) = j - \deg_W(t_k) + \deg_W(t_k) = j$. Thus Corollary 4.3.11 yields that, for every $k \in \{1, \dots, \mathrm{HF}_{\mathbb{X}}(j)\}$, the matrix $\mathcal{T}_k$ represents the K-vector space homomorphism $(\omega_R)_{-j} \to (\omega_R)_0$, $\varphi \mapsto x_0^{d_{j,k}} \bar{t}_k \cdot \varphi$, the multiplication by $x_0^{d_{j,k}} \bar{t}_k \in R_j$, with respect to the

K-vector space bases $(\overline{\mathcal{O}}^*)_{-j}$ of $(\omega_R)_{-j}$ and $(\overline{\mathcal{O}}^*)_0$ of $(\omega_R)_0$. Moreover, for all $\gamma \in \mathbb{Z}$ and $k \in \{1, \dots, \mu\}$, we let $d_{\gamma,k} = \gamma - \deg_W(t_k)$.

Now we prove that claim i) implies claim ii). Assume that there exists an element $(c_1, \dots, c_{\mathrm{HF}_{\mathbb{X}}(j)}) \in \mathcal{Z}_K(J) \setminus \{0\}$. Let $r = c_1 x_0^{d_{j,1}} \overline{t}_1 + \cdots + c_{\mathrm{HF}_{\mathbb{X}}(j)} x_0^{d_{j,\mathrm{HF}_{\mathbb{X}}(j)}} \overline{t}_{\mathrm{HF}_{\mathbb{X}}(j)}$ Then Proposition 4.1.15 yields $r \in R_j \setminus \{0\}$. As $\mathbb{X}$ is (i,j)-uniform, Theorem 5.2.7 shows that the K-vector space homomorphism $\widehat{\pi}_r^* : (\omega_R)_{-j} \to (\omega_R)_0$, $\varphi \mapsto r \cdot \varphi$, the multiplication by r, satisfies $\dim_K(\ker(\widehat{\pi}_r^*)) \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$. Moreover, Proposition 4.3.7 shows that $\dim_K((\omega_R)_{-j}) = \mathrm{HF}_{\omega_R}(-j) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$. Let $\mathcal{M}_r \in \mathrm{Mat}_{\mu-1,\mu-\mathrm{HF}_{\mathbb{X}}(j)}(K)$ denote the matrix representing $\widehat{\pi}_r^*$ with respect to $(\overline{\mathcal{O}}^*)_{-j}$ and $(\overline{\mathcal{O}}^*)_0$. As we have shown above, $\mathcal{T}_k$ represents the multiplication by $x_0^{d_{j,k}} \overline{t}_k$ with respect to the same K-vector space bases for every $k \in \{1, \dots, \mathrm{HF}_{\mathbb{X}}(j)\}$. Thus $\mathcal{M}_r = c_1 \mathcal{T}_1 + \cdots + c_{\mathrm{HF}_{\mathbb{X}}(j)} \mathcal{T}_{\mathrm{HF}_{\mathbb{X}}(j)}$. Altogether, $\mathrm{rk}_K(\mathcal{M}_r) = \dim_K((\omega_R)_{-j}) - \dim_K(\ker(\widehat{\pi}_r^*)) \geq i$ and hence there exists a non-vanishing i-minor of $\mathcal{M}_r$. This clearly contradicts $(c_1, \dots, c_{\mathrm{HF}_{\mathbb{X}}(j)}) \in \mathcal{Z}_K(J)$ and thus it follows that $\mathcal{Z}_K(J) \subseteq \{0\}$.

As the element 0 is obviously an element of the affine zero set $\mathcal{Z}_K(J)$, claim iii) is obviously equivalent to ii). Thus it remains to prove that ii) implies i). If $R_j = \{0\}$, the claim follows trivially from Theorem 5.2.7. Suppose that $R_j \neq \{0\}$ and let $r \in R_j \setminus \{0\}$. According to Proposition 4.1.15, there exists an element $(c_1, \dots, c_{\mathrm{HF}_{\mathbb{X}}(j)}) \in \mathbb{A}_K^{\mathrm{HF}_{\mathbb{X}}(j)} \setminus \{0\}$ such that $r = c_1 x_0^{d_{j,1}} \overline{t}_1 + \cdots + c_{\mathrm{HF}_{\mathbb{X}}(j)} x_0^{d_{j,\mathrm{HF}_{\mathbb{X}}(j)}} \overline{t}_{\mathrm{HF}_{\mathbb{X}}(j)}$. Let $\mathcal{M}_r \in \mathrm{Mat}_{\mu-1,\mu-\mathrm{HF}_{\mathbb{X}}(j)}(K)$ be the matrix representing the K-vector space homomorphism $\widehat{\pi}_r^* : (\omega_R)_{-j} \to (\omega_R)_0$, $\varphi \mapsto r \cdot \varphi$, the multiplication by the element r, with respect to the K-vector space bases $(\overline{\mathcal{O}}^*)_{-j}$ of $(\omega_R)_{-j}$ and $(\overline{\mathcal{O}}^*)_0$ of $(\omega_R)_0$. Then, just as we have shown above, we get $\mathcal{M}_r = c_1 \mathcal{T}_1 + \cdots + c_{\mathrm{HF}_{\mathbb{X}}(j)} \mathcal{T}_{\mathrm{HF}_{\mathbb{X}}(j)}$. Assume that $\dim_K(\ker(\widehat{\pi}_r^*)) > \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$. By Proposition 4.3.7, we have $\dim_K((\omega_R)_{-j}) = \mathrm{HF}_{\omega_R}(-j) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$. Thus we see that $\mathrm{rk}_K(\mathcal{M}_r) = \dim_K((\omega_R)_{-j}) - \dim_K(\ker(\widehat{\pi}_r^*)) < i$ and hence every i-minor of $\mathcal{M}_r$ vanishes. This is a contradiction to $(c_1, \dots, c_{\mathrm{HF}_{\mathbb{X}}(j)}) \in \mathcal{Z}_K(J) \setminus \{0\} = \emptyset$ by ii). Thus $\dim_K(\ker(\widehat{\pi}_r^*)) \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - i$ and $\mathbb{X}$ is (i,j)-uniform by Theorem 5.2.7.

For the remaining conditions, assume that K is algebraically closed. According to [KR05, Exmp. 6.3.2], we have $\mathcal{I}(\{0\}) = \langle y_1, \dots, y_{\mathrm{HF}_{\mathbb{X}}(j)} \rangle$. Thus the Ideal-Variety Correspondence [CLO07, Thm. 4.§2.7] and Hilbert's Nullstellensatz [KR00, Thm. 2.6.16] yield the equivalence of ii) and iv) and the equivalence of iii) and v). □

Example 5.2.10. Consider the zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ of Example 5.2.8, again. Recall that the polynomial ring $\overline{P} = \mathbb{Q}[x_0, x_1, x_2]$ was standard graded, that $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P} = \mathbb{Q}[x_0, x_1, x_2]$ possessed a projective $\mathcal{O}$-border basis G, where the order ideal $\mathcal{O}$ was of the form $\mathcal{O} = \{t_1, \dots, t_9\} = \{1, x_2, x_1, x_2^2, x_1 x_2, x_1^2, x_2^3, x_1 x_2^2, x_1^3\}$, and that the formal multiplication matrices of the $\mathcal{O}$-border basis $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P = \mathbb{Q}[x_1, x_2]$

were of the form

$$\mathcal{X}_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&-1&0&1&0&0\\ 0&0&0&0&0&1&0&0&2 \end{pmatrix}, \quad \mathcal{X}_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&-1&0&0\\ 0&0&1&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&2&0&0\\ 0&0&0&0&1&-1&0&1&-1\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Moreover, $\mathcal{O}$ determines both $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R$ by Proposition 4.1.15 and HF_{ω_R} by Proposition 4.3.7. For all $k \in \{1, \dots, 6\}$, we let $\mathcal{T}_k \in \mathrm{Mat}_{8,3}(\mathbb{Q})$ be the submatrix of $t_k(\mathcal{X}_1^{\mathrm{tr}}, \mathcal{X}_2^{\mathrm{tr}})$ obtained by deleting the first row and the first $\mathrm{HF}_{\mathbb{X}}(2) = 6$ columns. Then we have

$$\mathcal{T}_1 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}, \quad \mathcal{T}_2 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0\\ 0&1&0\\ 0&-1&0\\ 2&0&0\\ 0&1&0\\ 0&-1&0 \end{pmatrix}, \quad \mathcal{T}_3 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&-1&0\\ 0&0&1\\ 0&1&0\\ 0&0&0\\ 0&0&2 \end{pmatrix},$$

$$\mathcal{T}_4 = \begin{pmatrix} 1&0&0\\ 0&1&0\\ 2&0&0\\ 0&1&0\\ 0&0&0\\ 3&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}, \quad \mathcal{T}_5 = \begin{pmatrix} 0&1&0\\ 0&-1&0\\ 0&1&0\\ 0&0&0\\ 0&-1&0\\ 0&1&0\\ 0&0&0\\ 0&-1&0 \end{pmatrix}, \quad \mathcal{T}_6 = \begin{pmatrix} 0&-1&0\\ 0&0&1\\ 0&0&0\\ 0&-1&0\\ 0&0&2\\ 0&0&0\\ 0&0&0\\ 0&0&3 \end{pmatrix}.$$

Let $\{y_1, \dots, y_6\}$ be a set of further indeterminates. Then

$$y_1\mathcal{T}_1 + \cdots + y_6\mathcal{T}_6 = \begin{pmatrix} y_4 & y_5 - y_6 & 0\\ 0 & y_4 - y_5 & y_6\\ y_2 + 2y_4 & y_3 + y_5 & 0\\ 0 & y_2 - y_3 + y_4 - y_6 & 0\\ 0 & -y_2 - y_5 & y_3 + 2y_6\\ y_1 + 2y_2 + 3y_4 & y_3 + y_5 & 0\\ 0 & y_1 + y_2 + y_4 & 0\\ 0 & -y_2 - y_5 & y_1 + 2y_3 + 3y_6 \end{pmatrix}.$$

Let $J \subseteq \mathbb{Q}[y_1, \dots, y_6]$ be the ideal that is generated by the set of all 2-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_6\mathcal{T}_6 \in \mathrm{Mat}_{8,3}(\mathbb{Q}[y_1,\dots,y_6])$. Then $J = \langle f_1,\dots,f_{41}\rangle$ with

$f_1 = y_4^2 - y_4y_5,$

$f_2 = y_4y_6,$

$f_3 = y_5y_6 - y_6^2,$

$f_4 = y_3y_4 - y_2y_5 - y_4y_5 + y_2y_6 + 2y_4y_6,$

$f_5 = y_2y_4 - y_3y_4 + y_4^2 - y_4y_6,$

$f_6 = -y_2y_4 - y_4y_5,$

$f_7 = y_3y_4 + 2y_4y_6,$

$f_8 = y_3y_5 - y_3y_6 + 2y_5y_6 - 2y_6^2,$

$f_9 = y_3y_4 - y_1y_5 - 2y_2y_5 - 2y_4y_5 + y_1y_6 + 2y_2y_6 + 3y_4y_6,$

$f_{10} = y_1y_4 + y_2y_4 + y_4^2,$

$f_{11} = y_1y_4 + 2y_3y_4 + 3y_4y_6,$

$f_{12} = y_1y_5 + 2y_3y_5 - y_1y_6 - 2y_3y_6 + 3y_5y_6 - 3y_6^2,$

$f_{13} = -y_2y_4 - 2y_4^2 + y_2y_5 + 2y_4y_5,$

$f_{14} = -y_2y_6 - 2y_4y_6,$

$f_{15} = -y_3y_6 - y_5y_6,$

$f_{16} = -y_2y_6 + y_3y_6 - y_4y_6 + y_6^2,$

$f_{17} = y_3y_4 - y_3y_5 + y_2y_6 + 2y_4y_6 - y_5y_6,$

$f_{18} = -y_1y_4 - 2y_2y_4 - 3y_4^2 + y_1y_5 + 2y_2y_5 + 3y_4y_5,$

$f_{19} = -y_1y_6 - 2y_2y_6 - 3y_4y_6,$

$f_{20} = -y_1y_6 - y_2y_6 - y_4y_6,$

$f_{21} = y_1y_4 + 2y_3y_4 - y_1y_5 - 2y_3y_5 + y_2y_6 + 3y_4y_6 - 2y_5y_6,$

$f_{22} = y_2^2 - y_2y_3 + 3y_2y_4 - 2y_3y_4 + 2y_4^2 - y_2y_6 - 2y_4y_6,$

$f_{23} = -y_2^2 - 2y_2y_4 - y_2y_5 - 2y_4y_5,$

$f_{24} = y_2y_3 + 2y_3y_4 + 2y_2y_6 + 4y_4y_6,$

$f_{25} = y_3^2 + y_3y_5 + 2y_3y_6 + 2y_5y_6,$

$f_{26} = -y_1y_3 - y_2y_3 - y_3y_4 - y_1y_5 - y_2y_5 - y_4y_5,$

$f_{27} = y_1y_2 + y_2^2 + 2y_1y_4 + 3y_2y_4 + 2y_4^2,$

$f_{28} = y_1y_2 + 2y_2y_3 + 2y_1y_4 + 4y_3y_4 + 3y_2y_6 + 6y_4y_6,$

$f_{29} = y_1y_3 + 2y_3^2 + y_1y_5 + 2y_3y_5 + 3y_3y_6 + 3y_5y_6,$

$f_{30} = y_2y_3 - y_3^2 + y_3y_4 + 2y_2y_6 - 3y_3y_6 + 2y_4y_6 - 2y_6^2,$

$f_{31} = -y_1y_2 - 2y_2^2 + y_1y_3 + 2y_2y_3 - y_1y_4 - 5y_2y_4 + 3y_3y_4 - 3y_4^2 + y_1y_6 + 2y_2y_6 + 3y_4y_6,$

$f_{32} = y_1y_2 - y_1y_3 + 2y_2y_3 - 2y_3^2 + y_1y_4 + 2y_3y_4 - y_1y_6 + 3y_2y_6 - 5y_3y_6 + 3y_4y_6 - 3y_6^2,$

$f_{33} = y_1y_2 + 2y_2^2 + 3y_2y_4 + y_1y_5 + 2y_2y_5 + 3y_4y_5,$

$f_{34} = -y_1y_3 - 2y_2y_3 - 3y_3y_4 - 2y_1y_6 - 4y_2y_6 - 6y_4y_6,$

$f_{35} = -y_3^2 - y_3y_5 - 2y_3y_6 - 2y_5y_6,$

$$
\begin{aligned}
f_{36} &= -y_1y_3 - y_2y_3 - y_3y_4 - 2y_1y_6 - 2y_2y_6 - 2y_4y_6,\\
f_{37} &= -y_1y_2 - y_2y_3 - y_1y_5 - y_3y_5 - y_2y_6 - y_5y_6,\\
f_{38} &= y_1^2 + 3y_1y_2 + 2y_2^2 + 4y_1y_4 + 5y_2y_4 + 3y_4^2,\\
f_{39} &= -y_1y_2 - 2y_2^2 - 3y_2y_4 - y_1y_5 - 2y_2y_5 - 3y_4y_5,\\
f_{40} &= y_1^2 + 2y_1y_2 + 2y_1y_3 + 4y_2y_3 + 3y_1y_4 + 6y_3y_4 + 3y_1y_6 + 6y_2y_6 + 9y_4y_6,\\
f_{41} &= y_1^2 + y_1y_2 + 2y_1y_3 + 2y_2y_3 + y_1y_4 + 2y_3y_4 + 3y_1y_6 + 3y_2y_6 + 3y_4y_6.
\end{aligned}
$$

Since $f_k((0,0,1,0,-1,-1)) = 0$ for every index $k \in \{1,\dots,41\}$, we see that the point $0 \neq (0,0,1,0,-1,-1) \in \mathcal{Z}(J)$ and thus $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ is not $(2,2)$-uniform according to Corollary 5.2.9. Note that this result coincides with the result of Example 5.2.8.

The remaining part of this subsection is devoted to specific uniformity conditions, namely the Cayley-Bacharach property and i-uniformity condition, cf. Definition 5.2.1.

Definition 5.2.11. Let $\pi : U \otimes_K V \to W$ be a K-vector space homomorphism of finite dimensional K-vector spaces U, V, and W. We call π **biinjective** if for all $u \in U$ and $v \in V$, $\pi(u \otimes v) = 0$ implies $u = 0$ or $v = 0$.

Remark 5.2.12. Let ω_R denote the canonical module of R.

a) Let $\pi : R_{\sigma_{\mathbb{X}}} \otimes_K (\omega_R)_{-\sigma_{\mathbb{X}}} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r \cdot \varphi$ be the multiplication map. Since $x_0 \in R$ is non-zero divisor for R according to Proposition 5.1.8, we have $x_0^{\sigma_{\mathbb{X}}} \cdot \varphi \neq 0$ for all $\varphi \in (\omega_R)_{-\sigma_{\mathbb{X}}} \setminus \{0\}$, i.e. π is non-degenerate in the second argument. By Definition 5.2.1 and Theorem 5.2.7, the scheme $\mathbb{X}$ is a Cayley-Bacharach scheme if and only if for every element $r \in R_{\sigma_{\mathbb{X}}} \setminus \{0\}$, the K-vector space homomorphism $\widehat{\pi}_r^* : (\omega_R)_{-\sigma_{\mathbb{X}}} \to (\omega_R)_0$, $\varphi \mapsto r \cdot \varphi$, the multiplication by r, satisfies $\dim_K(\ker(\widehat{\pi}_r^*)) \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}}) - 1$. According to Proposition 4.3.7, $\dim_K((\omega_R)_{-\sigma_{\mathbb{X}}}) = \mathrm{HF}_{\omega_R}(-\sigma_{\mathbb{X}}) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}})$. Thus $\mathbb{X}$ is (i,j)-uniform if and only if there is no $r \in R_{\sigma_{\mathbb{X}}}$ that satisfies $r \cdot (\omega_R)_{-\sigma_{\mathbb{X}}} = \{0\}$, i.e. if and only if π is non-degenerate in the first argument. Altogether, it follows that $\mathbb{X}$ is a Cayley-Bacharach scheme if and only if the multiplication map π is non-degenerate. In particular, with the assumption that the base field K is algebraically closed and that $\overline{P}$ is standard graded, this coincides with a previous result in [Kre94, Thm. 2.6].

b) Let $i \in \mathbb{N}$. By Lemma 5.2.5 and Theorem 5.2.7, $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i))$-uniform if and only if for all $k \in \{i,\dots,\sigma_{\mathbb{X}}\}$ and for all $r \in R_k \setminus \{0\}$, the K-vector space homomorphism $\widehat{\pi}_r^* : (\omega_R)_k \to (\omega_R)_0$, $\varphi \mapsto r \cdot \varphi$, the multiplication by r, satisfies $\dim_K(\ker(\widehat{\pi}_r^*)) = 0$. Equivalently, for all $k \in \{i,\dots,\sigma_{\mathbb{X}}\}$, $r \in R_k$, and $\varphi \in (\omega_R)_{-k}$, $r \cdot \varphi = 0$ implies $r = 0$ or $\varphi = 0$. In other words, $\mathbb{X}$ is $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i))$-uniform if and only if the multiplication map $R_k \otimes_K (\omega_R)_{-k} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r \cdot \varphi$ is biinjective for every $k \in \{i,\dots,\sigma_{\mathbb{X}}\}$. Note that, with the assumption that the base field K is algebraically closed and that $\overline{P}$ is standard graded, this result coincides with [Kre94, Thm. 3.2].

5.2.2 The Reduced Case

In this subsection, we additionally assume that the subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is reduced. Since $\mathbb{X}$ is supposed to have a K-rational support, Proposition 5.1.7 yields that $\mathrm{Supp}(\mathbb{X}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{\deg(\mathbb{X})}\}$ where for all $i \in \{1, \ldots, \deg(\mathbb{X})\}$, $\mathfrak{p}_i = \mathcal{I}^+(\{p_i\})$ is the homogeneous vanishing ideal of a K-rational point $p_i \in \mathbb{P}_K(\overline{W})$. Moreover, for every $i \in \{1, \ldots, \deg(\mathbb{X})\}$, there is a subscheme $\widehat{\mathbb{X}}_i \subseteq \mathbb{X}$ with $\mathrm{Supp}(\widehat{\mathbb{X}}_i) = \mathrm{Supp}(\mathbb{X}) \setminus \{\mathfrak{p}_i\}$ and it satisfies $\deg(\widehat{\mathbb{X}}_i) = \deg(\mathbb{X}) - 1$. For all $i \in \{1, \ldots, \deg(\mathbb{X})\}$, we let $f_i \in R_{\sigma_{\mathbb{X}}+1}$ be a separator of the subscheme $\widehat{\mathbb{X}}_i \subseteq \mathbb{X}$ as defined in Definition 5.1.17.

The following proposition states that the separators $f_1, \ldots, f_{\deg(\mathbb{X})} \in R_{\sigma_{\mathbb{X}}+1}$ allow us to describe the structure of the subschemes of $\mathbb{X}$. Note that the second result has been proven in [GKR93, Prop. 1.13] for the base field K being algebraically closed and $\overline{P}$ being standard graded

Proposition 5.2.13. *Let $i \in \{1, \ldots, \deg(\mathbb{X})\}$, let $\nu_1, \ldots, \nu_i \in \{1, \ldots, \deg(\mathbb{X})\}$ be pairwise distinct elements, and let $\mathbb{Y} \subseteq \mathbb{X}$ be the subscheme with* $\mathrm{Supp}(\mathbb{Y}) = \{\mathfrak{p}_{\nu_1}, \ldots, \mathfrak{p}_{\nu_i}\}$.

a) We have $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) = \langle f_{\nu_1}, \ldots, f_{\nu_i}\rangle^{\mathrm{sat}} \subseteq R$.

b) For every $j \in \mathbb{N}$ with $j \geq \sigma_{\mathbb{X}} + 1$, the set $\{x_0^{j-(\sigma_{\mathbb{X}}+1)} f_{\nu_1}, \ldots, x_0^{j-(\sigma_{\mathbb{X}}+1)} f_{\nu_i}\}$ is a K-vector space basis of $(\langle f_{\nu_1}, \ldots, f_{\nu_i}\rangle)_j$.

Proof. For all $k \in \{1, \ldots, \deg(\mathbb{X})\}$, let $F_k \in \overline{P}_{\sigma_{\mathbb{X}}+1}$ be such that $f_k = \overline{F}_k$. According to Definition 5.1.17, we have $F_k(p_\ell) = \delta_{k\ell}$ for all $k, \ell \in \{1, \ldots, \deg(\mathbb{X})\}$, i.e. the set $\{f_{\nu_1}, \ldots, f_{\nu_i}\} \subseteq R_{\sigma_{\mathbb{X}}+1}$ is K-linearly independent.

First we prove a). Let $\mathbb{Y}' \subseteq \mathbb{X}$ be the subscheme that is scheme-theoretically defined by $\langle f_{\nu_1}, \ldots, f_{\nu_i}\rangle \subseteq \mathcal{I}^+_{\mathbb{X}}(\mathbb{Y})$. Then $\mathbb{Y}' \supseteq \mathbb{Y}$. Since $\{f_{\nu_1}, \ldots, f_{\nu_i}\} \subseteq R_{\sigma_{\mathbb{X}}+1}$ is K-linearly independent and since $\mathrm{Supp}(\mathbb{Y}) = \{\mathfrak{p}_{\nu_1}, \ldots, \mathfrak{p}_{\nu_i}\}$, we get $\deg(\mathbb{Y}') = i = \deg(\mathbb{Y})$. Therefore, $\mathbb{Y}' = \mathbb{Y}$ and Proposition 5.1.2 yields that $\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) = \langle f_{\nu_1}, \ldots, f_{\nu_i}\rangle^{\mathrm{sat}}$.

In order to prove claim b), let $j \in \mathbb{N}$ be such that $j \geq \sigma_{\mathbb{X}} + 1$. Since $\mathbb{Y} \subseteq \mathbb{X}$ implies that $\mathbb{Y} \cap H^{\mathrm{inf}} \subseteq \mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, Proposition 5.1.8 and the Second Noetherian Isomorphism Theorem imply that the element $x_0 \in \overline{P}$ is a non-zero divisor for $R/\mathcal{I}^+_{\mathbb{X}}(\mathbb{Y}) \cong \overline{P}/\mathcal{I}^+(\mathbb{Y})$. Therefore, the set $\{x_0^{j-(\sigma_{\mathbb{X}}+1)} f_{\nu_1}, \ldots, x_0^{j-(\sigma_{\mathbb{X}}+1)} f_{\nu_i}\} \subseteq R_j$ is K-linearly independent, too. As $\sigma_{\mathbb{Y}} + 1 \leq \sigma_{\mathbb{X}} + 1 \leq j$, we get $\mathrm{HF}_{\mathbb{Y}}(j) = \deg(\mathbb{Y}) = i$ and the claim follows. □

Now we can characterize reduced (i,j)-uniform schemes by means of the separators.

Theorem 5.2.14. (Characterization of Reduced (i,j)-Uniform Schemes)
Let $i \in \{1, \ldots, \deg(\mathbb{X}) - 1\}$ and let $j \in \{1, \ldots, \sigma_{\mathbb{X}}\}$. Then the following conditions are equivalent.

i) The reduced zero-dimensional scheme $\mathbb{X}$ is (i,j)-uniform.

ii) Every subset of i elements from $\{\overline{f}_1, \ldots, \overline{f}_{\deg(\mathbb{X})}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ is K-linearly independent.

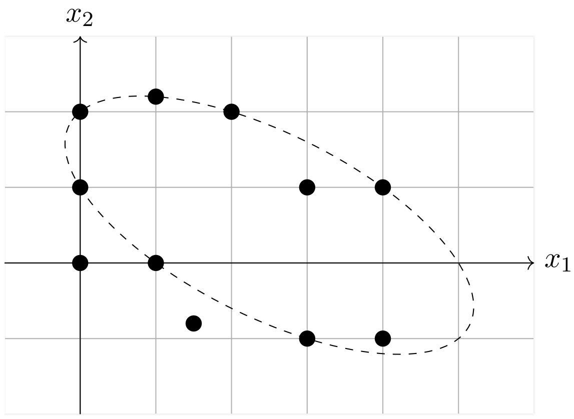
Proof. First we prove that i) implies ii). Suppose that the reduced zero-dimensional scheme $\mathbb{X}$ is (i,j)-uniform. Let $\nu_1,\dots,\nu_i \in \{1,\dots,\deg(\mathbb{X})\}$ be pairwise distinct elements and let $\mathbb{Y} \subseteq \mathbb{X}$ be the subscheme that satisfies $\mathrm{Supp}(\mathbb{Y}) = \mathrm{Supp}(\mathbb{X}) \setminus \{\mathfrak{p}_{\nu_1},\dots,\mathfrak{p}_{\nu_i}\}$. Then we have $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$. For a contradiction, assume that the set of residue classes $\{\overline{f}_{\nu_1},\dots,\overline{f}_{\nu_i}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ is K-linearly dependent. Then there exist a $g \in R_j \setminus \{0\}$ and $\lambda_1,\dots,\lambda_i \in K$ such that $x_0^{\sigma_{\mathbb{X}}+1-j} g = \lambda_1 f_{\nu_1} + \cdots + \lambda_i f_{\nu_i}$. In this situation, we have $\langle f_{\nu_1},\dots,f_{\nu_i}\rangle \subseteq \mathcal{I}_{\mathbb{X}}^+(\mathbb{Y})$ by Proposition 5.2.13. As the scheme $\mathbb{X}$ is (i,j)-uniform, $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ according to Definition 5.2.1 and thus $(\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_j = \{0\}$. Moreover, $\langle f_{\nu_1},\dots,f_{\nu_i}\rangle \subseteq \langle f_{\nu_1},\dots,f_{\nu_i}\rangle^{\mathrm{sat}} = \mathcal{I}_{\mathbb{X}}^+(\mathbb{Y})$ by the Propositions 5.1.2 and 5.2.13 and the ideal $\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y})$ is saturated by Proposition 5.1.2. Hence $x_0^{\sigma_{\mathbb{X}}+1-j} g \in \langle f_{\nu_1},\dots,f_{\nu_i}\rangle$ together with Lemma 5.1.13 imply $g \in (\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_j \setminus \{0\}$ in contradiction to $(\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_j = \{0\}$. Altogether, it follows that $\{\overline{f}_{\nu_1},\dots,\overline{f}_{\nu_i}\} \subseteq R/\langle x_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ is K-linearly independent.

Next we prove that ii) implies i). Suppose that all subsets of i elements from the set $\{\overline{f}_1,\dots,\overline{f}_{\deg(\mathbb{X})}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ are K-linearly independent and let $\mathbb{Y} \subseteq \mathbb{X}$ be a subscheme with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$. As $\mathbb{X}$ is reduced and $\mathrm{Supp}(\mathbb{X}) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_{\deg(\mathbb{X})}\}$, there are pairwise distinct elements $\nu_1,\dots,\nu_i \in \{1,\dots,\deg(\mathbb{X})\}$ with the property that $\mathrm{Supp}(\mathbb{Y}) = \mathrm{Supp}(\mathbb{X}) \setminus \{\mathfrak{p}_{\nu_1},\dots,\mathfrak{p}_{\nu_i}\}$. By Proposition 5.2.13, $\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}) = \langle f_{\nu_1},\dots,f_{\nu_i}\rangle^{\mathrm{sat}}$. Assume that $\mathrm{HF}_{\mathbb{Y}}(j) \neq \mathrm{HF}_{\mathbb{X}}(j)$. Then we have $\mathrm{HF}_{\mathbb{Y}}(j) < \mathrm{HF}_{\mathbb{X}}(j)$ as $\mathbb{Y} \subseteq \mathbb{X}$ and this implies the existence of an element $g \in (\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}))_j \setminus \{0\}$. In particular, Lemma 5.1.13 and Proposition 5.2.13 yield $x_0^k g \in \langle f_{\nu_1},\dots,f_{\nu_i}\rangle$ for some $k \in \mathbb{N}$. Since $f_r \in R_{\sigma_{\mathbb{X}}+1}$ for all $r \in \{1,\dots,\deg(\mathbb{X})\}$ by Definition 5.1.17, we have $(\langle f_{\nu_1},\dots,f_{\nu_i}\rangle)_\ell = \{0\}$ for all $\ell \in \{0,\dots,\sigma_{\mathbb{X}}\}$. As $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, x_0 is a non-zero divisor for R by Proposition 5.1.8. Thus it follows that $k \geq \sigma_{\mathbb{X}} + 1 - j$, i.e. $k + j \geq \sigma_{\mathbb{X}} + 1$. By Proposition 5.2.13, there exist scalars $\lambda_1,\dots,\lambda_i \in K$ such that $x_0^k g = \lambda_1 x_0^{k+j-(\sigma_{\mathbb{X}}+1)} f_{\nu_1} + \cdots + \lambda_i x_0^{k+j-(\sigma_{\mathbb{X}}+1)} f_{\nu_i}$. Since we have $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$, x_0 is a non-zero divisor for R by Proposition 5.1.8. Thus $x_0^{\sigma_{\mathbb{X}}+1-j} g = x_0^{k-(k+j-(\sigma_{\mathbb{X}}+1))} g = \lambda_1 f_{\nu_1} + \cdots + \lambda_i f_{\nu_i}$ and this implies $0 = \lambda_1 \overline{f}_{\nu_1} + \cdots + \lambda_i \overline{f}_{\nu_i}$ in $R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$. As $\{\overline{f}_{\nu_1},\dots,\overline{f}_{\nu_i}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ is K-linearly independent, we get $\lambda_1 = \cdots = \lambda_i = 0$ and, therefore, $x_0^{\sigma_{\mathbb{X}}+1-j} g = 0$. Since x_0 is a non-zero divisor for R, we get the contradiction $g = 0$. Altogether, it follows that $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ and $\mathbb{X}$ is (i,j)-uniform by Definition 5.2.1. □

Remark 5.2.15. Let $i \in \{1,\dots,\Delta\,\mathrm{HF}_{\mathbb{X}}(\sigma_{\mathbb{X}}+1)\}$. According to Lemma 5.2.4, $\mathbb{X}$ is i-uniform if and only if $\mathbb{X}$ is $(i,\sigma_{\mathbb{X}})$-uniform. By Theorem 5.2.14, this is equivalent for every subset of i elements from $\{\overline{f}_1,\dots,\overline{f}_{\deg(\mathbb{X})}\} \subseteq R/\langle \overline{x}_0\rangle$ to be K-linearly independent. Note that, with the additional assumption that the base field K is algebraically closed and that $\overline{P}$ is standard graded, this result coincides with [Kre94, Prop. 3.4].

The next example shows a reduced subscheme that is not $(4,2)$-uniform with respect to the standard grading but $(4,2)$-uniform with respect to a certain non-standard grading and give a geometric interpretation of (i,j)-uniform schemes in the reduced case.

Example 5.2.16. Let $P = \mathbb{Q}[x_1,x_2]$ be $\mathbb{Z}$-graded by the matrix $(w_1,w_2) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ with $w_1,w_2 \in \mathbb{N}\setminus\{0\}$ and let $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$ be $\mathbb{Z}$-graded by $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$. Moreover, let $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}(\overline{W}) = \mathrm{Proj}(\overline{P})$ be the reduced zero-dimensional subscheme of degree $\deg(\mathbb{X}) = 11$ with $\operatorname{Supp}(\mathbb{X}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_{11}\}$ where $\mathfrak{p}_i = \mathcal{I}^+(p_i)$ for all $i \in \{1, \dots, 11\}$ and $p_1 = (1:0:0)$, $p_2 = (1:0:1)$, $p_3 = (1:0:2)$, $p_4 = (1:1:0)$, $p_5 = (1:1:\frac{11}{5})$, $p_6 = (1:\frac{3}{2}:-\frac{4}{5})$, $p_7 = (1:2:2)$, $p_8 = (1:3:-1)$, $p_9 = (1:3:1)$, $p_{10} = (1:4:-1)$, and $p_{11} = (1:4:1)$. Then $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$ and all the points above are $\mathbb{Q}$-rational. By Definition 5.1.3 we can identify the points of $\mathbb{X}$, with the corresponding affine part in $\mathbb{A}^2_{\mathbb{Q}} = \operatorname{Spec}(P)$ as in the following picture.



Let $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$. Using Proposition 5.1.6 and [KR00, Prop. 3.2.7], we can compute $\mathcal{I}^+(\mathbb{X}) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{11}$. Therefore, we can compute the Hilbert function $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R$ of the scheme $\mathbb{X}$ using Proposition 4.1.15 and Corollary 4.1.14. We check whether $\mathbb{X}$ is $(4,2)$-uniform in two cases.

a) First let $W = (1,1) \in \operatorname{Mat}_{1,2}(\mathbb{Z})$, i.e. we equip $\overline{P}$ with the standard grading. Then $\mathrm{HF}_{\mathbb{X}} : 1, 3, 6, 10, 11, 11, \dots$ and thus $\sigma_{\mathbb{X}} = 3$. Let $F_1, F_6, F_9, F_{10} \in \overline{P}_4$ be defined by

$$\begin{aligned}
F_1 ={}& \tfrac{115}{2893}x_2^4 - \tfrac{302}{2893}x_0x_1^3 - \tfrac{203}{2893}x_0x_1^2x_2 - \tfrac{137}{2893}x_0x_1x_2^2 + \tfrac{809}{5786}x_0x_2^3 \\
&+ \tfrac{2317}{2893}x_0^2x_1^2 + \tfrac{1421}{2893}x_0^2x_1x_2 - \tfrac{52}{263}x_0^2x_2^2 - \tfrac{4908}{2893}x_0^3x_1 - \tfrac{5681}{5786}x_0^3x_2 + x_0^4, \\
F_6 ={}& -\tfrac{4375}{52074}x_2^4 + \tfrac{2600}{26037}x_0x_1^3 - \tfrac{4000}{26037}x_0x_1^2x_2 - \tfrac{2800}{8679}x_0x_1x_2^2 - \tfrac{3875}{8679}x_0x_2^3 \\
&- \tfrac{14200}{26037}x_0^2x_1^2 + \tfrac{28000}{26037}x_0^2x_1x_2 + \tfrac{9125}{4734}x_0^2x_2^2 + \tfrac{11600}{26037}x_0^3x_1 - \tfrac{12125}{8679}x_0^3x_2, \\
F_9 ={}& \tfrac{9775}{416592}x_2^4 - \tfrac{1966}{26037}x_0x_1^3 - \tfrac{6322}{26037}x_0x_1^2x_2 - \tfrac{7115}{17358}x_0x_1x_2^2 - \tfrac{8135}{23144}x_0x_2^3 \\
&+ \tfrac{11405}{26037}x_0^2x_1^2 + \tfrac{62471}{52074}x_0^2x_1x_2 + \tfrac{33715}{37872}x_0^2x_2^2 - \tfrac{9439}{26037}x_0^3x_1 - \tfrac{39035}{69432}x_0^3x_2, \\
F_{10} ={}& -\tfrac{7475}{46288}x_2^4 + \tfrac{142}{2893}x_0x_1^3 - \tfrac{441}{2893}x_0x_1^2x_2 - \tfrac{695}{5786}x_0x_1x_2^2 + \tfrac{4935}{23144}x_0x_2^3 \\
&- \tfrac{553}{2893}x_0^2x_1^2 + \tfrac{3281}{5786}x_0^2x_1x_2 + \tfrac{2065}{4208}x_0^2x_2^2 + \tfrac{411}{2893}x_0^3x_1 - \tfrac{12555}{23144}x_0^3x_2
\end{aligned}$$

and for all $i \in \{1,6,9,10\}$, let $f_i = \overline{F}_i \in R_4$. For all $i \in \{1,6,9,10\}$ and for all $j \in \{1, \dots, 11\}$, we have $F_i(p_j) = \delta_{ij}$, i.e. the element $f_i \in R_4$ is a separator of the subscheme $\widehat{\mathbb{X}}_i \subseteq \mathbb{X}$ with $\operatorname{Supp}(\widehat{\mathbb{X}}_i) = \operatorname{Supp}(\mathbb{X}) \setminus \{\mathfrak{p}_i\}$ by Definition 5.1.17. In the residue class ring $R/\langle \overline{x}_0^2 \rangle$, we have $100\overline{f}_1 + 69\overline{f}_6 - 60\overline{f}_9 - 20\overline{f}_{10} = 0$, i.e. $\{\overline{f}_1, \overline{f}_6, \overline{f}_9, \overline{f}_{10}\} \subseteq R/\langle \overline{x}^2 \rangle$ is $\mathbb{Q}$-linearly dependent. According to Theorem 5.2.14, the zero-dimensional subscheme $\mathbb{X}$ is thus not $(4,2)$-uniform with respect to the

standard grading.
Geometrically, we can interpret this result as follows. As indicated in the picture above, there exists an ellipse that passes through all projective points in the set $\{p_1,\dots,p_{11}\}\setminus\{p_1,p_6,p_9,p_{10}\}$. But since this ellipse that does not pass through all points in $\{p_1,\dots,p_{11}\}$, the value of the Hilbert function of the subscheme $\mathbb{Y}\subseteq\mathbb{X}$ with $\mathrm{Supp}(\mathbb{Y})=\mathrm{Supp}(\mathbb{X})\setminus\{\mathfrak{p}_1,\mathfrak{p}_6,\mathfrak{p}_9,\mathfrak{p}_{10}\}$ at position 2 is less than the value of the Hilbert function of $\mathbb{X}$ at position 2. Therefore, $\mathbb{X}$ is not $(4,2)$-uniform by Definition 5.2.1.

b) Second we let $W=(1,2)\in\mathrm{Mat}_{1,2}(\mathbb{Z})$. Then $\mathrm{HF}_{\mathbb{X}}: 1,2,4,6,9,11,11,\dots$ and thus $\sigma_{\mathbb{X}}=4$. Let $F_1,\dots,F_{11}\in\overline{P}_5$ be with

$$\begin{aligned}
F_1 &= \tfrac{551}{3000}x_1^5-\tfrac{7}{120}x_1^3x_2-\tfrac{2389}{1200}x_0x_1^4+\tfrac{17}{60}x_0x_1^2x_2+\tfrac{1}{2}x_0x_2^2\\
&\quad+\tfrac{381}{50}x_0^2x_1^3+\tfrac{7}{40}x_0^2x_1x_2-\tfrac{2807}{240}x_0^3x_1^2-\tfrac{3}{2}x_0^4x_2+\tfrac{4883}{1000}x_0^4x_1+x_0^5,\\
F_2 &= -\tfrac{3053}{4500}x_1^5+\tfrac{1}{5}x_1^3x_2+\tfrac{12767}{1800}x_0x_1^4-\tfrac{37}{30}x_0x_1^2x_2-x_0x_2^2+\tfrac{37}{30}x_0^2x_1x_2\\
&\quad-\tfrac{11833}{450}x_0^2x_1^3+\tfrac{14501}{360}x_0^3x_1^2+2x_0^3x_2-\tfrac{30599}{1500}x_0^4x_1,\\
F_3 &= \tfrac{467}{1000}x_1^5-\tfrac{17}{120}x_1^3x_2-\tfrac{1913}{400}x_0x_1^4+\tfrac{19}{20}x_0x_1^2x_2+\tfrac{1}{2}x_0x_2^2\\
&\quad+\tfrac{2593}{150}x_0^2x_1^3-\tfrac{169}{120}x_0^2x_1x_2-\tfrac{2059}{80}x_0x_1^2-\tfrac{1}{2}x_0x_2+\tfrac{38299}{3000}x_0x_1,\\
F_4 &= \tfrac{29}{33}x_1^5-\tfrac{5}{66}x_1^3x_2-\tfrac{581}{66}x_0x_1^4+\tfrac{35}{66}x_0x_1^2x_2\\
&\quad+\tfrac{683}{22}x_0^2x_1^3-\tfrac{10}{11}x_0^2x_1x_2-\tfrac{1498}{33}x_0^3x_1^2+\tfrac{256}{11}x_0^4x_1,\\
F_5 &= -\tfrac{6}{11}x_1^5+\tfrac{5}{66}x_1^3x_2+\tfrac{175}{33}x_0x_1^4-\tfrac{35}{66}x_0x_1^2x_2\\
&\quad-\tfrac{590}{33}x_0^2x_1^3+\tfrac{10}{11}x_0^2x_1x_2+\tfrac{805}{33}x_0^3x_1^2-\tfrac{124}{11}x_0^4x_1,\\
F_6 &= -\tfrac{32}{45}x_1^5+\tfrac{64}{9}x_0x_1^4-\tfrac{224}{9}x_0^2x_1^3+\tfrac{320}{9}x_0^3x_1^2-\tfrac{256}{15}x_0^4x_1,\\
F_7 &= \tfrac{1}{2}x_1^5-\tfrac{19}{4}x_0x_1^4+\tfrac{31}{2}x_0^2x_1^3-\tfrac{81}{4}x_0^3x_1^2+9x_0^4x_1,\\
F_8 &= \tfrac{11}{30}x_1^5+\tfrac{1}{12}x_1^3x_2-\tfrac{43}{12}x_0x_1^4-\tfrac{5}{12}x_0x_1^2x_2\\
&\quad+\tfrac{145}{12}x_0^2x_1^3+\tfrac{1}{3}x_0^2x_1x_2-\tfrac{49}{3}x_0^3x_1^2+\tfrac{112}{15}x_0^4x_1,\\
F_9 &= -\tfrac{43}{90}x_1^5-\tfrac{1}{12}x_1^3x_2+\tfrac{163}{36}x_0x_1^4+\tfrac{5}{12}x_0x_1^2x_2\\
&\quad-\tfrac{533}{36}x_0^2x_1^3-\tfrac{1}{3}x_0^2x_1x_2+\tfrac{176}{9}x_0^3x_1^2-\tfrac{44}{5}x_0^4x_1,\\
F_{10} &= -\tfrac{61}{600}x_1^5-\tfrac{1}{24}x_1^3x_2+\tfrac{239}{240}x_0x_1^4+\tfrac{1}{6}x_0x_1^2x_2\\
&\quad-\tfrac{67}{20}x_0^2x_1^3-\tfrac{1}{8}x_0^2x_1x_2+\tfrac{217}{48}x_0^3x_1^2-\tfrac{413}{200}x_0^4x_1,\\
F_{11} &= \tfrac{71}{600}x_1^5+\tfrac{1}{24}x_1^3x_2-\tfrac{269}{240}x_0x_1^4-\tfrac{1}{6}x_0x_1^2x_2\\
&\quad+\tfrac{221}{60}x_0^2x_1^3+\tfrac{1}{8}x_0^2x_1x_2-\tfrac{235}{48}x_0^3x_1^2+\tfrac{443}{200}x_0^4x_1
\end{aligned}$$

and for all $i\in\{1,\dots,11\}$, let $f_i=\overline{F}_i\in R_5$. For all $i,j\in\{1,\dots,11\}$, we have $F_i(p_j)=\delta_{ij}$, i.e. $f_i\in R_5$ is a separator of the subscheme $\widehat{\mathbb{X}}_i\subseteq\mathbb{X}$ that satisfies $\mathrm{Supp}(\widehat{\mathbb{X}}_i)=\mathrm{Supp}(\mathbb{X})\setminus\{\mathfrak{p}_i\}$ by Definition 5.1.17. Using linear algebra techniques, we can easily check that all subsets of four elements of $\{\overline{f}_1,\dots,\overline{f}_{11}\}\subseteq R/\langle\overline{x}_0^3\rangle$ are $\mathbb{Q}$-linearly independent. Thus the zero-dimensional subscheme $\mathbb{X}$ is $(4,2)$-uniform in this non-standard graded case according to Theorem 5.2.14.

Geometrically, we can interpret this result, too. Consider the above picture, again. Since x_2 has degree 2 and x_1 has degree 1, the only possible type of a curve of degree 2 that destroys the $(4,2)$-uniformity is a hyperbola. But we see that there is obviously no hyperbola passing through all but four points of $\mathbb{X}$. Therefore, the value of the Hilbert function of any subscheme $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg(\mathbb{Y}) = 7$ at position 2 equals the value of the Hilbert function of $\mathbb{X}$ at position 2, i.e. $\mathbb{X}$ is $(4,2)$-uniform by Definition 5.2.1.

5.3 Algorithms for Checking Uniformity Conditions

In this section, we turn the results of the previous section into algorithms that check uniformity conditions of non-empty zero-dimensional subschemes of $\mathbb{P}_K(\overline{W})$. These algorithms provide an affirmative answer to (generalizations of) [Kre01, Questions 1 and 3]. Whereas [Kre01, Question 1] was already affirmatively answered in the reduced case in [MP04] using Chow forms, cf. Remark 5.3.6, no answer to [Kre01, Question 3] has been known up to now.

As in the previous section, we fix the following notation. We let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ always be a non-empty zero-dimensional projective subscheme with K-rational support and $\mathbb{X} \cap H^{\inf} = \emptyset$. Moreover, we let $R = \overline{P}/\mathcal{I}^+(\mathbb{X})$ be the projective coordinate ring of $\mathbb{X}$ and ω_R be the canonical module of R. The multigraded Hilbert function of $\mathbb{X}$ is denoted by $H = \mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_R : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \dim_K(R_\gamma)$, its Castelnuovo function by $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma - 1)$, and we let $\sigma_{\mathbb{X}} = \max\{i \in \mathbb{Z} \mid H(i) < \deg(\mathbb{X})\}$.

First we study the general case. In this situation, we assume that the subscheme $\mathbb{X}$ is given by a generating set of its defining ideal $\mathcal{I}^+(\mathbb{X})$. Then for an arbitrary index $k \in \{1, \dots, \sigma_{\mathbb{X}}\}$, we determine necessary and sufficient conditions for the multiplication map $R_k \otimes_K (\omega_R)_{-k} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r \cdot \varphi$ to be biinjective in Proposition 5.3.1. If K is algebraically closed, we can use these characterizations to check whether $\mathbb{X}$ is $(\deg(\mathbb{X}) - H(j))$-uniform for arbitrary $j \in \mathbb{N}$ as shown in Remark 5.3.2. After that, we turn Corollary 5.2.9 into an algorithm in Proposition 5.3.3. This allows us to check whether $\mathbb{X}$ is (i,j)-uniform for arbitrary $j \in \mathbb{N}$ and $i \in \{1, \dots, \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)\}$ in the algebraically closed case.

