



Dissertation

# Various Differents for 0-Dimensional Schemes and Applications

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*To my parents, my wife and my daughter*



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# Contents

<b>Acknowledgements</b>	<b>iii</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>9</b>
2.1 Noether Differents of Algebras . . . . .	10
2.2 Fitting Ideals of Finitely Generated Modules . . . . .	22
2.3 First Properties of 0-Dimensional Schemes . . . . .	28
2.4 Trace Maps for 0-Dimensional Schemes . . . . .	41
<b>3 Various Differents for 0-Dimensional Schemes</b>	<b>49</b>
3.1 Noether Differents for 0-Dimensional Schemes . . . . .	50
3.2 Dedekind Differents for 0-Dimensional Schemes . . . . .	62
3.2.1 Dedekind Differents for Locally Gorenstein Schemes . . . . .	63
3.2.2 Dedekind Differents for 0-Dimensional Smooth Schemes . . . . .	72
3.2.3 Computing the Dedekind Differents . . . . .	75
3.3 Kähler Differents for 0-Dimensional Schemes . . . . .	85
<b>4 Differents and Uniformity of 0-Dimensional Schemes</b>	<b>101</b>
4.1 Differents and the Cayley-Bacharach Property . . . . .	103
4.2 Differents for Schemes Having Generic Hilbert Function . . . . .	121
4.3 Cayley-Bacharach Properties and Liaison . . . . .	130
4.4 Differents and Higher Uniformity of 0-Dimensional Schemes . . . . .	143
<b>5 Differents for Some Special Cases and Applications</b>	<b>157</b>
5.1 Differents for Almost Complete Intersections . . . . .	158

5.2	The Kähler Differents of the Coordinate Ring of 0-Dimensional Schemes	168
5.3	Differents for Fat Point Schemes . . . . .	175
	<b>Appendix</b>	<b>189</b>
	<b>Bibliography</b>	<b>201</b>

# Introduction

Let  $R_o$  be a commutative ring with identity. To any commutative algebra  $R/R_o$  we can associate various ideals in  $R$  which we call the differents. The most well known different is the Dedekind different which was introduced in algebraic number theory by Richard Dedekind in 1882. The other differents which are closely related to the Dedekind different are the Noether and Kähler differents. It is useful to compare these differents to each other and to relate them to properties of the algebra  $R/R_o$ . This direction of research was initiated by Emmy Noether [Noe], and pursued by many authors in the last half century (see for instance [AB], [Fos], [Her], [Ku1], [Ku5], [Mac], [SS], and [Wal]). It turned out that many structural properties of the algebra  $R/R_o$  can be phrased as properties of its differents. For example, one can obtain several interesting criteria for the algebra in terms of these differents such as the Ramification Criterion, the Regularity Criterion, and the Smoothness Criterion (see [Ku5]).

The goal of this thesis is to study the Noether, Dedekind, and Kähler differents of a particular class of algebras. More precisely, we investigate these differents for the homogeneous coordinate ring of a 0-dimensional scheme  $\mathbb{X}$  in the projective  $n$ -space  $\mathbb{P}_K^n$  over an arbitrary field  $K$ . This approach is inspired by the work of M. Kreuzer and his coworkers ([GKR], [Kr2], [Kr4]) which suggests that, in order to study the geometry of the scheme  $\mathbb{X}$ , one may start by considering related algebraic objects. Given such a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , let  $\mathcal{I}_{\mathbb{X}}$  denote the homogeneous vanishing ideal of  $\mathbb{X}$  in  $P = K[X_0, \dots, X_n]$ . The homogeneous coordinate ring of  $\mathbb{X}$  is then given by  $R = P/\mathcal{I}_{\mathbb{X}}$ . We always assume that no point of the support of  $\mathbb{X}$  lies on the hyperplane at infinity  $\mathcal{Z}^+(X_0)$ . Then the image  $x_0$  of  $X_0$  in  $R$  is a non-zerodivisor of  $R$  and the algebra  $R/K[x_0]$  is graded-free of rank  $\deg(\mathbb{X})$  (see [Kr3] and [KR3, Section 4.3]). In order to get more information about  $R/K[x_0]$ , and thus also about  $\mathbb{X}$ , it is useful to define and to consider the algebraic structure of the Noether, Dedekind,

and Kähler differentials of  $R/K[x_0]$ . In particular, it is interesting to study the relations between the algebraic structure of these differentials and geometric properties of the scheme  $\mathbb{X}$ . Furthermore, we think that the Noether, Dedekind, and Kähler differentials for 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  provide a source of excellent examples in which computer algebra methods can be applied to examine subtle structural properties of these objects.

## Motivating Questions

Let us briefly recall the definitions of the Noether and Kähler differentials for a 0-dimensional scheme  $\mathbb{X}$  in  $\mathbb{P}_K^n$ . As above, let  $\mathcal{I}_{\mathbb{X}}$  be the homogeneous vanishing ideal of  $\mathbb{X}$ , and let  $R = P/\mathcal{I}_{\mathbb{X}}$  be the homogeneous coordinate ring of  $\mathbb{X}$ . In the standard graded  $K[x_0]$ -algebra  $R \otimes_{K[x_0]} R = \bigoplus_{i \geq 0} (\bigoplus_{j+k=i} R_j \otimes R_k)$ , we have the homogeneous ideal  $\mathcal{J} = \text{Ker}(R \otimes_{K[x_0]} R \xrightarrow{\mu} R)$ , where  $\mu$  is the homogeneous  $R$ -linear map of degree zero given by  $\mu(f \otimes g) = fg$ . The Noether differential of the algebra  $R/K[x_0]$  (or for  $\mathbb{X}$  w.r.t.  $x_0$ ), denoted by  $\vartheta_N(R/K[x_0])$ , is the image of the annihilator of  $\mathcal{J}$  in  $R \otimes_{K[x_0]} R$  under the map  $\mu$ . It is clear that  $\vartheta_N(R/K[x_0])$  is a homogeneous ideal of  $R$ .

Notice that the module of Kähler differentials of the algebra  $R/K[x_0]$  is the finitely generated graded  $R$ -module  $\Omega_{R/K[x_0]}^1 = \mathcal{J}/\mathcal{J}^2$ . Moreover, we can associate with  $\Omega_{R/K[x_0]}^1$  an ascending sequence of homogeneous ideals of  $R$  which are known as the Fitting invariants or Fitting ideals of  $\Omega_{R/K[x_0]}^1$ . The most important one is the initial Fitting ideal of  $\Omega_{R/K[x_0]}^1$  which is the homogeneous ideal of  $R$  generated by the images of the maximal minors of the Jacobian matrix of the generators of  $\mathcal{I}_{\mathbb{X}}$ . We denote this invariant by  $\vartheta_K(R/K[x_0])$  and call it the Kähler differential of the algebra  $R/K[x_0]$  (or for  $\mathbb{X}$  w.r.t.  $x_0$ ). In order to define the Dedekind differential  $\vartheta_D(R/K[x_0])$  of the algebra  $R/K[x_0]$  (or for  $\mathbb{X}$  w.r.t.  $x_0$ ), it is necessary that the scheme  $\mathbb{X}$  satisfies some restrictive hypotheses. For example the definition in [DK] of  $\vartheta_D(R/K[x_0])$  requires that  $\mathbb{X}$  is reduced and has  $K$ -rational support. When these differentials are well-defined, we ask ourselves the following questions.

**Question 1.0.1.** *Can one compute  $\vartheta_N(R/K[x_0])$ ,  $\vartheta_K(R/K[x_0])$  and  $\vartheta_D(R/K[x_0])$  by using the existing functionality in a computer algebra system such as ApCoCoA?*

**Question 1.0.2.** *What can one say about these differentials and their relations for some special classes of schemes  $\mathbb{X}$  (for instance, for complete intersections, arithmetically Gorenstein schemes, locally Gorenstein schemes, or smooth schemes)?*

Furthermore, since all three kinds of differentials are homogeneous ideals of  $R$ , a natural question to ask is:

**Question 1.0.3.** *What are the Hilbert functions of these differentials?*

In particular, we would like to determine their Hilbert polynomials and their regularity indices. Apart from some special cases, to exactly determine the Hilbert functions of the Noether, Dedekind, and Kähler differentials of the algebra  $R/K[x_0]$  is not an easy problem, so we may try at least to find (possibly sharp) lower and upper bounds for their regularity indices.

Now let us turn these questions around and ask about algebraic consequences of geometric properties of  $\mathbb{X}$  for the Noether, Dedekind, and Kähler differentials. Suppose that  $K$  is an algebraically closed field, and let  $r_{\mathbb{X}} \geq 0$  be the regularity index of the Hilbert function  $\text{HF}_{\mathbb{X}}$  of  $R$ . We say that  $\mathbb{X}$  is a **Cayley-Bacharach scheme** (in short, **CB-scheme**) if every hypersurface of degree  $r_{\mathbb{X}} - 1$  which contains a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$  automatically contains  $\mathbb{X}$ . This notion was introduced for a finite set of distinct  $K$ -rational points in [GKR], and then was generalized for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an algebraically closed field  $K$  in [Kr2]. The main results in these papers characterize CB-schemes in terms of the structure of the canonical module of their homogeneous coordinate ring. This suggests the following questions:

**Question 1.0.4.** *Let  $K$  be an arbitrary field. Can one generalize the Cayley-Bacharach property for a 0-dimensional scheme  $\mathbb{X}$  in  $\mathbb{P}_K^n$ ? If the answer is yes, can one characterize the Cayley-Bacharach property in terms of the algebraic structure of the Noether, Dedekind, and Kähler differentials?*

It is worth noticing here that the definition of the Cayley-Bacharach property as above does not work for arbitrary 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  if  $K$  is not an algebraically closed field. For instance, the 0-dimensional reduced scheme  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  with  $\mathcal{I}_{\mathbb{X}} = \langle X_1^3 + 2X_0^3, X_2 \rangle$  has degree 3 but it does not have any subscheme of degree 2. The reason is that no closed point of  $\mathbb{X}$  is  $K$ -rational. Thus, for the first part of Question 1.0.4, we need more information about the scheme  $\mathbb{X}$ . Moreover, the last part of Question 1.0.4 can be generalized as follows:

**Question 1.0.5.** *How is the geometry (e.g., the properties of being in  $(i, j)$ -uniform, in uniform position) of  $\mathbb{X}$  reflected in the algebraic structure of these differentials?*

Another topic we examine in this thesis is to construct new applications of the

Noether, Dedekind, and Kähler differentials of  $R/K[x_0]$ . For instance, we investigate the relationships between these differentials and the  $i$ -th Fitting ideals of the module of Kähler differentials  $\Omega_{R/K}^1$ , and we consider these differentials for almost complete intersections. Also, we look more closely at the Hilbert functions and the regularity indices of these differentials for fat point schemes.

## Overview

Throughout the thesis we use the notation and terminology introduced in [KR2], [KR3], [Kr3] and [Ku5] unless stated otherwise. The main work of the thesis will be to examine a number of cases in which the questions posed above can be answered. We now give an overview of the individual contributions of the thesis. Since every chapter starts with an explanation of its organization, we omit such descriptions here.

Chapter 2 contains background results that we will need in the subsequent chapters. We fix some terminology and notation as well. The starting point of this chapter is the Noether different of an arbitrary algebra. We show how one can compute the Noether different of an algebra of finite type (see Proposition 2.1.21). Then we recall several well-known properties of the Fitting ideals of a finitely generated module and provide examples to illustrate these properties. Next, we focus on considering 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an arbitrary field  $K$  with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . We introduce the concept of maximal  $p_j$ -subschemes of  $\mathbb{X}$  (see Definition 2.3.16) and give some descriptions of them and their Hilbert functions. We also extend from [GKR] the definition of the degree of each point of  $\mathbb{X}$  (see Definition 2.3.23). Furthermore, we generalize the notion and some results found in [Kr2] of a separator of a subscheme of  $\mathbb{X}$  of degree  $\deg(\mathbb{X}) - 1$  to a set of separators of a maximal  $p_j$ -subscheme of  $\mathbb{X}$ . Lastly, we collect some facts concerning the trace maps for 0-dimensional schemes in  $\mathbb{P}_K^n$ .

In Chapter 3 we explore the Noether, Dedekind, and Kähler differentials for 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$ . In other words, we define these differentials for the algebra  $R/K[x_0]$  (or for  $\mathbb{X}$  w.r.t.  $x_0$ ) and then give answers to Question 1.0.1, Question 1.0.2, and Question 1.0.3 in some cases. Since the Noether different  $\vartheta_N(R/K[x_0])$  and the Kähler different  $\vartheta_K(R/K[x_0])$  are already defined in general, it remains to define the Dedekind different  $\vartheta_D(R/K[x_0])$ . We show that  $\vartheta_D(R/K[x_0])$  is well-defined if  $\mathbb{X}$  is locally Gorenstein (see Section 3.2).

Next we take a look at how to compute these differentials and examine their relations. Since  $\vartheta_K(R/K[x_0])$  is a homogeneous ideal in  $R$  generated by the images of the maximal

minors of the Jacobian matrix of the generators of  $\mathcal{I}_{\mathbb{X}}$ , it is not hard to compute  $\vartheta_K(R/K[x_0])$ . In order to compute  $\vartheta_N(R/K[x_0])$  one can use the algorithm given in Proposition 3.1.6. Moreover, we construct an algorithm for computing a homogeneous minimal system of generators of  $\vartheta_D(R/K[x_0])$  (see Proposition 3.2.29). The approach for computing the Dedekind different is based on the description of the Dedekind complementary module of the algebra  $R/K[x_0]$  in terms of the sets of separators. As in the general case, the relation between the Kähler different and the Noether different is  $\vartheta_N(R/K[x_0])^n \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_N(R/K[x_0])$ . In particular, the last inclusion becomes an equality if  $\mathbb{X}$  is a complete intersection. We also show that if  $\mathbb{X}$  is smooth then  $\vartheta_N(R/K[x_0]) = \vartheta_D(R/K[x_0])$  (see Theorem 3.2.17). However, the Noether and Dedekind differentials can be different even when  $\mathbb{X}$  is a complete intersection (see Example 3.3.6).

One of our main tasks in this chapter is to give an answer to Question 1.0.3 in some cases. In Proposition 3.2.5, we show that if  $\mathbb{X}$  is locally Gorenstein then  $\text{HP}_{\vartheta_D(R/K[x_0])}(z) = \text{deg}(\mathbb{X})$  and  $\text{ri}(\vartheta_D(R/K[x_0])) \leq 2r_{\mathbb{X}}$ , where  $r_{\mathbb{X}}$  is the regularity index of  $\text{HF}_{\mathbb{X}}$ . It is not easy to exactly determine the Hilbert polynomials of the Noether and Kähler differentials in general. Fortunately, we can give (sharp) lower and upper bounds for  $\text{HP}_{\vartheta_K(R/K[x_0])}(z)$  (see Proposition 3.3.19) by using sets of separators. As a consequence, we use the relations between these differentials mentioned above to derive bounds for  $\text{HP}_{\vartheta_N(R/K[x_0])}(z)$ . Also, we can completely describe the Hilbert functions of some of these differentials for several special classes of schemes  $\mathbb{X}$  such as complete intersections and arithmetically Gorenstein schemes (see Proposition 3.2.8 and Corollaries 3.3.5 and 3.3.7). Furthermore, Proposition 3.2.11 and Proposition 3.3.14 indicate that if  $\mathbb{X}$  is a projective point set, then  $\text{HP}_{\vartheta_N(R/K[x_0])}(z) = \text{HP}_{\vartheta_D(R/K[x_0])}(z) = \text{HP}_{\vartheta_K(R/K[x_0])}(z) = \text{deg}(\mathbb{X})$ ,  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$  and  $2r_{\mathbb{X}} \leq \text{ri}(\vartheta_K(R/K[x_0])) \leq nr_{\mathbb{X}}$ .

The principal results of Chapter 4 are related to Question 1.0.4 and Question 1.0.5. We first generalize the Cayley-Bacharach property to 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an arbitrary field  $K$  (see Section 4.1). This gives a positive answer to the first part of Question 1.0.4. In Theorem 4.1.7, we characterize CB-schemes in terms of their Dedekind differentials. Using this theorem, we derive some consequences for the Hilbert functions and the regularity indices of the Dedekind different for a 0-dimensional locally Gorenstein CB-scheme as well as the Noether different for a 0-dimensional smooth CB-scheme. This theorem is also an important tool to prove Proposition 4.1.15 which not only provides a criterion for a 0-dimensional locally Gorenstein scheme to be arithmetically Gorenstein but also provides an answer to the question of P. Griffiths

and J. Harris [GH]: **CB-scheme**  $+(?) = \mathbf{Complete\ intersection?}$  in the case of a 0-dimensional smooth subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^n$ .

Via the Kähler different we are able to describe the Cayley-Bacharach property for projective point sets  $\mathbb{X}$  in  $\mathbb{P}_K^n$  which have generic Hilbert function (i.e.,  $\mathrm{HF}_{\mathbb{X}}(i) = \min\{\deg(\mathbb{X}), \binom{i+n}{n}\}$  for  $i \in \mathbb{Z}$ ). We also generalize some results found in [GKR] such as Dedekind's formula (see Proposition 4.1.27) and the characterization of the Cayley-Bacharach property by using Liaison theory (see Theorem 4.3.6).

When a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  has  $K$ -rational support, we say that  $\mathbb{X}$  is  $(i, j)$ -**uniform** if every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$  satisfies  $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ . Notice that if  $\mathbb{X}$  has  $K$ -rational support then  $\mathbb{X}$  is  $(1, r_{\mathbb{X}} - 1)$ -uniform if and only if it is a CB-scheme. In the case that  $\mathbb{X}$  is locally Gorenstein and is  $(i, j)$ -uniform, we describe relations between the Dedekind different and the homogeneous saturated ideals of subschemes  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{X}) - i$  (see Proposition 4.4.10). Based on these relations, we give a characterization of a projective point set  $\mathbb{X}$  with  $\Delta_{\mathbb{X}} = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) \geq 2$  to be  $(2, r_{\mathbb{X}} - 1)$ -uniform (see Proposition 4.4.12). In addition, several propositions on the uniformities of a level scheme and cohomological uniformity are proven.

In Chapter 5 we are interested in studying the Noether, Dedekind, and Kähler differentials for finite special classes of schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  and finding out some applications of these differentials. First, we investigate these differentials for reduced 0-dimensional almost complete intersections  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over a perfect field  $K$ . We provide an explicit presentation of the Kähler different and derive a connection between these differentials (see also in [Her]). Furthermore, in the projective plane we can precisely compute the first syzygy module of the Kähler different by constructing a homogeneous system of generators of the normal module  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$  (see Proposition 5.1.12 and Corollary 5.1.13).

Next we establish a relation between the Kähler different  $\vartheta_K(R/K[x_0])$  and the first Kähler different  $\vartheta^{(1)}(R/K)$  of the algebra  $R/K$  (see Lemma 5.2.1). From this relation we derive bounds for the Hilbert polynomial and the regularity index of  $\vartheta^{(1)}(R/K)$ , and give some characterizations of a complete intersection in terms of  $\vartheta^{(1)}(R/K)$  (see Propositions 5.2.5 and 5.2.8).

Finally, we address the question what happens to these differentials for fat point schemes  $\mathbb{W}$  in  $\mathbb{P}_K^n$ . Let  $S$  denote the homogeneous coordinate ring of  $\mathbb{W}$ . We show that  $\vartheta_K(S/K[x_0])$  is non-zero if and only if some of the points of  $\mathbb{W}$  are reduced (see Lemma 5.3.4). In Theorem 5.3.6, we determine the Hilbert polynomial of  $\vartheta_K(S/K[x_0])$  and give upper bound for its regularity index. This bound enables us to use the upper



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bound for the regularity index of a fat point scheme to estimate the regularity index of  $\vartheta_K(S/K[x_0])$ . Many results about sharp bounds for the regularity index of a fat point scheme can be found in the literatures (see e.g. [CTV], [DG], [FL], [Thi], or [TV] *et al*). When the supporting set of points  $\mathbb{X}$  is arithmetically Gorenstein (which implies the Cayley-Bacharach property) or satisfies some uniformity condition, we can improve the bounds for the index of regularity of the Kähler different (see Propositions 5.3.13 and 5.3.15). Returning to the algebra  $S/K$ , we use the Hilbert functions of  $\vartheta^{(i)}(S/K)$  ( $i = 1, \dots, n - 1$ ) to formulate a criterion for  $\mathbb{W}$  to be reduced, compute their Hilbert polynomials and bound their regularity indices (see Proposition 5.3.16). At the same time we provide bounds for the Hilbert polynomials of  $\vartheta^{(n)}(S/K)$  and  $\vartheta_N(S/K[x_0])$ .

The computer algebra system ApCoCoA [ApC] was used during our work to perform the computations in most examples included in this thesis. In the appendix the ApCoCoA functions which implement algorithms and procedures for computing the Noether, Dedekind, and Kähler differentials and their Hilbert functions are provided. We also explain their usage with some examples.



# Chapter 2

## Preliminaries

In this chapter we lay the mathematical foundation for the thesis by collecting and extending some basic concepts, results, and techniques which are useful for the later chapters.

The chapter is divided into 4 sections. In Section 2.1 we discuss one of the main objects in this thesis, that is the Noether different of an algebra. We start this section by providing some elementary notions in algebra and several well known results of the Noether different of an algebra. We then establish other formulas for the Noether different of an algebra of finite type (see Propositions 2.1.12 and 2.1.14). Moreover, we use some techniques from the theory of Gröbner bases over rings to work out an algorithm to compute the Noether different of an algebra of finite type (see Proposition 2.1.21). Section 2.2 introduces the Fitting ideals of a finitely generated module that we use to define the Kähler different in Chapter 3. Some well known results of the Fitting ideals are collected in this section. We also give many examples to illustrate their properties. In Section 2.3 we introduce 0-dimensional schemes in the projective  $n$ -space  $\mathbb{P}_K^n$  over an arbitrary field  $K$ . Explicitly, we first recall from [Kr3, Section 1] several first results on a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ . If we write  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$  for some  $s \geq 1$ , then we define  $p_j$ -subschemes and maximal  $p_j$ -subschemes of  $\mathbb{X}$  (see Definition 2.3.16). By using the techniques that are inspired by those of [Kr2] and [Kr3], we establish a 1-1 correspondence between a socle element of the local ring at a point  $p_j \in \text{Supp}(\mathbb{X})$  and a maximal  $p_j$ -subscheme of  $\mathbb{X}$  (see Proposition 2.3.17). This correspondence also tells us that the scheme  $\mathbb{X}$  contains a subscheme of degree  $\deg(\mathbb{X}) - 1$  if and only if its support contains at least one  $K$ -rational point. In addition, we extend the definition of a separator of a subscheme of  $\mathbb{X}$  of degree  $\deg(\mathbb{X}) - 1$  to a set of separators of a maximal  $p_j$ -subscheme of  $\mathbb{X}$  as well as the definition of the degree of a  $K$ -rational point to a closed point in the support of  $\mathbb{X}$ . The extension of the degrees of closed

points of  $\mathbb{X}$  is a useful tool to define the Cayley-Bacharach property of a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an arbitrary field  $K$  in Chapter 4. We also derive some descriptions of the Hilbert function and the homogeneous vanishing ideal of a maximal  $p_j$ -subscheme of the scheme  $\mathbb{X}$  (see Proposition 2.3.21). We end this section with some generalizations of results in [Kr3, Section 1] of a set of separators of a maximal  $p_j$ -subscheme of the scheme  $\mathbb{X}$  (see Lemmata 2.3.25 and 2.3.26). The final section is a collection of facts concerning trace maps for a 0-dimensional scheme  $\mathbb{X}$  in  $\mathbb{P}_K^n$ . We also add some descriptions of the canonical module of the homogeneous coordinate ring of  $\mathbb{X}$  (see Proposition 2.4.6). Furthermore, we discuss some conditions equivalent to a 0-dimensional scheme being arithmetically Gorenstein (see Theorem 2.4.14 and Proposition 2.4.18).

## 2.1 Noether Differentials of Algebras

In the following by a ring  $R_o$  we shall always mean a commutative ring with identity element  $1_{R_o}$ . A **ring homomorphism** from a ring  $R_o$  to a ring  $R$  is a map  $\varphi : R_o \rightarrow R$  such that  $\varphi(1_{R_o}) = 1_R$  and for all elements  $a, b \in R_o$  we have  $\varphi(a + b) = \varphi(a) + \varphi(b)$  and  $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$  (i.e.,  $\varphi$  preserves the ring operations). An  $R_o$ -**module**  $M$  is a commutative group  $(M, +)$  with a scalar multiplication  $\cdot : R_o \times M \rightarrow M$  such that  $1_{R_o} \cdot m = m$  for all  $m \in M$ , and such that the associative and distributive laws are satisfied. An  $R_o$ -**submodule** of  $M$  is a (commutative) subgroup  $N \subseteq M$  such that  $R_o \cdot N \subseteq N$ . In particular, an  $R_o$ -submodule of the  $R_o$ -module  $R_o$  is called an **ideal** of  $R_o$ . This can be rephrased by saying that a subset  $I$  of  $R_o$  is an ideal if it is an additive subgroup of  $R_o$  such that if  $a \in R_o$  and  $b \in I$ , then  $ab \in I$ .

Some ideals of  $R_o$  have special properties. For instance, an ideal  $I \subsetneq R_o$  is called a **prime ideal** if  $ab \in I$  implies  $a \in I$  or  $b \in I$  for all  $a, b \in R_o$ , and it is called a **maximal ideal** of  $R_o$  if there is no ideal  $J$  such that  $I \subsetneq J \subsetneq R_o$ . Note that  $I$  is a prime ideal if and only if  $R_o/I$  is an integral domain, that  $I$  is a maximal ideal if and only if  $R_o/I$  is a field, and hence that maximal ideals are prime ideals.

Let  $M$  be an  $R_o$ -module. Given two  $R_o$ -submodules  $N$  and  $N'$  of  $M$ , the set

$$N :_{R_o} N' = \{a \in R_o \mid a \cdot N' \subseteq N\}$$

is an ideal of  $R_o$ . It is called the **colon ideal** (or the **ideal quotient** if  $M = R_o$ ) of  $N$  by  $N'$ . The ideal  $\langle 0 \rangle :_{R_o} M$  is called the **annihilator** of  $M$  and denoted by  $\text{Ann}_{R_o}(M)$ . For  $m \in M$  we write  $\text{Ann}_{R_o}(m) = \{a \in R_o \mid a \cdot m = 0\}$  and call it the **annihilator**

of  $m$ . It is clear that  $\text{Ann}_{R_o}(R_o/I) = I$  for every ideal  $I$  of  $R_o$ , and that a non-zero element  $a \in R_o$  is a non-zerodivisor of  $R_o$  if and only if  $\text{Ann}_{R_o}(a) = \langle 0 \rangle$ . For further information on module theory, see for instance [Ei1], [KR2] and [Mat].

**Definition 2.1.1.** An **algebra** is a triple  $(R, R_o, \varphi)$  where  $R$  and  $R_o$  are rings and  $\varphi : R_o \rightarrow R$  is a ring homomorphism, called the **structural homomorphism** of the algebra. The algebra is usually denoted by  $R/R_o$  or simply by  $R$ , and we sometimes call it an  $R_o$ -**algebra**  $R$  or an **algebra**  $R$  **over**  $R_o$ .

An algebra  $R'/R_o$  is called a **subalgebra** of  $R/R_o$  if  $R'$  is a subring of  $R$  that contains the image of  $R_o$ .

Given two algebras  $R/R_o$  and  $S/R_o$  with structural homomorphisms  $\varphi$  and  $\psi$ , a ring homomorphism  $\theta : R \rightarrow S$  is called an  $R_o$ -**algebra homomorphism** if we have  $\theta(\varphi(f) \cdot g) = \psi(f) \cdot \theta(g)$  for all  $f \in R_o$  and all  $g \in R$ .

From the definition we see that for two  $R_o$ -algebra homomorphisms  $\theta_1 : R \rightarrow S$  and  $\theta_2 : S \rightarrow T$ , the composition map  $\theta_2 \circ \theta_1 : R \rightarrow T$  is also an  $R_o$ -algebra homomorphism and  $\text{Ker}(\theta_2 \circ \theta_1) = \theta_1^{-1}(\text{Ker}(\theta_2))$ . Moreover, if  $R/R_o$  and  $S/R$  are algebras, then  $S/R_o$  is an algebra.

**Definition 2.1.2.** Let  $R/R_o$  be an algebra and let  $\Lambda$  be a set.

- (i) A set  $\{x_\lambda \mid \lambda \in \Lambda\}$  of elements of  $R$  is called a **system of generators** of  $R/R_o$  if for every element  $f \in R$  there is a finite subset  $\{\lambda_1, \dots, \lambda_n\}$  of  $\Lambda$  and a polynomial  $P(X_1, \dots, X_n) \in R_o[X_1, \dots, X_n]$  such that  $f = P(x_{\lambda_1}, \dots, x_{\lambda_n})$ . In this case we write  $R = R_o[x_\lambda \mid \lambda \in \Lambda]$ .
- (ii) The algebra  $R/R_o$  is said to be an **algebra of finite type** if it has a finite system of generators. An algebra of finite type over a field  $K$  is also called an **affine  $K$ -algebra**.
- (iii) The algebra  $R/R_o$  is called **finitely generated** (respectively, **free**, **projective**, **flat**, ...) if  $R$ , considered as an  $R_o$ -module, has the corresponding property.

For an algebra  $R/R_o$ , the tensor product  $R \otimes_{R_o} R$  is a ring (and hence an  $R_o$ -algebra) with respect to the multiplication  $(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1 f_2 \otimes g_1 g_2$  for all  $f_1, f_2, g_1, g_2 \in R$ . We call  $R \otimes_{R_o} R$  the **enveloping algebra** of the algebra  $R/R_o$ . Let  $\mathcal{J}$  denote the kernel of the canonical multiplication map  $\mu : R \otimes_{R_o} R \rightarrow R$  given by  $\mu(f \otimes g) = fg$  for  $f, g \in R$ . It is clear that  $\mathcal{J}$  is an ideal of  $R \otimes_{R_o} R$ .

**Lemma 2.1.3.** *We have*

$$\mathcal{J} = \langle f \otimes 1 - 1 \otimes f \mid f \in R \rangle_{R \otimes_{R_o} R}.$$

If  $R$  is generated as an  $R_o$ -algebra by  $\{x_\lambda \mid \lambda \in \Lambda\}$ , then

$$\mathcal{J} = \langle x_\lambda \otimes 1 - 1 \otimes x_\lambda \mid \lambda \in \Lambda \rangle_{R \otimes_{R_o} R}.$$

*Proof.* See [Ku7, G.7]. □

Now let  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J}) = \{f \in R \otimes_{R_o} R \mid f \cdot \mathcal{J} = 0\}$  be the annihilator of the ideal  $\mathcal{J}$ . The ring  $R \otimes_{R_o} R$  can be considered as an  $R$ -module in two ways, namely via

$$\alpha_1 : R \longrightarrow R \otimes_{R_o} R, \quad \alpha_1(f) = f \otimes 1 \quad \text{and} \quad \alpha_2 : R \longrightarrow R \otimes_{R_o} R, \quad \alpha_2(f) = 1 \otimes f.$$

Similarly,  $\mathcal{J}$  and  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$  are  $R$ -modules in two ways. However, on  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$  these two  $R$ -module structures coincide, since  $(f \otimes 1 - 1 \otimes f) \cdot \text{Ann}_{R \otimes_{R_o} R}(\mathcal{J}) = 0$  by the definition of the annihilator of  $\mathcal{J}$ . We can thus consider  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$  as an  $R$ -module in a unique way. Note that the image of  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$  under  $\mu$  is an ideal of  $R$ .

**Definition 2.1.4.** Let  $R/R_o$  be an algebra. The ideal  $\vartheta_N(R/R_o) = \mu(\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J}))$  is called the **Noether different** of the algebra  $R/R_o$ .

**Example 2.1.5.** Let  $R_o$  be a ring, and let  $R/R_o$  be the algebra given by the presentation  $R = R_o[X]/\langle X^2 \rangle$ . Let  $x$  denote the residue class of  $X$  in  $R$ . Then  $x^2 = 0$  and  $\mathcal{J} = \text{Ker}(\mu) = \langle x \otimes 1 - 1 \otimes x \rangle_{R \otimes_{R_o} R}$  (see Lemma 2.1.3). We also see that  $x \otimes x$  and  $x \otimes 1 + 1 \otimes x$  belong to  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$ . Let  $h = \sum_{i=1}^n (a_i + b_i x) \otimes (c_i + d_i x) \in \text{Ann}_{R \otimes_{R_o} R}(\mathcal{J})$  with  $a_i, b_i, c_i, d_i \in R_o$  and  $n \geq 1$ . We have

$$\begin{aligned} 0 &= h \cdot (x \otimes 1 - 1 \otimes x) = \sum_{i=1}^n (a_i x \otimes (c_i + d_i x) - (a_i + b_i x) \otimes c_i x) \\ &= \sum_{i=1}^n a_i c_i (x \otimes 1 - 1 \otimes x) + \sum_{i=1}^n (a_i d_i - b_i c_i) (x \otimes x). \end{aligned}$$

This implies  $\sum_{i=1}^n a_i c_i = 0$  and  $\sum_{i=1}^n a_i d_i = \sum_{i=1}^n b_i c_i$ , since  $R \otimes_{R_o} R$  is a free  $R_o$ -module with a basis  $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ . Thus  $h = \sum_{i=1}^n (a_i + b_i x) \otimes (c_i + d_i x) = \sum_{i=1}^n (a_i d_i \otimes x + b_i c_i x \otimes 1 + b_i d_i x \otimes x) = \sum_{i=1}^n a_i d_i (x \otimes 1 + 1 \otimes x) + \sum_{i=1}^n b_i d_i x \otimes x$ , and so  $h$  is contained in  $\langle x \otimes x, x \otimes 1 + 1 \otimes x \rangle_{R \otimes_{R_o} R}$ . Therefore we get the equality  $\text{Ann}_{R \otimes_{R_o} R}(\mathcal{J}) = \langle x \otimes x, x \otimes 1 + 1 \otimes x \rangle_{R \otimes_{R_o} R}$ , and hence the Noether different of  $R/R_o$  is  $\vartheta_N(R/R_o) = \langle 2x \rangle_R$ .

More generally, let  $R/R_o$  be an algebra of the form  $R = R_o[X]/\langle F \rangle$  with a monic polynomial  $F \in R_o[X]$ , and let  $\frac{\partial F}{\partial x}$  denote the image of the formal derivative  $\frac{\partial F}{\partial X}$  in  $R$ . Then the Noether different of the algebra  $R/R_o$  is given by  $\vartheta_N(R/R_o) = \langle \frac{\partial F}{\partial x} \rangle_R$ . This result is an application of Proposition 2.1.8 that we will see later.

Now we collect some properties of the Noether different of an algebra  $R/R_o$  which is given by generators and relations. Let  $\{x_\lambda \mid \lambda \in \Lambda\}$  be a system of generators of  $R/R_o$ , and let  $I$  denote the kernel of the surjective  $R_o$ -algebra homomorphism

$$\pi : R_o[X_\lambda \mid \lambda \in \Lambda] \longrightarrow R = R_o[x_\lambda \mid \lambda \in \Lambda]$$

defined by  $\pi(X_\lambda) = x_\lambda$  for all  $\lambda \in \Lambda$ . Then  $\pi$  induces an  $R_o$ -algebra isomorphism

$$R \cong R_o[X_\lambda \mid \lambda \in \Lambda]/I \quad (2.1)$$

The isomorphism (2.1) is called a **presentation** of  $R/R_o$  by generators and relations, and the ideal  $I$  is called the **ideal of algebraic relations** among  $\{x_\lambda \mid \lambda \in \Lambda\}$  with coefficients in  $R_o$ .

By setting  $A = R_o[X_\lambda \mid \lambda \in \Lambda]$ , we have the following composed map

$$\begin{array}{ccccc} \rho : S = R[X_\lambda \mid \lambda \in \Lambda] & \xrightarrow{\varphi} & R \otimes_{R_o} A & \xrightarrow{\text{id}_R \otimes \pi} & R \otimes_{R_o} R \\ X_\lambda & \longmapsto & 1 \otimes X_\lambda & \longmapsto & 1 \otimes x_\lambda \\ x_\lambda & \longmapsto & x_\lambda \otimes 1 & \longmapsto & x_\lambda \otimes 1 \end{array}$$

where  $\text{id}_R \otimes \pi$  is an  $R_o$ -algebra epimorphism with  $\text{Ker}(\text{id}_R \otimes \pi) = R \otimes_{R_o} I$  (cf. [Ku7, G.2]), and where  $\varphi$  is an  $R_o$ -algebra isomorphism. Also, we get

$$\text{Ker}(\rho) = \varphi^{-1}(\text{Ker}(\text{id}_R \otimes \pi)) = \varphi^{-1}(R \otimes_{R_o} I) = IS \subseteq S.$$

The following lemma gives us a formula for the Noether different of  $R/R_o$ .

**Lemma 2.1.6.** *Let  $R/R_o$  be an algebra which has a system of generators  $\{x_\lambda \mid \lambda \in \Lambda\}$ , let  $S = R[X_\lambda \mid \lambda \in \Lambda]$ , and let  $I$  be the ideal of  $A = R_o[X_\lambda \mid \lambda \in \Lambda]$  such that  $R \cong A/I$ . Then we have*

$$\vartheta_N(R/R_o) = \{ F(\{x_\lambda \mid \lambda \in \Lambda\}) \in R \mid F \in IS :_S \langle X_\lambda - x_\lambda \mid \lambda \in \Lambda \rangle_S \} \quad (2.2)$$

*Proof.* See [Ku1, Section 4, p. 178]. □

The localization of the Noether different of an algebra can be described as follows.

**Proposition 2.1.7.** *Let  $R/R_o$  be an algebra, let  $\mathcal{J}$  be the kernel of  $\mu : R \otimes_{R_o} R \rightarrow R$ ,  $f \otimes g \mapsto fg$  for  $f, g \in R$ , and let  $U \subseteq R$  be a multiplicatively closed subset. Assume that the ideal  $\mathcal{J}$  is finitely generated. Then we have  $\vartheta_N(R_U/R_o) = \vartheta_N(R/R_o)_U$ .*

*Proof.* See [SS, 15.6]. □

Our next proposition presents a well-known formula for the Noether different of an algebra in a special case.

**Proposition 2.1.8.** *Let  $R_o$  be a Noetherian ring, let  $R/R_o$  be a flat algebra of finite type which has a presentation  $R = R_o[X_1, \dots, X_n]/\langle F_1, \dots, F_n \rangle$ , where  $\{F_1, \dots, F_n\}$  forms an  $R_o[X_1, \dots, X_n]$ -regular sequence, and let  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  denote the image of the Jacobian determinant  $\det\left(\frac{\partial F_j}{\partial X_i}\right)_{1 \leq i, j \leq n}$  in  $R$ . Then we have*

$$\vartheta_N(R/R_o) = \left\langle \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right\rangle_R.$$

*Proof.* See [Ku5, G.3]. □

**Example 2.1.9.** Let  $R/R_o$  be an algebra which has a presentation

$$R = R_o[X_1, \dots, X_n]/\langle X_1^{\alpha_1} - a_1, \dots, X_n^{\alpha_n} - a_n \rangle$$

with  $a_i \in R_o$  and  $\alpha_i > 0$  for all  $i = 1, \dots, n$ . Let  $x_i$  be the image of  $X_i$  in  $R$  for  $i = 1, \dots, n$ . Then  $R$  is a free (hence flat)  $R_o$ -module, since  $\{X_1^{\beta_1} \cdots X_n^{\beta_n} \mid i = 1, \dots, n; 0 \leq \beta_i < \alpha_i\}$  is an  $R_o$ -basis of  $R$ . An application of Proposition 2.1.8 yields

$$\vartheta_N(R/R_o) = \left\langle \left(\prod_{i=1}^n \alpha_i\right) \cdot x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} \right\rangle_R.$$

**Example 2.1.10.** Let  $R$  be the affine  $K$ -algebra of the form  $R = K[X_0, X_1, X_2]/I$ , where  $I = \langle F_1, F_2 \rangle$  with  $F_1 = X_0 - X_2$  and  $F_2 = X_1^3 - 3X_1^2X_2 + 2X_1X_2^2$ . Clearly,  $I$  is a homogeneous ideal and  $\{F_1, F_2\}$  is a  $K[X_0, X_1, X_2]$ -regular sequence (since  $K[X_0, X_1, X_2]/\langle F_1 \rangle \cong K[X_1, X_2]$  and the image of  $F_2$  in  $K[X_1, X_2]$  is  $F_2$ ). The dehomogenization of  $I$  with respect to  $X_0$  is  $I^{\text{deh}} = \langle F_1^{\text{deh}}, F_2^{\text{deh}} \rangle = \langle 1 - X_2, X_1^3 - 3X_1^2X_2 + 2X_1X_2^2 \rangle \subset K[X_1, X_2]$ . Thus the homogenization of  $I^{\text{deh}}$  with respect to  $X_0$  is  $(I^{\text{deh}})^{\text{hom}} = I$ . By [KR3, Proposition 4.3.22], the ring  $R$  is a free  $K[X_0]$ -module. Let  $x_i$  denote the image of  $X_i$  in  $R$  for  $i = 0, 1, 2$ . By Proposition 2.1.8, the Noether different of the algebra  $R/K[x_0]$  is

$$\vartheta_N(R/K[x_0]) = \left\langle \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} \right\rangle_R = \langle 3x_1^2 - 6x_1x_2 + 2x_2^2 \rangle_R.$$

Before we go on, we collect some basic properties of ideals in residue class rings.

**Lemma 2.1.11.** *Let  $R$  be a ring, let  $I$  be a proper ideal of  $R$ , and let  $I_1, I_2$  be ideals in  $R$  such that  $I_1, I_2$  contain  $I$ . Then the following properties hold true.*



(i)  $(R/I)/(I_1/I) \cong R/I_1$  (Third Isomorphism Theorem for Rings).

(ii)  $I_1/I :_{R/I} I_2/I = (I_1 :_R I_2)/I$ .

(iii) Let  $\pi : R \rightarrow R/I$  be the canonical epimorphism, let  $n \geq 1$ , and let  $X_1, \dots, X_n$  be indeterminates. Then the map  $\varphi : R[X_1, \dots, X_n] \rightarrow (R/I)[X_1, \dots, X_n]$  defined by  $\varphi(\sum_{\alpha} f_{\alpha} T^{\alpha}) = \sum_{\alpha} \pi(f_{\alpha}) T^{\alpha}$ , where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ ,  $T^{\alpha} = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ ,  $f_{\alpha} \in \tilde{R}$ , and  $f_{\alpha} \neq 0$  for only finitely many  $\alpha \in \mathbb{N}^n$ , is a ring homomorphism which extends  $\pi$  and satisfies

$$\text{Ker}(\varphi) = IR[X_1, \dots, X_n] = \left\{ \sum_{\alpha} f_{\alpha} T^{\alpha} \mid \begin{array}{l} \alpha \in \mathbb{N}^n, f_{\alpha} \in I, \\ f_{\alpha} \neq 0 \text{ for only finitely many } \alpha \in \mathbb{N}^n \end{array} \right\}.$$

*Proof.* See [Mat, Chapter 1, Section 1] and [KR2, Proposition 3.2.18].  $\square$

Now let us consider an algebra  $R/R_o$  of finite type. There exists a presentation

$$R = R_o[X_1, \dots, X_n]/I = R_o[x_1, \dots, x_n]$$

where  $n \geq 1$ , where  $I$  is an ideal of the polynomial  $R_o$ -algebra  $A = R_o[X_1, \dots, X_n]$ , and where  $x_i$  is the image of  $X_i$  in  $R$  for  $i = 1, \dots, n$ . Suppose that  $I$  is generated by  $\{F_1, \dots, F_r\}$  with  $r \geq 1$ , and put  $S := R[X_1, \dots, X_n]$ . Then the formula (2.2) can be rewritten as follows

$$\vartheta_N(R/R_o) = \{ F(x_1, \dots, x_n) \mid F \in \langle F_1, \dots, F_r \rangle_S :_S \langle X_1 - x_1, \dots, X_n - x_n \rangle_S \} \quad (2.3)$$

Our next goal is to find out other formulas for the Noether different  $\vartheta_N(R/R_o)$ . For that, let  $Y_1, \dots, Y_n$  be new indeterminates, let  $J = \langle \tilde{F}_1, \dots, \tilde{F}_r \rangle_{R_o[Y_1, \dots, Y_n]}$ , where  $\tilde{F}_i = F_i(Y_1, \dots, Y_n)$  for  $i = 1, \dots, r$ , and let  $Q := A[Y_1, \dots, Y_n] = R_o[X_1, \dots, X_n, Y_1, \dots, Y_n]$ . Let  $IQ$  and  $JQ$  denote the ideals of  $Q$  generated by  $\{F_1, \dots, F_r\}$  and  $\{\tilde{F}_1, \dots, \tilde{F}_r\}$ , respectively. By Lemma 2.1.11(iii), we have  $R_o$ -algebra isomorphisms

$$Q/JQ \cong (R_o[Y_1, \dots, Y_n]/J)[X_1, \dots, X_n] \cong S = R[X_1, \dots, X_n].$$

Explicitly, the  $R_o$ -algebra isomorphism  $\phi : Q/JQ \xrightarrow{\sim} S$  is given by  $\phi(X_i + JQ) = X_i$  and  $\phi(Y_i + JQ) = x_i$  for  $i = 1, \dots, n$ . Thus we have the composed map

$$\begin{array}{ccccccccccc} \theta : & Q & \xrightarrow{\tilde{\pi}} & Q/JQ & \xrightarrow{\phi} & S & \xrightarrow{\varphi} & R \otimes_{R_o} A & \xrightarrow{\text{id}_R \otimes \pi} & R \otimes_{R_o} R & \xrightarrow{\mu} & R \\ & X_i & \mapsto & X_i + JQ & \mapsto & X_i & \mapsto & 1 \otimes X_i & \mapsto & 1 \otimes x_i & \mapsto & x_i \\ & Y_i & \mapsto & Y_i + JQ & \mapsto & x_i & \mapsto & x_i \otimes 1 & \mapsto & x_i \otimes 1 & \mapsto & x_i \end{array}$$

where  $\tilde{\pi}$  is the canonical  $R_o$ -algebra epimorphism, where  $\phi$  and  $\varphi$  are  $R_o$ -algebra isomorphisms, and where  $\text{id}_R \otimes \pi$  and  $\mu$  are  $R_o$ -algebra epimorphisms.

Another formula for the Noether different  $\vartheta_N(R/R_o)$  of the algebra  $R/R_o$  is given by the following proposition.

**Proposition 2.1.12.** *Using the notation as above, we have the formula*

$$\vartheta_N(R/R_o) = \{ \theta(F) \mid F \in (IQ + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q \}.$$

*Proof.* Since  $\phi : Q/JQ \rightarrow S$  is an isomorphism of  $R_o$ -algebras, this implies

$$\phi^{-1}(IS :_S I_1) = \phi^{-1}(IS) :_{Q/JQ} \phi^{-1}(I_1) \subseteq Q/JQ$$

where  $I_1 = \langle X_1 - x_1, \dots, X_n - x_n \rangle_S$  is an ideal of  $S$ . In the residue class ring  $Q/JQ$ , we see that

$$\phi^{-1}(IS) = (IQ + JQ)/JQ \quad \text{and} \quad \phi^{-1}(I_1) = (\langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q + JQ)/JQ.$$

We let  $I_2$  be the colon ideal  $(IQ + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q$  in the polynomial ring  $Q$ . According to Lemma 2.1.11(ii), we have

$$\begin{aligned} \phi^{-1}(IS) :_{Q/JQ} \phi^{-1}(I_1) &= [(IQ + JQ) :_Q (\langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q + JQ)]/JQ \\ &= I_2/JQ. \end{aligned}$$

It follows that  $\phi^{-1}(IS :_S I_1) = I_2/JQ$  and  $(\mu \circ \rho)(IS :_S I_1) = (\mu \circ \rho \circ \phi)(I_2/JQ) = \theta(I_2)$ , where  $\rho = (\text{id}_R \otimes \pi) \circ \varphi$ . Therefore we obtain the desired formula for the Noether different of  $R/R_o$  by using Formula (2.3).  $\square$

**Example 2.1.13.** Let  $R_o = \mathbb{Z}$  be the ring of integers, and let  $R/\mathbb{Z}$  be the algebra given by the presentation  $R = \mathbb{Z}[X_1, X_2]/I = \mathbb{Z}[x_1, x_2]$ , where  $I$  is the ideal of  $\mathbb{Z}[X_1, X_2]$  generated by  $F_1 = X_1X_2 - 2X_1 - X_2 + 2$ ,  $F_2 = 2X_1^2 - X_2^2 - 4X_1 + X_2 + 2$ , and  $F_3 = X_2^3 - 3X_2^2 + 2X_2$ , and where  $x_i$  is the image of  $X_i$  in  $R$  for  $i = 1, 2$ . We want to compute the Noether different  $\vartheta_N(R/\mathbb{Z})$  of  $R/\mathbb{Z}$  using Proposition 2.1.12.

We first form the polynomial ring  $Q = \mathbb{Z}[X_1, X_2, Y_1, Y_2]$  with new indeterminates  $Y_1$  and  $Y_2$ . In  $Q$ , we let  $IQ = \langle F_1, F_2, F_3 \rangle_Q$  and  $JQ = \langle \tilde{F}_1, \tilde{F}_2, \tilde{F}_3 \rangle_Q$ , where  $\tilde{F}_1 = Y_1Y_2 - 2Y_1 - Y_2 + 2$ ,  $\tilde{F}_2 = 2Y_1^2 - Y_2^2 - 4Y_1 + Y_2 + 2$ , and  $\tilde{F}_3 = Y_2^3 - 3Y_2^2 + 2Y_2$ . Then we compute a system of generators of the colon ideal  $(IQ + JQ) :_Q \langle X_1 - Y_1, X_2 - Y_2 \rangle_Q$  and get  $\{F_1, F_2, F_3, \tilde{F}_1, \tilde{F}_2, \tilde{F}_3, G_1, G_2, G_3\}$ , where  $G_1 = X_1^2Y_2^2 + X_1^2Y_2 - 2X_1Y_2^2 + X_2Y_2^2 + 6X_1Y_1 - 2X_1Y_2 - 2X_2Y_2 + Y_2^2 - 6X_1 - 6Y_1 + Y_2 + 6$ ,  $G_2 = 2X_1^2 + 2X_1Y_1 + X_2Y_2 + Y_2^2 - 6X_1 - 2X_2 - 2Y_1 - 3Y_2 + 6$ , and  $G_3 = X_2^2 + 2X_1Y_1 + X_2Y_2 + Y_2^2 - 2X_1 - 3X_2 - 2Y_1 - 3Y_2 + 4$ .

Now we calculate the images of  $G_1$ ,  $G_2$ , and  $G_3$  in  $R$  under the map  $\theta$ . We find

$$\begin{aligned}\theta(G_1) &= x_1^2 x_2^2 + x_1^2 x_2 - 2x_1 x_2^2 + x_2^3 + 6x_1^2 - 2x_1 x_2 - x_2^2 - 12x_1 + x_2 + 6, \\ \theta(G_2) &= 4x_1^2 + 2x_2^2 - 8x_1 - 5x_2 + 6, \\ \theta(G_3) &= 2x_1^2 + 3x_2^2 - 4x_1 - 6x_2 + 4.\end{aligned}$$

Note that the images of  $F_i$  and  $\tilde{F}_i$  in  $R$  under  $\theta$  are zero for  $i = 1, 2, 3$ . Thus we conclude that  $\vartheta_N(R/\mathbb{Z}) = \langle \theta(G_1), \theta(G_2), \theta(G_3) \rangle_R$ .

**Proposition 2.1.14.** *Using the same notation as in Proposition 2.1.12, we let*

$$\psi : Q = R_o[X_1, \dots, X_n, Y_1, \dots, Y_n] \rightarrow A = R_o[X_1, \dots, X_n] \quad (2.4)$$

be the  $R_o$ -algebra epimorphism defined by  $\psi(X_i) = X_i$  and  $\psi(Y_i) = X_i$  for  $i = 1, \dots, n$ , and let  $\{G_1, \dots, G_s\}$  be a system of generators of  $(IQ + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q$ . Then we have  $\theta = \pi \circ \psi$  and

$$\vartheta_N(R/R_o) = \langle \psi(G_1), \dots, \psi(G_s) \rangle_{A/I}.$$

*Proof.* Let us consider the following diagram

$$\begin{array}{ccccccc} Q & \xrightarrow{\psi} & & & A & & \\ \downarrow \tilde{\pi} & & & & \downarrow \pi & & \\ Q/JQ & \xrightarrow{\phi} & S & \xrightarrow{\varphi} & R \otimes_{R_o} A & \xrightarrow{\text{id}_R \otimes \pi} & R \otimes_{R_o} R & \xrightarrow{\mu} & R \end{array}$$

It is easy to check that this diagram is commutative. Since  $\theta = \mu \circ (\text{id}_R \otimes \pi) \circ \varphi \circ \phi \circ \tilde{\pi}$ , we have  $\theta = \pi \circ \psi$ . Moreover, since all homomorphisms in the diagram are surjective, we can apply Proposition 2.1.12 and get  $\vartheta_N(R/R_o) = (\pi \circ \psi)(\langle G_1, \dots, G_s \rangle_Q) = \langle \psi(G_1), \dots, \psi(G_s) \rangle_{A/I}$ , as desired.  $\square$

To use above tools for the computation of a system of generators of the Noether different of the algebra  $R/R_o$  of finite type, we require some results from Gröbner basis theory. In the following we give a brief introduction to Gröbner bases over rings and algorithms to compute them. For more details about Gröbner bases and their applications we refer to [KR2], [KR3] and [AL]. We begin by a prerequisite of the base ring  $R_o$  being computable.

**Definition 2.1.15.** Let  $R_o$  be a Noetherian ring. We will say that **linear equations are solvable in  $R_o$**  provided that

- (i) Given  $a, a_1, \dots, a_m \in R_o$ , there is an algorithm to determine whether the element  $a$  is contained in the ideal  $\langle a_1, \dots, a_m \rangle_{R_o}$  and if it is, to compute  $b_1, \dots, b_m \in R_o$  such that  $a = a_1b_1 + \dots + a_mb_m$ .
- (ii) Given  $a_1, \dots, a_m \in R_o$ , there is an algorithm that computes a system of generators of the  $R_o$ -module

$$\text{Syz}_{R_o}(a_1, \dots, a_m) = \{ (b_1, \dots, b_m) \in R_o^m \mid a_1b_1 + \dots + a_mb_m = 0 \}.$$

**Remark 2.1.16.** The Noetherian rings in which linear equations are solvable include  $\mathbb{Z}$ ,  $\mathbb{Z}/\langle m \rangle$  where  $m \geq 1$ ,  $\mathbb{Z}[i]$  where  $i^2 = -1$ ,  $\mathbb{Z}[\sqrt{-5}]$ , and  $K[X_1, \dots, X_n]$  where  $n \geq 1$  and where  $K$  is a field.

In what follows, we let  $R_o$  be a Noetherian ring such that linear equations are solvable in  $R_o$ , and let  $A = R_o[X_1, \dots, X_n]$ . The set of all terms of  $A$  is denoted by  $\mathbb{T}^n$  (or  $\mathbb{T}(X_1, \dots, X_n)$ ). We assume that we have a term ordering  $<_\sigma$  on  $\mathbb{T}^n$ . Each polynomial  $F \in A \setminus \{0\}$  has a unique representation as a linear combination of terms  $F = \sum_{i=1}^s a_i T_i$ , where  $a_1, \dots, a_s \in R_o \setminus \{0\}$ ,  $T_1, \dots, T_s \in \mathbb{T}^n$  with  $T_1 >_\sigma T_2 >_\sigma \dots >_\sigma T_s$ . As in Section 1.5 of [KR2], we define  $\text{LT}_\sigma(F) = T_1$ ,  $\text{LC}_\sigma(F) = a_1$  and  $\text{LM}_\sigma(F) = a_1 T_1$  (called the leading term, leading coefficient and leading monomial of  $F$  with respect to  $<_\sigma$ , respectively). Moreover, for an ideal  $I \subseteq A$ , the leading monomial ideal of  $I$  with respect to  $<_\sigma$  is given by  $\text{LM}_\sigma(I) = \langle \text{LM}_\sigma(F) \mid F \in I \rangle_A$ . We now present the Division Algorithm over rings.

**Theorem 2.1.17. (The Division Algorithm over Rings)** *Let  $s \geq 1$ , and let  $F, G_1, \dots, G_s \in A \setminus \{0\}$ . Consider the following sequence of instructions.*

- 1) Let  $Q_1 = \dots = Q_s = 0$ ,  $G = 0$  and  $V = F$ .
- 2) We check the following two conditions
  - a) There exists  $i \in \{1, \dots, s\}$  such that  $\text{LT}_\sigma(V)$  is a multiple of  $\text{LT}_\sigma(G_i)$ .
  - b) There are  $c_1, \dots, c_s \in R_o$  and  $T_1, \dots, T_s \in \mathbb{T}^n$  such that

$$\text{LM}_\sigma(V) = c_1 T_1 \text{LM}_\sigma(G_1) + \dots + c_s T_s \text{LM}_\sigma(G_s)$$

and  $\text{LT}_\sigma(V) = T_i \text{LT}_\sigma(G_i)$  for all  $i$  such that  $c_i \neq 0$ .

If these conditions are satisfied, then replace  $Q_i$  by  $Q_i + c_i T_i$  and replace  $V$  by  $V - c_1 T_1 G_1 - \dots - c_s T_s G_s$ .

3) Repeat step 2) until one of two conditions a) and b) is not satisfied. Then replace  $G$  by  $G + \text{LM}_\sigma(V)$  and replace  $V$  by  $V - \text{LM}_\sigma(V)$ .

4) If now  $V \neq 0$ , start again with step 2). Otherwise, return  $(G, Q_1, \dots, Q_s) \in A^{s+1}$ .

This is an algorithm which returns a tuple  $(G, Q_1, \dots, Q_s) \in A^{s+1}$  such that

$$F = \sum_{i=1}^s Q_i G_i + G$$

and such that the following conditions are satisfied.

a) If  $G \neq 0$ , then  $\text{LT}_\sigma(F) \geq_\sigma \text{LT}_\sigma(G)$  and no monomial of  $G$  lies in the leading monomial ideal  $\langle \text{LM}_\sigma(G_1), \dots, \text{LM}_\sigma(G_s) \rangle_A$ . The polynomial  $G$  is called an **irreducible polynomial** of  $F$  with respect to  $\{G_1, \dots, G_s\}$ .

b) If  $Q_i \neq 0$  for some  $i \in \{1, \dots, s\}$ , then  $\text{LT}_\sigma(F) \geq_\sigma \text{LT}_\sigma(Q_i G_i)$ .

*Proof.* See [AL], Theorem 4.1.10 and Algorithm 4.1.1. □

**Definition 2.1.18.** A finite set  $\{G_1, \dots, G_s\}$  of non-zero polynomials contained in an ideal  $I$  of  $A$  is called a  **$<_\sigma$ -Gröbner basis** of  $I$  if

$$\langle \text{LM}_\sigma(G_1), \dots, \text{LM}_\sigma(G_s) \rangle_A = \text{LM}_\sigma(I).$$

Note that a non-zero ideal  $I$  of  $A$  always has a  $<_\sigma$ -Gröbner basis, and if  $\{G_1, \dots, G_s\}$  is a  $<_\sigma$ -Gröbner basis of  $I$  then  $I = \langle G_1, \dots, G_s \rangle_A$ . For the proofs of these claims, see [AL, Corollaries 4.1.15 and 4.1.17]. To compute a  $<_\sigma$ -Gröbner basis of  $I$  we can use the following algorithm.

**Theorem 2.1.19. (Möller's Technique)** Let  $\{G_1, \dots, G_s\} \subseteq A \setminus \{0\}$  be a system of generators of an ideal  $I$  of  $A$ . Consider the following sequence of instructions.

1) Let  $\mathcal{G} = (G_1, \dots, G_s)$ ,  $r = 1$  and  $m = s$ .

2) If  $r > m$ , then return the result  $\mathcal{G}$ . Otherwise, compute

$$\mathcal{S} = \left\{ J \subseteq \{1, \dots, r\} \mid \begin{array}{l} r \in J \text{ and for all } i \in \{1, \dots, r\} : \\ \text{LT}_\sigma(G_i) \mid \text{lcm}(\text{LT}_\sigma(G_j) : j \in J) \Rightarrow i \in J \end{array} \right\}.$$

3) If  $\mathcal{S} = \emptyset$ , then go to step 6). Otherwise, choose  $J \in \mathcal{S}$  and delete it from  $\mathcal{S}$ . Set  $T_J = \text{lcm}(\text{LT}_\sigma(G_j) : j \in J)$ . Compute the set  $\mathcal{B}_J = \{b_{1J}, \dots, b_{n_J J}\}$  of generators of  $\langle \text{LC}_\sigma(G_j) \mid j \in J \setminus \{r\} \rangle_{R_\sigma} :_{R_\sigma} \langle \text{LC}_\sigma(G_r) \rangle_{R_\sigma}$ .

- 4) If  $\mathcal{B}_J \neq \emptyset$ , choose the smallest index  $i \in \{1, \dots, n_J\}$  such that  $b_{iJ} \in \mathcal{B}_J$  and delete  $b_{iJ}$  from  $\mathcal{B}_J$ . Compute  $b_j \in R_o$ ,  $j \in J \setminus \{r\}$  such that  $\sum_{j \in J \setminus \{r\}} b_j \text{LC}_\sigma(G_j) + b_{iJ} \text{LC}_\sigma(G_r) = 0$ . If  $\mathcal{B}_J = \emptyset$ , then continue with step 3).
- 5) Set  $F_{iJ} := \sum_{j \in J \setminus \{r\}} b_j \frac{T_j}{\text{LT}_\sigma(G_j)} G_j + b_{iJ} \frac{T_r}{\text{LT}_\sigma(G_r)} G_r$ , and use the Division Algorithm 2.1.17 to compute an irreducible polynomial  $\tilde{F}_{iJ}$  of  $F_{iJ}$  with respect to  $\{G \mid G \in \mathcal{G}\}$ . If  $\tilde{F}_{iJ} \neq 0$ , then append  $G_{m+1} = \tilde{F}_{iJ}$  to  $\mathcal{G}$  and increase  $m$  by one. Then continue with step 4).
- 6) Increase  $r$  by one and continue with step 2).

This is an algorithm which returns a tuple  $\mathcal{G}$  of polynomials which form a  $<_\sigma$ -Gröbner basis of the ideal  $I$ .

*Proof.* See [AL, Algorithm 4.2.2]. □

The next proposition indicates how one can compute the intersection  $I \cap J$  of two ideals  $I, J$  of  $A$  using elimination. This also enables us to compute the colon ideal  $I : J$  if  $J$  is generated by a set of non-zerodivisors in  $A$ .

**Proposition 2.1.20.** *Let  $I = \langle F_1, \dots, F_r \rangle_A$  and  $J = \langle G_1, \dots, G_s \rangle_A$  be ideals of  $A$ .*

- (i) *Let  $Y$  be a new indeterminate, let  $<_{\text{Elim}(Y)}$  be the elimination ordering on the set of terms of  $A[Y]$ , and let  $<_\sigma$  be the restriction of  $<_{\text{Elim}(Y)}$  on  $\mathbb{T}^n$  (see [KR2, Section 3.4]). Let*

$$U = \langle YF_1, \dots, YF_r, (1 - Y)G_1, \dots, (1 - Y)G_s \rangle_{A[Y]},$$

*and let  $\mathcal{G}$  be a  $<_{\text{Elim}(Y)}$ -Gröbner basis of  $U$ . Then  $I \cap J = U \cap A$  and  $\mathcal{G} \cap A$  is a  $<_\sigma$ -Gröbner basis of  $I \cap J$ .*

- (ii) *We have  $I : J = (I : \langle G_1 \rangle_A) \cap \dots \cap (I : \langle G_s \rangle_A)$ . If  $I \cap \langle G_j \rangle_A = \langle H_1 G_j, \dots, H_t G_j \rangle_A$  and  $G_j$  is not a zerodivisor in  $A$ , then  $I : \langle G_j \rangle_A = \langle H_1, \dots, H_t \rangle_A$ .*

*Proof.* See [AL], Theorem 4.3.6 and Propositions 4.3.9 and 4.3.11. □

At this point we have assembled all tools that we need to establish an algorithm for computing the Noether different of an algebra of finite type.

**Proposition 2.1.21.** *Let  $R_o$  be a Noetherian ring such that linear equations are solvable in  $R_o$ , and let  $R/R_o$  be an algebra of finite type. Suppose  $R$  has a presentation*

$$R = R_o[X_1, \dots, X_n]/I = R_o[x_1, \dots, x_n]$$

where  $n \geq 1$ , and where  $I$  is an ideal of the polynomial  $R_o$ -algebra  $A = R_o[X_1, \dots, X_n]$  given by a set of generators. Let  $<_\sigma$  be a term ordering on  $\mathbb{T}^n$ . Consider the following sequence of instructions.

- 1) Compute a tuple  $\mathcal{F} = (F_1, \dots, F_r)$  whose elements are a  $<_\sigma$ -Gröbner basis of the ideal  $I$  by using Möller's Technique 2.1.19.
- 2) Form the polynomial ring  $Q = A[Y_1, \dots, Y_n] = R_o[X_1, \dots, X_n, Y_1, \dots, Y_n]$ , and form the ideal  $JQ = \langle \tilde{F}_1, \dots, \tilde{F}_r \rangle_Q$ , where  $\tilde{F}_i = F_i(Y_1, \dots, Y_n)$  for  $i = 1, \dots, r$ .
- 3) Let  $<_{\tilde{\sigma}}$  be a term ordering on the set of terms  $\mathbb{T}^{2n} = \mathbb{T}(X_1, \dots, X_n, Y_1, \dots, Y_n)$  of  $Q$ . Compute a  $<_{\tilde{\sigma}}$ -Gröbner basis  $\hat{\mathcal{G}}$  of the colon ideal

$$\hat{I} = (IQ + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q = \bigcap_{i=1}^n ((IQ + JQ) :_Q \langle X_i - Y_i \rangle_Q)$$

by using Möller's Technique 2.1.19 and Proposition 2.1.20.

- 4) Take the image of  $\hat{\mathcal{G}}$  under  $\psi$ , where  $\psi$  is given by (2.4), and form the ideal  $\hat{J} = \psi(\hat{I}) = \langle \psi(G) \mid G \in \hat{\mathcal{G}} \rangle_A$ . Again using Möller's Technique 2.1.19, we compute a  $<_\sigma$ -Gröbner basis  $\hat{\mathcal{H}}$  of  $\hat{J}$ .
- 5) Let  $\mathcal{G}$  be the tuple of all polynomials of  $\hat{\mathcal{H}}$  which are not contained in  $\mathcal{F}$ , let  $\mathcal{V} = \emptyset$ , and let  $\mathcal{H} = \emptyset$ .
- 6) If  $\mathcal{G} = \emptyset$ , return the tuple  $\mathcal{H}$  and stop. Otherwise, choose a polynomial  $G$  in  $\mathcal{G}$  of the smallest degree and remove it from  $\mathcal{G}$ .
- 7) Compute an irreducible polynomial  $\tilde{G}$  of  $G$  w.r.t.  $\{F_1, \dots, F_r\} \cup \{H \mid H \in \mathcal{V}\}$  by using the Division Algorithm 2.1.17. If  $\tilde{G} = 0$ , continue with step 6).
- 8) Append  $\tilde{G}$  to the tuple  $\mathcal{V}$  and append the image of  $\tilde{G}$  in  $R$  to the tuple  $\mathcal{H}$ . Continue with step 6).

This is an algorithm which computes a tuple  $\mathcal{H}$  whose elements are a system of generators of the Noether different  $\vartheta_N(R/R_o)$ .

*Proof.* The correctness of this algorithm follows from Propositions 2.1.12 and 2.1.14. The finiteness of this algorithm is clear.  $\square$

**Example 2.1.22.** Let us consider again the algebra  $R/\mathbb{Z}$  given in Example 2.1.13. We let  $<_{\sigma}$  be the term ordering DegRevLex. Using Proposition 2.1.21, we compute a tuple  $\mathcal{H}$  whose elements are a system of generators of  $\vartheta_N(R/\mathbb{Z})$  and get  $\mathcal{H} = (4, x_2 + 2)$ . We see that  $4 \in \vartheta_N(R/\mathbb{Z})$ , but this is not a unit of  $R$ , and so  $\vartheta_N(R/\mathbb{Z}) = \langle 4, x_2 + 2 \rangle_R \neq \langle 1 \rangle_R$ . If we replace  $\mathbb{Z}$  by the field of rational numbers  $\mathbb{Q}$ , an application of Proposition 2.1.21 gives us  $\mathcal{H} = (1)$  and  $\vartheta_N(R/\mathbb{Q}) = \langle 1 \rangle_R$ .

**Remark 2.1.23.** Let  $R$  be an affine  $K$ -algebra of the form  $R = K[X_1, \dots, X_n]/I = K[x_1, \dots, x_n]$ , and let  $d = \dim(R)$ . By Noether's Normalization Theorem (cf. [Ku4, Theorem 3.1] or [KR3, Tutorial 78]), we can find algebraically independent elements  $X'_1, \dots, X'_d$  in  $K[X_1, \dots, X_n]$  such that the canonical map  $\iota : K[X'_1, \dots, X'_d] \hookrightarrow R$  is an injection and turns  $R$  into a finitely generated  $K[X'_1, \dots, X'_d]$ -module. Such a subalgebra  $K[X'_1, \dots, X'_d]$  is called a **Noether normalization** of  $R$ .

If the field  $K$  is infinite, there exists a linear change of coordinates given by  $(Y_1, \dots, Y_n) = (X_1, \dots, X_n)\mathcal{A}$ , where  $\mathcal{A} \in \text{Mat}_n(K)$  is a lower triangular matrix having units on the main diagonal, with the property that  $K[Y_1, \dots, Y_d]$  is a Noether normalization of  $R$ . To compute a Noether normalization of  $R$  one may use a randomized algorithm or a Las Vegas algorithm, see [KR3, Tutorial 78].

Now we assume that  $R$  has a Noether normalization of the form  $K[X_1, \dots, X_d]$ , and  $I$  is defined by a system of generators  $\{F_1, \dots, F_r\}$  with  $r \geq 1$ . Let  $x_i$  denote the image of  $X_i$  in  $R$  for  $i = 1, \dots, n$ , and let  $R_o = K[x_1, \dots, x_d]$ . Then an application of Proposition 2.1.21 yields a system of generators of the Noether different  $\vartheta_N(R/R_o)$  of the algebra  $R/R_o$ .

## 2.2 Fitting Ideals of Finitely Generated Modules

Let  $R$  be a ring. Given any finitely generated  $R$ -module  $M$ , we can associate to  $M$  a sequence of ideals of  $R$  which are determinantal ideals of a relation matrix for  $M$ . These ideals are well known and called the Fitting ideals or Fitting invariants of  $M$ . Many structural properties of the  $R$ -module  $M$  are reflected in its Fitting ideals. In this section we recall several facts about these Fitting ideals which will be used in the later chapters. Our main references are [Ku5] and [Nor].

Let  $\{m_1, \dots, m_n\}$  be a system of generators of a finitely generated  $R$ -module  $M$ . If  $M = \langle 0 \rangle$ , then we take  $n = 1$  and  $m_1 = 0$ . Every element  $m \in M$  can be written (not necessarily uniquely) as  $m = f_1 m_1 + \dots + f_n m_n$ , where  $f_1, \dots, f_n \in R$ . Let  $\mathcal{E} = \{e_1, \dots, e_n\}$  be the canonical basis of  $R^n$ . Let  $\varphi : R^n \rightarrow M$  denote the  $R$ -module



homomorphism determined by  $\mathcal{E}$  and by  $\{m_1, \dots, m_n\}$ , i.e.,  $\varphi(\sum_{i=1}^n f_i e_i) = \sum_{i=1}^n f_i m_i$ . By  $\mathcal{K}$  we denote the kernel of  $\varphi$ . Then we have a short exact sequence of  $R$ -modules

$$0 \longrightarrow \mathcal{K} \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0 \quad (2.5)$$

Let  $\{v_\lambda \mid \lambda \in \Lambda\}$  be a system of generators of the  $R$ -module  $\mathcal{K}$  with  $v_\lambda = (x_{1\lambda}, \dots, x_{n\lambda})$  in  $R^n$ . Notice that  $\mathcal{K} = \langle 0 \rangle$  if and only if  $M$  is a free  $R$ -module of rank  $n$  with basis  $\{m_1, \dots, m_n\}$ . In this case, we take  $\Lambda = \{1\}$  and  $v_1 = (0, \dots, 0)$ . Then the matrix  $(x_{i\lambda})_{\substack{i=1, \dots, n \\ \lambda \in \Lambda}}$  is called a **relation matrix** of  $M$  with respect to  $\{m_1, \dots, m_n\}$ . For  $i \in \{0, \dots, n-1\}$ , we let  $\mathcal{F}_i(M)$  denote the ideal of  $R$  generated by all  $(n-i)$ -rowed subdeterminants of  $(x_{i\lambda})_{\substack{i=1, \dots, n \\ \lambda \in \Lambda}}$ , and let

$$\mathcal{F}_i(M) = \begin{cases} R & \text{for } i \geq n, \\ \langle 0 \rangle & \text{for } i < 0. \end{cases}$$

**Definition 2.2.1.** Let  $i \in \mathbb{Z}$ . The ideal  $\mathcal{F}_i(M)$  is called the  **$i$ -th Fitting ideal** (or  **$i$ -th Fitting invariant**) of  $M$ . In particular, the ideal  $\mathcal{F}_0(M)$  is said to be the **initial Fitting ideal** (or **initial Fitting invariant**) of  $M$ .

We observe that

$$\langle 0 \rangle \subseteq \mathcal{F}_0(M) \subseteq \mathcal{F}_1(M) \subseteq \mathcal{F}_2(M) \subseteq \dots \subseteq \mathcal{F}_n(M) = R.$$

Moreover, the Fitting ideals of  $M$  have the following property.

**Lemma 2.2.2.** *Let  $i \in \mathbb{Z}$ . The  $i$ -th Fitting ideal  $\mathcal{F}_i(M)$  depends neither on the choice of the generating system  $\{m_1, \dots, m_n\}$  of  $M$  nor on the special choice of a relation matrix of  $M$  with respect to  $\{m_1, \dots, m_n\}$ .*

*Proof.* See [Ku5, D.1 and D.2] or [Nor, Chapter 3, Theorem 1]. □

**Example 2.2.3.** Let  $R = K[X]$  be a polynomial ring over a field  $K$ , and let  $M = R/\langle X \rangle \oplus R/\langle X^2 + 1 \rangle$ . It is clear that  $M$  is a finitely generated  $R$ -module. A system of generators of  $M$  is  $\{m_1, m_2\}$ , where  $m_1 = (\bar{1}, 0)$  and  $m_2 = (0, \bar{1})$ . Then we have a short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow R^2 \xrightarrow{\varphi} M \longrightarrow 0.$$

Here  $\varphi$  is defined by  $\varphi(f_1 e_1 + f_2 e_2) = f_1 m_1 + f_2 m_2 = (\bar{f}_1, \bar{f}_2)$ . It is not difficult to check that  $\mathcal{K} = \text{Ker}(\varphi)$  is given by  $\mathcal{K} = \langle v_1, v_2 \rangle_R$ , where  $v_1 = (X, 0)$  and  $v_2 = (0, X^2 + 1)$ .

Thus the relation matrix of  $M$  with respect to  $\{m_1, m_2\}$  is

$$\mathcal{M} = \begin{pmatrix} X & 0 \\ 0 & X^2 + 1 \end{pmatrix}$$

and

$$\mathcal{F}_i(M) = \begin{cases} \langle 0 \rangle & \text{for } i < 0, \\ \langle X^3 + X \rangle & \text{for } i = 0, \\ R & \text{for } i \geq 1. \end{cases} \quad (2.6)$$

Now we consider another system of generators of  $M = \langle m_1, m_2, m_3 \rangle_R$  given by  $m_1 = (\bar{1}, 0)$ ,  $m_2 = (0, \bar{1})$ , and  $m_3 = (\bar{1}, \bar{X})$ . The corresponding short exact sequence is

$$0 \longrightarrow \mathcal{K}' \longrightarrow R^3 \xrightarrow{\varphi'} M \longrightarrow 0.$$

The map  $\varphi'$  is defined by  $\varphi'(f_1e_1 + f_2e_2 + f_3e_3) = f_1m_1 + f_2m_2 + f_3m_3$  and  $\mathcal{K}' = \text{Ker}(\varphi')$ . We shall show that  $\mathcal{K}' = \langle v_1, v_2, v_3 \rangle_R$ , where  $v_1 = (X, 0, 0)$ ,  $v_2 = (0, X^2 + 1, 0)$ , and  $v_3 = (-1, -X, 1)$ . It is clear that  $v_i \in \mathcal{K}'$  for  $i = 1, 2, 3$ . Let  $v = f_1e_1 + f_2e_2 + f_3e_3 \in \mathcal{K}'$ . Since  $m_3 = m_1 + Xm_2$ , we have

$$0 = \varphi'(v) = f_1m_1 + f_2m_2 + f_3m_3 = (f_1 + f_3)m_1 + (f_2 + Xf_3)m_2.$$

This implies  $(f_1 + f_3)e_1 + (f_2 + Xf_3)e_2 \in \langle v_1, v_2 \rangle_R$ , and so

$$v = f_1e_1 + f_2e_2 + f_3e_3 = ((f_1 + f_3)e_1 + (f_2 + Xf_3)e_2) - f_3(e_1 + Xe_2 - e_3) \in \langle v_1, v_2, v_3 \rangle_R.$$

Thus the set  $\{v_1, v_2, v_3\}$  is a system of generators of  $\mathcal{K}'$ , as claimed.

Consequently, the relation matrix of  $M$  with respect to  $\{m_1, m_2, m_3\}$  is

$$\mathcal{M}' = \begin{pmatrix} X & 0 & -1 \\ 0 & X^2 + 1 & -X \\ 0 & 0 & 1 \end{pmatrix}$$

and  $\mathcal{F}_i(M)$  is the same as (2.6). Note that the ideal of  $R$  generated by all 2-rowed subdeterminants of  $\mathcal{M}$  is different from that one of  $\mathcal{M}'$ .

Recall that an  $R$ -module  $M$  is finitely presentable if there are a natural number  $n \in \mathbb{N}$  and a short exact sequence of  $R$ -modules

$$0 \longrightarrow \mathcal{K} \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

where  $\mathcal{K}$  is a finitely generated  $R$ -module. For example, finitely generated modules over Noetherian rings are always finitely presentable (cf. [Ku4, Chapter I, Proposition 2.17]). If  $M$  is finitely presentable, we see that the Fitting ideals  $\mathcal{F}_i(M)$ ,  $i \in \mathbb{Z}$ , are finitely generated ideals of  $R$ .

Now we collect some basic properties of Fitting ideals.

**Proposition 2.2.4.** *Let  $R$  be a ring, let  $M$  be a finitely generated  $R$ -module, and let  $i \in \mathbb{Z}$ .*

- (i) *For every algebra  $S/R$ , we have  $\mathcal{F}_i(S \otimes_R M) = S \cdot \mathcal{F}_i(M)$ .*
- (ii) *If  $U \subset R$  is a multiplicatively closed subset, then  $\mathcal{F}_i(M_U) = \mathcal{F}_i(M)_U$ .*
- (iii) *If  $I$  is an ideal of  $R$ , then  $\mathcal{F}_i(M/IM) = \overline{\mathcal{F}_i(M)}$ , where  $\overline{\mathcal{F}_i(M)}$  denotes the image of  $\mathcal{F}_i(M)$  in  $R/I$ .*

*Proof.* See [Ku5, D.4]. □

The next proposition says that the initial Fitting ideal  $\mathcal{F}_0(M)$  and  $\text{Ann}_R(M)$  have the same radical.

**Proposition 2.2.5.** *Let  $M$  be a finitely generated  $R$ -module which can be generated by  $n$  elements. Then*

$$(\text{Ann}_R(M))^n \subseteq \mathcal{F}_0(M) \subseteq \text{Ann}_R(M).$$

*In particular, if  $M$  can be generated by a single element, we have  $\mathcal{F}_0(M) = \text{Ann}_R(M)$ .*

*Proof.* See [Ku5, D.14] or [Nor, Chapter 3, Theorem 5]. □

**Example 2.2.6.** Let us go back to Example 2.2.3. The initial Fitting ideal of  $M$  is given by  $\mathcal{F}_0(M) = \langle X^3 + X \rangle \subseteq R = K[X]$ . Moreover, it is not hard to see that the annihilator of  $M$  is  $\text{Ann}_R(M) = \langle X \rangle \cap \langle X^2 + 1 \rangle$ . Since  $\langle X \rangle$  and  $\langle X^2 + 1 \rangle$  are coprime, we obtain  $\text{Ann}_R(M) = \langle X \rangle \cap \langle X^2 + 1 \rangle = \langle X^3 + X \rangle = \mathcal{F}_0(M)$ .

Next we consider a further finitely generated  $R$ -module  $M' = R/\langle X \rangle \oplus R/\langle X^3 \rangle$ . Then a minimal system of generators of  $M'$  is  $\{(\bar{1}, 0), (0, \bar{1})\}$  and the relation matrix of  $M'$  with respect to this system is

$$\mathcal{M}' = \begin{pmatrix} X & 0 \\ 0 & X^3 \end{pmatrix}.$$

Thus we obtain

$$\mathcal{F}_i(M') = \begin{cases} \langle 0 \rangle & \text{for } i < 0, \\ \langle X^4 \rangle & \text{for } i = 0, \\ \langle X \rangle & \text{for } i = 1, \\ R & \text{for } i \geq 2. \end{cases}$$

In this case we have  $\text{Ann}_R(M') = \langle X^3 \rangle$  and  $(\text{Ann}_R(M'))^2 \subsetneq \mathcal{F}_0(M') \subsetneq \text{Ann}_R(M')$ .

Observe that the Fitting ideals of a free  $R$ -module  $R^n$  with  $n \geq 1$  are given by

$$\mathcal{F}_i(R^n) = \begin{cases} \langle 0 \rangle & \text{if } i \leq n - 1, \\ R & \text{if } i \geq n. \end{cases}$$

More general, we have the following proposition.

**Proposition 2.2.7.** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be a short exact sequence of finitely generated  $R$ -modules and let  $i \geq 0$  and  $j \geq 0$  be integers. Then we have*

$$\mathcal{F}_i(M_1) \cdot \mathcal{F}_j(M_3) \subseteq \mathcal{F}_{i+j}(M_2).$$

Moreover, if the above exact sequence splits, i.e., if  $M_2 \cong M_1 \oplus M_3$ , then

$$\mathcal{F}_i(M_2) = \mathcal{F}_i(M_1 \oplus M_3) = \sum_{j+k=i} \mathcal{F}_j(M_1) \cdot \mathcal{F}_k(M_3)$$

for all  $i \in \mathbb{N}$ .

*Proof.* See [Ku5, D.15 and D.17] or [Nor, pages 90-93]. □

**Corollary 2.2.8.** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be a short exact sequence of finitely generated  $R$ -modules. Suppose  $M_3$  has a system of  $n_3$  generators such that the kernel of the corresponding presentation is also generated by  $n_3$  elements. Then  $\mathcal{F}_0(M_3)$  is a principal ideal and  $\mathcal{F}_0(M_1) \cdot \mathcal{F}_0(M_3) = \mathcal{F}_0(M_2)$ .*

*Proof.* See [Ku5, D.17] or [Nor, Chapter 3, Theorem 22]. □

**Example 2.2.9.** Let  $n \geq 1$ , let  $I_1, \dots, I_n$  be ideals of  $R$ , and let  $M = R/I_1 \oplus R/I_2 \oplus \dots \oplus R/I_n$ . By Proposition 2.2.7, the initial Fitting ideal of  $M$  is

$$\mathcal{F}_0(M) = \mathcal{F}_0(R/I_1) \cdot \mathcal{F}_0(R/I_2) \cdots \mathcal{F}_0(R/I_n).$$

Since  $\mathcal{F}_0(R/I_i) = I_i \subseteq R$  for  $i = 1, \dots, n$ , we have  $\mathcal{F}_0(M) = I_1 \cdot I_2 \cdots I_n$ . Suppose that the family  $\{I_i \mid i = 1, \dots, n\}$  is pairwise coprime, i.e.,  $I_i + I_j = R$  for  $i, j \in \{1, \dots, n\}$

and  $i \neq j$ . Then we have  $I_1 \cdot I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$  (cf. [Mat, Theorem 1.3]). In this case we obtain

$$\mathcal{F}_0(M) = I_1 \cap I_2 \cap \cdots \cap I_n = \text{Ann}_R(M).$$

Now we want to review some local properties of Fitting ideals. For that, let  $(R, \mathfrak{m})$  be a local ring, let  $M$  be a finitely generated  $R$ -module with a minimal system of generators  $\{m_1, \dots, m_n\}$ , and let (2.5) be the presentation defined by this system. Let  $v = (x_1, \dots, x_n) \in R^n$  be an element of  $\mathcal{K}$ . If there exists an index  $i \in \{1, \dots, n\}$  such that  $x_i \notin \mathfrak{m}$ , then  $x_i$  is a unit of  $R$ , so it follows from  $\sum_{j=1}^n x_j m_j = 0$  that  $m_i = -x_i^{-1} \sum_{j \neq i} x_j m_j$ . Thus we obtain a contradiction to the fact that  $\{m_1, \dots, m_n\}$  is a minimal system of generators of  $M$ . Hence the coefficients of the relation matrix with respect to  $\{m_1, \dots, m_n\}$  of  $M$  are elements of  $\mathfrak{m}$ , and therefore  $\mathcal{F}_i(M) \subseteq \mathfrak{m}$  for  $i \leq n - 1$ . From this we deduce the following property (cf. [Ku5, D.8]).

**Proposition 2.2.10.** *Let  $(R, \mathfrak{m})$  be a local ring, and let  $M$  be a finitely generated  $R$ -module. Then the number of minimal generators of  $M$  is  $\min\{i \mid \mathcal{F}_i(M) = R\}$ . Moreover, if  $\mathcal{F}_r(M)$  is the smallest non-zero Fitting ideal of  $M$ , then  $M$  is free of rank  $r$  if and only if  $\mathcal{F}_r(M) = R$ .*

Our next corollary is an immediate consequence of Proposition 2.2.10 and the local-global principle (see [Ku4, Chapter IV, Rule 1.1]).

**Corollary 2.2.11.** *For a ring  $R$  and a finitely generated  $R$ -module  $M$  the following statements are equivalent:*

- (i)  $M$  is locally free of rank  $r$ .
- (ii)  $\mathcal{F}_i(M) = \langle 0 \rangle$  for  $i = 0, \dots, r - 1$ , and  $\mathcal{F}_i(M) = R$  for  $i \geq r$ .

**Remark 2.2.12.** Let  $R$  be a ring,  $M$  a finitely generated  $R$ -module. We recall that

$$\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid M_{\mathfrak{p}} \neq \langle 0 \rangle\}.$$

For  $\mathfrak{p} \in \text{Spec}(R)$ , we have  $\mathfrak{p} \in \text{Supp}(M)$  if the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  contains at least one non-zero element. This is equivalent to the condition that  $\mathcal{F}_0(M) \subseteq \mathfrak{p}$ . Thus we have

$$\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathcal{F}_0(M) \subseteq \mathfrak{p}\}.$$

We conclude this section with the following proposition.

**Proposition 2.2.13.** *Let  $R$  be a reduced Noetherian ring, let  $M$  be a finitely generated  $R$ -module, let  $\text{pd}_R(M)$  denote the projective dimension of  $M$  (see for instance [Ku4, Chapter VII, Definition 1.3]), and let  $\mathcal{T}(M)$  be the set of all torsion elements in  $M$ , i.e.,  $\mathcal{T}(M) = \{m \in M \mid fm = 0 \text{ for some non-zero-divisor } f \in R\}$ . The following statements are equivalent:*

- (i) *The smallest non-zero Fitting ideal  $\mathcal{F}_r(M)$  of  $M$  is an invertible ideal.*
- (ii)  *$M/\mathcal{T}(M)$  is a projective  $R$ -module of rank  $r$ , and  $\text{pd}_R(M) \leq 1$ .*

*In particular,  $\mathcal{F}_0(M)$  is an invertible ideal if and only if  $\mathcal{T}(M) = M$  and  $\text{pd}_R(M) \leq 1$ .*

*Proof.* Since  $R$  is a reduced Noetherian ring, we have  $\text{Min}(R) = \text{Ass}(R)$ . The claim follows from [Ku5, D.19].  $\square$

## 2.3 First Properties of 0-Dimensional Schemes

Throughout this section, we work over an arbitrary field  $K$ . Let  $n \geq 0$  and let  $P$  be the polynomial ring  $P = K[X_0, \dots, X_n]$  equipped with its standard grading  $\deg(X_i) = 1$ .

**Definition 2.3.1.** Let  $I \subseteq P$  be a homogeneous ideal. The set

$$I^{\text{sat}} = \{F \in P \mid \langle X_0, \dots, X_n \rangle^i \cdot F \subseteq I \text{ for some } i \in \mathbb{N}\}$$

is called the **saturation** of  $I$ . The ideal  $I$  is called **saturated** if  $I = I^{\text{sat}}$ .

It is easy to see that the saturation  $I^{\text{sat}}$  of a given homogeneous ideal  $I \subseteq P$  is a homogeneous saturated ideal of  $P$  and  $I \subseteq I^{\text{sat}}$ .

In what follows, we let  $\mathbb{P}_K^n$  be the projective  $n$ -space over  $K$  (i.e.,  $\mathbb{P}_K^n = \text{Proj}(P)$ ). Here we identify the set of  $K$ -rational points of  $\mathbb{P}_K^n$  with the set of equivalence classes  $\mathbb{P}(K^{n+1}) = (K^{n+1} \setminus \{0\}) / \sim$ , where  $\sim$  is an equivalence relation on  $K^{n+1} \setminus \{0\}$  by letting  $(c_0, \dots, c_n) \sim (c'_0, \dots, c'_n)$  if and only if there exists an element  $\lambda \in K$  such that  $(c'_0, \dots, c'_n) = (\lambda c_0, \dots, \lambda c_n)$ . Also, we shall say “a subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^n$ ” or “a scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ ” when we really mean “a closed subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^n$  together with a fixed embedding  $\mathbb{X} \subseteq \mathbb{P}_K^n$ ”. For more details about the theory of schemes we refer to [Har, Chapter II]. Our main objects of study are 0-dimensional schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$ . We let  $\mathcal{I}_{\mathbb{X}} \subseteq \mathcal{O}_{\mathbb{P}_K^n}$  be the ideal sheaf of  $\mathbb{X}$  on  $\mathbb{P}_K^n$  (see [Har, Chapter II, Section 5, p. 115]). It is well known that the homogeneous ideal  $\Gamma_*(\mathcal{I}_{\mathbb{X}}) = \bigoplus_{i \geq 0} H^0(\mathbb{P}_K^n, \mathcal{I}_{\mathbb{X}}(i))$  is an ideal in  $P$  defining the scheme  $\mathbb{X}$ . Moreover, this ideal is saturated, and hence it is the largest homogeneous ideal in  $P$  defining the scheme  $\mathbb{X}$ .

**Definition 2.3.2.** Let  $\mathbb{X}$  be a 0-dimensional subscheme of  $\mathbb{P}_K^n$ , and let  $\mathcal{I}_{\mathbb{X}}$  be the ideal sheaf of  $\mathbb{X}$  on  $\mathbb{P}_K^n$ . Then the homogeneous saturated ideal  $\Gamma_*(\mathcal{I}_{\mathbb{X}}) = \bigoplus_{i \geq 0} H^0(\mathbb{P}_K^n, \mathcal{I}_{\mathbb{X}}(i))$  is called the **homogeneous vanishing ideal** of  $\mathbb{X}$  and is denoted by  $\mathcal{I}_{\mathbb{X}}$ , and the residue class ring  $R = P/\mathcal{I}_{\mathbb{X}}$  is called the **homogeneous coordinate ring** of  $\mathbb{X}$ .

Notice that the ring  $R = P/\mathcal{I}_{\mathbb{X}} = \bigoplus_{i \geq 0} R_i$  is a standard graded  $K$ -algebra (i.e.,  $R_0 = K$  and  $R = K[R_1]$  with  $\dim_K(R_1) < \infty$ ), its homogeneous maximal ideal is denoted by  $\mathfrak{m} := \bigoplus_{i \geq 1} R_i$ . We let  $\mathcal{Z}^+(X_0)$  be the subscheme of  $\mathbb{P}_K^n$  defined by  $\langle X_0 \rangle$  and call it the **hyperplane at infinity** of  $\mathbb{P}_K^n$ . The set of closed points of  $\mathbb{P}_K^n$  in  $\mathbb{X}$  will be denoted by  $\text{Supp}(\mathbb{X})$  and called the **support** of  $\mathbb{X}$ .

**Assumption 2.3.3.** Once and for all, we assume that no point of the support of  $\mathbb{X}$  lies on the hyperplane at infinity  $\mathcal{Z}^+(X_0)$ .

Let  $x_i$  be the image of  $X_i$  in  $R$  for  $i = 0, \dots, n$ . By Assumption 2.3.3,  $x_0$  is not a zerodivisor of  $R$ . Hence  $R$  is a 1-dimensional Cohen-Macaulay ring and  $\bar{R} = R/\langle x_0 \rangle$  is a 0-dimensional local ring with maximal ideal  $\bar{\mathfrak{m}} = (\mathfrak{m} + \langle x_0 \rangle)/\langle x_0 \rangle$  and  $\dim_K(\bar{R}) < \infty$ .

We write  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$  for some  $s \geq 1$ . For every  $j \in \{1, \dots, s\}$ , let  $\mathfrak{P}_j$  be the homogeneous prime ideal of  $P$  corresponding to  $p_j$ . Its image in  $R$  will be denoted by  $\mathfrak{p}_j$ . The ideals  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  are minimal homogeneous prime ideals of  $R$ . The local ring  $\mathcal{O}_{\mathbb{X}, p_j}$  is then the homogeneous localization of  $R$  at  $\mathfrak{p}_j$ , i.e.,  $\mathcal{O}_{\mathbb{X}, p_j} \cong R_{(\mathfrak{p}_j)}$ . The degree of the scheme  $\mathbb{X}$  is given by  $\deg(\mathbb{X}) = \dim_K H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) = \sum_{j=1}^s \dim_K(\mathcal{O}_{\mathbb{X}, p_j})$ .

Let  $\Gamma$  be the affine coordinate ring of  $\mathbb{X}$  in the affine space  $D_+(X_0) \cong \mathbb{A}_K^n$ . We then have  $\Gamma = H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$ . Moreover, the ring  $\Gamma$  is a 0-dimensional Artinian ring and  $\Gamma \cong R/\langle x_0 - 1 \rangle$  (cf. [Kr3, Lemma 1.2]). The canonical epimorphism  $\theta : R \rightarrow R/\langle x_0 - 1 \rangle \cong \Gamma$  is given by dehomogenization  $x_0 \mapsto 1$ . We set  $\mathcal{I}_{\mathbb{X}}^a = \mathcal{I}_{\mathbb{X}}/(\mathcal{I}_{\mathbb{X}} \cap \langle X_0 - 1 \rangle)$  and call it the **affine ideal** of  $\mathbb{X}$ , so we get  $\Gamma \cong A/\mathcal{I}_{\mathbb{X}}^a$ , where  $A = K[X_1, \dots, X_n]$ .

By using the homogenization theory and Theorem 4.3.22 in [KR3], we have the following properties.

**Lemma 2.3.4.** (i) We have  $\mathcal{I}_{\mathbb{X}}^a = (\mathcal{I}_{\mathbb{X}})^{\text{deh}}$  and  $\mathcal{I}_{\mathbb{X}} = (\mathcal{I}_{\mathbb{X}}^a)^{\text{hom}}$ .

(ii) The ring  $R$  is a graded free  $K[x_0]$ -module of rank  $\deg(\mathbb{X})$ .

**Remark 2.3.5.** Let  $I_j \subseteq P$  be the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$  for all  $j \in \{1, \dots, s\}$ . In order to compute the homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^s I_j$  of  $\mathbb{X}$  we use the results in the paper of J. Abbott, M. Kreuzer and L. Robbiano [AKR]. More precisely, we can either proceed degree by degree to compute the reduced Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$  by using the GPBM-Algorithm (cf. [AKR, Theorem 4.6]), or we can apply

the GBM-Algorithm (cf. [AKR, Theorem 3.1]). In case we apply the GBM-Algorithm to compute the reduced Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$ , we proceed as follows. Let  $<_{\sigma}$  be a degree compatible term ordering on  $\mathbb{T}^n = \mathbb{T}(X_1, \dots, X_n)$ , and let  $<_{\bar{\sigma}}$  be the extension of  $<_{\sigma}$  on  $\mathbb{T}^{n+1} = \mathbb{T}(X_0, \dots, X_n)$  (see [KR3, Definition 4.3.13]). Apply the GBM-Algorithm to compute the reduced  $<_{\bar{\sigma}}$ -Gröbner basis  $\{G_1, \dots, G_r\}$  of the affine ideal  $\mathcal{I}_{\mathbb{X}}^a$ . For  $k = 1, \dots, r$ , we then compute the homogenization  $G_k^{\text{hom}}$  of  $G_k$  with respect to  $X_0$ . Observe that  $\text{LT}_{\bar{\sigma}}(G_k^{\text{hom}}) = \text{LT}_{\sigma}(G_k)$  for all  $k = 1, \dots, r$ . Thus [KR3, Proposition 4.3.21] and Lemma 2.3.4(i) yield that the set  $\{G_1^{\text{hom}}, \dots, G_r^{\text{hom}}\}$  is the reduced  $<_{\bar{\sigma}}$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$ .

Next we introduce the Hilbert function of the homogeneous coordinate ring of  $\mathbb{X}$ . We start as follows.

**Definition 2.3.6.** Let  $M$  be a finitely generated graded  $R$ -module. It decomposes as a direct sum of its homogeneous components  $M = \bigoplus_{i \in \mathbb{Z}} M_i$ . The **Hilbert function** of  $M$  is defined by

$$\text{HF}_M(i) = \dim_K(M_i) \quad \text{for all } i \in \mathbb{Z}.$$

In particular, the Hilbert function of  $R$  is given by  $\text{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$  for all  $i \in \mathbb{Z}$ .

The Hilbert function of  $M$  is invariant under a homogeneous linear change of coordinates. Many properties of  $M$  are encoded in this invariant, for example, dimension and multiplicity (cf. [KR3], [BH]). We say that an integer function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is **of polynomial type** if there exists a number  $i_0 \in \mathbb{Z}$  and an integer valued polynomial  $q \in \mathbb{Q}[z]$  such that  $f(i) = q(i)$  for all  $i \geq i_0$ . This polynomial is uniquely determined and denoted by  $\text{HP}_f(z)$ . The Hilbert function of a finitely generated graded  $R$ -module  $M$  is an integer function of polynomial type (see [BH, Theorem 4.1.3]).

**Definition 2.3.7.** Let  $M$  be a finitely generated graded  $R$ -module, and let  $z$  be an indeterminate over  $\mathbb{Q}$ .

- (i) The integer valued polynomial associated to  $\text{HF}_M$  is called the **Hilbert polynomial** of  $M$  and is denoted by  $\text{HP}_M(z)$ . In other words, we have  $\text{HF}_M(i) = \text{HP}_M(i)$  for  $i \gg 0$ .

- (ii) The number

$$\text{ri}(M) = \min\{i \in \mathbb{Z} \mid \text{HF}_M(j) = \text{HP}_M(j) \text{ for all } j \geq i\}$$

is called the **regularity index** of  $M$  (or of  $\text{HF}_M$ ). Whenever  $\text{HF}_M(i) = \text{HP}_M(i)$  for all  $i \in \mathbb{Z}$ , we let  $\text{ri}(M) = -\infty$ . The regularity index of  $\text{HF}_{\mathbb{X}}$  will be denoted by  $r_{\mathbb{X}}$ .



In the following proposition we collect some descriptions of the Hilbert function and the **Castelnuovo function** of  $\mathbb{X}$ . Here the Castelnuovo function of  $\mathbb{X}$  is defined to be the first difference function of  $\text{HF}_{\mathbb{X}}$  given by  $\Delta \text{HF}_{\mathbb{X}}(i) = \text{HF}_{\mathbb{X}}(i) - \text{HF}_{\mathbb{X}}(i-1)$  for all  $i \in \mathbb{Z}$ .

**Proposition 2.3.8.** (i) For  $i < 0$ , we have  $\text{HF}_{\mathbb{X}}(i) = 0$ , and we have  $\text{HF}_{\mathbb{X}}(0) = 1$ .

(ii) We have  $\text{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})$  for  $i \geq r_{\mathbb{X}}$ .

(iii) We have  $\text{HF}_{\mathbb{X}}(0) < \text{HF}_{\mathbb{X}}(1) < \cdots < \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) < \text{HF}_{\mathbb{X}}(r_{\mathbb{X}})$ .

(iv)  $\Delta \text{HF}_{\mathbb{X}}(i) \neq 0$  if and only if  $i \in \{0, \dots, r_{\mathbb{X}}\}$ .

(v) We have  $\deg(\mathbb{X}) = \sum_{i=0}^{r_{\mathbb{X}}} \Delta \text{HF}_{\mathbb{X}}(i) = \sum_{i \in \mathbb{Z}} \Delta \text{HF}_{\mathbb{X}}(i)$ .

*Proof.* See [Kr3, Lemma 1.3]. □

For  $p_j \in \text{Supp}(\mathbb{X})$ , the **residue field** of  $\mathbb{X}$  at  $p_j$  is denoted by  $K(p_j) := \mathcal{O}_{\mathbb{X}, p_j} / \mathfrak{m}_{\mathbb{X}, p_j}$ . Obviously, the residue field  $K(p_j)$  is in general a finite dimensional  $K$ -vector space and  $K(p_j) = K$  if and only if  $p_j \in \text{Supp}(\mathbb{X})$  is a  $K$ -rational point of  $\mathbb{X}$ . By  $\mathbb{X}(K)$  we denote the set of all  $K$ -rational points of  $\mathbb{X}$ . It is clear that  $\mathbb{X}(K)$  is a subset of  $\text{Supp}(\mathbb{X})$ .

**Definition 2.3.9.** (i) We say that  $\mathbb{X}$  has  **$K$ -rational support** if  $\mathbb{X}(K) = \text{Supp}(\mathbb{X})$  or, equivalently, if each closed point of  $\mathbb{X}$  is also a  $K$ -rational point of  $\mathbb{X}$ .

(ii) We say that  $\mathbb{X}$  is **reduced at**  $p_j$  if the local ring  $\mathcal{O}_{\mathbb{X}, p_j}$  is reduced (i.e., if 0 is the only nilpotent element of  $\mathcal{O}_{\mathbb{X}, p_j}$ ). The scheme  $\mathbb{X}$  is called **reduced** if it is reduced at every point  $p_j \in \text{Supp}(\mathbb{X})$ .

(iii) The scheme  $\mathbb{X}$  is called a **projective point set** in  $\mathbb{P}_K^n$  if it is reduced and has  $K$ -rational support. In this case we write  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  with  $s = \deg(\mathbb{X})$ .

Remark that, if  $K$  is an algebraically closed field, then  $\mathbb{X}$  always has  $K$ -rational support, since all closed points of  $\mathbb{X}$  are  $K$ -rational by Hilbert's Nullstellensatz (see for instance [KR2, Chapter 2, Section 6]). In this situation,  $\mathbb{X}$  is reduced if and only if it is a projective point set in  $\mathbb{P}_K^n$ .

Many local properties of the scheme  $\mathbb{X}$  are based on the following result.

**Lemma 2.3.10.** Let  $\theta : R \rightarrow R/\langle x_0 - 1 \rangle = \Gamma$  be the canonical epimorphism given by dehomogenization. Then the  $K$ -linear map  $\theta|_{R_i} : R_i \rightarrow R/\langle x_0 - 1 \rangle = \Gamma$  is an isomorphism if  $i \geq r_{\mathbb{X}}$ , and it is injective if  $0 \leq i < r_{\mathbb{X}}$ . In particular, the  $K$ -linear map  $\iota : R_{r_{\mathbb{X}}} \rightarrow \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  given by  $\iota(f) = (f_{p_1}, \dots, f_{p_s})$  for all  $f \in R_{r_{\mathbb{X}}}$ , where  $f_{p_j}$  is the germ of  $f$  at the point  $p_j$  of  $\text{Supp}(\mathbb{X})$ , is an isomorphism.

*Proof.* See [Kr2, Lemma 1.1]. □

Notice that the isomorphism  $\iota : R_{r_{\mathbb{X}}} \xrightarrow{\sim} \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  satisfies  $\iota(x_0^{r_{\mathbb{X}}}) = (1, \dots, 1)$ . The first application of this isomorphism is to characterize a non-zerodivisor of  $R$  as follows.

**Lemma 2.3.11.** *Let  $f \in R_i$  with  $i \geq 0$  be a homogeneous element of  $R$ . If  $i \leq r_{\mathbb{X}}$  we set  $\tilde{f} = x_0^{r_{\mathbb{X}}-i} f$ , and if  $i > r_{\mathbb{X}}$  we write  $f = x_0^{i-r_{\mathbb{X}}} \tilde{f}$  for some  $\tilde{f} \in R_{r_{\mathbb{X}}}$ .*

*Then  $f$  is a non-zerodivisor of  $R$  if and only if  $\iota(\tilde{f})$  is a unit of  $\prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$ .*

*Proof.* See [Kr3, Lemma 1.5]. □

To investigate subschemes of the scheme  $\mathbb{X}$ , we require the following lemma.

