

Asymptotic optimality of scalar Gersho quantizers

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Abstract

In his famous paper [7] Gersho stressed that the codecells of optimal quantizers asymptotically make an equal contribution to the distortion of the quantizer. Motivated by this fact, we investigate in this paper quantizers in the scalar case, where each codecell contributes with exactly the same portion to the quantization error. We show that such quantizers of Gersho type - or Gersho quantizers for short - exist for non-atomic scalar distributions. As a main result we prove that Gersho quantizers are asymptotically optimal.

Keywords Asymptotically optimal quantization · Quantization error · Scalar quantization · Gersho quantizer · High rate quantization

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1 Introduction and notation

Let μ be a Borel probability measure on \mathbb{R}^d . A Borel-measurable mapping $q : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called quantizer if $\text{card}(q(\mathbb{R}^d)) \leq \text{card}(\mathbb{N})$, where card denotes cardinality. For any quantizer q , norm $\|\cdot\|$ and norm exponent $r > 0$ we define the distortion or quantization error

$$D(\mu, q, r) = \int \|x - q(x)\|^r d\mu(x) = \sum_{a \in q(\mathbb{R}^d)} \int_{q^{-1}(a)} \|x - a\|^r d\mu(x),$$

where $q^{-1}(a)$ is called codecell of the codepoint $a \in q(\mathbb{R}^d)$. The set $q(\mathbb{R}^d)$ is called codebook. The quantization error can be interpreted as a measure for the distance between μ and the image $\mu \circ q^{-1}$ of μ under q . Indeed, if \mathbb{R}^d is equipped with the Euclidean norm and if μ is vanishing on continuously differentiable $(d-1)$ -dimensional submanifolds of \mathbb{R}^d , then the quantization error equals the L_r -Wasserstein distance (see e.g. [15, Theorem 2.6]). Let us denote the set of

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all quantizers on \mathbb{R}^d by \mathcal{Q} . For $n \in \mathbb{N}$ we define the optimal n -th quantization error for μ of order r as

$$D_{n,r}(\mu) = \inf\{D(\mu, q, r) : q \in \mathcal{Q} \text{ and } \text{card}(q(\mathbb{R}^d)) \leq n\}.$$

The problem of optimal quantization is to find an optimal n -level quantizer for μ , i.e. a $q \in \mathcal{Q}$ with $D(\mu, q, r) = D_{n,r}(\mu)$. Although an optimal quantizer exists under weak assumptions on μ (cf. [10, Theorem 4.12]), the determination of the optimal quantizers has been achieved so far only for a few distributions (see e.g. [10, p.69ff], [6], [13], [14]). As another difficulty, often more than one optimal quantizer exists. This phenomenon of non-uniqueness can happen for distributions which are absolutely continuous with respect to the Lebesgue measure (cf. [1] resp. [10, Example 5.2]) but also for distributions which are singular [9, 14].

Due to these difficulties and also in view of aspects in applications (see [12, 8] for an excellent overview), one is more interested in asymptotics of the optimal quantization error for large quantization levels n . For distributions which have a non-vanishing part that is absolutely continuous with respect to the Lebesgue measure and which are satisfying a certain moment condition, these asymptotics are well-known [23, 3, 10]. Now n -level quantizers are of interest, which induce the same error asymptotics as the optimal ones if n tends to infinity. Such a sequence of quantizers is called asymptotically optimal.

Delattre et al. [5] have shown for a large class of scalar distributions, that for any sequence of asymptotically optimal quantizers, which in addition satisfy a condition of stationarity, the codecells contribute asymptotically with equal portion to the distortion of the quantizer. This behavior was first mentioned by Panter and Dite [19] for optimal scalar quantizers under high rates. Tóth [21] and Gersho [7] conjectured this asymptotical behavior of optimal quantizers also for higher-dimensional distributions. Now one can question if it is possible to construct asymptotically optimal quantizers by using this property of asymptotically equal moments on each codecell. Apart from [5], the formulation of this uniformity is quite vague or only conjectured. To start with a thorough mathematical formulation we consider quantizers where each codecell contributes with exactly the same portion to the quantization error. In the following section we will show that such quantizers of Gersho type - or Gersho quantizers for short - exist for non-atomic scalar distributions and all levels n . Moreover, we will show that Gersho quantizers are even unique for all levels n if μ has an interval as support.

Then, one can ask if such Gersho quantizers are asymptotically optimal. As a main result of this paper we will give in section three a positive answer to this question for a large class of scalar probability distributions (cf. Theorem 3.10). In the last section we provide concluding remarks which are mainly consisting of several remaining open questions.

For the rest of this paper let us assume that μ is one-dimensional and non-atomic, i.e. $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Moreover, we assume throughout the paper that μ has a finite r -th moment. For any Borel-measurable $A \subset \mathbb{R}$ with

$\mu(A) > 0$ we denote by $\mu(\cdot|A)$ the conditional probability of μ with respect to A . We denote by $\mathcal{C}(n, \mu, r)$ the set of all n -optimal quantizers for μ of order r .

Definition 1.1. We call $q \in \mathcal{Q}$ an n -level Gersho-quantizer of order r for μ if

(G1) $\text{card}(q(\mathbb{R})) = n$,

(G2) $q^{-1}(a)$ is an interval for every $a \in q(\mathbb{R})$,

(G3) every $a \in q(\mathbb{R})$ is optimal for its own codecell, i.e. the mapping $\mathbb{R} \ni x \rightarrow a$ is an element of $\mathcal{C}(1, \mu(\cdot|q^{-1}(a)), r)$ and

(G4) every codecell of q contributes an equal amount to the overall distortion, i.e. $\int_{q^{-1}(a)} |x - a|^r d\mu(x) = \frac{1}{n} D(\mu, q, r)$ for every $a \in q(\mathbb{R})$.

Let us denote by $\mathcal{G}(n, \mu, r)$ the set of all n -level Gersho-quantizers of order r for μ .

Remark 1.2. Because μ is non-atomic, we know that $D(\mu, q, r) > 0$ for every $q \in \mathcal{G}(n, \mu, r)$. Consequently, property (G4) implies that $\mu(q^{-1}(a)) > 0$ for every $a \in q(\mathbb{R})$, i.e. all codecells have non-vanishing μ -mass and, therefore, $\mu(\cdot|q^{-1}(a))$ in (G3) is well defined.

Remark 1.3. Let $q \in \mathcal{C}(n, \mu, r)$ be an n -optimal quantizer. If we consider the codebook $q(\mathbb{R})$ and a Voronoi partition $\{A_a : a \in q(\mathbb{R})\}$ of \mathbb{R} with respect to $q(\mathbb{R})$, i.e.

$$A_a \subset \{x \in \mathbb{R} : |x - a| = \min_{b \in q(\mathbb{R})} |x - b|\} \quad \mu - \text{a.s. for every } a \in q(\mathbb{R}),$$

then the quantizer $q : \mathbb{R} \rightarrow \mathbb{R}$ with $q(x) = a$ if $x \in A_a$ is also an n -optimal one (cf. [10, Lemma 3.1]). Due to this fact, the determination of optimal quantizers is reduced to the determination of an optimal n -codebook, where the codecells are specified by an arbitrary Voronoi partition related to this set. Let us call such quantizers Voronoi quantizers. Insofar we can and will assume w.l.o.g. that every n -optimal quantizer is a Voronoi quantizer and satisfies (G1), (G2) and (G3), see e.g. [10, Lemma 3.1] and [10, Theorem 4.1]. Gersho quantizers and optimal quantizers even coincide for some distributions. E.g. if μ is the uniform distribution on $[0, 1]$, then it follows straightforward from property (G4) and [10, Example 5.5] that $\mathcal{G}(n, \mu, r) = \mathcal{C}(n, \mu, r)$ for every $n \in \mathbb{N}$ and $r > 1$. In general, this is not the case. Even

$$\mathcal{G}(n, \mu, r) \cap \mathcal{C}(n, \mu, r) = \emptyset$$

and

$$D_{n,r}(\mu) < \inf\{D(\mu, q, r) : q \in \mathcal{G}(n, \mu, r)\}$$

for certain distributions μ , quantization levels n and norm exponent r is possible, see e.g. μ as from [1] resp. [10, Example 5.2] with $n = 2$ and $r = 2$. It remains open to characterize the Gersho quantizers which are Voronoi quantizers.

Asymptotically optimal quantizers in the scalar case can be constructed by companding techniques [17]. Alternatively, quantizers - which are unique and optimal for strongly unimodal scalar distributions (cf. [22]) - can be numerically determined by the famous Lloyd algorithm [18]. To get an overview of the historical development and the numerous modifications of the Lloyd algorithm the user is referred to [12]. One drawback of all these Lloyd methods so far is that at any level n the numerical calculation of the quantizer starts from scratch, i.e. the calculation results from lower quantization levels can not be used to reduce calculation complexity for level n . Moreover, the limit point of the Lloyd algorithm is not always necessarily an optimal one.

Clearly, the main goal of this paper is a theoretical one. In the following sections we investigate existence, uniqueness and asymptotic optimality of Gersho quantizers. Nevertheless, let us mention here that Gersho quantizers are also of practical relevance in constructing asymptotically optimal quantizers.

Using the results of this paper, we can determine (for a large class of probabilities) numerically the sequence of asymptotically optimal Gersho quantizers where the calculations from lower quantization levels are incorporated. In more detail, let $k \in \mathbb{N}$ and $n = 2^k$. For $k = 1$ Proposition 2.5 ensures the existence of a unique Gersho quantizer. We can determine this quantizer numerically e.g. by a bisection algorithm (cf. Remark 2.8). If $k > 1$, then the uniqueness of Gersho quantizers (cf. Proposition 2.5) enables us to determine the Gersho quantizer for level $n = 2^k$ by dividing all codecells of the quantizer for level 2^{k-1} into two cells having the same moment, such that (G4) is satisfied. This dispartment of the codecells is also done numerically by application of a bisection algorithm. Theorem 3.10 yields the asymptotic optimality of the quantizer sequence.

As a main advantage in contrast to any Lloyd algorithm we always determine a unique solution. Moreover, this algorithm is also amenable to a parallel implementation. Of course, we can determine numerically also the Gersho quantizers for level $n = n_0 2^k$ with fixed $n_0 \geq 2$ and $k \in \mathbb{N}$ or even any arbitrary level n . But in this last general case we are unable to use the calculation results from lower quantization levels for the actual one.

2 Existence and uniqueness of Gersho-quantizers

As already mentioned in the introduction, optimal quantizers exist under weak assumptions on μ , see [2], [20] and [10, Theorem 4.12]. In this section we will show by Proposition 2.5, that Gersho quantizers also always exist for non-atomic distributions. Before studying uniqueness we need to make some limitations for Gersho quantizers. Let $\mathcal{F}(n, \mu, r)$ be the set of all scalar quantizers where all codecells of such a quantizer have non-vanishing μ -mass and (G1), (G2) and (G3) are satisfied. By definition and in view of Remark 1.3 resp. Remark 1.2 we know that

$$\mathcal{G}(n, \mu, r) \cup \mathcal{C}(n, \mu, r) \subset \mathcal{F}(n, \mu, r).$$

Let $q \in \mathcal{F}(n, \mu, r)$ and let $I_1(q), \dots, I_n(q)$ be the codecells of q in increasing order. Let $a_i = \inf(I_i(q))$ and $b_i = \sup(I_i(q))$ for every $i \in \{1, \dots, n\}$. Clearly,

$$a_1 = -\infty, b_n = \infty \text{ and } a_i = b_{i-1} \text{ for every } i \in \{2, \dots, n\}. \quad (1)$$

Because μ is non-atomic let us assume w.l.o.g. throughout this paper that

$$I_1(q) = (a_1, b_1), \quad I_i(q) = [a_i, b_i] \quad \text{for every } i \in \{2, \dots, n\}.$$

Thus, the codecells of any quantizer $q \in \mathcal{F}(n, \mu, r)$ are completely characterized by their boundary points. With these conventions we will also investigate in this section the uniqueness of Gersho quantizers. Important for our subsequent argumentation is the following well-known result.