In the second subsection, we restrict ourselves to the reduced case. In this situation, we assume that $\mathbb{X}$ is given by its support, i.e. the set of all points vanishing on $\mathcal{I}^+(\mathbb{X})$. We start with an appropriate version of the Buchberger-Möller Algorithm in Proposition 5.3.7 for border bases that also takes the underlying grading W into account in order to compute the defining ideal of $\mathbb{X}$. After that, we turn Theorem 5.2.14 into an algorithm in Proposition 5.3.9.

5.3.1 The General Case

For the whole subsection, we assume that the base field K is algebraically closed. This is necessary since all the methods of this subsection are based on Hilbert's Nullstellensatz [KR00, Thm. 2.6.16]. Recall that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$ yields the existence of a projective $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})$ according to Proposition 5.1.8.

First we give two algorithms that allow us to check whether for $k \in \mathbb{N}$, the multiplication map $\pi : R_k \otimes_K (\omega_R)_{-k} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r \cdot \varphi$ is biinjective, i.e. an algorithm that checks whether $r \cdot \varphi = \pi(r \otimes \varphi) = 0$ implies $r = 0$ or $\varphi = 0$ for all $r \in R_k$ and $\varphi \in (\omega_R)_{-k}$ according to Definition 5.2.11. The first one of the following two methods of Proposition 5.3.1 generalizes [BK96, Rem. 8.3] to our setting. The second one applies the results of [Eis88, Sect. 1] to the special map π and thus generalizes the method described in [BK96, Rem. 8.4] to our setting. For the sake of completeness, we provide detailed proofs of both methods. Recall that for a square matrix $A = (a_{ij})_{1 \le i,j \le s} \in \mathrm{Mat}_s(K)$ of size $s \in \mathbb{N} \setminus \{0\}$, the trace of A is defined by $\mathrm{trace}(A) = a_{11} + \cdots + a_{ss} \in K$.

Proposition 5.3.1. (Biinjectivity Tests)
Let $G \subseteq \overline{P}$ be a projective $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})$. Moreover, we write $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ with $\deg_W(t_1) \le \cdots \le \deg_W(t_\mu)$ and let $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ denote the canonical basis of the K-vector space K^μ, let $k \in \{1, \ldots, \sigma_{\mathbb{X}}\}$, and let $\pi : R_k \otimes_K (\omega_R)_{-k} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r \cdot \varphi$ be the multiplication map. Furthermore, for every index $\ell \in \{2, \ldots, \mu\}$, let $\mathcal{C}_\ell = (c_{rs\ell})_{1 \le r \le H(k), H(k)+1 \le s \le \mu} \in \mathrm{Mat}_{H(k), \mu - H(k)}(K)$ be the matrix with the entries $c_{rs\ell} = \mathcal{E}_\ell \cdot t_r(\mathcal{X}_1^{\mathrm{tr}}, \ldots, \mathcal{X}_n^{\mathrm{tr}}) \cdot \mathcal{E}_s^{\mathrm{tr}} \in K$ for all $r \in \{1, \ldots, H(k)\}$ and $s \in \{H(k)+1, \ldots, \mu\}$ and let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and $i \in \{1, \ldots, \mu\}$.

a) Let $Q = K[y_1, \ldots, y_{H(k)+1}, z_{H(k)+1}, \ldots, z_\mu]$ be with a set of further indeterminates $\{y_1, \ldots, y_{H(k)}, z_{H(k)+1}, \ldots, z_\mu\}$ and for all indices $\ell \in \{2, \ldots, \mu\}$, we let $f_\ell = (y_1, \ldots, y_{H(k)}) \cdot \mathcal{C}_\ell \cdot (z_{H(k)+1}, \ldots, z_\mu)^{\mathrm{tr}} \in Q$. Then the multiplication map π is biinjective if and only if $\langle y_1 z_{H(k)+1}, \ldots, y_{H(k)} z_\mu \rangle \subseteq \sqrt{\langle f_2, \ldots, f_\mu \rangle}$.

b) Let $\{\varphi_1, \ldots, \varphi_d\} \subseteq (\omega_R)_0$ with a natural number $d \in \mathbb{N}$ be a K-vector space basis of $\mathrm{im}(\pi)$. For all $i \in \{1, \ldots, d\}$, we write $\varphi_i = b_{i2} x_0^{-d_{1,2}} \overline{t}_2^* + \cdots + b_{i\mu} x_0^{-d_{1,\mu}} \overline{t}_\mu^*$ with scalars $b_{i2}, \ldots, b_{i\mu} \in K$ and we let $\mathcal{B}_i = b_{i2}\mathcal{C}_2 + \cdots + b_{i\mu}\mathcal{C}_\mu$. Moreover, we let $\{\mathcal{D}_1, \ldots, \mathcal{D}_m\} \subseteq \mathrm{Mat}_{\mu - H(k), H(k)}(K)$ be a K-vector space basis of the set $M^\perp = \{\mathcal{D} \in \mathrm{Mat}_{\mu - H(k), H(k)}(K) \mid \mathrm{trace}(\mathcal{D}\mathcal{B}_2) = \cdots = \mathrm{trace}(\mathcal{D}\mathcal{B}_\mu) = 0\}$ and we let $Q = K[y_1, \ldots, y_m]$ be where $\{y_1, \ldots, y_m\}$ is a set of further indeterminates. Let $J \subseteq Q$ denote the ideal generated by the set of all 2-minors of the matrix $y_1\mathcal{D}_1 + \cdots + y_m\mathcal{D}_m \in \mathrm{Mat}_{\mu - H(k), H(k)}(Q)$. Then the map π is biinjective if and only if $\langle y_1, \ldots, y_m \rangle \subseteq \sqrt{J}$.

Proof. According to Proposition 4.1.15, the set $\{x_0^{d_{k,1}} \overline{t}_1, \ldots, x_0^{d_{k,H(k)}} \overline{t}_{H(k)}\}$ is a K-vector space basis of R_k, and by Proposition 4.3.7, the sets $\{x_0^{-d_{k+1,H(k)+1}} \overline{t}_{H(k)+1}^*, \ldots, x_0^{-d_{k+1,\mu}} \overline{t}_\mu^*\}$ and $\{x_0^{-d_{1,2}} \overline{t}_2^*, \ldots, x_0^{-d_{1,\mu}} \overline{t}_\mu^*\}$ are K-vector spaces of $(\omega_R)_{-k}$ and $(\omega_R)_0$, respectively.

First we show a). For all $r \in \{1,\dots,H(k)\}$ and $s \in \{H(k)+1,\dots,\mu\}$, we have

$$\begin{aligned}
x_0^{d_{k,r}}\bar{t}_r \cdot x_0^{-d_{k+1,s}}\bar{t}_s^* &= (0, x_0^{-d_{1,2}}\bar{t}_2^*,\dots,x_0^{-d_{1,\mu}}\bar{t}_\mu^*) \cdot (x_0^{d_{k,r}} t_r)(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot \mathcal{E}_s^{\mathrm{tr}} \\
&= (0, x_0^{-d_{1,2}}\bar{t}_2^*,\dots,x_0^{-d_{1,\mu}}\bar{t}_\mu^*) \cdot t_r(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot \mathcal{E}_s^{\mathrm{tr}} \\
&= \sum_{\ell=2}^{\mu} (\mathcal{E}_\ell \cdot t_r(\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \cdot \mathcal{E}_s^{\mathrm{tr}}) x_0^{-d_{1,\ell}}\bar{t}_\ell^* \\
&= \sum_{\ell=2}^{\mu} c_{rs\ell} x_0^{-d_{1,\ell}}\bar{t}_\ell^*
\end{aligned}$$

according to Proposition 4.3.10. Let $a_1,\dots,a_{H(k)}, b_{H(k)+1},\dots,b_\mu, \in K$. Then

$$\begin{aligned}
&\pi\left(\left(\sum_{r=1}^{H(k)} a_r x_0^{d_{k,r}}\bar{t}_r\right) \otimes \left(\sum_{s=H(k)+1}^{\mu} b_s x_0^{-d_{k+1,s}}\bar{t}_s^*\right)\right) \\
&= \left(\sum_{r=1}^{H(k)} a_r x_0^{d_{k,r}}\bar{t}_r\right) \cdot \left(\sum_{s=H(k)+1}^{\mu} b_s x_0^{-d_{k+1,s}}\bar{t}_s^*\right) \\
&= \sum_{r=1}^{H(k)} \sum_{s=H(k)+1}^{\mu} a_r b_s (x_0^{d_{k,r}}\bar{t}_r \cdot x_0^{-d_{k+1,s}}\bar{t}_s^*) \\
&= \sum_{r=1}^{H(k)} \sum_{s=H(k)+1}^{\mu} a_r b_s \sum_{\ell=2}^{\mu} c_{rs\ell} x_0^{-d_{1,\ell}}\bar{t}_\ell^* \\
&= \sum_{\ell=2}^{\mu} \left(\sum_{r=1}^{H(k)} \sum_{s=H(k)+1}^{\mu} a_r c_{rs\ell} b_s\right) x_0^{-d_{1,\ell}}\bar{t}_\ell^* \\
&= \sum_{\ell=2}^{\mu} ((a_1,\dots,a_{H(k)}) \cdot \mathcal{C}_\ell \cdot (b_{H(k)+1},\dots,b_\mu)^{\mathrm{tr}}) x_0^{-d_{1,\ell}}\bar{t}_\ell^* \\
&= \sum_{\ell=2}^{\mu} f_\ell(a_1,\dots,a_{H(k)}, b_{H(k)+1},\dots,b_\mu) x_0^{-d_{1,\ell}}\bar{t}_\ell^*.
\end{aligned}$$

By Definition 5.2.11, π is biinjective if and only if $f_\ell(a_1,\dots,a_{H(k)}, b_{H(k)+1},\dots,b_\mu) = 0$ for all $\ell \in \{2,\dots,\mu\}$ implies that $a_1 = \cdots = a_{H(k)} = 0$ or $b_{H(k)+1} = \cdots = b_\mu = 0$. In other words, π is biinjective if and only if

$$\mathcal{Z}(\langle f_2,\dots,f_\mu\rangle) \subseteq \mathcal{Z}(\langle y_1,\dots,y_{H(k)}\rangle) \cup \mathcal{Z}(\langle z_{H(k)+1},\dots,z_\mu\rangle).$$

By [CLO07, Lemma 1.§2.2], we have

$$\mathcal{Z}(\langle y_1,\dots,y_{H(k)}\rangle) \cup \mathcal{Z}(\langle z_{H(k)+1},\dots,z_\mu\rangle) = \mathcal{Z}(\langle y_1 z_{H(k)+1},\dots,y_{H(k)} z_\mu\rangle).$$

Now the Ideal-Variety Correspondence [CLO07, Thm. 4.§2.7], Hilbert's Nullstellensatz [KR00, Thm. 2.6.16], and [KR05, Coro. 4.1.12] yield that π is biinjective if and only if

$$\sqrt{\langle f_2, \ldots, f_\mu \rangle} \supseteq \sqrt{\langle y_1 z_{H(k)+1}, \ldots, y_{H(k)} z_\mu \rangle} = \langle y_1 z_{H(k)+1}, \ldots, y_{H(k)} z_\mu \rangle.$$

Next we prove claim b). By taking adjoints, the map π induces the dual K-vector space monomorphism $\pi^* : {(\omega_R)_0}^* \to (R_k \otimes_K (\omega_R)_{-k})^*$, defined by $(x_0^{-d_{1,\ell}} \bar{t}_\ell^*)^* \mapsto (x_0^{-d_{1,\ell}} \bar{t}_\ell^*)^* \circ \pi$ for all $\ell \in \{2, \ldots, \mu\}$. As above, for all $a_1, \ldots, a_{H(k)}, b_{H(k)+1}, \ldots, b_\mu \in K$, we have

$$\begin{aligned}
&\pi\left(\left(\sum_{r=1}^{H(k)} a_r x_0^{d_{k,r}} \bar{t}_r\right) \otimes \left(\sum_{s=H(k)+1}^{\mu} b_s x_0^{-d_{k+1,s}} \bar{t}_s^*\right)\right) \\
&= \sum_{\ell=2}^{\mu} ((a_1, \ldots, a_{H(k)}) \cdot \mathcal{C}_\ell \cdot (b_{H(k)+1}, \ldots, b_\mu)^{\mathrm{tr}}) x_0^{-d_{1,\ell}} \bar{t}_\ell^*.
\end{aligned}$$

Thus for all indices $\ell \in \{2, \ldots, \mu\}$, the matrix $\mathcal{C}_\ell \in \mathrm{Mat}_{H(k),\mu-H(k)}(K)$ represents the K-vector space homomorphism $\pi^*((x_0^{-d_{1,\ell}} \bar{t}_\ell^*)^*)$ with respect to the K-vector space bases $\{x_0^{d_{k,1}} \bar{t}_1, \ldots, x_0^{d_{k,H(k)}} \bar{t}_{H(k)}\}$ of R_k and $\{x_0^{-d_{1,2}} \bar{t}_2^*, \ldots, x_0^{-d_{1,\mu}} \bar{t}_\mu^*\}$ of $(\omega_R)_{-k}$. Since the map $\pi : R_k \otimes_K (\omega_R)_{-k} \twoheadrightarrow \mathrm{im}(\pi)$ is surjective, the restriction $\pi^*|_{(\mathrm{im}(\pi))^*}$ of π^* to $(\mathrm{im}(\pi))^*$ is injective according to [Bou89, Prop. II.§2.6.9]. As $\{\varphi_1, \ldots, \varphi_d\}$ is a K-vector space basis of $\mathrm{im}(\pi)$, and since $\pi^*|_{(\mathrm{im}(\pi))^*}$ is injective, it follows that

$$\begin{aligned}
(\mathrm{im}(\pi))^* &= \left(\bigoplus_{i=1}^{d} K\varphi_i\right)^* \\
&= \bigoplus_{i=1}^{d} K(b_{i2} x_0^{-d_{1,2}} \bar{t}_2^* + \cdots + b_{i\mu} x_0^{-d_{1,\mu}} \bar{t}_\mu^*) \\
&\cong \bigoplus_{i=1}^{d} K(b_{i2} x_0^{-d_{1,2}} \mathcal{C}_2 + \cdots + b_{i\mu} x_0^{-d_{1,\mu}} \mathcal{C}_\mu) \\
&= \bigoplus_{i=1}^{d} K\mathcal{B}_i \\
&\subseteq \mathrm{Mat}_{H(k),\mu-H(k)}(K).
\end{aligned}$$

Moreover, since we have seen above that R_k, $(\omega_R)_{-k}$, and $(\omega_R)_0$ are all finite-dimensional K-vector spaces, [Bou89, Subsection II.§4.2, p. 271] yields the canonical K-vector space isomorphism

$$R_k \otimes_K (\omega_R)_{-k} \cong (R_k^*)^* \otimes_K (\omega_R)_{-k} \cong \mathrm{Hom}_K(R_k^*, (\omega_R)_{-k}) \cong \mathrm{Mat}_{\mu-H(k),H(k)}(K).$$

We now prove that $(\mathrm{Mat}_{H(k),\mu-H(k)}(K))^* \cong \mathrm{Mat}_{\mu-H(k),H(k)}(K)$. Let

$$\Phi : \mathrm{Mat}_{\mu-H(k),H(k)}(K) \to (\mathrm{Mat}_{H(k),\mu-H(k)}(K))^*, \quad \mathcal{A} \mapsto (\mathcal{B} \mapsto \mathrm{trace}(\mathcal{A}\mathcal{B})).$$

For all $\mathcal{A}, \mathcal{A}' \in \mathrm{Mat}_{\mu-H(k),H(k)}(K)$, $\lambda \in K$, and $\mathcal{B} \in \mathrm{Mat}_{H(k),\mu-H(k)}(K)$, we have

$$\begin{aligned}\Phi(\lambda\mathcal{A}+\mathcal{A}')(\mathcal{B}) &= \mathrm{trace}((\lambda\mathcal{A}+\mathcal{A}')\mathcal{B})\\ &= \mathrm{trace}(\lambda\mathcal{A}\mathcal{B}+\mathcal{A}'\mathcal{B})\\ &= \lambda\,\mathrm{trace}(\mathcal{A}\mathcal{B})+\mathrm{trace}(\mathcal{A}'\mathcal{B})\\ &= \lambda\Phi(\mathcal{A})(\mathcal{B})+\Phi(\mathcal{A}')(\mathcal{B}),\end{aligned}$$

i.e. $\Phi(\lambda\mathcal{A}+\mathcal{A}') = \lambda\Phi(\mathcal{A})+\Phi(\mathcal{A}')$. Thus Φ is a K-vector space homomorphism. Let $\mathcal{A} \in \ker(\Phi)$. Let $i \in \{1,\dots,\mu-H(k)\}$, $j \in \{1,\dots,H(k)\}$, and let $a_{ij} \in K$ be the entry of $\mathcal{A}$ in the i^{th} row and the j^{th} column. Let $\mathcal{M}_{ij} \in \mathrm{Mat}_{H(k),\mu-H(k)}$ be the matrix with 1 in the entry of the j^{th} row and i^{th} column and 0 in all other entries. Then $a_{ij} = \mathrm{trace}(\mathcal{A}\mathcal{M}_{ij}) = 0$ since $\mathcal{A} \in \ker(\Phi)$. We see that $\mathcal{A} = 0$ and thus Φ is injective. Since $\dim_K(\mathrm{Mat}_{\mu-H(k),H(k)}(K)) = (\mu-H(k))\cdot H(k) = \dim_K((\mathrm{Mat}_{H(k),\mu-H(k)}(K))^*)$, the map Φ is a K-vector space isomorphism.
Altogether, we see that we can consider π as a K-vector space epimorphism

$$\pi : R_k \otimes_K (\omega_R)_{-k} \cong (\mathrm{Mat}_{H(k),\mu-H(k)}(K))^* \twoheadrightarrow \left(\bigoplus_{i=1}^{d} K\mathcal{B}_i\right)^* = (\mathrm{im}(\pi)^*)^* \cong \mathrm{im}(\pi)$$

with $M = \bigoplus_{i=1}^{d} K\mathcal{B}_i \subseteq \mathrm{Mat}_{H(k),\mu-H(k)}(K)$. By [Eis88, Prop. 1.1], it follows that π is biinjective if and only if $(M^{\perp})_1 = 0$, i.e. if and only if for all scalars $c_1,\dots,c_m \in K$, $\mathrm{rk}_K(c_1\mathcal{D}_1+\cdots+c_m\mathcal{D}_m) \le 1$ implies that $c_1 = \cdots = c_m = 0$. In other words, the map π is biinjective if and only if $\mathcal{Z}(J) \subseteq \mathcal{Z}(\langle y_1,\dots,y_m\rangle)$ and by the Ideal-Variety Correspondence [CLO07, Thm. 4.§2.7], Hilbert's Nullstellensatz [KR00, Thm. 2.6.16], and [KR05, Coro. 4.1.12], this is equivalent to $\sqrt{J} \supseteq \langle y_1,\dots,y_m\rangle$. □

The previous proposition can be used to check i-uniformity conditions in special cases.

Remark 5.3.2. Let $j \in \{1,\dots,\sigma_{\mathbb{X}}\}$. According to Remark 5.2.12, the scheme $\mathbb{X}$ is $(\deg(\mathbb{X})-\mathrm{HF}_{\mathbb{X}}(j))$-uniform if and only if the multiplication map $R_k \otimes_K (\omega_R)_{-k} \to (\omega_R)_0$, $r \otimes \varphi \mapsto r\cdot\varphi$ is biinjective for all $k \in \{j,\dots,\sigma_{\mathbb{X}}\}$. Since we can algorithmically check radical membership by [KR00, Coro. 3.5.15], we can use the methods of Proposition 5.3.1 to algorithmically check whether the scheme $\mathbb{X}$ is $(\deg(\mathbb{X})-\mathrm{HF}_{\mathbb{X}}(j))$-uniform. In Remark 5.3.5, we describe a new method to check whether $\mathbb{X}$ is i-uniform where $i \in \mathbb{N}$ can be an arbitrary natural number, which is based on Corollary 5.2.9.

Next we turn Corollary 5.2.9 into an explicit algorithm. Note that we assume that the subscheme $\mathbb{X}$ is given by a generating set of its defining ideal $\mathcal{I}^+(\mathbb{X})$.

Algorithm 8: `CheckUniformity`$(\{f_1, \dots, f_r\}, i, j)$

Input:
$r \in \mathbb{N}$ and $\{f_1, \dots, f_r\} \subseteq \overline{P}$ is a generating set of the homogeneous vanishing ideal $\mathcal{I}^+(\mathbb{X})$ of a non-empty zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ over an algebraically closed field K satisfying $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$,
$i, j \in \mathbb{N} \setminus \{0\}$

Compute a projective $\mathcal{O}$-border basis G of $\langle f_1, \dots, f_r \rangle \subseteq \overline{P}$.
Let $\mu \in \mathbb{N}$ and $t_1, \dots, t_\mu \subseteq \mathbb{T}^n$ be terms such that $\mathcal{O} = \{t_1, \dots, t_\mu\}$ and such that $\deg_W(t_1) \leq \dots \leq \deg_W(t_\mu)$.
Let $\mathcal{X}_1, \dots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of the $\mathcal{O}$-border basis $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$.
$\sigma_{\mathbb{X}} := \deg_W(t_\mu) - 1$
$H_j := \max\{k \in \{1, \dots, \mu\} \mid \deg_W(t_k) \leq j\}$
if $i > \mu - H_j$ **or** $j > \sigma_{\mathbb{X}}$ **then**
    return false
end
for $k := 1$ **to** H_j **do**
    Let $\mathcal{T}_k \in \mathrm{Mat}_{\mu-1,\mu-H_j}(K)$ be the submatrix of $t_k(\mathcal{X}_1^{\mathrm{tr}}, \dots, \mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K)$ obtained by deleting the first row and the first H_j columns.
end
Let $\{y_1, \dots, y_{H_j}\}$ be a set of further indeterminates.
Let $J \subseteq K[y_1, \dots, y_{H_j}]$ be the ideal generated by the set of all i-minors of the matrix $y_1\mathcal{T}_1 + \dots + y_{H_j}\mathcal{T}_{H_j} \in \mathrm{Mat}_{\mu-1,\mu-H_j}(K[y_1, \dots, y_{H_j}])$.
for $k := 1$ **to** H_j **do**
    if $y_k \notin \sqrt{J}$ **then**
        return false
    end
end
return true

Proposition 5.3.3. **(The** (i,j)**-Uniformity Test – The General Case)**
Let $\{f_1, \dots, f_r\} \subseteq \overline{P}$ *with* $r \in \mathbb{N}$ *be a generating set of the defining ideal* $\mathcal{I}^+(\mathbb{X})$ *and let* $i, j \in \mathbb{N} \setminus \{0\}$*. Then Algorithm 8 is actually an algorithm and the result*

$$\texttt{CheckUniformity}(\{f_1, \dots, f_r\}, i, j)$$

of Algorithm 8 applied to the input data $\{f_1, \dots, f_r\}$*,* i*, and* j *is a boolean value that determines whether the scheme* $\mathbb{X}$ *is* (i,j)*-uniform.*

Proof. First we show that Algorithm 8 is actually an algorithm. As $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is a non-empty zero-dimensional subscheme, we can compute a projective $\mathcal{O}$-border basis G of $\mathcal{I}^+(\mathbb{X}) \subset \overline{P}$ as in line 1 according to Proposition 5.1.8 and Corollary 4.1.14. By Proposition 4.1.7, the set $\{g^{\mathrm{deh}} \mid g \in G\} \subseteq P$ is an $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})^{\mathrm{deh}}$ and thus we can compute the corresponding formal multiplication matrices in line 3. In line 13, we have $i \leq \mu - H_j$ since otherwise the procedure would have stopped in line 6. As $\mathbb{X}$ is non-empty, $1 \in \mathcal{O}$ by Definitions 2.1.6 and 4.1.2 and hence $\mu - H_j \leq \mu - 1$. Thus we can compute the i-minors of $y_1\mathcal{T}_1 + \cdots + y_m\mathcal{T}_{H_j} \in \mathrm{Mat}_{\mu-H_j,\mu-1}(K[y_1,\dots,y_{H_j}])$, and hence also J as in line 13 using linear algebra techniques. The condition of the if-clause in line 15 can be algorithmically checked, for instance, with the Radical Membership Test [KR00, Coro. 3.5.15]. Since all the other steps of the procedure can be trivially computed and since the procedure obviously stops after a finite amount of time, it is actually an algorithm.

Next we prove the correctness. Since $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is a non-empty subscheme, we have $\langle f_1,\dots,f_r\rangle \subset \overline{P}$. Let $G \subseteq \overline{P}$ be the projective $\mathcal{O}$-border basis of $\mathcal{I}^+(\mathbb{X})$ as in line 1, let $\mu \in \mathbb{N}$ and $t_1,\dots,t_\mu \subseteq \mathbb{T}^n$ be such that $\mathcal{O} = \{t_1,\dots,t_\mu\}$ and $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$ as in line 2, and let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ denote the formal multiplication matrices of $\{g^{\mathrm{deh}} \mid g \in G\}$ as in line 3. Since $\langle f_1,\dots,f_r\rangle \neq \overline{P}$ and G is a projective $\mathcal{O}$-border basis of the ideal $\mathcal{I}^+(\mathbb{X})$, it follows $\mathcal{O} \neq \emptyset$ and thus $\mu \geq 1$ from Definition 4.1.2. Therefore, $\sigma_{\mathbb{X}} = \deg_W(t_\mu) - 1$ as in line 4 and $H_j = \max\{k \in \{1,\dots,\mu\} \mid \deg_W(t_k) \leq j\}$ as in line 5 exist. Since we have $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$, it follows $\mu = \deg(\mathbb{X})$, $\sigma_{\mathbb{X}} = \max\{k \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{X}}(k) < \deg(\mathbb{X})\}$, and $H_j = \mathrm{HF}_{\mathbb{X}}(j)$ from Proposition 4.1.15 and Definition 5.1.17. If $i > \mu - H_j = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$ or $j > \sigma_{\mathbb{X}}$, the subscheme $\mathbb{X}$ is not (i,j)-uniform according to Proposition 5.2.3 and Definition 5.2.1 and the correct boolean value is returned in line 7. For the other case, assume that $i \leq \mu - H_j$. Let $\{y_1,\dots,y_{H_j}\}$ be a set of further indeterminates as in line 12, $\mathcal{T}_k \in \mathrm{Mat}_{\mu-1,\mu-H_j}(K)$ be the submatrix of $t_k(\mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K)$ obtained by deleting the first row and the first H_j columns as in line 10, and let $J \subseteq K[y_1,\dots,y_{H_j}]$ be the ideal generated by the set of all i-minors of $y_1\mathcal{T}_1 + \cdots + y_{H_j}\mathcal{T}_{H_j} \in \mathrm{Mat}_{\mu-1,\mu-H_j}(K[y_1,\dots,y_{H_j}])$ as in line 13. If the algorithm stops in line 16, then there exists a $k \in \{1,\dots,H_j\}$ such that $y_k \notin \sqrt{J}$ by line 15. In this situation, $\langle y_1,\dots,y_{H_j}\rangle \not\subseteq \sqrt{J}$ and Corollary 5.2.9 imply that $\mathbb{X}$ is not (i,j)-uniform and the correct boolean value is returned in line 16. If the algorithm stops in line 19, then $y_k \in \sqrt{J}$ for all $k \in \{1,\dots,H_j\}$ by line 15. In this situation, $\langle y_1,\dots,y_{H_j}\rangle \subseteq \sqrt{J}$ and Corollary 5.2.9 imply that $\mathbb{X}$ is (i,j)-uniform and the correct boolean value is returned in line 19. □

Example 5.3.4. Let $\mathbb{X}_{\mathbb{C}} = \mathbb{X} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{C}) \subseteq \mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0,x_1,x_2])$ be the zero-dimensional subscheme $\mathbb{X}$ regarded over $\mathbb{C}$ where $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ is the zero-dimensional subscheme of the Examples 5.2.2, 5.2.8, and 5.2.10. Recall that $\mathbb{X}_{\mathbb{C}}$ was defined by the ideal $I = \mathfrak{p}_0^2 \cap \mathfrak{p}_1^2 \cap \mathfrak{p}_2^2$ where $\mathfrak{p}_0 = \mathcal{I}^+(\{(1:0:0)\})$, $\mathfrak{p}_1 = \mathcal{I}^+(\{(1:0:1)\})$, and $\mathfrak{p}_2 = \mathcal{I}^+(\{(1:1:0)\})$ were homogeneous vanishing ideals in $\mathbb{C}[x_0,x_1,x_2]$. In particular, we have $\mathrm{Supp}(\mathbb{X}_{\mathbb{C}}) = \{\mathfrak{p}_0,\mathfrak{p}_1,\mathfrak{p}_2\}$ and $\mathbb{X}_{\mathbb{C}} \cap H^{\mathrm{inf}} = \emptyset$. Applying the method described in Example 5.2.2, we can compute the homogeneous generating set $\{f_1,\dots,f_4\} \subseteq \mathbb{C}[x_0,x_1,x_2]$ of the ideal I where we have $f_1 = x_1^2x_2 + x_1x_2^2 - x_0x_1x_2$, $f_2 = x_2^4 - 2x_0x_2^3 + x_0^2x_2^2$,

$f_3 = x_1x_2^3 - x_0x_1x_2^2$, and $f_4 = x_1^4 - 2x_0x_1^3 + x_0^2x_1^2$. Altogether, the input data $\{f_1, \ldots, f_4\}$, $i = 2$, and $j = 2$ satisfy the assumptions of Algorithm 8. We now consider the steps of Algorithm 8 applied to these input data in detail.

In the initialization process of the lines 1 to 3, we compute the projective $\mathcal{O}$-border basis G where $\mathcal{O} = \{t_1, \ldots, t_9\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_2^3, x_1x_2^2, x_1^3\}$ and the corresponding formal multiplication matrices $\mathcal{X}_1, \mathcal{X}_2 \in \operatorname{Mat}_8(\mathbb{C})$ just as we did in Example 5.2.10. In line 4, we have $\sigma_{\mathbb{X}_\mathbb{C}} = \deg_W(x_1^3) - 1 = 2$. Since we have $\deg_W(t_6) = \deg(x_1^2) = 2$ and $\deg_W(t_7) = \deg_W(x_2^3) = 3 > 2$, we get $H_j = 6$ in line 5. As $i = 2 \leq 9 - H_j$ and $j = 2 = \sigma_{\mathbb{X}_\mathbb{C}}$, the if-clause in line 6 is not executed. Let $\mathcal{T}_1, \ldots, \mathcal{T}_6 \in \operatorname{Mat}_{9,3}(\mathbb{C})$ be the matrices that are computed in line 10 during the for-loop starting in line 9, let $\{y_1, \ldots, y_6\}$ be a set of further indeterminates as in line 12, and let $J \subseteq \mathbb{C}[y_1, \ldots, y_6]$ be the ideal generated by the set of all 2-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_6\mathcal{T}_6 \in \operatorname{Mat}_{8,3}(\mathbb{C}[y_1, \ldots, y_6])$. Recall that we already computed the matrices $\mathcal{T}_1, \ldots, \mathcal{T}_6$ and the ideal J in Example 5.2.10. Using the Radical Membership Test [KR00, Coro. 3.5.15], we see that $y_1 \in \sqrt{J}$ whereas $y_2 \notin \sqrt{J}$. Thus the condition of the if-clause in line 15 is satisfied and the algorithm returns the boolean value `false` in line 16.

By Proposition 5.3.3, the zero-dimensional subscheme $\mathbb{X}_\mathbb{C} \subseteq \mathbb{P}^2_\mathbb{C}$ is not $(2,2)$-uniform. Note that this result coincides with Example 5.2.10.

Finally, we sum up all the methods introduced in this subsection to check i-uniformity and (i,j)-uniformity conditions in the general, i.e. non-reduced, case. Recall again, that all these methods assume that the base field K is algebraically closed.

Remark 5.3.5. Let $i, j \in \mathbb{N} \setminus \{0\}$. Then we have the following possibilities to check whether the scheme $\mathbb{X}$ is i- respectively (i,j)-uniform.

a) According to Proposition 5.3.3, we can algorithmically check whether the scheme $\mathbb{X}$ is (i,j)-uniform. Note that, we do not need to assume that $\mathbb{X}$ is reduced but only that K is algebraically closed. In this general setting, no algorithm to check the (i,j)-uniformity conditions has been known by now.

b) Combining the Propositions 5.2.6 and 5.3.3, we can algorithmically check whether the scheme $\mathbb{X}$ is i-uniform. Note that by now it was only possible to check the i-uniformity condition if we had $i = \deg(\mathbb{X}) - \operatorname{HF}_\mathbb{X}(k)$ for some $k \in \mathbb{N}$ by applying the method described in Remark 5.3.2.

5.3.2 The Reduced Case

In this subsection, we do no longer assume that the base field K is algebraically closed but instead assume that the subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is reduced. Before we start to transform the results of Section 5.2 into explicit algorithms, we recall the results of [MP04] about (i,j)-uniform subschemes of $\mathbb{P}^n_K$. Note that the authors of that paper use a slightly different but equivalent notion of (i,j)-uniformity, cf. [MP04, Rem. 2], compared to ours in Definition 5.2.1, which is due to [Kre98, Defn. 7.12] and [Kre01, Defn. 2.1].

Remark 5.3.6. Let $I \subseteq \overline{P}$ be an ideal that set-theoretically defines a zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^n_K$ and assume that $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > \deg(\mathbb{X})$ holds. In [MP04, Prop. 18], the authors describe an algorithm that checks whether $\mathbb{X}$ is (i,j)-uniform for arbitrary $i, j \in \mathbb{N}$. The authors of that paper use a different approach than the one we will use later, namely they use Chow forms. The benefit of their method is that, in contrast to ours, they do not need to know the zero set of the ideal $\mathcal{I}^+(\mathbb{X})$ but only an ideal that defines $\mathbb{X}$ set-theoretically and they do not need to assume that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. However, if the ideal I is not a radical ideal, they need to compute its radical ideal $\sqrt{I} = \mathcal{I}^+(\mathbb{X})$. If K is finitely generated over a perfect field, this task can be algorithmically done, cf. [KL91] if $\mathrm{char}(K) = 0$ and [Kem02, Mat01] if $\mathrm{char}(K) > 0$. Note that if $0 < \mathrm{char}(K) \leq \deg(\mathbb{X})$ or if $\overline{P}$ is not standard graded, no algorithm to check the (i,j)-uniformity condition for arbitrary natural numbers $i \in \{1, \dots, \deg(\mathbb{X}) - 1\}$ and $j \in \{1, \dots, \sigma_{\mathbb{X}}\}$ is known by now.

As before, the reduced subscheme $\mathbb{X}$ is assumed to have a K-rational support. Thus the support of $\mathbb{X}$ consists of $\deg(\mathbb{X})$ vanishing ideals of K-rational projective points in $\mathbb{P}_K(\overline{W})$ according to Proposition 5.1.7. For the whole subsection, we assume that $\mathbb{X}$ is given by these K-rational projective points.

Before we are able to transform Theorem 5.2.14 into an explicit algorithm, we need to determine an algorithm for the computation of the separators of the subschemes $\mathbb{Y} \subseteq \mathbb{X}$ with $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$, cf. Definition 5.1.17. This task can be done with a suitable generalization of the Buchberger-Möller Algorithm introduced in [MB82]. The original Buchberger-Möller Algorithm computes a reduced Gröbner basis of the vanishing ideal of a finite set of affine points in $\mathbb{A}^n_K = \mathrm{Spec}(P)$. Though a version that computes a border basis instead of a reduced Gröbner basis was stated in [KP11, Algo. 4.1] and explicitly proved in [Lim14, Thm. 3.4.1], all these versions use the standard grading. This can sometimes lead to problems. For instance, if we compute an $\mathcal{O}$-border basis G with one of these versions of the Buchberger-Möller Algorithm, it can happen that a border term $b \in \partial\mathcal{O}$ is not contained in the degree form with respect to W of the corresponding border basis element $g_b \in G$, i.e. that there is a term $t \in \mathrm{Supp}(g_b)$ such that $\deg_W(t) > \deg_W(b)$. In this situation, homogenization of G does not yield a projective $\mathcal{O}$-border bases according to Corollary 4.1.9. Since we are particularly interested in projective $\mathcal{O}$-border bases and need them in Algorithm 10, we generalize the Buchberger-Möller Algorithm to be capable of handling gradings given by the positive matrix W.

Algorithm 9: $\texttt{BM}((p_1,\dots,p_\mu),(w_1,\dots,w_n),\sigma)$

Input:
$(p_1,\dots,p_\mu) \in (\mathbb{A}^n_K)^\mu$ is a tuple consisting of $\mu \in \mathbb{N}\setminus\{0\}$ pairwise distinct affine points in $\mathbb{A}^n_K = \operatorname{Spec}(P)$,
$(w_1,\dots,w_n) \in \operatorname{Mat}_{1,n}(\mathbb{Z})$ such that $w_i > 0$ for all $i \in \{1,\dots,n\}$
σ is a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_{(w_1,\dots,w_n)}$

$i := 0$
$\mathcal{O} := \emptyset$
$\mathcal{M} \in \operatorname{Mat}_{\mu,i}(K)$
$G := \emptyset$
$d := 0$
while $(\partial\mathcal{O})_{\geq d} \neq \emptyset$ **do**
  Let $r \in \mathbb{N}$ and let $\ell_1,\dots,\ell_r \in \mathbb{T}^n$ be terms such that $(\partial\mathcal{O})_d = \{\ell_1,\dots,\ell_r\}$ and such that $\ell_r >_\sigma \cdots >_\sigma \ell_1$.
  $\mathcal{A} := ((\ell_r(p_1),\dots,\ell_r(p_\mu))^{\mathrm{tr}} \mid \cdots \mid (\ell_1(p_1),\dots,\ell_1(p_\mu))^{\mathrm{tr}} \mid \mathcal{M}) \in \operatorname{Mat}_{\mu,i+r}(K)$
  Compute a matrix $\mathcal{B} \in \operatorname{Mat}_{k,i+r}(K)$ with $k \in \mathbb{N}$ rows such that the rows of $\mathcal{B}$ form a K-vector space basis of $\ker(\mathcal{A})$.
  Let $\mathcal{C} \in \operatorname{Mat}_{k,i+r}(K)$ be the reduced row echolon form of $\mathcal{B}$.
  Let $g_1,\dots,g_k \in P$ be such that $\mathcal{C}\cdot(\ell_r,\dots,\ell_1,t_i,\dots,t_1)^{\mathrm{tr}} = (g_1,\dots,g_k)^{\mathrm{tr}}$.
  $G := G \cup \{g_1,\dots,g_k\}$
  for $j := 1$ **to** r **do**
    if $\ell_j \notin \{\operatorname{LT}_\sigma(g_1),\dots,\operatorname{LT}_\sigma(g_k)\}$ **then**
      $i := i+1$
      $t_i := \ell_j$
      $\mathcal{O} := \mathcal{O} \cup \{t_i\}$
      $\mathcal{M} := ((t_i(p_1),\dots,t_i(p_\mu))^{\mathrm{tr}} \mid \mathcal{M}) \in \operatorname{Mat}_{\mu,i}(K)$
    end
  end
  $d := d+1$
end
$\mathcal{S} := (t_i,\dots,t_1)\cdot\mathcal{M}^{-1}$
return $(\mathcal{O},G,\mathcal{S})$

Proposition 5.3.7. (The Buchberger-Möller Algorithm)
Let $(p_1,\dots,p_\mu) \in (\mathbb{A}^n_K)^\mu$ be a tuple consisting of $\mu \in \mathbb{N}\setminus\{0\}$ pairwise distinct affine points in $\mathbb{A}^n_K = \operatorname{Spec}(P)$, let $\mathbb{X}^a = \{p_1,\dots,p_\mu\}$, and let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$. Then Algorithm 9 is actually an algorithm and the result

$$(\mathcal{O},G,\mathcal{S}) := \texttt{BM}((p_1,\dots,p_\mu),W,\sigma)$$

of Algorithm 9 applied to the input data $(p_1,\dots,p_\mu)$, W, and σ satisfies the following conditions.

1) The set $\mathcal{O} \subseteq \mathbb{T}^n$ is an order ideal in $\mathbb{T}^n$.

2) The set $G \subseteq P$ is the $\mathcal{O}$-border basis of the vanishing ideal $\mathcal{I}(\mathbb{X}^a)$ of $\mathbb{X}^a$.

3) For all $g \in G$, we have $\mathrm{LT}_{\sigma\,\mathrm{Pos}}(g) \in \partial\mathcal{O}$. In particular, the degree form of an element of G with respect to W contains the corresponding border term.

4) We have $\mathcal{S} = (s_1, \ldots, s_\mu) \in (\langle\mathcal{O}\rangle_K)^\mu$ and for all $i \in \{1, \ldots, \mu\}$, the polynomial s_i is a separator of p_i from the affine point set $\mathbb{X}^a \setminus \{p_i\}$, i.e. an element satisfying $s_i(p_j) = \delta_{ij}$ for all $j \in \{1, \ldots, \mu\}$.

Proof. First we show that Algorithm 9 is actually an algorithm. Later in the proof of the correctness, we will show that $\mathcal{O}$ is an order ideal in $\mathbb{T}^n$ during the whole procedure. Hence we can compute its border in the lines 6 and 7. The matrices $\mathcal{B}, \mathcal{C} \in \mathrm{Mat}_{\mu,i+r}(K)$ as in the lines 9 and 10 can be computed using standard linear algebra techniques. Moreover, we will see in the proof of the correctness that after the while-loop starting in line 6, we have $i = \mu$ and the rows of the matrix $\mathcal{M} \in \mathrm{Mat}_\mu(K)$ are K-linearly independent. Therefore, we can compute $(t_\mu, \ldots, t_1) \cdot \mathcal{M}^{-1}$ in line 23 using standard linear algebra techniques. All the other steps of the procedure can trivially be computed.
In order to show that the procedure stops after a finite amount of time, it suffices to show that the while-loop starting in line 6 eventually stops. The natural number i is initiated with 0 in line 1. Thus $\mathcal{O}$ has $i = 0$ elements in line 2 and the matrix $\mathcal{M}$ has $i = 0$ columns in line 3. Every time i is increased by 1 in line 15, the element t_i is added to $\mathcal{O}$ in line 17 and the evaluation vector $(t_i(p_1), \ldots, t_i(p_\mu))^{\mathrm{tr}} \in K^\mu$ is appended to $\mathcal{M}$ as a new first column in line 18. As $\mathcal{O}$ is only changed in line 17 and as $\mathcal{M}$ is only changed in line 18, we see that during the whole algorithm, $\mathcal{O} = \{t_1, \ldots, t_i\}$ and the columns of $\mathcal{M} \in \mathrm{Mat}_{\mu,i}(K)$ are the precisely the evaluation vectors of the elements in $\mathcal{O}$ at $(p_1, \ldots, p_\mu)$. Moreover, we will later prove that the columns of $\mathcal{M}$ are K-linearly independent during the whole algorithm. Since $\mathcal{M}$ has exactly $\mu \in \mathbb{N}$ rows, it hence follows that $\mathcal{O}$ can at most consist of μ terms. As d is increased at the end of every iteration of the while-loop in line 21, we eventually have $(\partial\mathcal{O})_{\geq d} = \emptyset$ in line 6 and the while-loop terminates. Therefore, the procedure stops after a finite amount of time and is thus an algorithm.

