



Holomorphic Extensions in the O-Minimal Structure $\mathbb{R}_{\text{an},\text{exp}}$

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1 Introduction

1.1 Motivation

The theory of complex analysis has its beginnings in the first part of the 19th century. The main cofounders of this area are Augustin-Louis Cauchy, Bernhard Riemann and Karl Weierstraß. It is the theory that investigates functions on complex numbers and sets itself strictly apart from the theory of real analysis. A very remarkable result is given by William Fogg Osgood in the year 1899. He proved the famous theorem that a function defined on an open subset of \mathbb{C}^n is holomorphic if it is continuous and holomorphic in every single variable (see [42]). This is in general not true in the real setting. (Compare with [26] for general properties of real analytic functions.)

Example

Consider the function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Then f is continuous at $(0, 0)$ and real analytic in every single variable, but it is not real analytic at $(0, 0)$.

Friedrich Moritz Hartogs could even strengthen the result of Osgood. He showed that a function defined on an open subset of \mathbb{C}^n is holomorphic if it is holomorphic in every single variable (see [42]). So one may even dispense with the continuity property. Furthermore Hartogs recognized that there are many phenomenas in complex analysis which hold in several variables but not in one variable. They are called Hartogs phenomena. An example is the following extension theorem which can be found in [29] (see [18] for the original version for polydiscs).

Theorem [Hartogs's Extension Theorem]

Let $n \geq 2$, $U \subset \mathbb{C}^n$ be open and $K \subset U$ be compact. Let $f : U \setminus K \rightarrow \mathbb{C}$ be holomorphic. If $U \setminus K$ is connected then there is a unique holomorphic $F : U \rightarrow \mathbb{C}$ such that $F|_{U \setminus K} = f$.

Consequently one sees that isolated singularities of meromorphic functions in several variables are removeable. Examples for non-Hartogs phenomenas are for example the Cauchy's integral formula and consequently the theorem of Liouville and the Maximum principle (see [42]). A theorem which holds in one variable, but not in several variables is for example the Riemann mapping theorem. One can show that the unit ball in higher dimensions is not biholomorphic to polydiscs.

The study of complex analysis has many applications in commutative algebra, functional analysis, algebraic topology or sheaf theory (see [42]) and also in physics, for example to give formal solutions for partial differential equations like the heat conduction equation.

We combine the theory of complex analysis with o-minimality. The designation o-minimal is the abbreviation for order-minimal. O-minimality was cofounded by Lou van den Dries in the early 1980's and connects different areas of pure mathematics like the area of analysis, real algebra, algebraic geometry, number theory and originates in mathematical logic and model theory (see [11]). An o-minimal structure \mathcal{M} is a family of sets $(\mathcal{M}_n)_{n \in \mathbb{N}}$ which fulfill specific axioms in which the sets \mathcal{M}_1 are precisely the finite unions of intervals and points on the reals. They are an elegant and surprisingly efficient generalization of the category of semialgebraic and of globally subanalytic sets. Lou van den Dries writes in [11]: "I had noticed that many properties of semialgebraic sets and maps could be derived from a few simple axioms". He also recognized that an o-minimal structure exhibits nice topological properties. This includes for example the cell decomposition theorem, the fact that every definable set is homeomorphic to a finite union of hypercubes and that every definable set has finitely many connected components which are again definable. Definable sets can also be triangulated and trivialized. So o-minimal structures have a kind of tame character, nice finiteness properties and offer great potential for research in different areas.

The major line of research was based on discovering o-minimal structures on the reals. For example the pure real field \mathbb{R} is o-minimal. The definable sets are exactly the semialgebraic sets since they are closed under projections by Alfred Tarski (see [41], this was popularized by Abraham Seidenberg in [36]). But there are much bigger o-minimal structures on \mathbb{R} . Together with a theorem of Gabrielov, Van den Dries could prove the remarkable result that \mathbb{R}_{an} , the expansion of the real field by all restricted analytic functions, is o-minimal (see [10]). The definable sets in \mathbb{R}_{an} are precisely the globally subanalytic ones. For example the restriction of the global sine function on a compact interval is globally subanalytic. In [10] it is shown that \mathbb{R}_{an} is polynomially bounded. So the global exponential function is not definable in there. Lou van den Dries and Chris Miller could even show in [15] that we stay in the o-minimal context if we add the global exponential function (see also [13]). Then we obtain the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. This is one of the most important o-minimal structures, because in addition to the restricted analytic functions all elementary functions like polynomial functions, the global arctangent function, the global logarithm, the global exponential function and hyperbolic functions are definable in there. They have crucial application in diophantine geometry (see for example [44] where definable functions with special diophantine properties and growth rates are investigated or [35] for the treatment of the André-Oort

conjecture for \mathbb{C}^n from the viewpoint of o-minimality).

Kobi Peterzil and Sergei Stepanovich Starchenko designed the development of complex analysis within the o-minimal framework also in non-standard setting (see [34]). One obtains analogies of classical results as well as strong differences, due to the o-minimality assumption. For example the maximum principle, the open mapping theorem, the theorem of Liouville, the identity theorem and Riemann's removeable theorem hold in this setting (see [34]). However the key feature is that a definable holomorphic function in an o-minimal expansion of \mathbb{R} does not have an essential singularity. So the entire definable holomorphic functions are exactly the polynomials. Further results of Peterzil and Starchenko concern analytic geometry. For example they gave some stronger version of Chow's theorem which states that a definable analytic subset of \mathbb{C}^n is already an algebraic variety (see [33]). All this theory can be formulated for an arbitrary real closed field R with algebraic closure $K = R[\sqrt{-1}]$. But to describe such features in this non-standard setting it is necessary to use "Topological Analysis" instead of power series and integration (see [34]).

So it would be desirable to understand definability in the complex setting outgoing from definability on the reals. A very interesting question in this context is the following. Are real analytic functions definable in an o-minimal structure on the reals reducts of definable holomorphic functions? To investigate this question one has to find a definable holomorphic extension for a given definable real analytic function. Of course finding a holomorphic extension is not difficult simply by power series expansion, but the crucial point is that this extension needs not to be definable. Tobias Kaiser affirmed this question for the o-minimal structure \mathbb{R}_{an} . In his paper [19] he proved the deep technical result that every real analytic globally subanalytic function extends in a globally subanalytic way to a holomorphic function. He could even establish a parametric version of this result. However there is no general concept to solve such difficult problems and this can't be generalized easily to arbitrary o-minimal structures on the reals. For example for $\mathbb{R}_{\text{an,exp}}$ issues regarding global complexification are not solved at present. But there is the deep model theoretical fact that every definable function in $\mathbb{R}_{\text{an,exp}}$ is piecewise given by $\mathcal{L}_{\text{an}}(\text{exp, log})$ -terms. Kaiser used this fact to show that univariate real analytic functions which are definable in $\mathbb{R}_{\text{an,exp}}$ extend in a definably way to a holomorphic function. A quantitative version of this result was given by Kaiser and Speissegger in [24] (compare also with Wilkie in [44]). But especially for this big structure it would be desirable to have an analogous result as in the globally subanalytic case to cover a huge class of real analytic functions, because there are many fields of application.

1.2 Outline and Results

In the next paragraphs we give a short overview of this thesis and its research results. More detailed information can be found in the respective chapter and section. Chapter 2 presents the preliminaries. First we investigate o-minimal structures on the reals and take a closer look at global complexification in such structures, give the most important definitions and facts. The dividing line from the structure \mathbb{R}_{an} and the structure $\mathbb{R}_{\text{an,exp}}$ comes from the global exponential function. The exponential function gives non-flatness, C^∞ but not real analyticity and definable holomorphic extensions in one variable on small areas as an example at the end of this chapter shows.

So the idea is to consider compositions of globally subanalytic functions and the global logarithm at first. We call such functions log-analytic. Then we take a closer look at compositions of log-analytic functions and exponentials whose arguments are locally bounded which we call restricted log-exp-analytic (compare Chapter 3).

Example

The function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto \arctan(\log(\max\{\log(x^4 + \log(y^2 + 2)), 1\})),$$

is log-analytic. The function

$$g :]0, 1[^2 \rightarrow \mathbb{R}, (x, y) \mapsto \arctan(\log(e^{1/x \cdot \log^2(1/y)} + \log(e^{e^{1/x}} + 2))),$$

is restricted log-exp-analytic, but the function

$$h : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} e^{-1/x} & x > 0, \\ 0, & \text{else,} \end{cases}$$

is not.

In Chapter 4, the first main chapter, we will formulate and prove preparation theorems for log-analytic functions, definable functions in $\mathbb{R}_{\text{an,exp}}$ and restricted log-exp-analytic functions. This theorems come from Lion and Rolin (see [28]) and give very nice representations for definable functions in one variable.

Examining this preparation theorems we obtain nice differentiability properties for restricted log-exp-analytic functions like strong quasianalyticity, non-flatness and the following parametric version of Tamm's theorem which has been formulated by Van den Dries and Miller in [14] for globally subanalytic functions (compare Chapter 5).

Theorem A

Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable such that X_t is open for $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x)$, be a restricted log-exp-analytic function in x . Then there is $N \in \mathbb{N}$ such that for all $t \in \mathbb{R}^n$ if $f(t, -)$ is C^N at x then $f(t, -)$ is real analytic at x .

In Chapter 6 we show that a real analytic restricted log-exp-analytic function has a holomorphic extension which is again restricted log-exp-analytic. This is the main result of our research.

Theorem B

Let $n \in \mathbb{N}$ and let $U \subset \mathbb{R}^n$ be open. Let $f : U \rightarrow \mathbb{R}$ be a real analytic restricted log-exp-analytic function. Then there is an open definable $Z \subset \mathbb{C}^n$ with $U \subset Z$ and a holomorphic restricted log-exp-analytic $F : Z \rightarrow \mathbb{C}$ with $F|_U = f$.

The idea of the proof is considering unary functions with parameters and then doing an induction on the number of variables (see [19]). This requires a rather sophisticated set-up. The arguments for the univariate case require the preparation theorems from Chapter 4 which are deep geometrical results for restricted log-exp-analytic functions, integration of restricted log-exp-analytic functions and methods from complex analysis. The arguments for the multivariate case require theorem A. We finally obtain a parametric version of theorem B. (see also Kaiser in [19] for a version of theorem B and C in the globally subanalytic setting.)