Proposition 2.1. (Uniqueness of 1-optimal quantizers) *If $r > 1$, then always exactly one 1-optimal quantizer for μ exists, i.e. $\text{card}(\mathcal{C}(1, \mu, r)) = 1$.*

Proof. The assertion follows from the strict convexity of the mapping

$$\mathbb{R} \ni a \rightarrow \int |x - a|^r d\mu(x) \in (0, \infty).$$

For a complete proof see e.g. [10, Theorem 2.4]. \square

In the following, Proposition 2.4 shows that every n -level Gersho-quantizer induces the same quantization error. Moreover, Proposition 2.5 states that exactly one n -level Gersho quantizer exists, i.e. $\text{card}(\mathcal{G}(n, \mu, r)) = 1$ if the support of μ is an (possibly unbounded) interval.

For any $-\infty \leq a < b \leq \infty$ we define $I_{a,b} = (a, b)$. If $\mu(I_{a,b}) > 0$ and $r > 1$, then Proposition 2.1 implies that always exactly one 1-optimal quantizer for $\mu(\cdot|I_{a,b})$ of order r exists, i.e. $\mathcal{C}(1, \mu(\cdot|I_{a,b}), r)$ is not empty and consists of exactly one element. We denote by $c_{a,b} \in \mathbb{R}$ the point which represents this unique optimal quantizer of $\mu(\cdot|I_{a,b})$, i.e.

$$\int |x - c_{a,b}|^r d\mu(\cdot|I_{a,b})(x) = D_{1,r}(\mu(\cdot|I_{a,b})).$$

In view of [10, Lemma 2.6(a)] we additionally know that $c_{a,b} \in [a, b]$. If we want to stress the dependency of $c_{a,b}$ on μ we write $c_{a,b}(\mu)$. Let us denote by $\text{supp}(\mu)$ the support of μ . For any set $I \subset \mathbb{R}$ we denote by 1_I the indicator function of I . Now let us consider the r -th moment $\int_{I_{a,b}} |x - c_{a,b}|^r d\mu(x)$ on $I_{a,b}$. It seems to be obvious that this moment increases with b . Let us prove this important result.

Lemma 2.2. *Let $-\infty \leq a < b < \infty$ and*

$$J = (a - b, \infty) \cap (\inf(\text{supp}(\mu)), \sup(\text{supp}(\mu))).$$

If $\mu(I_{a,b}) > 0$ and $r > 1$, then the mapping

$$J \ni z \rightarrow W_{a,b,\mu}(z) = \int_{I_{a,b+z}} |x - c_{a,b+z}|^r d\mu(x) \quad (2)$$

is continuous and increasing. Moreover, for every $x, y \in J$ with $x < y$

$$W_{a,b,\mu}(x) < W_{a,b,\mu}(y) \text{ if and only if } \mu([x+b, y+b]) > 0.$$

Proof. Note that $\mu(I_{a,b+z}) > 0$ for every $z \in J$.

1. First we will show that $J \ni z \rightarrow c_{a,b+z}$ is continuous.

Let $z \in J$. We proceed indirectly and assume that an $M > 0$ and a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ exist with $z + \varepsilon_n \in J$ for every $n \in \mathbb{N}$, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$|c_{a,b+z+\varepsilon_n} - c_{a,b+z}| \geq M \quad (3)$$

for every $n \in \mathbb{N}$. Since the mapping $\mathbb{R} \ni w \rightarrow \int_{I_{a,b+z}} |x-w|^r d\mu(z)$ is strictly convex (see proof of Proposition 2.1) we deduce that

$$\begin{aligned} C_1(M) &:= \max\left\{\int_{I_{a,b+z}} |x-w|^r d\mu(x) : w \in \{c_{a,b+z} - M, c_{a,b+z} + M\}\right\} \\ &= \inf\left\{\int_{I_{a,b+z}} |x-w|^r d\mu(x) : |w - c_{a,b+z}| \geq M\right\}. \end{aligned} \quad (4)$$

Let

$$C_2(M) = C_1(M) - \int_{I_{a,b+z}} |x - c_{a,b+z}|^r d\mu(x) > 0.$$

Now choose $n_0 \in \mathbb{N}$ such that

$$\int_{I_{a,b+z+\varepsilon_n}} |x - c_{a,b+z}|^r d\mu(x) - \int_{I_{a,b+z}} |x - c_{a,b+z}|^r d\mu(x) < C_2(M) \quad (5)$$

for every $n \geq n_0$. Now let $n \geq n_0$. Combining (3) and (4) we observe that

$$\int_{I_{a,b+z+\varepsilon_n}} |x - c_{a,b+z+\varepsilon_n}|^r d\mu(x) \geq \int_{I_{a,b+z}} |x - c_{a,b+z+\varepsilon_n}|^r d\mu(x) \geq C_1(M).$$

Using (5) we get

$$\begin{aligned} &\int_{I_{a,b+z+\varepsilon_n}} |x - c_{a,b+z+\varepsilon_n}|^r d\mu(x) \\ &\geq C_2(M) + \int_{I_{a,b+z}} |x - c_{a,b+z}|^r d\mu(x) > \int_{I_{a,b+z+\varepsilon_n}} |x - c_{a,b+z}|^r d\mu(x), \end{aligned}$$

which contradicts the optimality of $c_{a,b+z+\varepsilon_n}$. Hence, $J \ni z \rightarrow c_{a,b+z}$ is continuous.

2. Rest of the proof.

Let $\varepsilon > 0$ and $z \in J$. For any $\delta > 0$ we define

$$C_1(\delta) = \sup\{|c_{a,b+v}| : v \in J, |v - z| < \delta\}.$$

By step 1 we can find a $\delta_1 > 0$ such that $C_1(\delta_1) < \infty$. For every $v \in J$ with $|z - v| < \delta_1$ we have

$$|x - c_{a,b+v}|^r \leq (2 \max(|x|, |c_{a,b+v}|))^r \leq 2^r (|x|^r + C_1(\delta_1)^r),$$

where the right hand side is μ -integrable, because the r -th moment of μ is finite. Thus a $\delta_2 \in (0, \delta_1)$ exists with

$$\int_{[b+z-\delta_2, b+z+\delta_2]} |x - c_{a,b+v}|^r d\mu(x) < \frac{\varepsilon}{2} \quad (6)$$

for every $v \in J$ with $|z - v| < \delta_2$. For every $\delta_3 \in (0, \delta_2)$ we define

$$C_2(\delta_3) = \sup\{|c_{a,b+z} - c_{a,b+v}| : v \in J, |v - z| < \delta_3\} < \infty.$$

Applying step 1 we deduce that

$$C_2(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (7)$$

For any $v \in J$ with $|v - z| < \delta_3$ and $x \in I_{a,b+z+\delta_3}$ we define

$$f(x, z, v) = ||x - c_{a,b+z}|^r - |x - c_{a,b+v}|^r|. \quad (8)$$

Clearly,

$$f(x, z, v) \leq (|x| + |c_{a,b+z}|)^r + (|x| + |c_{a,b+z}| + C_2(\delta_3))^r. \quad (9)$$

Due to $C_2(\delta_3) < \infty$ the right hand side of (9) is μ -integrable. From (7) and (8) we deduce that $f(x, z, v) \rightarrow 0$ as $v \rightarrow z$. Hence, by dominated convergence a $\delta_4 \in (0, \delta_3)$ exists, such that

$$\int_{I_{a,b+z}} f(x, z, v) d\mu(x) < \varepsilon/2, \quad (10)$$

if $|v - z| < \delta_4$. Now let $v \in J$ such that $|v - z| < \delta_4$. We obtain

$$\begin{aligned} & |W_{a,b,\mu}(z) - W_{a,b,\mu}(v)| \\ & \leq \left| \int_{I_{a,b+\max(v,z)}} |x - c_{a,b+\max(v,z)}|^r d\mu(x) \right. \\ & \quad \left. - \int_{I_{a,b+\min(v,z)}} |x - c_{a,b+\max(v,z)}|^r d\mu(x) \right| \\ & + \left| \int_{I_{a,b+\min(v,z)}} |x - c_{a,b+\max(v,z)}|^r d\mu(x) \right. \\ & \quad \left. - \int_{I_{a,b+\min(v,z)}} |x - c_{a,b+\min(v,z)}|^r d\mu(x) \right| \\ & \leq \int_{[b+z-\delta_4, b+z+\delta_4]} |x - c_{a,b+\max(v,z)}|^r d\mu(x) + \int_{I_{a,b+z}} f(x, z, v) d\mu(x). \end{aligned}$$

From (6) and (10) we deduce that

$$|W_{a,b,\mu}(z) - W_{a,b,\mu}(v)| < \varepsilon,$$

which yields that the mapping is continuous. Finally we observe for any $x, y \in J$ with $x < y$ that

$$\begin{aligned} & W_{a,b,\mu}(y) - W_{a,b,\mu}(x) \\ &= \int_{I_{a,b+y}} |z - c_{a,b+y}|^r d\mu(z) - \int_{I_{a,b+x}} |z - c_{a,b+x}|^r d\mu(z) \\ &\geq \int_{I_{b+x,b+y}} |z - c_{a,b+y}|^r d\mu(z) \geq 0, \end{aligned} \tag{11}$$

which implies monotony. Recall that μ is non-atomic. Hence, inequality (11) is strict if and only if $\mu([x + b, y + b]) > 0$. \square

Because we want to show now that every n -level Gersho quantizer induces the same quantization error, the following definition makes sense.

Definition 2.3. *Let $n \in \mathbb{N}$ and assume that $\mathcal{G}(n, \mu, r) \neq \emptyset$. We define*

$$D_{n,r}^G(\mu) = \inf\{D(\mu, q, r) : q \in \mathcal{G}(n, \mu, r)\}$$

as the optimal n -th Gersho-quantization error for μ of order r .

Let

$$\mathbb{R} \ni x \rightarrow T(x) = -x$$

be the reflection with respect to the origin.

Clearly, the distortion of every n -optimal quantizer $q \in \mathcal{C}(n, \mu, r)$ equals $D_{n,r}(\mu)$. Therefore, it is natural to ask if a similar result holds for Gersho quantizers. The following proposition gives a positive answer.

Proposition 2.4. (Uniqueness of overall distortion)

Let $n \in \mathbb{N}$ and $r > 1$. Assume that an n -level Gersho quantizer exists, i.e. $\mathcal{G}(n, \mu, r) \neq \emptyset$. If $q \in \mathcal{G}(n, \mu, r)$, then the distortion induced by q equals the n -th Gersho-quantization error, i.e.

$$D(\mu, q, r) = D_{n,r}^G(\mu).$$

Proof. If $n = 1$, then the assertion follows immediately from (G3) and Proposition 2.1. Let $n \geq 2$. Let $I_1(q), \dots, I_n(q)$ be the codecells of q in increasing order of \mathbb{R} . Moreover, let $p \in \mathcal{G}(n, \mu, r)$ and denote by $I_1(p), \dots, I_n(p)$ the codecells of p in increasing order, too. We proceed indirectly and assume w.l.o.g. that

$$D(\mu, q, r) < D(\mu, p, r). \tag{12}$$

The idea of the proof is as follows. Using (12), (G4) and Lemma 2.2 one can show that the moment and the right endpoint of the first codecell $I_1(q)$ of q is strictly

smaller than the one of $I_1(p)$. By induction this holds up to the penultimate codecell of q and p . Consequently, the last codecell $I_n(q)$ of q is strictly larger (according to size and moment) than the last one of p , which contradicts (12). Now let us elaborate this idea in detail. We proceed in two steps.