Next we prove the correctness of the algorithm. We consider one iteration of the while-loop starting in line 6 in detail. Let $d \in \mathbb{N}$ and $\mathcal{O} = \{t_1, \ldots, t_i\} \subseteq \mathbb{T}^n$ be such that $(\partial\mathcal{O})_{\geq d} \neq \emptyset$ as in line 6. Since the elements of $\mathcal{O}$ have been added in a previous iteration of the while-loop and since d is increased at the end of every iteration of the while-loop in line 21, we see that $\deg_W(t_k) < d$ for all $k \in \{1, \ldots, i\}$. Let $r \in \mathbb{N}$ and $\ell_1, \ldots, \ell_r \in \mathbb{T}^n$ be terms such that $(\partial\mathcal{O})_d = \{\ell_1, \ldots, \ell_r\}$ and such that $\ell_r >_\sigma \cdots >_\sigma \ell_1$ as in line 7. Consider the iterations of the for-loop starting in line 13. As the for-loop runs from 1 to r, as $\ell_r >_\sigma \cdots >_\sigma \ell_1$, and as the algorithm only adds elements to $\mathcal{O}$ in line 17, it follows that $\ell_r >_\sigma \cdots >_\sigma \ell_1 >_\sigma t_i >_\sigma \cdots >_\sigma t_1$ at the beginning of the for-loop in line 13. Let $\mathcal{A} = ((\ell_r(p_1), \ldots, \ell_r(p_\mu))^{\mathrm{tr}} \mid \cdots \mid (\ell_1(p_1), \ldots, \ell_1(p_\mu))^{\mathrm{tr}} \mid \mathcal{M}) \in \mathrm{Mat}_{\mu,i+r}(K)$ be as in line 8 and let $\mathcal{C} \in \mathrm{Mat}_{k,i+r}(K)$ be a matrix in reduced row echolon form with $k \in \mathbb{N}$ rows such that the rows of $\mathcal{C}$ form a K-vector space basis of $\ker(\mathcal{A})$ as in line 10. Moreover, let $g_1, \ldots, g_k \in P$ be such that $\mathcal{C} \cdot (\ell_r, \ldots, \ell_1, t_i, \ldots, t_1)^{\mathrm{tr}} = (g_1, \ldots, g_k)^{\mathrm{tr}}$ as

in line 11. By construction, there is a K-linear dependency between the evaluation vectors of $\ell_r, \dots, \ell_1, t_i, \dots, t_1$ if and only if the corresponding columns of the matrix $\mathcal{A}$ are K-linearly dependent and this equivalent for the same columns of the matrix $\mathcal{C}$ to be K-linearly dependent. An element t is added to $\mathcal{O}$ in line 17 if and only if it is not the leading term with respect to σ of one of the elements in $\{g_1, \dots, g_k\}$ by line 14. And since $\ell_r >_\sigma \cdots >_\sigma \ell_1 >_\sigma t_i >_\sigma \cdots >_\sigma t_1$, this is equivalent for the evaluation vector $(t(p_1), \dots, t(p_\mu))^{\mathrm{tr}}$ of t at $(p_1, \dots, p_\mu)^{\mathrm{tr}}$ as in line 18 to be K-linearly independent of $\mathcal{M}$. Thus induction on the loop-variable j of line 13 and induction on d of line 6 yield that $\mathcal{M}$ consists of K-linearly independent columns during the whole algorithm.
Now we prove that $\mathcal{O}$ is an order ideal in $\mathbb{T}^n$. Let $t \in \mathcal{O}$ and $t', t'' \in \mathbb{T}^n$ be with $t = t't''$. According to Definition 2.1.1, it suffices to prove that $t' \in \mathcal{O}$. For a contradiction, assume that $t' \notin \mathcal{O}$. As the algorithm considers the terms degree-by-degree and as the weight vector W satisfies $w_i > 0$ for all $i \in \{1, \dots, n\}$, it follows $\deg_W(t') < \deg_W(t)$. Thus the algorithm considers t' before t in the while-loop. Since we have $t' \notin \mathcal{O}$, $t' = \mathrm{LT}_\sigma(g)$ for some $g \in G$ by the lines 14 and 17. As we have already seen above, every element $g_1, \dots, g_k \in P$ of line 11 vanishes on $(p_1, \dots, p_\mu)$. The set G is only changed in line 12. Hence the elements g and gt'' both vanish on $(p_1, \dots, p_\mu)$. As σ is a term ordering on $\mathbb{T}^n$, we have $\mathrm{LT}_\sigma(gt'') = \mathrm{LT}_\sigma(g)t'' = t't'' = t$ by [KR00, Prop. 1.5.3]. In particular, σ is also $\deg_W$-compatible and hence $\deg_W(t) = \deg_W(gt'')$ by [KR05, Defn. 4.2.1]. Therefore, the element gt'' yields a K-linearly dependency of the corresponding columns of $\mathcal{C}$ and is thus an element of $\{g_1, \dots, g_k\}$ in line 11 during the corresponding iteration of the while-loop. As $t = \mathrm{LT}_\sigma(gt'') = \mathrm{LT}_\sigma(g_s)$ for some $s \in \{1, \dots, k\}$, the condition of the if-clause in line 14 is not fulfilled and t is not added to $\mathcal{O}$ in line 17. This is clearly a contradiction to our assumption and thus it follows that $\mathcal{O}$ is an order ideal.
Next we show that G is the $\mathcal{O}$-border basis of the vanishing ideal $\mathcal{I}(\mathbb{X}^a)$. Obviously every border term $b \in \partial\mathcal{O}$ is considered once during the algorithm and is the leading term of a polynomial in G by the lines 10 to 12 and the lines 14 and 17. Moreover, as the matrix $\mathcal{C}$ in line 10 is in reduced row echolon form and as all the elements that are not leading terms of the elements of G are added to $\mathcal{O}$ by the lines 14 and 17, the set G is an $\mathcal{O}$-border prebasis by Definition 2.1.14. We now use Buchberger's Criterion for border bases 2.4.31 to show that G is the $\mathcal{O}$-border basis of $\mathcal{I}(\mathbb{X}^a)$. Let $b, b' \in \partial\mathcal{O}$ be neighbors with respect to $\mathcal{O}$ and let $g_b, g_{b'} \in G$ be the corresponding elements of G. Then the normal remainder $\mathrm{NR}_G(\mathrm{S}(g_b, g_{b'})) \in \langle \mathcal{O} \rangle_K$ of the S-vector $\mathrm{S}(g_b, g_{b'}) \in \langle G \rangle$ of g_b and $g_{b'}$ is an element of $\langle G \rangle \cap \langle \mathcal{O} \rangle_K$ according to the Definitions 2.2.2 and 2.4.30. We have already seen above that all the elements in G vanish on $(p_1, \dots, p_\mu)$ and that the evaluation vectors of the elements of $\mathcal{O}$ at $(p_1, \dots, p_\mu)$ are K-linearly independent. As an element of $\langle G \rangle$, the polynomial $\mathrm{NR}_G(\mathrm{S}(g_b, g_{b'}))$ vanishes at $(p_1, \dots, p_\mu)$. As $\mathrm{NR}_G(\mathrm{S}(g_b, g_{b'}) \in \langle \mathcal{O} \rangle_K$ and as the evaluation vectors of the elements of $\mathcal{O}$ at $(p_1, \dots, p_\mu)$ are K-linearly independent, it follows that $\mathrm{NR}_G(\mathrm{S}(g_b, g_{b'})) = 0$. Hence Buchberger's Criterion for border bases 2.4.31 yields that G is the $\mathcal{O}$-border basis of $\langle G \rangle$. We have already seen above that $\mathcal{M}$ is a matrix consisting of μ rows whose columns are the K-linear independent evaluation vectors of the elements in $\mathcal{O}$ at $(p_1, \dots, p_\mu)$ and that, therefore, $\#\mathcal{O} \leq \mu$. Recall that $\mathcal{O}_\sigma(\mathcal{I}(\mathbb{X})) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma(\mathcal{I}(\mathbb{X}))$. We have seen that all the elements in $\partial\mathcal{O}$ are leading terms of elements in $G \subseteq \mathcal{I}(\mathbb{X}^a)$, i.e. $\partial\mathcal{O} \subseteq \mathrm{LT}_\sigma\{\langle G \rangle\} \subseteq \mathrm{LT}_\sigma\{\mathcal{I}(\mathbb{X}^a)\}$. In particular, we get

$\bigcup_{k=1}^{\infty} \partial^k \mathcal{O} \subseteq \mathrm{LT}_\sigma\{\langle G \rangle\} \subseteq \mathrm{LT}_\sigma\{\mathcal{I}(\mathbb{X}^a)\} \subseteq \mathbb{T}^n$ and $\mathcal{O} = \mathbb{T}^n \setminus \bigcup_{k=1}^{\infty} \partial^k \mathcal{O} \supseteq \mathcal{O}_\sigma(\mathcal{I}(\mathbb{X}^a))$ by Definition 2.1.7 and Proposition 2.1.10. Macaulay's Basis Theorem [KR00, Thm. 1.5.7] and [KR05, Prop. 6.3.3] hence yield $\mu \geq \#\mathcal{O} \geq \#\mathcal{O}_\sigma(\mathcal{I}(\mathbb{X}^a)) = \mu$. Altogether, we see that $\#\mathcal{O} = \mu = i$ and $\mathcal{O} = \mathcal{O}_\sigma(\mathcal{I}(\mathbb{X}))$ at the end of the while-loop starting in line 6 and that we have $\mathrm{LT}_\sigma\{\langle G \rangle\} \supseteq \bigcup_{k=1}^{\infty} \partial^k \mathcal{O} = \mathbb{T}^n \setminus \mathcal{O} = \mathrm{LT}_\sigma\{\mathcal{I}(\mathbb{X}^a)\}$. As we also have $G \subseteq \mathcal{I}(\mathbb{X}^a)$, [KR05, Prop. 2.4.10] yields $\langle G \rangle = \mathcal{I}(\mathbb{X}^a)$. Thus we have proved that G is the $\mathcal{O}$-border basis of $\langle G \rangle = \mathcal{I}(\mathbb{X}^a)$.
In order to prove claim 3), let $b \in \partial\mathcal{O}$ and let $g_b \in G$ be the corresponding $\mathcal{O}$-border basis element. Then we had $b = \mathrm{LT}_\sigma(g_b)$ during the algorithm in line 14 since otherwise b would have been added to $\mathcal{O}$ in line 17. As σ is compatible with $\deg_W$, it follows that $b \in \mathrm{DF}_W(g_b)$ and claim 3) follows.
Finally, we show 4). Let $i \in \{1, \ldots, \mu\}$. We have already seen above that the columns of $\mathcal{M}$ are the evaluation vectors of the elements in $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ at $(p_1, \ldots, p_\mu)$ and that these vectors are K-linearly independent at the end of the algorithm in line 23, More precisely, we have

$$\mathcal{M} = \begin{pmatrix} t_\mu(p_1) & \cdots & t_1(p_1) \\ \vdots & \ddots & \vdots \\ t_\mu(p_\mu) & \cdots & t_1(p_\mu) \end{pmatrix} \in \mathrm{Mat}_\mu(K)$$

and $\mathcal{M}$ is invertible in this situation. Thus we have $\mathcal{S} = (t_\mu, \ldots, t_1) \cdot \mathcal{M}^{-1} \in (\langle \mathcal{O} \rangle_K)^\mu$ in line 23. Let $s_1, \ldots, s_\mu \in P$ be such that $\mathcal{S} = (s_1, \ldots, s_\mu)$ and let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ be the canonical K-vector space basis of K^μ. Write $\mathcal{M}^{-1} \cdot \mathcal{E}_i^{\mathrm{tr}} = (c_\mu, \ldots, c_1)^{\mathrm{tr}}$ with scalars $c_1, \ldots, c_\mu \in K$. Then we have $s_i = c_1 t_1 + \cdots + c_\mu t_\mu$ and

$$\mathcal{E}_i^{\mathrm{tr}} = \begin{pmatrix} t_\mu(p_1) & \cdots & t_1(p_1) \\ \vdots & \ddots & \vdots \\ t_\mu(p_\mu) & \cdots & t_1(p_\mu) \end{pmatrix} \cdot \begin{pmatrix} c_\mu \\ \vdots \\ c_1 \end{pmatrix} = \begin{pmatrix} c_1 t_1(p_1) + \cdots + c_\mu t_\mu(p_1) \\ \vdots \\ c_1 t_1(p_\mu) + \cdots + c_\mu t_\mu(p_\mu) \end{pmatrix} = \begin{pmatrix} s_i(p_1) \\ \vdots \\ s_i(p_\mu) \end{pmatrix},$$

i.e. s_i is a separator of p_i from the affine point set $\mathbb{X}^a \setminus \{p_i\}$ and claim 4) follows. □

Example 5.3.8. Consider the polynomial ring $P = \mathbb{Q}[x_1, x_2]$ which is graded by the matrix $W = (1, 3) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ and let $\mathbb{X}^a = \{p_1, \ldots, p_7\} \subseteq \mathbb{A}^2_{\mathbb{Q}}$ where $\mathbb{A}^2_{\mathbb{Q}} = \mathrm{Spec}(P)$ be the affine point set with $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (2, 0)$, $p_4 = (3, 0)$, $p_5 = (0, 1)$, $P_6 = (1, 2)$, and $p_7 = (2, 3)$. Moreover, let $\sigma = \mathtt{Ord}(\begin{smallmatrix} 1 & 3 \\ 1 & 1 \end{smallmatrix})$. Then σ is a term ordering by [KR00, Prop. 1.4.12] and it is compatible with $\deg_W$ by [KR05, Exmp. 4.2.2]. Thus $\mathbb{X}^a$, W, and σ satisfy the assumption of Algorithm 9. We now consider all the steps of the Buchberger-Möller Algorithm 9 applied to the input data $(p_1, \ldots, p_7)$, W, and σ in detail.

After the initialization process in the lines 1 to 5, we have $i = 0$, $\mathcal{O} = \emptyset$, the empty matrix $\mathcal{M} = (\ ) \in \mathrm{Mat}_{7,0}(\mathbb{Q})$ with seven rows and zero columns, $G = \emptyset$, and $d = 0$. Since $\mathcal{O} = \emptyset$, we have $\partial\mathcal{O} = \{1\}$ by Definition 2.1.2 and hence $(\partial\mathcal{O})_{\geq d} = \{1\} \neq \emptyset$. Thus the while-loop starting in line 6 is executed.

In line 7, we have $(\partial\mathcal{O})_d = (\partial\mathcal{O})_0 = \{1\}$, i.e. $r = 1$ and $\ell_1 = 1$. Thus we have

$$\mathcal{A} = \begin{pmatrix} \ell_1(0,0) \\ \ell_1(1,0) \\ \ell_1(2,0) \\ \ell_1(3,0) \\ \ell_1(0,1) \\ \ell_1(1,2) \\ \ell_1(2,3) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathrm{Mat}_{7,1}(\mathbb{Q})$$

after line 8 and therefore $\mathcal{B} = \mathcal{C} = \begin{pmatrix} \end{pmatrix} \in \mathrm{Mat}_{0,1}(\mathbb{Q})$, i.e. $k = 0$, in the lines 9 and 10. As $k = 0$, nothing needs to be done in the lines 11 and 12. Since $\ell_1 = 1$ is not the leading term with respect to σ of an element in $\emptyset$, the if-clause starting in line 14 is executed. After the if-clause in line 19, we have $i = 1$, $t_1 = 1$, $\mathcal{O} = \{1\}$, and

$$\mathcal{M} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathrm{Mat}_{7,1}(\mathbb{Q}).$$

In particular, the for-loop is finished as $r = 1$ and we get $d = 1$ after line 21. In this situation, $\partial\mathcal{O} = \{x_2, x_1\}$ and thus $(\partial\mathcal{O})_{\geq d} = \{x_2, x_1\} \neq \emptyset$ and the while-loop is executed, once again.
Now we have $(\partial\mathcal{O})_d = (\partial\mathcal{O})_1 = \{x_1\}$, i.e. $r = 1$ and $\ell_1 = x_1$. Thus we get

$$\mathcal{A} = \begin{pmatrix} \ell_1(0,0) & 1 \\ \ell_1(1,0) & 1 \\ \ell_1(2,0) & 1 \\ \ell_1(3,0) & 1 \\ \ell_1(0,1) & 1 \\ \ell_1(1,2) & 1 \\ \ell_1(2,3) & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,2}(\mathbb{Q})$$

in line 8 and, therefore, $\mathcal{B} = \mathcal{C} = \begin{pmatrix} \end{pmatrix} \in \mathrm{Mat}_{0,2}(\mathbb{Q})$, i.e. $k = 0$, after the lines 9 and 10. Since $k = 0$, we see that nothing changes in the lines 11 and 12. Moreover, $\ell_1 = x_1$ is not the leading term of an element of $\emptyset$ with respect to σ and the if-clause in line 14 is executed. After this if-clause, we have $i = 2$, $t_2 = x_1$, $\mathcal{O} = \{1, x_1\}$, and

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,2}(\mathbb{Q}),$$

and after the for-loop starting in line 13, $d = 2$. In this situation, $\partial\mathcal{O} = \{x_1^2, x_2, x_1x_2\}$ and thus $(\partial\mathcal{O})_{\geq d} = \{x_1^2, x_2, x_1x_2\} \neq \emptyset$ and the while-loop is executed, again.
The next two iterations of the while-loop are handled similarly. At the end of these two iterations, we then have $i = 5$, $t_3 = x_1^2$, $t_4 = x_2$, $t_5 = x_1^3$, $\mathcal{O} = \{1, x_1, x_1^2, x_2, x_1^3\}$,

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 8 & 0 & 4 & 2 & 1 \\ 27 & 0 & 9 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,5}(\mathbb{Q}),$$

and $d = 4$. As $(\partial\mathcal{O})_{\geq d} = (\{x_1x_2, x_1^4, x_1^2x_2, x_2^2, x_1^3x_2\})_{\geq 4} = \{x_1x_2, x_1^4, x_1^2x_2, x_2^2, x_1^3x_2\} \neq \emptyset$, the while-loop is iterated another time.
In line 7 of this next iteration, we have $(\partial\mathcal{O})_4 = \{x_1x_2, x_1^4\}$, i.e. $r = 2$, $\ell_1 = x_1x_2$, and $\ell_2 = x_1^4$. The computation of the matrices in the lines 8 to 10 yields

$$\mathcal{A} = \begin{pmatrix} \ell_2(0,0) & \ell_1(0,0) & 0 & 0 & 0 & 0 & 1 \\ \ell_2(1,0) & \ell_1(1,0) & 1 & 0 & 1 & 1 & 1 \\ \ell_2(2,0) & \ell_1(2,0) & 8 & 0 & 4 & 2 & 1 \\ \ell_2(3,0) & \ell_1(3,0) & 27 & 0 & 9 & 3 & 1 \\ \ell_2(0,1) & \ell_1(0,1) & 0 & 1 & 0 & 0 & 1 \\ \ell_2(1,2) & \ell_1(1,2) & 1 & 2 & 1 & 1 & 1 \\ \ell_2(2,3) & \ell_1(2,3) & 8 & 3 & 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 16 & 0 & 8 & 0 & 4 & 2 & 1 \\ 81 & 0 & 27 & 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & 1 \\ 16 & 6 & 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,7}(\mathbb{Q}),$$

and

$$\mathcal{B} = \mathcal{C} = \begin{pmatrix} 1 & 0 & -6 & 0 & 11 & -6 & 0 \end{pmatrix} \in \mathrm{Mat}_{1,7}(\mathbb{Q}),$$

i.e. $k = 1$. After the lines 11 and 12, we have $G = \{g_1\}$ with $g_1 = x_1^4 - 6x_1^3 + 11x_1^2 - 6x_1$. As $\ell_1 = x_1x_2 \notin \{x_1^4\} = \{\mathrm{LT}_\sigma(g_1)\}$ and as $\ell_2 = x_1^4 = \mathrm{LT}_\sigma(g_1)$, it follows that $i = 6$, $t_6 = x_1x_2$, $\mathcal{O} = \{1, x_1, x_1^2, x_2, x_1^3, x_1x_2\}$,

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 8 & 0 & 4 & 2 & 1 \\ 0 & 27 & 0 & 9 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 & 1 \\ 6 & 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,6}(\mathbb{Q}),$$

and $d = 5$ at the end of this iteration of the while-loop.

After another iteration, we have $i = 7$, $t_7 = x_1^2 x_2$, $\mathcal{O} = \{1, x_1, x_1^2, x_2, x_1^3, x_1x_2, x_1^2x_2\}$,

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 8 & 0 & 4 & 2 & 1 \\ 0 & 0 & 27 & 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 12 & 6 & 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,7}(\mathbb{Q}),$$

$d = 6$, and $(\partial\mathcal{O})_{\geq d} = (\{x_1^4, x_2^2, x_1^3x_2, x_1x_2^2, x_1^2x_2^2\})_{\geq 6} = \{x_2^2, x_1^3x_2, x_1x_2^2, x_1^2x_2^2\} \neq \emptyset$. In the following iteration, we have $(\partial\mathcal{O})_6 = \{x_2^2, x_1^3x_2\}$, i.e. $r = 2$, $\ell_1 = x_2^2$, and $\ell_2 = x_1^3x_2$ after line 7. The computation of the matrices in the lines 8 to 9 yields

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 27 & 0 & 9 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 4 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 24 & 9 & 12 & 6 & 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,9}(\mathbb{Q})$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & 2 & -3 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2,9}(\mathbb{Q}),$$

i.e. $k = 2$. The reduced row echolon form of $\mathcal{B}$ as in line 10 is then

$$\mathcal{C} = \begin{pmatrix} 1 & 0 & -3 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2,9}(\mathbb{Q})$$

and after line 12 we have $G = \{g_1, g_2, g_3\}$ with the polynomials $g_2 = x_2^2 - x_1x_2 - x_2$ and $g_3 = x_1^3x_2 - 3x_1^2x_2 + 2x_1x_2$. Moreover, no new elements are added to $\mathcal{O}$ in the if-clause.
After three further iterations of the while-loop, we have the following situation: $i = 7$, $\mathcal{O} = \{1, x_1, x_1^2, x_2, x_1^3, x_1x_2, x_1^2x_2\}$,

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 8 & 0 & 4 & 2 & 1 \\ 0 & 0 & 27 & 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 12 & 6 & 8 & 3 & 4 & 2 & 1 \end{pmatrix} \in \mathrm{Mat}_{7,7}(\mathbb{Q}),$$

$d = 9$, $G = \{g_1, \dots, g_5\}$ with $g_4 = x_1x_2^2 - x_1^2x_2 - x_1x_2$ and $g_5 = x_1^2x_2^2 - 4x_1^2x_2 + 2x_1x_2$, and $(\partial\mathcal{O})_{\geq d} = (\{x_1^4, x_2^2, x_1^3x_2, x_1x_2^2, x_1^2x_2^2\})_{\geq 9} = \emptyset$. Thus the while-loop terminates. Finally, since

$$\mathcal{M}^{-1} = \frac{1}{6} \cdot \begin{pmatrix} -3 & 3 & -1 & 0 & 3 & -3 & 1 \\ 9 & -6 & 1 & 0 & -9 & 6 & -1 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 6 & 0 & 0 \\ 6 & -15 & 12 & -3 & 0 & 0 & 0 \\ -11 & 18 & -9 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{7,7}(\mathbb{Q}),$$

we get $\mathcal{S} = (s_1, \dots, s_7) \in (\langle\mathcal{O}\rangle_K)^7$ in line 23 where the polynomials in $\mathcal{S}$ are of the form

$$\begin{aligned}
s_1 &= -\tfrac{1}{2}x_1^2x_2 + \tfrac{3}{2}x_1x_2 - \tfrac{1}{6}x_1^3 - x_2 + x_1^2 - \tfrac{11}{6}x_1 + 1,\\
s_2 &= \tfrac{1}{2}x_1^2x_2 - x_1x_2 + \tfrac{1}{2}x_1^3 - \tfrac{5}{2}x_1^2 + 3x_1,\\
s_3 &= -\tfrac{1}{6}x_1^2x_2 + \tfrac{1}{6}x_1x_2 - \tfrac{1}{2}x_1^3 + 2x_1^2 - \tfrac{3}{2}x_1,\\
s_4 &= \tfrac{1}{6}x_1^3 - \tfrac{1}{2}x_1^2 + \tfrac{1}{3}x_1,\\
s_5 &= \tfrac{1}{2}x_1^2x_2 - \tfrac{3}{2}x_1x_2 + x_2,\\
s_6 &= -\tfrac{1}{2}x_1^2x_2 + x_1x_2,\\
s_7 &= \tfrac{1}{6}x_1^2x_2 - \tfrac{1}{6}x_1x_2.
\end{aligned}$$

By Proposition 5.3.7, the set $G = \{g_1, \dots, g_5\}$ with

$$\begin{aligned}
g_1 &= x_1^4 - 6x_1 + 11x_1^2 - 6x_1^3,\\
g_2 &= x_2^2 - x_2 - x_1x_2,\\
g_3 &= x_1^3x_2 + 2x_1x_2 - 3x_1^2x_2,\\
g_4 &= x_1x_2^2 - x_1x_2 - x_1^2x_2,\\
g_5 &= x_1^2x_2^2 + 2x_1x_2 - 4x_1^2x_2
\end{aligned}$$

is the $\mathcal{O}$-border basis of the vanishing ideal $\mathcal{I}(\mathbb{X}^a)$ and for every $i \in \{1, \dots, 7\}$, the element s_i is a separator of the point p_i from the affine point set $\mathbb{X}^a \setminus \{p_i\}$.

Now we have all ingredients to transform Theorem 5.2.14 into an algorithm. Note that we assume that $\mathbb{X}$ is given by K-rational points $p_1, \dots, p_{\deg(\mathbb{X})} \in \mathbb{P}_K(\overline{W})$ such that the points are not contained in the hyperplane at infinity $H^{\mathrm{inf}} = \mathcal{Z}^+(x_0)$ and such that $\mathrm{Supp}(\mathbb{X}) = \{\mathcal{I}^+(\{p_1\}), \dots, \mathcal{I}^+(\{p_{\deg(\mathbb{X})}\})\}$, cf. Proposition 5.1.7.

Algorithm 10: `CheckUniformityReduced`$((p_1,\dots,p_\mu), i, j, (w_1,\dots,w_n), \sigma)$

Input:
$(p_1,\dots,p_\mu) \in (\mathbb{P}_K(\overline{W}))^\mu$ is a tuple consisting of $\mu \in \mathbb{N}\setminus\{0\}$ pairwise distinct K-rational projective points in $\mathbb{P}_K(\overline{W})$ satisfying $p_k \notin H^{\inf}$ for all $k \in \{1,\dots,\mu\}$,
$i, j \in \mathbb{N}\setminus\{0\}$,
$(w_1,\dots,w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ such that $w_i > 0$ for all $i \in \{1,\dots,n\}$,
σ is a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_{(w_1,\dots,w_n)}$

for $k := 1$ **to** μ **do**
  Let $p_{k1},\dots,p_{kn} \in K$ be such that $p_k = (1 : p_{k1} : \dots : p_{kn})$.
  $p_k^a := (p_{k1},\dots,p_{kn}) \in \mathbb{A}_K^n = \mathrm{Spec}(P)$
end
$(\mathcal{O}, G, \mathcal{S}) :=$ `BM`$((p_1^a,\dots,p_\mu^a), (w_1,\dots,w_n), \sigma)$
Let $t_1,\dots,t_\mu \in \mathbb{T}^n$ be with $\mathcal{O} = \{t_1,\dots,t_\mu\}$ and $\deg_W(t_1) \le \dots \le \deg_W(t_\mu)$.
Let $s_1,\dots,s_\mu \in \langle \mathcal{O} \rangle_K$ be such that $\mathcal{S} = (s_1,\dots,s_\mu)$.
$\sigma_{\mathbb{X}} := \deg_W(t_\mu) - 1$
$H_j := \max\{k \in \{1,\dots,\mu\} \mid \deg_W(t_k) \le j\}$
if $i > \mu - H_j$ **or** $j > \sigma_{\mathbb{X}}$ **then**
  return false
end
foreach subset $\{\nu_1,\dots,\nu_i\} \subseteq \{1,\dots,\mu\}$ **do**
  for $k := H_j + 1$ **to** μ **do**
    for $\ell := 1$ **to** i **do**
      Let $c_{k\ell} \in K$ denote the coefficient of t_k in s_{ν_ℓ}.
    end
  end
  $\mathcal{C} := (c_{k\ell})_{H_j+1 \le k \le \mu, 1 \le \ell \le i} \in \mathrm{Mat}_{\mu - H_j, i}(K)$
  if $\ker(\mathcal{C}) \neq \{0\}$ **then**
    return false
  end
end
return true

Proposition 5.3.9. (The (i,j)-Uniformity Test – The Reduced Case)
Let σ be a term ordering on $\mathbb{T}^n$ that is compatible with $\deg_W$ and $(p_1,\dots,p_\mu) \in (\mathbb{P}_K(\overline{W}))^\mu$ be a tuple of $\mu \in \mathbb{N}\setminus\{0\}$ pairwise distinct K-rational projective points in $\mathbb{P}_K(\overline{W})$. Assume that $p_k \notin H^{\inf}$ and let $\mathfrak{p}_k = \mathcal{I}^+(\{p_k\})$ denote the homogeneous vanishing ideal of p_k for all $k \in \{1,\dots,\mu\}$. Moreover, we let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be the reduced zero-dimensional subscheme with the property that $\mathrm{Supp}(\mathbb{X}) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_\mu\}$ and we let $i, j \in \mathbb{N}\setminus\{0\}$. Then Algorithm 10 is actually an algorithm and the result

$$\texttt{CheckUniformityReduced}((p_1,\dots,p_\mu), i, j, W, \sigma)$$

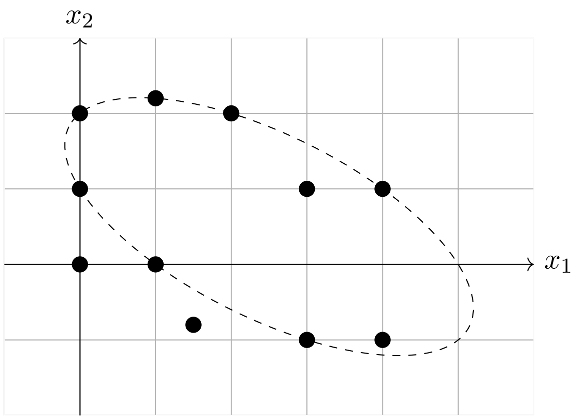
of Algorithm 10 applied to the input data $(p_1, \dots, p_\mu)$, i, j, W, and σ is a boolean value that determines whether the reduced scheme $\mathbb{X}$ is (i,j)-uniform.

Proof. First we show that the procedure is an algorithm. For all $k \in \{1, \dots, \mu\}$, we have $p_k \notin H^{\inf}$ and thus there are $p_{k1}, \dots, p_{kn} \in K$ with $p_k = (1 : p_{k1} : \cdots : p_{kn})$ as in line 2. For all $k \in \{1, \dots, \mu\}$, let $p_k^a = (p_{k1}, \dots, p_{kn}) \in \mathbb{A}_K^n = \operatorname{Spec}(P)$ be as in line 3. Since the projective points $p_1, \dots, p_\mu \in \mathbb{P}_K(\overline{W})$ are pairwise distinct, the corresponding affine points $p_1^a, \dots, p_n^a \in \mathbb{A}_K^n$ are also pairwise distinct. Hence the input data $(p_1^a, \dots, p_n^a)$, W, and σ in line 5 satisfy the requirements of Algorithm 9. Let $(\mathcal{O}, G, \mathcal{S})$ be the result of Algorithm 9 as in line 5 and let $I = \mathcal{I}(\{p_1^a, \dots, p_n^a\}) \subseteq P$ be the vanishing ideal of the affine point set $\{p_1^a, \dots, p_n^a\} \subseteq \mathbb{A}_K^n$. Then Proposition 5.3.7 yields that the set $G \subseteq P$ is the $\mathcal{O}$-border basis of I. By [KR05, Prop. 6.3.3], we have $\dim_K(P/I) = \mu$. Thus we see that $\#\mathcal{O} = \mu$ according to Definition 2.1.14, i.e. we can find terms $t_1, \dots, t_\mu \in \mathbb{T}^n$ such that $\mathcal{O} = \{t_1, \dots, t_\mu\}$ and such that $\deg_W(t_1) \le \cdots \le \deg_W(t_\mu)$ as in line 6. Since $\mu \ge 1$, $\sigma_{\mathbb{X}} = \deg_W(t_\mu) - 1$ as in line 8 and $H_j = \max\{k \in \{1, \dots, \mu\} \mid \deg_W(t_k) \le j\}$ as in line 9 exist. Moreover, $\mu = \deg(\mathbb{X})$, $\sigma_{\mathbb{X}} = \max\{k \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{X}}(k) < \deg(\mathbb{X})\}$, and $H_j = \mathrm{HF}_{\mathbb{X}}(j)$ according to Definition 5.1.17 and Proposition 4.1.15. By Proposition 5.3.7, there exist $s_1, \dots, s_\mu \in \langle \mathcal{O} \rangle_K$ such that $\mathcal{S} = (s_1, \dots, s_\mu)$. Moreover, we can check whether the kernel of a rectangular matrix over K is trivial or not as in line 20 using standard linear algebra techniques. All the other steps of the procedure can obviously be computed. Since the procedure obviously stops after a finite amount of time, it is actually an algorithm.

Next we prove the correctness. Since $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ is reduced, there is a subscheme $\widehat{\mathbb{X}}_k \subseteq \mathbb{X}$ with $\operatorname{Supp}(\widehat{\mathbb{X}}_k) = \operatorname{Supp}(\mathbb{X}) \setminus \{\mathfrak{p}_k\}$ and it satisfies $\deg(\widehat{\mathbb{X}}_i) = \deg(\mathbb{X}) - 1$ for all $k \in \{1, \dots, \mu\}$. For all $k, \ell \in \{1, \dots, \mu\}$, the polynomial $s_k \in \langle \mathcal{O} \rangle_K$ satisfies $s_k(p_\ell^a) = \delta_{k\ell}$ by Proposition 5.3.7. In particular, $s_k \neq 0$ and $\deg_W(s_k) \le \sigma_{\mathbb{X}} + 1$ for all $k \in \{1, \dots, \mu\}$. Let $F_k = x_0^{\sigma_{\mathbb{X}}+1-\deg_W(s_k)} s_k^{\mathrm{hom}}$ be for all $k \in \{1, \dots, \mu\}$. For all $k, \ell \in \{1, \dots, \mu\}$, $F_k(p_\ell) = (x_0^{\sigma_{\mathbb{X}}+1-\deg_W(s_k)} s_k^{\mathrm{hom}})(1, p_{\ell 1}, \dots, p_{\ell n}) = s_k(p_{\ell 1}, \dots, p_{\ell n}) = s_k(p_\ell^a) = \delta_{k\ell}$, i.e. $F_k \in \mathcal{I}^+(\widehat{\mathbb{X}}_k) \setminus \mathcal{I}^+(\mathbb{X})$. Hence for all $k \in \{1, \dots, \mu\}$, the residue class $f_k = \overline{F}_k \in \overline{P}_{\sigma_{\mathbb{X}}+1}$ in R is a separator of $\widehat{\mathbb{X}}_k$ according to Definition 5.1.17.

Finally, we consider all the different situation that can occur for the algorithm to stop. If the algorithm stops in line 11, then $i > \mu - H_j$ or $j > \sigma_{\mathbb{X}}$ in line 10. In both cases, the scheme $\mathbb{X}$ is trivially not (i,j)-uniform by Proposition 5.2.3 and Definition 5.2.1 and the correct boolean value is returned. Suppose that the algorithm does not stop in line 11. Then we have $i \in \{1, \dots, \mu - H_j\}$ and $j \in \{1, \dots, \sigma_{\mathbb{X}}\}$ according to line 10. Let $\{\nu_1, \dots, \nu_i\} \subseteq \{1, \dots, \mu\}$ be a subset consisting of i elements as in line 13. For all $k \in \{H_j + 1, \dots, \mu\}$, and $\ell \in \{1, \dots, i\}$, let $c_{k\ell} \in K$ be the coefficient of t_k in s_{ν_ℓ} as in line 16. Since we have $F_{\nu_\ell} = x_0^{\sigma_{\mathbb{X}}+1-\deg_W(s_{\nu_\ell})} s_{\nu_\ell}^{\mathrm{hom}}$, $c_{k\ell} \in K$ is also the coefficient of $x_0^{\sigma_{\mathbb{X}}+1-\deg_W(t_k)} t_k$ in F_{ν_ℓ} for all $k \in \{H_j + 1, \dots, \mu\}$ and $\ell \in \{1, \dots, i\}$. Let the matrix $\mathcal{C} = (c_{k\ell})_{H_j+1 \le k \le \mu, 1 \le \ell \le i} \in \operatorname{Mat}_{\mu - H_j, i}(K)$ be defined as in line 19. As we have $\deg_W(t_1) \le \cdots \le \deg_W(t_\mu)$ according to line 6, we see that for all $k \in \{1, \dots, \mu\}$, $\sigma_{\mathbb{X}} + 1 - \deg_W(t_k) < \sigma_{\mathbb{X}} + 1 - j$ holds if and only if $k > H_j$. Since the residue classes of the elements of $\{x_0^{\sigma_{\mathbb{X}}+1-\deg_W(t_1)} t_1, \dots, x_0^{\sigma_{\mathbb{X}}+1-\deg_W(t_\mu)} t_\mu\}$ in R form a K-vector space basis of $R_{\sigma_{\mathbb{X}}+1}$ according to Proposition 4.1.15, it follows that the residue classes of the elements of $\{x_0^{\sigma_{\mathbb{X}}+1-\deg_W(t_{H_j+1})}\overline{t}_{H_j+1}, \dots, x_0^{\sigma_{\mathbb{X}}+1-\deg_W(t_\mu)}\overline{t}_\mu\} \subseteq R_{\sigma_{\mathbb{X}}+1}$ in $R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ form a K-vector space basis of $(R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle)_{\sigma_{\mathbb{X}}+1}$. In particular, we see that for all $\ell \in \{1,\dots,i\}$, $(c_{\ell,H_j+1},\dots,c_{\ell\mu}) \in K^{\mu-H_j}$ is the corresponding coordinate tuple of the residue class of $f_{\nu_\ell} = \overline{F}_{\nu_\ell} \in R_{\sigma_{\mathbb{X}}+1}$ in $R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$. Hence the residue classes of the elements of $\{\overline{f}_{\nu_1},\dots,\overline{f}_{\nu_i}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ are K-linearly dependent if and only if $\ker(\mathcal{C}) \neq \{0\}$. In other words, the algorithm stops in line 21 if and only if there exists a subset $\{\nu_1,\dots,\nu_i\} \subseteq \{1,\dots,\mu\}$ such that the residue classes of the elements of $\{\overline{f}_{\nu_1},\dots,\overline{f}_{\nu_i}\} \subseteq R/\langle \overline{x}_0^{\sigma_{\mathbb{X}}+1-j}\rangle$ are K-linearly dependent. By Theorem 5.2.14, this is equivalent for $\mathbb{X}$ to be not (i,j)-uniform and the correct boolean value is returned in line 20. If no such K-linearly dependent subset exists, the algorithm stops in line 24. In this situation, the scheme $\mathbb{X}$ is (i,j)-uniform by Theorem 5.2.14 and the correct boolean value is returned in line 24. □

Example 5.3.10. Consider the reduced zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ of Example 5.2.16.a, again. Recall that $\mathbb{X} \subseteq \mathbb{P}^2_{\mathbb{Q}}$ was the reduced zero-dimensional subscheme of degree $\deg(\mathbb{X}) = 11$ with $\operatorname{Supp}(\mathbb{X}) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_{11}\}$ where $\mathfrak{p}_i = \mathcal{I}^+(p_i)$ for all $i \in \{1,\dots,11\}$ and $p_1 = (1:0:0)$, $p_2 = (1:0:1)$, $p_3 = (1:0:2)$, $p_4 = (1:1:0)$, $p_5 = (1:1:\frac{11}{5})$, $p_6 = (1:\frac{3}{2}:-\frac{4}{5})$, $p_7 = (1:2:2)$, $p_8 = (1:3:-1)$, $p_9 = (1:3:1)$, $p_{10} = (1:4:-1)$, and $p_{11} = (1:4:1)$. We have already seen in Example 5.2.16 that $\mathbb{X} \cap H^{\inf} = \emptyset$, that $\mathbb{X}$ is not $(4,2)$-uniform with respect to the standard grading defined by $W = (1,1) \in \operatorname{Mat}_{1,2}(\mathbb{Z})$, and that the reason for that was that an ellipse passes through the points in $\{p_1,\dots,p_{11}\} \setminus \{p_1,p_6,p_9,p_{10}\}$. By Definition 5.1.3, we can identify the points of $\mathbb{X}$, with the corresponding affine part in $\mathbb{A}^2_{\mathbb{Q}} = \operatorname{Spec}(\mathbb{Q}[x_1,x_2])$ as in the following picture.



Let $\sigma = \texttt{DegRevLex}$. Then the term ordering σ on $\mathbb{T}^n$ is compatible with $\deg_W$. Moreover, we let $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$ also be $\mathbb{Z}$-graded by the matrix $\overline{W} = (1 \mid W) \in \operatorname{Mat}_{1,3}(\mathbb{Z})$. In order to illustrate Algorithm 10, we apply it to the input data $(p_1,\dots,p_{11})$, $i = 4$, $j = 2$, W, and σ.

After the for-loop starting in line 1, $(p_1^a,\dots,p_{11}^a) \in (\mathbb{A}^2_{\mathbb{Q}})^{11}$ with $p_1^a = (0,0)$, $p_2^a = (0,1)$, $p_3^a = (0,2)$, $p_4^a = (1,0)$, $p_5^a = (1,\frac{11}{5})$, $p_6^a = (\frac{3}{2},-\frac{4}{5})$, $p_7^a = (2,2)$, $p_8^a = (3,-1)$, $p_9^a = (3,1)$,

$p^a_{10} = (4,-1)$, and $p^a_{11} = (4,1)$. The result of the Buchberger-Möller Algorithm 9 applied to the input data $(p^a_1,\dots,p^a_{11})$, W, and σ is

$$\mathcal{O} = \{t_1,\dots,t_{11}\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_2^3, x_1x_2^2, x_1^2x_2, x_1^3, x_2^4\} \subseteq \mathbb{T}^2$$

and $\mathcal{S} = (s_1,\dots,s_{11}) \in (\langle \mathcal{O}\rangle_{\mathbb{Q}})^{11}$ where

$$\begin{aligned}
s_1 &= \tfrac{115}{2893}x_1^4 - \tfrac{302}{2893}x_1^3 - \tfrac{203}{2893}x_1^2x_2 - \tfrac{137}{2893}x_1x_2^2 + \tfrac{809}{5786}x_2^3 \\
&\quad + \tfrac{2317}{2893}x_1^2 + \tfrac{1421}{2893}x_1x_2 - \tfrac{52}{263}x_2^2 - \tfrac{4908}{2893}x_1 - \tfrac{5681}{5786}x_2 + 1, \\
s_6 &= -\tfrac{4375}{52074}x_2^4 + \tfrac{2600}{26037}x_1^3 - \tfrac{4000}{26037}x_1^2x_2 - \tfrac{2800}{8679}x_1x_2^2 - \tfrac{3875}{8679}x_2^3 \\
&\quad - \tfrac{14200}{26037}x_1^2 + \tfrac{28000}{26037}x_1x_2 + \tfrac{9125}{4734}x_2^2 + \tfrac{11600}{26037}x_1 - \tfrac{12125}{8679}x_2, \\
s_9 &= \tfrac{9775}{416592}x_2^4 - \tfrac{1966}{26037}x_1^3 - \tfrac{6322}{26037}x_1^2x_2 - \tfrac{7115}{17358}x_1x_2^2 - \tfrac{8135}{23144}x_2^3 \\
&\quad + \tfrac{11405}{26037}x_1^2 + \tfrac{62471}{52074}x_1x_2 + \tfrac{33715}{37872}x_2^2 - \tfrac{9439}{26037}x_1 - \tfrac{39035}{69432}x_2, \\
s_{10} &= -\tfrac{7475}{46288}x_2^4 + \tfrac{142}{2893}x_1^3 - \tfrac{441}{2893}x_1^2x_2 - \tfrac{695}{5786}x_1x_2^2 + \tfrac{4935}{23144}x_2^3 \\
&\quad - \tfrac{553}{2893}x_1^2 + \tfrac{3281}{5786}x_1x_2 + \tfrac{2065}{4208}x_2^2 + \tfrac{411}{2893}x_1 - \tfrac{12555}{23144}x_2.
\end{aligned}$$

We have $\sigma_{\mathbb{X}} = \deg_W(t_{11}) - 1 = 3$ and $H_j = \max\{k \in \{1,\dots,11\} \mid \deg_W(t_k) \le 2\} = 6$ after the lines 8 and 9. As $i = 4 \le 5 = 11 - H_j$ and as $j = 2 \le 3 = \sigma_{\mathbb{X}}$, the if-clause in line 10 is not executed. Let $\{\nu_1,\dots,\nu_4\} \subseteq \{1,\dots,11\}$ with $\nu_1 = 1$, $\nu_2 = 6$, $\nu_3 = 9$, and $\nu_4 = 10$ be chosen in line 13. After the two for-loops starting in the lines 14 and 15, we get

$$\mathcal{C} = \tfrac{1}{416592} \cdot \begin{pmatrix} 58248 & -186000 & -146430 & 88830 \\ -19728 & -134400 & -170760 & -50040 \\ -29232 & -64000 & -101152 & -63504 \\ -43488 & 41600 & -31456 & 20448 \\ 16560 & -35000 & 9775 & -67275 \end{pmatrix} \in \mathrm{Mat}_{5,4}(\mathbb{Q})$$

in line 19. Since we have $\ker(\mathcal{C}) = \langle (100, 69, -60, -20)^{\mathrm{tr}} \rangle \neq \{0\}$, the if-clause starting in line 20 is executed and the boolean value `false` is returned in line 21. Note that this result coincides with the result of Example 5.2.16.a.

Finally, we sum up all the methods introduced in this section to check i-uniformity and (i,j)-uniformity.

Remark 5.3.11. Let $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$ be a non-empty (not necessarily reduced) zero-dimensional subscheme and assume that $\mathbb{X} \cap H^{\mathrm{inf}} = \emptyset$. Let $i, j \in \mathbb{N} \setminus \{0\}$. Then we have the following possibilities to check whether $\mathbb{X}$ is i- respectively (i,j)-uniform.

a) If K is algebraically closed and $\mathbb{X}$ is given by a generating set of its defining ideal $\mathcal{I}^+(\mathbb{X})$, we can use Proposition 5.3.3 to check whether $\mathbb{X}$ is (i,j)-uniform. In particular, combined with Proposition 5.2.6, we can use this to check whether $\mathbb{X}$ is i-uniform. Note that by now it was only possible to check the i-uniformity condition in the case that $i = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k)$ for some $k \in \mathbb{N}$ by applying the method described in Remark 5.3.2.

b) If $\mathbb{X}$ is reduced and given by the set of points in $\mathbb{P}_K(\overline{W})$ in its support, we can use Proposition 5.3.9 to check whether $\mathbb{X}$ is (i, j)-uniform. In particular, combined with Proposition 5.2.6, we can check whether $\mathbb{X}$ is i-uniform. Note that by now it was only possible to check the i-uniformity condition in the reduced case if K is algebraically closed and $i \in \{1, \ldots, \Delta H(\sigma_{\mathbb{X}} + 1)\}$ respectively $i = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k)$ for some $k \in \mathbb{N}$ as described in a) respectively Remark 5.2.15, or if $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) > \deg(\mathbb{X})$ as described in Remark 5.3.6. Moreover, note that we do not need to restrict the base field K, at all. Thus our approach extends the method described in Remark 5.3.6.

6 The (Projective) Border Basis Scheme

In this final chapter, we combine the results of the Chapters 4 and 5. In Chapter 4, we have introduced projective border bases. In particular, we proved the (Projective) Border Bases Correspondence in Corollary 4.1.9, which states that there is a one-to-one correspondence between projective $\mathcal{O}$-border bases and specific $\mathcal{O}$-border bases. Then in Chapter 5, we have studied zero-dimensional subschemes of weighted projective spaces and uniformity conditions that these subschemes satisfy. In particular, we have seen that there is a one-to-one correspondence between projective border bases and specific zero-dimensional subschemes of weighted projective spaces in Proposition 5.1.8. Let K be a field, $n \in \mathbb{N} \setminus \{0\}$, $P = K[x_1, \dots, x_n]$, and $\overline{P} = P[x_0] = K[x_0, \dots, x_n]$. Then the following figure shows the intersection of the previous two sections.

zero-dimensional closed subschemes $\mathbb{X}$ of weighted projective spaces $\mathbb{P}_K(\overline{W})$
that have no point on the hyperplane at infinity

projective border basis of the defining ideal $\mathcal{I}^+(\mathbb{X}) \subseteq \overline{P}$ $\downarrow$ $\quad$ $\uparrow$ projective subscheme defined by the homogeneous ideal

projective $\mathcal{O}$-border bases of homogeneous ideals in $\overline{P}$
where $\overline{P}$ is graded by $\overline{W} = (1 \mid W)$ with a positive matrix $W \in \mathrm{Mat}_{1,n}(\mathbb{Z})$

dehomogenization $\downarrow$ $\quad$ $\uparrow$ homogenization

$\mathcal{O}$-border bases of ideals in P with $b \in \mathrm{DF}_W(g_b)$ for all $b \in \partial\mathcal{O}$
where g_b denotes the $\mathcal{O}$-border basis element corresponding to b

The above correspondence is the starting point of this chapter. Let $\mathcal{O}$ be a non-empty order ideal in $\mathbb{T}^n$. In [KR08] and [Rob09], the authors parametrized all zero-dimensional ideals in P that possess a $\mathcal{O}$-border basis by introducing the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$. We introduce the subscheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, called the projective $\mathcal{O}$-border basis scheme, that parametrizes all one-dimensional homogeneous ideal in $\overline{P}$ that possess a projective $\mathcal{O}$-border basis in Definition 6.1.5. Then we introduce the corresponding universal projective $\mathcal{O}$-border basis family $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ and show that the residue classes of the elements of $\mathcal{O}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$ in

Theorem 6.1.13. This allows us to explicitly describe the multiplicative structure of the coordinate ring of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and its dual module by means of multiplication matrices in the Sections 6.2 and 6.3. In Section 6.4, we then explicitly describe the points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to an (i,j)-uniform zero-dimensional subscheme. The main result of this chapter is Theorem 6.4.4 and yields that for $i,j \in \mathbb{N}$, the set of all points of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the corresponding zero-dimensional subscheme is (i,j)-uniform is a constructible set with respect to the Zariski topology.

As in the previous two chapters, let the K-algebra $P = \bigoplus_{\gamma\in\mathbb{Z}} P_\gamma$ be $\mathbb{Z}$-graded by a matrix $W = (w_1,\dots,w_n) \in \mathrm{Mat}_{1,n}(\mathbb{Z})$ with $w_i > 0$ for all $i \in \{1,\dots,n\}$. Moreover, let x_0 denote the homogenizing indeterminate and let $\overline{P} = P[x_0] = K[x_0,\dots,x_n]$. Then $\overline{P} = \bigoplus_{\gamma\in\mathbb{Z}} \overline{P}_\gamma$ is positively $\mathbb{Z}$-graded by the matrix $\overline{W} = (1 \mid W) \in \mathrm{Mat}_{1,n+1}(K)$ according to [KR05, Defn. 4.1.6 and 4.2.4]. Let $\mathcal{O} = \{t_1,\dots,t_\mu\}$ with $\mu \in \mathbb{N}\setminus\{0\}$ be a finite order ideal in $\mathbb{T}^n$ and assume that $\deg_W(t_1) \le \cdots \le \deg_W(t_\mu)$. We denote its border by $\partial\mathcal{O} = \{b_1,\dots,b_\nu\}$ with $\nu \in \mathbb{N}\setminus\{0\}$. As before, we define the integer function $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1,\dots,\mu\} \mid \deg_W(t_k) \le \gamma\}$ and its first difference function $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma-1)$.

6.1 The (Projective) Border Basis Scheme

In this section, we introduce and study the projective $\mathcal{O}$-border basis scheme and the corresponding universal projective $\mathcal{O}$-border basis family $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$. The main result of this section is Theorem 6.1.13 and yields that the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$.

First we recall the definition of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$, which is due to [KR08], in Definition 6.1.1. The key idea of the $\mathcal{O}$-border basis scheme is to replace the scalars $a_{ij} \in K$ of the elements of a $\mathcal{O}$-border prebases as in Definition 2.1.14 by newly introduced indeterminates c_{ij}. This yields the so-called generic $\mathcal{O}$-border prebasis and allows us to consider all $\mathcal{O}$-border prebasis at once. In particular, the generic $\mathcal{O}$-border prebasis induces generic multiplication matrices. The image of these matrices under the substitution $c_{ij} \mapsto a_{ij}$ are pairwise commuting if and only if the image of the generic $\mathcal{O}$-border prebasis under the substitution $c_{ij} \mapsto a_{ij}$ is actually an $\mathcal{O}$-border basis. This way, the generic multiplication matrices define an affine scheme $\mathbb{B}_{\mathcal{O}}$ which parametrizes all $\mathcal{O}$-border bases. One of the main results about $\mathbb{B}_{\mathcal{O}}$ is that the residue classes of the elements of $\mathcal{O}$ form a module basis of the corresponding universal family, cf. Theorem 6.1.3.