**Lemma 2.3.12.** *Let  $J \subseteq R$  be a homogeneous ideal, and let  $J^{\text{sat}}$  be the saturation of  $J$ , i.e.,  $J^{\text{sat}} := \{f \in R \mid \mathfrak{m}^i f \subseteq J \text{ for some } i \geq 0\}$ . Then we have*

$$J^{\text{sat}} = \{f \in R \mid x_0^i f \in J \text{ for some } i \geq 0\}.$$

*Proof.* See [Kr3, Lemma 1.6]. □

Now we examine subschemes  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) \leq \deg(\mathbb{X}) - 1$ . By  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  we denote the saturated ideal of  $\mathbb{Y}$  in  $R$ . Then the homogeneous coordinate ring of  $\mathbb{Y}$  is  $R_{\mathbb{Y}} = R/\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . Because  $\dim_K (R_{\mathbb{Y}})_i = \deg(\mathbb{Y}) < \deg(\mathbb{X}) = \dim_K R_i$  for  $i \gg 0$ , we define  $\alpha_{\mathbb{Y}/\mathbb{X}} := \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_i \neq 0\}$  and call it the **initial degree** of  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . As  $x_0$  is a non-zerodivisor of  $R$ , Proposition 2.3.8 yields  $R_i = x_0^{i-r_{\mathbb{X}}} R_{r_{\mathbb{X}}}$  for all  $i \geq r_{\mathbb{X}}$ . Thus the description of the saturation of a homogeneous ideal of  $R$  given in Lemma 2.3.12 tells us that  $\alpha_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$ .

**Remark 2.3.13.** Suppose that  $\mathbb{Y} \subseteq \mathbb{X}$  is a subscheme of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$ .

(a) The Hilbert function of  $\mathbb{Y}$  is given by

$$\text{HF}_{\mathbb{Y}}(i) = \begin{cases} \text{HF}_{\mathbb{X}}(i) & \text{for } i < \alpha_{\mathbb{Y}/\mathbb{X}}, \\ \text{HF}_{\mathbb{X}}(i) - 1 & \text{for } i \geq \alpha_{\mathbb{Y}/\mathbb{X}}. \end{cases}$$

For the proof of this claim see [Kr2] or [Kr3, Lemma 1.7].

(b) There exists exactly one point  $p_j \in \mathbb{X}(K)$  such that  $\mathcal{O}_{\mathbb{Y}, p_j} \neq \mathcal{O}_{\mathbb{X}, p_j}$ . Indeed, it is easily seen that there is exactly one point  $p_j \in \text{Supp}(\mathbb{X})$  such that  $\mathcal{O}_{\mathbb{Y}, p_j} \neq \mathcal{O}_{\mathbb{X}, p_j}$ . If  $p_j \in \text{Supp}(\mathbb{X}) \setminus \mathbb{X}(K)$ , then we have  $\dim_K K(p_j) = \varkappa \geq 2$ , and we may write  $K(p_j) = K \oplus K v_1 \oplus \dots \oplus K v_{\varkappa-1}$ , where  $\{1, v_1, \dots, v_{\varkappa-1}\}$  is a  $K$ -basis of  $K(p_j)$ . Let  $f_{\mathbb{Y}} \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ . Then  $\iota(f_{\mathbb{Y}}) = (0, \dots, 0, s_j, 0, \dots, 0) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  for

some element  $s_j \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}$ . It is not difficult to check that  $s_j, v_1 s_j$  are  $K$ -linear independent. By setting  $f_1 := \iota^{-1}((0, \dots, 0, v_1 s_j, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ , we have  $\dim_K \langle f_1, f_{\mathbb{Y}} \rangle_K = 2$ . Also, we observe that  $x_0^{r_{\mathbb{X}}} f_1 = f_2 f_{\mathbb{Y}} \in \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  with  $f_2 = \iota^{-1}((0, \dots, 0, v_1, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$  and  $x_0$  is a non-zerodivisor of  $R_{\mathbb{Y}}$ . This implies  $f_1 \in \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . Hence we get  $2 = \dim_K \langle f_1, f_{\mathbb{Y}} \rangle_K \leq \dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} = 1$ , a contradiction.

**Example 2.3.14.** Let  $\mathbb{X} = \mathcal{Z}^+(2X_0^4 + X_0^2 X_1^2 - X_1^4)$  be the 0-dimensional scheme of degree 4 in the projective line  $\mathbb{P}_{\mathbb{Q}}^1$ . The affine ideal of  $\mathbb{X}$  is

$$\mathcal{I}_{\mathbb{X}}^a = \langle 2 + X_1^2 - X_1^4 \rangle = \langle (X_1^2 + 1)(X_1^2 - 2) \rangle = \mathfrak{q}_1 \cap \mathfrak{q}_2 \subseteq \mathbb{Q}[X_1]$$

where  $\mathfrak{q}_1 = \langle X_1^2 + 1 \rangle$  and  $\mathfrak{q}_2 = \langle X_1^2 - 2 \rangle$ . In  $\mathbb{Q}[X_1]$ , both  $\mathfrak{q}_1$  and  $\mathfrak{q}_2$  are maximal, and hence  $\text{Supp}(\mathbb{X}) = \{p_1, p_2\}$  where  $p_i$  is the closed point of  $\mathbb{X}$  corresponding to  $\mathfrak{q}_i$ , but  $\mathbb{X}(\mathbb{Q}) = \emptyset$ . Thus Remark 2.3.13(b) yields that  $\mathbb{X}$  has no subscheme of degree 3.

Recall from [Kr3, Section 2] that the  $\mathbb{Z}$ -graded ring  $\Gamma_*(\mathcal{O}_{\mathbb{X}}) = \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(i))$  has a presentation as follows

$$\Gamma_*(\mathcal{O}_{\mathbb{X}}) = \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(i)) \cong \bigoplus_{i \in \mathbb{Z}} \Gamma = \bigoplus_{i \in \mathbb{Z}} \left( \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} \right) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} [T_j, T_j^{-1}] \cong R_{x_0}$$

where  $T_1, \dots, T_s$  are indeterminates with  $\deg(T_1) = \dots = \deg(T_s) = 1$ . We denote  $\tilde{R} := \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} [T_j]$  the subring of  $\Gamma_*(\mathcal{O}_{\mathbb{X}})$ . According to Lemma 2.3.10, we have an injection

$$\tilde{\iota}: R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} [T_j] \hookrightarrow \Gamma_*(\mathcal{O}_{\mathbb{X}}), f \mapsto (f_{p_1} T_1^i, \dots, f_{p_s} T_s^i) \quad (2.7)$$

for  $f \in R_i$  with  $i \geq 0$ . Moreover,  $\tilde{\iota}(x_0) = (T_1, \dots, T_s)$  and the restriction  $\tilde{\iota}|_{R_i}: R_i \rightarrow \tilde{R}_i$  is an isomorphism of  $K$ -vector spaces for  $i \geq r_{\mathbb{X}}$ . Here and subsequently, let us denote

$$\varkappa_j := \dim_K K(p_j) = \dim_K(\mathcal{O}_{\mathbb{X}, p_j} / \mathfrak{m}_{\mathbb{X}, p_j})$$

for  $j = 1, \dots, s$ . Given an element  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ , we define

$$\mu(a) := \min\{i \in \mathbb{N} \mid (0, \dots, 0, aT_j^i, 0, \dots, 0) \in \tilde{\iota}(R)\}$$

and

$$\nu(a) := \max\{\mu(ab) \mid b \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\}.$$

Notice that  $\mu(0) = \nu(0) = 0$  and if  $s \geq 2$  and  $a \neq 0$  then  $\nu(a) \geq \mu(a) \geq 1$ , since  $\tilde{\iota}(R_0) = \langle (1, \dots, 1) \rangle_K$ . Some more precise rules for these values are provided by our next lemma.

**Lemma 2.3.15.** *Assume that we are in the setting introduced.*

- (i) We have  $\mu(a + b) \leq \max\{\mu(a), \mu(b)\}$  for all  $a, b \in \mathcal{O}_{\mathbb{X}, p_j}$ .
- (ii) Let  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  denote the socle of  $\mathcal{O}_{\mathbb{X}, p_j}$ , i.e.,  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) = \text{Ann}_{\mathcal{O}_{\mathbb{X}, p_j}}(\mathfrak{m}_{\mathbb{X}, p_j})$ , let  $s_j$  be a non-zero element of  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$ , and let  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  be such that whose residue classes form a  $K$ -basis of  $K(p_j)$ . Then we have

$$\nu(s_j) = \max\{\mu(e_{jk_j} s_j) \mid k_j = 1, \dots, \kappa_j\}.$$

- (iii) Let  $s_j, s'_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ . If  $s_j = as'_j$  for some  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ , then  $\nu(s_j) = \nu(s'_j)$ .

*Proof.* (i) It suffices to show the claim in the case  $\mu(a) \leq \mu(b)$ . Let

$$f = \tilde{\iota}^{-1}((0, \dots, 0, aT_j^{\mu(a)}, 0, \dots, 0)), \quad g = \tilde{\iota}^{-1}((0, \dots, 0, bT_j^{\mu(b)}, 0, \dots, 0)).$$

Then we have  $f, g \in R$  and

$$\tilde{\iota}(x_0^{\mu(b)-\mu(a)} f + g) = (0, \dots, 0, (a+b)T_j^{\mu(b)}, 0, \dots, 0).$$

It follows that  $\mu(a+b) \leq \mu(b)$ .

(ii) Observe that

$$\nu(s_j) = \max\{\mu(as_j) \mid a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\} \geq \max\{\mu(e_{jk_j} s_j) \mid k_j = 1, \dots, \kappa_j\}.$$

Now let  $a \in \mathcal{O}_{\mathbb{X}, p_j}$  be such that  $as_j \neq 0$ . Then  $a \notin \mathfrak{m}_{\mathbb{X}, p_j}$ , since  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ . So we may write  $a = c_{j1}e_{j1} + \dots + c_{j\kappa_j}e_{j\kappa_j} \pmod{\mathfrak{m}_{\mathbb{X}, p_j}}$  for  $c_{j1}, \dots, c_{j\kappa_j}$  in  $K$ , not all equal to zero. We deduce  $as_j = c_{j1}e_{j1}s_j + \dots + c_{j\kappa_j}e_{j\kappa_j}s_j$ . It follows from (i) that  $\mu(as_j) = \mu(c_{j1}e_{j1}s_j + \dots + c_{j\kappa_j}e_{j\kappa_j}s_j) \leq \max\{\mu(e_{jk_j} s_j) \mid k_j = 1, \dots, \kappa_j\}$ . This implies  $\nu(s_j) \leq \max\{\mu(e_{jk_j} s_j) \mid k_j = 1, \dots, \kappa_j\}$ . Therefore the claimed equality is proved.

(iii) Since  $s_j = as'_j$  and  $s_j, s'_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ , the element  $a$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$ . Thus we have

$$\begin{aligned} \nu(s_j) &= \max\{\mu(s_j b) \mid b \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\} = \max\{\mu(s'_j a b) \mid b \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\} \\ &\leq \nu(s'_j) = \max\{\mu(s'_j c) \mid c \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\} = \max\{\mu(s_j a^{-1} c) \mid c \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\} \\ &\leq \nu(s_j). \end{aligned}$$

Hence we get the equality  $\nu(s'_j) = \nu(s_j)$ . □

For abbreviation, we introduce the following notion of a special subscheme of  $\mathbb{X}$ .

**Definition 2.3.16.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and let  $j \in \{1, \dots, s\}$ . We say that a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  is a  $p_j$ -**subscheme** if the following conditions are satisfied:

- (i)  $\mathcal{O}_{\mathbb{Y}, p_k} = \mathcal{O}_{\mathbb{X}, p_k}$  for  $k \neq j$ .
- (ii) The map  $\mathcal{O}_{\mathbb{X}, p_j} \rightarrow \mathcal{O}_{\mathbb{Y}, p_j}$  is an epimorphism.

A  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  is called **maximal** if  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$ .

For example, let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme which has  $K$ -rational support. A maximal  $p_j$ -subscheme of  $\mathbb{X}$  is nothing but a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$  with  $\mathcal{O}_{\mathbb{Y}, p_j} \neq \mathcal{O}_{\mathbb{X}, p_j}$ .

From now on, if no ambiguity arises, we say “a  $p_j$ -subscheme of  $\mathbb{X}$ ” without specifying  $j \in \{1, \dots, s\}$ .

Given a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , at each  $p_j \in \text{Supp}(\mathbb{X})$  we define a relation  $\sim$  on the socle  $\mathfrak{S}(\mathcal{O}_{\mathbb{X}, p_j})$  by  $s_j \sim s'_j \Leftrightarrow s_j = as'_j$  for some  $a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . It is easy to see that  $\sim$  is an equivalence relation. The equivalence class of  $s_j$  under  $\sim$ , denoted by  $[s_j]$ , is defined as  $[s_j] = \{s'_j \in \mathfrak{S}(\mathcal{O}_{\mathbb{X}, p_j}) \mid s_j \sim s'_j\}$ . We set

$$U_{s_j} = \left\{ (0, \dots, 0, u_j, 0, \dots, 0) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} \mid u_j \in [s_j] \right\}.$$

**Proposition 2.3.17.** *Let  $\mathbb{X}$  be a 0-dimensional subscheme of  $\mathbb{P}_K^n$  with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . There is a 1-1 correspondence*

$$\left\{ \begin{array}{l} \text{maximal } p_j\text{-subschemes} \\ \text{of the scheme } \mathbb{X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subsets } U_{s_j} \subseteq \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} \\ \text{with } s_j \in \mathfrak{S}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\} \end{array} \right\}$$

*Proof.* Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, and let  $\{e_{j1}, \dots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  be such that whose residue classes form a  $K$ -basis of  $K(p_j)$ . Let  $f_{\mathbb{Y}} \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}} \setminus \{0\}$ , and write  $\tilde{v}(f_{\mathbb{Y}}) = (0, \dots, 0, s_j T_j^{\alpha_{\mathbb{Y}/\mathbb{X}}}, 0, \dots, 0) \in \tilde{R}$ . Clearly,  $\alpha_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$  and  $s_j \neq 0$ . We claim that  $s_j \in \mathfrak{S}(\mathcal{O}_{\mathbb{X}, p_j})$ . Indeed, if otherwise, then there is an element  $a \in \mathfrak{m}_{\mathbb{X}, p_j}$  such that  $as_j \neq 0$ . Suppose there are  $c_1, \dots, c_{\varkappa_j+1} \in K$  such that

$$c_1 e_{j1} s_j + \dots + c_{\varkappa_j} e_{j\varkappa_j} s_j + c_{\varkappa_j+1} a s_j = (c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j} + c_{\varkappa_j+1} a) s_j = 0.$$

If  $c_1 \bar{e}_{j1} + \dots + c_{\varkappa_j} \bar{e}_{j\varkappa_j} \neq 0$  in  $K(p_j)$ , then  $c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j}$  is a unit element, so is  $c_1 e_{j1} + \dots + c_{\varkappa_j} e_{j\varkappa_j} + c_{\varkappa_j+1} a$  (as  $a \in \mathfrak{m}_{\mathbb{X}, p_j}$ ). It follows from the above equality that  $s_j = 0$ , it is impossible. So, we must have  $c_1 \bar{e}_{j1} + \dots + c_{\varkappa_j} \bar{e}_{j\varkappa_j} = 0$ . This implies  $c_1 = \dots = c_{\varkappa_j} = 0$ , since  $\{\bar{e}_{j1}, \dots, \bar{e}_{j\varkappa_j}\}$  is a  $K$ -basis of  $K(p_j)$ . From this we deduce

$c_{\varkappa_j+1}as_j = 0$ , hence  $c_{\varkappa_j+1} = 0$  (as  $as_j \neq 0$ ). Therefore the set  $\{e_{j1}s_j, \dots, e_{j\varkappa_j}s_j, as_j\}$  is  $K$ -linearly independent. Let  $f_{jk_j} = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j}s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$  for  $k_j = 1, \dots, \varkappa_j$  and  $f_{as_j} = \tilde{\nu}^{-1}((0, \dots, 0, as_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ . Then we have  $x_0^{\alpha_{\mathbb{Y}/\mathbb{X}}} f_{as_j} = f_a f_{\mathbb{Y}}$ , where  $f_a = \tilde{\nu}^{-1}((0, \dots, 0, a T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ , and so Lemma 2.3.12 yields  $f_{as_j} \in \langle f_{\mathbb{Y}} \rangle_R^{\text{sat}} \subseteq \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . Thus we get

$$\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \geq \dim_K \langle f_{j1}, \dots, f_{j\varkappa_j}, f_{as_j} \rangle_K = \varkappa_j + 1$$

and hence  $\deg(\mathbb{Y}) < \deg(\mathbb{X}) - \varkappa_j$ , a contradiction.

Next we consider  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_i \setminus \{0\}$  with  $i \geq \alpha_{\mathbb{Y}/\mathbb{X}}$ . The previous claim also tells us that  $f_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ . If  $f_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus [s_j]$ , then it is not difficult to check that  $\{f_{p_j}, e_{j1}s_j, \dots, e_{j\varkappa_j}s_j\}$  is  $K$ -linearly independent. This implies  $\deg(\mathbb{Y}) < \deg(\mathbb{X}) - \varkappa_j$ , and it is impossible. Hence we have  $f_{p_j} \in [s_j]$ .

Let  $g \in R_i \setminus \{0\}$  with  $i \geq \alpha_{\mathbb{Y}/\mathbb{X}}$  be such that  $\tilde{\nu}(g) = (0, \dots, 0, g_{p_j} T_j^i, 0, \dots, 0)$  and  $g_{p_j} \in [s_j]$ . We are able to write  $g_{p_j} = as_j$  for some  $a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . Using a similar argument as the previous part we get  $g \in \langle f_{\mathbb{Y}} \rangle_R^{\text{sat}} \subseteq \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . Therefore the image of  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} \setminus \{0\}$  in  $\prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  is  $U_{s_j}$  with  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ , as was to be shown.

Conversely, let  $(0, \dots, 0, s_j, 0, \dots, 0) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  with  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ , and let  $f = \tilde{\nu}^{-1}((0, \dots, 0, s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ . We set  $\mathbb{Y} := \mathcal{Z}^+(f) \subseteq \mathbb{X}$ . Then we have  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f \rangle_R^{\text{sat}}$ . Obviously, the scheme  $\mathbb{Y}$  is a  $p_j$ -subscheme of  $\mathbb{X}$ . It suffices to prove  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$ . Let  $f_{jk_j} = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j}s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$  and  $g_{jk_j} = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$  for  $k_j = 1, \dots, \varkappa_j$ . We see that  $x_0^{r_{\mathbb{X}}} f_{jk_j} = g_{jk_j} f$  for every  $k_j \in \{1, \dots, \varkappa_j\}$ , and so Lemma 2.3.12 implies  $f_{jk_j} \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$ . Thus we get the inequality

$$\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \geq \dim_K \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K = \varkappa_j.$$

Moreover, for  $h \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ , there is a number  $m \in \mathbb{N}$  such that  $x_0^m h \in \langle f \rangle_R$ . This clearly forces  $h_{p_j} = as_j$  for some  $a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$  and  $h_{p_k} = 0$  for  $k \neq j$ . Let us write  $as_j = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} e_{jk_j} s_j$  for some  $c_{j1}, \dots, c_{j\varkappa_j} \in K$ . Then  $\tilde{\nu}(h) = \tilde{\nu}(\sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j})$ . Since the map  $\tilde{\nu}$  is injective, we have  $h = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j} \in \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K$ . This implies  $\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \leq \varkappa_j$ , and therefore this inequality becomes an equality. Hence we obtain  $\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{i+r_{\mathbb{X}}} = \varkappa_j$  for  $i \geq 0$  or  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - \varkappa_j$ , as desired.  $\square$

**Corollary 2.3.18.** *A 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  contains a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{X}) - 1$  if and only if  $\mathbb{X}(K) \neq \emptyset$ .*

*Proof.* This follows from Remark 2.3.13(b) and Proposition 2.3.17.  $\square$

Let  $\mathbb{Y}$  be a maximal  $p_j$ -subscheme of  $\mathbb{X}$ , and let  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$  be a socle element corresponding to a non-zero element of  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}}$ . We shall also say that  $s_j$  is a socle element of  $\mathcal{O}_{\mathbb{X}, p_j}$  corresponding to  $\mathbb{Y}$ . Let  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  such that whose residue classes form a  $K$ -basis of  $K(p_j)$ . For  $k_j = 1, \dots, \kappa_j$ , we set

$$f_{jk_j}^* := \tilde{v}^{-1}((0, \dots, 0, e_{jk_j} s_j T_j^{\mu(e_{jk_j} s_j)}, 0, \dots, 0)).$$

It follows from Lemma 2.3.15 that the maximal degree of  $\{f_{j1}^*, \dots, f_{j\kappa_j}^*\}$  depends neither on the choice of the element  $s_j$  nor on the specific choice of  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ .

**Definition 2.3.19.** (i) The set  $\{f_{j1}^*, \dots, f_{j\kappa_j}^*\}$  is called a **set of minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$**  (with respect to  $s_j$  and  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ ).

(ii) The set  $\{f_{j1}, \dots, f_{j\kappa_j}\}$ , where  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \mu(e_{jk_j} s_j)} f_{jk_j}^*$  for  $k_j = 1, \dots, \kappa_j$ , is said to be a **set of separators of  $\mathbb{Y}$  in  $\mathbb{X}$**  (with respect to  $s_j$  and  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ ).

(iii) The number

$$\nu_{\mathbb{Y}/\mathbb{X}} := \max\{\deg(f_{jk_j}^*) \mid k_j = 1, \dots, \kappa_j\}$$

is called the **maximal degree of a minimal separator of  $\mathbb{Y}$  in  $\mathbb{X}$** .

**Remark 2.3.20.** If  $\mathbb{X}$  has  $K$ -rational support, then  $\kappa_1 = \dots = \kappa_s = 1$  and, for every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$ , a minimal separator  $f_{\mathbb{Y}}^*$  of  $\mathbb{Y}$  in  $\mathbb{X}$  is exactly a non-zero element of  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}}$ , i.e.,  $f_{\mathbb{Y}}^*$  is a minimal separator of  $\mathbb{Y}$  in  $\mathbb{X}$  in the sense of [Kr2]. We also see that  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}} + i} = \langle x_0^i \cdot f_{\mathbb{Y}}^* \rangle_K$  for  $i \geq 0$ , and an element  $f_{\mathbb{Y}} \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$  is a separator of  $\mathbb{Y}$  in  $\mathbb{X}$ . Especially, when  $\mathbb{X} = \{p_1, \dots, p_s\}$  is a projective point set in  $\mathbb{P}_K^n$ , for  $j \in \{1, \dots, s\}$  we write  $p_j = (1 : p_{j1} : \dots : p_{jn})$  with  $p_{jk} \in K$ , and for  $f \in R$  we set  $f(p_j) := F(1, p_{j1}, \dots, p_{jn})$ , where  $F$  is any representative of  $f$  in  $P$ . Then  $f \in R_{r_{\mathbb{X}}}$  is a separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  if and only if  $f(p_j) \neq 0$  and  $f(p_k) = 0$  for  $k \neq j$ . We shall say that a separator  $f_j \in R_{r_{\mathbb{X}}}$  of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  is a **normal separator** if  $f_j(p_k) = \delta_{jk}$  for  $1 \leq k \leq s$ .

**Proposition 2.3.21.** *Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, let  $s_j$  be a socle element of  $\mathcal{O}_{\mathbb{X}, p_j}$  corresponding to  $\mathbb{Y}$ , let  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  such that whose residue classes form a  $K$ -basis of  $K(p_j)$ , and let  $\{f_{j1}^*, \dots, f_{j\kappa_j}^*\}$  be a set of minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$  with respect to  $s_j$  and  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ . The following assertions hold true.*

(i) *We have  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f \rangle_R^{\text{sat}}$  for every  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_i \setminus \{0\}$  with  $i \geq \alpha_{\mathbb{Y}/\mathbb{X}}$ .*

(ii) We have  $\nu_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$  and the Hilbert function of  $\mathbb{Y}$  satisfies

$$\mathrm{HF}_{\mathbb{Y}}(i) = \begin{cases} \mathrm{HF}_{\mathbb{X}}(i) & \text{if } i < \alpha_{\mathbb{Y}/\mathbb{X}}, \\ \leq \mathrm{HF}_{\mathbb{X}}(i) - 1 & \text{if } \alpha_{\mathbb{Y}/\mathbb{X}} \leq i < \nu_{\mathbb{Y}/\mathbb{X}}, \\ \mathrm{HF}_{\mathbb{X}}(i) - \varkappa_j & \text{if } i \geq \nu_{\mathbb{Y}/\mathbb{X}}. \end{cases}$$

(iii) There is a special choice of the set  $\{e_{j1}, \dots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  such that whose residue classes form a  $K$ -basis of  $K(p_j)$ ,  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_{j1}^*, \dots, f_{j\varkappa_j}^* \rangle_R$ , and for all  $i \in \mathbb{Z}$  we have

$$\Delta \mathrm{HF}_{\mathbb{Y}}(i) = \Delta \mathrm{HF}_{\mathbb{X}}(i) - \#\{f_{jk_j}^* \in \{f_{j1}^*, \dots, f_{j\varkappa_j}^*\} \mid \deg(f_{jk_j}^*) = i\}.$$

*Proof.* (i) It is clear that  $\langle f \rangle_R^{\mathrm{sat}} \subseteq \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . For the other inclusion, we write

$$\tilde{v}(f) = (0, \dots, 0, a s_j T_j^i, 0, \dots, 0) \in \tilde{R}$$

for some  $a \in \mathcal{O}_{\mathbb{Y}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . Similarly, for every  $g \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_k$  with  $k \geq \alpha_{\mathbb{Y}/\mathbb{X}}$  we have  $\tilde{v}(g) = (0, \dots, 0, b s_j T_j^k, 0, \dots, 0)$  with  $b \in \mathcal{O}_{\mathbb{X}, p_j}$ . If  $b$  is not unit of  $\mathcal{O}_{\mathbb{X}, p_j}$ , then  $b s_j = 0$ , so  $g = 0 \in \langle f \rangle_R^{\mathrm{sat}}$ . Otherwise, let  $h = \tilde{v}^{-1}((0, \dots, 0, b a^{-1} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ . Then  $x_0^{r_{\mathbb{X}}+i} g = x_0^k h f \in \langle f \rangle_R$ , so  $g \in \langle f \rangle_R^{\mathrm{sat}}$  by Lemma 2.3.12. Hence we have  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f \rangle_R^{\mathrm{sat}}$ .

(ii) Observe that  $\alpha_{\mathbb{Y}/\mathbb{X}} \leq \nu_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$  and  $\mathrm{HF}_{\mathbb{Y}}(i) \leq \mathrm{HF}_{\mathbb{X}}(i) - 1$  for  $\alpha_{\mathbb{Y}/\mathbb{X}} \leq i < \nu_{\mathbb{Y}/\mathbb{X}}$ . It remains to show that  $\mathrm{HF}_{\mathbb{Y}}(i + \nu_{\mathbb{Y}/\mathbb{X}}) = \mathrm{HF}_{\mathbb{X}}(i + \nu_{\mathbb{Y}/\mathbb{X}}) - \varkappa_j$  for  $i \geq 0$ . We set  $g_{jk_j} := x_0^{\nu_{\mathbb{Y}/\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^* \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\nu_{\mathbb{Y}/\mathbb{X}}}$  for all  $k_j = 1, \dots, \varkappa_j$ . Then we have  $\tilde{v}(g_{jk_j}) = (0, \dots, 0, e_{jk_j} s_j T_j^{\nu_{\mathbb{Y}/\mathbb{X}}}, 0, \dots, 0) \in \tilde{R}$ . Since  $\{e_{j1} s_j, \dots, e_{j\varkappa_j} s_j\}$  is  $K$ -linearly independent, this implies

$$\varkappa_j = \dim_K \langle g_{j1}, \dots, g_{j\varkappa_j} \rangle_K \leq \dim_K (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\nu_{\mathbb{Y}/\mathbb{X}}} \leq \varkappa_j.$$

So  $\dim_K (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\nu_{\mathbb{Y}/\mathbb{X}}} = \dim_K (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{i + \nu_{\mathbb{Y}/\mathbb{X}}} = \varkappa_j$  for all  $i \geq 0$ . Therefore  $\mathrm{HF}_{\mathbb{Y}}(i + \nu_{\mathbb{Y}/\mathbb{X}}) = \mathrm{HF}_{\mathbb{X}}(i + \nu_{\mathbb{Y}/\mathbb{X}}) - \varkappa_j$  for all  $i \geq 0$ , in particular,  $\mathrm{HF}_{\mathbb{Y}}(i + r_{\mathbb{X}}) = \deg(\mathbb{X}) - \varkappa_j = \deg(\mathbb{Y})$  for all  $i \geq 0$ .

(iii) We may construct the set  $\{e_{j1}, \dots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  with the desired properties as follows. Let  $d_{\alpha_{\mathbb{Y}/\mathbb{X}}} = \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(\alpha_{\mathbb{Y}/\mathbb{X}})$ ,  $d_{\alpha_{\mathbb{Y}/\mathbb{X}}+i} = \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(\alpha_{\mathbb{Y}/\mathbb{X}} + i) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(\alpha_{\mathbb{Y}/\mathbb{X}} + i - 1)$  for  $i = 1, \dots, \nu_{\mathbb{Y}/\mathbb{X}} - \alpha_{\mathbb{Y}/\mathbb{X}}$ . Then we have  $\varkappa_j = d_{\alpha_{\mathbb{Y}/\mathbb{X}}} + d_{\alpha_{\mathbb{Y}/\mathbb{X}}+1} + \dots + d_{\nu_{\mathbb{Y}/\mathbb{X}}}$ . We begin taking  $f_{j1}^*, \dots, f_{j d_{\alpha_{\mathbb{Y}/\mathbb{X}}}}^*$  a  $K$  basis of  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}}$ . For  $i = 1, \dots, \nu_{\mathbb{Y}/\mathbb{X}} - \alpha_{\mathbb{Y}/\mathbb{X}}$ , if  $d_{\alpha_{\mathbb{Y}/\mathbb{X}}+i} > 0$ , we choose  $f_{j \sum_{0 \leq k < i} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k+1}}^*, \dots, f_{j \sum_{0 \leq k \leq i} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k}}^*$  such that the set

$$\left\{ x_0^i f_{j1}^*, \dots, x_0^i f_{j d_{\alpha_{\mathbb{Y}/\mathbb{X}}}}^*, \dots, x_0 f_{j \sum_{0 \leq k < i-1} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k+1}}^*, \dots, x_0 f_{j \sum_{0 \leq k \leq i-1} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k}}^*, \right. \\ \left. f_{j \sum_{0 \leq k < i} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k+1}}^*, \dots, f_{j \sum_{0 \leq k \leq i} d_{\alpha_{\mathbb{Y}/\mathbb{X}}+k}}^* \right\}$$



forms a  $K$ -basis of  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{\alpha_{\mathbb{Y}/\mathbb{X}}+i}$ . Then the ideal  $J = \langle f_{j_1}^*, \dots, f_{j_{\nu_j}}^* \rangle_R$  is a subideal of  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  and  $\mathrm{HF}_J(i) = \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i)$  for all  $i \leq \nu_{\mathbb{Y}/\mathbb{X}}$ . By (ii) we have  $\mathrm{HF}_J(i) = \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i) = \nu_j$  for  $i \geq \nu_{\mathbb{Y}/\mathbb{X}}$ . This implies  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = J = \langle f_{j_1}^*, \dots, f_{j_{\nu_j}}^* \rangle_R$ . Moreover, it follows from the construction of the set  $\{f_{j_1}^*, \dots, f_{j_{\nu_j}}^*\}$  that for all  $i \in \mathbb{Z}$  we have

$$\mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i) = \#\{f_{jk_j}^* \in \{f_{j_1}^*, \dots, f_{j_{\nu_j}}^*\} \mid \deg(f_{jk_j}^*) \leq i\}.$$

Thus we obtain

$$\begin{aligned} \Delta \mathrm{HF}_{\mathbb{Y}}(i) &= \mathrm{HF}_{\mathbb{Y}}(i) - \mathrm{HF}_{\mathbb{Y}}(i-1) \\ &= (\mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i)) - (\mathrm{HF}_{\mathbb{X}}(i-1) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i-1)) \\ &= \Delta \mathrm{HF}_{\mathbb{X}}(i) - (\mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i) - \mathrm{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{X}}}(i-1)) \\ &= \Delta \mathrm{HF}_{\mathbb{X}}(i) - \#\{f_{jk_j}^* \in \{f_{j_1}^*, \dots, f_{j_{\nu_j}}^*\} \mid \deg(f_{jk_j}^*) = i\}. \end{aligned}$$

Now let us write  $\tilde{v}(f_{jk_j}^*) = (0, \dots, 0, e_{jk_j} s_j T_j^{\deg(f_{jk_j}^*)}, 0, \dots, 0)$  for  $k_j = 1, \dots, \nu_j$ . Obviously, the set  $\{e_{j_1} s_j, \dots, e_{j_{\nu_j}} s_j\}$  is  $K$ -linearly independent. It remains to show that the residue classes  $\{\bar{e}_{j_1}, \dots, \bar{e}_{j_{\nu_j}}\}$  form a  $K$ -basis of  $K(p_j)$ . Suppose there are  $c_{j_1}, \dots, c_{j_{\nu_j}} \in K$  such that  $c_{j_1} \bar{e}_{j_1} + \dots + c_{j_{\nu_j}} \bar{e}_{j_{\nu_j}} = 0$ . It follows that the element  $c_{j_1} e_{j_1} + \dots + c_{j_{\nu_j}} e_{j_{\nu_j}}$  is contained in  $\mathfrak{m}_{\mathbb{X}, p_j}$ . This implies  $c_{j_1} e_{j_1} s_j + \dots + c_{j_{\nu_j}} e_{j_{\nu_j}} s_j = 0$ . Since  $\{e_{j_1} s_j, \dots, e_{j_{\nu_j}} s_j\}$  is  $K$ -linearly independent, we deduce  $c_{j_1} = \dots = c_{j_{\nu_j}} = 0$ . Therefore the set  $\{\bar{e}_{j_1}, \dots, \bar{e}_{j_{\nu_j}}\}$  is a  $K$ -basis of  $K(p_j)$ , and the conclusion follows.  $\square$

Note that the set  $\{f_{j_1}^*, \dots, f_{j_{\nu_j}}^*\}$  of minimal separators of a maximal  $p_j$ -subscheme  $\mathbb{Y}$  in  $\mathbb{X}$  as in Proposition 2.3.21(iii) is not necessarily a homogeneous minimal system of generators of  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ , as the following example shows.

**Example 2.3.22.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the 0-dimensional reduced complete intersection with  $\mathcal{I}_{\mathbb{X}} = \langle X_2, X_0^5 X_1 - \frac{11}{6} X_0^4 X_1^2 + 2X_0^3 X_1^3 - 2X_0^2 X_1^4 + X_0 X_1^5 - \frac{1}{6} X_1^6 \rangle$ . Then  $\mathbb{X}$  contains the projective point set  $\mathbb{Y} = \{(1 : 0 : 0), (1 : 1 : 0), (1 : 2 : 0), (1 : 3 : 0)\}$  which is a maximal  $p$ -subscheme, where  $p$  is the closed point corresponding to the homogeneous prime ideal  $\mathfrak{P} = \langle X_1^2 + X_0^2, X_2 \rangle$ . We see that  $\deg(\mathbb{Y}) = 4 = \deg(\mathbb{X}) - 2$ , and two minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$  are  $f_1^* = x_0^3 x_1 - \frac{11}{6} x_0^2 x_1^2 + x_0 x_1^3 - \frac{1}{6} x_1^4$  and  $f_2^* = x_1 f_1^*$ . Moreover, the equality of the Castelnuovo function of  $\mathbb{Y}$  holds true, while  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_1^* \rangle_R$ .

**Definition 2.3.23.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{K}}^n$  be a 0-dimensional scheme, and let  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . For every  $p_j \in \mathrm{Supp}(\mathbb{X})$ , the **degree of  $p_j$  in  $\mathbb{X}$**  is defined as

$$\deg_{\mathbb{X}}(p_j) := \min \{ \nu_{\mathbb{Y}/\mathbb{X}} \mid \mathbb{Y} \text{ is a maximal } p_j\text{-subscheme of } \mathbb{X} \}.$$

**Remark 2.3.24.** (a) In the sense of the above definition, the degree of  $p_j$  in  $\mathbb{X}$  is given by

$$\deg_{\mathbb{X}}(p_j) = \min\{\nu(s_j) \mid s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}\}.$$

(b) If  $\mathbb{X}$  has  $K$ -rational support, then we have

$$\deg_{\mathbb{X}}(p_j) = \min\{\alpha_{\mathbb{Y}/\mathbb{X}} \mid \mathbb{Y} \text{ is a maximal } p_j\text{-subscheme of } \mathbb{X}\}.$$

If, in addition,  $\mathbb{X}$  is reduced, then  $\deg_{\mathbb{X}}(p_j) = \alpha_{\mathbb{X} \setminus \{p_j\}/\mathbb{X}}$  for all  $j = 1, \dots, s$ .

(c) We have  $0 \leq \deg_{\mathbb{X}}(p_j) \leq r_{\mathbb{X}}$  for all  $p_j \in \text{Supp}(\mathbb{X})$ . In particular, if  $\mathbb{X}$  is a projective point set in  $\mathbb{P}_K^n$ , then there always exists a point  $p_j \in \mathbb{X}$  with maximal degree  $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$  (cf. [GKR, Proposition 1.14]).

We end this section with following two lemmata in which we generalize some results found in [Kr3, Section 1].

**Lemma 2.3.25.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $f \in R_i$  with  $i \geq 0$ , let  $\mathbb{Y}$  be a maximal  $p_j$ -subscheme of  $\mathbb{X}$ , and let  $\{f_{j1}, \dots, f_{j\kappa_j}\} \subseteq R_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{Y}$  in  $\mathbb{X}$ .*

- (i) *We have  $f \cdot f_{jl} = \sum_{k_j=1}^{\kappa_j} c_{jk_j}^l x_0^i f_{jk_j}$  for some  $c_{jk_j}^l \in K$  and  $l \in \{1, \dots, \kappa_j\}$ .*
- (ii) *If  $f \cdot f_{jl} = 0$  for some  $l \in \{1, \dots, \kappa_j\}$ , then  $f \cdot f_{j\lambda} = 0$  for all  $\lambda \in \{1, \dots, \kappa_j\}$ . Moreover,  $f \cdot f_{jl} \neq 0$  if and only if  $f_{p_j} \notin \mathfrak{m}_{\mathbb{X}, p_j}$ .*
- (iii) *Let  $\mathbb{Y}'$  be a maximal  $p_{j'}$ -subscheme of  $\mathbb{X}$ , and let  $\{f_{j'1}, \dots, f_{j'\kappa_{j'}}\} \subseteq R_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{Y}'$  in  $\mathbb{X}$ . Then we have*

$$f_{jk_j} \cdot f_{j'k_{j'}} \in \begin{cases} x_0^{r_{\mathbb{X}}} \langle f_{j1}, \dots, f_{j\kappa_j} \rangle_K & \text{if } j = j' \text{ and } \dim_{K(p_j)}(\mathcal{O}_{\mathbb{X}, p_j}) = 1, \\ \langle 0 \rangle & \text{otherwise.} \end{cases}$$

*Proof.* We derive claim (i) from the fact that

$$f \cdot f_{jl} \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}+i} = \langle x_0^i f_{j1}, \dots, x_0^i f_{j\kappa_j} \rangle_K$$

for  $l = 1, \dots, \kappa_j$ . Claims (ii) and (iii) follow by using the injection  $\tilde{\nu}$  and  $(f_{jk_j})_{p_j}$  is a socle element of  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  for  $k_j = 1, \dots, \kappa_j$ .  $\square$

**Lemma 2.3.26.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , let  $\{f_{j1}, \dots, f_{j\kappa_j}\} \subseteq R_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ .*

(i) For every  $i \geq r_{\mathbb{X}}$  the set  $\{x_0^{i-r_{\mathbb{X}}}f_{11}, \dots, x_0^{i-r_{\mathbb{X}}}f_{1\kappa_1}, \dots, x_0^{i-r_{\mathbb{X}}}f_{s1}, \dots, x_0^{i-r_{\mathbb{X}}}f_{s\kappa_s}\}$  is a  $K$ -basis of  $R_i$ .

(ii) If  $f \in R_i$  and  $a_{11}x_0^j f_{11} + \dots + a_{s\kappa_s}x_0^j f_{s\kappa_s} \in R_{j+r_{\mathbb{X}}}$  for some  $i, j \geq 0$  and  $a_{11}, \dots, a_{s\kappa_s} \in K$ , then

$$f \cdot (a_{11}x_0^j f_{11} + \dots + a_{s\kappa_s}x_0^j f_{s\kappa_s}) = \sum_{l=1}^{\kappa_1} a_{1l}c_{11}^l x_0^{i+j} f_{11} + \dots + \sum_{l=1}^{\kappa_s} a_{sl}c_{s\kappa_s}^l x_0^{i+j} f_{s\kappa_s}$$

where  $c_{j1}^l, \dots, c_{j\kappa_j}^l \in K$  satisfy  $f \cdot f_{jl} = \sum_{k_j=1}^{\kappa_j} c_{jk_j}^l x_0^i f_{jk_j}$  for  $j = 1, \dots, s$  and  $l \in \{1, \dots, \kappa_j\}$ .

*Proof.* (i) Since  $\mathbb{X}$  is reduced, we have  $\mathcal{O}_{\mathbb{X}, p_j} = K(p_j) = \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  for  $j = 1, \dots, s$ . For  $i \geq r_{\mathbb{X}}$ , we see that

$$\tilde{v}(x_0^{i-r_{\mathbb{X}}}f_{jk_j}) = (0, \dots, 0, e_{jk_j}T_j^i, 0, \dots, 0) \in \tilde{R},$$

where  $\{e_{j1}, \dots, e_{j\kappa_j}\}$  is the  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ , for  $j = 1, \dots, s$  and for  $k_j = 1, \dots, \kappa_j$ . Then it is not difficult to show that the set  $\{\tilde{v}(x_0^{i-r_{\mathbb{X}}}f_{11}), \dots, \tilde{v}(x_0^{i-r_{\mathbb{X}}}f_{s\kappa_s})\}$  is  $K$ -linearly independent. Thus this set forms a  $K$ -basis of  $\tilde{R}_i$  for all  $i \geq r_{\mathbb{X}}$ . Since for every  $i \geq r_{\mathbb{X}}$  the restriction  $\tilde{v}|_{R_i} : R_i \rightarrow \tilde{R}_i$  is an isomorphism of  $K$ -vector spaces, this implies  $\{x_0^{i-r_{\mathbb{X}}}f_{11}, \dots, x_0^{i-r_{\mathbb{X}}}f_{s\kappa_s}\}$  is a  $K$ -basis of  $R_i$ .

(ii) This follows from Lemma 2.3.25(i).  $\square$

## 2.4 Trace Maps for 0-Dimensional Schemes

As in the previous section we let  $K$  be an arbitrary field, let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ , let  $\mathcal{I}_{\mathbb{X}}$  be the homogeneous vanishing ideal of  $\mathbb{X}$  in  $P = K[X_0, \dots, X_n]$ , and let  $R = P/\mathcal{I}_{\mathbb{X}}$  be the homogeneous coordinate ring of  $\mathbb{X}$ . Also, we denote the image of  $X_i$  in  $R$  by  $x_i$  for  $i = 0, \dots, n$ . Then  $x_0$  is not a zerodivisor of  $R$  and  $R$  is a graded-free  $K[x_0]$ -module of rank  $m = \deg(\mathbb{X})$ .

Let  $<_{\sigma}$  be a degree-compatible term ordering on  $\mathbb{T}^n$ . The affine ideal of  $\mathbb{X}$  is  $\mathcal{I}_{\mathbb{X}}^a = (\mathcal{I}_{\mathbb{X}})^{\text{deh}} \subseteq A = K[X_1, \dots, X_n]$ . Then we have  $\mathcal{O} = \mathbb{T}^n \setminus \text{LT}_{\sigma}(\mathcal{I}_{\mathbb{X}}^a) = \{T_1, \dots, T_m\}$  with  $T_j = X_1^{\alpha_{j1}} \cdots X_n^{\alpha_{jn}}$  and  $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn}) \in \mathbb{N}^n$  for  $j = 1, \dots, m$ . W.l.o.g. we assume that  $T_1 <_{\sigma} \cdots <_{\sigma} T_m$ . Let  $t_j = T_j + \mathcal{I}_{\mathbb{X}} \in R$  and set  $\deg(t_j) := \deg(T_j) = n_j$  for  $j = 1, \dots, m$ . Then  $n_1 \leq \cdots \leq n_m$  and the set  $\mathcal{B} = \{t_1, \dots, t_m\}$  is a  $K[x_0]$ -basis of  $R$ . From now on, if not stated otherwise, as a  $K[x_0]$ -basis of  $R$  we always choose the above  $K[x_0]$ -basis  $\mathcal{B} = \{t_1, \dots, t_m\}$ .

**Definition 2.4.1.** Let  $\varphi \in \text{End}(R)$  and let  $\mathcal{B} = \{t_1, \dots, t_m\}$  be a  $K[x_0]$ -basis of  $R$ . Write  $\varphi(t_j) = \sum_{k=1}^m a_{jk} t_k$  with  $a_{j1}, \dots, a_{jm} \in K[x_0]$  for  $j = 1, \dots, m$ . The **trace** and **norm** of  $\varphi$  (independent of choice of the basis) are given by

$$\text{Tr}(\varphi) = \sum_{j=1}^m a_{jj} \quad \text{and} \quad \text{N}(\varphi) = \det(a_{jk}).$$

**Definition 2.4.2.** Let  $f \in R$ , and let  $\mu_f : R \rightarrow R$  denote the multiplication by  $f$ . The map  $\text{Tr}_{R/K[x_0]} : R \rightarrow K[x_0]$  defined by  $\text{Tr}_{R/K[x_0]}(f) = \text{Tr}(\mu_f)$  is called the **canonical trace map** of  $R/K[x_0]$ , and  $\text{N}_{R/K[x_0]} : R \rightarrow K[x_0]$  defined by  $\text{N}_{R/K[x_0]}(f) = \text{N}(\mu_f)$  is called the **canonical norm map** of  $R/K[x_0]$ .

It follows from the definition that  $\text{Tr}_{R/K[x_0]} \in \text{Hom}_{K[x_0]}(R, K[x_0])$ ,  $\text{Tr}_{R/K[x_0]}(1_R) = m$ ,  $\text{N}_{R/K[x_0]}(f_1 f_2) = \text{N}_{R/K[x_0]}(f_1) \cdot \text{N}_{R/K[x_0]}(f_2)$  for  $f_1, f_2 \in R$ , and  $\text{N}_{R/K[x_0]}(f)$  is a unit if and only if  $f$  is a unit of  $R$ . We collect some results about canonical trace maps and canonical norm maps in the following proposition (cf. [Ku5, F.4,5,6,7]).

**Proposition 2.4.3.** (i) *We have  $\text{Tr}_{R/K[x_0]} = \sum_{i=1}^m t_i t_i^*$ , where the set  $\{t_1^*, \dots, t_m^*\}$  in  $\text{Hom}_{K[x_0]}(R, K[x_0])$  is the dual basis of  $\{t_1, \dots, t_m\}$ .*

(ii) Base change: *If  $S/K[x_0]$  is an algebra, then  $\text{Tr}_{(S \otimes_{K[x_0]} R)/S} = \text{id}_S \otimes_{K[x_0]} \text{Tr}_{R/K[x_0]}$  and  $\text{N}_{(S \otimes_{K[x_0]} R)/S}(1 \otimes f) = 1_S \cdot \text{N}_{R/K[x_0]}(f)$  for every  $f \in R$ .*

(iii) Direct products: *Let  $S = S_1 \times \dots \times S_n$  be a direct product of finitely generated free  $K[x_0]$ -algebras  $S_1, \dots, S_n$ . Then for  $s = (s_1, \dots, s_n) \in S$  we have*

$$\text{Tr}_{S/K[x_0]}(s) = \sum_{i=1}^n \text{Tr}_{S_i/K[x_0]}(s_i) \quad \text{and} \quad \text{N}_{S/K[x_0]}(s) = \prod_{i=1}^n \text{N}_{S_i/K[x_0]}(s_i).$$

(iv) Transitive law: *Let  $T/R$  be another finitely generated free algebra. Then  $\text{Tr}_{T/K[x_0]}$  and  $\text{N}_{T/K[x_0]}$  are defined, and*

$$\text{Tr}_{T/K[x_0]} = \text{Tr}_{R/K[x_0]} \circ \text{Tr}_{T/R} \quad \text{and} \quad \text{N}_{T/K[x_0]} = \text{N}_{R/K[x_0]} \circ \text{N}_{T/R}.$$

**Definition 2.4.4.** Let  $S/K$  be a standard graded algebra, and let  $M$  and  $N$  be graded  $S$ -modules. We say that a homomorphism  $\varphi : M \rightarrow N$  has **degree**  $i$  if  $\deg(\varphi(v)) = i + \deg(v)$  for every homogeneous element  $v \in M$ . The  $K$ -vector space of all homomorphisms of degree  $i$  from  $M$  to  $N$  is denoted by  $\underline{\text{Hom}}_S(M, N)_i$ . A homomorphism  $\phi : M \rightarrow N$  is called **homogeneous** if  $\phi \in \underline{\text{Hom}}_S(M, N)_i$  for some  $i$ , we also say that  $\phi$  is a **(homogeneous) homomorphism of graded modules**. We put

$$\underline{\text{Hom}}_S(M, N) = \bigoplus_{i \in \mathbb{Z}} \underline{\text{Hom}}_S(M, N)_i,$$

and consider it as a graded  $S$ -module with  $\{\underline{\text{Hom}}_S(M, N)_i\}_{i \in \mathbb{Z}}$  as its grading.

Notice that 0 has arbitrary degree, thus in order to check whether  $\varphi \in \underline{\text{Hom}}_S(M, N)$  has degree  $i$ , it suffices to check the condition  $\deg(\varphi(v)) = i + \deg(v)$  only on the homogeneous elements outside  $\text{Ker}(\varphi)$ . In general, the graded  $S$ -module  $\underline{\text{Hom}}_S(M, N)$  is a submodule of  $\text{Hom}_S(M, N)$ . We have equality in the following case.

**Proposition 2.4.5.** *If  $M$  is a finitely generated graded  $S$ -module and  $N$  is a graded  $S$ -module, then  $\underline{\text{Hom}}_S(M, N) = \text{Hom}_S(M, N)$ .*

*Proof.* See [Pev, Proposition 2.7]. □

Now let us turn back to consider the algebra  $R/K[x_0]$ . We denote  $\omega_{R/K[x_0]} := \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])(-1)$  and call it the **canonical module** of  $R/K[x_0]$ . For  $i \in \mathbb{Z}$ , the  $K$ -vector space  $(\omega_{R/K[x_0]})_i$  consists of all  $K[x_0]$ -linear homomorphisms  $\varphi : R \rightarrow K[x_0]$  with  $\varphi(R_j) \subseteq K[x_0]_{j+i}$  for all  $j \in \mathbb{Z}$ . The  $R$ -module structure of  $\omega_{R/K[x_0]}$  is defined by setting  $(f \cdot \varphi)(f') = \varphi(ff')$  for all  $f, f' \in R$  and  $\varphi \in \omega_{R/K[x_0]}$ . Several properties of  $\omega_{R/K[x_0]}$  are collected as follows.

**Proposition 2.4.6.** (i) *The graded  $R$ -module  $\omega_{R/K[x_0]}$  is finitely generated.*

(ii) *The element  $x_0$  is not a zerodivisor on  $\omega_{R/K[x_0]}$ , i.e., for  $\varphi \in \omega_{R/K[x_0]}$  if  $x_0 \cdot \varphi = 0$  then  $\varphi = 0$ .*

(iii) *The Hilbert function of  $\omega_{R/K[x_0]}$  satisfies*

$$\text{HF}_{\omega_{R/K[x_0]}}(i) = \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(-i) \quad \text{for all } i \in \mathbb{Z}.$$

(iv) *If  $\mathbb{Y} \subseteq \mathbb{X}$  is a subscheme and  $R_{\mathbb{Y}} = R/\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  is its homogeneous coordinate ring, then there is a canonical isomorphism of graded  $R$ -modules*

$$\omega_{R_{\mathbb{Y}}/K[x_0]} \cong \{ \varphi \in \omega_{R/K[x_0]} \mid \mathcal{I}_{\mathbb{Y}/\mathbb{X}} \cdot \varphi = 0 \}.$$

*Proof.* See [Kr3, Satz 2.3-4] or [GW]. □

Observe that the dual basis  $\mathcal{B}^* = \{t_1^*, \dots, t_m^*\}$  of  $\mathcal{B} = \{t_1, \dots, t_m\}$  consists of homogeneous homomorphisms in  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  with  $\deg(t_j^*) = -\deg(t_j) = -n_j$  for  $j = 1, \dots, m$ . If we write  $\alpha = \sum_{j=1}^m g_j t_j^* \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  for  $g_1, \dots, g_m \in R$ , then  $\alpha$  is homogeneous of degree  $i$  if and only if  $g_j$  is homogeneous of degree  $n_j + i$  for all  $j = 1, \dots, m$ . It is clear that the canonical trace map  $\text{Tr}_{R/K[x_0]}$  is homogeneous of degree zero.

**Definition 2.4.7.** The algebra  $R/K[x_0]$  has a **trace map**  $\sigma$  if there exists an element  $\sigma \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  such that

$$\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) = R \cdot \sigma.$$

If the algebra  $R/K[x_0]$  has a trace map  $\sigma$ , it is called a **homogeneous trace map** if it is a homogeneous element of  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$ .

**Remark 2.4.8.** (a) In general, the canonical trace map is not a trace map, i.e., we do not always have  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) = R \cdot \text{Tr}_{R/K[x_0]}$ .

(b) If  $R/K[x_0]$  has a trace map  $\sigma$ , then  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is a free  $R$ -module with basis  $\{\sigma\}$ . For this we have to prove  $\text{Ann}_R(\sigma) = \langle 0 \rangle$ . Let  $f \in R$  be such that  $f \cdot \sigma = 0$ . We have  $0 = (f \cdot \sigma)(f') = \sigma(ff') = (f' \cdot \sigma)(f)$  for every  $f' \in R$ . This implies  $\alpha(f) = 0$  for every  $\alpha \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$ . If we write  $f = \sum_{j=1}^m g_j t_j$  for some  $g_1, \dots, g_m \in K[x_0]$ , then we get  $g_j = t_j^*(\sum_{j=1}^m g_j t_j) = t_j^*(f) = 0$  for  $j = 1, \dots, m$ , and hence  $f = 0$ .

(c) Let  $\sigma$  be a trace map of  $R/K[x_0]$ . An element  $\sigma' \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is a trace map of  $R/K[x_0]$  if and only if there exists a unit  $u \in R$  with  $\sigma' = u \cdot \sigma$ .

**Proposition 2.4.9.** *If  $\sigma$  is a trace of the algebra  $R/K[x_0]$ , there exists a dual basis  $\{t'_1, \dots, t'_m\}$  to the basis  $\mathcal{B} = \{t_1, \dots, t_m\}$  with respect to  $\sigma$ ; i.e., there are elements  $t'_1, \dots, t'_m \in R$  such that  $\sigma(t_j t'_k) = \delta_{jk}$  for  $j, k = 1, \dots, m$ . In this case, we have*

$$\text{Tr}_{R/K[x_0]} = \left( \sum_{j=1}^m t_j t'_j \right) \cdot \sigma.$$

*Proof.* (See [Ku5, F.4]) Write  $t_j^* = t'_j \cdot \sigma$  with  $t'_j \in R$  and  $j = 1, \dots, m$  for the elements of the dual basis of  $\mathcal{B}$ . Then  $\sigma(t_j t'_k) = t_k^*(t_j) = \delta_{jk}$  for  $j, k = 1, \dots, m$ . Being the images of the  $t_j^*$  under the isomorphism  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) \cong R$ , the  $t'_j$  form a basis of  $R/K[x_0]$ . Thus Proposition 2.4.3(i) yields  $\text{Tr}_{R/K[x_0]} = \sum_{j=1}^m t_j t_j^* = \left( \sum_{j=1}^m t_j t'_j \right) \cdot \sigma$ .  $\square$

Next we take a look at the enveloping algebra  $R \otimes_{K[x_0]} R$  of  $R/K[x_0]$ . For  $i \in \mathbb{Z}$ , we let  $(R \otimes_{K[x_0]} R)_i$  denote the subgroup of  $R \otimes_{K[x_0]} R$  generated by the elements of the form  $f \otimes f'$  where  $f \in R_j$  and  $f' \in R_k$  with  $j + k = i$ . It is easy to check that

$$R \otimes_{K[x_0]} R = \bigoplus_{i \in \mathbb{Z}} (R \otimes_{K[x_0]} R)_i$$

and  $(R \otimes_{K[x_0]} R)_i \cdot (R \otimes_{K[x_0]} R)_j \subseteq (R \otimes_{K[x_0]} R)_{i+j}$  for all  $i, j \in \mathbb{Z}$ . Hence  $R \otimes_{K[x_0]} R$  is a graded  $K[x_0]$ -algebra with its grading  $\{(R \otimes_{K[x_0]} R)_i\}_{i \in \mathbb{Z}}$ .

Let  $\mathcal{J}$  be the kernel of the canonical multiplication map  $\mu : R \otimes_{K[x_0]} R \rightarrow R$  given by  $\mu(f \otimes g) = fg$  for all  $f, g \in R$ . The following lemma gives us a relation between the  $R$ -module  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  and  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  which follows from [Ku5, F.9].

**Proposition 2.4.10.** *We have an isomorphism of graded  $K[x_0]$ -modules of degree zero*

$$\begin{aligned} \Theta : R \otimes_{K[x_0]} R &\longrightarrow \underline{\text{Hom}}_{K[x_0]}(\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]), R) \\ f = \sum_k a_k \otimes b_k &\longmapsto \Theta(f) \text{ with } \Theta(f)(\alpha) \mapsto \sum_k \alpha(a_k) b_k, \quad \alpha \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0]). \end{aligned}$$

In particular, the homomorphism  $\Theta$  induces an isomorphism of graded  $R$ -modules

$$\Theta : \text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J}) \xrightarrow{\sim} \underline{\text{Hom}}_R(\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]), R).$$

**Corollary 2.4.11.** *If  $R/K[x_0]$  has a trace map, the following assertions hold true.*

- (i)  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  is a free  $R$ -module of rank 1.
- (ii) The isomorphism  $\Theta$  induces a bijection between the set of all trace maps of  $R/K[x_0]$  and the set of all generators of the  $R$ -module  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$ : Each trace map  $\sigma$  in  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is mapped to the unique element  $\Delta_\sigma = \sum_{j=1}^m t'_j \otimes t_j$  in  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  such that  $\sum_{j=1}^m \sigma(t'_j) t_j = 1$ .
- (iii) There is a one-to-one correspondence between the homogeneous traces of  $R/K[x_0]$  and the homogeneous generators of  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$ .
- (iv) If  $\sum_{j=1}^m g_j \otimes t_j$  generates the  $R$ -module  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  and  $\sigma \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is such that  $\sum_{j=1}^m \sigma(g_j) t_j = 1$ , then  $\sigma$  is a trace map and  $\Delta_\sigma = \sum_{j=1}^m g_j \otimes t_j$ ; hence  $\{g_1, \dots, g_m\}$  is the dual basis to  $\{t_1, \dots, t_m\}$  with respect to  $\sigma$ .

*Proof.* See [Ku5, F.10] and [Ku7, H.20]. □

**Corollary 2.4.12.** *Suppose  $R/K[x_0]$  has a trace map  $\sigma$ . Let  $\{t'_1, \dots, t'_m\}$  be the dual basis to  $\{t_1, \dots, t_m\}$  with respect to  $\sigma$ , and let  $\Delta_\sigma = \sum_{j=1}^m t'_j \otimes t_j$ . Then we have*

$$\text{Tr}_{R/K[x_0]} = \mu(\Delta_\sigma) \cdot \sigma \quad \text{and} \quad \mu(\Delta_\sigma) = \sum_{j=1}^m \text{Tr}_{R/K[x_0]}(t_j) t'_j.$$

In particular,  $\text{Tr}_{R/K[x_0]}$  is a trace map if and only if  $\mu(\Delta_\sigma)$  is a unit of  $R$ .

*Proof.* By Proposition 2.4.9, we have

$$\text{Tr}_{R/K[x_0]} = \left( \sum_{j=1}^m t_j t'_j \right) \cdot \sigma = \mu \left( \sum_{j=1}^m t'_j \otimes t_j \right) \cdot \sigma = \mu(\Delta_\sigma) \cdot \sigma.$$

Moreover, for  $k = 1, \dots, m$ , we see that

$$\begin{aligned} \left( \left( \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j t'_j) \right) \cdot \sigma \right) (t_k) &= \sigma \left( \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j) t'_j t_k \right) = \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j) \sigma(t'_j t_k) \\ &= \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j) \delta_{jk} = \mathrm{Tr}_{R/K[x_0]}(t_k). \end{aligned}$$

Thus  $\mu(\Delta_\sigma) \cdot \sigma = \mathrm{Tr}_{R/K[x_0]} = \left( \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j) t'_j \right) \cdot \sigma$ . Since  $\mathrm{Ann}_R(\sigma) = \langle 0 \rangle$ , we get the equality  $\mu(\Delta_\sigma) = \sum_{j=1}^m \mathrm{Tr}_{R/K[x_0]}(t_j) t'_j$ . Finally, the additional claim follows from Remark 2.4.8(c).  $\square$

Recall that the 0-dimensional local ring  $\overline{R} = R/\langle x_0 \rangle$  is a standard graded  $K$ -algebra and has  $\mathrm{HF}_{\overline{R}}(i) = \Delta \mathrm{HF}_{\mathbb{X}}(i)$  for all  $i \in \mathbb{Z}$ . For simplicity of notation, we denote the residue class of elements of  $R$  in  $\overline{R}$  with a bar. If  $\mathcal{B} = \{t_1, \dots, t_m\}$  is the  $K[x_0]$ -basis of  $R$ , then  $\overline{\mathcal{B}} = \{\overline{t}_1, \dots, \overline{t}_m\}$  is a  $K$ -basis of  $\overline{R}$  as a  $K$ -vector space. For  $\alpha \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$ , we see that  $\alpha(x_0 R) = \langle x_0 \rangle_{K[x_0]}$ . So,  $\alpha$  induces an element  $\overline{\alpha} \in \underline{\mathrm{Hom}}_K(\overline{R}, K)$  with  $\overline{\alpha}(\overline{f}) = \overline{\alpha(f)}$  for all  $f \in R$ . The dual basis  $\mathcal{B}^*$  of  $\mathcal{B}$  is thereby mapped to the dual basis  $\{\overline{t}_1^*, \dots, \overline{t}_m^*\}$  of  $\overline{\mathcal{B}}$ . [Ku7, H.5] shows that the  $R$ -linear map  $\varphi : \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \rightarrow \underline{\mathrm{Hom}}_K(\overline{R}, K)$  defined by  $\varphi(\alpha) = \overline{\alpha}$  for all  $\alpha \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$  induces an isomorphism of graded  $\overline{R}$ -modules

$$\underline{\mathrm{Hom}}_K(\overline{R}, K) \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) / \langle x_0 \rangle \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]).$$

An application of this isomorphism and Nakayama's Lemma for graded modules implies the following result (cf. [Ku7, H.13]).

**Lemma 2.4.13.** *Let  $\sigma \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$  be a homogeneous element. Then  $\sigma$  is a trace map of  $R/K[x_0]$  if and only if the induced element  $\overline{\sigma} \in \underline{\mathrm{Hom}}_K(\overline{R}, K)$  is a trace map of  $\overline{R}/K$ .*

Our next theorem provides characterizations of the existence of a homogeneous trace map of the algebra  $R/K[x_0]$ .

**Theorem 2.4.14.** *Let  $\mathfrak{S}(\overline{R})$  denote the socle of  $\overline{R}$ , i.e.,  $\mathfrak{S}(\overline{R}) = \{x \in \overline{R} \mid \overline{\mathfrak{m}} \cdot x = 0\}$ . The following conditions are equivalent.*

- (i) *The algebra  $\overline{R}/K$  has a trace map.*
- (ii) *The algebra  $\overline{R}/K$  has a homogeneous trace map.*
- (iii) *We have  $\dim_K(\mathfrak{S}(\overline{R})) = 1$ .*
- (iv) *The algebra  $R/K[x_0]$  has a homogeneous trace map.*



*Proof.* The equivalence of conditions (i), (ii) and (iii) follows from [Ku7, H.14 and H.16]. It remains to show that those conditions are equivalent to (iv). By Lemma 2.4.13, it suffices to show that for a homogeneous trace map  $\bar{\sigma} \in \underline{\text{Hom}}_K(\bar{R}, K)$ , there exists a homogeneous element  $\sigma \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  such that its induced element is  $\bar{\sigma}$ . This follows from the fact that the  $R$ -linear map  $\varphi : \underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) \rightarrow \underline{\text{Hom}}_K(\bar{R}, K)$  given by  $\alpha \mapsto \bar{\alpha}$  is a surjective homogeneous homomorphism of graded modules of degree zero. Furthermore, given a homogeneous trace map  $\bar{\sigma} \in \underline{\text{Hom}}_K(\bar{R}, K)$ , we can construct a homogeneous trace map  $\sigma \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  in detail as follows. Let  $r_{\mathbb{X}}$  be the regularity index of  $\text{HF}_{\mathbb{X}}$ . It is easy to check that  $\deg(\bar{\sigma}) = -r_{\mathbb{X}}$  and  $\mathfrak{G}(\bar{R}) = \bar{R}_{r_{\mathbb{X}}}$ . Thus  $\bar{\sigma}(\mathfrak{G}(\bar{R})) \neq \langle 0 \rangle$  and  $\bar{\sigma}(\bar{R}_i) = \langle 0 \rangle$  for  $i < r_{\mathbb{X}}$ . Let  $\{\bar{t}_1, \dots, \bar{t}_m\}$  be the  $K$ -basis of  $\bar{R}$  corresponding to the  $K[x_0]$ -basis  $\{t_1, \dots, t_m\}$  of  $R$ . We write  $\bar{\sigma}(\bar{t}_j) = c_j \in K$  for  $j = 1, \dots, m$ . If  $\deg(\bar{t}_j) = \deg(t_j) < r_{\mathbb{X}}$ , then  $c_j = 0$ . Let  $\sigma : R \rightarrow K[x_0]$  be a  $K[x_0]$ -linear map with  $\sigma(t_j) = c_j$  for  $j = 1, \dots, m$ . Clearly, we have  $\sigma \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) \setminus \{0\}$  and  $\sigma(t_j) = 0$  if  $\deg(t_j) < r_{\mathbb{X}}$ . This implies  $\sigma \in (\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]))_{-r_{\mathbb{X}}} \setminus \{0\}$  and its induced element is  $\bar{\sigma}$ .  $\square$

**Definition 2.4.15.** A  $d$ -dimensional Noetherian local ring  $(S, \mathfrak{m})$  is called a **Gorenstein local ring** if it is a Cohen-Macaulay local ring of type  $r = 1$ , where the type  $r$  is given by

$$r = \dim_K(\mathfrak{G}(S/\langle a_1, \dots, a_d \rangle)).$$

Here  $\{a_1, \dots, a_d\}$  is any parameter system of  $S$  (i.e., it generates an  $\mathfrak{m}$ -primary ideal). A Noetherian ring  $S$  is called a **Gorenstein ring** if  $S_{\mathfrak{m}}$  is a Gorenstein local ring for all  $\mathfrak{m} \in \text{Max}(S)$ .

Let  $S$  be a graded Noetherian ring and let  $\mathfrak{p}$  be any homogeneous prime ideal of  $S$ . Then  $\mathfrak{p}$  is contained in a homogeneous maximal ideal  $\mathfrak{m}$  of  $S$  and  $S_{\mathfrak{p}}$  is a localization of  $S_{\mathfrak{m}}$  (i.e.,  $S_{\mathfrak{p}} \cong (S_{\mathfrak{m}})_{\mathfrak{p}S_{\mathfrak{m}}}$ ). Thus if  $S_{\mathfrak{m}}$  is a Cohen-Macaulay (resp. Gorenstein) local ring for every homogeneous maximal ideal  $\mathfrak{m}$  of  $S$ , then  $S_{\mathfrak{p}}$  is a Cohen-Macaulay (resp. Gorenstein) local ring for every homogeneous prime ideal  $\mathfrak{p}$  of  $S$ .

**Theorem 2.4.16.** *Let  $S$  be a graded Noetherian ring. If  $S_{\mathfrak{p}}$  is a Cohen-Macaulay local ring of type  $\leq r$  for every homogeneous prime ideal  $\mathfrak{p}$ , then  $S$  is a Cohen-Macaulay ring of global type  $\leq r$ . In particular, if  $S_{\mathfrak{p}}$  is Gorenstein for every homogeneous prime ideal  $\mathfrak{p}$ , then  $S$  is Gorenstein.*

*Proof.* See [AG, Theorem 3.1].  $\square$

**Definition 2.4.17.** Let  $S$  be a ring, and let  $T/S$  be an algebra.

- (i) The algebra  $T/S$  is called a **Gorenstein algebra** if it is flat, and if for all  $\mathfrak{P} \in \text{Spec}(T)$  and  $\mathfrak{p} = \mathfrak{P} \cap S$  the ring  $T_{\mathfrak{P}}/\mathfrak{p}T_{\mathfrak{P}}$  is a Gorenstein local ring.
- (ii) The algebra  $T/S$  is called **unramified** if
  - For all  $\mathfrak{P} \in \text{Spec}(T)$  and  $\mathfrak{p} = \mathfrak{P} \cap S$ , we have  $\mathfrak{P}T_{\mathfrak{P}} = \mathfrak{p}T_{\mathfrak{P}}$ .
  - $k(\mathfrak{P})/k(\mathfrak{p})$  is a separable algebraic field extension, where  $k(\mathfrak{P}) = T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}$  and  $k(\mathfrak{p}) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ .

The algebra  $T/S$  is said to be **étale** if it is flat and unramified.

It is well-known that the algebra  $R/K[x_0]$  is graded-free of rank  $\deg(\mathbb{X})$ . We also have  $\langle x_0 \rangle_{K[x_0]} = \mathfrak{m} \cap K[x_0]$  and  $R_{\mathfrak{m}}/\langle x_0 \rangle_{R_{\mathfrak{m}}} = (R/\langle x_0 \rangle_R)_{\overline{\mathfrak{m}}} = \overline{R}_{\overline{\mathfrak{m}}} = \overline{R}$ . Hence  $R/K[x_0]$  is a Gorenstein algebra if and only if  $\overline{R}$  is a Gorenstein local ring. Moreover, we have the following property.

**Proposition 2.4.18.** *The following assertions are equivalent.*

- (i)  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) = R \cdot \sigma \cong R(r_{\mathbb{X}})$  with  $\sigma \in (\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]))_{-r_{\mathbb{X}}}$ .
- (ii)  $\overline{R}$  is a Gorenstein local ring (i.e.,  $\dim_K(\mathfrak{G}(\overline{R})) = 1$ ).
- (iii) The algebra  $R/K[x_0]$  is a Gorenstein algebra.
- (iv)  $R$  is a Gorenstein ring.

*Proof.* The equivalence of conditions (i), (ii) and (iii) follows from Theorem 2.4.14 and the preceding argument. Now we show that the conditions (i) and (iv) are equivalent. By Proposition 2.1.3 in [GW], the Cohen-Macaulay ring  $R$  is a Gorenstein ring if and only if  $\omega_{R/K[x_0]} \cong R(d)$  for some  $d \in \mathbb{Z}$ . Since  $\text{HF}_{\omega_{R/K[x_0]}}(i) = \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(-i)$  for  $i \in \mathbb{Z}$  by Proposition 2.4.6(iii), this is equivalent to  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) \cong R(r_{\mathbb{X}})$ . Therefore the conclusion follows.  $\square$

**Corollary 2.4.19.** *Suppose the algebra  $R/K[x_0]$  has a (homogeneous) trace map. Then  $\text{Tr}_{R/K[x_0]}$  is a trace map if and only if  $R/K[x_0]$  is étale.*

*Proof.* The claim follows from Corollary 2.4.12 and [Ku5, F.8].  $\square$

# Chapter 3

## Various Differentials for 0-Dimensional Schemes

The goal of this chapter is to define and study various differentials, namely the Noether, Dedekind and Kähler differentials, for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an arbitrary field  $K$ . In algebra, these differentials have been known for several decades. Many structural properties of an algebra are encoded in those invariants, or in similar ones derived from them (cf. [Ku1], [Ku5], [Mac], [SS]). In this chapter we look more closely at these differentials for the scheme  $\mathbb{X}$  by working out algorithms for the computation of them, examining and relating their algebraic structures, and investigating their Hilbert functions.

The chapter contains three sections and each section introduces one kind of these differentials. Section 3.1 begins with a definition of the Noether differential (see Definition 3.1.1). We show that the Noether differential is a principal ideal of the homogeneous coordinate ring  $R$  of  $\mathbb{X}$  when  $\mathbb{X}$  is arithmetically Gorenstein (see Proposition 3.1.2), and deduce from Section 2.1 its representations (see Corollary 3.1.5). Additionally, we indicate how we can explicitly compute a minimal homogeneous system of generators of a homogeneous ideal of  $R$  (see Proposition 3.1.10) and apply it to compute that of the Noether differential (see Proposition 3.1.13).

Section 3.2 is devoted to exploring the Dedekind differential. Explicitly, we first present some results on the homogeneous quotient ring  $Q^h(R)$  of  $R$  (see Propositions 3.2.1 and 3.2.3). If we restrict to 0-dimensional locally Gorenstein schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , the graded-free algebra  $Q^h(R)/K[x_0, x_0^{-1}]$  has a homogeneous trace map. This enables us to define the Dedekind differential for 0-dimensional locally Gorenstein schemes (see Definition 3.2.4). In Proposition 3.2.5, we describe the Hilbert function of the Dedekind differential and give an upper bound for its regularity index. In particular,

this upper bound is sharp if  $\mathbb{X}$  is a projective point set in  $\mathbb{P}_K^n$  (see Proposition 3.2.11). We also characterize 0-dimensional arithmetically Gorenstein schemes in terms of their Dedekind differents (see Proposition 3.2.8) and give a description of the Dedekind complementary module (see Proposition 3.2.9). When  $\mathbb{X}$  is smooth, we show that the Noether different and the Dedekind different agree (see Theorem 3.2.17). So the Dedekind different in this case can be computed using the method introduced in Section 3.1. In the remainder of this section, we modify slightly the GBM-algorithm which was proposed by J. Abbott *et al* [AKR] to compute some special  $K$ -bases of the vector space  $R_{r_{\mathbb{X}}}$  where  $r_{\mathbb{X}}$  is the regularity index of  $R$  (see Propositions 3.2.22 and 3.2.25) and apply it to formulate an algorithm for computing the Dedekind different for a locally Gorenstein scheme  $\mathbb{X}$  under an additional hypothesis (see Proposition 3.2.29).

The Kähler different for 0-dimensional schemes is introduced in Section 3.3. It is defined as the first Fitting ideal of the module of Kähler differentials  $\Omega_{R/K[x_0]}^1$ . We first take a look at relations of the Kähler different and two other differents in the special case of a complete intersection  $\mathbb{X}$  (see Corollaries 3.3.5 and 3.3.7). We then show that the Noether different and the Dedekind different are not equal even when  $\mathbb{X}$  is a complete intersection (see Example 3.3.6) and present some general relations between these differents (see Propositions 3.3.9 and 3.3.12). Moreover, we describe the Hilbert function of the Kähler different and bound its regularity index when  $\mathbb{X}$  is a projective point set in  $\mathbb{P}_K^n$  (see Proposition 3.3.14). Next, we relate it to the minimal separators of  $\mathbb{X}$  (see Corollary 3.3.16). Finally, we bound the Hilbert polynomial of the Kähler different for an arbitrary 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  (see Proposition 3.3.19) and derive from these bounds some consequences (see Corollaries 3.3.21 and 3.3.22).

Throughout this chapter, we let  $K$  be an arbitrary field, let  $n \geq 1$ , and let  $\mathbb{P}_K^n$  be the projective  $n$ -space over  $K$ . The homogeneous coordinate ring of  $\mathbb{P}_K^n$  is  $P = K[X_0, \dots, X_n]$  equipped with its standard grading  $\deg(X_i) = 1$ . Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ . By  $\mathcal{I}_{\mathbb{X}}$  we denote the homogeneous vanishing ideal of  $\mathbb{X}$  in  $P$ . The homogeneous coordinate ring of  $\mathbb{X}$  is then given by  $R = P/\mathcal{I}_{\mathbb{X}}$ . The image of  $X_i$  in  $R$  is denoted by  $x_i$  for  $i = 0, \dots, n$ .

### 3.1 Noether Differents for 0-Dimensional Schemes

In this section we address the problem of defining and computing the Noether different for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ . We know that  $x_0$  is a non-zerodivisor of  $R$  and the algebra  $R/K[x_0]$  is graded-free of rank  $\deg(\mathbb{X})$ . Furthermore, the enveloping

algebra  $R \otimes_{K[x_0]} R$  of  $R/K[x_0]$  is a graded ring. Let  $\mathcal{J}$  be the kernel of the canonical multiplication map  $\mu : R \otimes_{K[x_0]} R \rightarrow R$  given by  $\mu(f \otimes g) = f \cdot g$  for  $f, g \in R$ . Note that  $\mu$  is an  $R$ -linear map preserving degrees and

$$\mathcal{J} = \langle x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n \rangle_{R \otimes_{K[x_0]} R}.$$

This implies that  $\mathcal{J}$  and  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  are homogeneous ideals of  $R \otimes_{K[x_0]} R$ . Consequently,  $\mu(\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J}))$  is a homogeneous ideal of  $R$ .

**Definition 3.1.1.** The homogeneous ideal  $\vartheta_N(R/K[x_0]) = \mu(\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J}))$  is called the **Noether different** of the algebra  $R/K[x_0]$  (or for  $\mathbb{X}$  with respect to  $x_0$ ).

First we give descriptions of the Noether different  $\vartheta_N(R/K[x_0])$  when the scheme  $\mathbb{X}$  is a complete intersection or an arithmetically Gorenstein scheme. Here  $\mathbb{X}$  is called a **complete intersection** if  $\mathcal{I}_{\mathbb{X}}$  can be generated by  $n$  homogeneous polynomials, and it is called **arithmetically Gorenstein** if  $R$  is a Gorenstein ring. Note that if  $\mathbb{X}$  is a complete intersection then it is arithmetically Gorenstein, but the converse is not true in general (see Example 3.1.4).

**Proposition 3.1.2.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $r_{\mathbb{X}}$  denote the regularity index of  $\text{HF}_{\mathbb{X}}$ .*

- (i) *If  $\mathbb{X}$  is arithmetically Gorenstein and  $\text{char}(K) \nmid \deg(\mathbb{X})$ , then  $\vartheta_N(R/K[x_0])$  is a principal ideal of  $R$  generated by a non-zero homogeneous element of degree  $r_{\mathbb{X}}$ .*
- (ii) *If  $\mathbb{X}$  is a complete intersection with  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_n \rangle$ , then we have*

$$\vartheta_N(R/K[x_0]) = \left\langle \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right\rangle_R.$$

- (iii) *We have  $\vartheta_N(R_U/K[x_0]) = \vartheta_N(R/K[x_0])_U$  for every multiplicatively closed subset  $U \subseteq R$ .*

*Proof.* Since  $\mathbb{X}$  is arithmetically Gorenstein, the algebra  $R/K[x_0]$  has a homogeneous trace map  $\sigma$  of degree  $-r_{\mathbb{X}}$  such that  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) = R \cdot \sigma$  (see Proposition 2.4.18). In this case, by Corollary 2.4.11, the graded  $R$ -module  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J})$  is free of rank 1 and it is generated by  $\Delta_{\sigma}$ , where  $\Delta_{\sigma}$  is the preimage under the isomorphism  $\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J}) \cong \underline{\text{Hom}}_R(\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]), R)$  of  $\psi : \underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) \rightarrow R$  given by  $\psi(\sigma) = 1$ . Hence the Noether different  $\vartheta_N(R/K[x_0]) = \mu(\text{Ann}_{R \otimes_{K[x_0]} R}(\mathcal{J}))$  is a principal ideal generated by  $\mu(\Delta_{\sigma})$ . Obviously,  $\text{Tr}_{R/K[x_0]}(1) = \deg(\mathbb{X}) \neq 0$  (as  $\text{char}(K) \nmid \deg(\mathbb{X})$ ). It follows from  $\text{Tr}_{R/K[x_0]} = \mu(\Delta_{\sigma}) \cdot \sigma$  (see Corollary 2.4.12)

that  $\mu(\Delta_\sigma) \neq 0$ . Since the maps  $\mu$  and  $\Theta^{-1}$  are homogeneous of degree zero and  $\deg(\sigma) = -r_{\mathbb{X}}$ , we get  $\deg(\mu(\Delta_\sigma)) = r_{\mathbb{X}}$ . Thus claim (i) follows. Claim (ii) follows from Proposition 2.1.8, and claim (iii) is an immediate consequence of Proposition 2.1.7.  $\square$

If  $n = 1$ , then every 0-dimensional subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^1$  is always a complete intersection. The homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}}$  is generated by a non-zero homogeneous polynomial  $F_1 \in P$ . In this situation, the Noether different of the algebra  $R/K[x_0]$  is  $\vartheta_N(R/K[x_0]) = \langle \frac{\partial F_1}{\partial x_1} \rangle$ . If  $n > 1$  and  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is a projective point set of degree 1, then it is clear that  $\vartheta_N(R/K[x_0]) = R$ .

Let us illustrate the results of this proposition with two more examples.

**Example 3.1.3.** Let  $\mathbb{X} = \{p_1, \dots, p_9\} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the projective point set given by  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0)$ ,  $p_3 = (1 : 2 : 0)$ ,  $p_4 = (1 : 0 : 1)$ ,  $p_5 = (1 : 1 : 1)$ ,  $p_6 = (1 : 0 : 2)$ ,  $p_7 = (1 : 2 : 1)$ ,  $p_8 = (1 : 1 : 2)$ , and  $p_9 = (1 : 2 : 2)$ . The vanishing ideal of  $\mathbb{X}$  in  $P = \mathbb{Q}[X_0, X_1, X_2]$  is  $\mathcal{I}_{\mathbb{X}} = \langle X_0^2 X_1 - \frac{3}{2} X_0 X_1^2 + \frac{1}{2} X_1^3, X_0^2 X_2 - \frac{3}{2} X_0 X_2^2 + \frac{1}{2} X_2^3 \rangle$ , and so  $\mathbb{X}$  is a complete intersection. By Proposition 3.1.2, the Noether different of  $R/\mathbb{Q}[x_0]$  is given by

$$\begin{aligned} \vartheta_N(R/\mathbb{Q}[x_0]) &= \left\langle \det \begin{pmatrix} x_0^2 - 3x_0x_1 + \frac{3}{2}x_1^2 & 0 \\ 0 & x_0^2 - 3x_0x_2 + \frac{3}{2}x_2^2 \end{pmatrix} \right\rangle \\ &= \langle (x_0^2 - 3x_0x_1 + \frac{3}{2}x_1^2)(x_0^2 - 3x_0x_2 + \frac{3}{2}x_2^2) \rangle. \end{aligned}$$

**Example 3.1.4.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set consisting of five points  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 0 : 1 : 0)$ ,  $p_3 = (1 : 0 : 0 : 1)$ ,  $p_4 = (1 : 1 : 0 : 1)$ , and  $p_5 = (1 : 2 : 2 : 1)$ . Then a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$  is  $\{X_0X_1 - X_1X_3, X_1^2 - X_1X_3 - X_2X_3, X_0X_2 - X_2^2 + X_2X_3, X_1X_2 - 2X_2X_3, X_0X_3 - X_3^2\} \subseteq \mathbb{Q}[X_0, \dots, X_3]$ . So, the scheme  $\mathbb{X}$  is not a complete intersection. However, a simple calculation gives us  $\text{HF}_{\mathbb{X}} : 1 \ 4 \ 5 \ 5 \cdots$  and  $\text{HF}_{\mathbb{X} \setminus \{p_j\}} : 1 \ 4 \ 4 \cdots$  for  $j = 1, \dots, 5$ . Thus it follows from Theorem 7 in [GO] that  $\mathbb{X}$  is arithmetically Gorenstein. In this case, the Noether different of  $R/\mathbb{Q}[x_0]$  is  $\vartheta_N(R/\mathbb{Q}[x_0]) = \langle x_0^2 - 2x_2^2 - 3x_1x_3 + 7x_2x_3 + x_3^2 \rangle$ .

In Section 2.1 we already discussed a method for computing a system of generators of the Noether different of an algebra of finite type. As an application of this method, we can show how to compute a homogeneous system of generators of the Noether different of the algebra  $R/K[x_0]$ . It is based on the representations of the Noether different of  $R/K[x_0]$  which we present below. Let us write

$$R = P/\mathcal{I}_{\mathbb{X}} = K[X_0, X_1, \dots, X_n]/\mathcal{I}_{\mathbb{X}} = K[x_0, x_1, \dots, x_n].$$

Let  $\{F_1, \dots, F_r\}$  be a homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$ , where  $r \geq n$ . Let  $Y_1, \dots, Y_n$  be new indeterminates, and let  $\gamma : K[X_0, X_1, \dots, X_n] \rightarrow K[X_0, Y_1, \dots, Y_n]$  be a  $K[X_0]$ -algebra homomorphism given by  $\gamma(X_i) = Y_i$  for  $i = 1, \dots, n$ . By  $J$  we denote the homogeneous ideal of  $K[X_0, Y_1, \dots, Y_n]$  generated by  $\{\gamma(F_1), \dots, \gamma(F_r)\}$ . We form the standard graded polynomial ring  $Q = K[X_0, X_1, \dots, X_n, Y_1, \dots, Y_n]$  with  $\deg(X_0) = \dots = \deg(X_n) = \deg(Y_1) = \dots = \deg(Y_n) = 1$ . In  $Q$ , we denote by  $\mathcal{I}_{\mathbb{X}}Q$  and  $JQ$  the homogeneous ideals generated by  $\{F_1, \dots, F_r\}$  and  $\{\gamma(F_1), \dots, \gamma(F_r)\}$ , respectively. As in the proof of Proposition 2.1.14, we have a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{\psi} & P \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ Q/JQ & \xrightarrow{\phi} S = R[X_1, \dots, X_n] \xrightarrow{\rho} R \otimes_{K[x_0]} R \xrightarrow{\mu} R & \end{array} \quad (3.1)$$

where  $\rho : S \rightarrow R \otimes_{K[x_0]} R$  is a  $K[x_0]$ -algebra epimorphism given by  $\rho(X_i) = 1 \otimes x_i$  for  $i = 1, \dots, n$ ,  $\rho(x_i) = x_i \otimes 1$  for  $i = 0, \dots, n$ ;  $\phi : Q/JQ \rightarrow S$  is a  $K[x_0]$ -algebra isomorphism defined by  $\phi(X_i + JQ) = X_i$  for  $i = 0, \dots, n$  and  $\phi(Y_j + JQ) = x_j$  for  $j = 1, \dots, n$ ;  $\psi : Q \rightarrow P$  is a  $P$ -epimorphism defined by  $\psi(Y_j) = X_j$  for  $j = 1, \dots, n$ ;  $\tilde{\pi}$  and  $\pi$  are the canonical  $K[x_0]$ -algebra epimorphisms. Notice that all maps in the above diagram are homogeneous of degree zero.

**Corollary 3.1.5.** *Using the notation as above. In  $Q = K[X_0, X_1, \dots, X_n, Y_1, \dots, Y_n]$ , we form the homogeneous colon ideal*

$$\widehat{I} := (\mathcal{I}_{\mathbb{X}}Q + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q.$$

Then the Noether different of  $R/K[x_0]$  is given by

$$\begin{aligned} \vartheta_N(R/K[x_0]) &= \{F(x_1, \dots, x_n) \in R \mid F \in \mathcal{I}_{\mathbb{X}}S :_S \langle X_1 - x_1, \dots, X_n - x_n \rangle_S\} \\ &= (\mu \circ \rho \circ \phi \circ \tilde{\pi})(\widehat{I}) = \psi(\widehat{I})/\mathcal{I}_{\mathbb{X}}. \end{aligned}$$

*Proof.* This follows from Lemma 2.1.6 and Propositions 2.1.12 and 2.1.14.  $\square$

As we mentioned above, a homogeneous system of generators of  $\vartheta_N(R/K[x_0])$  can be found by using its representations in Corollary 3.1.5. Explicitly, we first compute a homogeneous generating system  $\{H_1, \dots, H_t, \gamma(F_1), \dots, \gamma(F_r)\}$  of the colon ideal  $\widehat{I} = (\mathcal{I}_{\mathbb{X}}Q + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q$  with  $H_i \notin JQ$  for  $i = 1, \dots, t$ . Secondly, we compute a homogeneous generating system  $\{H'_1, \dots, H'_t, F_1, \dots, F_r\}$  of the homogeneous ideal  $\widehat{J} = \psi(\widehat{I})$  of  $P$ , where  $H'_i = \psi(H_i)$  for  $i = 1, \dots, t$ . Finally, by taking  $h_i$

the image of  $H'_i$  in  $R$  for  $i = 1, \dots, t$ , we obtain a homogeneous system of generators  $\{h_1, \dots, h_t\}$  of  $\vartheta_N(R/K[x_0])$ .

Next we would like to describe explicitly a minimal homogeneous system of generators of  $\vartheta_N(R/K[x_0])$  which is a subset of  $\{h_1, \dots, h_t\}$ . For this purpose, we discuss below how to compute a minimal homogeneous system of generators of a homogeneous ideal in the residue class ring  $R = P/\mathcal{I}_{\mathbb{X}}$ .

In what follows, we let  $<_{\sigma}$  be a term ordering on the monoid  $\mathbb{T}^{n+1} = \mathbb{T}(X_0, \dots, X_n)$  of terms of  $P$  and let  $\mathcal{F} = \{F_1, \dots, F_r\}$  be a homogeneous  $<_{\sigma}$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$ . Recall from [KR2, Section 2.4] that the normal form of a (homogeneous) polynomial  $F \in P$  with respect to  $<_{\sigma}$  is  $\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(F) = \text{NR}_{\sigma, \mathcal{F}}(F)$  which can be computed by using the Division Algorithm given in [KR2, Theorem 1.6.4]. We say that  $F$  is a **normal (homogeneous) polynomial modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$**  if  $F = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(F)$ . In order to perform division in  $R$ , we present the following algorithm.

**Proposition 3.1.6. (The Division Algorithm)** *Let  $\mathcal{F} = \{F_1, \dots, F_r\}$  be a homogeneous  $<_{\sigma}$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$ , and let  $F, G_1, \dots, G_s \in P \setminus \{0\}$  be normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$ , where  $s \geq 1$ . Consider the following sequence of instructions.*

- 1) Let  $Q_1 = \dots = Q_s = 0$ ,  $G = H = 0$  and  $V = F$ .
- 2) If there exists  $i \in \{1, \dots, s\}$  such that  $\text{LT}_{\sigma}(V) = T \cdot \text{LT}_{\sigma}(G_i)$  for some term  $T \in \mathbb{T}^{n+1}$ , then replace  $Q_i$  by  $Q_i + \frac{\text{LC}_{\sigma}(V)}{\text{LC}_{\sigma}(G_i)}T$ ,  $H$  by  $H + \frac{\text{LC}_{\sigma}(V)}{\text{LC}_{\sigma}(G_i)}(\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(TG_i) - TG_i)$ , and  $V$  by  $V - \frac{\text{LC}_{\sigma}(V)}{\text{LC}_{\sigma}(G_i)}\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(TG_i)$ .
- 3) Repeat step 2) until there is no more  $i \in \{1, \dots, s\}$  such that  $\text{LT}_{\sigma}(V)$  is a multiple of  $\text{LT}_{\sigma}(G_i)$ . Then replace  $G$  by  $G + \text{LM}_{\sigma}(V)$  and  $V$  by  $V - \text{LM}_{\sigma}(V)$ .
- 4) If  $V = 0$ , return the tuple  $(G, H, Q_1, \dots, Q_s) \in P^{s+2}$  and stop. Otherwise, start again with step 2).

This is an algorithm which returns a tuple  $(G, H, Q_1, \dots, Q_s) \in P^{s+2}$  such that

$$F = \sum_{i=1}^s Q_i G_i + G + H$$

and such that the following conditions are satisfied.

- a) We have  $H \in \mathcal{I}_{\mathbb{X}}$  and if  $H \neq 0$  then  $\text{LT}_{\sigma}(F) >_{\sigma} \text{LT}_{\sigma}(H)$ .
- b) If  $G \neq 0$ , then no elements of  $\text{Supp}(G)$  is contained in the monomial ideal  $\langle \{\text{LT}_{\sigma}(G_1), \dots, \text{LT}_{\sigma}(G_s)\} \cup \text{LT}_{\sigma}\{\mathcal{I}_{\mathbb{X}}\} \rangle$  and  $\text{LT}_{\sigma}(F) \geq_{\sigma} \text{LT}_{\sigma}(G)$ .



- c) For  $i \in \{1, \dots, s\}$ ,  $Q_i$  is a normal homogeneous polynomial modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  in  $P$ . If  $Q_i \neq 0$  for some  $i \in \{1, \dots, s\}$ , then  $\text{LT}_{\sigma}(F) \geq_{\sigma} \text{LT}_{\sigma}(Q_i G_i)$ .
- d) For any index  $i = 1, \dots, s$  and all term  $T \in \text{Supp}(Q_i)$ , we have

$$T \cdot \text{LT}_{\sigma}(G_i) \notin \langle \{\text{LT}_{\sigma}(G_1), \dots, \text{LT}_{\sigma}(G_{i-1})\} \cup \text{LT}_{\sigma}\{\mathcal{I}_{\mathbb{X}}\} \rangle.$$

Moreover, the tuple  $(G, H, Q_1, \dots, Q_s) \in P^{s+2}$  satisfies the above conditions is uniquely determined by the tuple  $(F, G_1, \dots, G_s) \in P^{s+1}$ .

*Proof.* Analogous to [KR2, Theorem 1.6.4].  $\square$

Let  $F, G_1, \dots, G_s \in P \setminus \{0\}$  be normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$ , where  $s \geq 1$ , and let  $\mathcal{G}$  be the tuple  $(G_1, \dots, G_s)$ . We denote a polynomial  $G$  obtained in Proposition 3.1.6 by  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F)$ . For  $F = 0$ , we let  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F) = 0$ . As an immediate consequence of Proposition 3.1.6, we have the following corollary.

**Corollary 3.1.7.** *Let  $J \subseteq P$  be a homogeneous ideal which contains  $\mathcal{I}_{\mathbb{X}}$ , let  $\mathcal{G} = \{G_1, \dots, G_s\}$  be a set of non-zero normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  such that  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous system of generators of  $J$ , and let  $\mathcal{G} = (G_1, \dots, G_s)$ . Then the following conditions are equivalent.*

- (i) *The set  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$ .*
- (ii) *A normal homogeneous polynomial  $F \in P \setminus \{0\}$  modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  satisfies  $F \in J$  if and only if  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F) = 0$ .*
- (iii) *For every normal homogeneous polynomial  $F \in J \setminus \mathcal{I}_{\mathbb{X}}$  modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$ , there are  $H_1, \dots, H_s \in P$  and  $H \in \mathcal{I}_{\mathbb{X}}$  such that  $F = \sum_{i=1}^s H_i G_i + H$ ,  $\text{LT}_{\sigma}(F) >_{\sigma} \text{LT}_{\sigma}(H)$  if  $H \neq 0$ , and  $\text{LT}_{\sigma}(F) \geq_{\sigma} \text{LT}_{\sigma}(H_i G_i)$  for all  $i = 1, \dots, s$  such that  $H_i G_i \neq 0$ .*

Let  $\mathcal{G} = \{G_1, \dots, G_s\}$  be a set of non-zero normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  such that  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous system of generators of a homogeneous ideal  $J \subseteq P$  containing  $\mathcal{I}_{\mathbb{X}}$ . We set  $\mathbb{S}_{\mathcal{G} \cup \mathcal{F}} = \{S_{ij} \in P \mid (i, j) \in \{1, \dots, s\} \times \{1, \dots, s+r\}, i < j\}$ , where

$$S_{ij} = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}} \left( \frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(G_j))}{\text{LM}_{\sigma}(G_i)} G_i - \frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(G_j))}{\text{LM}_{\sigma}(G_j)} G_j \right)$$

if  $j \leq s$  and

$$S_{ij} = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}} \left( \frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(F_{j-s}))}{\text{LM}_{\sigma}(G_i)} G_i - \frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(F_{j-s}))}{\text{LM}_{\sigma}(F_{j-s})} F_{j-s} \right)$$

if  $j > s$ , and call it the **set of normal S-polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$**  of  $\mathcal{G} \cup \mathcal{F}$ . Clearly, all normal S-polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  in  $\mathbb{S}_{\mathcal{G} \cup \mathcal{F}}$  are homogeneous and if  $j \leq s$  then  $\deg(S_{ij}) \geq \max\{\deg(G_i), \deg(G_j)\}$  and if  $j > s$  then  $\deg(S_{ij}) > \deg(G_i)$ .

**Proposition 3.1.8. (Buchberger's Criterion)** *Let  $J \subseteq P$  be a homogeneous ideal which contains  $\mathcal{I}_{\mathbb{X}}$ , let  $\mathcal{G} = \{G_1, \dots, G_s\}$  be a set of non-zero normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  such that  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous system of generators of  $J$ , and let  $\mathcal{G} = (G_1, \dots, G_s)$ . Then  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$  if and only if  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F) = 0$  for all  $F \in \mathbb{S}_{\mathcal{G} \cup \mathcal{F}}$ .*

*Proof.* If  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$ , then  $F \in \mathbb{S}_{\mathcal{G} \cup \mathcal{F}} \subset J$  implies  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F) = 0$  by Corollary 3.1.7. Conversely, suppose that  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F) = 0$  for all  $F \in \mathbb{S}_{\mathcal{G} \cup \mathcal{F}}$ . To prove the set  $\mathcal{G} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$ , it suffices to prove that all  $S$ -polynomials of  $\mathcal{G} \cup \mathcal{F}$  reduce to zero with respect to  $\mathcal{G} \cup \mathcal{F}$  by [KR2, Theorem 2.4.1 and Proposition 2.5.2]. But this follows directly from the assumption and  $\mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$ .  $\square$

Our next lemma gives a characterization of a minimal homogeneous system of generators of a homogeneous ideal  $J/\mathcal{I}_{\mathbb{X}}$  of  $R$ .

**Lemma 3.1.9.** *Let  $J/\mathcal{I}_{\mathbb{X}}$  be a homogeneous ideal of  $R$  generated by non-zero homogeneous elements  $\{h_1, \dots, h_t\}$ . For  $i = 1, \dots, t$ , let  $H_i \in P$  be the normal representative modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  of  $h_i$ , let  $m_i = \deg(H_i)$ , and assume that  $m_1 \leq \dots \leq m_t$ . Then the following conditions are equivalent.*

- (i) *The set  $\{h_1, \dots, h_t\}$  is a minimal homogeneous system of generators of  $J/\mathcal{I}_{\mathbb{X}}$ .*
- (ii)  *$H_i \notin \langle \{H_1, \dots, H_{i-1}\} \cup \mathcal{F} \rangle$  for  $i = 1, \dots, t$ .*
- (iii)  *$H_i \notin \langle \{H_1, \dots, \widehat{H}_i, \dots, H_t\} \cup \mathcal{F} \rangle$  for  $i = 1, \dots, t$ , where  $\widehat{H}_i$  denotes an element that is not included.*

*Proof.* (i) $\Rightarrow$ (ii): Assume that  $\{h_1, \dots, h_t\}$  is a minimal set of generators of  $J/\mathcal{I}_{\mathbb{X}}$  and  $H_i \in \langle \{H_1, \dots, H_{i-1}\} \cup \mathcal{F} \rangle$  for some  $i \in \{1, \dots, t\}$ . Then there is a relation  $H_i = \sum_{j=1}^{i-1} G_j H_j + \sum_{j=1}^r G_{i-1+j} F_j$  with  $G_j \in P$ . This implies  $h_i = \sum_{j=1}^{i-1} g_j h_j \in R$ , where  $g_j$  is the image of  $G_j$  in  $R$  for  $j = 1, \dots, i-1$ . We get  $J/\mathcal{I}_{\mathbb{X}} = \langle h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_t \rangle_R$ , it is impossible. Hence we must have  $H_i \notin \langle \{H_1, \dots, H_{i-1}\} \cup \mathcal{F} \rangle$  for all  $i \in \{1, \dots, t\}$ .

(ii) $\Rightarrow$ (iii): We shall show that  $H_i \notin \langle \{H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_t\} \cup \mathcal{F} \rangle$  for all  $i \in \{1, \dots, t\}$  if  $H_i \notin \langle \{H_1, \dots, H_{i-1}\} \cup \mathcal{F} \rangle$  for all  $i \in \{1, \dots, t\}$ . Otherwise, we obtain a representation

$$H_i = G_1 H_1 + \dots + G_{i-1} H_{i-1} + G_{i+1} H_{i+1} + \dots + G_t H_t + G_{t+1} F_1 + \dots + G_{t+r} F_r$$

where  $G_j \in P$  is a homogeneous polynomial of degree  $m_i - m_j$  for  $j \in \{1, \dots, i-1, i+1, \dots, t\}$  and of degree  $m_i - \deg(F_{j-t})$  for  $j \in \{t+1, \dots, t+r\}$  (cf. [KR2, Corollary 1.7.11]). This implies  $G_j = 0$  if  $m_i < m_j$ . Hence there are two possibilities.

+ If  $m_i > m_j$  for all  $j$  such that  $G_j \neq 0$ , then  $j < i$  and  $H_i = G_1 H_1 + \cdots + G_{i-1} H_{i-1} + G_{t+1} F_1 + \cdots + G_{t+r} F_r$ . It follows that  $H_i \in \langle \{H_1, \dots, H_{i-1}\} \cup \mathcal{F} \rangle$ , a contradiction.

+ There exists some index  $j$  with  $G_j \in K \setminus \{0\}$  such that  $m_i = m_j$ . We define  $j_0 = \max\{j \in \{1, \dots, t\} \mid G_j \in K \setminus \{0\}\}$ . Then we get  $H_{j_0} \in \langle \{H_1, \dots, H_{j_0-1}\} \cup \mathcal{F} \rangle$ . This is also a contradiction.

(iii) $\Rightarrow$ (i): If  $\{h_1, \dots, h_t\}$  is not a minimal set of generators of  $J/\mathcal{I}_{\mathbb{X}}$ , then there is an index  $i \in \{1, \dots, t\}$  such that  $h_i = \sum_{j \neq i} g_j h_j$  with  $g_j \in R$ . This implies that  $h_i - \sum_{j \neq i} g_j h_j = 0$  or  $H_i - \sum_{j \neq i} G_j H_j \in \mathcal{I}_{\mathbb{X}}$ , where  $G_j$  is a representative of  $g_j$  in  $P$ . Therefore we have  $H_i \in \langle \{H_1, \dots, \widehat{H}_i, \dots, H_t\} \cup \mathcal{F} \rangle$ , a contradiction.  $\square$

Notice that the two last conditions of the lemma can be checked effectively using the Submodule Membership Test (cf. [KR2, Proposition 2.4.10]). This gives us a method for computing a minimal homogeneous system of generators of  $J/\mathcal{I}_{\mathbb{X}}$ . However, this method requires a large number of Gröbner basis computations. In the following proposition, we present a procedure which computes a minimal homogeneous system of generators of  $J/\mathcal{I}_{\mathbb{X}}$  and avoids some unnecessary computations of Gröbner bases.

**Proposition 3.1.10. (Buchberger's Algorithm with Minimalization)** *Let  $J$  be a homogeneous ideal of  $P$  containing  $\mathcal{I}_{\mathbb{X}}$ , let  $\mathcal{H} = (h_1, \dots, h_s)$  be a tuple of homogeneous elements of  $R$  which generate  $J/\mathcal{I}_{\mathbb{X}}$ . For  $j = 1, \dots, s$ , let  $H_j \in P$  be the normal representative modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  of  $h_j$ , and assume that  $\deg(H_1) \leq \cdots \leq \deg(H_s)$ . Consider the following sequence of instructions.*

- 1) Let  $\mathbb{S} = \emptyset$ ,  $\mathcal{V} = (H_1, \dots, H_s)$ ,  $\mathcal{G} = \emptyset$ ,  $s' = 0$ , and  $\mathcal{H}_{\min} = \emptyset$ .
- 2) Let  $d$  be the smallest degree of an element in  $\mathbb{S}$  or in  $\mathcal{V}$ . Form the subset  $\mathbb{S}_d$  of  $\mathbb{S}$  consisting of elements  $F$  such that  $\deg(F) = d$ , form the subtuple  $\mathcal{V}_d$  of  $\mathcal{V}$  consisting of the elements  $H_i$  such that  $\deg(H_i) = d$ , and delete their entries from  $\mathbb{S}$  and  $\mathcal{V}$ , respectively.
- 3) If  $\mathbb{S}_d = \emptyset$ , continue with step 6). Otherwise, choose an element  $S \in \mathbb{S}_d$  and delete it from  $\mathbb{S}_d$ .
- 4) Compute  $S' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(S)$  by using the Division Algorithm 3.1.6. If  $S' = 0$ , continue with step 3).
- 5) Increase  $s'$  by one, append  $G_{s'} = S'$  to the tuple  $\mathcal{G}$ , and append the set  $\{S_{is'} \mid 1 \leq i < s'\} \cup \{\widetilde{S}_{s'j} \mid 1 \leq j \leq r\}$  to the set  $\mathbb{S}$ , where

$$S_{is'} = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}\left(\frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(G_{s'}))}{\text{LM}_{\sigma}(G_i)} G_i - \frac{\text{lcm}(\text{LT}_{\sigma}(G_i), \text{LT}_{\sigma}(G_{s'}))}{\text{LM}_{\sigma}(G_{s'})} G_{s'}\right)$$

and

$$\tilde{S}_{s'j} = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}} \left( \frac{\text{lcm}(\text{LT}_{\sigma}(G_{s'}), \text{LT}_{\sigma}(F_j))}{\text{LM}_{\sigma}(G_{s'})} G_{s'} - \frac{\text{lcm}(\text{LT}_{\sigma}(G_{s'}), \text{LT}_{\sigma}(F_j))}{\text{LM}_{\sigma}(F_j)} F_j \right).$$

Then continue with step 3).