1. We will show that

$$\sup I_{n-1}(q) < \sup I_{n-1}(p) \quad \text{and} \quad \mu([\sup I_{n-1}(q), \sup I_{n-1}(p)]) > 0.$$

We proceed by induction and assume first that $n = 2$. According to (1), (12) and by (G4) we know that

$$\begin{aligned} & W_{-\infty,0,\mu}(\sup I_1(q)) \\ &= \int_{I_1(q)} |x - c_{-\infty, \sup I_1(q)}|^r d\mu(x) < \int_{I_1(p)} |x - c_{-\infty, \sup I_1(p)}|^r d\mu(x) \\ &= W_{-\infty,0,\mu}(\sup I_1(p)). \end{aligned}$$

Applying Lemma 2.2 we obtain

$$\sup I_1(q) < \sup I_1(p) \quad \text{and} \quad \mu([\sup I_1(q), \sup I_1(p)]) > 0.$$

Now let us assume that $n > 2$,

$$\sup I_{n-2}(q) < \sup I_{n-2}(p) \quad \text{and} \quad \mu([\sup I_{n-2}(q), \sup I_{n-2}(p)]) > 0.$$

Again from (12) and (G4) we deduce that

$$\begin{aligned} & W_{\inf I_{n-1}(q),0,\mu}(\sup I_{n-1}(q)) \\ &= \int_{I_{n-1}(q)} |x - c_{\inf I_{n-1}(q), \sup I_{n-1}(q)}|^r d\mu(x) \\ &< \int_{I_{n-1}(p)} |x - c_{\inf I_{n-1}(p), \sup I_{n-1}(p)}|^r d\mu(x) \\ &= \int_{T(I_{n-1}(p))} |x - c_{-\sup I_{n-1}(p), -\inf I_{n-1}(p)}(\mu \circ T^{-1})|^r d\mu \circ T^{-1}(x) \\ &= W_{-\sup I_{n-1}(p),0,\mu \circ T^{-1}}(-\inf I_{n-1}(p)). \end{aligned} \tag{13}$$

By (1) we know that $\inf I_{n-1}(q) = \sup I_{n-2}(q) < \sup I_{n-2}(p) = \inf I_{n-1}(p)$. Thus, Lemma 2.2 implies

$$\begin{aligned} W_{-\sup I_{n-1}(p),0,\mu \circ T^{-1}}(-\inf I_{n-1}(p)) &\leq W_{-\sup I_{n-1}(p),0,\mu \circ T^{-1}}(-\inf I_{n-1}(q)) \\ &= W_{\inf I_{n-1}(q),0,\mu}(\sup I_{n-1}(p)). \end{aligned} \tag{14}$$

Combing (13) and (14) we get

$$W_{\inf I_{n-1}(q),0,\mu}(\sup I_{n-1}(q)) < W_{\inf I_{n-1}(q),0,\mu}(\sup I_{n-1}(p)).$$

Now Lemma 2.2 yields

$$\sup I_{n-1}(q) < \sup I_{n-1}(p) \quad \text{and} \quad \mu([\sup I_{n-1}(q), \sup I_{n-1}(p)]) > 0.$$

2. Rest of the proof.

Applying (1) and definition (2) we compute

$$\int_{I_n(p)} |x - c_{\inf I_n(p), \infty}|^r d\mu(x) = W_{-\infty, 0, \mu \circ T^{-1}}(-\sup I_{n-1}(p)). \quad (15)$$

Now, step 1, Lemma 2.2 and definition (2) implies

$$\begin{aligned} W_{-\infty, 0, \mu \circ T^{-1}}(-\sup I_{n-1}(p)) &< W_{-\infty, 0, \mu \circ T^{-1}}(-\sup I_{n-1}(q)) \\ &= \int_{I_n(q)} |x - c_{\inf I_n(q), \infty}|^r d\mu(x). \end{aligned} \quad (16)$$

From (12) and (G4) we deduce

$$\int_{I_n(q)} |x - c_{\inf I_n(q), \infty}|^r d\mu(x) < \int_{I_n(p)} |x - c_{\inf I_n(p), \infty}|^r d\mu(x). \quad (17)$$

Combining (17), (16) and (15) we get the contradiction

$$\int_{I_n(p)} |x - c_{\inf I_n(p), \infty}|^r d\mu(x) < \int_{I_n(p)} |x - c_{\inf I_n(p), \infty}|^r d\mu(x),$$

which proves the assertion. \square

Now we can state and prove the main result of this section.

Proposition 2.5. (Existence and Uniqueness of Gersho quantizers)

Let $r > 1$. For any $n \in \mathbb{N}$ the set $\mathcal{G}(n, \mu, r)$ of Gersho-quantizers is not empty. Moreover, if the support of μ is an interval, then exactly one Gersho quantizer exists, i.e. $\text{card}(\mathcal{G}(n, \mu, r)) = 1$.

Proof. We proceed by induction. Assume first that $n = 1$. In this case the assertion follows directly from Proposition 2.1. Now assume that for every $z \in V = \{w \in \mathbb{R} : \mu(I_{w, \infty}) \in (0, 1)\}$ the set $\mathcal{G}(n-1, \mu(\cdot|I_{z, \infty}), r)$ is not empty and that $\text{card}(\mathcal{G}(n-1, \mu(\cdot|I_{z, \infty}), r)) = 1$ if the support of $\mu(\cdot|I_{z, \infty})$ is an interval. Let us consider the mapping

$$\Phi_{n-1}(z) = \frac{\mu(I_{z, \infty}) D_{n-1, r}^G(\mu(\cdot|I_{z, \infty}))}{n-1}. \quad (18)$$

The idea of the proof is quite straightforward. First, one constructs a quantizer which consists of n cells, where the leftmost cell ends at z and all other cells are designed such that every moment of one of these $n-1$ cells equals $\Phi_{n-1}(z)$. Using the induction hypothesis it suffices to show that only one z_0 exists, such that the moment of the leftmost cell and $\Phi_{n-1}(z_0)$ coincide. Indeed, this can be achieved by showing that $\Phi_{n-1}(\cdot)$ is decreasing and continuous. We divide the remaining part of the proof into several steps.

1. We construct quantizers $q_{n,z}$ and $\tilde{q}_{n,z}$ and show that $\tilde{q}_{n,z} \in \mathcal{G}(n-1, \mu(\cdot|I_{z,\infty}), r)$. We define for every $z \in V$ recursively

$$w_0(z) = z, \quad w_{n-1}(z) = \infty$$

and

$$w_i(z) = \inf \left\{ w > w_{i-1}(z) : \int_{[w_{i-1}(z), w]} |x - c_{w_{i-1}(z), w}|^r d\mu(x) = \Phi_{n-1}(z) \right\}.$$

for every $i \in \{1, \dots, n-2\}$. Let $p_z \in \mathcal{G}(n-1, \mu(\cdot|I_{z,\infty}), r)$ and denote by $I_1(p_z), \dots, I_{n-1}(p_z)$ the codecells of p_z in increasing order. Note that

$$D_{n-1, r}^G(\mu(\cdot|I_{z,\infty})) = D(\mu(\cdot|I_{z,\infty}), p_z, r)$$

according to Proposition 2.4. Thus we deduce by (G4) and (18) that

$$\Phi_{n-1}(z) = \int_{[z, \sup I_1(p_z)]} |x - c_{z, \sup I_1(p_z)}|^r d\mu(x).$$

Hence, the definition of $w_1(z)$ implies $w_1(z) \leq \sup I_1(p_z)$. Moreover,

$$\mu([w_1(z), \sup I_1(p_z)]) = 0,$$

because otherwise we would get together with Lemma 2.2 that

$$\begin{aligned} \frac{\mu(I_{z,\infty})D(\mu(\cdot|I_{z,\infty}), p_z, r)}{n-1} &= \Phi_{n-1}(z) \\ &= \int_{[z, w_1(z)]} |x - c_{z, w_1(z)}|^r d\mu(x) \\ &< \int_{[z, \sup I_1(p_z)]} |x - p_z(x)|^r d\mu(x), \end{aligned}$$

which would contradict $p_z \in \mathcal{G}(n-1, \mu(\cdot|I_{z,\infty}), r)$. Hence we obtain inductively that

$$\mu([w_i(z), \sup I_i(p_z)]) = 0 \text{ for every } i \in \{1, \dots, n-2\},$$

which implies together with (G3), (G4) and (1) that

$$\begin{aligned} &\int_{[w_{n-2}(z), w_{n-1}(z)]} |x - c_{w_{n-2}(z), w_{n-1}(z)}|^r d\mu(x) \\ &= \int_{I_{n-1}(p_z)} |x - p_z(x)|^r d\mu(x) = \Phi_{n-1}(z) > 0. \end{aligned} \quad (19)$$

From (19) we know that $\mu([w_{n-2}(z), w_{n-1}(z)]) > 0$. Moreover, the definition of w_i and Lemma 2.2 imply for $i \in \{1, \dots, n-2\}$ that

$$\int_{[w_{i-1}(z), w_i(z)]} |x - c_{w_{i-1}(z), w_i(z)}|^r d\mu(x) = \Phi_{n-1}(z) > 0, \quad (20)$$

which yields $\mu([w_{i-1}(z), w_i(z)]) > 0$ for every $i \in \{1, \dots, n-2\}$. Thus we can define for $x \in \mathbb{R}$ the quantizers

$$q_{n,z}(x) = c_{-\infty,z} \mathbf{1}_{(-\infty,z)}(x) + \sum_{i=0}^{n-2} c_{w_i(z), w_{i+1}(z)} \mathbf{1}_{[w_i(z), w_{i+1}(z)]}(x)$$

and

$$\tilde{q}_{n,z}(x) = c_{-\infty, w_1(z)} \mathbf{1}_{(-\infty, w_1(z))}(x) + \sum_{i=1}^{n-2} c_{w_i(z), w_{i+1}(z)} \mathbf{1}_{[w_i(z), w_{i+1}(z)]}(x).$$

The definition of $q_{n,z}$ and $\tilde{q}_{n,z}$ imply together with (19) and (20) that

$$\tilde{q}_{n,z} \in \mathcal{G}(n-1, \mu(\cdot | I_{z,\infty}), r).$$

We will show that $\Phi_{n-1}(\cdot)$ is decreasing and continuous.