After this summary of the $\mathcal{O}$-border basis scheme, we newly introduce the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ in Definition 6.1.5. The projective $\mathcal{O}$-border basis scheme is the subscheme of $\mathbb{B}_{\mathcal{O}}$ that parametrizes all projective $\mathcal{O}$-border bases. The remaining part of this section is then dedicated to the study of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. In particular, we generalize Theorem 6.1.3 to the projective setting in Theorem 6.1.13.

Before we introduce the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, we recall the definition of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ as introduced in [KR08, Defn. 3.1 and 3.3].

Definition 6.1.1. Let $\{c_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}\}$ be a set of further indeterminates.

a) The set $G = \{g_1,\dots,g_\nu\}$ defined by

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i \in P[c_{11},\dots,c_{\mu\nu}]$$

is called the **generic $\mathcal{O}$-border prebasis**.

b) Let $G \subseteq P[c_{11},\dots,c_{\mu\nu}]$ be the generic $\mathcal{O}$-border prebasis. For every $r \in \{1,\dots,n\}$, let $\mathcal{X}_r = (\xi_{k\ell}^{(r)})_{1\le k,\ell\le\mu} \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ be the r^{th} formal multiplication matrix of G as defined in Definition 2.4.15, i.e.

$$\xi_{k\ell}^{(r)} = \begin{cases} \delta_{ki} & \text{if } x_r t_\ell = t_i \in \mathcal{O}, \\ c_{kj} & \text{if } x_r t_\ell = b_j \in \partial\mathcal{O} \end{cases}$$

for all $k,\ell \in \{1,\dots,\mu\}$. It is also called the $r^{\mathbf{th}}$ **generic multiplication matrix** with respect to $\mathcal{O}$.

c) Let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ be the generic multiplication matrices with respect to $\mathcal{O}$. The affine scheme $\mathbb{B}_{\mathcal{O}} \subseteq \mathbb{A}_K^{\mu\nu} = \mathrm{Spec}(K[c_{11},\dots,c_{\mu\nu}])$ defined by the ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ that is generated by the entries of the matrices $\mathcal{X}_r\mathcal{X}_s - \mathcal{X}_s\mathcal{X}_r$ with $r,s \in \{1,\dots,n\}$ is called the **$\mathcal{O}$-border basis scheme**. Its coordinate ring $K[c_{11},\dots,c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ is denoted by $B_{\mathcal{O}}$.

d) Let $G \subseteq P[c_{11},\dots,c_{\mu\nu}]$ be the generic $\mathcal{O}$-border prebasis, $B_{\mathcal{O}}$ be the coordinate ring of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$, and $U_{\mathcal{O}} = B_{\mathcal{O}}[x_1,\dots,x_n]/\langle G\rangle B_{\mathcal{O}}[x_1,\dots,x_n]$. Then the canonical K-algebra homomorphism $B_{\mathcal{O}} \hookrightarrow B_{\mathcal{O}}[x_1,\dots,x_n] \twoheadrightarrow U_{\mathcal{O}}$ is called the **universal $\mathcal{O}$-border basis family**.

Example 6.1.2. Let $P = \mathbb{Q}[x_1,x_2]$ be equipped with the standard grading, i.e. graded by $W = (1,1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$. The set $\mathcal{O} = \{t_1,\dots,t_6\} = \{1, x_2, x_1, x_1x_2, x_1^2, x_1^3\}$ is an order ideal in $\mathbb{T}^2$ and the set $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^2, x_1x_2^2, x_1^2x_2, x_1^3x_2, x_1^4\}$ is the corresponding border according to the Definitions 2.1.6 and 2.1.7. Moreover, we let $\{c_{ij} \mid i \in \{1,\dots,6\}, j \in \{1,\dots,5\}\}$ be a set of further indeterminates. Then the set $G = \{g_1,\dots,g_5\} \subseteq \mathbb{Q}[c_{11},\dots,c_{65}]$ with $g_j = b_j - c_{1j} - c_{2j}x_2 - c_{3j}x_1 - c_{4j}x_1x_2 - c_{5j}x_1^2 - c_{6j}x_1^3$ for all $j \in \{1,\dots,5\}$ is the generic $\mathcal{O}$-border prebasis according to Definition 6.1.1. Then

$$\mathcal{X}_1 = \begin{pmatrix} 0 & 0 & 0 & c_{13} & 0 & c_{15} \\ 0 & 0 & 0 & c_{23} & 0 & c_{25} \\ 1 & 0 & 0 & c_{33} & 0 & c_{35} \\ 0 & 1 & 0 & c_{43} & 0 & c_{45} \\ 0 & 0 & 1 & c_{53} & 0 & c_{55} \\ 0 & 0 & 0 & c_{63} & 1 & c_{65} \end{pmatrix}, \qquad \mathcal{X}_2 = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & c_{13} & c_{14} \\ 1 & c_{21} & 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{31} & 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{41} & 1 & c_{42} & c_{43} & c_{44} \\ 0 & c_{51} & 0 & c_{52} & c_{53} & c_{54} \\ 0 & c_{61} & 0 & c_{62} & c_{63} & c_{64} \end{pmatrix}$$

are the generic multiplication matrices $\mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Mat}_6(\mathbb{Q}[c_{11}, \ldots, c_{65}])$ with respect to $\mathcal{O}$. By Definition 6.1.1, the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}} \subseteq \mathbb{A}^{30}_{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[c_{11}, \ldots, c_{65}])$ is defined by the ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) = \langle f_1, \ldots, f_{24} \rangle \subseteq \mathbb{Q}[c_{11}, \ldots, c_{65}]$ with

$$
\begin{aligned}
f_1 &= c_{13}c_{41} + c_{15}c_{61} - c_{12}\\
f_2 &= -c_{11}c_{23} + c_{13}c_{42} - c_{12}c_{43} - c_{13}c_{53} + c_{15}c_{62} - c_{14}c_{63}\\
f_3 &= c_{13}c_{43} + c_{15}c_{63} - c_{14}\\
f_4 &= -c_{11}c_{25} + c_{13}c_{44} - c_{12}c_{45} - c_{13}c_{55} + c_{15}c_{64} - c_{14}c_{65}\\
f_5 &= c_{23}c_{41} + c_{25}c_{61} - c_{22}\\
f_6 &= -c_{21}c_{23} + c_{23}c_{42} - c_{22}c_{43} - c_{23}c_{53} + c_{25}c_{62} - c_{24}c_{63} - c_{13}\\
f_7 &= c_{23}c_{43} + c_{25}c_{63} - c_{24}\\
f_8 &= -c_{21}c_{25} + c_{23}c_{44} - c_{22}c_{45} - c_{23}c_{55} + c_{25}c_{64} - c_{24}c_{65} - c_{15}\\
f_9 &= c_{33}c_{41} + c_{35}c_{61} + c_{11} - c_{32}\\
f_{10} &= -c_{23}c_{31} + c_{33}c_{42} - c_{32}c_{43} - c_{33}c_{53} + c_{35}c_{62} - c_{34}c_{63} + c_{12}\\
f_{11} &= c_{33}c_{43} + c_{35}c_{63} + c_{13} - c_{34}\\
f_{12} &= -c_{25}c_{31} + c_{33}c_{44} - c_{32}c_{45} - c_{33}c_{55} + c_{35}c_{64} - c_{34}c_{65} + c_{14}\\
f_{13} &= c_{41}c_{43} + c_{45}c_{61} + c_{21} - c_{42}\\
f_{14} &= -c_{23}c_{41} - c_{43}c_{53} + c_{45}c_{62} - c_{44}c_{63} + c_{22} - c_{33}\\
f_{15} &= c_{43}^2 + c_{45}c_{63} + c_{23} - c_{44}\\
f_{16} &= -c_{25}c_{41} + c_{43}c_{44} - c_{42}c_{45} - c_{43}c_{55} + c_{45}c_{64} - c_{44}c_{65} + c_{24} - c_{35}\\
f_{17} &= c_{41}c_{53} + c_{55}c_{61} + c_{31} - c_{52}\\
f_{18} &= -c_{23}c_{51} - c_{43}c_{52} + c_{42}c_{53} - c_{53}^2 + c_{55}c_{62} - c_{54}c_{63} + c_{32}\\
f_{19} &= c_{43}c_{53} + c_{55}c_{63} + c_{33} - c_{54}\\
f_{20} &= -c_{25}c_{51} - c_{45}c_{52} + c_{44}c_{53} - c_{53}c_{55} + c_{55}c_{64} - c_{54}c_{65} + c_{34}\\
f_{21} &= c_{41}c_{63} + c_{61}c_{65} + c_{51} - c_{62}\\
f_{22} &= -c_{23}c_{61} - c_{43}c_{62} + c_{42}c_{63} - c_{53}c_{63} - c_{63}c_{64} + c_{62}c_{65} + c_{52}\\
f_{23} &= c_{43}c_{63} + c_{63}c_{65} + c_{53} - c_{64}\\
f_{24} &= -c_{25}c_{61} - c_{45}c_{62} + c_{44}c_{63} - c_{55}c_{63} + c_{54}
\end{aligned}
$$

and its coordinate ring is $B_{\mathcal{O}} = \mathbb{Q}[c_{11}, \ldots, c_{65}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})$.

The order ideal is not only used to define the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$, the residue classes of its elements in $U_{\mathcal{O}}$ also form a $B_{\mathcal{O}}$-module basis of $U_{\mathcal{O}}$ by [KR08, Thm. 3.4].

Theorem 6.1.3. (The Universal Border Basis Family)
Let $B_{\mathcal{O}} \to U_{\mathcal{O}}$ be the universal $\mathcal{O}$-border basis family. Then the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}$ form a $B_{\mathcal{O}}$-module basis of $U_{\mathcal{O}}$.

Proof. This theorem was shown in [KR08, Thm. 3.4]. □

The $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ parametrizes all $\mathcal{O}$-border bases. An $\mathcal{O}$-border basis induces a projective $\mathcal{O}$-border basis if and only if every border term is contained in the support of the degree form with respect to W of the corresponding border basis element according to the (Projective) Border Bases Correspondence 4.1.9. The following proposition makes use of this fact.

Proposition 6.1.4. *Let $B_{\mathcal{O}}$ be the coordinate ring of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ and let $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ be the subscheme of $\mathbb{B}_{\mathcal{O}}$ defined by the ideal*

$$I_{\mathcal{O}}^{\mathrm{proj}} = \langle \bar{c}_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}, \deg_W(t_i) > \deg_W(b_j) \rangle \subseteq B_{\mathcal{O}}.$$

Then the points of the scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ are in one-to-one correspondence to the homogeneous ideals in $\overline{P}$ that possess a projective $\mathcal{O}$-border basis. In particular, there is a canonical K-algebra isomorphism between the coordinate ring $B_{\mathcal{O}}^{\mathrm{proj}}$ of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and the K-algebra $K[c_{11},\dots,c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$ where

$$\widetilde{I}_{\mathcal{O}}^{\mathrm{proj}} = \langle c_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}, \deg_W(t_i) > \deg_W(b_j) \rangle \subseteq K[c_{11},\dots,c_{\mu\nu}].$$

Proof. By Lemma 2.6.7 and the Second Noetherian Isomorphism Theorem, there is a canonical K-algebra isomorphism

$$B_{\mathcal{O}}^{\mathrm{proj}} = B_{\mathcal{O}}/I_{\mathcal{O}}^{\mathrm{proj}} = (K[c_{11},\dots,c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}))/I_{\mathcal{O}}^{\mathrm{proj}} \cong K[c_{11},\dots,c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}.$$

Next we prove that every point of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ induces a unique homogeneous ideal in $\overline{P}$ that possesses a projective $\mathcal{O}$-border basis. Let $a = (a_{11},\dots,a_{\mu\nu}) \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. Then a is a zero of the ideal $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$. Let $G = \{g_1,\dots,g_\nu\}$ with $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ for all $j \in \{1,\dots,\nu\}$ be the $\mathcal{O}$-border prebasis induced by a and let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K)$ be the formal multiplication matrices of G as in Definition 2.4.15. Since a is a zero of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$, the matrices $\mathcal{X}_1,\dots,\mathcal{X}_n$ are pairwise commuting according to Definition 6.1.1. Hence Theorem 2.4.19 and Proposition 2.3.2 yield that G is the unique $\mathcal{O}$-border basis of $\langle G \rangle$. Moreover, as a is also a zero of $\widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$, we have $a_{ij} = 0$ for all $\deg_W(t_i) > \deg_W(b_j)$, i.e. $b_j \in \mathrm{DF}_W(g_j)$. By Corollary 4.1.9, the corresponding homogeneous ideal $\langle G \rangle^{\mathrm{hom}} \subseteq \overline{P}$ possesses a projective $\mathcal{O}$-border basis.

For the converse correspondence, let $I \subseteq \overline{P}$ be a homogeneous ideal that possesses a projective $\mathcal{O}$-border basis $G = \{g_1,\dots,g_\nu\}$. For all $i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$, let $\Delta_{i,j} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{Z}$. According to Proposition 4.1.7, there exists a point $a = (a_{11},\dots,a_{\mu\nu}) \in \mathbb{A}_K^{\mu\nu}$ such that $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} x_0^{\Delta_{i,j}} t_i$ for all $j \in \{1,\dots,\nu\}$ and such that $a_{ij} = 0$ for all $i \in \{1,\dots,\mu\}$ with $\deg_W(t_i) > \deg_W(b_j)$. Hence a is a zero of $\widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$. Moreover, Proposition 4.1.7 also yields that $\{g_1^{\mathrm{deh}},\dots,g_\nu^{\mathrm{deh}}\} \subseteq P[c_{11},\dots,c_{\mu\nu}]$ is an $\mathcal{O}$-border basis. Thus Theorem 2.4.19 implies the commutativity of the formal multiplication matrices of $\{g_1^{\mathrm{deh}},\dots,g_\nu^{\mathrm{deh}}\}$ and thus a is also a zero of $\mathcal{I}(\mathbb{B}_{\mathcal{O}})$ by Definition 6.1.1. Altogether, a is a zero of $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$ and thus a point of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$.

Since all the operations above were uniquely determined, the claim follows. □

Now we are able to define the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. Recall that the integer function H is defined by $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1,\dots,\mu\} \mid \deg_W(t_k) \leq \gamma\}$ and its first difference function is $\Delta H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto H(\gamma) - H(\gamma - 1)$.

Definition 6.1.5. a) The subscheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ defined by the ideal

$$I_{\mathcal{O}}^{\mathrm{proj}} = \langle \overline{c}_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}, \deg_W(t_i) > \deg_W(b_j) \rangle \subseteq B_{\mathcal{O}}$$

is called the **projective $\mathcal{O}$-border basis scheme**. Its coordinate ring is denoted by $B_{\mathcal{O}}^{\mathrm{proj}}$.

b) For all $i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$, we let $\Delta_{i,j} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{Z}$. Then the set $G = \{g_1,\dots,g_\nu\}$ defined by

$$g_j = b_j - \sum_{i=1}^{H(\deg_W(b_j))} c_{ij} x_0^{\Delta_{i,j}} t_i = b_j - \sum_{\substack{i \in \{1,\dots,\mu\} \\ \deg_W(b_j) \geq \deg_W(t_i)}} c_{ij} x_0^{\Delta_{i,j}} t_i \in \overline{P}[c_{11},\dots,c_{\mu\nu}]$$

for all $j \in \{1,\dots,\nu\}$ is called the **generic projective $\mathcal{O}$-border prebasis**.

c) Let $G \subseteq \overline{P}[c_{11},\dots,c_{\mu\nu}]$ be the generic projective $\mathcal{O}$-border prebasis, let $B_{\mathcal{O}}^{\mathrm{proj}}$ denote the coordinate ring of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, and we let $U_{\mathcal{O}}^{\mathrm{proj}} = B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]/\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$. Then the canonical K-algebra homomorphism $B_{\mathcal{O}}^{\mathrm{proj}} \hookrightarrow B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n] \twoheadrightarrow U_{\mathcal{O}}^{\mathrm{proj}}$ is called the **universal projective $\mathcal{O}$-border basis family**.

Example 6.1.6. Consider the order ideal $\mathcal{O}$ in $\mathbb{T}^2$ the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$, and the universal $\mathcal{O}$-border basis family $B_{\mathcal{O}} \to U_{\mathcal{O}}$ of Example 6.1.2, again. Recall that $\mathcal{O}$ lived in the standard graded polynomial ring $P = \mathbb{Q}[x_1,x_2]$, i.e. that $\overline{P}$ graded by the matrix $W = (1,1) \in \mathrm{Mat}_{1,2}(\mathbb{Q})$, that $\mathcal{O} = \{t_1,\dots,t_6\} = \{1, x_2, x_1, x_1x_2, x_1^2, x_1^3\}$, and that its border was $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^2, x_1x_2^2, x_1^2x_2, x_1^3x_2, x_1^4\}$. For all $i \in \{1,\dots,6\}$ and $j \in \{1,\dots,5\}$ with $(i,j) \neq (6,1)$, we have $\deg_W(t_i) = \deg(t_i) \leq \deg(b_j) = \deg_W(b_j)$ and we have $\deg_W(t_6) = \deg(x_1^3) = 3 > 2 = \deg(x_2^2) = \deg_W(b_1)$. By Definition 6.1.5, the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ is thus the subscheme of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$ defined by $I_{\mathcal{O}}^{\mathrm{proj}} = \langle \overline{c}_{61} \rangle \subseteq B_{\mathcal{O}} = \mathbb{Q}[c_{11},\dots,c_{65}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}})$. Let the $\mathbb{Q}$-algebra $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$ be standard graded. Then $G = \{g_1,\dots,g_5\} \subseteq \overline{P}[c_{11},\dots,c_{65}]$ where the polynomials are of the form $g_1 = x_2^2 - c_{11}x_0^2 - c_{21}x_0x_2 - c_{31}x_0x_1 - c_{41}x_1x_2 - c_{51}x_1^2$, $g_j = b_j - c_{1j}x_0^3 - c_{2j}x_0^2x_2 - c_{3j}x_0^2x_1 - c_{4j}x_0x_1x_2 - c_{5j}x_0x_1^2 - c_{6j}x_1^3$ for all indices $j \in \{2,3\}$, and $g_j = b_j - c_{1j}x_0^4 - c_{2j}x_0^3x_2 - c_{3j}x_0^3x_1 - c_{4j}x_0^2x_1x_2 - c_{5j}x_0^2x_1^2 - c_{6j}x_0x_1^3$ for all $j \in \{4,5\}$ is the generic projective $\mathcal{O}$-border prebasis according to Definition 6.1.5. Moreover, there is a canonical $\mathbb{Q}$-algebra isomorphism between the coordinate ring $B_{\mathcal{O}}^{\mathrm{proj}}$ of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and $\mathbb{Q}[c_{11},\dots,c_{65}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \langle c_{61} \rangle$ by Proposition 6.1.4.

To ease the notation of this section, we let $\{c_{11},\dots,c_{\mu\nu}\}$ be a set of further indeterminates, $Q = P[c_{11},\dots,c_{\mu\nu}]$, and $\overline{Q} = \overline{P}[c_{11},\dots,c_{\mu\nu}]$. Then $Q = \bigoplus_{\gamma \in \mathbb{Z}} Q_\gamma$ and

$\overline{Q} = \bigoplus_{\gamma\in\mathbb{Z}} \overline{Q}_\gamma$ are non-negatively $\mathbb{Z}$-graded by the matrices $V = (W \mid 0) \in \mathrm{Mat}_{1,n+\mu\nu}(\mathbb{Z})$ and $\overline{V} = (\overline{W} \mid 0) \in \mathrm{Mat}_{1,n+1+\mu\nu}(\mathbb{Z})$, respectively, by [KR05, Defn. 4.1.6 and 4.2.4]. Let $\mathbb{B}_\mathcal{O}$ denote the $\mathcal{O}$-border basis scheme and $B_\mathcal{O} \to U_\mathcal{O}$ the universal $\mathcal{O}$-border basis family as in Definition 6.1.1. For all $i \in \{1,\dots,\mu\}$ and all $j \in \{1,\dots,\nu\}$, we let $\Delta_{i,j} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{Z}$. Moreover, let $G = \{g_1,\dots,g_\nu\} \subseteq Q$ be the generic projective $\mathcal{O}$-border prebasis, let $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ be the projective $\mathcal{O}$-border basis scheme, $I_\mathcal{O}^{\mathrm{proj}} \subseteq B_\mathcal{O}$ be its defining ideal, and let $B_\mathcal{O}^{\mathrm{proj}} \to U_\mathcal{O}^{\mathrm{proj}}$ be the universal projective $\mathcal{O}$-border basis family as in Definition 6.1.5.

The projective $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ differs from the homogeneous $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{hom}}$ introduced in [KR08, Section 5] as the following remark shows.

Remark 6.1.7. In [KR08, Section 5], the authors introduced the **homogeneous $\mathcal{O}$-border basis scheme** $\mathbb{B}_\mathcal{O}^{\mathrm{hom}}$ in the standard graded case. In our setting, the homogeneous $\mathcal{O}$-border basis scheme $\mathbb{B}_\mathcal{O}^{\mathrm{hom}}$ is is the subscheme of $\mathbb{B}_\mathcal{O}$ defined by the ideal

$$I_\mathcal{O}^{\mathrm{hom}} = \langle \bar{c}_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}, \deg_W(t_i) \neq \deg_W(b_j)\rangle \subseteq B_\mathcal{O}.$$

The idea behind this definitions was to parametrize all the zero-dimensional homogeneous ideals of P that possess an $\mathcal{O}$-border basis. To reach this goal, they defined the generic homogeneous $\mathcal{O}$-border prebasis to be of the form $G = \{g_1,\dots,g_\nu\} \subseteq Q$ with the homogeneous polynomials

$$g_j = b_j - \sum_{i=H(\deg_W(b_j)-1)+1}^{H(\deg_W(b_j))} c_{ij} t_i = b_j - \sum_{\substack{i\in\{1,\dots,\mu\}\\ \deg_W(b_j)=\deg_W(t_i)}} c_{ij} t_i \in Q_{\deg_W(b_j)}$$

for all $j \in \{1,\dots \nu\}$. Note hat their approach strongly differs from ours. They regard all $\mathcal{O}$-border bases in P with the additional property to be homogeneous whereas we regard all $\mathcal{O}$-border bases in P with the additional property to be Macaulay bases with respect to the grading given by W, cf. Corollary 4.1.9 and Proposition 4.1.7. Since $I_\mathcal{O}^{\mathrm{proj}} \subseteq I_\mathcal{O}^{\mathrm{hom}}$, the homogeneous $\mathcal{O}$-border basis scheme is related to the (projective) $\mathcal{O}$-border basis scheme via the following chain of subschemes $\mathbb{B}_\mathcal{O}^{\mathrm{hom}} \subseteq \mathbb{B}_\mathcal{O}^{\mathrm{proj}} \subseteq \mathbb{B}_\mathcal{O}$.

The remaining part of this section is devoted to prove an analogous version of Theorem 6.1.3 for the projective $\mathcal{O}$-border scheme $\mathbb{B}_\mathcal{O}^{\mathrm{proj}}$, i.e. to prove that the residue classes of the elements of $\mathcal{O}$ in $U_\mathcal{O}^{\mathrm{proj}}$ form a $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-module basis of $U_\mathcal{O}^{\mathrm{proj}}$, in Theorem 6.1.13.

Proposition 6.1.8. *The map*

$$B_\mathcal{O}^{\mathrm{proj}}[x_0] \times U_\mathcal{O}^{\mathrm{proj}} \to U_\mathcal{O}^{\mathrm{proj}}, \quad (p,r) \mapsto \overline{p}r$$

turns the ring $U_\mathcal{O}^{\mathrm{proj}}$ into a $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-algebra.

Proof. The canonical $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-algebra homomorphism $B_\mathcal{O}^{\mathrm{proj}}[x_0] \twoheadrightarrow B_\mathcal{O}^{\mathrm{proj}}[\overline{x}_0] \hookrightarrow U_\mathcal{O}^{\mathrm{proj}}$, where we have $\overline{x}_0 \in U_\mathcal{O}^{\mathrm{proj}}$, equips $U_\mathcal{O}^{\mathrm{proj}}$ with the given $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-module structure. $\square$

Proposition 6.1.9. *There are canonical K-algebra isomorphisms*

$$U_{\mathcal{O}}^{\mathrm{proj}} \cong \overline{Q}/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle$$

and

$$B_{\mathcal{O}}^{\mathrm{proj}}[x_0] \cong K[x_0, c_{11}, \ldots, c_{\mu\nu},]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}].$$

Proof. The substitution K-algebra epimomorphism $\varphi : \overline{Q} \twoheadrightarrow B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]$ defined by $c_{ij} \mapsto \overline{c}_{ij} \in B_{\mathcal{O}}^{\mathrm{proj}}$ for all $i \in \{1, \ldots, \mu\}$ and $j \in \{1, \ldots, \nu\}$ satisfies $\ker(\varphi) = \mathcal{I}(\mathbb{B}_{\mathcal{O}})\overline{Q}$. Moreover, we have $\varphi^{-1}(\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]) = \ker(\varphi) + \langle G \rangle = \mathcal{I}(\mathbb{B}_{\mathcal{O}})\overline{Q} + \langle G \rangle$ by Lemma 2.6.7. The Second Noetherian Isomorphism Theorem and Definition 6.1.5 thus yield the first canonical K-algebra isomorphism

$$\begin{aligned} U_{\mathcal{O}}^{\mathrm{proj}} &= B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]/\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n] \\ &= \varphi(\overline{Q})/\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n] \\ &\cong \overline{Q}/\varphi^{-1}(\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]) \\ &= \overline{Q}/\mathcal{I}(\mathbb{B}_{\mathcal{O}})\overline{Q} + \langle G \rangle. \end{aligned}$$

Since the restriction $\varphi|_{K[x_0, c_{11}, \ldots, c_{\mu\nu}]}$ of φ to the K-algebra $K[x_0, c_{11}, \ldots, c_{\mu\nu}]$ satisfies $\ker(\varphi|_{K[x_0, c_{11}, \ldots, c_{\mu\nu}]}) = \mathcal{I}(\mathbb{B}_{\mathcal{O}})K[x_0, c_{11}, \ldots, c_{\mu\nu}]$ and $\mathrm{im}(\varphi|_{K[x_0, c_{11}, \ldots, c_{\mu\nu}]}) = B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$, the second canonical K-algebra isomorphism follows from the Homomorphism Theorem. □

Lemma 6.1.10. *Assume that K is algebraically closed and let $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ be a maximal ideal. Then the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}}$ form a K-vector space basis of $U_{\mathcal{O}}^{\mathrm{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}}$.*

Proof. For the whole proof, we identify the elements of $U_{\mathcal{O}}^{\mathrm{proj}}$ and $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ with their images under the canonical K-algebra isomorphisms $U_{\mathcal{O}}^{\mathrm{proj}} \cong \overline{Q}/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle$ and $B_{\mathcal{O}}^{\mathrm{proj}}[x_0] \cong K[x_0, c_{11}, \ldots, c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}]$ of Proposition 6.1.9, respectively, without mention.

Let $\mathfrak{m} = \widetilde{\mathfrak{m}}/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}]$ be with a maximal ideal $\widetilde{\mathfrak{m}} \subseteq K[x_0, c_{11}, \ldots, c_{\mu\nu}]$ that satisfies $\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}] \subseteq \widetilde{\mathfrak{m}}$. As K is algebraically closed and as $\widetilde{\mathfrak{m}}$ is a maximal ideal, [KR05, Coro. 2.6.9] implies the existence of $d, a_{11}, \ldots, a_{\mu\nu} \in K$ such that $\widetilde{\mathfrak{m}} = \langle x_0 - d, c_{11} - a_{11}, \ldots, c_{\mu\nu} - a_{\mu\nu} \rangle$. Let $\widehat{\mathfrak{m}} = \langle c_{11} - a_{11}, \ldots, c_{\mu\nu} - a_{\mu\nu} \rangle \subseteq K[c_{11}, \ldots, c_{\mu\nu}]$. As $\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}] \subseteq \widetilde{\mathfrak{m}}$, it follows that $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) \subseteq \mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}) \subseteq \widehat{\mathfrak{m}}$. Thus we get

$$\begin{aligned} \mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}} &= (\widetilde{\mathfrak{m}}/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})K[x_0, c_{11}, \ldots, c_{\mu\nu}])(\overline{Q}/\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle) \\ &= (\widetilde{\mathfrak{m}}\overline{Q} + \mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle)/(\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle) \\ &= ((\widehat{\mathfrak{m}} + \langle x_0 - d \rangle)\overline{Q} + \langle G \rangle)/(\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle) \end{aligned}$$

and the Second Noetherian Isomorphism Theorem yields the canonical K-algebra isomorphism

$$\begin{aligned} U_{\mathcal{O}}^{\mathrm{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}} &= (\overline{Q}/(\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle))/(((\widehat{\mathfrak{m}} + \langle x_0 - d \rangle)\overline{Q} + \langle G \rangle)/(\mathcal{I}(\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}})\overline{Q} + \langle G \rangle)) \\ &\cong \overline{Q}/(\widehat{\mathfrak{m}} + \langle x_0 - d \rangle)\overline{Q} + \langle G \rangle. \end{aligned}$$

Consider the substitution K-algebra epimorphism $\varphi : \overline{Q} \twoheadrightarrow \overline{P}$, defined by $c_{ij} \mapsto a_{ij}$ for all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$. As $\ker(\varphi) = \widehat{\mathfrak{m}}\overline{Q}$, Lemma 2.6.7 yields

$$\begin{aligned}\varphi^{-1}(\langle x_0 - d\rangle + \langle \varphi(G)\rangle) &= \varphi^{-1}(\langle x_0 - d\rangle + \varphi(\langle G\rangle))\\ &= \ker(\varphi) + (\langle x_0 - d\rangle\overline{Q} + \langle G\rangle)\\ &= \widehat{\mathfrak{m}}\overline{Q} + \langle x_0 - d\rangle\overline{Q} + \langle G\rangle\\ &= (\widehat{\mathfrak{m}} + \langle x_0 - d\rangle)\overline{Q} + \langle G\rangle.\end{aligned}$$

Hence the canonical K-algebra isomorphism

$$\begin{aligned}U_{\mathcal{O}}^{\mathrm{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}} &\cong \overline{Q}/(\widehat{\mathfrak{m}} + \langle x_0 - d\rangle)\overline{Q} + \langle G\rangle\\ &= \overline{Q}/\varphi^{-1}(\langle x_0 - d\rangle + \langle \varphi(G)\rangle)\\ &\cong \varphi(\overline{Q})/(\langle x_0 - d\rangle + \langle \varphi(G)\rangle)\\ &= \overline{P}/(\langle x_0 - d\rangle + \langle \varphi(G)\rangle)\end{aligned}$$

is a consequence of the Second Noetherian Isomorphism Theorem. Next we show that the image $\varphi(G) \subseteq \overline{P}$ of the generic projective $\mathcal{O}$-border prebasis G under φ is the projective $\mathcal{O}$-border basis of $\langle \varphi(G)\rangle \subseteq \overline{P}$. For all $k \in \{1, \dots, n\}$, let $\overline{\mathcal{X}}_k \in \mathrm{Mat}_\mu(K)$ be the image of the k^{th} generic multiplication matrix $\mathcal{X}_k \in \mathrm{Mat}_\mu(K[c_{11}, \dots, c_{\mu\nu}])$ with respect to $\mathcal{O}$ as defined in Definition 6.1.1 under φ. Then for all $k \in \{1, \dots, n\}$, $\overline{\mathcal{X}}_k$ is exactly the k^{th} formal multiplication matrix of $\{\varphi(g_1)^{\mathrm{deh}}, \dots, \varphi(g_\nu)^{\mathrm{deh}}\} \subseteq P$. Since $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) \subseteq \widehat{\mathfrak{m}} \subseteq \ker(\varphi)$, $\overline{\mathcal{X}}_k\overline{\mathcal{X}}_\ell = \overline{\mathcal{X}}_\ell\overline{\mathcal{X}}_k$ for all $k, \ell \in \{1, \dots, n\}$ by Definition 6.1.1. Thus $\{\varphi(g_1)^{\mathrm{deh}}, \dots, \varphi(g_\nu)^{\mathrm{deh}}\}$ is the $\mathcal{O}$-border basis of $\langle \varphi(g_1)^{\mathrm{deh}}, \dots, \varphi(g_\nu)^{\mathrm{deh}}\rangle = \langle \varphi(G)\rangle^{\mathrm{deh}}$ by Theorem 2.4.19. Moreover, for all $j \in \{1, \dots, \nu\}$, we have

$$\varphi(g_j)^{\mathrm{deh}} = \left(b_j - \sum_{i=1}^{H(\deg_W(b_j))} a_{ij}x_0^{\Delta_{i,j}}t_i\right)^{\mathrm{deh}} = b_j - \sum_{i=1}^{H(\deg_W(b_j))} a_{ij}t_i$$

by Definition 6.1.5 and thus $b_j \in \mathrm{Supp}(\mathrm{DF}_W(\varphi(g_j)^{\mathrm{deh}}))$ and $(\varphi(g_j)^{\mathrm{deh}})^{\mathrm{hom}} = \varphi(g_j)$. Since $\{\varphi(g_1)^{\mathrm{deh}}, \dots, \varphi(g_\nu)^{\mathrm{deh}}\}$ is the $\mathcal{O}$-border basis of $\langle \varphi(G)\rangle^{\mathrm{deh}}$ and since $\mathcal{O} \neq \emptyset$, we have $\langle \varphi(G)\rangle^{\mathrm{deh}} \subset P$ by Definition 2.1.14. According to Proposition 4.1.8, $\varphi(G)$ is the projective $\mathcal{O}$-border basis of $(\langle \varphi(G)\rangle^{\mathrm{deh}})^{\mathrm{hom}}$. By Proposition 4.1.7, $\{\varphi(g_1)^{\mathrm{deh}}, \dots, \varphi(g_\nu)^{\mathrm{deh}}\}$ is a Macaulay basis of $\langle \varphi(G)\rangle^{\mathrm{deh}}$ with respect to the grading given by W. Therefore, [KR05, Thm. 4.3.19] yields $\langle \varphi(G)\rangle = (\langle \varphi(G)\rangle^{\mathrm{deh}})^{\mathrm{hom}}$, i.e. $\varphi(G) \subseteq \overline{P}$ is the projective $\mathcal{O}$-border basis of $\langle \varphi(G)\rangle$. Altogether, [KR05, Thm. 4.3.22] yields

$$U_{\mathcal{O}}^{\mathrm{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\mathrm{proj}} \cong \overline{P}/(\langle x_0 - d\rangle + (\langle \varphi(G)\rangle^{\mathrm{deh}})^{\mathrm{hom}}) \cong \begin{cases} P/\,\mathrm{DF}_W(\langle \varphi(G)\rangle^{\mathrm{deh}}) & \text{if } d = 0,\\ P/\langle \varphi(G)\rangle^{\mathrm{deh}} & \text{if } d \neq 0.\end{cases}$$

Since the K-algebra isomorphism on the left side is canonical and since the proof of [KR05, Thm. 4.3.22] shows that the K-algebra isomorphisms on the right side satisfy $\bar{t}_i \mapsto c_i\bar{t}_i$ with $c_i \in K \setminus \{0\}$ for all $i \in \{1, \dots, \mu\}$, it suffices to show that the residue classes of the elements of $\mathcal{O}$ in $P/\,\mathrm{DF}_W(\langle \varphi(G)\rangle^{\mathrm{deh}})$ if $d = 0$ respectively in $P/\langle \varphi(G)\rangle^{\mathrm{deh}}$

if $d \neq 0$ form a K-vector space basis of $P/\operatorname{DF}_W(\langle\varphi(G)\rangle^{\text{deh}})$ respectively of $P/\langle\varphi(G)\rangle^{\text{deh}}$. We have already seen above that $\langle\varphi(G)\rangle^{\text{deh}}$ possesses an $\mathcal{O}$-border basis, i.e. the residue classes of the elements of $\mathcal{O}$ in $P/\langle\varphi(G)\rangle^{\text{deh}}$ form a K-vector space basis of $P/\langle\varphi(G)\rangle^{\text{deh}}$ by Definition 2.1.14. If $d = 0$, [KR08, Thm. 2.4] yields that $\operatorname{DF}_W(\langle\varphi(G)\rangle^{\text{deh}}) \subseteq P$ possesses an $\mathcal{O}$-border basis. Thus if $d = 0$, the residue classes of the elements of $\mathcal{O}$ in $P/\operatorname{DF}_W(\langle\varphi(G)\rangle^{\text{deh}})$ form a K-vector space basis of $P/\operatorname{DF}_W(\langle\varphi(G)\rangle^{\text{deh}})$ according to Definition 2.1.14. □

Lemma 6.1.11. *The residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\text{proj}}$ generate $U_{\mathcal{O}}^{\text{proj}}$ as a $B_{\mathcal{O}}^{\text{proj}}[x_0]$-module.*

Proof. For all $i \in \{1, \ldots, \mu\}$, let $\bar{t}_i \in U_{\mathcal{O}}^{\text{proj}}$ denote the residue class of $t_i \in \mathcal{O}$ in $U_{\mathcal{O}}^{\text{proj}}$. According to [Kun85, Coro. IV.1.6], we have $U_{\mathcal{O}}^{\text{proj}} = \langle \bar{t}_1, \ldots, \bar{t}_\mu\rangle$ as $B_{\mathcal{O}}^{\text{proj}}[x_0]$-modules if and only if for all maximal ideals $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$, $(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}} = (\langle \bar{t}_1, \ldots, \bar{t}_\mu\rangle)_{\mathfrak{m}} = \langle \frac{\bar{t}_1}{1}, \ldots, \frac{\bar{t}_\mu}{1}\rangle$ as $(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$-modules. For all maximal ideals $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$, the localisation $(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{m}(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$ according to [Kun85, Exmp. d in Section III.3]. Thus a corollary of the Lemma of Nakayama [Kun85, Coro. IV.2.3] yields that we have $U_{\mathcal{O}}^{\text{proj}} = \langle \bar{t}_1, \ldots, \bar{t}_\mu\rangle$ as $B_{\mathcal{O}}^{\text{proj}}[x_0]$-modules if and only if for all maximal ideals $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$, we have $(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}/\mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}} = \langle \frac{\bar{t}_1}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}, \ldots, \frac{\bar{t}_\mu}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}\rangle$ as $(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}/\mathfrak{m}(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$-vector spaces.

By [Kun85, Rule III.4.15], there exists a canonical $(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$-module isomorphism $((B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}})/(\mathfrak{m}(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}) \cong (B_{\mathcal{O}}^{\text{proj}}[x_0]/\mathfrak{m})_{\mathfrak{m}}$. Thus we get the chain of field extensions $K \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]/\mathfrak{m} \subseteq (B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}})/(\mathfrak{m}(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}})$. Therefore, $U_{\mathcal{O}}^{\text{proj}} = \langle \bar{t}_1, \ldots, \bar{t}_\mu\rangle$ as $B_{\mathcal{O}}^{\text{proj}}[x_0]$-module follows if we show that for all maximal ideals $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$, we have $(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}/\mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}} = \langle \frac{\bar{t}_1}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}, \ldots, \frac{\bar{t}_\mu}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}\rangle$ as K-vector spaces. In other words, it suffices to show that for every maximal ideal $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$, the identity map $\langle \frac{\bar{t}_1}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}}), \ldots, \frac{\bar{t}_\mu}{1} + \mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})\rangle \hookrightarrow (U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}/\mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}$ is a K-algebra epimorphism. According to [KR00, Prop. 3.6.6], a K-algebra homomorphism is surjective if and only if a specific Gröbner basis contains specific elements. Since Gröbner bases do not change under field extensions by [KR00, Lemma 2.4.16], we can without loss of generality assume that K is algebraically closed.

Let $\mathfrak{m} \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0]$ be a maximal ideal. According to [Kun85, Rule III.4.15], there is a canonical $(B_{\mathcal{O}}^{\text{proj}}[x_0])_{\mathfrak{m}}$-module isomorphism $(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}/\mathfrak{m}(U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}} \cong (U_{\mathcal{O}}^{\text{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}$. Using this isomorphism, we see that the claim follows if we show that the identity map $\langle \frac{\bar{t}_1 + \mathfrak{m}U_{\mathcal{O}}^{\text{proj}}}{1}, \ldots, \frac{\bar{t}_\mu + \mathfrak{m}U_{\mathcal{O}}^{\text{proj}}}{1}\rangle \hookrightarrow (U_{\mathcal{O}}^{\text{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\text{proj}})_{\mathfrak{m}}$ is a K-algebra epimorphism. Since the identity map $\langle \bar{t}_1 + \mathfrak{m}U_{\mathcal{O}}^{\text{proj}}, \ldots, \bar{t}_\mu + \mathfrak{m}U_{\mathcal{O}}^{\text{proj}}\rangle \hookrightarrow U_{\mathcal{O}}^{\text{proj}}/\mathfrak{m}U_{\mathcal{O}}^{\text{proj}}$ is a K-algebra isomorphism according to Lemma 6.1.10, the claim follows from [Kun85, Rule III.4.7]. □

Lemma 6.1.12. *For all $\gamma \in \mathbb{Z}$ and $i \in \{1, \ldots, \mu\}$, we let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$. Let $M = \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\text{proj}}[x_0] \cdot t_i \subseteq B_{\mathcal{O}}^{\text{proj}}[x_0, \ldots, x_n]$, i.e. the $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\text{proj}}[x_0]$-submodule of $B_{\mathcal{O}}^{\text{proj}}[x_0, \ldots, x_n]$ with $B_{\mathcal{O}}^{\text{proj}}[x_0]$-module basis $\mathcal{O}$. Let $\overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n \in \operatorname{Mat}_\mu(B_{\mathcal{O}}^{\text{proj}})$ be the images of the generic multiplication matrices with respect to $\mathcal{O}$ in $\operatorname{Mat}_\mu(B_{\mathcal{O}}^{\text{proj}})$. For all*

$\gamma, \gamma' \in \mathbb{Z}$, for all $c_1, \ldots, c_{H(\gamma)} \in B_{\mathcal{O}}^{\mathrm{proj}}$, and for all $q \in \overline{Q}_{\gamma'}$, we define

$$\begin{aligned} & q * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) \\ & = (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\ & \quad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}. \end{aligned}$$

Then $*$ equips M with the structure of a $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]$-module and M is cyclically generated by $t_1 = 1$ as a $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]$-module. In particular, for every $i \in \{1, \ldots, \mu\}$, we have $t_i * t_1 = t_i$.

Proof. The $\mathbb{Z}$-grading of $\overline{Q}$ defined by the matrix $\overline{V} = (\overline{W} \mid 0)$ induces a $\mathbb{Z}$-grading of $M \subseteq B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]$. By Definition 6.1.1, the matrices $\overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ are pairwise commuting. Let $c_1, \ldots, c_{H(\gamma)}, c'_1, \ldots, c'_{H(\gamma)} \in B_{\mathcal{O}}^{\mathrm{proj}}$, $q, q' \in \overline{Q}_{\gamma'}$, and $\widetilde{q} \in \overline{Q}_{\gamma''}$, where $\gamma, \gamma', \gamma'' \in \mathbb{Z}$. Since the matrices $\overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n$ are pairwise commuting, we get

$$\begin{aligned} & 1 * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) \\ & = (x_0^{d_{\gamma,1}} t_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, 0, \ldots, 0) \cdot 1(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\ & = (x_0^{d_{\gamma,1}} t_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, 0, \ldots, 0) \cdot \mathcal{I}_\mu \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\ & = c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, \end{aligned}$$

and

$$\begin{aligned} & (q \cdot \widetilde{q}) * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})) \\ & = (x_0^{d_{\gamma+\gamma'+\gamma'',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma'+\gamma'',H(\gamma+\gamma'+\gamma'')}} t_{H(\gamma+\gamma'+\gamma'')}, 0, \ldots, 0) \\ & \quad \cdot (q\widetilde{q})(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\ & = (x_0^{d_{\gamma+\gamma'+\gamma'',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma'+\gamma'',H(\gamma+\gamma'+\gamma'')}} t_{H(\gamma+\gamma'+\gamma'')}, 0, \ldots, 0) \\ & \quad \cdot \widetilde{q}(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\ & = \widetilde{q} * ((x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\ & \qquad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}) \\ & = q * (\widetilde{q} * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})), \end{aligned}$$

and

$$\begin{aligned}
&(q+q') * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\quad \cdot (q+q')(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\quad \cdot (q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) + q'(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n)) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\qquad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&\quad + (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\qquad \cdot q'(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (q * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})) \\
&\quad + (q' * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})),
\end{aligned}$$

and

$$\begin{aligned}
&q * ((c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) + (c'_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c'_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})) \\
&= q * ((c_1 + c'_1) x_0^{d_{\gamma,1}} t_1 + \cdots + (c_{H(\gamma)} + c'_{H(\gamma)}) x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma)}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\quad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1 + c'_1, \ldots, c_{H(\gamma)} + c'_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\qquad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&\quad + (x_0^{d_{\gamma+\gamma',1}} t_1, \ldots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \ldots, 0) \\
&\qquad \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot (c'_1, \ldots, c'_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (q * (c_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})) \\
&\quad + (q * (c'_1 x_0^{d_{\gamma,1}} t_1 + \cdots + c'_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)})).
\end{aligned}$$

Thus $(M, +, *)$ is a $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \ldots, x_n]$-module.

Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ be the canonical $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}})^\mu$. We prove by induction on $i \in \{1, \ldots, \mu\}$ that $t_i * t_1 = t_i$. For $i = 1$, we have

$$t_1 * t_1 = 1 * t_1 = (t_1, 0, \ldots, 0) \cdot 1(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \ldots, \overline{\mathcal{X}}_n) \cdot \mathcal{E}_1^{\mathrm{tr}} = (t_1, 0, \ldots, 0) \cdot \mathcal{I}_\mu \cdot \mathcal{E}_1^{\mathrm{tr}} = t_1.$$

For the induction step, we let $i > 1$. As $\mathcal{O} \subseteq \mathbb{T}^n$ is an order ideal in $\mathbb{T}^n$ and as $\deg_W(t_1) \leq \cdots \leq \deg_W(t_\mu)$, there exist a $k \in \{1, \ldots, n\}$ and an $\ell \in \{1, \ldots, i-1\}$ such that $t_i = x_k t_\ell$. Let $\widetilde{\gamma} = \deg_W(t_i) = \deg_W(t_\ell) + w_k$ and write the k^{th} generic multiplication matrix $\mathcal{X}_k = (\xi_{rs}^{(k)})_{1\le r,s\le\mu} \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ as in Definition 6.1.1. Since $\widetilde{\gamma} = \deg_W(t_i)$, we have $i \le H(\widetilde{\gamma})$ and the induction hypothesis yields

$$\begin{aligned}
t_i * t_1 &= (x_k t_\ell) * t_1 \\
&= x_k * (t_\ell * t_1) \\
&= x_k * t_\ell \\
&= (x_0^{d_{\widetilde{\gamma},1}} t_1, \dots, x_0^{d_{\widetilde{\gamma},H(\widetilde{\gamma})}} t_{H(\widetilde{\gamma})}, 0, \dots 0) \cdot x_k(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot \mathcal{E}_\ell^{\mathrm{tr}} \\
&= (x_0^{d_{\widetilde{\gamma},1}} t_1, \dots, x_0^{d_{\widetilde{\gamma},H(\widetilde{\gamma})}} t_{H(\widetilde{\gamma})}, 0, \dots 0) \cdot \overline{\mathcal{X}}_k \cdot \mathcal{E}_\ell^{\mathrm{tr}} \\
&= \overline{\xi}_{1\ell}^{(k)} x_0^{d_{\widetilde{\gamma},1}} t_1 + \dots + \overline{\xi}_{H(\widetilde{\gamma}),\ell}^{(k)} x_0^{d_{\widetilde{\gamma},H(\widetilde{\gamma})}} t_{H(\widetilde{\gamma})} \\
&= \delta_{1i} x_0^{d_{\widetilde{\gamma},1}} t_1 + \dots + \delta_{H(\widetilde{\gamma}),i} x_0^{d_{\widetilde{\gamma},H(\widetilde{\gamma})}} t_{H(\widetilde{\gamma})} \\
&= x_0^{d_{\widetilde{\gamma},i}} t_i \\
&= t_i.
\end{aligned}$$

Altogether, we see that the $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$-module M is cyclically generated by $t_1 = 1$. □

Finally, we have all ingredients to prove an analogous version of Theorem 6.1.3 for $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$.