- 6) If  $\mathcal{V}_d = \emptyset$ , continue with step 9). Otherwise, choose an element  $H \in \mathcal{V}_d$  and delete it from  $\mathcal{V}_d$ .
- 7) Compute  $H' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(H)$ . If  $H' = 0$ , continue with step 6).
- 8) Increase  $s'$  by one, append  $G_{s'} = H'$  to the tuple  $\mathcal{G}$ , append the image of  $H$  in  $R$  to the tuple  $\mathcal{H}_{\min}$ , and append the set  $\{S_{is'} \mid 1 \leq i < s'\} \cup \{\tilde{S}_{s'j} \mid 1 \leq j \leq r\}$ , which is established in an analogous way as in step 5), to the set  $\mathbb{S}$ . Continue with step 6).
- 9) If  $\mathbb{S} = \emptyset$  and  $\mathcal{V} = \emptyset$ , return the pair  $(\mathcal{G}, \mathcal{H}_{\min})$  and stop. Otherwise, continue with step 2).

This is an algorithm which returns a pair  $(\mathcal{G}, \mathcal{H}_{\min})$  such that the set  $\{G \mid G \in \mathcal{G}\} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$ , and  $\mathcal{H}_{\min}$  is a subtuple of  $\mathcal{H}$  which minimally generates  $J/\mathcal{I}_{\mathbb{X}}$ .

*Proof.* The finiteness of this algorithm and the claim that when the algorithm stops, the set  $\{G \mid G \in \mathcal{G}\} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $J$ , follow by using a similar argument as in [KR3, Theorem 4.5.5] and by using Buchberger's Criterion 3.1.8. The minimality of resulting tuple  $\mathcal{H}_{\min}$  can be shown in the same way as in [KR3, Theorem 4.6.3]. We note that after the procedure has finished looping through steps 3), 4), and 5), the tuple  $\mathcal{G}$  satisfies the property that there are non-zero homogeneous normal polynomials  $G'_1, \dots, G'_t$  modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  such that  $\deg(G'_j) > d$  for all  $j = 1, \dots, t$  and such that the set  $\{G \mid G \in \mathcal{G}\} \cup \{G'_1, \dots, G'_t\} \cup \mathcal{F}$  is a homogeneous  $<_{\sigma}$ -Gröbner basis of  $\langle \{G \mid G \in \mathcal{G}\} \cup \mathcal{F} \rangle$ . Every time an element  $G_{s'}$  of degree  $d$  is added to  $\mathcal{G}$  in step 8), the property is also true for the new tuple  $\mathcal{G}$ . This enables us to check whether a new normal polynomial  $H \in \mathcal{V}_d$  modulo  $\mathcal{I}_{\mathbb{X}}$  w.r.t.  $<_{\sigma}$  which is chosen in step 6) is contained in  $\langle \{G \mid G \in \mathcal{G}\} \cup \mathcal{F} \rangle$  via step 7), and hence Lemma 3.1.9 guarantees that  $\mathcal{H}_{\min}$  is always a minimal system of generators of  $\langle \{G \mid G \in \mathcal{G}\} \cup \mathcal{F} \rangle / \mathcal{I}_{\mathbb{X}}$  at the end of an iteration of step 8).  $\square$

**Remark 3.1.11.** When we are only interested in computing a minimal homogeneous system  $\mathcal{H}_{\min}$  of generators of  $J/\mathcal{I}_{\mathbb{X}}$ , we can speed this algorithm up by stopping the execution of the algorithm after degree  $d_{\max} = \deg(H_s)$  is finished and appending only the set  $\{S_{is'} \mid 1 \leq i < s', \deg(S_{is'}) \leq d_{\max}\} \cup \{\tilde{S}_{s'j} \mid 1 \leq j \leq r, \deg(\tilde{S}_{s'j}) \leq d_{\max}\}$  to the

set  $\mathbb{S}$  in steps 5) and 8). The implementation of the algorithm computing only the set  $\mathcal{H}_{\min}$  is given in the appendix.

**Example 3.1.12.** Let  $\mathbb{X} = \{p_1, \dots, p_5\}$  be the projective point set of  $\mathbb{P}_{\mathbb{F}_3}^2$ , where  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 1)$ ,  $p_3 = (1 : 2 : 0)$ ,  $p_4 = (1 : 0 : 2)$ , and  $p_5 = (1 : 2 : 2)$ . Let  $<_{\sigma}$  be the term ordering DegRevLex. Then a homogeneous  $<_{\sigma}$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$  is  $\mathcal{F} = \{F_1, F_2, F_3\}$  with  $F_1 = X_0X_1 + X_1^2 - X_0X_2 - X_2^2$ ,  $F_2 = X_1^2X_2 - X_1X_2^2$ , and  $F_3 = X_0^2X_2 - X_2^3$ . Let  $\mathcal{H}$  be the tuple of homogeneous elements  $(h_1, h_2, h_3, h_4)$ , where  $h_1 = x_1^3 - x_2^3$ ,  $h_2 = x_0^3 - x_2^3$ ,  $h_3 = x_0x_2^3 - x_2^4$ , and  $h_4 = x_2^4$ , and let  $J/\mathcal{I}_{\mathbb{X}}$  be the homogeneous ideal of  $R$  generated by the elements in  $\mathcal{H}$ . Let  $H_1 = X_1^3 - X_2^3$ ,  $H_2 = X_0^3 - X_2^3$ ,  $H_3 = X_0X_2^3 - X_2^4$ , and  $H_4 = X_2^4$ . We follow the Buchberger Algorithm with Minimalization 3.1.10 to compute a minimal homogeneous system of generators of  $J/\mathcal{I}_{\mathbb{X}}$ . We stop the computation after degree  $d_{\max} = \deg(H_4) = 4$  is finished and append only the set  $\{S_{is'} \mid 1 \leq i < s', \deg(S_{is'}) \leq 4\} \cup \{\tilde{S}_{s'j} \mid 1 \leq j \leq r, \deg(\tilde{S}_{s'j}) \leq 4\}$  to the set  $\mathbb{S}$  in steps 5) and 8).

- 1) Let  $\mathbb{S} = \emptyset$ ,  $\mathcal{V} = (H_1, H_1, H_3, H_4)$ ,  $\mathcal{G} = \emptyset$ ,  $s' = 0$ , and  $\mathcal{H}_{\min} = \emptyset$ .
- 2) Let  $d = \deg(H_1) = 3$ ,  $\mathcal{V}_3 = (H_1, H_2)$ ,  $\mathcal{V} = (H_3, H_4)$ ,  $\mathbb{S}_3 = \emptyset$ .
- 3) Since  $\mathbb{S}_3 = \emptyset$ , we continue with step 6).
- 6) Choose  $H = H_1$  and let  $\mathcal{V}_3 = (H_2)$ .
- 7) Compute  $H' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(H) = H_1$ .
- 8) Let  $s' = 1$ ,  $\mathcal{G} = (G_1)$  with  $G_1 = H_1$ ,  $\mathcal{H}_{\min} = (h_1)$ , and  $\mathbb{S} = \{\tilde{S}_{11}, \tilde{S}_{12}\}$ , where  $\tilde{S}_{11} = X_1^4 - X_2^4$ ,  $\tilde{S}_{12} = X_1X_2^3 - X_2^4$ . Since  $\deg(\tilde{S}_{13}) = 6 > d_{\max}$ , we do not add  $\tilde{S}_{13}$  to  $\mathbb{S}$ .
- 6) Choose  $H = H_2$  and let  $\mathcal{V}_3 = \emptyset$ .
- 7) Compute  $H' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(H) = H_2$ .
- 8) Let  $s' = 2$ ,  $\mathcal{G} = (G_1, G_2)$  with  $G_2 = H_2$ ,  $\mathcal{H}_{\min} = (h_1, h_2)$ . Since  $S_{12}$  and  $\tilde{S}_{22}$  have degree  $> d_{\max}$ , we let  $\mathbb{S} = \{\tilde{S}_{11}, \tilde{S}_{12}, \tilde{S}_{21}, \tilde{S}_{23}\}$ , where  $\tilde{S}_{21} = -X_1^4 + X_0X_2^3 - X_1X_2^3 + X_2^4$  and  $\tilde{S}_{23} = X_0X_2^3 - X_2^4$ .
- 6) Since  $\mathcal{V}_3 = \emptyset$ , we continue with step 9).
- 9) Since  $\mathbb{S} \neq \emptyset$  and  $\mathcal{V} \neq \emptyset$ , we continue with step 2).
- 2) We have  $\deg(H_3) = \deg(\tilde{S}_{11}) = 4$ . Let  $d = 4$ ,  $\mathcal{V}_4 = (H_3, H_4)$ ,  $\mathcal{V} = \emptyset$ ,  $\mathbb{S}_4 = \{\tilde{S}_{11}, \tilde{S}_{12}, \tilde{S}_{21}, \tilde{S}_{23}\}$ , and  $\mathbb{S} = \emptyset$ .
- 3) Choose  $S = \tilde{S}_{11}$  and let  $\mathbb{S}_4 = \{\tilde{S}_{12}, \tilde{S}_{21}, \tilde{S}_{23}\}$ .
- 4) Compute  $S' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(S) = X_1X_2^3 - X_2^4$ .

- 5) Let  $s' = 3$ ,  $\mathcal{G} = (G_1, G_2, G_3)$  with  $G_3 = S' = X_1X_2^3 - X_2^4$ . Since  $S_{13}$ ,  $S_{23}$ ,  $\tilde{S}_{31}$ ,  $\tilde{S}_{32}$ , and  $\tilde{S}_{33}$  are either zero or homogeneous of degree  $> d_{\max}$ , we let  $\mathbb{S} = \emptyset$ .
- 3) Choose  $S = \tilde{S}_{12}$  and let  $\mathbb{S}_4 = \{\tilde{S}_{21}, \tilde{S}_{23}\}$ .
- 4) Compute  $S' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(S) = 0$ . Since  $S' = 0$ , we continue with step 3).
- 3) Choose  $S = \tilde{S}_{21}$  and let  $\mathbb{S}_4 = \{\tilde{S}_{23}\}$ .
- 4) Compute  $S' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(S) = X_0X_2^3 - X_2^4$ .
- 5) Let  $s' = 4$ ,  $\mathcal{G} = (G_1, G_2, G_3, G_4)$  with  $G_4 = S' = X_0X_2^3 - X_2^4$ , and  $\mathbb{S} = \emptyset$ . Note that  $S_{14}$ ,  $S_{24}$ ,  $S_{34}$ ,  $\tilde{S}_{41}$ ,  $\tilde{S}_{42}$ , and  $\tilde{S}_{43}$  are either zero or homogeneous of degree  $> d_{\max}$ .
- 3) Choose  $S = \tilde{S}_{23}$  and let  $\mathbb{S}_4 = \emptyset$ .
- 4) Compute  $S' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(S) = 0$  and continue with step 3).
- 3) Since  $\mathbb{S}_4 = \emptyset$ , we continue with step 6).
- 6) Choose  $H = H_3$  and let  $\mathcal{V}_4 = (H_4)$ .
- 7) Compute  $H' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(H) = 0$  and continue with step 6).
- 6) Choose  $H = H_4$  and let  $\mathcal{V}_4 = \emptyset$ .
- 7) Compute  $H' = \text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(H) = X_2^4$ .
- 8) Let  $s' = 5$ ,  $\mathcal{G} = (G_1, G_2, G_3, G_4, G_5)$  with  $G_5 = H_4$ ,  $\mathcal{H}_{\min} = (h_1, h_2, h_4)$ , and  $\mathbb{S} = \emptyset$ . Note that all  $S_{15}, \dots, S_{45}$ ,  $\tilde{S}_{51}$ ,  $\tilde{S}_{52}$ , and  $\tilde{S}_{53}$  have degree  $> d_{\max}$ .
- 6) Since  $\mathcal{V}_4 = \emptyset$ , we continue with step 9).
- 9) Since  $\mathbb{S} = \emptyset$  and  $\mathcal{V} = \emptyset$ , return  $\mathcal{H}_{\min}$  and stop.

The returned tuple  $\mathcal{H}_{\min} = (h_1, h_2, h_4)$  of this algorithm is a minimal homogeneous system of generators of the homogeneous ideal  $J/\mathcal{I}_{\mathbb{X}}$ .

Now we are ready to formulate the following algorithm for the computation of a minimal homogeneous system of generators of the Noether different for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ .

**Proposition 3.1.13. (Computation of Noether Differents)** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and for  $j \in \{1, \dots, s\}$  let  $I_j \subseteq P$  be the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$ . Let  $<_{\sigma}$  be a term ordering on  $\mathbb{T}^{n+1}$ . Consider the following sequences of instructions.*

- 1) Compute the reduced  $<_{\sigma}$ -Gröbner basis  $\mathcal{F} = \{F_1, \dots, F_r\}$  of  $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^s I_j$  by using the GPBM-Algorithm (see [AKR, Theorem 4.6]).

- 2) Form the polynomial ring  $Q = K[X_0, X_1, \dots, X_n, Y_1, \dots, Y_n]$  which is equipped with the standard grading ( $\deg(X_0) = \dots = \deg(X_n) = \deg(Y_1) = \dots = \deg(Y_n) = 1$ ), and form  $JQ = \langle F_1(X_0, Y_1, \dots, Y_n), \dots, F_r(X_0, Y_1, \dots, Y_n) \rangle_Q$ .
- 3) Let  $<_{\bar{\sigma}}$  be a term ordering on  $\mathbb{T}^{2n+1} = \mathbb{T}(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n)$ . Compute a homogeneous  $<_{\bar{\sigma}}$ -Gröbner basis  $\mathcal{G} = \{G_1, \dots, G_t\}$  of the homogeneous colon ideal

$$\widehat{I} = (\mathcal{I}_{\mathbb{X}}Q + JQ) :_Q \langle X_1 - Y_1, \dots, X_n - Y_n \rangle_Q.$$

- 4) Compute a homogeneous system of generators  $\{H_1, \dots, H_t\}$  of  $\widehat{J} = \psi(\widehat{I})$ , where  $\psi$  is given by (3.1), by taking  $H_i = \psi(G_i)$  for  $i = 1, \dots, t$ . Then sort the set  $\{H_1, \dots, H_t\}$  such that  $\deg(H_1) \leq \dots \leq \deg(H_t)$ .
- 5) Set  $\mathcal{H} = \{H_1, \dots, H_t\} \setminus \mathcal{F}$ . For each polynomial  $H \in \mathcal{H}$ , compute its normal polynomial  $\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(H)$ . If  $\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(H) \neq 0$ , replace  $H$  by  $\text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(H)$ . Otherwise, delete  $H$  from  $\mathcal{H}$ .
- 6) Apply Buchberger's Algorithm with Minimalization 3.1.10 to compute a tuple  $\mathcal{H}_{\min}$  which generates  $\widehat{J}/\mathcal{I}_{\mathbb{X}}$  minimally.
- 7) Return the tuple  $\mathcal{H}_{\min}$  and stop.

This is an algorithm which computes a tuple  $\mathcal{H}_{\min}$  whose elements are a minimal homogeneous system of generators of the Noether different  $\vartheta_N(R/K[x_0])$ .

*Proof.* The finiteness of this algorithm is obviously true. The correctness of this algorithm follows by combining Corollary 3.1.5 and Proposition 3.1.10.  $\square$

In the following examples we use the term ordering DegRevLex for the computations. For more details about implementation see the appendix.

**Example 3.1.14.** Let  $\mathbb{X} = \{p_1, \dots, p_4\} \subseteq \mathbb{P}_K^3$  be the projective point set given by  $p_1 = (1 : 1 : 1 : 1)$ ,  $p_2 = (1 : 2 : 3 : 4)$ ,  $p_3 = (2 : 3 : 4 : 5)$ , and  $p_4 = (3 : 4 : 5 : 1)$ . First we consider the case  $K = \mathbb{Q}$ . An application of Proposition 3.1.6 yields a minimal homogeneous system of generators  $\{h_1, h_2, h_3\}$  of  $\vartheta_N(R/\mathbb{Q}[x_0])$ , where  $h_1 = x_1x_3 - 2x_2x_3 + x_3^2$ ,  $h_2 = x_2x_3^2 - \frac{66}{85}x_3^3$ , and  $h_3 = x_2^2x_3 - \frac{154}{255}x_3^3$ . This also tells us that  $\mathbb{X}$  is not arithmetically Gorenstein over  $\mathbb{Q}$ .

Next we take  $K = \mathbb{F}_5$ . In this situation, the homogeneous vanishing ideal of  $\mathbb{X}$  can be generated by three homogeneous polynomials. This implies that  $\mathbb{X}$  is a complete intersection (and hence it is arithmetically Gorenstein). A calculation tells us that the Noether different of  $R/\mathbb{F}_5[x_0]$  is given by  $\vartheta_N(R/\mathbb{F}_5[x_0]) = \langle x_2^3 + x_2^2x_3 + 2x_2x_3^2 - 2x_3^3 \rangle$ .

**Example 3.1.15.** Let us consider the 0-dimensional subscheme  $\mathbb{X}$  of  $\mathbb{P}_{\mathbb{Q}}^2$  of degree 16 with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_7\}$  and its homogeneous vanishing ideal given by  $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^7 I_j$ , where  $I_1 = \langle X_0 - X_1, X_2 \rangle$ ,  $I_2 = \langle X_1, X_0 + X_2 \rangle$ ,  $I_3 = \langle X_0 - X_1, 2X_0 - X_2 \rangle$ ,  $I_4 = \langle X_0 - X_1, X_0 - X_2 \rangle$ ,  $I_5 = \langle X_1, X_2 \rangle^2$ ,  $I_6 = \langle 3X_0^2 + X_1^2, X_2 \rangle^2$ , and  $I_7 = \langle X_1, 2X_0^3 + X_2^3 \rangle$  (Notice that  $I_j$  is the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$  for  $j = 1, \dots, 7$ ). By applying Proposition 3.1.6, a minimal homogeneous generating system of the Noether different  $\vartheta_N(R/\mathbb{Q}[x_0])$  is computed as  $\{x_1^2x_2^4 - \frac{3}{5}x_1x_2^5, x_0^2x_2^6 + \frac{23}{25}x_0x_2^7 + \frac{1}{25}x_2^8, x_0^3x_2^5 - \frac{3}{25}x_0x_2^7 + \frac{14}{25}x_2^8, x_0^4x_1^5 + \frac{2}{3}x_0^2x_1^7 + \frac{1}{9}x_1^9\}$ .

## 3.2 Dedekind Differents for 0-Dimensional Schemes

As usual, let  $K$  be an arbitrary field, and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ . Let us write  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$  for some  $s \geq 1$ . We know that  $\Gamma_*(\mathcal{O}_{\mathbb{X}}) = \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(i)) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}] \cong R_{x_0}$ , where  $T_1, \dots, T_s$  are indeterminates with  $\deg(T_1) = \dots = \deg(T_s) = 1$ . Let  $\tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$ , and let  $\tilde{\iota}: R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j] \hookrightarrow \Gamma_*(\mathcal{O}_{\mathbb{X}})$  be the injection given by (2.7). Note that  $\tilde{\iota}$  is a homogeneous injection of degree zero and  $\tilde{\iota}(x_0) = (T_1, \dots, T_s)$ .

The localization of  $R$  with respect to the set of all homogeneous non-zero-divisors of  $R$  is called the **homogeneous quotient ring** of  $R$ , and given by

$$Q^h(R) = \left\{ \frac{f}{g} \mid f, g \in R, g \text{ is a homogeneous non-zero-divisor} \right\}.$$

We have the following identification for the homogeneous quotient ring of  $R$  which is a generalized version of [Kr4, Proposition 1.4].

**Proposition 3.2.1.** *The map  $\tilde{\iota}$  extends to an isomorphism of graded  $R$ -modules*

$$\Psi: Q^h(R) \xrightarrow{\sim} \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}],$$

where for every element  $f/g \in Q^h(R)$  with  $f \in R_k$  and a non-zero-divisor  $g \in R_l$

$$\Psi\left(\frac{f}{g}\right) = \frac{\tilde{\iota}(f)}{\tilde{\iota}(g)} = \left(\frac{f_{p_1}}{g_{p_1}}T_1^{k-l}, \dots, \frac{f_{p_s}}{g_{p_s}}T_s^{k-l}\right).$$

In particular, we have  $Q^h(R) \cong R_{x_0}$ .

*Proof.* For a non-zero-divisor  $g \in R_i$ , the element  $g_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j}$  is a unit element for all  $j = 1, \dots, s$  (see Lemma 2.3.11). Let  $f/g \in Q^h(R)$  with  $f \in R_k$  and a non-zero-divisor



$g \in R_l$ . Then  $\tilde{v}(f) = (f_{p_1} T_1^k, \dots, f_{p_s} T_s^k)$  and  $\tilde{v}(g) = (g_{p_1} T_1^l, \dots, g_{p_s} T_s^l)$ , so we get

$$\Psi\left(\frac{f}{g}\right) = \frac{\tilde{v}(f)}{\tilde{v}(g)} = \left(\frac{f_{p_1} T_1^{k-l}}{g_{p_1}}, \dots, \frac{f_{p_s} T_s^{k-l}}{g_{p_s}}\right) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}].$$

Thus the map  $\Psi : Q^h(R) \rightarrow \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]$  is well defined. It is clearly true that  $\Psi$  is  $R$ -linear, homogeneous of degree zero. If  $\Psi\left(\frac{f}{g}\right) = 0$ , then  $\frac{f_{p_j}}{g_{p_j}} = 0 \in \mathcal{O}_{\mathbb{X}, p_j}$  for all  $j = 1, \dots, s$ . This implies  $f_{p_j} = 0$  for all  $j = 1, \dots, s$ , and so  $f = 0$ , since the map  $\tilde{v}$  is injective. Hence the map  $\Psi$  is an injection. Now we show that the map  $\Psi$  is surjective. Let  $(g_1, \dots, g_s) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]$ . For  $i \gg 0$ , we have  $\dim_K(R_i) = \deg(\mathbb{X}) = \dim_K(\prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]_i)$ . Thus, for  $i \gg 0$ ,  $(T_1^i g_1, \dots, T_s^i g_s)$  is of the form  $\Psi(f)$  for some  $f \in R$ . Therefore the element  $(g_1, \dots, g_s)$  is the image of  $f/x_0^i$ , and the claim follows.  $\square$

### 3.2.1 Dedekind Differents for Locally Gorenstein Schemes

**Definition 3.2.2.** A 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is called **locally Gorenstein** if at each point  $p_j \in \text{Supp}(\mathbb{X})$  the local ring  $\mathcal{O}_{\mathbb{X}, p_j}$  is a Gorenstein ring.

**Proposition 3.2.3.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . Set  $L := Q^h(R)$  and  $L_0 := K[x_0, x_0^{-1}]$ . Then the following statements hold true.*

- (i) *The algebra  $L/L_0$  has a homogeneous trace map  $\sigma_{L/L_0}$  of degree zero.*
- (ii) *There is an isomorphism of graded  $L$ -modules  $\Sigma : L \rightarrow \underline{\text{Hom}}_{L_0}(L, L_0)$  given by  $\Sigma(1) = \sigma_{L/L_0}$ .*
- (iii) *A homogeneous element  $\sigma \in \underline{\text{Hom}}_{L_0}(L, L_0)$  is a trace map of the algebra  $L/L_0$  if and only if there exists a unit  $u \in L$  such that  $\sigma = u \cdot \sigma_{L/L_0}$ .*

*Proof.* Since  $\mathbb{X}$  is locally Gorenstein, the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  is a finite Gorenstein algebra for every  $j \in \{1, \dots, s\}$ . It then follows from [Ku5, E.16] that there is a trace map  $\bar{\sigma}_j \in \text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K)$  such that  $\text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K) = \mathcal{O}_{\mathbb{X}, p_j} \cdot \bar{\sigma}_j$  for  $j = 1, \dots, s$ . Clearly,  $L_0/K[x_0]$  and  $\mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]/L_0$  are algebras, and

$$\mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}] \cong \mathcal{O}_{\mathbb{X}, p_j} \otimes_K K[T_j, T_j^{-1}] \cong \mathcal{O}_{\mathbb{X}, p_j} \otimes_K L_0.$$

By [Ku5, F.16(a)], the map  $\sigma_j = \bar{\sigma}_j \otimes \text{id}_{L_0} : \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}] \rightarrow K[T_j, T_j^{-1}] \cong L_0$  is a trace map of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]/L_0$ . Moreover,  $\sigma_j$  is a homogeneous trace map of degree zero and

$$\underline{\text{Hom}}_{L_0}(\mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}], L_0) = \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}] \cdot \sigma_j.$$

Based on Proposition 3.2.1, we may identify  $L = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]$ . Therefore an application of [Ku5, F.16(b)] yields that the algebra  $L/L_0$  has a homogeneous trace map  $\sigma_{L/L_0} = (\sigma_1, \dots, \sigma_s)$  of degree zero given by  $\sigma_{L/L_0}(g) = \sum_{j=1}^s \sigma_j(g_j)$  for any element  $g = (g_1, \dots, g_s) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}] = L$ , i.e.,

$$\underline{\mathrm{Hom}}_{L_0}(L, L_0) = \underline{\mathrm{Hom}}_{L_0}\left(\prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}], L_0\right) = \left(\prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]\right) \cdot \sigma_{L/L_0}.$$

Hence claim (i) follows.

In order to prove (ii) and (iii), it suffices to prove only claim (ii). To this end, we may write  $\underline{\mathrm{Hom}}_{L_0}(L, L_0) = L \cdot \sigma_{L/L_0}$ , where  $\sigma_{L/L_0}$  is a homogeneous trace map of degree zero as above. It is enough to show that  $\sigma_{L/L_0} \in \underline{\mathrm{Hom}}_{L_0}(L, L_0)$  satisfies  $\mathrm{Ann}_L(\sigma_{L/L_0}) = \langle 0 \rangle$ . Assume that  $f \cdot \sigma_{L/L_0} = 0$  for some homogeneous element  $f \in L$ . We have  $f \cdot \sigma_{L/L_0}(g) = \sigma_{L/L_0}(fg) = g \cdot \sigma_{L/L_0}(f) = 0$  for all  $g \in L$ . This implies  $\alpha(f) = 0$  for every  $\alpha \in \underline{\mathrm{Hom}}_{L_0}(L, L_0)$ . Since the algebra  $R/K[x_0]$  is free of rank  $\deg(\mathbb{X})$  and  $L \cong R_{x_0} \cong R \otimes_{K[x_0]} L_0$ , it follows that the algebra  $L/L_0$  is also free of rank  $\deg(\mathbb{X})$ . Let  $m = \deg(\mathbb{X})$ , let  $\{b_1, \dots, b_m\}$  be a  $L_0$ -basis of  $L$ , and let  $\{b_1^*, \dots, b_m^*\}$  be the dual basis of  $\{b_1, \dots, b_m\}$ . We write  $f = \sum_{j=1}^m g_j b_j \in L$  ( $g_j \in L_0$ ). Then  $g_j = b_j^*(\sum_{j=1}^m g_j b_j) = b_j^*(f) = 0$  for all  $j = 1, \dots, m$ . Hence we obtain  $f = 0$ , and so  $\mathrm{Ann}_L(\sigma_{L/L_0}) = \langle 0 \rangle$ .  $\square$

In what follows, let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with support  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and let  $L := Q^h(R)$  and  $L_0 := K[x_0, x_0^{-1}]$ . It follows from Proposition 3.2.3 that  $\underline{\mathrm{Hom}}_{L_0}(L, L_0) = \underline{\mathrm{Hom}}_{L_0}(L, L_0) = L \cdot \sigma_{L/L_0}$ . Moreover, we observe that  $L \cong R \otimes_{K[x_0]} L_0$ , and  $L_0$  is a flat  $K[x_0]$ -module (cf. [Bo2, II, §3, Proposition 13]). According to [Ei1, Proposition 2.10], we have isomorphisms of graded  $L_0$ -modules

$$\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \otimes_{K[x_0]} L_0 \cong \underline{\mathrm{Hom}}_{L_0}(L_0 \otimes_{K[x_0]} R, L_0 \otimes_{K[x_0]} K[x_0]) \cong \underline{\mathrm{Hom}}_{L_0}(L, L_0)$$

and

$$\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \otimes_{K[x_0]} L_0 \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \otimes_R L.$$

Thus there are canonical isomorphisms

$$\underline{\mathrm{Hom}}_{L_0}(L, L_0) \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \otimes_{K[x_0]} L_0 \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \otimes_R L$$

and the canonical map  $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \hookrightarrow \underline{\mathrm{Hom}}_{L_0}(L, L_0)$  is injective.

**Definition 3.2.4.** The image of the monomorphism of  $R$ -graded modules

$$\begin{aligned} \Phi : \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) &\hookrightarrow \underline{\mathrm{Hom}}_{L_0}(L, L_0) = L \cdot \sigma_{L/L_0} \xrightarrow{\Sigma^{-1}} L \\ \varphi &\longmapsto \varphi \otimes \mathrm{id}_{L_0} \end{aligned} \quad (3.2)$$

is a homogeneous fractional  $R$ -ideal  $\mathfrak{C}_{R/K[x_0]}$  of  $L$ . It is called the **Dedekind complementary module** of  $R/K[x_0]$ . Its inverse, the  $R$ -ideal

$$\vartheta_D(R/K[x_0]) = \mathfrak{C}_{R/K[x_0]}^{-1} = \{f \in L \mid f \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R\},$$

is called the **Dedekind different** of  $R/K[x_0]$  (or for  $\mathbb{X}$  with respect to  $x_0$ ).

From the above definition, we have an isomorphism of graded  $R$ -modules  $\mathfrak{C}_{R/K[x_0]} \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])$ . It then follows from Proposition 2.4.6 that the graded  $R$ -module  $\mathfrak{C}_{R/K[x_0]}$  is finitely generated and

$$\mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i - 1) \text{ for all } i \in \mathbb{Z}.$$

As usual, we let  $r_{\mathbb{X}}$  denote the regularity index of  $\mathrm{HF}_{\mathbb{X}}$ , i.e.,  $r_{\mathbb{X}} = \min \{i \in \mathbb{Z} \mid \mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})\}$ . Some properties of the Dedekind different of the algebra  $R/K[x_0]$  are given in the following proposition.

**Proposition 3.2.5.** (i) *The Dedekind different  $\vartheta_D(R/K[x_0])$  is a homogeneous ideal of  $R$  and  $x_0^{2r_{\mathbb{X}}} \in \vartheta_D(R/K[x_0])$ .*

(ii) *The Hilbert function of  $\vartheta_D(R/K[x_0])$  satisfies  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < 0$ ,  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \deg(\mathbb{X})$  for  $i \geq 2r_{\mathbb{X}}$ , and*

$$0 \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(0) \leq \cdots \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) = \deg(\mathbb{X}).$$

(iii) *The regularity index of  $\vartheta_D(R/K[x_0])$  satisfies  $r_{\mathbb{X}} \leq \mathrm{ri}(\vartheta_D(R/K[x_0])) \leq 2r_{\mathbb{X}}$ .*

*Proof.* It is obviously true that  $(\mathfrak{C}_{R/K[x_0]})_0 \subseteq (L)_0$  as  $K$ -vector spaces. By the above argument, we have  $\mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(0) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-1) = \deg(\mathbb{X}) = \mathrm{HF}_L(0)$ . This implies the equality  $(\mathfrak{C}_{R/K[x_0]})_0 = (L)_0$ . Thus we obtain  $R_0 \subseteq (L)_0 = (\mathfrak{C}_{R/K[x_0]})_0$ , and in particular,  $1 \in \mathfrak{C}_{R/K[x_0]}$ . Hence  $\vartheta_D(R/K[x_0])$  is a homogeneous ideal of  $R$ .

Now let  $m = \deg(\mathbb{X})$ , let  $\{t_1, \dots, t_m\}$  be a  $K[x_0]$ -basis of  $R$  (see Section 2.4), and let  $\{t_1^*, \dots, t_m^*\}$  be the dual basis of  $\{t_1, \dots, t_m\}$ . Note that  $\deg(t_j) = n_j$  for  $j = 1, \dots, m$ . Then  $t_j^* \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-n_j}$  for every  $j \in \{1, \dots, m\}$ . By letting  $g_j = \Phi(t_j^*)$  for  $j = 1, \dots, m$ , we obtain  $\mathfrak{C}_{R/K[x_0]} = \langle g_1, \dots, g_m \rangle_R \subseteq L$ . Here  $g_j$  is also homogeneous of degree  $\deg(g_j) = -n_j$  (since  $\Phi$  is homogeneous of degree zero).

We claim that, for  $j \in \{1, \dots, m\}$ , there is a homogeneous element  $g'_j \in R_{r_{\mathbb{X}}}$  such that  $g_j = x_0^{-r_{\mathbb{X}}-n_j} g'_j \in \mathfrak{C}_{R/K[x_0]}$ . Indeed, since  $g_j \in L \cong R_{x_0}$ , there exist  $g''_j \in R$  and  $d_j \in \mathbb{N}$  such that  $g_j = x_0^{-d_j} g''_j$ . If  $\deg(g''_j) = d_j - n_j \leq r_{\mathbb{X}}$ , then we set  $g'_j = x_0^{r_{\mathbb{X}}-d_j+n_j} g''_j \in R_{r_{\mathbb{X}}}$ . If  $\deg(g''_j) = d_j - n_j > r_{\mathbb{X}}$ , then we write  $g''_j = x_0^{d_j-n_j-r_{\mathbb{X}}} g'_j$  for some  $g'_j \in R_{r_{\mathbb{X}}}$ , since  $R_i = x_0^{i-r_{\mathbb{X}}} R_{r_{\mathbb{X}}}$  for all  $i \geq r_{\mathbb{X}}$ . Thus we get  $g_j = x_0^{-r_{\mathbb{X}}-n_j} g'_j$ , as claimed.

Consequently, we have  $\mathfrak{C}_{R/K[x_0]} = \langle x_0^{-r_{\mathbb{X}}-n_1} g'_1, \dots, x_0^{-r_{\mathbb{X}}-n_m} g'_m \rangle_R$ . Now it is clear that  $x_0^{2r_{\mathbb{X}}} \in \vartheta_D(R/K[x_0])$ , since  $n_j \leq r_{\mathbb{X}}$  and  $x_0^{2r_{\mathbb{X}}} \cdot (x_0^{-r_{\mathbb{X}}-n_j} g'_j) = x_0^{r_{\mathbb{X}}-n_j} g'_j \in R_{2r_{\mathbb{X}}-n_j}$  for all  $j = 1, \dots, m$ . Hence claim (i) follows.

Next we prove claim (ii). It is clear that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(i+1)$  for all  $i \in \mathbb{Z}$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < 0$ , since  $\vartheta_D(R/K[x_0])$  is a homogeneous ideal of  $R$  by (i). Note that  $\mathrm{HF}_{\mathbb{X}}(i) = m = \deg(\mathbb{X})$  for all  $i \geq r_{\mathbb{X}}$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i)$  for all  $i \in \mathbb{Z}$ . So the Hilbert function of  $\vartheta_D(R/K[x_0])$  satisfies  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq m$  for all  $i \in \mathbb{Z}$ . We write  $\mathfrak{C}_{R/K[x_0]} = \langle x_0^{-r_{\mathbb{X}}-n_1} g'_1, \dots, x_0^{-r_{\mathbb{X}}-n_m} g'_m \rangle_R$  with  $g'_1, \dots, g'_m \in R_{r_{\mathbb{X}}}$  as above, and let  $\{f_1, \dots, f_m\}$  be a  $K$ -basis of  $R_{r_{\mathbb{X}}}$ . Then  $f_i g'_j \in R_{2r_{\mathbb{X}}}$ . There is  $\tilde{f}_{ij} \in R_{r_{\mathbb{X}}}$  such that  $f_i g'_j = x_0^{r_{\mathbb{X}}} \tilde{f}_{ij}$  for all  $i, j \in \{1, \dots, m\}$ . Thus  $(x_0^{r_{\mathbb{X}}} f_i) \cdot (x_0^{-r_{\mathbb{X}}-n_j} g'_j) = x_0^{-n_j} f_i g'_j = x_0^{r_{\mathbb{X}}-n_j} \tilde{f}_{ij} \in R_{2r_{\mathbb{X}}-n_j}$  for all  $i, j$ . It follows that  $\{x_0^{r_{\mathbb{X}}} f_1, \dots, x_0^{r_{\mathbb{X}}} f_m\} \subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}} \subseteq R_{2r_{\mathbb{X}}}$ . On the other hand, we see that

$$\begin{aligned} m &= \mathrm{HF}_{\mathbb{X}}(2r_{\mathbb{X}}) = \dim_K \langle x_0^{r_{\mathbb{X}}} f_1, \dots, x_0^{r_{\mathbb{X}}} f_m \rangle_K \\ &\leq \dim_K \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}} = \mathrm{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) \leq \mathrm{HF}_{\mathbb{X}}(2r_{\mathbb{X}}) = m. \end{aligned}$$

Therefore we obtain the equalities  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = m = \deg(\mathbb{X})$  for all  $i \geq 2r_{\mathbb{X}}$ .

Finally, claim (iii) is an immediate consequence of claims (i) and (ii).  $\square$

**Remark 3.2.6.** Given a 0-dimensional locally Gorenstein scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  with support  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , we have  $\dim_K(p_j) \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) = 1$  for  $j = 1, \dots, s$ , where  $K(p_j) = \mathcal{O}_{\mathbb{X}, p_j} / \mathfrak{m}_{\mathbb{X}, p_j}$  is the residue field of  $\mathbb{X}$  at  $p_j$ . It follows from Proposition 2.3.17 that there is a unique maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  corresponding to a socle element of  $\mathcal{O}_{\mathbb{X}, p_j}$ . In this case, the local ring  $\mathcal{O}_{\mathbb{Y}_j, p_j}$  has a presentation as  $\mathcal{O}_{\mathbb{Y}_j, p_j} = \mathcal{O}_{\mathbb{X}, p_j} / \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$ . Let  $\varkappa_j = \dim_K K(p_j)$ , and let  $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$  be a set of minimal separators of  $\mathbb{Y}_j$  in  $\mathbb{X}$  (see Definition 2.3.19). Then we have

$$\deg_{\mathbb{X}}(p_j) = \max\{\deg(f_{jk_j}^*) \mid k_j = 1, \dots, \varkappa_j\} \leq r_{\mathbb{X}}.$$

**Lemma 3.2.7.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, and let  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . For  $j \in \{1, \dots, s\}$ , let  $\mathbb{Y}_j \subseteq \mathbb{X}$  be the maximal  $p_j$ -subscheme, and let  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$  be a set of separators of  $\mathbb{Y}_j$  in  $\mathbb{X}$ . If there exists a point  $p_j$*

in  $\text{Supp}(\mathbb{X})$  with  $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$  and an element  $g$  in  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  with  $f_{jk_j}g \neq 0$  for some  $k_j \in \{1, \dots, \varkappa_j\}$ , then  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ .

*Proof.* Since  $\text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) = m = \deg(\mathbb{X})$  and  $\text{ri}(\vartheta_D(R/K[x_0])) \leq 2r_{\mathbb{X}}$  by Proposition 3.2.5, it suffices to prove  $\text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}} - 1) < m$ . This is equivalent to proving  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1} \subsetneq R_{2r_{\mathbb{X}}-1}$ . Suppose for contradiction that  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1} = R_{2r_{\mathbb{X}}-1}$ . By the assumption that  $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$ , there is an index  $k_j \in \{1, \dots, \varkappa_j\}$  such that  $x_0 \nmid f_{jk_j}$ , say  $f_{j1}$ . Of course, we have  $x_0^{r_{\mathbb{X}}-1}f_{jk_j} \in \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$  for  $k_j = 1, \dots, \varkappa_j$ . We may write  $g = x_0^{-2r_{\mathbb{X}}}g' \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  with  $g' \in R_{r_{\mathbb{X}}} \setminus \{0\}$ . Since  $f_{jk_j}g \neq 0$  for some  $k_j \in \{1, \dots, \varkappa_j\}$ , it follows that  $(g')_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$  is a unit element (see Lemma 2.3.25). This also implies that  $fg' \neq 0$  for all  $f \in \mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} \setminus \{0\}$ . Let us consider the injection  $\tilde{\iota} : R \rightarrow \tilde{R}$  and write  $\tilde{\iota}(f_{jk_j}) = (0, \dots, 0, e_{jk_j}s_jT_j^{r_{\mathbb{X}}}, 0, \dots, 0) \in \tilde{R}$  for  $k_j = 1, \dots, \varkappa_j$ , where  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$  is a socle element corresponding to  $\mathbb{Y}_j$  and  $\{e_{j1}, \dots, e_{j\varkappa_j}\}$  is a set of elements in  $\mathcal{O}_{\mathbb{X}, p_j}$  whose residue classes form a  $K$ -basis of  $K(p_j)$ . Then the residue classes of elements  $\{(g')_{p_j}e_{j1}, \dots, (g')_{p_j}e_{j\varkappa_j}\}$  also form a  $K$ -basis of  $K(p_j)$ . There are  $c_{j1}, \dots, c_{j\varkappa_j} \in K$  such that  $e_{j1} = c_{j1}(g')_{p_j}e_{j1} + \dots + c_{j\varkappa_j}(g')_{p_j}e_{j\varkappa_j} \pmod{\mathfrak{m}_{\mathbb{X}, p_j}}$ . This implies  $e_{j1}s_j = c_{j1}(g')_{p_j}e_{j1}s_j + \dots + c_{j\varkappa_j}(g')_{p_j}e_{j\varkappa_j}s_j$ . We put  $f = c_{j1}f_{j1} + \dots + c_{j\varkappa_j}f_{j\varkappa_j} \in \mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} \setminus \{0\}$ . Observe that  $f \cdot g' = x_0^{r_{\mathbb{X}}}f_{j1}$ , since  $\tilde{\iota}$  is injective. Thus we have  $x_0^{r_{\mathbb{X}}-1}f \cdot g = x_0^{-r_{\mathbb{X}}-1}f \cdot g' = x_0^{-1}f_{j1} \in R_{r_{\mathbb{X}}-1}$  (as  $x_0^{r_{\mathbb{X}}-1}f \in \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ ). It follows that  $f_{j1} \in x_0R_{r_{\mathbb{X}}-1}$ , in contradiction to the fact that  $x_0 \nmid f_{j1}$ .  $\square$

**Proposition 3.2.8.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. The following conditions are equivalent.*

- (i)  $\mathbb{X}$  is arithmetically Gorenstein.
- (ii)  $\mathbb{X}$  is locally Gorenstein and  $\mathfrak{C}_{R/K[x_0]} \cong R(r_{\mathbb{X}})$  (i.e., there exists  $g \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $\mathfrak{C}_{R/K[x_0]} = \langle g \rangle_R$  and  $\text{Ann}_R(g) = \langle 0 \rangle$ ).
- (iii)  $\mathbb{X}$  is locally Gorenstein and there exists an element  $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$  such that  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$ .
- (iv)  $\mathbb{X}$  is locally Gorenstein and there exists an element  $h \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}}$  with  $\text{Ann}_R(h) = \langle 0 \rangle$ .

If these conditions are satisfied, then the Hilbert function of the Dedekind different satisfies  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for all  $i \in \mathbb{Z}$ , and  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ .

*Proof.* We remark that if  $\mathbb{X}$  is arithmetically Gorenstein then it is also locally Gorenstein. The equivalence of conditions (i) and (ii) is deduced from Proposition 2.4.18.

(ii) $\Rightarrow$ (iii): Assume that  $\mathfrak{C}_{R/K[x_0]} = \langle g \rangle_R \cong R(r_{\mathbb{X}})$  and  $\text{Ann}_R(g) = \langle 0 \rangle$ . Then  $g$  is a unit element of  $L$ . Obviously,  $h = g^{-1} \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}} \subseteq R_{r_{\mathbb{X}}}$  (as  $h \in L$  and  $h \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R$ ). We shall prove that  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$ . Let  $f \in \vartheta_D(R/K[x_0])_i$  for some  $i \geq 0$ . If  $i < r_{\mathbb{X}}$ , then  $fg \in R_{i-r_{\mathbb{X}}} = \langle 0 \rangle$ , and hence  $f = 0$  (as  $\text{Ann}_R(g) = \langle 0 \rangle$ ). Thus  $\vartheta_D(R/K[x_0])_i = \langle 0 \rangle$  for all  $i < r_{\mathbb{X}}$ . In the case  $i \geq r_{\mathbb{X}}$ , there is an element  $f_1 \in R_{i-r_{\mathbb{X}}}$  such that  $fg = f_1$ . This implies  $(f - f_1h)g = 0$ . Since  $\text{Ann}_R(g) = \langle 0 \rangle$  and  $f - f_1h \in R_i$ , we have  $f = f_1h$ , and consequently  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$ .

(iii) $\Rightarrow$ (ii): Assume that  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$  with  $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$ . Proposition 3.2.5 yields that there is an element  $f \in R_{r_{\mathbb{X}}}$  such that  $fh = x_0^{2r_{\mathbb{X}}} \in \vartheta_D(R/K[x_0])$ , and so  $h$  must be a non-zerodivisor of  $R$  or  $\text{Ann}_R(h) = \langle 0 \rangle$ . Because  $\dim_K((\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}) = \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) = \Delta_{\mathbb{X}} > 0$ , there exists a homogeneous element  $g = x_0^{-2r_{\mathbb{X}}}g' \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \setminus \{0\}$  for some  $g' \in R_{r_{\mathbb{X}}}$ . Then  $gh = x_0^{-2r_{\mathbb{X}}}g'h = c \in R_0 = K$ . If  $c = 0$  then  $g'h = 0$ , and hence  $g' = 0$ , since  $h$  and  $x_0$  are non-zerodivisors of  $R$ . This implies  $g = x_0^{-2r_{\mathbb{X}}}g' = 0$ , a contradiction. Hence it follows that  $c \neq 0$ , and  $\mathfrak{C}_{R/K[x_0]} = \langle c^{-1}g \rangle_R$  with  $\text{Ann}_R(c^{-1}g) = \langle 0 \rangle$ .

(iii) $\Rightarrow$ (iv): This has been shown in the proof of “(iii) implies (ii)”.

(iv) $\Rightarrow$ (iii): Suppose  $h \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}}$  with  $\text{Ann}_R(h) = \langle 0 \rangle$ . Let  $g$  be a non-zero homogeneous element of degree  $-r_{\mathbb{X}}$  of  $\mathfrak{C}_{R/K[x_0]}$ . (Such an element always exists, since  $\text{HF}_{\mathfrak{C}_{R/K[x_0]}(-r_{\mathbb{X}})} = \Delta_{\mathbb{X}} \geq 1$ .) We can write  $g = x_0^{-2r_{\mathbb{X}}}g'$  with  $g' \in R_{r_{\mathbb{X}}}$  as before. Since  $x_0$  and  $h$  are not zerodivisors of  $R$ , we have  $gh = x_0^{-2r_{\mathbb{X}}}g'h \neq 0$ . It follows that  $gh = c \in K \setminus \{0\}$  and  $\text{Ann}_R(g) = \langle 0 \rangle$ . As the proof of “(ii) implies (iii)”, we obtain  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$ .

The additional claim follows from the fact that  $\vartheta_D(R/K[x_0]) \cong R(-r_{\mathbb{X}})$ .  $\square$

Now let us consider the injection  $\tilde{\iota} : R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  given by (2.7). For every  $p_j \in \text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , let  $\nu_j := \dim_K(\mathcal{O}_{\mathbb{X}, p_j})$  and let  $\{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ . We set  $f_{jk_j} := \tilde{\iota}^{-1}((0, \dots, 0, e_{jk_j}T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$  for  $k_j = 1, \dots, \nu_j$ . Note that  $f_{jk_j} \cdot f_{j'k_{j'}} = 0$  for  $j \neq j'$  and  $R_{r_{\mathbb{X}}} = \langle f_{11}, \dots, f_{1\nu_1}, \dots, f_{s1}, \dots, f_{s\nu_s} \rangle_K$ . Since  $\mathbb{X}$  is locally Gorenstein, as in the proof of Proposition 3.2.3 we see that the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  has a trace map  $\bar{\sigma}_j \in \text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K)$ . An argument similar to that given in Proposition 2.4.9 shows that there is a  $K$ -basis  $\{e'_{j1}, \dots, e'_{j\nu_j}\}$  of  $\mathcal{O}_{\mathbb{X}, p_j}$  such that  $\bar{\sigma}_j(e_{jk_j}e'_{jk'_j}) = e_{jk'_j}^*(e_{jk_j}) = \delta_{k_jk'_j}$  for all  $k_j, k'_j = 1, \dots, \nu_j$  and  $\text{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K} = (\sum_{k=1}^{\nu_j} e_{jk}e'_{jk}) \cdot \bar{\sigma}_j$ . We say that the  $K$ -basis  $\{e'_{j1}, \dots, e'_{j\nu_j}\}$  is a **dual basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  to the  $K$ -basis  $\{e_{j1}, \dots, e_{j\nu_j}\}$  with respect to  $\bar{\sigma}_j$** .

When  $\mathbb{X}$  is a projective point set in  $\mathbb{P}_K^n$ , [Kr4, Proposition 1.6] provides an explicit description of the Dedekind complementary module in terms of the set of separators. This can be generalized for a 0-dimensional locally Gorenstein scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  by using the above tools as follows.

**Proposition 3.2.9.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, let  $\Phi$  be the monomorphism of  $R$ -graded modules defined by (3.2), let  $i \geq 0$ , and let  $\varphi \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{i-r_{\mathbb{X}}}$ . We write  $\varphi(f_{jk_j}) = c_{jk_j} x_0^i$  with  $c_{jk_j} \in K$ . Then we have*

$$\Phi(\varphi) = \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1} T_1^{i-r_{\mathbb{X}}}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} T_s^{i-r_{\mathbb{X}}} \right) \in (\mathfrak{C}_{R/K[x_0]})_{i-r_{\mathbb{X}}}.$$

In particular,  $\Phi(\varphi)$  can be identified with the element  $x_0^{i-2r_{\mathbb{X}}} (\sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j} \tilde{f}_{jk_j})$  of  $R_{x_0} \cong L$ , where  $\tilde{f}_{jk_j} = \tilde{\iota}^{-1}((0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$  for all  $j = 1, \dots, s$  and for all  $k_j = 1, \dots, \nu_j$ .

*Proof.* We set  $\epsilon_{jk_j} := (0, \dots, 0, e_{jk_j}, 0, \dots, 0) \in \prod_{l=1}^s \mathcal{O}_{\mathbb{X}, p_l}$  for  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . Due to Proposition 3.2.1, the set  $\{\epsilon_{11}, \dots, \epsilon_{1\nu_1}, \dots, \epsilon_{s1}, \dots, \epsilon_{s\nu_s}\}$  is a  $L_0$ -basis of  $L = Q^h(R)$ , where  $L_0 = K[x_0, x_0^{-1}]$ . So, the mapping  $\varphi \otimes \text{id}_{L_0} : L \cong R \otimes_{K[x_0]} L_0 \rightarrow L_0$  satisfies

$$(\varphi \otimes \text{id}_{L_0})(x_0^{r_{\mathbb{X}}} \epsilon_{jk_j}) = (\varphi \otimes \text{id}_{L_0})((0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) = \varphi(f_{jk_j}) = c_{jk_j} x_0^i$$

for  $j = 1, \dots, s$ ;  $k_j = 1, \dots, \nu_j$ . Thus  $(\varphi \otimes \text{id}_{L_0})(\epsilon_{jk_j}) = c_{jk_j} x_0^{i-r_{\mathbb{X}}}$  for all  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . On the other hand, the algebra  $L/L_0$  has a homogeneous trace map  $\sigma_{L/L_0}$  of degree zero by Proposition 3.2.3. We have

$$\begin{aligned} \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} \right) \cdot \sigma_{L/L_0}(\epsilon_{jk_j}) &= \sigma_{L/L_0} \left( (0, \dots, 0, \sum_{k'_j=1}^{\nu_j} c_{jk'_j} e'_{jk'_j} e_{jk_j}, 0, \dots, 0) \right) \\ &= \bar{\sigma}_j \left( \sum_{k'_j=1}^{\nu_j} c_{jk'_j} e'_{jk'_j} e_{jk_j} \right) = \sum_{k'_j=1}^{\nu_j} c_{jk'_j} \bar{\sigma}_j(e'_{jk'_j} e_{jk_j}) \\ &= \sum_{k'_j=1}^{\nu_j} c_{jk'_j} \delta_{k_j k'_j} = c_{jk_j}. \end{aligned}$$

This implies that  $(\varphi \otimes \text{id}_{L_0}) = x_0^{i-r_{\mathbb{X}}} \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} \right) \cdot \sigma_{L/L_0}$  in  $\underline{\text{Hom}}_{L_0}(L, L_0)$ . Hence

$$\Phi(\varphi) = \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1} T_1^{i-r_{\mathbb{X}}}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} T_s^{i-r_{\mathbb{X}}} \right) \in (\mathfrak{C}_{R/K[x_0]})_{i-r_{\mathbb{X}}}.$$

Additionally, we observe that

$$\begin{aligned} x_0^{2r_{\mathbb{X}}}\Phi(\varphi) &= x_0^{2r_{\mathbb{X}}} \cdot \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1} T_1^{i-r_{\mathbb{X}}}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} T_s^{i-r_{\mathbb{X}}} \right) \\ &= \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1} T_1^{r_{\mathbb{X}}+i}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} T_s^{r_{\mathbb{X}}+i} \right) = \tilde{v}(x_0^i \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j} \tilde{f}_{jk_j} \right)). \end{aligned}$$

Thus the conclusion follows.  $\square$

Next we determine a set of generators of the  $K$ -vector space  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  for a 0-dimensional locally Gorenstein scheme  $\mathbb{X}$ . This is a similar property as [Kr4, Corollary 1.10]. To shorten notation, we let  $m = \deg(\mathbb{X})$ ,  $f_1 = f_{11}, \dots, f_{\nu_1} = f_{1\nu_1}$ ,  $f_{\nu_1+1} = f_{21}, \dots, f_{\nu_1+\nu_2} = f_{2\nu_2}, \dots, f_{\sum_{j=1}^{s-1} \nu_j+1} = f_{s1}, \dots, f_m = f_{s\nu_s}$ . Similarly, we let  $\tilde{f}_1 = \tilde{f}_{11}, \dots, \tilde{f}_m = \tilde{f}_{s\nu_s}$ . The image of  $f_j$  in  $\bar{R} := R/\langle x_0 \rangle$  is denoted by  $Lf_j$  and is called the **leading form** of  $f_j$  for  $j = 1, \dots, m$ . Since  $\{Lf_1, \dots, Lf_m\}$  generates the  $K$ -vector space  $\bar{R}_{r_{\mathbb{X}}}$ , we can renumber  $\{f_1, \dots, f_m\}$  in such a way that  $\{Lf_1, \dots, Lf_{\Delta_{\mathbb{X}}}\}$  is a  $K$ -basis of  $\bar{R}_{r_{\mathbb{X}}}$ , where  $\Delta_{\mathbb{X}} = \text{HF}_{\mathbb{X}}(r_{\mathbb{X}}) - \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1)$  is the last non-zero difference of  $\text{HF}_{\mathbb{X}}$ . Then we write

$$Lf_{\Delta_{\mathbb{X}}+j} = \beta_{j1}Lf_1 + \dots + \beta_{j\Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}$$

for  $j = 1, \dots, m - \Delta_{\mathbb{X}}$ ,  $\beta_{jk} \in K$ .

**Lemma 3.2.10.** (i) Let  $h_1, \dots, h_{m-\Delta_{\mathbb{X}}} \in R_{r_{\mathbb{X}}-1}$  be elements which satisfy

$$x_0 h_j = f_{\Delta_{\mathbb{X}}+j} - \beta_{j1}f_1 - \dots - \beta_{j\Delta_{\mathbb{X}}}f_{\Delta_{\mathbb{X}}}.$$

Then  $\{h_1, \dots, h_{m-\Delta_{\mathbb{X}}}\}$  forms a  $K$ -basis of  $R_{r_{\mathbb{X}}-1}$ .

(ii) The set  $\{\tilde{g}_1, \dots, \tilde{g}_{\Delta_{\mathbb{X}}}\}$  forms a  $K$ -basis of  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$ , where

$$\tilde{g}_j = x_0^{-2r_{\mathbb{X}}}(\tilde{f}_j + \beta_{1j}\tilde{f}_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{m-\Delta_{\mathbb{X}}j}\tilde{f}_m)$$

for every  $j \in \{1, \dots, \Delta_{\mathbb{X}}\}$ .

*Proof.* (i) Clearly, the set  $\{f_1, \dots, f_m\}$  forms a  $K$ -basis of  $R_{r_{\mathbb{X}}}$ . Thus claim (i) follows from the fact that the elements  $f_{\Delta_{\mathbb{X}}+j} - \beta_{j1}f_1 - \dots - \beta_{j\Delta_{\mathbb{X}}}f_{\Delta_{\mathbb{X}}}$  with  $j = 1, \dots, m - \Delta_{\mathbb{X}}$  are linearly independent elements of  $x_0 R_{r_{\mathbb{X}}-1}$ .

(ii) Notice that  $\dim_K \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}} = \dim_K (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \Delta_{\mathbb{X}} \geq 1$ . The  $K$ -vector space  $\text{Hom}_K(\bar{R}_{r_{\mathbb{X}}}, K)$  has a  $K$ -basis  $\{(Lf_1)^*, \dots, (Lf_{\Delta_{\mathbb{X}}})^*\}$ . By [Kr4, Lemma 1.5], there is a 1-1 correspondence between elements  $\varphi$  of  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}}$



and  $K$ -linear maps  $\bar{\varphi} : R_{r_{\mathbb{X}}} \rightarrow K$  with  $\bar{\varphi}(x_0 R_{r_{\mathbb{X}}-1}) = 0$ . Thus we can lift the  $K$ -basis  $\{(Lf_1)^*, \dots, (Lf_{\Delta_{\mathbb{X}}})^*\}$  to obtain a  $K$ -basis  $\{\varphi_1, \dots, \varphi_{\Delta_{\mathbb{X}}}\}$  of  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}}$  with  $\varphi_j(f_k) = \delta_{jk}$  for  $k = 1, \dots, \Delta_{\mathbb{X}}$  and  $\varphi_j(f_{\Delta_{\mathbb{X}}+k}) = \varphi_j(\beta_{k1}f_1 + \dots + \beta_{k\Delta_{\mathbb{X}}}f_{\Delta_{\mathbb{X}}}) = \beta_{kj}$  for  $k = 1, \dots, m - \Delta_{\mathbb{X}}$ . Hence the claim follows by Proposition 3.2.9.  $\square$

If  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  is a projective point set, then  $\nu_1 = \dots = \nu_s = 1$ ,  $f_j$  is the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$ , and  $f_j = \tilde{f}_j$  for all  $j = 1, \dots, s$ . The following proposition gives us a description of the Hilbert function of  $\vartheta_D(R/K[x_0])$  and its regularity index for projective point sets.

**Proposition 3.2.11.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set. Then we have  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < 0$ ,  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = s$  for  $i \geq 2r_{\mathbb{X}}$  and*

$$0 \leq \text{HF}_{\vartheta_D(R/K[x_0])}(0) \leq \dots \leq \text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}} - 1) < \text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) = s.$$

*In particular, the regularity index of  $\vartheta_D(R/K[x_0])$  is  $2r_{\mathbb{X}}$ .*

*Proof.* It is clearly true that  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < 0$  and  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \text{HF}_{\vartheta_D(R/K[x_0])}(i+1)$  for all  $i \in \mathbb{Z}$ , since  $\vartheta_D(R/K[x_0])$  is a homogeneous ideal of  $R$ . By Proposition 3.2.5, we have  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = s$  for  $i \geq 2r_{\mathbb{X}}$ . Now we shall prove the last inequality  $\text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}} - 1) < \text{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) = s$ . This is equivalent to proving  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ . Since  $\Delta_{\mathbb{X}} \geq 1$ , we may assume that  $Lf_1 \neq 0$  in  $\bar{R}_{r_{\mathbb{X}}}$ . This means that  $\deg_{\mathbb{X}}(p_1) = r_{\mathbb{X}}$ . By Lemma 3.2.10(ii), the elements  $\tilde{g}_j = x_0^{-2r_{\mathbb{X}}}(f_j + \beta_{1j}f_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{s-\Delta_{\mathbb{X}}j}f_s)$  such that  $1 \leq j \leq \Delta_{\mathbb{X}}$  form a  $K$ -basis of  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$ . We have

$$f_1 \cdot \tilde{g}_1 = f_1 \cdot x_0^{-2r_{\mathbb{X}}}(f_1 + \beta_{11}f_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{s-\delta_1}f_s) = x_0^{-r_{\mathbb{X}}}f_1 \neq 0.$$

Therefore Lemma 3.2.7 implies  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ , as desired.  $\square$

Obviously, the Dedekind different  $\vartheta_D(R/K[x_0])$  is a homogeneous ideal of  $R$  which depends on the choice of the linear non-zerodivisor  $x_0$  of  $R$ . In the case that  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is a projective point set, we have the following corollary.

**Corollary 3.2.12.** *Let  $\mathbb{X}$  be a projective point set in  $\mathbb{P}_K^n$ , and let  $\ell \in R_1$  be a non-zerodivisor for  $R$ . Then we have*

$$x_0 \mathfrak{C}_{R/K[x_0]} = \ell \mathfrak{C}_{R/K[\ell]} \quad \text{and} \quad \ell \vartheta_D(R/K[x_0]) = x_0 \vartheta_D(R/K[\ell]).$$

*Proof.* The first equality is due to Proposition 1.8 in [Kr4]. It remains to prove the second one. Let  $h \in \vartheta_D(R/K[x_0])$ . By the first equality, we have

$$h \cdot \mathfrak{C}_{R/K[x_0]} = \frac{h}{x_0}(x_0 \mathfrak{C}_{R/K[x_0]}) = \frac{h\ell}{x_0} \mathfrak{C}_{R/K[\ell]} \subseteq R.$$

This implies that  $\frac{h\ell}{x_0} \in \vartheta_D(R/K[\ell]) \subseteq R$ , and so  $h\ell \in x_0\vartheta_D(R/K[\ell])$ . Similarly, we get  $gx_0 \in \ell\vartheta_D(R/K[x_0])$  for all  $g \in \vartheta_D(R/K[\ell])$ . Hence we obtain the equality  $\ell\vartheta_D(R/K[x_0]) = x_0\vartheta_D(R/K[\ell])$ , as we wanted to show.  $\square$

### 3.2.2 Dedekind Differents for 0-Dimensional Smooth Schemes

**Definition 3.2.13.** A Noetherian local ring  $S$  containing the field  $K$  is called **geometrically regular over  $K$**  if the ring  $L \otimes_K S$  is regular for every finite field extension  $L/K$ .

Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . We say that a point  $p_j \in \text{Supp}(\mathbb{X})$  is a **smooth point of  $\mathbb{X}$** , or that  $\mathbb{X}$  is **smooth at  $p_j$** , if the local ring  $\mathcal{O}_{\mathbb{X}, p_j}$  is geometrically regular over  $K$ . We say that  $\mathbb{X}$  is **smooth** if it is smooth at all of its points.

**Remark 3.2.14.** Notice that if  $p_j$  is a smooth point of  $\mathbb{X}$  then it is a reduced point of  $\mathbb{X}$ . The converse is true if  $K$  is a perfect field (cf. [Ku5, Propositions 5.18 and 7.12]). Moreover, if  $p_j$  is a reduced point of  $\mathbb{X}$  and  $K(p_j)/K$  is a separable field extension, then  $p_j$  is a smooth point of  $\mathbb{X}$  (cf. [Ku5, Corollary 7.16]).

For every  $p_j \in \text{Supp}(\mathbb{X})$ , the local ring  $\mathcal{O}_{\mathbb{X}, p_j}$  is a finite dimensional  $K$ -vector space. The module of Kähler differentials of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  will be denoted by  $\Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1$  (see [Ku5, Section 1]). It is clear that  $\Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1$  is a finitely generated  $\mathcal{O}_{\mathbb{X}, p_j}$ -module.

The smoothness criterion for the 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is provided by our next theorem, which follows from [Ku6, Theorem 5.16].

**Theorem 3.2.15.** *Let  $\overline{K}$  be the algebraic closure of  $K$ . The following assertions are equivalent.*

- (i)  $\mathbb{X}$  is smooth at  $p_j$ .
- (ii)  $\overline{K} \otimes_K \mathcal{O}_{\mathbb{X}, p_j}$  is regular.
- (iii)  $\Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1 = 0$ .
- (iv)  $K' \otimes_K \mathcal{O}_{\mathbb{X}, p_j}$  is regular for any field extension  $K'/K$ .

**Proposition 3.2.16.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme, and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . Let  $Q(R)$  be the full ring of quotients of  $R$ , and set  $L := Q^h(R)$  and  $L_0 := K[x_0, x_0^{-1}]$ .*

- (i) *The canonical trace map  $\text{Tr}_{L/L_0}$  is a homogeneous trace map of degree zero of the algebra  $L/L_0$ , especially,  $L/L_0$  is étale.*

(ii) *The canonical trace map  $\mathrm{Tr}_{Q(R)/K(x_0)}$  is a trace map of the algebra  $Q(R)/K(x_0)$ , especially,  $Q(R)/K(x_0)$  is étale.*

*Proof.* Since the scheme  $\mathbb{X}$  is smooth, it follows from Theorem 3.2.15 that  $\Omega_{\mathcal{O}_{\mathbb{X},p_j}/K}^1 = 0$  and  $\mathcal{O}_{\mathbb{X},p_j} = K(p_j)$  for every  $p_j \in \mathrm{Supp}(\mathbb{X})$ . [Ku5, Proposition 6.8] yields that  $\mathcal{O}_{\mathbb{X},p_j}/K$  is a finite separable field extension. So, the canonical trace map  $\mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}$  is a trace map of  $\mathcal{O}_{\mathbb{X},p_j}/K$ , i.e.,

$$\mathrm{Hom}_K(\mathcal{O}_{\mathbb{X},p_j}, K) = \mathcal{O}_{\mathbb{X},p_j} \cdot \mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}.$$

It follows that the trace map  $\sigma_{L/L_0}$  in Proposition 3.2.3(i) is exactly the canonical trace map  $\mathrm{Tr}_{L/L_0}$  if we choose  $\bar{\sigma}_j = \mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}$  for  $j = 1, \dots, s$  in the proceed constructing the trace map  $\sigma_{L/L_0}$ . The additional claim of (i) follows by [Ku5, Proposition F.8]. Thus claim (i) follows.

For (ii), it suffices to show that  $\mathrm{Tr}_{Q(R)/K(x_0)}$  is a trace map of  $Q(R)/K(x_0)$ . Since  $\mathbb{X}$  is reduced, it follows from [Ku4, III, Proposition 4.23] and [Bo2, V, §1, Proposition 9] that

$$Q(R) \cong \prod_{j=1}^s Q(R/\mathfrak{p}_j) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}(T_j).$$

As above,  $\mathcal{O}_{\mathbb{X},p_j}/K$  is a finite separable field extension for every  $p_j \in \mathrm{Supp}(\mathbb{X})$ . Then  $\mathcal{O}_{\mathbb{X},p_j}$  and  $K(x_0)$  are linearly disjoint over  $K$  (cf. [Mor, V, Section 20]). This implies  $\mathcal{O}_{\mathbb{X},p_j} \otimes_K K(x_0) \cong \mathcal{O}_{\mathbb{X},p_j}K(x_0) = \mathcal{O}_{\mathbb{X},p_j}(x_0)$ . Thus  $Q(R) \cong K(x_0) \otimes_K \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}$ , in particular,  $Q(R)$  is a free  $K(x_0)$ -module of rank  $\mathrm{deg}(\mathbb{X})$ . Let  $\nu_j = \dim_K \mathcal{O}_{\mathbb{X},p_j}$ , let  $\{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X},p_j}$  for  $j = 1, \dots, s$ , and set

$$\epsilon_{jk_j} = (0, \dots, 0, e_{jk_j}, 0, \dots, 0) \in \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}$$

for  $j = 1, \dots, s$  and for  $k_j = 1, \dots, \nu_j$ . Then the set  $\{\epsilon_{11}, \dots, \epsilon_{1\nu_1}, \dots, \epsilon_{s1}, \dots, \epsilon_{s\nu_s}\}$  is a  $K$ -basis of  $\prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}$ . Thus this set is also a  $K(x_0)$ -basis of  $Q(R)$ . By [Ku5, F.5 and F.6], the canonical trace map of  $Q(R)/K(x_0)$  is defined by

$$\mathrm{Tr}_{Q(R)/K(x_0)} : Q(R) \rightarrow K(x_0), \quad \mathrm{Tr}_{Q(R)/K(x_0)}(\epsilon_{jk_j}) = \mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}(e_{jk_j})$$

for all  $j \in \{1, \dots, s\}$  and  $k_j \in \{1, \dots, \nu_j\}$ . Consider the homomorphism of  $Q(R)$ -modules  $\Theta : Q(R) \rightarrow \mathrm{Hom}_{K(x_0)}(Q(R), K(x_0))$  given by  $\Theta(1) = \mathrm{Tr}_{Q(R)/K(x_0)}$ . For every  $j \in \{1, \dots, s\}$  we let  $\{e'_{j1}, \dots, e'_{j\nu_j}\}$  be the dual basis of  $\mathcal{O}_{\mathbb{X},p_j}$  to the  $K$ -basis  $\{e_{j1}, \dots, e_{j\nu_j}\}$  with respect to  $\mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}$ . Then  $\sum_{k_j=1}^{\nu_j} e_{jk_j} e'_{jk_j} = 1$  and

$$\mathrm{Tr}_{\mathcal{O}_{\mathbb{X},p_j}/K}(e_{jk_j} e'_{jk'_j}) = e_{jk'_j}^*(e_{jk_j}) = \delta_{k_j k'_j} \quad (k_j, k'_j = 1, \dots, \nu_j).$$

Let  $\varphi : Q(R) \rightarrow K(x_0)$  be a  $K(x_0)$ -linear mapping. For  $j \in \{1, \dots, s\}$  and for  $k_j \in \{1, \dots, \nu_j\}$ , we have  $\varphi(\epsilon_{jk_j}) = g_{jk_j}(x_0)/h_{jk_j}(x_0)$  for some  $g_{jk_j}(x_0), h_{jk_j}(x_0) \in K[x_0]$  and  $h_{jk_j}(x_0) \neq 0$ . Observe that

$$\begin{aligned} & \left( \sum_{k_1=1}^{\nu_1} e'_{1k_1} \frac{g_{1k_1}(T_1)}{h_{1k_1}(T_1)}, \dots, \sum_{k_s=1}^{\nu_s} e'_{sk_s} \frac{g_{sk_s}(T_s)}{h_{sk_s}(T_s)} \right) \cdot \mathrm{Tr}_{Q(R)/K(x_0)}(\epsilon_{jk'_j}) \\ &= \sum_{k_j=1}^{\nu_j} \frac{g_{jk_j}(x_0)}{h_{jk_j}(x_0)} \cdot \mathrm{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K}(e'_{jk_j} e_{jk'_j}) = \sum_{k_j=1}^{\nu_j} \frac{g_{jk_j}(x_0)}{h_{jk_j}(x_0)} \cdot \delta_{k_j k'_j} = \frac{g_{jk'_j}(x_0)}{h_{jk'_j}(x_0)}. \end{aligned}$$

It follows that

$$\varphi = \left( \sum_{k_1=1}^{\nu_1} e'_{1k_1} \frac{g_{1k_1}(T_1)}{h_{1k_1}(T_1)}, \dots, \sum_{k_s=1}^{\nu_s} e'_{sk_s} \frac{g_{sk_s}(T_s)}{h_{sk_s}(T_s)} \right) \cdot \mathrm{Tr}_{Q(R)/K(x_0)}$$

and that  $\Theta$  is surjective. Next we show that  $\Theta$  is injective. For  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ , we let  $g_{jk_j}, h_{jk_j} \in K[T_j]$  be polynomials such that  $h_{jk_j} \neq 0$  and

$$\psi = \left( \sum_{k_1=1}^{\nu_1} e'_{1k_1} \frac{g_{1k_1}}{h_{1k_1}}, \dots, \sum_{k_s=1}^{\nu_s} e'_{sk_s} \frac{g_{sk_s}}{h_{sk_s}} \right) \cdot \mathrm{Tr}_{Q(R)/K(x_0)} = 0.$$

Thus we have  $0 = \psi(\epsilon_{jk_j}) = g_{jk_j}(x_0)/h_{jk_j}(x_0)$  for  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . This implies  $(\sum_{k_1=1}^{\nu_1} e'_{1k_1} \frac{g_{1k_1}}{h_{1k_1}}, \dots, \sum_{k_s=1}^{\nu_s} e'_{sk_s} \frac{g_{sk_s}}{h_{sk_s}}) = \mathbf{0}$ , and hence  $\Theta$  is injective. Therefore the homomorphism  $\Theta$  is an isomorphism of  $Q(R)$ -modules, in particular,  $\mathrm{Tr}_{Q(R)/K(x_0)}$  is a trace map of  $Q(R)/K(x_0)$ .  $\square$

It is clear that every 0-dimensional smooth scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is locally Gorenstein. So, the Dedekind complement module  $\mathfrak{C}_{R/K[x_0]}$  and Dedekind different  $\vartheta_D(R/K[x_0])$  are well-defined for a 0-dimensional smooth scheme  $\mathbb{X}$ . Moreover, the Dedekind different and the Noether different of  $R/K[x_0]$  agree, as the following theorem shows.

**Theorem 3.2.17.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. Then we have*

$$\vartheta_D(R/K[x_0]) = \vartheta_N(R/K[x_0]).$$

*Proof.* Notice that the algebra  $R/K[x_0]$  is free of rank  $\deg(\mathbb{X})$ , so it is flat. Thus the equality  $\vartheta_N(R/K[x_0]) = \vartheta_D(R/K[x_0])$  follows from [Ku5, G.11] with a similar argument if the algebra  $L/L_0$  is étale, where  $L = Q^h(R)$  and  $L_0 = K[x_0, x_0^{-1}]$ . But this is proved in Proposition 3.2.16, and hence the conclusion follows.  $\square$

**Definition 3.2.18.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. We let

$$\vartheta_{\mathbb{X}}(R/K[x_0]) := \vartheta_N(R/K[x_0]) = \vartheta_D(R/K[x_0])$$

and call  $\vartheta_{\mathbb{X}}(R/K[x_0])$  the **Noether-Dedekind different** (in short, **ND-different**) of  $R/K[x_0]$  (or for  $\mathbb{X}$  with respect to  $x_0$ ).

**Remark 3.2.19.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme.

- (i) The ND-different satisfies  $\text{HP}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(z) = \deg(\mathbb{X})$  and  $\text{ri}(\vartheta_{\mathbb{X}}(R/K[x_0])) \leq 2r_{\mathbb{X}}$ .
- (ii) In order to compute  $\vartheta_{\mathbb{X}}(R/K[x_0])$ , one can use Proposition 3.1.6. Another efficient method for computing  $\vartheta_{\mathbb{X}}(R/K[x_0])$  will be provided in the next subsection.

**Corollary 3.2.20.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. Then the following assertions are equivalent.

- (i) The scheme  $\mathbb{X}$  is arithmetically Gorenstein.
- (ii) There exists an element  $h \in R_{r_{\mathbb{X}}} \setminus \{0\}$  such that  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle h \rangle_R$ .
- (iii) There exists an element  $h \in \vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}}$  with  $\text{Ann}_R(h) = \langle 0 \rangle$ .

If these assertions are satisfied, then  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for all  $i \in \mathbb{Z}$  and  $\text{ri}(\vartheta_{\mathbb{X}}(R/K[x_0])) = 2r_{\mathbb{X}}$ .

*Proof.* This is an immediate consequence of Proposition 3.2.8. □

**Example 3.2.21.** Let  $\mathbb{X} = \{p_1, \dots, p_5\} \subseteq \mathbb{P}_{\mathbb{F}_7}^3$  be the projective point set with  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 1 : 1)$ ,  $p_3 = (1 : -1 : 1 : -1)$ ,  $p_4 = (1 : 2 : 4 : 8)$ , and  $p_5 = (8 : 4 : 2 : 1)$ . Notice that  $\mathbb{X}$  consists of five points on the twisted cubic curve  $C = \{(u^3 : u^2v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}_{\mathbb{F}_7}^1\}$  in  $\mathbb{P}_{\mathbb{F}_7}^3$ . We have  $\text{HF}_{\mathbb{X}} : 1 \ 4 \ 5 \ 5 \dots$  and  $r_{\mathbb{X}} = 2$ . The Noether-Dedekind different  $\vartheta_{\mathbb{X}}(R/\mathbb{F}_7[x_0])$  of  $R/\mathbb{F}_7[x_0]$  is a principal ideal generated by  $h_1 = x_0^2 - 2x_0x_3 - 3x_1x_3 - 2x_3^2$ . Here we computed  $h_1$  by using Proposition 3.1.6. It is not hard to check that the value of  $h_1$  at each point of  $\mathbb{X}$  is not zero. Lemma 2.3.11 yields that  $h_1$  is a non-zero-divisor of  $R$ . So,  $\mathbb{X}$  is arithmetically Gorenstein and the Hilbert function of  $\vartheta_{\mathbb{X}}(R/\mathbb{F}_7[x_0])$  is  $\text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{F}_7[x_0])} : 0 \ 0 \ 1 \ 4 \ 5 \ 5 \dots$ . However,  $\mathbb{X}$  is not a complete intersection since the number of elements in a minimal homogeneous system of generators of the homogeneous vanishing ideal of  $\mathbb{X}$  is 5.

### 3.2.3 Computing the Dedekind Differents

In this subsection we consider the problem of computing a minimal homogeneous system of generators of  $\vartheta_D(R/K[x_0])$  for a 0-dimensional locally Gorenstein scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ . The good approach of this problem is to use the  $K$ -basis  $\{f_{11}, \dots, f_{s\nu_s}\}$  of the  $K$ -vector space  $R_{r_{\mathbb{X}}}$  which was explained in Subsection 3.2.1 for the computation. This approach is based on the GBM-Algorithm (see [AKR, Theorem 3.1]) and the description of the Dedekind complementary module given by Proposition 3.2.9.

As above, we let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . For  $j \in \{1, \dots, s\}$ , let  $I_j$  be the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$ , and let  $\mathfrak{q}_j = I_j^{\text{deh}} \subseteq A = K[X_1, \dots, X_n]$ . Then  $\mathcal{I}_{\mathbb{X}} = I_1 \cap \dots \cap I_s$  and the affine ideal of  $\mathbb{X}$  is given by  $\mathcal{I}_{\mathbb{X}}^a = \mathcal{I}_{\mathbb{X}}^{\text{deh}} = \bigcap_{j=1}^s \mathfrak{q}_j \subseteq A$ . Let  $\nu_j = \dim_K(\mathcal{O}_{\mathbb{X}, p_j})$ , let  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j} \cong A/\mathfrak{q}_j$ , and let  $\pi_j : A \rightarrow A/\mathfrak{q}_j$  be the canonical map for  $j = 1, \dots, s$ . Combining the canonical isomorphism  $A/\mathfrak{q}_j \rightarrow K^{\nu_j}$  with  $\pi_j$  we obtain a  $K$ -linear, surjective map  $\text{NFV}_{\mathcal{O}_j} : A \rightarrow K^{\nu_j}$  which sends every polynomial  $F \in A$  to the uniquely defined tuple  $(a_1, \dots, a_{\nu_j}) \in K^{\nu_j}$  such that  $\pi_j(F) = \sum_{k=1}^{\nu_j} a_k e_{jk}$ . The map  $\text{NFV}_{\mathcal{O}_j}$  is called a **normal form vector map** with respect to  $\mathcal{O}_j$ . Obviously, we have  $\mathfrak{q}_j = \text{Ker}(\text{NFV}_{\mathcal{O}_j})$ .

Now we make a small alteration of the GBM-Algorithm to compute a tuple  $\mathcal{S} = (s_1, \dots, s_m)$  of polynomials of  $A$  such that

$$\text{NFV}_{\mathcal{O}_1}(s_i) \oplus \dots \oplus \text{NFV}_{\mathcal{O}_s}(s_i) = (0, \dots, 0, \underset{[i]}{1}, 0, \dots, 0) \in K^m,$$

where  $m = \sum_{j=1}^s \nu_j$  is the degree of the scheme  $\mathbb{X}$ , as the following proposition shows.

**Proposition 3.2.22.** *In the situation as above, and let  $<_{\sigma}$  be a degree-compatible term ordering on  $\mathbb{T}^n$ . Consider the following sequence of instructions.*

- 1) Let  $\mathcal{G} = \emptyset$ ,  $\mathcal{O} = \emptyset$ ,  $\mathcal{S} = \emptyset$ ,  $L = \{1\}$ , and let  $\mathcal{M} = (m_{jk})$  be a matrix over  $K$  with  $m = \sum_{j=1}^s \nu_j$  columns and initially zero rows.
- 2) If  $L$  is empty, then row reduce  $\mathcal{M}$  to a diagonal matrix and mimic these row operations on the elements of  $\mathcal{S}$  (considered as a column vector). Next replace  $\mathcal{S}$  by  $\mathcal{M}^{-1}\mathcal{S}$  then return the triple  $(\mathcal{G}, \mathcal{O}, \mathcal{S})$  and stop. Otherwise choose the term  $T = \min_{\sigma}(L)$  and remove it from  $L$ .
- 3) Compute the vector  $\mathbf{v} = \text{NFV}_{\mathcal{O}_1}(T) \oplus \dots \oplus \text{NFV}_{\mathcal{O}_s}(T) \in K^m$ .
- 4) Reduce  $\mathbf{v}$  against the rows of  $\mathcal{M}$  to obtain

$$\mathbf{v}^* = \mathbf{v} - \sum_j a_j (m_{j1}, \dots, m_{jm}) \quad \text{with } a_j \in K.$$

- 5) If  $\mathbf{v}^* = \mathbf{0}$  then append the polynomial  $T - \sum_j a_j s_j$  to the tuple  $\mathcal{G}$ , where  $s_j$  is the  $j$ -th element of the tuple  $\mathcal{S}$ . Remove from  $L$  all multiples of  $T$ . Then continue with step 2).
- 6) Otherwise  $\mathbf{v}^* \neq \mathbf{0}$  so append  $\mathbf{v}^*$  as a new row to  $\mathcal{M}$ , and  $T - \sum_j a_j s_j$  as a new element to  $\mathcal{S}$ . Append the corresponding term  $T$  as a new element to  $\mathcal{O}$ . Add to  $L$  those elements of  $\{X_1 T, \dots, X_n T\}$  which are neither multiples of an element of  $L$  nor multiples of  $\{\text{LT}_{\sigma}(g) \mid g \in \mathcal{G}\}$ . Continue with step 2).

This is an algorithm which returns a triple  $(\mathcal{G}, \mathcal{O}, \mathcal{S})$  such that the following assertions are satisfied.

- (i)  $\mathcal{G}$  is the reduced  $<_\sigma$ -Gröbner basis of  $\mathcal{I}_{\mathbb{X}}^a = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ .
- (ii)  $\mathcal{O}$  is a tuple whose components are precisely the elements of  $\mathbb{T}^n \setminus \text{LT}_\sigma(\mathcal{I}_{\mathbb{X}}^a)$ .
- (iii)  $\mathcal{S} = (s_1, \dots, s_m)$  is a tuple of polynomials of  $A$  such that  $s_i$  has degree  $\deg(s_i) \leq r_{\mathbb{X}}$  and satisfies

$$\text{NFV}_{\mathcal{O}_1}(s_i) \oplus \cdots \oplus \text{NFV}_{\mathcal{O}_s}(s_i) = (0, \dots, 0, \underset{[i]}{1}, 0, \dots, 0) \in K^m.$$

*Proof.* By [AKR, Theorem 3.1], we need only prove the correctness of (iii). We observe that at each point in the procedure the matrix  $\mathcal{M}$  always has linearly independent rows, and the rows of  $\mathcal{M}$  are the evaluation vectors  $\text{NFV}_{\mathcal{O}_1}(s_i) \oplus \cdots \oplus \text{NFV}_{\mathcal{O}_s}(s_i)$  of the polynomials in  $\mathcal{S}$ . At the end of the procedure, i.e., when  $L$  is empty, the tuple  $\mathcal{O}$  contains  $m$  terms and the terms of  $\mathcal{O}$  are merely the leading terms of the corresponding elements in  $\mathcal{S}$ . Since  $<_\sigma$  is a degree compatible term ordering on  $\mathbb{T}^n$ , the degrees of elements in  $\mathcal{S}$  are smaller than or equal to  $r_{\mathbb{X}}$ . At this point we also see that  $\mathcal{M}$  is an invertible square matrix of size  $m \times m$ . Moreover, step 6) shows that  $\mathcal{M}$  is an upper triangular matrix after a permutation of rows. When we diagonalize  $\mathcal{M}$  and mimic these row operations on the elements of  $\mathcal{S}$ , we have a tuple  $\mathcal{S} = (s'_1, \dots, s'_m)$  such that

$$\text{NFV}_{\mathcal{O}_1}(s'_i) \oplus \cdots \oplus \text{NFV}_{\mathcal{O}_s}(s'_i) = (0, \dots, 0, \underset{[i]}{m_{ii}}, 0, \dots, 0) \in K^m.$$

Therefore, by replacing  $\mathcal{S}$  by  $\mathcal{M}^{-1}\mathcal{S}$ , we obtain a tuple with the desired property.  $\square$

Recall that the  $K$ -vector space  $R_{r_{\mathbb{X}}}$  has a  $K$ -basis  $\{f_{11}, \dots, f_{s\nu_s}\}$  with

$$f_{jk_j} = \tilde{\iota}^{-1}((0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$$

for  $k_j = 1, \dots, \nu_j$ , where  $\tilde{\iota} : R \rightarrow \tilde{R}$  is the injection given by (2.7). We can apply Proposition 3.2.22 to compute this basis as follows.

**Remark 3.2.23.** Suppose that  $\mathcal{S} = (s_1, \dots, s_m)$  is already computed by Proposition 3.2.22. Let  $G_{jk_j} = (s_{\sum_{k < j} \nu_k + k_j})^{\text{hom}} \in P$  and  $F_{jk_j} = X_0^{r_{\mathbb{X}} - \deg(G_{jk_j})} G_{jk_j}$  for  $j = 1, \dots, s$ ,  $k_j = 1, \dots, \nu_j$ . Since  $\pi_i(s_{\sum_{k < j} \nu_k + k_j}) = 0$  for  $i \neq j$  and  $\pi_j(s_{\sum_{k < j} \nu_k + k_j}) = e_{jk_j}$ , we have  $\tilde{\iota}(F_{jk_j} + \mathcal{I}_{\mathbb{X}}) = (0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0) \in \tilde{R}$  for  $j = 1, \dots, s$  and for  $k_j = 1, \dots, \nu_j$ . Thus the set  $\{F_{jk_j} + \mathcal{I}_{\mathbb{X}} \mid j = 1, \dots, s; k_j = 1, \dots, \nu_j\}$  is the  $K$ -basis of  $R_{r_{\mathbb{X}}}$  that we wanted to compute. Notice that when  $\mathbb{X}$  is reduced, the set  $\{F_{j1} + \mathcal{I}_{\mathbb{X}}, \dots, F_{j\nu_j} + \mathcal{I}_{\mathbb{X}}\}$  is a set of separators of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ .

According to the hypothesis, for  $j \in \{1, \dots, s\}$  the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  has a trace map  $\bar{\sigma}_j \in \text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K)$  with  $\text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K) = \mathcal{O}_{\mathbb{X}, p_j} \cdot \bar{\sigma}_j$ . Let  $\mathcal{O}'_j = \{e'_{j1}, \dots, e'_{j\nu_j}\}$  be the dual  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  to the  $K$ -basis  $\mathcal{O}_j$  with respect to  $\bar{\sigma}_j$ , and let  $\tilde{f}_{j1}, \dots, \tilde{f}_{j\nu_j}$  be homogeneous elements of  $R_{r_{\mathbb{X}}}$  such that

$$\tilde{f}_{jk_j} = \tilde{v}^{-1}((0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$$

for  $k_j = 1, \dots, \nu_j$ . We set  $\mathcal{A}_j = (a_{kl}^j)_{1 \leq k, l \leq \nu_j} = (\bar{\sigma}_j(e_{jk} e_{jl}))_{1 \leq k, l \leq \nu_j} \in \text{Mat}_{\nu_j \times \nu_j}(K)$  and  $\mathcal{E}_{jk} = (0 \ \dots \ 0 \ 1 \ 0 \ \dots \ 0)_{[k]}^{\text{tr}} \in \text{Mat}_{\nu_j \times 1}(K)$  for  $k = 1, \dots, \nu_j$ . In the case the matrix  $\mathcal{A}_j$  is computable, the  $K$ -basis  $\mathcal{O}'_j$  can be computed in terms of the  $K$ -basis  $\mathcal{O}_j$  by our next lemma.

**Lemma 3.2.24.** *The following assertions hold.*

- (i) *The matrix  $\mathcal{A}_j = (a_{kl}^j)_{1 \leq k, l \leq \nu_j}$  is invertible, i.e.,  $\det(\mathcal{A}_j) \neq 0$ .*
- (ii) *Let  $\mathcal{X} = (x_1, \dots, x_{\nu_j})^{\text{tr}}$ , where  $x_1, \dots, x_{\nu_j}$  are indeterminates. For  $k = 1, \dots, \nu_j$ , the system of linear equations  $\mathcal{A}_j \cdot \mathcal{X} = \mathcal{E}_{jk}$  has exactly one solution  $(\alpha_{k1}, \dots, \alpha_{k\nu_j})$  and  $e'_{jk} = \alpha_{k1} e_{j1} + \dots + \alpha_{k\nu_j} e_{j\nu_j}$ .*

*Proof.* Let  $\Phi : \mathcal{O}_{\mathbb{X}, p_j} \times \mathcal{O}_{\mathbb{X}, p_j} \rightarrow K$  be the bilinear form on the vector space  $\mathcal{O}_{\mathbb{X}, p_j}$  over  $K$  defined by  $\Phi(a, b) = \bar{\sigma}_j(ab)$  for all  $a, b \in \mathcal{O}_{\mathbb{X}, p_j}$ . Then  $\mathcal{A}_j$  is the associated matrix of  $\Phi$ , and [Bo1, Chapter 5, § 8, Lemma 1] implies that  $\det(\mathcal{A}_j) \neq 0$  if and only if  $\Phi$  is non-degenerate, i.e., for every  $a \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$  there exists  $b$  in  $\mathcal{O}_{\mathbb{X}, p_j}$  such that  $\Phi(a, b) \neq 0$ . Since  $\bar{\sigma}_j$  is a trace map of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$ , we have  $\text{Ann}_{\mathcal{O}_{\mathbb{X}, p_j}}(\bar{\sigma}_j) = \langle 0 \rangle$ . Thus, for every  $a \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ , we have  $a \cdot \bar{\sigma}_j \neq 0$ . In other words, there is an element  $b \in \mathcal{O}_{\mathbb{X}, p_j}$  such that  $(a \cdot \bar{\sigma}_j)(b) = \bar{\sigma}_j(ab) \neq 0$ . This shows that  $\Phi$  is non-degenerate, and hence claim (i) follows.

Now we prove (ii). We write  $e'_{jk} = \beta_{k1} e_{j1} + \dots + \beta_{k\nu_j} e_{j\nu_j}$  with  $\beta_{k1}, \dots, \beta_{k\nu_j} \in K$ . For every  $l \in \{1, \dots, \nu_j\}$ , we have  $\delta_{kl} = e_{jk}^*(e_{jl}) = (e'_{jk} \cdot \bar{\sigma}_j)(e_{jl}) = \bar{\sigma}_j(e_{jl} e'_{jk}) = \sum_{k_j=1}^{\nu_j} \beta_{kk_j} \bar{\sigma}_j(e_{jl} e_{jk_j}) = \sum_{k_j=1}^{\nu_j} \beta_{kk_j} a_{lk_j}^j$ . It follows that the tuple  $(\beta_{k1}, \dots, \beta_{k\nu_j})$  is a solution of the system of linear equations  $\mathcal{A}_j \cdot \mathcal{X} = \mathcal{E}_{jk}$ . But by (i) the system of linear equations  $\mathcal{A}_j \cdot \mathcal{X} = \mathcal{E}_{jk}$  has exactly one solution  $(\alpha_{k1}, \dots, \alpha_{k\nu_j})$ . Therefore we obtain  $(\beta_{k1}, \dots, \beta_{k\nu_j}) = (\alpha_{k1}, \dots, \alpha_{k\nu_j})$ , as desired.  $\square$

In order to apply Proposition 3.2.9 for computing the Dedekind complementary module, it is necessary to compute the set  $\{\tilde{f}_{11}, \dots, \tilde{f}_{1\nu_1}, \dots, \tilde{f}_{s1}, \dots, \tilde{f}_{s\nu_s}\}$ . For this, we present the following algorithm which is based on Proposition 3.2.22 and Lemma 3.2.24.



**Proposition 3.2.25.** *Let  $\mathfrak{q}_j \subseteq A = K[X_1, \dots, X_n]$  be the affine ideal of  $\mathbb{X}$  at  $p_j$ , let  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j} \cong A/\mathfrak{q}_j$ . Suppose that the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  has a computable trace map  $\bar{\sigma}_j$  for  $j = 1, \dots, s$ . Let  $m = \deg(\mathbb{X})$ , and let  $<_\sigma$  be a degree compatible term ordering on  $\mathbb{T}^n$ . Consider the following sequence of instructions.*

- 1) For  $j = 1, \dots, s$  and for  $k = 1, \dots, \nu_j$ , form matrices  $\mathcal{A}_j = (\bar{\sigma}_j(e_{jk}e_{jl}))_{1 \leq k, l \leq \nu_j} \in \text{Mat}_{\nu_j \times \nu_j}(K)$  and  $\mathcal{E}_{jk} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^{\text{tr}} \in \text{Mat}_{\nu_j \times 1}(K)$ .
- 2) Compute the dual  $K$ -basis  $\mathcal{O}'_j = \{e'_{j1}, \dots, e'_{j\nu_j}\}$  of  $\mathcal{O}_{\mathbb{X}, p_j}$  to  $\mathcal{O}_j$  with respect to  $\bar{\sigma}_j$  for  $j \in \{1, \dots, s\}$ . Here  $e'_{jk} = \alpha_{k1}e_{j1} + \cdots + \alpha_{k\nu_j}e_{j\nu_j}$  where  $(\alpha_{k1}, \dots, \alpha_{k\nu_j})$  is the solution of the system of linear equations  $\mathcal{A}_j \mathcal{X} = \mathcal{E}_{jk}$  (see Lemma 3.2.24).
- 3) Compute a tuple  $\mathcal{S}' = (s'_1, \dots, s'_m)$  such that  $s'_i$  has degree  $\deg(s'_i) \leq r_{\mathbb{X}}$  and satisfies

$$\text{NFV}_{\mathcal{O}'_1}(s'_1) \oplus \cdots \oplus \text{NFV}_{\mathcal{O}'_s}(s'_s) = (0, \dots, 0, 1, 0, \dots, 0) \in K^m$$

by using Proposition 3.2.22.

- 4) Compute  $G_{jk_j} = (s'_{\sum_{k < j} \nu_k + k_j})^{\text{hom}} \in P$  and  $\tilde{F}_{jk_j} = X_0^{r_{\mathbb{X}} - \deg(G_{jk_j})} G_{jk_j}$  for  $j = 1, \dots, s$ ;  $k_j = 1, \dots, \nu_j$ . Return the tuple  $\tilde{\mathcal{S}} = ((\tilde{F}_{jk_j} \mid k_j = 1, \dots, \nu_j) \mid j = 1, \dots, s)$  and stop.

This is an algorithm which computes a tuple  $\tilde{\mathcal{S}} = (\tilde{\mathcal{S}}_1, \dots, \tilde{\mathcal{S}}_s)$  such that the set of all images of  $\tilde{\mathcal{S}}_j$  in  $R$  is  $\{\tilde{f}_{j1}, \dots, \tilde{f}_{j\nu_j}\}$  with  $\tilde{f}_{jk_j} = \tilde{v}^{-1}((0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$  for  $k_j = 1, \dots, \nu_j$ .

*Proof.* By assumption, we compute  $b_{jk_j} = \bar{\sigma}_j(e_{jk_j})$  and present  $e_{jk}e_{jl} = c_{j1}^{kl}e_{j1} + \cdots + c_{j\nu_j}^{kl}e_{j\nu_j}$  for some  $c_{jk_j}^{kl} \in K$  and for  $k, l, k_j = 1, \dots, \nu_j$ . Then we have

$$\alpha_{kl}^j = \bar{\sigma}_j(e_{jk}e_{jl}) = \sum_{k_j=1}^{\nu_j} c_{jk_j}^{kl} \bar{\sigma}_j(e_{jk_j}) = \sum_{k_j=1}^{\nu_j} c_{jk_j}^{kl} b_{jk_j}$$

for  $k, l \in \{1, \dots, \nu_j\}$ . This shows us how to form the matrix  $\mathcal{A}_j = (a_{kl}^j)_{1 \leq k, l \leq \nu_j}$  for  $j = 1, \dots, s$  in step 1).

The correctness of this algorithm follows from the observation that for  $j \in \{1, \dots, s\}$  and  $k_j \in \{1, \dots, \nu_j\}$  we have  $\tilde{v}(\tilde{F}_{jk_j} + \mathcal{I}_{\mathbb{X}}) = \tilde{v}(\tilde{f}_{jk_j}) = (0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)$ , since  $\pi_i(s'_{\sum_{k < j} \nu_k + k_j}) = 0$  for  $i \neq j$  and  $\pi_j(s'_{\sum_{k < j} \nu_k + k_j}) = e'_{jk_j}$ , where  $\pi_i : A \rightarrow A/\mathfrak{q}_i$  is the canonical projection for  $i = 1, \dots, s$ . Additionally, the finiteness of this algorithm is clearly true.  $\square$

**Remark 3.2.26.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme. We would like to make some following comments about the existence of a computable trace map of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  for  $j = 1, \dots, s$ .

- (a) If  $\mathbb{X}$  has  $K$ -rational support, for every  $j \in \{1, \dots, s\}$  let  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  and assume  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) = K \cdot h_j$ . Then an element  $\tau \in \text{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K)$  is a trace map of  $\mathcal{O}_{\mathbb{X}, p_j}/K$  if and only if  $\tau(h_j) \neq 0$  (see [Ku8, Lemma 8.7]). Now we write  $h_j = a_1 e_{j1} + \dots + a_k e_{jk}$  with  $a_1, \dots, a_k \in K$  and  $a_k \neq 0$ . Then the trace map  $\bar{\sigma}_j$  can be chosen to be the map  $e_{jk}^* : \mathcal{O}_{\mathbb{X}, p_j} \rightarrow K$  defined by  $e_{jk}^*(e_{jk_j}) = \delta_{kk_j}$  for  $k_j = 1, \dots, \nu_j$ .
- (b) When  $\mathbb{X}$  is smooth, the trace map  $\bar{\sigma}_j$  of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  can be chosen to be the canonical trace map  $\text{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K}$  for  $j = 1, \dots, s$ . In this case, we let  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  and write  $e_{jk} e_{jl} = c_{j1}^{kl} e_{j1} + \dots + c_{j\nu_j}^{kl} e_{j\nu_j}$  and  $\mathcal{C}_{kl} = \begin{pmatrix} c_{j1}^{kl} & \dots & c_{j\nu_j}^{kl} \end{pmatrix} \in \text{Mat}_{1 \times \nu_j}(K)$  with  $c_{jk_j}^{kl} \in K$  for  $k, l, k_j = 1, \dots, \nu_j$ . Then the values of the canonical trace map is computed by

$$\text{Tr}_{\mathcal{O}_{\mathbb{X}, p_j}/K}(e_{jk}) = \text{Tr} \begin{pmatrix} \mathcal{C}_{k1} \\ \vdots \\ \mathcal{C}_{k\nu_j} \end{pmatrix} = c_{j1}^{k1} c_{j2}^{k2} \dots c_{j\nu_j}^{k\nu_j}$$

for  $k = 1, \dots, \nu_j$ .

- (c) Suppose that for every  $j \in \{1, \dots, s\}$  we have  $\mathfrak{q}_j = \langle G_{j1}, \dots, G_{jn} \rangle$ , where

$$G_{jk} = X_k^{m_{jk}} + H_{jk, m_{jk}-1}(X_1, \dots, X_{k-1}) X_k^{m_{jk}-1} + \dots + H_{jk, 0}(X_1, \dots, X_{k-1})$$

are polynomials in  $X_1, \dots, X_k$  monic in  $X_k$  for  $k = 1, \dots, n$ . Then

$$\mathcal{O}_{\mathbb{X}, p_j} = A/\mathfrak{q}_j = \bigoplus_{0 \leq \alpha_k < m_{jk}} K x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

is a complete intersection over  $K$ . According to [Ku8, Example 8.16], the trace map  $\bar{\sigma}_j$  of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  can be chosen to be the map

$$\tau_j(x_1^{\alpha_1} \dots x_n^{\alpha_n}) = \begin{cases} 1 & \text{for } \alpha_1 = m_{j1} - 1, \dots, \alpha_n = m_{jn} - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, this can be applied if the scheme  $\mathbb{X}$  is reduced but not smooth as follows. For every  $j \in \{1, \dots, s\}$ , let  $L_j$  be the separable closure of  $K$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ .

By Primitive Element Theorem (see for instance [Mor, Theorem 5.6]), there is  $\beta_{j1} \in L_j$  such that  $L_j = K(\beta_{j1})$ . Let  $G_{j1} \in K[X_1]$  be the minimal polynomial of  $\beta_{j1}$ . By [Kar, Proposition 3.21], the field extension  $\mathcal{O}_{\mathbb{X}, p_j}/L_j$  is purely inseparable. We may write  $\mathcal{O}_{\mathbb{X}, p_j} = L_j(\beta_{j2}, \dots, \beta_{jd}) = K(\beta_{j1}, \dots, \beta_{jd})$  for some  $d \geq 2$ . Let  $q = \text{char}(K) > 0$ . The minimal polynomial of  $\beta_{jk}$  over  $K$  is of the form  $G_{jk} = (X_k - \beta_{jk})^{q^{m_k}}$  for  $2 \leq k \leq d$  and for some  $m_k \geq 1$ . Thus  $\mathcal{O}_{\mathbb{X}, p_j} = K[X_1, \dots, X_d]/\langle G_{j1}, \dots, G_{jd} \rangle$  is a complete intersection over  $K$  of the above form.

Now let  $<_\sigma$  be a degree-compatible term ordering on  $\mathbb{T}^n$ , let  $m = \text{deg}(\mathbb{X})$ , and let  $\mathcal{O} = \mathbb{T}^n \setminus \text{LT}_\sigma(\mathcal{I}_{\mathbb{X}}^a) = \{T'_1, \dots, T'_m\}$  with  $T'_i = X_1^{\alpha_{i1}} \dots X_n^{\alpha_{in}}$  and  $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in}) \in \mathbb{N}^n$  for  $i = 1, \dots, m$ . Let  $t_i := T'_i + \mathcal{I}_{\mathbb{X}} \in R$  and set  $\text{deg}(t_i) := \text{deg}(T'_i) = n_i$  for  $i = 1, \dots, m$ . We assume that  $T'_1 <_\sigma \dots <_\sigma T'_m$ , it follows that the degrees  $n_i$  satisfy  $n_1 \leq \dots \leq n_m$ . Then the set  $\mathcal{B} = \{t_1, \dots, t_m\}$  is a  $K[x_0]$ -basis of  $R$  and  $R_i = \bigoplus_{k=1}^m K[x_0]_{i-n_k} t_k$  for all  $i \in \mathbb{Z}$  (a direct sum of  $K$ -vector spaces). Let  $\mathcal{B}^* = \{t_1^*, \dots, t_m^*\}$  be the dual basis of  $\mathcal{B}$ . It is clear that  $t_i^* \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is homogeneous of degree  $\text{deg}(t_i^*) = -n_i$  for  $i = 1, \dots, m$ , and  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]) = \langle t_1^*, \dots, t_m^* \rangle_{K[x_0]}$ .

Recall that  $\{f_{11}, \dots, f_{s\nu_s}\}$  and  $\{\tilde{f}_{11}, \dots, \tilde{f}_{s\nu_s}\}$  are the  $K$ -bases of  $R_{r_{\mathbb{X}}}$  such that  $\tilde{v}(f_{jk_j}) = (0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)$  and  $\tilde{v}(\tilde{f}_{jk_j}) = (0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)$ , where  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  is a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ , and  $\mathcal{O}'_j = \{e'_{j1}, \dots, e'_{j\nu_j}\}$  is the dual  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  to the  $K$ -basis  $\mathcal{O}_j$  with respect to the trace map  $\bar{\sigma}_j$ , for  $j = 1, \dots, s$  and for  $k_j = 1, \dots, \nu_j$ . Using this notation, we get the following property.

**Proposition 3.2.27.** (i) *If we write  $t_i^*(f_{jk_j}) = c_{jk_j}^i x_0^{r_{\mathbb{X}} - n_i}$  for some  $c_{jk_j}^i \in K$ , then  $\mathfrak{C}_{R/K[x_0]} = \langle g_1, \dots, g_m \rangle_{K[x_0]}$ , where*

$$g_i = x_0^{-n_i - r_{\mathbb{X}}} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^i \tilde{f}_{jk_j} \right) \in R_{x_0} \quad (3.3)$$

for  $j = 1, \dots, m$ .

(ii) *Let  $h_i = x_0^{2r_{\mathbb{X}}} g_i \in R$  for  $i = 1, \dots, m$ . We have*

$$\vartheta_D(R/K[x_0]) = \bigcap_{i=1}^m (R :_R \langle g_i \rangle_R) = \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle h_1, \dots, h_m \rangle_R.$$

*Proof.* Let  $g_i = \Phi(t_i^*) \in (\mathfrak{C}_{R/K[x_0]})_{-n_i}$  for  $i = 1, \dots, m$ . Proposition 3.2.9 enables us to write  $g_i = x_0^{-n_i - r_{\mathbb{X}}} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^i \tilde{f}_{jk_j} \right) \in R_{x_0} \cong L$ , and hence claim (i) follows.