Theorem C

Let $n \in \mathbb{N}_0, m \in \mathbb{N}$ and $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable such that X_t is open for all $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x)$, be restricted log-exp-analytic in x such that f_t is real analytic for every $t \in \mathbb{R}^n$. Then there is a definable $Z \subset \mathbb{R}^n \times \mathbb{C}^m$ with $X \subset Z$ such that Z_t is open for all $t \in \mathbb{R}^n$ and a function $F : Z \rightarrow \mathbb{C}, (t, z) \mapsto F(t, z)$, which is restricted log-exp-analytic in z such that F_t is holomorphic for every $t \in \mathbb{R}^n$ and $F|_X = f$.

Finally, in Chapter 7 we give a short conclusion and discuss some open questions.

Notations

The empty sum is by definition 0 and the empty product is by definition 1. By \mathbb{Q} we denote the set of rational numbers, by \mathbb{R} we denote the set of real numbers, by \mathbb{C} the set of complex numbers, by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the set of nonnegative integers. Let $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $a \in \mathbb{R}$ we set $\mathbb{R}_{>a} := \{x \in \mathbb{R} \mid x > a\}$. Given $x \in \mathbb{R}$ let $\lceil x \rceil$ be the smallest integer which is not smaller than x and let $\text{sign}(x) \in \{\pm 1\}$ its sign if $x \neq 0$. For $a \in \mathbb{R}$ we have $a < \infty$, $-\infty < a$ and we set $a + \infty := \infty$ and $a - \infty := -\infty$. For $a, b \in \mathbb{R}$ with $a \leq b$ we denote by $[a, b]$ the closed interval and by $]a, b[$ the open interval with endpoints a, b , respectively. Given a subset A of \mathbb{R}^n we denote by \overline{A} its closure. By \log_k we denote the k -times iterated of the natural logarithm and by \exp_k the k -times iterated of the natural exponential (where $\log_0 = \exp_0 = \text{id}$). Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $r > 0$, we define the box

$$Q^n(x, r) := \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid |y_j - x_j| < r \text{ for all } j \in \{1, \dots, n\}\}.$$

For $m \in \mathbb{N}$ and $X \subset \mathbb{R}^m$ we consider the following: For $x \in \mathbb{R}^m$ let $\text{dist}(x, X) := \inf\{|x - y| \mid y \in X\}$. For two functions $f, g : X \rightarrow \mathbb{R}$ we say that $f > g$ if $f(x) > g(x)$ for every $x \in X$. For a set E of positive real valued functions on X we set $\log(E) := \{\log(g) \mid g \in E\}$. For $C \subset \mathbb{R}^m$ with $C \subset X$ and a set E of real valued functions on X we set $E|_C := \{g|_C \mid g \in E\}$.

For $X \subset \mathbb{R}^n \times \mathbb{R}$ let $X_{\neq 0} := \{(x, y) \in X \mid y > 0\}$.

For $X \subset \mathbb{R}^n \times \mathbb{R}^m$ and $t \in \mathbb{R}^n$ we set $X_t := \{x \in \mathbb{R}^m \mid (t, x) \in X\}$ and for a function $f : X \rightarrow \mathbb{R}$, $(t, x) \mapsto f(t, x)$, and $t \in \mathbb{R}^n$ let $f_t : X_t \rightarrow \mathbb{R}$, $x \mapsto f(t, x)$.

We set $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$. For $m, n \in \mathbb{N}$ we denote by $M(m \times n, \mathbb{Q})$ the set of $m \times n$ -matrices with rational entries. For $P \in M(m \times n, \mathbb{Q})$ we denote by ${}^tP \in M(n \times m, \mathbb{Q})$ its transpose. By the symbol \sim we denote asymptotic equivalence.

We set $\mathbb{C}^- := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $\mathbb{C}_a^- := \mathbb{C} \setminus \mathbb{R}_{\leq a}$ and $\mathbb{C}_a^+ := \mathbb{C} \setminus \mathbb{R}_{\geq a}$ for $a \in \mathbb{R}$. Let z range over \mathbb{C} . We denote by $\arg : \mathbb{C} \setminus \{0\} \rightarrow]-\pi, \pi]$ the standard argument function and define $\log : \mathbb{C}^- \rightarrow \mathbb{C}$, $z \mapsto \log(z) := \log(|z|) + i\arg(z)$, (i.e. z is mapped on the principal value of the complex logarithm) and $z^q := e^{q\log(z)}$ for $z \in \mathbb{C}^-$ and $q \in \mathbb{Q}$. For $r_1, r_2 \in \mathbb{R}_{\geq 0}$ and $c \in \mathbb{C}$ we set

$$A(c, r_1, r_2) := \{z \in \mathbb{C} \mid r_1 < |z - c| < r_2\}.$$

For $r \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$ we set

$$B(c, r) := \{z \in \mathbb{C} \mid |z - c| < r\}.$$

Given $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $r > 0$, we define the polydisc

$$D^n(z, r) := \{(w_1, \dots, w_n) \in \mathbb{C}^n \mid |w_j - z_j| < r \text{ for all } j \in \{1, \dots, n\}\}.$$

Terminology from model theory

By $\mathcal{L} := (\leq, +, \cdot, -, 0, 1)$ we denote the language of ordered rings, by \mathcal{L}_{an} the language of ordered rings augmented by symbols for all restricted analytic functions (compare with Definition 2.7 for the notion of a restricted analytic function), by $\mathcal{L}_{\text{an}}(\exp, \log)$ the language \mathcal{L}_{an} augmented by symbols for the global exponential function and the global logarithm and by $\mathcal{L}_{\text{an}}(^{-1}, (\sqrt[n]{\dots})_{n=2,3,\dots}, \log)$ the language \mathcal{L}_{an} augmented by a function symbol $^{-1}$ for taking reciprocals with respect to " \cdot ", by function symbols $\sqrt[n]{\dots}$ for taking the n 'th root for $n \in \mathbb{N}$ with $n \geq 2$, and by a symbol for the global logarithm.

Let $\mathcal{L}' \in \{\mathcal{L}_{\text{an}}, \mathcal{L}_{\text{an}}(^{-1}, (\sqrt[n]{\dots})_{n=2,3,\dots}, \log), \mathcal{L}_{\text{an}}(\exp, \log)\}$. An \mathcal{L}' -term is inductively defined as follows.

- (i) 0 and 1 (i.e. every constant symbol in \mathcal{L}') are \mathcal{L}' -terms.
- (ii) Let x_1, x_2, \dots be the infinite list of variables in \mathcal{L}' . Then x_1, x_2, \dots are \mathcal{L}' -terms.
- (iii) If f is an m -ary function symbol in \mathcal{L}' for $m \in \mathbb{N}$ and t_1, \dots, t_m are \mathcal{L}' -terms then $f(t_1, \dots, t_m)$ is an \mathcal{L}' -term.

2 Preliminaries

In this chapter we take up some necessary preliminaries. First we will introduce o-minimal structures, give basic definitions, theorems, explain its "tame character" and speak about parameterized integrals. Then we will combine this theory with complex analysis and take holomorphic extensions of definable real analytic functions within an o-minimal structure on the reals into account, give facts, definitions and some results.

2.1 O-Minimal Structures

2.1 Definition

A subset A of \mathbb{R}^n , $n \geq 1$, is called **semialgebraic** if there are $k, l \in \mathbb{N}_0$ and real polynomials $f_i, g_{i,1}, \dots, g_{i,k} \in \mathbb{R}[X_1, \dots, X_n]$ for $1 \leq i \leq l$ such that

$$A = \bigcup_{i=1}^l \{x \in \mathbb{R}^n \mid f_i(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,k}(x) > 0\}.$$

A map is called semialgebraic if its graph is semialgebraic.

2.2 Definition

A subset A of \mathbb{R}^n , $n \geq 1$, is called **semianalytic** if for each $a \in \mathbb{R}^n$ there are open neighbourhoods U, V of a with $\bar{U} \subset V$, $k, l \in \mathbb{N}_0$ and real analytic functions $f_i, g_{i,1}, \dots, g_{i,k}$ on V for $1 \leq i \leq l$, such that

$$A \cap U = \bigcup_{i=1}^l \{x \in U \mid f_i(x) = 0, g_{i,1}(x) > 0, \dots, g_{i,k}(x) > 0\}.$$

A map is called semianalytic if its graph is semianalytic.

2.3 Definition

A subset B of \mathbb{R}^n , $n \geq 1$, is called **subanalytic** if for each $a \in \mathbb{R}^n$ there is an open neighbourhood U of a , some $p \geq n$ and some bounded semianalytic set $A \subset \mathbb{R}^p$ such that $B \cap U = \pi_n(A)$ where $\pi_n : \mathbb{R}^p \rightarrow \mathbb{R}^n$, $(x_1, \dots, x_p) \mapsto (x_1, \dots, x_n)$, is the projection on the first n coordinates. A map is called subanalytic if its graph is subanalytic.

2.4 Remark

A semialgebraic set is semianalytic. A semianalytic set is subanalytic.

See [2] and [37] for geometrical descriptions of semianalytic resp. subanalytic sets and functions.

2.5 Definition

A subset B of \mathbb{R}^n , $n \geq 1$, is called **globally subanalytic** if it is subanalytic after applying the semialgebraic homeomorphism

$$\mathbb{R}^n \rightarrow]-1, 1[^n, x_i \mapsto \frac{x_i}{\sqrt{1+x_i^2}},$$

for $i \in \{1, \dots, n\}$. A map is called globally subanalytic if its graph is globally subanalytic.

2.6 Example

- (1) A semialgebraic set is globally subanalytic.
- (2) The restriction of the global sine function on a compact interval is globally subanalytic.

2.7 Definition

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called **restricted analytic** if there is a real convergent power series p in n variables which converges on an open neighbourhood of $[-1, 1]^n$ such that

$$f(x) = \begin{cases} p(x), & x \in [-1, 1]^n, \\ 0, & \text{else.} \end{cases}$$

2.8 Definition

For $n \in \mathbb{N}$ let M_n be a set of subsets of \mathbb{R}^n and let $\mathcal{M} := (M_n)_{n \in \mathbb{N}}$. Then \mathcal{M} is a **structure** on \mathbb{R} if the following holds for all $m, n, p \in \mathbb{N}$.