2. We will show that $\Phi_{n-1}(\cdot)$ is decreasing.

Let $\delta > 0$ such that $z + \delta < w_1(z)$ and $\mu([z + \delta, w_1(z)]) > 0$. If $\mu([z, z + \delta]) = 0$, then definition (18) implies $\Phi_{n-1}(z) = \Phi_{n-1}(z + \delta)$. Hence we can assume w.l.o.g. that $\mu([z, z + \delta]) > 0$. Thus we get from Lemma 2.2

$$\begin{aligned} \Phi_{n-1}(z) &= \int_{[z, w_1(z)]} |x - c_{z, w_1(z)}|^r d\mu(x) \\ &> \int_{[z+\delta, w_1(z)]} |x - c_{z, w_1(z)}|^r d\mu(x) \\ &\geq \int_{[z+\delta, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) > 0. \end{aligned} \quad (21)$$

Now let us assume that

$$\Phi_{n-1}(z) < \Phi_{n-1}(z + \delta). \quad (22)$$

In this case we obtain

$$\begin{aligned} &\int_{[z+\delta, w_1(z+\delta)]} |x - c_{z+\delta, w_1(z+\delta)}|^r d\mu(x) \\ &\geq \int_{[z+\delta, w_1(z+\delta)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) \\ &= \Phi_{n-1}(z + \delta) > \Phi_{n-1}(z). \end{aligned} \quad (23)$$

Combining (21) and (23) we get

$$\int_{[z+\delta, w_1(z+\delta)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) > \int_{[z+\delta, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x),$$

which implies that $w_1(z) < w_1(z + \delta)$ and $\mu([w_1(z), w_1(z + \delta)]) > 0$. Inductively one obtains that $\mu([w_{n-2}(z), w_{n-2}(z + \delta)]) > 0$, yielding that

$$\begin{aligned}\Phi_{n-1}(z + \delta) &= \int_{[w_{n-2}(z+\delta), \infty)} |x - c_{w_{n-2}(z+\delta), \infty}|^r d\mu(x) \\ &< \int_{[w_{n-2}(z), \infty)} |x - c_{w_{n-2}(z), \infty}|^r d\mu(x) = \Phi_{n-1}(z),\end{aligned}$$

which contradicts assumption (22). Thus we have proved that $\Phi_{n-1}(\cdot)$ is decreasing.

3. We will show that $\Phi_{n-1}(\cdot)$ is continuous.

Note that $w_1(\cdot)$ is increasing. Indeed, Lemma 2.2 and Step 2 yields

$$\begin{aligned}\int_{[z, w_1(z+\delta)]} |x - c_{z, w_1(z+\delta)}|^r d\mu(x) &= \Phi_{n-1}(z + \delta) \leq \Phi_{n-1}(z) \\ &= \int_{[z, w_1(z)]} |x - c_{z, w_1(z)}|^r d\mu(x) \\ &\leq \int_{[z, w_1(z)]} |x - c_{z, w_1(z+\delta)}|^r d\mu(x),\end{aligned}$$

implying $w_1(z + \delta) \leq w_1(z)$. Hence, $w_1(\cdot)$ is increasing and consequently we get

$$\begin{aligned}\Phi_{n-1}(z + \delta) &= \int_{[z+\delta, w_1(z+\delta)]} |x - c_{z+\delta, w_1(z+\delta)}|^r d\mu(x) \\ &\geq \int_{[z+\delta, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x),\end{aligned}$$

which implies

$$\begin{aligned}0 &\leq \Phi_{n-1}(z) - \Phi_{n-1}(z + \delta) \\ &\leq \int_{[z, w_1(z)]} |x - c_{z, w_1(z)}|^r d\mu(x) - \int_{[z+\delta, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) \\ &\leq \int_{[z, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) - \int_{[z+\delta, w_1(z)]} |x - c_{z+\delta, w_1(z)}|^r d\mu(x) \\ &\leq \mu([z, z + \delta])|z - w_1(z)|^r.\end{aligned}$$

Because the right hand side tends to 0 as $\delta \rightarrow 0$ we obtain that $\Phi_{n-1}(\cdot)$ is continuous.

4. Rest of the proof.

Now we consider the mapping

$$\Psi_1(z) = W_{-\infty, 0, \mu}(z) = \int_{(-\infty, z)} |x - c_{-\infty, z}(\mu)|^r d\mu(x).$$

From Lemma 2.2 we know that $\Psi_1(\cdot)$ is increasing and continuous. Moreover, $\lim_{z \rightarrow -\infty} \Psi_1(z) = 0$ and $\lim_{z \rightarrow \infty} \Psi_1(z) = D_{1,r}(\mu) > 0$. On the other hand we

know from step 2 and 3 that $\Phi_{n-1}(\cdot)$ is decreasing and continuous. Moreover, $\lim_{z \rightarrow -\infty} \Phi_{n-1}(z) > 0$ and $\lim_{z \rightarrow \infty} \Phi_{n-1}(z) = 0$. Hence a $z_0 \in V$ exists, such that $\Phi_{n-1}(z_0) = \Psi_1(z_0)$. It is easy to check that $q_{n,z_0} \in \mathcal{G}(n, \mu, r)$. Finally, assume that the support of μ is an interval. By Lemma 2.2 the mapping $\Psi_1(\cdot)$ is strictly increasing in this case. Hence, exactly one $z_0 \in \mathbb{R}$ exists, such that $\Phi_{n-1}(z_0) = \Psi_1(z_0)$. Moreover, we know by assumption that

$$\text{card}(\mathcal{G}(n-1, \mu(\cdot|_{I_{z_0, \infty}}), r)) = 1.$$

From $\tilde{q}_{n,z_0} \in \mathcal{G}(n-1, \mu(\cdot|_{I_{z_0, \infty}}), r)$ and by the definition of q_{n,z_0} we deduce, that q_{n,z_0} is the only optimal n -th Gersho quantizer for μ , i.e. $\text{card}(\mathcal{G}(n, \mu, r)) = 1$, which finishes the proof. \square

Remark 2.6. *If μ is the uniform distribution on*

$$[-1, -1/2] \cup [1/2, 1],$$

then $\mathcal{G}(2, \mu, r)$ is not countable, because

$$q_\varepsilon(\cdot) = -\frac{3}{4} \cdot \mathbf{1}_{(-\infty, -1/2+\varepsilon)}(\cdot) + \frac{3}{4} \cdot \mathbf{1}_{[-1/2+\varepsilon, \infty)}(\cdot)$$

is a Gersho quantizer for every $\varepsilon \in (0, 1)$. Let us identify two Gersho quantizers as equal if they have the same codebook and for every codepoint the codecells are μ -a.s. identical. Insofar, we recognize that the quantizers $q_\varepsilon(\cdot)$ are all equal. Now it remains open, if a non-atomic distribution μ exists, where for some fixed n and r

- (a) *at least two Gersho quantizers exist which are not equal but have the same codebook or*
- (b) *at least two Gersho quantizers exist which have not the same codebook.*

Moreover, it remains open to characterize those non-atomic scalar distributions where for every n and r the corresponding Gersho quantizers are all equal.

Remark 2.7. *Looking again at the proof of Proposition 2.5 we observe that $F(z) \rightarrow D_{n-1,r}^G(\mu)$ as $z \rightarrow -\infty$ and $F(z_0) = D_{n,r}^G(\mu)$, where the mapping F is defined as*

$$(-\infty, z_0] \ni z \rightarrow F(z) = \Psi_1(z) + (n-1)\Phi_{n-1}(z).$$

Unfortunately, the author was not able to prove or disprove the monotonicity of F . If F would be decreasing, then the overall distortion would be decreasing, i.e.

$$D_{n-1,r}^G(\mu) \geq D_{n,r}^G(\mu) \tag{24}$$

would be true. Clearly, $D_{n,r}^G(\mu) \geq D_{2n,r}^G(\mu)$, but it remains open if (24) is true. Nevertheless, we will prove the weaker result that the distortion per codecell is decreasing (cf. Lemma 3.7 (b)).

Remark 2.8. Assume that the support of μ is a (possibly unbounded) interval and denote by I the interior of the support, which is an open interval. Now we consider the mapping

$$\begin{aligned} I \ni x \rightarrow V(x) &= \int_{(-\infty, x)} |z - c_{-\infty, x}|^r d\mu(z) - \int_{(x, \infty)} |z - c_{x, \infty}|^r d\mu(z) \\ &= W_{-\infty, 0, \mu}(x) - W_{-\infty, 0, \mu \circ T^{-1}}(-x). \end{aligned}$$

From Lemma 2.2 we know that $V(\cdot)$ is continuous and strictly increasing. Let $x_0 \in I$ be the unique point with $V(x_0) = 0$. Clearly, x_0 is equal to the right resp. left boundary point of the codecells of the unique 2-level Gersho quantizer for μ . As sketched in the introduction, it is of practical relevance to determine x_0 by a bisection algorithm. If $r = 2$, then it is well-known (cf. [10, Example 2.3(b)]) that $c_{-\infty, x}(\mu)$ equals the expected value of $\mu(\cdot | (-\infty, x])$ for every $x \in I$. Additionally, if μ has a continuous Lebesgue density, then $V(\cdot)$ becomes differentiable on I and one can use even a Newton type method for the numerical determination of x_0 .

3 High-rate quantization of scalar distributions with Gersho-quantizers

We denote by λ the Lebesgue measure on \mathbb{R} . As already mentioned in the introduction, the asymptotic behavior of optimal quantizers - even in higher dimensions - is well-known. For scalar distributions the following holds. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to λ , where $\mu_a = h_a \lambda$ with Lebesgue density h_a .

Theorem 3.1 ([23, 3, 10]). Assume that μ_a does not vanish and that

$$\int |x|^{r+\delta} d\mu(x) < \infty \quad (25)$$

for some $\delta > 0$. Then

$$n^r D_{n,r}(\mu) \rightarrow 2^{-r}(1+r)^{-1} \left(\int h_a^{1/(1+r)} d\lambda \right)^{1+r} \quad \text{if } n \rightarrow \infty. \quad (26)$$

Because we assume throughout this section that μ is absolutely continuous with respect to λ , we make the following definition with h as the Lebesgue density of μ . Let $Q(r) = 2^{-r}(1+r)^{-1}$. If $\int h^{1/(1+r)} d\lambda < \infty$, then we define the constant

$$C_0 = Q(r) \left(\int h^{1/(1+r)} d\lambda \right)^{1+r}.$$

Remark 3.2. Note that $C_0 < \infty$ follows from (25) by applying a Hölder argument, see [10, Remark 6.3(a)].

In view of (26) one is interested in quantizer sequences where the quantization error converges to C_0 as the quantization level tends to infinity. To this end the following definition makes sense.

Definition 3.3. *Assume that $C_0 < \infty$. We call a sequence $(q_n)_{n \in \mathbb{N}}$ of quantizers asymptotically optimal if $\text{card}(q_n(\mathbb{R})) \leq n$ and*

$$n^r D(\mu, q_n, r) \rightarrow C_0$$

for $n \rightarrow \infty$.

Remark 3.4. *Another definition of asymptotic optimality would be*

$$\frac{D(\mu, q_n, r)}{D_{n,r}(\mu)} \rightarrow 1$$

as $n \rightarrow \infty$, which is equivalent to Definition 3.3 if (25) is satisfied. Slightly different from other definitions (cf. [10, p.93]), we require in Definition (3.3) only the finiteness of C_0 , which is by [10, Remark 6.3(b)] weaker than condition (25). Clearly, if $(q_n)_{n \in \mathbb{N}}$ is asymptotically optimal, then $D(\mu, q_n, r) \rightarrow 0$ as $n \rightarrow \infty$.

Now it is natural to ask if Gersho quantizers are asymptotically optimal. As a main result of this paper (cf. Theorem 3.10) we will prove in this section, that the answer is positive at least for distributions with a so-called weakly unimodal density.

Definition 3.5. *We call a probability density function h weakly unimodal, if h is continuous on its support and there exists an $l_0 > 0$ such that $\{x : h(x) \geq l\}$ is a compact interval for every $l \in (0, l_0)$.*

Remark 3.6. *Many scalar probability densities are weakly unimodal, e.g. the ones of Gaussian, Laplace or exponential distributions. Clearly, every distribution μ with a weakly unimodal density has an (possibly unbounded) interval as support. Hence, Proposition 2.5 implies $\text{card}(\mathcal{G}(n, \mu, r)) = 1$ if $r > 1$.*

In the proof of Theorem 3.1 the assertion is proved first for distributions with piecewise constant density. Then the general case is proved by an approximation argument. Unfortunately, we cannot use this technique for Gersho quantizers, because any amendments of the density will destroy property (G4) in general. Therefore, we use a different approach in this paper. We split the quantization error into the contribution of the codecells which are lying inside a compact interval and the contribution of the remaining ones. The asymptotic behavior inside the interval can be determined by the uniform continuity of h on compact intervals. The number and the contribution to the error of the codecells which are outside the interval can be controlled by (G4).