Now we prove (ii). We have  $\vartheta_D(R/K[x_0]) = R :_{R_{x_0}} \mathfrak{C}_{R/K[x_0]} \subseteq R$ . This implies

$$\vartheta_D(R/K[x_0]) = R :_R \langle g_1, \dots, g_m \rangle_R = \bigcap_{i=1}^m (R :_R \langle g_i \rangle_R).$$

Since  $x_0$  is a non-zero-divisor of  $R$  and  $x_0^{2r_{\mathbb{X}}} \mathfrak{C}_{R/K[x_0]} \subseteq R$ , we get

$$\begin{aligned} R :_R \langle g_i \rangle_R &= \{ f \in R \mid fg_i \in R \} = \{ f \in R \mid fx_0^{-n_i - r_{\mathbb{X}}} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^i \tilde{f}_{jk_j} \right) \in R \} \\ &= \{ f \in R \mid fx_0^{r_{\mathbb{X}} - n_i} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^i \tilde{f}_{jk_j} \right) \in x_0^{2r_{\mathbb{X}}} R \} \\ &= \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle x_0^{2r_{\mathbb{X}}} g_i \rangle_R = \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle h_i \rangle_R. \end{aligned}$$

Therefore we obtain

$$\vartheta_D(R/K[x_0]) = \bigcap_{i=1}^m (R :_R \langle g_i \rangle_R) = \bigcap_{i=1}^m \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle h_i \rangle_R = \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle h_1, \dots, h_m \rangle_R$$

as we wanted to show.  $\square$

Next we set  $\text{soc}(\mathcal{O}) := \{ T'_i \in \mathcal{O} \mid X_j T'_i \notin \mathcal{O} \text{ for all } j = 1, \dots, n \}$ . The elements of the set  $\text{soc}(\mathcal{O})$  are called the **socle monomials** of  $\mathbb{X}$ . Let  $q \geq 1$  and  $i_1, \dots, i_q \in \{1, \dots, m\}$  such that  $\text{soc}(\mathcal{O}) = \{T'_{i_1}, \dots, T'_{i_q}\}$ . [BK, Proposition 4.4] tells us that the set  $\{t_{i_1}^*, \dots, t_{i_q}^*\}$  is a minimal Gröbner basis of the  $R$ -module  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  with respect to a suitable filtration. Therefore we get  $\mathfrak{C}_{R/K[x_0]} = \langle g_{i_1}, \dots, g_{i_q} \rangle_R \subset R_{x_0}$ , where  $g_{i_l} = x_0^{-n_{i_l} - r_{\mathbb{X}}} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^{i_l} \tilde{f}_{jk_j} \right)$  for all  $l = 1, \dots, q$ . As a consequence of Proposition 3.2.27, we immediately have the following corollary.

**Corollary 3.2.28.** (i) Let  $h_l = x_0^{2r_{\mathbb{X}}} g_{i_l} \in R$  for  $l = 1, \dots, q$ . We have

$$\vartheta_D(R/K[x_0]) = \bigcap_{l=1}^q (R :_R \langle g_{i_l} \rangle_R) = \langle x_0^{2r_{\mathbb{X}}} \rangle_R :_R \langle h_1, \dots, h_q \rangle_R.$$

(ii) Let  $H_i$  be a representative of  $h_i$  in  $P$  for  $i = 1, \dots, q$ . We have

$$\vartheta_D(R/K[x_0]) = ((\langle X_0^{2r_{\mathbb{X}}} \rangle + \mathcal{I}_{\mathbb{X}}) :_P \langle H_1, \dots, H_q \rangle) / \mathcal{I}_{\mathbb{X}}.$$

Our next proposition shows how we can compute minimal homogeneous systems of generators of the Dedekind differents for 0-dimensional locally Gorenstein schemes  $\mathbb{X} \subseteq \mathbb{P}_K^n$  when the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  has a computable trace map  $\bar{\sigma}_j$  for  $j = 1, \dots, s$ .

**Proposition 3.2.29. (Computation of Dedekind Differents)** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , let  $m = \deg(\mathbb{X})$ , and let  $I_j \subseteq P = K[X_0, X_1, \dots, X_n]$  be the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$  for  $j = 1, \dots, s$ . Let  $<_{\sigma}$  be a degree compatible term ordering on  $\mathbb{T}^n$ , and let  $<_{\bar{\sigma}}$  be the extension of  $<_{\sigma}$  on  $\mathbb{T}^{n+1}$  (cf. [KR3, Definition 4.3.13]). Suppose that the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$  has a computable trace map  $\bar{\sigma}_j$  for  $j = 1, \dots, s$ . Consider the following sequence of instructions.

- 1) For every  $j \in \{1, \dots, s\}$ , we form the affine ideal  $\mathfrak{q}_j \subseteq A = K[X_1, \dots, X_n]$  of  $\mathbb{X}$  at  $p_j$  by taking the dehomogenization of  $I_j$  and compute a  $K$ -basis  $\mathcal{O}_j = \{e_{j1}, \dots, e_{j\nu_j}\}$  of  $\mathcal{O}_{\mathbb{X}, p_j} \cong A/\mathfrak{q}_j$  for  $j = 1, \dots, s$ .
- 2) Compute the reduced  $<_{\sigma}$ -Gröbner basis  $\{G_1, \dots, G_r\}$  of  $\mathcal{I}_{\mathbb{X}}^a$  and a  $K$ -basis  $\mathcal{O} = \{T'_1, \dots, T'_m\}$  of the affine  $K$ -algebra  $A/\mathcal{I}_{\mathbb{X}}^a$  and a tuple  $\mathcal{S} = (s_1, \dots, s_m)$  with  $\text{NFV}_{\mathcal{O}_1}(s_i) \oplus \dots \oplus \text{NFV}_{\mathcal{O}_s}(s_i) = (0, \dots, \underset{[i]}{1}, \dots, 0) \in K^m$  for  $j = 1, \dots, m$  by using Proposition 3.2.22.
- 3) Compute the set of socle monomials  $\text{soc}(\mathcal{O}) = \{T'_{i_1}, \dots, T'_{i_q}\} \subseteq \mathcal{O}$ .
- 4) Compute the reduced  $<_{\bar{\sigma}}$ -Gröbner basis  $\{F_1, \dots, F_r\}$  of  $\mathcal{I}_{\mathbb{X}}$  by taking the homogenization  $F_i = G_i^{\text{hom}}$  of  $G_i$  with respect to  $X_0$  for  $i = 1, \dots, r$  (see Remark 2.3.5).
- 5) Use Proposition 3.2.25 to compute a tuple  $\tilde{\mathcal{S}} = (\tilde{\mathcal{S}}_1, \dots, \tilde{\mathcal{S}}_s)$  such that the subtuple  $\tilde{\mathcal{S}}_j = (\tilde{f}_{j1}, \dots, \tilde{f}_{j\nu_j})$  consists of elements of  $R_{r_{\mathbb{X}}}$  with

$$\tilde{f}_{jk_j} = \tilde{v}^{-1}((0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$$

for  $k_j = 1, \dots, \nu_j$ , where the set  $\mathcal{O}'_j = \{e'_{j1}, \dots, e'_{j\nu_j}\}$  is the dual  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  to  $\mathcal{O}_j$  with respect to  $\bar{\sigma}_j$  for  $j = 1, \dots, s$ .

- 6) Compute the homogeneous generating system  $\{g_{i_1}, \dots, g_{i_q}\}$  of the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$  corresponding to  $\text{soc}(\mathcal{O})$ . Form the set  $\{h_1, \dots, h_q\}$  with  $h_l = x_0^{2r_{\mathbb{X}}} g_{i_l} \in R$  for  $l = 1, \dots, q$ .
- 7) Form  $J_1 = \langle X_0^{2r_{\mathbb{X}}}, F_1, \dots, F_r \rangle$  and  $J_2 = \langle H_1, \dots, H_q, F_1, \dots, F_r \rangle$  two homogeneous ideals of  $P$ , where  $H_k$  is a representative of  $h_k$  in  $P$  for  $k = 1, \dots, q$ . Compute the reduced  $<_{\bar{\sigma}}$ -Gröbner basis  $\{H'_1, \dots, H'_u\}$  ( $u \geq 1$ ) of the colon ideal  $J_1 :_P J_2$ . Sort the set  $\{H'_1, \dots, H'_u\}$  such that  $\deg(H'_1) \leq \dots \leq \deg(H'_u)$ .
- 8) Set  $\mathcal{H} = \{H'_1, \dots, H'_u\} \setminus \{F_1, \dots, F_r\}$ . For every polynomial  $H \in \mathcal{H}$ , compute its normal polynomial  $H' = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(H)$ . If  $H' \neq 0$ , replace  $H$  by  $H'$ . Otherwise, delete  $H$  from  $\mathcal{H}$ .
- 9) Apply Buchberger's Algorithm with Minimalization 3.1.10 to compute a tuple  $\mathcal{H}_{\min}$  which generates  $(J_1 :_P J_2)/\mathcal{I}_{\mathbb{X}}$  minimally.
- 10) Return the tuple  $\mathcal{H}_{\min}$  and stop.

This is an algorithm which returns a tuple  $\mathcal{H}_{\min}$  whose elements are a minimal homogeneous system of generators of the Dedekind different  $\vartheta_D(R/K[x_0])$ .

*Proof.* The finiteness of this procedure is obviously true. The correctness of this procedure follows by combining Proposition 3.1.10 and Corollary 3.2.28. We explain how to compute the homogeneous generating system  $\{g_{i_1}, \dots, g_{i_q}\}$  of the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$  in step 6). After step 2) we computed  $\mathcal{S} = (s_1, \dots, s_m)$  with  $\text{NFV}_{\mathcal{O}_1}(s_i) \oplus \dots \oplus \text{NFV}_{\mathcal{O}_s}(s_i) = (0, \dots, 1, \dots, 0) \in K^m$  for  $i = 1, \dots, m$ . For  $i \in \{1, \dots, m\}$ , we see that all terms of  $s_i$  belong to  $\mathcal{O} = \{T'_1, \dots, T'_m\}$ . This enables us to write  $s_i = s_{\sum_{k < j} \nu_k + k_j} = c_{jk_j}^1 T'_1 + \dots + c_{jk_j}^m T'_m$  with  $i = \sum_{k < j} \nu_k + k_j$  and  $1 \leq k_j \leq \nu_j$ . We then form a matrix  $\mathcal{C} = (c_{jk_j}^i) \in \text{Mat}_{m \times m}(K)$ . Now we assume that the algorithm has already executed step 5). For every  $j \in \{1, \dots, s\}$ , let  $\tilde{\mathcal{S}}_j = (\tilde{f}_{j1}, \dots, \tilde{f}_{j\nu_j})$  be the tuple of elements of  $R_{r_{\mathbb{X}}}$  with  $\tilde{f}_{jk_j} = \tilde{v}^{-1}((0, \dots, 0, e'_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$  for  $k_j = 1, \dots, \nu_j$ .

For every  $l \in \{1, \dots, q\}$ , we claim that

$$h_l = x_0^{2r_{\mathbb{X}}} g_{i_l} = x_0^{r_{\mathbb{X}} - n_{i_l}} \left( \sum_{j=1}^s \sum_{k_j=1}^{\nu_j} c_{jk_j}^{i_l} \tilde{f}_{jk_j} \right) \in R.$$

Indeed, for  $j \in \{1, \dots, s\}$  and for  $k_j \in \{1, \dots, \nu_j\}$ , let  $F_{jk_j} \in P_{r_{\mathbb{X}}}$  be the homogeneous polynomial such that  $F_{jk_j} = X_0^{r_{\mathbb{X}} - \text{deg}(s_{\sum_{k < j} \nu_k + k_j})} (s_{\sum_{k < j} \nu_k + k_j})^{\text{hom}}$ . Set  $f_{jk_j} := F_{jk_j} + \mathcal{I}_{\mathbb{X}} \in R_{r_{\mathbb{X}}}$  for all  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . Then

$$\tilde{v}(f_{jk_j}) = \tilde{v}(F_{jk_j} + \mathcal{I}_{\mathbb{X}}) = (0, \dots, 0, e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0) \in \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$$

for  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . In  $R$ , we get  $f_{jk_j} = c_{jk_j}^1 x_0^{r_{\mathbb{X}} - n_1} t_1 + \dots + c_{jk_j}^m x_0^{r_{\mathbb{X}} - n_m} t_m$  for  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$ . Hence we have  $t_{i_l}^*(f_{jk_j}) = t_{i_l}^*(c_{jk_j}^1 x_0^{r_{\mathbb{X}} - n_1} t_1 + \dots + c_{jk_j}^m x_0^{r_{\mathbb{X}} - n_m} t_m) = c_{jk_j}^{i_l} x_0^{r_{\mathbb{X}} - n_{i_l}}$  for  $j = 1, \dots, s$  and  $k_j = 1, \dots, \nu_j$  and  $l = 1, \dots, q$ . Thus the equalities for  $h_l$  and  $g_{i_l}$  as above follow.  $\square$

**Remark 3.2.30.** In order to compute the Hilbert function of the Dedekind different  $\vartheta_D(R/K[x_0])$  for a 0-dimensional locally Gorenstein scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  we perform the following steps.

- 1) Compute a homogeneous generating system  $\{h_1, \dots, h_t\}$  of  $\vartheta_D(R/K[x_0])$  by using Proposition 3.2.29.
- 2) Form the homogeneous ideal  $J = \mathcal{I}_{\mathbb{X}} + \langle H_1, \dots, H_t \rangle$  of  $P$ , where  $H_i$  is the normal representative of  $h_i$  in  $P$  for  $i = 1, \dots, t$ .
- 3) Compute the regularity index  $r_{\mathbb{X}}$  of  $\text{HF}_{\mathbb{X}}$  and then return the Hilbert function of Dedekind different  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i) - \text{HF}_{P/J}(i)$  for all  $i \in \mathbb{Z}$ . Notice

that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < 0$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \deg(\mathbb{X})$  for  $i \geq 2r_{\mathbb{X}}$  (see Proposition 3.2.5).

We end this section with an application of Proposition 3.2.29 to a concrete case.

**Example 3.2.31.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the reduced 0-dimensional scheme with support  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_6\}$ , where  $p_1 = (1 : 1 : 0 : 0)$ ,  $p_2 = (1 : 0 : 1 : 0)$ ,  $p_3 = (1 : 0 : 0 : 1)$ ,  $p_4 = (1 : 0 : 1 : 1)$ ,  $p_5$  corresponds to  $\mathfrak{P}_5 = \langle 3X_0^2 + X_1^2, X_2, X_3 \rangle$ , and  $p_6$  corresponds to  $\mathfrak{P}_6 = \langle X_1, 2X_0^3 + X_2^3, X_3 \rangle$ . Let  $<_{\sigma}$  be the term ordering DegRevLex on  $\mathbb{T}^4$ . We have  $\mathrm{HF}_{\mathbb{X}} : 1 \ 4 \ 7 \ 9 \ 9 \ \dots$  and  $r_{\mathbb{X}} = 3$ . By using Proposition 3.2.29, a minimal homogeneous system of generators of the Dedekind different  $\vartheta_D(R/\mathbb{Q}[x_0])$  is given by

$$\begin{aligned} h_1 &= x_2x_3^2 - \frac{1}{2}x_3^3, \\ h_2 &= x_0^2x_1^2 - \frac{2}{3}x_0x_1^3 + x_1^4, \\ h_3 &= x_0^3x_2^2 - \frac{3}{2}x_0x_2^4 + 2x_2^5, \\ h_4 &= x_1^5 - \frac{27}{16}x_0^2x_2^3 + \frac{27}{8}x_0x_2^4 - \frac{27}{16}x_2^5, \\ h_5 &= x_2^6. \end{aligned}$$

The Hilbert function of  $\vartheta_D(R/\mathbb{Q}[x_0])$  is  $\mathrm{HF}_{\vartheta_D(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 1 \ 3 \ 6 \ 9 \ 9 \ \dots$ , and its regularity index is  $\mathrm{ri}(\vartheta_D(R/\mathbb{Q}[x_0])) = 2r_{\mathbb{X}} = 6$ .

### 3.3 Kähler Differents for 0-Dimensional Schemes

The main goal of this section is to describe the module structure and Hilbert function of the Kähler different for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over an arbitrary field  $K$  with  $\mathrm{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ .

In the enveloping algebra  $R \otimes_{K[x_0]} R$  we have the homogeneous ideal  $\mathcal{J} = \mathrm{Ker}(\mu)$ , where  $\mu : R \otimes_{K[x_0]} R \rightarrow R$  is the homogeneous  $R$ -linear map given by  $\mu(f \otimes g) = fg$ . The **module of Kähler differentials** of the algebra  $R/K[x_0]$  is defined to be the finitely generated graded  $R$ -module  $\Omega_{R/K[x_0]}^1 = \mathcal{J}/\mathcal{J}^2$ . The homogeneous  $K[x_0]$ -linear map  $d_{R/K[x_0]} : R \rightarrow \Omega_{R/K[x_0]}^1$  given by  $f \mapsto f \otimes 1 - 1 \otimes f + \mathcal{J}^2$  is called the **universal derivation** of  $R/K[x_0]$ . More generally, for any algebra  $T/S$  we can define in the same way the module of Kähler differentials  $\Omega_{T/S}^1 = \tilde{\mathcal{J}}/\tilde{\mathcal{J}}^2$ , where  $\tilde{\mathcal{J}} = \mathrm{Ker}(T \otimes_S T \rightarrow T)$ , and the universal derivation  $d_{T/S} : T \rightarrow \Omega_{T/S}^1$  ( $t \mapsto t \otimes 1 - 1 \otimes t + \tilde{\mathcal{J}}^2$ ) (cf. [Ku5, Section 1]).

By using the notion of Fitting ideals which was introduced in Section 2.2, we give a definition of the Kähler different for  $\mathbb{X}$  as follows.

**Definition 3.3.1.** For  $i \geq 0$ , the  $i$ -th Fitting ideal of  $\Omega_{R/K[x_0]}^1$  is denoted by

$$\vartheta^{(i)}(R/K[x_0]) = \mathcal{F}_i(\Omega_{R/K[x_0]}^1)$$

and is called the  $i$ -th **Kähler different** of  $R/K[x_0]$  (or for  $\mathbb{X}$  with respect to  $x_0$ ). In particular, the ideal  $\vartheta^0(R/K[x_0])$  is called the **Kähler different** of  $R/K[x_0]$  (or for  $\mathbb{X}$  with respect to  $x_0$ ) and is also denoted by  $\vartheta_K(R/K[x_0])$ .

From the definition we see that there is a chain of inclusions

$$\langle 0 \rangle \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta^{(1)}(R/K[x_0]) \subseteq \cdots \subseteq \vartheta^{(i)}(R/K[x_0]) \subseteq \cdots \subseteq R$$

and  $\vartheta^{(i)}(R/K[x_0]) = R$  for  $i \geq m$ , where  $m$  is the minimal number of generators of  $\Omega_{R/K[x_0]}^1$ . Let  $\{F_1, \dots, F_r\}$  be a homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$ , where  $r \geq n$ . By [Ku5, Proposition 4.19], we have the following exact sequence for the module of Kähler differentials:

$$0 \rightarrow \mathcal{K} \rightarrow R dX_1 \oplus \cdots \oplus R dX_n \rightarrow \Omega_{R/K[x_0]}^1 \rightarrow 0 \quad (3.4)$$

Here  $\mathcal{K}$  is generated by the elements  $\sum_{i=1}^n \frac{\partial F_j}{\partial x_i} dX_i$  such that  $j \in \{1, \dots, r\}$ . The Jacobian matrix  $\left( \frac{\partial F_j}{\partial x_i} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, r}}$  is a relation matrix of  $\Omega_{R/K[x_0]}^1$  with respect to  $\{dx_1, \dots, dx_n\}$ . Thus  $\vartheta^{(i)}(R/K[x_0])$  is the ideal of  $R$  generated by all  $(n-i)$ -minors of the Jacobian matrix. In particular,  $\vartheta_K(R/K[x_0])$  is generated by the  $n$ -minors of that matrix. This implies  $\vartheta^{(i)}(R/K[x_0])$  with  $i \geq 1$  and  $\vartheta_K(R/K[x_0])$  are homogeneous ideals of  $R$ .

**Remark 3.3.2.** Let  $I_j \subseteq P$  be the homogeneous vanishing ideal of the scheme  $\mathbb{X}$  at  $p_j$  for  $j = 1, \dots, s$ , and let  $<_{\sigma}$  be a term ordering on  $\mathbb{T}^{n+1}$ . Then a minimal homogeneous system of generators of the Kähler different  $\vartheta_K(R/K[x_0])$  of  $R/K[x_0]$  can be computed by performing the following sequence of instructions.

- 1) Compute the reduced  $<_{\sigma}$ -Gröbner basis  $\{F_1, \dots, F_r\}$  of  $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^s I_j$  by using the GPBM-Algorithm (see [AKR, Theorem 4.6]).
- 2) Form the Jacobian matrix  $\mathfrak{J} = \left( \frac{\partial F_j}{\partial X_i} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, r}}$ . Compute the set  $\{H_1, \dots, H_t\}$  of all non-zero  $n$ -minors of  $\mathfrak{J}$ , and sort the set  $\{H_1, \dots, H_t\}$  such that  $\deg(H_1) \leq \cdots \leq \deg(H_t)$ .
- 3) Set  $\mathcal{H} = \{H_1, \dots, H_t\} \setminus \{F_1, \dots, F_r\}$ . For every polynomial  $H \in \mathcal{H}$ , compute its normal polynomial  $H' = \text{NF}_{\sigma, \mathcal{I}_{\mathbb{X}}}(H)$ . If  $H' \neq 0$ , replace  $H$  by  $H'$ . Otherwise, delete  $H$  from  $\mathcal{H}$ .



- 4) Compute a tuple  $\mathcal{H}_{\min}$  which generates  $(\langle \mathcal{H} \rangle + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}}$  minimally by using the Buchberger Algorithm with Minimalization 3.1.10.
- 5) Return the tuple  $\mathcal{H}_{\min}$  which is a minimal homogeneous system of generators of  $\vartheta_K(R/K[x_0])$ .

Recall that the affine ideal of a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is  $\mathcal{I}_{\mathbb{X}}^a = \mathcal{I}_{\mathbb{X}}^{\text{deh}} \subseteq A = K[X_1, \dots, X_n]$ , and the affine coordinate ring of  $\mathbb{X}$  is  $\Gamma = A/\mathcal{I}_{\mathbb{X}}^a \cong R/\langle x_0 - 1 \rangle$ . By  $\vartheta^{(i)}(\Gamma/K)$  we denote the  $i$ -th Kähler different of the algebra  $\Gamma/K$ , where  $i \in \mathbb{N}$ .

**Proposition 3.3.3.** *Let  $i$  be a positive integer.*

- (i) *Let  $\overline{\vartheta^{(i)}(R/K[x_0])}$  denote the image of  $\vartheta^{(i)}(R/K[x_0])$  in  $\Gamma$ . Then we have*

$$\vartheta^{(i)}(\Gamma/K) = \overline{\vartheta^{(i)}(R/K[x_0])}.$$

- (ii) *For every multiplicatively closed set  $U \subseteq R$ , we have*

$$\vartheta^{(i)}(R_U/K[x_0]) = \vartheta^{(i)}(R/K[x_0])_U.$$

*Proof.* This follows from Proposition 2.2.4. □

Given a homogeneous system  $\{F_1, \dots, F_r\}$  of generators of  $\mathcal{I}_{\mathbb{X}}$  and  $p_j \in \text{Supp}(\mathbb{X})$ , we define the Jacobian matrix at the point  $p_j$  of  $\mathbb{X}$  by  $\mathfrak{J}(p_j) := \left( \left( \frac{\partial F_k}{\partial x_i} \right)_{p_j} \right)_{\substack{i=1, \dots, n \\ k=1, \dots, r}}$  where  $\left( \frac{\partial F_k}{\partial x_i} \right)_{p_j}$  is the image of  $\frac{\partial F_k}{\partial x_i}$  under the map

$$\begin{aligned} \phi_j : R &\rightarrow R/\langle x_0 - 1 \rangle = \Gamma \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j} \xrightarrow{\text{proj}} \mathcal{O}_{\mathbb{X}, p_j} \xrightarrow{\text{can}} \mathcal{O}_{\mathbb{X}, p_j} / \mathfrak{m}_{\mathbb{X}, p_j} = K(p_j) \\ f &\mapsto f^{\text{deh}} \mapsto (f_{p_1}, \dots, f_{p_s}) \mapsto \bar{f}_{p_j}. \end{aligned}$$

All entries of  $\mathfrak{J}(p_j)$  are elements of  $K(p_j)$ . The Jacobian criterion for smoothness of 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is provided by our next proposition, which follows from [Ku5, Corollary 7.18 and Theorem 10.12].

**Proposition 3.3.4.** *The following conditions are equivalent.*

- (i)  $\mathbb{X}$  is smooth at  $p_j$ .
- (ii)  $\text{rank}(\mathfrak{J}(p_j)) = n - \dim \mathcal{O}_{\mathbb{X}, p_j} = n$ .
- (iii)  $\vartheta_K(R/K[x_0]) \not\subseteq \mathfrak{p}_j$ , where  $\mathfrak{p}_j$  is the prime ideal of  $R$  corresponding to  $p_j$ .
- (iv)  $\vartheta^{(0)}(\Gamma/K) \not\subseteq \mathfrak{p}'_j$ , where  $\mathfrak{p}'_j$  the image of  $\mathfrak{p}_j$  in  $\Gamma$ .

**Corollary 3.3.5.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional complete intersection, and let  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_n \rangle$  where  $F_j \in P$  is a homogeneous polynomial of degree  $d_j$  for  $j = 1, \dots, n$ . Then we have*

$$\vartheta_K(R/K[x_0]) = \vartheta_N(R/K[x_0]) = \left\langle \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right\rangle_R.$$

In addition, if  $\mathbb{X}$  is smooth, then

- (i)  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0]) = \left\langle \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right\rangle_R$ , where the element  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  is a non-zerodivisor of  $R$  of degree  $r_{\mathbb{X}} = \sum_{i=1}^n d_i - n$ .
- (ii)  $\mathfrak{C}_{R/K[x_0]} = \left\langle \left( \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right)^{-1} \right\rangle_R$ .
- (iii)  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for all  $i \in \mathbb{Z}$ .

*Proof.* This follows from Corollary 3.2.20 and Propositions 3.1.2 and 3.3.4.  $\square$

The following example shows that the smoothness of  $\mathbb{X}$  in the additional claim of Corollary 3.3.5 is necessary, and that the Noether and Dedekind differents are not equal even when  $\mathbb{X}$  is a complete intersection.

**Example 3.3.6.** Let  $K$  be a field with  $\mathrm{char}(K) \neq 2, 3$ , and let  $\mathbb{X} \subseteq \mathbb{P}_K^2$  be the 0-dimensional complete intersection defined by  $\mathcal{I}_{\mathbb{X}} = \langle F, G \rangle$ , where  $F = X_1(X_1 - 2X_0)(X_1 + 2X_0)$  and  $G = (X_2 - X_0)(X_1^2 + X_2^2 - 4X_0^2)$ . Then  $\mathbb{X}$  has degree 9 and the support of  $\mathbb{X}$  is  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_7\}$  with  $p_1 = (1 : 0 : 1)$ ,  $p_2 = (1 : 0 : 2)$ ,  $p_3 = (1 : 0 : -2)$ ,  $p_4 = (1 : 2 : 1)$ ,  $p_5 = (1 : 2 : 0)$ ,  $p_6 = (1 : -2 : 1)$ , and  $p_7 = (1 : -2 : 0)$ . A homogeneous primary decomposition of the homogeneous vanishing ideal of  $\mathbb{X}$  is  $\mathcal{I}_{\mathbb{X}} = I_1 \cap \dots \cap I_7$ , where  $I_i = \mathfrak{P}_i$  for  $i \neq 5, 7$ ,  $I_5 = \langle X_1 - 2X_0, X_2^2 \rangle$ , and  $I_7 = \langle X_1 + 2X_0, X_2^2 \rangle$ . This means that  $\mathbb{X}$  is not reduced at  $p_5$  and  $p_7$ , hence  $\mathbb{X}$  is not smooth at those points. By Corollary 3.3.5, the Kähler and Noether differents are

$$\begin{aligned} \vartheta_K(R/K[x_0]) &= \vartheta_N(R/K[x_0]) = \left\langle \frac{\partial(F, G)}{\partial(x_1, x_2)} \right\rangle \\ &= \left\langle 4x_0x_1^2x_2 - 16x_0^2x_2^2 - 3x_1^2x_2^2 - 2x_0x_2^3 + 6x_2^4 \right\rangle. \end{aligned}$$

Observe that  $\frac{\partial(F, G)}{\partial(x_1, x_2)}(p_5) = \frac{\partial(F, G)}{\partial(x_1, x_2)}(p_7) = 0$ . This implies  $\frac{\partial(F, G)}{\partial(x_1, x_2)} \in \mathfrak{P}_5 \cap \mathfrak{P}_7$ , and hence  $\frac{\partial(F, G)}{\partial(x_1, x_2)}$  is a zero-divisor of  $R$ . Moreover, we have  $\mathrm{HF}_{\vartheta_K(R/K[x_0])} : 0 \ 0 \ 0 \ 0 \ 1 \ 3 \ 6 \ 7 \ \dots$ ,  $\mathrm{ri}(\vartheta_K(R/K[x_0])) = 7$ , and  $\mathrm{HP}_{\vartheta_K(R/K[x_0])}(z) = 7 < 9 = \mathrm{deg}(\mathbb{X})$ . Therefore we cannot remove the assumption that  $\mathbb{X}$  is smooth in the additional claim of Corollary 3.3.5. Also, we have  $\vartheta_N(R/K[x_0]) \neq \vartheta_D(R/K[x_0])$ , since  $\mathrm{HP}_{\vartheta_D(R/K[x_0])}(z) = \mathrm{deg}(\mathbb{X}) = 9 > 7 = \mathrm{HP}_{\vartheta_N(R/K[x_0])}(z)$ .

For a 0-dimensional reduced complete intersection  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , we can describe the Hilbert function of the Kähler different as follows.

**Corollary 3.3.7.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional reduced complete intersection, and let  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_n \rangle$ , where  $F_j \in P$  is a homogeneous polynomial of degree  $d_j$  for  $j = 1, \dots, n$ . Let  $d := \sum_{j=1}^n d_j - n$ , and let  $\mathbb{Y} \subseteq \mathbb{X}$  be a subscheme defined by  $\mathcal{I}_{\mathbb{Y}} = \bigcap_{p_j \in \text{Supp}(\mathbb{X}): \text{smooth}} \mathfrak{P}_j$ . Then, for all  $i \in \mathbb{Z}$ , we have*

$$\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \text{HF}_{\vartheta_N(R/K[x_0])}(i) = \text{HF}_{\mathbb{Y}}(i - d).$$

*Proof.* Let  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  be the ideal of  $\mathbb{Y}$  in  $R$  and put  $\Delta := \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$ . By Proposition 3.3.4, the element  $\Delta + \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  is a non-zerodivisor of  $R_{\mathbb{Y}} = R/\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  and  $\Delta \in \bigcap_{p_j \in \text{Supp}(\mathbb{X}) \setminus \text{Supp}(\mathbb{Y})} \mathfrak{p}_j$ . Fix the degree  $i \geq 0$  and suppose  $\text{HF}_{\mathbb{Y}}(i) = t$ . Let  $\{g_1 + \mathcal{I}_{\mathbb{Y}/\mathbb{X}}, \dots, g_t + \mathcal{I}_{\mathbb{Y}/\mathbb{X}}\}$  be a  $K$ -basis of the vector space  $(R_{\mathbb{Y}})_i$ . Then the set  $\{\Delta \cdot g_1 + \mathcal{I}_{\mathbb{Y}/\mathbb{X}}, \dots, \Delta \cdot g_t + \mathcal{I}_{\mathbb{Y}/\mathbb{X}}\} \subseteq (R_{\mathbb{Y}})_{i+d}$  is  $K$ -linearly independent. It follows that the vector space  $(\Delta \cdot R)_{i+d}$  has  $K$ -dimension greater than or equal to  $t$ . In other words, we have  $\text{HF}_{\vartheta_K(R/K[x_0])}(i+d) \geq \text{HF}_{\mathbb{Y}}(i)$ .

On the other hand, we observe that  $\Delta \cdot h = 0$  in  $R$  for every homogeneous element  $h \in \mathcal{I}_{\mathbb{Y}/\mathbb{X}} \setminus \{0\}$ , since  $\mathbb{X}$  is reduced. For every  $f \in R_i$ , we write  $f = a_1 g_1 + \dots + a_t g_t + h$  for some  $a_1, \dots, a_t \in K$  and  $h \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_i$ . Then  $\Delta \cdot f = \Delta \cdot (a_1 g_1 + \dots + a_t g_t + h) = a_1 \Delta \cdot g_1 + \dots + a_t \Delta \cdot g_t \in \langle \Delta \cdot g_1, \dots, \Delta \cdot g_t \rangle_K$  (as  $\Delta \cdot h = 0$  in  $R$ ). Thus we have  $\vartheta_K(R/K[x_0])_{d+i} = (\Delta \cdot R)_{i+d} \subseteq \langle \Delta \cdot g_1, \dots, \Delta \cdot g_t \rangle_K$ , and hence  $\text{HF}_{\vartheta_K(R/K[x_0])}(i+d) \leq t = \text{HF}_{\mathbb{Y}}(i)$ . Therefore the conclusion follows.  $\square$

**Remark 3.3.8.** In the setting of Corollary 3.3.7, the regularity indices of the Kähler and Noether differentials are  $d + r_{\mathbb{Y}}$ . If we remove the condition that  $\mathbb{X}$  is reduced, then we have  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \text{HF}_{\vartheta_N(R/K[x_0])}(i) \geq \text{HF}_{\mathbb{Y}}(i - d)$  for all  $i \in \mathbb{Z}$ .

Our next proposition collects some relations between the Kähler and Noether differentials of the algebra  $R/K[x_0]$  and the annihilator of the module of Kähler differentials  $\Omega_{R/K[x_0]}^1$ , which follow from [Ku5, Proposition 10.18].

**Proposition 3.3.9.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $m$  be the number of minimal generators of  $\Omega_{R/K[x_0]}^1$ . Then we have*

$$(\text{Ann}_R(\Omega_{R/K[x_0]}^1))^m \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_N(R/K[x_0]) \subseteq \text{Ann}_R(\Omega_{R/K[x_0]}^1).$$

*In particular, if  $\mathbb{X}$  is smooth, then  $\vartheta_{\mathbb{X}}(R/K[x_0])^m \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

- Definition 3.3.10.** (i) A local ring  $(S, \mathfrak{m})$  is called a **complete intersection** if it is Noetherian and its  $\mathfrak{m}$ -adic completion  $\widehat{S}$  is a quotient of a regular local ring  $A$  by an ideal generated by an  $A$ -regular sequence.
- (ii) Given a ring  $S$  and an algebra  $T/S$ , we say that  $T/S$  is **locally a complete intersection** if for all  $\mathfrak{P} \in \text{Spec}(T)$  the algebra  $T_{\mathfrak{P}}/S_{\mathfrak{p}}$  with  $\mathfrak{p} = \mathfrak{P} \cap S$  is flat and the local ring  $T_{\mathfrak{P}}/\mathfrak{p}T_{\mathfrak{P}}$  is a complete intersection.

It is well known (cf. [BH, Theorem 2.3.3]) that if  $S$  is a Noetherian local ring and  $S = A/I$  with a regular local ring  $A$ , then  $S$  is a complete intersection if and only if  $I$  is generated by an  $A$ -regular sequence. We refer to [BH, Section 2.3] for more properties of complete intersection rings, and refer to [Ku5, Appendix C] for further information on complete intersection algebras.

**Lemma 3.3.11.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. Then the following statements are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is a complete intersection.*
- (ii) *The algebra  $R/K[x_0]$  is locally a complete intersection.*
- (iii) *The local ring  $\overline{R} = R/\langle x_0 \rangle$  is a complete intersection.*

*Proof.* Let  $\{F_1, \dots, F_r\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$ , where  $r \geq n$ . If  $\mathbb{X}$  is a complete intersection, then  $r = n$ ,  $\{F_1, \dots, F_n\}$  is an  $P$ -regular sequence, and  $R = K[X_0][X_1, \dots, X_n]/\langle F_1, \dots, F_n \rangle$ . Hence  $R/K[x_0]$  is locally a complete intersection by [Ku5, Corollary C.7]. Thus we have “(i) implies (ii)”. Moreover, “(ii) implies (iii)” follows from the observations that  $\langle x_0 \rangle_{K[x_0]} = \mathfrak{m} \cap K[x_0]$  and  $R_{\mathfrak{m}}/\langle x_0 \rangle R_{\mathfrak{m}} = (R/\langle x_0 \rangle)_{\overline{\mathfrak{m}}} = \overline{R}_{\overline{\mathfrak{m}}} = \overline{R}$ , where  $\overline{\mathfrak{m}} = \mathfrak{m}/\langle x_0 \rangle$  is the maximal ideal of  $\overline{R}$ .

It remains to prove “(iii) implies (i)”. Observe that if  $\{F_1, \dots, F_r, X_0\}$  is a minimal homogeneous system of generators of the ideal  $\mathcal{I}_{\mathbb{X}} + \langle X_0 \rangle$  then we write

$$\overline{R} = \overline{R}_{\overline{\mathfrak{m}}} \cong (P/\mathcal{I}_{\mathbb{X}} + \langle X_0 \rangle)_{\overline{\mathfrak{m}}} \cong P_{P_+}/(\langle F_1, \dots, F_r, X_0 \rangle)_{P_+}.$$

Since  $\overline{R}$  is a complete intersection, the set  $\{F_1, \dots, F_r, X_0\}$  is a  $P_{P_+}$ -regular sequence (see [BH, Theorem 2.1.2]). [Ku5, Lemma C.28] implies that  $\{F_1, \dots, F_r, X_0\}$  is a  $P$ -regular sequence, and hence  $r = n$  or  $\mathbb{X}$  is a complete intersection. Therefore we only need to show that  $\{F_1, \dots, F_r, X_0\}$  is a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}} + \langle X_0 \rangle$ . Clearly, we have  $X_0 \notin \mathcal{I}_{\mathbb{X}}$ . If there is an index  $i \in \{1, \dots, r\}$  such that  $F_i \in \langle F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_r, X_0 \rangle$ , then we get a representation  $F_i = \sum_{j \neq i} G_j F_j + G X_0$  where  $G_j \in P$  is a homogeneous polynomial of degree  $\deg(F_i) - \deg(F_j)$  for  $j \neq i$

and where  $G \in P$  is a homogeneous polynomial of degree  $\deg(F_i) - 1$  (cf. [KR2, Corollary 1.7.11]). This implies  $GX_0 = F_i - \sum_{j \neq i} G_j F_j \in \mathcal{I}_{\mathbb{X}}$ , and so  $G \in \mathcal{I}_{\mathbb{X}}$  (as  $x_0$  is a non-zerodivisor of  $R$ ). Thus there are homogeneous polynomials  $H_1, \dots, H_r \in P$  such that  $G = \sum_{j=1}^r H_j F_j$  and  $\deg(H_j) = \deg(G) - \deg(F_j)$ . Note that  $H_i = 0$  (as  $\deg(G) < \deg(F_i)$ ). Hence we have  $F_i = \sum_{j \neq i} (G_j + H_j X_0) F_j$ , in contradiction to the minimality of  $\{F_1, \dots, F_r\}$ .  $\square$

Let us go back for a moment to Example 3.2.21. We saw that the projective point set  $\mathbb{X} = \{(1 : 0 : 0 : 0), (1 : 1 : 1 : 1), (1 : -1 : 1 : -1), (1 : 2 : 4 : 8), (8 : 4 : 2 : 1)\}$  in  $\mathbb{P}_{\mathbb{F}_7}^3$  consisting of 5 points on the twisted cubic curve is arithmetically Gorenstein, but it is not a complete intersection. Furthermore, the Noether-Dedekind different of  $R/\mathbb{F}_7[x_0]$  is given by  $\vartheta_{\mathbb{X}}(R/\mathbb{F}_7[x_0]) = \langle x_0^2 - 2x_0x_3 - 3x_1x_3 - 2x_3^2 \rangle$ . On the other hand, a calculation provides us with  $\vartheta_K(R/\mathbb{F}_7[x_0]) = \langle x_2x_3^2 - 3x_3^3, x_1x_3^2, x_0x_3^2 - 3x_3^3, x_0^3 \rangle$ . Thus we get  $\vartheta_K(R/\mathbb{F}_7[x_0]) \subsetneq \vartheta_{\mathbb{X}}(R/\mathbb{F}_7[x_0])$ . This shows that the condition that a projective point set  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is arithmetically Gorenstein does not suffice to imply  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ . However, in case  $\mathbb{X}$  is arithmetically Gorenstein we have the following property.

**Proposition 3.3.12.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme which is arithmetically Gorenstein. The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is a complete intersection.*
- (ii) *The Hilbert function of  $\vartheta_K(R/K[x_0])$  satisfies  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ .*
- (iii)  *$\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

*Proof.* (i) $\Rightarrow$ (ii): If  $\mathbb{X}$  is a complete intersection, then let  $\{F_1, \dots, F_n\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$ . By Corollary 3.3.5, we have

$$\vartheta_K(R/K[x_0]) = \left\langle \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \right\rangle_R = \vartheta_{\mathbb{X}}(R/K[x_0])$$

where  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  is a non-zerodivisor of  $R$  with  $\deg\left(\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}\right) = r_{\mathbb{X}}$ . Thus we get  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ .

(ii) $\Rightarrow$ (iii): By Proposition 3.3.9, we have  $\vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ . We now prove the equality. Since  $\mathbb{X}$  is arithmetically Gorenstein, we get  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle h \rangle_R \cong R(-r_{\mathbb{X}})$  with  $h \in R_{r_{\mathbb{X}}}$  by Corollary 3.2.20. Also, we have

$$\langle 0 \rangle \neq \vartheta_K(R/K[x_0])_{r_{\mathbb{X}}} \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}} = h \cdot R_0 = h \cdot K.$$

This implies  $h \in \vartheta_K(R/K[x_0])$ , and hence  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .

(iii) $\Rightarrow$ (i): Suppose that  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ . Since  $\mathbb{X}$  is arithmetically Gorenstein and by Corollary 3.2.20, we have  $\vartheta_K(R/K[x_0]) = \langle h \rangle_R$  for some non-zerodivisor  $h \in R_{r_{\mathbb{X}}}$ . In particular,  $\vartheta_K(R/K[x_0])$  is an invertible ideal. Moreover, if the module of Kähler differentials  $\Omega_{Q(R)/K[x_0]}^1 = \langle 0 \rangle$ , where  $Q(R)$  is the full ring of quotients of  $R$ , then it follows from [Ku5, Theorem 10.14] that the algebra  $R/K[x_0]$  is locally a complete intersection, and hence  $\mathbb{X}$  is a complete intersection by Lemma 3.3.11. Therefore it suffices to prove  $\Omega_{Q(R)/K[x_0]}^1 = \langle 0 \rangle$ . According to Proposition 3.2.16, the algebra  $Q(R)/K(x_0)$  is étale, and free of rank  $\deg(\mathbb{X})$ . Thus [Ku5, Proposition 6.8] yields  $\Omega_{Q(R)/K(x_0)}^1 = \langle 0 \rangle$ . Also, it is not hard to see that  $\text{Ker}(K(x_0) \otimes_{K[x_0]} K(x_0) \xrightarrow{\mu} K(x_0)) = \langle \{f \otimes 1 - 1 \otimes f \mid f \in K(x_0)\} \rangle_{K(x_0) \otimes_{K[x_0]} K(x_0)} = \langle 0 \rangle$ , and so  $\Omega_{K(x_0)/K[x_0]}^1 = \langle 0 \rangle$ . On the other hand, we have  $Q(R) \cong K(x_0) \otimes_K \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}$  (as  $\mathbb{X}$  is smooth). This implies

$$Q(R) \cong K(x_0) \otimes_{K[x_0]} (K[x_0] \otimes_K \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}) \cong K(x_0) \otimes_{K[x_0]} \tilde{R}$$

where  $\tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  and  $T_1, \dots, T_s$  are indeterminates. By [Ku5, Formulas 4.4], we obtain  $\Omega_{Q(R)/K(x_0)}^1 \cong K(x_0) \otimes_{K[x_0]} \Omega_{\tilde{R}/K[x_0]}^1$  and

$$\begin{aligned} \Omega_{Q(R)/K[x_0]}^1 &\cong K(x_0) \otimes_{K[x_0]} \Omega_{\tilde{R}/K[x_0]}^1 \oplus \tilde{R} \otimes_{K[x_0]} \Omega_{K(x_0)/K[x_0]}^1 \\ &\cong \Omega_{Q(R)/K(x_0)}^1 \oplus \tilde{R} \otimes_{K[x_0]} \Omega_{K(x_0)/K[x_0]}^1 = \langle 0 \rangle. \end{aligned}$$

This completes the proof.  $\square$

**Remark 3.3.13.** We make the following two remarks on the Hilbert function and the regularity index of the Kähler different in two special cases of the scheme  $\mathbb{X}$ .

- (a) If  $n = 1$ , then every 0-dimensional subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^1$  is always a complete intersection. The homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}}$  is generated by a non-zero homogeneous polynomial  $F_1 \in P$ . In this situation, the Kähler and Noether differentials are  $\vartheta_K(R/K[x_0]) = \vartheta_N(R/K[x_0]) = \langle \frac{\partial F_1}{\partial x_1} \rangle$ . Additionally, if  $\mathbb{X}$  is smooth, then  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i - s + 1)$  for all  $i \in \mathbb{Z}$  and  $\text{ri}(\vartheta_K(R/K[x_0])) = 2r_{\mathbb{X}} = (n + 1)r_{\mathbb{X}} = (n + 1)(s - 1)$ .
- (b) Let  $\mathbb{X} = \{p_1\} \subseteq \mathbb{P}_K^n$  be a projective point set of degree 1. We write  $p_1 = (1 : p_{11} : \dots : p_{1n})$  for some  $p_{11}, \dots, p_{1n} \in K$ . Then the homogeneous vanishing ideal of  $\mathbb{X}$  is  $\mathcal{I}_{\mathbb{X}} = \mathfrak{P}_1 = \langle L_{11}, \dots, L_{1n} \rangle$  where  $L_{1i} = X_i - p_{1i}X_0$  for  $i = 1, \dots, n$ . It follows that

$$\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0]) = \langle \partial(L_{11}, \dots, L_{1n}) / \partial(x_1, \dots, x_n) \rangle = \langle 1 \rangle.$$

Also, we have  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i)$  for all  $i \in \mathbb{N}$  and  $\mathrm{ri}(\vartheta_K(R/K[x_0])) = 2r_{\mathbb{X}} = 0$ .

Our next proposition describes the Hilbert function of  $\vartheta_K(R/K[x_0])$  and gives lower and upper bounds for its regularity index when  $\mathbb{X}$  is a projective point set in  $\mathbb{P}_K^n$ .

**Proposition 3.3.14.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, and let  $m$  be the number of minimal generators of  $\Omega_{R/K[x_0]}^1$ .*

(i) *The Hilbert function of  $\vartheta_K(R/K[x_0])$  satisfies*

$$\dots = 0 \leq \mathrm{HF}_{\vartheta_K(R/K[x_0])}(0) \leq \mathrm{HF}_{\vartheta_K(R/K[x_0])}(1) \leq \mathrm{HF}_{\vartheta_K(R/K[x_0])}(2) \leq \dots \leq s$$

and  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = s$  for  $i \gg 0$ .

(ii) *If  $n \geq 2$ , then we have*

$$2r_{\mathbb{X}} \leq \mathrm{ri}(\vartheta_K(R/K[x_0])) \leq \min\{nr_{\mathbb{X}}, 2mr_{\mathbb{X}}\} \leq \min\{nr_{\mathbb{X}}, 2sr_{\mathbb{X}}\}.$$

*In particular, the first inequality becomes an equality if  $\mathbb{X}$  is a complete intersection or  $n = 2$ .*

*Proof.* It is obviously true that  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_K(R/K[x_0])}(i+1) \leq s$  for all  $i \in \mathbb{N}$  and  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = 0$  for  $i < 0$ . By Proposition 3.3.9, we have

$$\vartheta_{\mathbb{X}}(R/K[x_0])^m \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0]).$$

Let  $f_j \in R_{r_{\mathbb{X}}}$  be the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . We remark that  $R_{r_{\mathbb{X}}+i} = x_0^i R_{r_{\mathbb{X}}} = \langle x_0^i f_1, \dots, x_0^i f_s \rangle_K$  for  $i \geq 0$ . Since  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = s$  for  $i \geq 2r_{\mathbb{X}}$  by Proposition 3.2.11, we get  $\vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}} = R_{2r_{\mathbb{X}}} = \langle x_0^{r_{\mathbb{X}}} f_1, \dots, x_0^{r_{\mathbb{X}}} f_s \rangle_K$ . This implies  $\langle (x_0^{r_{\mathbb{X}}} f_1)^m, \dots, (x_0^{r_{\mathbb{X}}} f_s)^m \rangle_K \subseteq (\vartheta_{\mathbb{X}}(R/K[x_0])^m)_{2mr_{\mathbb{X}}} \subseteq \vartheta_K(R/K[x_0])_{2mr_{\mathbb{X}}} \subseteq R_{2mr_{\mathbb{X}}}$ . For  $j \in \{1, \dots, s\}$ , we see that  $f_j^m = x_0^{r_{\mathbb{X}}} f_j(p_j) f_j^{m-1} = x_0^{r_{\mathbb{X}}} f_j^{m-1} = x_0^{(m-1)r_{\mathbb{X}}} f_j$ , and hence  $(x_0^{r_{\mathbb{X}}} f_j)^m = x_0^{(2m-1)r_{\mathbb{X}}} f_j$ . Thus we have

$$\begin{aligned} R_{2mr_{\mathbb{X}}} &= \langle (x_0^{r_{\mathbb{X}}} f_1)^m, \dots, (x_0^{r_{\mathbb{X}}} f_s)^m \rangle_K = \langle x_0^{(2m-1)r_{\mathbb{X}}} f_1, \dots, x_0^{(2m-1)r_{\mathbb{X}}} f_s \rangle_K \\ &= (\vartheta_{\mathbb{X}}(R/K[x_0])^m)_{2mr_{\mathbb{X}}} = \vartheta_K(R/K[x_0])_{2mr_{\mathbb{X}}}. \end{aligned}$$

Therefore we obtain  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])^m}(i) = s$  for all  $i \geq 2mr_{\mathbb{X}}$ , and claim (i) follows.

Now we prove (ii). It follows from the proof of (i) that

$$\mathrm{ri}(\vartheta_K(R/K[x_0])) \leq 2mr_{\mathbb{X}} \leq \min\{2nr_{\mathbb{X}}, 2sr_{\mathbb{X}}\}.$$

Here the inequality  $m \leq \min\{n, s\}$  holds true, since  $\Omega_{R/K[x_0]}^1 = \langle dx_1, \dots, dx_n \rangle_R = \langle dt_1, \dots, dt_s \rangle_R$  where  $t_1, \dots, t_s$  form a  $K[x_0]$ -basis of  $R$ . Moreover, Proposition 3.2.11 shows that the regularity index of  $\vartheta_{\mathbb{X}}(R/K[x_0])$  is  $2r_{\mathbb{X}}$ , and so the regularity index of the Kähler different must satisfy  $2r_{\mathbb{X}} \leq \text{ri}(\vartheta_K(R/K[x_0]))$ . Thus we get

$$2r_{\mathbb{X}} \leq \text{ri}(\vartheta_K(R/K[x_0])) \leq 2mr_{\mathbb{X}} \leq \min\{2nr_{\mathbb{X}}, 2sr_{\mathbb{X}}\}.$$

Next we prove the inequality  $\text{ri}(\vartheta_K(R/K[x_0])) \leq nr_{\mathbb{X}}$  if  $n \geq 2$ . Notice that the inequality does not hold when  $n = 1$  (see Remark 3.3.13(a)). We see that if we have  $f_j^n \in \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$  for all  $j = 1, \dots, s$  then

$$R_{nr_{\mathbb{X}}} \supseteq \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}} \supseteq \langle f_1^n, \dots, f_s^n \rangle_K = \langle x_0^{(n-1)r_{\mathbb{X}}} f_1, \dots, x_0^{(n-1)r_{\mathbb{X}}} f_s \rangle_K = R_{nr_{\mathbb{X}}}.$$

This implies  $\vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}} = R_{nr_{\mathbb{X}}}$ , and hence the inequality  $\text{ri}(\vartheta_K(R/K[x_0])) \leq nr_{\mathbb{X}}$  holds true. Using this observation, we only need to prove  $f_j^n \in \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$  for all  $j = 1, \dots, s$ . It suffices to prove that  $f_1^n$  belongs to  $\vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$ , since the other cases follow similarly. We let  $F_j \in P_{r_{\mathbb{X}}}$  be a representative of  $f_j$  for every  $j \in \{1, \dots, s\}$ , and write  $\mathfrak{B}_1 = \langle L_{11}, \dots, L_{1n} \rangle \subseteq P$  as in Remark 3.3.13(b). It is clearly true that  $F_1 L_{1i} \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}+1}$  for all  $i = 1, \dots, n$ , so the element  $\partial(F_1 L_{11}, \dots, F_1 L_{1n}) / \partial(x_1, \dots, x_n)$  is contained in the  $K$ -vector space  $\vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$ . We calculate

$$\begin{aligned} \frac{\partial(F_1 L_{11}, \dots, F_1 L_{1n})}{\partial(X_1, \dots, X_n)} &= \det \begin{pmatrix} F_1 \frac{\partial L_{11}}{\partial X_1} + L_{11} \frac{\partial F_1}{\partial X_1} & \cdots & F_1 \frac{\partial L_{11}}{\partial X_n} + L_{11} \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ F_1 \frac{\partial L_{1n}}{\partial X_1} + L_{1n} \frac{\partial F_1}{\partial X_1} & \cdots & F_1 \frac{\partial L_{1n}}{\partial X_n} + L_{1n} \frac{\partial F_1}{\partial X_n} \end{pmatrix} \\ &\stackrel{(*)}{=} \det \begin{pmatrix} F_1 \frac{\partial L_{11}}{\partial X_1} & \cdots & F_1 \frac{\partial L_{11}}{\partial X_n} \\ \vdots & \ddots & \vdots \\ F_1 \frac{\partial L_{1n}}{\partial X_1} & \cdots & F_1 \frac{\partial L_{1n}}{\partial X_n} \end{pmatrix} + \det \begin{pmatrix} L_{11} \frac{\partial F_1}{\partial X_1} & \cdots & L_{11} \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ L_{1n} \frac{\partial F_1}{\partial X_1} & \cdots & L_{1n} \frac{\partial F_1}{\partial X_n} \end{pmatrix} + G \\ &= \det \begin{pmatrix} F_1 \frac{\partial L_{11}}{\partial X_1} & \cdots & F_1 \frac{\partial L_{11}}{\partial X_n} \\ \vdots & \ddots & \vdots \\ F_1 \frac{\partial L_{1n}}{\partial X_1} & \cdots & F_1 \frac{\partial L_{1n}}{\partial X_n} \end{pmatrix} + \frac{\partial F_1}{\partial X_1} \cdots \frac{\partial F_1}{\partial X_n} \det \begin{pmatrix} L_{11} & \cdots & L_{11} \\ \vdots & \ddots & \vdots \\ L_{1n} & \cdots & L_{1n} \end{pmatrix} + G \\ &\stackrel{(**)}{=} F_1^n \frac{\partial(L_{11}, \dots, L_{1n})}{\partial(X_1, \dots, X_n)} + G = F_1^n + G \end{aligned}$$

for some polynomial  $G \in \mathcal{I}_{\mathbb{X}}$ . Here the equality  $(*)$  holds because all other determinants contain at least one column of the form  $(F_1 \frac{\partial L_{11}}{\partial X_j} \cdots F_1 \frac{\partial L_{1n}}{\partial X_j})^{\text{tr}}$  and one column of the form  $(L_{11} \frac{\partial F_1}{\partial X_k} \cdots L_{1n} \frac{\partial F_1}{\partial X_k})^{\text{tr}}$ . Since  $F_1 L_{1j} \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}+1}$  for all  $j = 1, \dots, n$ , they are contained in  $\mathcal{I}_{\mathbb{X}}$ . The equality  $(**)$  follows from  $n \geq 2$  and the last equality holds since



$\frac{\partial(L_{11}, \dots, L_{1n})}{\partial(X_1, \dots, X_n)} = 1$  by Remark 3.3.13(b). Hence we obtain  $f_1^n \in \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$ , as we wanted to show.

The last claim follows from the inequalities  $2r_{\mathbb{X}} \leq \text{ri}(\vartheta_K(R/K[x_0])) \leq nr_{\mathbb{X}}$  and the equality  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$  if  $\mathbb{X}$  is a complete intersection.  $\square$

When  $n \geq 3$ , the upper bound for the regularity index of  $\vartheta_K(R/K[x_0])$  given in Proposition 3.3.14 is sharp, as the following example shows.

**Example 3.3.15.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set consisting of the following ten points:  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 0 : 1 : 0)$ ,  $p_3 = (1 : 0 : 0 : 1)$ ,  $p_4 = (1 : 1 : 0 : 1)$ ,  $p_5 = (1 : 2 : 0 : 0)$ ,  $p_6 = (1 : 2 : 1 : 2)$ ,  $p_7 = (1 : 2 : 2 : 1)$ ,  $p_8 = (1 : 2 : 2 : 2)$ ,  $p_9 = (1 : 3 : 1 : 1)$ , and  $p_{10} = (1 : 2 : 3 : 1)$ . By calculation, we have  $\text{HF}_{\mathbb{X}} : 1 \ 4 \ 10 \ 10 \ \dots$  and  $r_{\mathbb{X}} = 2$ . The Hilbert function of  $\vartheta_K(R/\mathbb{Q}[x_0])$  is  $\text{HF}_{\vartheta_K(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 10 \ 10 \ \dots$ , and its regularity index is  $\text{ri}(\vartheta_K(R/\mathbb{Q}[x_0])) = 6$ . In this case,  $\mathbb{X}$  spans  $\mathbb{P}_{\mathbb{Q}}^3$ , this implies the elements  $\{dx_1, dx_2, dx_3\}$  form a minimal homogeneous system of generators of  $\Omega_{R/K[x_0]}^1$  (cf. [DK, Corollary 1.7]), so we get  $m = n = 3$ . Therefore we obtain  $\text{ri}(\vartheta_K(R/\mathbb{Q}[x_0])) = 6 = \min\{nr_{\mathbb{X}}, 2mr_{\mathbb{X}}\}$ .

**Corollary 3.3.16.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, where  $s \geq 2$ . Let  $\tilde{R}$  denote the integral closure of  $R$  in its full ring of quotients, and let  $\tilde{\mathfrak{F}}_{\tilde{R}/R}$  be the conductor of  $R$  in  $\tilde{R}$ .*

(i) *We have*

$$(\tilde{\mathfrak{F}}_{\tilde{R}/R})^{\max\{2, n\}} \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0]) \subseteq \tilde{\mathfrak{F}}_{\tilde{R}/R}.$$

(ii) *We have  $\vartheta_K(R/K[x_0]) = \langle \vartheta_K(R/K[x_0])_{i \leq nr_{\mathbb{X}}} \rangle_R$ .*

*Proof.* Let  $f_j^*$  be a minimal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . By [GKR, Proposition 3.13], the conductor  $\tilde{\mathfrak{F}}_{\tilde{R}/R}$  as an ideal of  $R$  is generated by the set  $\{f_1^*, \dots, f_s^*\}$ . First we consider the case  $n = 1$ . Then  $\max\{2, n\} = 2$  and it is not hard to check that  $\tilde{\mathfrak{F}}_{\tilde{R}/R}^2 \subseteq \vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0]) \subseteq \tilde{\mathfrak{F}}_{\tilde{R}/R}$ . Now we assume  $n \geq 2$ . Using the same argument as in the proof of Proposition 3.3.14, we can prove  $(f_j^*)^n \in \vartheta_K(R/K[x_0])$  for all  $j = 1, \dots, s$ . Since  $f_j^* f_k^* = 0$  if  $j \neq k$ , we get  $\tilde{\mathfrak{F}}_{\tilde{R}/R}^n \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ . On the other hand, we have the inclusions  $\tilde{R} \subseteq \mathfrak{C}_{R/K[x_0]} \subseteq L = Q^h(R)$ . It follows that

$$\vartheta_{\mathbb{X}}(R/K[x_0]) = R :_L \mathfrak{C}_{R/K[x_0]} \subseteq R :_L \tilde{R} = \tilde{\mathfrak{F}}_{\tilde{R}/R}.$$

Hence claim (i) follows.

Claim (ii) is obviously true if  $n = 1$ . If  $n \geq 2$ , then the claim (ii) is an immediate consequence of Proposition 3.3.14, since we have

$$\vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}+i} = R_{nr_{\mathbb{X}}+i} = x_0^i R_{nr_{\mathbb{X}}} = x_0^i \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}}$$

for all  $i \geq 0$ . □

**Corollary 3.3.17.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^2$  be a projective point set. If the Hilbert function of the Noether-Dedekind different satisfies  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}} - 1) = 0$ , then  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

*Proof.* Since  $n = 2$ , Proposition 3.3.14 yields that  $\text{ri}(\vartheta_K(R/K[x_0])) = nr_{\mathbb{X}} = 2r_{\mathbb{X}}$ . Moreover, we have  $\vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ . Thus it follows from the equality  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}} - 1) = 0$  that  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0]) = \bigoplus_{i \geq 2r_{\mathbb{X}}} R_i$ , and the conclusion follows. □

**Example 3.3.18.** Let  $\mathbb{Y} = \{(1 : 0 : 0), (1 : 0 : 1), (1 : 1 : 1), (1 : 2 : 0), (1 : 2 : 1), (1 : 0 : 2)\} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the projective point set, sketched in the affine plane  $\mathbb{A}^2 = D_+(X_0) = \{(c_0 : c_1 : c_2) \in \mathbb{P}_{\mathbb{Q}}^2 \mid c_0 \neq 0\}$  as follows:

$$\begin{array}{ccc} (0, 2) & \bullet & (2, 2) \\ & \bullet \bullet \bullet & \\ (0, 0) & \bullet & (2, 0) \end{array}$$

The Hilbert function of  $\mathbb{Y}$  is  $\text{HF}_{\mathbb{Y}} : 1 \ 3 \ 6 \ 6 \cdots$  and the regularity index of  $\text{HF}_{\mathbb{Y}}$  is  $r_{\mathbb{Y}} = 2$ . We also have  $\vartheta_{\mathbb{Y}}(R/K[x_0]) = \langle x_2^4, x_1 x_2^3, x_0 x_2^3, x_1^4, x_0 x_1^3, x_0^4 \rangle = \bigoplus_{i \geq 4} R_i$ . By Corollary 3.3.17, we get  $\vartheta_K(R/\mathbb{Q}[x_0]) = \vartheta_{\mathbb{Y}}(R/\mathbb{Q}[x_0])$  and  $\text{ri}(\vartheta_K(R/\mathbb{Q}[x_0])) = 4 = nr_{\mathbb{Y}}$ .

However, Corollary 3.3.17 is not true in general. For that, we turn back to Example 3.3.15. Observe that the Hilbert function of the Noether-Dedekind different is  $\text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 10 \ 10 \cdots$ , and so  $\text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(2r_{\mathbb{X}} - 1) = \text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(3) = 0$ . But it is clearly true that  $\vartheta_K(R/\mathbb{Q}[x_0]) \neq \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])$ .

Apart from some other special cases, to exactly determine the Hilbert polynomial of the Kähler different for an arbitrary 0-dimensional subscheme  $\mathbb{X}$  of  $\mathbb{P}_K^n$  is not an easy question, so we try at least to find (possibly sharp) lower and upper bounds for it.

**Proposition 3.3.19.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and let  $\mathbb{X}_{\text{sm}}$  be the set of smooth points of  $\mathbb{X}$  in  $\text{Supp}(\mathbb{X})$ . Then we have*

$$\sum_{p_j \in \mathbb{X}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j}) \leq \text{HP}_{\vartheta_K(R/K[x_0])}(z) \leq \text{deg}(\mathbb{X}) - (s - \#\mathbb{X}_{\text{sm}}).$$

*Proof.* Let  $\mathfrak{P}_j \subseteq P$  be the associated prime ideal of  $p_j$  for  $j = 1, \dots, s$ , and set

$$I := \bigcap_{p_j \in \text{Supp}(\mathbb{X}) \setminus \mathbb{X}_{\text{sm}}} \mathfrak{P}_j.$$

It follows from Proposition 3.3.4 that  $\vartheta_K(R/K[x_0]) \subseteq \mathfrak{p}_j = \mathfrak{P}_j/\mathcal{I}_{\mathbb{X}}$  for every point  $p_j \in \text{Supp}(\mathbb{X}) \setminus \mathbb{X}_{\text{sm}}$ . Hence we get  $\vartheta_K(R/K[x_0]) \subseteq I/\mathcal{I}_{\mathbb{X}}$ , and consequently

$$\text{HF}_{\vartheta_K(R/K[x_0])}(i) \leq \text{HF}_{I/\mathcal{I}_{\mathbb{X}}}(i) = \text{HF}_{\mathbb{X}}(i) - \text{HF}_{P/I}(i) \leq \text{deg}(\mathbb{X}) - \text{deg}(\mathbb{Y})$$

for all  $i \in \mathbb{N}$ , where  $\mathbb{Y}$  is the 0-dimensional subscheme of  $\mathbb{P}_K^n$  defined by  $I$ . Observe that the scheme  $\mathbb{Y}$  has degree  $\text{deg}(\mathbb{Y}) \geq s - \#\mathbb{X}_{\text{sm}}$ . Thus we obtain  $\text{HP}_{\vartheta_K(R/K[x_0])}(z) \leq \text{deg}(\mathbb{X}) - \text{deg}(\mathbb{Y}) \leq \text{deg}(\mathbb{X}) - (s - \#\mathbb{X}_{\text{sm}})$ .

Next we prove the first inequality of  $\text{HP}_{\vartheta_K(R/K[x_0])}(z)$ . If  $\mathbb{X}_{\text{sm}} = \emptyset$ , then there is nothing to prove, since we always have  $\text{HP}_{\vartheta_K(R/K[x_0])}(z) \geq 0$ . Now let us consider the case  $\varrho := \#\mathbb{X}_{\text{sm}} \geq 1$ . Without loss of generality, we may assume that  $\mathbb{X}_{\text{sm}} = \{p_1, \dots, p_\varrho\}$ . For  $j \in \{1, \dots, \varrho\}$ , we have  $\vartheta_K(R/K[x_0]) \not\subseteq \mathfrak{p}_j = \mathfrak{P}_j/\mathcal{I}_{\mathbb{X}}$  by Proposition 3.3.4. It follows from Homogeneous Prime Avoidance (see for instance [KR3, Proposition 5.6.22]) that there exists a homogeneous element  $h \in \vartheta_K(R/K[x_0])_d \setminus \{0\}$  for some  $d \geq 0$  such that  $h \notin \bigcup_{j=1}^{\varrho} (\mathfrak{p}_j)_d$ . Fix  $j \in \{1, \dots, \varrho\}$ . We set  $\nu_j = \dim_K(\mathcal{O}_{\mathbb{X}, p_j})$ , and let  $\{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ . Since  $p_j$  is a smooth point of  $\mathbb{X}$ ,  $\mathcal{O}_{\mathbb{X}, p_j}/K$  is a finite separable field extension. In particular, we have  $h_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ , where  $h_{p_j}$  is the germ of  $h$  at  $p_j$ . For any non-zero element  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ , it is not difficult to verify that  $\{ae_{j1}, \dots, ae_{j\nu_j}\}$  is a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ , then so is  $\{h_{p_j}e_{jk}e_{j1}, \dots, h_{p_j}e_{jk}e_{j\nu_j}\}$ , where  $1 \leq k \leq \nu_j$ .

Now we consider the isomorphism of  $K$ -vector spaces  $\iota : R_{r_{\mathbb{X}}} \rightarrow \prod_{j=1}^{\varrho} \mathcal{O}_{\mathbb{X}, p_j}$  given by  $\iota(f) = (f_{p_1}, \dots, f_{p_s})$ , where  $f_{p_j}$  is the germ of  $f$  at  $p_j$ . For  $j = 1, \dots, \varrho$  and for  $k_j = 1, \dots, \nu_j$ , we let  $f_{jk_j} = \iota^{-1}((0, \dots, 0, e_{jk_j}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ . Then we get

$$\langle hf_{11}, \dots, hf_{1\nu_1}, \dots, hf_{\varrho 1}, \dots, hf_{\varrho \nu_\varrho} \rangle_K \subseteq \vartheta_K(R/K[x_0])_{r_{\mathbb{X}}+d} \subseteq R_{r_{\mathbb{X}}+d}.$$

Now we show that the elements  $\{hf_{11}, \dots, hf_{1\nu_1}, \dots, hf_{\varrho 1}, \dots, hf_{\varrho \nu_\varrho}\}$  are  $K$ -linearly independent. Remark that for  $j_1, j_2 \in \{1, \dots, \varrho\}$  and for  $k_i \in \{1, \dots, \nu_{j_i}\}$ , where  $i = 1, 2$ , we have  $f_{j_1 k_1} \cdot f_{j_2 k_2} \neq 0$  if  $j_1 = j_2$  and  $f_{j_1 k_1} \cdot f_{j_2 k_2} = 0$  if  $j_1 \neq j_2$ , and  $hf_{j_1 k_1}^2 \neq 0$  in  $R_{2r_{\mathbb{X}}+d}$ . Suppose for a contradiction that there are  $c_{11}, \dots, c_{1\nu_1}, \dots, c_{\varrho 1}, \dots, c_{\varrho \nu_\varrho} \in K$ , not all equal to zero, such that  $\sum_{j=1}^{\varrho} \sum_{k_j=1}^{\nu_j} c_{jk_j} hf_{jk_j} = 0$ . W.l.o.g. we may assume  $c_{11} \neq 0$ . We then have

$$hf_{11}^2 = \frac{1}{c_{11}} \left( \sum_{k_1=2}^{\nu_1} c_{1k_1} hf_{1k_1} f_{11} + \sum_{j=2}^{\varrho} \sum_{k_j=1}^{\nu_j} c_{jk_j} hf_{jk_j} f_{11} \right) = \frac{1}{c_{11}} \sum_{k_1=2}^{\nu_1} c_{1k_1} hf_{11} f_{1k_1}.$$

Thus, in  $\mathcal{O}_{\mathbb{X}, p_1}$ , we get the equality  $h_{p_1} e_{11}^2 = \frac{1}{c_{11}} \sum_{k_1=2}^{\nu_1} c_{1k_1} h_{p_1} e_{11} e_{1k_1}$ , in contradiction to the fact that  $\{h_{p_1} e_{11}^2, h_{p_1} e_{11} e_{12}, \dots, h_{p_1} e_{11} e_{1\nu_1}\}$  is a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_1}$ . Therefore we obtain

$$\begin{aligned} \text{HP}_{\vartheta_K(R/K[x_0])}(z) &\geq \dim_K \langle hf_{11}, \dots, hf_{1\nu_1}, \dots, hf_{\varrho 1}, \dots, hf_{\varrho \nu_\varrho} \rangle_K \\ &= \sum_{j=1}^{\varrho} \nu_j = \sum_{p_j \in \mathbb{X}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j}) \end{aligned}$$

and the proposition is completely proved.  $\square$

**Example 3.3.20.** Let us go back to Example 3.3.6. We see that the 0-dimensional complete intersection  $\mathbb{X} \subseteq \mathbb{P}_K^2$  is not smooth. Also, we have  $\deg(\mathbb{X}) = 9$ ,  $s = 7$ ,  $\varrho = 5$  (the number of smooth points of  $\mathbb{X}$ ), and the Kähler different is generated by one element which vanishes at two points in support of  $\mathbb{X}$ . It is not difficult to calculate the following Hilbert functions:

$$\begin{aligned} \text{HF}_{\mathbb{X}} &: 1 \ 3 \ 6 \ 8 \ 9 \ 9 \ \dots \\ \text{HF}_{\vartheta_K(R/K[x_0])} &: 0 \ 0 \ 0 \ 0 \ 1 \ 3 \ 6 \ 7 \ 7 \ \dots \end{aligned}$$

Thus the Hilbert polynomial of the Kähler different  $\vartheta_K(R/K[x_0])$  satisfies

$$\text{HP}_{\vartheta_K(R/K[x_0])}(z) = \deg(\mathbb{X}) - (s - \varrho) = 7 > 5 = \varrho = \sum_{p_j \in \mathbb{X}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j})$$

and its regularity index is  $\text{ri}(\vartheta_K(R/K[x_0])) = 7 < 2r_{\mathbb{X}} = 8$ .

According to [GM, Proposition 1.1], we can find a minimal system  $\{F_1, \dots, F_r\}$  of generators of the homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}}$  such that  $\deg(F_j) \leq r_{\mathbb{X}} + 1$  for all  $j = 1, \dots, r$ . Since  $\vartheta_K(R/K[x_0])$  is generated by  $n$ -minors of the Jacobian matrix  $\mathcal{J} = \left( \frac{\partial F_j}{\partial x_i} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, r}}$ , it follows that  $\vartheta_K(R/K[x_0])$  is generated in degree  $\leq nr_{\mathbb{X}}$ . In the special case that  $\mathbb{X}$  is a 0-dimensional smooth subscheme of  $\mathbb{P}_K^n$ , we can find a homogeneous element  $h \in \vartheta_K(R/K[x_0])_d$  for some  $d \geq 0$  such that  $h \notin \bigcup_{j=1}^s (\mathfrak{p}_j)_d$  by a similar argument as in the proof of Proposition 3.3.19. By Lemma 2.3.11, such an element  $h$  is a non-zerodivisor of  $R$ .

If  $d > nr_{\mathbb{X}}$  and  $\langle \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}} \rangle_R \subseteq \bigcup_{j=1}^s (\mathfrak{p}_j)$ , then  $\langle \vartheta_K(R/K[x_0])_{nr_{\mathbb{X}}} \rangle_R \subseteq \mathfrak{p}_j$  for some  $j \in \{1, \dots, s\}$ , and hence the element  $h$  cannot exist. Thus  $h$  can be chosen such that  $\deg(h) = d \leq nr_{\mathbb{X}}$ . Moreover, if  $\{f_1, \dots, f_{\deg(\mathbb{X})}\}$  is a  $K$ -basis of  $R_{r_{\mathbb{X}}}$ , then the set  $\{hf_1, \dots, hf_{\deg(\mathbb{X})}\}$  is a  $K$ -basis of  $\vartheta_K(R/K[x_0])_{d+r_{\mathbb{X}}}$ .

We summarize these arguments here.

**Corollary 3.3.21.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. Then we have  $\text{HP}_{\vartheta_K(R/K[x_0])}(z) = \deg(\mathbb{X})$  and  $\text{ri}(\vartheta_K(R/K[x_0])) \leq r_{\mathbb{X}}(n + 1)$ .*

**Corollary 3.3.22.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme such that  $\mathcal{I}_{\mathbb{X}} = \langle (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}} \rangle_P$ . Then  $\vartheta_K(R/K[x_0])$  is generated by elements of degree  $n(\alpha_{\mathbb{X}} - 1)$  and*

$$n(\alpha_{\mathbb{X}} - 1) \leq \text{ri}(\vartheta_K(R/K[x_0])) \leq n(\alpha_{\mathbb{X}} - 1) + r_{\mathbb{X}}.$$

*Proof.* If  $n = 1$ , then  $\mathbb{X}$  is a complete intersection with  $\mathcal{I}_{\mathbb{X}} = \langle F_1 \rangle \subseteq P$ . This implies  $\vartheta_K(R/K[x_0]) = \langle \partial F_1 / \partial x_1 \rangle$  and  $\partial F_1 / \partial x_1$  is a non-zero-divisor of  $R$  of degree  $r_{\mathbb{X}} = \alpha_{\mathbb{X}} - 1$  (see Corollary 3.3.5). Thus we have  $\text{ri}(\vartheta_K(R/K[x_0])) = 2r_{\mathbb{X}} = 2(\alpha_{\mathbb{X}} - 1)$ , and the claim follows. Now we suppose that  $n \geq 2$ . Clearly, we have  $\vartheta_K(R/K[x_0]) = \langle \vartheta_K(R/K[x_0])_{n(\alpha_{\mathbb{X}}-1)} \rangle_R$  and  $n(\alpha_{\mathbb{X}}-1) \leq \text{ri}(\vartheta_K(R/K[x_0]))$ . Notice that  $n(\alpha_{\mathbb{X}}-1) + r_{\mathbb{X}} \leq r_{\mathbb{X}}(n+1)$  holds true, since  $\alpha_{\mathbb{X}} \leq r_{\mathbb{X}} + 1$  (see [GM, Proposition 1.1]). In this case, there is a non-zero-divisor  $h$  of  $R$  contained in  $\vartheta_K(R/K[x_0])_{n(\alpha_{\mathbb{X}}-1)}$  by the argument before Corollary 3.3.21. Let  $f_1, \dots, f_{\deg(\mathbb{X})} \in R_{r_{\mathbb{X}}}$  be a  $K$ -basis of  $R_{r_{\mathbb{X}}}$ . Then we have  $hf_j \in \vartheta_K(R/K[x_0])_{n(\alpha_{\mathbb{X}}-1)+r_{\mathbb{X}}}$  for all  $j \in \{1, \dots, \deg(\mathbb{X})\}$ , and

$$\deg(\mathbb{X}) = \dim_K \langle hf_1, \dots, hf_{\deg(\mathbb{X})} \rangle_K \leq \text{HF}_{\vartheta_K(R/K[x_0])}(n(\alpha_{\mathbb{X}} - 1) + r_{\mathbb{X}}) \leq \deg(\mathbb{X}).$$

This implies that  $\text{HF}_{\vartheta_K(R/K[x_0])}(n(\alpha_{\mathbb{X}} - 1) + r_{\mathbb{X}}) = \deg(\mathbb{X})$ , and therefore we get  $\text{ri}(\vartheta_K(R/K[x_0])) \leq n(\alpha_{\mathbb{X}} - 1) + r_{\mathbb{X}}$ , as desired.  $\square$

**Example 3.3.23.** Let us consider the 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  with the homogeneous vanishing ideal given by  $\mathcal{I}_{\mathbb{X}} = \langle F_1, F_2, F_3 \rangle \subseteq \mathbb{Q}[X_0, X_1, X_2]$ , where  $F_1 = X_0X_1X_2 - X_0X_2^2 - X_1X_2^2 + X_2^3$ ,  $F_2 = X_0^2X_2 - 2X_0X_2^2 + X_2^3$ , and  $F_3 = X_0^3X_1 - X_0^2X_1^2 + X_0X_1^3 - X_1^4 + 2X_1^2X_2^2 - 2X_0X_2^3 - 2X_1X_2^3 + 2X_2^4$ . A homogeneous primary decomposition of  $\mathcal{I}_{\mathbb{X}}$  is given by

$$\mathcal{I}_{\mathbb{X}} = \mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \mathfrak{P}_3 \cap \mathfrak{P}_4 \cap \mathfrak{P}_5^2 \cap \mathfrak{P}_6$$

where  $\mathfrak{P}_1 = \langle X_1, X_2 \rangle$ ,  $\mathfrak{P}_2 = \langle X_1 - X_0, X_2 \rangle$ ,  $\mathfrak{P}_3 = \langle X_1 + X_0, X_2 - X_0 \rangle$ ,  $\mathfrak{P}_4 = \langle X_1, X_2 - X_0 \rangle$ ,  $\mathfrak{P}_5 = \langle X_1 - X_0, X_2 - X_0 \rangle$ , and  $\mathfrak{P}_6 = \langle X_1^2 + X_0^2, X_2 \rangle$ . So, we get  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_6\}$ ,  $\mathbb{X}_{\text{sm}} = \{p_1, \dots, p_4, p_6\}$ ,  $\varrho = 5$ , and  $\deg(\mathbb{X}) = 9$ . In particular,  $\mathbb{X}$  is not smooth at  $p_5$ , and so it is not smooth. A calculation gives us the following Hilbert functions:

$$\begin{aligned} \text{HF}_{\mathbb{X}} & : 1 \ 3 \ 6 \ 8 \ 9 \ 9 \ \dots \\ \text{HF}_{\vartheta_K(R/K[x_0])} & : 0 \ 0 \ 0 \ 0 \ 0 \ 2 \ 4 \ 6 \ 6 \ \dots \end{aligned}$$

Thus we obtain  $\text{HP}_{\vartheta_K(R/K[x_0])}(z) = \sum_{p_j \in \mathbb{X}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j}) = 6 < \deg(\mathbb{X}) - (s - \varrho) = 8$ , and  $\text{ri}(\vartheta_K(R/K[x_0])) = 7 < 2r_{\mathbb{X}} = 8$ .

Let  $\mathbb{Y} \subseteq \mathbb{X}$  be the subscheme defined by  $\mathcal{I}_{\mathbb{Y}} = \mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \mathfrak{P}_3 \cap \mathfrak{P}_4 \cap \mathfrak{P}_6 \subseteq P$ . Then  $\mathbb{Y}$  is a smooth subscheme of  $\mathbb{X}$  of degree 6. Moreover, we have

$$\begin{aligned} \text{HF}_{\mathbb{Y}} & : 1 \ 3 \ 5 \ 6 \ 6 \ \cdots \\ \text{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])} & : 0 \ 0 \ 0 \ 1 \ 3 \ 5 \ 6 \ 6 \ \cdots \end{aligned}$$

Hence we get  $\text{HP}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(z) = \deg(\mathbb{Y}) = 6$  and  $\text{ri}(\vartheta_K(R_{\mathbb{Y}}/K[x_0])) = 6 = 2r_{\mathbb{Y}}$ .

**Remark 3.3.24.** Let  $K$  be a perfect field, let  $n \geq 2$ , let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional scheme, and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . It follows from Corollary 3.3.21 that  $x_0^{r_{\mathbb{X}}(n+1)} \in \vartheta_K(R/K[x_0])$ . However, another non-zerodivisor of  $R$  contained in  $\vartheta_K(R/K[x_0])$  of smaller degree can be found in terms of a minimal system of generators of  $\mathcal{I}_{\mathbb{X}}$  as follows. In view of [DK, Proposition 5.1], we suppose that  $\mathcal{I}_{\mathbb{X}}$  has a minimal homogeneous system of generators  $\{F_1, \dots, F_r\}$  such that  $\{F_1, \dots, F_n\}$  is a  $P$ -regular sequence and  $\langle F_1, \dots, F_n \rangle P_{\mathfrak{P}_j} = \mathfrak{P}_j P_{\mathfrak{P}_j}$  for  $j = 1, \dots, s$ . It follows that the ideal  $J = \langle F_1, \dots, F_n \rangle$  defines a 0-dimensional complete intersection  $\mathbb{W} \subseteq \mathbb{P}_K^n$  which is reduced at the points of  $\mathbb{X}$ . Since  $K$  is perfect, the scheme  $\mathbb{W}$  is also smooth at the points of  $\mathbb{X}$ . By the Jacobian criterion for smoothness 3.3.4, we have  $(\partial(F_1, \dots, F_n)/\partial(x_1, \dots, x_n))_{p_j} \neq 0$  in  $K(p_j)$  for  $j = 1, \dots, s$ . Put  $\Delta = \partial(F_1, \dots, F_n)/\partial(x_1, \dots, x_n) \in \vartheta_K(R/K[x_0])_d \subseteq R_d$  with  $d = \sum_{j=1}^n \deg(F_j) - n$ . Since  $\mathcal{I}_{\mathbb{X}}$  is generated in degree  $\leq r_{\mathbb{X}} + 1$ , we have  $d \leq nr_{\mathbb{X}}$ . An application of Lemma 2.3.11 yields that  $\Delta$  is a non-zerodivisor of  $R$ .

If  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  is a projective point set, we may omit the assumption that  $K$  is perfect. Because if the scheme  $\mathbb{W}$  is reduced at the point  $p_j$  of  $\mathbb{X}$  then  $\mathcal{O}_{\mathbb{W}, p_j} = K(p_j) = K$ , and hence  $\mathbb{W}$  is also smooth at  $p_j$ .

# Differents and Uniformity of 0-Dimensional Schemes

Given a 0-dimensional scheme  $\mathbb{X}$  in the projective  $n$ -space  $\mathbb{P}_K^n$  over an arbitrary field  $K$  such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ , we are interested in studying relations between the geometry of the scheme and the algebraic structure of the Noether, Dedekind, and Kähler differentials. Explicitly, we want to find out some characterizations of uniformity properties of  $\mathbb{X}$  in terms of these differentials. The techniques we use in this chapter are inspired by those of Martin Kreuzer and his coworkers in their study of 0-dimensional schemes in  $\mathbb{P}_K^n$  (cf. [GKR], [KK], [Kr2], [Kr3], [Kr4], [KR1], and [GK]).

In Section 4.1, we start by generalizing the definition of the Cayley-Bacharach scheme (CB-scheme) in  $\mathbb{P}_K^n$  over an arbitrary field  $K$ . This notion was introduced and studied for projective point sets in [GKR] and for 0-dimensional schemes over an algebraically closed field in [Kr2]. The first main result is then Theorem 4.1.7 which gives a characterization of CB-schemes in terms of their Dedekind differentials. This characterization also shows that the CB-scheme property can be shown by checking a particular homogeneous component of the Dedekind differential (see Corollary 4.1.9), provides some descriptions of the Hilbert function, and determines the regularity index of the Dedekind differential for a 0-dimensional locally Gorenstein CB-scheme (see Propositions 4.1.11 and 4.1.12). Furthermore, when  $\mathbb{X}$  is a 0-dimensional smooth CB-scheme, we can determine  $\mathbb{X}$  to be an arithmetically Gorenstein scheme (respectively, a complete intersection) by looking at the Hilbert function of the Dedekind differential (respectively, the Kähler differential) at degree  $r_{\mathbb{X}}$ , where  $r_{\mathbb{X}}$  is the regularity index of  $\text{HF}_{\mathbb{X}}$  (see Proposition 4.1.15). The remainder of this section is devoted to discussing the Cayley-Bacharach property of degree  $d$  (CBP( $d$ )) of the scheme  $\mathbb{X}$ , where  $0 \leq d \leq r_{\mathbb{X}} - 1$ . We give a sufficient condition for CBP( $d$ ) in terms of the Dedekind

different (see Proposition 4.1.23). Also, we characterize  $\text{CBP}(d)$  in terms of the conductor of  $R$  in a particular extension ring which is the integral closure of  $R$  if  $\mathbb{X}$  is reduced (see Proposition 4.1.26) and present a generalization of the Dedekind formula for the conductor and the Dedekind complementary module which is found in [GKR] (see Proposition 4.1.27).

Section 4.2 is concerned with the study of a particular class of 0-dimensional schemes in  $\mathbb{P}_K^n$  whose Hilbert functions are as large as possible, namely the schemes having generic Hilbert function. More precisely, we first indicate, in Lemma 4.2.2, that a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , which has  $K$ -rational support and generic Hilbert function, has  $\text{CBP}(d)$  for  $d = 0, \dots, r_{\mathbb{X}} - 2$ . However, not all such schemes are CB-schemes (see Remark 4.2.3). Then we generalize some characterizations of arithmetically Gorenstein property for projective point sets found in [GO, Section 3-4] to reduced 0-dimensional schemes (see Proposition 4.2.4 and Corollary 4.2.5). Next we restrict our attention to projective point sets  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$ . We demonstrate that  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$  if and only if it is a CB-scheme and its Kähler different equals the  $n$ -th power of the conductor of  $R$  in its integral closure (see Proposition 4.2.7). Moreover, we use the Kähler different to provide some sufficient conditions for Cayley-Bacharach properties (see Propositions 4.2.9 and 4.2.11) and we characterize projective point sets having generic Hilbert function under some additional hypotheses (see Propositions 4.2.7 and 4.2.13).

In Section 4.3, we use Liaison theory to explore Cayley-Bacharach properties of 0-dimensional schemes  $\mathbb{X}$  in  $\mathbb{P}_K^n$ . This approach was initiated by A.V. Geramita, M. Kreuzer, and L. Robbiano [GKR]. They showed that a projective point set  $\mathbb{X}$ , which is contained in a reduced 0-dimensional complete intersection  $\mathbb{W}$ , is a CB-scheme if and only if there exists a homogeneous polynomial of degree  $r_{\mathbb{W}} - r_{\mathbb{X}}$  in the homogeneous vanishing ideal of the complement of  $\mathbb{X}$  in  $\mathbb{W}$  which does not vanish at any point of  $\mathbb{X}$ . This result is generalized in Theorem 4.3.6 for 0-dimensional schemes contained in a 0-dimensional arithmetically Gorenstein scheme. Due to this theorem, we characterize  $\text{CBP}(d)$  in terms of the Dedekind complementary module (see Corollary 4.3.9 and Theorem 4.3.10) and describe the Hilbert function of the Dedekind different (Proposition 4.3.12). We also show that if two 0-dimensional schemes  $\mathbb{X}$  and  $\mathbb{Y}$  are linked by a 0-dimensional arithmetically Gorenstein scheme  $\mathbb{W}$  and  $p_j \in \text{Supp}(\mathbb{X})$ , then there is a 1-1 correspondence between  $p_j$ -subschemes of  $\mathbb{X}$  and schemes containing  $\mathbb{Y}$  as a  $p_j$ -subscheme (see Lemma 4.3.14). Furthermore, we apply this correspondence to generalize a result of [KR1] (see Theorem 4.3.15).



In the final section 4.4 we investigate higher uniformities for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  with  $K$ -rational support. We say that  $\mathbb{X}$  is  $(i, j)$ -**uniform** if every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$  satisfies  $\mathrm{HF}_{\mathbb{Y}}(j) = \mathrm{HF}_{\mathbb{X}}(j)$ . When  $\mathbb{X}$  is locally Gorenstein and  $(i, j)$ -uniform, we describe relations between the Dedekind different and the homogeneous saturated ideals of maximal  $p_j$ -subschemes of  $\mathbb{X}$  (see Propositions 4.4.7 and 4.4.10). By using these relations, we give some characterizations of  $(2, j)$ -uniformity (see Propositions 4.4.12 and 4.4.15). Additionally, we find a class of CB-schemes which are not  $(2, r_{\mathbb{X}} - 1)$ -uniformity (see Proposition 4.4.18). We end this section with some results (see Propositions 4.4.20 and 4.4.21) about cohomological uniformity.

Throughout this chapter we work over an arbitrary field  $K$ . We let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $\mathcal{I}_{\mathbb{X}} \subseteq P = K[X_0, \dots, X_n]$  be the homogeneous vanishing ideal of  $\mathbb{X}$ , and let  $R = P/\mathcal{I}_{\mathbb{X}}$  be the homogeneous coordinate ring of  $\mathbb{X}$  in  $\mathbb{P}_K^n$ . We always assume that no point of the support of  $\mathbb{X}$  lies on  $\mathcal{Z}^+(X_0)$ , and we denote the image of  $X_i$  in  $R$  by  $x_i$  for  $i = 0, \dots, n$ . It is necessary to keep in mind that  $x_0$  is not a zerodivisor of  $R$  and  $R$  is a graded-free  $K[x_0]$ -module of rank  $m = \deg(\mathbb{X})$ .

## 4.1 Differents and the Cayley-Bacharach Property

Let us start this section by recalling from Section 2.3 the following notation. Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$  for some  $s \geq 1$ . We let  $\varkappa_j := \dim_K K(p_j) = \dim_K(\mathcal{O}_{\mathbb{X}, p_j}/\mathfrak{m}_{\mathbb{X}, p_j})$  for  $j = 1, \dots, s$ . For any element  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ , we set  $\mu(a) = \min\{i \in \mathbb{N} \mid (0, \dots, 0, aT_j^i, 0, \dots, 0) \in \tilde{\nu}(R)\}$  and  $\nu(a) = \max\{\mu(ab) \mid b \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\}$ , where  $\tilde{\nu}: R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  is the injection given by  $\tilde{\nu}(f) = (f_{p_1}T_1^i, \dots, f_{p_s}T_s^i)$  for  $f \in R_i$  with  $i \geq 0$ .

Given a maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$ , we let  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  be a socle element of  $\mathcal{O}_{\mathbb{X}, p_j}$  corresponding to  $\mathbb{Y}$ . According to Proposition 2.3.21(iv), we can take a set  $\{e_{j1}, \dots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  such that whose residue classes form a  $K$ -basis of  $K(p_j)$  and  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_{j1}^*, \dots, f_{j\varkappa_j}^* \rangle_R$ , where  $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$  is the set of minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$  (w.r.t.  $s_j$  and  $\{e_{j1}, \dots, e_{j\varkappa_j}\}$ ), i.e.,

$$f_{jk_j}^* = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j}s_j T_j^{\mu(e_{jk_j}s_j)}, 0, \dots, 0))$$

for all  $k_j \in \{1, \dots, \varkappa_j\}$ . The set  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$ , where  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \mu(e_{jk_j}s_j)} f_{jk_j}^*$  for  $k_j = 1, \dots, \varkappa_j$ , is the set of separators of  $\mathbb{Y}$  in  $\mathbb{X}$  (w.r.t.  $s_j$  and  $\{e_{j1}, \dots, e_{j\varkappa_j}\}$ ). Obviously, we have  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} = \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K$ . The maximal degree of minimal separators of  $\mathbb{Y}$

in  $\mathbb{X}$  is  $\nu_{\mathbb{Y}/\mathbb{X}} = \max\{\deg(f_{jk_j}^*) \mid k_j = 1, \dots, \varkappa_j\}$ . Notice that  $\nu_{\mathbb{Y}/\mathbb{X}}$  does not depend on a specific choice of a set of minimal separators and  $\nu_{\mathbb{Y}/\mathbb{X}} = \nu(s'_j)$  for every element  $s'_j \in s_j \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}$  (see Lemma 2.3.15).

Recall from Definition 2.3.23 that, for every point  $p_j \in \text{Supp}(\mathbb{X})$ , the degree of  $p_j$  in  $\mathbb{X}$  is given by

$$\deg_{\mathbb{X}}(p_j) = \min \left\{ \nu_{\mathbb{Y}/\mathbb{X}} \mid \mathbb{Y} \text{ is a maximal } p_j\text{-subscheme of } \mathbb{X} \right\}.$$

It is clear that  $\deg_{\mathbb{X}}(p_j) \leq r_{\mathbb{X}}$  for all  $j = 1, \dots, s$ . In case all points of  $\text{Supp}(\mathbb{X})$  have the maximum possible degree  $r_{\mathbb{X}}$ , we have the following notion.

**Definition 4.1.1.** A 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is called a **Cayley-Bacharach scheme** (in short, **CB-scheme**) if every point  $p_j \in \text{Supp}(\mathbb{X})$  has degree  $\deg_{\mathbb{X}}(p_j) = r_{\mathbb{X}}$ .

First of all, we give an example which shows that a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  with  $\mathbb{X}(K) \subsetneq \text{Supp}(\mathbb{X})$  can be a CB-scheme.

**Example 4.1.2.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the reduced 0-dimensional scheme of degree 14 with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_{12}\}$ , where  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0)$ ,  $p_3 = (1 : 1 : 1)$ ,  $p_4 = (1 : 0 : 1)$ ,  $p_5 = (1 : -1 : 1)$ ,  $p_6 = (1 : 1 : -1)$ ,  $p_7 = (1 : 0 : -1)$ ,  $p_8 = (1 : 2 : 0)$ ,  $p_9 = (1 : 2 : 1)$ ,  $p_{10} = (1 : 2 : -1)$ ,  $p_{11}$  corresponds to  $\mathfrak{P}_{11} = \langle 2X_0^2 + X_1^2, X_2 \rangle$ , and  $p_{12}$  corresponds to  $\mathfrak{P}_{12} = \langle X_1, X_0^2 + 7X_2^2 \rangle$ . Clearly,  $\mathbb{X}$  does not have  $\mathbb{Q}$ -rational support, since the two points  $p_{11}$  and  $p_{12}$  are not contained in  $\mathbb{X}(\mathbb{Q})$ . A calculation gives us

$$\begin{aligned} \text{HF}_{\mathbb{X}} : & \quad 1 \ 3 \ 6 \ 10 \ 14 \ 14 \ \dots \\ \text{HF}_{\mathbb{X} \setminus \{p_j\}} : & \quad 1 \ 3 \ 6 \ 10 \ 13 \ 13 \ \dots \quad (j = 1, \dots, 10) \\ \text{HF}_{\mathbb{X} \setminus \{p_{11}\}} : & \quad 1 \ 3 \ 6 \ 10 \ 12 \ 12 \ \dots \\ \text{HF}_{\mathbb{X} \setminus \{p_{12}\}} : & \quad 1 \ 3 \ 6 \ 9 \ 12 \ 12 \ \dots \end{aligned}$$

We have  $\alpha_{\mathbb{X} \setminus \{p_j\}/\mathbb{X}} = r_{\mathbb{X} \setminus \{p_j\}} = r_{\mathbb{X}} = 4$  for  $j = 1, \dots, 11$ . This implies  $\deg_{\mathbb{X}}(p_j) = 4$  for  $j = 1, \dots, 11$ . We also see that  $\alpha_{\mathbb{X} \setminus \{p_{12}\}/\mathbb{X}} = 3 < r_{\mathbb{X} \setminus \{p_{12}\}} = r_{\mathbb{X}} = 4$ . However, it follows from  $\text{HF}_{\mathcal{I}_{\mathbb{X} \setminus \{p_{12}\}}/\mathbb{X}}(3) = 1 < \varkappa_{12} = \dim_{\mathbb{Q}} \mathcal{O}_{\mathbb{X}, p_{12}} = 2$  that there is a minimal separator of  $\mathbb{X} \setminus \{p_{12}\}$  in  $\mathbb{X}$  having maximal degree 4. In other words, we have  $\deg_{\mathbb{X}}(p_{12}) = 4$ . Hence the scheme  $\mathbb{X}$  is a CB-scheme.

**Remark 4.1.3.** Given a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , we consider the following two statements:

- (a) The scheme  $\mathbb{X}$  is a CB-scheme.
- (b) Every hypersurface of degree  $r_{\mathbb{X}} - 1$  which contains a maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  automatically contains  $\mathbb{X}$ .

When the scheme  $\mathbb{X}$  has  $K$ -rational support, then the statements (a) and (b) are equivalent. Indeed, we have  $\varkappa_j = \dim_K K(p_j) = 1$  for all  $j = 1, \dots, s$  and

$$\deg_{\mathbb{X}}(p_j) = \min\{\alpha_{\mathbb{Y}/\mathbb{X}} \mid \mathbb{Y} \text{ is a maximal } p_j\text{-subscheme of } \mathbb{X}\}.$$

It follows that  $\mathbb{X}$  is a CB-scheme if and only if, for each point  $p_j \in \text{Supp}(\mathbb{X})$ , every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  has  $\alpha_{\mathbb{Y}/\mathbb{X}} = r_{\mathbb{X}}$ . Thus (a) is clearly equivalent to (b).

In general, we observe that (b) implies (a), but (a) does not imply (b). For example, the reduced 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  given in Example 4.1.2 is a CB-scheme. But  $\alpha_{\mathbb{X} \setminus \{p_{12}\}/\mathbb{X}} = 3 < r_{\mathbb{X} \setminus \{p_{12}\}} = r_{\mathbb{X}} = 4$ , and so the statement (b) is not satisfied.

The following proposition gives a simple criterion for detecting whether a given 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is a CB-scheme.

**Proposition 4.1.4.** *The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is a CB-scheme.*
- (ii) *For all  $j \in \{1, \dots, s\}$  and  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$  we have  $\nu(s_j) = r_{\mathbb{X}}$ .*
- (iii) *If  $\mathbb{Y} \subseteq \mathbb{X}$  is a maximal  $p_j$ -subscheme and  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$  is a set of separators of  $\mathbb{Y}$  in  $\mathbb{X}$ , then there exists  $k_j \in \{1, \dots, \varkappa_j\}$  such that  $x_0 \nmid f_{jk_j}$ .*
- (iv) *For all  $p_j \in \text{Supp}(\mathbb{X})$ , every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  satisfies*

$$\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} < \varkappa_j.$$

*Proof.* (i) $\Leftrightarrow$ (ii): This follows from Definition 4.1.1 and Proposition 2.3.17.

(i) $\Leftrightarrow$ (iii): If we write  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^*$  with  $f_{jk_j}^* \in R_{\deg(f_{jk_j}^*)} \setminus x_0 R_{\deg(f_{jk_j}^*)-1}$  for  $k_j = 1, \dots, \varkappa_j$ , then the set  $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$  is a set of minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$ . Hence the claim is clearly true.

(i) $\Leftrightarrow$ (iv): For a maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$ , we always have  $\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_i \leq \varkappa_j$  for  $i \geq 0$ . Also, we see that  $\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} = \varkappa_j$  if and only if  $\deg(f_{jk_j}^*) \leq r_{\mathbb{X}} - 1$  for all  $k_j = 1, \dots, \varkappa_j$ . Thus the conclusion follows.  $\square$

**Example 4.1.5.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the reduced 0-dimensional scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_7\}$ , where  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0)$ ,  $p_3 = (1 : 1 : 1)$ ,  $p_4 = (1 : 0 : 1)$ ,  $p_5 = (1 : -1 : 1)$ ,  $p_6 = (1 : 2 : 3)$ , and  $p_7$  corresponds to  $\mathfrak{P}_7 = \langle 2X_0^2 + X_1^2, X_2 \rangle$ . We have  $\varkappa_1 = \dots = \varkappa_6 = 1$  and  $\varkappa_7 = 2$ . The Hilbert functions of  $\mathbb{X}$  and its subschemes are computed as follows

$$\begin{aligned} \text{HF}_{\mathbb{X}} : & \quad 1 \ 3 \ 6 \ 8 \ 8 \ \dots \\ \text{HF}_{\mathbb{X} \setminus \{p_j\}} : & \quad 1 \ 3 \ 6 \ 7 \ 7 \ \dots \quad (j = 1, \dots, 6) \\ \text{HF}_{\mathbb{X} \setminus \{p_7\}} : & \quad 1 \ 3 \ 6 \ 6 \ 6 \ \dots \end{aligned}$$

From this we deduce  $(\mathcal{I}_{\mathbb{X} \setminus \{p_j\}/\mathbb{X}})_{r_{\mathbb{X}}-1} = (\mathcal{I}_{\mathbb{X} \setminus \{p_j\}/\mathbb{X}})_2 = \langle 0 \rangle$  for all  $j = 1, \dots, 7$ . Hence Proposition 4.1.4 yields that  $\mathbb{X}$  is a CB-scheme.

Next we consider the subscheme  $\mathbb{Y} = \mathbb{X} \setminus \{p_6\}$  of  $\mathbb{X}$ . Then the support of  $\mathbb{Y}$  is  $\text{Supp}(\mathbb{Y}) = \{p_1, \dots, p_5, p_7\}$ . The Hilbert functions of  $\mathbb{Y}$  and its subschemes are

$$\begin{aligned} \text{HF}_{\mathbb{Y}} &: 1 \ 3 \ 6 \ 7 \ 7 \ \dots \\ \text{HF}_{\mathbb{Y} \setminus \{p_j\}} &: 1 \ 3 \ 5 \ 6 \ 6 \ \dots \quad (j = 1, 3, 5) \\ \text{HF}_{\mathbb{Y} \setminus \{p_j\}} &: 1 \ 3 \ 6 \ 6 \ 6 \ \dots \quad (j = 2, 4) \\ \text{HF}_{\mathbb{Y} \setminus \{p_7\}} &: 1 \ 3 \ 5 \ 5 \ 5 \ \dots \end{aligned}$$

We see that  $\dim_K(\mathcal{I}_{\mathbb{Y} \setminus \{p_j\}/\mathbb{Y}})_{r_{\mathbb{Y}}-1} = \dim_K(\mathcal{I}_{\mathbb{Y} \setminus \{p_j\}/\mathbb{Y}})_2 = 1 = \dim_K \mathcal{O}_{\mathbb{Y}, p_j}$  for  $j = 1, 3, 5$ . Thus the scheme  $\mathbb{Y}$  is not a CB-scheme by Proposition 4.1.4.

**Lemma 4.1.6.** *A homogeneous element  $\varphi \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  satisfies  $\text{Ann}_R(\varphi) = \langle 0 \rangle$  if and only if for every  $p_j \in \text{Supp}(\mathbb{X})$  and for every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  we have  $f \cdot \varphi \neq 0$  for any element  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ .*

*Proof.* If  $\text{Ann}_R(\varphi) = \langle 0 \rangle$ , then it is clear that  $f \cdot \varphi \neq 0$  for all  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ . Conversely, if  $g \cdot \varphi = 0$  for some  $g \in R_i \setminus \{0\}$  with  $i \geq 0$ , then we may assume that  $g_{p_j} \neq 0$  for some  $j \in \{1, \dots, s\}$ . Let  $U = \langle g_{p_j} \rangle_{\mathcal{O}_{\mathbb{X}, p_j}}$ . It is clear that  $U \neq \langle 0 \rangle$ , and so  $U \cap \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \neq \langle 0 \rangle$ , since  $\text{Ass}(U) = \{\mathfrak{m}_{\mathbb{X}, p_j}\}$  (see [Ku4, Chapter IV, §3, p. 189]). Thus we can find an element  $a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}$  such that  $s_j = ag_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$ . Now we let  $f = \tilde{\iota}^{-1}((0, \dots, 0, s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$  and  $h = \iota^{-1}((0, \dots, 0, a T_j^{r_{\mathbb{X}}}, 0, \dots, 0)) \in R_{r_{\mathbb{X}}}$ . Then  $gh = x_0^i f$  and  $x_0^i f \cdot \varphi = 0$ . Since  $x_0$  is a non-zerodivisor on  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  (cf. [Kr2, Lemma 1.3]), we get  $f \cdot \varphi = 0$ . Moreover, the ideal  $\langle f \rangle_R$  defines a maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  by Proposition 2.3.17. Hence the conclusion follows.  $\square$

At this point we are able to characterize Cayley-Bacharach schemes in terms of their Dedekind differents.

**Theorem 4.1.7.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is a CB-scheme.*
- (ii) *For each  $p_j \in \text{Supp}(\mathbb{X})$ , every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  satisfies*

$$x_0^{r_{\mathbb{X}}-1} (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \not\subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}.$$

*If, moreover, the field  $K$  is infinite, the above conditions are equivalent to:*

- (iii) *There is an element  $g \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $\text{Ann}_R(g) = \langle 0 \rangle$ .*

*Proof.* (i) $\Rightarrow$ (ii): Since  $\mathbb{X}$  is locally Gorenstein, there is for each point  $p_j$  a uniquely determined maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  corresponding to a socle element  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  of  $\mathcal{O}_{\mathbb{X}, p_j}$ . Let  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  be such that whose residue classes form a  $K$ -basis of  $K(p_j)$ , and let  $\{f_{j1}^*, \dots, f_{j\kappa_j}^*\}$  (resp.  $\{f_{j1}, \dots, f_{j\kappa_j}\}$ ) be a set of minimal separators (resp. separators) of  $\mathbb{Y}_j$  in  $\mathbb{X}$  with respect to  $s_j$  and  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ . Since  $\mathbb{X}$  is a CB-scheme, for every  $j \in \{1, \dots, s\}$  there is an index  $k_j \in \{1, \dots, \kappa_j\}$  such that  $f_{jk_j} = f_{jk_j}^* \notin x_0 R_{r_{\mathbb{X}}-1}$ . We assume without loss of generality that  $f_{j1} = f_{j1}^* \notin x_0 R_{r_{\mathbb{X}}-1}$  for  $j = 1, \dots, s$ . Let us fix an index  $j \in \{1, \dots, s\}$ . Then we can define a  $K$ -linear map  $\bar{\varphi}_j : R_{r_{\mathbb{X}}} \rightarrow K$  such that  $\bar{\varphi}_j(x_0 R_{r_{\mathbb{X}}-1}) = \langle 0 \rangle$  and  $\bar{\varphi}_j(f_{j1}) \neq 0$ . By [Kr2, Lemma 1.5], we may lift  $\bar{\varphi}_j$  to obtain a  $K[x_0]$ -linear map  $\varphi_j : R \rightarrow K[x_0]$  of degree  $-r_{\mathbb{X}}$ , i.e.,  $\varphi_j$  is an element of  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}}$  such that  $\varphi_j|_{R_{r_{\mathbb{X}}}} = \bar{\varphi}_j$ , especially,  $\varphi_j(f_{j1}) \neq 0$ .