Theorem 6.1.13. (The Universal Projective Border Basis Family) *The residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$.*

Proof. Let $\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ be the images of the generic multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$. The matrices $\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n$ are pairwise commuting according to Definition 6.1.1. Let M, $*$, and $d_{\gamma,i}$ for all $\gamma\in\mathbb{Z}$ and $i\in\{1,\dots,\mu\}$ be defined as in Lemma 6.1.12. Then $\Theta : B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n] \twoheadrightarrow M$, $p \mapsto p * t_1 = p * 1$ satisfies $\Theta(t_i) = t_i * t_1 = t_i$ for all $i \in \{1,\dots,\mu\}$ and is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$-module epimorphism by Lemma 6.1.12. Moreover, for all $\gamma,\gamma' \in \mathbb{Z}$ and $c_1,\dots,c_{H(\gamma)} \in B_{\mathcal{O}}^{\mathrm{proj}}$, we have

$$\begin{aligned}
&x_0^{\gamma'} * (c_1 x_0^{d_{\gamma,1}} t_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}) \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot x_0^{\gamma'} (\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1,\dots,c_{H(\gamma)},0,\dots,0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} t_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} t_{H(\gamma+\gamma')}, 0, \dots, 0) \cdot \mathcal{I}_\mu^{\gamma'} \cdot (c_1,\dots,c_{H(\gamma)},0,\dots,0)^{\mathrm{tr}} \\
&= c_1 x_0^{d_{\gamma+\gamma',1}} t_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma+\gamma',H(\gamma)}} t_{H(\gamma)}.
\end{aligned}$$

Let $j \in \{1,\dots,\nu\}$. Then there exist indices $k\in\{1,\dots,n\}$ and $\ell\in\{1,\dots,\mu\}$ such that $b_j = x_k t_\ell$ by Definition 2.1.7. Let $\{\mathcal{E}_1,\dots,\mathcal{E}_\mu\}$ be the canonical $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}})^\mu$, let $\gamma = \deg_W(b_j)$, and let $\mathcal{X}_k = (\xi_{rs}^{(k)})_{1\le r,s\le\mu} \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ be as in Definition 6.1.1. Then $\deg_W(t_1) \le \dots \le \deg_W(t_\mu)$ yields that $\gamma \ge \deg_W(t_i)$ for all

$i \in \{1,\dots,H(\gamma)\}$. Let $\overline{g}_j \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$ be the image of $g_j \in \overline{Q}$ in $B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$. By Definition 6.1.5, we have $\overline{c}_{ij} = 0 \in B_{\mathcal{O}}^{\mathrm{proj}}$ for all $i \in \{i+1,\dots,\mu\}$ and thus get

$$\begin{aligned}
\Theta(\overline{g}_j) &= \Theta\left(b_j - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{\Delta_{i,j}} t_i\right)\\
&= \Theta(b_j) - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{\Delta_{i,j}} * \Theta(t_i)\\
&= b_j * t_1 - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{\Delta_{i,j}} * t_i\\
&= x_k * (t_\ell * t_1) - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{\Delta_{i,j}} t_i\\
&= x_k * t_\ell - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{\gamma - \deg_W(t_i)} t_i\\
&= (x_0^{d_{\gamma,1}} t_1, \dots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, 0, \dots, 0) \cdot \overline{\mathcal{X}}_k \cdot \mathcal{E}_\ell^{\mathrm{tr}} - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{d_{\gamma,i}} t_i\\
&= (x_0^{d_{\gamma,1}} t_1, \dots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, 0, \dots, 0) \cdot (\overline{\xi}_{1\ell}^{(k)}, \dots, \overline{\xi}_{\mu\ell}^{(k)})^{\mathrm{tr}} - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{d_{\gamma,i}} t_i\\
&= (x_0^{d_{\gamma,1}} t_1, \dots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}, 0, \dots, 0) \cdot (\overline{c}_{1j}, \dots, \overline{c}_{H(\gamma),j}, 0, \dots, 0)^{\mathrm{tr}} - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{d_{\gamma,i}} t_i\\
&= \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{d_{\gamma,i}} t_i - \sum_{i=1}^{H(\gamma)} \overline{c}_{ij} x_0^{d_{\gamma,i}} t_i\\
&= 0,
\end{aligned}$$

i.e. we have $\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n] \subseteq \ker(\Theta)$. This induces the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$-module epimorphism $\overline{\Theta} : B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]/\langle G \rangle B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n] = U_{\mathcal{O}}^{\mathrm{proj}} \twoheadrightarrow M$, $\overline{p} \mapsto p * t_1$ by Definition 6.1.5. Let $\overline{\mathcal{O}}$ denote the image of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$. Since $\overline{\Theta}(\overline{\mathcal{O}}) = \mathcal{O}$ and since $\mathcal{O} \subseteq M$ is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of M, it follows that $\overline{\mathcal{O}} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ is $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-linearly independent. The claim now follows from Lemma 6.1.11, which yields that the set $\overline{\mathcal{O}}$ also generates $U_{\mathcal{O}}^{\mathrm{proj}}$ as a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module. □

As a direct consequence, we can give explicit $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases of $U_{\mathcal{O}}^{\mathrm{proj}}$ and its homogeneous components.

Corollary 6.1.14. *For all $\gamma \in \mathbb{Z}$ and $i \in \{1,\dots,\mu\}$, we let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$.*

a) The element $x_0 \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ is a non-zero divisor for $U_{\mathcal{O}}^{\mathrm{proj}}$.

b) There are canonical isomorphisms of $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-modules

$$U_{\mathcal{O}}^{\mathrm{proj}} \cong \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)) \cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\gamma))^{\Delta H(\gamma)}.$$

In particular, $\mathrm{rk}_{B_{\mathcal{O}}^{\mathrm{proj}}}((U_{\mathcal{O}}^{\mathrm{proj}})_\gamma) = H(\gamma)$ *for all* $\gamma \in \mathbb{Z}$.

c) Let $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$. Then the residue classes of the elements of $\mathcal{O}_0$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$. In particular, for every $\gamma \in \mathbb{Z}$, the residue classes of the elements of $(\mathcal{O}_0)_\gamma = \{x_0^{d_{\gamma,1}} t_1, \dots, x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)}\}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$.

Proof. In order to prove a), let $r \in U_{\mathcal{O}}^{\mathrm{proj}}$ be such that $x_0 \cdot r = 0$. By Theorem 6.1.13, there exist $c_1, \dots, c_\mu \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ such that $r = c_1\bar{t}_1 + \cdots + c_\mu \bar{t}_\mu$ in $U_{\mathcal{O}}^{\mathrm{proj}}$. Thus we get $0 = x_0 \cdot r = c_1 x_0 \bar{t}_1 + \cdots + c_\mu x_0 \bar{t}_\mu$. Since the residue classes of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$, we see that $c_1 x_0 = \cdots = c_\mu x_0 = 0$ in $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$. Thus we get $c_1 = \cdots = c_\mu = 0$ and $r = 0$, i.e. $x_0 \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ is a non-zero divisor for $U_{\mathcal{O}}^{\mathrm{proj}}$.

The claims b) and c) are direct consequences of Theorem 6.1.13 and the definitions of the maps H and ΔH. □

The final remark compares the projective $\mathcal{O}$-border basis scheme with the Gröbner basis scheme as introduced in [Rob09].

Remark 6.1.15. Let σ be a term ordering and, without loss of generality, assume that the elements of the border of $\mathcal{O}$ are ordered such that $\{b_1, \dots, b_\eta\} \subseteq \partial\mathcal{O}$ with $\eta \in \{1, \dots, \nu\}$ contains precisely the corners of $\mathcal{O}$, cf. Definition 2.3.3. Moreover, let

$$S_{\mathcal{O},\sigma} = \{c_{ij} \mid i \in \{1, \dots, \mu\}, j \in \{1, \dots, \eta\}, b_j >_\sigma t_i\} \subseteq \{c_{11}, \dots, c_{\mu\nu}\}$$

and

$$L_{\mathcal{O},\sigma} = \langle c_{ij} \mid i \in \{1, \dots, \mu\}, j \in \{1, \dots, \nu\}, t_i >_\sigma b_j\} \subseteq K[c_{11}, \dots, c_{\mu\nu}].$$

In [Rob09, Defn. 2.4], the author introduces the **$(\mathcal{O}, \sigma)$-Gröbner basis scheme** $\mathbb{G}_{\mathcal{O},\sigma}$ as the subscheme of $\mathbb{A}_K^{\#S_{\mathcal{O},\sigma}} = \mathrm{Spec}(K[S_{\mathcal{O},\sigma}])$ defined by the ideal

$$I_{\mathcal{O},\sigma} = (\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + L_{\mathcal{O},\sigma}) \cap K[S_{\mathcal{O},\sigma}] \subseteq K[S_{\mathcal{O},\sigma}]$$

The idea behind that was to parametrize all the zero-dimensional ideals of P that possess a σ-Gröbner basis with the property that the corresponding leading term ideal is $\langle \partial\mathcal{O} \rangle$. To reach this goal, the author defined the generic $(\mathcal{O}, \sigma)$-Gröbner prebasis to be of the form $G' = \{g'_1, \dots, g'_\eta\} \subseteq P$ where for all $j \in \{1, \dots \eta\}$,

$$g'_j = b_j - \sum_{\substack{i \in \{1,\dots,\mu\} \\ b_j >_\sigma t_i}} c_{ij} t_i \in P[S_{\mathcal{O},\sigma}].$$

Similar to [Rob09, Defn. 2.4 and 2.6], we denote the coordinate ring of $\mathbb{G}_{\mathcal{O},\sigma}$ by $G_{\mathcal{O},\sigma}$, let $U_{\mathcal{O},\sigma} = G_{\mathcal{O},\sigma}[x_1,\dots,x_n]/\langle G'\rangle G_{\mathcal{O},\sigma}[x_1,\dots,x_n]$, and call the the canonical K-algebra homomorphism $G_{\mathcal{O},\sigma} \hookrightarrow G_{\mathcal{O},\sigma}[x_1,\dots,x_n] \twoheadrightarrow U_{\mathcal{O},\sigma}$ the **universal $(\mathcal{O},\sigma)$-Gröbner basis family**. Note that, as done in Proposition 6.1.9 for the canonical K-algebra isomorphism $U_{\mathcal{O}}^{\mathrm{proj}} \cong \overline{Q}/\mathcal{I}(\mathbb{B}_{\mathcal{O}})\overline{Q} + \langle G\rangle$, one can prove that there exists a canonical K-algebra isomorphism $U_{\mathcal{O},\sigma} \cong P[S_{\mathcal{O},\sigma}]/I_{\mathcal{O},\sigma}P[S_{\mathcal{O},\sigma}] + \langle G'\rangle$, i.e. the above definition of $U_{\mathcal{O},\sigma}$ indeed coincides with the one in [Rob09, Defn. 2.6]. Then [Rob09, Thm. 2.9] yields that the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O},\sigma}$ form a $G_{\mathcal{O},\sigma}$-module basis of $U_{\mathcal{O},\sigma}$. Moreover, if the term ordering σ is compatible with $\deg_W$, the σ-Gröbner bases parametrized by $\mathbb{G}_{\mathcal{O},\sigma}$ are also Macaulay bases with respect to W. Thus we see that $\mathbb{G}_{\mathcal{O},\sigma}$ shares all the crucial properties of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that we explicitly needed to study uniformity conditions of zero-dimensional projective subschemes of weighted projective spaces $\mathbb{P}_K(\overline{W})$ given by their defining ideal as we have done in Chapter 5.

But using the $(\mathcal{O},\sigma)$-Gröbner basis scheme $\mathbb{G}_{\mathcal{O},\sigma}$ instead of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ has also some disadvantages.
First of all, projective $\mathcal{O}$-border bases are more general as the following example shows. Consider the set $H = \{x_1x_2 - x_2^2 - x_1^2, x_1^3, x_1x_2^2, x_1^2x_2, x_1^3\} \subseteq \mathbb{Q}[x_1,x_2]$ and the order ideal $\mathcal{O} = \{1, x_2, x_1, x_2^2, x_1^2\}$ in $\mathbb{T}^2$. Then H is an $\mathcal{O}$-border basis of the ideal $\langle H\rangle \subseteq \mathbb{Q}[x_1,x_2]$ as one can easily check using Theorem 2.4.19. Furthermore, H is a Macaulay basis with respect to the grading given by W as one can easily check using [KR05, Thm. 4.3.19] and [KR00, Prop. 2.4.19]. But there exists no term ordering σ on $\mathbb{T}^2$ such that H is a σ-Gröbner basis of $\langle H\rangle$ with the property that $\mathrm{LT}_\sigma(\langle H\rangle) = \langle \partial\mathcal{O}\rangle$. In other words, the set H does not correspond to a point on the $(\mathcal{O},\sigma)$-Gröbner basis scheme $\mathbb{G}_{\mathcal{O},\sigma}$ where σ is any arbitrary term ordering on $\mathbb{T}^2$. But since the set H is the projective $\mathcal{O}$-border basis of $\langle H\rangle^{\mathrm{hom}}$ by Proposition 4.1.8, the set H corresponds to a point on the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$.
Another benefit of projective $\mathcal{O}$-border bases is that their multiplicative structure is fully determined by the corresponding (projective) multiplication matrices as we have seen in Section 4.2 and that these matrices can immediately be read off from a given projective $\mathcal{O}$-border basis, cf. the Definitions 2.4.15 and 4.2.1. This enables us to study uniformity conditions using the multiplicative structure of the coordinate ring of a zero-dimensional projective subscheme as we have done in Section 5.2 much more explicitly.

6.2 The Multiplicative Structure of the Universal Projective Border Basis Family

In this section, we study the multiplicative structure of the universal projective $\mathcal{O}$-border basis family $U_{\mathcal{O}}^{\mathrm{proj}}$. Similar to Section 4.2, we first introduce the generic projective multiplication matrices with respect to $\mathcal{O}$ in Definition 6.2.1 and show that they are homogeneous matrices in Proposition 6.2.3. Then we show that the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module structure of $U_{\mathcal{O}}^{\mathrm{proj}}$ can be completely described by the generic projective multiplication matrices with respect to $\mathcal{O}$ in Proposition 6.2.4 and Corollary 6.2.5. After that, we prove that we can also use the generic multiplication matrices with respect to $\mathcal{O}$ to explicitly describe the multiplicative structure of $U_{\mathcal{O}}^{\mathrm{proj}}$ in Proposition 6.2.7 and Corollary 6.2.8.

To ease the notation of this section, we let $Q = P[c_{11}, \dots, c_{\mu\nu}]$ and $\overline{Q} = \overline{P}[c_{11}, \dots, c_{\mu\nu}]$, again. Then $Q = \bigoplus_{\gamma \in \mathbb{Z}} Q_\gamma$ and $\overline{Q} = \bigoplus_{\gamma \in \mathbb{Z}} \overline{Q}_\gamma$ are non-negatively $\mathbb{Z}$-graded by the matrices $V = (W \mid 0) \in \mathrm{Mat}_{1,n+\mu\nu}(\mathbb{Z})$ and $\overline{V} = (\overline{W} \mid 0) \in \mathrm{Mat}_{1,n+1+\mu\nu}(\mathbb{Z})$, respectively, according to [KR05, Defn. 4.1.6 and 4.2.4]. We let $\mathbb{B}_{\mathcal{O}}$ denote the $\mathcal{O}$-border basis scheme and $B_{\mathcal{O}} \to U_{\mathcal{O}}$ the universal $\mathcal{O}$-border basis family as in Definition 6.1.1. For all indices $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$, we let $\Delta_{ij} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{Z}$. Moreover, let $G = \{g_1, \dots, g_\nu\} \subseteq Q$ denote the generic projective $\mathcal{O}$-border prebasis, let $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{B}_{\mathcal{O}}$ be the projective $\mathcal{O}$-border basis scheme, let $I_{\mathcal{O}}^{\mathrm{proj}} \subseteq B_{\mathcal{O}}$ be its defining ideal, and let $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ be the universal projective $\mathcal{O}$-border basis family as in Definition 6.1.5.

First we introduce the generic projective multiplication matrices with respect to $\mathcal{O}$.

Definition 6.2.1. For every index $r \in \{0, \dots, n\}$, let $\mathcal{X}_r^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0, c_{11}, \dots, c_{\mu\nu}])$ be the r^{th} formal projective multiplication matrix of G as defined in Definition 4.2.1, i.e. $\mathcal{X}_0^{\mathrm{proj}} = x_0 \mathcal{I}_\mu$ and $\mathcal{X}_r^{\mathrm{proj}} = (\xi_{k\ell}^{(r,\mathrm{proj})})_{1 \le k,\ell,\le \mu}$ with

$$\xi_{k\ell}^{(r,\mathrm{proj})} = \begin{cases} \delta_{ki} & \text{if } x_r t_\ell = t_i \in \mathcal{O} \\ c_{kj} x_0^{\Delta_{kj}} & \text{if } x_r t_\ell = b_j \in \partial\mathcal{O} \text{ and } \deg_W(t_k) \le \deg_W(b_j) \\ 0 & \text{if } x_r t_\ell = b_j \in \partial\mathcal{O} \text{ and } \deg_W(t_k) > \deg_W(b_j) \end{cases}$$

for all $r \in \{1, \dots, n\}$ and $k, \ell \in \{1, \dots, \mu\}$. For all $r \in \{0, \dots, n\}$, we call the matrix $\mathcal{X}_r^{\mathrm{proj}}$ the $r^{\mathbf{th}}$ **generic projective multiplication matrix** with respect to $\mathcal{O}$.

Example 6.2.2. Consider the order ideal $\mathcal{O}$ of Example 6.1.6, again. Recall that the order ideal was $\mathcal{O} = \{t_1, \dots, t_6\} = \{1, x_2, x_1, x_1x_2, x_1^2, x_1^3\}$ and that its border was of the form $\partial\mathcal{O} = \{b_1, \dots, b_5\} = \{x_2^2, x_1x_2^2, x_1^2x_2, x_1^3x_2, x_1^4\}$. Moreover, $G = \{g_1, \dots, g_5\} \subseteq \overline{Q}$ where the polynomials are of the form $g_1 = x_2^2 - c_{11}x_0^2 - c_{21}x_0x_2 - c_{31}x_0x_1 - c_{41}x_1x_2 - c_{51}x_1^2$, $g_j = b_j - c_{1j}x_0^3 - c_{2j}x_0^2x_2 - c_{3j}x_0^2x_1 - c_{4j}x_0x_1x_2 - c_{5j}x_0x_1^2 - c_{6j}x_1^3$ for all indices $j \in \{2, 3\}$, and $g_j = b_j - c_{1j}x_0^4 - c_{2j}x_0^3x_2 - c_{3j}x_0^3x_1 - c_{4j}x_0^2x_1x_2 - c_{5j}x_0^2x_1^2 - c_{6j}x_0x_1^3$ for all $j \in \{4, 5\}$ was the generic projective $\mathcal{O}$-border prebasis. Then the generic projective multiplication matrices $\mathcal{X}_0^{\mathrm{proj}}, \mathcal{X}_1^{\mathrm{proj}}, \mathcal{X}_2^{\mathrm{proj}} \in \mathrm{Mat}_6(\mathbb{Q}[x_0, c_{11}, \dots, c_{65}])$ with respect to $\mathcal{O}$ are

$$\mathcal{X}_0^{\mathrm{proj}} = x_0 \mathcal{I}_6 = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 \end{pmatrix}, \quad \mathcal{X}_1^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & c_{13}x_0^3 & 0 & c_{15}x_0^4 \\ 0 & 0 & 0 & c_{23}x_0^2 & 0 & c_{25}x_0^3 \\ 1 & 0 & 0 & c_{33}x_0^2 & 0 & c_{35}x_0^3 \\ 0 & 1 & 0 & c_{43}x_0 & 0 & c_{45}x_0^2 \\ 0 & 0 & 1 & c_{53}x_0 & 0 & c_{55}x_0^2 \\ 0 & 0 & 0 & c_{63} & 1 & c_{65}x_0 \end{pmatrix},$$

and

$$\mathcal{X}_2^{\text{proj}} = \begin{pmatrix} 0 & c_{11}x_0^2 & 0 & c_{12}x_0^3 & c_{13}x_0^3 & c_{14}x_0^4 \\ 1 & c_{21}x_0 & 0 & c_{22}x_0^2 & c_{23}x_0^2 & c_{24}x_0^3 \\ 0 & c_{31}x_0 & 0 & c_{32}x_0^2 & c_{33}x_0^2 & c_{34}x_0^3 \\ 0 & c_{41} & 1 & c_{42}x_0 & c_{43}x_0 & c_{44}x_0^2 \\ 0 & c_{51} & 0 & c_{52}x_0 & c_{53}x_0 & c_{54}x_0^2 \\ 0 & 0 & 0 & c_{62} & c_{63} & c_{64}x_0 \end{pmatrix}$$

according to Definition 6.2.1.

Next we show that the generic projective multiplication matrices with respect to $\mathcal{O}$ are homogeneous degree-ordered matrices as defined in Definition 4.2.3.

Proposition 6.2.3. *Let $\mathcal{X}_0^{\text{proj}}, \dots, \mathcal{X}_n^{\text{proj}} \in \operatorname{Mat}_\mu(K[x_0, c_{11}, \dots, c_{\mu\nu}])$ be the formal projective multiplication matrices of G.*

a) For every index $r \in \{0, \dots, n\}$, the matrix $\mathcal{X}_r^{\text{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{V}}(t_1), \dots, \deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r t_1), \dots, \deg_{\overline{V}}(x_r t_\mu))$ for the columns.

b) For all $r, s \in \{0, \dots, n\}$, the matrices $\mathcal{X}_r^{\text{proj}} \mathcal{X}_s^{\text{proj}}$ and $\mathcal{X}_r^{\text{proj}} \mathcal{X}_s^{\text{proj}} - \mathcal{X}_s^{\text{proj}} \mathcal{X}_r^{\text{proj}}$ are both homogeneous degree-ordered matrices with respect to the degree pair given by $(\deg_{\overline{V}}(t_1), \dots, \deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r x_s t_1), \dots, \deg_{\overline{V}}(x_r x_s t_\mu))$ for the columns.

Proof. For all $r \in \{0, \dots, n\}$, let $\mathcal{X}_r^{\text{proj}} = (\xi_{k\ell}^{(r,\text{proj})})_{1 \le i,j \le \mu} \in \operatorname{Mat}_\mu(K[x_0, c_{11}, \dots, c_{\mu\nu}])$. Note that all entries of the matrices $\mathcal{X}_0^{\text{proj}}, \dots, \mathcal{X}_n^{\text{proj}}$ are homogeneous.

For the proof of a), we let $r \in \{0, \dots, n\}$. If $\mathcal{X}_r^{\text{proj}} = 0$, nothing needs to be shown. Thus assume that there are $k, \ell \in \{1, \dots, \mu\}$ such that $\xi_{k\ell}^{(r,\text{proj})} \neq 0$. If $r = 0$, $\xi_{k\ell}^{r,\text{proj}} \neq 0$ implies $k = \ell$ and thus

$$\deg_{\overline{V}}(\xi_{k\ell}^{(0,\text{proj})}) = \deg_{\overline{V}}(x_0) = \deg_{\overline{V}}(x_0 t_\ell) - \deg_{\overline{V}}(t_k)$$

by Definition 6.2.1. Now suppose that $r \neq 0$. If $x_r t_\ell = t_i$ for some $i \in \{1, \dots, \mu\}$, $\delta_{ki} = \xi_{k\ell}^{(r,\text{proj})} \neq 0$ implies $k = i$ by Definition 6.2.1. Thus we get

$$\deg_{\overline{V}}(\xi_{k\ell}^{(r,\text{proj})}) = \deg_{\overline{V}}(\delta_{ki}) = \deg_{\overline{V}}(1) = 0 = \deg_{\overline{V}}(x_r t_\ell) - \deg_{\overline{V}}(t_k)$$

according to Definition 6.2.1. If $x_r t_\ell = b_j$ for some $j \in \{1, \dots, \nu\}$, we get

$$\deg_{\overline{V}}(\xi_{k\ell}^{(r,\text{proj})}) = \deg_{\overline{V}}(c_{kj} x_0^{\Delta_{kj}}) = \deg_{\overline{V}}(b_j) - \deg_{\overline{V}}(t_k) = \deg_{\overline{V}}(x_r t_\ell) - \deg_{\overline{V}}(t_k)$$

by Definition 6.2.1. In all cases, $\mathcal{X}_r^{\text{proj}}$ is homogeneous with respect to the degree pair given by $(\deg_{\overline{V}}(t_1), \dots, \deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r t_1), \dots, \deg_{\overline{V}}(x_r t_\mu))$ for the columns and as $\deg_{\overline{V}}(t_1) \le \dots \le \deg_{\overline{V}}(t_\mu)$, it is also degree-ordered by Definition 4.2.3.

In order to prove b), we let now $r,s \in \{0,\dots,n\}$. By a), $\mathcal{X}_r^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{V}}(t_1),\dots,\deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r t_1),\dots,\deg_{\overline{V}}(x_r t_\mu))$ for the columns and $\mathcal{X}_s^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{V}}(t_1),\dots,\deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_s t_1),\dots,\deg_{\overline{V}}(x_s t_\mu))$ for the columns. In particular, $\mathcal{X}_s^{\mathrm{proj}}$ is also a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{V}}(x_r t_1),\dots,\deg_{\overline{V}}(x_r t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r x_s t_1),\dots,\deg_{\overline{V}}(x_r x_s t_\mu))$ for the columns according to Definition 4.2.3. If $\mathcal{X}_r^{\mathrm{proj}}\mathcal{X}_s^{\mathrm{proj}} = 0$, the matrix $\mathcal{X}_r^{\mathrm{proj}}\mathcal{X}_s^{\mathrm{proj}}$ trivially satisfies the claimed properties. Thus assume that there are $k,\ell \in \{1,\dots,\mu\}$ with $\sum_{m=1}^{\mu} \xi_{km}^{(r,\mathrm{proj})}\xi_{m\ell}^{(s,\mathrm{proj})} \neq 0$. Let $m \in \{1,\dots,\mu\}$ be such that $\xi_{km}^{(r,\mathrm{proj})} \neq 0$ and $\xi_{m\ell}^{(s,\mathrm{proj})} \neq 0$. As the matrices $\mathcal{X}_r^{\mathrm{proj}}$ and $\mathcal{X}_s^{\mathrm{proj}}$ are homogeneous degree-ordered matrices with respect to the above degree pairs, we get

$$\begin{aligned}\deg_{\overline{V}}(\xi_{km}^{(r,\mathrm{proj})}\xi_{m\ell}^{(s,\mathrm{proj})}) &= (\deg_{\overline{V}}(x_r t_m) - \deg_{\overline{V}}(t_k)) + (\deg_{\overline{V}}(x_r x_s t_\ell) - \deg_{\overline{V}}(x_r t_m))\\ &= \deg_{\overline{V}}(x_r x_s t_\ell) - \deg_{\overline{V}}(t_k)\end{aligned}$$

and $\mathcal{X}_r^{\mathrm{proj}}\mathcal{X}_s^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the degree pair given by $(\deg_{\overline{V}}(t_1),\dots,\deg_{\overline{V}}(t_\mu))$ for the rows and $(\deg_{\overline{V}}(x_r x_s t_1),\dots,\deg_{\overline{V}}(x_r x_s t_\mu))$ for the columns according to Definition 4.2.3. By interchanging the roles of r and s, we see that $\mathcal{X}_s^{\mathrm{proj}}\mathcal{X}_r^{\mathrm{proj}}$ is a homogeneous degree-ordered matrix with respect to the same degree pair, too. In particular, it follows that $\mathcal{X}_r^{\mathrm{proj}}\mathcal{X}_s^{\mathrm{proj}} - \mathcal{X}_s^{\mathrm{proj}}\mathcal{X}_r^{\mathrm{proj}}$ is also a homogeneous degree-ordered matrix according to Definition 4.2.3. $\square$

The next two propositions show that the generic projective multiplication matrices with respect to $\mathcal{O}$ can be used to explicitly describe the multiplicative structure of the universal projective $\mathcal{O}$-border basis family $U_{\mathcal{O}}^{\mathrm{proj}}$.

Proposition 6.2.4. *Let $\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ denote the images of the generic projective formal multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$, let $c_1,\dots,c_\mu \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$, and let $q = c_1 t_1 + \dots + c_\mu t_\mu \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$.*

a) For every $k \in \{0,\dots,n\}$, we have

$$\overline{x}_k \cdot \overline{q} = (\overline{t}_1,\dots,\overline{t}_\mu)\cdot \overline{\mathcal{X}}_k^{\mathrm{proj}} \cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}.$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. In particular, the matrices $\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}}$ are pairwise commuting.

b) For every $q' \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n]$, we have

$$\overline{q}' \cdot \overline{q} = (\overline{t}_1,\dots,\overline{t}_\mu)\cdot q'(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})\cdot (c_1,\dots,c_\mu)^{\mathrm{tr}}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$.

Proof. Let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])^\mu$.

In order to prove the equality of a), let $\ell \in \{1, \dots, \mu\}$. Then we have

$$x_0 \cdot c_\ell \bar{t}_\ell = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot x_0 \mathcal{I}_\mu \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \overline{\mathcal{X}}_0^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. Let $k \in \{1, \dots, n\}$. If $x_k t_\ell = t_i \in \mathcal{O}$ for some $i \in \{1, \dots, \mu\}$, we have

$$\overline{x}_k \cdot c_\ell \bar{t}_\ell = c_\ell \bar{t}_i = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot (\delta_{1i} c_\ell, \dots, \delta_{\mu i} c_\ell)^{\mathrm{tr}} = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \overline{\mathcal{X}}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. If $x_k t_\ell = b_j$ for some $j \in \{1, \dots, \nu\}$, we have

$$\begin{aligned}
\overline{x}_k \cdot c_\ell \bar{t}_\ell &= c_\ell \bar{b}_j \\
&= \bar{c}_{1j} x_0^{\Delta_{1j}} c_\ell \bar{t}_1 + \dots + \bar{c}_{H(\deg_W(b_j)),j} x_0^{\Delta_{H(\deg_W(b_j)),j}} c_\ell \bar{t}_{H(\deg_W(b_j))} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot (\bar{c}_{1j} x_0^{\Delta_{1j}} c_\ell, \dots, \bar{c}_{H(\deg_W(b_j)),j} x_0^{\Delta_{H(\deg_W(b_j)),j}} c_\ell, 0, \dots, 0)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \overline{\mathcal{X}}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. Altogether, we see that for all $k \in \{0, \dots, n\}$,

$$\overline{x}_k \cdot \overline{q} = \sum_{\ell=1}^{\mu} \overline{x}_k \cdot c_\ell \bar{t}_\ell = \sum_{\ell=1}^{\mu} (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \overline{\mathcal{X}}_k^{\mathrm{proj}} \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} = (\bar{t}_1, \dots, \bar{t}_\mu) \cdot \overline{\mathcal{X}}_k^{\mathrm{proj}} \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. As the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ form a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$ according to Theorem 6.1.13, we have also proved that for all $k \in \{0, \dots, n\}$, $\overline{\mathcal{X}}_k^{\mathrm{proj}}$ represents the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-algebra endomorphism of $U_{\mathcal{O}}^{\mathrm{proj}}$ defined by $\overline{q}' \mapsto \overline{x}_k \overline{q}'$ for all $q' \in \overline{Q}$, i.e. the multiplication by $\overline{x}_k$ in $U_{\mathcal{O}}^{\mathrm{proj}}$, with respect to this $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis. In particular, the commutativity of $U_{\mathcal{O}}^{\mathrm{proj}}$ thus yields that the matrices $\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}$ are pairwise commuting.

Next we prove b). We start to prove the claim for homogeneous polynomials. Let $q' \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \dots, x_n])_\gamma$ be with $\gamma \in \mathbb{Z}$. If $\gamma \le 0$, we have $q' \in B_{\mathcal{O}}^{\mathrm{proj}}$ and thus

$$\begin{aligned}
\overline{q}' \cdot \overline{q} &= q' \cdot (c_1 \bar{t}_1 + \dots + c_\mu \bar{t}_\mu) \\
&= q' c_1 \bar{t}_1 + \dots + q' c_\mu \bar{t}_\mu \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot q' \mathcal{I}_\mu \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\bar{t}_1, \dots, \bar{t}_\mu) \cdot q'(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. We prove the claim for $\gamma \in \mathbb{N} \setminus \{0\}$ by induction on γ. For the induction start

$\gamma = 1$, there exist $d_0, \dots, d_n \in B_{\mathcal{O}}^{\mathrm{proj}}$ such that $q = d_0 x_0 + \cdots d_n x_n$. By a), we get

$$\begin{aligned}
\overline{q}' \cdot \overline{q} &= \sum_{m=1}^{n} d_m \overline{x}_m \cdot \overline{q} \\
&= \sum_{m=1}^{n} (\overline{t}_1, \dots, \overline{t}_\mu) \cdot (d_m x_m)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot \left(\sum_{m=1}^{n} d_m x_m \right) (\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot q(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ and the induction start is proven. For the induction step, suppose now that $\gamma > 1$. Then we can write $q' = c_1' t_1' + \cdots + c_s' t_s'$ with $s \in \mathbb{N}$, $c_1', \dots, c_s' \in B_{\mathcal{O}}^{\mathrm{proj}}$, and $t_1', \dots, t_s' \in \mathbb{T}_\gamma^{n+1}$. For every $r \in \{1, \dots, s\}$, there exist a $k_r \in \{0, \dots, n\}$ and a term $u_r \in \mathbb{T}_{<\gamma}^{n+1}$ such that $x_{k_r} u_r = t_r'$. Then the induction hypothesis and a) yield

$$\begin{aligned}
c_r' \overline{t}_r' \cdot \overline{q} &= \overline{x}_{k_r} \cdot c_r' \overline{u}_r \overline{q} \\
&= \overline{x}_{k_r} \cdot ((\overline{t}_1, \dots, \overline{t}_\mu) \cdot (c_r' u_r q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}) \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot \overline{\mathcal{X}}_{k_r}^{\mathrm{proj}} \cdot (c_r' u_r q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot (x_{k_r} \cdot c_r' u_r q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot (c_r' t_r' q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ for every $r \in \{1, \dots, s\}$. Altogether, we get

$$\begin{aligned}
\overline{q}' \cdot \overline{q} &= \sum_{r=1}^{s} c_r' \overline{t}_r' \cdot \overline{q} \\
&= \sum_{r=1}^{s} (\overline{t}_1, \dots, \overline{t}_\mu) \cdot (c_r' t_r' q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot \left(\sum_{r=1}^{s} c_r' t_r' q \right) (\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}} \\
&= (\overline{t}_1, \dots, \overline{t}_\mu) \cdot (q' q)(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{proj}}) \cdot (c_1, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ and the claim follows in the homogeneous case.
In order to prove the claim for arbitrary $q' \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \dots, x_n]$, we let $q' = \sum_{\gamma \in \mathbb{Z}} q_\gamma'$ be

the decomposition of q' into its homogeneous components. Then

$$\begin{aligned}\overline{q}' \cdot \overline{q} &= \sum_{\gamma\in\mathbb{Z}} \overline{q}'_\gamma \cdot \overline{q} \\ &= \sum_{\gamma\in\mathbb{Z}} (\bar{t}_1,\dots,\bar{t}_\mu)\cdot q'_\gamma(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})\cdot(c_1,\dots,c_\mu)^{\mathrm{tr}} \\ &= (\bar{t}_1,\dots,\bar{t}_\mu)\cdot\left(\sum_{\gamma\in\mathbb{Z}} q'_\gamma\right)(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})\cdot(c_1,\dots,c_\mu)^{\mathrm{tr}} \\ &= (\bar{t}_1,\dots,\bar{t}_\mu)\cdot q'(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})\cdot(c_1,\dots,c_\mu)^{\mathrm{tr}}\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ from the homogeneous case. □

Corollary 6.2.5. *Let $\overline{\mathcal{O}} = \{\bar{t}_1,\dots,\bar{t}_\mu\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ denote the image of the order ideal $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$. Moreover, let $\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ be the images of the generic projective multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$, $\gamma\in\mathbb{Z}$, and $q \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_\gamma$. Then $q(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}}) \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-algebra homomorphism $\pi_{\overline{q}} : U_{\mathcal{O}}^{\mathrm{proj}}(-\gamma) \to U_{\mathcal{O}}^{\mathrm{proj}}$, $r \mapsto \overline{q}\cdot r$, the multiplication by $\overline{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$, with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}$ of $U_{\mathcal{O}}^{\mathrm{proj}}$.*

Proof. By Theorem 6.1.13, $\overline{\mathcal{O}}$ is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$. Let $\{\mathcal{E}_1,\dots,\mathcal{E}_\mu\}$ denote the canonical $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])^\mu$. Then the claim follows since we have $\pi_{\overline{q}}(\bar{t}_i) = \overline{q}\cdot\bar{t}_i = (\bar{t}_1,\dots,\bar{t}_\mu)\cdot q(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})\cdot\mathcal{E}_i^{\mathrm{tr}}$ for every $i\in\{1,\dots,\mu\}$ by Proposition 6.2.4. □

Example 6.2.6. Let $P = \mathbb{Q}[x_1,x_2]$ be standard graded, i.e. graded by the matrix $W = (1,1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ and let $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$ also be standard graded, i.e. graded by the matrix $\overline{W} = (1 \mid W) = (1,1,1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$. According to the Definitions 2.1.6, 2.1.7, and 4.1.11, the set $\mathcal{O} = \{t_1,\dots,t_8\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_1^2x_2, x_1^3\}$ is an order ideal in $\mathbb{T}^2$ with maxdeg_W-border $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^3, x_1x_2^2, x_1^2x_2^2, x_1^3x_2, x_1^4\}$. Let $\{c_{ij} \mid i\in\{1,\dots,8\}, j\in\{1,\dots,5\}\}$, be a set of further indeterminates. let $\mathbb{B}_{\mathcal{O}}$ be the $\mathcal{O}$-border basis scheme and $B_{\mathcal{O}} \to U_{\mathcal{O}}$ be the universal $\mathcal{O}$-border basis family as defined in Definition 6.1.1. Furthermore, we let $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{B}_{\mathcal{O}}$ be the projective $\mathcal{O}$-border basis scheme, $I_{\mathcal{O}}^{\mathrm{proj}} \subseteq B_{\mathcal{O}}$ its defining ideal, and $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ be the universal projective $\mathcal{O}$-border basis family as defined in Definition 6.1.5. Since the order ideal $\mathcal{O}$ has a maxdeg_W-border, we have $I_{\mathcal{O}}^{\mathrm{proj}} = \{0\}$, i.e. $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ coincides with $\mathbb{B}_{\mathcal{O}}$.

Let $G = \{g_1,\dots,g_5\} \subseteq \overline{P}[c_{11},\dots,c_{85}]$ be the generic projective $\mathcal{O}$-border prebasis. Then $g_j = b_j - c_{1j}x_0^3 - c_{2j}x_0^2x_2 - c_{3j}x_0^2x_1 - c_{4j}x_0x_2^2 - c_{5j}x_0x_1x_2 - c_{6j}x_0x_1^2 - c_{7j}x_1^2x_2 - c_{8j}x_1^3$ for every index $j\in\{1,2\}$, and for every index j in the set $\{3,4,5\}$, the polynomial $g_j = b_j - c_{1j}x_0^4 - c_{2j}x_0^3x_2 - c_{3j}x_0^3x_1 - c_{4j}x_0^2x_2^2 - c_{5j}x_0^2x_1x_2 - c_{6j}x_0^2x_1^2 - c_{7j}x_0x_1^2x_2 - c_{8j}x_0x_1^3$. The images $\overline{\mathcal{X}}_0^{\mathrm{proj}}, \overline{\mathcal{X}}_1^{\mathrm{proj}}, \overline{\mathcal{X}}_2^{\mathrm{proj}} \in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ of the generic projective multiplication matrices $\mathcal{X}_0^{\mathrm{proj}}, \mathcal{X}_1^{\mathrm{proj}}, \mathcal{X}_2^{\mathrm{proj}} \in \mathrm{Mat}_8(K[x_0,c_{11},\dots,c_{85}])$ with respect to $\mathcal{O}$ in the ring $\mathrm{Mat}_8(B^{\mathrm{proj}}_{\mathcal{O}}[x_0])$ are

$$\overline{\mathcal{X}}^{\mathrm{proj}}_0 = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 \end{pmatrix},$$

$$\overline{\mathcal{X}}^{\mathrm{proj}}_1 = \begin{pmatrix} 0 & 0 & 0 & \overline{c}_{12}x_0^3 & 0 & 0 & \overline{c}_{14}x_0^4 & \overline{c}_{15}x_0^4 \\ 0 & 0 & 0 & \overline{c}_{22}x_0^2 & 0 & 0 & \overline{c}_{24}x_0^3 & \overline{c}_{25}x_0^3 \\ 1 & 0 & 0 & \overline{c}_{32}x_0^2 & 0 & 0 & \overline{c}_{34}x_0^3 & \overline{c}_{35}x_0^3 \\ 0 & 0 & 0 & \overline{c}_{42}x_0 & 0 & 0 & \overline{c}_{44}x_0^2 & \overline{c}_{45}x_0^2 \\ 0 & 1 & 0 & \overline{c}_{52}x_0 & 0 & 0 & \overline{c}_{54}x_0^2 & \overline{c}_{55}x_0^2 \\ 0 & 0 & 1 & \overline{c}_{62}x_0 & 0 & 0 & \overline{c}_{64}x_0^2 & \overline{c}_{65}x_0^2 \\ 0 & 0 & 0 & \overline{c}_{72} & 1 & 0 & \overline{c}_{74}x_0 & \overline{c}_{75}x_0 \\ 0 & 0 & 0 & \overline{c}_{82} & 0 & 1 & \overline{c}_{84}x_0 & \overline{c}_{85}x_0 \end{pmatrix},$$

and

$$\overline{\mathcal{X}}^{\mathrm{proj}}_2 = \begin{pmatrix} 0 & 0 & 0 & \overline{c}_{11}x_0^3 & \overline{c}_{12}x_0^3 & 0 & \overline{c}_{13}x_0^4 & \overline{c}_{14}x_0^4 \\ 1 & 0 & 0 & \overline{c}_{21}x_0^2 & \overline{c}_{22}x_0^2 & 0 & \overline{c}_{23}x_0^3 & \overline{c}_{24}x_0^3 \\ 0 & 0 & 0 & \overline{c}_{31}x_0^2 & \overline{c}_{32}x_0^2 & 0 & \overline{c}_{33}x_0^3 & \overline{c}_{34}x_0^3 \\ 0 & 1 & 0 & \overline{c}_{41}x_0 & \overline{c}_{42}x_0 & 0 & \overline{c}_{43}x_0^2 & \overline{c}_{44}x_0^2 \\ 0 & 0 & 1 & \overline{c}_{51}x_0 & \overline{c}_{52}x_0 & 0 & \overline{c}_{53}x_0^2 & \overline{c}_{54}x_0^2 \\ 0 & 0 & 0 & \overline{c}_{61}x_0 & \overline{c}_{62}x_0 & 0 & \overline{c}_{63}x_0^2 & \overline{c}_{64}x_0^2 \\ 0 & 0 & 0 & \overline{c}_{71} & \overline{c}_{72} & 1 & \overline{c}_{73}x_0 & \overline{c}_{74}x_0 \\ 0 & 0 & 0 & \overline{c}_{81} & \overline{c}_{82} & 0 & \overline{c}_{83}x_0 & \overline{c}_{84}x_0 \end{pmatrix}$$

by Definition 6.1.5. Let $\overline{\mathcal{O}}$ be the image of $\mathcal{O}$ in $U^{\mathrm{proj}}_{\mathcal{O}}$, $q = x_1^2 + x_0x_2 \in (B^{\mathrm{proj}}_{\mathcal{O}}[x_0,\dots,x_n])_2$, and let $\pi_{\overline{q}} : U^{\mathrm{proj}}_{\mathcal{O}}(-2) \to U^{\mathrm{proj}}_{\mathcal{O}}$, $r \mapsto \overline{q}\cdot r$ be the multiplication by $\overline{q} \in (U^{\mathrm{proj}}_{\mathcal{O}})_2$. Then by

Corollary 6.2.5, the matrix

$$
\begin{aligned}
&q(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \overline{\mathcal{X}}_1^{\mathrm{proj}}, \overline{\mathcal{X}}_2^{\mathrm{proj}}) \\
&= (\overline{\mathcal{X}}_1^{\mathrm{proj}})^2 + \overline{\mathcal{X}}_0^{\mathrm{proj}} \cdot \overline{\mathcal{X}}_2^{\mathrm{proj}} \\
&= \begin{pmatrix}
0 & 0 & 0 & m_{14} & m_{15} & m_{16} & m_{17} & m_{18} \\
x_0 & 0 & 0 & m_{24} & m_{25} & m_{26} & m_{27} & m_{28} \\
0 & 0 & 0 & m_{34} + \overline{c}_{12}x_0^3 & m_{35} & m_{36} & m_{37} + \overline{c}_{14}x_0^4 & m_{38} + \overline{c}_{15}x_0^4 \\
0 & x_0 & 0 & m_{44} & m_{45} & m_{46} & m_{47} & m_{48} \\
0 & 0 & x_0 & m_{54} + \overline{c}_{22}x_0^2 & m_{55} & m_{56} & m_{57} + \overline{c}_{24}x_0^3 & m_{58} + \overline{c}_{25}x_0^3 \\
1 & 0 & 0 & m_{64} + \overline{c}_{32}x_0^2 & m_{65} & m_{66} & m_{67} + \overline{c}_{34}x_0^3 & m_{68} + \overline{c}_{35}x_0^3 \\
0 & 1 & 0 & m_{74} + \overline{c}_{52}x_0 & m_{75} & m_{76} + x_0 & m_{77} + \overline{c}_{54}x_0^2 & m_{78} + \overline{c}_{55}x_0^2 \\
0 & 0 & 1 & m_{84} + \overline{c}_{62}x_0 & m_{85} & m_{86} & m_{87} + \overline{c}_{64}x_0^2 & m_{88} + \overline{c}_{65}x_0^2
\end{pmatrix} \\
&\in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])
\end{aligned}
$$

where we have

$$
\begin{aligned}
m_{i4} &= (\overline{c}_{i2}\overline{c}_{42} + \overline{c}_{i4}\overline{c}_{72} + \overline{c}_{i5}\overline{c}_{82} + \overline{c}_{i1})x_0^{4-\deg(t_i)}, \\
m_{i5} &= (\overline{c}_{i2} + \overline{c}_{i4})x_0^{4-\deg(t_i)}, \\
m_{i6} &= \overline{c}_{i5}x_0^{4-\deg(t_i)}, \\
m_{i7} &= (\overline{c}_{i2}\overline{c}_{44} + \overline{c}_{i4}\overline{c}_{74} + \overline{c}_{i5}\overline{c}_{84} + \overline{c}_{i3})x_0^{5-\deg(t_i)}, \\
m_{i8} &= (\overline{c}_{i2}\overline{c}_{45} + \overline{c}_{i4}\overline{c}_{75} + \overline{c}_{i5}\overline{c}_{85} + \overline{c}_{i4})x_0^{5-\deg(t_i)}
\end{aligned}
$$

for all $i \in \{1, \dots, 8\}$ represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-algebra homomorphism $\pi_{\overline{q}}$ with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}$ of $U_{\mathcal{O}}^{\mathrm{proj}}$.

In the remaining part of this section, we prove that the generic multiplication matrices with respect to $\mathcal{O}$ can also be used to explicitly describe the multiplicative structure of the universal projective $\mathcal{O}$-border basis family $U_{\mathcal{O}}^{\mathrm{proj}}$.

Proposition 6.2.7. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and $i \in \{1, \dots, \mu\}$, let $\overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ denote the images of the generic multiplication matrices with respect to the order ideal $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$, let $c_1, \dots, c_{H(\gamma)} \in B_{\mathcal{O}}^{\mathrm{proj}}$, and we let $q = c_1 x_0^{d_{\gamma,1}} t_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} t_{H(\gamma)} \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \dots, x_n])_\gamma$.*

a) We have

$$
\begin{aligned}
x_0 \cdot \overline{q} = (&x_0^{d_{\gamma+1,1}}\overline{t}_1, \dots, x_0^{d_{\gamma+1,H(\gamma+1)}}\overline{t}_{H(\gamma+1)}, 0, \dots, 0) \\
&\cdot \mathcal{I}_\mu \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ and for every $k \in \{1, \dots, n\}$, we have

$$
\begin{aligned}
\overline{x}_k \cdot \overline{q} = (&x_0^{d_{\gamma+w_k,1}}\overline{t}_1, \dots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}}\overline{t}_{H(\gamma+w_k)}, 0, \dots, 0) \\
&\cdot \overline{\mathcal{X}}_k \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}
$$

in $U_{\mathcal{O}}^{\text{proj}}$.

b) *For every* $\gamma' \in \mathbb{Z}$ *and every* $q' \in (B_{\mathcal{O}}^{\text{proj}}[x_0, \dots, x_n])_{\gamma'}$, *we have*

$$\begin{aligned}\overline{q}' \cdot \overline{q} = (&x_0^{d_{\gamma+\gamma',1}}\overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\ &\cdot q'(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\text{tr}}\end{aligned}$$

in $U_{\mathcal{O}}^{\text{proj}}$.