Let $-\infty < u < v < \infty$ and $I = [u, v]$. Assume that $\mu(I) > 0$. For every $n \in \mathbb{N}$ let $n_2^I(n)$ be the number of codepoints whose codecell is located outside (u, v) , i.e.

$$n_2^I(n) = \text{card}(\{c \in q_n(\mathbb{R}) : q_n^{-1}(c) \subset \mathbb{R} \setminus (u, v)\}). \quad (27)$$

It should be noted at this point that $n_1^I(n)$ will be defined and used later, cf. (30).

The following result shows that $n_2^I(\cdot)$ is increasing. By doing this, we first prove that the distortion per codecell is decreasing (cf. Lemma 3.7 (b)). As already mentioned in Remark 2.7 this result is weaker than relation (24).

Lemma 3.7. *Let μ be weakly unimodal, $r > 1$ and $(q_n)_{n \in \mathbb{N}}$ the sequence of n -level Gersho quantizers for μ . Then,*

- (a) $n_2^I(n+1) \geq n_2^I(n)$ and
- (b) $\frac{D(\mu, q_n, r)}{n} \geq \frac{D(\mu, q_{n+1}, r)}{n+1}$ for every $n \in \mathbb{N}$.

Proof. First we will prove assertion (b).

If $n = 1$, then let $I_1(q_2)$ and $I_2(q_2)$ in increasing order. Applying (1) and (G3) we calculate

$$\begin{aligned}
& D(\mu, q_1, r) \\
&= \int |x - q_1(x)|^r d\mu(x) \\
&= \int_{(-\infty, \sup I_1(q_2))} |x - q_1(x)|^r d\mu(x) + \int_{[\sup I_1(q_2), \infty)} |x - q_1(x)|^r d\mu(x) \\
&\geq \int_{I_1(q_2)} |x - c_{-\infty, \sup I_1(q_2)}|^r d\mu(x) + \int_{I_2(q_2)} |x - c_{\inf I_2(q_2), \infty}|^r d\mu(x) \\
&= D(\mu, q_2, r) > D(\mu, q_2, r)/2.
\end{aligned}$$

Now let $n \geq 2$. In this case we proceed indirectly and assume the contrary, i.e. we assume that

$$\frac{D(\mu, q_n, r)}{n} < \frac{D(\mu, q_{n+1}, r)}{n+1}. \quad (28)$$

Now let $q_n(\mathbb{R}) = \{c_1, \dots, c_n\}$ with $-\infty < c_1 < \dots < c_n < \infty$. Moreover, let $q_{n+1}(\mathbb{R}) = \{d_1, \dots, d_{n+1}\}$ with $-\infty < d_1 < \dots < d_{n+1} < \infty$. Let $y_1 = \sup q_n^{-1}(c_1)$ and $z_1 = \sup q_{n+1}^{-1}(d_1)$. From (28) we obtain together with (G4) that

$$\int_{(-\infty, y_1)} |x - c_1|^r d\mu(x) < \int_{(-\infty, z_1)} |x - d_1|^r d\mu(x).$$

Consequently, Lemma 2.2 implies together with (G3) that $y_1 < z_1$. If $n > 2$, then we obtain inductively that

$$\sup q_n^{-1}(c_i) < \sup q_{n+1}^{-1}(d_i) \leq \sup q_{n+1}^{-1}(d_{i+1}) \quad (29)$$

for every $i \in \{1, \dots, n-1\}$. On the other hand, applying again (28) and (G4) we get

$$\int_{(\inf q_{n+1}^{-1}(d_{n+1}), \infty)} |x - d_{n+1}|^r d\mu(x) > \int_{(\inf q_n^{-1}(c_n), \infty)} |x - c_n|^r d\mu(x).$$

Thus, (1) and Lemma 2.2 yields

$$\sup q_{n+1}^{-1}(d_n) = \inf q_{n+1}^{-1}(d_{n+1}) < \inf q_n^{-1}(c_n) = \sup q_n^{-1}(c_{n-1}),$$

which contradicts (29). Thus, assertion (b) is proved.

Now we will prove assertion (a).

For any $n \in \mathbb{N}$ the number $n_2^I(n)$ can be divided into two parts. The number of codecells on the left of interval I and right to I . Thus we define

$$n_2^{I,l}(n) = \text{card}\{c \in q_n(\mathbb{R}) : \sup q_n^{-1}(c) \leq \min I\}$$

and

$$n_2^{I,r}(n) = \text{card}\{c \in q_n(\mathbb{R}) : \inf q_n^{-1}(c) \geq \max I\}.$$

If $n_2^{I,l}(n) = 0$, then we immediately obtain $n_2^{I,l}(n) \leq n_2^{I,l}(n+1)$. Now let us assume that $n_2^{I,l}(n) > 0$. Let $a = \min q_n(\mathbb{R})$ and $b = \min q_{n+1}(\mathbb{R})$. From (b) we obtain

$$\int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \geq \int_{q_{n+1}^{-1}(b)} |x - b|^r d\mu(x).$$

Applying (1) and (G3) we obtain from Lemma 2.2 that $\sup q_n^{-1}(a) \geq \sup q_{n+1}^{-1}(b)$. Now let us define a' resp. b' as the largest codepoint of $q_n(\mathbb{R})$ resp. $q_{n+1}(\mathbb{R})$ which is located left to I . More exactly,

$$a' = \max\{c \in q_n(\mathbb{R}) : \sup q_n^{-1}(c) \leq \min I\}$$

and

$$b' = \max\{c \in q_{n+1}(\mathbb{R}) : \sup q_{n+1}^{-1}(c) \leq \min I\}.$$

Thus, using Lemma 2.2 again, we obtain by induction that $\sup q_n^{-1}(a') \geq \sup q_{n+1}^{-1}(b')$, which implies $n_2^{I,l}(n) \leq n_2^{I,l}(n+1)$. By the same argumentation we deduce that $n_2^{I,r}(n) \leq n_2^{I,r}(n+1)$. Assertion (b) follows now by the observation $n_2^{I,l}(n) + n_2^{I,r}(n) = n_2^I(n)$ and the proof is finished. \square

For any n -level Gersho quantizer q_n we define

$$J_{n,I} = \{a \in q_n(\mathbb{R}) : q_n^{-1}(a) \subset I\} \quad \text{and} \quad n_1^I(n) = \text{card } J_{n,I}. \quad (30)$$

Moreover, we denote by $S(\mu)$ the interior of $\text{supp}(\mu)$ and define $T_{n,I}$ as the closure of the union of all the n -level codecells in I , i.e.

$$T_{n,I} = [\inf q_n^{-1}(\min J_{n,I}), \sup q_n^{-1}(\max J_{n,I})]$$

if $J_{n,I}$ is nonempty. As already depicted above, we first prove the asymptotic optimality for the part of the quantization error where codecells are located inside a compact interval (cf. Lemma 3.9). As a motivation for our approach

let us consider the following heuristic argumentation. For large n and $a \in J_{n,I}$ we observe that

$$\begin{aligned} Q(r) \left(\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda \right)^{1+r} &\approx \text{diam}(q_n^{-1}(a))^{1+r} h(a) Q(r) \\ &= \int_{q_n^{-1}(a)} h(a) |x-a|^r d\lambda(x) \approx \int_{q_n^{-1}(a)} |x-a|^r d\mu(x) = n^{-1} D_{n,r}^G(\mu), \end{aligned} \quad (31)$$

where the last equation is due to (G4). The right hand side is independent of $a \in J_{n,I}$ and, therefore, also approximately the left hand side. But this implies that

$$Q(r) \left(\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda \right)^{1+r} \approx Q(r) \left((n_1^I(n))^{-1} \int_I h^{1/(1+r)} d\lambda \right)^{1+r}. \quad (32)$$

Now (31) and (32) indicate that

$$n^r D_{n,r}^G(\mu) \approx Q(r) n^{1+r} (n_1^I(n))^{-(1+r)} \left(\int_I h^{1/(1+r)} d\lambda \right)^{1+r}. \quad (33)$$

To show exactly that asymptotic optimality holds, we first have to justify the approximation in (31) and (32). This is done by Lemma 3.8. Moreover, we must show that the right hand side of (33) converges against a constant $C(I)$ and that $|C(I) - C_0|$ is arbitrarily small according to a suitable choice of $I \subset \text{supp}(\mu)$. This is the content of Lemma 3.9 and the proof of Theorem 3.10.

Lemma 3.8. *Let $r > 1$. Let μ be weakly unimodal and $(q_n)_{n \in \mathbb{N}}$ the sequence of n -level Gersho quantizers for μ . Assume that $I \subset S(\mu)$. Then an $m \in \mathbb{N}$ exists, such that $J_{n,I}$ is nonempty for every $n \geq m$. Let $n \geq m$ and*

$$\begin{aligned} \varepsilon_1(n) &= \max \left\{ \left| \frac{\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda}{\int_{T_{n,I}} h^{1/(1+r)} d\lambda} - \frac{1}{n_1^I(n)} \right| : a \in J_{n,I} \right\}, \\ \varepsilon_2(n) &= \max \left\{ \left| \int_{q_n^{-1}(a)} |x-a|^r d\mu(x) - \text{diam}(q_n^{-1}(a))^{1+r} h(a) Q(r) \right| : a \in J_{n,I} \right\}, \\ \varepsilon_3(n) &= \max \left\{ \left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \text{diam}(q_n^{-1}(a)) h(a)^{1/(1+r)} \right| : a \in J_{n,I} \right\}. \end{aligned}$$

Then,

$$\text{diam}(T_{n,I}) \rightarrow \text{diam}(I), \quad (34)$$

$$n_1^I(n) \varepsilon_1(n) \rightarrow 0, \quad (35)$$

$$(n_1^I(n))^{1+r} \varepsilon_2(n) \rightarrow 0, \quad (36)$$

$$n_1^I(n) \varepsilon_3(n) \rightarrow 0, \quad (37)$$

as $n \rightarrow \infty$. Moreover,

$$\limsup_{n \rightarrow \infty} \frac{\varepsilon_3(n)}{\min \{ \text{diam}(q_n^{-1}(a)) : a \in J_{n,I} \}} = 0. \quad (38)$$

Proof. Because μ is weakly unimodal and due to $I \subset S(\mu)$ we get

$$\begin{aligned} 0 < M_{1,I} &:= \min\{h(x) : x \in I\} \text{ and} \\ \infty > M_{2,I} &:= \max\{h(x) : x \in I\} \geq M_{1,I}. \end{aligned} \quad (39)$$

We divide the remaining proof into six steps.

1. We will show that $J_{n,I}$ is nonempty for every $n \geq m$.

Note that

$$\begin{aligned} D(\mu, q_n, r) &= \sum_{a \in q_n(\mathbb{R})} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ &\leq \sum_{a \in q_n(\mathbb{R})} \int_{q_n^{-1}(a)} |x - q_1(x)|^r d\mu(x) = D(\mu, q_1, r) < \infty, \end{aligned} \quad (40)$$

which implies

$$D(\mu, q_n, r)/n \rightarrow 0 \quad (41)$$

as $n \rightarrow \infty$. If $J_{n,I} = \emptyset$, then a $c \in q_n(\mathbb{R})$ exists, such that

$$\text{diam}(q_n^{-1}(c) \cap I) \geq \text{diam}(I)/2 \text{ and } q_n^{-1}(c) \setminus I \neq \emptyset.$$

Recall $I = [u, v]$. Hence, (G4) yields

$$D(\mu, q_n, r)/n \geq \min\{e_1, e_2\} > 0 \quad (42)$$

with

$$e_1 = \int_{[u, (u+v)/2]} |x - (\frac{3}{4}u + \frac{1}{4}v)|^r d\mu(x)$$

and

$$e_2 = \int_{[(u+v)/2, v]} |x - (\frac{1}{4}u + \frac{3}{4}v)|^r d\mu(x).$$

Note, that the right hand side of (42) is independent of n . Thus, we deduce from (41) and (42) that an $m \in \mathbb{N}$ exists, such that $J_{n,I}$ is nonempty for all $n \geq m$. Let us assume w.l.o.g. for the rest of this proof that $m = 1$.