Given a homogeneous element  $f \in (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ , we proceed to show that  $f \cdot \varphi_j \neq 0$ . According to Proposition 2.3.21, we have  $\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} = \langle f \rangle^{\text{sat}} = \langle f_{j1} \rangle^{\text{sat}}$ . This implies that  $x_0^k f_{j1} \in \langle f \rangle$  for some  $k \geq 0$ , and so we may write  $x_0^k f_{j1} = fh$  for some  $h \in R_k \setminus \{0\}$ . Consequently, we have  $(f \cdot \varphi_j)(h) = \varphi_j(hf) = \varphi_j(x_0^k f_{j1}) = x_0^k \varphi_j(f_{j1}) \neq 0$ . From this we conclude  $f \cdot \varphi_j \neq 0$  for all  $f \in (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ .

Since  $\mathfrak{C}_{R/K[x_0]} = \Phi(\underline{\text{Hom}}_{K[x_0]}(R, K[x_0]))$ , where  $\Phi$  is the monomorphism of graded  $R$ -modules in Definition 3.2.4, we find  $g_j^* = \Phi(\varphi_j) \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $f \cdot g_j^* \neq 0$  for all  $f \in (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ . By Proposition 3.2.5, we have  $x_0^{2r_{\mathbb{X}}} \in \vartheta_D(R/K[x_0])$ . This enables us to write  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \subseteq (R_{x_0})_{-r_{\mathbb{X}}}$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}} \setminus \{0\}$ . We have  $f_{jk_j} \cdot \tilde{g}_j^* \neq 0$  for  $k_j = 1, \dots, \kappa_j$ . It follows from Lemma 2.3.25 that  $(\tilde{g}_j^*)_{p_j}$  is a unit element of  $\mathcal{O}_{\mathbb{X}, p_j}$ . Therefore, for every  $j \in \{1, \dots, s\}$ , we have constructed an element  $g_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}} \setminus \{0\}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ .

Now we assume for a contradiction that there is a maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  such that

$$x_0^{r_{\mathbb{X}}-1} (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}.$$

For such an index  $j$ , let  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  be constructed as above. To get a contradiction, it suffices to show  $x_0 \mid f_{j1}$ . We write  $\tilde{v}(f_{j1}) = (0, \dots, 0, e_{j1} s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0)$  and  $f = \tilde{v}^{-1}((0, \dots, 0, e_{j1} (\tilde{g}_j^*)_{p_j}^{-1} s_j T_j^{r_{\mathbb{X}}}, 0, \dots, 0))$ . Then  $0 \neq x_0^{r_{\mathbb{X}}-1} f \in x_0^{r_{\mathbb{X}}-1} (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}}$  and  $f \tilde{g}_j^* = x_0^{r_{\mathbb{X}}} f_{j1}$ , especially,  $x_0^{r_{\mathbb{X}}-1} f \in \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ . Also, we observe that

$$x_0^{r_{\mathbb{X}}-1} f \cdot g_j^* = x_0^{r_{\mathbb{X}}-1} f \cdot (x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^*) = x_0^{-r_{\mathbb{X}}-1} f \tilde{g}_j^* = x_0^{-r_{\mathbb{X}}-1+r_{\mathbb{X}}} f_{j1} = x_0^{-1} f_{j1}.$$

Thus it follows from  $\mathfrak{C}_{R/K[x_0]} \cdot \vartheta_D(R/K[x_0]) \subseteq R$  that  $x_0^{-1} f_{j1} \in R_{r_{\mathbb{X}}-1} \setminus \{0\}$ , and hence we obtain  $f_{j1} \in x_0 R_{r_{\mathbb{X}}-1}$  or  $x_0 \mid f_{j1}$ , as desired.

(ii) $\Rightarrow$ (i): Suppose that the scheme  $\mathbb{X}$  is not a CB-scheme. Then there is a maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  such that  $\deg(f_{jk_j}^*) \leq r_{\mathbb{X}} - 1$  for all  $k_j = 1, \dots, \kappa_j$ . Notice

that  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^*$  in  $x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} R_{\deg(f_{jk_j}^*)}$  for all  $k_j = 1, \dots, \varkappa_j$ . We let  $m = \deg(\mathbb{X})$ , we let  $\{t_1, \dots, t_m\}$  be the  $K[x_0]$ -basis of  $R$  introduced in Section 2.4, and we let  $\{t_1^*, \dots, t_m^*\} \subseteq \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  be its dual basis. We remark that the element  $t_k^*$  is homogeneous of degree  $\deg(t_k^*) = -\deg(t_k) = -n_k$  and  $n_k \leq r_{\mathbb{X}}$  for  $k = 1, \dots, m$ . Also, we may write  $\mathfrak{C}_{R/K[x_0]} = \langle g_1, \dots, g_m \rangle_{K[x_0]} \subseteq R_{x_0}$  where  $g_k = \Phi(t_k^*) = x_0^{-2r_{\mathbb{X}}} \tilde{g}_k$  with  $\tilde{g}_k \in R_{2r_{\mathbb{X}} - n_k}$  for  $k = 1, \dots, m$ . By Lemma 2.3.25, there are  $c_{j1}, \dots, c_{j\varkappa_j} \in K$  such that  $f_{j1} \cdot \tilde{g}_k = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{2r_{\mathbb{X}} - n_k} f_{jk_j}$ . We calculate

$$\begin{aligned} x_0^{r_{\mathbb{X}}-1} f_{j1} \cdot g_k &= x_0^{r_{\mathbb{X}}-1} f_{j1} \cdot (x_0^{-2r_{\mathbb{X}}} \tilde{g}_k) = x_0^{-r_{\mathbb{X}}-1} f_{j1} \tilde{g}_k = x_0^{r_{\mathbb{X}}-n_k-1} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j} \\ &= x_0^{r_{\mathbb{X}}-n_k} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*) - 1} f_{jk_j}^* \in R_{2r_{\mathbb{X}} - n_k - 1}. \end{aligned}$$

This implies  $x_0^{r_{\mathbb{X}}-1} f_{j1} g_k \in R_{2r_{\mathbb{X}} - n_k - 1}$  for every  $k \in \{1, \dots, m\}$ . Hence the element  $x_0^{r_{\mathbb{X}}-1} f_{j1}$  is contained in  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ . Similarly, we can show that  $x_0^{r_{\mathbb{X}}-1} f_{jk_j}$  is a homogeneous element of degree  $2r_{\mathbb{X}} - 1$  of  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$  for all  $k_j = 2, \dots, \varkappa_j$ .

Therefore we obtain

$$x_0^{r_{\mathbb{X}}-1} (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} = \langle x_0^{r_{\mathbb{X}}-1} f_{j1}, \dots, x_0^{r_{\mathbb{X}}-1} f_{j\varkappa_j} \rangle_K \subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1},$$

in contradiction to the assumption that  $x_0^{r_{\mathbb{X}}-1} (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \not\subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ .

Next we assume that the field  $K$  is infinite. We want to prove “(i)  $\Leftrightarrow$  (iii)”. Suppose that  $\mathbb{X}$  is a CB-scheme and  $f_{j1} = f_{j1}^* \notin x_0 R_{r_{\mathbb{X}}-1}$  for  $j = 1, \dots, s$ . Then there are elements  $\bar{\varphi}_1, \dots, \bar{\varphi}_s \in \text{Hom}_K(R_{r_{\mathbb{X}}}, K)$  such that  $\bar{\varphi}_j(x_0 R_{r_{\mathbb{X}}-1}) = \langle 0 \rangle$  and  $\bar{\varphi}_j(f_{j1}) \neq 0$  for all  $j \in \{1, \dots, s\}$ . Since the field  $K$  is infinite, there are  $\lambda_1, \dots, \lambda_s \in K$  such that the  $K$ -linear map  $\bar{\varphi} = \sum_{j=1}^s \lambda_j \bar{\varphi}_j$  satisfies  $\bar{\varphi}(x_0 R_{r_{\mathbb{X}}-1}) = \langle 0 \rangle$  and  $\bar{\varphi}(f_{j1}) \neq 0$  for all  $j = 1, \dots, s$ . Again [Kr2, Lemma 1.5] yields an element  $\varphi \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}}$  such that  $\varphi|_{R_{r_{\mathbb{X}}}} = \bar{\varphi}$ . For  $j \in \{1, \dots, s\}$ , we can argue as in the proof of “(i)  $\Rightarrow$  (ii)” to get  $f \cdot \varphi \neq 0$ , where  $f \in (\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \setminus \{0\}$ . By Lemma 4.1.6,  $\varphi \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-r_{\mathbb{X}}}$  satisfies  $\text{Ann}_R(\varphi) = \langle 0 \rangle$ . Thus we find  $g = \Phi(\varphi) \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $\text{Ann}_R(g) = \langle 0 \rangle$ .

Conversely, let  $g \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  with  $\text{Ann}_R(g) = \langle 0 \rangle$ , and let  $\varphi = \Phi^{-1}(g)$ . Clearly, we have  $\text{Ann}_R(\varphi) = \langle 0 \rangle$ . Assume that  $\mathbb{X}$  is not a CB-scheme. Then there is an index  $j \in \{1, \dots, s\}$  such that  $\deg_{\mathbb{X}}(p_j) < r_{\mathbb{X}}$ . For such an index  $j$ , we shall show that  $f_{j1} \cdot \varphi = 0$ . Indeed, the assumption implies  $\deg(f_{jk_j}^*) < r_{\mathbb{X}}$  and  $\varphi(f_{jk_j}) = \varphi(x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^*) = x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} \varphi(f_{jk_j}^*) = 0$  for all  $k_j = 1, \dots, \varkappa_j$ . Let  $i \geq 0$ , and let  $h \in R_i$  be a non-zero homogeneous element. If  $h f_{j1} = 0$ , then  $(f_{j1} \cdot \varphi)(h) = 0$ . Suppose that  $h f_{j1} \neq 0$ . In this case, Lemma 2.3.25 enables us to write  $h f_{j1} = \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^i f_{jk_j}$  for some elements

$c_{j_1}, \dots, c_{j_{\neq j}} \in K$ . Thus we have  $(f_{j_1} \cdot \varphi)(h) = \varphi(hf_{j_1}) = \varphi(\sum_{k_j=1}^{\neq j} c_{jk_j} x_0^i f_{jk_j}) = \sum_{k_j=1}^{\neq j} c_{jk_j} x_0^i \varphi(f_{jk_j}) = 0$ . Hence we obtain  $f_{j_1} \cdot \varphi = 0$ , and therefore  $\text{Ann}_R(\varphi) \neq \langle 0 \rangle$ , a contradiction.  $\square$

Given an arbitrary field  $K$ , if the condition (iii) of Theorem 4.1.7 is satisfied, then  $\mathbb{X}$  is a CB-scheme. However, the converse is not true, as the following example shows.

**Example 4.1.8.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{F}_2}^n$  be the projective point set consisting of three points  $p_1 = (1 : 1 : 0)$ ,  $p_2 = (1 : 0 : 1)$ , and  $p_3 = (1 : 1 : 1)$ . We have  $\text{HF}_{\mathbb{X}} : 1 \ 3 \ 3 \cdots$  and  $r_{\mathbb{X}} = 1$ . It is not difficult to check that  $\mathbb{X}$  is a CB-scheme. A calculation gives us  $(\mathfrak{C}_{R/\mathbb{F}_2[x_0]})_{-1} = \langle g_1, g_2 \rangle_{\mathbb{F}_2}$ , where  $g_1 = x_0^{-2}x_1$  and  $g_2 = x_0^{-2}x_2$ . If  $g \in (\mathfrak{C}_{R/\mathbb{F}_2[x_0]})_{-1}$ , then  $g$  is one of three forms:  $g_1$ ,  $g_2$ , and  $g_1 + g_2$ . We see that  $x_0 + x_1 \in \text{Ann}_R(g_1)$ ,  $x_0 + x_2 \in \text{Ann}_R(g_2)$ , and  $x_0 + x_1 + x_2 \in \text{Ann}_R(g_1 + g_2)$ . Thus  $(\mathfrak{C}_{R/\mathbb{F}_2[x_0]})_{-1}$  cannot contain an element  $g$  such that  $\text{Ann}_R(g) = \langle 0 \rangle$ . Hence the condition (iii) of Theorem 4.1.7 is not satisfied in this case.

The following corollary is an immediate consequence of Theorem 4.1.7.

**Corollary 4.1.9.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme.*

- (i) *If  $\mathbb{X}$  has  $K$ -rational support then it is a CB-scheme if and only if for every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\text{deg}(\mathbb{Y}) = \text{deg}(\mathbb{X}) - 1$  and for every separator  $f_{\mathbb{Y}}$  of  $\mathbb{Y}$  in  $\mathbb{X}$  we have  $x_0^{r_{\mathbb{X}}-1} f_{\mathbb{Y}} \notin \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ .*
- (ii) *If the Dedekind different satisfies  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1} = \langle 0 \rangle$  then the scheme  $\mathbb{X}$  is a CB-scheme.*

**Example 4.1.10.** Let  $\mathbb{X} = \{p_1, \dots, p_6\}$  be the projective point set in  $\mathbb{P}_{\mathbb{Q}}^2$ , where  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 2 : 0)$ ,  $p_3 = (1 : 2 : 1)$ ,  $p_4 = (1 : 0 : 2)$ ,  $p_5 = (1 : 1 : 2)$ , and  $p_6 = (1 : 2 : 2)$ . We sketch  $\mathbb{X}$  in the affine plane  $D_+(X_0) = \mathbb{A}_{\mathbb{Q}}^2$  as follows:

$$\begin{array}{ccc} (0, 2) & \bullet & \bullet & \bullet & (2, 2) \\ & & & \bullet & \\ (0, 0) & \bullet & & \bullet & (2, 0) \end{array}$$

Then  $\mathbb{X}$  has the Hilbert function  $\text{HF}_{\mathbb{X}} : 1 \ 3 \ 6 \ 6 \cdots$  and the regularity index  $r_{\mathbb{X}} = 2$ . An application of Proposition 3.2.29 gives us the Dedekind different

$$\vartheta_D(R/\mathbb{Q}[x_0]) = \langle x_2^4, x_1x_2^3, x_0x_2^3, x_1^4, x_0x_1^3, x_0^4 \rangle = \bigoplus_{i \geq 4} R_i.$$

This implies  $\vartheta_D(R/\mathbb{Q}[x_0])_{2r_{\mathbb{X}}-1} = \vartheta_D(R/\mathbb{Q}[x_0])_3 = \langle 0 \rangle$ . Therefore Corollary 4.1.9(ii) yields that  $\mathbb{X}$  is a CB-scheme.

**Proposition 4.1.11.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein CB-scheme. Then the Hilbert function of  $\vartheta_D(R/K[x_0])$  satisfies  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < r_{\mathbb{X}}$ ,  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \deg(\mathbb{X})$  for  $i \geq 2r_{\mathbb{X}}$  and*

$$0 \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(r_{\mathbb{X}}) \leq \cdots \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}} - 1) < \mathrm{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}}) = \deg(\mathbb{X}).$$

*In this case, the regularity index of  $\vartheta_D(R/K[x_0])$  is exactly  $2r_{\mathbb{X}}$ .*

*Proof.* Since  $\mathbb{X}$  is a CB-scheme, there are elements  $g_1^*, \dots, g_s^*$  in  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$  (as in the proof of Theorem 4.1.7). Let  $h \in \vartheta_D(R/K[x_0])_i$  with  $i < r_{\mathbb{X}}$ . Then we have  $h \cdot g_j^* = x_0^{-2r_{\mathbb{X}}} h \tilde{g}_j^* \in R_{i-r_{\mathbb{X}}} = \langle 0 \rangle$  for  $j = 1, \dots, s$ . This implies  $h \tilde{g}_j^* = 0$ , in particular,  $h_{p_j} \cdot (\tilde{g}_j^*)_{p_j} = 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$  for all  $j \in \{1, \dots, s\}$ . Since  $(\tilde{g}_j^*)_{p_j}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$  for  $j = 1, \dots, s$ , we have to get  $h_{p_j} = 0$  for all  $j = 1, \dots, s$ . In other words, we have  $\tilde{\iota}(h) = 0$ . So, we get  $h = 0$  (as  $\tilde{\iota}$  is an injection). Hence the Hilbert function of the Dedekind different satisfies  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = 0$  for  $i < r_{\mathbb{X}}$ . Moreover, Proposition 3.2.5 yields that  $0 \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(i+1)$  for  $i \geq r_{\mathbb{X}}$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \deg(\mathbb{X})$  for  $i \geq 2r_{\mathbb{X}}$ . Now we need to show that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(2r_{\mathbb{X}} - 1) < \deg(\mathbb{X})$ , i.e.,  $\vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1} \subsetneq R_{2r_{\mathbb{X}}-1}$ . But this follows from Theorem 4.1.7, since otherwise we have  $x_0^{r_{\mathbb{X}}-1}(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \subseteq \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$  for every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  and thus  $\mathbb{X}$  is not a CB-scheme.  $\square$

Now let  $\alpha_{\vartheta_D} := \min\{i \in \mathbb{Z} \mid \vartheta_D(R/K[x_0])_i \neq \langle 0 \rangle\}$  be the initial degree of the Dedekind different  $\vartheta_D(R/K[x_0])$ . If  $\mathbb{X}$  is a locally Gorenstein CB-scheme then we have  $r_{\mathbb{X}} \leq \alpha_{\vartheta_D} \leq 2r_{\mathbb{X}}$ . Our next proposition gives a bound for the Hilbert function of the Dedekind different when  $\mathbb{X}$  is a locally Gorenstein CB-scheme.

**Proposition 4.1.12.** *Let  $K$  be an infinite field, and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein CB-scheme. Then for all  $i \in \mathbb{Z}$  we have*

$$\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}}).$$

*Moreover, let  $i_0$  be the smallest number such that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0) = \mathrm{HF}_{\mathbb{X}}(i_0 - r_{\mathbb{X}}) > 0$ . Then we have*

- (i)  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for  $i \geq i_0$ .
- (ii)  $\vartheta_D(R/K[x_0]) = \left\langle \vartheta_D(R/K[x_0])_{\alpha_{\vartheta_D}}, \vartheta_D(R/K[x_0])_{\alpha_{\vartheta_D}+1}, \dots, \vartheta_D(R/K[x_0])_{i_0} \right\rangle_R$ .



*Proof.* Because  $\mathbb{X}$  is a CB-scheme and  $K$  is infinite, there is a non-zerodivisor  $\tilde{g} \in R_{r_{\mathbb{X}}}$  such that  $g = x_0^{-2r_{\mathbb{X}}}\tilde{g} \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  (see Theorem 4.1.7). For  $i < r_{\mathbb{X}}$  or  $i \geq 2r_{\mathbb{X}}$ , it follows from Proposition 4.1.11 that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$ . It remains to consider the case  $r_{\mathbb{X}} \leq i < 2r_{\mathbb{X}}$ . In this case we have  $\tilde{g} \cdot \vartheta_D(R/K[x_0])_i \subseteq x_0^{2r_{\mathbb{X}}}R_{i-r_{\mathbb{X}}}$ . Thus we obtain

$$\begin{aligned} \mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) &= \dim_K(\vartheta_D(R/K[x_0])_i) = \dim_K(\tilde{g} \cdot \vartheta_D(R/K[x_0])_i) \\ &\leq \dim_K(x_0^{2r_{\mathbb{X}}}R_{i-r_{\mathbb{X}}}) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}}). \end{aligned}$$

Now we prove the additional claims. For (i), it suffices to prove the equality  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0 + 1) = \mathrm{HF}_{\mathbb{X}}(i_0 + 1 - r_{\mathbb{X}})$ . Let  $f \in R_{i_0+1-r_{\mathbb{X}}} \setminus \{0\}$ . We may write  $f = x_0g_0 + x_1g_1 + \cdots + x_ng_n$  for some homogeneous elements  $g_0, \dots, g_n \in R_{i_0-r_{\mathbb{X}}}$ . By assumption, we have  $\tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0} = x_0^{2r_{\mathbb{X}}}R_{i_0-r_{\mathbb{X}}}$ . There is  $h_j \in \vartheta_D(R/K[x_0])_{i_0}$  such that  $x_0^{2r_{\mathbb{X}}}g_j = \tilde{g}h_j$  for every  $j \in \{0, \dots, n\}$ . Hence

$$\begin{aligned} x_0^{2r_{\mathbb{X}}}f &= x_0^{2r_{\mathbb{X}}}(x_0g_0 + x_1g_1 + \cdots + x_ng_n) = x_0\tilde{g}h_0 + x_1\tilde{g}h_1 + \cdots + x_n\tilde{g}h_n \\ &= \tilde{g}(x_0h_0 + x_1h_1 + \cdots + x_nh_n) \end{aligned}$$

and so  $x_0^{2r_{\mathbb{X}}}f \in \tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0+1}$ . Thus we have  $x_0^{2r_{\mathbb{X}}}R_{i_0+1-r_{\mathbb{X}}} = \tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0+1}$ . In other words, we have  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0 + 1) = \mathrm{HF}_{\mathbb{X}}(i_0 + 1 - r_{\mathbb{X}})$ .

To prove (ii), we let

$$W = \left\{ h \in \vartheta_D(R/K[x_0])_{i_0+1} \mid h = \sum_{j=0}^n x_j h_j, h_j \in \vartheta_D(R/K[x_0])_{i_0} \right\}$$

be the subspace of  $\vartheta_D(R/K[x_0])_{i_0+1}$  generated by  $\vartheta_D(R/K[x_0])_{i_0}$ . We need to show that  $W = \vartheta_D(R/K[x_0])_{i_0+1}$ . Let  $h \in \vartheta_D(R/K[x_0])_{i_0+1} \setminus \{0\}$ . Then there is an element  $f \in R_{i_0+1-r_{\mathbb{X}}}$  such that  $\tilde{g}h = x_0^{2r_{\mathbb{X}}}f$ . Using the same argument as in the proof of (i), we get  $\tilde{g}h = x_0^{2r_{\mathbb{X}}}f = \tilde{g}(x_0h_0 + x_1h_1 + \cdots + x_nh_n)$  for some  $h_0, h_1, \dots, h_n \in \vartheta_D(R/K[x_0])_{i_0}$ . This implies  $\tilde{g}(h - \sum_{j=0}^n x_j h_j) = 0$ . Since  $\mathrm{Ann}_R(\tilde{g}) = \langle 0 \rangle$ , we have  $h = \sum_{j=0}^n x_j h_j \in W$ . Hence  $W = \vartheta_D(R/K[x_0])_{i_0+1}$ , and the claim follows by induction.  $\square$

**Corollary 4.1.13.** *Let  $K$  be an infinite field, and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth CB-scheme. Then we have*

$$\mathrm{HF}_{\vartheta_K(R/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$$

for all  $i \in \mathbb{Z}$ . If let  $i_0$  be the smallest number such that  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i_0) = \mathrm{HF}_{\mathbb{X}}(i_0 - r_{\mathbb{X}}) > 0$ . Then the following claims hold:

- (i)  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for  $i \geq i_0$ .
- (ii)  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle \vartheta_{\mathbb{X}}(R/K[x_0])_{\alpha_{\vartheta_{\mathbb{X}}}}, \vartheta_{\mathbb{X}}(R/K[x_0])_{\alpha_{\vartheta_{\mathbb{X}}+1}}, \dots, \vartheta_{\mathbb{X}}(R/K[x_0])_{i_0} \rangle_R$ .
- (iii) If, in addition, we have  $\vartheta_K(R/K[x_0])_i = \vartheta_{\mathbb{X}}(R/K[x_0])_i$  for some  $i \in \{i_0, \dots, 2r_{\mathbb{X}}\}$ , then  $\mathrm{ri}(\vartheta_K(R/K[x_0])) = 2r_{\mathbb{X}}$ .

*Proof.* This is a consequence of Proposition 4.1.12 and the fact that  $\vartheta_K(R/K[x_0])$  is a subideal of  $\vartheta_{\mathbb{X}}(R/K[x_0])$ .  $\square$

**Example 4.1.14.** Let  $\mathbb{X} = \{p_1, \dots, p_7\} \subseteq \mathbb{P}_{\mathbb{Q}}^3$ , where  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 1)$ ,  $p_3 = (1 : 1 : 1 : 1)$ ,  $p_4 = (1 : 3 : 0 : 1)$ ,  $p_5 = (1 : 4 : 0 : 1)$ ,  $p_6 = (1 : 5 : 0 : 1)$ , and  $p_7 = (1 : 6 : 0 : 0)$ . Then we have  $\mathrm{HF}_{\mathbb{X}} : 1 \ 4 \ 6 \ 7 \ 7 \dots$  and  $r_{\mathbb{X}} = 3$ . By using Proposition 3.2.29, we see that the Noether-Dedekind different of  $R/\mathbb{Q}[x_0]$  is given by

$$\begin{aligned} \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0]) = \langle & x_2x_3, x_0^3 - \frac{1}{18}x_0x_1^2 + \frac{1}{18}x_1^2x_3 - x_3^3, \\ & x_1^3x_3 - \frac{39}{4}x_1^2x_3^2 + \frac{59}{2}x_1x_3^3 - \frac{107}{4}x_3^4, x_1x_3^4 - \frac{41}{11}x_3^5 \rangle. \end{aligned}$$

The Hilbert function of  $\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])$  is  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 1 \ 2 \ 4 \ 6 \ 7 \ 7 \dots$ . Since  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(r_{\mathbb{X}} - 1) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(2) = 1 \neq 0$ , we see that  $\mathbb{X}$  is not a CB-scheme. We also see that  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(3) = 2 > 1 = \mathrm{HF}_{\mathbb{X}}(3 - r_{\mathbb{X}})$ . Hence we cannot omit the hypothesis that  $\mathbb{X}$  is a CB-scheme in Proposition 4.1.12.

Our next proposition provides a characterization of 0-dimensional arithmetically Gorenstein schemes. Moreover, it gives an affirmative answer to a question posed in [GH],[DM]: **CB-scheme** + (?) = **Complete intersection?** if  $\mathbb{X}$  is a 0-dimensional smooth subscheme of  $\mathbb{P}_K^n$ .

**Proposition 4.1.15.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme.*

- (i) *The scheme  $\mathbb{X}$  is arithmetically Gorenstein if and only if it is a locally Gorenstein CB-scheme and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ .*
- (ii) *If  $\mathbb{X}$  is smooth, then  $\mathbb{X}$  is a complete intersection if and only if it is a CB-scheme and  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ .*

*Proof.* (i) Suppose that the scheme  $\mathbb{X}$  is arithmetically Gorenstein. Then it is clearly true that  $\mathbb{X}$  is locally Gorenstein. By Proposition 3.2.8, there is a non-zero-divisor  $h$  of  $R$  contained in  $\vartheta_D(R/K[x_0])_{r_{\mathbb{X}}}$  and an element  $g \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $\mathrm{Ann}_R(g) = \langle 0 \rangle$ . Hence  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$  and the scheme  $\mathbb{X}$  is a CB-scheme by Theorem 4.1.7.

Conversely, we remark that if  $\mathbb{X}$  is locally Gorenstein, then  $\vartheta_D(R/K[x_0])$  is well-defined. Since  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ , we let  $h \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}} \setminus \{0\}$ . Then there

is a point  $p_j \in \text{Supp}(\mathbb{X})$  such that  $h_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ . Because  $\mathbb{X}$  is a CB-scheme, we can argue as in the proof of Theorem 4.1.7 to get an element  $g_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \setminus \{0\}$  such that  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . In  $\mathcal{O}_{\mathbb{X}, p_j}$ , we have  $h_{p_j} \cdot (\tilde{g}_j^*)_{p_j} \neq 0$ . This implies  $h \cdot \tilde{g}_j^* \neq 0$ , and hence  $h \cdot g_j^* \neq 0$  (as  $x_0$  is a non-zero-divisor of  $R$ ). Furthermore, there is an element  $c \in K \setminus \{0\}$  such that  $c = h \cdot g_j^* \in R_0 = K$ . Thus  $h$  is a non-zero-divisor of  $R$ , and therefore the scheme  $\mathbb{X}$  is arithmetically Gorenstein by Proposition 3.2.8.

(ii) Obviously, if  $\mathbb{X}$  is a complete intersection then it is a CB-scheme. Moreover, it follows from Corollary 3.3.5 that  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ . Next we prove the converse. Since  $\vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$  and  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ , we have  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ . By (i), the 0-dimensional smooth scheme  $\mathbb{X}$  is arithmetically Gorenstein. Therefore, by Proposition 3.3.12, the scheme  $\mathbb{X}$  is a complete intersection, as we wanted to show.  $\square$

A similar result as [Kr2, Corollary 2.5] is given by the following corollary.

**Corollary 4.1.16.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. Then  $\mathbb{X}$  is arithmetically Gorenstein if and only if it is a locally Gorenstein CB-scheme and  $\text{HF}_{\mathbb{X}}(i) + \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) = \text{deg}(\mathbb{X})$  for all  $i \in \mathbb{Z}$ .*

*Proof.* This result follows from Proposition 3.2.8 and Theorem 4.1.7. Here we notice that if  $\mathbb{X}$  is a locally Gorenstein CB-scheme and  $\Delta_{\mathbb{X}} = \text{deg}(\mathbb{X}) - \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) = 1$  then  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle g \rangle_K$  with  $\text{Ann}_R(g) = \langle 0 \rangle$ . Indeed, it follows from  $\Delta_{\mathbb{X}} = 1$  that the  $K$ -vector space  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  is generated by one element  $g = x_0^{-2r_{\mathbb{X}}} \tilde{g}$ , where  $\tilde{g} \in R_{r_{\mathbb{X}}} \setminus \{0\}$ . Since  $\mathbb{X}$  is a CB-scheme, a similar argument as in the proof of Theorem 4.1.7 implies that the element  $\tilde{g}_{p_j}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$  for every  $j \in \{1, \dots, s\}$ . By Lemma 2.3.11, we have that  $\tilde{g}$  is a non-zero-divisor of  $R$ , and hence  $\text{Ann}_R(g) = \langle 0 \rangle$ .  $\square$

**Corollary 4.1.17.** *Let  $\mathbb{X}$  be a 0-dimensional smooth CB-scheme in the projective plane  $\mathbb{P}_K^2$ . If  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}} + 1) = \text{HF}_{\mathbb{X}}(1)$ , then  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

*Proof.* We see that if  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ , then the scheme  $\mathbb{X}$  is a complete intersection by Proposition 4.1.15, and so we get  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ . Now we suppose  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) = 0$ . Then we have to get  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}}) = 0$ . Otherwise, it follows from Proposition 4.1.15 that  $\mathbb{X}$  is arithmetically Gorenstein. Since  $n = 2$ , it is well known (cf. [Ser, Proposition 7]) that  $\mathbb{X}$  is arithmetically Gorenstein if and only if it is a complete intersection, and hence  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ , a contradiction.

Next, by the hypothesis and Corollary 4.1.13, we have  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}} + 1) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}} + 1) = \mathrm{HF}_{\mathbb{X}}(1)$ . From this we argue as in the proof of Proposition 4.1.12 to obtain  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}} + i) = \mathrm{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(r_{\mathbb{X}} + i)$  for all  $i \geq 1$ . Therefore the conclusion follows.  $\square$

The following two examples show how one can apply Proposition 4.1.15 in practice.

**Example 4.1.18.** Let  $\mathbb{X} = \{p_1, \dots, p_8\} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be given by  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 0)$ ,  $p_3 = (1 : -1 : 0 : 0)$ ,  $p_4 = (1 : 2 : 0 : 0)$ ,  $p_5 = (1 : 0 : 1 : 1)$ ,  $p_6 = (1 : 0 : 0 : 1)$ ,  $p_7 = (1 : 0 : -1 : 1)$ , and  $p_8 = (1 : 0 : 2 : 1)$ . Here  $\{p_1, p_2, p_3, p_4\} \subseteq L_1 = \mathcal{Z}^+(X_2, X_3)$  and  $\{p_5, p_6, p_7, p_8\} \subseteq L_2 = \mathcal{Z}^+(X_3 - X_0, X_1)$ , where  $L_1$  and  $L_2$  are two skew lines in  $\mathbb{P}_{\mathbb{Q}}^3$ . Then we have  $\mathrm{HF}_{\mathbb{X}} : 1 \ 4 \ 6 \ 8 \ 8 \ \dots$  and  $r_{\mathbb{X}} = 3$ . It is easy to check that  $\mathbb{X}$  is a CB-scheme. Also, we have

$$\begin{aligned} \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0]) = \langle & x_2^2 x_3^2 - x_2 x_3^3 - \frac{1}{5} x_3^4, x_2^3 x_3 - 2x_2 x_3^3 + \frac{1}{5} x_3^4, \\ & x_0^2 x_1^2 + 4x_0 x_1^3 - 3x_1^4, x_0^4 + 15x_0 x_1^3 - 10x_1^4 - x_3^4 \rangle \end{aligned}$$

and  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 4 \ 6 \ 8 \ 8 \ \dots$ , especially,  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(r_{\mathbb{X}}) = 0$ . Thus it follows from Proposition 4.1.15 that  $\mathbb{X}$  is not arithmetically Gorenstein.

**Example 4.1.19.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set consisting of eight points  $p_1, \dots, p_8$  on the twisted cubic curve  $C = \{(u^3 : u^2 v : uv^2 : v^3) \mid (u : v) \in \mathbb{P}_{\mathbb{Q}}^1\}$ , where  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 1 : 1)$ ,  $p_3 = (1 : -1 : 1 : -1)$ ,  $p_4 = (1 : 2 : 4 : 8)$ ,  $p_5 = (1 : -2 : 4 : -8)$ ,  $p_6 = (8 : 4 : 2 : 1)$ ,  $p_7 = (1 : 3 : 9 : 27)$ , and  $p_8 = (1 : -3 : 9 : -27)$ . Here  $\mathrm{HF}_{\mathbb{X}} : 1 \ 4 \ 7 \ 8 \ 8 \ \dots$  and  $r_{\mathbb{X}} = 3$ . It is well known (cf. [GO, Section 4]) that  $\mathbb{X}$  is a CB-scheme. Moreover, we have  $\vartheta_K(R/\mathbb{Q}[x_0]) = \langle h_1, h_2, h_3, h_4 \rangle$ , where

$$\begin{aligned} h_1 &= x_0 x_2 x_3^2 + \frac{834905041}{46905556898} x_0 x_3^3 - \frac{10864852683}{23452778449} x_1 x_3^3 - \frac{118190989}{46905556898} x_2 x_3^3 + \frac{985439798}{23452778449} x_3^4, \\ h_2 &= x_0 x_1 x_3^2 - \frac{9195042601}{23452778449} x_0 x_3^3 - \frac{1669810082}{23452778449} x_1 x_3^3 + \frac{749057820}{23452778449} x_2 x_3^3 + \frac{236381978}{23452778449} x_3^4, \\ h_3 &= x_0^2 x_3^2 + \frac{648448323}{23452778449} x_0 x_3^3 - \frac{5828244873}{23452778449} x_1 x_3^3 - \frac{91795767}{23452778449} x_2 x_3^3 + \frac{712547456}{23452778449} x_3^4, \\ h_4 &= x_0^4 + \frac{2300442431639}{3377200096656} x_0 x_3^3 - \frac{973727660323}{844300024164} x_1 x_3^3 - \frac{325655676731}{3377200096656} x_2 x_3^3 + \frac{137268654523}{844300024164} x_3^4. \end{aligned}$$

Notice that  $\deg(h_j) = 4$  for all  $j = 1, \dots, 4$ . Thus the Hilbert function of the Kähler different  $\vartheta_K(R/\mathbb{Q}[x_0])$  satisfies  $\mathrm{HF}_{\vartheta_K(R/\mathbb{Q}[x_0])}(r_{\mathbb{X}}) = \mathrm{HF}_{\vartheta_K(R/\mathbb{Q}[x_0])}(3) = 0$ , and hence  $\mathbb{X}$  is not a complete intersection by Proposition 4.1.15. On the other hand, the Noether-Dedekind different  $\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])$  is a principal ideal of  $R$  generated by

$$h = x_0^3 - \frac{337}{18} x_0^2 x_3 - \frac{1561}{432} x_0 x_1 x_3 + \frac{6433}{216} x_0 x_2 x_3 + \frac{301}{216} x_0 x_3^2 - \frac{979}{108} x_1 x_3^2 - \frac{49}{432} x_2 x_3^2 + \frac{145}{216} x_3^3.$$

Therefore Proposition 4.1.15 implies that the scheme  $\mathbb{X}$  is arithmetically Gorenstein and  $\mathrm{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i - r_{\mathbb{X}})$  for all  $i \in \mathbb{Z}$ .

**Proposition 4.1.20.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . If there exists an index  $j \in \{1, \dots, s\}$  such that  $p_j \notin \mathbb{X}(K)$  and the maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  satisfies  $\alpha_{\mathbb{Y}_j/\mathbb{X}} = r_{\mathbb{X}}$ , then  $\mathbb{X}$  cannot be arithmetically Gorenstein.*

*Proof.* We only need to show that  $\Delta_{\mathbb{X}} \geq 2$ . For this purpose, we let  $Lf$  denote the image of each element  $f \in R$  in  $\bar{R} = R/\langle x_0 \rangle$ . Let  $\varkappa_j = \dim_K K(p_j)$ , let  $\{f_{j1}, \dots, f_{j\varkappa_j}\} \subseteq R_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{Y}_j$  in  $\mathbb{X}$ . Since  $p_j \notin \mathbb{X}(K)$ , we have  $\varkappa_j \geq 2$ . We claim that the set  $\{Lf_{j1}, \dots, Lf_{j\varkappa_j}\}$  is  $K$ -linearly independent. Indeed, suppose that there are  $c_{j1}, \dots, c_{j\varkappa_j} \in K$ , not all equal to zero, such that  $c_{j1}Lf_{j1} + \dots + c_{j\varkappa_j}Lf_{j\varkappa_j} = 0$ . It follows from  $\dim_K \langle f_{j1}, \dots, f_{j\varkappa_j} \rangle_K = \varkappa_j$  that  $f = c_{j1}f_{j1} + \dots + c_{j\varkappa_j}f_{j\varkappa_j}$  is a non-zero element of  $(\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}}$ . Clearly, we have  $Lf = 0$ , and so we may write  $f = x_0h$  for some  $h \in R_{r_{\mathbb{X}}-1} \setminus \{0\}$ . We also observe that  $h \in \langle f \rangle^{\text{sat}} = \mathcal{I}_{\mathbb{Y}_j/\mathbb{X}}$  (see Lemma 2.3.12 and Proposition 2.3.21). This tells us that  $\alpha_{\mathbb{Y}_j/\mathbb{X}} < r_{\mathbb{X}}$ , in contradiction to our hypothesis that  $\alpha_{\mathbb{Y}_j/\mathbb{X}} = r_{\mathbb{X}}$ . Thus the set  $\{Lf_{j1}, \dots, Lf_{j\varkappa_j}\}$  is  $K$ -linearly independent. From this we obtain  $\Delta_{\mathbb{X}} = \dim_K \bar{R}_{r_{\mathbb{X}}} \geq \dim_K \langle Lf_{j1}, \dots, Lf_{j\varkappa_j} \rangle = \varkappa_j \geq 2$ , as we wanted.  $\square$

**Definition 4.1.21.** Let  $d \geq 0$ , let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . We say that  $\mathbb{X}$  has the **Cayley-Bacharach property of degree  $d$**  (in short,  $\mathbb{X}$  has  $\text{CBP}(d)$ ) if every point  $p_j \in \text{Supp}(\mathbb{X})$  has  $\deg_{\mathbb{X}}(p_j) \geq d + 1$ .

In this terminology,  $\mathbb{X}$  is a CB-scheme if and only if  $\mathbb{X}$  has  $\text{CBP}(r_{\mathbb{X}} - 1)$ . Moreover, if  $\mathbb{X}$  has  $\text{CBP}(d)$ , then  $\mathbb{X}$  has  $\text{CBP}(d - 1)$ , and every 0-dimensional scheme  $\mathbb{X}$  with  $\deg(\mathbb{X}) \geq 2$  has  $\text{CBP}(0)$ . Since  $\deg_{\mathbb{X}}(p_j) \leq r_{\mathbb{X}}$  for every point  $p_j \in \text{Supp}(\mathbb{X})$ , this implies that the number  $r_{\mathbb{X}} - 1$  is the largest degree  $d \geq 0$  such that  $\mathbb{X}$  can have the Cayley-Bacharach property of degree  $d$ . Therefore it is enough to consider the Cayley-Bacharach property in degree  $d \in \{0, \dots, r_{\mathbb{X}} - 1\}$ .

Let  $\tilde{\nu} : R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  be the injection given by  $\tilde{\nu}(f) = (f_{p_1}T_1^i, \dots, f_{p_s}T_s^i)$  for  $f \in R_i$  with  $i \geq 0$ . Recall that, for all  $j \in \{1, \dots, s\}$  and  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ , we have  $\mu(a) = \min\{i \in \mathbb{N} \mid (0, \dots, 0, aT_j^i, 0, \dots, 0) \in \tilde{\nu}(R)\}$  and  $\nu(a) = \max\{\mu(ab) \mid b \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}\}$ .

**Proposition 4.1.22.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, let  $0 \leq d \leq r_{\mathbb{X}} - 1$ , and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . Then the following statements are equivalent.*

- (i) *The scheme  $\mathbb{X}$  has  $\text{CBP}(d)$ .*
- (ii) *For all  $j \in \{1, \dots, s\}$  and  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$ , we have  $\nu(s_j) \geq d + 1$ .*
- (iii) *If  $\mathbb{Y} \subseteq \mathbb{X}$  is a maximal  $p_j$ -subscheme and  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$  is a set of separators of  $\mathbb{Y}$  in  $\mathbb{X}$ , then there exists  $k_j \in \{1, \dots, \varkappa_j\}$  such that  $x_0^{r_{\mathbb{X}}-d} \nmid f_{jk_j}$ .*

(iv) For each  $p_j \in \text{Supp}(\mathbb{X})$ , every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  satisfies

$$\dim_K(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_d < \varkappa_j.$$

*Proof.* This follows by a similar argument as in the proof of Proposition 4.1.4.  $\square$

**Proposition 4.1.23.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . If for every  $p_j \in \text{Supp}(\mathbb{X})$  the maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  satisfies*

$$x_0^d(\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{r_{\mathbb{X}}} \not\subseteq \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+d}$$

then  $\mathbb{X}$  has CBP( $d$ ).

In particular, if  $\mathbb{X}$  satisfies  $\vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+d} = \langle 0 \rangle$  then  $\mathbb{X}$  has CBP( $d$ ).

*Proof.* Suppose for contradiction that  $\mathbb{X}$  does not have CBP( $d$ ). By Proposition 4.1.22, there is an index  $j \in \{1, \dots, s\}$  such that the maximal  $p_j$ -subscheme  $\mathbb{Y}_j \subseteq \mathbb{X}$  has  $\dim_K(\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_d = \varkappa_j$ . Let  $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$  (resp.  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$ ) be a set of minimal separators (resp. separators) of  $\mathbb{Y}_j$  in  $\mathbb{X}$ . Then  $\deg(f_{jk_j}^*) \leq d$  and  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^*$  for all  $k_j \in \{1, \dots, \varkappa_j\}$ . Let  $m = \deg(\mathbb{X})$ , let  $\{t_1, \dots, t_m\}$  be the  $K[x_0]$ -basis of  $R$  as in Section 2.4, and let  $\{t_1^*, \dots, t_m^*\} \subseteq \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  be its dual basis. So, the Dedekind complementary module can be written as  $\mathfrak{C}_{R/K[x_0]} = \langle g_1, \dots, g_m \rangle_{K[x_0]}$ , where  $g_k = \Phi(t_k^*) = x_0^{-r_{\mathbb{X}} - n_k} \tilde{g}_k$  with  $\tilde{g}_k \in R_{r_{\mathbb{X}}}$  and  $n_k = \deg(t_k) \leq r_{\mathbb{X}}$  for  $k = 1, \dots, m$ . Thus we have

$$\begin{aligned} (x_0^d f_{jk}) \cdot (x_0^{-r_{\mathbb{X}} - n_l} \tilde{g}_l) &= x_0^{d - r_{\mathbb{X}} - n_l} f_{jk} \tilde{g}_l = x_0^{d - n_l} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} f_{jk_j} \\ &= x_0^{d - n_l} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^* \\ &= x_0^{r_{\mathbb{X}} - n_l} \sum_{k_j=1}^{\varkappa_j} c_{jk_j} x_0^{d - \deg(f_{jk_j}^*)} f_{jk_j}^* \end{aligned}$$

for some  $c_{j1}, \dots, c_{j\varkappa_j} \in K$ . Since  $r_{\mathbb{X}} - n_l \geq 0$  and  $d - \deg(f_{jk_j}^*) \geq 0$ , this implies that  $(x_0^d f_{jk}) \cdot (x_0^{-r_{\mathbb{X}} - n_l} \tilde{g}_l) \in R_{r_{\mathbb{X}}+d-n_l}$  for all  $l = 1, \dots, m$ . Consequently, the element  $x_0^d f_{jk}$  is contained in  $\vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}+d}$  for all  $k = 1, \dots, \varkappa_j$ , and therefore we get  $x_0^d(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \subseteq \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+d}$ , in contradiction to our assumption.  $\square$

The following example shows that the converse of Proposition 4.1.23 is not true in the general case (except for the case  $d = r_{\mathbb{X}} - 1$ ).

**Example 4.1.24.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the projective point set consisting of the points  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0)$ ,  $p_3 = (1 : 2 : 0)$ ,  $p_4 = (1 : 3 : 1)$ ,  $p_5 = (1 : 4 : 0)$ ,  $p_6 = (1 : 5 : 0)$ ,  $p_7 = (1 : 6 : 1)$ , and  $p_8 = (1 : 1 : 1)$ . It is easy to see that  $\text{HF}_{\mathbb{X}} : 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ \cdots$  and  $r_{\mathbb{X}} = 4$ . The Noether-Dedekind different is computed by

$$\begin{aligned} \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0]) = \langle & x_1^2 x_2^2 - \frac{20}{3} x_1 x_2^3 + 9x_2^4, x_0 x_1^6 - \frac{857}{3675} x_1^7, \\ & x_0^5 - \frac{393}{100} x_0^3 x_1^2 + \frac{1431}{400} x_0^2 x_1^3 - \frac{209}{200} x_0 x_1^4 + \frac{39}{400} x_1^5 - \frac{3919}{760} x_2^5 \rangle \end{aligned}$$

and its Hilbert function is  $\text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ \cdots$ . Clearly,  $\mathbb{X}$  is not arithmetically Gorenstein and  $\text{HF}_{\vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])}(r_{\mathbb{X}}) \neq 0$ . Hence  $\mathbb{X}$  is not a CB-scheme by Proposition 4.1.15(i). By applying Proposition 4.1.22, we can check that  $\mathbb{X}$  has CBP( $d$ ) for  $0 \leq d \leq 2$ . Now  $\mathbb{Y}_4 := \mathbb{X} \setminus \{p_4\}$  has a separator of the form  $f_4 = x_0 x_1^2 x_2 - 7x_0 x_1 x_2^2 + 6x_0 x_2^3$ . It is not difficult to verify that  $x_0^{r_{\mathbb{X}}-2} f_4 \in \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])_{2r_{\mathbb{X}}-2}$ . Thus  $\mathbb{X}$  has CBP(2), but  $x_0^2(\mathcal{I}_{\mathbb{Y}_4/\mathbb{X}})_{r_{\mathbb{X}}} \subseteq \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+2}$ .

Now we let  $\mathfrak{F}_{\tilde{R}/R}$  denote the conductor of  $R$  in the ring  $\tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$ , i.e.,  $\mathfrak{F}_{\tilde{R}/R} = \{f \in \tilde{R} \mid f\tilde{R} \subseteq R\}$ . When the scheme  $\mathbb{X}$  is reduced,  $\tilde{R}$  is the integral closure of  $R$  in its full quotient ring, and hence  $\mathfrak{F}_{\tilde{R}/R}$  is the conductor of  $R$  in its integral closure in the traditional sense. Notice that  $\mathfrak{F}_{\tilde{R}/R}$  is an ideal of both  $R$  and  $\tilde{R}$ . Some explicit descriptions of this ideal are presented by the following proposition which follows from [Kr3, Proposition 2.9]. Here it is not necessary to assume that the field  $K$  is an algebraically closed field.

**Proposition 4.1.25.** *We have the following assertions.*

- (i)  $\mathfrak{F}_{\tilde{R}/R} = \langle (0, \dots, 0, aT_j^{\nu(a)}, 0, \dots, 0) \mid 1 \leq j \leq s, a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\} \rangle$  as an ideal of  $\tilde{R}$ .
- (ii)  $\mathfrak{F}_{\tilde{R}/R} = \langle f_a \mid 1 \leq j \leq s, a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\} \rangle$  as an ideal of  $R$ , where  $f_a$  is the preimage of  $(0, \dots, 0, aT_j^{\nu(a)}, 0, \dots, 0)$  under the injection  $\tilde{\iota} : R \rightarrow \tilde{R}$ .

**Proposition 4.1.26.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . Then  $\mathbb{X}$  has CBP( $d$ ) if and only if  $\mathfrak{F}_{\tilde{R}/R} \subseteq \bigoplus_{i \geq d+1} R_i$ . In particular,  $\mathbb{X}$  is a CB-scheme if and only if  $\mathfrak{F}_{\tilde{R}/R} = \bigoplus_{i \geq r_{\mathbb{X}}} R_i$ .*

*Proof.* Let  $\mathbb{X}$  have CBP( $d$ ). For a contradiction, we assume that  $\mathfrak{F}_{\tilde{R}/R} \not\subseteq \bigoplus_{i \geq d+1} R_i$ . By Proposition 4.1.25(ii), we have  $\mathfrak{F}_{\tilde{R}/R} = \langle f_a \mid 1 \leq j \leq s, a \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\} \rangle$  as an ideal of  $R$ . It follows that there is a homogeneous element  $f_a \in \mathfrak{F}_{\tilde{R}/R} \setminus \{0\}$  such that  $\tilde{\iota}(f_a) = (0, \dots, 0, aT_j^{\nu(a)}, 0, \dots, 0)$  and  $\nu(a) \leq d$ . So, we can find an element  $b \in \mathcal{O}_{\mathbb{X}, p_j}$  such that  $s_j := ab \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  is non-zero. We then have

$$\nu(s_j) = \nu(ab) = \max\{\mu(abc) \mid c \in \mathcal{O}_{\mathbb{X}, p_j}, abc \neq 0\} \leq \nu(a) \leq d.$$

Thus Proposition 4.1.22 yields that  $\mathbb{X}$  does not have  $\text{CBP}(d)$ , a contradiction.

Conversely, we suppose that  $\mathfrak{F}_{\tilde{R}/R} \subseteq \bigoplus_{i \geq d+1} R_i$ . Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, and let  $\{f_{j1}^*, \dots, f_{j\kappa_j}^*\}$  be a set of minimal separators of  $\mathbb{Y}$  in  $\mathbb{X}$ . Recall that  $\tilde{\iota}(f_{jk_j}^*) = (0, \dots, 0, e_{jk_j} s_j T_j^{\mu(e_{jk_j} s_j)}, 0, \dots, 0)$  for  $k_j = 1, \dots, \kappa_j$ , where  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$  is a socle element corresponding to  $\mathbb{Y}$ , and where  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  is such that whose residue classes form a  $K$ -basis of  $K(p_j)$ . By Lemma 2.3.15, we have

$$\nu(s_j) = \max\{\mu(s_j b) \mid b \in \mathcal{O}_{\mathbb{X}, p_j}, s_j b \neq 0\} = \max\{\deg(f_{jk_j}^*) \mid 1 \leq k_j \leq \kappa_j\}.$$

Without loss of generality, we can assume that  $\nu(s_j) = \deg(f_{j1}^*) = \mu(e_{j1} s_j)$ . Thus we have  $\nu(s_j) = \nu(e_{j1} s_j)$  and  $f_{j1}^* \in \mathfrak{F}_{\tilde{R}/R}$ . Since  $\mathfrak{F}_{\tilde{R}/R} \subseteq \bigoplus_{i \geq d+1} R_i$ , it follows that  $\nu(s_j) = \deg(f_{j1}^*) \geq d+1$ . From this we conclude that  $\deg_{\mathbb{X}}(p_j) \geq d+1$  for all  $j = 1, \dots, s$ , in other words, the scheme  $\mathbb{X}$  has  $\text{CBP}(d)$ .

Moreover, if we identify  $R$  with its image under  $\tilde{\iota}$ , we have  $R_i = \tilde{R}_i$  for all  $i \geq r_{\mathbb{X}}$ . Thus the ideal  $\bigoplus_{i \geq r_{\mathbb{X}}} R_i$  is an ideal of both  $R$  and  $\tilde{R}$ , and it is contained in the conductor  $\mathfrak{F}_{\tilde{R}/R}$ . Hence the additional claim follows.  $\square$

Our next proposition presents a generalization of Dedekind's formula for the conductor  $\mathfrak{F}_{\tilde{R}/R}$  and the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$ . We use the notation  $\nu_j = \dim_K \mathcal{O}_{\mathbb{X}, p_j}$  for all  $j = 1, \dots, s$ .

**Proposition 4.1.27.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , let  $I_j$  be the homogeneous vanishing ideal of  $\mathbb{X}$  at  $p_j$ , and let  $\mathbb{Y}_j$  be the subscheme of  $\mathbb{X}$  defined by  $\mathcal{I}_{\mathbb{Y}_j} = \bigcap_{k \neq j} I_k$  for  $j = 1, \dots, s$ . The formula*

$$\mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]} = \tilde{R}$$

holds true if one of the following conditions is satisfied:

- (i) The scheme  $\mathbb{X}$  is a CB-scheme.
- (ii) For all  $j \in \{1, \dots, s\}$ , the Hilbert function of  $\mathbb{Y}_j$  is of the form

$$\text{HF}_{\mathbb{Y}_j}(i) = \begin{cases} \text{HF}_{\mathbb{X}}(i) & \text{if } i < \alpha_{\mathbb{Y}_j/\mathbb{X}}, \\ \text{HF}_{\mathbb{X}}(i) - \nu_j & \text{if } i \geq \alpha_{\mathbb{Y}_j/\mathbb{X}}. \end{cases}$$

In particular, if  $\mathbb{X}$  is arithmetically Gorenstein, then we have

$$\mathfrak{F}_{\tilde{R}/R} = \langle \tilde{\iota}(\vartheta_D(R/K[x_0])) \rangle_{\tilde{R}} \subseteq \tilde{R}.$$



*Proof.* Since  $\mathbb{X}$  is a locally Gorenstein scheme, we have  $\mathrm{Hom}_K(\mathcal{O}_{\mathbb{X}, p_j}, K) \cong \mathcal{O}_{\mathbb{X}, p_j}$  for all  $j = 1, \dots, s$ . This implies the isomorphism  $\tilde{R} \cong \underline{\mathrm{Hom}}_{K[x_0]}(\tilde{R}, K[x_0])$ . Hence we get  $\mathrm{HF}_{\underline{\mathrm{Hom}}_{K[x_0]}(\tilde{R}, K[x_0])}(i) = 0$  for  $i < 0$  and  $\mathrm{HF}_{\underline{\mathrm{Hom}}_{K[x_0]}(\tilde{R}, K[x_0])}(i) = \deg(\mathbb{X})$  for  $i \geq 0$ . Let  $f \in (\mathfrak{F}_{\tilde{R}/R})_i$ , let  $g \in (\mathfrak{C}_{R/K[x_0]})_k$ , and let  $\varphi \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_k$  such that  $g = \Phi(\varphi)$  where  $\Phi$  was defined in Definition 3.2.4. Observe that  $(f \cdot \varphi)(\tilde{R}) = \varphi(f\tilde{R}) \subseteq \varphi(R) \subseteq K[x_0]$ . This yields  $f \cdot \varphi \in \underline{\mathrm{Hom}}_{K[x_0]}(\tilde{R}, K[x_0])_{i+k}$ . If  $f \cdot \varphi \neq 0$ , then  $\deg(f \cdot \varphi) = i + k \geq 0$ . Thus we have  $f \cdot g = f \cdot \Phi(\varphi) = \Phi(f \cdot \varphi) \in \tilde{R}$ , and hence we get the inclusion  $\mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]} \subseteq \tilde{R}$ .

Now we prove the reverse inclusion when either (i) or (ii) is satisfied.

(i) For every  $j \in \{1, \dots, s\}$  we let  $\{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  and set  $\epsilon_{jk_j} := (0, \dots, 0, e_{jk_j}, 0, \dots, 0) \in \tilde{R}$ , where  $k_j \in \{1, \dots, \nu_j\}$ . Then the elements  $\{\epsilon_{11}, \dots, \epsilon_{s\nu_s}\}$  form a  $K[x_0]$ -basis of  $\tilde{R}$ . Thus it is enough to show that  $\epsilon_{11}, \dots, \epsilon_{s\nu_s}$  are contained in  $\mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]}$ . Since  $\mathbb{X}$  is a CB-scheme, for  $j = 1, \dots, s$  we argue as in the proof of Theorem 4.1.7 to get  $g_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $g_j^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_j^*$ , where  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}}$  and  $(\tilde{g}_j^*)_{p_j}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$ . By identifying  $R$  with its image in  $\tilde{R}$  under  $\tilde{\nu}$ , we have that the element  $h_{jk_j} := (0, \dots, 0, (\tilde{g}_j^*)_{p_j}^{-1} e_{jk_j} T_j^{r_{\mathbb{X}}}, 0, \dots, 0)$  is contained in  $R_{r_{\mathbb{X}}} \setminus \{0\}$  for all  $j \in \{1, \dots, s\}$  and all  $k_j \in \{1, \dots, \nu_j\}$ . We see that

$$h_{jk_j} \cdot g_j^* = x_0^{-2r_{\mathbb{X}}} h_{jk_j} \tilde{g}_j^* = x_0^{-2r_{\mathbb{X}}} (0, \dots, 0, e_{jk_j} T_j^{2r_{\mathbb{X}}}, 0, \dots, 0) = \epsilon_{jk_j} \in \tilde{R}.$$

By Proposition 4.1.26, we have  $\mathfrak{F}_{\tilde{R}/R} = \bigoplus_{i \geq r_{\mathbb{X}}} R_i$ . This implies  $h_{11}, \dots, h_{s\nu_s} \in \mathfrak{F}_{\tilde{R}/R}$ . Therefore we obtain  $\epsilon_{11}, \dots, \epsilon_{s\nu_s} \in \mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]}$ , and the claim follows.

(ii) In a similar fashion, we proceed to show that  $\epsilon_{11}, \dots, \epsilon_{s\nu_s} \in \mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]}$ . For  $j = 1, \dots, s$ , let  $\bar{\sigma}_j$  denote the trace map of the algebra  $\mathcal{O}_{\mathbb{X}, p_j}/K$ , and let  $\{e'_{j1}, \dots, e'_{j\nu_j}\}$  be the dual  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$  to the  $K$ -basis  $\{e_{j1}, \dots, e_{j\nu_j}\}$  with respect to  $\bar{\sigma}_j$ , i.e.,  $e'_{j1}, \dots, e'_{j\nu_j} \in \mathcal{O}_{\mathbb{X}, p_j}$  such that  $\bar{\sigma}_j(e_{jk_j} e'_{jk'_j}) = e_{jk'_j}^*(e_{jk_j}) = \delta_{k_j k'_j}$  for  $k_j, k'_j \in \{1, \dots, \nu_j\}$ . We may assume that  $e'_{j1}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$  for all  $j \in \{1, \dots, s\}$ . Notice that the subscheme  $\mathbb{Y}_j$  has degree  $\deg(\mathbb{Y}_j) = \deg(\mathbb{X}) - \nu_j$  for all  $j = 1, \dots, s$ . It follows from the assumption that  $\alpha_{\mathbb{Y}_j/\mathbb{X}} = \mu(e_{j1}) = \dots = \mu(e_{j\nu_j})$ . Then we have  $\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} = \langle f_{j1}^*, \dots, f_{j\nu_j}^* \rangle$ , where  $f_{jk_j}^* = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j} T_j^{\alpha_{\mathbb{Y}_j/\mathbb{X}}}, 0, \dots, 0))$  for  $k_j = 1, \dots, \nu_j$ . We also see that the image of  $(\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{\alpha_{\mathbb{Y}_j/\mathbb{X}}}$  in  $\tilde{R}$  is  $\{(0, \dots, 0, a T_j^{\alpha_{\mathbb{Y}_j/\mathbb{X}}}, 0, \dots, 0) \mid a \in \mathcal{O}_{\mathbb{X}, p_j}\}$ . This implies  $\nu(a) = \mu(a) = \alpha_{\mathbb{Y}_j/\mathbb{X}}$  for every non-zero element  $a \in \mathcal{O}_{\mathbb{X}, p_j}$ . Thus Proposition 4.1.25 yields that  $\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} \subseteq \mathfrak{F}_{\tilde{R}/R}$ .

Obviously, we have  $f_{jk_j}^* \notin \langle x_0 \rangle$  and its image  $Lf_{jk_j}^*$  in  $\bar{R}$  is a non-zero element for  $k_j = 1, \dots, \nu_j$ . If there exist elements  $a_{j1}, \dots, a_{j\nu_j}$  in  $K$ , not all equal to zero, such that  $\sum_{k_j=1}^{\nu_j} a_{jk_j} Lf_{jk_j}^* = 0$ , then  $f = \sum_{k_j=1}^{\nu_j} a_{jk_j} f_{jk_j}^*$  is contained in  $(\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}})_{\alpha_{\mathbb{Y}_j/\mathbb{X}}} \setminus \{0\}$  and we

get  $Lf = 0$ . So, we have  $f \in x_0 R_{\alpha_{\mathbb{Y}_j/\mathbb{X}}-1}$ . Let  $h \in R_{\alpha_{\mathbb{Y}_j/\mathbb{X}}-1} \setminus \{0\}$  be such that  $f = x_0 h$ . Since the ideal  $\mathcal{I}_{\mathbb{Y}_j/\mathbb{X}}$  is saturated, Lemma 2.3.12 implies  $h \in \mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} \setminus \{0\}$ , a contradiction. Thus we have shown that the set  $\{Lf_{j1}^*, \dots, Lf_{j\nu_j}^*\}$  is  $K$ -linearly independent.

Consequently, there is a homogeneous  $K$ -linear map  $\bar{\varphi}_{j1} : \bar{R} \rightarrow K$  of degree  $-\alpha_{\mathbb{Y}_j/\mathbb{X}}$  with  $\bar{\varphi}_{j1}(Lf_{j1}^*) \neq 0$  and  $\bar{\varphi}_{j1}(Lf_{jk_j}^*) = 0$  for  $k_j = 2, \dots, \nu_j$ . Using the epimorphism  $\underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) \rightarrow \underline{\mathrm{Hom}}_K(\bar{R}, K)$ , we can lift  $\bar{\varphi}_{j1}$  to obtain a homogeneous element  $\varphi_{j1} \in \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])_{-\alpha_{\mathbb{Y}_j/\mathbb{X}}}$  with  $\varphi_{j1}(f_{j1}^*) \neq 0$  and  $\varphi_{j1}(f_{jk_j}^*) = 0$  for  $k_j = 2, \dots, \nu_j$ . Clearly, the set  $\{x_0^{r_{\mathbb{X}}-\mu(e_{11})} f_{11}^*, \dots, x_0^{r_{\mathbb{X}}-\mu(e_{s\nu_s})} f_{s\nu_s}^*\}$  forms a  $K$ -basis of the  $K$ -vector space  $R_{r_{\mathbb{X}}}$ . We write  $\varphi_{j1}(x_0^{r_{\mathbb{X}}-\mu(e_{j'k_{j'}})} f_{j'k_{j'}}^*) = c_{j'k_{j'}} x_0^{r_{\mathbb{X}}-\alpha_{\mathbb{Y}_j/\mathbb{X}}}$  for all  $j' = 1, \dots, s$  and  $k_{j'} = 1, \dots, \nu_{j'}$ . By Proposition 3.2.9, we have

$$g_{j1} := \Phi(\varphi_{j1}) = \left( \sum_{k_1=1}^{\nu_1} c_{1k_1} e'_{1k_1} T_1^{-\alpha_{\mathbb{Y}_j/\mathbb{X}}}, \dots, \sum_{k_s=1}^{\nu_s} c_{sk_s} e'_{sk_s} T_s^{-\alpha_{\mathbb{Y}_j/\mathbb{X}}} \right) \in \mathfrak{C}_{R/K[x_0]}.$$

Since  $e'_{j1}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$  and  $c_{j1} \in K \setminus \{0\}$ , for  $k_j = 1, \dots, \nu_j$  we set

$$h_{jk_j} = \tilde{\iota}^{-1}((0, \dots, 0, (e'_{j1} c_{j1})^{-1} e_{jk_j} T_j^{\alpha_{\mathbb{Y}_j/\mathbb{X}}}, 0, \dots, 0)).$$

Then  $h_{j1}, \dots, h_{j\nu_j} \in \mathcal{I}_{\mathbb{Y}_j/\mathbb{X}} \subseteq \mathfrak{F}_{\tilde{R}/R}$ . In  $\tilde{R}$ , we have

$$\begin{aligned} h_{jk_j} \cdot g_{j1} &= (0, \dots, 0, (e'_{j1} c_{j1})^{-1} e_{jk_j} \sum_{l_j=1}^{\nu_j} c_{jl_j} e'_{jl_j}, 0, \dots, 0) \\ &= (0, \dots, 0, e_{jk_j}, 0, \dots, 0) = \epsilon_{jk_j}, \end{aligned}$$

since  $c_{j2} = \dots = c_{j\nu_j} = 0$ . Thus we obtain  $\epsilon_{jk_j} \in \mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]}$ , as was to be shown.

Finally, if  $\mathbb{X}$  is arithmetically Gorenstein, then  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$  for a nonzerodivisor  $h \in R_{r_{\mathbb{X}}}$  by Proposition 3.2.8. Observe that  $\tilde{\iota}(h) = (h_{p_1} T_1^{r_{\mathbb{X}}}, \dots, h_{p_1} T_1^{r_{\mathbb{X}}})$  in  $\tilde{R}$ , where  $h_{p_j}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$  for  $j = 1, \dots, s$  (see Lemma 2.3.11). Therefore, in  $\tilde{R}$ , we get  $\mathfrak{F}_{\tilde{R}/R} = \bigoplus_{i \geq r_{\mathbb{X}}} \tilde{R}_i = \langle \tilde{\iota}(\vartheta_D(R/K[x_0])) \rangle_{\tilde{R}} \subseteq \tilde{R}$ .  $\square$

When we specialize to the case of projective point sets, the condition (ii) of Proposition 4.1.27 is satisfied. Therefore we recover the following result of A.V. Geramita *et al* (see [GKR, Proposition 3.15]).

**Corollary 4.1.28.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set in  $\mathbb{P}_K^n$ . Then we have  $\mathfrak{F}_{\tilde{R}/R} \cdot \mathfrak{C}_{R/K[x_0]} = \tilde{R}$ .*

**Proposition 4.1.29.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme.*

- (i) *We have  $\mathfrak{F}_{\tilde{R}/R}^2 \subseteq \vartheta_D(R/K[x_0]) \subseteq \mathfrak{F}_{\tilde{R}/R}$ .*
- (ii) *If  $\mathbb{X}$  has CBP( $d$ ), then we have  $\vartheta_D(R/K[x_0])_d = \langle 0 \rangle$ .*

*Proof.* (i) Let  $L = Q^h(R) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]$ . Since  $(\mathfrak{C}_{R/K[x_0]})_i = (L)_i = (\tilde{R})_i$  for all  $i \geq 0$ , this implies  $\tilde{R} \subseteq \mathfrak{C}_{R/K[x_0]}$ . Thus we get

$$\vartheta_D(R/K[x_0]) = R :_L \mathfrak{C}_{R/K[x_0]} \subseteq R :_L \tilde{R} = R :_{\tilde{R}} \tilde{R} = \mathfrak{F}_{\tilde{R}/R}.$$

On the other hand, we see that  $\mathfrak{F}_{\tilde{R}/R}^2 \cdot \mathfrak{C}_{R/K[x_0]} \subseteq \mathfrak{F}_{\tilde{R}/R} \tilde{R} \subseteq R$ . This yields the inclusion  $\mathfrak{F}_{\tilde{R}/R}^2 \subseteq \vartheta_D(R/K[x_0])$ . Therefore the claim follows.

(ii) If  $\vartheta_D(R/K[x_0])_d \neq \langle 0 \rangle$  then there exists a non-zero homogeneous element  $h$  of degree  $d$  in  $\vartheta_D(R/K[x_0])_d$ , and hence  $h \in (\mathfrak{F}_{\tilde{R}/R})_d$ . By Proposition 4.1.26, the scheme  $\mathbb{X}$  does not have CBP( $d$ ), a contradiction.  $\square$

**Corollary 4.1.30.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, and let  $f_j$  be the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . Then  $\mathbb{X}$  is a CB-scheme if and only if  $x_0^{r_{\mathbb{X}}-2} f_j \notin \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-2}$  for all  $j = 1, \dots, s$ .*

*Proof.* It is clear that  $x_0^{r_{\mathbb{X}}-1} f_j \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$  if  $x_0^{r_{\mathbb{X}}-2} f_j \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-2}$ . Thus we get  $x_0^{r_{\mathbb{X}}-2} f_j \notin \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-2}$  for every  $j \in \{1, \dots, s\}$  if  $\mathbb{X}$  is a CB-scheme by Corollary 4.1.9(i). Conversely, if  $\mathbb{X}$  is not a CB-scheme, then we can find a minimal separator  $f_j^* \in R$  with  $d_j = \deg(f_j^*) \leq r_{\mathbb{X}} - 1$ . Moreover, we may choose the minimal separator  $f_j^*$  such that  $f_j^*(p_j) = 1$ . Notice that  $f_j^* \in \mathfrak{F}_{\tilde{R}/R}$  (see Proposition 4.1.25(ii)). By Proposition 4.1.29(i), we get  $(f_j^*)^2 \in \mathfrak{F}_{\tilde{R}/R}^2 \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ . Since  $\mathbb{X}$  is a projective point set, we have  $x_0^{d_j} f_j^* = (f_j^*)^2$  and  $f_j = x_0^{r_{\mathbb{X}}-d_j} f_j^* \in R_{r_{\mathbb{X}}}$ . This implies that  $x_0^{r_{\mathbb{X}}-2} f_j = x_0^{2r_{\mathbb{X}}-2d_j-2} (x_0^{d_j} f_j^*) \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-2}$ . Therefore the corollary is proved.  $\square$

## 4.2 Differents for Schemes Having Generic Hilbert Function

In this section we are interested in studying the following special class of 0-dimensional schemes in the projective  $n$ -space  $\mathbb{P}_K^n$ .

**Definition 4.2.1.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. We say that  $\mathbb{X}$  has **generic Hilbert function** if  $\text{HF}_{\mathbb{X}}(i) = \min\{\deg(\mathbb{X}), \binom{i+n}{n}\}$  for all  $i \in \mathbb{Z}$ .

Let  $\alpha_{\mathbb{X}} = \min\{i \in \mathbb{N} \mid (\mathcal{I}_{\mathbb{X}})_i \neq 0\}$  be the initial degree of the homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}}$ . It is well known (cf. [GO] and [BK]) that if  $\mathbb{X}$  has generic Hilbert function, then  $\mathcal{I}_{\mathbb{X}} = \langle (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}}, (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}+1} \rangle_P$  and  $\alpha_{\mathbb{X}}$  is the unique integer such that

$$\binom{n + \alpha_{\mathbb{X}} - 1}{n} \leq \deg(\mathbb{X}) < \binom{n + \alpha_{\mathbb{X}}}{n}.$$

We have the following observation.

**Lemma 4.2.2.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme which has generic Hilbert function. Suppose that  $\mathbb{X}$  has  $K$ -rational support and  $\deg(\mathbb{X}) \geq 2$  and  $r_{\mathbb{X}} \geq 2$ .*

(i) *The scheme  $\mathbb{X}$  has  $\text{CBP}(d)$  for all  $d = 0, \dots, r_{\mathbb{X}} - 2$ .*

(ii) *If  $\deg(\mathbb{X}) = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ , then  $\mathbb{X}$  is a CB-scheme.*

*Proof.* Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, where  $p_j \in \text{Supp}(\mathbb{X})$ . Observe that if  $F \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{X}}}$  and  $L \in P_1$  is a linear form through the point  $p_j$ , then  $FL \in \mathcal{I}_{\mathbb{X}}$ . Thus we obtain  $\alpha_{\mathbb{X}} - 1 \leq \alpha_{\mathbb{Y}/\mathbb{X}} \leq r_{\mathbb{X}}$ . On the other hand, it follows from the equality  $r_{\mathbb{X}} = \min\{i \in \mathbb{Z} \mid \text{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})\}$  that  $\mathbb{X}$  has generic Hilbert function if and only if  $\alpha_{\mathbb{X}} \geq r_{\mathbb{X}}$ . This implies  $\alpha_{\mathbb{Y}/\mathbb{X}} \geq \alpha_{\mathbb{X}} - 1 \geq r_{\mathbb{X}} - 1$ . By Proposition 4.1.22,  $\mathbb{X}$  has  $\text{CBP}(d)$  for all  $d = 0, \dots, r_{\mathbb{X}} - 2$ , and claim (i) follows.

In addition, if  $\deg(\mathbb{X}) = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ , then  $\alpha_{\mathbb{X}} = r_{\mathbb{X}} + 1$ . So, for every  $p_j \in \text{Supp}(\mathbb{X})$  and for every maximal  $p_j$ -subscheme  $\mathbb{Y} \subseteq \mathbb{X}$ , we have  $\alpha_{\mathbb{Y}/\mathbb{X}} \geq \alpha_{\mathbb{X}} - 1 \geq r_{\mathbb{X}}$ . Therefore claim (ii) follows by Proposition 4.1.22 again.  $\square$

**Remark 4.2.3.** Given a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$ , the condition that  $\mathbb{X}$  has generic Hilbert function does not imply that it is a CB-scheme. For instance, consider the projective point set  $\mathbb{X} = \{p_1, \dots, p_{11}\} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  of degree 11 with  $p_1 = (1 : 1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 1)$ ,  $p_3 = (1 : 0 : 1 : 1)$ ,  $p_4 = (1 : 1 : 1 : 1)$ ,  $p_5 = (1 : 2 : 2 : 1)$ ,  $p_6 = (1 : 3 : 1 : 1)$ ,  $p_7 = (1 : 3 : 2 : 0)$ ,  $p_8 = (1 : 3 : 2 : 1)$ ,  $p_9 = (1 : 3 : 3 : 0)$ ,  $p_{10} = (1 : 3 : 3 : 1)$ , and  $p_{11} = (1 : 3 : 3 : 3)$ . A calculation gives us  $r_{\mathbb{X}} = 3 = \alpha_{\mathbb{X}}$ . Hence  $\mathbb{X}$  has generic Hilbert function and  $\text{CBP}(1)$ . However,  $\mathbb{X}$  is not a CB-scheme, since there is a separator  $f_1^* = 3x_0^2 - x_0x_1 - 3x_0x_3 + x_1x_3$  of  $\mathbb{X} \setminus \{p_1\}$  in  $\mathbb{X}$  of degree 2.

In the following proposition we present a generalization of a result for projective point sets found in [GO] for the special case of reduced 0-dimensional subschemes of  $\mathbb{P}_K^n$ .

**Proposition 4.2.4.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional CB-scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ .*

(i) *If  $\mathbb{X}$  has generic Hilbert function, then  $\mathbb{X}$  is arithmetically Gorenstein if and only if  $\deg(\mathbb{X}) \cdot r_{\mathbb{X}} = 2 \binom{n+r_{\mathbb{X}}}{n+1}$ .*

(ii) *If there is a maximal  $p_j$ -subscheme  $\mathbb{Y}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$  such that  $\mathbb{Y}$  has generic Hilbert function, but  $\mathbb{X}$  does not have generic Hilbert function, then  $\mathbb{X}$  is arithmetically Gorenstein if and only if  $\deg(\mathbb{X}) \cdot r_{\mathbb{X}} = 2 \left( \binom{n+r_{\mathbb{X}}-1}{n+1} + \deg(\mathbb{X}) - 1 \right)$ .*

*Proof.* Since  $\mathbb{X}$  is reduced, then  $\tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  is the integral closure of  $R$  in its full ring of quotients  $Q(R)$ . It is well known that the scheme  $\mathbb{X}$  is arithmetically Gorenstein if and only if  $\ell(\tilde{R}/\mathfrak{F}_{\tilde{R}/R}) = \dim_K(\tilde{R}/\mathfrak{F}_{\tilde{R}/R}) = 2 \dim_K(R/\mathfrak{F}_{\tilde{R}/R}) = 2\ell(R/\mathfrak{F}_{\tilde{R}/R})$  (see [Bas, Corollary 6.5]). Since  $\mathbb{X}$  is a CB-scheme, it follows from Proposition 4.1.26 that  $\mathfrak{F}_{\tilde{R}/R} = \bigoplus_{i \geq r_{\mathbb{X}}} R_i$ . Hence we get the equalities  $\dim_K(\tilde{R}/\mathfrak{F}_{\tilde{R}/R}) = \deg(\mathbb{X}) \cdot r_{\mathbb{X}}$  and  $\dim_K(R/\mathfrak{F}_{\tilde{R}/R}) = \sum_{i=0}^{r_{\mathbb{X}}-1} \text{HF}_{\mathbb{X}}(i)$ .

Suppose that  $\mathbb{X}$  has generic Hilbert function. It follows that  $\dim_K(R/\mathfrak{F}_{\tilde{R}/R}) = \sum_{i=0}^{r_{\mathbb{X}}-1} \binom{n+i}{n} = \binom{n+r_{\mathbb{X}}}{n+1}$ . Therefore claim (i) follows.

Now we show claim (ii). We know that  $r_{\mathbb{X}} - 1 \leq r_{\mathbb{Y}} \leq r_{\mathbb{X}}$ . If  $r_{\mathbb{Y}} = r_{\mathbb{X}}$ , then  $\alpha_{\mathbb{X}} \geq \alpha_{\mathbb{Y}} \geq r_{\mathbb{Y}} = r_{\mathbb{X}}$  (as  $\mathbb{Y}$  has generic Hilbert function), and so  $\mathbb{X}$  has generic Hilbert function, in contradiction to the hypothesis. Thus we conclude that  $r_{\mathbb{Y}} = r_{\mathbb{X}} - 1$  and  $\text{HF}_{\mathbb{X}}(r_{\mathbb{X}}-1) = \deg(\mathbb{X})-1$ . It follows that  $\dim_K(R/\mathfrak{F}_{\tilde{R}/R}) = \sum_{i=0}^{r_{\mathbb{X}}-2} \binom{n+i}{n} + (\deg(\mathbb{X})-1) = \binom{n+r_{\mathbb{X}}-1}{n+1} + \deg(\mathbb{X}) - 1$ , and this completes the proof.  $\square$

Note that if  $n = 1$  or  $\deg(\mathbb{X}) = 1$ , then  $\mathbb{X}$  is always arithmetically Gorenstein. Our next corollary provides some characterizations of 0-dimensional arithmetically Gorenstein schemes  $\mathbb{X}$  when  $n \geq 2$  and  $\deg(\mathbb{X}) > 1$ .

**Corollary 4.2.5.** *Let  $n \geq 2$  and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional CB-scheme of degree  $\deg(\mathbb{X}) > 1$ .*

- (i) *If  $\mathbb{X}$  has generic Hilbert function, then*
  - (a)  *$\mathbb{X}$  is arithmetically Gorenstein if and only if  $\deg(\mathbb{X}) = 2$  or  $\deg(\mathbb{X}) = n + 2$ .*
  - (b) *If  $\mathbb{X}$  is arithmetically Gorenstein, then  $\text{ri}(\vartheta_D(R/K[x_0])) \in \{2, 4\}$ .*
- (ii) *If  $\deg(\mathbb{X}) > n + 1$  and there is a maximal  $p_j$ -subscheme  $\mathbb{Y}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$  such that  $\mathbb{Y}$  has generic Hilbert function, but  $\mathbb{X}$  does not have generic Hilbert function, then  $\mathbb{X}$  is arithmetically Gorenstein if and only if  $\deg(\mathbb{X}) = 2(n + 2)$  or  $\deg(\mathbb{X}) = (n + 3)(n + 2)/2 - 1$ .*

*Proof.* Part (a) of claim (i) follows from Proposition 4.2.4(i) and by an argument analogous to the proof of Theorem 7 in [GO]. Part (b) of claim (i) is a consequence of (a) and the fact that  $\text{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ . Claim (ii) follows from Proposition 4.2.4(ii) and in the same way as the proof of Theorem 9 in [GO]. We note that in the claim (ii) it suffices to treat the case  $\deg(\mathbb{X}) > n + 1$ , since if  $\deg(\mathbb{X}) \leq n + 1$  then  $\mathbb{X}$  is contained in a hyperplane  $H \cong \mathbb{P}_K^{n-1}$ . Moreover, there do not exist reduced 0-dimensional CB-schemes  $\mathbb{X}$  in  $\mathbb{P}_K^2$  of degree 17 or 22 such that there is a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 1$  having generic Hilbert function, but  $\mathbb{X}$  does not have generic

Hilbert function. Otherwise, such a scheme  $\mathbb{X}$  would be arithmetically Gorenstein (by Proposition 4.2.4(ii)), and hence a complete intersection of type  $(d_1, d_2)$  for some  $1 \leq d_1 \leq d_2$ . Then we have  $\deg(\mathbb{X}) = d_1 d_2$  and  $(\mathcal{I}_{\mathbb{X}})_4 \subseteq (\mathcal{I}_{\mathbb{Y}})_4 = \langle 0 \rangle$  (as  $\mathbb{Y}$  has generic Hilbert function). If  $\deg(\mathbb{X}) = 17 = d_1 d_2$ , then  $(d_1, d_2) = (1, 17)$ , and hence  $(\mathcal{I}_{\mathbb{X}})_1 \neq \langle 0 \rangle$ . This is impossible. If  $\deg(\mathbb{X}) = 22 = d_1 d_2$ , then  $(d_1, d_2) = (1, 22)$  or  $(2, 11)$ . This implies that  $(\mathcal{I}_{\mathbb{X}})_2 \neq \langle 0 \rangle$ , a contradiction.  $\square$

**Example 4.2.6.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the reduced 0-dimensional scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_4\}$ , where  $p_1 = (1 : 0 : 1 : 0)$ ,  $p_2 = (1 : 1 : 0 : 1)$ ,  $p_3 = (1 : 1 : 1 : 1)$ , and where  $p_4$  corresponds to  $\mathfrak{P}_4 = \langle 2X_0^2 + X_1^2, X_2, X_3 \rangle$ . Then  $\deg(\mathbb{X}) = n + 2 = 5$  and  $\alpha_{\mathbb{X}} = r_{\mathbb{X}} = 2$ . Thus  $\mathbb{X}$  has generic Hilbert function. Moreover, it is easy to check that  $\mathbb{X}$  is a CB-scheme. Therefore  $\mathbb{X}$  is arithmetically Gorenstein by Corollary 4.2.5(i). In this case, we have  $\vartheta_D(R/K[x_0]) = \langle x_1^2 + x_2^2 - 3x_2x_3 \rangle \cong R(-2)$  and  $\text{ri}(\vartheta_D(R/K[x_0])) = 4$ .

Observer that if a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  has generic Hilbert function, then the Hilbert function of the Kähler different satisfies  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = 0$  for  $i < n(r_{\mathbb{X}} - 1)$ . Furthermore, if  $\alpha_{\mathbb{X}} = r_{\mathbb{X}} + 1$ , then  $\vartheta_K(R/K[x_0])$  is generated by homogeneous elements of degree  $nr_{\mathbb{X}}$ . In the following, we want to examine whether the converse of this statement is true. If  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  is a projective point set, an affirmative answer for this problem is given by our next proposition. Notice that if  $n = 1$  or  $(n \geq 2$  and  $s = 1)$  then the projective point set  $\mathbb{X}$  is a complete intersection, and so  $\vartheta_K(R/K[x_0])$  is generated by a non-zerodivisor of  $R$  of degree  $nr_{\mathbb{X}}$  and we have  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i - s + 1)$  for all  $i \in \mathbb{N}$ . Also, it is not hard to see that  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n + \alpha_{\mathbb{X}} - 1}{n}$ . Thus it suffices to consider the case  $n \geq 2$  and  $s \geq 2$ .

**Proposition 4.2.7.** *Let  $n \geq 2$  and let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set of degree  $s \geq 2$ . Suppose that  $\text{char}(K) = 0$  or  $\text{char}(K) > r_{\mathbb{X}}$ . The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n + \alpha_{\mathbb{X}} - 1}{n}$ .*
- (ii) *The Hilbert function of Kähler different is given by*

$$\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \begin{cases} 0 & \text{if } i < nr_{\mathbb{X}}, \\ s & \text{if } i \geq nr_{\mathbb{X}}. \end{cases}$$

- (iii) *The scheme  $\mathbb{X}$  is a CB-scheme and  $\vartheta_K(R/K[x_0]) = \mathfrak{F}_{R/R}^n$ .*

*Proof.* (i) $\Leftrightarrow$ (ii): Suppose that  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ . Then we have  $\alpha_{\mathbb{X}} = r_{\mathbb{X}} + 1$ . Thus  $\vartheta_K(R/K[x_0])$  is generated by homogeneous elements of degree  $nr_{\mathbb{X}}$ . By Proposition 3.3.14, we have  $\text{ri}(\vartheta_K(R/K[x_0])) = nr_{\mathbb{X}}$ . Therefore  $\vartheta_K(R/K[x_0]) = \bigoplus_{i \geq nr_{\mathbb{X}}} R_i$  and  $\text{HF}_{\vartheta_K(R/K[x_0])}$  is as above.

Conversely, suppose that  $\vartheta_K(R/K[x_0]) = \bigoplus_{i \geq nr_{\mathbb{X}}} R_i$  and  $\alpha_{\mathbb{X}} \leq r_{\mathbb{X}}$  (i.e.,  $\mathbb{X}$  does not have generic Hilbert function or  $\mathbb{X}$  has generic Hilbert function but  $s \neq \binom{n+\alpha_{\mathbb{X}}-1}{n}$ ). By Remark 3.3.24, we suppose that  $\mathcal{I}_{\mathbb{X}}$  has a minimal homogeneous system of generators  $\{F_1, \dots, F_r\}$  such that  $\deg(F_j) \leq r_{\mathbb{X}} + 1$ ,  $\{F_1, \dots, F_n\}$  is a  $P$ -regular sequence, and  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  is a non-zerodivisor of  $R$  (i.e., it does not vanish at any point of  $\mathbb{X}$ ). If there is an index  $j \in \{1, \dots, n\}$  such that  $\deg(F_j) \leq r_{\mathbb{X}}$ , then  $\deg(\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}) \leq nr_{\mathbb{X}} - 1$ . Hence we have  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)} \in \vartheta_K(R/K[x_0]) \setminus \{0\}$ , and this is a contradiction.

Now we consider the case  $\deg(F_j) = r_{\mathbb{X}} + 1$  for  $j = 1, \dots, n$ . Since  $\alpha_{\mathbb{X}} \leq r_{\mathbb{X}}$ , there is an element  $F_j \in (\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}}$  for some  $j \geq n + 1$ , say  $F_{n+1}$ . By the assumption on the characteristic of  $K$  and by Euler's rule, we may assume that  $\frac{\partial F_{n+1}}{\partial X_1} \neq 0$ . Then it is clear that  $\frac{\partial F_{n+1}}{\partial X_1} \notin \mathcal{I}_{\mathbb{X}}$ , since  $\deg(\frac{\partial F_{n+1}}{\partial X_1}) < \alpha_{\mathbb{X}}$ . So, there is a point  $p_k$  of  $\mathbb{X}$  such that  $\frac{\partial F_{n+1}}{\partial x_1}(p_k) \neq 0$ . Without loss of generality, we can assume that  $\frac{\partial F_{n+1}}{\partial x_1}(p_1) \neq 0$ . Set  $\mathcal{V}_i := (\frac{\partial F_i}{\partial x_1}(p_1), \dots, \frac{\partial F_i}{\partial x_n}(p_1)) \in K^n$  for  $i = 1, \dots, n + 1$  and

$$\mathcal{V} := \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p_1) & \cdots & \frac{\partial F_1}{\partial x_n}(p_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(p_1) & \cdots & \frac{\partial F_n}{\partial x_n}(p_1) \end{pmatrix}.$$

The matrix  $\mathcal{V}$  is invertible, since  $\det(\mathcal{V}) = \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}(p_1) \neq 0$ . By using elementary row operations, we can transform the matrix  $\mathcal{V}$  into an upper-triangular matrix  $\mathcal{W} = (w_{ij})$  such that its diagonal entries are all non-zero (i.e.,  $w_{ik} = 0$  for  $k < i$  and  $w_{ii} \neq 0$  for all  $i = 1, \dots, n$ ). Let  $\mathcal{W}_i$  denote the  $i$ -th row of the matrix  $\mathcal{W}$ . Then there are  $\lambda_{ij} \in K$ ,  $i, j \in \{1, \dots, n\}$ , such that

$$\begin{aligned} \mathcal{W}_i &= (w_{i1}, \dots, w_{in}) = \lambda_{i1}\mathcal{V}_1 + \cdots + \lambda_{in}\mathcal{V}_n \\ &= \left( \sum_{j=1}^n \lambda_{ij} \frac{\partial F_j}{\partial x_1}(p_1), \dots, \sum_{j=1}^n \lambda_{ij} \frac{\partial F_j}{\partial x_n}(p_1) \right). \end{aligned}$$

For every  $i \in \{2, \dots, n\}$ , let  $G_i := \lambda_{i1}F_1 + \cdots + \lambda_{in}F_n \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}+1} \setminus \{0\}$ . Then we have  $\frac{\partial G_i}{\partial x_k}(p_1) = \sum_{j=1}^n \lambda_{ij} \frac{\partial F_j}{\partial x_k}(p_1) = w_{ik}$  for all  $i = 2, \dots, n$  and  $k = 1, \dots, n$ . Notice that

$\frac{\partial G_i}{\partial x_k}(p_1) = w_{ik} = 0$  for  $k < i$  and  $\frac{\partial G_i}{\partial x_i}(p_1) = w_{ii} \neq 0$ . Thus we get

$$\begin{aligned} \frac{\partial(F_{n+1}, G_2, \dots, G_n)}{\partial(x_1, \dots, x_n)}(p_1) &= \det \begin{pmatrix} \frac{\partial F_{n+1}}{\partial x_1}(p_1) & \frac{\partial F_{n+1}}{\partial x_2}(p_1) & \cdots & \frac{\partial F_{n+1}}{\partial x_n}(p_1) \\ 0 & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{nn} \end{pmatrix} \\ &= \frac{\partial F_{n+1}}{\partial x_1}(p_1) w_{22} \cdots w_{nn} \neq 0. \end{aligned}$$

Hence we obtain  $\frac{\partial(F_{n+1}, G_2, \dots, G_n)}{\partial(x_1, \dots, x_n)} \in \vartheta_K(R/K[x_0])_{\leq nr_{\mathbb{X}}-1} \setminus \{0\}$ , and this is a contradiction again. Therefore it must be the case  $\alpha_{\mathbb{X}} = r_{\mathbb{X}} + 1$  or  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ , as was to be shown.

(ii) $\Leftrightarrow$ (iii): If  $\mathbb{X}$  is a CB-scheme and  $\mathfrak{F}_{\tilde{R}/R}^n = \vartheta_K(R/K[x_0])$ , then we have

$$\mathfrak{F}_{\tilde{R}/R}^n = \langle f_1^n, \dots, f_s^n \rangle_R = \bigoplus_{i \geq nr_{\mathbb{X}}} R_i = \vartheta_K(R/K[x_0])$$

where  $f_j$  is the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . Thus the Hilbert function of  $\vartheta_K(R/K[x_0])$  is given as claim (ii).

Conversely, if  $\vartheta_K(R/K[x_0]) = \bigoplus_{i \geq nr_{\mathbb{X}}} R_i$ , then  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ , and so  $\mathbb{X}$  is a CB-scheme by Lemma 4.2.2(ii). For every  $j \in \{1, \dots, s\}$  we let  $f_j^*$  be the minimal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  with  $f_j^*(p_j) = 1$ . It follows from Proposition 4.1.25 that  $\mathfrak{F}_{\tilde{R}/R}^n = \langle (f_1^*)^n, \dots, (f_s^*)^n \rangle_R$ . Since  $\mathbb{X}$  is a CB-scheme, this implies that  $\deg(f_j^*) = r_{\mathbb{X}}$  or  $f_j^* = f_j$  for all  $j = 1, \dots, s$ . Therefore we get  $\mathfrak{F}_{\tilde{R}/R}^n = \bigoplus_{i \geq nr_{\mathbb{X}}} R_i = \vartheta_K(R/K[x_0])$ , as was to be shown.  $\square$

**Corollary 4.2.8.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set. If  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$  then  $\text{ri}(\vartheta_K(R/K[x_0])) = nr_{\mathbb{X}}$ .*

*Proof.* This is an immediate consequence of Proposition 4.2.7.  $\square$

From Proposition 4.2.7 we see that if the Hilbert function of  $\vartheta_K(R/K[x_0])$  in degree  $nr_{\mathbb{X}} - 1$  is zero then  $\mathbb{X}$  is a CB-scheme. This observation can be generalized as follows.

**Proposition 4.2.9.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, let  $f_j$  be the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ , and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ .*

(i) *If  $\text{HF}_{\vartheta_K(R/K[x_0])}(nd) = 0$  then  $\mathbb{X}$  has CBP( $d$ ).*

(ii) *Suppose that  $n = 2$ . Then  $\mathbb{X}$  is a CB-scheme if and only if for every  $j \in \{1, \dots, s\}$  we have  $x_0^{r_{\mathbb{X}}-2} f_j \notin \vartheta_K(R/K[x_0])_{2r_{\mathbb{X}}-2}$ . In particular, if  $\text{HF}_{\vartheta_K(R/K[x_0])}(2r_{\mathbb{X}}-2) = 0$ , then  $\mathbb{X}$  is a CB-scheme.*



*Proof.* If  $r_{\mathbb{X}} \leq 1$ , then it is easy to check that  $\mathbb{X}$  is always a CB-scheme. Therefore we may assume that  $r_{\mathbb{X}} \geq 2$ . For every  $j \in \{1, \dots, s\}$ , let  $f_j^* \in R$  be a minimal separator of  $\mathbb{Y}_j$  in  $\mathbb{X}$ . Suppose that  $\mathbb{X}$  does not have CBP( $d$ ). Then there exists an index  $j \in \{1, \dots, s\}$  such that  $\deg(f_j^*) \leq d$ . By Corollary 3.3.16(i), we have the inclusion  $\mathfrak{F}_{\tilde{R}/R}^n \subseteq \vartheta_K(R/K[x_0]) \subseteq \mathfrak{F}_{\tilde{R}/R}$ . This implies  $0 \neq (f_j^*)^n \in \vartheta_K(R/K[x_0])$ . Hence we see that  $0 < \text{HF}_{\vartheta_K(R/K[x_0])}(n \deg(f_j^*)) \leq \text{HF}_{\vartheta_K(R/K[x_0])}(nd)$ . We get a contradiction to our assumption that  $\text{HF}_{\vartheta_K(R/K[x_0])}(nd) = 0$ . Thus claim (i) is proved completely.

Now we prove claim (ii). If  $x_0^{r_{\mathbb{X}}-2} f_j \in \vartheta_K(R/K[x_0]) \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$  for some  $j \in \{1, \dots, s\}$ , then  $\mathbb{X}$  is not a CB-scheme by Corollary 4.1.30. Conversely, we assume that  $x_0^{r_{\mathbb{X}}-2} f_j \notin \vartheta_K(R/K[x_0])$  for all  $j = 1, \dots, s$  and  $\mathbb{X}$  is not a CB-scheme. Then we can find a minimal separator  $f_j^* \in R$  with  $d_j = \deg(f_j^*) \leq r_{\mathbb{X}} - 1$  and  $f_j^*(p_j) = 1$ . We have  $x_0^{r_{\mathbb{X}}-2} f_j = x_0^{2r_{\mathbb{X}}-2d_j-2} (f_j^*)^2 \in \mathfrak{F}_{\tilde{R}/R}^2$ . Therefore Corollary 3.3.16(i) yields that  $x_0^{r_{\mathbb{X}}-2} f_j \in \vartheta_K(R/K[x_0])$ , a contradiction.  $\square$

**Remark 4.2.10.** The converse of Proposition 4.2.9(i) is not true. For example, we take  $\mathbb{X} = \{(1 : 0 : 0), (1 : 1 : 0), (1 : 0 : 1), (1 : 1 : 1)\} \subseteq \mathbb{P}_K^2$ . Then  $\mathbb{X}$  is a complete intersection with  $r_{\mathbb{X}} = 2$ . Of course,  $\mathbb{X}$  is a CB-scheme. But we see that  $\text{HF}_{\vartheta_K(R/K[x_0])}(n(r_{\mathbb{X}} - 1)) = \text{HF}_{\vartheta_K(R/K[x_0])}(2) = \text{HF}_{\mathbb{X}}(2 - 2) = \text{HF}_{\mathbb{X}}(0) = 1 \neq 0$ .

Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, where  $n \geq 2$ . Suppose that  $\mathbb{X}$  has generic Hilbert function. Let  $t_1$  and  $t_2$  be the numbers of minimal generators of degree  $r_{\mathbb{X}}$  and of degree  $r_{\mathbb{X}} + 1$  of  $\mathcal{I}_{\mathbb{X}}$ , respectively. Notice that if  $s = \binom{n+\alpha_{\mathbb{X}}-1}{n}$ , then  $r_{\mathbb{X}} = \alpha_{\mathbb{X}} - 1$ , and so  $t_1 = 0$ ; and if  $\binom{n+\alpha_{\mathbb{X}}-1}{n} < s < \binom{n+\alpha_{\mathbb{X}}}{n}$ , then  $t_1 = \dim_K(\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} = \binom{n+\alpha_{\mathbb{X}}}{n} - s$ . We assume that  $t_1 \geq n$ , and let  $F_1, \dots, F_{t_1}$  be the minimal generators of  $\mathcal{I}_{\mathbb{X}}$  of degree  $r_{\mathbb{X}}$ . Let  $\{h_1, \dots, h_{\binom{t_1}{n}}\} \subseteq R_{n(r_{\mathbb{X}}-1)}$  be a set of generators of the  $K$ -vector space  $\vartheta_K(R/K[x_0])_{n(r_{\mathbb{X}}-1)}$ , where  $h_i = \frac{\partial(F_{i_1}, \dots, F_{i_n})}{\partial(x_1, \dots, x_n)}$  for a subset  $\{i_1, \dots, i_n\} \subseteq \{1, \dots, t_1\}$ . For every  $j \in \{1, \dots, s\}$ , we denote

$$A = \begin{pmatrix} h_1(p_1) & h_2(p_1) & \cdots & h_{\binom{t_1}{n}}(p_1) \\ h_1(p_2) & h_2(p_2) & \cdots & h_{\binom{t_1}{n}}(p_2) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(p_s) & h_2(p_s) & \cdots & h_{\binom{t_1}{n}}(p_s) \end{pmatrix}_{s \times \binom{t_1}{n}}, \quad \mathcal{E}_j = \begin{pmatrix} 0 \\ \vdots \\ 1_{[j]} \\ \vdots \\ 0 \end{pmatrix}_{s \times 1}, \quad \mathcal{A}_j = \begin{pmatrix} A & \mathcal{E}_j \end{pmatrix}_{s \times \binom{t_1}{n} + 1}.$$

**Proposition 4.2.11.** *Let  $n \geq 2$ , and let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set which has generic Hilbert function. Let  $t_1$  and  $t_2$  be the numbers of minimal generators of degree  $r_{\mathbb{X}}$  and of degree  $r_{\mathbb{X}} + 1$  of  $\mathcal{I}_{\mathbb{X}}$ , respectively.*

(i) If  $t_1 < n$ , then  $\mathbb{X}$  is a CB-scheme.

(ii) Suppose that  $t_1 \geq n$  and  $r_{\mathbb{X}} \geq 2$ . If  $\text{rank}(\mathcal{A}_j) > \text{rank}(\mathcal{A})$  for all  $j = 1, \dots, s$ , then  $\mathbb{X}$  is a CB-scheme. Conversely, if  $\mathbb{X}$  is a CB-scheme and  $n(r_{\mathbb{X}} - 1) \leq 2r_{\mathbb{X}} - 1$ , then  $\text{rank}(\mathcal{A}_j) > \text{rank}(\mathcal{A})$  for all  $j = 1, \dots, s$ .

*Proof.* (i) It is clear that  $\mathbb{X}$  is a CB-scheme if  $r_{\mathbb{X}} = 1$ . Thus we may assume that  $r_{\mathbb{X}} \geq 2$ . If  $t_1 < n$ , then the Kähler different  $\vartheta_K(R/K[x_0])$  is generated by homogeneous element of degree at least  $nr_{\mathbb{X}} - n + 1$ , and so  $\text{HF}_{\vartheta_K(R/K[x_0])}(n(r_{\mathbb{X}} - 1)) = 0$ . Hence Proposition 4.2.9 implies that  $\mathbb{X}$  is a CB-scheme.

(ii) Let  $f_j \in R_{r_{\mathbb{X}}}$  be the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . Notice that  $n(r_{\mathbb{X}} - 1) \geq r_{\mathbb{X}}$  as  $r_{\mathbb{X}} \geq 2$ . For every  $i \in \{1, \dots, \binom{t_1}{n}\}$ , we may write

$$h_i = x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}(h_i(p_1)f_1 + \cdots + h_i(p_s)f_s) = x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}\sum_{j=1}^s h_i(p_j)f_j.$$

Suppose that  $\text{rank}(\mathcal{A}_j) > \text{rank}(\mathcal{A})$  for all  $j = 1, \dots, s$ , and  $\mathbb{X}$  is not a CB-scheme. Then there is a minimal separator  $f_j^*$  of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  such that  $\deg(f_j^*) \leq r_{\mathbb{X}} - 1$  and  $f_j = x_0^{r_{\mathbb{X}}-\deg(f_j^*)}f_j^*$ . Without loss of generality, we may assume that  $\deg(f_1^*) \leq r_{\mathbb{X}} - 1$  and  $f_1 = x_0^{r_{\mathbb{X}}-\deg(f_1^*)}f_1^*$ . It follows from Corollary 3.3.16(i) that  $x_0^{n\deg(f_1^*)-r_{\mathbb{X}}}f_1 = (f_1^*)^n \in \vartheta_K(R/K[x_0])_{n\deg(f_1^*)}$ , and consequently  $x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}f_1 \in \vartheta_K(R/K[x_0])_{n(r_{\mathbb{X}}-1)}$ . Then there exist  $c_1, \dots, c_{\binom{t_1}{n}} \in K$ , not all equal to zero, such that

$$\begin{aligned} x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}f_1 &= c_1h_1 + \cdots + c_{\binom{t_1}{n}}h_{\binom{t_1}{n}} \\ &= x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}\left(c_1\sum_{j=1}^s h_1(p_j)f_j + \cdots + c_{\binom{t_1}{n}}\sum_{j=1}^s h_{\binom{t_1}{n}}(p_j)f_j\right) \\ &= x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}}\left(f_1\sum_{k=1}^{\binom{t_1}{n}} h_k(p_1)c_k + \cdots + f_s\sum_{k=1}^{\binom{t_1}{n}} h_k(p_s)c_k\right). \end{aligned}$$

Thus there are linear equations in  $c_1, \dots, c_{\binom{t_1}{n}}$  as follows

$$\begin{aligned} \sum_{k=1}^{\binom{t_1}{n}} h_k(p_1)c_k &= h_1(p_1)c_1 + \cdots + h_{\binom{t_1}{n}}(p_1)c_{\binom{t_1}{n}} = 1, \\ \sum_{k=1}^{\binom{t_1}{n}} h_k(p_j)c_k &= h_1(p_j)c_1 + \cdots + h_{\binom{t_1}{n}}(p_j)c_{\binom{t_1}{n}} = 0, \quad j = 2, \dots, s. \end{aligned}$$

This implies  $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{A}_1)$ , in contradiction to our assumption.