Proof. For all $s \in \{1, \dots, n\}$, let $\overline{\mathcal{X}}_s = (\overline{\xi}_{k\ell}^{(s)})_{1 \le k, \ell \le \mu} \in \text{Mat}_\mu(B_{\mathcal{O}}^{\text{proj}})$ be as in Definitions 6.1.1 and 6.1.5 and let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ be the canonical $B_{\mathcal{O}}^{\text{proj}}$-module basis of $(B_{\mathcal{O}}^{\text{proj}})^\mu$. Moreover, we denote $w_0 = 1$ and $\mathcal{X}_0 = \mathcal{I}_\mu$. If $\gamma < 0$, we have $(B_{\mathcal{O}}^{\text{proj}}[x_0, \dots, x_n])_\gamma = \{0\}$ and the claims follow trivially. Thus suppose that $\gamma \ge 0$.

The first equality of a) follows as we have

$$\begin{aligned}x_0 \cdot \overline{q} &= x_0 \cdot (c_1 x_0^{d_{\gamma,1}}\overline{t}_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}}\overline{t}_{H(\gamma)}) \\ &= c_1 x_0^{d_{\gamma,1}+1}\overline{t}_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}+1}\overline{t}_{H(\gamma)} \\ &= c_1 x_0^{d_{\gamma+1,1}}\overline{t}_1 + \dots + c_{H(\gamma)} x_0^{d_{\gamma+1,H(\gamma)}}\overline{t}_{H(\gamma)} \\ &= (x_0^{d_{\gamma+1,1}}\overline{t}_1, \dots, x_0^{d_{\gamma+1,H(\gamma+1)}}\overline{t}_{H(\gamma+1)}, 0, \dots, 0) \cdot \mathcal{I}_\mu \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\text{tr}}\end{aligned}$$

in $U_{\mathcal{O}}^{\text{proj}}$. In order to prove the second equality, let $k \in \{1, \dots, n\}$ and $\ell \in \{1, \dots, H(\gamma)\}$. If $x_k t_\ell = t_i \in \mathcal{O}$ for some $i \in \{1, \dots, \mu\}$, we have $\deg_W(t_i) = \deg_W(t_\ell) + w_k$ and hence

$$\begin{aligned}\overline{x}_k \cdot x_0^{d_{\gamma,\ell}}\overline{t}_\ell &= x_0^{d_{\gamma+w_k,i}}\overline{t}_i \\ &= \sum_{m=1}^{H(\gamma+w_k)} \delta_{mi} x_0^{d_{\gamma+w_k,m}}\overline{t}_m \\ &= \sum_{m=1}^{H(\gamma+w_k)} \overline{\xi}_{m\ell}^{(k)} x_0^{d_{\gamma+w_k,m}}\overline{t}_m \\ &= (x_0^{d_{\gamma+w_k,1}}\overline{t}_1, \dots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}}\overline{t}_{H(\gamma+w_k)}, 0, \dots, 0) \cdot \overline{\mathcal{X}}_k \cdot \mathcal{E}_\ell^{\text{tr}}\end{aligned}$$

in $U_{\mathcal{O}}^{\text{proj}}$. If $x_k t_\ell = b_j$ for some $j \in \{1, \dots, \nu\}$, we get $\deg_W(b_j) = \deg_W(t_\ell) + w_k$ and for every $m \in \{1, \dots, H(\deg_W(b_j))\}$, it follows that

$$\begin{aligned}d_{\gamma,\ell} + \Delta_{mj} &= \gamma - \deg_W(t_\ell) + \deg_W(b_j) - \deg_W(t_m) \\ &= \gamma - \deg_W(t_\ell) + \deg_W(t_\ell) + w_k - \deg_W(t_m) \\ &= \gamma + w_k - \deg_W(t_m) \\ &= d_{\gamma+w_k,m}.\end{aligned}$$

Altogether, we get

$$\begin{aligned}
\overline{x}_k \cdot x_0^{d_{\gamma,\ell}} \overline{t}_\ell &= x_0^{d_{\gamma,\ell}} \overline{b}_j \\
&= \sum_{m=1}^{H(\deg_W(b_j))} \overline{c}_{mj} x_0^{d_{\gamma,\ell}+\Delta_{mj}} \overline{t}_m \\
&= \sum_{m=1}^{H(\gamma+w_k)} \overline{\xi}_{m\ell}^{(k)} x_0^{d_{\gamma+w_k,m}} \overline{t}_m \\
&= (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \overline{\mathcal{X}}_k \cdot \mathcal{E}_\ell^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. Claim a) now follows as

$$\begin{aligned}
\overline{x}_k \cdot \overline{q} &= \sum_{\ell=1}^{H(\gamma)} \overline{x}_k \cdot c_\ell x_0^{d_{\gamma,\ell}} \overline{t}_\ell \\
&= \sum_{\ell=1}^{H(\gamma)} (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \overline{\mathcal{X}}_k \cdot c_\ell \mathcal{E}_\ell^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+w_k,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma+w_k,H(\gamma+w_k)}} \overline{t}_{H(\gamma+w_k)}, 0, \ldots, 0) \cdot \overline{\mathcal{X}}_k \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$.

Next we prove b). For all $\gamma' \in \mathbb{Z}$ with $\gamma' < 0$, we have $\overline{Q}_{\gamma'} = \{0\}$ and b) holds trivially. We prove by induction on $\gamma' \in \mathbb{N}$ that b) holds in general. For $\gamma' = 0$, it follows that $q' \in B_{\mathcal{O}}^{\mathrm{proj}}$ and thus

$$\begin{aligned}
\overline{q}' \cdot \overline{q} &= q' c_1 x_0^{d_{\gamma,1}} \overline{t}_1 + \cdots + q' c_{H(\gamma)} x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)} \\
&= (x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot q' \mathcal{I}_\mu \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma,1}} \overline{t}_1, \ldots, x_0^{d_{\gamma,H(\gamma)}} \overline{t}_{H(\gamma)}, 0, \ldots, 0) \cdot q' (\mathcal{I}_\mu, \mathcal{X}_1, \ldots, \mathcal{X}_n) \cdot (c_1, \ldots, c_{H(\gamma)}, 0, \ldots, 0)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. This shows the claim for $\gamma' = 0$. For the induction step, we let $\gamma' > 0$. Then there exist an $s \in \mathbb{N} \setminus \{0\}$, $c'_1, \ldots, c'_s \in B_{\mathcal{O}}^{\mathrm{proj}}$, and $t'_1, \ldots, t'_s \in \mathbb{T}_{\gamma'}^{n+1}$ such that $q' = c'_1 t'_1 + \cdots + c'_s t'_s$. For all indices $r \in \{1, \ldots, s\}$, there are $k_r \in \{0, \ldots, n\}$ and $u_r \in \mathbb{T}_{<\gamma'}^{n+1}$ such that $t'_r = x_{k_r} u_r$. Hence the induction hypothesis, the commutativity of the matrices $\overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n$, which follows from Definition 6.1.1, together with a) yield

$$\begin{aligned}
&\overline{x}_{k_r} \cdot (c'_r \overline{u}_r \cdot \overline{q}) \\
&= \overline{x}_{k_r} \cdot ((x_0^{d_{\gamma+\gamma'-w_{k_r},1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma'-w_{k_r},H(\gamma+\gamma'-w_{k_r})}} \overline{t}_{H(\gamma+\gamma'-w_{k_r})}, 0, \dots, 0) \\
&\quad \cdot (c'_r u_r)(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}) \\
&= (x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot (\overline{\mathcal{X}}_{k_r} \cdot (c'_r u_r)(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n)) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot (c'_r x_{k_r} u_r)(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$ for all $r \in \{1, \dots, s\}$. The claim for $\gamma' > 0$ hence follows as

$$\begin{aligned}
\overline{q}' \cdot \overline{q} &= \sum_{r=1}^{s} \overline{x}_{k_r} \cdot (c'_r \overline{u}_r \cdot \overline{q}) \\
&= \sum_{r=1}^{s} ((x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\qquad \cdot (c'_r x_{k_r} u_r)(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}) \\
&= (x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot \left(\sum_{r=1}^{s} c'_r x_{k_r} u_r \right) (\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}} \\
&= (x_0^{d_{\gamma+\gamma',1}} \overline{t}_1, \dots, x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}} \overline{t}_{H(\gamma+\gamma')}, 0, \dots, 0) \\
&\quad \cdot q'(\mathcal{I}_\mu, \overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n) \cdot (c_1, \dots, c_{H(\gamma)}, 0, \dots, 0)^{\mathrm{tr}}
\end{aligned}$$

in $U_{\mathcal{O}}^{\mathrm{proj}}$. $\square$

Corollary 6.2.8. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and all $i \in \{1, \dots, \mu\}$, let $\overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ denote the images of the generic multiplication matrices with respect to the order ideal $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$, and we let $\overline{\mathcal{O}}_0$ denote the image of the set $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$. Moreover, we let $\gamma, \gamma' \in \mathbb{Z}$ and $q \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0, \dots, x_n])_{\gamma'}$. Then the submatrix of the matrix $q(\mathcal{I}_\mu, \mathcal{X}_1, \dots, \mathcal{X}_n) \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ consisting of the first $H(\gamma + \gamma')$ rows and the first $H(\gamma)$ columns represents the $B_{\mathcal{O}}^{\mathrm{proj}}$-algebra homomorphism $\widehat{\pi}_{\overline{q}} : (U_{\mathcal{O}}^{\mathrm{proj}})_\gamma \to (U_{\mathcal{O}}^{\mathrm{proj}})_{\gamma+\gamma'}$, $r \mapsto \overline{q} \cdot r$, the multiplication by $\overline{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_{\gamma'}$, with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases $(\overline{\mathcal{O}}_0)_\gamma$ of $(U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$ and $(\overline{\mathcal{O}}_0)_{\gamma+\gamma'}$ of $(U_{\mathcal{O}}^{\mathrm{proj}})_{\gamma+\gamma'}$.*

Proof. Let $\{\mathcal{E}_1, \dots, \mathcal{E}_\mu\}$ denote the canonical $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}})^\mu$. According to Corollary 6.1.14, the sets $(\overline{\mathcal{O}}_0)_\gamma$ and $(\overline{\mathcal{O}}_0)_{\gamma+\gamma'}$ are $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases of $(U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$

and $(U_{\mathcal{O}}^{\mathrm{proj}})_{\gamma+\gamma'}$, respectively. For all $k \in \{1,\dots,H(\gamma)\}$, Proposition 6.2.7 yields

$$\begin{aligned}\widehat{\pi}_{\overline{q}}(x_0^{d_{\gamma,k}}\overline{t}_k) &= \overline{q}\cdot x_0^{d_{\gamma,k}}\overline{t}_k\\ &= (x_0^{d_{\gamma+\gamma',1}}\overline{t}_1,\dots,x_0^{d_{\gamma+\gamma',H(\gamma+\gamma')}}\overline{t}_{H(\gamma+\gamma')},0,\dots,0)\cdot q(\mathcal{I}_\mu,\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n)\cdot\mathcal{E}_k^{\mathrm{tr}}\end{aligned}$$

and the claim is a direct consequence of this. □

Example 6.2.9. Consider the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{B}_{\mathcal{O}}$ and the corresponding universal projective $\mathcal{O}$-border basis family $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ of Example 6.2.6, again. Recall that we had $\mathcal{O} = \{t_1,\dots,t_8\} = \{1,x_2,x_1,x_2^2,x_1x_2,x_1^2,x_1^2x_2,x_1^3\}$ with the maxdeg_W-border $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^3,x_1x_2^2,x_1^2x_2^2,x_1^3x_2,x_1^4\}$ and that the polynomials lived in the standard graded rings $P = \mathbb{Q}[x_1,x_2]$ respectively $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$. The generic projective $\mathcal{O}$-border basis was of the form $G = \{g_1,\dots,g_5\} \subseteq \overline{P}[c_{11},\dots,c_{85}]$ with $g_j = b_j - c_{1j}x_0^3 - c_{2j}x_0^2x_2 - c_{3j}x_0^2x_1 - c_{4j}x_0x_2^2 - c_{5j}x_0x_1x_2 - c_{6j}x_0x_1^2 - c_{7j}x_1^2x_2 - c_{8j}x_1^3$ for all indices $j \in \{1,2\}$ and for every index j in the set $\{3,4,5\}$, the polynomial $g_j = b_j - c_{1j}x_0^4 - c_{2j}x_0^3x_2 - c_{3j}x_0^3x_1 - c_{4j}x_0^2x_2^2 - c_{5j}x_0^2x_1x_2 - c_{6j}x_0^2x_1^2 - c_{7j}x_0x_1^2x_2 - c_{8j}x_0x_1^3$. According to Definition 6.1.1, the images $\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2 \in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}})$ of the generic multiplication matrices $\mathcal{X}_1,\mathcal{X}_2 \in \mathrm{Mat}_8(\mathbb{Q}[c_{11},\dots,c_{85}])$ with respect to $\mathcal{O}$ in $\mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}})$ are

$$\overline{\mathcal{X}}_1 = \begin{pmatrix} 0&0&0&\overline{c}_{12}&0&0&\overline{c}_{14}&\overline{c}_{15}\\ 0&0&0&\overline{c}_{22}&0&0&\overline{c}_{24}&\overline{c}_{25}\\ 1&0&0&\overline{c}_{32}&0&0&\overline{c}_{34}&\overline{c}_{35}\\ 0&0&0&\overline{c}_{42}&0&0&\overline{c}_{44}&\overline{c}_{45}\\ 0&1&0&\overline{c}_{52}&0&0&\overline{c}_{54}&\overline{c}_{55}\\ 0&0&1&\overline{c}_{62}&0&0&\overline{c}_{64}&\overline{c}_{65}\\ 0&0&0&\overline{c}_{72}&1&0&\overline{c}_{74}&\overline{c}_{75}\\ 0&0&0&\overline{c}_{82}&0&1&\overline{c}_{84}&\overline{c}_{85} \end{pmatrix},\quad \overline{\mathcal{X}}_2 = \begin{pmatrix} 0&0&0&\overline{c}_{11}&\overline{c}_{12}&0&\overline{c}_{13}&\overline{c}_{14}\\ 1&0&0&\overline{c}_{21}&\overline{c}_{22}&0&\overline{c}_{23}&\overline{c}_{24}\\ 0&0&0&\overline{c}_{31}&\overline{c}_{32}&0&\overline{c}_{33}&\overline{c}_{34}\\ 0&1&0&\overline{c}_{41}&\overline{c}_{42}&0&\overline{c}_{43}&\overline{c}_{44}\\ 0&0&1&\overline{c}_{51}&\overline{c}_{52}&0&\overline{c}_{53}&\overline{c}_{54}\\ 0&0&0&\overline{c}_{61}&\overline{c}_{62}&0&\overline{c}_{63}&\overline{c}_{64}\\ 0&0&0&\overline{c}_{71}&\overline{c}_{72}&1&\overline{c}_{73}&\overline{c}_{74}\\ 0&0&0&\overline{c}_{81}&\overline{c}_{82}&0&\overline{c}_{83}&\overline{c}_{84} \end{pmatrix}.$$

Consider the polynomial $q = x_1^2 + x_0x_2 \in (\overline{P}[c_{11},\dots,c_{85}])_2$ and the map $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1,\dots,8\} \mid \deg(t_k) \le \gamma\}$. Then

$$\begin{aligned} q(\mathcal{I}_8,\overline{\mathcal{X}}_1,\overline{\mathcal{X}}_2) &= \overline{\mathcal{X}}_1^2 + \mathcal{I}_8\cdot\overline{\mathcal{X}}_2\\ &= \begin{pmatrix} 0&0&0&\widehat{m}_{14}&\widehat{m}_{15}&\widehat{m}_{16}&\widehat{m}_{17}&\widehat{m}_{18}\\ 1&0&0&\widehat{m}_{24}&\widehat{m}_{25}&\widehat{m}_{26}&\widehat{m}_{27}&\widehat{m}_{28}\\ 0&0&0&\widehat{m}_{34}+\overline{c}_{12}&\widehat{m}_{35}&\widehat{m}_{36}&\widehat{m}_{37}+\overline{c}_{14}&\widehat{m}_{38}+\overline{c}_{15}\\ 0&1&0&\widehat{m}_{44}&\widehat{m}_{45}&\widehat{m}_{46}&\widehat{m}_{47}&\widehat{m}_{48}\\ 0&0&1&\widehat{m}_{54}+\overline{c}_{22}&\widehat{m}_{55}&\widehat{m}_{56}&\widehat{m}_{57}+\overline{c}_{24}&\widehat{m}_{58}+\overline{c}_{25}\\ 1&0&0&\widehat{m}_{64}+\overline{c}_{32}&\widehat{m}_{65}&\widehat{m}_{66}&\widehat{m}_{67}+\overline{c}_{34}&\widehat{m}_{68}+\overline{c}_{35}\\ 0&1&0&\widehat{m}_{74}+\overline{c}_{52}&\widehat{m}_{75}&\widehat{m}_{76}+1&\widehat{m}_{77}+\overline{c}_{54}&\widehat{m}_{78}+\overline{c}_{55}\\ 0&0&1&\widehat{m}_{84}+\overline{c}_{62}&\widehat{m}_{85}&\widehat{m}_{86}&\widehat{m}_{87}+\overline{c}_{64}&\widehat{m}_{88}+\overline{c}_{65} \end{pmatrix}\\ &\in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}) \end{aligned}$$

where we have

$$\begin{aligned}
\widehat{m}_{i4} &= \overline{c}_{i2}\overline{c}_{42} + \overline{c}_{i4}\overline{c}_{72} + \overline{c}_{i5}\overline{c}_{82} + \overline{c}_{i1},\\
\widehat{m}_{i5} &= \overline{c}_{i2} + \overline{c}_{i4},\\
\widehat{m}_{i6} &= \overline{c}_{i5},\\
\widehat{m}_{i7} &= \overline{c}_{i2}\overline{c}_{44} + \overline{c}_{i4}\overline{c}_{74} + \overline{c}_{i5}\overline{c}_{84} + \overline{c}_{i3},\\
\widehat{m}_{i8} &= \overline{c}_{i2}\overline{c}_{45} + \overline{c}_{i4}\overline{c}_{75} + \overline{c}_{i5}\overline{c}_{85} + \overline{c}_{i4}
\end{aligned}$$

for all $i \in \{1,\dots,8\}$. By Corollary 6.2.8, the submatrix

$$\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{8,3}(B_{\mathcal{O}}^{\mathrm{proj}})$$

of $q(\mathcal{I}_8, \overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2)$ consisting of the first $H(3) = 8$ rows and the first $H(1) = 3$ columns represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}$-algebra homomorphism $\widehat{\pi}_{\overline{q}} : (U_{\mathcal{O}}^{\mathrm{proj}})_1 \to (U_{\mathcal{O}}^{\mathrm{proj}})_3$, $r \mapsto \overline{q}\cdot r$, the multiplication by $\overline{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_2$, with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases $\{x_0, \overline{x}_2, \overline{x}_1\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ of $(U_{\mathcal{O}}^{\mathrm{proj}})_1$ and $\{x_0^3, x_0^2\overline{x}_2, x_0^2\overline{x}_1, x_0\overline{x_2^2}, x_0\overline{x_1x_2}, x_0\overline{x_1^2}, \overline{x_1^2x_2}, \overline{x_1^3}\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ of $(U_{\mathcal{O}}^{\mathrm{proj}})_3$. Note that since all the entries of the above matrix are in the base field $\mathbb{Q}$, the matrix is independent of the coefficients of the generic projective $\mathcal{O}$-border prebasis G. Therefore, this matrix describes the multiplication by $\overline{q}$ for any arbitrary projective $\mathcal{O}$-border basis.

6.3 The Generic Canonical Module of the Universal Projective Border Basis Family

After the study of the multiplicative structure of the universal projective $\mathcal{O}$-border basis family $U_{\mathcal{O}}^{\mathrm{proj}}$ in Section 6.2, we now want to consider the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module of $U_{\mathcal{O}}^{\mathrm{proj}}$. Recall Section 4.3. For every homogeneous ideal $I \subseteq \overline{P}$ with the property that it possesses a projective $\mathcal{O}$-border basis, we can explicitly describe the dual $K[x_0]$-module of the quotient ring $R = \overline{P}/I$, which is precisely the canonical module ω_R of R by Remark 4.3.5, and its multiplicative structure, cf. the Propositions 4.3.8 and 4.3.10 and Corollary 4.3.11. Moreover, we have seen that projective $\mathcal{O}$-border bases are unique by Proposition 4.1.7 and that thus the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ parametrizes the one-dimensional homogeneous ideals in $\overline{P}$ that possess a projective $\mathcal{O}$-border basis in Section 6.1 This yields the question whether we can parametrize all canonical modules ω_R of quotient rings $R = \overline{P}/I$ where I possesses a projective $\mathcal{O}$-border basis using $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. In this section, we show that this can actually be done. More precisely, we do the following.

First we introduce the generic canonical module of $U_{\mathcal{O}}^{\mathrm{proj}}/J$ where $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ is a homogeneous ideal as the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module of $U_{\mathcal{O}}^{\mathrm{proj}}/J$ in Definition 6.3.2. Then we prove that the residue classes of the elements of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ immediately yield $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$- respectively $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases of the generic canonical module $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ of $U_{\mathcal{O}}^{\mathrm{proj}}$ respectively its homogeneous components in Proposition 6.3.4. After that, we study the multiplicative structure of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ and describe it as explicitly as possible by means of the generic (projective) multiplication matrices with respect to $\mathcal{O}$ in the Propositions 6.3.5 and 6.3.7 and Corollary 6.3.8. In particular, Corollary 6.3.8 is the crucial part of the proof of the main result of this chapter, namely Theorem 6.4.4, which states that the set of all points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to (i,j)-uniform zero-dimensional subschemes of $\mathbb{P}_K(\overline{W})$ is constructible.

As in the previous section, we denote $Q = P[c_{11}, \dots, c_{\mu\nu}]$ and $\overline{Q} = \overline{P}[c_{11}, \dots, c_{\mu\nu}]$. Then $Q = \bigoplus_{\gamma \in \mathbb{Z}} Q_\gamma$ and $\overline{Q} = \bigoplus_{\gamma \in \mathbb{Z}} \overline{Q}_\gamma$ are non-negatively $\mathbb{Z}$-graded by the matrices $V = (W \mid 0) \in \mathrm{Mat}_{1,n+\mu\nu}(\mathbb{Z})$ and $\overline{V} = (\overline{W} \mid 0) \in \mathrm{Mat}_{1,n+1+\mu\nu}(\mathbb{Z})$, respectively. For all $i \in \{1, \dots, \mu\}$ and $j \in \{1, \dots, \nu\}$, let $\Delta_{ij} = \deg_W(b_j) - \deg_W(t_i) \in \mathbb{Z}$. Moreover, let $G = \{g_1, \dots, g_\nu\} \subseteq \overline{Q}$ be the generic projective $\mathcal{O}$-border prebasis, let $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ be the projective $\mathcal{O}$-border basis scheme with coordinate ring $B_{\mathcal{O}}^{\mathrm{proj}}$, and let $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ be the universal projective $\mathcal{O}$-border basis family as defined in Definition 6.1.5.

Before we start, we briefly recall Definition 4.3.1 and Proposition 4.3.2. For a $\mathbb{Z}$-graded ring R and $\mathbb{Z}$-graded R-modules M and N, we defined $\mathrm{Hom}_R(M,N)$ in Definition 4.3.1 to be the set of all homogeneous R-module homomorphisms $M \to N$ and we saw that $\underline{\mathrm{Hom}}_R(M,N) = \bigoplus_{\gamma \in \mathbb{Z}} (\underline{\mathrm{Hom}}_R(M,N))_\gamma = \bigoplus_{\gamma \in \mathbb{Z}} \mathrm{Hom}_R(M, N(\gamma))$ is a $\mathbb{Z}$-graded R-module.

Lemma 6.3.1. *Let $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ be a homogeneous ideal and let $S = U_{\mathcal{O}}^{\mathrm{proj}}/J$. Moreover, we let $M = \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(S, B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$.*

a) The map

$$U_{\mathcal{O}}^{\mathrm{proj}} \times S \to S, \quad (r, r' + J) \mapsto rr' + J \quad (r, r' \in U_{\mathcal{O}}^{\mathrm{proj}})$$

equips the $\mathbb{Z}$-graded ring S with the structure of a $\mathbb{Z}$-graded $U_{\mathcal{O}}^{\mathrm{proj}}$-algebra.

b) The map

$$U_{\mathcal{O}}^{\mathrm{proj}} \times M \to M, \quad (r, \varphi) \mapsto (r' \mapsto \varphi(r \cdot r'))$$

turns the $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module M into a $\mathbb{Z}$-graded $U_{\mathcal{O}}^{\mathrm{proj}}$-module. In particular, the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module structure of M as in Proposition 4.3.2 is compatible with this $U_{\mathcal{O}}^{\mathrm{proj}}$-structure.

Proof. Claim a) follows from [KR00, Rem. 1.7.9].

In order to prove b), let $r, r', r'' \in U_{\mathcal{O}}^{\mathrm{proj}}$ and $\varphi, \varphi' \in M$. Using a), we see that

$$(1 \cdot \varphi)(\overline{r}'') = \varphi(1 \cdot \overline{r}'') = \varphi(\overline{r}''),$$

i.e. $1 \cdot \varphi = \varphi$,

$$((rr') \cdot \varphi)(\overline{r}'') = \varphi((rr') \cdot \overline{r}'') = \varphi(r' \cdot (r \cdot \overline{r}'')) = (r' \cdot \varphi)(r \cdot \overline{r}'') = (r \cdot (r' \cdot \varphi))(\overline{r}'')$$

i.e. $(rr') \cdot \varphi = r \cdot (r' \cdot \varphi)$,

$$((r + r') \cdot \varphi)(\overline{r}'') = \varphi((r + r') \cdot \overline{r}'') = \varphi(r \cdot \overline{r}'') + \varphi(r' \cdot \overline{r}'') = (r \cdot \varphi)(\overline{r}'') + (r' \cdot \varphi)(\overline{r}'')$$

i.e. $(r + r') \cdot \varphi = r \cdot \varphi + r' \cdot \varphi$, and

$$\begin{aligned}(r \cdot (\varphi + \varphi'))(\overline{r}') &= (\varphi + \varphi')(r \cdot \overline{r}') \\ &= \varphi(r \cdot \overline{r}') + \varphi'(r \cdot \overline{r}') \\ &= (r \cdot \varphi)(\overline{r}') + (r \cdot \varphi')(\overline{r}') \\ &= (r \cdot \varphi + r \cdot \varphi')(\overline{r}'),\end{aligned}$$

i.e. $r \cdot (\varphi + \varphi') = r \cdot \varphi + r \cdot \varphi'$. Thus the given map turns M into a $U_{\mathcal{O}}^{\mathrm{proj}}$-module, which is obviously compatible with the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module structure of M given in Proposition 4.3.2. □

Next we define the generic canonical module of $U_{\mathcal{O}}^{\mathrm{proj}}/J$ where $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ is a homogeneous ideal as the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module of $U_{\mathcal{O}}^{\mathrm{proj}}$. Note that after applying the substition $c_{ij} \mapsto a_{ij}$ for some point $(a_{11}, \ldots, a_{\mu\nu}) \in B_{\mathcal{O}}^{\mathrm{proj}}$, this construction coincides with the one in Definition 4.3.4. In other words, $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ parametrizes the canonical modules of all the residue class rings $\overline{P}/I$ where I is a homogeneous ideal that possesses a projective $\mathcal{O}$-border basis.

Definition 6.3.2. Let $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ be a homogeneous ideal and $S = U_{\mathcal{O}}^{\mathrm{proj}}/J$. Then

$$\omega_S = \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(S, B_{\mathcal{O}}^{\mathrm{proj}}[x_0])(-1)$$

is called the **generic canonical module** of S. If $J = \{0\}$, we identify $U_{\mathcal{O}}^{\mathrm{proj}}$ with $U_{\mathcal{O}}^{\mathrm{proj}}/J$ and also write $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ instead of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}/J}$.

This definition immediately yields the following properties of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Proposition 6.3.3. *a) The element $x_0 \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ is a non-zero divisor for $U_{\mathcal{O}}^{\mathrm{proj}}$.*

b) The element $x_0 \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ is a non-zero divisor for $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

c) *Let $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ be a homogeneous ideal and let $S = U_{\mathcal{O}}^{\mathrm{proj}}/J$. Then there exists a canonical homogeneous $U_{\mathcal{O}}^{\mathrm{proj}}$-module isomorphism*

$$\omega_S \cong \{\varphi \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}} \mid J \cdot \varphi = \{0\}\}.$$

Proof. First we prove a). Assume that there exists a homomorphism $\varphi \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}} \setminus \{0\}$ such that $x_0\varphi = 0$. Then there is an element $r \in U_{\mathcal{O}}^{\mathrm{proj}}$ such that $\varphi(r) \neq 0$ and Lemma 6.3.1 yields $0 = 0(r) = (x_0\varphi)(r) = \varphi(x_0 \cdot r) = x_0 \cdot \varphi(r)$. Since $\varphi(r) \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0] \setminus \{0\}$, this is a contradiction to Corollary 6.1.14. Thus x_0 is a non-zero divisor for $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Next we prove claim b). Let $M = \{\varphi \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}} \mid J \cdot \varphi = \{0\}\}$. For all $\varphi_1, \varphi_2 \in M$, $r_1, r_2 \in U_{\mathcal{O}}^{\mathrm{proj}}$, and $s \in J$, we have

$$\begin{aligned}
(s \cdot (\varphi_1 - \varphi_2))(r_1) &= (\varphi_1 - \varphi_2)(sr_1) \\
&= \varphi_1(sr) - \varphi_2(sr_1) \\
&= (s \cdot \varphi_1)(r) - (s \cdot \varphi_2)(r_1) \\
&= 0(r_1) + 0(r_1) \\
&= 0,
\end{aligned}$$

i.e. $J \cdot (\varphi_1 - \varphi_2) = \{0\}$ and thus $\varphi_1 - \varphi_2 \in M$, and since $r_1 s \in J$, we have

$$(s \cdot (r_1 \cdot \varphi_1))(r_2) = (r_1 \cdot \varphi_1)(sr_2) = \varphi_1(r_1 s r_2) = ((r_1 s) \cdot \varphi_1)(r_2) = 0(r_2) = 0,$$

i.e. $J \cdot (r_1 \cdot \varphi_1) = \{0\}$ and thus $r_1 \cdot \varphi_1 \in M$. Altogether, it follows that the set M is a $U_{\mathcal{O}}^{\mathrm{proj}}$-submodule of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

For every $\varphi \in M$ and all $s \in J$, we have

$$\varphi(s) = \varphi(s \cdot 1) = (s \cdot \varphi)(1) = 0(1) = 0$$

and hence $J \subseteq \ker(\varphi)$. Let $\varepsilon : U_{\mathcal{O}}^{\mathrm{proj}} \twoheadrightarrow S = U_{\mathcal{O}}^{\mathrm{proj}}/J$ be the canonical $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module epimorphism. Then for every $\varphi \in M$, the Universal Property of the Residue Class Module S induces a unique $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module homomorphism $\overline{\varphi} : S \to B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$ which satisfies $\varphi = \overline{\varphi} \circ \varepsilon$ and $\overline{\varphi} \in \omega_S$. In other words, every element $\varphi \in M$ induces a unique element $\overline{\varphi} \in \omega_S$. Thus the map

$$\phi : M \to \omega_S, \quad \varphi \mapsto \overline{\varphi} \quad \text{such that} \quad \varphi = \overline{\varphi} \circ \varepsilon$$

is well-defined. For all $\varphi_1, \varphi_2 \in M$ and $r_1, r_2 \in U_{\mathcal{O}}^{\mathrm{proj}}$, we have

$$\begin{aligned}
\phi(r_1 \cdot (\varphi_1 - \varphi_2))(r_2 + J) &= \overline{r_1 \cdot (\varphi_1 - \varphi_2)}(r_2 + J) \\
&= (r_1 \cdot (\varphi_1 - \varphi_2))(r_2) \\
&= (\varphi_1 - \varphi_2)(r_1 r_2) \\
&= \varphi_1(r_1 r_2) - \varphi_2(r_1 r_2) \\
&= \overline{\varphi}_1(r_1 r_2 + J) - \overline{\varphi}_2(r_1 r_2 + J) \\
&= (\overline{\varphi}_1 - \overline{\varphi}_2)(r_1 r_2 + J) \\
&= (\phi(\varphi_1) - \phi(\varphi_2))(r_1 r_2 + J) \\
&= (r_1 \cdot (\phi(\varphi_1) - \phi(\varphi_2)))(r_2 + J),
\end{aligned}$$

i.e. $\phi(r_1 \cdot (\varphi_1 - \varphi_2)) = r_1 \cdot (\phi(\varphi_1) - \phi(\varphi_2))$. Altogether, it follows that the map ϕ is a $U_{\mathcal{O}}^{\mathrm{proj}}$-module homomorphism. Since for all $\gamma, \gamma' \in \mathbb{Z}$, $r \in (U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$, and $\varphi \in M_{\gamma'}$, we also have

$$\phi(\varphi)(r + J) = \overline{\varphi}(r + J) = \varphi(r) \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\gamma' - 1))_\gamma = (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-1))_{\gamma+\gamma'},$$

i.e. $\phi(\varphi) \in (\omega_S)_{\gamma'}$, the $U_{\mathcal{O}}^{\mathrm{proj}}$-module homomorphism ϕ is homogeneous.
Next we prove that ϕ is bijective. For every $\overline{\varphi} \in \omega_S$, the $U_{\mathcal{O}}^{\mathrm{proj}}$-module homomorphism $\varphi = \overline{\varphi} \circ \varepsilon \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ satisfies $(s \cdot \varphi)(1) = \varphi(s) = \overline{\varphi}(s + J) = \overline{\varphi}(0) = 0$ for every $s \in J$, i.e. $s \cdot \varphi = 0$ and thus $J \cdot \varphi = \{0\}$. In particular, for every $\overline{\varphi} \in \omega_S$, we have $\overline{\varphi} \circ \varepsilon \in M$ and $\phi(\overline{\varphi} \circ \varepsilon) = \overline{\varphi}$. Altogether, it follows that ϕ is surjective. For every $\varphi \in \ker(\phi)$, we have $\varphi = \phi(\varphi) \circ \varepsilon = 0 \circ \varepsilon = 0$, i.e. $\ker(\phi) = \{0\}$ and ϕ is hence injective. □

In the remaining parts of this thesis, for all homogeneous ideals $J \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$, we identify the elements of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}/J}$ with the corresponding elements in $\{\varphi \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}} \mid J \cdot \varphi = \{0\}\}$ as proven in Proposition 6.3.3 without mention.

Next we show that the residue classes of the elements of $\mathcal{O}$ yield $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$- respectively $B_{\mathcal{O}}^{\mathrm{proj}}$-module bases of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Proposition 6.3.4. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and all $i \in \{1, \ldots, \mu\}$, let $\overline{\mathcal{O}} = \{\bar{t}_1, \ldots, \bar{t}_\mu\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ denote the image of the order ideal $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$, and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*, \ldots, \bar{t}_\mu^*\} \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ be the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\overline{\mathcal{O}}$, i.e. we have*

$$\bar{t}_i^* : U_{\mathcal{O}}^{\mathrm{proj}} \to B_{\mathcal{O}}^{\mathrm{proj}}[x_0], \quad \bar{t}_j \mapsto \delta_{ij} \quad (j \in \{1, \ldots, \mu\})$$

for every $i \in \{1, \ldots, \mu\}$.

a) The set $\overline{\mathcal{O}}^*$ is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$. In particular, there are canonical isomorphisms of $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-modules

$$\omega_{U_{\mathcal{O}}^{\mathrm{proj}}} \cong \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\deg_W(t_i) - 1) \cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\gamma - 1))^{\Delta H(\gamma)}.$$

In particular, $\mathrm{rk}_{B_{\mathcal{O}}^{\mathrm{proj}}}((\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}) = \mu - H(\gamma) = \mu - \mathrm{rk}_{B_{\mathcal{O}}^{\mathrm{proj}}}((U_{\mathcal{O}}^{\mathrm{proj}})_\gamma)$ for all $\gamma \in \mathbb{Z}$.

b) Let $\overline{\mathcal{O}}_0$ denote the image of $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k \mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ and let $\overline{\mathcal{O}}_0^* \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ be the set of dual objects of the elements in $\overline{\mathcal{O}}_0$. For every integer $\gamma \in \mathbb{Z}$, the set $(\overline{\mathcal{O}}_0^*)_{-\gamma} = \{x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}_{H(\gamma)+1}^*, \ldots, x_0^{-d_{\gamma+1,\mu}} \bar{t}_\mu^*\} \subseteq (\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$ is a $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$.

Proof. First we prove a). Since $\overline{\mathcal{O}} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $U_{\mathcal{O}}^{\mathrm{proj}}$ according to Lemma 6.1.12, the dual basis $\overline{\mathcal{O}}^*$ is a $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ according to [Bou89, II.§2.6, Prop. 11 and Defn. 7]. By [Bou89, II.§1.6, Coro. 1 and II.§11.6, Rem.]), there is a canonical homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module isomorphism

$$\begin{aligned}&\underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}\left(\bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)), B_{\mathcal{O}}^{\mathrm{proj}}[x_0]\right)\\ &\cong \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)), B_{\mathcal{O}}^{\mathrm{proj}}[x_0]).\end{aligned}$$

Moreover, there exists a canonical homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module isomorphism

$$U_{\mathcal{O}}^{\mathrm{proj}} = \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0] \cdot \bar{t}_i \cong \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_{\overline{W}}(t_i))$$

by Corollary 6.1.14 and a canonical homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module isomorphism

$$\underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(B_{\mathcal{O}}^{\mathrm{proj}}[x_0], B_{\mathcal{O}}^{\mathrm{proj}}[x_0]) \cong B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$$

by [Bou89, II.§11.6, p. 376]. Altogether, we get the canonical isomorphism

$$\begin{aligned}\omega_{U_{\mathcal{O}}^{\mathrm{proj}}} &= \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(U_{\mathcal{O}}^{\mathrm{proj}}, B_{\mathcal{O}}^{\mathrm{proj}}[x_0])(-1)\\ &\cong \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}\left(\bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)), B_{\mathcal{O}}^{\mathrm{proj}}[x_0]\right)(-1)\\ &\cong \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)), B_{\mathcal{O}}^{\mathrm{proj}}[x_0])(-1)\\ &= \bigoplus_{i=1}^{\mu} \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(B_{\mathcal{O}}^{\mathrm{proj}}[x_0], B_{\mathcal{O}}^{\mathrm{proj}}[x_0])(\deg_W(t_i)-1)\\ &\cong \bigoplus_{i=1}^{\mu} B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\deg_W(t_i)-1)\\ &\cong \bigoplus_{\gamma=0}^{\deg_W(t_\mu)} (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\gamma-1))^{\Delta H(\gamma)}\end{aligned}$$

of $\mathbb{Z}$-graded $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-modules and the claim follows. In particular, for all $\gamma \in \mathbb{Z}$, we have

$$\begin{aligned}
(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma} &\cong \left(\bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\tilde{\gamma}-1))^{\Delta H(\tilde{\gamma})} \right)_{-\gamma} \\
&= \bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} ((B_{\mathcal{O}}^{\mathrm{proj}}[x_0](\tilde{\gamma}-1))_{-\gamma})^{\Delta H(\tilde{\gamma})} \\
&= \bigoplus_{\tilde{\gamma}=0}^{\deg_W(t_\mu)} (B_{\mathcal{O}}^{\mathrm{proj}}[x_0]_{\tilde{\gamma}-1-\gamma})^{\Delta H(\tilde{\gamma})}.
\end{aligned}$$

and as $\mathcal{O}$ is ordered increasingly with respect to $\deg_W$, we get

$$\mathrm{rk}_{B_{\mathcal{O}}^{\mathrm{proj}}}((\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}) = \bigoplus_{\tilde{\gamma}=\gamma+1}^{\deg_W(t_\mu)} \Delta H(\tilde{\gamma}) = H(\deg_W(t_\mu)) - H(\gamma) = \mu - H(\gamma).$$

As $\mathrm{rk}_{(U_{\mathcal{O}}^{\mathrm{proj}})_\gamma} = H(\gamma)$ for all $\gamma \in \mathbb{Z}$ by Corollary 6.1.14, the remaining equality follows.

In order to prove b), let $\gamma \in \mathbb{Z}$. For all $i \in \{1,\dots,\mu\}$, we have

$$\bar{t}_i^*(\bar{t}_i) = 1 \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]_0 = (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)))_{\deg_W(t_i)},$$

and $\bar{t}_i^*(\bar{t}_j) = 0 \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0](-\deg_W(t_i)))_{\deg_W(t_i)}$ for all indices $j \in \{1,\dots,\mu\}\setminus\{i\}$, i.e. we have $\bar{t}_i^* \in (\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\deg_W(t_i)+1}$. Moreover, for every index $i \in \{H(\gamma)+1,\dots,\mu\}$, we get $-d_{\gamma+1,i} = -\gamma-1+\deg_W(t_i) \in \mathbb{N}$ and thus $x_0^{-d_{\gamma+1,i}}\bar{t}_i^* \in (\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$. Since $\bar{t}_i^*(\bar{t}_j) = \delta_{ij}$ for all $i,j \in \{1,\dots,\mu\}$ and since $\mathrm{rk}_{B_{\mathcal{O}}^{\mathrm{proj}}}((\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}) = \mu - H(\gamma)$ according to a), it follows that $(\overline{\mathcal{O}}_0^*)_{-\gamma} = \{x_0^{-d_{\gamma+1,H(\gamma)+1}}\bar{t}_{H(\gamma)+1}^*,\dots,x_0^{-d_{\gamma+1,\mu}}\bar{t}_\mu^*\} \subseteq (\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$ is a K-vector space basis of $(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$. □

Next we explicitly describe the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module structure of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ by means of the generic projective multiplication matrices with respect to $\mathcal{O}$.

Proposition 6.3.5. *Let $\overline{\mathcal{O}} = \{\bar{t}_1,\dots,\bar{t}_\mu\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ denote the image of the order ideal $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*,\dots,\bar{t}_\mu^*\} \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ be the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, let $\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}} \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ be the images of the generic projective multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$, let $c_1,\dots,c_\mu \in B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$, and let $\varphi = c_1\bar{t}_1^* + \cdots + c_\mu\bar{t}_\mu^* \in \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$. Then for all $\gamma \in \mathbb{Z}$ and $q \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_\gamma$, we have*

$$\bar{q}\cdot\varphi = (\bar{t}_1^*,\dots,\bar{t}_\mu^*)\cdot q((\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}},\dots,(\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}})\cdot(c_1,\dots,c_\mu)^{\mathrm{tr}},$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$, i.e. the matrix $q((\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}},\dots,(\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module homomorphism $\pi_{\bar{q}}^* : \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}(-\gamma) \to \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$, $\varphi \mapsto \bar{q}\cdot\varphi$, the multiplication by $\bar{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_\gamma$, with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}^*$ of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Proof. Let $\gamma \in \mathbb{Z}$ and we let $q \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_\gamma$. According to Corollary 6.2.5, $q(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}}) \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module homomorphism $\pi_{\overline{q}} : U_{\mathcal{O}}^{\mathrm{proj}}(-\gamma) \mapsto U_{\mathcal{O}}^{\mathrm{proj}}$, $r \mapsto \overline{q}\cdot r$, the multiplication by $\overline{q}$, with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}$. The dual module of the free $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module $U_{\mathcal{O}}^{\mathrm{proj}}$ is $(U_{\mathcal{O}}^{\mathrm{proj}})^* = \underline{\mathrm{Hom}}_{B_{\mathcal{O}}^{\mathrm{proj}}[x_0]}(U_{\mathcal{O}}^{\mathrm{proj}}, B_{\mathcal{O}}^{\mathrm{proj}}[x_0]) = \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}(1)$. Thus the induced homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module homomorphism $\pi_{\overline{q}}^* : \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}(-\gamma) \to \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$, $\varphi \mapsto \varphi\circ\pi_{\overline{q}} = \overline{q}\cdot\varphi$ with respect to the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}^*$ of $\overline{\mathcal{O}}$ is represented by the matrix $q(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}} \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ according to [Bou89, II.§10.4, Prop. 3] and Corollary 6.2.5. Moreover, the matrices $\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}}$, and hence also the matrices $(\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}},\dots,(\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}}$, are pairwise commuting by Proposition 6.2.4 and the claim follows from $q(\overline{\mathcal{X}}_0^{\mathrm{proj}},\dots,\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}} = q((\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}},\dots,(\overline{\mathcal{X}}_n^{\mathrm{proj}})^{\mathrm{tr}})$. □

Example 6.3.6. Consider the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and the corresponding universal projective $\mathcal{O}$-border family $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ of Example 6.2.6, again. Recall that the polynomial rings $P = \mathbb{Q}[x_1,x_2]$ respectively $\overline{P} = \mathbb{Q}[x_0,x_1,x_2]$ were standard graded, i.e. graded by the gradings given by $W = (1,1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ respectively $\overline{W} = (1,1,1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$, and that $\mathcal{O} = \{t_1,\dots,t_8\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_1^2x_2, x_1^3\}$ had the maxdeg_W-border $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^3, x_1x_2^2, x_1^2x_2^2, x_1^3x_2, x_1^4\}$. Moreover, we saw that the images $\overline{\mathcal{X}}_0^{\mathrm{proj}}, \overline{\mathcal{X}}_1^{\mathrm{proj}}, \overline{\mathcal{X}}_2^{\mathrm{proj}} \in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ of the generic projective multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ are

$$\overline{\mathcal{X}}_0^{\mathrm{proj}} = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 \end{pmatrix},$$

$$\overline{\mathcal{X}}_1^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & \overline{c}_{12}x_0^3 & 0 & 0 & \overline{c}_{14}x_0^4 & \overline{c}_{15}x_0^4 \\ 0 & 0 & 0 & \overline{c}_{22}x_0^2 & 0 & 0 & \overline{c}_{24}x_0^3 & \overline{c}_{25}x_0^3 \\ 1 & 0 & 0 & \overline{c}_{32}x_0^2 & 0 & 0 & \overline{c}_{34}x_0^3 & \overline{c}_{35}x_0^3 \\ 0 & 0 & 0 & \overline{c}_{42}x_0 & 0 & 0 & \overline{c}_{44}x_0^2 & \overline{c}_{45}x_0^2 \\ 0 & 1 & 0 & \overline{c}_{52}x_0 & 0 & 0 & \overline{c}_{54}x_0^2 & \overline{c}_{55}x_0^2 \\ 0 & 0 & 1 & \overline{c}_{62}x_0 & 0 & 0 & \overline{c}_{64}x_0^2 & \overline{c}_{65}x_0^2 \\ 0 & 0 & 0 & \overline{c}_{72} & 1 & 0 & \overline{c}_{74}x_0 & \overline{c}_{75}x_0 \\ 0 & 0 & 0 & \overline{c}_{82} & 0 & 1 & \overline{c}_{84}x_0 & \overline{c}_{85}x_0 \end{pmatrix},$$

and

$$\overline{\mathcal{X}}_2^{\mathrm{proj}} = \begin{pmatrix} 0 & 0 & 0 & \bar{c}_{11}x_0^3 & \bar{c}_{12}x_0^3 & 0 & \bar{c}_{13}x_0^4 & \bar{c}_{14}x_0^4 \\ 1 & 0 & 0 & \bar{c}_{21}x_0^2 & \bar{c}_{22}x_0^2 & 0 & \bar{c}_{23}x_0^3 & \bar{c}_{24}x_0^3 \\ 0 & 0 & 0 & \bar{c}_{31}x_0^2 & \bar{c}_{32}x_0^2 & 0 & \bar{c}_{33}x_0^3 & \bar{c}_{34}x_0^3 \\ 0 & 1 & 0 & \bar{c}_{41}x_0 & \bar{c}_{42}x_0 & 0 & \bar{c}_{43}x_0^2 & \bar{c}_{44}x_0^2 \\ 0 & 0 & 1 & \bar{c}_{51}x_0 & \bar{c}_{52}x_0 & 0 & \bar{c}_{53}x_0^2 & \bar{c}_{54}x_0^2 \\ 0 & 0 & 0 & \bar{c}_{61}x_0 & \bar{c}_{62}x_0 & 0 & \bar{c}_{63}x_0^2 & \bar{c}_{64}x_0^2 \\ 0 & 0 & 0 & \bar{c}_{71} & \bar{c}_{72} & 1 & \bar{c}_{73}x_0 & \bar{c}_{74}x_0 \\ 0 & 0 & 0 & \bar{c}_{81} & \bar{c}_{82} & 0 & \bar{c}_{83}x_0 & \bar{c}_{84}x_0 \end{pmatrix}.$$

Let $q = x_1^2 + x_0x_2 \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_2$ and let $\pi_{\bar{q}}^* : \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}(-2) \to \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$, $\varphi \mapsto \bar{q}\cdot\varphi$ be the multiplication by $\bar{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_2$. Furthermore, let $\overline{\mathcal{O}} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ be the image of $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$ and let $\overline{\mathcal{O}}^* \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ be the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\mathcal{O}$. Then the matrix

$$\begin{aligned}
&q((\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}}, (\overline{\mathcal{X}}_1^{\mathrm{proj}})^{\mathrm{tr}}, (\overline{\mathcal{X}}_2^{\mathrm{proj}})^{\mathrm{tr}}) \\
&= ((\overline{\mathcal{X}}_1^{\mathrm{proj}})^{\mathrm{tr}})^2 + (\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}} \cdot (\overline{\mathcal{X}}_2^{\mathrm{proj}})^{\mathrm{tr}} \\
&= \begin{pmatrix} 0 & x_0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & 0 & 1 \\ m_{14} & m_{24} & m_{34} + \bar{c}_{12}x_0^3 & m_{44} & m_{54} + \bar{c}_{22}x_0^2 & m_{64} + \bar{c}_{32}x_0^2 & m_{74} + \bar{c}_{52}x_0 & m_{84} + \bar{c}_{62}x_0 \\ m_{15} & m_{25} & m_{35} & m_{45} & m_{55} & m_{65} & m_{75} & m_{85} \\ m_{16} & m_{26} & m_{36} & m_{46} & m_{56} & m_{66} & m_{76} + x_0 & m_{86} \\ m_{17} & m_{27} & m_{37} + \bar{c}_{14}x_0^4 & m_{47} & m_{57} + \bar{c}_{24}x_0^3 & m_{67} + \bar{c}_{34}x_0^3 & m_{77} + \bar{c}_{54}x_0^2 & m_{87} + \bar{c}_{64}x_0^2 \\ m_{18} & m_{28} & m_{38} + \bar{c}_{15}x_0^4 & m_{48} & m_{58} + \bar{c}_{25}x_0^3 & m_{68} + \bar{c}_{35}x_0^3 & m_{78} + \bar{c}_{55}x_0^2 & m_{88} + \bar{c}_{65}x_0^2 \end{pmatrix} \\
&\in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])
\end{aligned}$$

where we have

$$\begin{aligned}
m_{i4} &= (\bar{c}_{i2}\bar{c}_{42} + \bar{c}_{i4}\bar{c}_{72} + \bar{c}_{i5}\bar{c}_{82} + \bar{c}_{i1})x_0^{4-\deg(t_i)}, \\
m_{i5} &= (\bar{c}_{i2} + \bar{c}_{i4})x_0^{4-\deg(t_i)}, \\
m_{i6} &= \bar{c}_{i5}x_0^{4-\deg(t_i)}, \\
m_{i7} &= (\bar{c}_{i2}\bar{c}_{44} + \bar{c}_{i4}\bar{c}_{74} + \bar{c}_{i5}\bar{c}_{84} + \bar{c}_{i3})x_0^{5-\deg(t_i)}, \\
m_{i8} &= (\bar{c}_{i2}\bar{c}_{45} + \bar{c}_{i4}\bar{c}_{75} + \bar{c}_{i5}\bar{c}_{85} + \bar{c}_{i4})x_0^{5-\deg(t_i)}
\end{aligned}$$

for all indices $i \in \{1,\dots,8\}$ represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module homomorphism $\pi_{\bar{q}}^*$ with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis $\overline{\mathcal{O}}^*$ of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$. Note that the matrices $\overline{\mathcal{X}}_0^{\mathrm{proj}}, \overline{\mathcal{X}}_1^{\mathrm{proj}}, \overline{\mathcal{X}}_2^{\mathrm{proj}} \in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}}[x_0])$ are pairwise commuting and, therefore, we see that $q((\overline{\mathcal{X}}_0^{\mathrm{proj}})^{\mathrm{tr}}, (\overline{\mathcal{X}}_1^{\mathrm{proj}})^{\mathrm{tr}}, (\overline{\mathcal{X}}_2^{\mathrm{proj}})^{\mathrm{tr}}) = (q(\overline{\mathcal{X}}_0^{\mathrm{proj}}, \overline{\mathcal{X}}_1^{\mathrm{proj}}, \overline{\mathcal{X}}_2^{\mathrm{proj}})^{\mathrm{tr}}$. In particular, the above result coincides with the result of Example 6.2.6.