- (S1) If $A, B \in M_n$ then $A \cup B$, $A \cap B$ and $\mathbb{R}^n \setminus A \in M_n$. (So M_n is a Boolean algebra of subsets of M_n .)
- (S2) If $A \in M_n$ and $B \in M_m$ then $A \times B \in M_{n+m}$.
- (S3) If $A \in M_p$ and $p \geq n$ then $\pi_n(A) \in M_n$ where $\pi_n : \mathbb{R}^p \rightarrow \mathbb{R}^n$, $(x_1, \dots, x_p) \mapsto (x_1, \dots, x_n)$, denotes the projection on the first n coordinates.
- (S4) M_n contains the semialgebraic subsets of \mathbb{R}^n .

The structure $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ on \mathbb{R} is called **o-minimal** if additionally the following holds.

- (O) The sets in M_1 are exactly the finite unions of intervals and points.

2.9 Definition

Let $\mathcal{M} = (M_n)_{n \in \mathbb{N}}$ be a structure. Let $n \in \mathbb{N}$.

- a) A subset A of \mathbb{R}^n is called **definable in \mathcal{M}** if $A \in M_n$.
- b) Let $B \subset \mathbb{R}^n$ and $m \in \mathbb{N}$. A function $f : B \rightarrow \mathbb{R}^m$ is **definable in \mathcal{M}** if its graph $\{(x, f(x)) \mid x \in B\}$ is definable in \mathcal{M} .

2.10 Remark

The global sine or cosine function is not definable in an o-minimal expansion of \mathbb{R} .

2.11 Example

- (1) The smallest o-minimal structure on \mathbb{R} is given by the semialgebraic sets (see [41] and [36]). This is the pure real field and it is denoted by \mathbb{R} .
- (2) \mathbb{R}_{exp} , the structure generated on the real field by the global exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ (i.e. the smallest structure containing the semi-algebraic sets and the graph of the exponential function), is o-minimal. (With a theorem of Khovanskii from [25] we obtain o-minimality together with a result from [45] that the theory of \mathbb{R}_{exp} is model complete.)
- (3) \mathbb{R}_{an} , the structure generated on the real field by the restricted analytic functions, is o-minimal (see [10] together with Gabrielov's theorem from [17] that the complement of a globally subanalytic function is again globally subanalytic; see also [28], Section 1 for an analytic proof). The sets definable in \mathbb{R}_{an} are precisely the globally subanalytic ones (see [11]).
- (4) $\mathbb{R}_{\text{an,exp}}$, the structure generated by \mathbb{R}_{an} and \mathbb{R}_{exp} , is o-minimal (see for example [13] for a model theoretic proof respectively [28], Section 2 for an analytic proof).

Especially the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ is very important, because a huge class of elementary functions is definable in there. With the following fact we understand completely how the definable functions in $\mathbb{R}_{\text{an,exp}}$ look like.

2.12 Fact (Van den Dries/Macintyre/Marker, [13], Corollary 4.7)

*Let $n \in \mathbb{N}$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be definable in $\mathbb{R}_{\text{an,exp}}$. Then there are $s \in \mathbb{N}$ and $\mathcal{L}_{\text{an}}(\text{exp}, \text{log})$ -terms t_1, \dots, t_s such that for every $x \in \mathbb{R}^n$ there is $j \in \{1, \dots, s\}$ with $f(x) = t_j(x)$. So f is **piecewise given by $\mathcal{L}_{\text{an}}(\text{exp}, \text{log})$ -terms**.*

Subsequently, we exhibit the tame geometric behaviour of o-minimal structures. An essential role in this context plays the cell decomposition theorem which states that we can decompose every definable set in finitely many disjoint subsets of a special form called cells. This concept is very helpful for technical proofs: The idea is to do calculation on every single cell at first and finally obtain the result by considering all of them.

For the rest of the section "definable" means always "definable in \mathcal{M} " if the underlying o-minimal structure \mathcal{M} on \mathbb{R} is clear from the context.

2.13 Definition

Let \mathcal{M} be an o-minimal structure on \mathbb{R} . By induction on n we define a **definable cell** $C \subset \mathbb{R}^n$ as follows.

$n = 1$: C is either a singleton or an open interval.

$n \rightarrow n + 1$: C has one of the following form.

- i) $C = \text{graph}(f)$,
- ii) $C = \{(t, x) \in B \times \mathbb{R} \mid f(t) < x < g(t)\} :=]f, g[_B$,
- iii) $C = \{(t, x) \in B \times \mathbb{R} \mid f(t) < x < \infty\} :=]f, \infty[_B$,
- iv) $C = \{(t, x) \in B \times \mathbb{R} \mid -\infty < x < f(x)\} :=]-\infty, f[_B$,
- v) $C = \{(t, x) \in B \times \mathbb{R} \mid -\infty < x < \infty\} :=]-\infty, \infty[_B$,

where $B \subset \mathbb{R}^n$ is a definable cell called the **base** of C in \mathbb{R}^n and $f, g : B \rightarrow \mathbb{R}$ are continuous definable functions with additionally $f < g$ in ii).

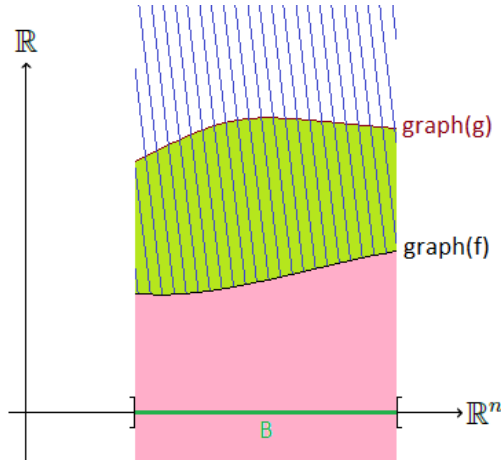


Figure 2.1 Different cell types in \mathbb{R}^{n+1} on the base B

2.14 Definition

Let \mathcal{M} be an o-minimal structure on \mathbb{R} . Let $m \in \mathbb{N}$ and $A \subset \mathbb{R}^m$ be non-empty and definable. A finite partition \mathcal{C} of A into definable cells is called a **definable cell decomposition** \mathcal{C} of A .

2.15 Theorem

Let \mathcal{M} be an o-minimal structure on \mathbb{R} . Let $n \in \mathbb{N}$ and $A \subset \mathbb{R}^n$ be definable. The following holds.

- (1) There is a definable cell decomposition \mathcal{C} of A .
- (2) Let $m \in \mathbb{N}$. If $f : A \rightarrow \mathbb{R}^m$ is a definable function then the definable cell composition \mathcal{C} of A can be chosen in this way that $f|_C$ is continuous for every $C \in \mathcal{C}$.

Proof

We refer to Van den Dries (see [11], Chapter 3). ■

Since a definable cell is definably connected an immediate consequence from this theorem is that every definable set is the union of finitely many connected components and these are again definable. We refer to the book of Van den Dries in [11], Coste in [9] or Miller/Rolin/Speissegger in [32] for more on the general properties of o-minimal structures.

To close this chapter we pick up results on definability of parameterized integrals over the reals which will be important for our purposes in Chapter 6.2. The class of constructible functions which was introduced by Cluckers and Miller in [7] plays a major role in this context.

2.16 Definition

Let $n \in \mathbb{N}$. Let $A \subset \mathbb{R}^n$ be a globally subanalytic set. A function $f : A \rightarrow \mathbb{R}$ is called **constructible** if there are $k \in \mathbb{N}$, $l_i \in \mathbb{N}$ for $i \in \{1, \dots, k\}$ and globally subanalytic functions $g_i : A \rightarrow \mathbb{R}$ and $h_{ij} : A \rightarrow \mathbb{R}_{>0}$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l_i\}$ such that

$$f(x) = \sum_{i=1}^k g_i(x) \prod_{j=1}^{l_i} \log(h_{ij}(x))$$

for every $x \in A$.

2.17 Remark

A globally subanalytic function is constructible.

Let $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let u range over \mathbb{R}^n and v over \mathbb{R}^m .

2.18 Definition

Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ be globally subanalytic. We set

$$\text{Fin}(f) := \left\{ u \in \mathbb{R}^n \mid \int_{\mathbb{R}^m} |f(u, v)| dv < \infty \right\}.$$

2.19 Fact (Comte/Lion/Rolin, [8], Theorem 1)

Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ be globally subanalytic. The following holds.

- (1) The set $\text{Fin}(f)$ is globally subanalytic.
- (2) The function

$$h : \text{Fin}(f) \rightarrow \mathbb{R}, u \mapsto \int_{\mathbb{R}^m} f(u, v) dv,$$

is constructible.

2.20 Fact (Cluckers/Miller, [6], Theorem 2.5)

The class of constructible functions is stable under parametric integration.

This means that the class of constructible functions is the smallest subclass of all definable functions in $\mathbb{R}_{\text{an,exp}}$ which contains all the globally subanalytic ones and is stable under parametric integration. Further and more deep theory on integration of constructible functions where loci of integrability and Lebesgue classes of constructible functions are studied can be found in [5] and [6].

However it is in general not possible to extend this result within $\mathbb{R}_{\text{an,exp}}$ beyond the constructible setting as the following fact indicates.

2.21 Fact (Van den Dries/Macintyre/Marker, [12] Theorem 5.11)

The function $\mathbb{R} \rightarrow \mathbb{R}, u \mapsto \int_0^u e^{-v^2} dv$, is not definable in $\mathbb{R}_{\text{an,exp}}$.

An important consequence of this fact is that parameterized integrals of functions which are piecewise given by $\mathcal{L}_{\text{an}}(-1, (\sqrt{\dots})_{n=2,3,\dots}, \log)$ -terms are in general not definable in $\mathbb{R}_{\text{an,exp}}$. (Compare Chapter 3 for an analytic definition of this class of functions.)