Now let  $\mathbb{X}$  be a CB-scheme and  $n(r_{\mathbb{X}} - 1) \leq 2r_{\mathbb{X}} - 1$ . We assume for contradiction that  $\text{rank}(\mathcal{A}_j) = \text{rank}(\mathcal{A})$  for some  $j \in \{1, \dots, s\}$ , may say  $j = 1$ . Let  $\mathcal{C} = (c_1, \dots, c_{\binom{t_1}{n}}) \in$

$K^{\binom{t_1}{n}}$  be a root of the system of linear equations  $\mathcal{A} \cdot \mathcal{Y} = \mathcal{E}_1$  where  $\mathcal{Y} = (y_1, \dots, y_{\binom{t_1}{n}})^{\text{tr}}$ . It follows that  $x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}} f_1 - (c_1 h_1 + \dots + c_{\binom{t_1}{n}} h_{\binom{t_1}{n}}) = 0$  in  $R$ . So,  $x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}} f_1$  is a non-zero element of  $\vartheta_K(R/K[x_0])_{n(r_{\mathbb{X}}-1)}$ . Since  $\mathbb{X}$  is a CB-scheme, there exists an element  $g_1 = x_0^{-2r_{\mathbb{X}}}(a_1 f_1 + \dots + a_s f_s) \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  with  $a_1, \dots, a_s \in K$  and  $a_1 \neq 0$  (as in the proof of Theorem 4.1.7). We have  $x_0^{n(r_{\mathbb{X}}-1)-r_{\mathbb{X}}} f_1 \cdot g_1 = x_0^{n(r_{\mathbb{X}}-1)-3r_{\mathbb{X}}} a_1 f_1 f_1 = x_0^{n(r_{\mathbb{X}}-1)-2r_{\mathbb{X}}} a_1 f_1 \in R$ . Thus we get  $f_1 \in x_0 R_{r_{\mathbb{X}}-1}$ , since  $n(r_{\mathbb{X}}-1) \leq 2r_{\mathbb{X}}-1$ . This means that  $\mathbb{X}$  is not a CB-scheme, a contradiction.  $\square$

Let  $s$  be an integer such that

$$\binom{n + (\alpha - 1)}{n} < s < \binom{n + \alpha}{n} \text{ with } n \geq 2, \alpha \geq 2$$

and let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set satisfying  $\text{HF}_{\mathbb{X}}(\alpha) = s$ . If  $\dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha} \leq n$ , then  $\mathbb{X}$  has generic Hilbert function and  $s = \binom{n+\alpha}{n} - \dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha}$ . Indeed, we see that  $r_{\mathbb{X}} \leq \alpha$  and  $\text{HF}_{\mathbb{X}}(\alpha-1) \leq \text{HF}_{\mathbb{P}_K^n}(\alpha-1) = \binom{n+(\alpha-1)}{n} < s$ . This implies  $r_{\mathbb{X}} = \alpha$ . Suppose that  $\alpha_{\mathbb{X}} < r_{\mathbb{X}}$ . Let  $F$  be a non-zero form of degree  $r_{\mathbb{X}} - 1$  in  $(\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}-1}$ . Then  $X_0 F, X_1 F, \dots, X_n F$  are  $n+1$  linearly independent forms of degree  $r_{\mathbb{X}}$  in  $(\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}}$ . Thus we obtain  $\dim_K(\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} = \dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha} > n$ , a contradiction. Therefore  $\alpha_{\mathbb{X}} \geq r_{\mathbb{X}}$  or  $\mathbb{X}$  has generic Hilbert function and  $s = \binom{n+\alpha}{n} - \dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha}$ , as we wished.

**Proposition 4.2.12.** *Let  $s$  be an integer such that*

$$\binom{n + (\alpha - 1)}{n} < s \leq \binom{n + \alpha}{n} - n \text{ with } n \geq 2, \alpha \geq 2.$$

*Then there is a non-empty open set in  $(\mathbb{P}_K^n)^s$  whose each point corresponds to a projective point set  $\mathbb{X} = \{p_1, \dots, p_s\}$  of  $\mathbb{P}_K^n$  (not necessary  $\mathbb{X} \cap \mathcal{Z}^+(X_0) = \emptyset$ ) such that*

- (i)  $\dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha} \geq n$ ,
- (ii) *there are  $n$  independent forms in  $(\mathcal{I}_{\mathbb{X}})_{\alpha}$  which meet in precisely  $\alpha^n$  distinct  $K$ -rational points.*

*Proof.* The result follows immediately from [GM, Proposition 4.4].  $\square$

**Proposition 4.2.13.** *Let  $s$  be an integer such that*

$$\binom{n + (\alpha - 1)}{n} < s \leq \binom{n + \alpha}{n} - n \text{ with } n \geq 2, \alpha \geq 2,$$

*and suppose  $\text{char}(K) \nmid \alpha$ . Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set satisfying the following conditions:*

- (i)  $\mathrm{HF}_{\mathbb{X}}(\alpha) = s$ ,
- (ii)  $\dim_K(\mathcal{I}_{\mathbb{X}})_{\alpha} \geq n$ ,
- (iii) there are  $n$  independent forms in  $(\mathcal{I}_{\mathbb{X}})_{\alpha}$  which meet in precisely  $\alpha^n$  distinct  $K$ -rational points.

Then  $\mathbb{X}$  has generic Hilbert function if and only if  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(n\alpha - n - 1) = 0$ .

*Proof.* Notice that the condition  $\mathrm{HF}_{\mathbb{X}}(\alpha) = s$  implies  $r_{\mathbb{X}} = \alpha$ . If  $\mathbb{X}$  has generic Hilbert function, then  $r_{\mathbb{X}} = \alpha = \alpha_{\mathbb{X}}$ . Thus  $\vartheta_K(R/K[x_0])$  is generated by homogeneous elements of degree  $\geq n\alpha - n$ . It follows that  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(n\alpha - n - 1) = 0$ . Conversely, suppose that  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(n\alpha - n - 1) = 0$ . According to the condition (iii), we let  $F_1, \dots, F_n \in (\mathcal{I}_{\mathbb{X}})_{\alpha}$  be  $n$  forms of degree  $\alpha$  having exactly  $\alpha^n$  distinct common zeros in  $\mathbb{P}_K^n$ . Then  $F_1, \dots, F_n$  form a  $P$ -regular sequence (cf. [KK, Theorem 1.12]), and so the ideal  $\langle F_1, \dots, F_n \rangle$  defines a complete intersection  $\mathbb{W}$  of degree  $\alpha^n$  containing  $\mathbb{X}$ . By Corollary 3.3.5, the element  $\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  is a non-zerodivisor of  $P/\langle F_1, \dots, F_n \rangle$  of degree  $n\alpha - n$ . In particular, it does not vanish at any point of  $\mathbb{W}$  (so of  $\mathbb{X}$ ). If  $\alpha_{\mathbb{X}} < \alpha$ , then we can argue analogously as in the proof of the part “(ii) $\Rightarrow$ (i)” of Proposition 4.2.7 to obtain a non-zero homogeneous element of degree  $n\alpha - n - 1$  in  $\vartheta_K(R/K[x_0])$ . This implies  $\mathrm{HF}_{\vartheta_K(R/K[x_0])}(n\alpha - n - 1) \neq 0$ , in contradiction to the assumption. Therefore we must have  $\alpha_{\mathbb{X}} = \alpha = r_{\mathbb{X}}$ . In other words,  $\mathbb{X}$  has generic Hilbert function.  $\square$

**Corollary 4.2.14.** *Let  $n \geq 2$  and let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set. If  $\mathbb{X}$  has generic Hilbert function with  $s = \binom{n+\alpha_{\mathbb{X}}}{n} - n$  and  $(\mathcal{I}_{\mathbb{X}})_{\alpha_{\mathbb{X}}}$  contains a  $P$ -regular sequence of length  $n$  which meet in precisely  $\alpha_{\mathbb{X}}^n$  distinct  $K$ -rational points, then  $\mathbb{X}$  is a CB-scheme.*

*Proof.* This follows from Proposition 4.2.11.  $\square$

### 4.3 Cayley-Bacharach Properties and Liaison

Throughout this section let  $K$  be an arbitrary field, let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme such that  $\mathrm{Supp}(\mathbb{W}) \cap \mathcal{Z}^+(X_0) = \emptyset$ , let  $\mathcal{I}_{\mathbb{W}}$  be the homogeneous vanishing ideal of  $\mathbb{W}$  in  $P$ , and let  $S := P/\mathcal{I}_{\mathbb{W}}$  be the homogeneous coordinate ring of  $\mathbb{W}$ . We shall use “ $-$ ” to denote residue classes modulo  $X_0$ . For a 0-dimensional subscheme  $\mathbb{X} \subseteq \mathbb{W}$ , the ideal  $\mathrm{Ann}_S(\mathcal{I}_{\mathbb{X}/\mathbb{W}}) \subseteq S$  is saturated and defines a 0-dimensional subscheme  $\mathbb{Y}$  of  $\mathbb{W}$  (see [Kr3, Section 16]).

**Definition 4.3.1.** The subscheme  $\mathbb{Y} \subseteq \mathbb{W}$  which is defined by the homogeneous ideal  $\mathcal{I}_{\mathbb{Y}/\mathbb{W}} = \text{Ann}_S(\mathcal{I}_{\mathbb{X}/\mathbb{W}})$  is said to be the **residual scheme** of  $\mathbb{X}$  in  $\mathbb{W}$ .

We collect from [DGO] or [Kr3, Section 16] some useful results.

**Proposition 4.3.2.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, and let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ . The following assertions hold true.*

- (i)  $\mathcal{I}_{\mathbb{X}/\mathbb{W}} = \text{Ann}_S(\mathcal{I}_{\mathbb{Y}/\mathbb{W}})$ .
- (ii)  $\deg(\mathbb{W}) = \deg(\mathbb{X}) + \deg(\mathbb{Y})$ .
- (iii)  $\Delta \text{HF}_{\mathbb{W}}(i) = \Delta \text{HF}_{\mathbb{X}}(i) + \Delta \text{HF}_{\mathbb{Y}}(r_{\mathbb{W}} - i)$  for  $i = 0, \dots, r_{\mathbb{W}}$ .
- (iv)  $r_{\mathbb{W}} = r_{\mathbb{X}} + \alpha_{\mathbb{Y}/\mathbb{W}} = r_{\mathbb{Y}} + \alpha_{\mathbb{X}/\mathbb{W}}$ .

**Lemma 4.3.3.** *For every  $d \in \{1, \dots, r_{\mathbb{X}}\}$ , we have*

$$(\overline{\mathcal{I}_{\mathbb{W}}})_{r_{\mathbb{W}}} : (\overline{\mathcal{I}_{\mathbb{Y}}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)} = (\overline{\mathcal{I}_{\mathbb{X}}})_d.$$

*Proof.* Clearly, we have  $\mathcal{I}_{\mathbb{X}} \cdot \mathcal{I}_{\mathbb{Y}} \subseteq \mathcal{I}_{\mathbb{W}}$ . This implies that  $(\overline{\mathcal{I}_{\mathbb{X}}})_d \subseteq (\overline{\mathcal{I}_{\mathbb{W}}})_{r_{\mathbb{W}}} : (\overline{\mathcal{I}_{\mathbb{Y}}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}$ . For the other inclusion, let  $f \in (\overline{\mathcal{I}_{\mathbb{W}}})_{r_{\mathbb{W}}} : (\overline{\mathcal{I}_{\mathbb{Y}}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}$ . In  $\overline{S} = S/\langle x_0 \rangle$ , we have  $\overline{f} \in (\text{Ann}_{\overline{S}}((\overline{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}))_d$ . Since  $\mathbb{W}$  is arithmetically Gorenstein, the ring  $\overline{S}$  is a 0-dimensional local Gorenstein ring with socle  $\overline{S}_{r_{\mathbb{W}}} \cong K$ . Thus we can argue in the same way as Lemma 4.1 and Proposition 4.3.a of [GKR] to get

$$\begin{aligned} (\text{Ann}_{\overline{S}}((\overline{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}))_d &= (\text{Ann}_{\overline{S}}((\overline{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}})_{r_{\mathbb{W}} - d}))_d \\ &= (\text{Ann}_{\overline{S}}(\overline{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}}))_d = (\overline{\mathcal{I}_{\mathbb{X}/\mathbb{W}}})_d. \end{aligned}$$

Consequently, we have  $\overline{f} \in (\overline{\mathcal{I}_{\mathbb{X}/\mathbb{W}}})_d$ , and hence  $f \in (\overline{\mathcal{I}_{\mathbb{X}}})_d$ , as desired.  $\square$

As usual, we let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$  and set  $\varkappa_j = \dim_K K(p_j)$  for  $j = 1, \dots, s$ . Given a maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$ , let  $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$  be a set of minimal separators of  $\mathbb{X}'$  in  $\mathbb{X}$ . Then a set of separators of  $\mathbb{X}'$  in  $\mathbb{X}$  is  $\{f_{j1}, \dots, f_{j\varkappa_j}\}$ , where  $f_{jk_j} = x_0^{r_{\mathbb{X}} - \deg(f_{jk_j}^*)} f_{jk_j}^*$  for  $k_j = 1, \dots, \varkappa_j$ . Let  $F_{jk_j}^* \in P_{\deg(f_{jk_j}^*)}$  and  $F_{jk_j} \in P_{r_{\mathbb{X}}}$  be representatives of  $f_{jk_j}^*$  and  $f_{jk_j}$ , respectively. We also say that the set  $\{F_{j1}^*, \dots, F_{j\varkappa_j}^*\}$  is a set of minimal separators of  $\mathbb{X}'$  in  $\mathbb{X}$ , and the set  $\{F_{j1}, \dots, F_{j\varkappa_j}\}$  is a set of separators of  $\mathbb{X}'$  in  $\mathbb{X}$ . Let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . In view of Proposition 4.1.22, the scheme  $\mathbb{X}$  has CBP( $d$ ) if and only if the following condition holds: for every point  $p_j \in \text{Supp}(\mathbb{X})$  and for every maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$ ; letting  $\{F_{j1}, \dots, F_{j\varkappa_j}\}$  be a set of separators of  $\mathbb{X}'$  in  $\mathbb{X}$ , we have  $F_{jk_j} \notin \langle X_0^{r_{\mathbb{X}} - d}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$  for some  $k_j \in \{1, \dots, \varkappa_j\}$ .

**Lemma 4.3.4.** *Let  $\mathbb{X}$  have  $\text{CBP}(d)$ , let  $0 \leq d \leq r_{\mathbb{X}} - 1$ , let  $\mathbb{X}' \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, and let  $\{F_{j1}, \dots, F_{j\alpha_j}\} \subseteq P_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{X}'$  in  $\mathbb{X}$ .*

(i) *If  $\langle F_{j1}, \dots, F_{j\alpha_j} \rangle_K \subseteq \langle X_0^{r_{\mathbb{X}}-d-1}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$  and if we write  $F_{jk_j} = F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-1} G_{jk_j}$  with  $F'_{jk_j} \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}}$  and  $G_{jk_j} \in P_{d+1}$ , then there is  $k_j \in \{1, \dots, \alpha_j\}$  such that*

$$G_{jk_j} \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}.$$

(ii) *If  $d = r_{\mathbb{X}} - 1$ , then  $F_{jk_j} \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}}$  for some  $k_j \in \{1, \dots, \alpha_j\}$ .*

*Proof.* Suppose that  $G_{jk_j} \in (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  for all  $k_j = 1, \dots, \alpha_j$ . Then we have  $G_{jk_j}(\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1} \subseteq (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ . Thus Lemma 4.3.3 yields that  $\overline{G}_{jk_j} \in (\overline{\mathcal{I}_{\mathbb{X}}})_{d+1}$ . This allows us to write  $G_{jk_j} = G'_{jk_j} + X_0 H_{jk_j}$  with  $G'_{jk_j} \in (\mathcal{I}_{\mathbb{X}})_{d+1}$  and  $H_{jk_j} \in P_d$ . Note that  $H_{jk_j} \in (\mathcal{I}_{\mathbb{X}'})_d$ . From this we rewrite  $F_{jk_j} = (F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-1} G'_{jk_j}) + X_0^{r_{\mathbb{X}}-d} H_{jk_j}$  for all  $k_j = 1, \dots, \alpha_j$ . It follows that  $F_{jk_j} \in \langle X_0^{r_{\mathbb{X}}-d}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$  for all  $k_j = 1, \dots, \alpha_j$ . Thus  $\mathbb{X}$  does not have  $\text{CBP}(d)$ , in contradiction to our hypothesis. Therefore claim (i) is completely proved. The proof of claim (ii) is similar.  $\square$

**Definition 4.3.5.** Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, and let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ . We say that  $\mathbb{X}$  and  $\mathbb{Y}$  are **geometrically linked** (by  $\mathbb{W}$ ) if they have no common irreducible component.

From the point of view of the saturated ideals, the schemes  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked by  $\mathbb{W}$  if and only if  $\mathcal{I}_{\mathbb{W}} = \mathcal{I}_{\mathbb{X}} \cap \mathcal{I}_{\mathbb{Y}}$  and neither  $\mathcal{I}_{\mathbb{X}}$  nor  $\mathcal{I}_{\mathbb{Y}}$  is contained in any associated prime of the other (see [Mig, Section 5.2]). In this case, if  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  and  $\text{Supp}(\mathbb{Y}) = \{p'_1, \dots, p'_t\} \subseteq \mathbb{P}_K^n$ , then we have  $\text{Supp}(\mathbb{W}) = \text{Supp}(\mathbb{X}) \cup \text{Supp}(\mathbb{Y}) = \{p_1, \dots, p_s, p'_1, \dots, p'_t\}$  and  $\text{Supp}(\mathbb{X}) \cap \text{Supp}(\mathbb{Y}) = \emptyset$ . This also implies that  $\mathcal{O}_{\mathbb{W}, p_j} = \mathcal{O}_{\mathbb{X}, p_j}$  for  $j = 1, \dots, s$  and  $\mathcal{O}_{\mathbb{W}, p'_j} = \mathcal{O}_{\mathbb{Y}, p'_j}$  for  $j = 1, \dots, t$ .

Let  $p_j \in \text{Supp}(\mathbb{X})$ . Using the map  $\Theta_j : P \longrightarrow R = P/\mathcal{I}_{\mathbb{X}} \xrightarrow{\theta} \prod_{k=1}^s \mathcal{O}_{\mathbb{X}, p_k} \xrightarrow{\text{proj}} \mathcal{O}_{\mathbb{X}, p_j}$  we can form the element  $H_{p_j} = \Theta_j(H) \in \mathcal{O}_{\mathbb{X}, p_j}$  for each  $H \in P$ . Then a homogeneous polynomial  $H$  is contained in  $\mathcal{I}_{\mathbb{X}}$  if and only if  $H_{p_j} = 0$  for all  $j = 1, \dots, s$ .

Now we present a characterization of  $\text{CBP}(d)$  using the homogeneous vanishing ideal of a geometrically linked scheme. This result is a generalized version of [GKR, Theorem 4.6].

**Theorem 4.3.6.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, and let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ . Let*

$0 \leq d \leq r_{\mathbb{X}} - 1$ , let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and suppose that  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked. The following conditions are equivalent.

- (i) The scheme  $\mathbb{X}$  has CBP( $d$ ).
- (ii) For every  $p_j \in \text{Supp}(\mathbb{X})$ , there exists an element  $H_j$  of  $(\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$  such that  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \{0\}$  is a unit.

*Proof.* (ii) $\Rightarrow$ (i): Let  $\mathbb{X}' \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, let  $s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j}) \setminus \{0\}$  be a socle element corresponding to  $\mathbb{X}'$ , and let  $\{e_{j1}, \dots, e_{j\kappa_j}\} \subseteq \mathcal{O}_{\mathbb{X}, p_j}$  be such that whose residue classes form a  $K$ -basis of  $K(p_j)$ . Let  $\{F_{j1}^*, \dots, F_{j\kappa_j}^*\}$  be a set of minimal separators of  $\mathbb{X}'$  in  $\mathbb{X}$  (with respect to  $s_j$  and  $\{e_{j1}, \dots, e_{j\kappa_j}\}$ ), i.e., let

$$F_{jk_j}^* + \mathcal{I}_{\mathbb{X}} = \tilde{\nu}^{-1}((0, \dots, 0, e_{jk_j} s_j T_j^{\mu(e_{jk_j} s_j)}, 0, \dots, 0))$$

for all  $k_j = 1, \dots, \kappa_j$ , where  $\tilde{\nu} : R \rightarrow \tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  is the injection given by  $\tilde{\nu}(F + \mathcal{I}_{\mathbb{X}}) = (F_{p_1} T_1^i, \dots, F_{p_s} T_s^i)$  for  $F \in P_i$  ( $i \geq 0$ ). Set

$$d_j := \nu(s_j) = \max\{\deg(F_{jk_j}^*) \mid 1 \leq k_j \leq \kappa_j\}.$$

We proceed to show that  $d_j \geq d + 1$ . By assumption, let  $H_j \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$  be an element such that  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . It is clear that  $(F_{jk_j}^* H_j)_p = (F_{jk_j}^*)_p (H_j)_p = 0$  for every point  $p$  in  $\text{Supp}(\mathbb{W}) \setminus \{p_j\}$  and  $(F_{jk_j}^* H_j)_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{W}, p_j}$ . Since  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked, we have  $\mathcal{O}_{\mathbb{X}, p_j} = \mathcal{O}_{\mathbb{W}, p_j}$  for all  $j = 1, \dots, s$ . So, it follows from  $(F_{jk_j}^*)_{p_j} = e_{jk_j} s_j \in \mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$  that  $(F_{jk_j}^* H_j)_{p_j} = e_{jk_j} s_j (H_j)_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{W}, p_j})$ .

For  $k_j \in \{1, \dots, \kappa_j\}$ , we denote  $G_{jk_j} + \mathcal{I}_{\mathbb{X}} := X_0^{d_j - \deg(F_{jk_j}^*)} F_{jk_j}^* + \mathcal{I}_{\mathbb{X}}$ . Let  $0 \leq i \leq n$  and  $1 \leq k \leq \kappa_j$ . Then we have  $X_i G_{jk_j} + \mathcal{I}_{\mathbb{X}} = \sum_{k_j=1}^{\kappa_j} c_{jk_j} X_0 G_{jk_j} + \mathcal{I}_{\mathbb{X}}$  for some  $c_{j1}, \dots, c_{j\kappa_j} \in K$ . This implies  $X_i G_{jk_j} H_j + \mathcal{I}_{\mathbb{W}} = \sum_{k_j=1}^{\kappa_j} c_{jk_j} X_0 G_{jk_j} H_j + \mathcal{I}_{\mathbb{W}}$ . Because the set  $\{e_{j1} s_j (H_j)_{p_j}, \dots, e_{j\kappa_j} s_j (H_j)_{p_j}\}$  is  $K$ -linear independent, we have

$$\begin{aligned} \dim_K (\langle F_{jk_j}^* H_j + \mathcal{I}_{\mathbb{W}} \mid 1 \leq k_j \leq \kappa_j \rangle_S)_i \\ = \dim_K (\langle G_{jk_j} H_j + \mathcal{I}_{\mathbb{W}} \mid 1 \leq k_j \leq \kappa_j \rangle_S)_i = \kappa_j \end{aligned}$$

for every  $i \geq d_j + \alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1 = r_{\mathbb{W}} + d_j - d - 1$ . Hence the homogeneous ideal  $\langle F_{jk_j}^* H_j + \mathcal{I}_{\mathbb{W}} \mid 1 \leq k_j \leq \kappa_j \rangle_S$  defines a maximal  $p_j$ -subscheme  $\mathbb{W}' \subseteq \mathbb{W}$  and the homogeneous vanishing ideal of  $\mathbb{W}'$  satisfies  $\dim_K (\mathcal{I}_{\mathbb{W}'/\mathbb{W}})_{r_{\mathbb{W}} + d_j - d - 1} = \kappa_j$ . Moreover, the scheme  $\mathbb{W}$  is arithmetically Gorenstein. In particular, it is a CB-scheme. It follows from Proposition 4.1.22 that  $r_{\mathbb{W}} + d_j - d - 1 \geq r_{\mathbb{W}}$ , and hence  $d_j \geq d + 1$ . Consequently, the scheme  $\mathbb{X}$  has CBP( $d$ ).

(i)⇒(ii): Let  $\mathbb{X}' \subseteq \mathbb{X}$  be a maximal  $p_j$ -subscheme, and let  $\{F_{j1}, \dots, F_{j\kappa_j}\} \subseteq P_{r_{\mathbb{X}}}$  be a set of separators of  $\mathbb{X}'$  in  $\mathbb{X}$ . Since  $\mathbb{X}$  has CBP( $d$ ), there exists  $k_j \in \{1, \dots, \kappa_j\}$  such that  $F_{jk_j} \notin \langle X_0^{r_{\mathbb{X}}-d}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$ . Now we need to find an element  $H_j \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  which satisfies  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$ .

First we consider the case  $\langle F_{j1}, \dots, F_{jk_j} \rangle_K \subseteq \langle X_0^{r_{\mathbb{X}}-d-1}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$ . If we write  $F_{jk_j} = F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-1}G_{jk_j}$  with  $F'_{jk_j} \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}}$  and  $G_{jk_j} \in P_{d+1}$ , then Lemma 4.3.4 implies that there is an index  $k_j \in \{1, \dots, \kappa_j\}$  such that  $G_{jk_j} \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$ , say  $G_{j1}$ . So, there is an element  $H_j$  of  $(\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  such that  $G_{j1}H_j \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ . It is clear that  $\mathcal{I}_{\mathbb{X}} \cdot \mathcal{I}_{\mathbb{Y}} \subseteq \mathcal{I}_{\mathbb{W}}$  and  $G_{jk_j} \in \mathcal{I}_{\mathbb{X}'}$  for all  $k_j = 1, \dots, \kappa_j$ . We deduce  $(G_{j1}H_j)_p = 0$  in  $\mathcal{O}_{\mathbb{W},p}$  for every point  $p \in \text{Supp}(\mathbb{W}) \setminus \{p_j\}$ . Since  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked, we get  $(G_{j1})_{p_j} \in \mathfrak{G}(\mathcal{O}_{\mathbb{W},p_j})$ . If  $(H_j)_{p_j} \in \mathfrak{m}_{\mathbb{X},p_j}$ , then in  $\mathcal{O}_{\mathbb{X},p_j}$  we have  $(G_{j1}H_j)_{p_j} = 0$ , and so  $(G_{j1}H_j)_p = 0$  in  $\mathcal{O}_{\mathbb{W},p}$  for all points  $p \in \text{Supp}(\mathbb{W})$ . Since the map  $\iota_S : S_{r_{\mathbb{W}}} \xrightarrow{\sim} \prod_{p \in \text{Supp}(\mathbb{W})} \mathcal{O}_{\mathbb{W},p}$  is an isomorphism, we deduce  $G_{j1}H_j \in (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ , a contradiction. Hence we obtain  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$ .

Next we consider the case  $\langle F_{j1}, \dots, F_{jk_j} \rangle_K \not\subseteq \langle X_0^{r_{\mathbb{X}}-d-1}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$ . Without loss of generality, we assume that  $F_{j1} \notin \langle X_0^{r_{\mathbb{X}}-d-1}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$ . If  $\langle F_{j1}, \dots, F_{jk_j} \rangle_K$  is contained in  $\langle X_0^{r_{\mathbb{X}}-d-2}, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$ . Then we write  $F_{jk_j} = F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-2}G_{jk_j}$  with  $F'_{jk_j} \in (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}}$  and  $G_{jk_j} \in (\mathcal{I}_{\mathbb{X}'})_{d+2}$ . Using the same argument as in the proof of Lemma 4.3.4, we can check that  $G_{j1} \cdot (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-2} \not\subseteq (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ . Therefore there exists an element  $H_j \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-2}$  such that  $G_{j1}H_j \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ . It follows by the same reasoning as above that  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$ , and therefore  $X_0H_j \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  and  $(X_0H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{p_j}$ .

By repeating this process for ideals of the form  $\langle X_0^\delta, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$  with  $\delta \leq r_{\mathbb{X}} - d - 2$ , we can reach the case  $\langle F_{j1}, \dots, F_{jk_j} \rangle_K \not\subseteq \langle X_0, (\mathcal{I}_{\mathbb{X}})_{r_{\mathbb{X}}} \rangle_P$  eventually. Thus there is an index  $k_j \in \{1, \dots, \kappa_j\}$  such that  $F_{jk_j} \notin (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}} \subseteq (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$ . From this we can find an element  $H_j$  of  $(\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  such that  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$  in exactly the same way as above. Altogether, for every point  $p_j$  of  $\text{Supp}(\mathbb{X})$ , we can find an element  $H_j \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  such that  $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$ .  $\square$

**Remark 4.3.7.** When the field  $K$  is infinite, the scheme  $\mathbb{X}$  has CBP( $d$ ) if and only if there is an element  $H \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+(r_{\mathbb{X}}-d)-1}$  such that its image in  $R$  is a non-zerodivisor. This follows from Theorem 4.3.6 and Lemma 2.3.11.

**Example 4.3.8.** Going back to Example 3.3.6, we have seen that the scheme  $\mathbb{X}$  is a complete intersection of degree 9 with  $\text{Supp}(\mathbb{X}) = \mathbb{X}(K) = \{p_1, \dots, p_7\}$ , but it is not reduced at  $p_5$  and  $p_7$ . A homogeneous primary decomposition of  $\mathcal{I}_{\mathbb{X}}$  is  $\mathcal{I}_{\mathbb{X}} = I_1 \cap \dots \cap I_7$ ,



where  $I_i = \mathfrak{P}_i$  for  $i \neq 5, 7$ ,  $I_5 = \langle X_1 - 2X_0, X_2^2 \rangle$ , and  $I_7 = \langle X_1 + 2X_0, X_2^2 \rangle$ .

- (a) Let  $\mathbb{X}_1$  denote the 0-dimensional subscheme of  $\mathbb{X}$  of degree 5 defined by the ideal  $\mathcal{I}_{\mathbb{X}_1} = I_1 \cap I_3 \cap I_4 \cap I_5 \subseteq P$ . Then  $\mathbb{X}_1$  is not reduced, but it is locally Gorenstein. Let  $\mathbb{Y}_1$  be the residual scheme of  $\mathbb{X}_1$  in  $\mathbb{X}$ . It is not hard to see that  $\mathbb{X}_1$  and  $\mathbb{Y}_1$  are geometrically linked. We calculate  $r_{\mathbb{X}} = 4$  and  $r_{\mathbb{X}_1} = \alpha_{\mathbb{X}_1/\mathbb{X}} = r_{\mathbb{Y}_1} = \alpha_{\mathbb{Y}_1/\mathbb{X}} = 2$ . In this case there is a homogeneous polynomial  $H \in (\mathcal{I}_{\mathbb{Y}_1})_2$  such that its image in the homogeneous coordinate ring  $R_{\mathbb{X}_1}$  of  $\mathbb{X}_1$  is a non-zerodivisor, for instance,  $H = X_0^2 + X_0X_1 + \frac{1}{4}X_1^2 - \frac{1}{2}X_0X_2 - \frac{1}{4}X_1X_2$ . Therefore Theorem 4.3.6 implies that the scheme  $\mathbb{X}_1$  is a CB-scheme.
- (b) Let  $\mathbb{X}_2$  be the projective point set in  $\mathbb{X}$  with its vanishing ideal  $\mathcal{I}_{\mathbb{X}_2} = \mathfrak{P}_1 \cap \mathfrak{P}_3 \cap \mathfrak{P}_4 \cap \mathfrak{P}_5$ , and let  $\mathbb{Y}_2$  be the residual scheme of  $\mathbb{X}_2$  in  $\mathbb{X}$ . Then the homogeneous saturated ideal of  $\mathbb{Y}_2$  is  $\mathcal{I}_{\mathbb{Y}_2} = \mathfrak{P}_2 \cap \mathfrak{P}_5 \cap \mathfrak{P}_6 \cap I_7$ . It is clear that  $r_{\mathbb{X}_2} = \alpha_{\mathbb{X}_2/\mathbb{X}} = r_{\mathbb{Y}_2} = \alpha_{\mathbb{Y}_2/\mathbb{X}} = 2$  and  $(\mathcal{I}_{\mathbb{Y}_2})_2 = \langle X_0^2 - \frac{1}{4}X_1^2 - \frac{1}{2}X_0X_2 - \frac{1}{4}X_1X_2 \rangle_K$ . In this case  $\mathbb{X}_2$  and  $\mathbb{Y}_2$  are not geometrically linked, since  $\text{Supp}(\mathbb{X}_2) \cap \text{Supp}(\mathbb{Y}) = \{p_5\}$ . Also, it is not difficult to verify that  $\mathbb{X}_2$  is a CB-scheme. However, there is no element  $H$  in  $(\mathcal{I}_{\mathbb{Y}_2})_2$  such that  $H_{p_5} \neq 0$  in  $\mathcal{O}_{\mathbb{X}_2, p_5}$ . Moreover, we see that the element  $F_5 = X_1^2 - 2X_1X_2$  is a minimal separator of  $\mathbb{X}_2 \setminus \{p_5\}$  in  $\mathbb{X}_2$  and  $(F_5H_5)_{p_5}$  is a socle element of  $\mathcal{O}_{\mathbb{X}, p_5}$ , where  $H_5 = X_0^2 - \frac{1}{4}X_1^2 - \frac{1}{2}X_0X_2 - \frac{1}{4}X_1X_2 \in (\mathcal{I}_{\mathbb{Y}_2})_2$ .

According to Proposition 2.2.9 in [GW] or Remark 4.7 in [GKR], we get the following sequence of isomorphism of graded  $R$ -modules

$$\begin{aligned} \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])(-r_{\mathbb{X}}) &\cong \text{Hom}_S(R, \underline{\text{Hom}}_{K[x_0]}(S, K[x_0]))(-r_{\mathbb{X}}) \\ &\cong \text{Hom}_S(S/\mathcal{I}_{\mathbb{X}/\mathbb{W}}, S(r_{\mathbb{W}}))(-r_{\mathbb{X}}) \\ &\cong \text{Hom}_S(S/\mathcal{I}_{\mathbb{X}/\mathbb{W}}, S)(r_{\mathbb{W}} - r_{\mathbb{X}}) \cong \text{Ann}_S(\mathcal{I}_{\mathbb{X}/\mathbb{W}})(\alpha_{\mathbb{Y}/\mathbb{W}}) \\ &\cong \mathcal{I}_{\mathbb{Y}/\mathbb{W}}(\alpha_{\mathbb{Y}/\mathbb{W}}). \end{aligned}$$

Since  $\mathbb{W}$  is arithmetically Gorenstein and  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked, this implies that  $\mathbb{X}$  is locally Gorenstein. So, the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$  of the algebra  $R/K[x_0]$  is well-defined and we have the isomorphism of graded  $R$ -modules  $\mathfrak{C}_{R/K[x_0]}(-r_{\mathbb{X}}) \cong \mathcal{I}_{\mathbb{Y}/\mathbb{W}}(\alpha_{\mathbb{Y}/\mathbb{W}})$ . Based on this isomorphism and Theorem 4.3.6, we get the following consequence.

**Corollary 4.3.9.** *In the setting of Theorem 4.3.6, the following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  has CBP( $d$ ).*
- (ii) *For every  $p_j \in \text{Supp}(\mathbb{X})$ , there exists an element  $g_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$  such that  $g_j^* = x_0^{-r_{\mathbb{X}}-d-1}\tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ .*

Moreover, if  $K$  is infinite, then the above conditions are equivalent to:

(iii) There exists an element  $g$  of  $(\mathfrak{C}_{R/K[x_0]})_{-d-1}$  such that  $\text{Ann}_R(g) = \langle 0 \rangle$ .

**Theorem 4.3.10.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X}$  be a subscheme of  $\mathbb{W}$  such that  $\mathbb{X}$  and its residual scheme in  $\mathbb{W}$  are geometrically linked, and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . The following conditions are equivalent.*

(i) The scheme  $\mathbb{X}$  has CBP( $d$ ).

(ii) The multiplication map  $R_{r_{\mathbb{X}}} \otimes (\mathfrak{C}_{R/K[x_0]})_{-d-1} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{r_{\mathbb{X}}-d-1}$  is nondegenerate.

(iii) For all  $i, j \geq 0$ , the multiplication map

$$R_i \otimes (\mathfrak{C}_{R/K[x_0]})_{-d-1+j} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-d-1+i+j}$$

is nondegenerate.

(iv) The multiplication map  $R_d \otimes (\mathfrak{C}_{R/K[x_0]})_{-d-1} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-1}$  is nondegenerate.

*Proof.* (i) $\Rightarrow$ (ii): If  $g \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$  and  $g \cdot R_i = \langle 0 \rangle$  for some  $i \geq 0$  then  $x_0^i g = 0$ , and hence  $g = 0$ . Now we let  $f \in R_{r_{\mathbb{X}}} \setminus \{0\}$ . Then there is a point  $p_j$  of  $\text{Supp}(\mathbb{X})$  such that  $f_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ . We want to prove  $f \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} \neq \langle 0 \rangle$ . Since  $\mathbb{X}$  has CBP( $d$ ), Corollary 4.3.9 yields that there is an element  $g_j^* \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$  such that  $g_j^* = x_0^{-r_{\mathbb{X}}-d-1} \tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$ . Then we have  $f_{p_j} \cdot (\tilde{g}_j^*)_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ . This implies that  $f \tilde{g}_j^* \neq 0$ , and so  $f \cdot g_j^* \neq 0$ . Therefore the multiplication map  $R_{r_{\mathbb{X}}} \otimes (\mathfrak{C}_{R/K[x_0]})_{-d-1} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{r_{\mathbb{X}}-d-1}$  is nondegenerate.

(ii) $\Rightarrow$ (iii): Since  $g \cdot R_i \neq \langle 0 \rangle$  for every  $g \in (\mathfrak{C}_{R/K[x_0]})_{-d-1+j} \setminus \{0\}$ , it is enough to show that if  $f \in R_i$  satisfies  $f \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1+j} = \langle 0 \rangle$  then  $f = 0$ . Suppose that  $f \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1+j} = \langle 0 \rangle$  for  $f \in R_i$ . This implies  $f x_0^j \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} = \langle 0 \rangle$ , and so  $f \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} = \langle 0 \rangle$ . If  $i \leq r_{\mathbb{X}}$ , then  $f x_0^{r_{\mathbb{X}}-i} \in R_{r_{\mathbb{X}}}$  and  $f x_0^{r_{\mathbb{X}}-i} \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} = \langle 0 \rangle$ , and hence  $f = 0$ . If  $i > r_{\mathbb{X}}$ , then we may write  $f = x_0^{i-r_{\mathbb{X}}} \tilde{f}$  with  $\tilde{f} \in R_{r_{\mathbb{X}}}$ . It follows from  $\tilde{f} \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} = \langle 0 \rangle$  that  $\tilde{f} = 0$ , and therefore  $f = x_0^{i-r_{\mathbb{X}}} \tilde{f} = 0$ .

(iii) $\Rightarrow$ (iv): This is clearly true.

(iv) $\Rightarrow$ (i): Suppose that  $\mathbb{X}$  does not have CBP( $d$ ), i.e., that there exists a maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$  and a set of separators  $\{f_{j1}, \dots, f_{j\kappa_j}\} \subseteq R_{r_{\mathbb{X}}}$  of  $\mathbb{X}'$  in  $\mathbb{X}$  such that  $x_0^{r_{\mathbb{X}}-d} \mid f_{jk_j}$  for all  $k_j = 1, \dots, \kappa_j$ . We write  $f_{jk_j} = x_0^{r_{\mathbb{X}}-d} h_{jk_j}$  with  $h_{jk_j} \in R_d$  for  $k_j = 1, \dots, \kappa_j$ . For  $g \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$ , let  $\varphi \in \underline{\text{Hom}}_{K[x_0]}(R, K[x_0])_{-d-1}$  be a homogeneous element such that  $g = \Phi(\varphi)$ , where  $\Phi$  is the homomorphism of graded  $R$ -modules given in Definition 3.2.4. Then  $\varphi(R_i) = \langle 0 \rangle$  for  $i \leq d$ , in particular, we have  $\varphi(f_{jk_j}) = x_0^{r_{\mathbb{X}}-d} \varphi(h_{jk_j}) = 0$  for all  $k_j = 1, \dots, \kappa_j$ . Now we show that  $f_{j1} \cdot \varphi = 0$ .

Let  $i \geq 0$  and let  $f \in R_i$ . By Lemma 2.3.25, we may write  $ff_{j_1} = \sum_{k_j=1}^{z_j} x_0^i c_{jk_j} f_{jk_j}$  for some  $c_{jk_j} \in K$ . Then we have  $(f_{j_1} \cdot \varphi)(f) = \varphi(ff_{j_1}) = \varphi(\sum_{k_j=1}^{z_j} x_0^i c_{jk_j} f_{jk_j}) = \sum_{k_j=1}^{z_j} x_0^i c_{jk_j} \varphi(f_{jk_j}) = 0$ , and so  $f_{j_1} \cdot \varphi = 0$ . Thus  $0 = \Phi(f_{j_1} \cdot \varphi) = f_{j_1} \Phi(\varphi) = f_{j_1} g$ . This implies that  $h_{j_1} \cdot g = 0$  for an arbitrary element  $g \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$ , and therefore we obtain  $h_{j_1} \cdot (\mathfrak{C}_{R/K[x_0]})_{-d-1} = \langle 0 \rangle$ , a contradiction.  $\square$

**Corollary 4.3.11.** *In the setting of Theorem 4.3.10, assume that  $K$  is infinite and  $\mathbb{X}$  has CBP( $d$ ).*

(i) *For all  $i \in \mathbb{Z}$  we have  $\mathrm{HF}_{\mathbb{X}}(i) + \mathrm{HF}_{\mathbb{X}}(d-i) \leq \deg(\mathbb{X})$ .*

(ii) *For  $i = 0, \dots, r_{\mathbb{X}}$  we have  $h_0 + \dots + h_i \leq h_{d-i+1} + \dots + h_{r_{\mathbb{X}}}$ , where  $h_i = \Delta \mathrm{HF}_{\mathbb{X}}(i)$ .*

*Proof.* Since  $K$  is infinite and  $\mathbb{X}$  has CBP( $d$ ), there is an element  $g \in (\mathfrak{C}_{R/K[x_0]})_{-d-1}$  such that  $\mathrm{Ann}_R(g) = \langle 0 \rangle$  by Corollary 4.3.9. Thus the map  $\mu_g : R \rightarrow \mathfrak{C}_{R/K[x_0]}(-d-1)$  given by  $f \mapsto fg$  is injective, and hence we have

$$\mathrm{HF}_{\mathbb{X}}(i) \leq \mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}(-d-1)}(i) = \mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(i-d-1) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(d-i)$$

for all  $i \in \mathbb{Z}$ . This completes the proof of claim (i). Claim (ii) follows from the claim (i) and the fact that  $\mathrm{HF}_{\mathbb{X}}(i) = h_0 + \dots + h_i$  and  $\deg(\mathbb{X}) = h_0 + \dots + h_{r_{\mathbb{X}}}$ .  $\square$

Corollary 4.3.9 can be used to describe the Hilbert function and the regularity index of the Dedekind different as follows. We use the notation  $\alpha_{\vartheta_D} = \min\{i \in \mathbb{N} \mid \vartheta_D(R/K[x_0])_i \neq \langle 0 \rangle\}$ .

**Proposition 4.3.12.** *Let  $K$  be an infinite field, let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, and let  $\mathbb{X}$  be a subscheme of  $\mathbb{W}$  such that  $\mathbb{X}$  and its residual scheme in  $\mathbb{W}$  are geometrically linked. Suppose that  $\mathbb{X}$  has CBP( $d$ ) with  $0 \leq d \leq r_{\mathbb{X}} - 1$ .*

(i) *We have  $d+1 \leq \alpha_{\vartheta_D} \leq 2r_{\mathbb{X}}$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i-d-1)$  for all  $i \in \mathbb{Z}$ .*

(ii) *Let  $i_0$  be the smallest number such that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0) = \mathrm{HF}_{\mathbb{X}}(i_0-d-1) > 0$ . Then we have  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) = \mathrm{HF}_{\mathbb{X}}(i-d-1)$  for all  $i \geq i_0$  and*

$$\mathrm{ri}(\vartheta_D(R/K[x_0])) = \max\{i_0, r_{\mathbb{X}} + d + 1\}.$$

(iii) *If  $\mathbb{X}$  is a projective point set and  $d \leq r_{\mathbb{X}} - 2$ , then  $i_0 = 2r_{\mathbb{X}}$ , in particular,  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) < \mathrm{HF}_{\mathbb{X}}(i-d-1)$  for  $i \in \{\alpha_{\vartheta_D}, \dots, 2r_{\mathbb{X}} - 1\}$ .*

*Proof.* By Corollary 4.3.9, there is a non-zero-divisor  $\tilde{g} \in R_{r_{\mathbb{X}}}$  such that  $g = x_0^{-r_{\mathbb{X}}-d-1}\tilde{g}$  in  $(\mathfrak{C}_{R/K[x_0]})_{-d-1}$ . We observe that  $\tilde{g} \cdot \vartheta_D(R/K[x_0])_i \subseteq x_0^{r_{\mathbb{X}}+d+1}R_{i-d-1}$ . This implies  $\vartheta_D(R/K[x_0])_i = \langle 0 \rangle$  for  $i \leq d$ , and so  $d+1 \leq \alpha_{\vartheta_D}$ . We calculate

$$\begin{aligned} \mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) &= \dim_K \vartheta_D(R/K[x_0])_i = \dim_K(\tilde{g} \cdot \vartheta_D(R/K[x_0])_i) \\ &\leq \dim_K(x_0^{r_{\mathbb{X}}+d+1}R_{i-d-1}) = \mathrm{HF}_{\mathbb{X}}(i-d-1). \end{aligned}$$

Thus claim (i) is completely proved.

Now we prove claim (ii). Clearly, we have  $d+1 \leq i_0 \leq 2r_{\mathbb{X}}$ . We only need to show that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0+1) = \mathrm{HF}_{\mathbb{X}}(i_0-d) > 0$ . Let  $f \in R_{i_0-d} \setminus \{0\}$ . There are  $g_0, \dots, g_n \in R_{i_0-d-1}$  such that  $f = x_0g_0 + x_1g_1 + \dots + x_n g_n$ . By assumption, we have  $\tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0} = x_0^{r_{\mathbb{X}}+d+1}R_{i_0-d-1}$ . This enables us to write  $x_0^{r_{\mathbb{X}}+d+1}g_j = \tilde{g}h_j$  for some  $h_j \in \vartheta_D(R/K[x_0])_{i_0}$ , where  $j \in \{0, \dots, n\}$ . Thus we have

$$\begin{aligned} x_0^{r_{\mathbb{X}}+d+1}f &= x_0^{r_{\mathbb{X}}+d+1}(x_0g_0 + x_1g_1 + \dots + x_n g_n) = x_0\tilde{g}h_0 + x_1\tilde{g}h_1 + \dots + x_n\tilde{g}h_n \\ &= \tilde{g}(x_0h_0 + x_1h_1 + \dots + x_n h_n) \end{aligned}$$

and so  $x_0^{r_{\mathbb{X}}+d+1}f \in \tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0+1}$ . Hence we get  $x_0^{r_{\mathbb{X}}+d+1}R_{i_0-d} = \tilde{g} \cdot \vartheta_D(R/K[x_0])_{i_0+1}$ . In other words, we have  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i_0+1) = \mathrm{HF}_{\mathbb{X}}(i_0-d)$ .

Let  $k = \max\{i_0, r_{\mathbb{X}} + d + 1\}$ . In order to prove the equality  $\mathrm{ri}(\vartheta_D(R/K[x_0])) = k$ , we consider the following two cases.

**Case (1)** Let  $i_0 \geq r_{\mathbb{X}} + d + 1$ . Then we have  $k = i_0$ . Observe that

$$\mathrm{deg}(\mathbb{X}) \geq \mathrm{HF}_{\vartheta_D(R/K[x_0])}(k) = \mathrm{HF}_{\mathbb{X}}(k-d-1) \geq \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \mathrm{deg}(\mathbb{X}).$$

It follows that  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(k) = \mathrm{deg}(\mathbb{X})$ , and hence  $k \geq \mathrm{ri}(\vartheta_D(R/K[x_0]))$ . Moreover, for  $i < k = i_0$ , we have  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) < \mathrm{HF}_{\mathbb{X}}(i-d-1) \leq \mathrm{HF}_{\mathbb{X}}(k-d-1) = \mathrm{deg}(\mathbb{X})$ . Thus we get  $\mathrm{ri}(\vartheta_D(R/K[x_0])) = k$ .

**Case (2)** Let  $i_0 < r_{\mathbb{X}} + d + 1$ . Then we have  $k = r_{\mathbb{X}} + d + 1$  and  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(k) = \mathrm{HF}_{\mathbb{X}}(k-d-1) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \mathrm{deg}(\mathbb{X})$ . This implies  $k \geq \mathrm{ri}(\vartheta_D(R/K[x_0]))$ . For  $i < k$ , we have  $\mathrm{HF}_{\vartheta_D(R/K[x_0])}(i) \leq \mathrm{HF}_{\mathbb{X}}(i-d-1) \leq \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}-1) < \mathrm{deg}(\mathbb{X})$ . Hence we obtain  $\mathrm{ri}(\vartheta_D(R/K[x_0])) = k$  again.

Finally, claim (iii) follows from (ii) and the fact that  $\mathrm{ri}(\vartheta_D(R/K[x_0])) = 2r_{\mathbb{X}}$ .  $\square$

**Lemma 4.3.13.** *Let  $A/K$  be a finite Gorenstein algebra.*

- (i) *There is a non-degenerate  $K$ -bilinear form  $\Phi : A \times A \rightarrow K$  with the property that  $\Phi(xy, z) = \Phi(x, yz)$  for all  $x, y, z \in A$ .*

(ii) Let  $I$  be a non-zero ideal of  $A$ , and let  $I^0 = \{x \in A \mid \Phi(I, x) = 0\}$ . Then we have  $\text{Ann}_A(I) = I^0$  and  $\dim_K I + \dim_K \text{Ann}_A(I) = \dim_K A$ .

*Proof.* Claim (i) follows from [Lam, Theorem 3.15], and claim (ii) follows from [Lam, Lemma 16.38 and Theorem 16.40].  $\square$

**Lemma 4.3.14.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X}$  (resp.  $\mathbb{X}'$ ) be a subscheme of  $\mathbb{W}$ , and let  $\mathbb{Y}$  (resp.  $\mathbb{Y}'$ ) be the residual scheme of  $\mathbb{X}$  (resp.  $\mathbb{X}'$ ) in  $\mathbb{W}$ . Let  $p_j \in \text{Supp}(\mathbb{X})$ . Then  $\mathbb{X}'$  is a (maximal)  $p_j$ -subscheme of  $\mathbb{X}$  if and only if  $\mathbb{Y}'$  contains  $\mathbb{Y}$  as a (maximal)  $p_j$ -subscheme.*

*Proof.* As sets,  $\text{Supp}(\mathbb{W}) = \text{Supp}(\mathbb{X}) \cup \text{Supp}(\mathbb{Y})$  by [Mig, Proposition 5.2.2]. Hence we may write  $\text{Supp}(\mathbb{W}) = \{p_1, \dots, p_s, p_{s+1}, \dots, p_t, p_{t+1}, \dots, p_u\}$ ,  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_t\}$ , and  $\text{Supp}(\mathbb{Y}) = \{p_{s+1}, \dots, p_u\}$ . Then there are ideals  $\mathfrak{q}_{s+1}, \dots, \mathfrak{q}_t \subseteq \mathcal{O}_{\mathbb{W}, p_j}$  such that

$$\mathcal{O}_{\mathbb{X}, p_j} = \begin{cases} \mathcal{O}_{\mathbb{W}, p_j} & \text{for } j = 1, \dots, s, \\ \mathcal{O}_{\mathbb{W}, p_j} / \mathfrak{q}_j & \text{for } j = s+1, \dots, t, \\ \langle 0 \rangle & \text{for } j = t+1, \dots, u. \end{cases}$$

We consider the map  $\theta : S \rightarrow \prod_{j=1}^u \mathcal{O}_{\mathbb{W}, p_j}$ . Notice that  $\theta|_{S_i}$  is an injection for  $0 \leq i < r_{\mathbb{W}}$  and  $\theta|_{S_i}$  is an isomorphism for all  $i \geq r_{\mathbb{W}}$ . We see that

$$\theta(\mathcal{I}_{\mathbb{X}/\mathbb{W}}) = \theta((\mathcal{I}_{\mathbb{X}/\mathbb{W}})_{r_{\mathbb{W}}}) = \{0\} \times \dots \times \{0\} \times \mathfrak{q}_{s+1} \times \dots \times \mathfrak{q}_t \times \mathcal{O}_{\mathbb{W}, p_{t+1}} \times \dots \times \mathcal{O}_{\mathbb{W}, p_u}$$

and  $\dim_K \theta(\mathcal{I}_{\mathbb{X}/\mathbb{W}}) = \deg(\mathbb{W}) - \deg(\mathbb{X})$ . In  $\prod_{j=1}^u \mathcal{O}_{\mathbb{W}, p_j}$ , we set

$$\Lambda := \mathcal{O}_{\mathbb{W}, p_1} \times \dots \times \mathcal{O}_{\mathbb{W}, p_s} \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_{s+1}}}(\mathfrak{q}_{s+1}) \times \dots \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_t}}(\mathfrak{q}_t) \times \{0\} \times \dots \times \{0\}.$$

By Lemma 4.3.13, we have  $\deg(\mathbb{W}) = \dim_K \Lambda + \dim_K \theta(\mathcal{I}_{\mathbb{X}/\mathbb{W}})$ . This implies that  $\dim_K \Lambda = \deg(\mathbb{W}) - \deg(\mathbb{Y})$ . We want to show that  $\theta(\mathcal{I}_{\mathbb{Y}/\mathbb{W}}) = \Lambda$ . Let  $i \geq \alpha_{\mathbb{Y}/\mathbb{W}}$  and let  $f$  be a non-zero homogeneous element in  $(\mathcal{I}_{\mathbb{Y}/\mathbb{W}})_i$ . If  $\theta(f) = (f_{p_1}, \dots, f_{p_u}) \notin \Lambda$ , then we consider the following two cases:

**Case (a<sub>1</sub>)** There is an index  $j \in \{s+1, \dots, t\}$  such that  $f_{p_j} \notin \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}_j)$ .

In this case we find a non-zero element  $a \in \mathfrak{q}_j$  such that  $a \cdot f_{p_j} \neq 0$ . Let  $g = \theta^{-1}((0, \dots, 0, a, 0, \dots, 0)) \in (\mathcal{I}_{\mathbb{X}/\mathbb{W}})_{r_{\mathbb{W}}} \setminus \{0\}$ . Then  $f \cdot g \neq 0$  in  $S$ , since  $(f \cdot g)_{p_j} \neq 0$ . This is a contradiction to the fact that  $\mathcal{I}_{\mathbb{X}/\mathbb{W}} \cdot \mathcal{I}_{\mathbb{Y}/\mathbb{W}} = \langle 0 \rangle$ .

**Case (a<sub>2</sub>)** Suppose that  $f_{p_j} \in \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}_j)$  for all  $j = 1, \dots, t$ , and that there is an index  $k \in \{t+1, \dots, u\}$  such that  $f_{p_k} \neq 0$ . In this case we argue as in case (a<sub>1</sub>) to obtain  $g \in (\mathcal{I}_{\mathbb{X}/\mathbb{W}})_{r_{\mathbb{W}}} \setminus \{0\}$  such that  $f \cdot g \neq 0$ . This is impossible.

Thus we have shown that  $\theta(f) \in \Lambda$  for every  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{W}})_i \setminus \{0\}$ , where  $i \geq \alpha_{\mathbb{Y}/\mathbb{W}}$ . In other words, we get  $\theta(\mathcal{I}_{\mathbb{Y}/\mathbb{W}}) \subseteq \Lambda$ . For the other inclusion, let  $i \geq 0$ , let  $f \in S_i \setminus \{0\}$  such that  $\theta(f) \in \Lambda$ , and let  $g \in (\mathcal{I}_{\mathbb{X}/\mathbb{W}})_k \setminus \{0\}$  with  $k \geq \alpha_{\mathbb{X}/\mathbb{W}}$ . Then we have  $f \cdot g \in S_{i+k}$  and  $\theta(f \cdot g) = \mathbf{0}$ . This implies that  $f \cdot g = 0$ , and hence  $f \in \text{Ann}_S(\mathcal{I}_{\mathbb{X}/\mathbb{W}}) = \mathcal{I}_{\mathbb{Y}/\mathbb{W}}$ . Altogether, we have  $\theta(\mathcal{I}_{\mathbb{Y}/\mathbb{W}}) = \Lambda$ , as wanted.

Now we assume that  $\mathbb{X}'$  is a  $p_j$ -subscheme of  $\mathbb{X}$ . Then we have  $\mathcal{O}_{\mathbb{X}', p_k} = \mathcal{O}_{\mathbb{X}, p_k}$  for all  $k \in \{1, \dots, u\} \setminus \{j\}$  and  $\mathcal{O}_{\mathbb{X}', p_j} = \mathcal{O}_{\mathbb{W}, p_j} / \mathfrak{q}'_j$ . We distinguish the following two cases.

**Case (b<sub>1</sub>)** Suppose that  $1 \leq j \leq s$ .

We see that  $\mathfrak{q}'_j \neq \langle 0 \rangle$  and

$$\begin{aligned} \theta(\mathcal{I}_{\mathbb{X}'/\mathbb{W}}) &= \{0\} \times \cdots \times \mathfrak{q}'_j \times \cdots \times \{0\} \times \mathfrak{q}_{s+1} \times \cdots \times \mathfrak{q}_t \times \mathcal{O}_{\mathbb{W}, p_{t+1}} \times \cdots \times \mathcal{O}_{\mathbb{W}, p_u} \\ \theta(\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}) &= \mathcal{O}_{\mathbb{W}, p_1} \times \cdots \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}'_j) \times \cdots \times \mathcal{O}_{\mathbb{W}, p_s} \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_{s+1}}}(\mathfrak{q}_{s+1}) \times \cdots \\ &\quad \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_t}}(\mathfrak{q}_t) \times \{0\} \times \cdots \times \{0\}. \end{aligned}$$

This implies  $\mathcal{O}_{\mathbb{Y}', p_k} = \mathcal{O}_{\mathbb{Y}, p_k}$  for  $k \neq j$  and  $\mathcal{O}_{\mathbb{Y}', p_j} = \mathcal{O}_{\mathbb{W}, p_j} / \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}'_j) \neq \langle 0 \rangle = \mathcal{O}_{\mathbb{Y}, p_j}$ . Hence  $\mathbb{Y}$  is a  $p_j$ -subscheme of  $\mathbb{Y}'$ .

**Case (b<sub>2</sub>)** Suppose that  $s+1 \leq j \leq t$ .

We have  $\mathfrak{q}'_j \supseteq \mathfrak{q}_j$  and

$$\begin{aligned} \theta(\mathcal{I}_{\mathbb{X}'/\mathbb{W}}) &= \{0\} \times \cdots \times \{0\} \times \mathfrak{q}_{s+1} \times \cdots \times \mathfrak{q}'_j \times \cdots \times \mathfrak{q}_t \times \mathcal{O}_{\mathbb{W}, p_{t+1}} \times \cdots \times \mathcal{O}_{\mathbb{W}, p_u} \\ \theta(\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}) &= \mathcal{O}_{\mathbb{W}, p_1} \times \cdots \times \mathcal{O}_{\mathbb{W}, p_s} \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_{s+1}}}(\mathfrak{q}_{s+1}) \times \cdots \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}'_j) \times \cdots \\ &\quad \times \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_t}}(\mathfrak{q}_t) \times \{0\} \times \cdots \times \{0\}. \end{aligned}$$

This implies that  $\mathcal{O}_{\mathbb{Y}', p_k} = \mathcal{O}_{\mathbb{Y}, p_k}$  for  $k \neq j$  and  $\mathcal{O}_{\mathbb{Y}', p_j} = \mathcal{O}_{\mathbb{W}, p_j} / \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}'_j) \neq \mathcal{O}_{\mathbb{W}, p_j} / \text{Ann}_{\mathcal{O}_{\mathbb{W}, p_j}}(\mathfrak{q}_j) = \mathcal{O}_{\mathbb{Y}, p_j}$ . Hence  $\mathbb{Y}$  is a  $p_j$ -subscheme of  $\mathbb{Y}'$ .

Conversely, if  $\mathbb{Y}$  is a  $p_j$ -subscheme of  $\mathbb{Y}'$ , where  $p_j \in \text{Supp}(\mathbb{X})$ , we can argue analogously as above to get that  $\mathbb{X}'$  is a  $p_j$ -subscheme of  $\mathbb{X}$ .  $\square$

The next theorem provides a characterization of the Cayley-Bacharach property of degree  $d$  which is a generalization of [KR1, Theorem 4.1].

**Theorem 4.3.15.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ , and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . Then the following statements are equivalent.*

- (i) *The scheme  $\mathbb{X}$  has CBP( $d$ ).*
- (ii) *Every subscheme  $\mathbb{Y}' \subseteq \mathbb{W}$  containing  $\mathbb{Y}$  as a maximal  $p_j$ -subscheme, where  $p_j$  is a point in  $\text{Supp}(\mathbb{X})$ , satisfies  $\text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) > 0$ .*

*Proof.* Let  $p_j \in \text{Supp}(\mathbb{X})$ , let  $\mathbb{X}'$  be a maximal  $p_j$ -subscheme of  $\mathbb{X}$ , let  $R_{\mathbb{X}'}$  be the homogeneous coordinate ring of  $\mathbb{X}'$ , and let  $\mathbb{Y}'$  be the residual scheme of  $\mathbb{X}'$  in  $\mathbb{W}$ . By Proposition 4.3.2, we have  $\deg(\mathbb{Y}') = \deg(\mathbb{Y}) + \varkappa_j$ ,  $\mathbb{Y} \subseteq \mathbb{Y}'$ , and  $r_{\mathbb{X}'} + \alpha_{\mathbb{Y}'/\mathbb{W}} = r_{\mathbb{W}} = r_{\mathbb{X}} + \alpha_{\mathbb{Y}/\mathbb{W}}$ . Moreover, we see that  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])(-r_{\mathbb{X}}) \cong \mathcal{I}_{\mathbb{Y}/\mathbb{W}}(\alpha_{\mathbb{Y}/\mathbb{W}})$  and  $\underline{\text{Hom}}_{K[x_0]}(R_{\mathbb{X}'}, K[x_0])(-r_{\mathbb{X}'}) \cong \mathcal{I}_{\mathbb{Y}'/\mathbb{W}}(\alpha_{\mathbb{Y}'/\mathbb{W}})$ . This implies  $\text{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - i) - 1) = \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(i)$  and  $\text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - i) - 1) = \text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}'/\mathbb{W}} + (r_{\mathbb{X}'} - i) - 1) = (\deg(\mathbb{X}) - \varkappa_j) - \text{HF}_{\mathbb{X}'}(i)$ . Thus we get

$$\text{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - i) - 1) - \text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - i) - 1) = \varkappa_j - \text{HF}_{\mathcal{I}_{\mathbb{X}'/\mathbb{X}}}(i).$$

According to Proposition 4.1.22,  $\mathbb{X}$  has CBP( $d$ ) if and only if  $\text{HF}_{\mathcal{I}_{\mathbb{X}'/\mathbb{X}}}(d) < \varkappa_j$  for every maximal  $p_j$ -subscheme  $\mathbb{X}'$  of  $\mathbb{X}$ . This is equivalent to  $\text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) = \text{HF}_{\mathcal{I}_{\mathbb{Y}/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) - \text{HF}_{\mathcal{I}_{\mathbb{Y}'/\mathbb{W}}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) > 0$  for every residual scheme  $\mathbb{Y}' \subseteq \mathbb{W}$  of a maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$ . By Lemma 4.3.14, there is a 1-1 correspondence between a maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$  and a subscheme  $\mathbb{Y}' \subseteq \mathbb{W}$  containing  $\mathbb{Y}$  as a maximal  $p_j$ -subscheme. In particular,  $\mathbb{Y}'$  is exactly the residual scheme of  $\mathbb{X}'$  in  $\mathbb{W}$ . Therefore the proof of the proposition is complete.  $\square$

**Corollary 4.3.16.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ , and let  $0 \leq d \leq r_{\mathbb{W}} - \alpha_{\mathbb{Y}/\mathbb{W}} - 1$ . If  $\mathcal{I}_{\mathbb{Y}/\mathbb{W}}$  is generated by its elements of degree less than or equal to  $r_{\mathbb{W}} - d - 1$  then  $\mathbb{X}$  has CBP( $d$ ).*

*Proof.* This follows by Theorem 4.3.15 using a similar argument as in the proof of [Kr3, Corollary 16.5].  $\square$

**Corollary 4.3.17.** *Let  $\mathbb{W} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, let  $\mathbb{X} \subseteq \mathbb{W}$  be a subscheme, let  $\mathbb{Y} \subseteq \mathbb{W}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ , and let  $0 \leq d \leq r_{\mathbb{X}} - 1$ . Then the following conditions are equivalent.*

- (i)  $\mathbb{X}$  has CBP( $d$ ).
- (ii)  $\mathcal{I}_{\mathbb{W}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1} = \mathcal{I}_{\mathbb{X}}$
- (iii)  $\text{Ann}_S((\mathcal{I}_{\mathbb{Y}/\mathbb{W}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}) = \mathcal{I}_{\mathbb{X}/\mathbb{W}}$
- (iv)  $(\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}} - 1} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1} = (\mathcal{I}_{\mathbb{X}})_d$
- (v)  $(\text{Ann}_S((\mathcal{I}_{\mathbb{Y}/\mathbb{W}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}))_d = (\mathcal{I}_{\mathbb{X}/\mathbb{W}})_d$

*Proof.* Observe that “(ii) $\Leftrightarrow$ (iii)” and “(iv) $\Leftrightarrow$ (v)” are obviously true. Moreover, the implication “(ii) $\Rightarrow$ (iv)” is clear. It remains to prove “(i) $\Rightarrow$ (ii)” and “(iv) $\Rightarrow$ (i)”. First we show “(i) $\Rightarrow$ (ii)”. We have  $\mathcal{I}_{\mathbb{X}} \subseteq \mathcal{I}_{\mathbb{W}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$ . To prove the reverse inclusion, we let  $F \in \mathcal{I}_{\mathbb{W}} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$ . Suppose for a contradiction that  $F \notin \mathcal{I}_{\mathbb{X}}$ . There is a point  $p_j \in \text{Supp}(\mathbb{X})$  such that  $F_{p_j} \neq 0$ . Then we can find an element  $a_j \in \mathcal{O}_{\mathbb{X}, p_j}$  such that  $a_j \cdot F_{p_j}$  is a socle element in  $\mathfrak{G}(\mathcal{O}_{\mathbb{X}, p_j})$ . By Proposition 2.3.17, there is a maximal  $p_j$ -subscheme  $\mathbb{X}'$  of  $\mathbb{X}$  corresponding to  $(0, \dots, 0, a_j \cdot F_{p_j}, 0, \dots, 0) \in \prod_{p_k \in \text{Supp}(\mathbb{X})} \mathcal{O}_{\mathbb{X}, p_k}$ . Let  $\mathbb{Y}'$  be the residual scheme of  $\mathbb{X}'$  in  $\mathbb{W}$ . Since  $\mathbb{X}$  has CBP( $d$ ), Theorem 4.3.15 tells us that  $\text{HF}_{\mathcal{I}_{\mathbb{Y}'}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) > 0$ . Let  $G \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$ . It is clear that  $FG \in \mathcal{I}_{\mathbb{W}}$  and  $G \cdot \mathcal{I}_{\mathbb{X}} \subseteq \mathcal{I}_{\mathbb{W}}$ . Moreover, since  $\mathcal{I}_{\mathbb{W}}$  is saturated, we get  $G \cdot \langle F, \mathcal{I}_{\mathbb{X}} \rangle^{\text{sat}} \subseteq \mathcal{I}_{\mathbb{W}}$ . It follows from the inclusion  $\mathcal{I}_{\mathbb{X}'} \subseteq \langle F, \mathcal{I}_{\mathbb{X}} \rangle^{\text{sat}}$  that  $G \cdot \mathcal{I}_{\mathbb{X}'} \subseteq \mathcal{I}_{\mathbb{W}}$  or  $G \in (\mathcal{I}_{\mathbb{Y}'})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$ . Thus we obtain  $\text{HF}_{\mathcal{I}_{\mathbb{Y}'}}(\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1) = 0$ , a contradiction.

Now we prove “(iv) $\Rightarrow$ (i)”. For a contradiction, assume that the scheme  $\mathbb{X}$  does not have CBP( $d$ ). Then there is a maximal  $p_j$ -subscheme  $\mathbb{X}' \subseteq \mathbb{X}$  such that  $\deg(F_{j k_j}^*) \leq d$  for all  $k_j = 1, \dots, \varkappa_j$ , where  $\{F_{j 1}^*, \dots, F_{j \varkappa_j}^*\}$  is a set of minimal separators of  $\mathbb{X}'$  in  $\mathbb{X}$ . Let  $G_{j k_j} = X_0^{d - \deg(F_{j k_j}^*)} F_{j k_j}^*$  for  $k_j = 1, \dots, \varkappa_j$ . Since  $G_{j k_j} \notin (\mathcal{I}_{\mathbb{X}})_d$ , the hypothesis implies  $G_{j k_j} \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}} - 1} : (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1}$  for  $k_j = 1, \dots, \varkappa_j$ . Let  $H \in (\mathcal{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d) - 1} \setminus \{0\}$  be such that  $G_{j 1} H \notin (\mathcal{I}_{\mathbb{W}})_{r_{\mathbb{W}} - 1}$ . As sets, we have  $\text{Supp}(\mathbb{W}) = \text{Supp}(\mathbb{X}) \cup \text{Supp}(\mathbb{Y})$ . For  $p \in \text{Supp}(\mathbb{W}) \setminus \{p_j\}$ , we see that  $(G_{j 1} H)_p = 0$  in  $\mathcal{O}_{\mathbb{W}, p}$  and  $(G_{j 1} H)_{p_j} \neq 0$  in  $\mathcal{O}_{\mathbb{W}, p_j}$ . By writing  $\mathcal{O}_{\mathbb{X}, p_j} = \mathcal{O}_{\mathbb{W}, p_j} / \mathfrak{q}_j$  for some ideal  $\mathfrak{q}_j$  of  $\mathcal{O}_{\mathbb{W}, p_j}$ , we get  $\mathfrak{q}_j \cdot H_{p_j} = 0$  in  $\mathcal{O}_{\mathbb{W}, p}$  and  $a \cdot (G_{j 1})_{p_j} \in \mathfrak{q}_j$  for every  $a \in \mathfrak{m}_{\mathbb{W}, p_j}$ . This implies  $\mathfrak{m}_{\mathbb{W}, p_j} \cdot (G_{j 1} H)_{p_j} = \langle 0 \rangle$  in  $\mathcal{O}_{\mathbb{W}, p}$ . Thus the element  $(G_{j 1} H)_{p_j}$  is a socle element in  $\mathfrak{G}(\mathcal{O}_{\mathbb{W}, p_j})$ .

Notice that if  $G \in \mathcal{I}_{\mathbb{X}'} \setminus \mathcal{I}_{\mathbb{X}}$  then Proposition 2.3.17 implies  $G_{p_j} = x \cdot (G_{j 1})_{p_j} + y$  for some  $x \notin \mathfrak{m}_{\mathbb{W}, p_j}$  and some  $y \in \mathfrak{q}_j$ . Hence we have  $(GH)_{p_j} = (x \cdot (G_{j 1})_{p_j} + y)H_{p_j} = x \cdot (G_{j 1} H)_{p_j} \neq 0$ . Now we prove that the set  $\{(G_{j 1} H)_{p_j}, \dots, (G_{j \varkappa_j} H)_{p_j}\} \subseteq \mathcal{O}_{\mathbb{W}, p_j}$  is  $K$ -linearly independent. Suppose that there are elements  $a_{j 1}, \dots, a_{j \varkappa_j} \in K$  such that  $a_{j 1}(G_{j 1} H)_{p_j} + \dots + a_{j \varkappa_j}(G_{j \varkappa_j} H)_{p_j} = 0$ . Then we have  $(a_{j 1}G_{j 1} + \dots + a_{j \varkappa_j}G_{j \varkappa_j})_{p_j} \cdot H_{p_j} = 0$ . This shows that  $a_{j 1}G_{j 1} + \dots + a_{j \varkappa_j}G_{j \varkappa_j} \in \mathcal{I}_{\mathbb{X}}$ . Furthermore, we have

$$\dim_K(\mathcal{I}_{\mathbb{X}'/\mathbb{X}})_d = \dim_K(\langle G_{j k_j} + \mathcal{I}_{\mathbb{X}} \mid k_j = 1, \dots, \varkappa_j \rangle_R)_d = \varkappa_j.$$

From this we get  $a_{j 1} = \dots = a_{j \varkappa_j} = 0$ . Hence the set  $\{(G_{j 1} H)_{p_j}, \dots, (G_{j \varkappa_j} H)_{p_j}\}$  is  $K$ -linearly independent, as was to be shown.

Let  $J := \langle G_{j k} H + \mathcal{I}_{\mathbb{W}} \mid k = 1, \dots, \varkappa_j \rangle_S$ . It is clearly true that  $\dim_K J_{r_{\mathbb{W}} - 1 + i} \geq \varkappa_j$  for all  $i \geq 0$ . Moreover, we can write  $X_i G_{j l} + \mathcal{I}_{\mathbb{X}} = \sum_{k_j=1}^{\varkappa_j} c_{j k_j}^l X_0 G_{j k_j} + \mathcal{I}_{\mathbb{X}}$  for some  $c_{j 1}^l, \dots, c_{j \varkappa_j}^l \in K$ , where  $0 \leq i \leq n$  and  $1 \leq l \leq \varkappa_j$ . This implies  $X_i G_{j l} H + \mathcal{I}_{\mathbb{W}} =$



$\sum_{k_j=1}^{\varkappa_j} c_{jk_j}^l X_0 G_{jk_j} H + \mathcal{I}_{\mathbb{W}}$ . Hence we obtain  $\dim_K J_{r_{\mathbb{W}}-1+i} \leq \varkappa_j$  for all  $i \geq 0$ . Altogether, we get  $\dim_K J_{r_{\mathbb{W}}-1+i} = \varkappa_j$  for all  $i \geq 0$ . Consequently, the ideal  $J$  defines a maximal  $p_j$ -subscheme  $\mathbb{W}' \subseteq \mathbb{W}$  of degree  $\deg(\mathbb{W}') = \deg(\mathbb{W}) - \varkappa_j$ . In particular, we have  $\dim_K (\mathcal{I}_{\mathbb{W}'/\mathbb{W}})_{r_{\mathbb{W}}-1} = \varkappa_j$ . Therefore Proposition 4.1.22 yields that the scheme  $\mathbb{W}$  is not a CB-scheme, a contradiction.  $\square$

## 4.4 Differents and Higher Uniformity of 0-Dimensional Schemes

In this section we let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme which has  $K$ -rational support, i.e.,  $\mathbb{X}(K) = \text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . In this situation the Cayley-Bacharach property is merely the weakest of a whole series of uniformity conditions which a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  can satisfy. The following uniformity concepts were introduced in [Kr4, Section 5] and [GK].

**Definition 4.4.1.** Let  $1 \leq i \leq \deg(\mathbb{X}) - 1$  and  $1 \leq j < r_{\mathbb{X}}$ . We say that  $\mathbb{X}$  is  $(i, j)$ -uniform if every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$  satisfies  $\text{HF}_{\mathbb{Y}}(j) = \text{HF}_{\mathbb{X}}(j)$ .

**Remark 4.4.2.** (a) The scheme  $\mathbb{X}$  has CBP( $d$ ) if and only if it is  $(1, d)$ -uniform.

(b) If  $\mathbb{X}$  is  $(i, j)$ -uniform, then it is  $(i - 1, j)$ -uniform and  $(i, j - 1)$ -uniform.

(c) If  $\mathbb{X}$  is  $(i, r_{\mathbb{X}} - 1)$ -uniform, then every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{X}) - i \leq \deg(\mathbb{Y}) \leq \deg(\mathbb{X})$  has Hilbert function  $\text{HF}_{\mathbb{Y}}(j) = \min\{\deg(\mathbb{Y}), \text{HF}_{\mathbb{X}}(j)\}$  for all  $j \in \mathbb{Z}$ . The converse is true for  $i \in \{1, \dots, \Delta_{\mathbb{X}}\}$ , where  $\Delta_{\mathbb{X}} = \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1)$  is the last non-zero difference of  $\text{HF}_{\mathbb{X}}$ .

(d) The scheme  $\mathbb{X}$  is in uniform position if and only if it is  $(i, j)$ -uniform for all  $1 \leq j \leq r_{\mathbb{X}} - 1$  and all  $1 \leq i \leq \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(j)$ .

(e) If  $\mathbb{X}$  is  $(i, j)$ -uniform, then  $i \leq \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(j)$ .

(f) If  $\mathbb{X}$  is arithmetically Gorenstein, then it cannot be  $(2, r_{\mathbb{X}} - 1)$ -uniform. This follows from  $\Delta_{\mathbb{X}} = 1$  and by [Kr2, Proposition 2.8] and (e).

**Corollary 4.4.3.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. If  $\mathbb{X}$  is a CB-scheme with  $\Delta_{\mathbb{X}} = 1$ , then it is  $(2, r_{\mathbb{X}} - 2)$ -uniform.

*Proof.* Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a subscheme of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - 2$ . By [Kr2, Lemma 2.2], there is a subscheme  $\mathbb{Y}_1$  of  $\mathbb{X}$  of degree  $\deg(\mathbb{X}) - 1$  such that  $\mathbb{Y} \subseteq \mathbb{Y}_1 \subseteq \mathbb{X}$ . It follows

from [Kr2, Proposition 2.8] that the scheme  $\mathbb{Y}_1$  is a CB-scheme. Since  $\Delta_{\mathbb{X}} = 1$ , we have  $\mathrm{HF}_{\mathbb{Y}_1}(r_{\mathbb{X}} - 1) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1) = \deg(\mathbb{Y}_1)$  and  $r_{\mathbb{Y}_1} = r_{\mathbb{X}} - 1$ . Thus  $\mathrm{HF}_{\mathbb{Y}}(r_{\mathbb{X}} - 2) = \mathrm{HF}_{\mathbb{Y}}(r_{\mathbb{Y}_1} - 1) = \mathrm{HF}_{\mathbb{Y}_1}(r_{\mathbb{Y}_1} - 1) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 2)$ , and hence  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 2)$ -uniform.  $\square$

A characterization of the uniformity of  $\mathbb{X}$  in terms of the Dedekind complementary module is given by the next proposition which follows from [Kr2, Theorem 3.2].

**Proposition 4.4.4.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, and let  $j \in \{1, \dots, r_{\mathbb{X}} - 1\}$ . The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is  $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j), j)$ -uniform.*
- (ii) *The multiplication map  $\mu : R_j \otimes (\mathfrak{C}_{R/K[x_0]})_{-j-1} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-1}$  is biinjective, i.e., if  $\mu(f \otimes g) = 0$  implies  $f = 0$  or  $g = 0$  for all  $f \in R_j$  and  $g \in (\mathfrak{C}_{R/K[x_0]})_{-j-1}$ .*

We have the following observation.

**Lemma 4.4.5.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme, let  $j \in \{1, \dots, r_{\mathbb{X}} - 2\}$ , and let  $\Delta \mathrm{HF}_{\mathbb{X}}(j + 1) = k \in \{1, \dots, \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - 1\}$ . If the multiplication map  $R_j \otimes (\mathfrak{C}_{R/K[x_0]})_{-j-2} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-2}$  is biinjective, then  $\mathbb{X}$  is  $(s - \mathrm{HF}_{\mathbb{X}}(j) - k, j)$ -uniform.*

*Proof.* Assume that  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \mathrm{HF}_{\mathbb{X}}(j) + k$  satisfies  $\mathrm{HF}_{\mathbb{Y}}(j) < \mathrm{HF}_{\mathbb{X}}(j)$ . Then  $r_{\mathbb{Y}} \geq j + 1$  and there is an element  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j \setminus \{0\}$ . If  $r_{\mathbb{Y}} = j + 1$ , then  $\mathrm{HF}_{\mathbb{Y}}(j + 1) = \deg(\mathbb{Y}) = \mathrm{HF}_{\mathbb{X}}(j) + k = \mathrm{HF}_{\mathbb{X}}(j + 1)$  by assumption. This implies  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{j+1} = \langle 0 \rangle$ , contradicting  $0 \neq x_0 f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{j+1}$ . Hence we must have  $r_{\mathbb{Y}} \geq j + 2$ . In this case Proposition 2.4.6 yields that there is  $g \in (\mathfrak{C}_{R/K[x_0]})_{-j-2} \setminus \{0\}$  such that  $fg = 0$ , a contradiction.  $\square$

**Proposition 4.4.6.** *If  $K$  is algebraically closed and  $\mathbb{X}$  is a 0-dimensional locally Gorenstein scheme such that  $\Delta \mathrm{HF}_{\mathbb{X}}(j + 1) = 1$  for some  $j \in \{1, \dots, r_{\mathbb{X}} - 2\}$ , the following statements are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is  $(i, j)$ -uniform for some  $1 \leq i \leq \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j)$ .*
- (ii) *The Hilbert function of  $\mathbb{X}$  satisfies  $\mathrm{HF}_{\mathbb{X}}(1) = 2$ , i.e., the support of  $\mathbb{X}$  lies on a line.*

*Proof.* It is clear that (ii) implies (i). Now we assume that  $\mathbb{X}$  is  $(\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - k, j)$ -uniform for some  $k \in \{0, \dots, \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - 1\}$ . By Remark 4.4.2(b), we may assume that  $k \neq 0$ . Then the multiplication map  $R_j \otimes (\mathfrak{C}_{R/K[x_0]})_{-j-k-1} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-k-1}$  is biinjective by a similar argument as in the proof of [Kr2, Theorem 3.2]. Since

$K$  is an algebraically closed field, the Biinjective Map Lemma (cf. [Kr2, Section 3]) yields  $\mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(-k-1) \geq \mathrm{HF}_{\mathbb{X}}(j) + \mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(-j-k-1) - 1$ . This implies  $\mathrm{deg}(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k) \geq \mathrm{HF}_{\mathbb{X}}(j) + (\mathrm{deg}(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j+k)) - 1$ . Since  $\Delta \mathrm{HF}_{\mathbb{X}}(j+1) = 1$ , we have  $\mathrm{HF}_{\mathbb{X}}(j+k) = \mathrm{HF}_{\mathbb{X}}(j) + k$  (cf. [KR3, Corollary 5.5.28]). Hence we get  $\mathrm{deg}(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(k) \geq \mathrm{HF}_{\mathbb{X}}(j) + (\mathrm{deg}(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(j) - k) - 1 = \mathrm{deg}(\mathbb{X}) - (k+1)$ , or  $\mathrm{HF}_{\mathbb{X}}(k) \leq k+1$ . Moreover, the Hilbert function of  $\mathbb{X}$  satisfies  $0 < \mathrm{HF}_{\mathbb{X}}(1) < \cdots < \mathrm{HF}_{\mathbb{X}}(k) \leq k+1$ . Therefore it must be the case that  $\mathrm{HF}_{\mathbb{X}}(1) = 2$ .  $\square$

In view of Lemma 2.3.12, the saturation of a non-zero homogeneous ideal  $J \subseteq R$  is given by

$$J^{\mathrm{sat}} = \{f \in R \mid x_0^n f \subseteq J \text{ for some } n \geq 0\}.$$

In particular, the ideal  $J$  is saturated if and only if  $x_0 f \in J$  implies  $f \in J$ . Using this description, we prove the following proposition.

**Proposition 4.4.7.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein scheme with support  $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and let  $1 \leq m \leq \Delta_{\mathbb{X}}$ . If  $\mathbb{X}$  is  $(m, r_{\mathbb{X}} - 1)$ -uniform, then any subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\mathrm{deg}(\mathbb{Y}) = \mathrm{deg}(\mathbb{X}) - m$  satisfies*

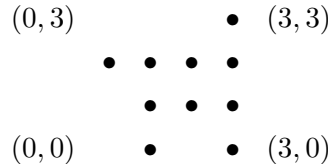
$$x_0^{r_{\mathbb{X}}-1}(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1} = \langle 0 \rangle.$$

*Proof.* Suppose for contradiction that there is a non-zero element  $h \in x_0^{r_{\mathbb{X}}-1}(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{2r_{\mathbb{X}}-1}$ , where  $\mathbb{Y} \subseteq \mathbb{X}$  is a subscheme of degree  $\mathrm{deg}(\mathbb{Y}) = \mathrm{deg}(\mathbb{X}) - m$ . Since  $\mathbb{X}$  is  $(m, r_{\mathbb{X}} - 1)$ -uniform, we have  $\mathrm{HF}_{\mathbb{Y}}(i) = \mathrm{HF}_{\mathbb{X}}(i)$  for  $i < r_{\mathbb{X}}$  and  $\mathrm{HF}_{\mathbb{Y}}(i) = \mathrm{deg}(\mathbb{X}) - m$  for  $i \geq r_{\mathbb{X}}$ . Let  $f_1, \dots, f_m$  be a  $K$ -basis of the  $K$ -vector space  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$ . Then we have  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_1, \dots, f_m \rangle$  and  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}+i} = \langle x_0^i f_1, \dots, x_0^i f_m \rangle_K$  for all  $i \geq 0$ . So, we may write  $h = x_0^{r_{\mathbb{X}}-1}(a_1 f_1 + \cdots + a_m f_m)$  for some  $a_1, \dots, a_m \in K$ . On the other hand, the scheme  $\mathbb{X}$  is also a CB-scheme. As in the proof of Theorem 4.1.7 we find elements  $g_1^*, \dots, g_s^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $g_j^* = x_0^{-2r_{\mathbb{X}}}\tilde{g}_j^*$  with  $\tilde{g}_j^* \in R_{r_{\mathbb{X}}} \setminus \{0\}$  and  $(\tilde{g}_j^*)_{p_j} \in \mathcal{O}_{\mathbb{X}, p_j} \setminus \mathfrak{m}_{\mathbb{X}, p_j}$  for  $j = 1, \dots, s$ . Given  $j \in \{1, \dots, s\}$ , we have

$$\begin{aligned} h \cdot g_j^* &= x_0^{r_{\mathbb{X}}-1}(a_1 f_1 + \cdots + a_m f_m) \cdot x_0^{-2r_{\mathbb{X}}}\tilde{g}_j^* = x_0^{-r_{\mathbb{X}}-1}(a_1 f_1 + \cdots + a_m f_m)\tilde{g}_j^* \\ &= x_0^{-1}(b_{j1} f_1 + \cdots + b_{jm} f_m) \in R_{r_{\mathbb{X}}-1} \end{aligned}$$

for some  $b_{j1}, \dots, b_{jm} \in K$ . This yields that  $b_{j1} f_1 + \cdots + b_{jm} f_m \in x_0 R_{r_{\mathbb{X}}-1}$ . Since  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  is a saturated ideal of  $R$  and  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} = \langle 0 \rangle$ , the condition  $b_{j1} f_1 + \cdots + b_{jm} f_m \in x_0 R_{r_{\mathbb{X}}-1}$  implies  $b_{j1} = \cdots = b_{jm} = 0$ , especially,  $h \cdot g_j^* = h \cdot \tilde{g}_j^* = 0$ . It follows that  $h_{p_j} \cdot (\tilde{g}_j^*)_{p_j} = 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$ . Since the element  $(\tilde{g}_j^*)_{p_j}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_j}$ , we get  $h_{p_j} = 0$ . Thus we have shown that  $h_{p_j} = 0$  in  $\mathcal{O}_{\mathbb{X}, p_j}$  for all  $j = 1, \dots, s$ . Therefore Lemma 2.3.10 yields that  $h = 0$ , in contradiction to  $h \neq 0$ .  $\square$

**Example 4.4.8.** Let  $K$  be a field with  $\text{char}(K) = 0$  or  $\text{char}(K) \geq 5$ . We consider the projective point set  $\mathbb{X} = \{p_1, \dots, p_{10}\} \subseteq \mathbb{P}_K^2$  with  $p_1 = (1 : 1 : 0)$ ,  $p_2 = (1 : 3 : 0)$ ,  $p_3 = (1 : 1 : 1)$ ,  $p_4 = (1 : 2 : 1)$ ,  $p_5 = (1 : 3 : 1)$ ,  $p_6 = (1 : 0 : 2)$ ,  $p_7 = (1 : 1 : 2)$ ,  $p_8 = (1 : 2 : 2)$ ,  $p_9 = (1 : 3 : 2)$ , and  $p_{10} = (1 : 3 : 3)$ . Sketched in the affine plane  $\mathbb{A}^2 = D_+(X_0) = \{(c_0 : c_1 : c_2) \in \mathbb{P}_\mathbb{Q}^2 \mid c_0 \neq 0\}$  the set  $\mathbb{X}$  codes as follows:



The Hilbert function of  $\mathbb{X}$  is  $\text{HF}_\mathbb{X} : 1 \ 3 \ 6 \ 10 \ 10 \ \dots$ , so that  $r_\mathbb{X} = 3$  and  $\alpha_\mathbb{X} = 4 = \Delta_\mathbb{X}$ . Furthermore, the Noether-Dedekind different is computed by  $\vartheta_D(R/K[x_0]) = \bigoplus_{i \geq 6} R_i$ , and so  $\vartheta_D(R/K[x_0])_{2r_\mathbb{X}-1} = \langle 0 \rangle$ . In this case,  $\mathbb{X}$  is  $(1, r_\mathbb{X} - 1)$ -uniform by Corollary 4.1.9(ii). Let  $\mathbb{Y} = \mathbb{X} \setminus \{p_4, p_6, p_8\}$ . Using the figure, we see that the hypersurface defined by  $F = (X_1 - X_0)(X_1 - 3X_0)$  contains  $\mathbb{Y}$ . This implies  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_2 \neq 0$  (as  $\alpha_\mathbb{X} = 4$ ). Thus we obtain  $\text{HF}_\mathbb{Y}(r_\mathbb{X} - 1) = \text{HF}_\mathbb{Y}(2) < \text{HF}_\mathbb{X}(2) = \text{HF}_\mathbb{X}(r_\mathbb{X} - 1) = 6$ . Explicitly, the Hilbert function of  $\mathbb{Y}$  is given by  $\text{HF}_\mathbb{Y} : 1 \ 3 \ 5 \ 7 \ 7 \ \dots$ , and  $\text{HF}_\mathbb{Y}(r_\mathbb{X} - 1) = 5 < 6 = \text{HF}_\mathbb{X}(r_\mathbb{X} - 1)$ . It follows from Definition 4.4.1 that  $\mathbb{X}$  is neither  $(3, 2)$ -uniform nor  $(\Delta_\mathbb{X}, r_\mathbb{X} - 1)$ -uniform. Hence the converse of Proposition 4.4.7 is not true in general.

**Corollary 4.4.9.** *If a 0-dimensional locally Gorenstein scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  satisfies  $r_\mathbb{X} = 1$ , then either  $\mathbb{X}$  has  $\vartheta_D(R/K[x_0])_1 = \langle 0 \rangle$  or  $\mathbb{X}$  is arithmetically Gorenstein.*

*Proof.* Since  $r_\mathbb{X} = 1$ , we have  $\text{HF}_\mathbb{X} : 1 \ \text{deg}(\mathbb{X}) \ \dots$  and  $\Delta_\mathbb{X} = \text{deg}(\mathbb{X}) - 1$ . Thus it is easy to check that  $\mathbb{X}$  is in uniform position. Now it follows from Proposition 4.4.7 that  $\vartheta_D(R/K[x_0])_1 \cap (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_1 = \langle 0 \rangle$  for any subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\text{deg}(\mathbb{Y}) = \text{deg}(\mathbb{X}) - 1$ . Hence we get  $\vartheta_D(R/K[x_0])_0 = \langle 0 \rangle$ . If  $\vartheta_D(R/K[x_0])_1 \neq \langle 0 \rangle$ , then the scheme  $\mathbb{X}$  is arithmetically Gorenstein by Proposition 4.1.15.  $\square$

Under the assumption that  $\mathbb{X}$  is a CB-scheme, Proposition 4.4.7 can be generalized as follows.

**Proposition 4.4.10.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional locally Gorenstein CB-scheme, and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . If  $\mathbb{X}$  is  $(i, j)$ -uniform, then any subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\text{deg}(\mathbb{Y}) = \text{deg}(\mathbb{X}) - i$  satisfies*

$$x_0^j (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_\mathbb{X}} \cap \vartheta_D(R/K[x_0])_{r_\mathbb{X}+j} = \langle 0 \rangle.$$

*Proof.* Since  $\mathbb{X}$  is a CB-scheme, there are elements  $g_1^*, \dots, g_s^* \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  such that  $g_k^* = x_0^{-2r_{\mathbb{X}}} \tilde{g}_k^*$  with  $\tilde{g}_k^* \in R_{r_{\mathbb{X}}} \setminus \{0\}$  and  $(\tilde{g}_k^*)_{p_k} \in \mathcal{O}_{\mathbb{X}, p_k} \setminus \mathfrak{m}_{\mathbb{X}, p_k}$  for  $k = 1, \dots, s$  (as in the proof of Theorem 4.1.7). Let  $\mathbb{Y} \subseteq \mathbb{X}$  be a subscheme of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$ , and let  $h \in x_0^j (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j}$ . We want to show that  $h = 0$ . Let  $\{f_1, \dots, f_i\}$  be a  $K$ -basis of the  $K$ -vector space  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}}$ . There are  $a_1, \dots, a_i \in K$  such that  $h = a_1 x_0^j f_1 + \dots + a_i x_0^j f_i \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j}$ . Let  $1 \leq k \leq s$ . We see that

$$h \cdot g_k^* = x_0^{-2r_{\mathbb{X}}+j} (a_1 f_1 + \dots + a_i f_i) \tilde{g}_k^* = x_0^{-r_{\mathbb{X}}+j} (b_{k1} f_1 + \dots + b_{ki} f_i) \in R_j$$

for some  $b_{k1}, \dots, b_{ki} \in K$ . It follows that  $b_{k1} f_1 + \dots + b_{ki} f_i \in x_0^{r_{\mathbb{X}}-j} R_j$ . So, we may write  $b_{k1} f_1 + \dots + b_{ki} f_i = x_0^{r_{\mathbb{X}}-j} \tilde{h}_k$  for some  $\tilde{h}_k \in R_j$ . Since  $\mathbb{X}$  is  $(i, j)$ -uniform, we have  $\text{HF}_{\mathbb{Y}}(j) = \text{HF}_{\mathbb{X}}(j)$ , so  $\alpha_{\mathbb{Y}/\mathbb{X}} \geq j + 1$ . Furthermore, the ideal  $\langle f_1, \dots, f_i \rangle$  of  $R$  defines the subscheme  $\mathbb{Y}$  of  $\mathbb{X}$  scheme-theoretically. This implies  $\langle f_1, \dots, f_i \rangle \subseteq \langle f_1, \dots, f_i \rangle^{\text{sat}} = \mathcal{I}_{\mathbb{Y}/\mathbb{X}}$ . Because  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}}$  is saturated and  $x_0^{r_{\mathbb{X}}-j} \tilde{h}_k = b_{k1} f_1 + \dots + b_{ki} f_i \in \langle f_1, \dots, f_i \rangle_{r_{\mathbb{X}}}$ , we deduce that  $\tilde{h}_k \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j = \langle 0 \rangle$ . Thus we get  $h \cdot g_k^* = x_0^{-r_{\mathbb{X}}+j} (b_{k1} f_1 + \dots + b_{ki} f_i) = \tilde{h}_k = 0$ , and hence  $h \cdot \tilde{g}_k^* = 0$ . In particular, we have  $h_{p_k} \cdot (\tilde{g}_k^*)_{p_k} = 0$  in  $\mathcal{O}_{\mathbb{X}, p_k}$ . Since the element  $(\tilde{g}_k^*)_{p_k}$  is a unit of  $\mathcal{O}_{\mathbb{X}, p_k}$ , we get  $h_{p_k} = 0$ . Thus  $h_{p_k} = 0$  in  $\mathcal{O}_{\mathbb{X}, p_k}$  for all  $k = 1, \dots, s$ . Hence Lemma 2.3.10 yields  $h = 0$ , and therefore

$$x_0^j (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j} = \langle 0 \rangle$$

for any subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$ .  $\square$

**Corollary 4.4.11.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional arithmetically Gorenstein scheme, and assume  $i \leq \deg(\mathbb{X}) - \text{HF}_{\mathbb{X}}(j)$ . The following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is  $(i, j)$ -uniform.*
- (ii) *Every subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{Y}) = \deg(\mathbb{X}) - i$  satisfies*

$$x_0^j (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j} = \langle 0 \rangle.$$

*Proof.* Since the scheme  $\mathbb{X}$  is arithmetically Gorenstein, it is also a CB-scheme and  $\Delta_{\mathbb{X}} = 1$ . The implication “(i) $\Rightarrow$ (ii)” follows from Proposition 4.4.10. Now we prove “(ii) $\Rightarrow$ (i)”. Suppose that  $\mathbb{X}$  is not  $(i, j)$ -uniform, i.e., there is a subscheme  $\mathbb{Y} \subseteq \mathbb{X}$  of degree  $\deg(\mathbb{X}) - i$  such that  $\text{HF}_{\mathbb{Y}}(j) < \text{HF}_{\mathbb{X}}(j)$ . Let us write  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} = \langle f_1, \dots, f_i \rangle_K$ . We deduce that  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j = (\langle f_1, \dots, f_i \rangle^{\text{sat}})_j \neq \langle 0 \rangle$ . Let  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j \setminus \{0\}$ , and write  $x_0^{r_{\mathbb{X}}-j} f = a_1 f_1 + \dots + a_i f_i$  for some  $a_1, \dots, a_i \in K$ . By Proposition 3.2.8, there is a non-zerodivisor  $h \in R_{r_{\mathbb{X}}}$  such that  $\vartheta_D(R/K[x_0]) = \langle h \rangle_R$ . Then we have

$$0 \neq fh = x_0^{j-r_{\mathbb{X}}} (a_1 f_1 + \dots + a_i f_i) \cdot h = x_0^j (b_1 f_1 + \dots + b_i f_i) \in \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j}$$

for some elements  $b_1, \dots, b_i \in K$ . Hence we get  $x_0^j(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}} \cap \vartheta_D(R/K[x_0])_{r_{\mathbb{X}}+j} \neq \langle 0 \rangle$ , a contradiction.  $\square$

Now let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set of degree  $s$ , and let  $f_j \in R_{r_{\mathbb{X}}}$  be the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$  for  $j = 1, \dots, s$ . By  $Lf_j$  we denote the leading form (i.e. the residue class) of  $f_j$  in  $\overline{R} := R/\langle x_0 \rangle$  ( $j = 1, \dots, s$ ). Notice that we can renumber  $\{p_1, \dots, p_s\}$  in such a way that  $\{Lf_1, \dots, Lf_{\Delta_{\mathbb{X}}}\}$  is a  $K$ -basis of  $\overline{R}_{r_{\mathbb{X}}}$ . We write

$$Lf_{\Delta_{\mathbb{X}}+j} = \beta_{j1}Lf_1 + \dots + \beta_{j\Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}$$

for  $j = 1, \dots, s - \Delta_{\mathbb{X}}$ , and form the matrix  $\mathcal{B} := (\beta_{ji})^{\text{tr}}$ . Recall from Lemma 3.2.10 that the elements  $h_1, \dots, h_{s-\Delta_{\mathbb{X}}} \in R_{r_{\mathbb{X}}-1}$  which satisfy  $x_0h_k = f_{\Delta_{\mathbb{X}}+k} - \beta_{k1}f_1 - \dots - \beta_{k\Delta_{\mathbb{X}}}f_{\Delta_{\mathbb{X}}}$  form a  $K$ -basis of  $R_{r_{\mathbb{X}}-1}$ , and  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1, \dots, \tilde{g}_{\Delta_{\mathbb{X}}} \rangle_K$  with  $\tilde{g}_j = x_0^{-2r_{\mathbb{X}}}(f_j + \beta_{1j}f_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{s-\Delta_{\mathbb{X}}j}f_s)$  for  $j = 1, \dots, \Delta_{\mathbb{X}}$ .

We observe that  $x_0^{r_{\mathbb{X}}-i}f_j \notin \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-i}$  for all  $j \in \{1, \dots, \Delta_{\mathbb{X}}\}$  and for all  $i \in \{1, \dots, r_{\mathbb{X}} - 1\}$ . Otherwise, there is an index  $j \in \{1, \dots, \Delta_{\mathbb{X}}\}$  such that  $x_0^{r_{\mathbb{X}}-i}f_j \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-i}$ . Then we obtain  $x_0^{r_{\mathbb{X}}-i}f_j\tilde{g}_j = x_0^{-i}f_j \in R_{r_{\mathbb{X}}-i}$ , and so  $f_j \in x_0^i R_{r_{\mathbb{X}}-i}$ , contradicting the hypothesis  $Lf_j \neq 0$ .

**Proposition 4.4.12.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set with  $\Delta_{\mathbb{X}} \geq 2$ . Then the following conditions are equivalent.*

(i)  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform.

(ii) For any subset of two elements  $\{j_1, j_2\} \subseteq \{1, \dots, s\}$  we have

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} \cap \langle x_0^{r_{\mathbb{X}}-1}f_{j_1}, x_0^{r_{\mathbb{X}}-1}f_{j_2} \rangle_K = \langle 0 \rangle.$$

*Proof.* If  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform, then  $\mathbb{X}$  is also a CB-scheme. Thus the implication “(i) $\Rightarrow$ (ii)” follows from Proposition 4.4.10. Now we prove “(ii) $\Rightarrow$ (i)”. We suppose for a contradiction that  $\mathbb{X}$  is not  $(2, r_{\mathbb{X}} - 1)$ -uniform. Then there exist  $j_1, j_2 \in \{1, \dots, s\}$  such that  $j_1 < j_2$  and such that the subscheme  $\mathbb{Y} = \mathbb{X} \setminus \{p_{j_1}, p_{j_2}\}$  of degree  $s - 2$  satisfies  $\text{HF}_{\mathbb{Y}}(r_{\mathbb{X}} - 1) < \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1)$ . Since the ideal  $\langle f_{j_1}, f_{j_2} \rangle$  defines  $\mathbb{Y}$  in  $\mathbb{X}$  schematically, we have  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_{j_1}, f_{j_2} \rangle^{\text{sat}}$ . It follows from  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1} \neq \langle 0 \rangle$  that there is a non-zero homogeneous element  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_{r_{\mathbb{X}}-1}$  such that  $x_0f = a_1f_{j_1} + a_2f_{j_2}$  for some  $a_1, a_2 \in K$ . Because  $\mathbb{X}$  is a CB-scheme, we have  $a_1 \neq 0$  and  $a_2 \neq 0$ . Letting  $\alpha = -a_2/a_1$ , we get  $f_{j_1} - \alpha f_{j_2} = x_0f/a_1 \in x_0R_{r_{\mathbb{X}}-1}$  and  $Lf_{j_1} = \alpha Lf_{j_2}$ . As above, we assume that the set  $\{Lf_1, \dots, Lf_{\Delta_{\mathbb{X}}}\}$  is a  $K$ -basis of  $\overline{R}_{r_{\mathbb{X}}}$ . We consider the following three cases.

**Case (a)** Suppose that  $1 \leq j_1 < j_2 \leq \Delta_{\mathbb{X}}$ . In this case we observe that  $\{Lf_{j_1}, Lf_{j_2}\}$  is linearly independent in  $\overline{R}_{r_{\mathbb{X}}}$ . But we also have  $Lf_{j_1} - \alpha Lf_{j_2} = 0$ . Thus this is impossible.

**Case (b)** Suppose that  $1 \leq j_1 \leq \Delta_{\mathbb{X}} < j_2 \leq s$ . Let us write  $j_2 = \Delta_{\mathbb{X}} + i_2$ . In  $\overline{R}_{r_{\mathbb{X}}}$ , we have  $Lf_{j_2} = \alpha^{-1}Lf_{j_1} = \beta_{i_2 1}Lf_1 + \cdots + \beta_{i_2 \Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}$ . We deduce that

$$\beta_{i_2 1}Lf_1 + \cdots + \beta_{i_2 j_1 - 1}Lf_{j_1 - 1} + (\beta_{i_2 j_1} - \alpha^{-1})Lf_{j_1} + \beta_{i_2 j_1 + 1}Lf_{j_1 + 1} + \cdots + \beta_{i_2 \Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}} = 0.$$

This implies  $\beta_{i_2 j_1} = \alpha^{-1}$  and  $\beta_{i_2 k} = 0$  for every  $k \neq j_1$ . We write  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1, \dots, \tilde{g}_{\Delta_{\mathbb{X}}} \rangle_K$  with  $\tilde{g}_k = x_0^{-2r_{\mathbb{X}}}(f_k + \beta_{1k}f_{\Delta_{\mathbb{X}}+1} + \cdots + \beta_{s-\Delta_{\mathbb{X}}k}f_s)$  for all  $k = 1, \dots, \Delta_{\mathbb{X}}$ . Now we compute

$$\begin{aligned} x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot \tilde{g}_k &= x_0^{-r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2})(f_k + \beta_{1k}f_{\Delta_{\mathbb{X}}+1} + \cdots + \beta_{s-\Delta_{\mathbb{X}}k}f_s) \\ &= \begin{cases} -x_0^{-1}\beta_{i_2 k}\alpha^2 f_{j_2} & \text{if } k \neq j_1 \\ x_0^{-1}(f_{j_1} - \beta_{i_2 j_1}\alpha^2 f_{j_2}) & \text{if } k = j_1 \end{cases} \\ &= \begin{cases} 0 & \text{if } k \neq j_1 \\ x_0^{-1}(f_{j_1} - \alpha f_{j_2}) & \text{if } k = j_1 \end{cases} \\ &= \begin{cases} 0 & \text{if } k \neq j_1 \\ f/a_1 & \text{if } k = j_1. \end{cases} \end{aligned}$$

This implies that  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot \tilde{g}_k \in R_{r_{\mathbb{X}}-1}$  for all  $k = 1, \dots, \Delta_{\mathbb{X}}$ . Moreover, a homogeneous element  $g \in (\mathfrak{C}_{R/K[x_0]})_i$  with  $i > -r_{\mathbb{X}}$  can be written as  $g = x_0^{-r_{\mathbb{X}}+i}g'$  with some  $g' \in R_{r_{\mathbb{X}}}$ . It follows that  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot g = x_0^{i-1}(f_{j_1} - \alpha^2 f_{j_2})g' = x_0^{r_{\mathbb{X}}+i-1}(g'(p_{j_1})f_{j_1} - \alpha^2 g'(p_{j_2})f_{j_2}) \in R_{2r_{\mathbb{X}}+i-1}$ . Thus we get  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R$ , and hence  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$ , in contradiction to our assumption.