Finally, we explicitly describe the $B_{\mathcal{O}}^{\mathrm{proj}}$-module structure of $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ by means of the generic multiplication matrices with respect to $\mathcal{O}$. In particular, Corollary 6.3.8 will turn out to be the crucial part of the proof of the main result of this chapter, namely Theorem 6.4.4, which states that the set of all points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to (i,j)-uniform zero-dimensional subschemes of $\mathbb{P}_K(\overline{W})$ is constructible with respect to the Zariski topology.

Proposition 6.3.7. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and $i \in \{1,\dots,\mu\}$, let $\overline{\mathcal{O}} = \{\bar{t}_1,\dots,\bar{t}_\mu\} \subseteq U_{\mathcal{O}}^{\mathrm{proj}}$ denote the image of the order ideal $\mathcal{O}$ in $U_{\mathcal{O}}^{\mathrm{proj}}$, and let $\overline{\mathcal{O}}^* = \{\bar{t}_1^*,\dots,\bar{t}_\mu^*\} \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ be the dual $B_{\mathcal{O}}^{\mathrm{proj}}[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, we let $\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ denote the images of the generic multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$, let $\gamma \in \mathbb{Z}$, let $c_{H(\gamma)+1},\dots,c_\mu \in B_{\mathcal{O}}^{\mathrm{proj}}$, and let $\varphi = c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}_{H(\gamma)+1}^* + \dots + c_\mu x_0^{-d_{\gamma+1,\mu}} \bar{t}_\mu^* \in (\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma}$.*

a) We have

$$x_0 \cdot \varphi = (0,\dots,0, x_0^{-d_{\gamma,H(\gamma-1)+1}} \bar{t}_{H(\gamma-1)+1}^*, \dots, x_0^{-d_{\gamma,\mu}} \bar{t}_\mu^*) \cdot \mathcal{I}_\mu \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ and for every $k \in \{1,\dots,n\}$, we have

$$\bar{x}_k \cdot \varphi = (0,\dots,0, x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \bar{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \bar{t}_\mu^*) \cdot \overline{\mathcal{X}}_k^{\mathrm{tr}} \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

b) For every $\gamma' \in \mathbb{Z}$ and every $q' \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_{\gamma'}$, we have

$$\bar{q}' \cdot \varphi = (0,\dots,0, \bar{x}_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}} \bar{t}_{H(\gamma-\gamma')+1}^*, \dots, \bar{x}_0^{-d_{\gamma-\gamma'+1,\mu}} \bar{t}_\mu^*) \cdot q'(\mathcal{I}_\mu, \overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Proof. For every $s \in \{1,\dots,n\}$, we denote $\overline{\mathcal{X}}_s = (\bar{\xi}_{k\ell}^{(s)})_{1\le k,\ell\le\mu} \in \mathrm{Mat}_\mu(B_{\mathcal{O}}^{\mathrm{proj}})$ as in Definitions 6.1.1 and 6.1.5 and we let $\{\mathcal{E}_1,\dots,\mathcal{E}_\mu\}$ denote the canonical $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis of $(B_{\mathcal{O}}^{\mathrm{proj}})^\mu$. If $\gamma \ge \deg_W(t_\mu)$, we have $(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-\gamma} = \{0\}$ and the claims follow trivially. Thus we suppose that $\gamma < \deg_W(t_\mu)$.

First we prove a). For every $i \in \{1,\dots,\mu\}$, we have

$$-d_{\gamma+1,i} + 1 = -(\gamma + 1 - \deg_W(t_i)) + 1 = -(\gamma - \deg_W(t_i)) = -d_{\gamma,i}$$

and get

$$\begin{aligned}
x_0 \cdot \varphi &= x_0 \cdot (c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}} \bar{t}^*_{H(\gamma)+1} + \cdots + c_\mu x_0^{-d_{\gamma+1,\mu}} \bar{t}^*_\mu) \\
&= c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}+1} \bar{t}^*_{H(\gamma)+1} + \cdots + c_\mu x_0^{-d_{\gamma+1,\mu}+1} \bar{t}^*_\mu \\
&= c_{H(\gamma)+1} x_0^{-d_{\gamma,H(\gamma)+1}} \bar{t}^*_{H(\gamma)+1} + \cdots + c_\mu x_0^{-d_{\gamma,\mu}} \bar{t}^*_\mu \\
&= (0, \dots, 0, x_0^{-d_{\gamma,H(\gamma-1)+1}} \bar{t}^*_{H(\gamma-1)+1}, \dots, x_0^{d_{\gamma,\mu}} \bar{t}^*_\mu) \\
&\quad \cdot \mathcal{I}_\mu \cdot (0, \dots, 0, c_{H(\gamma)+1}, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$. For the second part, let $k \in \{1, \dots, n\}$. Moreover, let $i \in \{H(\gamma)+1, \dots, \mu\}$, $\ell \in \{1, \dots, \mu\}$, and $\tilde{\gamma} = \deg_W(t_\ell) + w_k$. Then Proposition 6.2.7 yields

$$\begin{aligned}
(\bar{x}_k \cdot \bar{t}^*_i)(\bar{t}_\ell) &= \bar{t}^*_i(\bar{x}_k \bar{t}_\ell) \\
&= \bar{t}^*_i((x_0^{d_{\tilde{\gamma},1}} \bar{t}_1, \dots, x_0^{d_{\tilde{\gamma},H(\tilde{\gamma})}} \bar{t}_{H(\tilde{\gamma})}, 0, \dots, 0) \cdot \overline{\mathcal{X}}_k \cdot \mathcal{E}^{\mathrm{tr}}_\ell) \\
&= \begin{cases} 0 & \text{if } \deg_W(t_i) > \tilde{\gamma} \\ \bar{\xi}^{(k)}_{i\ell} x_0^{d_{\tilde{\gamma},i}} & \text{if } \deg_W(t_i) \le \tilde{\gamma} \end{cases} \\
&= \begin{cases} 0 & \text{if } \deg_W(t_\ell) < \deg_W(t_i) - w_k \\ \bar{\xi}^{(k)}_{i\ell} x_0^{d_{\tilde{\gamma},i}} & \text{if } \deg_W(t_\ell) \ge \deg_W(t_i) - w_k \end{cases}.
\end{aligned}$$

Assume that there is an $\tilde{\ell} \in \{1, \dots, \mu\}$ such that $\gamma + 1 - w_k \le \deg_W(t_{\tilde{\ell}}) < \deg_W(t_i) - w_k$ and such that $\bar{\xi}^{(k)}_{i\tilde{\ell}} \neq 0$. Then we have $\gamma + 1 \le \deg_W(t_{\tilde{\ell}}) + w_k < \deg_W(t_i)$. We distinguish two cases. If $x_k t_{\tilde{\ell}} = t_r \in \mathcal{O}$ for some $r \in \{1, \dots, \mu\}$, we get $\deg_W(t_{\tilde{\ell}}) + w_k = \deg_W(t_r)$. In particular, we see that $r \neq i$ and thus $\bar{\xi}^{(k)}_{i\tilde{\ell}} = \delta_{ir} = 0$ according to Definition 6.1.1 in this situation. If we have $x_k t_{\tilde{\ell}} = b_s \in \partial\mathcal{O}$ for some index $s \in \{1, \dots, \nu\}$, then $\deg_W(t_i) > \deg_W(t_{\tilde{\ell}}) + w_k = \deg_W(b_s)$. Since G is the generic projective $\mathcal{O}$-border basis, Definition 6.1.5 implies that $g_s = b_s - \sum_{m=1}^{H(\deg_W(b_s))} c_{ms} x_0^{\Delta_{ms}} t_m$ and thus we have $\bar{\xi}^{(k)}_{i\tilde{\ell}} = 0$ by Definition 6.1.1 in this situation. Since both cases yield a contradiction, there is no such index $\tilde{\ell}$. In particular, this implies that

$$(\bar{x}_k \cdot \bar{t}^*_i)(\bar{t}_\ell) = \begin{cases} 0 & \text{if } \deg_W(t_\ell) < \gamma - w_k + 1 \\ \xi^{(k)}_{i\ell} x_0^{d_{\tilde{\gamma},i}} & \text{if } \deg_W(t_\ell) \ge \gamma - w_k + 1 \end{cases}.$$

Moreover, we have

$$\begin{aligned}
-d_{\gamma+1,i} + d_{\tilde{\gamma},i} &= -(\gamma + 1 - \deg_W(t_i)) + \tilde{\gamma} - \deg_W(t_i) \\
&= -\gamma - 1 + \deg_W(t_\ell) + w_k \\
&= -d_{\gamma - w_k + 1, \ell}.
\end{aligned}$$

Altogether, we get

$$\begin{aligned}
&\overline{x}_k \cdot c_i x_0^{-d_{\gamma+1,i}} \overline{t}_i^* \\
&= \sum_{\ell=H(\gamma-w_k)+1}^{\mu} \overline{\xi}_{i\ell}^{(k)} c_i x_0^{-d_{\gamma-w_k+1,\ell}} \overline{t}_\ell^* \\
&= (0,\dots,0, x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*) \cdot \overline{\mathcal{X}}_k^{\mathrm{tr}} \cdot c_i \mathcal{E}_i^{\mathrm{tr}}
\end{aligned}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ and thus

$$\begin{aligned}
\overline{x}_k \cdot \varphi &= \sum_{i=H(\gamma)+1}^{\mu} \overline{x}_k \cdot c_i x_0^{-d_{\gamma+1,i}} \overline{t}_i^* \\
&= \sum_{i=H(\gamma)+1}^{\mu} (0,\dots,0, x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*) \cdot \overline{\mathcal{X}}_k^{\mathrm{tr}} \cdot c_i \mathcal{E}_i^{\mathrm{tr}} \\
&= (0,\dots,0, x_0^{-d_{\gamma-w_k+1,H(\gamma-w_k)+1}} \overline{t}_{H(\gamma-w_k)+1}^*, \dots, x_0^{-d_{\gamma-w_k+1,\mu}} \overline{t}_\mu^*) \\
&\quad \cdot \overline{\mathcal{X}}_k^{\mathrm{tr}} \cdot (0,\dots,0, c_{H(\gamma)+1}, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$.

Next we prove b). Let $\gamma' \in \mathbb{Z}$ and $q' \in (B_{\mathcal{O}}^{\mathrm{proj}}[x_0,\dots,x_n])_{\gamma'}$. For all $\gamma' \in \mathbb{Z}$ with $\gamma' < 0$, we have $q' = 0$ and b) holds trivially. We prove by induction on $\gamma' \in \mathbb{N}$ that b) holds in general. For $\gamma' = 0$, it follows $q' \in B_{\mathcal{O}}^{\mathrm{proj}}$ and thus

$$\begin{aligned}
\overline{q}' \cdot \varphi &= q' c_{H(\gamma)+1} x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^* + \dots + q' c_\mu x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^* \\
&= (0,\dots,0, x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^*, \dots, x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^*) \\
&\quad \cdot q' \mathcal{I}_\mu \cdot (0,\dots,0, c_{H(\gamma)+1}, \dots, c_\mu)^{\mathrm{tr}} \\
&= (0,\dots,0, x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}_{H(\gamma)+1}^*, \dots, x_0^{-d_{\gamma+1,\mu}} \overline{t}_\mu^*) \\
&\quad \cdot q' (\mathcal{I}_\mu, \overline{\mathcal{X}}_1^{\mathrm{tr}}, \dots, \overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0, c_{H(\gamma)+1}, \dots, c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $\omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$. For the induction step, let now $\gamma' > 0$. Then there exist a natural number $s \in \mathbb{N} \setminus \{0\}$, $c_1',\dots,c_s' \in B_{\mathcal{O}}^{\mathrm{proj}}$, and $t_1',\dots,t_s' \in \mathbb{T}_{\gamma'}^{n+1}$ such that $q' = c_1't_1' + \dots + c_s't_s'$. Then for every $r \in \{1,\dots,s\}$, there are a $k_r \in \{0,\dots,n\}$ and a $u_r \in \mathbb{T}_{<\gamma'}^{n+1}$ such that $t_r' = x_{k_r} u_r$. Let $w_0 = 1$, $\overline{\mathcal{X}}_0 = \mathcal{I}_\mu$, and for all $r \in \{1,\dots,s\}$, let $\hat{\gamma}_r = \gamma - \gamma' + w_{k_r}$. Then the induction hypothesis, the commutativity of the matrices $\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n$, which follows from Definition 6.1.1, together with a) yield

$$\begin{aligned}
c_r'\overline{t}_r' \cdot \varphi &= \overline{x}_{k_r} \cdot (c_r'\overline{u}_r \cdot \varphi) \\
&= \overline{x}_{k_r} \cdot ((0,\dots,0,x_0^{-d_{\hat{\gamma}_r+1,H(\hat{\gamma}_r)+1}}\overline{t}^*_{H(\hat{\gamma}_r)+1},\dots,x_0^{-d_{\hat{\gamma}_r+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c'u_r)(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\hat{\gamma}_r-w_{k_r}+1,H(\hat{\gamma}_r-w_{k_r})+1}}\overline{t}^*_{H(\hat{\gamma}_r-w_{k_r})+1},\dots,x_0^{-d_{\hat{\gamma}_r-w_{k_r}+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (\overline{\mathcal{X}}_{k_r}^{\mathrm{tr}} \cdot (c_r'u_r)(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}})) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c_r'x_{k_r}u_r)(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot (c_r't_r')(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}})
\end{aligned}$$

in $\omega_{U_\mathcal{O}^{\mathrm{proj}}}$ for all $r \in \{1,\dots,s\}$. Thus it follows that

$$\begin{aligned}
\overline{q}' \cdot \varphi &= \sum_{r=1}^{s} c_r'\overline{t}_r' \cdot \varphi \\
&= \sum_{r=1}^{s}((0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\qquad \cdot (c_r't_r')(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}) \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot \left(\sum_{r=1}^{s} c_r'\overline{t}_r'\right)(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}} \\
&= (0,\dots,0,x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}}\overline{t}^*_{H(\gamma-\gamma')+1},\dots,x_0^{-d_{\gamma-\gamma'+1,\mu}}\overline{t}^*_\mu) \\
&\quad \cdot q'(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \cdot (0,\dots,0,c_{H(\gamma)+1},\dots,c_\mu)^{\mathrm{tr}}
\end{aligned}$$

in $\omega_{U_\mathcal{O}^{\mathrm{proj}}}$ and the claim follows for $\gamma' > 0$. □

Corollary 6.3.8. *Let $d_{\gamma,i} = \gamma - \deg_W(t_i) \in \mathbb{Z}$ be for all $\gamma \in \mathbb{Z}$ and all $i \in \{1,\dots,\mu\}$, let $\overline{\mathcal{O}} = \{\overline{t}_1,\dots,\overline{t}_\mu\} \subseteq U_\mathcal{O}^{\mathrm{proj}}$ denote the image of the order ideal $\mathcal{O}$ in $U_\mathcal{O}^{\mathrm{proj}}$, and let $\overline{\mathcal{O}}^* = \{\overline{t}_1^*,\dots,\overline{t}_\mu^*\} \subseteq \omega_{U_\mathcal{O}^{\mathrm{proj}}}$ be the dual $B_\mathcal{O}^{\mathrm{proj}}[x_0]$-module basis of $\overline{\mathcal{O}}$. Moreover, we let $\overline{\mathcal{O}}_0$ be the image of $\mathcal{O}_0 = \bigcup_{k=0}^{\infty} x_0^k\mathcal{O}$ in $U_\mathcal{O}^{\mathrm{proj}}$, we let $\overline{\mathcal{O}}_0^* \subseteq \omega_{U_\mathcal{O}^{\mathrm{proj}}}$ be the dual objects of the elements in $\overline{\mathcal{O}}_0$, and we let $\overline{\mathcal{X}}_1,\dots,\overline{\mathcal{X}}_n \in \mathrm{Mat}_\mu(B_\mathcal{O}^{\mathrm{proj}})$ denote the images of the generic multiplication matrices with respect to $\mathcal{O}$ in $\mathrm{Mat}_\mu(B_\mathcal{O}^{\mathrm{proj}})$. Furthermore, we let $\gamma,\gamma' \in \mathbb{Z}$ and $q \in (B_\mathcal{O}^{\mathrm{proj}}[x_0,\dots,x_n])_{\gamma'}$. Then the submatrix obtained from deleting the first $H(\gamma-\gamma')$ rows and the first $H(\gamma)$ columns of $q(\mathcal{I}_\mu,\overline{\mathcal{X}}_1^{\mathrm{tr}},\dots,\overline{\mathcal{X}}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(B_\mathcal{O}^{\mathrm{proj}})$*

represents the $B^{\mathrm{proj}}_{\mathcal{O}}$-module homomorphism $\widehat{\pi}^*_{\overline{q}} : (\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma} \to (\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma+\gamma'}$, $\varphi \mapsto \overline{q} \cdot \varphi$, i.e. the multiplication by the element $\overline{q} \in (U^{\mathrm{proj}}_{\mathcal{O}})_{\gamma'}$, with respect to the $B^{\mathrm{proj}}_{\mathcal{O}}$-module bases $(\overline{\mathcal{O}}^*_0)_{-\gamma}$ of $(\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma}$ and $(\overline{\mathcal{O}}^*_0)_{-\gamma+\gamma'}$ of $(\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma+\gamma'}$.

Proof. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_\mu\}$ denote the canonical $B^{\mathrm{proj}}_{\mathcal{O}}$-module basis of $(B^{\mathrm{proj}}_{\mathcal{O}})^\mu$. According to Proposition 6.3.4, the sets $(\overline{\mathcal{O}}^*_0)_{-\gamma} = \{x_0^{-d_{\gamma+1,H(\gamma)+1}} \overline{t}^*_{H(\gamma)+1}, \ldots, x_0^{-d_{\gamma+1,\mu}} \overline{t}^*_\mu\}$ and $(\overline{\mathcal{O}}^*_0)_{-\gamma+\gamma'} = \{x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}} \overline{t}^*_{H(\gamma-\gamma')+1}, \ldots, x_0^{-d_{\gamma-\gamma'+1,\mu}} \overline{t}^*_\mu\}$ are $B^{\mathrm{proj}}_{\mathcal{O}}$-module bases of $(\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma}$ and $(\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-\gamma+\gamma'}$, respectively. For all indices $k \in \{H(\gamma)+1, \ldots, \mu\}$, Proposition 6.3.7 yields

$$\begin{aligned}
&\widehat{\pi}^*_{\overline{q}}(x_0^{-d_{\gamma+1,k}} \overline{t}^*_k) \\
&= \overline{q} \cdot x_0^{-d_{\gamma+1,k}} \overline{t}^*_k \\
&= (0, \ldots, 0, x_0^{-d_{\gamma-\gamma'+1,H(\gamma-\gamma')+1}} \overline{t}^*_{H(\gamma-\gamma')+1}, \ldots, x_0^{-d_{\gamma-\gamma'+1,\mu}} \overline{t}^*_\mu) \cdot q(\mathcal{I}_\mu, \overline{\mathcal{X}}^{\mathrm{tr}}_1, \ldots, \overline{\mathcal{X}}^{\mathrm{tr}}_n) \cdot \mathcal{E}^{\mathrm{tr}}_k
\end{aligned}$$

and the claim is a direct consequence of this. □

Example 6.3.9. Consider the projective $\mathcal{O}$-border basis scheme $\mathbb{B}^{\mathrm{proj}}_{\mathcal{O}}$ and its universal projective $\mathcal{O}$-border basis family $B^{\mathrm{proj}}_{\mathcal{O}} \to U^{\mathrm{proj}}_{\mathcal{O}}$ of Example 6.2.9, again. Recall that the polynomial rings $P = \mathbb{Q}[x_1, x_2]$ respectively $\overline{P} = \mathbb{Q}[x_0, x_1, x_2]$ were standard graded, i.e. graded by $W = (1,1) \in \mathrm{Mat}_{1,2}(\mathbb{Z})$ respectively $\overline{W} = (1,1,1) \in \mathrm{Mat}_{1,3}(\mathbb{Z})$, that the order ideal was of the form $\mathcal{O} = \{t_1, \ldots, t_8\} = \{1, x_2, x_1, x_2^2, x_1x_2, x_1^2, x_1^2x_2, x_1^3\}$, and that $\mathcal{O}$ had the maxdeg_W-border $\partial\mathcal{O} = \{b_1, \ldots, b_5\} = \{x_2^3, x_1x_2^2, x_1^2x_2^2, x_1^3x_2, x_1^4\}$. We also saw that the images $\overline{\mathcal{X}}_1, \overline{\mathcal{X}}_2 \in \mathrm{Mat}_8(B^{\mathrm{proj}}_{\mathcal{O}})$ of the generic multiplication matrices $\mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Mat}_8(\mathbb{Q}[c_{11}, \ldots, c_{85}])$ with respect to $\mathcal{O}$ in $\mathrm{Mat}_8(B^{\mathrm{proj}}_{\mathcal{O}})$ were

$$\overline{\mathcal{X}}_1 = \begin{pmatrix}
0 & 0 & 0 & \overline{c}_{12} & 0 & 0 & \overline{c}_{14} & \overline{c}_{15} \\
0 & 0 & 0 & \overline{c}_{22} & 0 & 0 & \overline{c}_{24} & \overline{c}_{25} \\
1 & 0 & 0 & \overline{c}_{32} & 0 & 0 & \overline{c}_{34} & \overline{c}_{35} \\
0 & 0 & 0 & \overline{c}_{42} & 0 & 0 & \overline{c}_{44} & \overline{c}_{45} \\
0 & 1 & 0 & \overline{c}_{52} & 0 & 0 & \overline{c}_{54} & \overline{c}_{55} \\
0 & 0 & 1 & \overline{c}_{62} & 0 & 0 & \overline{c}_{64} & \overline{c}_{65} \\
0 & 0 & 0 & \overline{c}_{72} & 1 & 0 & \overline{c}_{74} & \overline{c}_{75} \\
0 & 0 & 0 & \overline{c}_{82} & 0 & 1 & \overline{c}_{84} & \overline{c}_{85}
\end{pmatrix} \quad
\overline{\mathcal{X}}_2 = \begin{pmatrix}
0 & 0 & 0 & \overline{c}_{11} & \overline{c}_{12} & 0 & \overline{c}_{13} & \overline{c}_{14} \\
1 & 0 & 0 & \overline{c}_{21} & \overline{c}_{22} & 0 & \overline{c}_{23} & \overline{c}_{24} \\
0 & 0 & 0 & \overline{c}_{31} & \overline{c}_{32} & 0 & \overline{c}_{33} & \overline{c}_{34} \\
0 & 1 & 0 & \overline{c}_{41} & \overline{c}_{42} & 0 & \overline{c}_{43} & \overline{c}_{44} \\
0 & 0 & 1 & \overline{c}_{51} & \overline{c}_{52} & 0 & \overline{c}_{53} & \overline{c}_{54} \\
0 & 0 & 0 & \overline{c}_{61} & \overline{c}_{62} & 0 & \overline{c}_{63} & \overline{c}_{64} \\
0 & 0 & 0 & \overline{c}_{71} & \overline{c}_{72} & 1 & \overline{c}_{73} & \overline{c}_{74} \\
0 & 0 & 0 & \overline{c}_{81} & \overline{c}_{82} & 0 & \overline{c}_{83} & \overline{c}_{84}
\end{pmatrix}.$$

Moreover, $q = x_1^2 + x_0x_2 \in (B^{\mathrm{proj}}_{\mathcal{O}}[x_0, x_1, x_2])_2$ and we denoted the map $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1, \ldots, 8\} \mid \deg(t_k) \le \gamma\}$. Let $\widehat{\pi}^*_{\overline{q}} : (\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_{-2} \to (\omega_{U^{\mathrm{proj}}_{\mathcal{O}}})_0$, $\varphi \mapsto \overline{q} \cdot \varphi$ be the multiplication by $\overline{q} \in (U_{\mathcal{O}}^{\mathrm{proj}})_2$. Then

$$\begin{aligned}
&q(\mathcal{I}_8, \overline{\mathcal{X}}_1^{\mathrm{tr}}, \overline{\mathcal{X}}_2^{\mathrm{tr}}) \\
&= (\overline{\mathcal{X}}_1^{\mathrm{tr}})^2 + \mathcal{I}_8 \cdot \overline{\mathcal{X}}_2^{\mathrm{tr}} \\
&= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\widehat{m}_{14} & \widehat{m}_{24} & \widehat{m}_{34} + \overline{c}_{12} & \widehat{m}_{44} & \widehat{m}_{54} + \overline{c}_{22} & \widehat{m}_{64} + \overline{c}_{32} & \widehat{m}_{74} + \overline{c}_{52} & \widehat{m}_{84} + \overline{c}_{62} \\
\widehat{m}_{15} & \widehat{m}_{25} & \widehat{m}_{35} & \widehat{m}_{45} & \widehat{m}_{55} & \widehat{m}_{65} & \widehat{m}_{75} & \widehat{m}_{85} \\
\widehat{m}_{16} & \widehat{m}_{26} & \widehat{m}_{36} & \widehat{m}_{46} & \widehat{m}_{56} & \widehat{m}_{66} & \widehat{m}_{76} + 1 & \widehat{m}_{86} \\
\widehat{m}_{17} & \widehat{m}_{27} & \widehat{m}_{37} + \overline{c}_{14} & \widehat{m}_{47} & \widehat{m}_{57} + \overline{c}_{24} & \widehat{m}_{67} + \overline{c}_{34} & \widehat{m}_{77} + \overline{c}_{54} & \widehat{m}_{87} + \overline{c}_{64} \\
\widehat{m}_{18} & \widehat{m}_{28} & \widehat{m}_{38} + \overline{c}_{15} & \widehat{m}_{48} & \widehat{m}_{58} + \overline{c}_{25} & \widehat{m}_{68} + \overline{c}_{35} & \widehat{m}_{78} + \overline{c}_{55} & \widehat{m}_{88} + \overline{c}_{65}
\end{pmatrix} \\
&\in \mathrm{Mat}_8(B_{\mathcal{O}}^{\mathrm{proj}})
\end{aligned}$$

where we have

$$\begin{aligned}
\widehat{m}_{i4} &= \overline{c}_{i2}\overline{c}_{42} + \overline{c}_{i4}\overline{c}_{72} + \overline{c}_{i5}\overline{c}_{82} + \overline{c}_{i1}, \\
\widehat{m}_{i5} &= \overline{c}_{i2} + \overline{c}_{i4}, \\
\widehat{m}_{i6} &= \overline{c}_{i5}, \\
\widehat{m}_{i7} &= \overline{c}_{i2}\overline{c}_{44} + \overline{c}_{i4}\overline{c}_{74} + \overline{c}_{i5}\overline{c}_{84} + \overline{c}_{i3}, \\
\widehat{m}_{i8} &= \overline{c}_{i2}\overline{c}_{45} + \overline{c}_{i4}\overline{c}_{75} + \overline{c}_{i5}\overline{c}_{85} + \overline{c}_{i4}
\end{aligned}$$

for all $i \in \{1, \dots, 8\}$. By Corollary 6.3.8, the submatrix

$$\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\widehat{m}_{74} + \overline{c}_{52} & \widehat{m}_{84} + \overline{c}_{62} \\
\widehat{m}_{75} & \widehat{m}_{85} \\
\widehat{m}_{76} + 1 & \widehat{m}_{86} \\
\widehat{m}_{77} + \overline{c}_{54} & \widehat{m}_{87} + \overline{c}_{64} \\
\widehat{m}_{78} + \overline{c}_{55} & \widehat{m}_{88} + \overline{c}_{65}
\end{pmatrix} \in \mathrm{Mat}_{7,2}(B_{\mathcal{O}}^{\mathrm{proj}})$$

of the matrix $q(\mathcal{I}_8, \overline{\mathcal{X}}_1^{\mathrm{tr}}, \overline{\mathcal{X}}_2^{\mathrm{tr}})$ obtained by deleting the first $H(0) = 1$ rows and the first $H(2) = 6$ columns represents the homogeneous $B_{\mathcal{O}}^{\mathrm{proj}}$-module homomorphism $\widehat{\pi}_{\overline{q}}^*$ with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis $\{\overline{x_1^2 x_2}^*, \overline{x_1^3}^*\} \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ of $(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_{-2}$ and with respect to the $B_{\mathcal{O}}^{\mathrm{proj}}$-module basis $\{\overline{x_2}^*, \overline{x_1}^*, x_0\overline{x_2^2}^*, x_0\overline{x_1 x_2}^*, x_0\overline{x_1^2}^*, x_0^2\overline{x_1^2 x_2}^*, x_0^2\overline{x_1^3}^*\} \subseteq \omega_{U_{\mathcal{O}}^{\mathrm{proj}}}$ of $(\omega_{U_{\mathcal{O}}^{\mathrm{proj}}})_0$.

6.4 Uniformity Conditions

In the final section, we use the results of the previous two sections in order to explicitly describe the points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to an (i, j)-uniform zero-dimensional subscheme. The main result of this chapter is Theorem 6.4.4 and yields that for $i, j \in \mathbb{N}$, the set of all points of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the corresponding zero-dimensional subscheme is (i, j)-uniform is a constructible set with respect to the Zariski topology.

In this section, we additionally assume that the base field K is algebraically closed. This assumption is crucial since we want to apply Corollary 5.2.9 and thus need to ensure that every zero-dimensional subscheme of $\mathbb{P}_K(\overline{W})$ that corresponds to a point of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ has a K-rational support. Since K is assumed to be algebraically closed, this condition trivially holds. As before, we denote the polynomial rings $Q = P[c_{11}, \ldots, c_{\mu\nu}]$ and $\overline{Q} = \overline{P}[c_{11}, \ldots, c_{\mu\nu}]$. Then the polynomial rings $Q = \bigoplus_{\gamma\in\mathbb{Z}} Q_\gamma$ and $\overline{Q} = \bigoplus_{\gamma\in\mathbb{Z}} \overline{Q}_\gamma$ are non-negatively $\mathbb{Z}$-graded by the matrices $V = (W \mid 0) \in \mathrm{Mat}_{1,n+\mu\nu}(K)$ and $\overline{V} = (\overline{W} \mid 0) \in \mathrm{Mat}_{1,n+1+\mu\nu}(K)$, respectively. Let $\mathbb{B}_{\mathcal{O}}$ denote the $\mathcal{O}$-border basis scheme and let $B_{\mathcal{O}} \to U_{\mathcal{O}}$ the universal $\mathcal{O}$-border basis family. As in Definition 6.1.5, we let $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{B}_{\mathcal{O}}$ denote the projective $\mathcal{O}$-border basis scheme, i.e. the subscheme of the $\mathcal{O}$-border basis scheme defined by the ideal

$$I_{\mathcal{O}}^{\mathrm{proj}} = \langle \overline{c}_{ij} \mid i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, \nu\}, \deg_W(t_i) > \deg_W(b_j)\rangle \subseteq B_{\mathcal{O}},$$

we let $G = \{g_1, \ldots, g_\nu\} \subseteq \overline{Q}$ be the generic projective $\mathcal{O}$-border prebasis, and we let $B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$ be the universal projective $\mathcal{O}$-border basis family.

First we prove that every point of the projective $\mathcal{O}$-border scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ induces a zero-dimensional subscheme of $\mathbb{P}_K(\overline{W})$ with the property that none of its points lies on the hyperplane at infinity H^{inf}.

Proposition 6.4.1. *Every point $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ induces a unique zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ with K-rational support and with the property that $\mathbb{X}_a \cap H^{\mathrm{inf}} = \emptyset$.*

Proof. Let $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. According to Proposition 6.1.4, there is a unique homogeneous ideal $I_a \subseteq \overline{P}$ that possesses a projective $\mathcal{O}$-border basis corresponding to a. Since $\mathcal{O} \neq \emptyset$, we have $I_a \subset \overline{P}$. Let $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ be the subscheme that is scheme-theoretically defined by the ideal I_a. According to Corollary 4.1.16, $\overline{P}/I_a$ is a one-dimensional ring and thus $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ is a zero-dimensional subscheme. Moreover, the Propositions 4.1.7 and 5.1.8 yield that $I_a = I_a :_{\overline{P}} \langle x_0\rangle^\infty$, i.e. I_a is saturated by Lemma 5.1.13, and that $\mathbb{X}_a \cap H^{\mathrm{inf}} = \emptyset$. Since the base field K is algebraically closed, $\mathbb{X}_a$ trivially has a K-rational support by Definition 5.1.5. In particular, since there is a one-to-one correspondence between the subschemes of $\mathbb{P}_K(\overline{W})$ and the saturated homogeneous ideals of $\overline{P}$ by Proposition 5.1.2 and since border bases are unique by Proposition 4.1.7, the uniqueness of the induced zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ follows. □

In the Propositions 6.1.4 and 6.4.1, we have seen that every zero-dimensional subscheme of $\mathbb{P}_K(\overline{W})$ with the property that it has a K-rational support and that none of its points lies on the hyperplane at infinity H^{inf} induces a unique point of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and vice versa. The following definition is based on this one-to-one correspondence.

Definition 6.4.2. a) For every homogeneous ideal $I \subseteq \overline{P}$ that possesses a projective $\mathcal{O}$-border basis, the unique point $a_I \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ corresponding to I as in Proposition 6.1.4 is called the **point on the projective $\mathcal{O}$-border basis scheme induced by I**.

b) For every point $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ on the projective $\mathcal{O}$-border basis scheme, the unique zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ corresponding to a as in Proposition 6.4.1 is called the **zero-dimensional scheme induced by a**.

The remaining part of this section is used to characterize and study the points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to an (i,j)-uniform subscheme of $\mathbb{P}_K(\overline{W})$. Before we do that, we need the definition of constructible sets as, for instance, in [CLO05, Section I.1.1].

Definition 6.4.3. Let (X, τ) be a Noetherian topologial space. A subset $E \subseteq X$ is called **constructible** if there is an $m \in \mathbb{N}$ and closed subsets $U_1, V_1, \ldots, U_m, V_m \subseteq X$ with respect to τ such that $E = \bigcup_{i=1}^{m} (U_i \setminus V_i)$.

At last, we are now able to state and prove the main result of this chapter, namely that the set of all points of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the corresponding zero-dimensional subscheme of $\mathbb{P}_K(\overline{W})$ is (i,j)-uniform is constructible with respect to the Zariski topology.

Theorem 6.4.4. (The (i,j)-Uniform Subschemes Form a Constructible Set) *Let $i \in \{1, \ldots, \mu - H(j)\}$, let $j \in \{1, \ldots, \deg_W(t_\mu) - 1\}$, let $\{y_1, \ldots, y_{H(j)}\}$ be a set of further indeterminates, and let $T = K[c_{11}, \ldots, c_{\mu\nu}, y_1, \ldots, y_{H(j)}]$. Moreover, we let $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathrm{Mat}_\mu(K[c_{11}, \ldots, c_{\mu\nu}])$ denote the generic multiplication matrices with respect to $\mathcal{O}$ and for every $k \in \{1, \ldots, H(j)\}$, we let $\mathcal{T}_k \in \mathrm{Mat}_{\mu-1,\mu-H(j)}(K[c_{11}, \ldots, c_{\mu\nu}])$ be the submatrix of $t_k(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}}, \ldots, \mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K[c_{11}, \ldots, c_{\mu\nu}])$ obtained by deleting the first row and the first $H(j)$ columns. Moreover, let $J \subseteq T$ be the ideal generated by the set of all i-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_{H(j)}\mathcal{T}_{H(j)} \in \mathrm{Mat}_{\mu-1,\mu-H(j)}(T)$ and let $\pi_{\mu\nu} : \mathrm{Spec}(K[c_{11}, \ldots, c_{\mu\nu}, y_1, \ldots, y_{H(j)}] = \mathbb{A}_K^{\mu\nu+H(j)} \to \mathbb{A}_K^{\mu\nu} = \mathrm{Spec}(K[c_{11}, \ldots, c_{\mu\nu}])$ be the projection to the first $\mu\nu$ components. Then for every point $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, the following conditions are equivalent.*

i) We have $a \in \pi_{\mu\nu}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1, \ldots, y_{H(j)} \rangle))$.

ii) The zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a, as defined in Definition 6.1.5, is not (i,j)-uniform.

In particular, the set of all points $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is (i,j)-uniform is a constructible set with respect to the Zariski topology.

Proof. Let $a = (a_{11}, \ldots, a_{\mu\nu}) \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, let $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ be the zero-dimensional scheme induced by a as in Definition 6.1.5, and let $J_a \subseteq K[y_1, \ldots, y_{H(j)}]$ be the image of J under the substitution K-algebra epimorphism $T \twoheadrightarrow K[y_1, \ldots, y_{H(j)}]$ defined by $c_{ij} \mapsto a_{ij}$ for all

$i \in \{1,\dots,\mu\}$ and $j \in \{1,\dots,\nu\}$. As $\mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle) = \mathbb{A}_K^{\mu\nu} \times \{0\} \subseteq \mathbb{A}_K^{\mu\nu+H(j)}$, we have $a \in \pi_{\mu\nu}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle))$ if and only if there is a $(c_1,\dots,c_{H(j)}) \in \mathbb{A}_K^{H(j)} \setminus \{0\}$ and $(a_{11},\dots,a_{\mu\nu},c_1,\dots,c_{H(j)}) \in \mathcal{Z}(J)$, where $\mathbb{A}_K^{H(j)} = \operatorname{Spec}(K[y_1,\dots,y_{H(j)}])$. Moreover, $(a_{11},\dots,a_{\mu\nu},c_1,\dots,c_{H(j)}) \in \mathcal{Z}(J)$ if and only if $(c_1,\dots,c_{H(j)}) \in \mathcal{Z}(J_a)$. Altogether, $a \in \pi_{\mu\nu}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle))$ if and only if there is a $(c_1,\dots,c_{H(j)}) \in \mathcal{Z}(J_a) \setminus \{0\}$. By Proposition 6.4.1, the zero-dimensional scheme $\mathbb{X}_a$ induced by a has a K-rational support and satisfies $\mathbb{X}_a \cap H^{\mathrm{inf}} = \emptyset$. Hence Corollary 5.2.9 shows that the existence of an element $(c_1,\dots,c_{H(j)}) \in \mathcal{Z}(J_a) \setminus \{0\}$ is equivalent for the subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ to be not (i,j)-uniform.

Thus it remains to prove that the set of all points $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is (i,j)-uniform is a constructible set with respect to the Zariski topology. The zero sets $\mathcal{Z}(J)$ and $\mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle)$ are closed with respect to the Zariski topology. Thus the set $\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle)$ is a constructible set by Definition 6.4.3. The projection $\pi_{\mu\nu}$ is obviously a polynomial map. Thus the image $\pi_{\mu\nu}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle))$ is also a constructible set with respect to the Zariski topology according to [Kem07, Algo. 1.6]. Moreover, the points of the subscheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{A}^{\mu\nu}$ are given by $\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \subseteq K[c_{11},\dots,c_{\mu\nu}]$, where we let $\widetilde{I}_{\mathcal{O}}^{\mathrm{proj}} = \langle c_{ij} \mid \deg_W(t_i) > \deg_W(b_j), i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}\rangle$, and also form a constructible set by Definition 6.4.3 with respect to the Zariski topology. Since $(U_1 \setminus V_1) \cap (U_2 \setminus V_2) = (U_1 \cap U_2) \setminus (V_1 \cup V_2)$ for any arbitrary sets U_1, V_1, U_2, and V_2, the set $\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \cap \pi_{\mu\nu}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_{H(j)}\rangle))$ is also a constructible set with respect to the Zariski topology by Definition 6.4.3 and the claim follows. $\square$

Example 6.4.5. Consider the situation of Examples 6.1.2 and 6.1.6 over $\mathbb{C}$, again. Recall that $\mathbb{C}[x_0,x_1,x_2]$ was standard graded, i.e. graded by $\overline{W} = (1,1,1) \in \operatorname{Mat}_{1,3}(\mathbb{Z})$, that the order ideal in $\mathbb{T}^2$ was $\mathcal{O} = \{t_1,\dots,t_6\} = \{1, x_2, x_1, x_1x_2, x_1^2, x_1^3\}$, with the border $\partial\mathcal{O} = \{b_1,\dots,b_5\} = \{x_2^2, x_1x_2^2, x_1^2x_2, x_1^3x_2, x_1^4\}$. Let $\{y_1,\dots,y_5\}$ be a set of further indeterminates, let $T = \mathbb{C}[c_{11},\dots,c_{65},y_1,\dots,y_5]$, and let $\mathcal{X}_1, \mathcal{X}_2 \in \operatorname{Mat}_6(\mathbb{C}[c_{11},\dots,c_{65}])$ denote the generic multiplication matrices with respect to $\mathcal{O}$. As in Example 6.1.2,

$$\mathcal{X}_1 = \begin{pmatrix} 0 & 0 & 0 & c_{13} & 0 & c_{15} \\ 0 & 0 & 0 & c_{23} & 0 & c_{25} \\ 1 & 0 & 0 & c_{33} & 0 & c_{35} \\ 0 & 1 & 0 & c_{43} & 0 & c_{45} \\ 0 & 0 & 1 & c_{53} & 0 & c_{55} \\ 0 & 0 & 0 & c_{63} & 1 & c_{65} \end{pmatrix}, \qquad \mathcal{X}_2 = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & c_{13} & c_{14} \\ 1 & c_{21} & 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{31} & 0 & c_{32} & c_{33} & c_{34} \\ 0 & c_{41} & 1 & c_{42} & c_{43} & c_{44} \\ 0 & c_{51} & 0 & c_{52} & c_{53} & c_{54} \\ 0 & c_{61} & 0 & c_{62} & c_{63} & c_{64} \end{pmatrix}.$$

Let $f_1,\dots,f_{24} \in \mathbb{C}[c_{11},\dots,c_{65}]$ be the polynomials of Example 6.1.2 with the property that $\mathcal{I}(\mathbb{B}_{\mathcal{O}}) = \langle f_1,\dots,f_{24}\rangle$ and let $a = (a_{11},\dots,a_{65})$ be with $a_{51} = a_{62} = -1$, $a_{63} = 1$, and $a_{ij} = 0$ for all $i \in \{1,\dots,6\}$ and $j \in \{1,\dots,5\}$ such that $(i,j) \notin \{(5,1),(6,2),(6,3)\}$. Then $a \in \mathcal{Z}(\{f_1,\dots,f_{24}\}) = \mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}))$. For all $i \in \{1,\dots,6\}$ and $j \in \{1,\dots,5\}$, we have $\deg_W(t_i) > \deg_W(b_j)$ if and only if $(i,j) = (6,5)$. Thus we get $a \in \mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \langle c_{61}\rangle)$, i.e. $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ by Proposition 6.1.4.