2.22 Example

Consider

$$f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}, (u, v) \mapsto \begin{cases} \frac{1}{2v^2 \sqrt{\log(v)}}, & 1 < v < u, \\ 0, & \text{else.} \end{cases}$$

We obtain for $1 < u$

$$\begin{aligned} \int_0^{\sqrt{\log(u)}} e^{-v^2} dv &= \int_0^{\log(u)} \frac{e^{-v}}{2\sqrt{v}} dv \\ &= \int_1^u \frac{1}{2v^2 \sqrt{\log(v)}} dv. \end{aligned}$$

For $u \in \mathbb{R}$ let

$$g(u) := \int_{\mathbb{R}} f(u, v) dv.$$

We have $g(u) = 0$ for $u \leq 1$ and

$$g(u) = \int_0^{\sqrt{\log(u)}} e^{-v^2} dv \leq \sqrt{\pi}$$

for $u > 1$. So we see that $\text{Fin}(f) = \mathbb{R}$. By Fact 2.21 we obtain that $\mathbb{R} \rightarrow \mathbb{R}, u \mapsto \int_0^u e^{-v^2} dv$, is not definable in $\mathbb{R}_{\text{an,exp}}$. Therefore $\mathbb{R} \rightarrow \mathbb{R}, u \mapsto \int_{\mathbb{R}} f(u, v) dv$, is also not definable in $\mathbb{R}_{\text{an,exp}}$. \blacksquare

Note that the antiderivative of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be considered as a special parameterized integral:

$$\int_0^x f(t) dt = \int_{\mathbb{R}} \tilde{f}(x, t) dt$$

where $\tilde{f}(x, t) = f(t) \mathbb{1}_{[0, x]}(t)$. For an o-minimal structure \mathcal{M} on \mathbb{R} consider $\mathcal{P}(\mathcal{M})$, the Pfaffian closure of \mathcal{M} , which is an o-minimal expansion of \mathcal{M} . It goes beyond the scope of this thesis to give the exact definition. For more details we refer to [32] and [39].

2.23 Fact (Speissegger, [39], Corollary on p.1)

Let $n = 1$ and $m = 1$. Let \mathcal{M} be an o-minimal structure on \mathbb{R} and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be definable in \mathcal{M} . Then the function $F : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \int_0^x f(t) dt$, is definable in $\mathcal{P}(\mathcal{M})$.

This shows that the parameterized integrals from Fact 2.21 and Example 2.22 are indeed definable in $\mathcal{P}(\mathbb{R}_{\text{an,exp}})$. Questions about this matter regarding definability of such integrals in more than one variable are quite open. Compare for example [23] for parameterized exponential integrals given by the Brownian motion on globally subanalytic sets.

2.2 Complexification and Global Complexification

In this section \mathcal{M} denotes a fixed o-minimal structure on \mathbb{R} and the expression "definable" means always "definable in \mathcal{M} " if not otherwise mentioned. We identify \mathbb{C} with \mathbb{R}^2 via $x + iy \mapsto (x, y)$. So "definable in \mathbb{C}^m " means "definable in \mathbb{R}^{2m} " for $m \in \mathbb{N}$.

2.24 Definition

We say that \mathcal{M} has **complexification** if the following holds. Let $l \in \mathbb{N}$. Let $U \subset \mathbb{R}^l$ be open and let $f : U \rightarrow \mathbb{R}$ be a definable real analytic function. Then for every $x \in U$ there is a definable open neighbourhood $B \subset \mathbb{R}^l$ of x , an open set $V \subset \mathbb{C}^l$ with $B \subset V$, and a definable holomorphic function $F : V \rightarrow \mathbb{C}$ such that $F|_B = f|_B$.

2.25 Example

- (1) Let \mathcal{M} be an o-minimal expansion of \mathbb{R}_{an} (e.g. $\mathcal{M} = \mathbb{R}_{\text{an,exp}}$). Then \mathcal{M} has complexification.
- (2) The o-minimal structure \mathbb{R}_{exp} does not have complexification.

Proof

(1): This follows simply by power series expansion of a real power series in the complex numbers.

(2): We denote by f the real exponential function which is definable in \mathbb{R}_{exp} . Consider

$$\exp : \mathbb{C} \rightarrow \mathbb{C}, x + iy \mapsto \exp(x)(\cos(y) + i \sin(y)),$$

which is holomorphic. Let $x \in \mathbb{R}$ and let V be an open ball around x in \mathbb{C} . By the identity theorem we see that $\exp|_V$ is the unique holomorphic extension of $f|_{V \cap \mathbb{R}}$ on V . But Bianconi showed in [1] that no restriction of the global sine function on a non-empty open interval is definable in \mathbb{R}_{exp} . ■

We mention that such questions are not answered completely. A field of open problems in this context are for example o-minimal structures which are generated by so called convergent Weierstrass systems. We refer to [30] for the details.

2.26 Definition

Let $l \in \mathbb{N}$. We say that \mathcal{M} has **l -ary global complexification** if the following holds. Let $U \subset \mathbb{R}^l$ be open and let $f : U \rightarrow \mathbb{R}$ be a definable real analytic function. Then there is a definable open set V in \mathbb{C}^l with $U \subset V$ and a definable holomorphic function $F : V \rightarrow \mathbb{C}$ such that $F|_U = f$. Moreover we call F a **global complexification** of f .

2.27 Remark

Let $l \in \mathbb{N}$. If \mathcal{M} has $(l+1)$ -ary global complexification then it has l -ary global complexification.

By Fact 2.12 we have that every definable function in $\mathbb{R}_{\text{an,exp}}$ is piecewise given by $\mathcal{L}_{\text{an}}(\exp, \log)$ -terms. A consequence is the following.

2.28 Fact (Kaiser, [19] Theorem C)

The o-minimal structure $\mathbb{R}_{\text{an,exp}}$ has unary global complexification.

2.29 Definition

We say that \mathcal{M} has **global complexification** if \mathcal{M} has l -ary global complexification for every $l \in \mathbb{N}$.

2.30 Remark

If \mathcal{M} has global complexification then it has complexification.

It is not known to us whether there is an o-minimal structure which has complexification but no global complexification. The problem is that there is no general concept how definable functions in o-minimal structures look like. But for the case that \mathcal{M} is the pure real field all definable real analytic functions are so called Nash functions. For this class of functions there are strong results like the implicit function theorem or the Artin-Mazur description (compare [3], Chapter 8).

2.31 Fact (Kaiser, [21] Theorem B)

The pure real field \mathbb{R} has global complexification.

(See also Shiota [38], Chapter I.6.7.)

For the o-minimal structure \mathbb{R}_{an} the situation is much more complicated since there are only piecewise nice representations for definable functions (compare Fact 2.12). Fortunately there are deep geometrical results for globally subanalytic functions. These are the preparation theorems of Lion and Rolin which give a nice representation of a globally subanalytic function in one variable and are precise enough to establish global complexification: The idea is to consider unary functions at first and then do a non-trivial induction on the number of variables. Therefore in [19] Tobias Kaiser set up the notion of parametric global complexification.

2.32 Definition

Let $l \in \mathbb{N}$. We say that \mathcal{M} has **l -ary parametric global complexification** if the following holds. Let $n \in \mathbb{N}_0$, $X \subset \mathbb{R}^n \times \mathbb{R}^l$ be definable such that X_t is open for all $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}$ be definable such that $f_t : X_t \rightarrow \mathbb{R}$ is real analytic for all $t \in \mathbb{R}^n$. Then there is a definable $Z \subset \mathbb{R}^n \times \mathbb{C}^l$ with $X \subset Z$ such that $Z_t = \{z \in \mathbb{C}^l \mid (t, z) \in Z\}$ is open for every $t \in \mathbb{R}^n$ and a definable function $F : Z \rightarrow \mathbb{C}$ such that $F_t : Z_t \rightarrow \mathbb{C}, z \mapsto F(t, z)$, is holomorphic for all $t \in \mathbb{R}^n$ and $F|_X = f$. We call $F : Z \rightarrow \mathbb{C}$ an **l -ary parametric global complexification** of f .

2.33 Remark

Let $l \in \mathbb{N}$.

- (1) If \mathcal{M} has $(l + 1)$ -ary parametric global complexification then it has l -ary parametric global complexification.
- (2) If \mathcal{M} has l -ary parametric global complexification then it has l -ary global complexification.

2.34 Definition

We say that \mathcal{M} has **parametric global complexification** if \mathcal{M} has l -ary parametric global complexification for every $l \in \mathbb{N}$.

2.35 Remark

If \mathcal{M} has parametric global complexification then it has global complexification.

If one deals with parametric global complexification there is the problem to "lift" the open property and the holomorphy from the univariate case into the multivariate case since piecewise open does not imply open. So we set up the concept of high parametric global complexification.

For $m, n, l \in \mathbb{N}$ let t range over \mathbb{R}^n , u over \mathbb{R}^m and x over \mathbb{R}^l .

2.36 Definition

- (a) Let $l \in \mathbb{N}$. We say that \mathcal{M} has **l -ary high parametric global complexification** if the following holds. Let $n, m \in \mathbb{N}_0$ and $X \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$ be definable such that $X_t = \{(u, x) \in \mathbb{R}^m \times \mathbb{R}^l \mid (t, u, x) \in X\}$ is open in $\mathbb{R}^m \times \mathbb{R}^l$ for every $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}$ be definable such that $f_t : X_t \rightarrow \mathbb{R}, (u, x) \mapsto f(t, u, x)$, is real analytic for every $t \in \mathbb{R}^n$. Then there is an l -ary parametric global complexification $F : Z \rightarrow \mathbb{C}, (t, u, z) \mapsto F(t, u, z)$, of f such that $Z_t = \{(u, z) \in \mathbb{R}^m \times \mathbb{C}^l \mid (t, u, z) \in Z\}$ is open in $\mathbb{R}^m \times \mathbb{C}^l$ for $t \in \mathbb{R}^n$. We call F an **l -ary high parametric global complexification of f with respect to (u, x)** .
- (b) We say that \mathcal{M} has **high parametric global complexification** if \mathcal{M} has l -ary high parametric global complexification for every $l \in \mathbb{N}$.

2.37 Remark

If \mathcal{M} has high parametric global complexification then it has parametric global complexification and therefore global complexification.

2.38 Fact (Kaiser, [19], Theorem 2.10)

The o -minimal structure \mathbb{R}_{an} has high parametric global complexification. Therefore \mathbb{R}_{an} has parametric global complexification and global complexification.

For the rest of the chapter let $\mathcal{M} = \mathbb{R}_{\text{an,exp}}$.

2.39 Example

Let $X := \mathbb{R} \times \mathbb{R}$ and consider the definable function

$$f : X \rightarrow \mathbb{R}, (u, x) \mapsto \begin{cases} \exp(\frac{x}{u}), & u > 0, \\ 0, & u \leq 0. \end{cases}$$

Let

$$Z := \{(u, z) \in \mathbb{R} \times \mathbb{C} \mid |\text{Im}(z)| < \pi u \text{ if } u > 0\}.$$

Then the function

$$G : Z \rightarrow \mathbb{C}, (u, z) \mapsto \begin{cases} \exp(\frac{z}{u}), & u > 0, \\ 0, & u \leq 0, \end{cases}$$

is a unary parametric global complexification of f .