**Case (c)** Suppose that  $\Delta_{\mathbb{X}} < j_1 < j_2 \leq s$ . In this case we write  $j_1 = \Delta_{\mathbb{X}} + i_1$ ,  $j_2 = \Delta_{\mathbb{X}} + i_2$ ,

$$\begin{aligned} Lf_{j_1} &= Lf_{\Delta_{\mathbb{X}}+i_1} = \beta_{i_1 1}Lf_1 + \cdots + \beta_{i_1 \Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}, \\ Lf_{j_2} &= Lf_{\Delta_{\mathbb{X}}+i_2} = \beta_{i_2 1}Lf_1 + \cdots + \beta_{i_2 \Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}. \end{aligned}$$

It follows from the equality  $Lf_{j_1} = \alpha Lf_{j_2}$  that

$$(\beta_{i_1 1} - \alpha\beta_{i_2 1})Lf_1 + \cdots + (\beta_{i_1 \Delta_{\mathbb{X}}} - \alpha\beta_{i_2 \Delta_{\mathbb{X}}})Lf_{\Delta_{\mathbb{X}}} = 0.$$

Hence we get  $\beta_{i_1 k} = \alpha \beta_{i_2 k}$  for all  $k = 1, \dots, \Delta_{\mathbb{X}}$ . We calculate

$$\begin{aligned} (f_{j_1} - \alpha^2 f_{j_2}) \cdot \tilde{g}_k &= x_0^{-2r_{\mathbb{X}}}(f_{\Delta_{\mathbb{X}+i_1}} - \alpha^2 f_{\Delta_{\mathbb{X}+i_2}})(f_k + \beta_{1k} f_{\Delta_{\mathbb{X}+1}} + \dots + \beta_{s-\Delta_{\mathbb{X}}k} f_s) \\ &= x_0^{-r_{\mathbb{X}}}(\beta_{i_1 k} f_{\Delta_{\mathbb{X}+i_1}} - \beta_{i_2 k} \alpha^2 f_{\Delta_{\mathbb{X}+i_2}}) \\ &= x_0^{-r_{\mathbb{X}}} \beta_{i_1 k} (f_{\Delta_{\mathbb{X}+i_1}} - \alpha f_{\Delta_{\mathbb{X}+i_2}}) \\ &= x_0^{-r_{\mathbb{X}}} \beta_{i_1 k} x_0 f / a_1 = x_0^{-r_{\mathbb{X}}+1} \beta_{i_1 k} f / a_1. \end{aligned}$$

From this we deduce  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot \tilde{g}_k \in R_{r_{\mathbb{X}}-1}$  for all  $k = 1, \dots, \Delta_{\mathbb{X}}$ . Using the same reasoning as in the second case, we obtain  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R$ , and hence  $x_0^{r_{\mathbb{X}}-1}(f_{j_1} - \alpha^2 f_{j_2}) \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$ , in contradiction to our assumption again.

Altogether, the claim is completely proved.  $\square$

**Corollary 4.4.13.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set with  $\Delta_{\mathbb{X}} \geq 2$ . If  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}} - 1) = 0$ , then  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform.*

**Example 4.4.14.** Let us go back to Example 4.4.8. We observed that the projective point set  $\mathbb{X} = \{p_1, \dots, p_{10}\} \subseteq \mathbb{P}_K^2$  has  $r_{\mathbb{X}} = 3$  and  $\Delta_{\mathbb{X}} = 4$ , and the Noether-Dedekind different is given by  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}} \rangle_R = \bigoplus_{i \geq 6} R_i$ . In particular, we have  $\text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(2r_{\mathbb{X}} - 1) = 0$ . Thus Corollary 4.4.13 yields that  $\mathbb{X}$  is  $(2, 2)$ -uniform.

Next we use the above method to characterize the uniformities of a level scheme. Here we say that a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is **level** if the Artinian local ring  $\bar{R}$  satisfies  $\mathfrak{G}(\bar{R}) = \bar{R}_{r_{\mathbb{X}}}$ . It is well known (cf. [Kr3, Section 11]) that  $\mathbb{X}$  is a level scheme if and only if the graded  $R$ -module  $\underline{\text{Hom}}_{K[x_0]}(R, K[x_0])$  is generated by homogeneous elements of degree  $-r_{\mathbb{X}}$ . In particular, if the scheme  $\mathbb{X}$  is a locally Gorenstein scheme, then this is also equivalent to the condition that  $\mathfrak{C}_{R/K[x_0]} = \langle (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \rangle_R$ .

**Proposition 4.4.15.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set which is a level scheme. If  $\Delta_{\mathbb{X}} = 1$ , we let  $j \in \{1, \dots, r_{\mathbb{X}} - 2\}$ . Otherwise, we let  $j \in \{1, \dots, r_{\mathbb{X}} - 1\}$ . Then the following conditions are equivalent.*

(i)  $\mathbb{X}$  is  $(2, j)$ -uniform.

(ii) For any subset of two elements  $\{j_1, j_2\} \subseteq \{1, \dots, s\}$  we have

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}+j} \cap \langle x_0^j f_{j_1}, x_0^j f_{j_2} \rangle_K = \langle 0 \rangle.$$

*Proof.* First we observe that if  $\Delta_{\mathbb{X}} = 1$ , then  $\mathbb{X}$  is not  $(2, r_{\mathbb{X}} - 1)$ -uniform, since  $2 > \Delta_{\mathbb{X}} = s - \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - 1)$  and Remark 4.4.2(e) applies. That is why we take  $j < r_{\mathbb{X}} - 1$  in this



case. Now let  $j$  satisfy our hypothesis. Since  $\mathbb{X}$  is a level scheme, [GK, Proposition 6.1] tells us that  $\mathbb{X}$  is a CB-scheme. Hence, if  $\mathbb{X}$  is  $(2, j)$ -uniform, we have

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}+j} \cap \langle x_0^j f_{j_1}, x_0^j f_{j_2} \rangle_K = \langle 0 \rangle$$

for any subset of two elements  $\{j_1, j_2\} \subseteq \{1, \dots, s\}$  by Proposition 4.4.10.

Conversely, suppose for a contradiction that  $\mathbb{X}$  is not  $(2, j)$ -uniform, i.e., there is a subscheme  $\mathbb{Y} = \mathbb{X} \setminus \{p_{j_1}, p_{j_2}\}$  of degree  $s - 2$  such that  $\text{HF}_{\mathbb{Y}}(j) < \text{HF}_{\mathbb{X}}(j)$ . Then  $\mathcal{I}_{\mathbb{Y}/\mathbb{X}} = \langle f_{j_1}, f_{j_2} \rangle^{\text{sat}} \subseteq R$  and  $(\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j \neq \langle 0 \rangle$ . Let  $f \in (\mathcal{I}_{\mathbb{Y}/\mathbb{X}})_j \setminus \{0\}$ . We may write  $x_0^{r_{\mathbb{X}}-j} f = f(p_1)f_1 + f(p_2)f_2 + \dots + f(p_s)f_s = f(p_{j_1})f_{j_1} + f(p_{j_2})f_{j_2}$  with  $f(p_{j_1}), f(p_{j_2}) \in K$ . Since  $\mathbb{X}$  is a CB-scheme, we must get  $f(p_{j_1}), f(p_{j_2}) \in K \setminus \{0\}$ . By the same argument as in the proof of Proposition 4.4.12, we obtain  $x_0^j(f(p_{j_1})^2 f_{j_1} - f(p_{j_2})^2 f_{j_2}) \cdot \tilde{g}_k \in R_j$  for all  $k = 1, \dots, \Delta_{\mathbb{X}}$ . Moreover, we have  $\mathfrak{C}_{R/K[x_0]} = \langle (\mathfrak{C}_{R/K[x_0]})_{r_{\mathbb{X}}} \rangle_R$ , because  $\mathbb{X}$  is a level scheme. It follows that  $x_0^j(f(p_{j_1})^2 f_{j_1} - f(p_{j_2})^2 f_{j_2}) \cdot \mathfrak{C}_{R/K[x_0]} \subseteq R$ . In other words, we obtain  $x_0^j(f(p_{j_1})^2 f_{j_1} - f(p_{j_2})^2 f_{j_2}) \in \vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}+j}$ , a contradiction.  $\square$

**Corollary 4.4.16.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set which is a level scheme. If  $\Delta_{\mathbb{X}} = 1$ , we let  $j \in \{1, \dots, r_{\mathbb{X}} - 2\}$ . Otherwise, we let  $j \in \{1, \dots, r_{\mathbb{X}} - 1\}$ . Suppose that for any subset of two elements  $\{j_1, j_2\} \subseteq \{1, \dots, s\}$  we have*

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{r_{\mathbb{X}}+j} \cap \langle x_0^j f_{j_1}, x_0^j f_{j_2} \rangle_K = \langle 0 \rangle.$$

*Then  $\mathbb{X}$  is  $(2 + k, j - k)$ -uniform for  $k = 1, \dots, j - 1$ .*

*Proof.* This follows from Proposition 4.4.15 and [GK, Proposition 6.1].  $\square$

**Corollary 4.4.17.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set.*

- (i) *If  $\mathbb{X}$  is a CB-scheme, if  $\Delta_{\mathbb{X}} = 1$ , and if  $\text{char}(K) \nmid s$ , then  $x_0^{r_{\mathbb{X}}} R_{r_{\mathbb{X}}-1} \not\subseteq \vartheta_{\mathbb{X}}(R/K[x_0])$ . In particular, we have  $x_0^{r_{\mathbb{X}}} \notin \vartheta_{\mathbb{X}}(R/K[x_0])$ .*
- (ii) *If  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform, then we have  $x_0^{r_{\mathbb{X}}} h_k \notin \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$  for every  $k \in \{1, \dots, s - \Delta_{\mathbb{X}}\}$ .*

*Proof.* (i) Notice that  $Lf_j \neq 0$  for all  $j = 1, \dots, s$  if  $\mathbb{X}$  is a CB-scheme. Suppose that  $x_0^{r_{\mathbb{X}}} h_k \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$  for all  $k = 1, \dots, s-1$ . Because  $\mathbb{X}$  is a CB-scheme and  $\Delta_{\mathbb{X}} = 1$ , we have  $x_0 h_k = f_{1+k} - \beta_{k1} f_1$ ,  $Lf_{1+k} = \beta_{k1} Lf_1$  with  $\beta_{k1} \neq 0$ , and  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1 \rangle_K$  with  $\tilde{g}_1 = x_0^{-2r_{\mathbb{X}}}(f_1 + \beta_{11} f_2 + \dots + \beta_{s-11} f_s)$ . Then  $x_0^{r_{\mathbb{X}}} h_k \cdot \tilde{g}_1 = \beta_{k1} x_0^{-1}(f_{1+k} - f_1) \in R_{r_{\mathbb{X}}-1}$ . This implies  $\beta_{k1}(f_{1+k} - f_1) \in x_0 R_{r_{\mathbb{X}}-1}$  and  $\beta_{k1}(Lf_{1+k} - Lf_1) = (\beta_{k1}^2 - \beta_{k1})Lf_1 = 0$ , and so  $\beta_{k1} = 1$  for  $k = 1, \dots, s - 1$ . Thus we get  $\beta_{11} + \dots + \beta_{s-11} = s - 1 \neq -1$ , in contradiction to [Kr4, Lemma 1.2.a].

(ii) Suppose for a contradiction that there is an index  $k \in \{1, \dots, s - \Delta_{\mathbb{X}}\}$  such that  $x_0^{r_{\mathbb{X}}} h_k \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$ . For any  $j \in \{1, \dots, \Delta_{\mathbb{X}}\}$ , we calculate  $x_0^{r_{\mathbb{X}}} h_k \cdot \tilde{g}_j = \beta_{kj} x_0^{-1} (f_{\Delta_{\mathbb{X}}+k} - f_j) \in R_{r_{\mathbb{X}}-1}$ . This implies that  $\beta_{kj} (f_{\Delta_{\mathbb{X}}+k} - f_j) \in x_0 R_{r_{\mathbb{X}}-1}$ , and so  $\beta_{kj} (Lf_{\Delta_{\mathbb{X}}+k} - Lf_j) = 0$ . Since  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform, we have  $\Delta_{\mathbb{X}} \geq 2$  and the set  $\{Lf_{\Delta_{\mathbb{X}}+k}, Lf_j\}$  is linearly independent in  $\overline{R}_{r_{\mathbb{X}}}$  by [Kr2, Proposition 3.4]. Hence it follows from  $\beta_{kj} (Lf_{\Delta_{\mathbb{X}}+k} - Lf_j) = 0$  that  $\beta_{kj} = 0$  for all  $j = 1, \dots, \Delta_{\mathbb{X}}$ , and  $Lf_{\Delta_{\mathbb{X}}+k} = \beta_{k1} Lf_1 + \dots + \beta_{k\Delta_{\mathbb{X}}} Lf_{\Delta_{\mathbb{X}}} = 0$ , contradicting the fact that  $\mathbb{X}$  is a CB-scheme.  $\square$

**Proposition 4.4.18.** *Let  $K$  be an infinite field, let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set which is a CB-scheme, and let  $\alpha_{\vartheta_{\mathbb{X}}} := \min\{i \in \mathbb{N} \mid \text{HF}_{\vartheta_{\mathbb{X}}(R/K[x_0])}(i) = \text{HF}_{\mathbb{X}}(i - r_{\mathbb{X}}) > 0\}$ . Suppose that  $\alpha_{\vartheta_{\mathbb{X}}} \leq 2r_{\mathbb{X}} - 1$ . Then  $\mathbb{X}$  is not  $(2, r_{\mathbb{X}} - 1)$ -uniform. In particular, if  $\Delta_{\mathbb{X}} \geq 2$  then  $\mathbb{X}$  is not  $(\Delta_{\mathbb{X}}, r_{\mathbb{X}} - 1)$ -uniform.*

*Proof.* If  $\Delta_{\mathbb{X}} = 1$  then  $\mathbb{X}$  is not  $(2, r_{\mathbb{X}} - 1)$ -uniform by Remark 4.4.2(e). Now we consider the case  $\Delta_{\mathbb{X}} \geq 2$  and  $\alpha_{\vartheta_{\mathbb{X}}} \leq 2r_{\mathbb{X}} - 1$ . Since  $\alpha_{\vartheta_{\mathbb{X}}} \leq 2r_{\mathbb{X}} - 1$  and  $\mathbb{X}$  is a CB-scheme, there is  $\tilde{g} = c_1 f_1 + \dots + c_s f_s \in R_{r_{\mathbb{X}}} \setminus \{0\}$  such that  $x_0^{-2r_{\mathbb{X}}} \tilde{g} \in \mathfrak{C}_{R/K[x_0]}$ , such that  $\text{Ann}_R(\tilde{g}) = \langle 0 \rangle$ , and such that  $\tilde{g} \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} = x_0^{2r_{\mathbb{X}}} R_{r_{\mathbb{X}}-1}$  (see Proposition 4.1.12). Notice that  $c_j \neq 0$  for all  $j = 1, \dots, s$ . We have  $R_{r_{\mathbb{X}}-1} = \langle h_1, \dots, h_{s-\Delta_{\mathbb{X}}} \rangle_K$ , where the elements  $h_k$  satisfy  $x_0 h_k = f_{\Delta_{\mathbb{X}}+k} - \beta_{k1} f_1 - \dots - \beta_{k\Delta_{\mathbb{X}}} f_{\Delta_{\mathbb{X}}}$ . For every  $k \in \{1, \dots, s - \Delta_{\mathbb{X}}\}$ , we can write  $x_0^{2r_{\mathbb{X}}} h_k = \tilde{g} v_k = (c_1 f_1 + \dots + c_s f_s) v_k$  with  $v_k \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1}$ . This implies  $\vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} = \langle v_1, \dots, v_{s-\Delta_{\mathbb{X}}} \rangle_K$  and

$$v_k = x_0^{r_{\mathbb{X}}-1} \left( \frac{1}{c_{\Delta_{\mathbb{X}}+k}} f_{\Delta_{\mathbb{X}}+k} - \frac{\beta_{k1}}{c_1} f_1 - \dots - \frac{\beta_{k\Delta_{\mathbb{X}}}}{c_{\Delta_{\mathbb{X}}}} f_{\Delta_{\mathbb{X}}} \right).$$

Moreover, we have

$$x_0^{2r_{\mathbb{X}}} (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} = \tilde{g} \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} = x_0^{2r_{\mathbb{X}}} R_{r_{\mathbb{X}}-1}.$$

Let us write  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1, \dots, \tilde{g}_{\Delta_{\mathbb{X}}} \rangle_K$  with  $\tilde{g}_j = x_0^{-2r_{\mathbb{X}}} (f_j + \beta_{1j} f_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{s-\Delta_{\mathbb{X}}j} f_s)$  for  $j = 1, \dots, \Delta_{\mathbb{X}}$ . We obtain  $x_0^{2r_{\mathbb{X}}} \tilde{g}_j v_k = x_0^{2r_{\mathbb{X}}-1} \frac{\beta_{kj}}{c_j c_{\Delta_{\mathbb{X}}+k}} (c_j f_{\Delta_{\mathbb{X}}+k} - c_{\Delta_{\mathbb{X}}+k} f_j) \in x_0^{2r_{\mathbb{X}}} R_{r_{\mathbb{X}}-1}$  for all  $k \in \{1, \dots, s - \Delta_{\mathbb{X}}\}$  and  $j \in \{1, \dots, \Delta_{\mathbb{X}}\}$ . Hence, if  $\beta_{kj} \neq 0$ , then we have  $(c_j f_{\Delta_{\mathbb{X}}+k} - c_{\Delta_{\mathbb{X}}+k} f_j) \in x_0 R_{r_{\mathbb{X}}-1}$ . Because of  $\beta_{1j} + \dots + \beta_{s-\Delta_{\mathbb{X}}j} = -1$  for  $j = 1, \dots, \Delta_{\mathbb{X}}$ , we may assume without loss of generality that  $\beta_{11} \neq 0$ . Thus we get  $(c_1 f_{\Delta_{\mathbb{X}}+1} - c_{\Delta_{\mathbb{X}}+1} f_1) \in x_0 R_{r_{\mathbb{X}}-1}$ . On the other hand, we can write  $x_0^{2r_{\mathbb{X}}-1} (c_1 f_{\Delta_{\mathbb{X}}+1} - c_{\Delta_{\mathbb{X}}+1} f_1) = \tilde{g} v$  for some  $v \in \vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} \setminus \{0\}$ . This implies that for all  $j \in \{2, \dots, \Delta_{\mathbb{X}}, \Delta_{\mathbb{X}} + 2, \dots, s\}$  we have  $v(p_j) = 0$ . Hence we deduce

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} \cap \left\langle x_0^{r_{\mathbb{X}}-1} f_1, x_0^{r_{\mathbb{X}}-1} f_{j_{\Delta_{\mathbb{X}}+1}} \right\rangle_K \neq \langle 0 \rangle.$$

Therefore, by Proposition 4.4.10,  $\mathbb{X}$  is not  $(2, r_{\mathbb{X}} - 1)$ -uniform.  $\square$

In the remainder of this section, we discuss the following notion (cf. [Kr1], [Kr4]).

**Definition 4.4.19.** A projective point set  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  is said to **split cohomologically** if we can decompose  $\mathbb{X} = \mathbb{Y} \cup \mathbb{Y}'$  such that  $\mathbb{Y} \neq \emptyset$ ,  $\mathbb{Y}' \neq \emptyset$ ,  $\mathbb{Y} \cap \mathbb{Y}' = \emptyset$  and  $\sum_{p_j \in \mathbb{Y}} K \cdot Lf_j \cap \sum_{p_j \in \mathbb{Y}'} K \cdot Lf_j = \langle 0 \rangle$  in  $\overline{R}_{r_{\mathbb{X}}}$ . If  $\mathbb{X}$  does not split cohomologically, we say that  $\mathbb{X}$  is **cohomologically uniform**.

In [Kr4], Theorem 2.4 tells us that  $\mathbb{X}$  is cohomologically uniform if and only if the multiplication map  $\mu : R_{r_{\mathbb{X}}-1} \otimes (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-1}$  is nondegenerate and surjective. It follows that  $(\Delta_{\mathbb{X}}, r_{\mathbb{X}} - 1)$ -uniform schemes are cohomologically uniform. Moreover, if  $\mathbb{X}$  is cohomologically uniform, then  $\mathbb{X}$  is a CB-scheme, and the converse is true if  $\Delta_{\mathbb{X}} = 1$ .

**Proposition 4.4.20.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set. Suppose that  $2 \leq \Delta_{\mathbb{X}} \leq 3$  and that for any subset of two elements  $\{j_1, j_2\} \subseteq \{1, \dots, s\}$  we have*

$$\vartheta_{\mathbb{X}}(R/K[x_0])_{2r_{\mathbb{X}}-1} \cap \langle x_0^{r_{\mathbb{X}}-1} f_{j_1}, x_0^{r_{\mathbb{X}}-1} f_{j_2} \rangle_K = \langle 0 \rangle.$$

*Then  $\mathbb{X}$  is cohomologically uniform.*

*Proof.* By assumption and Proposition 4.4.12,  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform. If  $\Delta_{\mathbb{X}} = 2$ , then  $\mathbb{X}$  is  $(\Delta_{\mathbb{X}}, r_{\mathbb{X}} - 1)$ -uniform, so it is cohomologically uniform via the above argument. Now we consider the case  $\Delta_{\mathbb{X}} = 3$ . Obviously,  $\mathbb{X}$  is a CB-scheme. Thus it follows from Theorem 4.3.10 that the multiplication map  $\mu : R_{r_{\mathbb{X}}-1} \otimes (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-1}$  is nondegenerate. So, we need to prove that this map is surjective. For this, we assume that  $\{Lf_1, \dots, Lf_3\}$  is a  $K$ -basis of  $\overline{R}_{r_{\mathbb{X}}}$ , we write  $Lf_{3+j} = \beta_{j1}Lf_1 + \beta_{j2}Lf_2 + \beta_{j3}Lf_3$  for  $j = 1, \dots, s - 3$ , and we form the matrix

$$\mathcal{B} := (\beta_{ji})^{\text{tr}} = \begin{pmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{s-31} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{s-32} \\ \beta_{13} & \beta_{23} & \cdots & \beta_{s-33} \end{pmatrix}.$$

Now we fix an index  $j \in \{1, \dots, s - 3\}$ . If  $\beta_{j1} = \beta_{j2} = \beta_{j3} = 0$  or if two of the elements  $\beta_{j1}, \beta_{j2}, \beta_{j3}$  are zero, say  $\beta_{j1} \neq 0$  and  $\beta_{j2} = \beta_{j3} = 0$ , then we have  $Lf_{3+j} = \beta_{j1}Lf_1 + \beta_{j2}Lf_2 + \beta_{j3}Lf_3 = 0$  or  $Lf_{3+j} = \beta_{j1}Lf_1 + \beta_{j2}Lf_2 + \beta_{j3}Lf_3 = \beta_{j1}Lf_1$ . Both cases cannot occur, since  $\mathbb{X}$  is  $(2, r_{\mathbb{X}} - 1)$ -uniform and by [Kr2, Proposition 3.4]. Thus each column of  $\mathcal{B}$  contains at least two non-zero entries. On the other hand, no row of  $\mathcal{B}$  can be zero. Otherwise, we have  $\beta_{1k} = \cdots = \beta_{s-3k} = 0$  for some  $k \in \{1, 2, 3\}$ , and it follows from [Kr4, Lemma 1.2] that  $0 = \beta_{1k} + \cdots + \beta_{s-3k} = -1$ , a contradiction.

Therefore the matrix  $\mathcal{B}$  has no zero row and each column of  $\mathcal{B}$  contains at least two non-zero entries.

Let  $h_1, \dots, h_{s-3} \in R_{r_{\mathbb{X}}-1}$  form a  $K$ -basis of  $R_{r_{\mathbb{X}}-1}$  such that  $x_0 h_j = f_{3+j} - \beta_{j1} f_1 - \beta_{j2} f_2 - \beta_{j3} f_3$  for  $j = 1, \dots, s-3$ . We also write  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1, \tilde{g}_2, \tilde{g}_3 \rangle_K$  with  $\tilde{g}_k = x_0^{-2r_{\mathbb{X}}}(f_k + \beta_{1k} f_4 + \dots + \beta_{s-3k} f_s)$  for  $k = 1, 2, 3$ . Then  $h_j \cdot \tilde{g}_k = \beta_{jk} x_0^{-r_{\mathbb{X}}-1}(f_{3+j} - f_k)$  in  $(\mathfrak{C}_{R/K[x_0]})_{-1}$  for  $j = 1, \dots, s-3$  and  $k = 1, 2, 3$ . We define a relation  $\sim$  on the set  $\{1, \dots, s\}$  by  $j \sim k$  if and only if  $x_0^{-r_{\mathbb{X}}-1}(f_j - f_k) \in \text{Im}(\mu)$ . Clearly, the relation  $\sim$  is an equivalence relation. Furthermore, if  $\beta_{j-3k} \neq 0$ , then in  $(\mathfrak{C}_{R/K[x_0]})_{-1}$  we have  $h_{j-3} \tilde{g}_k = \beta_{j-3k} x_0^{-r_{\mathbb{X}}-1}(f_j - f_k)$  or  $j \sim k$ . According to [Kr4, Corollary 1.11], the elements  $x_0^{-r_{\mathbb{X}}-1}(f_{i+1} - f_i)$  with  $i = 1, \dots, s-1$  form a  $K$ -basis of  $(\mathfrak{C}_{R/K[x_0]})_{-1}$ , so in order to prove that  $\mu$  is surjective, it suffices to show that  $1 \sim 2 \sim \dots \sim s$ .

Let  $\{j_1, \dots, j_{s-3}\}$  be a permutation of  $\{4, \dots, s\}$  such that  $\beta_{j_k-31} \neq 0$  for  $k = 1, \dots, t$  and  $\beta_{j_k-31} = 0$  for  $k = t+1, \dots, s-3$ . Here we have  $1 \leq t \leq s-3$  and  $1 \sim j_1 \sim \dots \sim j_t$ . If  $t = s-3$ , then  $1 \sim j_1 \sim \dots \sim j_{s-3} \sim 2 \sim 3$  (i.e.,  $1 \sim \dots \sim s$ ) because each row of  $\mathcal{B}$  has a non-zero entry. Now we consider the case  $t < s-3$  and we rewrite the matrix  $\mathcal{B}$  into the following matrix

$$\begin{pmatrix} \beta_{j_1-31} & \beta_{j_2-31} & \cdots & \beta_{j_t-31} & 0 & \cdots & 0 \\ \beta_{j_1-32} & \beta_{j_2-32} & \cdots & \beta_{j_t-32} & \beta_{j_{t+1}-32} & \cdots & \beta_{j_{s-3}-32} \\ \beta_{j_1-33} & \beta_{j_2-33} & \cdots & \beta_{j_t-33} & \beta_{j_{t+1}-33} & \cdots & \beta_{j_{s-3}-33} \end{pmatrix}.$$

Since each column of new matrix also contains at least two non-zero entries, this yields a number  $k \in \{2, 3\}$  such that not all elements of  $\{\beta_{j_1-3k}, \dots, \beta_{j_t-3k}\}$  are zero. Without loss of generality, let  $k = 2$  and  $\beta_{j_v-32} \neq 0$  for some  $v \in \{1, \dots, t\}$ . This implies that  $2 \sim j_v \sim 1$ . Moreover, we also have  $\beta_{j_l-3k} \neq 0$  for  $l = t+1, \dots, s-3$  and  $k = 2, 3$ . This yields  $2 \sim j_{t+1} \sim \dots \sim j_{s-3} \sim 3$ . Therefore we obtain  $1 \sim 2 \sim 3 \sim j_1 \sim \dots \sim j_{s-3}$  or  $1 \sim 2 \sim \dots \sim s$ , as required.  $\square$

**Proposition 4.4.21.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set, and let  $m := \lfloor \frac{\Delta_{\mathbb{X}}}{2} \rfloor + 1$  where, for a rational number  $\alpha$ , we let  $\lfloor \alpha \rfloor$  be the greatest integer less than or equal to  $\alpha$ . If  $\mathbb{X}$  is  $(m, r_{\mathbb{X}} - 1)$ -uniform, then it is cohomologically uniform. If, in addition,  $\text{char}(K) = 0$  or  $\text{char}(K) \geq r_{\mathbb{X}}$ , then we have*

$$\text{HF}_{\mathbb{X}}(i) + \text{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) \leq s - \Delta_{\mathbb{X}} + 1$$

for all  $i \in \{0, \dots, r_{\mathbb{X}} - 1\}$ .

*Proof.* First we note that if  $\Delta_{\mathbb{X}} \leq 3$ , then the claim is proved by Proposition 4.4.20. So, we may assume that  $\Delta_{\mathbb{X}} \geq 4$  and  $m \geq 3$ . By assumption,  $\mathbb{X}$  is a CB-scheme. Thus

Theorem 4.3.10 implies the multiplication map  $\mu: R_{r_{\mathbb{X}}-1} \otimes (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} \rightarrow (\mathfrak{C}_{R/K[x_0]})_{-1}$  is nondegenerate. Due to [Kr4, Theorem 2.4], it is enough to prove that  $\mu$  is surjective. We assume that  $\{Lf_1, \dots, Lf_{\Delta_{\mathbb{X}}}\}$  is a  $K$ -basis of  $\overline{R}_{r_{\mathbb{X}}}$ . We write  $Lf_{\Delta_{\mathbb{X}}+j} = \beta_{j1}Lf_1 + \dots + \beta_{j\Delta_{\mathbb{X}}}Lf_{\Delta_{\mathbb{X}}}$  for  $j = 1, \dots, s - \Delta_{\mathbb{X}}$  and form the matrix

$$\mathcal{B} := (\beta_{ji})^{\text{tr}} = \begin{pmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{s-\Delta_{\mathbb{X}}1} \\ \beta_{12} & \beta_{22} & \cdots & \beta_{s-\Delta_{\mathbb{X}}2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1\Delta_{\mathbb{X}}} & \beta_{2\Delta_{\mathbb{X}}} & \cdots & \beta_{s-\Delta_{\mathbb{X}}\Delta_{\mathbb{X}}} \end{pmatrix}.$$

Since  $\mathbb{X}$  is  $(m, r_{\mathbb{X}} - 1)$ -uniform, it follows from [Kr2, Proposition 3.4] that every subset of  $m$  elements from  $\{Lf_1, \dots, Lf_s\}$  is linearly independent in  $\overline{R}_{r_{\mathbb{X}}}$ . Thus, for every  $j \in \{1, \dots, s - \Delta_{\mathbb{X}}\}$ , at least  $m$  elements of  $\{\beta_{j1}, \dots, \beta_{j\Delta_{\mathbb{X}}}\}$  are non-zero. In other words, each column of  $\mathcal{B}$  has at least  $m$  non-zero entries. Furthermore, in [Kr4, Lemma 1.2] it is shown that  $\beta_{1k} + \dots + \beta_{s-\Delta_{\mathbb{X}}k} = -1$  for  $k = 1, \dots, \Delta_{\mathbb{X}}$ . Thus  $\mathcal{B}$  has no zero row.

As before, we let  $\{h_1, \dots, h_{s-\Delta_{\mathbb{X}}}\} \in R_{r_{\mathbb{X}}-1}$  be a  $K$ -basis of  $R_{r_{\mathbb{X}}-1}$  such that  $x_0h_j = f_{\Delta_{\mathbb{X}}+j} - \beta_{j1}f_1 - \dots - \beta_{j\Delta_{\mathbb{X}}}f_{\Delta_{\mathbb{X}}}$  for  $j = 1, \dots, s - \Delta_{\mathbb{X}}$ , and we write  $(\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}} = \langle \tilde{g}_1, \dots, \tilde{g}_{\Delta_{\mathbb{X}}} \rangle_K$  with  $\tilde{g}_k = x_0^{-2r_{\mathbb{X}}}(f_k + \beta_{1k}f_{\Delta_{\mathbb{X}}+1} + \dots + \beta_{s-\Delta_{\mathbb{X}}k}f_s)$  for  $k = 1, \dots, \Delta_{\mathbb{X}}$ . We see that  $h_j\tilde{g}_k = \beta_{jk}x_0^{-r_{\mathbb{X}}-1}(f_{\Delta_{\mathbb{X}}+j} - f_k) \in (\mathfrak{C}_{R/K[x_0]})_{-1}$  for  $j = 1, \dots, s - \Delta_{\mathbb{X}}$  and  $k = 1, \dots, \Delta_{\mathbb{X}}$ . As in the proof of Proposition 4.4.20, we define an equivalence relation  $\sim$  on the set  $\{1, \dots, s\}$  by  $j \sim k$  if and only if  $x_0^{-r_{\mathbb{X}}-1}(f_j - f_k) \in \text{Im}(\mu)$ . Notice that if  $\beta_{j-\Delta_{\mathbb{X}}k} \neq 0$ , then  $h_{j-\Delta_{\mathbb{X}}} \tilde{g}_k = \beta_{j-\Delta_{\mathbb{X}}k} x_0^{-r_{\mathbb{X}}-1}(f_j - f_k) \in (\mathfrak{C}_{R/K[x_0]})_{-1}$  or  $j \sim k$ . By [Kr4, Corollary 1.11], we need to show that the elements  $x_0^{-r_{\mathbb{X}}-1}(f_{i+1} - f_i)$  with  $i = 1, \dots, s-1$  are in the image of  $\mu$ . It suffices to show that  $1 \sim 2 \sim \dots \sim s$ . First we renumber the points  $p_{\Delta_{\mathbb{X}}+1}, \dots, p_s$  such that  $\beta_{j1} \neq 0$  for  $j = 1, \dots, t_1$  and  $\beta_{j1} = 0$  for  $j = t_1 + 1, \dots, s - \Delta_{\mathbb{X}}$ . Here we have  $1 \leq t_1 \leq s - \Delta_{\mathbb{X}}$  and  $1 \sim \Delta_{\mathbb{X}} + 1 \sim \dots \sim \Delta_{\mathbb{X}} + t_1$ . If  $t_1 = s - \Delta_{\mathbb{X}}$ , then we have  $1 \sim \Delta_{\mathbb{X}} + 1 \sim \dots \sim \Delta_{\mathbb{X}} + t_1 = s \sim 2 \sim \dots \sim \Delta_{\mathbb{X}}$  because each row of  $\mathcal{B}$  has a non-zero entry. Thus we have  $1 \sim \dots \sim s$ , as desired.

Next, we consider the case  $t_1 < s - \Delta_{\mathbb{X}}$ . Because each column of the matrix  $\mathcal{B}$  contains at least  $m$  non-zero entries, we can renumber  $p_2, \dots, p_{\Delta_{\mathbb{X}}}$  such that  $\beta_{t_1k} \neq 0$  for  $k = 2, \dots, m$ . So, we get  $1 \sim \Delta_{\mathbb{X}} + t_1 \sim 2 \sim \dots \sim m$ . Also, the equality  $m = \lfloor \frac{\Delta_{\mathbb{X}}}{2} \rfloor + 1$  implies  $\Delta_{\mathbb{X}} - m \leq m - 1$ . This allows us to assume  $\beta_{t_1+12} \neq 0$ , since if  $\beta_{t_1+12} = \dots = \beta_{t_1+1m} = 0$  then the  $(t_1 + 1)$ -th column of  $\mathcal{B}$  contains at most  $m - 1$  non-zero entries, which is impossible. Consequently, we may renumber  $p_{\Delta_{\mathbb{X}}+t_1+1}, \dots, p_s$  such that  $\beta_{t_1+12} \neq 0, \dots, \beta_{t_22} \neq 0$  and  $\beta_{t_2+12} = \dots = \beta_{s-\Delta_{\mathbb{X}}2} = 0$  for some  $t_1 + 1 \leq t_2 \leq s - \Delta_{\mathbb{X}}$ . Then we have  $1 \sim \dots \sim m \sim \Delta_{\mathbb{X}} + t_1 \sim \dots \sim \Delta_{\mathbb{X}} + t_2$ . If  $t_2 = s - \Delta_{\mathbb{X}}$ , then  $1 \sim \dots \sim s$  (as each row of  $\mathcal{B}$  has a non-zero entry). In the

case  $t_2 < s - \Delta_{\mathbb{X}}$ , we renumber  $p_3, \dots, p_{\Delta_{\mathbb{X}}}$  such that  $\beta_{t_2 k} \neq 0$  for  $k = 3, \dots, m + 1$ ,  $\beta_{t_2+13} \neq 0$ , and we get  $m \sim \Delta_{\mathbb{X}} + t_2 \sim m + 1$ . Continuing this process, we eventually find a number  $k \in \{1, \dots, m\}$  such that  $t_k = s - \Delta_{\mathbb{X}}$ , and the matrix  $\mathcal{B}$  looks as follows:

$$\mathcal{B} = \begin{pmatrix} \beta_{11} & \cdots & \beta_{t_1 1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \beta_{12} & \cdots & \beta_{t_1 2} & \beta_{t_1+1 2} & \cdots & \beta_{t_2 2} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \beta_{1k} & \cdots & \beta_{t_1 k} & \beta_{t_1+1 k} & \cdots & \beta_{t_2 k} & \beta_{t_2+1 k} & \cdots & \beta_{t_{k-1}+1 k} & \cdots & \beta_{t_k k} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ \beta_{1\Delta_{\mathbb{X}}} & \cdots & \beta_{t_1\Delta_{\mathbb{X}}} & \beta_{t_1+1\Delta_{\mathbb{X}}} & \cdots & \beta_{t_2\Delta_{\mathbb{X}}} & \beta_{t_2+1\Delta_{\mathbb{X}}} & \cdots & \beta_{t_{k-1}+1\Delta_{\mathbb{X}}} & \cdots & \beta_{t_k\Delta_{\mathbb{X}}} \end{pmatrix}.$$

Here we get  $k \leq m$ , since if  $k \geq m + 1$  and  $t_m < s - \Delta_{\mathbb{X}}$ , then the  $(m + 1)$ -th column of the matrix  $\mathcal{B}$  has at most  $m - 1$  non-zero entries, in contradiction to the fact that each column of the matrix  $\mathcal{B}$  has at least  $m$  non-zero entries. Moreover, we obtain  $1 \sim \cdots \sim m + k - 1 \sim \Delta_{\mathbb{X}} + 1 \sim \cdots \sim \Delta_{\mathbb{X}} + t_k = s$ . Since each row of  $\mathcal{B}$  has a non-zero entry, this implies  $1 \sim 2 \sim \cdots \sim s$ , as we wanted to show. Finally, the inequality for the Hilbert function of  $\mathbb{X}$  follows from [Kr4, Theorem 3.1].  $\square$

# Chapter 5

## Differents for Some Special Cases and Applications

In the previous chapters we investigated many interesting properties of the Noether, Dedekind, and Kähler differentials for 0-dimensional schemes in  $\mathbb{P}_K^n$ . In particular, we saw that a lot of information about the geometry of the schemes is reflected in the structure of these differentials. In this chapter we proceed with the study of these differentials for several special classes of 0-dimensional schemes in  $\mathbb{P}_K^n$  and find out their applications.

In the first section of this chapter we look at the above differentials for 0-dimensional almost complete intersections in  $\mathbb{P}_K^n$ . The notion of an almost complete intersection has been extensively studied by many authors in the last thirty years. Among the people that have worked on it are E. Kunz, T. Matsuoka, J. Herzog, R. Waldi, M. Kreuzer, and G.D. Dominicus (see, for example, [Ku2], [Ku3], [Mats], [Her], [Wal], [DK]). In this section, we present a description of the Dedekind complementary module as in [DK] for a reduced 0-dimensional almost complete intersection  $\mathbb{X} \subseteq \mathbb{P}_K^n$  over a perfect field  $K$  (see Proposition 5.1.5). Using this result, we construct an explicit presentation of the Kähler differential (see Proposition 5.1.6) and derive a connection between the Kähler differential and the Noether-Dedekind differential (see Corollary 5.1.8) as in [Her, Satz 3.1]. Moreover, the first syzygy module of the Kähler differential for a reduced 0-dimensional almost complete intersection in  $\mathbb{P}_K^2$  is explicitly described by constructing a homogeneous system of generators of the normal module  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$  (see Proposition 5.1.12 and Corollary 5.1.13).

In the second section we apply our knowledge of the Noether, Dedekind, and Kähler differentials for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  to explore the Kähler differential  $\vartheta^{(1)}(R/K)$  of the algebra  $R/K$ , where  $R$  is the homogeneous coordinate ring of  $\mathbb{X}$ . In Lemma 5.2.1, we establish some relations between  $\vartheta^{(1)}(R/K)$  and the Kähler differential  $\vartheta_K(R/K[x_0])$ .

From these relations we derive bounds for the Hilbert polynomial and the regularity index of  $\vartheta^{(1)}(R/K)$  (see Proposition 5.2.2). We also prove some characterizations of a complete intersection in terms of  $\vartheta^{(1)}(R/K)$  (see Propositions 5.2.5 and 5.2.8). At the end of this section, we present a characterization for arithmetically Gorenstein schemes (see Corollary 5.2.10).

In the third section we are concerned with studying the differents for fat point schemes. We first recall the definition of fat point schemes  $\mathbb{W}$  in  $\mathbb{P}_K^n$  (see Definition 5.3.1). Then we describe the Kähler differents and their Hilbert functions when the scheme  $\mathbb{W}$  is contained in the projective line (see Lemma 5.3.3). Next we concentrate our attention on the Kähler different  $\vartheta_K(S/K[x_0])$  in the case  $n \geq 2$ , where  $S$  is the homogeneous coordinate ring of  $\mathbb{W}$ . We show that it is zero if and only if the multiplicity of each point of  $\mathbb{W}$  is greater than 1 (see Lemma 5.3.4), compute its Hilbert polynomial, bound its regularity index (see Theorem 5.3.6), and determine bounds for its Hilbert function (see Proposition 5.3.7). For some special positions of the support of the fat point scheme  $\mathbb{W}$ , we can improve the bounds for the regularity index of  $\vartheta_K(S/K[x_0])$  which involve other numerical invariants of the scheme (see Propositions 5.3.13 and 5.3.15). Finally, we provide some properties of the Kähler differents of the algebra  $S/K$  (see Proposition 5.3.16) and give bounds for the Hilbert polynomial of the Noether different  $\vartheta_N(S/K[x_0])$  (see Proposition 5.3.18).

As usual, let  $K$  be an arbitrary field, and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ . By  $\mathcal{I}_{\mathbb{X}}$  we denote the homogeneous vanishing ideal of  $\mathbb{X}$  in  $P$ . The homogeneous coordinate ring of  $\mathbb{X}$  is then given by  $R = P/\mathcal{I}_{\mathbb{X}}$ , and the image of  $X_i$  in  $R$  is denoted by  $x_i$  for  $i = 0, \dots, n$ .

## 5.1 Differents for Almost Complete Intersections

**Definition 5.1.1.** Let  $\mathcal{A} = (a_{ij})$  be a matrix in  $\text{Mat}_{r,s}(P)$ . We say that  $\mathcal{A}$  is a **homogeneous matrix** (or simply **homogeneous**) if there exist two tuples  $d_0 = (d_{01}, \dots, d_{0r}) \in \mathbb{Z}^r$  and  $d_1 = (d_{11}, \dots, d_{1s}) \in \mathbb{Z}^s$  such that the polynomial  $a_{ij}$  is homogeneous of degree  $\deg(a_{ij}) = d_{1j} - d_{0i}$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$ . In this case, the pair  $(d_0, d_1) \in \mathbb{Z}^r \times \mathbb{Z}^s$  is called a **degree pair** of  $\mathcal{A}$ .

Note that degree pairs are not uniquely determined, since if  $(d_0, d_1)$  is a degree pair of a homogeneous matrix  $\mathcal{A}$  then  $(d'_0, d'_1)$ , where  $d'_0 = (d_{01} + d, \dots, d_{0r} + d)$  and  $d'_1 = (d_{11} + d, \dots, d_{1s} + d)$  for some  $d \in \mathbb{Z}$ , is also a degree pair of  $\mathcal{A}$ . Moreover, when  $\mathcal{A}$  is a homogeneous square matrix of size  $r \times r$ , its determinant is a homogeneous



polynomial of degree  $\sum_{i=1}^r (d_{1i} - d_{0i})$ . We refer the reader to [KR3, Section 4.7] for the general theory of homogeneous matrices.

**Definition 5.1.2.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme.

- (i) We say that  $\mathbb{X}$  is an **almost complete intersection** if  $\mathcal{I}_{\mathbb{X}}$  is minimally generated by  $n + 1$  homogeneous polynomials  $F_1, \dots, F_{n+1} \in P$ .
- (ii) We say that  $\mathbb{X}$  is a **special almost complete intersection** if there exists a  $n$ -tuple of homogeneous polynomials  $\mathcal{G} = (G_1, \dots, G_n)$  and a homogeneous matrix  $\mathcal{A} = (a_{ij})_n$  over  $P$  such that  $\mathcal{I}_{\mathbb{X}} = \langle \{\mathcal{G} \cdot \mathcal{A}\} \cup \{\det(\mathcal{A})\} \rangle$ .

**Proposition 5.1.3.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional local Gorenstein scheme which is an almost complete intersection. If  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) \neq 0$  for some  $i \leq r_{\mathbb{X}}$ , then  $\mathbb{X}$  is not a CB-scheme.*

*Proof.* According to [Ku2, Corollary 1.2], a Noetherian local ring which is an almost complete intersection is not a Gorenstein ring. This implies that if  $\mathbb{X}$  is an almost complete intersection then  $\mathbb{X}$  is not arithmetically Gorenstein, since the Noetherian local ring  $\overline{R}$  cannot be both an almost complete intersection and a Gorenstein ring. Thus if  $\text{HF}_{\vartheta_D(R/K[x_0])}(i) \neq 0$  for some  $i \leq r_{\mathbb{X}}$ , then Propositions 4.1.11 and 4.1.15 imply that  $\mathbb{X}$  cannot be a CB-scheme.  $\square$

In what follows, we work over a perfect field  $K$ . Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional scheme which is an almost complete intersection, and let  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_{n+1} \rangle$  such that  $\{F_1, \dots, F_n\}$  is a  $P$ -regular sequence. For  $j \in \{1, \dots, n + 1\}$ , we denote  $d_j := \deg(F_j)$ . By Remark 3.3.24, we may assume that the ideal  $J = \langle F_1, \dots, F_n \rangle$  defines a 0-dimensional complete intersection  $\mathbb{W}$  which is smooth at the points of  $\mathbb{X}$ . In particular, the homogeneous element  $\Delta_{n+1} := \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$  contained in  $\vartheta_K(R/K[x_0])$  is a non-zerodivisor of  $R$ .

Let  $\mathbb{Y}$  be the residual scheme of  $\mathbb{X}$  in  $\mathbb{W}$ . Then we can check that  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked by  $\mathbb{W}$ . The following lemma provides a relation between the image of the homogeneous vanishing ideal of  $\mathbb{Y}$  in  $R$  and the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$  which we prove using arguments in the spirit of [Ku3, Section 3].

**Lemma 5.1.4.** *Using the notation introduced as above, we have*

$$(\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} = \Delta_{n+1} \mathfrak{C}_{R/K[x_0]}.$$

*Proof.* Let  $S$  and  $R_{\mathbb{Y}}$  denote the homogeneous coordinate rings of  $\mathbb{W}$  and  $\mathbb{Y}$  respectively. We set  $L = Q^h(R)$  and  $L_0 = K[x_0, x_0^{-1}]$ . Since  $\mathbb{X}$  and  $\mathbb{Y}$  are geometrically linked by  $\mathbb{W}$ , by Proposition 3.2.1, we have  $Q^h(S) = L_0 \otimes_{K[x_0]} S = L \times Q^h(R_{\mathbb{Y}})$ . So, the image of  $\mathcal{I}_{\mathbb{Y}/\mathbb{W}}$  in  $Q^h(S)$  is  $(\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \times \langle 0 \rangle$ . Moreover, we have the commutative diagram of graded  $K[x_0]$ -modules

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0]) & \xrightarrow{\phi_1} & \underline{\mathrm{Hom}}_{L_0}(L, L_0) \\ \downarrow \psi_1 & & \downarrow \phi_2 \\ \underline{\mathrm{Hom}}_{K[x_0]}(S, K[x_0]) & \xrightarrow{\psi_2} & \underline{\mathrm{Hom}}_{L_0}(S \otimes_{K[x_0]} L_0, L_0) \end{array}$$

where all maps are injective and homogeneous of degree zero. Since  $\mathbb{X}$  is smooth, the canonical trace map  $\mathrm{Tr}_{L/L_0}$  is a trace map of the algebra  $L/L_0$  (see Proposition 3.2.16). We also have  $\underline{\mathrm{Hom}}_{L_0}(L_0 \otimes_{K[x_0]} S, L_0) = \underline{\mathrm{Hom}}_{L_0}(L, L_0) \times \underline{\mathrm{Hom}}_{L_0}(Q^h(R_{\mathbb{Y}}), L_0)$ . Thus  $\phi_2$  is the canonical injection onto the first factor, and so  $\mathrm{Im}(\phi_2 \circ \phi_1) = \mathfrak{C}_{R/K[x_0]} \cdot \mathrm{Tr}_{L/L_0} \times \langle 0 \rangle$ . On the other hand, since  $\mathbb{W}$  is a complete intersection, [Ku5, F.23] yields that the algebra  $S/K[x_0]$  has the homogeneous trace map  $\tau$  of degree  $-r_{\mathbb{W}}$  such that  $\mathrm{Tr}_{S/K[x_0]} = \Delta_{n+1} \cdot \tau$ . We see that  $\mathcal{I}_{\mathbb{Y}/\mathbb{W}} \cong \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-r_{\mathbb{W}})$  and the map  $f \cdot \tau$  factors through  $R$  for all  $f \in \mathcal{I}_{\mathbb{Y}/\mathbb{W}}$ . This implies  $\mathrm{Im}(\psi_1) = \mathcal{I}_{\mathbb{Y}/\mathbb{W}} \cdot \tau$ . According to [Ku5, F.16], we have  $\underline{\mathrm{Hom}}_{L_0}(L_0 \otimes_{K[x_0]} S, L_0) = (L \times Q^h(R_{\mathbb{Y}})) \cdot (\mathrm{id}_{L_0} \otimes \tau)$ , and so  $\mathrm{Im}(\psi_2 \circ \psi_1) = \mathrm{Im}(\phi_2 \circ \phi_1)$  implies  $((\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \times \langle 0 \rangle) \cdot (\mathrm{id}_{L_0} \otimes \tau) = \mathfrak{C}_{R/K[x_0]} \cdot \mathrm{Tr}_{L/L_0} \times \langle 0 \rangle$ . Furthermore, we observe that  $\Delta_{n+1}((\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \times \langle 0 \rangle) \cdot (\mathrm{id}_{L_0} \otimes \tau) = ((\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \times \langle 0 \rangle) \cdot \mathrm{Tr}_{L_0 \otimes_{K[x_0]} S/L_0} = (\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \cdot \mathrm{Tr}_{L/L_0} \times \langle 0 \rangle$ . It follows that  $(\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} \cdot \mathrm{Tr}_{L/L_0} = \Delta_{n+1} \mathfrak{C}_{R/K[x_0]} \cdot \mathrm{Tr}_{L/L_0}$ . Therefore we get  $(\mathcal{I}_{\mathbb{Y}} + \mathcal{I}_{\mathbb{X}})/\mathcal{I}_{\mathbb{X}} = \Delta_{n+1} \mathfrak{C}_{R/K[x_0]}$ , since  $\mathrm{Ann}_L(\mathrm{Tr}_{L/L_0}) = \langle 0 \rangle$ .  $\square$

In [DK], Proposition 5.2 provides a description of the Dedekind complementary module  $\mathfrak{C}_{R/K[x_0]}$ . This description can be proved by using the above lemma as follows.

**Proposition 5.1.5.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional scheme which is an almost complete intersection as above. Then we have an exact sequence of graded  $R$ -modules*

$$0 \longrightarrow \mathfrak{C}_{R/K[x_0]} \left( n - \sum_{j=1}^{n+1} d_j \right) \xrightarrow{\varphi} \bigoplus_{j=1}^{n+1} R(-d_j) \xrightarrow{\psi} \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2 \longrightarrow 0$$

where  $\varphi$  is given by  $\varphi(g) = \sum_{j=1}^{n+1} g \Delta_j e_j$ ,  $\Delta_j = \frac{\partial(F_1, \dots, \widehat{F}_j, \dots, F_{n+1})}{\partial(x_1, \dots, x_n)}$  ( $j = 1, \dots, n+1$ ) are  $n$ -minors of the Jacobian matrix, and  $\psi$  is given by  $\psi(e_j) = F_j + \mathcal{I}_{\mathbb{X}}^2$  for  $j = 1, \dots, n+1$ .

*Proof.* We first observe  $g \Delta_j \in R$  for  $g \in \mathfrak{C}_{R/K[x_0]}$  and for  $j = 1, \dots, n+1$ , since  $\Delta_1, \dots, \Delta_{n+1} \in \mathfrak{v}_{\mathbb{X}}(R/K[x_0])$ . It is not difficult to verify that  $\varphi$  is well-defined and  $R$ -linear. Let  $g \in (\mathfrak{C}_{R/K[x_0]})_i$  for some  $i \geq -r_{\mathbb{X}}$ . Then  $\varphi(g) = \sum_{j=1}^{n+1} g \Delta_j e_j$  is homogeneous

of degree  $i + \sum_{j=1}^{n+1} d_j - n$ . (Note that  $\deg(\Delta_j) = \sum_{k \neq j} d_k - n$ ,  $\deg(e_j) = d_j$  for  $j = 1, \dots, n+1$ .) This implies that  $\varphi$  is homogeneous of degree zero.

Next we prove that  $\varphi$  is injective. Let  $g \in (\mathfrak{C}_{R/K[x_0]})_i$ , where  $i \geq -r_{\mathbb{X}}$ , and suppose that  $\varphi(g) = \sum_{j=1}^{n+1} g\Delta_j e_j = 0$ . Then  $g\Delta_j = 0$  for  $j = 1, \dots, n+1$ . Since  $x_0$  and  $\Delta_{n+1}$  are non-zerodivisors of  $R$  and  $g$  can be written as  $g = x_0^{-m}\tilde{g}$  with  $m \geq 0$  and  $\tilde{g} \in R$ , we deduce  $g = 0$ .

Now we check that  $\text{Ker}(\psi) = \text{Im}(\varphi)$ . For  $g \in (\mathfrak{C}_{R/K[x_0]})_i$  with  $i \geq -r_{\mathbb{X}}$ , Lemma 5.1.4 allows us to write  $g\Delta_{n+1} = G + \mathcal{I}_{\mathbb{X}}$  with  $G \in J : \mathcal{I}_{\mathbb{X}}$ , where  $J = \langle F_1, \dots, F_n \rangle_P$ . Also, we get a representation  $GF_{n+1} = \sum_{j=1}^n G_j F_j$  with homogeneous polynomials  $G_1, \dots, G_n \in P$  of degree  $\deg(G_j) = \deg(G) + d_{n+1} - d_j$  for  $j = 1, \dots, n$ . For  $k \in \{1, \dots, n\}$ , we see that  $\frac{\partial(GF_{n+1})}{\partial X_k} = \frac{\partial F_{n+1}}{\partial X_k} G + \frac{\partial G}{\partial X_k} F_{n+1} = \sum_{j=1}^n \left( \frac{\partial F_j}{\partial X_k} G_j + \frac{\partial G_j}{\partial X_k} F_j \right)$ . This yields

$$\sum_{j=1}^n \frac{\partial F_j}{\partial X_k} G_j - \frac{\partial F_{n+1}}{\partial X_k} G = \frac{\partial G}{\partial X_k} F_{n+1} - \sum_{j=1}^n \frac{\partial G_j}{\partial X_k} F_j \in \mathcal{I}_{\mathbb{X}}.$$

Thus the element  $\sum_{j=1}^n (G_j + \mathcal{I}_{\mathbb{X}})e_j - (G + \mathcal{I}_{\mathbb{X}})e_{n+1}$  is contained in the kernel of the  $R$ -linear map  $j : \bigoplus_{j=1}^{n+1} R(-d_j) \rightarrow \bigoplus_{j=1}^n R(-1)$  given by the Jacobian matrix  $\mathfrak{J} = \left( \frac{\partial F_j}{\partial x_i} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, n+1}}$ . Moreover, we have  $\text{Ker}(j) = \left\{ \sum_{j=1}^{n+1} h\Delta_j e_j \mid h \in \vartheta_K(R/K[x_0])^{-1} \right\}$  (cf. [Ku3, Lemma 1]). This implies that  $\sum_{j=1}^n (G_j + \mathcal{I}_{\mathbb{X}})e_j - (G + \mathcal{I}_{\mathbb{X}})e_{n+1} = \sum_{j=1}^{n+1} h\Delta_j e_j$  for some  $h \in \vartheta_K(R/K[x_0])^{-1} \subseteq R_{x_0}$ . Hence we deduce  $g\Delta_{n+1} = G + \mathcal{I}_{\mathbb{X}} = -h\Delta_{n+1}$ , and so we get  $h = -g$ , since  $\Delta_{n+1}$  is a non-zerodivisor of  $R$ . Also, for  $j = 1, \dots, n$ , we have  $G_j + \mathcal{I}_{\mathbb{X}} = -g\Delta_j$ . Thus we obtain

$$\begin{aligned} \psi(\varphi(g)) &= \psi\left(\sum_{j=1}^{n+1} g\Delta_j e_j\right) = \psi\left(-\sum_{j=1}^n (G_j + \mathcal{I}_{\mathbb{X}})e_j + (G + \mathcal{I}_{\mathbb{X}})e_{n+1}\right) \\ &= -\sum_{j=1}^n G_j F_j + GF_{n+1} + \mathcal{I}_{\mathbb{X}}^2 = 0, \end{aligned}$$

and hence  $\text{Im}(\varphi) \subseteq \text{Ker}(\psi)$ .

To prove the other inclusion, we suppose that there are homogeneous polynomials  $H_1, \dots, H_{n+1} \in P$  such that  $\psi(\sum_{j=1}^{n+1} (H_j + \mathcal{I}_{\mathbb{X}})e_j) = \sum_{j=1}^{n+1} H_j F_j + \mathcal{I}_{\mathbb{X}}^2 = 0$ . It follows from the equality  $\mathcal{I}_{\mathbb{X}}^2 = \mathcal{I}_{\mathbb{X}}J + \langle F_{n+1}^2 \rangle$  that there are homogeneous polynomials  $G_{jk}, G \in P$  such that  $\sum_{j=1}^{n+1} H_j F_j = \sum_{j=1}^n \sum_{k=1}^{n+1} G_{jk} F_k F_j + GF_{n+1}^2$ . This clearly forces  $\sum_{j=1}^n (H_j - \sum_{k=1}^{n+1} G_{jk} F_k) F_j + (H_{n+1} - GF_{n+1}) F_{n+1} = 0$ , and so the polynomial  $H_{n+1} - GF_{n+1}$  is contained in  $J : \mathcal{I}_{\mathbb{X}}$ . By Lemma 5.1.4, we see that  $H_{n+1} - GF_{n+1} + \mathcal{I}_{\mathbb{X}} = H_{n+1} + \mathcal{I}_{\mathbb{X}} \in \Delta_{n+1}\mathfrak{C}_{R/K[x_0]}$ . Thus there exists a homogeneous element  $g \in \mathfrak{C}_{R/K[x_0]}$  such that  $H_{n+1} + \mathcal{I}_{\mathbb{X}} = g\Delta_{n+1}$ . As above, we get  $g\Delta_j = H_j - \sum_{k=1}^{n+1} G_{jk} F_k + \mathcal{I}_{\mathbb{X}} = H_j + \mathcal{I}_{\mathbb{X}}$  for  $j = 1, \dots, n$ . Therefore we have  $\varphi(g) = \sum_{j=1}^{n+1} g\Delta_j e_j = \sum_{j=1}^{n+1} (H_j + \mathcal{I}_{\mathbb{X}})e_j$ , and consequently  $\text{Im}(\varphi) \supseteq \text{Ker}(\psi)$ .  $\square$

Our next proposition provides an explicit presentation of the Kähler different for a reduced 0-dimensional almost complete intersection.

**Proposition 5.1.6.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional almost complete intersection as above, and let  $\{\widehat{e}_1, \dots, \widehat{e}_{n+1}\}$  be the canonical basis of  $\bigoplus_{j=1}^{n+1} R(d_j)$ . Then we have an exact sequence of graded  $R$ -modules*

$$0 \longrightarrow (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* \xrightarrow{\widehat{\psi}} \bigoplus_{j=1}^{n+1} R(d_j) \xrightarrow{\widehat{\varphi}} \vartheta_K(R/K[x_0])\left(\sum_{j=1}^{n+1} d_j - n\right) \longrightarrow 0$$

where  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* = \underline{\text{Hom}}_R(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2, R)$ , the map  $\widehat{\psi}$  is given by  $\widehat{\psi}(\alpha) = \sum_{j=1}^{n+1} \alpha(F_j + \mathcal{I}_{\mathbb{X}}^2)\widehat{e}_j$ , and the map  $\widehat{\varphi}$  is given by  $\widehat{\varphi}(\sum_{j=1}^{n+1} g_j \widehat{e}_j) = \sum_{j=1}^{n+1} g_j \Delta_j$ . In particular, for all  $i \in \mathbb{Z}$  we have

$$\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \sum_{j=1}^{n+1} \text{HF}_{\mathbb{X}}(i + n - \sum_{k \neq j} d_k) - \text{HF}_{(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*}(i + n - \sum_{j=1}^{n+1} d_j).$$

*Proof.* Applying  $\underline{\text{Hom}}_R(\ , R)$  to the exact sequence in Proposition 5.1.5, we get the following exact sequence of graded  $R$ -modules

$$0 \longrightarrow (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* \xrightarrow{\psi^*} \underline{\text{Hom}}_R\left(\bigoplus_{j=1}^{n+1} R(-d_j), R\right) \xrightarrow{\varphi^*} \underline{\text{Hom}}_R(\mathfrak{C}_{R/K[x_0]}(n - \sum_{j=1}^{n+1} d_j), R).$$

Obviously, the maps  $\psi^*$  and  $\varphi^*$  are homogeneous homomorphisms of degree zero. Moreover, we have the isomorphism of graded  $R$ -modules

$$\theta : \underline{\text{Hom}}_R\left(\bigoplus_{j=1}^{n+1} R(-d_j), R\right) = \bigoplus_{j=1}^{n+1} \underline{\text{Hom}}_R(R(-d_j), R) \xrightarrow{\sim} \bigoplus_{j=1}^{n+1} R(d_j)$$

given by  $\theta(\beta) = \sum_{j=1}^{n+1} \beta(e_j)\widehat{e}_j$  and  $\theta^{-1}(\sum_{j=1}^{n+1} g_j \widehat{e}_j) = \mu_{\sum_{j=1}^{n+1} g_j e_j}$ , where  $\mu_{\sum_{j=1}^{n+1} g_j e_j}$  is the multiplication map given by  $\mu_{\sum_{j=1}^{n+1} g_j e_j}(\sum_{j=1}^{n+1} h_j e_j) = \sum_{j=1}^{n+1} g_j h_j$  for all elements  $\sum_{j=1}^{n+1} h_j e_j \in \bigoplus_{j=1}^{n+1} R(-d_j)$ . We deduce  $\widehat{\psi} = \theta \circ \psi^*$  and  $\widehat{\psi}(\alpha) = \sum_{j=1}^{n+1} \alpha(F_j + \mathcal{I}_{\mathbb{X}}^2)\widehat{e}_j$  for all  $\alpha \in (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$ . Furthermore, there is an isomorphism of graded  $R$ -modules  $\underline{\text{Hom}}_R(\mathfrak{C}_{R/K[x_0]}(n - \sum_{j=1}^{n+1} d_j), R) \xrightarrow{\sim} \vartheta_{\mathbb{X}}(R/K[x_0])\left(\sum_{j=1}^{n+1} d_j - n\right)$ ,  $\beta \mapsto \beta(1)$ . By taking the composition of this map and  $\varphi^* \circ \theta^{-1}$ , we get a homogeneous homomorphism of degree zero  $\widehat{\varphi} : \bigoplus_{j=1}^{n+1} R(d_j) \rightarrow \vartheta_{\mathbb{X}}(R/K[x_0])\left(\sum_{j=1}^{n+1} d_j - n\right)$  given by  $\widehat{\varphi}(\sum_{j=1}^{n+1} g_j \widehat{e}_j) = \sum_{j=1}^{n+1} g_j \Delta_j$  for all  $\sum_{j=1}^{n+1} g_j \widehat{e}_j \in \bigoplus_{j=1}^{n+1} R(d_j)$ . Hence we find

$$\text{Im}(\widehat{\varphi}) = \langle \Delta_1, \dots, \Delta_{n+1} \rangle \left(\sum_{j=1}^{n+1} d_j - n\right) = \vartheta_K(R/K[x_0])\left(\sum_{j=1}^{n+1} d_j - n\right).$$

Therefore we obtain the desired exact sequence.  $\square$

**Remark 5.1.7.** Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional almost complete intersection. We make some observations about the preceding proposition.

- (a) If  $\{\alpha_1, \dots, \alpha_u\}$  is a homogeneous system of generators of  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$ , then the first syzygy module  $\text{Syz}_R^1(\Delta_1, \dots, \Delta_{n+1})$  of the Kähler different  $\vartheta_K(R/K[x_0])$  is generated by the elements

$$\{(\alpha_j(F_1 + \mathcal{I}_{\mathbb{X}}^2), \dots, \alpha_j(F_{n+1} + \mathcal{I}_{\mathbb{X}}^2)) \mid j = 1, \dots, u\}.$$

- (b) Since  $\text{HF}_{(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*}(i) = 0$  for  $i < -d = -\max\{d_j \mid j = 1, \dots, n+1\}$ , Proposition 5.1.6 yields  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = 0$  for  $i < -d + \sum_{j=1}^{n+1} d_j - n$ .

The derived functors of  $\underline{\text{Hom}}_R(\ , \ )$  will be denoted by  $\underline{\text{Ext}}_R^i(\ , \ )$ , where  $i \in \mathbb{Z}$ . Notice that, for a finitely generated graded  $R$ -module  $M$  and a graded  $R$ -module  $N$ , the  $R$ -modules  $\underline{\text{Ext}}_R^i(M, N)$  admit a natural grading and  $\underline{\text{Ext}}_R^i(M, N) = \text{Ext}_R^i(M, N)$  for the underlying  $R$ -modules. In analogy with [Her, Satz 3.1], we get the following connection between the differents.

**Corollary 5.1.8.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a reduced 0-dimensional almost complete intersection as above. We have an exact sequence of graded  $R$ -modules*

$$0 \longrightarrow \vartheta_K(R/K[x_0]) \longrightarrow \vartheta_{\mathbb{X}}(R/K[x_0]) \xrightarrow{\delta} \underline{\text{Ext}}_R^1(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2, R)(n - \sum_{j=1}^{n+1} d_j) \longrightarrow 0,$$

where  $\delta$  is the connecting homomorphism. In particular, if  $\mathbb{X}$  is a special almost complete intersection, then  $\underline{\text{Ext}}_R^1(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2, R) = 0$  and  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .

*Proof.* Applying the functor  $\underline{\text{Hom}}_R(\ , R)$  to the exact sequence

$$0 \longrightarrow \mathfrak{C}_{R/K[x_0]}(n - \sum_{j=1}^{n+1} d_j) \xrightarrow{\varphi} \bigoplus_{j=1}^{n+1} R(-d_j) \xrightarrow{\psi} \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2 \longrightarrow 0$$

we get a long exact sequence of graded  $R$ -modules

$$\begin{aligned} 0 \longrightarrow (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* &\xrightarrow{\widehat{\psi}} \bigoplus_{j=1}^{n+1} R(d_j) \xrightarrow{\widehat{\varphi}} \vartheta_{\mathbb{X}}(R/K[x_0]) \left(\sum_{j=1}^{n+1} d_j - n\right) \\ &\xrightarrow{\delta} \underline{\text{Ext}}_R^1(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2, R) \longrightarrow \underline{\text{Ext}}_R^1\left(\bigoplus_{j=1}^{n+1} R(d_j), R\right) = 0, \end{aligned}$$

where  $\widehat{\psi}$  and  $\widehat{\varphi}$  are described as in Proposition 5.1.6. Thus the exact sequence of the corollary follows from the fact that  $\text{Im}(\widehat{\varphi}) = \vartheta_K(R/K[x_0])\left(\sum_{j=1}^{n+1} d_j - n\right)$ . The last statement follows from [Wal, Satz 4]. □

Note that the equality  $\vartheta_K(R/\mathbb{Q}[x_0]) = \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0])$  can occur without the assumption that  $\mathbb{X}$  is a special almost complete intersection, as our next example shows.

**Example 5.1.9.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set consisting of thirteen points  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 0)$ ,  $p_3 = (1 : 0 : 0 : 1)$ ,  $p_4 = (1 : 0 : 1 : 0)$ ,  $p_5 = (1 : 0 : -1 : 0)$ ,  $p_6 = (1 : 0 : 0 : -1)$ ,  $p_7 = (1 : 0 : 1 : -1)$ ,  $p_8 = (1 : 0 : -1 : 1)$ ,  $p_9 = (1 : 1 : -1 : 0)$ ,  $p_{10} = (1 : 1 : -1 : -1)$ ,  $p_{11} = (1 : 1 : 1 : -1)$ ,  $p_{12} = (1 : 1 : -1 : 1)$ , and  $p_{13} = (1 : 1 : 0 : -1)$ . Then the homogeneous vanishing ideal  $\mathcal{I}_{\mathbb{X}}$  is minimally generated by one quadratic form and three cubic forms, and the Cohen-Macaulay type of  $R$  is 4. Thus  $\mathbb{X}$  is an almost complete intersection, but it is not a special almost complete intersection, since  $\text{ht}(\mathcal{I}_{\mathbb{X}}) = 3 < 4 = \text{type}(R)$  (cf. [Wal, Bemerkung 1]). In this case, a calculation gives us

$$\begin{aligned} \vartheta_K(R/\mathbb{Q}[x_0]) &= \langle x_3^6, x_0x_2^4 - 1/2x_1x_2^4 - x_0x_3^4 + 1/2x_1x_3^4, \\ &\quad x_1^5 - 2x_1x_2^4 + 4x_0x_2x_3^3 - 6x_1x_2x_3^3 - 2x_2^2x_3^3 + 2x_0x_3^4 - 3x_1x_3^4 - 2x_2x_3^4, \\ &\quad x_0^5 + 1/2x_1x_2^4 - x_0x_2x_3^3 + 6x_1x_2x_3^3 + 5x_2^2x_3^3 - 2x_0x_3^4 + 3/2x_1x_3^4 + 5x_2x_3^4 \rangle \\ &= \vartheta_{\mathbb{X}}(R/\mathbb{Q}[x_0]). \end{aligned}$$

Therefore these differents are equal even when the almost complete intersection  $\mathbb{X}$  is not special.

Now we restrict our attention to the reduced 0-dimensional schemes  $\mathbb{X}$  in the projective plane  $\mathbb{P}_K^2$ . According to the Hilbert-Burch theorem (cf. [Ei1, Theorem 20.15] or [Pev, Theorem 24.2]), there exists a homogeneous non-zerodivisor  $u$  and a homogeneous matrix  $\mathcal{A} = (a_{jk})$  of size  $r \times (r - 1)$  such that the minimal graded free resolution of  $\mathcal{I}_{\mathbb{X}}$  has the form

$$0 \longrightarrow \bigoplus_{j=1}^{r-1} P(-d'_j) \xrightarrow{\varphi_1} \bigoplus_{j=1}^r P(-d_j) \xrightarrow{\varphi_0} \mathcal{I}_{\mathbb{X}} \longrightarrow 0 \tag{5.1}$$

Here  $\varphi_1$  is defined by  $\mathcal{A}$  and  $\varphi_0$  is defined by  $(uF_1 \ \cdots \ uF_r)$ , where  $F_j$  is the determinant of the matrix  $(\mathcal{E}_j \ \mathcal{A})$  with  $\mathcal{E}_j = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & [j] & & & \end{pmatrix}^{\text{tr}}$  for  $j = 1, \dots, r$ .

**Remark 5.1.10.** Note that the above non-zerodivisor  $u$  is contained in  $K$ , and so we may assume  $u = 1$ . Indeed, suppose for a contradiction that  $u \notin K$ . Then we have  $\text{deg}(u) \geq 1$ . Because  $\mathcal{I}_{\mathbb{X}}$  is a homogeneous ideal of height 2, we can assume that  $\{uF_1, uF_2\}$  is a  $P$ -regular sequence. We see that  $F_1 \notin \langle uF_1 \rangle$  or the image of  $F_1$  in  $P/\langle uF_1 \rangle$  is not zero. Moreover, the image of  $uF_2$  in  $P/\langle uF_1 \rangle$  is a non-zerodivisor. But the image of  $uF_2F_1$  in  $P/\langle uF_1 \rangle$  is zero, a contradiction.

Tensoring (5.1) with  $R$  we get the exact sequence of graded  $R$ -modules

$$0 \longrightarrow \mathrm{Tor}_1^P(\mathcal{I}_{\mathbb{X}}, R) \longrightarrow \bigoplus_{j=1}^{r-1} R(-d'_j) \xrightarrow{\bar{\varphi}_1} \bigoplus_{j=1}^r R(-d_j) \xrightarrow{\bar{\varphi}_0} \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2 \longrightarrow 0$$

where  $\bar{\varphi}_1$  is defined by the matrix  $\bar{\mathcal{A}} = (\bar{a}_{jk})$ ,  $\bar{a}_{jk} = a_{jk} + \mathcal{I}_{\mathbb{X}}$ , and  $\bar{\varphi}_0$  is given by  $\bar{\varphi}_0(e_j) = F_j + \mathcal{I}_{\mathbb{X}}^2$  for  $j = 1, \dots, r$ . Hence we deduce the following exact sequence of graded  $R$ -modules:

$$0 \longrightarrow (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* \xrightarrow{\hat{\varphi}_0} \bigoplus_{j=1}^r R(d_j) \xrightarrow{\hat{\varphi}_1} \bigoplus_{j=1}^{r-1} R(d'_j).$$

The map  $\hat{\varphi}_0$  is given by  $\hat{\varphi}_0(\alpha) = (\alpha(F_1 + \mathcal{I}_{\mathbb{X}}^2), \dots, \alpha(F_r + \mathcal{I}_{\mathbb{X}}^2))$ , and  $\hat{\varphi}_1$  is defined by  $\bar{\mathcal{A}}^{\mathrm{tr}}$ . Therefore a mapping  $\alpha : \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2 \rightarrow R$  is an  $R$ -module homomorphism if and only if the vector  $(\alpha(F_1 + \mathcal{I}_{\mathbb{X}}^2), \dots, \alpha(F_r + \mathcal{I}_{\mathbb{X}}^2))$  satisfies  $\bar{\mathcal{A}}^{\mathrm{tr}} \cdot (\alpha(F_1 + \mathcal{I}_{\mathbb{X}}^2), \dots, \alpha(F_r + \mathcal{I}_{\mathbb{X}}^2)) = \mathbf{0}$ . To describe the homogeneous generators of  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$ , we require the following lemma.

**Lemma 5.1.11.** *In the above situation, let  $\mathcal{V} = (G_1, \dots, G_r) \in (\bigoplus_{j=1}^r P(-d_j))_i$  be a non-zero homogeneous vector such that  $\mathcal{A}^{\mathrm{tr}} \cdot \mathcal{V} = \mathbf{0}$ . Then there exists a homogeneous polynomial  $G \in P_i$  such that  $\mathcal{V} = G \cdot (F_1, \dots, F_r)$ .*

*Proof.* It is not difficult to see that  $\mathrm{rank}(\mathcal{A}) = r - 1$ , and so  $\mathrm{rank}(\mathcal{A}^{\mathrm{tr}}) = r - 1$  (see for instance [Bro] for the definition of the rank of a matrix over a ring). Let  $Q(P)$  denote the quotient field of  $P$ , and let  $W = \{v \in \bigoplus_{j=1}^r P(-d_j) \mid \mathcal{A}^{\mathrm{tr}} \cdot v = \mathbf{0}\}$ . Then we get  $\dim_{Q(P)} W = 1$ . This implies that  $\mathcal{V}$  and  $(F_1, \dots, F_r)$  are proportional over  $Q(P)$ , and so  $G_j F_k = G_k F_j$  for all  $j, k$ . Obviously, we have  $F_1 \neq 0$  in  $Q(P)$ . This enables us to write  $G_j = G_1 F_1^{-1} F_j$  for all  $j = 1, \dots, r$ . Since  $\mathcal{V} \neq \mathbf{0}$ , the element  $G_1 F_1^{-1}$  is not zero. We see that  $G_1 F_j \in \langle F_1 \rangle_P$  for all  $j = 1, \dots, r$ . Thus  $G_1 \in \langle F_1 \rangle_P : \mathcal{I}_{\mathbb{X}}$ . Moreover, the multiplication map  $\mu_{F_1} : \mathcal{I}_{\mathbb{X}} \rightarrow \mathcal{I}_{\mathbb{X}}$  is injective and  $\mathcal{I}_{\mathbb{X}} \cong \bigoplus_{j=1}^r P(-d_j) / \mathrm{Im}(\varphi_1)$ . By [Bro, Lemma D.3], we have  $\langle F_1 \rangle_P = \langle F_1 \rangle_P : \mathcal{I}_{\mathbb{X}}$ . Hence  $G_1 = G \cdot F_1$  for some  $G \in P_i \setminus \{0\}$  and  $\mathcal{V} = G \cdot (F_1, \dots, F_r)$ , as we wanted.  $\square$

In analogy with [Sch, Corollary 3], a homogeneous system of generators of the normal module  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$  can be constructed as in the following proposition.

**Proposition 5.1.12.** *Using the notation as above, we let  $\mathcal{A}_k$  denote the  $k$ -th column of the matrix  $\mathcal{A}$  for  $k = 1, \dots, r - 1$ .*

(i) *The map  $\alpha_{jk} : \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2 \rightarrow R$  ( $j = 1, \dots, r; k = 1, \dots, r - 1$ ) defined by*

$$\alpha_{jk}(F_l + \mathcal{I}_{\mathbb{X}}^2) = \det \begin{pmatrix} \mathcal{E}_j & \mathcal{A}_1 & \cdots & \mathcal{A}_{k-1} & \mathcal{E}_l & \mathcal{A}_{k+1} & \cdots & \mathcal{A}_{r-1} \end{pmatrix} + \mathcal{I}_{\mathbb{X}}$$

*is a homogeneous element of degree  $d_j - d'_k$  of the  $R$ -module  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$ .*

(ii) We have

$$(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^* = \langle \alpha_{jk} \mid 1 \leq j \leq r, 1 \leq k \leq r-1 \rangle_R.$$

*Proof.* We set  $G_{jkl} = \det \begin{pmatrix} \mathcal{E}_j & \mathcal{A}_1 & \cdots & \mathcal{A}_{k-1} & \mathcal{E}_l & \mathcal{A}_{k+1} & \cdots & \mathcal{A}_{r-1} \end{pmatrix}$  for  $l = 1, \dots, r$ , and  $\mathcal{G}_{jk} = (G_{jk1}, \dots, G_{jkr})$  for  $j = 1, \dots, r$  and  $k = 1, \dots, r-1$ . Observe that

$$\begin{aligned} (\mathcal{A}_{k'})^{\text{tr}} \cdot \mathcal{G}_{jk} &= \sum_{l=1}^r a_{k'l} G_{jkl} = \sum_{l=1}^r \det \begin{pmatrix} \mathcal{E}_j & \mathcal{A}_1 & \cdots & \mathcal{A}_{k-1} & a_{k'l} \mathcal{E}_l & \mathcal{A}_{k+1} & \cdots & \mathcal{A}_{r-1} \end{pmatrix} \\ &= \det \begin{pmatrix} \mathcal{E}_j & \mathcal{A}_1 & \cdots & \mathcal{A}_{k-1} & \mathcal{A}_{k'} & \mathcal{A}_{k+1} & \cdots & \mathcal{A}_{r-1} \end{pmatrix}. \end{aligned}$$

This implies that  $(\mathcal{A}_{k'})^{\text{tr}} \cdot \mathcal{G}_{jk} = 0$  if  $k' \neq k$  and  $(\mathcal{A}_k)^{\text{tr}} \cdot \mathcal{G}_{jk} = F_j \in \mathcal{I}_{\mathbb{X}}$ . Thus we get  $\overline{\mathcal{A}}^{\text{tr}} \cdot (\alpha_{jk}(F_1 + \mathcal{I}_{\mathbb{X}}^2), \dots, \alpha_{jk}(F_r + \mathcal{I}_{\mathbb{X}}^2)) = \mathbf{0}$ . Therefore the element  $\alpha_{jk}$  is contained in  $(\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)^*$  for all  $j, k$ . Now we check that  $\alpha_{jk}$  is homogeneous of degree  $d_j - d'_k$ . Obviously,  $\alpha_{jk}(F_l + \mathcal{I}_{\mathbb{X}}^2) = 0$  if  $l = j$ . In order to compute the degree of  $\alpha_{jk}$ , we need to consider the degree of  $\alpha_{jk}(F_l + \mathcal{I}_{\mathbb{X}}^2)$  with  $l \neq j$ . Notice that  $\deg(a_{jk}) = d'_k - d_j$  for all  $j, k$ . We can exhibit the submatrix of the matrix  $\mathcal{A}$  by deleting the  $j$ -th and  $l$ -th rows ( $l > j$ ) and deleting the  $k$ -th column as follows:

$$\begin{pmatrix} d'_1 & \cdots & d'_{k-1} & d'_{k+1} & \cdots & d'_{r-1} \\ \left( \begin{array}{cccccc} a_{11} & \cdots & a_{1k-1} & a_{1k+1} & \cdots & a_{1r-1} \\ \vdots & & \vdots & \vdots & \cdots & \vdots \\ a_{j-11} & \cdots & a_{j-1k-1} & a_{j-1k+1} & \cdots & a_{j-1r-1} \\ a_{j+11} & \cdots & a_{j+1k-1} & a_{j+1k+1} & \cdots & a_{j+1r-1} \\ \vdots & & \vdots & \vdots & \cdots & \vdots \\ a_{l-11} & \cdots & a_{l-1k-1} & a_{l-1k+1} & \cdots & a_{l-1r-1} \\ a_{l+11} & \cdots & a_{l+1k-1} & a_{l+1k+1} & \cdots & a_{l+1r-1} \\ \vdots & & \vdots & \vdots & \cdots & \vdots \\ a_{r1} & \cdots & a_{rk-1} & a_{rk+1} & \cdots & a_{rr-1} \end{array} \right) & \begin{array}{l} -d_1 \\ \vdots \\ -d_{j-1} \\ -d_{j+1} \\ \vdots \\ -d_{l-1} \\ -d_{l+1} \\ \vdots \\ -d_r \end{array} \end{pmatrix}$$

Thus we have  $\deg(\alpha_{jk}(F_l + \mathcal{I}_{\mathbb{X}}^2)) = \sum_{i \neq k} d'_i - \sum_{i \neq j, l} d_i$  for all  $l \neq j$ . This implies that  $\deg(\alpha_{jk}) = \sum_{i \neq k} d'_i - \sum_{i \neq j, l} d_i - d_l = \sum_{i \neq k} d'_i - \sum_{i \neq j} d_i$ . It is well known (cf. [Ei2, Proposition 3.8]) that  $\sum_{i=1}^{r-1} d'_i = \sum_{i=1}^r d_i$ . Hence we obtain  $\deg(\alpha_{jk}) = d_j - d'_k$ , which completes the proof of (i).

Next we prove (ii). For this, let  $\alpha \in (\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{X}}^2)_i^*$  be a non-zero homogeneous element and let  $\mathcal{G} = (G_1, \dots, G_r)$  with  $\alpha(F_j + \mathcal{I}_{\mathbb{X}}^2) = G_j + \mathcal{I}_{\mathbb{X}}$ ,  $G_j \in P_{d_j+i}$ , for  $j = 1, \dots, r$ . It is clear that  $(\mathcal{A}_k)^{\text{tr}} \cdot \mathcal{G} \in \mathcal{I}_{\mathbb{X}}$ , and so  $(\mathcal{A}_k)^{\text{tr}} \cdot \mathcal{G} = \sum_{j=1}^r b_{jk} F_j$  for some  $b_{jk} \in P_{d'_k-d_j+i}$ . Put



$\mathcal{G}' = \sum_{j,k} b_{jk} \mathcal{G}_{jk}$ . For every  $k' \in \{1, \dots, r-1\}$ , we have

$$(\mathcal{A}_{k'})^{\text{tr}} \cdot (\mathcal{G} - \mathcal{G}') = \sum_{j=1}^r b_{jk'} F_j - \sum_{j,k} b_{jk} (\mathcal{A}_{k'})^{\text{tr}} \cdot \mathcal{G}_{jk} = \sum_{j=1}^r b_{jk'} F_j - \sum_{j=1}^r b_{jk'} F_j = 0.$$

Hence  $\mathcal{A}^{\text{tr}} \cdot (\mathcal{G} - \mathcal{G}') = 0$ . It follows from Lemma 5.1.11 that there is a homogeneous element  $G \in P_i$  such that  $\mathcal{G} - \mathcal{G}' = G \cdot (F_1, \dots, F_r)$ . Consequently, we obtain  $\alpha = \sum_{j,k} (b_{jk} + \mathcal{I}_{\mathbb{X}}) \cdot \alpha_{jk}$ , and the claim (ii) follows.  $\square$

Suppose  $\mathbb{X} \subseteq \mathbb{P}_K^2$  is an almost complete intersection. Then the minimal graded free resolution of  $\mathcal{I}_{\mathbb{X}}$  has the form

$$0 \longrightarrow P(-d'_1) \oplus P(-d'_2) \xrightarrow{\mathcal{A}} P(-d_1) \oplus P(-d_2) \oplus P(-d_3) \longrightarrow \mathcal{I}_{\mathbb{X}} \longrightarrow 0$$

and  $\mathcal{I}_{\mathbb{X}}$  is generated by all  $2 \times 2$ -minors of the homogeneous matrix  $\mathcal{A} = (a_{jk})_{\substack{j=1,2,3 \\ k=1,2}}$ . In this situation, we give an explicit description of the first syzygy module of the Kähler different  $\vartheta_K(R/K[x_0]) = \langle \Delta_1, \Delta_2, \Delta_3 \rangle$  of the algebra  $R/K[x_0]$  as follows.

**Corollary 5.1.13.** *Let  $\mathbb{X}$  be a reduced 0-dimensional scheme in  $\mathbb{P}_K^2$  which is an almost complete intersection. Then we have an exact sequence of graded  $R$ -modules*

$$0 \rightarrow \text{Syz}_R^1(\Delta_1, \Delta_2, \Delta_3) \longrightarrow R(d_1) \oplus R(d_2) \oplus R(d_3) \longrightarrow \vartheta_K(R/K[x_0]) \left( \sum_{j=1}^3 d_j - 2 \right) \rightarrow 0.$$

In particular, the module  $\text{Syz}_R^1(\Delta_1, \Delta_2, \Delta_3)$  is generated by the following homogeneous vectors:  $(0, a_{32} + \mathcal{I}_{\mathbb{X}}, -a_{22} + \mathcal{I}_{\mathbb{X}})$ ,  $(0, -a_{31} + \mathcal{I}_{\mathbb{X}}, a_{21} + \mathcal{I}_{\mathbb{X}})$ ,  $(-a_{32} + \mathcal{I}_{\mathbb{X}}, 0, a_{12} + \mathcal{I}_{\mathbb{X}})$ ,  $(-a_{31} + \mathcal{I}_{\mathbb{X}}, 0, a_{11} + \mathcal{I}_{\mathbb{X}})$ ,  $(a_{22} + \mathcal{I}_{\mathbb{X}}, -a_{12} + \mathcal{I}_{\mathbb{X}}, 0)$ , and  $(-a_{21} + \mathcal{I}_{\mathbb{X}}, a_{11} + \mathcal{I}_{\mathbb{X}}, 0)$ .

*Proof.* This follows from Remark 5.1.7(a) and Proposition 5.1.12.  $\square$

We close this section with the following observation.

**Corollary 5.1.14.** *Let  $\mathbb{X}$  be a reduced 0-dimensional subscheme of  $\mathbb{P}_K^2$ .*

(i) *If  $\mathbb{X}$  is an almost complete intersection, then  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

(ii) *If  $\mathbb{X}$  is contained in a complete intersection of degree  $\deg(\mathbb{X}) + 1$ , then we have  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ .*

*Proof.* (i) Let  $\mathcal{G} = (-a_{12}, a_{11})$  and  $\mathcal{A}' = \begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix}$ . Then it is clear that we have  $\mathcal{I}_{\mathbb{X}} = \langle \{\mathcal{G}\mathcal{A}'\} \cup \{\det(\mathcal{A}')\} \rangle \subseteq P$ . Thus  $\mathbb{X}$  is a special almost complete intersection, and therefore we have  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$  by Corollary 5.1.8.

(ii) This follows from (i) and the fact that  $\mathbb{X}$  is a complete intersection or an almost complete intersection.  $\square$

## 5.2 The Kähler Differents of the Coordinate Ring of 0-Dimensional Schemes

Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme such that  $\text{Supp}(\mathbb{X}) \cap \mathcal{Z}^+(X_0) = \emptyset$ . In the same way as Definition 3.3.1, we define the  $i$ -th Kähler different of the algebra  $R/K$  to be the  $i$ -th Fitting ideal of  $\Omega_{R/K}^1$ , and we denote it by  $\vartheta^{(i)}(R/K)$ . Clearly, we have  $\langle 0 \rangle \subseteq \vartheta^{(0)}(R/K) \subseteq \vartheta^{(1)}(R/K) \subseteq \vartheta^{(2)}(R/K) \subseteq \dots \subseteq R$  and  $\vartheta^{(i)}(R/K) = R$  for  $i \geq m$ , where  $m$  is the minimal number of generators of  $\Omega_{R/K}^1$ .

Notice that the set  $\{dx_0, \dots, dx_n\}$  is a homogeneous system of generators of the graded  $R$ -module  $\Omega_{R/K}^1$ . Moreover, we have  $\text{Ann}_R(\Omega_{R/K}^1) = \langle 0 \rangle$  by [Kr3, Satz 4.5]. Thus Proposition 3.3.9 implies  $\vartheta^{(0)}(R/K) = \vartheta_N(R/K) = \langle 0 \rangle$ . If  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is a 0-dimensional smooth scheme, then  $\vartheta^{(0)}(R/K) = \langle 0 \rangle$  and  $\vartheta^{(i)}(R/K) \neq \langle 0 \rangle$  for all  $i \geq 1$ .

The following lemma provides useful relations between the Kähler differents of  $R/K[x_0]$  and of  $R/K$ .

**Lemma 5.2.1.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme. Then we have*

$$x_0\vartheta^{(1)}(R/K) = \vartheta_K(R/K[x_0])\mathfrak{m} \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta^{(1)}(R/K).$$

*Proof.* By [Ku5, Proposition 3.24], we have an exact sequence of graded  $R$ -modules

$$0 \longrightarrow Rdx_0 \xrightarrow{\alpha} \Omega_{R/K}^1 \xrightarrow{\beta} \Omega_{R/K[x_0]}^1 \longrightarrow 0$$

where  $\alpha$  is the inclusion map and  $\beta$  is given by  $\beta(fd_{R/K}f') = fd_{R/K[x_0]}f'$ . So, it follows from Proposition 2.2.7 that

$$\vartheta_K(R/K[x_0]) = \mathcal{F}_1(Rdx_0) \cdot \vartheta_K(R/K[x_0]) \subseteq \vartheta^{(1)}(R/K).$$

Now we show that  $x_0\vartheta^{(1)}(R/K) = \vartheta_K(R/K[x_0])\mathfrak{m} \subseteq \vartheta_K(R/K[x_0])$ . Let  $\{F_1, \dots, F_r\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{X}}$ . The Jacobian matrix  $\left(\frac{\partial F_j}{\partial x_i}\right)_{\substack{i=0, \dots, n \\ j=1, \dots, r}}$  is a relation matrix of  $\Omega_{R/K}^1$  with respect to  $\{dx_0, \dots, dx_n\}$  (cf. [Ku5, Proposition 4.19]). Thus  $\vartheta^{(i)}(R/K)$  is the ideal of  $R$  generated by all  $(n+1-i)$ -minors of the Jacobian matrix. In particular, the ideal  $\vartheta^{(1)}(R/K)$  is generated by the minors of size  $n$  of that matrix. Let  $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, r\}$ , and let

$$\Delta_{I,j} := \frac{\partial(F_{i_1}, \dots, F_{i_n})}{\partial(x_0, \dots, \hat{x}_j, \dots, x_n)} \quad (j = 0, \dots, n).$$

We see that  $\Delta_{I,0} \in \vartheta_K(R/K[x_0])$  and  $\Delta_{I,j} \in \vartheta^{(1)}(R/K)$  for  $j = 0, \dots, n$ . If  $\Delta_{I,0} = 0$ , then we claim that  $\Delta_{I,j} = 0$  for all  $j \in \{1, \dots, n\}$ . Indeed, suppose that there is an

index  $j \in \{1, \dots, n\}$  such that  $\Delta_{I,j} \neq 0$ . By applying Euler's rule, we find relations

$$\frac{\partial F_{i_k}}{\partial x_0} x_0 + \frac{\partial F_{i_k}}{\partial x_1} x_1 + \dots + \frac{\partial F_{i_k}}{\partial x_n} x_n = 0 \quad \text{for } k = 1, \dots, n.$$

Thus it follows from Cramer's rule that  $\Delta_{I,j} x_0 = (-1)^{n+1-j} \Delta_{I,0} x_j$ . Since  $x_0$  is a non-zerodivisor of  $R$ , we have  $\Delta_{I,j} x_0 \neq 0$ , and so  $\Delta_{I,0} \neq 0$ . Moreover, we have  $\Delta_{I,0} x_j = (-1)^{n+1-j} \Delta_{I,j} x_0$  for all  $j \in \{1, \dots, n\}$ . Furthermore, the ideal  $x_0 \vartheta^{(1)}(R/K)$  (resp.  $\vartheta_K(R/K[x_0])\mathfrak{m}$ ) is generated by elements of the form  $\Delta_{I,j} x_0$  (resp.  $\Delta_{I,0} x_j$ ) for all  $j = 0, \dots, n$  and all subsets  $I = \{i_1, \dots, i_n\} \subseteq \{1, \dots, r\}$ . This implies that  $x_0 \vartheta^{(1)}(R/K) = \vartheta_K(R/K[x_0])\mathfrak{m} \subseteq \vartheta_K(R/K[x_0])$ , and the proof is complete.  $\square$

By applying the lemma, we can give bounds for the Hilbert polynomial and the regularity index of the Kähler different  $\vartheta^{(1)}(R/K)$ , as the following proposition shows.

**Proposition 5.2.2.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme with support  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , and let  $\mathbb{X}_{\text{sm}}$  be the set of smooth points of  $\mathbb{X}$  in  $\text{Supp}(\mathbb{X})$ . Then we have*

$$\sum_{p_j \in \mathbb{X}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j}) \leq \text{HP}_{\vartheta^{(1)}(R/K)}(z) = \text{HP}_{\vartheta_K(R/K[x_0])}(z) \leq \text{deg}(\mathbb{X}) - (s - \#\mathbb{X}_{\text{sm}})$$

and  $\text{ri}(\vartheta_K(R/K[x_0])) - 1 \leq \text{ri}(\vartheta^{(1)}(R/K)) \leq \text{ri}(\vartheta_K(R/K[x_0]))$ .

*Proof.* By Lemma 5.2.1, we have  $x_0 \vartheta^{(1)}(R/K) \subseteq \vartheta_K(R/K[x_0]) \subseteq \vartheta^{(1)}(R/K)$ . This implies the equalities  $\text{HP}_{x_0 \vartheta^{(1)}(R/K)}(z) = \text{HP}_{\vartheta_K(R/K[x_0])}(z) = \text{HP}_{\vartheta^{(1)}(R/K)}(z)$ , since  $x_0$  is a non-zerodivisor of  $R$ . Hence Proposition 3.3.19 yields the above bounds for the Hilbert polynomial of  $\vartheta^{(1)}(R/K)$ . Now we prove the claimed inequalities of regularity indices. Obviously, we have  $\text{ri}(\vartheta^{(1)}(R/K)) \leq \text{ri}(\vartheta_K(R/K[x_0]))$ . It follows from the inclusion  $x_0 \vartheta^{(1)}(R/K) \subseteq \vartheta_K(R/K[x_0])$  that  $\text{HF}_{\vartheta^{(1)}(R/K)}(i) \leq \text{HF}_{\vartheta_K(R/K[x_0])}(i + 1)$  for all  $i \in \mathbb{Z}$ . Therefore we obtain the inequalities  $\text{ri}(\vartheta_K(R/K[x_0])) - 1 \leq \text{ri}(\vartheta^{(1)}(R/K)) \leq \text{ri}(\vartheta_K(R/K[x_0]))$ .  $\square$

If  $\mathbb{X} \subseteq \mathbb{P}_K^n$  is a 0-dimensional smooth scheme, then the Hilbert polynomial of the Kähler different  $\vartheta^{(1)}(R/K)$  is  $\text{HP}_{\vartheta^{(1)}(R/K)}(z) = \text{deg}(\mathbb{X})$  and its regularity index satisfies  $\text{ri}(\vartheta^{(1)}(R/K)) \leq r_{\mathbb{X}}(n + 1)$  (see Corollary 3.3.21). In particular, if  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  is a projective point set, then we have  $\text{HP}_{\vartheta^{(1)}(R/K)}(z) = s$  and  $2r_{\mathbb{X}} - 1 \leq \text{ri}(\vartheta^{(1)}(R/K)) \leq nr_{\mathbb{X}}$ . Furthermore, we have the following corollary.

**Corollary 5.2.3.** *Let  $\mathbb{X}$  be a 0-dimensional reduced complete intersection in  $\mathbb{P}_K^n$ , and let  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_n \rangle$ , where  $F_j \in P$  is a homogeneous polynomial of degree  $d_j$  for*

$j = 1, \dots, n$ . Let  $d := \sum_{j=1}^n d_j - n$ , and let  $\mathbb{Y} \subseteq \mathbb{X}$  be the subscheme defined by  $\mathcal{I}_{\mathbb{Y}} = \bigcap_{p_j \in \text{Supp}(\mathbb{X}): \text{smooth}} \mathfrak{P}_j$ . Then, for all  $i \in \mathbb{Z}$ , we have

$$\text{HF}_{\vartheta^{(1)}(R/K)}(i) = \begin{cases} 0 & \text{if } i < d, \\ \text{HF}_{\mathbb{Y}}(i + 1 - d) & \text{if } i \geq d. \end{cases}$$

In particular, we have  $\text{ri}(\vartheta^{(1)}(R/K)) = \text{ri}(\vartheta_K(R/K[x_0])) - 1 = d + r_{\mathbb{Y}} - 1$ .

*Proof.* We set  $\Delta_0 = \frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}$ . Then  $\vartheta_K(R/K[x_0]) = \langle \Delta_0 \rangle_R$  by Corollary 3.3.5. It follows from Lemma 5.2.1 that  $x_0 \vartheta^{(1)}(R/K) = \Delta_0 \mathfrak{m}$ . Since  $x_0$  is a non-zerodivisor of  $R$ , the Hilbert function of  $\vartheta^{(1)}(R/K)$  satisfies  $\text{HF}_{\vartheta^{(1)}(R/K)}(i) = \text{HF}_{\Delta_0 \mathfrak{m}}(i + 1)$  for all  $i \in \mathbb{Z}$ . If  $i < d$ , then  $0 \leq \text{HF}_{\vartheta^{(1)}(R/K)}(i) \leq \text{HF}_{\mathfrak{m}}(i + 1 - d) = 0$ , and so  $\text{HF}_{\vartheta^{(1)}(R/K)}(i) = 0$ . For  $i \geq d$ , we see that  $\text{HF}_{\vartheta^{(1)}(R/K)}(i) = \text{HF}_{\Delta_0 \mathfrak{m}}(i + 1) = \text{HF}_{\vartheta_K(R/K[x_0])}(i + 1)$ . Furthermore, Corollary 3.3.7 yields that  $\text{HF}_{\vartheta_K(R/K[x_0])}(i) = \text{HF}_{\mathbb{Y}}(i - d)$  for all  $i \in \mathbb{Z}$ . This implies  $\text{HF}_{\vartheta^{(1)}(R/K)}(i) = \text{HF}_{\mathbb{Y}}(i + 1 - d)$  for all  $i \geq d$ .  $\square$

**Corollary 5.2.4.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme which is a complete intersection. Then we have*

$$\text{HF}_{\vartheta^{(1)}(R/K)}(i) = \begin{cases} 0 & \text{if } i < r_{\mathbb{X}}, \\ \text{HF}_{\mathbb{X}}(i + 1 - r_{\mathbb{X}}) & \text{if } i \geq r_{\mathbb{X}}. \end{cases}$$

In particular, we have  $\text{ri}(\vartheta^{(1)}(R/K)) = \text{ri}(\vartheta_K(R/K[x_0])) - 1 = 2r_{\mathbb{X}} - 1$ .

*Proof.* This is an immediate consequence of Corollary 5.2.3.  $\square$

Now we present a characterization for a smooth scheme  $\mathbb{X}$  being a complete intersection in terms of the Kähler different  $\vartheta^{(1)}(R/K)$  as follows.

**Proposition 5.2.5.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. Then the following conditions are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is a complete intersection.*
- (ii) *The scheme  $\mathbb{X}$  is a CB-scheme and  $\text{HF}_{\vartheta^{(1)}(R/K)}(r_{\mathbb{X}}) \neq 0$ .*
- (iii)  *$x_0 \vartheta^{(1)}(R/K) \cdot \mathfrak{C}_{R/K[x_0]} = \mathfrak{m}$ .*

*Proof.* (i)  $\Leftrightarrow$  (ii): According to Lemma 5.2.1, we infer that  $\text{HF}_{\vartheta^{(1)}(R/K)}(r_{\mathbb{X}}) \neq 0$  if and only if  $\text{HF}_{\vartheta_K(R/K[x_0])}(r_{\mathbb{X}}) \neq 0$ . Hence the claim follows from Proposition 4.1.15(ii).

(i) $\Rightarrow$ (iii): If  $\mathbb{X}$  is a complete intersection, then we set  $\Delta_j := \frac{\partial(F_1, \dots, F_n)}{\partial(x_0, \dots, \widehat{x_j}, \dots, x_n)}$  for  $j = 0, \dots, n$ , where  $\{F_1, \dots, F_n\}$  is a minimal system of generators of  $\mathcal{I}_{\mathbb{X}}$ . By Corollary 3.3.5, we get  $\vartheta_K(R/K[x_0]) = \langle \Delta_0 \rangle_R$ ,  $\mathfrak{C}_{R/K[x_0]} = \langle \Delta_0^{-1} \rangle_R$ , and  $\Delta_0$  is nonzerodivisor of  $R$ . We also have  $x_0\vartheta^{(1)}(R/K) = \Delta_0\mathfrak{m}$  by Lemma 5.2.1. Then multiplying by  $\mathfrak{C}_{R/K[x_0]}$ , we obtain  $x_0\vartheta^{(1)}(R/K) \cdot \mathfrak{C}_{R/K[x_0]} = \mathfrak{m}$ .

(iii) $\Rightarrow$ (i): Suppose that  $x_0\vartheta^{(1)}(R/K) \cdot \mathfrak{C}_{R/K[x_0]} = \mathfrak{m}$ . It follows from the equality  $x_0\vartheta^{(1)}(R/K) = \vartheta_K(R/K[x_0])\mathfrak{m}$  that  $\vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]}\mathfrak{m} = \mathfrak{m}$ . Since  $\vartheta_K(R/K[x_0])$  is a subideal of  $\vartheta_{\mathbb{X}}(R/K[x_0])$ , this implies that

$$\vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]} \subseteq \vartheta_{\mathbb{X}}(R/K[x_0])\mathfrak{C}_{R/K[x_0]} \subseteq R.$$

If  $\vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]} \subsetneq R$ , then  $\vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]}$  is a homogeneous ideal of  $R$  contained in  $\mathfrak{m}$  and

$$\langle 0 \rangle \subseteq \mathfrak{m} = \vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]}\mathfrak{m} \subseteq \langle 0 \rangle + \mathfrak{m}^2.$$

By Nakayama’s lemma (cf. [KR2, Proposition 1.7.15]), we have  $\mathfrak{m} = \langle 0 \rangle$ , which is impossible. Thus we must have  $\vartheta_K(R/K[x_0])\mathfrak{C}_{R/K[x_0]} = R$ . Consequently,  $\mathfrak{C}_{R/K[x_0]}$  is invertible and  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ . So, the scheme  $\mathbb{X}$  is arithmetically Gorenstein and  $\vartheta_K(R/K[x_0]) = \vartheta_{\mathbb{X}}(R/K[x_0])$ . Therefore Proposition 3.3.12 yields that  $\mathbb{X}$  is a complete intersection. □

As usual, for a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_K^n$  with  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ , we let  $\tilde{R} = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]$  be the subring of  $L := Q^h(R) = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]$ . Our next proposition gives us a smoothness criterion for  $\mathbb{X}$  in terms of the module of Kähler differentials  $\Omega_{\tilde{R}/K}^1$  of the  $K$ -algebra  $\tilde{R}$ .

**Proposition 5.2.6.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $\text{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ . The following assertions are equivalent:*

- (i) *The scheme  $\mathbb{X}$  is smooth.*
- (ii)  $\Omega_{\tilde{R}/K}^1 = 0$ .
- (iii)  $\Omega_{\tilde{R}/K}^1 = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]dT_j$ .

*Proof.* “(i) $\Leftrightarrow$ (ii)” follows from [Ku6, Corollary 4.12]. It remains to show “(i) $\Leftrightarrow$ (iii)”. On account of [Ku5, Corollary 4.8], we have  $\Omega_{\tilde{R}/K}^1 = \prod_{j=1}^s \Omega_{\mathcal{O}_{\mathbb{X}, p_j}[T_j]/K}^1$ . Also, [Ku5, Formula 4.4.b] implies

$$\begin{aligned} \Omega_{\mathcal{O}_{\mathbb{X}, p_j}[T_j]/K}^1 &= \Omega_{\mathcal{O}_{\mathbb{X}, p_j} \otimes_K K[T_j]/K}^1 = \mathcal{O}_{\mathbb{X}, p_j} \otimes_K \Omega_{K[T_j]/K}^1 \oplus K[T_j] \otimes_K \Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1 \\ &= \mathcal{O}_{\mathbb{X}, p_j}[T_j]dT_j \oplus K[T_j] \otimes_K \Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1. \end{aligned}$$

It follows that  $\Omega_{\mathcal{O}_{\mathbb{X}, p_j}[T_j]/K}^1 = \mathcal{O}_{\mathbb{X}, p_j}[T_j]dT_j$  if and only if  $\Omega_{\mathcal{O}_{\mathbb{X}, p_j}/K}^1 = 0$ . This is equivalent to the condition that  $p_j$  is a smooth point of  $\mathbb{X}$  (see Theorem 3.2.15). Therefore  $\Omega_{\tilde{R}/K}^1 = \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]dT_j$  if and only if  $\mathbb{X}$  is smooth.  $\square$

In the remainder of this section, we assume that the scheme  $\mathbb{X}$  is smooth. We consider the canonical inclusion  $\tilde{\iota} : R \hookrightarrow \tilde{R}$  given by  $\tilde{\iota}(f) = (f_{p_1}T_1^i, \dots, f_{p_s}T_s^i)$  for  $f \in R_i$  with  $i \geq 0$ . By Proposition 3.23 in [Ku5], we have an  $R$ -linear map

$$\begin{aligned} \Psi : \Omega_{R/K}^1 &\rightarrow \Omega_{\tilde{R}/K}^1 \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j]dT_j \hookrightarrow \Omega_{L/K}^1 \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X}, p_j}[T_j, T_j^{-1}]dT_j, \\ f dx_i &\mapsto (f_{p_1}(x_i)_{p_1}T_1^k dT_1, \dots, f_{p_s}(x_i)_{p_s}T_s^k dT_s) \end{aligned}$$

for  $f \in R_k$ . The map  $\Psi$  is homogeneous of degree zero and satisfies

$$(\Psi \circ d)(f) = (d_{\tilde{R}} \circ \tilde{\iota})(f) \quad \text{for all } f \in R$$

and  $\Psi(dx_0) = (dT_1, \dots, dT_s)$ .