Let $H : \mathbb{Z} \to \mathbb{Z}$, $\gamma \mapsto \#\{k \in \{1,\dots,6\} \mid \deg_W(t_k) \le \gamma\}$ and for all $k \in \{1,\dots,5\}$, let $\mathcal{T}_k \in \mathrm{Mat}_{5,1}(\mathbb{C}[c_{11},\dots,c_{\mu\nu}])$ be the submatrix of $t_k(\mathcal{I}_6, \mathcal{X}_1^{\mathrm{tr}}, \mathcal{X}_2^{\mathrm{tr}})$ obtained by deleting the first row and the first $H(2) = 5$ columns. Then we have

$$\mathcal{T}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathcal{T}_2 = \begin{pmatrix} 0 \\ 0 \\ c_{63} \\ 1 \\ c_{65} \end{pmatrix}, \qquad \mathcal{T}_3 = \begin{pmatrix} c_{61} \\ 0 \\ c_{62} \\ c_{63} \\ c_{64} \end{pmatrix},$$

and

$$\mathcal{T}_4 = \begin{pmatrix} c_{62} \\ c_{63} \\ c_{23}c_{61} + c_{43}c_{62} + c_{53}c_{63} + c_{63}c_{64} \\ c_{64} \\ c_{25}c_{61} + c_{45}c_{62} + c_{55}c_{63} + c_{64}c_{65} \end{pmatrix}, \qquad \mathcal{T}_5 = \begin{pmatrix} c_{63} \\ 1 \\ c_{43}c_{63} + c_{63}c_{65} + c_{53} \\ c_{65} \\ c_{45}c_{63} + c_{65}^2 + c_{55} \end{pmatrix}.$$

In particular, the ideal $J \subseteq \mathbb{Q}[c_{11},\dots,c_{65},y_1,\dots,y_5]$ generated by the 1-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_5\mathcal{T}_5 \in \mathbb{Q}[c_{11},\dots,c_{65},y_1,\dots,y_5]$ is of the form $J = \langle h_1,\dots,h_5\rangle$ with

$$\begin{aligned}
h_1 &= y_3c_{61} + y_4c_{62} + y_5c_{63},\\
h_2 &= y_4c_{63} + y_5,\\
h_3 &= y_2c_{63} + y_3c_{62} + y_4(c_{23}c_{61} + c_{43}c_{62} + c_{53}c_{63} + c_{63}c_{64}) + y_5(c_{43}c_{63} + c_{63}c_{65} + c_{53}),\\
h_4 &= y_2 + y_3c_{63} + y_4c_{64} + y_5c_{65},\\
h_5 &= y_1 + y_2c_{65} + y_3c_{64} + y_4(c_{25}c_{61} + c_{45}c_{62} + c_{55}c_{63} + c_{64}c_{65}) + y_5(c_{45}c_{63} + c_{65}^2 + c_{55}).
\end{aligned}$$

Let φ denote the substitution $\mathbb{C}$-algebra epimorphism defined by $c_{ij} \mapsto a_{ij}$ for all indices $i \in \{1,\dots,6\}$ and $j \in \{1,\dots,5\}$ and let $\pi_{30} : \mathbb{A}^{35}_{\mathbb{C}} \to \mathbb{A}^{30}_{\mathbb{C}}$ where we have $\mathbb{A}^{35}_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[c_{11},\dots,c_{65},y_1,\dots,y_5])$ and $\mathbb{A}^{30}_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[c_{11},\dots,c_{65}])$ be the projection to the first 30 components. Then $\varphi(h_1) = -y_4 + y_5$, $\varphi(h_2) = y_4 + y_5$, $\varphi(h_3) = y_2 - y_3$, $\varphi(h_4) = y_2 + y_3$, $\varphi(h_5) = y_1$, i.e. $\mathcal{Z}(\varphi(J)) = \{0\} \subseteq \mathbb{A}^5_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_5])$, and thus $a \notin \pi_{30}(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1,\dots,y_5\rangle))$. By Theorem 6.4.4, the zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}^2_{\mathbb{C}}$ induced by a is $(1,2)$-uniform, i.e. $\mathbb{X}_a$ is a Cayley-Bacharach scheme by Definition 5.2.1.

Corollary 6.4.6. *Let $i \in \{1,\dots,\mu - H(j)\}$ and let $j \in \{1,\dots,\deg_W(t_\mu) - 1\}$. We let $\{y_1,\dots,y_{H(j)}\}$ be a set of further indeterminates and $T = K[c_{11},\dots,c_{\mu\nu},y_1,\dots,y_{H(j)}]$. Let $\widetilde{I}^{\mathrm{proj}}_{\mathcal{O}} = \langle c_{ij} \mid i \in \{1,\dots,\mu\}, j \in \{1,\dots,\nu\}, \deg_W(t_i) > \deg_W(b_j)\rangle \subseteq K[c_{11},\dots,c_{\mu\nu}]$ and let $\mathcal{X}_1,\dots,\mathcal{X}_n \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ denote the generic multiplication matrices with respect to $\mathcal{O}$. For all $k \in \{1,\dots,H(j)\}$, let $\mathcal{T}_k \in \mathrm{Mat}_{\mu-1,\mu-H(j)}(K[c_{11},\dots,c_{\mu\nu}])$ be the submatrix of $t_k(\mathcal{I}_\mu, \mathcal{X}_1^{\mathrm{tr}},\dots,\mathcal{X}_n^{\mathrm{tr}}) \in \mathrm{Mat}_\mu(K[c_{11},\dots,c_{\mu\nu}])$ obtained by deleting the first row and the first $H(j)$ columns. Moreover, let $J \subseteq T$ be the ideal generated by the set of all i-minors of the matrix $y_1\mathcal{T}_1 + \cdots + y_{H(j)}\mathcal{T}_{H(j)} \in \mathrm{Mat}_{\mu-1,\mu-H(j)}(T)$.*

a) The zero set $\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}^{\mathrm{proj}}_{\mathcal{O}} + ((J :_T \langle y_1,\dots,y_{H(j)}\rangle) \cap K[c_{11},\dots,c_{\mu\nu}])) \subseteq \mathbb{A}^{\mu\nu}_K$ where $\mathbb{A}^{\mu\nu}_K = \mathrm{Spec}(K[c_{11},\dots,c_{\mu\nu}])$ contains all points $a \in \mathbb{B}^{\mathrm{proj}}_{\mathcal{O}}$ with the property that the zero-dimensional scheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is not (i,j)-uniform.

b) If $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ and $a \notin \mathcal{Z}\left((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]\right)$, then the zero-dimensional scheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is (i,j)-uniform.

c) If we have $\sqrt{(J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]} \nsubseteq \sqrt{\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}}$, there is an element $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ with the property that the zero-dimensional scheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is (i,j)-uniform.

Proof. First we prove a). According to Proposition 6.1.4, there is a canonical K-algebra isomorphism $B_{\mathcal{O}}^{\mathrm{proj}} \cong K[c_{11}, \dots, c_{\mu\nu}]/\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}$. Thus $\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \subseteq \mathbb{A}_K^{\mu\nu}$ where $\mathbb{A}_K^{\mu\nu} = \mathrm{Spec}(K[c_{11}, \dots, c_{\mu\nu}])$ contains precisely the points of the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$. Consider the projection to the first $\mu\nu$ components $\pi_{\mu\nu} : \mathbb{A}_K^{\mu\nu + H(j)} \to \mathbb{A}_K^{\mu\nu}$ where $\mathbb{A}_K^{\mu\nu+H(j)} = \mathrm{Spec}(K[c_{11}, \dots, c_{\mu\nu}, y_1, \dots, y_{H(j)}])$. According to Theorem 6.4.4,

$$\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \cap \pi_{\mu\nu}\left(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1, \dots, y_{H(j)}\rangle\right)$$

is exactly the set of all points $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ on the projective $\mathcal{O}$-border basis scheme such that the zero-dimensional scheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ induced by a is not (i,j)-uniform. By [CLO07, Thm. 4.§4.7], we have

$$\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1, \dots, y_{H(j)}\rangle) \subseteq \mathcal{Z}(J :_T \langle y_1, \dots, y_{H(j)}\rangle)$$

and [CLO07, Thm. 4.§4.2] yields

$$\pi_{\mu\nu}\left(\mathcal{Z}(J :_T \langle y_1, \dots, y_{H(j)}\rangle)\right) \subseteq \mathcal{Z}\left((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]\right).$$

Altogether, it follows

$$\begin{aligned}&\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \cap \pi_{\mu\nu}\left(\mathcal{Z}(J) \setminus \mathcal{Z}(\langle y_1, \dots, y_{H(j)}\rangle)\right)\\ &\subseteq \mathcal{Z}\left(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) + \left((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]\right)\right)\end{aligned}$$

from [CLO07, Thm. 4.§3.4].

Now we start to prove b). By [CLO07, Thm. 4.§3.4], we have

$$\begin{aligned}&\mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}) \cap \mathcal{Z}((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])\\ &= \mathcal{Z}(\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}} + ((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])).\end{aligned}$$

Therefore, claim a) yields that for every point $a \in \mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ such that

$$a \notin \mathcal{Z}(((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])),$$

the induced zero-dimensional scheme $\mathbb{X}_a \subseteq \mathbb{P}_K(\overline{W})$ is (i,j)-uniform.

For the proof of c), suppose that

$$\sqrt{((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])} \nsubseteq \sqrt{\mathcal{I}(\mathbb{B}_{\mathcal{O}}) + \widetilde{I}_{\mathcal{O}}^{\mathrm{proj}}}.$$

Then we have

$$\sqrt{\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}} + ((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])} \not\subseteq \sqrt{\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}}$$

and the Ideal-Variety Correspondence [CLO07, Thm. 4.§2.7] yields

$$\mathcal{Z}\left(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}} + \left((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]\right)\right) \not\supseteq \mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}).$$

By [CLO07, Thm. 4.§3.4], we have

$$\begin{aligned} &\mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}} + ((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}])) \\ &= \mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}) \cap \mathcal{Z}((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]) \end{aligned}$$

and this implies that

$$\mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}) \cap \mathcal{Z}((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]) \not\supseteq \mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}).$$

Since this is equivalent to

$$\mathcal{Z}(\mathcal{I}(\mathbb{B}_\mathcal{O}) + \widetilde{I}_\mathcal{O}^{\mathrm{proj}}) \setminus \mathcal{Z}((J :_T \langle y_1, \dots, y_{H(j)}\rangle) \cap K[c_{11}, \dots, c_{\mu\nu}]) \neq \emptyset,$$

claim c) follows from b). □

Example 6.4.7. Consider the situation of Example 6.4.5, again. Recall that we had the order ideal $\mathcal{O} = \{t_1, \dots, t_6\} = \{1, x_2, x_1, x_1x_2, x_1^2, x_1^3\}$, had the maxdeg_W-border $\partial\mathcal{O} = \{b_1, \dots, b_5\} = \{x_2^2, x_1x_2^2, x_1^2x_2, x_1^3x_2, x_1^4\}$. Moreover, $a = (a_{11}, \dots, a_{65}) \in \mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ was of the form $a_{51} = a_{62} = -1$, $a_{63} = 1$, and $a_{ij} = 0$ for all $i \in \{1, \dots, 6\}$ and $j \in \{1, \dots, 5\}$ with $(i, j) \notin \{(5, 1), (6, 2), (6, 3)\}$ and we had $J = \langle h_1, \dots, h_5\rangle \subseteq \mathbb{C}[c_{11}, \dots, c_{65}, y_1, \dots, y_5]$ with the polynomials

$$\begin{aligned} h_1 &= y_3c_{61} + y_4c_{62} + y_5c_{63}, \\ h_2 &= y_4c_{63} + y_5, \\ h_3 &= y_2c_{63} + y_3c_{62} + y_4(c_{23}c_{61} + c_{43}c_{62} + c_{53}c_{63} + c_{63}c_{64}) + y_5(c_{43}c_{63} + c_{63}c_{65} + c_{53}), \\ h_4 &= y_2 + y_3c_{63} + y_4c_{64} + y_5c_{65}, \\ h_5 &= y_1 + y_2c_{65} + y_3c_{64} + y_4(c_{25}c_{61} + c_{45}c_{62} + c_{55}c_{63} + c_{64}c_{65}) + y_5(c_{45}c_{63} + c_{65}^2 + c_{55}). \end{aligned}$$

The ideal $(J :_{\mathbb{C}[c_{11},\dots,c_{65},y_1,\dots,y_5]} \langle y_1, \dots, y_5\rangle) \cap \mathbb{C}[c_{11}, \dots, c_{65}] \subseteq \mathbb{C}[c_{11}, \dots, c_{65}]$ is a principal ideal generated by the polynomial $p = c_{43}c_{61}c_{63}^2 + c_{63}^4 - c_{23}c_{61}^2 - c_{43}c_{61}c_{62} - 2c_{62}c_{63}^2 + c_{62}^2$. Since $p(a) = 4 \neq 0$, Corollary 6.4.6 implies the existence of a point $a \in \mathbb{B}_\mathcal{O}^{\mathrm{proj}}$ such that the zero-dimensional subscheme $\mathbb{X}_a \subseteq \mathbb{P}_\mathbb{C}^2$ induced by a is a Cayley-Bacharach scheme. Note that this result coincides with Example 6.4.5.

Remark 6.4.8. Every algebraically closed field, and thus also K, is perfect. Therefore, we can compute radical ideals in polynomial rings over K, cf. [KL91] if $\mathrm{char}(K) = 0$ and [Mat01, Kem02] if $\mathrm{char}(K) > 0$. Moreover, we can compute sums of ideals, colon ideals, respectively elimination ideals, with standard Gröbner basis techniques, cf. [KR00, Rem. 3.2.1, Prop. 3.2.15, respectively Thm. 3.4.5]. Altogether, we see that condition c) of Theorem 6.4.4 can be checked algorithmically.

7 Conclusion and Outlook

In this thesis, we have generalized and applied border bases in various ways.

Before this thesis, the theory of border basis was restricted to zero-dimensional ideals in polynomial rings. In Chapter 2, we extended the theory to finitely generated modules over polynomial rings. Our contribution in the Sections 2.1 to 2.5 was to generalize all the well-known definitions, concepts, characterizations, and algorithms that hold for border bases of zero-dimensional ideals in a polynomial ring to border bases of submodules of free modules of finite rank over a polynomial ring with finite codimension. In Section 2.6, we established generalized border bases of submodules of arbitrary finitely generated modules over a polynomial ring with finite codimension. By a process of lifting these generalized border bases to the free setting, we were able to characterize and, under certain circumstances, also compute generalized border bases in this general setting. As a byproduct, we saw that this new border bases theory was applicable for subideal border bases and immediately yielded many new characterizations and an algorithm for their computation that uses linear algebra techniques. By now there has only been one characterization available for subideal border bases and the only algorithm for the computation of subideal border bases has needed much more effort than ours.

After the theoretical introduction of border bases for modules in this thesis, one could look for applications different from subideal border bases. Another possibility of further research could be to take a closer look at numerical aspects of these border bases. Just as it was done, for instance, in [Lim14] for border bases and in [KP11] for subideal border bases, (generalized) border bases might be a good tool to model real word applications in a data-driven setting. A natural question to ask is whether we can apply and make use of the newly introduced concepts and the more general bases in such numerical settings.

In Chapter 3, we proved Schreyer's Theorem for border bases in free modules of finite rank over polynomial rings. To this end, we generalized the results of [KK14] to the module setting. As a byproduct, we found a totally new proof of the characterization of border bases via liftings of border syzygies. This proof is quite remarkable since it does not depend on the characterization of border bases via commuting matrices but it only depends on the characterizations of border bases via the special generation property and via rewrite rules.

Many of the nice properties of border bases do use or are related to the characterization of border bases via commuting matrices. Obviously, this theorem cannot be generalized to a non-commutative setting. The study of border bases in a non-commutative setting started with [BTBQM00]: the authors of that paper introduced border bases in free associative algebras over a field and used them to compute Gröbner bases by FGLM techniques in a non-commutative setting. But the authors of that paper neither further developed the basic concepts of border bases nor characterized border bases in a non-commutative setting. By proving a non-commutative version of the characterizations via the special generation property and via rewrite rules, our new proof of the characterization of border bases via liftings of border syzygies might yield a possible way to develop a general border basis theory in a non-commutative setting.

Whereas the first two chapters of this thesis dealt with border bases in a module setting, the remaining part of this thesis was then dedicated to the effect of homogenization to border bases in a polynomial ring and to applications of such homogenizations of border bases. In Chapter 4, we introduced the notion of projective border bases of homogeneous ideals in a polynomial ring and we showed that projective border bases are related to specific border bases. Then we studied the multiplicative structure of a polynomial ring modulo a projective border basis and of its canonical module in detail. In particular, we described these multiplicative structures explicitly by means of formal multiplication matrices. These explicit descriptions turned out to be very useful in Chapter 5 for the study of zero-dimensional closed subschemes of weighted projective spaces. In Chapter 6, we introduced the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ as a subscheme of the $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}$. Then we showed that the points of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ are in one-to-one correspondence to specific zero-dimensional closed subschemes of a weighted projective space and studied the set of points on $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to (i,j)-uniform schemes.
Besides the introduction of projective border bases and the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$, the main contribution in the Chapters 4 to 6 are the characterizations of (i,j)-uniform zero-dimensional subschemes of weighted projective spaces if the defining ideals of these subschemes possess a projective border basis. We can prove these characterizations for arbitrary $i,j \in \mathbb{N}$ in weighted projective spaces over arbitrary fields whereas by now the only ways to characterize such (i,j)-uniform subschemes were restricted to special base fields, e.g. to algebraically closed fields, reduced subschemes, or to the standard grading. If the base field is algebraically closed or if the given zero-dimensional closed subscheme is reduced, our characterizations even yielded algorithms that allow us to check whether the subscheme is (i,j)-uniform or not. Furthermore, we proved that the set of all points on the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to (i,j)-uniform schemes is constructible with respect to the Zariski topology.
An obvious way to do further research in this area is to study the constructible set of all points on the projective $\mathcal{O}$-border basis scheme $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$ that correspond to (i,j)-uniform schemes. One could ask, for instance, whether this constructible set has any special algebraic or geometric properties. Another path could be to restrict oneself to subschemes that satisfy certain (i,j)-uniformity conditions, e.g. Cayley-Bacharach schemes or schemes in uniform position. Instead of regarding (i,j)-uniform schemes in weighted projective spaces, one could also transform the notion of (i,j)-uniform schemes to the affine setting. This could be done by carefully reformulating the definitions and results of this thesis. For instance, the defintion of (i,j)-uniform subschemes uses Hilbert functions of projective coordinate rings. By replacing these Hilbert functions by affine Hilbert functions, cf. [KR05, Section 5.6], of the coordinate rings of subschemes of affine spaces, one could define and study (i,j)-uniform schemes in the affine setting.

Notation

The following list gives a brief overview of most of the notation used in this thesis.

Sets and Maps

$\mathbb{N}$	set of natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{F}_q$	finite field with q elements
K	arbitrary field
$\overline{K}$	algebraic closure of a field K
$\mathrm{char}(K)$	characteristic of a field K

$A \subseteq B$	set A is a (not necessarily proper) subset of set B
$A \subset B$	set A is a proper subset of set B
$A \setminus B$	set difference of A and B
$\#A$	number of elements of a finite set A
$A \hookrightarrow B$	injective map $A \to B$
$A \twoheadrightarrow B$	surjective map $A \to B$
$A \xrightarrow{\sim} B$	bijective map $A \to B$
$\psi \circ \varphi$	composition of two maps $\varphi : A \to B$ and $\psi : B \to C$
id_A	identity map on a set A
φ^*	dual of a linear map φ

$\ker(\varphi)$	kernel of a homomorphism φ
$\mathrm{im}(\varphi)$	image of a map φ
$\varphi\|_U$	restriction of $\varphi : A \to B$ to $U \subseteq A$

Elements, Gradings, and Matrices

δ_{ij}	Kronecker delta, i.e. $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ else
p^*	dual element of p
$\overline{p}$	residue class of p

$\deg(p)$	degree of a polynomial $p \in P \setminus \{0\}$
$\deg_{x_i}(p)$	x_i-degree of a polynomial $p \in P \setminus \{0\}$
$\deg_W(p)$	degree of a polynomial $p \in P \setminus \{0\}$ with respect to the grading given by a matrix W
$\mathrm{DF}_W(p)$	degree form of a polynomial $p \in P \setminus \{0\}$ with respect to the grading given by a matrix W
p^{hom}	homogenization of an element p
p^{deh}	dehomogenization of an element p
$p(\mathcal{X}_1, \dots, \mathcal{X}_n)$	evaluation of a polynomial $p \in P$ at $(\mathcal{X}_1, \dots, \mathcal{X}_n)$, i.e. applying the substitution $x_k \mapsto \mathcal{X}_k$

$\mathrm{Mat}_k(K)$	set of all $k \times k$-matrices over K
$\mathrm{Mat}_{k,\ell}(K)$	set of all $k \times \ell$-matrices over K
$\mathcal{I}_k$	identity matrix of size $k \times k$
$\mathcal{A}^{\mathrm{tr}}$	transposed matrix of a matrix $\mathcal{A}$
$(\mathcal{A} \mid \mathcal{B})$	$\in \mathrm{Mat}_{k,\ell+m}(K)$, concatenation of a matrix $\mathcal{A} \in \mathrm{Mat}_{k,\ell}(K)$ and a matrix $\mathcal{B} \in \mathrm{Mat}_{k,m}(K)$
$\mathrm{rk}(\mathcal{A})$	rank of a matrix $\mathcal{A}$
$\mathrm{trace}(\mathcal{A}) = a_{11} + \cdots + a_{kk}$	trace of a square matrix $\mathcal{A} = (a_{ij})_{1 \le i,j \le k}$

Vector Spaces, Modules, and Ideals

$\dim_K(V)$	dimension of a K-vector space V
$\operatorname{codim}_K(V,W)$	codimension of a K-vector subspace $V \subseteq W$
$\langle \mathcal{O} \rangle_K$	K-vector subspace generated by $\mathcal{O}$
$\operatorname{rk}_R(M)$	rank of a free R-module M
$\langle v_1, \ldots, v_k \rangle$	submodule generated by $\{v_1, \ldots, v_k\}$

$M_1 \oplus M_2$	direct sum of two groups or modules
IM	submodule of module M generated by products of the form pm where $p \in I$, $m \in M$,
$N :_P M$	$\subseteq P$, colon ideal of a module N by a module M
$N :_M I^\infty$	$\subseteq M$, saturation of a module N by an ideal I in M
$\sqrt{I}$	radical of an ideal I
$\operatorname{Syz}_P(g_1, \ldots, g_\nu)$	first syzygy module of $(g_1, \ldots, g_\nu)$
$M_\mathfrak{p}$	localization of a module M at the prime ideal $\mathfrak{p}$
M^*	dual module of a module M
ω_R	canonical module of R
I^{hom}	homogenization of an ideal I
I^{deh}	dehomogenization of an ideal I

$M(\gamma')$	$= \bigoplus_{\gamma \in \mathbb{Z}} (M(\gamma'))_\gamma = \bigoplus_{\gamma \in \mathbb{Z}} M_{\gamma+\gamma'}$, $\mathbb{Z}$-graded module obtained from the $\mathbb{Z}$-graded module $M = \bigoplus_{\gamma \in \mathbb{Z}} M_\gamma$ by shifting degrees by $\gamma' \in \mathbb{Z}$
$S_\gamma = M_\gamma \cap S$	set of all homogeneous elements of degree γ of a subset S of a graded module $M = \bigoplus_{\gamma \in \mathbb{Z}} M_\gamma$
$S_{<\gamma}$	$= \bigoplus_{\gamma'=-\infty}^{\gamma-1} S_{\gamma'}$, set of all homogeneous elements of degree less than $\gamma \in \mathbb{Z}$ of a subset S of a $\mathbb{Z}$-graded module M
$S_{\geq\gamma}$	$= \bigoplus_{\gamma'=\gamma}^{\infty} S_{\gamma'}$, set of all homogeneous elements of degree greater than or equal to $\gamma \in \mathbb{Z}$ of a subset S of a $\mathbb{Z}$-graded module M

HF_M	$\gamma \mapsto \dim_K(M_\gamma)$, Hilbert function of a $\mathbb{Z}$-graded module M
$\Delta\,\mathrm{HF}_M$	$\gamma \mapsto \mathrm{HF}_M(\gamma) - \mathrm{HF}_M(\gamma-1)$, Castelnuovo function of HF_M of a $\mathbb{Z}$-graded module M

Polynomials, Vectors, and Term Orderings

σ	term ordering
`DegRevLex`	degree-reverse-lexicographic term ordering
$\mathtt{Ord}(V)$	term ordering associated to a matrix V
$\sigma\,\mathrm{Pos}$	term ordering "σ first, then position"

$P = K[x_1,\dots,x_n]$	polynomial ring over K in the indeterminates $x_1,\dots,x_n$, $n \in \mathbb{N}\setminus\{0\}$
$t = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$	term in P, $\alpha_1,\dots,\alpha_n \in \mathbb{N}$
$\log(t)$	logarithm of a term $t = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\log(t) = (\alpha_1,\dots,\alpha_n)$
$\mathbb{T}^n$	monoid of all terms in P
ct	monomial in P, $c \in K$, $t \in \mathbb{T}^n$
$\mathrm{LT}_\sigma(p)$	$\in \mathbb{T}^n$, leading term of a polynomial $p \in P\setminus\{0\}$ with respect to a term ordering σ
$\mathrm{LC}_\sigma(p)$	$\in K$, leading coefficient of a polynomial $p \in P\setminus\{0\}$ with respect to a term ordering σ
$\mathrm{LM}_\sigma(p)$	$= \mathrm{LC}_\sigma(T)\cdot\mathrm{LT}_\sigma(t)$, leading monomial of $p \in P\setminus\{0\}$ with respect to a term ordering σ
$\mathrm{LT}_\sigma(I)$	$= \langle \mathrm{LT}_\sigma(p) \mid p \in P\setminus\{0\}\rangle \subseteq P$, leading term ideal of an ideal $I \subseteq P$ with respect to a term ordering σ
$\mathrm{LT}_\sigma\{I\}$	$= \{\mathrm{LT}_\sigma(p) \mid p \in P\setminus\{0\}\} \subseteq \mathbb{T}^n$, monoideal of leading terms of an ideal $I \subseteq P$ with respect to a term ordering σ
$\mathrm{NF}_{\sigma,I}(p)$	$\in \langle \mathcal{O}_\sigma(I)\rangle_K$, normal formal of a polynomial $p \in P$ with respect to a term ordering σ and an ideal $I \subseteq P$

P^r	free P-module with canonical basis $\{e_1,\dots,e_r\}$, $r \in \mathbb{N}\setminus\{0\}$

te_k	term in P^r, $t \in \mathbb{T}^n$, $k \in \{1, \dots, r\}$
$\mathbb{T}^n\langle e_1, \dots, e_r \rangle$	monoid of all terms in P^r
cte_k	monomial in P^r, $c \in K$, $t \in \mathbb{T}^n$, $k \in \{1, \dots, r\}$
$\mathrm{LT}_\sigma(v)$	$\in \mathbb{T}^n\langle e_1, \dots, e_r \rangle$, leading term of a vector $v \in P^r \setminus \{0\}$ with respect to a term ordering σ
$\mathrm{LC}_\sigma(v)$	$\in K$, leading coefficient of a vector $v \in P^r \setminus \{0\}$ with respect to a term ordering σ
$\mathrm{LM}_\sigma(v)$	$= \mathrm{LC}_\sigma(v) \cdot \mathrm{LT}_\sigma(v)$, leading monomial of $v \in P^r \setminus \{0\}$ with respect to a term ordering σ
$\mathrm{LT}_\sigma(U)$	$= \langle \mathrm{LT}_\sigma(v) \mid v \in P^r \setminus \{0\} \rangle \subseteq P^r$, leading term module of a P-submodule $U \subseteq P^r$ with respect to a term ordering σ
$\mathrm{LT}_\sigma\{U\}$	$= \{\mathrm{LT}_\sigma(v) \mid v \in P^r \setminus \{0\}\} \subseteq \mathbb{T}^n\langle e_1, \dots, e_r \rangle$, monomodule of leading terms of a P-submodule $U \subseteq P^r$ with respect to a term ordering σ
$\mathcal{O}_\sigma(U)$	$= \mathbb{T}^n\langle e_1, \dots, e_r \rangle \setminus \mathrm{LT}_\sigma\{U\}$, order ideal of a P-submodule $U \subseteq P^r$ with respect to a term ordering σ
$\mathrm{NF}_{\sigma,U}(v)$	$\in \langle \mathcal{O}_\sigma(U) \rangle_K$, normal formal of a vector $v \in P^r$ with respect to a term ordering σ and a P-submodule $U \subseteq P^r$

(Projective) Border Bases

$\mathcal{O} = \{t_1, \dots, t_\mu\}$	order ideal in $\mathbb{T}^n$
$\partial\mathcal{O} = \{b_1, \dots, b_\nu\}$	border of an order ideal $\mathcal{O}$ in $\mathbb{T}^n$
$\mathcal{O} = \{t_1 e_{\alpha_1}, \dots, t_\mu e_{\alpha_\mu}\}$	order ideal in $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$
$\partial\mathcal{O} = \{b_1 e_{\beta_1}, \dots, b_\nu e_{\beta_\nu}\}$	border of an order ideal $\mathcal{O}$ in $\mathbb{T}^n\langle e_1, \dots, e_r \rangle$
$\mathcal{O}^\varphi = \{t_1 m_{\alpha_1}, \dots, t_\mu m_{\alpha_\mu}\}$	(generalized) order ideal in $\varphi(\mathbb{T}^n\langle e_1, \dots, e_r \rangle)$ with respect to $\varphi : P^r \twoheadrightarrow M = \langle m_1, \dots, m_r \rangle$, $e_i \mapsto m_i$
$\partial\mathcal{O}^\varphi = \{b_1 m_{\beta_1}, \dots, b_\nu m_{\beta_\nu}\}$	border of (generalized) order ideal in $\varphi(\mathbb{T}^n\langle e_1, \dots, e_r \rangle)$ with respect to $\varphi : P^r \twoheadrightarrow M = \langle m_1, \dots, m_r \rangle$, $e_i \mapsto m_i$
$\partial^k\mathcal{O}$	k^{th} border of an order ideal $\mathcal{O}$
$\overline{\partial^k\mathcal{O}}$	k^{th} border closure of an order ideal $\mathcal{O}$
$G = \{g_1, \dots, g_\nu\} \subseteq P$	$\mathcal{O}$-border prebasis, $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ where $a_{ij} \in K$

$G = \{g_1, \dots, g_\nu\} \subseteq P^r$	$\mathcal{O}$-border prebasis, $g_j = b_j e_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i e_{\alpha_i}$ where $a_{ij} \in K$
$G^\varphi = \{g_1^\varphi, \dots, g_\nu^\varphi\} \subseteq M$	(generalized) $\mathcal{O}$-border prebasis, $g_j^\varphi = b_j m_{\beta_j} - \sum_{i=1}^{\mu} a_{ij} t_i m_{\alpha_i}$ where $a_{ij} \in K$ with respect to $\varphi : P^r \twoheadrightarrow M = \langle m_1, \dots, m_r \rangle$, $e_i \mapsto m_i$
$G = \{g_1, \dots, g_\nu\} \subseteq P[x_0]$	projective $\mathcal{O}$-border prebasis, $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ where we have $a_{ij} \in K[x_0]$
$\mathrm{ind}_{\mathcal{O}}(v)$	index of v with respect to an order ideal $\mathcal{O}$
$\mathrm{NR}_G(v) \in \langle \mathcal{O} \rangle_K$	normal remainder of v with respect to an $\mathcal{O}$-border prebasis G
$\mathrm{NF}_{\mathcal{O},U}(v) \in \langle \mathcal{O} \rangle_K$	normal formal of v with respect to an order ideal $\mathcal{O}$ and a P-submodule U
$\mathrm{BF}_{\mathcal{O}}(v) \in \mathbb{T}^n \langle e_1, \dots, e_r \rangle$	border form of a vector $v \in P^r \setminus \{0\}$ with respect to an order ideal $\mathcal{O}$ in $\mathbb{T}^n \langle e_1, \dots, e_r \rangle$
$\mathrm{BF}_{\mathcal{O}}(U) \subseteq P^r$	border form module of a P-submodule $U \subseteq P^r$ with respect to an order ideal $\mathcal{O}$ in $\mathbb{T}^n \langle e_1, \dots, e_r \rangle$
$\xrightarrow{g_j}$	reduction step using g_j
$\xrightarrow{G}$	rewrite relation associated to G
$\xleftrightarrow{G}$	equivalence relation generated by $\xrightarrow{G}$
σ_{ij}	neighbor syzygy of the neighbors $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ with respect to $\mathcal{O}$
λ_{ij}	neighbor lifting of the neighbors $b_i e_{\beta_i}$ and $b_j e_{\beta_j}$ with respect to $\mathcal{O}$
Λ	set of all neighbor liftings with respect to $\mathcal{O}$
$\mathcal{X}_k \in \mathrm{Mat}_\mu(K)$	k^{th} formal multiplication matrix with respect to an $\mathcal{O}$-border prebasis
$\mathcal{X}_k^{\text{proj}} \in \mathrm{Mat}_\mu(K[x_0])$	k^{th} formal projective multiplication matrix with respect to a projective $\mathcal{O}$-border prebasis
Δ_{ij}	$= \deg_W(b_j) - \deg_W(t_i)$
$d_{\gamma,i}$	$= \gamma - \deg_W(t_i)$, $\gamma \in \mathbb{Z}$
$H : \mathbb{Z} \to \mathbb{Z}$	$\gamma \mapsto \#\{k \in \{1, \dots, \mu\} \mid \deg_W(t_k) \le \gamma\}$
$\Delta H : \mathbb{Z} \to \mathbb{Z}$	$\gamma \mapsto H(\gamma) - H(\gamma - 1)$, first difference function of H
$\mathrm{S}(g_i, g_j)$	S-vector of g_i and g_j

(Projective) Border Basis Schemes

$\mathbb{B}_{\mathcal{O}} \subseteq \mathbb{A}_K^{\mu\nu}$	$\mathcal{O}$-border basis scheme
$B_{\mathcal{O}}$	cordinate ring of $\mathbb{B}_{\mathcal{O}}$
$B_{\mathcal{O}} \to U_{\mathcal{O}}$	universal $\mathcal{O}$-border basis family
$\{c_{11}, \ldots, c_{\mu\nu}\}$	$= \{c_{ij} \mid i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, \nu\}\}$
$\mathcal{X}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$	k^{th} generic multiplication matrix with respect to an order ideal $\mathcal{O}$ in $\mathbb{T}^n$

$\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}} \subseteq \mathbb{B}_{\mathcal{O}}$	projective $\mathcal{O}$-border basis scheme
$B_{\mathcal{O}}^{\mathrm{proj}}$	cordinate ring of $\mathbb{B}_{\mathcal{O}}^{\mathrm{proj}}$
$B_{\mathcal{O}}^{\mathrm{proj}} \to U_{\mathcal{O}}^{\mathrm{proj}}$	universal projective $\mathcal{O}$-border basis family
$\mathcal{X}_k^{\mathrm{proj}} \in \mathrm{Mat}_\mu(K[x_0, c_{ij}])$	k^{th} generic projective multiplication matrix with respect to an order ideal $\mathcal{O}$ in $\mathbb{T}^n$

Algebraic Geometry

$\mathbb{A}_K^n$	n-dimensional affine space over K
$\mathbb{P}_K^n$	n-dimensional projective space over K
$\mathbb{P}_K(\overline{W})$	weighted projective space over K with respect to the matrix $\overline{W}$
$a \in \mathbb{X}$	closed point of the scheme $\mathbb{X}$
$\mathbb{Y} \subseteq \mathbb{X}$	closed subscheme of the scheme $\mathbb{X}$
$\mathrm{Spec}(R)$	spectrum of a ring R
$\mathrm{Proj}(R)$	homogeneous spectrum of a $\mathbb{Z}$-graded ring R

$\mathcal{I}^+(\mathbb{X}) \subseteq P[x_0]$	(saturated) defining ideal of a subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$
$\mathcal{I}_{\mathbb{X}}^+(\mathbb{Y}) \subseteq \overline{P}/\mathcal{I}^+(\mathbb{X})$	(saturated) defining ideal of a subscheme $\mathbb{Y} \subseteq \mathbb{X}$
$\mathrm{HF}_{\mathbb{X}}$	$\gamma \mapsto \dim_K((\overline{P}/\mathcal{I}^+(\mathbb{X}))_\gamma)$, Hilbert function of a closed subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$

$\Delta\,\mathrm{HF}_M$	$\gamma \mapsto \mathrm{HF}_M(\gamma) - \mathrm{HF}_M(\gamma - 1)$, Castelnuovo function of a closed subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$
$\deg(\mathbb{X})$	$= \max\{\mathrm{HF}_{\mathbb{X}}(\gamma) \mid \gamma \in \mathbb{Z}\}$ for a non-empty closed zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$
$\sigma_{\mathbb{X}}$	$= \max\{\gamma \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{X}}(\gamma) < \deg(\mathbb{X})\}$ for a non-empty closed zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$
$\alpha_{\mathbb{X}}(\mathbb{Y})$	$= \min\{\gamma \in \mathbb{Z} \mid (I^+_{\mathbb{X}}(\mathbb{Y}))_\gamma \neq 0\}$, initial degree of a non-empty subscheme $\mathbb{Y}$ of a zero-dimensional subscheme $\mathbb{X} \subseteq \mathbb{P}_K(\overline{W})$

$\mathcal{Z}_L(f) \subseteq \mathbb{A}^n_L$	set of zeros of a polynomial $f \in P$ in an extension field $K \subseteq L$
$\mathcal{Z}_L(I) \subseteq \mathbb{A}^n_L$	set of zeros of an ideal $I \subseteq P$ in an extension field $K \subseteq L$
$\mathcal{Z}(f) \subseteq \mathbb{A}^n_{\overline{K}}$	set of zeros of a polynomial $f \in P$ in the algebraic closure $\overline{K}$
$\mathcal{Z}(I) \subseteq \mathbb{A}^n_{\overline{K}}$	set of zeros of an ideal $I \subseteq P$ in the algebraic closure $\overline{K}$
$\mathcal{I}(S) \subseteq P$	vanishing ideal of a subset of affine points S in the affine space $\mathbb{A}^n_K$

$\mathcal{Z}^+_L(f) \subseteq \mathbb{P}_L(\overline{W})$	set consisting of all projective zeros of a homogeneous polynomial $f \in P[x_0]$ in an extension field $K \subseteq L$
$\mathcal{Z}^+_L(I) \subseteq \mathbb{P}_L(\overline{W})$	set of projective zeros of a homogeneous ideal $I \subseteq P[x_0]$ in an extension field $K \subseteq L$
$\mathcal{Z}^+(f) \subseteq \mathbb{P}_{\overline{K}}(\overline{W})$	set consisting of all projective zeros of a homogeneous polynomial $f \in P[x_0]$ in the algebraic closure $\overline{K}$
$\mathcal{Z}^+(I) \subseteq \mathbb{P}_{\overline{K}}(\overline{W})$	set of projective zeros of a homogeneous ideal $I \subseteq P[x_0]$ in the algebraic closure $\overline{K}$
$H^{\mathrm{inf}} = \mathcal{Z}^+(x_0)$	hyperplane at infinity
$\mathcal{I}^+(S) \subseteq P[x_0]$	homogeneous vanishing ideal of a subset of projective points S in the projective space $\mathbb{P}_K(\overline{W})$

Miscellaneous

$i \coloneqq i + 1$	assignment in an algorithm

Bibliography

[AL94] W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Math. **3**, Amer. Math. Soc., Providence 1994

[AM69] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading 1969

[Bac86] I. Bacharach, Über den Cayley'schen Schnittpunktsatz, *Math. Ann.* **26** (1886), 275–299

[BK96] S. Beck and M. Kreuzer, How to compute the canonical module of a set of points, in: T. Recio and L. Gonzalez-Vega (eds.), *Algorithms in Algebraic Geometry and Applications*, Proc. Conf. MEGA '94, Santander 1994, Progress in Math. **143**, Birkhäuser, Basel 1996, pp. 51–78

[Bou89] N. Bourbaki, *Elements of Mathematics, Algebra I, Chapters 1-3*, Springer-Verlag, New York 1989

[BTBQM00] M.A. Borges-Trenard, M. Borges-Quintana, and T. Mora, Computing Gröbner bases by FGLM techniques in a non-commutative setting, *J. Symbolic Comput.* **30** (2000), 429–449

[Buc65] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Doctoral Thesis, Leopold-Franzens-Universität Innsbruck 1965

[Buc06] B. Buchberger, Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal, *J. Symbolic Comput.* **41** (2006), 475–511

[Cay43] A. Cayley, On the intersection of curves, *Cambridge Math. J.* **3** (1843), 211–213

[CLO05] D. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry*, second ed., Graduate Texts in Math. **185**, Springer, New York 2005

[CLO07] David. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, third ed., Springer, New York 2007

[CS99] M. Caboara and M. Silvestri, Classification of compatible module orderings, *J. Pure Appl. Algebra* **142** (1999), 13–24.

[Dol82] I. Dolgachev, Weighted projective varieties, in: J.B. Carrell (ed.), *Group Actions and Vector Fields*, Lect. Notes in Math. **956**, Springer, Berlin Heidelberg 1982, pp. 34-71

[EH00] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Springer-Verlag, New York 2000

[Eis88] David Eisenbud, Linear sections of determinantal varieties, *Amer. J. Math.* **110** (1988), 541–575

[GH91] M. Giusti and J. Heintz, Algorithmes – disons rapides – pour la décomposition d'une variété algébrique en composantes irréductibles et équidimensionnelles, in: T. Mora and C. Traverso (eds.) *Effective Methods in Algebraic Geometry*, Proc. Conf. MEGA-'90 (Boston) Progress in Math. **94**, Birkhäuser, Boston 1991, pp. 169–194

[GKR93] A.V. Geramita, M. Kreuzer, and L. Robbiano, Cayley-Bacharach schemes and their canonical modules, *Trans. Amer. Math. Soc.* **339** (1993), 163–189

[GW78] S. Goto and K. Watanabe, On graded rings, I, *J. Math. Soc. Japan* **30** (1978), 179–213

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. **52**, Springer, New York 1977

[Har80] J. Harris, The genus of space curves, *Math. Ann.* **249** (1980), 191–204

[Har92] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Math **133**, Springer, New York 1992

[HK71] J. Herzog and E. Kunz, *Der kanonische Modul eines Cohen-Macaulay-Rings*, Lect. Notes in Math. **238**, Springer, Berlin 1971

[HKPP09] D. Heldt, M. Kreuzer, S. Pokutta, and H. Poulisse, Approximate computation of zero-dimensional polynomial ideals, *J. Symbolic Comput.* **44** (2009), 1566–1591

[Hui06] M. Huibregtse, An elementary construction of the multigraded Hilbert scheme of points, *Pacific J. Math.* **223** (2006), 269–315

[IF00] A.R. Iano-Fletcher, Working with weighted complete intersections, in: A. Corti and M. Reid (eds.), *Explicit birational geometry of* 3*-folds, (2000)*, Cambridge Univ. Press, Cambridge 2000, pp. 101–173

[IK99] A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lect. Notes in Math. **1721**, Springer, Berlin Heidelberg 1999

[Kem02] G. Kemper, The calculation of radical ideals in positive characteristic, *J. Symbolic Comput.* **34** (2002), 229–238

[Kem07] G. Kemper, Morphisms and Constructible Sets: Making Two Theorems of Chevalley Constructive, preprint (2007), available at: https://www-m11.ma.tum.de/kemper/publications/

[KK05] A. Kehrein and M. Kreuzer, Characterizations of border bases, *J. Pure Appl. Algebr.* **196** (2005), 251–270

[KK06] A. Kehrein and M. Kreuzer, Computing border bases, *J. Pure Appl. Algebr.* **205** (2006), 279–295

[KK14] M. Kreuzer and M. Kriegl, Gröbner bases for syzygy modules of border bases, *J. Algebra Appl.* **13** (2014), 145003 (32 pages)

[KKR03] A. Kehrein, M. Kreuzer, and L Robbiano, An algebraist's view on border bases, in: B. Bronstein, A.M. Cohen, H. Cohen, D. Eisenbud, B. Sturmfels, and A. Dickenstein, and I.Z. Emiris (eds.), *Solving Polynomial Equations*, Algorithms and Computation in Mathematics **14**, Springer, Berlin Heidelberg 2003, pp. 169–202

[KL91] T. Krick and A. Logar, An algorithm for the computation of the radical of an ideal in the ring of polynomials, in: H.F. Mattson, T. Mora, and T.R.N. Rao (eds.), *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, Lect. Notes in Comput. Sci. **539**, Springer, Berlin Heidelberg 1991, pp. 195–205

[KP11] M. Kreuzer and H. Poulisse, Subideal border bases, *Math. Comp.* **80** (2011), 1135–1154

[KPR10] M. Kreuzer, H. Poulisse, and L. Robbiano, From oil fields to Hilbert schemes, in: L. Robbiano and J. Abbott (eds.), *Approximate Commutative Algebra*, Springer-Verlag Wien 2010, pp. 1–54.

[KR00] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, Berlin Heidelberg 2000

[KR05] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 2*, Springer-Verlag, Berlin Heidelberg 2005

[KR08] M. Kreuzer and L. Robbiano, Deformations of border bases, *Collect. Math.* **59** (2008), 275–297

[Kre94] M. Kreuzer, On the canonical module of a 0-dimensional scheme, *Canad. J. Math.* **46** (1994), 357–379

[Kre98] M. Kreuzer, *Beiträge zur Theorie der nulldimensionalen Unterschemata projektiver Räume*, Habilitationsschrift 1997, Regensburger Math. Schriften **26**, Universität Regensburg 1998

[Kre00] M. Kreuzer, On the canonical ideal of a set of points, *Boll. Unione Mat. Ital.* **3-B.1** (2000), 221–261

[Kre01] M. Kreuzer, Algorithms for checking uniformity conditions and applications in coding theory, in: A.V. Geramita (ed.), *Proc. of the Workshop "Zero-Dimensional Schemes and Applications", Naples 2000*, Queen's Papers in Pure and Appl. Math. **123**, Queen's University, Kingston 2001, pp. C1-C9

[Kri13] M. Kriegl, Module border bases, preprint (2013), available at: http://arxiv.org/abs/1302.6383

[Kun85] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Boston 1985

[Lim14] Jan Limbeck, *Computation of Approximate Border Bases and Applications*, Doctoral Thesis, Universität Passau 2014

[Mat01] R. Matsumoto, Computing the radical of an ideal in positive characteristic, *J. Symbol. Comput.* **32** (2001), 263–271

[MB82] H.M. Möller and B. Buchberger The construction of multivariate polynomials with preassigned zeros, in: J. Calmet (ed.), *Computer Algebra*, EUROCAM '82, Proc. Conf. Marseille 1982, Lect. Notes Comput. Sci. **144** (1982), pp. 24–31

[MM82] E. Mayr, A. Meyer, The complexity of the word problems for commutative semi-groups and polynomials ideals, *Adv. Math.* **46** (1982), 305-329

[Mou99] B. Mourrain, A new criterion for normal form algorithms, in: M. Fossorier, H. Imai, S. Lin, and A. Poli (eds.), *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, Lect. Notes in Comput. Sci. **1719**, Springer, Berlin Heidelberg 1999 pp. 430–442

[MP04] J. Migliore and C. Peterson, A symbolic test for (i,j)-uniformity in reduced zero-dimensional schemes, *J. Symbolic Comput.* **37** (2004), 403–413

[MR93] T. Mora and L. Robbiano, Points in affine and projective spaces, in: D. Eisenbud and L. Robbiano (eds.), *Computational Algebraic Geometry and Commutative Algebra*, Cambridge Univ. Press, Cambridge 1993, pp. 106–150

[Mum66] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Ann. of Math. Stud. **59**, Princeton Univ. Press, Princeton 1966.

[Rat87] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.* **276** (1987), 565–579

[Rob09] L. Robbiano, On border basis and Gröbner basis schemes, *Collect. Math.* **60** (2009), 11–25

[Sch80] Frank-Olaf Schreyer, *Die Berechnung von Syzygien mit dem verallgemeinerten Weierstraßschen Divisionssatz und eine Anwendung auf analytische Cohen–Macaulay Stellenalgebren minimaler Multiplizität*, Diploma Thesis, Universität Hamburg 1980

[Ste04] H.J. Stetter, *Numerical Polynomial Algebra*, SIAM, Philadelphia 2004

Publications

Journal Papers

M. Kreuzer and M. Kriegl, Gröbner bases for syzygy modules of border bases, *J. Algebra Appl.* **13** (2014), 145003 (32 pages)

Preprint Papers

M. Kriegl, Module border bases, preprint (2013), available at: http://arxiv.org/abs/1302.6383