2.40 Remark

Let X and f be as in Example 2.39. Let $Z \subset \mathbb{R} \times \mathbb{C}$ be definable and let $F : Z \rightarrow \mathbb{C}$ be a unary parametric global complexification of f . Then there is no $r > 0$ such that $]0, r[\times (]0, r[+ i]0, r[) \subset Z$, i.e. F is no unary high parametric global complexification with respect to (u, x) .

Proof

Note that Z_t is open for all $t \in \mathbb{R}$. Assume the contrary. Fix $r > 0$ such that

$$Q :=]0, r[\times (]0, r[+ i]0, r[) \subset Z.$$

By the identity theorem we have for $(u, z) \in Z$ that $F(u, z) = \exp(z/u)$ if $u > 0$ and $F(u, z) = 0$ otherwise. Note that Q is definable, but $F|_Q$ is not definable, because for every $(u, x + iy) \in Q$ we have

$$F(u, x + iy) = \exp(x/u)(\cos(y/u) + i \sin(y/u))$$

and the global sine function (or cosine function) is not definable at infinity, a contradiction. ■

Nevertheless it is absolutely possible that $\mathbb{R}_{\text{an,exp}}$ has global complexification since the underlying function must be real analytic and the function f in Example 2.39 is not real analytic at zero.

In this thesis we investigate functions on open sets which are compositions of globally subanalytic functions, the global logarithm and exponentials whose arguments are locally bounded. We call them restricted log-exp-analytic. Since real analytic functions are locally definable in \mathbb{R}_{an} there should be a connection between real analytic functions and restricted log-exp-analytic functions. The strategy here is also to formulate, prove and use a version of the preparation theorem of Lion and Rolin, construct the holomorphic extension of a real analytic restricted log-exp-analytic function f by considering every single exponential term which occurs in its preparation, construct a unary high parametric global complexification for f which is again restricted log-exp-analytic and then do a technical induction on the number of variables similarly as Kaiser did in [19].

3 Definable Functions in $\mathbb{R}_{\text{an,exp}}$

3.1 Log-Analytic Functions and the Exponential Number

By Fact 2.12 every definable function is given piecewise by $\mathcal{L}_{\text{an}}(\text{exp}, \text{log})$ -terms. So every definable function is piecewise the composition of globally subanalytic functions, the global exponential function and the global logarithm. Let's look at logarithmic-analytic functions (log-analytic for short) at first, i.e. functions which are iterated compositions from either side of globally subanalytic functions and the global logarithm. By Kaiser and Opris they exhibit nice differentiability properties like non-flatness (see [22]). From the viewpoint of logic, log-analytic functions are definable in the o-minimal expansion $\mathbb{R}_{\text{an,exp}}$ of \mathbb{R}_{an} ; in fact they generate the whole structure $\mathbb{R}_{\text{an,exp}}$.

For the whole section definable means definable in $\mathbb{R}_{\text{an,exp}}$ if not otherwise mentioned. For Section 3.1 let $m \in \mathbb{N}$ and $X \subset \mathbb{R}^m$ be definable.

3.1 Definition

Let $f : X \rightarrow \mathbb{R}$ be a function.

- (a) Let $r \in \mathbb{N}_0$. By induction on r we define that f is **log-analytic of order at most r** .

Base case: The function f is log-analytic of order at most 0 if f is piecewise the restriction of globally subanalytic functions, i.e. there is a decomposition \mathcal{C} of X into finitely many definable sets such that for $C \in \mathcal{C}$ there is a globally subanalytic function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ such that $f|_C = F|_C$.

Inductive step: The function f is log-analytic of order at most r if the following holds: There is a decomposition \mathcal{C} of X into finitely many definable sets such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, a globally subanalytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$, and log-analytic functions $g_1, \dots, g_k : C \rightarrow \mathbb{R}, h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ of order at most $r - 1$ such that

$$f|_C = F(g_1, \dots, g_k, \log(h_1), \dots, \log(h_l)).$$

- (b) Let $r \in \mathbb{N}_0$. We call f **log-analytic of order r** if f is log-analytic of order at most r but not of order at most $r - 1$.
- (c) We call f **log-analytic** if f is log-analytic of order r for some $r \in \mathbb{N}_0$.

3.2 Remark

- (1) A log-analytic function is definable.
- (2) The log-analytic functions are precisely those definable functions which are piecewise given by $\mathcal{L}_{\text{an}}(-1, (\sqrt[\cdot]{\dots})_{n=2,3,\dots}, \log)$ -terms.
- (3) A function is log-analytic of order 0 if and only if it is piecewise the restriction of globally subanalytic functions.
- (4) A constructible function is log-analytic of order at most 1.

3.3 Remark

- (1) Let $r \in \mathbb{N}_0$. The set of log-analytic functions on X of order at most r is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.
- (2) The set of log-analytic functions on X is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.

3.4 Example

The function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto \arctan(\log(\max\{\log(x^4 + \log(y^2 + 2)), 1\})),$$

is log-analytic (of order 3).

3.5 Definition

Let $f : X \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on X .

- (a) By induction on $e \in \mathbb{N}_0$ we define that f has **exponential number at most e with respect to E** .

Base Case: The function f has exponential number at most 0 with respect to E if f is log-analytic.

Inductive Step: The function f has exponential number at most e with respect to E if the following holds: There are $k, l \in \mathbb{N}_0$, functions $g_1, \dots, g_k : X \rightarrow \mathbb{R}$ and $h_1, \dots, h_l : X \rightarrow \mathbb{R}$ with exponential number at most $e - 1$ with respect to E and a log-analytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f = F(g_1, \dots, g_k, \exp(h_1), \dots, \exp(h_l))$$

and $\exp(h_1), \dots, \exp(h_l) \in E$.

- (b) Let $e \in \mathbb{N}_0$. We say that f has **exponential number e with respect to E** if f has exponential number at most e with respect to E but not at most $e - 1$ with respect to E .
- (c) We say that f **can be constructed from E** if there is $e \in \mathbb{N}_0$ such that f has exponential number e with respect to E .

3.6 Remark

- (1) Let E be a set of positive definable functions on X . A function $f : X \rightarrow \mathbb{R}$ which has exponential number at most $e \in \mathbb{N}_0$ with respect to E is definable.
- (2) Let E be the set of all positive definable functions on X . Then every definable function $f : X \rightarrow \mathbb{R}$ can be constructed from E .

Proof

Property (1) is clear. We show property (2). Every function $g : X \rightarrow \mathbb{R}_{>0}$ of the form $g = \exp(h)$ for a definable $h : X \rightarrow \mathbb{R}$ is contained in E . Therefore with Definition 3.5 and Fact 2.12 we obtain that every definable function can be constructed from E . ■

3.7 Example

The function

$$g :]0, 1[\rightarrow \mathbb{R}, x \mapsto \arctan(\log(e^{1/x} + \log(e^{e^{1/x}} + 2))),$$

has exponential number (at most) 2 with respect to

$$E := \{\mathbb{R} \mapsto \mathbb{R}_{>0}, x \mapsto e^{1/x}, \mathbb{R} \mapsto \mathbb{R}_{>0}, x \mapsto e^{e^{1/x}}\}$$

and therefore can be constructed from E .

3.8 Remark

Let $X_1, X_2 \subset \mathbb{R}^m$ be definable and disjoint. Let $X = X_1 \cup X_2$. For $j \in \{1, 2\}$ let E_j be a set of positive definable functions on X_j and $f_j : X_j \rightarrow \mathbb{R}$ be a function. Let $e \in \mathbb{N}_0$ be such that f_j has exponential number at most e with respect to E_j for $j \in \{1, 2\}$. Let

$$E := \{g \mid g : X \rightarrow \mathbb{R} \text{ is a function with } g|_{X_j} \in E_j \text{ for } j \in \{1, 2\}\}.$$

Then

$$f : X \rightarrow \mathbb{R}, x \mapsto \begin{cases} f_1(x), & x \in X_1, \\ f_2(x), & x \in X_2, \end{cases}$$

has exponential number at most e with respect to E .

3.9 Remark

Let $e \in \mathbb{N}_0$. Let $C \subset X$ be definable and $f : X \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on X such that f has exponential number at most e with respect to E . Then $f|_C$ has exponential number at most e with respect to $E|_C$.

3.10 Remark

Let $f : X \rightarrow \mathbb{R}$ be a function. Let E_1 and E_2 be sets of definable functions on X with $E_1 \subset E_2$. Let $e \in \mathbb{N}_0$. If f has exponential number at most e with respect to E_1 then f has exponential number at most e with respect to E_2 .

3.11 Proposition

Let $e \in \mathbb{N}_0$. Let E be a set of positive definable functions on X .

- (1) Let $f : X \rightarrow \mathbb{R}$ be a function with exponential number at most e with respect to E . Then $\exp(f)$ has exponential number at most $e + 1$ with respect to $E \cup \{\exp(f)\}$.
- (2) Let $s \in \mathbb{N}_0$. Let $f_1, \dots, f_s : X \rightarrow \mathbb{R}$ be functions with exponential number at most e with respect to E and let $F : \mathbb{R}^s \rightarrow \mathbb{R}$ be log-analytic. Then $F(f_1, \dots, f_s)$ has exponential number at most e with respect to E .

Proof

(1): One sees with Definition 3.5 applied to $g := F(\exp(f))$ where $F = \text{id}_{\mathbb{R}}$ that $\exp(f)$ has exponential number at most $e + 1$ with respect to $E \cup \{\exp(f)\}$.

(2): We may assume $e > 0$. Let $k, l \in \mathbb{N}_0$, $g_1, \dots, g_k, h_1, \dots, h_l : X \rightarrow \mathbb{R}$ be functions with exponential number at most $e - 1$ with respect to E with $\exp(h_1), \dots, \exp(h_l) \in E$, and $G_j : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ be log-analytic such that $f_j = G_j(\beta)$ for $j \in \{1, \dots, s\}$ where $\beta := (g_1, \dots, g_k, \exp(h_1), \dots, \exp(h_l))$. Let v range over \mathbb{R}^{k+l} . Then

$$H : \mathbb{R}^{k+l} \rightarrow \mathbb{R}, v \mapsto F(G_1(v), \dots, G_s(v)),$$

is log-analytic such that $H(\beta) = F(f_1, \dots, f_s)$. ■

We need this concept also in the complex setting.