2. We will prove (34).

We proceed indirectly and assume that a constant $M \in [0, \text{diam}(I))$ and a subsequence of $(\text{diam}(T_{n,I}))_{n \in \mathbb{N}}$ exists, which we also denote by $(\text{diam}(T_{n,I}))_{n \in \mathbb{N}}$, such that

$$\text{diam}(T_{n,I}) \rightarrow M \quad \text{as } n \rightarrow \infty. \quad (43)$$

Let

$$\begin{aligned} s_n &= \max\{a \in q_n(\mathbb{R}) : \inf q_n^{-1}(a) < \min I\}, \\ t_n &= \min\{a \in q_n(\mathbb{R}) : \sup q_n^{-1}(a) > \max I\}. \end{aligned}$$

Let

$$s = \lim_{n \rightarrow \infty} (\sup q_n^{-1}(s_n) - \min I) \text{ and } t = \lim_{n \rightarrow \infty} (\max I - \inf q_n^{-1}(t_n)).$$

Due to (43) we know that s and t exist and that $s + t = \text{diam } I - M > 0$. If we assume w.l.o.g. that $s > 0$, then we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{q_n^{-1}(s_n)} |x - s_n|^r d\mu(x) \\ & \geq \liminf_{n \rightarrow \infty} \int_{I \cap q_n^{-1}(s_n)} |x - s_n|^r M_{1,I} d\lambda(x) \\ & \geq \liminf_{n \rightarrow \infty} \int_{I \cap q_n^{-1}(s_n)} |x - (\sup q_n^{-1}(s_n) + \min I)/2|^r M_{1,I} d\lambda(x) \\ & = \liminf_{n \rightarrow \infty} Q(r) M_{1,I} (\sup q_n^{-1}(s_n) - \min I)^{1+r} = Q(r) M_{1,I} s^{1+r} > 0. \end{aligned} \quad (44)$$

On the other hand we get from (G4) that

$$\limsup_{n \rightarrow \infty} \int_{q_n^{-1}(s_n)} |x - s_n|^r d\mu(x) = \limsup_{n \rightarrow \infty} \frac{D(\mu, q_n, r)}{n}. \quad (45)$$

Combining (45) and (40) we obtain

$$\limsup_{n \rightarrow \infty} \int_{q_n^{-1}(s_n)} |x - s_n|^r d\mu(x) = 0,$$

which contradicts (44).

3. We will show that $n_0 \in \mathbb{N}$ and constants $A_0, B_0 \in (0, \infty)$ exist, such that

$$\max\{\text{diam}(q_n^{-1}(a)) : a \in J_{n,I}\} \leq A_0 (n_1^I(n))^{-1}$$

and

$$\min\{\text{diam}(q_n^{-1}(a)) : a \in J_{n,I}\} \geq B_0 (n_1^I(n))^{-1}$$

for every $n \geq n_0$.

Let $a \in J_{n,I}$ and notice that

$$\begin{aligned} M_{1,I} & \leq M_{1,I}^a = \min\{h(x) : x \in q_n^{-1}(a)\}, \\ M_{2,I} & \geq M_{2,I}^a = \max\{h(x) : x \in q_n^{-1}(a)\}. \end{aligned} \quad (46)$$

Applying property (G3) of q_n we obtain that

$$\begin{aligned} Q(r) M_{1,I}^a \text{diam}(q_n^{-1}(a))^{1+r} & = \inf\left\{ \int_{q_n^{-1}(a)} |x - b|^r M_{1,I}^a d\lambda(x) : b \in \mathbb{R} \right\} \\ & \leq \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ & \leq \inf\left\{ \int_{q_n^{-1}(a)} |x - b|^r M_{2,I}^a d\lambda(x) : b \in \mathbb{R} \right\} \\ & = Q(r) M_{2,I}^a \text{diam}(q_n^{-1}(a))^{1+r}. \end{aligned} \quad (47)$$

Now (G4) and (46) are implying that

$$\begin{aligned}
& M_{1,I} \left(\max\{\text{diam}(q_n^{-1}(b)) : b \in J_{n,I}\} \right)^{1+r} \\
& \leq Q(r)^{-1} \max\left\{ \int_{q_n^{-1}(b)} |x-b|^r d\mu(x) : b \in J_{n,I} \right\} \\
& = Q(r)^{-1} \min\left\{ \int_{q_n^{-1}(b)} |x-b|^r d\mu(x) : b \in J_{n,I} \right\} \\
& \leq M_{2,I} \left(\min\{\text{diam}(q_n^{-1}(b)) : b \in J_{n,I}\} \right)^{1+r}. \tag{48}
\end{aligned}$$

According to (47) a $\xi_a \in q_n^{-1}(a) \subset I$ exists, such that

$$\int_{q_n^{-1}(a)} |x-a|^r d\mu(x) = Q(r)h(\xi_a) \text{diam}(q_n^{-1}(a))^{1+r}. \tag{49}$$

From (48) we deduce that

$$\begin{aligned}
& n_1^I(n) \cdot \min\{\text{diam}(q_n^{-1}(b)) : b \in J_{n,I}\} \leq \text{diam}(T_{n,I}) \\
& \leq n_1^I(n) \cdot \max\{\text{diam}(q_n^{-1}(b)) : b \in J_{n,I}\} \\
& \leq n_1^I(n) \cdot \left(\frac{M_{2,I}}{M_{1,I}} \right)^{1/(1+r)} \min\{\text{diam}(q_n^{-1}(b)) : b \in J_{n,I}\} \\
& \leq \left(\frac{M_{2,I}}{M_{1,I}} \right)^{1/(1+r)} \text{diam}(T_{n,I}), \tag{50}
\end{aligned}$$

where the last inequality follows from the first one. According to (34) and definition (30) choose $n_0 \in \mathbb{N}$ such that

$$\text{diam}(I)/2 \leq \text{diam}(T_{n,I}) \leq \text{diam}(I)$$

for every $n \geq n_0$. Hence, the setting

$$A_0 = \left(\frac{M_{2,I}}{M_{1,I}} \right)^{1/(1+r)} \text{diam}(I), \quad B_0 = \left(\frac{M_{1,I}}{M_{2,I}} \right)^{1/(1+r)} \text{diam}(I)/2$$

and relation (50) finishes the proof of this step.

4. We will prove (36).

Let us denote by κ_I the modulus of continuity of h restricted to I , i.e.

$$\kappa_I(w) = \sup\{|h(x) - h(y)| : x, y \in I, |x - y| \leq w\}.$$

Applying (49) and Step 3 we deduce for every $n \geq n_0$ that

$$\begin{aligned}
& \max\left\{ \left| \int_{q_n^{-1}(a)} |x-a|^r d\mu(x) - \text{diam}(q_n^{-1}(a))^{1+r} h(a) Q(r) \right| : a \in J_{n,I} \right\} \\
& \leq Q(r) \max\{\kappa_I(\text{diam}(q_n^{-1}(a))) \text{diam}(q_n^{-1}(a))^{1+r} : a \in J_{n,I}\} \\
& \leq Q(r) \kappa_I(A_0(n_1^I(n))^{-1})(A_0(n_1^I(n))^{-1})^{1+r}. \tag{51}
\end{aligned}$$

Clearly, according to (40) we know that

$$D(\mu, q_n, r)/n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (52)$$

This implies that

$$n_1^I(n) \rightarrow \infty \quad \text{as } n \rightarrow \infty. \quad (53)$$

Indeed, if we assume the contrary, then we can find a sequence $(v_n)_{n \in \mathbb{N}}$ with $v_n \in J_{n,I}$ such that

$$\int_{q_n^{-1}(v_n)} |x - v_n|^r d\mu \geq Q(r)M_{1,I} \liminf_{n \rightarrow \infty} \text{diam}(q_n^{-1}(v_n))^{1+r} > 0,$$

which implies according to property (G4) of a Gersho quantizer that

$$\liminf_{n \rightarrow \infty} D(\mu, q_n, r)/n > 0,$$

a contradiction to (52). Thus, (53) implies together with (51) and the definition of $\varepsilon_2(n)$ that $(n_1^I(n))^{1+r} \varepsilon_2(n) \rightarrow 0$ as $n \rightarrow \infty$.

5. We will prove (37) and (38).

For every $a, b \in J_{n,I}$ a $\tau_a \in q_n^{-1}(a)$ and $\tau_b \in q_n^{-1}(b)$ exist, such that

$$h(\tau_a)^{1/(1+r)} \text{diam}(q_n^{-1}(a)) = \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda \quad (54)$$

and

$$h(\tau_b)^{1/(1+r)} \text{diam}(q_n^{-1}(b)) = \int_{q_n^{-1}(b)} h^{1/(1+r)} d\lambda. \quad (55)$$

Moreover,

$$\begin{aligned} & \left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \text{diam}(q_n^{-1}(a))h(a)^{1/(1+r)} \right| \\ &= |h(\tau_a)^{1/(1+r)} - h(a)^{1/(1+r)}| \text{diam}(q_n^{-1}(a)). \end{aligned}$$

If we denote by ω_I the modulus of continuity of $h^{1/(1+r)}$ restricted to I , then step 3 yields for every $n \geq n_0$ that

$$\begin{aligned} \varepsilon_3(n) &= \max \left\{ \left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \text{diam}(q_n^{-1}(a))h(a)^{1/(1+r)} \right| : a \in J_{n,I} \right\} \\ &\leq \omega_I(A_0(n_1^I(n))^{-1})A_0(n_1^I(n))^{-1}. \end{aligned}$$

Because $h^{1/(1+r)}$ is uniformly continuous on I we have $n_1^I(n)\varepsilon_3(n) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, (38) follows immediately from step 3.