**Corollary 5.2.7.** *We have  $\text{Ker}(\Psi) = \{ \sum_{i=0}^n g_i dx_i \in \Omega_{R/K}^1 \mid \sum_{i=0}^n g_i x_i = 0 \}$ .*

*Proof.* Clearly, we have  $\{ \sum_{i=0}^n g_i dx_i \in \Omega_{R/K}^1 \mid \sum_{i=0}^n g_i x_i = 0 \} \subseteq \text{Ker}(\Psi)$ . Now let  $\omega = \sum_{i=0}^n g_i dx_i \in (\Omega_{R/K}^1)_k$  be a homogeneous element of degree  $k$  such that  $\Psi(\omega) = 0$ . Then we find  $0 = \Psi(\omega) = (\sum_{i=0}^n (g_i)_{p_1}(x_i)_{p_1}T_1^k dT_1, \dots, \sum_{i=0}^n (g_i)_{p_s}(x_i)_{p_s}T_s^k dT_s)$ . This implies  $\sum_{i=0}^n (g_i)_{p_j}(x_i)_{p_j} = \sum_{i=0}^n (g_i x_i)_{p_j} = 0$  for  $j = 1, \dots, s$ . Since the map  $\tilde{\iota}$  is injective, we get  $\sum_{i=0}^n g_i x_i = 0$ , and hence the claim follows.  $\square$

Given  $j \in \{1, \dots, s\}$ , we let  $\nu_j = \dim_K \mathcal{O}_{\mathbb{X}, p_j}$ , let  $\{e_{j1}, \dots, e_{j\nu_j}\}$  be a  $K$ -basis of  $\mathcal{O}_{\mathbb{X}, p_j}$ , and set  $\epsilon_{jk_j} = (0, \dots, 0, e_{jk_j}, 0, \dots, 0)$  for  $k_j = 1, \dots, \nu_j$ . We put  $L_0 := K[x_0, x_0^{-1}]$ . Then the set  $\{\epsilon_{11}, \dots, \epsilon_{1\nu_1}, \dots, \epsilon_{s1}, \dots, \epsilon_{s\nu_s}\}$  is a basis of the graded-free algebra  $L/L_0$ . Moreover,  $L/L_0$  is étale (see Proposition 3.2.16). Thus the canonical trace map  $\text{Tr}_{L/L_0}$  of  $L/L_0$  induces canonically a trace map  $\text{Tr}_{L/L_0}^1 : \Omega_{L/K}^1 \rightarrow \Omega_{L_0/K}^1$  given by  $\text{Tr}_{L/L_0}^1(\epsilon_{jk_j} \cdot (dT_1, \dots, dT_s)) = \text{Tr}_{L/L_0}(\epsilon_{jk_j})dx_0$  for  $j = 1, \dots, s$ ;  $k_j = 1, \dots, \nu_j$ . We set

$$\Omega_{\mathbb{X}} := \{ \omega \in \Omega_{L/K}^1 \mid \text{Tr}_{L/L_0}^1(\omega R) \subseteq \Omega_{K[x_0]/K}^1 \}.$$

Then it is easy to check that  $\Psi : \Omega_{R/K}^1 \rightarrow \Omega_{\mathbb{X}} \hookrightarrow \Omega_{L/K}^1$  and  $\Omega_{\mathbb{X}} = \mathfrak{C}_{R/K[x_0]}\Psi(dx_0)$ . Moreover, we have the following property.

**Proposition 5.2.8.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. Then the scheme  $\mathbb{X}$  is a complete intersection if and only if  $\Psi(\Omega_{R/K}^1) = \vartheta^{(1)}(R/K)\Omega_{\mathbb{X}}$ .*

*Proof.* Assume that  $\mathbb{X}$  is a complete intersection with  $\mathcal{I}_{\mathbb{X}} = \langle F_1, \dots, F_n \rangle$ . We set  $\Delta_j := \frac{\partial(F_1, \dots, F_n)}{\partial(x_0, \dots, \widehat{x_j}, \dots, x_n)}$  for  $j = 0, \dots, n$ . In  $\Omega_{R/K}^1$ , there are relations

$$\frac{\partial F_i}{\partial x_0} dx_0 + \frac{\partial F_i}{\partial x_1} dx_1 + \dots + \frac{\partial F_i}{\partial x_n} dx_n = 0 \quad \text{for } i = 1, \dots, n.$$

Due to Cramer's rule, for every  $j \in \{1, \dots, n\}$  we have  $\Delta_0 dx_j = (-1)^{n+1-j} \Delta_j dx_0$ . This implies that  $\Psi(dx_j) = (-1)^{n+1-j} \frac{\Delta_j}{\Delta_0} \Psi(dx_0)$  for all  $j = 1, \dots, n$ , since  $\Psi$  is  $R$ -linear. Therefore we deduce from  $\Omega_{R/K}^1 = Rdx_0 + \dots + Rdx_n$  and  $\mathfrak{C}_{R/K[x_0]} = \langle \Delta_0^{-1} \rangle_R$  that

$$\begin{aligned} \Psi(\Omega_{R/K}^1) &= \left\langle \frac{\Delta_0}{\Delta_0}, \frac{\Delta_1}{\Delta_0}, \dots, \frac{\Delta_n}{\Delta_0} \right\rangle_R \Psi(dx_0) \\ &= \vartheta^{(1)}(R/K) \langle \Delta_0^{-1} \rangle_R \Psi(dx_0) = \vartheta^{(1)}(R/K) \Omega_{\mathbb{X}}. \end{aligned}$$

Conversely, we consider the epimorphism of  $R$ -graded modules  $\varphi : \Omega_{R/K}^1 \rightarrow \mathfrak{m}$  given by  $dx_i \mapsto x_i$  for  $i = 0, \dots, n$ . By Corollary 5.2.7, we have  $\text{Ker}(\Psi) = \text{Ker}(\varphi)$  and the following diagram of graded  $R$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(\varphi) & \hookrightarrow & \Omega_{R/K}^1 & \xrightarrow{\varphi} & \mathfrak{m} & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \downarrow \gamma & & \\ 0 & \longrightarrow & \text{Ker}(\Psi) & \hookrightarrow & \Omega_{R/K}^1 & \xrightarrow{\Psi} & \Psi(\Omega_{R/K}^1) & \longrightarrow & 0 \end{array}$$

where  $\gamma$  is given by  $\gamma(f) = f \cdot \frac{\Psi(dx_0)}{x_0} \in \Psi(\Omega_{R/K}^1) \subseteq L \cdot \Psi(dx_0)$  for all  $f \in \mathfrak{m}$ . The map  $\gamma$  is well-defined, since in  $L$  we have  $x_i = ((x_i)_{p_1}, \dots, (x_i)_{p_s}) \cdot x_0$  for all  $i = 1, \dots, n$ . Moreover, for  $i \in \{0, \dots, n\}$  we see that  $\gamma \circ \varphi(dx_i) = \gamma(x_i) = ((x_i)_{p_1}, \dots, (x_i)_{p_s}) \cdot x_0 \frac{\Psi(dx_0)}{x_0} = ((x_i)_{p_1}, \dots, (x_i)_{p_s}) \cdot \Psi(dx_0) = \Psi(dx_i)$ . So, the diagram is commutative. It is not difficult to verify that  $\gamma$  is a homogeneous isomorphism of degree zero. Thus we have  $\Psi(\Omega_{R/K}^1) = \mathfrak{m} \cdot \frac{\Psi(dx_0)}{x_0} \cong \mathfrak{m}$ . It follows from  $\Psi(\Omega_{R/K}^1) = \vartheta^{(1)}(R/K) \Omega_{\mathbb{X}}$  that  $\mathfrak{m} \cdot \Psi(dx_0) = x_0 \vartheta^{(1)}(R/K) \mathfrak{C}_{R/K[x_0]} \cdot \Psi(dx_0)$ . Since  $\text{Ann}_R(\Psi(dx_0)) = \langle 0 \rangle$ , we obtain  $\mathfrak{m} = x_0 \vartheta^{(1)}(R/K) \mathfrak{C}_{R/K[x_0]}$ . Therefore the conclusion follows from Proposition 5.2.5.  $\square$

**Remark 5.2.9.** Let us make some following observations.

- (a) We have  $\Psi(\Omega_{R/K}^1) = \mathfrak{m} \cdot \frac{\Psi(dx_0)}{x_0} \cong \mathfrak{m}$ .
- (b) There is an exact sequence of graded  $R$ -modules

$$0 \rightarrow \text{Ker}(\Psi) \rightarrow \Omega_{R/K}^1 \rightarrow \Omega_{\mathbb{X}} \rightarrow \Omega_{\mathbb{X}} / \Psi(\Omega_{R/K}^1) \rightarrow 0.$$

We set  $J_{\mathbb{X}} = \Omega_{\mathbb{X}}/\Psi(\Omega_{R/K}^1)$ . Then we have  $\mathrm{HF}_{J_{\mathbb{X}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i) - \mathrm{HF}_{\mathfrak{m}}(i)$  for all  $i \in \mathbb{Z}$  and

$$\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1) \deg(\mathbb{X}) + 1 + \sum_{i=0}^{r_{\mathbb{X}}-1} (\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)).$$

**Corollary 5.2.10.** *Let  $K$  be an infinite field, and let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional smooth scheme. The following statements are equivalent.*

- (i) *The scheme  $\mathbb{X}$  is arithmetically Gorenstein.*
- (ii) *The scheme  $\mathbb{X}$  is a CB-scheme and  $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1) \deg(\mathbb{X}) + 1$ .*

*Proof.* If  $\mathbb{X}$  is an arithmetically Gorenstein scheme, then  $\mathbb{X}$  is a CB-scheme and we have  $\mathrm{HF}_{\mathbb{X}}(i) + \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1) = \deg(\mathbb{X})$  for all  $i = 0, \dots, r_{\mathbb{X}} - 1$  (see Corollary 4.1.16). Thus Remark 5.2.9(b) implies that  $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1) \deg(\mathbb{X}) + 1$ .

Conversely, we assume that  $\mathbb{X}$  is a CB-scheme and  $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1) \deg(\mathbb{X}) + 1$ . By Theorem 4.1.7, we find an element  $g \in (\mathfrak{C}_{R/K[x_0]})_{-r_{\mathbb{X}}}$  with  $\mathrm{Ann}_R(g) = \langle 0 \rangle$ . We see that  $R(-r_{\mathbb{X}}) \cong R \cdot g \subseteq \mathfrak{C}_{R/K[x_0]}$ . This implies

$$\mathrm{HF}_{\mathbb{X}}(i) \leq \mathrm{HF}_{\mathfrak{C}_{R/K[x_0]}}(i - r_{\mathbb{X}}) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)$$

for all  $i \in \mathbb{Z}$ . Since  $\dim_K(J_{\mathbb{X}}) = (r_{\mathbb{X}} - 1) \deg(\mathbb{X}) + 1$ , we deduce  $\sum_{i=0}^{r_{\mathbb{X}}-1} (\deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)) = 0$ . Hence we must have  $\mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}} - i - 1)$  for all  $i \in \mathbb{Z}$ . Therefore Corollary 4.1.16 yields that  $\mathbb{X}$  is arithmetically Gorenstein.  $\square$

**Corollary 5.2.11.** *If  $\mathbb{X}$  is arithmetically Gorenstein with  $\deg(\mathbb{X}) \neq \mathrm{char}(K)$ , then  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle h \rangle \not\subseteq \langle x_0 \rangle$ . In particular, we have  $\mathfrak{m}^{-1} = \langle 1, h/x_0 \rangle_R$ .*

*Proof.* Proposition 3.2.8 yields that  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle h \rangle$  and that  $\mathfrak{C}_{R/K[x_0]} = \langle h^{-1} \rangle_R$ , where  $h \in R_{r_{\mathbb{X}}}$  is a non-zero-divisor. Let  $\mathcal{B} = \{t_1, \dots, t_{\deg(\mathbb{X})}\}$  be a  $K[x_0]$ -basis of  $R$ , let  $\sigma$  be a trace map of  $R/K[x_0]$ , and let  $\{t'_1, \dots, t'_{\deg(\mathbb{X})}\}$  be the dual  $K[x_0]$ -basis of  $R$  to  $\mathcal{B}$  w.r.t.  $\sigma$ . We may assume  $t_1 = 1$ . Then  $\mathrm{Tr}_{R/K[x_0]}(t_1) = \deg(\mathbb{X}) \neq \mathrm{char}(K)$ . Also, Corollary 2.4.12 yields

$$h = \sum_{j=1}^{\deg(\mathbb{X})} \mathrm{Tr}_{R/K[x_0]}(t_j) t'_j = \deg(\mathbb{X}) t'_1 + \sum_{j=2}^{\deg(\mathbb{X})} \mathrm{Tr}_{R/K[x_0]}(t_j) t'_j.$$

Thus  $\vartheta_{\mathbb{X}}(R/K[x_0]) = \langle h \rangle \not\subseteq \langle x_0 \rangle$ . Moreover, we have  $x_0^{-1} \mathfrak{m} \Psi(dx_0) = \Psi(\Omega_{R/K}^1) \subseteq \Omega_{\mathbb{X}} = \langle h^{-1} \rangle_R \Psi(dx_0)$ . This implies  $\mathfrak{m} \vartheta_{\mathbb{X}}(R/K[x_0]) \subseteq \langle x_0 \rangle_R$ , and so  $\vartheta_{\mathbb{X}}(R/K[x_0]) \subseteq \langle x_0 \rangle_R : \mathfrak{m}$ . By [Kr3, Satz 4.21], there is a surjective multiplication map  $\mu_{1/x_0} : \langle x_0 \rangle_R : \mathfrak{m} \rightarrow \mathfrak{m}^{-1}$ . Since  $x_0 \nmid h$  and  $\ell(\mathfrak{m}^{-1}/R) = 1$ , we get  $\mathfrak{m}^{-1} = \langle 1, h/x_0 \rangle_R$ , as desired.  $\square$



### 5.3 Differents for Fat Point Schemes

In this section we work over an arbitrary field  $K$ . Let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set of  $\mathbb{P}_K^n$ . For  $j = 1, \dots, s$ , we let  $\mathfrak{P}_j$  be the associated homogeneous prime ideal of  $p_j$  in  $P = K[X_0, \dots, X_n]$ .

**Definition 5.3.1.** Given a sequence of positive integers  $m_1, \dots, m_s$ , the intersection  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1^{m_1} \cap \dots \cap \mathfrak{P}_s^{m_s}$  is a saturated homogeneous ideal in  $P$  and is therefore the vanishing ideal of a zero-dimensional subscheme  $\mathbb{W}$  of  $\mathbb{P}_K^n$ .

- (i) The scheme  $\mathbb{W}$  is called a **fat point scheme** in  $\mathbb{P}_K^n$  and is denoted by  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$ .
- (ii) If  $m_1 = \dots = m_s = m$ , we denote  $\mathbb{W}$  also by  $m\mathbb{X}$  and call it an **equimultiple fat point scheme**.

**Remark 5.3.2.** The degree of the fat point scheme  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  is given by the formula  $\deg(\mathbb{W}) = \sum_{j=1}^s \binom{m_j+n-1}{n}$ . In particular, if  $\mathbb{W}$  is an equimultiple fat point scheme with  $m_1 = \dots = m_s = m$ , then  $\deg(\mathbb{W}) = s \binom{m+n-1}{n}$ . Let  $r_{\mathbb{W}}$  be the regularity index the Hilbert function of  $\mathbb{W}$ . Then we have  $\text{HF}_{\mathbb{W}}(i) = 0$  for  $i < 0$ ,  $1 = \text{HF}_{\mathbb{W}}(0) < \text{HF}_{\mathbb{W}}(1) < \dots < \text{HF}_{\mathbb{W}}(r_{\mathbb{W}} - 1) < \deg(\mathbb{W})$ , and  $\text{HF}_{\mathbb{W}}(i) = \deg(\mathbb{W}) = \sum_{j=1}^s \binom{m_j+n-1}{n}$  for  $i \geq r_{\mathbb{W}}$ .

Notice that the support of  $\mathbb{W}$  is  $\mathbb{X} = \{p_1, \dots, p_s\}$ . In what follows, we always assume that no point of  $\mathbb{X}$  lies on the hyperplane  $\mathcal{Z}^+(X_0)$ . The homogeneous coordinate ring of  $\mathbb{W}$  is  $S = P/\mathcal{I}_{\mathbb{W}}$ . The image of  $X_i$  in  $S$  is denoted by  $x_i$  for  $i = 0, \dots, n$ . Then  $x_0$  is a non-zerodivisor of  $S$ , and the graded algebra  $S/K[x_0]$  is free of rank  $\deg(\mathbb{W}) = \sum_{j=1}^s \binom{m_j+n-1}{n}$ .

Now we look at the Hilbert functions and give upper bounds for the regularity indices of the Kähler differentials  $\vartheta_K(S/K[x_0])$  and  $\vartheta^{(i)}(S/K)$ . Our first case is  $n = 1$ , i.e., fat point schemes in the projective line  $\mathbb{P}_K^1$ .

**Lemma 5.3.3.** *Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^1$  with  $m_j \geq 1$  for  $j = 1, \dots, s$ , let  $\mu = m_1 + \dots + m_s$ , and suppose  $\text{char}(K) = 0$  or  $\text{char}(K) > \mu$ .*

- (i) *The ideal  $\vartheta_K(S/K[x_0])$  is a principal homogeneous ideal of  $S$  and its Hilbert function is of the form*

$$\text{HF}_{\vartheta_K(S/K[x_0])} : \begin{matrix} 0 & \cdots & 0 & 1 & 2 & 3 & \cdots & s-1 & s & s & \cdots \end{matrix}$$

$[\mu-1]$ 
 $[\mu+s-2]$

*In particular, its regularity index satisfies  $\text{ri}(\vartheta_K(S/K[x_0])) = \mu + s - 2$ .*

(ii) The ideal  $\vartheta^{(1)}(S/K)$  is non-zero and its Hilbert function is of the form

$$\text{HF}_{\vartheta^{(1)}(S/K)} : 0 \cdots 0 \underset{[\mu-1]}{2} \ 3 \ 4 \cdots s-1 \underset{[\mu+s-3]}{s} \ s \cdots .$$

In particular, its regularity index satisfies  $\text{ri}(\vartheta^{(1)}(S/K)) = \mu + s - 3$ .

*Proof.* Let us write  $p_j = (1 : p_{j1})$  and  $\mathfrak{P}_j = \langle X_1 - p_{j1}X_0 \rangle$  for all  $j = 1, \dots, s$  (note that  $p_{j1} \neq p_{k1}$  if  $k \neq j$ ). Then the homogeneous vanishing ideal of the support of  $\mathbb{W}$  is generated by  $F = (X_1 - p_{11}X_0) \cdots (X_1 - p_{s1}X_0)$  and we have  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1^{m_1} \cap \cdots \cap \mathfrak{P}_s^{m_s} = \langle G \rangle$  with  $G = (X_1 - p_{11}X_0)^{m_1} \cdots (X_1 - p_{s1}X_0)^{m_s}$ . Obviously, the Hilbert function of  $\mathbb{W}$  is of the form  $\text{HF}_{\mathbb{W}} : 1 \ 2 \ 3 \cdots \mu - 1 \ \mu \cdots$ . We have

$$\frac{\partial G}{\partial X_0} = \frac{\partial F}{\partial X_0} \cdot \prod_{j=1}^s (X_1 - p_{j1}X_0)^{m_j-1}, \quad \frac{\partial G}{\partial X_1} = \frac{\partial F}{\partial X_1} \cdot \prod_{j=1}^s (X_1 - p_{j1}X_0)^{m_j-1}.$$

Thus the Kähler different of  $\mathbb{W}$  is given by

$$\vartheta_K(S/K[x_0]) = \langle \frac{\partial G}{\partial x_1} \rangle = \langle \frac{\partial F}{\partial x_1} \cdot \prod_{j=1}^s (x_1 - p_{j1}x_0)^{m_j-1} \rangle.$$

We put  $\tilde{G} := \prod_{j=1}^s (X_1 - p_{j1}X_0)^{m_j-1}$  and  $J := \langle G, \tilde{G} \frac{\partial F}{\partial X_1} \rangle$ . In order to calculate the Hilbert function of  $\vartheta_K(S/K[x_0])$ , it is enough to calculate  $\text{HF}_{P/J}$  and then apply the equality  $\text{HF}_{\vartheta_K(S/K[x_0])}(i) = \text{HF}_{\mathbb{W}}(i) - \text{HF}_{P/J}(i)$  for all  $i \geq 0$ . Since  $\text{char}(K) = 0$  or  $\text{char}(K) > \mu$ , Euler's relation implies  $\mu G = \tilde{G}(X_0 \frac{\partial F}{\partial X_0} + X_1 \frac{\partial F}{\partial X_1})$ , and so  $\tilde{G}X_0 \frac{\partial F}{\partial X_0} \in J$ . Hence we may write  $J = \langle G, \tilde{G} \frac{\partial F}{\partial X_1} \rangle = I \cdot \langle X_0 \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1} \rangle$ , where  $I$  is the principal ideal of  $P$  generated by  $\tilde{G}$ . Moreover, it is not hard to show that  $\{X_0 \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1}\}$  is a  $P$ -regular sequence. Consequently, this sequence is also an  $I$ -regular sequence. (Here  $I$  is regarded as a graded  $P$ -module). Therefore we get  $\text{HF}_{I/\langle G_1, G_2 \rangle I}(i) = \text{HF}_I(i) - \text{HF}_I(i - s) - \text{HF}_I(i - s + 1) + \text{HF}_I(i - 2s + 1)$  and

$$\begin{aligned} \text{HF}_{P/J}(i) &= \text{HF}_P(i) - \text{HF}_J(i) = \text{HF}_P(i) - (\text{HF}_I(i) - \text{HF}_{I/\langle G_1, G_2 \rangle I}(i)) \\ &= \text{HF}_P(i) - \text{HF}_I(i - s) - \text{HF}_I(i - s + 1) + \text{HF}_I(i - 2s + 1). \end{aligned}$$

On the other hand, we have  $\text{HF}_I(i) = (i + 1) - \text{HF}_{P/I}(i)$  and  $\text{HF}_{P/I} : 1 \ 2 \ 3 \cdots \mu - s \cdots$ . Thus we obtain

$$\text{HF}_{P/J} : 1 \ 2 \ \cdots \ \mu - 2 \ \mu - 1 \ \underset{[\mu-1]}{\mu - 1} \ \mu - 2 \ \mu - 3 \cdots \ \mu - (s - 1) \ \underset{[\mu+s-2]}{\mu - s} \ \mu - s \cdots$$

and claim (i) follows. Claim (ii) follows by the same way as the proof of (i). □

Next we consider the case  $n \geq 2$ . In this case, the Kähler different of a fat point scheme  $\mathbb{W}$  is zero if the multiplicity of each point is greater than or equal to 2, as the following lemma shows.

**Lemma 5.3.4.** *Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}^n$ , where  $n \geq 2$ , and let  $s'$  be the number of indices  $j \in \{1, \dots, s\}$  such that  $m_j = 1$ .*

(i) *If  $s' = 0$ , i.e.,  $m_j \geq 2$  for  $j = 1, \dots, s$ , then  $\vartheta_K(S/K[x_0]) = \langle 0 \rangle$ .*

(ii) *If  $s' > 0$  then  $\vartheta_K(S/K[x_0]) \neq \langle 0 \rangle$ .*

*Proof.* (i) For  $j = 1, \dots, s$ , we write  $P_j = (1 : p_{j1} : \dots : p_{jn})$  with  $p_{jk} \in K$ . Then we have  $\mathfrak{P}_j = \langle L_{j1}, \dots, L_{jn} \rangle$ , where  $L_{j1} = X_1 - p_{j1}X_0, \dots, L_{jn} = X_n - p_{jn}X_0$  are linearly independent linear forms, and  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1^{m_1} \cap \dots \cap \mathfrak{P}_s^{m_s}$ . Let  $\{G_1, \dots, G_r\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{W}}$ .

For  $i \in \{1, \dots, r\}$  and  $j \in \{1, \dots, s\}$ , we have  $G_i \in \mathfrak{P}_j^{m_j} = \langle L_{j1}, \dots, L_{jn} \rangle^{m_j} = \langle H_1, \dots, H_t \rangle$  with  $t = \binom{n+m_j-1}{n-1}$ . There are polynomials  $G_{i1}, \dots, G_{it} \in P$  such that  $G_i = G_{i1}H_1 + \dots + G_{it}H_t$ . Observe that we have  $\partial H_k / \partial X_l \in \mathfrak{P}_j^{m_j-1}$  for  $k = 1, \dots, t$  and  $l = 1, \dots, n$ , since  $m_j \geq 2$ . This yields  $\partial G_i / \partial X_l \in \mathfrak{P}_j^{m_j-1}$ . Thus it follows from  $n \geq 2$  that, for every  $I = \{i_1, \dots, i_n\} \subset \{1, \dots, r\}$  and every  $j \in \{1, \dots, s\}$ , we have

$$\frac{\partial(G_{i_1}, \dots, G_{i_n})}{\partial(X_1, \dots, X_n)} = \det \begin{pmatrix} \frac{\partial G_{i_1}}{\partial X_1} & \dots & \frac{\partial G_{i_1}}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_{i_n}}{\partial X_1} & \dots & \frac{\partial G_{i_n}}{\partial X_n} \end{pmatrix} \in \mathfrak{P}_j^{m_j}.$$

Now the fact that  $\vartheta_K(S/K[x_0])$  is generated by the images in  $S$  of the elements of the form  $\frac{\partial(G_{i_1}, \dots, G_{i_n})}{\partial(X_1, \dots, X_n)}$  implies the claim  $\vartheta_K(S/K[x_0]) = \langle 0 \rangle$ .

(ii) Let  $u = s - s'$ , and let  $j_1, \dots, j_u \in \{1, \dots, s\}$  be the indices such that  $m_{j_k} > 1$  for  $k \in \{1, \dots, u\}$ . If  $u = 0$  then  $\mathbb{W}$  is a projective point set in  $\mathbb{P}_K^n$  and it is clearly true that  $\vartheta_K(S/K[x_0]) \neq \langle 0 \rangle$ .

Now we assume that  $0 < u < s$ . We write  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1^{m_1} \cap \dots \cap \mathfrak{P}_s^{m_s} = I \cap I'$ , where  $I = \mathfrak{P}_{j_1}^{m_{j_1}} \cap \dots \cap \mathfrak{P}_{j_u}^{m_{j_u}}$  and  $I' = \bigcap_{j \neq j_1, \dots, j_u} \mathfrak{P}_j$ . By  $\mathbb{Y}$  we denote the subscheme of  $\mathbb{W}$  defined by  $I'$ . Let  $R_{\mathbb{Y}}$  be the homogeneous coordinate ring of  $\mathbb{Y}$ . Then  $\vartheta_K(R_{\mathbb{Y}}/K[x_0]) \neq \langle 0 \rangle$  shows that there are homogeneous polynomials  $F_1, \dots, F_n \in I'$  such that  $\frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} \notin I'$ . Thus there is an index  $j_0 \in \{1, \dots, s\} \setminus \{j_1, \dots, j_u\}$  such that  $\frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} \notin \mathfrak{P}_{j_0}$ . Also, the Prime Avoidance Theorem (cf. [KR3, Proposition 5.6.22]) yields  $I \not\subseteq \mathfrak{P}_{j_0}$ , and consequently there exists a homogeneous polynomial  $F_0 \in I \setminus \mathfrak{P}_{j_0}$ . This implies that  $F_0 F_i \in \mathcal{I}_{\mathbb{W}}$  for  $i = 1, \dots, n$ .

In the following we show  $\frac{\partial(F_0F_1, \dots, F_0F_n)}{\partial(X_1, \dots, X_n)} \notin \mathfrak{P}_{j_0}$ . Then we get  $\frac{\partial(F_0F_1, \dots, F_0F_n)}{\partial(X_1, \dots, X_n)} \notin \mathcal{I}_{\mathbb{W}}$ , and thus  $\vartheta_K(R_{\mathbb{W}}/K[x_0]) \neq \langle 0 \rangle$ , as claimed. We have

$$\begin{aligned} \frac{\partial(F_0F_1, \dots, F_0F_n)}{\partial(X_1, \dots, X_n)} &= \det \begin{pmatrix} F_0 \frac{\partial F_1}{\partial X_1} + F_1 \frac{\partial F_0}{\partial X_1} & \cdots & F_0 \frac{\partial F_1}{\partial X_n} + F_1 \frac{\partial F_0}{\partial X_n} \\ \vdots & \ddots & \vdots \\ F_0 \frac{\partial F_n}{\partial X_1} + F_n \frac{\partial F_0}{\partial X_1} & \cdots & F_0 \frac{\partial F_n}{\partial X_n} + F_n \frac{\partial F_0}{\partial X_n} \end{pmatrix} \\ &= \det \begin{pmatrix} F_0 \frac{\partial F_1}{\partial X_1} & \cdots & F_0 \frac{\partial F_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ F_0 \frac{\partial F_n}{\partial X_1} & \cdots & F_0 \frac{\partial F_n}{\partial X_n} \end{pmatrix} + G \\ &= F_0^n \frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} + G \end{aligned}$$

for some polynomial  $G \in I' \subseteq \mathfrak{P}_{j_0}$ . Now if  $F_0^n \frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} \in \mathfrak{P}_{j_0}$  then  $F_0 \in \mathfrak{P}_{j_0}$  or  $\frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} \in \mathfrak{P}_{j_0}$ , in contradiction to the construction. Therefore we must have  $F_0^n \frac{\partial(F_1, \dots, F_n)}{\partial(X_1, \dots, X_n)} \notin \mathfrak{P}_{j_0}$ , and the conclusion follows.  $\square$

**Remark 5.3.5.** Given a fat point scheme  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  in  $\mathbb{P}_K^n$ , the number  $s'$  given in Lemma 5.3.4 is the number of smooth points of  $\mathbb{W}$ . In particular, it is also the number of  $K$ -rational points of  $\mathbb{W}$ . Thus Proposition 3.3.19 yields that the Hilbert polynomial of  $\vartheta_K(S/K[x_0])$  satisfies  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) \geq \sum_{p_j \in \mathbb{W}_{\text{sm}}} \dim_K(\mathcal{O}_{\mathbb{X}, p_j}) = s'$ , where  $\mathbb{W}_{\text{sm}}$  is the set of all smooth points of  $\mathbb{W}$ . Hence claim (ii) of Lemma 5.3.4 is an immediate consequence of this observation.

At this point we are ready to prove our first main result of this section. In a non-trivial case, we determine the Hilbert polynomial of the Kähler different and bound its regularity index.

**Theorem 5.3.6.** *Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$ , where  $n \geq 2$ , and assume that there exists an index  $s' \leq s$  such that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$ . Then the Kähler different  $\vartheta_K(S/K[x_0])$  has the constant Hilbert polynomial  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) = s'$  and its regularity index satisfies  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nr_{\mathbb{W}}$ .*

*Proof.* If  $s' = 0$  then  $m_j \geq 2$  for every  $j \in \{1, \dots, s\}$ , and we have  $\vartheta_K(S/K[x_0]) = \langle 0 \rangle$  by Lemma 5.3.4(i). Now we consider the case  $s' = s$ , i.e., the case when  $\mathbb{W} = \{p_1, \dots, p_s\}$  is a set of  $s$  distinct reduced  $K$ -rational points of  $\mathbb{P}_K^n$ . Then Proposition 3.3.14 implies that  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) = s$  and  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nr_{\mathbb{W}}$ , proving the theorem in the reduced case.

Consequently, we may assume that  $0 < s' < s$ . In this case, we first show  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) \leq s'$ . To this end, let  $\{h_1, \dots, h_t\}$  be a homogeneous system of generators of  $\vartheta_K(S/K[x_0])$ , and let  $H_i \in P$  be a representative of  $h_i$  for  $i = 1, \dots, t$ . We let  $J := \langle H_1, \dots, H_t \rangle + \mathcal{I}_{\mathbb{W}} \subseteq P$  and write  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_{s'} \cap \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s} = I \cap I'$ , where  $I' = \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_{s'}$  and  $I = \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s}$ . Using the same argument as in the proof of Lemma 5.3.4(i), we obtain  $J \subseteq \mathfrak{P}_j^{m_j}$  for  $j = s'+1, \dots, s$ , and it follows that  $\mathcal{I}_{\mathbb{W}} \subseteq J \subseteq I$ . Thus we have  $\text{HF}_{P/I}(i) \leq \text{HF}_{P/J}(i)$  for all  $i \geq 0$ . By Remark 5.3.2, the Hilbert polynomial of  $P/I$  is the constant polynomial  $\text{HP}_{P/I}(z) = \sum_{j=s'+1}^s \binom{m_j+n-1}{n}$ . We deduce that  $\text{HP}_{P/J}(z) \geq \text{HP}_{P/I}(z) = \sum_{j=s'+1}^s \binom{m_j+n-1}{n}$ . Hence we get

$$\begin{aligned} \text{HP}_{\vartheta_K(S/K[x_0])}(z) &= \text{HP}_{\mathbb{W}}(z) - \text{HP}_{P/J}(z) \leq \text{HP}_{\mathbb{W}}(z) - \text{HP}_{P/I}(z) \\ &= \sum_{j=1}^s \binom{m_j+n-1}{n} - \sum_{j=s'+1}^s \binom{m_j+n-1}{n} \\ &= \sum_{j=1}^{s'} \binom{m_j+n-1}{n} = s', \end{aligned}$$

as we wanted to show.

Secondly, we need to show that  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) \geq s'$ . But this inequality follows from Proposition 3.3.19. Altogether, we have shown that  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) = s'$ .

Finally, it remains to prove  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nr_{\mathbb{W}}$ . For every  $j \in \{1, \dots, s'\}$ , let  $f_j^* \in S \setminus \{0\}$  be a minimal separator of  $\mathbb{W} \setminus \{P_j\}$  in  $\mathbb{W}$ , and let  $F_j^* \in P$  be a representative of  $f_j^*$ . It is well known that  $\deg(F_j^*) \leq r_{\mathbb{W}}$  and  $(F_j^*)^k \notin \mathcal{I}_{\mathbb{W}}$  (or  $(f_j^*)^k \neq 0$ ) for  $j = 1, \dots, s'$  and  $k \geq 1$ . Let us write  $\mathfrak{P}_j = \langle L_{j1}, \dots, L_{jn} \rangle$  as in the proof of Lemma 5.3.4. We see that  $F_j^* L_{jk} \in \mathcal{I}_{\mathbb{W}}$  for  $k = 1, \dots, n$ . Moreover, we may argue as the proof of Proposition 3.3.14 to obtain  $\frac{\partial(F_j^* L_{j1}, \dots, F_j^* L_{jn})}{\partial(X_1, \dots, X_n)} \equiv (F_j^*)^n \pmod{\mathcal{I}_{\mathbb{W}}}$ , and this implies  $(f_j^*)^n \in \vartheta_K(S/K[x_0])_{n \deg(f_j^*)}$ . Thus we know that  $x_0^{n(r_{\mathbb{W}} - \deg(f_j^*))} (f_j^*)^n \in \vartheta_K(S/K[x_0])_{nr_{\mathbb{W}}}$  for all  $j = 1, \dots, s'$ .

Let us show that the elements  $x_0^{n(r_{\mathbb{W}} - \deg(f_1^*))} (f_1^*)^n, \dots, x_0^{n(r_{\mathbb{W}} - \deg(f_{s'}^*))} (f_{s'}^*)^n$  are  $K$ -linearly independent. Suppose there exist elements  $c_1, \dots, c_{s'}$  in  $K$ , not all equal to zero, such that  $\sum_{j=1}^{s'} c_j x_0^{n(r_{\mathbb{W}} - \deg(f_j^*))} (f_j^*)^n = 0$ . W.l.o.g., we can assume  $c_1 \neq 0$ . Then we have

$$x_0^{n(r_{\mathbb{W}} - \deg(f_1^*))} (f_1^*)^{n+1} = \frac{1}{c_1} \sum_{j=2}^{s'} c_j x_0^{n(r_{\mathbb{W}} - \deg(f_j^*))} f_1^* (f_j^*)^n = 0.$$

Since  $x_0$  is a non-zerodivisor of  $S$ , we get  $(f_1^*)^{n+1} = 0$ , a contradiction.

It follows that we have an inequality

$$\mathrm{HF}_{\vartheta_K(S/K[x_0])}(nr_{\mathbb{W}}) \geq \dim_K \left\langle x_0^{n(r_{\mathbb{W}} - \deg(f_1^*))} (f_1^*)^n, \dots, x_0^{n(r_{\mathbb{W}} - \deg(f_{s'}^*))} (f_{s'}^*)^n \right\rangle_K = s'.$$

Combining this with  $\mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) \leq s'$  for  $i \geq 0$  and  $\mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) = s'$  for  $i \gg 0$ , we obtain  $\mathrm{HF}_{\vartheta_K(S/K[x_0])}(nr_{\mathbb{W}}) = s'$  and  $\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq nr_{\mathbb{W}}$ , as desired.  $\square$

Our next proposition gives us another way to prove the equality in Theorem 5.3.6 for the Hilbert polynomial of a fat point scheme.

**Proposition 5.3.7.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set in  $\mathbb{P}_K^n$ , where  $n \geq 2$ , and let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$  with  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$ . Let  $\mathbb{Y} = \{p_1, \dots, p_{s'}\}$  be a subscheme of  $\mathbb{X}$ , and let  $R_{\mathbb{Y}}$  be the homogeneous coordinate ring of  $\mathbb{Y}$ . Then there exists a number  $d \geq 1$  such that*

$$\mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i - nd) \leq \mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i)$$

for all  $i \in \mathbb{N}$ .

*Proof.* Let us denote  $I := \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s}$  and  $I' := \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_{s'}$ . Then we have  $\mathcal{I}_{\mathbb{W}} = \mathfrak{P}_1 \cap \dots \cap \mathfrak{P}_{s'} \cap \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s} = I \cap I'$ . Since  $I \not\subseteq \mathfrak{P}_j$  for  $j \in \{1, \dots, s'\}$ , the Prime Avoidance Theorem yields  $I \not\subseteq \bigcup_{j=1}^{s'} \mathfrak{P}_j$ . Let  $F \in I \setminus \bigcup_{j=1}^{s'} \mathfrak{P}_j$  be a homogeneous polynomial of degree  $d$ , and let  $f$  be the image of  $F$  in  $R_{\mathbb{Y}}$ . By Lemma 2.3.11, the element  $f$  is a non-zerodivisor of  $R_{\mathbb{Y}}$ . Moreover, we observe that  $FG \in \mathcal{I}_{\mathbb{W}}$  for all  $G \in \mathcal{I}_{\mathbb{Y}}$ , and for  $G_1, \dots, G_n \in \mathcal{I}_{\mathbb{Y}}$  we have

$$\frac{\partial(FG_1, \dots, FG_n)}{\partial(X_1, \dots, X_n)} = F^n \frac{\partial(G_1, \dots, G_n)}{\partial(X_1, \dots, X_n)} + H$$

with a polynomial  $H \in \mathcal{I}_{\mathbb{Y}}$ . Notice that the homogeneous polynomial  $H$  belongs to  $\mathcal{I}_{\mathbb{W}} \subseteq \mathcal{I}_{\mathbb{Y}}$ , since  $n \geq 2$  and  $\frac{\partial F}{\partial X_i} \in \mathfrak{P}_j^{m_j-1}$  for  $j = s'+1, \dots, s$  and  $i = 1, \dots, n$ . From this we deduce  $f^n \vartheta_K(R_{\mathbb{Y}}/K[x_0]) \subseteq (J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}}$ , where  $J$  is the preimage in  $P$  of the Kähler different  $\vartheta_K(S/K[x_0])$ . It is obviously true that  $(J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}} \subseteq \vartheta_K(R_{\mathbb{Y}}/K[x_0])$ , and hence

$$f^n \vartheta_K(R_{\mathbb{Y}}/K[x_0]) \subseteq (J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}} \subseteq \vartheta_K(R_{\mathbb{Y}}/K[x_0]).$$

In particular, for every  $i \in \mathbb{N}$ , we have

$$\mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i - nd) = \mathrm{HF}_{f^n \vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i) \leq \mathrm{HF}_{(J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}}}(i) \leq \mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i).$$

Moreover, we have the exact sequence

$$0 \longrightarrow \frac{J \cap \mathcal{I}_{\mathbb{Y}}}{\mathcal{I}_{\mathbb{W}}} \longrightarrow \frac{J}{\mathcal{I}_{\mathbb{W}}} \longrightarrow \frac{J + \mathcal{I}_{\mathbb{Y}}}{\mathcal{I}_{\mathbb{Y}}} \longrightarrow 0.$$

Since  $\mathcal{I}_{\mathbb{W}} \subseteq J \subseteq I$ , this implies  $\mathcal{I}_{\mathbb{W}} \subseteq J \cap \mathcal{I}_{\mathbb{Y}} \subseteq I \cap \mathcal{I}_{\mathbb{Y}} = \mathcal{I}_{\mathbb{W}}$ . Consequently, we get an isomorphism  $J/\mathcal{I}_{\mathbb{W}} \cong (J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}}$ . It follows that  $\mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) = \mathrm{HF}_{J/\mathcal{I}_{\mathbb{W}}}(i) = \mathrm{HF}_{(J+\mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}}}(i)$  for all  $i \in \mathbb{N}$ . Therefore we obtain inequalities

$$\mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i - nd) \leq \mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) \leq \mathrm{HF}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(i)$$

for all  $i \in \mathbb{N}$ . □

It is easy to apply the preceding proposition in order to see that  $\mathrm{HP}_{\vartheta_K(S/K[x_0])}(z) = \mathrm{HP}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(z)$ . Since  $\mathbb{Y}$  is a projective point set of  $\mathbb{P}_K^n$  of degree  $\deg(\mathbb{Y}) = s'$ , Proposition 3.3.14 yields  $\mathrm{HP}_{\vartheta_K(R_{\mathbb{Y}}/K[x_0])}(z) = s'$ , and so the equality of the Hilbert polynomial of  $\mathbb{W}$  provided by Theorem 5.3.6 follows.

**Remark 5.3.8.** For a fat point scheme of the form  $\mathbb{W} = \{p_1, p_2, \dots, p_s; 1, m_2, \dots, m_s\}$  in  $\mathbb{P}^n$ , where  $n \geq 2$  and  $2 \leq m_2 \leq \dots \leq m_s$ , the Kähler different  $\vartheta_K(S/K[x_0])$  is a principal homogeneous ideal of  $S$  and is generated by a separator of the subscheme  $\mathbb{W} \setminus \{p_1\}$ . Indeed, letting  $\alpha_{\vartheta_K} := \min\{i \in \mathbb{N} \mid \vartheta_K(S/K[x_0])_i \neq 0\}$ , we have  $\mathrm{HF}_{\vartheta_K(S/K[x_0])}(i) = 1$  for all  $i \geq \alpha_{\vartheta_K}$  by Theorem 5.3.6. This means that  $\vartheta_K(S/K[x_0]) = \langle h \rangle_S$  for some homogeneous element  $h \in S$  of degree  $\alpha_{\vartheta_K}$ . Moreover, since  $h \in \mathfrak{P}_2^{m_2} \cap \dots \cap \mathfrak{P}_s^{m_s} \setminus \mathfrak{P}_1$ , it follows that  $h$  is a separator of  $\mathbb{W} \setminus \{p_1\}$ .

An upper bound for the regularity index of  $\vartheta_K(S/K[x_0])$  in terms of  $m_1, \dots, m_s$  is given by the following corollary.

**Corollary 5.3.9.** *Let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set of  $\mathbb{P}_K^n$ , where  $n \geq 2$ , and let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme supported on  $\mathbb{X}$  with  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$  for some  $1 \leq s' \leq s$ .*

(i) *We have  $\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq \min\{nm_s r_{\mathbb{X}}, n(\sum_{j=1}^s m_j - 1)\}$ .*

(ii) *If  $\mathbb{X}$  is in general position (i.e., if no  $n+1$  points of  $\mathbb{X}$  lie on a hyperplane of  $\mathbb{P}_K^n$ ), then we have*

$$\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq n \max\left\{m_s, \left\lfloor \frac{\sum_{j=1}^s m_j + n - 2}{n} \right\rfloor\right\}.$$

*Proof.* In [DG], E.D. Davis and A.V. Geramita proved that  $r_{\mathbb{W}} \leq \sum_{j=1}^s m_j - 1$ . Thus Theorem 5.3.6 yields  $\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq n(\sum_{j=1}^s m_j - 1)$ . For the proof of (i), it remains to show that  $\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq nm_s r_{\mathbb{X}}$ . For  $j = 1, \dots, s$ , let  $F_j \in P_{r_{\mathbb{X}}}$  be a representative of the normal separator of  $\mathbb{X} \setminus \{p_j\}$  in  $\mathbb{X}$ . For all  $j \in \{1, \dots, s'\}$ , we let

$$I^{(j)} = \mathfrak{P}_1 \cap \dots \cap \widehat{\mathfrak{P}_j} \cap \dots \cap \mathfrak{P}_{s'} \cap \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s}.$$

It is straightforward to check that  $F_j^{m_s} \in I^{(j)}$  and  $F_j^{m_s} \notin \mathfrak{P}_j$  for  $j = 1, \dots, s'$ . Therefore the image  $f_j^{m_s}$  of  $F_j^{m_s}$  in  $S$  is a separator of  $\mathbb{W} \setminus \{p_j\}$  in  $\mathbb{W}$ . Now we argue as in proof of Theorem 5.3.6 and obtain  $(f_j^{m_s})^n \in \vartheta_K(S/K[x_0])_{nm_s r_{\mathbb{X}}}$  for  $j = 1, \dots, s'$ , as well as  $\dim_K \langle (f_1^{m_s})^n, \dots, (f_{s'}^{m_s})^n \rangle_K = s'$ . Thus we have  $\text{HF}_{\vartheta_K(S/K[x_0])}(nm_s r_{\mathbb{X}}) = s'$  and  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nm_s r_{\mathbb{X}}$ , and claim (i) follows.

Notice that the bound  $\text{ri}(\vartheta_K(S/K[x_0])) \leq n(\sum_{j=1}^s m_j - 1)$  can be derived by using the set of separators. Explicitly, we can find a hyperplane  $\mathcal{Z}^+(L_{jk})$  which contains  $p_k$  and does not contain  $p_j$  for  $k = 1, \dots, j-1, j+1, \dots, s$ . We put  $G_j = \prod_{k \neq j} (L_{jk})^{m_k} \in P$  for  $j = 1, \dots, s'$ . Thus we have  $G_j \in I^{(j)} \setminus \mathfrak{P}_j$  and  $\deg(G_1) = \sum_{k \neq 1} m_k = \sum_{k=1}^s m_k - 1 = \deg(G_j)$  for  $j = 1, \dots, s'$ . Using a similar argument as above, we get the desired bound.

It remains to prove (ii). Since  $\mathbb{X}$  is in general position in  $\mathbb{P}_K^n$ , it follows from [CTV, Lemma 4] that we can find  $t$  linear forms, say  $L'_{j1}, \dots, L'_{jt}$ , defining hyperplanes which avoid  $p_j$  such that  $H_j = L'_{j1} \cdots L'_{jt} \in I^{(j)}$ . Here  $t$  is a positive integer such that  $nt \geq \sum_{j=1}^s m_j - 1$  and  $t \geq m_s$ . For instance, we may choose  $t = \max \{m_s, \lfloor (\sum_{j=1}^s m_j + n - 2)/n \rfloor\}$ , since  $\lfloor (\sum_{j=1}^s m_j + n - 2)/n \rfloor = \min \{t \mid nt \geq \sum_{j=1}^s m_j - 1\}$ . Thus the image of  $H_j$  in  $S$  is a separator of  $\mathbb{W} \setminus \{p_j\}$  in  $\mathbb{W}$  for  $j = 1, \dots, s'$ . Reasoning as above, we have  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nt = n \max \{m_s, \lfloor (\sum_{j=1}^s m_j + n - 2)/n \rfloor\}$ .  $\square$

The following examples show that the bound in this corollary is sometimes sharp, and sometimes it is not.

**Example 5.3.10.** Let  $\mathbb{X} = \{p_1, p_2, p_3, p_4\}$  be the projective point set of  $\mathbb{P}_{\mathbb{Q}}^2$  with  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0)$ ,  $p_3 = (1 : 0 : 1)$ , and  $p_4 = (1 : 1 : 1)$ . It is clear that  $\mathbb{X}$  is in general position in  $\mathbb{P}_{\mathbb{Q}}^2$ .

(a) The fat point scheme  $\mathbb{W}_1 = \{p_1, p_2, p_3, p_4; 1, 1, 1, 2\}$  has the Hilbert function  $\text{HF}_{\mathbb{W}_1} : 1 \ 3 \ 6 \ 6 \cdots$  and the regularity index  $r_{\mathbb{W}_1} = 2$ . Its Kähler different satisfies  $\text{HF}_{\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 3 \ 3 \cdots$ . This implies that

$$\text{ri}(\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])) = 4 = nr_{\mathbb{W}_1} = n \max \left\{ m_4, \left\lfloor \left( \sum_{j=1}^4 m_j + n - 2 \right) / n \right\rfloor \right\}.$$

(b) Now we consider the fat point scheme  $\mathbb{W}_2 = \{p_1, p_2, p_3, p_4; 1, 1, 2, 2\}$  in  $\mathbb{P}_{\mathbb{Q}}^2$ . Here we have  $\text{HF}_{\mathbb{W}_2} : 1 \ 3 \ 6 \ 8 \ 8 \cdots$  and  $r_{\mathbb{W}_2} = 3$ . Moreover, we calculate  $\text{HF}_{\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 1 \ 2 \ 2 \cdots$ . Thus we have

$$\text{ri}(\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])) = 5 < 6 = nr_{\mathbb{W}_2} = n \left\lfloor \left( \sum_{j=1}^4 m_j + n - 2 \right) / n \right\rfloor.$$



**Remark 5.3.11.** Given a fat point scheme  $\mathbb{W}$  whose support is in general position, if we apply [CTV, Theorem 6] and Theorem 5.3.6 then we get

$$\text{ri}(\vartheta_K(S/K[x_0])) \leq nr_{\mathbb{W}} \leq n \max \left\{ m_s + m_{s-1} - 1, \left\lfloor \left( \sum_{j=1}^s m_j + n - 2 \right) / n \right\rfloor \right\}.$$

In the case  $m_s \geq \left\lfloor \left( \sum_{j=1}^s m_j + n - 2 \right) / n \right\rfloor$ , Corollary 5.3.9(ii) gives us a better bound for  $\text{ri}(\vartheta_K(S/K[x_0]))$  which is given by  $\text{ri}(\vartheta_K(S/K[x_0])) \leq nm_s$ . For instance, we take  $\text{Supp}(\mathbb{W})$  to be in general position in  $\mathbb{P}_K^n$ ,  $s' = s - 2$ ,  $m_{s-1} = 2 \leq m_s$ , and  $s \leq (n - 1)m_s + 1$ .

Let us also check the remark for an example in  $\mathbb{P}_K^3$ .

**Example 5.3.12.** Let  $\mathbb{W} = \{p_1, p_2, p_3, p_4; 1, 1, 2, 2\}$  be the fat point scheme in  $\mathbb{P}_{\mathbb{Q}}^3$  with  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 1)$ ,  $p_3 = (1 : 0 : 1 : 1)$ , and  $p_4 = (1 : 1 : 1 : 1)$ . Its support  $\mathbb{X} = \{p_1, p_2, p_3, p_4\}$  is clearly in general position in  $\mathbb{P}_{\mathbb{Q}}^3$ . We calculate  $\text{HF}_{\mathbb{W}} : 1 \ 4 \ 9 \ 10 \ 10 \ \dots$  and  $r_{\mathbb{W}} = 3$ , as well as  $\text{HF}_{\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])} : 0 \ 0 \ 0 \ 0 \ 0 \ 2 \ 2 \ \dots$ . Thus the above inequality reads

$$\text{ri}(\vartheta_{\mathbb{Q}}(S/\mathbb{Q}[x_0])) = 5 < 6 = n \max \left\{ m_4, \left\lfloor \left( \sum_{j=1}^4 m_j + n - 2 \right) / n \right\rfloor \right\} < 9 = nr_{\mathbb{W}}.$$

Based on the notion of Cayley-Bacharach property and Liaison theory introduced in Section 4.1 and Section 4.3, we can now improve the bound for the regularity index of the Kähler different.

**Proposition 5.3.13.** *Let  $n \geq 2$ , and let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set of  $\mathbb{P}_K^n$  which is arithmetically Gorenstein. Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$  supported at  $\mathbb{X}$ . Assume that  $K$  is infinite and that there exists a number  $s' \in \{1, \dots, s\}$  such that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$ . Let  $\mathbb{Y}$  be the subscheme  $\mathbb{Y} = \{p_1, \dots, p_{s'}\}$  of  $\mathbb{X}$ , and let  $d \in \{1, \dots, r_{\mathbb{Y}} - 1\}$  be the greatest integer such that  $\mathbb{Y}$  has  $\text{CBP}(d)$ . Then we have*

$$\text{ri}(\vartheta_K(S/K[x_0])) \leq n(m_s(r_{\mathbb{X}} - d - 1) + r_{\mathbb{Y}}).$$

*Proof.* Let  $\mathbb{Y}' = \mathbb{X} \setminus \mathbb{Y}$  be the residual scheme of  $\mathbb{Y}$  in  $\mathbb{X}$ , and let  $\mathcal{I}_{\mathbb{Y}'}$  be the homogeneous vanishing ideal of  $\mathbb{Y}'$ . Since  $\mathbb{Y}$  has  $\text{CBP}(d)$ , Theorem 4.3.6 implies that there exists an element of  $(\mathcal{I}_{\mathbb{Y}'})_{r_{\mathbb{X}}-d-1}$ , namely  $H$ , which does not vanish at any point of  $\mathbb{Y}$ . Hence we have  $H^{m_s} \in \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s}$  and  $H^{m_s} \notin \mathfrak{P}_j$  for  $j = 1, \dots, s'$ . Therefore the image  $h$  of  $H$  in the homogeneous coordinate ring  $R_{\mathbb{Y}}$  of  $\mathbb{Y}$  is a non-zerodivisor (see Lemma 2.3.11). As in the proof of Proposition 5.3.7, we obtain  $(h^{m_s})^n \vartheta_K(R_{\mathbb{Y}}/K[x_0]) \subseteq (J + \mathcal{I}_{\mathbb{Y}})/\mathcal{I}_{\mathbb{Y}}$ ,

where  $J$  is the homogeneous ideal of  $P$  such that  $\vartheta_K(S/K[x_0]) = J/\mathcal{I}_{\mathbb{W}}$ , and where the regularity index of the ideal  $(h^{m_s})^n \vartheta_K(R_{\mathbb{Y}}/K[x_0])$  is greater than or equal to the regularity index of the Kähler different  $\vartheta_K(S/K[x_0])$ .

On the other hand, since  $\mathbb{Y}$  is a projective point set of  $\mathbb{P}_K^n$  of degree  $s'$ , we have  $\text{ri}(\vartheta_K(R_{\mathbb{Y}}/K[x_0])) \leq nr_{\mathbb{Y}}$  by Proposition 3.3.14. Altogether, we get

$$\begin{aligned} \text{ri}((h^{m_s})^n \vartheta_K(R_{\mathbb{Y}}/K[x_0])) &= \deg((h^{m_s})^n) + \text{ri}(\vartheta_K(R_{\mathbb{Y}}/K[x_0])) \\ &= nm_s(r_{\mathbb{X}} - d - 1) + \text{ri}(\vartheta_K(R_{\mathbb{Y}}/K[x_0])) \\ &\leq n(m_s(r_{\mathbb{X}} - d - 1) + r_{\mathbb{Y}}), \end{aligned}$$

and hence  $\text{ri}(\vartheta_K(S/K[x_0])) \leq \text{ri}((h^{m_s})^n \vartheta_K(R_{\mathbb{Y}}/K[x_0])) \leq n(m_s(r_{\mathbb{X}} - d - 1) + r_{\mathbb{Y}})$ , as we wanted to show.  $\square$

The following example shows that a strict inequality  $\text{ri}(\vartheta_K(S/K[x_0])) < nr_{\mathbb{W}}$  can occur even for a fat point scheme  $\mathbb{W}$  supported at a set of points  $\mathbb{X}$  which is contained in a line.

**Example 5.3.14.** Let  $n \geq 2$ , and let  $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_K^n$  be a projective point set which lies on a line. Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$  supported at  $\mathbb{X}$ . W.l.o.g. we may assume that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$  for some  $1 \leq s' < s$ . Recall that we have  $r_{\mathbb{W}} = \sum_{i=1}^s m_i - 1$ .

In this setting, the schemes  $\mathbb{X}$  and  $\mathbb{Y} = \{p_1, \dots, p_{s'}\}$  are complete intersections, and we have  $r_{\mathbb{X}} = s - 1$  and  $r_{\mathbb{Y}} = s' - 1$ . Therefore  $\mathbb{Y}$  is a CB-scheme and  $\text{ri}(\vartheta_K(R_{\mathbb{Y}}/K[x_0])) = 2r_{\mathbb{Y}}$ . Using the argument of the proof of Proposition 5.3.13, we get

$$\text{ri}(\vartheta_K(S/K[x_0])) \leq nm_s(r_{\mathbb{X}} - (r_{\mathbb{Y}} - 1) - 1) + 2r_{\mathbb{Y}} = nm_s(s - s') + 2(s' - 1).$$

This bound shows that the strict inequality  $\text{ri}(\vartheta_K(S/K[x_0])) < nr_{\mathbb{W}}$  can occur even when  $\mathbb{X}$  lies on a line.

For instance, consider the case  $n = 4$  and  $m_s = \dots = m_{s'+1} = 3$ . Then we have  $\text{ri}(\vartheta_K(S/K[x_0])) \leq 12(s - s') + 2(s' - 1) = 12s - 10s' - 2$  and  $r_{\mathbb{W}} = \sum_{i=1}^s m_i - 1 = 3(s - s') + s' - 1 = 3s - 2s' - 1$ . The inequality  $12s - 10s' - 2 < 4(3s - 2s' - 1)$  is equivalent to  $s' > 1$ . Therefore, if  $s' > 1$ , then we have  $\text{ri}(\vartheta_K(S/K[x_0])) < 4r_{\mathbb{W}}$ .

As in the previous proposition, in order to estimate  $\text{ri}(\vartheta_K(S/K[x_0]))$  we need to find a hypersurface  $\mathcal{Z}^+(H)$  of minimal degree which contains the fat point scheme  $\mathbb{Y}' = \{p_{s'+1}, \dots, p_s; m_{s'+1}, \dots, m_s\}$  but does not pass through any of the points  $p_1, \dots, p_{s'}$ . Using the technique in [TV, Theorem 2.2], we obtain the following upper bound for the regularity index  $\text{ri}(\vartheta_K(S/K[x_0]))$ .

**Proposition 5.3.15.** *Let  $K$  be a infinite field, let  $n \geq 2$ , and let  $\mathbb{X} = \{p_1, \dots, p_s\}$  be a projective point set of  $\mathbb{P}_K^n$ . Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$  supported at  $\mathbb{X}$ . W.l.o.g. we may assume that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$  for some  $1 \leq s' < s$ . Suppose that every subscheme  $\mathbb{X}' \subseteq \mathbb{X}$  of degree  $\leq s - s' + 1$  satisfies*

$$\mathrm{HF}_{\mathbb{X}'}(i) = \min\{\mathrm{HF}_{\mathbb{X}}(i), \deg(\mathbb{X}')\}$$

for all  $i \in \mathbb{Z}$ . Let  $\mathbb{X}'' \subseteq \mathbb{X}$  be a subscheme of degree  $s - s' + 1$ , let  $d$  be the regularity index of  $\mathbb{X}''$ , let  $h_i = \mathrm{HF}_{\mathbb{X}''}(i)$  for  $i = 0, \dots, d$ , and let  $\mathbb{Y} = \{p_1, \dots, p_{s'}\}$ . Then we have

$$\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq n \left( \sum_{i=0}^{d-1} m_{s-h_i+1} + r_{\mathbb{Y}} \right).$$

*Proof.* We first see that  $h_i = \mathrm{HF}_{\mathbb{X}}(i)$  for  $i = 0, \dots, d-1$  and  $h_d = \deg(\mathbb{X}'') = s - s' + 1 \leq \mathrm{HF}_{\mathbb{X}}(d)$ . So we get  $s - h_i + 2 \geq s' + 1$  for every  $i \in \{1, \dots, d\}$  and  $s - h_d + 2 = s' + 1$ . Now we let  $j \in \{1, \dots, s'\}$  and  $i \in \{1, \dots, d\}$ . For the sets  $\mathbb{V} = \{p_{s-h_i+2}, \dots, p_s\}$  and  $\mathbb{V}' = \mathbb{V} \cup \{p_j\}$ , we have  $\deg(\mathbb{V}) = h_i - 1$  and  $\deg(\mathbb{V}') = h_i$ . By assumption, we know that

$$\mathrm{HF}_{\mathbb{V}}(i) = \min\{\mathrm{HF}_{\mathbb{X}}(i), h_i - 1\} = h_i - 1 < h_i = \deg(\mathbb{V}').$$

Thus we can find a hypersurface  $\mathcal{Z}^+(H_i)$  of degree  $i$  passing through all points of  $\mathbb{V}$  but avoiding  $p_j$ . For  $k \in \{1, \dots, s - s'\}$ , let  $t$  be the least integer such that  $k \leq h_t - 1$ . Then we have  $s - k + 1 \leq s - h_{t-1} + 1$ ,  $t \leq d$ , and  $H_i \in \mathfrak{P}_{s-k+1}$  for all  $i \geq t$ . Hence we obtain

$$H_t^{m_{s-h_{t-1}+1} - m_{s-h_t+1}} \dots H_{d-1}^{m_{s-h_{d-2}+1} - m_{s-h_{d-1}+1}} H_d^{m_{s-h_{d-1}+1}} \in \mathfrak{P}_{s-k+1}^{m_{s-h_{t-1}+1}} \subseteq \mathfrak{P}_{s-k+1}^{m_{s-k+1}}.$$

Next we define  $\tilde{H}_j := H_1^{m_{s-h_0+1} - m_{s-h_1+1}} \dots H_{d-1}^{m_{s-h_{d-2}+1} - m_{s-h_{d-1}+1}} H_d^{m_{s-h_{d-1}+1}}$ . We have  $\tilde{H}_j \in \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s} \setminus \mathfrak{P}_j$ . From this we deduce that  $\tilde{H}_j(p_j) \neq 0$  for  $j = 1, \dots, s'$ . Since  $K$  is infinite, we can find elements  $\lambda_1, \dots, \lambda_{s'}$  in  $K$  such that  $H := \lambda_1 \tilde{H}_1 + \dots + \lambda_{s'} \tilde{H}_{s'}$  does not vanish at any point of  $\mathbb{Y} = \{p_1, \dots, p_{s'}\}$ . Clearly, we have  $H \in \mathfrak{P}_{s'+1}^{m_{s'+1}} \cap \dots \cap \mathfrak{P}_s^{m_s}$ , and the image of  $H$  in  $R_{\mathbb{Y}}$  is a non-zerodivisor. The degree of  $H$  satisfies  $\deg(H) = (m_{s-h_0+1} - m_{s-h_1+1}) + 2(m_{s-h_1+1} - m_{s-h_2+1}) + \dots + (d-1)(m_{s-h_{d-2}+1} - m_{s-h_{d-1}+1}) + dm_{s-h_{d-1}+1} = \sum_{i=0}^{d-1} m_{s-h_i+1}$ . Arguing as in the proof of Proposition 5.3.13, we obtain  $\mathrm{ri}(\vartheta_K(S/K[x_0])) \leq n \left( \sum_{i=0}^{d-1} m_{s-h_i+1} + r_{\mathbb{Y}} \right)$  as claimed.  $\square$

The Hilbert polynomials and the regularity indices of the Kähler differents  $\vartheta^{(i)}(S/K)$  can be derived from those for  $S/K[x_0]$  and are given by the following proposition.

**Proposition 5.3.16.** *Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}_K^n$ , where  $n \geq 2$ , such that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$  for some  $0 \leq s' \leq s$ . The following assertions hold true.*

- (i) *We have  $\vartheta^{(i)}(S/K) = \langle 0 \rangle$  for some  $i \in \{1, \dots, n-1\}$  if and only if  $s' = 0$ .*
- (ii) *For  $i \in \{1, \dots, n-1\}$ , we have  $\text{HP}_{\vartheta^{(i)}(S/K)}(z) = s'$ ,  $\text{ri}(\vartheta^{(i)}(S/K)) \leq \text{ri}(\vartheta^{(1)}(S/K))$  and  $\text{ri}(\vartheta_K(S/K[x_0])) - 1 \leq \text{ri}(\vartheta^{(1)}(S/K)) \leq \text{ri}(\vartheta_K(S/K[x_0]))$ .*
- (iii) *We have  $\vartheta^{(n)}(S/K) \neq \langle 0 \rangle$  and its Hilbert polynomial satisfies*

$$s' \leq \text{HP}_{\vartheta^{(n)}(S/K)}(z) \leq \sum_{j=1}^s \binom{m_j + n - 2}{n - 1}.$$

*Proof.* First we prove (i). Proposition 5.2.2 yields that  $\text{HP}_{\vartheta_K(S/K[x_0])}(z) = \text{HP}_{\vartheta^{(1)}(S/K)}(z)$ . Therefore Theorem 5.3.6 implies the equalities  $\text{HP}_{\vartheta^{(1)}(S/K)}(z) = \text{HP}_{\vartheta_K(S/K[x_0])}(z) = s'$ . If we have  $\vartheta^{(i)}(S/K) = \langle 0 \rangle$  for some  $i \in \{1, \dots, n-1\}$ , then we get  $\vartheta^{(1)}(S/K) = \langle 0 \rangle$ , and hence  $s' = 0$ .

Conversely, suppose that  $s' = 0$ . Let  $\{G_1, \dots, G_r\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{W}}$ . Since  $m_j \geq 2$  for  $j = 1, \dots, s$ , we know that  $\partial G_k / \partial X_l \in \mathfrak{P}_j^{m_j-1}$  for  $k \in \{1, \dots, r\}$  and  $l \in \{0, \dots, n\}$ . Now  $i \leq n-1$  implies that all  $(n+1-i)$ -minors of the Jacobian matrix  $\left(\frac{\partial G_k}{\partial X_l}\right)_{\substack{k=1, \dots, r \\ l=0, \dots, n}}$  are contained in  $\mathcal{I}_{\mathbb{W}}$ . Therefore we get  $\vartheta^{(i)}(S/K) = \langle 0 \rangle$  and claim (i) follows.

To prove (ii), we note that  $s' = \text{HP}_{\vartheta^{(1)}(S/K)}(z) \leq \text{HP}_{\vartheta^{(i)}(S/K)}(z)$ . Furthermore, we have  $\vartheta^{(i)}(S/K) \subseteq (\bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j}) / \mathcal{I}_{\mathbb{W}}$ . This implies

$$\text{HP}_{\vartheta^{(i)}(S/K)}(z) \leq \text{HP}_{(\bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j}) / \mathcal{I}_{\mathbb{W}}}(z) = \text{HP}_{\mathbb{W}}(z) - \text{HP}_{P / \bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j}}(z) = s'.$$

Thus we get  $\text{HP}_{\vartheta^{(i)}(S/K)}(z) = s'$  for  $i = 1, \dots, n-1$ . Moreover, it is clear that

$$\text{ri}(\vartheta^{(i)}(S/K)) \leq \text{ri}(\vartheta^{(1)}(S/K)) \leq \text{ri}(\vartheta_K(S/K[x_0])).$$

Also, the inequalities  $\text{ri}(\vartheta_K(S/K[x_0])) - 1 \leq \text{ri}(\vartheta^{(1)}(S/K)) \leq \text{ri}(\vartheta_K(S/K[x_0]))$  follows from Proposition 5.2.2.

Finally, we prove (iii). We have  $\vartheta^{(n)}(S/K) = \langle \frac{\partial G_k}{\partial x_l} \mid 1 \leq k \leq r, 0 \leq l \leq n \rangle_S \neq \langle 0 \rangle$ . Since it follows from (ii) that  $s' = \text{HP}_{\vartheta^{(n-1)}(S/K)}(z) \leq \text{HP}_{\vartheta^{(n)}(S/K)}(z)$ , it remains to show the last inequality. It is easy to check that  $\vartheta^{(n)}(S/K) \subseteq \mathfrak{P}_j^{m_j-1} / \mathcal{I}_{\mathbb{W}}$  for  $j = s'+1, \dots, s$ .

Hence we get  $\vartheta^{(n)}(S/K) \subseteq (\bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j-1})/\mathcal{I}_{\mathbb{W}}$ . Thus we find

$$\begin{aligned} \text{HP}_{\vartheta^{(n)}(S/K)}(z) &\leq \text{HP}_{(\bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j-1})/\mathcal{I}_{\mathbb{W}}}(z) = \text{HP}_{\mathbb{W}}(z) - \text{HP}_{P/\bigcap_{j=s'+1}^s \mathfrak{P}_j^{m_j-1}}(z) \\ &= \sum_{j=1}^s \binom{m_j+n-1}{n} - \sum_{j=s'+1}^s \binom{m_j+n-2}{n} \\ &= \sum_{j=1}^s \left[ \binom{m_j+n-1}{n} - \binom{m_j+n-2}{n} \right] = \sum_{j=1}^s \binom{m_j+n-2}{n-1}. \end{aligned}$$

and claim (iii) is proved. □

**Remark 5.3.17.** If  $s' < s$ , then there exists an index  $j \in \{1, \dots, s\}$  such that  $\langle 0 \rangle \neq \vartheta^{(n)}(S/K) \subseteq \mathfrak{P}_j$ . Proposition 2.2.10 implies that  $\mu(\Omega_{S/K}^1) = n + 1$ , and hence the set  $\{dx_0, \dots, dx_n\}$  is a minimal system of generators of  $\Omega_{S/K}^1$ .

Both in Section 3.2 and in Section 3.3, we have seen that the Noether different for a 0-dimensional smooth scheme in  $\mathbb{P}_K^n$  contains a homogeneous non-zerodivisor of the homogeneous coordinate ring of the scheme. However, this property is not true for an arbitrary fat point scheme in  $\mathbb{P}_K^n$ , as the following proposition shows.

**Proposition 5.3.18.** *Let  $\mathbb{W} = \{p_1, \dots, p_s; m_1, \dots, m_s\}$  be a fat point scheme in  $\mathbb{P}^n$  such that  $m_1 = \dots = m_{s'} = 1$  and  $2 \leq m_{s'+1} \leq \dots \leq m_s$  for some  $0 \leq s' \leq s$ . Then the Noether different of  $S/K[x_0]$  satisfies  $\vartheta_N(S/K[x_0]) \subseteq \vartheta^{(n)}(S/K)$  and*

$$s' \leq \text{HP}_{\vartheta_N(S/K[x_0])}(z) \leq \sum_{j=1}^s \binom{m_j+n-2}{n-1},$$

and it does not contain a homogeneous non-zerodivisor of  $S$ . In particular, if  $n = 1$  and we let  $\mu = \sum_{j=1}^s m_j$ , then  $\vartheta_N(S/K[x_0]) = \vartheta_K(S/K[x_0])$  and

$$\text{HF}_{\vartheta_N(S/K[x_0])} : 0 \cdots 0 \underset{[\mu-1]}{1} \ 2 \ 3 \cdots s-1 \underset{[\mu+s-2]}{s} \ s \cdots .$$

*Proof.* Let  $\{G_1, \dots, G_r\}$  be a minimal homogeneous system of generators of  $\mathcal{I}_{\mathbb{W}}$ . Let  $h \in \vartheta_N(S/K[x_0]) \setminus \{0\}$ , and let  $H$  be a representative of  $h$  in  $P$ . By Corollary 3.1.5, we have  $H \cdot (X_i - x_i) \in \mathcal{I}_{\mathbb{W}}S[X_1, \dots, X_n]$  for  $i = 1, \dots, n$ . So, there are homogeneous elements  $H_{i1}, \dots, H_{ir} \in S[X_1, \dots, X_n]$  such that  $H \cdot (X_i - x_i) = \sum_{j=1}^r H_{ij}G_j$ . Then  $\frac{\partial(H \cdot (X_i - x_i))}{\partial X_i} = (X_i - x_i) \frac{\partial H}{\partial X_i} + H = \sum_{j=1}^r (H_{ij} \frac{\partial G_j}{\partial X_i} + G_j \frac{\partial H_{ij}}{\partial X_i})$ . By taking the image in  $S$ , the last equality becomes  $h = \sum_{j=1}^t h_{ij} \frac{\partial G_j}{\partial x_i}$ , where  $h_{ij}$  is the image of  $H_{ij}$  in  $S$ . This implies  $h \in \langle \frac{\partial G_j}{\partial x_i} \mid 1 \leq j \leq t, 0 \leq i \leq n \rangle_S = \vartheta^{(n)}(S/K)$ . Hence we obtain the

inclusion  $\vartheta_N(S/K[x_0]) \subseteq \vartheta^{(n)}(S/K)$  and Proposition 5.3.16(iii) yields

$$\text{HP}_{\vartheta_N(S/K[x_0])}(z) \leq \text{HP}_{\vartheta^{(n)}(S/K)}(z) \leq \sum_{j=1}^s \binom{m_j + n - 2}{n - 1}.$$

On the other hand, if  $n = 1$ , then the scheme  $\mathbb{W}$  is a complete intersection. In this case, we have  $\vartheta_K(S/K[x_0]) = \vartheta_N(S/K[x_0])$  by Corollary 3.3.5. Consequently, the above description of  $\text{HF}_{\vartheta_N(S/K[x_0])}$  follows from Lemma 5.3.3. In particular, we have  $\text{HP}_{\vartheta_N(S/K[x_0])}(z) = \text{HP}_{\vartheta_K(S/K[x_0])}(z) = s \geq s'$ . In the case  $n \geq 2$ , it follows from Proposition 3.3.9 that  $\vartheta_K(S/K[x_0]) \subseteq \vartheta_N(S/K[x_0])$ . Then Theorem 5.3.6 yields  $\text{HP}_{\vartheta_N(S/K[x_0])}(z) \geq \text{HP}_{\vartheta_K(S/K[x_0])}(z) = s'$ . Thus we have shown that  $\text{HP}_{\vartheta_N(S/K[x_0])}(z) \geq s'$ . Furthermore, the ideal  $\vartheta_N(S/K[x_0])$  does not contain a homogeneous non-zerodivisor of  $S$ , since otherwise we would have  $\text{HP}_{\vartheta_N(S/K[x_0])}(z) = \text{deg}(\mathbb{W}) > \sum_{j=1}^s \binom{m_j + n - 2}{n - 1}$ . Therefore the proof of the proposition is complete.  $\square$

Finally, we give an example of a fat point scheme  $\mathbb{W}$  in  $\mathbb{P}_K^n$  supported at a complete intersection, and we compute  $\text{HF}_{\vartheta_K(S/K[x_0])}$ ,  $\text{HF}_{\vartheta^{(i)}(S/K)}$ , and  $\text{HF}_{\vartheta_N(S/K[x_0])}$  in this case. Moreover, this example shows that the upper bound of  $\text{HP}_{\vartheta^{(n)}(S/K)}(z)$  given in Proposition 5.3.16 is sharp, and that the Noether and Kähler differents of  $S/K[x_0]$  have the same Hilbert polynomial but do not agree.

**Example 5.3.19.** Let  $\mathbb{W} = \{p_1, \dots, p_9; m_1, \dots, m_9\}$  be the fat point scheme in  $\mathbb{P}_{\mathbb{Q}}^4$ , where  $p_1 = (1 : 0 : 0 : 1 : 1)$ ,  $p_2 = (1 : 1 : 0 : 1 : 1)$ ,  $p_3 = (1 : 0 : 1 : 1 : 1)$ ,  $p_4 = (1 : 1 : 1 : 1 : 1)$ ,  $p_5 = (1 : 2 : 0 : 1 : 1)$ ,  $p_6 = (1 : 0 : 2 : 1 : 1)$ ,  $p_7 = (1 : 1 : 2 : 1 : 1)$ ,  $p_8 = (1 : 2 : 1 : 1 : 1)$ , and  $p_9 = (1 : 2 : 2 : 1 : 1)$ , and where  $m_1 = \dots = m_5 = 1$  and  $m_6 = \dots = m_9 = 2$ . Then the support of  $\mathbb{W}$  is a complete intersection, and we have  $\text{HF}_{\mathbb{W}} : 1 \ 5 \ 12 \ 18 \ 23 \ 25 \ \dots$  and  $r_{\mathbb{W}} = 5$ . It is not difficult to calculate the following Hilbert functions

$$\begin{aligned} \text{HF}_{\vartheta_K(S/\mathbb{Q}[x_0])} &: 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 4 \ 5 \ 5 \ \dots \\ \text{HF}_{\vartheta^{(1)}(S/\mathbb{Q}[x_0])} &: 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 5 \ 5 \ 5 \ \dots \\ \text{HF}_{\vartheta^{(4)}(S/\mathbb{Q}[x_0])} &: 0 \ 2 \ 8 \ 14 \ 19 \ 21 \ 21 \ \dots \\ \text{HF}_{\vartheta_N(S/\mathbb{Q}[x_0])} &: 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 3 \ 5 \ 5 \ \dots \end{aligned}$$

Thus we get  $\text{HP}_{\vartheta^{(4)}(S/\mathbb{Q}[x_0])}(z) = \sum_{j=1}^s \binom{m_j + n - 2}{n - 1} = 21$  and  $\text{ri}(\vartheta^{(4)}(S/\mathbb{Q}[x_0])) = 5$ . This indicates that the upper bound of  $\text{HP}_{\vartheta^{(n)}(S/K[x_0])}(z)$  given in Proposition 5.3.16 is sharp. Also, we have  $\text{HP}_{\vartheta_K(S/\mathbb{Q}[x_0])}(z) = \text{HP}_{\vartheta^{(1)}(S/\mathbb{Q}[x_0])}(z) = \text{HP}_{\vartheta_N(S/\mathbb{Q}[x_0])}(z) = s' = 5$  and

$$\text{ri}(\vartheta_N(S/\mathbb{Q}[x_0])) = 9 < \text{ri}(\vartheta^{(1)}(S/\mathbb{Q}[x_0])) = 12 < \text{ri}(\vartheta_K(S/\mathbb{Q}[x_0])) = 13.$$

Hence the Hilbert polynomials of  $\vartheta_N(S/\mathbb{Q}[x_0])$  and  $\vartheta_K(S/\mathbb{Q}[x_0])$  are equal, but their Hilbert functions are not equal.

# Appendix

In Chapter 3 we have discussed the computation of the Noether, Dedekind and Kähler differentials for 0-dimensional schemes in  $\mathbb{P}_K^n$ . In particular, we have explained algorithms for computing these differentials and their Hilbert functions. In this appendix we provide the functions which implement these algorithms in ApCoCoA and describe their usage with some examples. The computer algebra system ApCoCoA is primarily designed for working with *real world problems* by using symbolic computation methods and by developing new libraries for related computations. It can be obtained for free via the ApCoCoA home page:

<http://www.apcocoa.org>

There are also a comprehensive manual and a series of tutorials at this web address.

The default term ordering for the rings in ApCoCoA is defined as DegRevLex. We use this term ordering for our computations in all examples of the appendix.

## A.1 Implementation of the Division Algorithm 3.1.6

```
-----
-- NRR(F,GG,FF): Perform division in the residual class ring R = P/I_X
--       where P=K[X_0,...,X_n] is the standard graded polynomial ring over
--       a field K with deg(X_i)=1 and I_X is a homogeneous ideal of P
--       defining a 0-dimensional scheme X in P^n_K.
-- Input: F a homogeneous normal polynomial modulo I_X = <FF>
--       FF a list of a homogeneous Groebner basis of I_X
--       GG a list of homogeneous normal polynomials modulo I_X
-- Output: A list [R1,R2,[Q_i]] such that F = sum_iG_iQ_i+ R1 + R2, R2 in I_X
-----

Define NRR(F,GG,FF)
  If GG <> [] Then
```

```

K1 := Len(GG); Q := NewList(K1,0);
R1 := 0; R2 := 0; V := F;
LTG := [LT(P) | P In GG];
Repeat
  If NR(LT(V), LTG) = 0 Then
    D := DivAlg(LT(V), LTG);
    DD := D.Quotients;
    For I := 1 To Len(DD) Do
      If DD[I] <> 0 Then
        C := LC(V)/LC(GG[I]);
        Q[I] := Q[I] + C*DD[I];
        R2 := R2 + C*(NR(DD[I]*GG[I],FF) - DD[I]*GG[I]);
        V := V - C*NR(DD[I]*GG[I],FF);
      EndIf;
    EndFor;
    Else R1 := R1 + LM(V); V := V - LM(V);
  EndIf;
Until V = 0;
Return [R1,R2,Q];
Else Return [F,0,Q];
EndIf;
EndDefine;

```

**Example A.1.1.** Consider the projective point set  $\mathbb{X} = \{p_1, \dots, p_5\} \subseteq \mathbb{P}_{\mathbb{F}_3}^2$ , where  $p_1 = (1 : 0 : 0)$ ,  $p_2 = (1 : 1 : 1)$ ,  $p_3 = (1 : 2 : 0)$ ,  $p_4 = (1 : 0 : 2)$ , and  $p_5 = (1 : 2 : 2)$ . Then a homogeneous Gröbner basis of  $\mathcal{I}_{\mathbb{X}}$  is  $\mathcal{F} = \{X_0X_1 + X_1^2 - X_0X_2 - X_2^2, X_1^2X_2 - X_1X_2^2, X_0^2X_2 - X_2^3\}$ . Let  $F = -X_1^4 + X_0X_2^3 - X_1X_2^3 + X_2^4$  be the homogeneous normal polynomial modulo  $\mathcal{I}_{\mathbb{X}}$ , and let  $\mathcal{G}$  be the tuple consisting of the following three normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$ :  $G_1 = X_1^3 - X_2^3$ ,  $G_2 = X_0^3 - X_2^3$ , and  $G_3 = X_1X_2^3 - X_2^4$ . To compute  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F)$ , we run the following commands in ApCoCoA:

```

Use ZZ/(3)[x[0..2]];
F := - x[1]^4 + x[0]x[2]^3 - x[1]x[2]^3 + x[2]^4;
FF := [x[0]x[1] + x[1]^2 - x[0]x[2] - x[2]^2, x[1]^2x[2] - x[1]x[2]^2,
       x[0]^2x[2] - x[2]^3];
GG := [x[1]^3 - x[2]^3, x[0]^3 - x[2]^3, x[1]x[2]^3 - x[2]^4];
NRR(F,GG,FF);

```

The output of these commands is the following list whose first element is  $\text{NR}_{\sigma, \mathcal{I}_{\mathbb{X}}, \mathcal{G}}(F)$ :

```
[x[0]x[2]^3 - x[2]^4, 0, [-x[1], 0, 1]].
```



## A.2 The Buchberger Algorithm with Minimalization 3.1.10

```

-----
-- HBAM(HH, FF): Compute a minimal homogeneous system of generators of J/I_X
-- Input: FF a list of a homogeneous Groebner basis of I_X = <FF>
--       HH = [H_1,...,H_s] a list of homogeneous normal polynomials
--       modulo I_X such that J = <HH> + I_X and deg(H_1) <=...<= deg(H_s)
-- Output: A list whose elements generate minimally J/I_X
-----

```

```

Define HBAM(HH, FF)
  SS := []; VV := HH; GG := []; N := 0; HHmin := [];
  D := 0; Dmax := Deg(HH[Len(HH)]);
  SSd := []; VVd := [];
Repeat
  CC := ConcatLists([SS,VV]);
  D := Deg(CC[1]);
  For I := 2 To Len(CC) Do
    If D > Deg(CC[I]) Then D := Deg(CC[I]); EndIf;
  EndFor;
  If SS <> [] Then
    SSd := [F | F In SS And Deg(F)=D];
    SS := [F | F In SS And Deg(F)>D];
  EndIf;
  If VV <> [] Then
    VVd := [F | F In VV And Deg(F)=D];
    VV := [F | F In VV And Deg(F)>D];
  EndIf;
  While SSd <> [] Do
    S1 := SSd[1]; Remove(SSd,1);
    NRS1 := NRR(S1,GG,FF); S11 := NRS1[1];
    If S11 <> 0 Then
      N := N+1; Append(GG,S11);
      For I := 1 To N Do
        SI := LCM(LT(GG[I]), LT(GG[N]))* GG[I]/LM(GG[I])
          - LCM(LT(GG[I]), LT(GG[N]))* GG[N]/LM(GG[N]);
        Sin := NR(SI, FF);
        If Sin <> 0 And Deg(Sin) <= Dmax Then
          Append(SS, Sin);
        EndIf;
      EndFor;
      For J := 1 To Len(FF) Do
        SJ := LCM(LT(GG[N]), LT(FF[J]))* GG[N]/LM(GG[N])
          - LCM(LT(GG[N]), LT(FF[J]))* FF[J]/LM(FF[J]);

```

```

        Snj := NR(SJ, FF);
        If Snj <> 0 And Deg(Snj) <= Dmax Then
            Append(SS, Snj);
        EndIf;
    EndFor;
EndIf;
EndWhile;
While VVd <> [] Do
    H1 := VVd[1]; Remove(VVd,1);
    NRH1 := NRR(H1,GG,FF);
    H11 := NRH1[1];
    If H11 <> 0 Then
        N := N+1; Append(GG,H11); Append(HHmin,H1);
        For I:= 1 To N Do
            SI := LCM(LT(GG[I]), LT(GG[N]))* GG[I]/LM(GG[I])
                - LCM(LT(GG[I]), LT(GG[N]))* GG[N]/LM(GG[N]);
            Sin := NR(SI, FF);
            If Sin <> 0 And Deg(Sin) <= Dmax Then
                Append(SS, Sin);
            EndIf;
        EndFor;
        For J:= 1 To Len(FF) Do
            SJ := LCM(LT(GG[N]), LT(FF[J]))* GG[N]/LM(GG[N])
                - LCM(LT(GG[N]), LT(FF[J]))* FF[J]/LM(FF[J]);
            Snj := NR(SJ, FF);
            If Snj <> 0 And Deg(Snj) <= Dmax Then
                Append(SS, Snj);
            EndIf;
        EndFor;
    EndIf;
EndWhile;
Until (SS = [] And VV = []);
Return HHmin;
EndDefine;

```

**Example A.2.1.** Going back to Example A.1.1, we have seen that  $\mathcal{F} = \{X_0X_1 + X_1^2 - X_0X_2 - X_2^2, X_1^2X_2 - X_1X_2^2, X_0^2X_2 - X_2^3\}$  is a homogeneous Gröbner basis of the vanishing ideal  $\mathcal{I}_{\mathbb{X}}$  of  $\mathbb{X}$ . Let  $J$  be the homogeneous ideal of  $P$  generated by  $\mathcal{F} \cup \{H_1, \dots, H_4\}$ , where  $H_1 = X_1^3 - X_2^3$ ,  $H_2 = X_0^3 - X_2^3$ ,  $H_3 = X_0X_2^3 - X_2^4$ , and  $H_4 = X_2^4$ . Note that all  $H_i$  are normal homogeneous polynomials modulo  $\mathcal{I}_{\mathbb{X}}$  and  $\deg(H_1) \leq \dots \leq \deg(H_4)$ . We can compute a minimal homogeneous system of generators of the ideal  $J/\mathcal{I}_{\mathbb{X}}$  by running the following commands in ApCoCoA:

```

Use ZZ/(3)[x[0..2]];
FF := [x[0]x[1] + x[1]^2 - x[0]x[2] - x[2]^2, x[1]^2x[2] - x[1]x[2]^2,
       x[0]^2x[2] - x[2]^3];
HH := [x[1]^3 - x[2]^3, x[0]^3 - x[2]^3, x[0]x[2]^3 - x[2]^4, x[2]^4];
HBAM(HH,FF);

```

The output of the above commands is the following list of minimal generators of  $J/\mathcal{I}_{\mathbb{X}}$ :

```
[x[1]^3 - x[2]^3, x[0]^3 - x[2]^3, x[2]^4].
```

### A.3 Computation of the Noether Different 3.1.13

```

-----
-- NoetherDifferentRel(PP): Compute the Noether different
-- Input: PP = a list of points in  $P^n_K$  which are not contained in  $Z^+(x_0)$ 
--          or a homogeneous vanishing ideal of a 0-dimensional scheme
--          in  $P^n_K$  which does not intersect  $Z^+(x_0)$ 
-- Output: A list of minimal generators of the Noether different
-----

```

```

Define NoetherDifferentRel(PP)
  If Type(PP) = IDEAL Then
    IP := Minimalized(PP);
  Else IP := IdealOfProjectivePoints(PP);
  EndIf;
  N := Len(Indets()) - 1;
  Qxy := CoeffRing[x[0..N],y[1..N]];
  Using Qxy Do
    L := ConcatLists([[x[0]], [y[I] | I In 1..N]]); F := RMap(L);
    L1 := [x[I] | I In 0..N]; F1 := RMap(L1);
    J1 := Image(IP,F); GJ1 := ReducedGBasis(J1);
    J2 := Image(IP,F1);
    J3 := Ideal([x[I]-y[I] | I In 1..N]);
    J := (J2 + J1) : (J3 + J1);
  EndUsing;
  L3 := Indets(); L4 := ConcatLists([L3, Last(L3,N)]);
  F2 := RMap(L4); PL := ReducedGBasis(Image(J,F2));
  If Len(PL) > 1 Then
    For I := 1 To Len(PL) Do
      For J := I + 1 To Len(PL) Do
        If LT(PL[I]) > LT(PL[J]) Then
          CH := PL[I]; PL[I] := PL[J]; PL[J] := CH;
        EndIf;
      EndFor;
    EndFor;
  EndFor;

```

```

EndIf;
GIP := GBasis(IP); D := Len(PL); T := 1;
While T < D + 1 Do
  If NR(PL[T],GIP) = 0 Then
    Remove(PL,T); D := Len(PL); T := 1;
  Else PL[T] := NR(PL[T],GIP); T := T + 1;
  EndIf;
EndWhile;
NL := HBAM(PL, GBasis(IP));
Return NL;
EndDefine;

```

**Example A.3.1.** Let us compute the Noether different for the 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  of degree 16 defined by  $\mathcal{I}_{\mathbb{X}} = \bigcap_{j=1}^7 I_j$ , where  $I_1 = \langle X_0 - X_1, X_2 \rangle$ ,  $I_2 = \langle X_1, X_0 + X_2 \rangle$ ,  $I_3 = \langle X_0 - X_1, 2X_0 - X_2 \rangle$ ,  $I_4 = \langle X_0 - X_1, X_0 - X_2 \rangle$ ,  $I_5 = \langle X_1, X_2 \rangle^2$ ,  $I_6 = \langle 3X_0^2 + X_1^2, X_2 \rangle^2$ , and  $I_7 = \langle X_1, 2X_0^3 + X_2^3 \rangle$ . We run the following commands in ApCoCoA:

```

Use QQ[x[0..2]];
I1 := Ideal([x[0]-x[1],x[2]]);
I2 := Ideal([x[1],x[0]+x[2]]);
I3 := Ideal([x[0]-x[1],2x[0]-x[2]]);
I4 := Ideal([x[0]-x[1],x[0]-x[2]]);
I5 := Ideal([x[1],x[2]])^2;
I6 := Ideal([3x[0]^2+x[1]^2,x[2]])^2;
I7 := Ideal([x[1], 2x[0]^3+x[2]^3]);
IP := Intersection(I1, I2, I3, I4, I5, I6, I7);
NoetherDifferentRel(IP);

```

The result of these commands is the following list of minimal generators of the Noether different.

```

[x[1]^2x[2]^4 - 3/5x[1]x[2]^5, x[0]^2x[2]^6 + 23/25x[0]x[2]^7 + 1/25x[2]^8,
x[0]^3x[2]^5 - 3/25x[0]x[2]^7 + 14/25x[2]^8, x[0]^4x[1]^5 + 2/3x[0]^2x[1]^7
+ 1/9x[1]^9].

```

## A.4 Computation of the Dedekind Different 3.2.29

```

-----
-- DedekindDifferentRel(PP): Compute the Dedekind different
-- Input: PP = a list of points in P^n_K which are not contained in Z^+(x_0)
-- Output: A list of minimal generators of the Dedekind different
-----

```

---

```

Define DedekindDifferentRel(PP)
  N := Len(PP[1]) - 1;
  PP := [(PP[I][1])^(-1)*PP[I] | I In 1..Len(PP)];
  RR := CoeffRing[x[1..N]];
  IP := IdealOfProjectivePoints(PP);
  S := SeparatorsOfProjectivePoints(PP);
  Rx := RegularityIndex(Hilbert(CurrentRing()/IP));
  P1 := [Last(PP[I],N) | I In 1..Len(PP)];
  Using RR Do
    I1 := IdealOfPoints(P1); OI := QuotientBasis(I1);
    L1 := ConcatLists([[1],Indets()]); F := RMap(L1);
    S1 := Image(S,F); GI1 := GBasis(I1);
    S2 := [NR(S1[I],GI1) | I In 1..Len(S1)];
    OA := NewList(0); D := 1;
    For J := 1 To Len(OI) Do
      For I := 1 To N Do
        If (x[I]*OI[J] IsIn OI) Then D := D + 1; EndIf;
      EndFor;
      If D = 1 Then Append(OA, OI[J]); Else D := 1; EndIf;
    EndFor;
  EndUsing;
  L2 := Indets(); F1 := RMap(Last(L2,N));
  OP := Image(OA,F1); S3 := Image(S2,F1);
  SI := ConcatLists([Homogenized(L2[1], [S3[I]]) | I In 1..Len(S3)]);
  S := [L2[1]^(Rx-Deg(F))*F | F In SI];
  L3 := []; V := NewList(Len(OP));
  For I := 1 To Len(OP) Do
    L3 := [CoeffOfTerm(L2[1]^(Rx-Deg(OP[I]))*OP[I], F) | F In S];
    V[I] := (L2[1]^(Rx-Deg(OP[I])))*Sum([S[J]*L3[J] | J In 1..Len(S)]);
  EndFor;
  PL := ReducedGBasis((Ideal(L2[1]^(2*Rx)) + IP): Ideal(V) + IP);
  If Len(PL) > 1 Then
    For I := 1 To Len(PL)-1 Do
      For J := I+1 To Len(PL) Do
        If LT(PL[I]) > LT(PL[J]) Then
          CH := PL[I]; PL[I] := PL[J]; PL[J] := CH;
        EndIf;
      EndFor;
    EndFor;
  EndIf;
  GIP := GBasis(IP); D := Len(PL); T := 1;
  While T < D + 1 Do
    If NR(PL[T],GIP) = 0 Then

```

```

        Remove(PL,T); D := Len(PL); T := 1;
    Else PL[T] := NR(PL[T],GIP); T := T + 1;
    EndIf;
EndWhile;
NL := HBAM(PL, GBasis(IP));
Return NL;
EndDefine;

```

**Example A.4.1.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set consisting of eight points  $p_1 = (1 : 0 : 0 : 0)$ ,  $p_2 = (1 : 1 : 0 : 0)$ ,  $p_3 = (1 : -1 : 0 : 0)$ ,  $p_4 = (1 : 2 : 0 : 0)$ ,  $p_5 = (1 : 0 : 1 : 1)$ ,  $p_6 = (1 : 0 : 0 : 1)$ ,  $p_7 = (1 : 0 : -1 : 1)$ , and  $p_8 = (1 : 0 : 2 : 1)$ . In order to compute the Noether and Dedekind differentials for  $\mathbb{X}$ , we run the following commands in ApCoCoA:

```

Use QQ[x[0..3]];
PP := [[1,0,0,0], [1,1,0,0], [1,-1,0,0], [1,2,0,0],
[1,0,0,1], [1,0,1,1], [1,0,-1,1], [1,0,2,1]];
NoetherDifferentRel(PP);
DedekindDifferentRel(PP);

```

The results of these commands are the following two lists of minimal generators of the Noether and Dedekind differentials respectively.

```

[x[2]^2x[3]^2 - x[2]x[3]^3 - 1/5x[3]^4, x[2]^3x[3] - 2x[2]x[3]^3 + 1/5x[3]^4,
x[0]^2x[1]^2 + 4x[0]x[1]^3 - 3x[1]^4, x[0]^4 + 15x[0]x[1]^3 - 10x[1]^4 - x[3]^4]
-----
[x[2]^2x[3]^2 - x[2]x[3]^3 - 1/5x[3]^4, x[2]^3x[3] - 2x[2]x[3]^3 + 1/5x[3]^4,
x[0]^2x[1]^2 + 4x[0]x[1]^3 - 3x[1]^4, x[0]^4 + 15x[0]x[1]^3 - 10x[1]^4 - x[3]^4]

```

Thus the Noether and Dedekind differentials are equal in this case.

## A.5 Computation of the Hilbert Function of the ND-Different

```

-----
-- HilbertNDDiff(PP): Compute the Hilbert function of the ND-different
-- Input: PP = a list of points in P^n_K which are not contained in Z^+(x_0)
-- Output: The Hilbert function of the ND-different
-----

```

```

Define HilbertNDDiff(PP)
    N := Len(PP[1]) - 1;
    PP := [(PP[I][1])^(-1)*PP[I] | I In 1..Len(PP)];
    RR := CoeffRing[x[1..N]];
    IP := IdealOfProjectivePoints(PP);

```

---

```

S := SeparatorsOfProjectivePoints(PP);
HF1 := HilbertFn(CurrentRing()/IP);
Rx := RegularityIndex(HF1);
P1 := [Last(PP[I],N) | I In 1..Len(PP)];
Using RR Do
  I1 := IdealOfPoints(P1); OI := QuotientBasis(I1);
  L1 := ConcatLists([[1],Indets()]);
  F := RMap(L1); S1 := Image(S,F); GI1 := GBasis(I1);
  S2 := [NR(S1[I],GI1) | I In 1..Len(S1)];
  OA := NewList(0); D := 1;
  For J := 1 To Len(OI) Do
    For I := 1 To N Do
      If (x[I]*OI[J] IsIn OI) Then D := D + 1; EndIf;
    EndFor;
    If D = 1 Then Append(OA, OI[J]); Else D := 1; EndIf;
  EndFor;
EndUsing;
L2 := Indets(); F1 := RMap(Last(L2,N));
OP := Image(OA,F1); S3 := Image(S2,F1);
SI := ConcatLists([Homogenized(L2[1], [S3[I]]) | I In 1..Len(S3)]);
S := [L2[1]^(Rx-Deg(F))*F | F In SI];
L3 := []; V := NewList(Len(OP));
For I := 1 To Len(OP) Do
  L3 := [CoeffOfTerm(L2[1]^(Rx-Deg(OP[I]))*OP[I], F) | F In S];
  V[I] := (L2[1]^(Rx-Deg(OP[I])))*Sum([S[J]*L3[J] | J In 1..Len(S)]);
EndFor;
PL := (Ideal(L2[1]^(2*Rx)) + IP): Ideal(V) + IP;
HF2 := HilbertFn(CurrentRing()/PL);
EvalHF := [EvalHilbertFn(HF1,K) - EvalHilbertFn(HF2,K) | K In 0..2*Rx];
Rvd := Len(EvalHF)-1;
For I := 1 To Rvd Do PrintLn "H(",I-1,") = ",EvalHF[I]
EndFor;
Using QQt Do
  Print "H(t) = ", EvalHF[Rvd+1], " for t >= ", Rvd
EndUsing;
EndDefine;

```

**Example A.5.1.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  be the projective point set given in Example A.4.1. The Hilbert function of the ND-different can be computed by running the following commands in ApCoCoA:

```

Use QQ[x[0..3]];
PP := [[1,0,0,0], [1,1,0,0], [1,-1,0,0], [1,2,0,0],

```

```
[1,0,0,1], [1,0,1,1], [1,0,-1,1], [1,0,2,1]];
HilbertNDDiff(PP);
```

The output of the above commands is the following Hilbert function of the ND-different:

```
H(0) = 0
H(1) = 0
H(2) = 0
H(3) = 0
H(4) = 4
H(5) = 6
H(t) = 8   for t >= 6
-----
```

## A.6 Computation of the Kähler Different 3.3.2

```
-----
-- KaehlerDifferentRel(PP): Compute the Kaehler different
-- Input: PP = a list of points in P^n_K which are not contained in Z^+(x_0)
--          or a homogeneous vanishing ideal of a 0-dimensional scheme
--          in P^n_K which does not intersect Z^+(x_0)
-- Output: A list of minimal generators of the Kaehler different
-----
```

```
Define KaehlerDifferentRel(PP)
  If Type(PP) = IDEAL Then
    IP := Minimalized(PP);
  Else IP := Minimalized(IdealOfProjectivePoints(PP));
  EndIf;
  M := Jacobian(Gens(IP)); K := Len(M); N := Len(M[1]);
  J1 := Submat(M,ConcatLists([1..K]),2..N);
  JJ := Minors(N-1, J1);
  PL := ReducedGBasis(Ideal(JJ) + IP);
  If Len(PL) > 1 Then
    For I := 1 To Len(PL) Do
      For J := I + 1 To Len(PL) Do
        If LT(PL[I]) > LT(PL[J]) Then
          CH := PL[I]; PL[I] := PL[J]; PL[J] := CH;
        EndIf;
      EndFor;
    EndFor;
  EndIf;
  GIP := GBasis(IP); D := Len(PL); T := 1;
  While T < D + 1 Do
    If NR(PL[T],GIP) = 0 Then
```



```

        Remove(PL,T); D := Len(PL); T := 1;
    Else PL[T] := NR(PL[T],GIP); T := T + 1;
    EndIf;
EndWhile;
NL := HBAM(PL, GBasis(IP));
Return NL;
EndDefine;

```

**Example A.6.1.** Let us compute the Kähler different for the projective point set  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^3$  given in Example A.4.1. We run the following commands in ApCoCoA:

```

Use QQ[x[0..3]];
PP := [[1,0,0,0], [1,1,0,0], [1,-1,0,0], [1,2,0,0],
[1,0,0,1], [1,0,1,1], [1,0,-1,1], [1,0,2,1]];
KaehlerDifferentRel(PP);

```

The output of the above commands is the following list of minimal generators of the Kähler different:

```

[x[2]x[3]^4 - 1/2x[3]^5, x[2]^2x[3]^3 - 7/10x[3]^5, x[2]^3x[3]^2 - 4/5x[3]^5,
x[0]x[1]^4 - 13/20x[1]^5, x[0]^2x[1]^3 - 2/5x[1]^5, x[0]^5 - 1/2x[1]^5 - x[3]^5]

```

In this case the Kähler different is a proper subideal of the ND-different, since the ND-different is generated by homogeneous elements of degree 4 (see Example A.4.1).

## A.7 Computation of the Hilbert Function of the Kähler Different

```

-----
-- HilbertKDiff(PP): Compute the Hilbert function of the Kaehler different
-- Input: PP = a list of points in P^n_K which are not contained in Z^+(x_0)
--           or a homogeneous vanishing ideal of a 0-dimensional scheme
--           in P^n_K which does not intersect Z^+(x_0)
-- Output: The Hilbert function of the Kaehler different
-----

```

```

Define HilbertKDiff(PP)
  If Type(PP) = IDEAL Then
    IP := Minimalized(PP);
  Else IP := Minimalized(IdealOfProjectivePoints(PP));
  EndIf;
  J := Jacobian(Gens(IP)); K := Len(J); N := Len(J[1]);
  J1 := Submat(J,ConcatLists([1..K]),2..N);
  JJ := Minors(N-1, J1); J2 := Ideal(JJ) + IP;
  HF1 := HilbertFn(CurrentRing()/IP); HF2 := HilbertFn(CurrentRing()/J2);
  KO := Max(RegularityIndex(HF1),RegularityIndex(HF2));

```

```

DH := EvalHilbertFn(HF1,K0)-EvalHilbertFn(HF2,K0); K1 := K0+1;
Repeat K1 := K1-1;
Until EvalHilbertFn(HF1,K1) - EvalHilbertFn(HF2,K1) <> DH Or K1 = 0;
If K1 = 0 Then EvalHF := [EvalHilbertFn(HF1,K1) - EvalHilbertFn(HF2,K1)];
Else EvalHF := [EvalHilbertFn(HF1,K) - EvalHilbertFn(HF2,K) | K In 0..(K1+1)];
EndIf;
Rvd := Len(EvalHF) - 1;
For I := 1 To Rvd Do PrintLn "H(",I-1,") = ",EvalHF[I]
EndFor;
Using QQt Do
    Print "H(t) = ", EvalHF[Rvd+1], " for t >= ", Rvd
EndUsing;
EndDefine;

```

**Example A.7.1.** Let  $\mathbb{X} \subseteq \mathbb{P}_{\mathbb{Q}}^2$  be the 0-dimensional scheme given in Example A.3.1. We compute the Hilbert function of the Kähler different by running the following commands in ApCoCoA:

```

Use QQ[x[0..2]];
I1 := Ideal([x[0]-x[1],x[2]]);
I2 := Ideal([x[1],x[0]+x[2]]);
I3 := Ideal([x[0]-x[1],2x[0]-x[2]]);
I4 := Ideal([x[0]-x[1],x[0]-x[2]]);
I5 := Ideal([x[1],x[2]])^2;
I6 := Ideal([3x[0]^2+x[1]^2,x[2]])^2;
I7 := Ideal([x[1], 2x[0]^3+x[2]^3]);
IP := Intersection(I1, I2, I3, I4, I5, I6, I7);
HilbertKDiff(IP);

```

The result of these commands is the following Hilbert function of the Kähler different:

```

H(0) = 0
H(1) = 0
H(2) = 0
H(3) = 0
H(4) = 0
H(5) = 0
H(6) = 1
H(7) = 2
H(8) = 4
H(t) = 7 for t >= 9
-----

```

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