We fix a definable $Z \subset \mathbb{C}^m$. Let $z := (z_1, \dots, z_m)$ range over \mathbb{C}^m . Let $x := (x_1, \dots, x_m)$ and $y := (y_1, \dots, y_m)$ range over \mathbb{R}^m such that $z = x + iy$. We fix a function $F : Z \rightarrow \mathbb{C}$ with $F(z) = u(x, y) + iv(x, y)$ where u is the real and v the imaginary part of F considered as real functions.

3.12 Definition

We call F constructible if u and v are constructible. We call F log-analytic if u and v are log-analytic.

3.13 Example

The function $\mathbb{C}^- \rightarrow \mathbb{C}, w \mapsto \log(w)$, is constructible and therefore log-analytic.

Proof

Note that for every $w \in \mathbb{C}^-$

$$\log(w) = \log(|w|) + i\arg(w).$$

We see that

$$\mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}, (u, v) \mapsto \log(\sqrt{u^2 + v^2}),$$

is constructible and

$$\mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}, (u, v) \mapsto \arg(u + iv),$$

is globally subanalytic. Therefore $\mathbb{C}^- \rightarrow \mathbb{C}, w \mapsto \log(|w|) + i\arg(w)$, is constructible. ■

3.14 Definition

Let E be a set of definable functions on Z without zeros. We say that F can be constructed from E if u and v can be constructed from

$$E^{\text{Re}} := \{\exp(\text{Re}(g)) \mid g \in \log(E)\}$$

where $\log(E) := \{\log(h) \mid h \in E\}$.

3.15 Remark

Let E be a set of positive real valued definable functions on Z . Assume that F takes only real values. Then Definition 3.14 coincides with Definition 3.5 if one considers F and every $g \in E$ as a real function.

Proof

We have $F = u, v = 0$ and $E^{\text{Re}} = \{\exp(\text{Re}(g)) \mid g \in \log(E)\} = E$ since $\text{Re}(g) = g$ for every $g \in \log(E)$. ■

We can formulate Proposition 3.11 also for the complex setting.

3.16 Proposition

Let $e \in \mathbb{N}_0$. Let E be a set of definable functions on Z .

- (1) Assume that $\text{Im}(F)$ is bounded and that $F : Z \rightarrow \mathbb{C}$ can be constructed from E . Then $\exp(F)$ can be constructed from $D := E \cup \{\exp(F)\}$.
- (2) Let $F_1, \dots, F_l : Z \rightarrow \mathbb{C}$ be functions which can be constructed from E and let $G : \mathbb{C}^l \rightarrow \mathbb{C}$ be log-analytic. Then $G(F_1, \dots, F_l)$ can be constructed from E .

Proof

(1): Note that $\exp(F)$ is definable. We have

$$\exp(F(x + iy)) = \exp(u(x, y))(\cos(v(x, y)) + i \sin(v(x, y)))$$

for $x + iy \in Z$. By Proposition 3.11(1) $\exp(u)$ can be constructed from $E^{\text{Re}} \cup \{\exp(u)\}$. Let $M \in \mathbb{R}_{>0}$ be such that $|v| \leq M$. Let $T \in \{\cos, \sin\}$. Then

$$T^* : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} T(x), & |x| \leq M, \\ 0, & \text{else,} \end{cases}$$

is globally subanalytic. Therefore by Proposition 3.11(2)

$$Z \mapsto \mathbb{R}, x + iy \mapsto \exp(u(x, y))T^*(v(x, y)),$$

can be constructed from $E^{\text{Re}} \cup \{\exp(u)\} = D^{\text{Re}}$.

(2): Let u_j be the real part and v_j be the imaginary part of F_j considered as real functions, i.e.

$$F_j(x + iy) = u_j(x, y) + iv_j(x, y)$$

for $j \in \{1, \dots, l\}$ and $x + iy \in Z$. Let \tilde{u} be the real part and \tilde{v} be the imaginary part of G considered as real functions. Then

$$G(F_1, \dots, F_l) = \tilde{u}(u_1, \dots, u_l, v_1, \dots, v_l) + i\tilde{v}(u_1, \dots, u_l, v_1, \dots, v_l).$$

We are done with Proposition 3.11(2). ■

For the rest of Section 3.1 let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $Z \subset \mathbb{R}^{m-1} \times \mathbb{K}$. Let t range over \mathbb{R}^{m-1} , z over \mathbb{K} and τ over \mathbb{R} . Let $\pi^+ : \mathbb{R}^{m-1} \times \mathbb{C} \rightarrow \mathbb{R}^{m-1}$, $(t, z) \mapsto t$, be the projection on the first $m - 1$ real coordinates.

3.17 Definition

Let $l \in \mathbb{N}_0$ and $g : \pi^+(Z) \rightarrow \mathbb{R}^l$ be definable. We call a function $f : Z \rightarrow \mathbb{C}$ **globally subanalytic in z with support function g** respectively **constructible in z with support function g** if there is a globally subanalytic respectively constructible function $F : \mathbb{R}^l \times \mathbb{K} \rightarrow \mathbb{C}$ such that

$$f(t, z) = F(g(t), z)$$

for every $(t, z) \in Z$.

3.18 Remark

Let $l \in \mathbb{N}_0$ and $g : \pi^+(Z) \rightarrow \mathbb{R}^l$ be definable. The following holds.

- (1) A globally subanalytic function resp. constructible function $Z \rightarrow \mathbb{C}$ in z with support function g is definable.
- (2) Let $q \in \mathbb{N}$ and let $G : \mathbb{C}^q \rightarrow \mathbb{C}$ be a globally subanalytic function. Let $J_1, \dots, J_q : Z \rightarrow \mathbb{C}$ be globally subanalytic in z with support function g . Then

$$H : Z \rightarrow \mathbb{C}, (t, z) \mapsto G(J_1(t, z), \dots, J_q(t, z)),$$

is globally subanalytic in z with support function g .

Proof

(1): Clear.

(2): Let $K_1, \dots, K_q : \mathbb{R}^l \times \mathbb{K} \rightarrow \mathbb{C}$ be globally subanalytic such that $J_j(t, z) = K_j(g(t), z)$ for every $(t, z) \in Z$ and $j \in \{1, \dots, q\}$. Let w range over \mathbb{R}^l . Then

$$F : \mathbb{R}^l \times \mathbb{K} \rightarrow \mathbb{C}, (w, z) \mapsto G(K_1(w, z), \dots, K_q(w, z)),$$

is globally subanalytic. We obtain

$$F(g(t), z) = G(J_1(t, z), \dots, J_q(t, z))$$

for every $(t, z) \in Z$. ■

3.19 Remark

Let $\mathbb{K} = \mathbb{R}$. Let $Y \subset \mathbb{R}^m$ be definable. Let $l \in \mathbb{N}_0$ and let $a, b \in \mathbb{R}$ with $a \leq b$. Let $Z = Y \times [a, b]$. Let τ range over \mathbb{R} and let $f : Z \rightarrow \mathbb{C}$ be globally subanalytic in τ with support function $g : Y \rightarrow \mathbb{R}^l$. Suppose that f_t is bounded for every $t \in Y$. Consider

$$F : Y \rightarrow \mathbb{C}, t \mapsto \int_a^b f(t, \tau) d\tau.$$

Then F is well-defined and there is a constructible $H : \mathbb{R}^l \rightarrow \mathbb{C}$ such that $F(t) = H(g(t))$ for every $t \in Y$.

Proof

Let w range over \mathbb{R}^l . Let $J : \mathbb{R}^l \times \mathbb{R} \rightarrow \mathbb{C}, (w, \tau) \mapsto J(w, \tau)$, be globally subanalytic such that

$$f(t, \tau) = J(g(t), \tau)$$

for every $(t, \tau) \in Z$. Let $J = J_1 + iJ_2$ where J_1 is the real part and J_2 the imaginary part of J . For $j \in \{1, 2\}$ we consider

$$G_j : \mathbb{R}^l \times \mathbb{R} \rightarrow \mathbb{R}, (w, \tau) \mapsto \begin{cases} J_j(w, \tau), & \tau \in [a, b], \\ 0, & \text{if} \\ & \text{else.} \end{cases}$$

Then G_j is globally subanalytic. Let $G := G_1 + iG_2$. By Fact 2.19(1) the set

$$\text{Fin}(G) := \{w \in \mathbb{R}^l \mid \int_{\mathbb{R}} |G_j(w, \tau)| d\tau < \infty \text{ for } j \in \{1, 2\}\}$$

is globally subanalytic. We obtain $g(Y) \subset \text{Fin}(G)$, because f_t is bounded for every $t \in Y$. With Fact 2.19(2) we obtain that the function $H : \mathbb{R}^l \rightarrow \mathbb{C}$ defined by

$$H(w) = \int_{\mathbb{R}} G(w, \tau) d\tau = \int_a^b J(w, \tau) d\tau$$

if $w \in \text{Fin}(G)$ and $H(w) = 0$ otherwise is constructible. For $t \in Y$ we have $F(t) = H(g(t))$. ■

3.2 Restricted Log-Exp-Analytic Functions

In this section we introduce a large non-trivial class of definable functions which can be constructed from a set of positive definable functions whose logarithms are locally bounded. This is a proper subclass of all definable functions, but contains the class of log-analytic functions properly.

For Section 3.2 we fix $m \in \mathbb{N}$. For Definition 3.20 to Definition 3.23 we fix an open definable set $X \subset \mathbb{R}^m$.

3.20 Definition

Let $C \subset X$ be a non-empty definable set. Let $f : C \rightarrow \mathbb{R}$ be a function. We call f **locally bounded with reference set X** if for every $x \in X$ there is an open neighbourhood U of x in X such that $U \cap C = \emptyset$ or $f|_{U \cap C}$ is bounded.

3.21 Definition

Let $C \subset X$ be a non-empty definable set. Let $f : C \rightarrow \mathbb{R}$ be a function.

- (a) Let $e \in \mathbb{N}_0$. We say that f is **restricted log-exp-analytic of order (at most) e with reference set X** if f has exponential number (at most) e with respect to a set E of positive definable functions on C such that every $h \in \log(E)$ is locally bounded with reference set X .
- (b) We say that f is **restricted log-exp-analytic with reference set X** if f can be constructed from a set E of positive definable functions on C such that every $h \in \log(E)$ is locally bounded with reference set X .