6. We will prove (35).

Let $n \geq n_0$ and $a, b \in J_{n,I}$ and let $\xi_a \in q_n^{-1}(a)$ and $\xi_b \in q_n^{-1}(b)$ according to (49). Let $\tau_a \in q_n^{-1}(a)$ and $\tau_b \in q_n^{-1}(b)$ according to (54) and (55). We define

$$f(n) = \max \left\{ \left| \left(\frac{h(\tau_c)}{h(\tau_d)} \right)^{1/(1+r)} \left(\frac{h(\xi_d)}{h(\xi_c)} \right)^{1/(1+r)} - 1 \right| : c, d \in J_{n,I} \right\}.$$

From (52) we obtain by the same argumentation as for (53) that

$$\sup\{\text{diam}(q_n^{-1}(c)) : c \in J_{n,I}\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But this implies together with the continuity of h that $f(n) \rightarrow 0$ as $n \rightarrow \infty$. Now we calculate with (54) and (55) that

$$\begin{aligned} & \left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \int_{q_n^{-1}(b)} h^{1/(1+r)} d\lambda \right| \\ &= |h(\tau_a)^{1/(1+r)} \text{diam}(q_n^{-1}(a)) - h(\tau_b)^{1/(1+r)} \text{diam}(q_n^{-1}(b))|. \end{aligned}$$

Let us assume w.l.o.g. that

$$h(\tau_a)^{1/(1+r)} \text{diam}(q_n^{-1}(a)) \geq h(\tau_b)^{1/(1+r)} \text{diam}(q_n^{-1}(b)).$$

Thus we obtain from property (G4), relation (49) and step 3 that

$$\begin{aligned} & \left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \int_{q_n^{-1}(b)} h^{1/(1+r)} d\lambda \right| \\ &= h(\tau_a)^{1/(1+r)} \text{diam}(q_n^{-1}(a)) - h(\tau_b)^{1/(1+r)} \text{diam}(q_n^{-1}(b)) \\ &= h(\tau_a)^{1/(1+r)} \text{diam}(q_n^{-1}(b)) \left(\frac{h(\xi_b)}{h(\xi_a)} \right)^{1/(1+r)} \\ & \quad - h(\tau_b)^{1/(1+r)} \text{diam}(q_n^{-1}(b)) \\ &\leq \text{diam}(q_n^{-1}(b)) h(\tau_b)^{1/(1+r)} f(n) \\ &\leq A_0 \frac{1}{n_1^I(n)} M_{2,I}^{1/(1+r)} f(n). \end{aligned} \tag{56}$$

Now we define for every $c \in J_{n,I}$

$$p_c = \frac{\int_{q_n^{-1}(c)} h^{1/(1+r)} d\lambda}{\int_{T_{n,I}} h^{1/(1+r)} d\lambda}.$$

Applying (56) we get

$$\begin{aligned} & \left| \frac{1}{n_1^I(n)} - p_a \right| = \left| \frac{1}{n_1^I(n)} \sum_{c \in J_{n,I}} (p_c - p_a) \right| \\ &\leq \frac{1}{n_1^I(n)} \sum_{c \in J_{n,I}} \frac{\left| \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \int_{q_n^{-1}(c)} h^{1/(1+r)} d\lambda \right|}{\int_{T_{n,I}} h^{1/(1+r)} d\lambda} \\ &\leq \left(\int_{T_{n,I}} h^{1/(1+r)} d\lambda \right)^{-1} \frac{A_0}{n_1^I(n)} M_{2,I}^{1/(1+r)} f(n). \end{aligned} \tag{57}$$

Now, (57) and (34) imply (35) and the proof is finished. \square

Recall $Q(r) = 2^{-r}(1+r)^{-1}$ and $S(\mu)$ as the interior of $\text{supp}(\mu)$.

Lemma 3.9. *Let μ be weakly unimodal and $(q_n)_{n \in \mathbb{N}}$ the sequence of n -level Gersho quantizers for μ . Assume that $I \subset S(\mu)$. Then,*

$$n_1^I(n)^r \sum_{a \in J_{n,I}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \rightarrow Q(r) \left(\int_I h^{1/(1+r)} d\lambda \right)^{1+r}$$

as $n \rightarrow \infty$.

Proof. We deduce from (36) and the definition of $\varepsilon_3(n)$ that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n_1^I(n)^r \sum_{a \in J_{n,I}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ & \leq \limsup_{n \rightarrow \infty} n_1^I(n)^r \left(n_1^I(n) \varepsilon_2(n) + \sum_{a \in J_{n,I}} \text{diam}(q_n^{-1}(a))^{1+r} h(a) Q(r) \right) \\ & = \limsup_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} \text{diam}(q_n^{-1}(a))^{1+r} h(a) \\ & \leq \limsup_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} \left(\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda + \varepsilon_3(n) \right)^{1+r}. \end{aligned} \quad (58)$$

By completely similar arguments we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} n_1^I(n)^r \sum_{a \in J_{n,I}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ & \geq \liminf_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} \left(\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda - \varepsilon_3(n) \right)^{1+r}. \end{aligned} \quad (59)$$

The definitions

$$v_a = \int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda, \quad m_0 = \min\{v_c : c \in J_{n,I}\}$$

imply together with (38) that

$$\begin{aligned} 1 & \leq \limsup_{n \rightarrow \infty} \frac{n_1^I(n)^r \sum_{a \in J_{n,I}} (v_a + \varepsilon_3(n))^{1+r}}{n_1^I(n)^r \sum_{a \in J_{n,I}} v_a^{1+r}} \\ & \leq \limsup_{n \rightarrow \infty} \frac{\left(1 + \frac{\varepsilon_3(n)}{m_0}\right)^{1+r} \sum_{a \in J_{n,I}} v_a^{1+r}}{\sum_{a \in J_{n,I}} v_a^{1+r}} \\ & \leq \limsup_{n \rightarrow \infty} \left(1 + \frac{\varepsilon_3(n)}{M_{1,I}^{1/(1+r)} \min\{\text{diam}(q_n^{-1}(a)) : a \in J_{n,I}\}} \right)^{1+r} = 1, \end{aligned} \quad (60)$$

with $M_{1,I}$ as defined in (39). Similarly,

$$\begin{aligned} 1 &\geq \liminf_{n \rightarrow \infty} \frac{n_1^I(n)^r \sum_{a \in J_{n,I}} (v_a - \varepsilon_3(n))^{1+r}}{n_1^I(n)^r \sum_{a \in J_{n,I}} v_a^{1+r}} \\ &\geq \liminf_{n \rightarrow \infty} \left(1 - \frac{\varepsilon_3(n)}{M_{1,I}^{1/(1+r)} \min\{\text{diam}(q_n^{-1}(a)) : a \in J_{n,I}\}} \right)^{1+r} = 1. \end{aligned} \quad (61)$$

Thus, (58) turns together with (60) and (35) into

$$\begin{aligned} &\limsup_{n \rightarrow \infty} n_1^I(n)^r \sum_{a \in J_{n,I}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ &\leq \limsup_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} \left(\int_{q_n^{-1}(a)} h^{1/(1+r)} d\lambda \right)^{1+r} \\ &\leq \limsup_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} (n_1^I(n)^{-1} + \varepsilon_1(n))^{1+r} \left(\int_{T_{n,I}} h^{1/(1+r)} d\lambda \right)^{1+r} \\ &= Q(r) \limsup_{n \rightarrow \infty} (1 + \varepsilon_1(n) n_1^I(n))^{1+r} \left(\int_{T_{n,I}} h^{1/(1+r)} d\lambda \right)^{1+r} \\ &= Q(r) \left(\int_I h^{1/(1+r)} d\lambda \right)^{1+r}. \end{aligned} \quad (62)$$

On the other hand, from (59) we deduce with (61) and (35) that

$$\begin{aligned} &\liminf_{n \rightarrow \infty} n_1^I(n)^r \sum_{a \in J_{n,I}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) \\ &\geq \liminf_{n \rightarrow \infty} Q(r) n_1^I(n)^r \sum_{a \in J_{n,I}} (n_1^I(n)^{-1} - \varepsilon_1(n))^{1+r} \left(\int_{T_{n,I}} h^{1/(1+r)} d\lambda \right)^{1+r} \\ &= Q(r) \left(\int_I h^{1/(1+r)} d\lambda \right)^{1+r}. \end{aligned} \quad (63)$$

Now, (62) and (63) yield the assertion. \square

Finally, we are able to state and prove the main result of the whole paper. In the proof we rely on Lemma 3.9. To this end we have to control $n/n_1^{I_0}(n)$ for a (suitable chosen) fixed interval I_0 . This is possible by the monotonicity of $n_2^{I_0}(\cdot)$ according to Lemma 3.7 (a).

Theorem 3.10. *Let μ be weakly unimodal and $(q_n)_{n \in \mathbb{N}}$ the sequence of n -level Gersho quantizers for μ . If $C_0 < \infty$, then $(q_n)_{n \in \mathbb{N}}$ is asymptotically optimal.*

Proof. Let $n \in \mathbb{N}$ and denote by $x_1(n)$ the right endpoint of the leftmost codecell of q_n . Similarly, let $x_2(n)$ the left endpoint of the rightmost codecell of q_n . More exactly,

$$x_1(n) = \sup q_n^{-1}(\min q_n(\mathbb{R})), \quad x_2(n) = \inf q_n^{-1}(\max q_n(\mathbb{R})).$$

Recall (cf.(40)) that $D(\mu, q_n, r) \leq D(\mu, q_1, r)$. Consequently, using (G4) we obtain that

$$\begin{aligned} & \int_{(-\infty, x_1(n))} |x - \min q_n(\mathbb{R})|^r d\mu(x) = \int_{(x_2(n), \infty)} |x - \max q_n(\mathbb{R})|^r d\mu(x) \\ & = n^{-1} D(\mu, q_n, r) \leq n^{-1} D(\mu, q_1, r) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, Lemma 2.2 implies that

$$x_1(n) \rightarrow \inf(\text{supp}(\mu)) \quad \text{and} \quad x_2(n) \rightarrow \sup(\text{supp}(\mu)) \quad \text{as } n \rightarrow \infty.$$

Now choose $n_0 > 6$ such that $\mu(I_0) > 0$ with

$$I_0 = [x_1(n_0), x_2(n_0)] \subset \text{supp}(\mu).$$

Now let $n \geq n_0$ and define $k(n) \in \{0, 1, \dots\}$ such that

$$n_0 2^{k(n)} \leq n < n_0 2^{k(n)+1}. \quad (64)$$

Recall definition (27). By the definition of I_0 we obtain $n_2^{I_0}(n_0) = 2$. Due to the uniqueness of q_n (cf. Proposition 2.5) we get inductively that

$$n_2^{I_0}(n_0 2^{k(n)}) = 2^{k(n)+1} \quad \text{resp.} \quad n_2^{I_0}(n_0 2^{k(n)+1}) = 2^{k(n)+2}.$$

Now, (64) and Lemma 3.7 (a) imply

$$2^{k(n)+1} \leq n_2^{I_0}(n) \leq 2^{k(n)+2},$$

yielding

$$\frac{n_2^{I_0}(n)}{n} \leq \frac{2^{k(n)+2}}{n_0 2^{k(n)}} = \frac{4}{n_0}$$

Thus we deduce from definition (27) and (30) that

$$1 \leq \frac{n}{n_1^{I_0}(n)} \leq \frac{n}{n - n_2^{I_0}(n) - 2} \leq \frac{1}{1 - \frac{4}{n_0} - \frac{2}{n}} \leq \frac{1}{1 - \frac{6}{n_0}}. \quad (65)$$

Applying (G4) we get

$$\begin{aligned} n^r D(\mu, q_n, r) &= n^{1+r} \frac{D(\mu, q_n, r)}{n} \\ &= \left(\frac{n}{n_1^{I_0}} \right)^{1+r} (n_1^{I_0})^r \sum_{a \in J_{n, I_0}} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x). \quad (66) \end{aligned}$$

Now, (65), (66) and Lemma 3.9 are yielding

$$\limsup_{n \rightarrow \infty} n^r D(\mu, q_n, r) \leq \left(\frac{1}{1 - \frac{6}{n_0}} \right)^{1+r} Q(r) \left(\int_{I_0} h^{1/(1+r)} d\lambda \right)^{1+r}. \quad (67)$$

By the same arguments we also deduce

$$\liminf_{n \rightarrow \infty} n^r D(\mu, q_n, r) \geq Q(r) \left(\int_{I_0} h^{1/(1+r)} d\lambda \right)^{1+r}. \quad (68)$$

Because the choice of $n_0 > 6$ was arbitrary, the assertion follows from (67) and (68). \square

Remark 3.11. *Alternatively, relation (68) follows also directly from Theorem 3.1.*