3.22 Example

The function

$$g :]0, 1[^2 \rightarrow \mathbb{R}, (x, y) \mapsto \arctan(\log(e^{1/x \cdot \log^2(1/y)} + \log(e^{e^{1/x}} + 2))),$$

is restricted log-exp-analytic (of order 2).

3.23 Definition

Let $f : X \rightarrow \mathbb{R}$ be a function.

- (a) Let $e \in \mathbb{N}_0$. The function f is called **restricted log-exp-analytic of order at most e** if f is restricted log-exp-analytic of order at most e with reference set X .
- (b) The function f is called **restricted log-exp-analytic** if f is restricted log-exp-analytic with reference set X .

The following example shows that not every definable function is restricted log-exp-analytic.

3.24 Example

The definable function

$$f : \mathbb{R} \rightarrow \mathbb{R}, v \mapsto \begin{cases} \exp(-1/v), & v > 0, \\ 0, & v \leq 0, \end{cases} \quad \text{if}$$

is not restricted log-exp-analytic, but $f|_{\mathbb{R}_{>0}}$ is.

Proof

In Chapter 5 non-flatness of non-zero restricted log-exp-analytic functions is shown, but f is flat at 0 and not the zero function. Clearly

$$g : \mathbb{R}_{>0} \rightarrow \mathbb{R}, v \mapsto -1/v,$$

is globally subanalytic and locally bounded. So $f|_{\mathbb{R}_{>0}}$ is restricted log-exp-analytic, because $f|_{\mathbb{R}_{>0}}$ can be constructed from $E := \{\exp(g)\}$ by Proposition 3.11(1). ■

We will formulate and prove the main results of our research in the parametric setting below. So we set up the concept of restricted log-exp-analytic functions in single variables.

For the rest of Section 3 let t range over \mathbb{R}^n and x over \mathbb{R}^m . Fix definable sets $C, X \subset \mathbb{R}^n \times \mathbb{R}^m$ with $C \subset X$. Suppose that X_t is open for every $t \in \mathbb{R}^n$.

3.25 Definition

Let $f : C \rightarrow \mathbb{R}$ be a function.

- (a) We call f **locally bounded in x with reference set X** if the following holds. For $t \in \mathbb{R}^n$ and $w \in X_t$ there is an open neighbourhood U of w in X_t such that $U \cap C_t = \emptyset$ or $f_t|_{U \cap C_t}$ is bounded.
- (b) Suppose that C_t is open for every $t \in \mathbb{R}^n$. We call f **locally bounded in x** if f is locally bounded in x with reference set C .

3.26 Remark

Let $Y \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable with $X \subset Y$ such that Y_t is open for every $t \in \mathbb{R}^n$. Let $f : C \rightarrow \mathbb{R}$ be locally bounded in x with reference set Y . Then f is locally bounded in x with reference set X .

3.27 Remark

The set of locally bounded functions in x with reference set X on C is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.

3.28 Remark

Let $C_1, C_2 \subset X$ be disjoint definable sets such that $C = C_1 \cup C_2$. Let $g_j : C_j \rightarrow \mathbb{R}$ be locally bounded in x with reference set X for $j \in \{1, 2\}$. Then

$$g : C \rightarrow \mathbb{R}, (t, x) \mapsto \begin{cases} g_1(t, x), & (t, x) \in C_1, \\ g_2(t, x), & (t, x) \in C_2, \end{cases}$$

is locally bounded in x with reference set X .

Proof

Let $t \in \mathbb{R}^n$ and let $w \in X_t$. Then there is an open neighbourhood U_1 of w in X_t such that $U_1 \cap C_1 = \emptyset$ or $(g_1)_t|_{U_1 \cap C_1}$ is bounded and an open neighbourhood U_2 of w in X_t such that $U_2 \cap C_2 = \emptyset$ or $(g_2)_t|_{U_2 \cap C_2}$ is bounded. Let $U := U_1 \cap U_2$. We have that $U \cap C = \emptyset$ or by the definition of g that $g_t|_{U \cap C}$ is bounded. ■

3.29 Definition

Let $f : C \rightarrow \mathbb{R}$ be a function.

- (a) Let $e \in \mathbb{N}_0$. We say that f is **restricted log-exp-analytic in x of order (at most) e with reference set X** if f has exponential number (at most) e with respect to a set E of positive definable functions on C such that every $g \in \log(E)$ is locally bounded in x with reference set X .
- (b) We say that f is **restricted log-exp-analytic in x with reference set X** if f can be constructed from a set E of positive definable functions

on C such that every $g \in \log(E)$ is locally bounded in x with reference set X .

3.30 Remark

Let $Y \subset \mathbb{R}^n \times \mathbb{R}^m$ be open with $C \subset Y$. A restricted log-exp-analytic function $f : C \rightarrow \mathbb{R}$ with reference set Y is restricted log-exp-analytic in x with reference set Y .

3.31 Remark

Let $e \in \mathbb{N}_0$.

- (1) Let $Y \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable with $X \subset Y$ such that Y_t is open for every $t \in \mathbb{R}^n$. Let $f : C \rightarrow \mathbb{R}$ be restricted log-exp-analytic in x of order at most e with reference set Y . Then f is restricted log-exp-analytic in x of order at most e with reference set X .
- (2) Let $f : C \rightarrow \mathbb{R}$ be restricted log-exp-analytic in x of order at most e with reference set X . Let $W \subset C$. Then $f|_W$ is restricted log-exp-analytic in x of order at most e with reference set X .

Proof

Let E be a set of positive definable functions on C such that every $h \in \log(E)$ is locally bounded in x with reference set Y and f has exponential number at most e with respect to E . We obtain property (1) immediately with Remark 3.26 applied to every $h \in \log(E)$. Property (2) follows immediately with Remark 3.9 applied to E . ■

3.32 Remark

Let $l \in \mathbb{N}$. For $j \in \{1, \dots, l\}$ let $f_j : C \rightarrow \mathbb{R}$ be a function which is restricted log-exp-analytic in x with reference set X . Let $F : \mathbb{R}^l \rightarrow \mathbb{R}$ be log-analytic. Then

$$C \rightarrow \mathbb{R}, x \mapsto F(f_1(x), \dots, f_l(x)),$$

is restricted log-exp-analytic in x with reference set X .

Proof

This follows from Proposition 3.11(2) applied to a set E of positive definable functions on C such that f can be constructed from E and every $g \in \log(E)$ is locally bounded in x with reference set X . ■

3.33 Remark

Let $C_1, C_2 \subset \mathbb{R}^m$ be disjoint and definable with $C_1 \cup C_2 = C$. For $j \in \{1, 2\}$ let $f_j : C_j \rightarrow \mathbb{R}$ be restricted log-exp-analytic in x with reference set X . Then

$$f : C \rightarrow \mathbb{R}, (t, x) \mapsto \begin{cases} f_1(t, x), & (t, x) \in C_1, \\ f_2(t, x), & (t, x) \in C_2, \end{cases}$$

is restricted log-exp-analytic in x with reference set X .

Proof

For $j \in \{1, 2\}$ let E_j be a set of positive definable functions on C_j such that every $g_j \in \log(E_j)$ is locally bounded in x with reference set X and f_j can be constructed from E_j . Let

$$E := \{g : C \rightarrow \mathbb{R} \mid g \text{ is a function with } g|_{C_j} \in E_j \text{ for } j \in \{1, 2\}\}.$$

We see with Remark 3.28 that every $g \in \log(E)$ is locally bounded in x with reference set X . We see with Remark 3.8 that f can be constructed from E . ■

3.34 Definition

A function $f : X \rightarrow \mathbb{R}$ is called **restricted log-exp-analytic in x** if f is restricted log-exp-analytic in x with reference set X .

3.35 Remark

Let $k \in \mathbb{N}_0$. Let $w := (w_1, \dots, w_k)$ range over \mathbb{R}^k . Let $g : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be log-analytic and continuous. Let

$$V := \{(t, x, w) \in X \times \mathbb{R}^k \mid (t, x + g(w)) \in X\}.$$

Let $f : X \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x)$, be restricted log-exp-analytic in x . Then $F : V \rightarrow \mathbb{R}, (t, x, w) \mapsto f(t, x + g(w))$, is restricted log-exp-analytic in (x, w) .

Proof

Note that V_t is open in $\mathbb{R}^m \times \mathbb{R}^k$ for every $t \in \mathbb{R}^n$. Let E be a set of positive definable functions on X such that $\log(E)$ consists only of locally bounded functions in x and f can be constructed from E . Consider

$$\tilde{E} := \{V \rightarrow \mathbb{R}_{>0}, (t, x, w) \mapsto h(t, x + g(w)) \mid h \in E\}.$$

Note that F can be constructed from \tilde{E} . ■

Claim

Let $\alpha : X \rightarrow \mathbb{R}$ be locally bounded in x . Then $\alpha^* : V \rightarrow \mathbb{R}, (t, x, w) \mapsto \alpha(t, x + g(w))$, is locally bounded in (x, w) .

Proof of the claim

Let $t \in \mathbb{R}^n$ and let $(x_0, w_0) \in V_t$. Note that $x_0 + g(w_0) \in X_t$. Then there is an open neighbourhood \hat{U} of $x_0 + g(w_0)$ in X_t such that $\alpha_t|_{\hat{U}}$ is bounded. Let

$$U := \{(x, w) \in X_t \times \mathbb{R}^k \mid x + g(w) \in \hat{U}\}.$$

Since g is continuous we see that U is an open neighbourhood of (x_0, w_0) in V_t such that $(\alpha^*)_t|_U$ is bounded. ■ Claim

By the claim we see that $\log(\tilde{E})$ is a set of locally bounded functions in (x, w) on V and we are done. ■

Restricted Log-Exp-Analytic Functions in the Complex Setting

Finally we describe briefly this class of functions in the complex setting.

Let $z := (z_1, \dots, z_m)$ range over \mathbb{C}^m . Let $x := (x_1, \dots, x_m)$ and $y := (y_1, \dots, y_m)$ range over \mathbb{R}^m such that $z = x + iy$.

3.36 Definition

Let $D \subset \mathbb{C}^m$ be definable and let $F : D \rightarrow \mathbb{C}$ be a function with $F(z) = u(x, y) + iv(x, y)$ where u is the real and v is the imaginary part of F considered as real functions.