4 Concluding remarks

Remark 4.1. *Although condition (25) in Theorem 3.1 cannot be dropped completely (cf. [10, Example 6.4]), it can be weakened into $C_0 < \infty$ for one-dimensional weakly unimodal distributions in view of Theorem 3.10. Indeed, (25) is only needed to prove in Theorem 3.1 that $n^r D_{n,r}(\mu)$ is upper-bounded by C_0 , but this follows from Theorem 3.10 if μ is weakly unimodal and $C_0 < \infty$.*

Remark 4.2. *It seems that Theorem 3.10 is also valid if the density h consists of a finite linear combination of weakly unimodal densities. Nevertheless it remains open, if Gersho quantizers are asymptotically optimal for arbitrary densities with finite r -th moment and $C_0 < \infty$. Moreover, it would be interesting to know if Theorem 3.10 remains valid for those distributions whose Gersho quantizers are unique.*

Remark 4.3. *The methods of this paper are strictly confined to the one-dimensional case. To find answers regarding the existence, construction and asymptotic optimality of Gersho quantizers in higher dimensions further research is necessary.*

Remark 4.4. *For dyadic homogeneous one-dimensional Cantor distributions [9, 13, 14] it is known that*

$$0 < \underline{C} = \liminf_{n \rightarrow \infty} n^r D_{n,r}(\mu) < \limsup_{n \rightarrow \infty} n^r D_{n,r}(\mu) = \overline{C} < \infty.$$

It is easy to check for these measures that for $n = 2^k$ a unique optimal quantizer q_k exists, these quantizers are also Gersho quantizers, and that $2^{kr} D_{2^k,r}(\mu)$ tends to \underline{C} . Hence,

$$\underline{C} = \liminf_{n \rightarrow \infty} n^r \inf \{ D(\mu, q, r) : q \in \mathcal{G}(n, \mu, r) \} \quad (69)$$

and by definition

$$\overline{C} \leq \limsup_{n \rightarrow \infty} n^r \inf\{D(\mu, q, r) : q \in \mathcal{G}(n, \mu, r)\}. \quad (70)$$

It remains open, if (70) is also an equation or not. Moreover, there exist distributions [11] which are non-atomic and singular to the Lebesgue measure but $\underline{C} = \overline{C}$. It remains also open, if (69) is also valid for these distributions or if even (70) turns into an equation for such measures.

Remark 4.5. Let $I \subset \mathbb{R}$ be a compact interval and let q_n be a quantizer with n codepoints. Recall definitions (30). If $(q_n)_{n \in \mathbb{N}}$ is asymptotically optimal for $\mu = h\lambda$ and (25) is satisfied, then Bucklew [4] has shown that

$$\frac{n_1^I(n)}{n} \rightarrow \frac{\int_I h^{\frac{1}{1+r}} d\lambda}{\int_{\mathbb{R}} h^{\frac{1}{1+r}} d\lambda} \quad \text{as } n \rightarrow \infty. \quad (71)$$

Hence, Remark (3.2) and Theorem 3.10 implies that (71) holds also for Gersho quantizers if μ is weakly unimodal and satisfies (25). Beside of this point density result, one is also interested in densities for the error and the mass of the codecells. More formally, let

$$E(n, I, q_n) = \frac{\int_I |x - q_n(x)|^r d\mu(x)}{D(\mu, q_n, r)}$$

and

$$M(n, I, q_n) = \sup\{|n\mu(q_n^{-1}(a)) - h^{r/(1+r)}(a) \cdot \left(\int h^{1/(1+r)} d\lambda\right)| : a \in J_{n,I}\}.$$

By analyzing the approach in [4] and [16], it is reasonable to conjecture that

$$E(n, I, q_n) \rightarrow \frac{\int_I h^{\frac{1}{1+r}} d\lambda}{\int_{\mathbb{R}} h^{\frac{1}{1+r}} d\lambda} \quad \text{as } n \rightarrow \infty, \quad (72)$$

i.e. that error and point density are asymptotically equal for asymptotically optimal quantizers, which would imply that (72) holds also for Gersho quantizers if μ is weakly unimodal and satisfies (25). Finally, due to [5, Theorem 4] we know that $M(n, I, q_n) \rightarrow 0$ if h is weakly unimodal and q_n is an asymptotically optimal stationary Voronoi quantizer. Note that a Voronoi quantizer is stationary if every codecell has non-vanishing μ -mass and (G3) is satisfied. Although, Gersho quantizers satisfy (G3) by definition they need not to be Voronoi quantizers. However, we conjecture that $M(n, I, q_n) \rightarrow 0$ still holds for Gersho quantizers if h is weakly unimodal and (25) is satisfied.

Remark 4.6. As already mentioned in the introduction, Delattre et al. [5] have shown that asymptotically optimal stationary quantizers q_n satisfy (G4) asymptotically, i.e.

$$G(n, I, q_n) = \sup\{|n^{1+r} \int_{q_n^{-1}(a)} |x - a|^r d\mu(x) - C_0| : a \in J_{n,I}\} \rightarrow 0 \quad (73)$$

as $n \rightarrow \infty$. Note, that asymptotic optimality does not imply (73). To see this let μ be the uniform distribution on $[0, 1]$, let $\varepsilon \in (0, 1)$ and consider the quantizer

$$\begin{aligned} & q_n^\varepsilon(x) \\ &= \frac{\varepsilon}{n} \cdot 1_{[0, \frac{\varepsilon}{n}]}(x) + \sum_{i=1}^{n-1} \left(\frac{\varepsilon}{n} + \frac{1 - \frac{\varepsilon}{n}}{n-1} \left(\frac{3}{2}^i - 1 \right) \right) \cdot 1_{[\frac{\varepsilon}{n} + (i-1) \cdot \frac{1 - \frac{\varepsilon}{n}}{n-1}, \frac{\varepsilon}{n} + i \cdot \frac{1 - \frac{\varepsilon}{n}}{n-1}]}(x). \end{aligned}$$

By a straightforward calculation one gets

$$n^r D(\mu, q_n^\varepsilon, r) = Q(r) \left(\varepsilon^{1+r} n^{-1} + \frac{n-1}{n} \left(1 + \frac{1-\varepsilon}{n-1} \right)^{1+r} \right) \rightarrow Q(r)$$

as $n \rightarrow \infty$. Hence, $(q_n^\varepsilon(x))_{n \in \mathbb{N}}$ is asymptotically optimal and satisfies (G1), (G2) and (G3), but satisfies neither (G4) nor (73).

Assume that (25) is satisfied and the distribution is weakly unimodal. By Theorem 3.1 and Theorem 3.10 we know that optimal quantizers and Gersho quantizers are asymptotically optimal. Nevertheless, quantizers can exist - as shown by this example - which are neither optimal nor Gersho quantizers but asymptotically optimal. Motivated by this and (73) let us define the condition

$$(G5) n^{1+r} \max\left\{ \left| \int_{q_n^{-1}(a)} |x-a|^r d\mu(x) - \frac{D(\mu, q_n, r)}{n} \right| : a \in q_n(\mathbb{R}) \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now consider the class of quantizers which are satisfying conditions (G1), (G2), (G3) and (G5). Of course, (G5) is weaker than (G4). Hence, Gersho quantizers are contained in this class but it is open if optimal quantizers are also a part of this class. Again, the example from above demonstrates that this class does not contain all asymptotically optimal quantizers. It remains open if every quantizer of this class is asymptotically optimal.

Remark 4.7. To the authors knowledge there are no general results available so far concerning the rate of convergence in (26). Fort and Pages [6, Theorem 5] have shown for three special families of scalar distributions (exponential, power and inverse power) that

$$\limsup_{n \rightarrow \infty} |C_0 - n^r D(\mu, q_n, r)| \frac{n}{\log(n)} < \infty, \quad (74)$$

if $q_n \in \mathcal{C}(n, \mu, r)$. It remains open, if (74) is still true for Gersho quantizers. Moreover, it is unclear if (74) is generally true for weakly unimodal densities and Gersho quantizers.

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References

- [1] Abaya, E.F., Wise, G.L.: Some notes on optimal quantization. Proceedings of the International Conference on Communications (Denver, Colorado), 30.7.1-10.7.5. IEEE Press, New York (1981)
- [2] Abaya, E.F., Wise, G.L.: On the existence of optimal quantizers. *IEEE Trans. Inf. Theory* **28**(6), 937–940 (1982)
- [3] Bucklew, J.A., Wise, G.L.: Multidimensional asymptotic quantization theory with r th power distortion measures. *IEEE Trans. Inf. Theory* **28**(2), 239–247 (1982)
- [4] Bucklew, J.A.: Two Results on the Asymptotic Performance of Quantizers. *IEEE Trans. Inf. Theory* **30**(2), 341–348 (1984)
- [5] Delattre, S., Fort, J.-C., Pagès, G.: Local distortion and μ -mass of the cells of one dimensional asymptotically optimal quantizers. *Commun. Stat., Theory Methods* **33**(5), 1087–1117 (2004)
- [6] Fort, J.-C., Pagès, G.: Asymptotics of optimal quantizers for some scalar distributions. *J. Comput. Appl. Math.* **146**(2), 253–275 (2002)
- [7] Gersho, A.: Asymptotically optimal block quantization. *IEEE Trans. Inf. Theory* **25**(4), 373–380 (1979)
- [8] Gersho, A., Gray, R.M.: Vector quantization and signal compression. Boston, Kluwer Academic Publishers. (1992)
- [9] Graf, S., Luschgy, H.: The quantization of the Cantor distribution. *Math. Nachr.* **183**, 113–133 (1997)
- [10] Graf, S., Luschgy, H.: Foundations of Quantization for Probability Distributions. Lecture Notes 1730 Springer, Berlin (2000)
- [11] Graf, S., Luschgy, H.: The point density measure in the quantization of self-similar probabilities. *Math. Proc. Camb. Philos. Soc.* **138**(3), 513–531 (2005)
- [12] Gray, R., Neuhoff, D.: Quantization. *IEEE Trans. Inf. Theory* **44**(6), 2325–2383 (1998)
- [13] Kesseböhmer M., Zhu, S.: Stability of quantization dimension and quantization for homogeneous Cantor measures. *Math. Nachr.* **280**(8), 866–881 (2007)
- [14] Kreitmeier, W.: Optimal quantization for dyadic homogeneous Cantor distributions. *Math. Nachr.* **281**(9), 1307–1327 (2008)
- [15] Kreitmeier, W.: Optimal vector quantization in terms of Wasserstein distance. *J. Multivariate Anal.* **102**(8), 1225–1239 (2011)

- [16] Kreitmeier, W., Linder, T.: Entropy Density and Mismatch in High-Rate Scalar Quantization with Rényi Entropy Constraint. *IEEE Trans. Inform. Theory* **58**, 4105–4116 (2012)
- [17] Linder, T.: On asymptotically optimal companding quantization. *Probl. Control Inf. Theory* **20**(6), 475–484 (1991)
- [18] Lloyd, S.P.: Least squares quantization in PCM. *IEEE Trans. Inf. Theory* **28**(2), 129–137 (1982)
- [19] Panter, P.F., Dite, W.: Quantization in pulse-count modulation with nonuniform spacing of levels. *Proc. IRE* **39**, 44–48 (1951)
- [20] Pollard, D.: Quantization and the method of k-means. *IEEE Trans. Inf. Theory* **28**(2), 199–205 (1982)
- [21] Tóth, L.F.: Sur la représentation d’une population infinie par un nombre fini d’éléments. *Acta Math. Acad. Sci. Hung.* **10**(3-4), 299–304 (1959)
- [22] Trushkin, A.V.: Monotony of Lloyd’s method II for log-concave density and convex error weighting function. *IEEE Trans. Inf. Theory* **30**(2), 380–383 (1984)
- [23] Zador, P.: Development and evaluation of procedures for quantizing multivariate distributions. PhD thesis, Stanford Univ. (1963)

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