- (a) Let $Z \subset \mathbb{C}^m$ be open and definable with $D \subset Z$. We say that F is **restricted log-exp-analytic with reference set Z** if u and v are restricted log-exp-analytic with reference set Z considered as a real set.
- (b) Assume that D is open in \mathbb{C}^m . We say that F is **restricted log-exp-analytic** if u and v are restricted log-exp-analytic with reference set D .

Let $l \in \mathbb{N}_0$ and let w range over \mathbb{R}^l . Let $D \subset \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{C}^m$ be a definable set and let $F : D \rightarrow \mathbb{C}$ be a function with $F(t, w, z) = u(t, w, x, y) + iv(t, w, x, y)$ where u is the real part and v is the imaginary part of F considered as real functions.

3.37 Definition

- (a) Let $Z \subset \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{C}^m$ be definable with $D \subset Z$ such that Z_t is open in $\mathbb{R}^l \times \mathbb{C}^m$ for every $t \in \mathbb{R}^n$. We say that f is **restricted log-exp-analytic in (w, z) with reference set Z** if u and v are restricted log-exp-analytic in (w, x, y) with reference set Z considered as a real set.
- (b) Suppose that D_t is open for every $t \in \mathbb{R}^n$. We say that f is **restricted log-exp-analytic in (w, z)** if u and v are restricted log-exp-analytic in (w, x, y) with reference set D .

4 Preparation Theorems in \mathbb{R}_{an} and $\mathbb{R}_{\text{an,exp}}$

One approach to investigate questions on definable functions in $\mathbb{R}_{\text{an,exp}}$ is to do an induction on the complexity of terms which are involved in the construction of f . For example one can show with this method that every definable function is piecewise real analytic. But the problem is that a definable function allows a representation by 'nice' terms only piecewise. One consequence is that it is hard to control a definable function on the boundary of a piece as the following example indicates.

Example

Let $f(t, x)$ be the definable function with $f(t, x) = x - t/\log(x)$ if $x > 0$ and zero else. Then the following asymptotics hold. For every $t \neq 0$ we have $f(t, -) \sim -t/\log(x)$ and $f(0, -) \sim x \searrow 0$.

Also deep technical proofs require a much better representation of definable functions than a piecewise description by terms. Such a good representation is given by the preparation theorems of Lion and Rolin in [28] which we will formulate for the o-minimal structures \mathbb{R}_{an} and $\mathbb{R}_{\text{an,exp}}$. These theorems are precise enough to obtain proofs that \mathbb{R}_{an} and $\mathbb{R}_{\text{an,exp}}$ are o-minimal without using model theory. In case of log-analytic functions it states that the log-analytic function $f(t, x)$ where x is the last variable can be piecewise written as $f(t, x) = a(t)|y_0(t, x)|^{q_0} \cdot \dots \cdot |y_r(t, x)|^{q_r} u(t, x)$ where $y_0(t, x) = x - \Theta_0(t)$, $y_1(t, x) = \log(|y_0(t, x)|) - \Theta_1(t), \dots$, the q_j 's are rational exponents and $u(t, x)$ is a unit of a special form. This gives roughly that the function $f(t, -)$ behaves piecewise as iterated logarithms independently of t where the order of iteration is bounded in terms of f (see also Van den Dries and Speissegger in [16]). But the problem is that the functions $a(t), \Theta_0(t), \dots, \Theta_r(t)$ although being definable in $\mathbb{R}_{\text{an,exp}}$ are in general not log-analytic anymore. We will present an example below. To be able to use the preparation theorems for deep technical proofs on differentiability and global complexification our first key result in this section is the observation that a log-analytic function can be prepared with data of a special form. We will call them nice functions which form a proper larger class than the log-analytic ones. For this one has to redo the proof of the existing preparation result. Our second goal for this chapter is to establish a preparation theorem for definable functions again by adapting the arguments of Lion/Rolin. The starting point are log-analytic functions and then we consider definable functions as compositions of log-analytic functions and the exponential function. Finally we formulate the preparation theorem for the class of restricted log-exp-analytic functions.

For the whole chapter let $n \in \mathbb{N}_0$, $t := (t_1, \dots, t_n)$ range over \mathbb{R}^n , x over \mathbb{R} and let $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, (t, x) \mapsto t$. Definable means definable in $\mathbb{R}_{\text{an,exp}}$ if not

otherwise mentioned.

4.1 Logarithmic Scales

Let $C \subset \mathbb{R}^{n+1}$ be definable.

4.1 Definition

Let $r \in \mathbb{N}_0$. A tuple $\mathcal{Y} := (y_0, \dots, y_r)$ of functions on C is called an r -**logarithmic scale** on C with **center** $\Theta = (\Theta_0, \dots, \Theta_r)$ if the following holds:

- (a) $y_j > 0$ or $y_j < 0$ for every $j \in \{0, \dots, r\}$.
- (b) Θ_j is a definable function on $\pi(C)$ for every $j \in \{0, \dots, r\}$.
- (c) For $(t, x) \in C$ we have $y_0(t, x) = x - \Theta_0(t)$ and $y_j(t, x) = \log(|y_{j-1}(t, x)|) - \Theta_j(t)$ for every $j \in \{1, \dots, r\}$.
- (d) We have $\Theta_0 = 0$ or there is $\epsilon_0 \in]0, 1[$ such that $0 < |y_0(t, x)| < \epsilon_0|x|$ for all $(t, x) \in C$. For $j \in \{1, \dots, r\}$ the following holds: We have $\Theta_j = 0$ or there is $\epsilon_j \in]0, 1[$ such that $0 < |y_j(t, x)| < \epsilon_j|\log(|y_{j-1}(t, x)|)|$ for all $(t, x) \in C$.

We also write y_0 instead of (y_0) for a 0-logarithmic scale.

4.2 Example

Let $n = 1$ and consider

$$C := \{(t, x) \in \mathbb{R} \times \mathbb{R} \mid t \in]0, 1[, \frac{1}{1+t} + e^{-t-1/t} < x < \frac{1}{1+t} + e^{-1/t}\}.$$

Let $\Theta_0 : \pi(C) \rightarrow \mathbb{R}, t \mapsto \frac{1}{1+t}$, $\Theta_1 : \pi(C) \rightarrow \mathbb{R}, t \mapsto -\frac{1}{t}$ and $\Theta_2 = 0$. Let $y_0 : C \rightarrow \mathbb{R}, (t, x) \mapsto x - \Theta_0(t)$, and inductively for $j \in \{1, 2\}$ let $y_j : C \rightarrow \mathbb{R}, (t, x) \mapsto \log(|y_{j-1}(t, x)|) - \Theta_j(t)$. Then $\mathcal{Y} := (y_0, y_1, y_2)$ is a 2-logarithmic scale with center $(\Theta_0, \Theta_1, \Theta_2)$.

Proof

Since $\Theta_0 < C$ we see that $y_0 > 0$. So we have

$$y_1(t, x) = \log(y_0(t, x)) - \Theta_1(t)$$

for $(t, x) \in C$. We have $\Theta_0 + e^{\Theta_1} > C$ and therefore $y_1 < 0$ since y_1 is strictly monotone increasing in x . This gives

$$y_2(t, x) = \log(-y_1(t, x)) - \Theta_2(t)$$

for $(t, x) \in C$. We have $\Theta_0 + e^{\Theta_1-1} < C$ and therefore $y_2 < 0$ and $\sigma_2 = -1$. An easy calculation shows that $x - \Theta_0(t) < \epsilon_0 x$ and that

$$|\log(x - \Theta_0(t)) - \Theta_1(t)| < \epsilon_1 |\log(x - \Theta_0(t))|$$

for $(t, x) \in C$ where $\epsilon_0 := 1/2$ and $\epsilon_1 := 1/2$. So one sees that (y_0, y_1, y_2) is a 2-logarithmic scale with center $(\Theta_0, \Theta_1, \Theta_2)$.

4.3 Remark

Let $r \in \mathbb{N}_0$. An r -logarithmic scale may not exist on C . For example there is no r -logarithmic scale on $\mathbb{R}^n \times \mathbb{R}_{\neq 0}$.

Proof

Suppose the contrary. Let \mathcal{Y} be such an r -logarithmic scale. Then $\Theta_0 = 0$ since $y_0 \neq 0$ on C . But then $y_0(t, x) > 0$ for $(t, x) \in \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ with $x > 0$ and $y_0(t, x) < 0$ for $(t, x) \in \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ with $x < 0$. This is a contradiction to Definition 4.1(a).

4.4 Definition

Let $r \in \mathbb{N}_0$. We call C **r -admissible** if there is an r -logarithmic scale on C . We call C **r -unique** if there is exactly one r -logarithmic scale on C .

4.5 Remark

Let $r \in \mathbb{N}_0$. Let C be r -admissible. Then $C \subset \mathbb{R}^n \times \mathbb{R}_{\neq 0}$.

Proof

Suppose the contrary. Then there is $t \in \pi(C)$ such that $(t, 0) \in C$. Since $y_0 \neq 0$ on C we have $\Theta_0 \neq 0$. So there is $\epsilon_0 \in]0, 1[$ such that $0 < |y_0(t, x)| < \epsilon_0|x|$ for $(t, x) \in C$. But then $0 < |y_0(t, 0)| < 0$, a contradiction. ■

Let $r \in \mathbb{N}_0$. For Chapter 4.1 we assume that C is r -admissible and we fix an r -logarithmic scale $\mathcal{Y} := (y_0, \dots, y_r)$ on C with center $(\Theta_0, \dots, \Theta_r)$.

4.6 Definition

We set

$$C^{\mathcal{Y}} := \{(t, y_0(t, x), \dots, y_r(t, x)) \mid (t, x) \in C\} \subset \mathbb{R}^n \times \mathbb{R}^{r+1}.$$

4.7 Definition

The **sign** $\text{sign}(\mathcal{Y}) \in \{-1, 1\}^{r+1}$ of \mathcal{Y} is defined by

$$\text{sign}(\mathcal{Y}) = (\text{sign}(y_0), \dots, \text{sign}(y_r)).$$

4.8 Definition

Let $q = (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$. We set

$$|\mathcal{Y}|^{\otimes q} := \prod_{j=0}^r |y_j|^{q_j}.$$

4.9 Definition

Let \mathcal{D} be a decomposition of C into finitely many definable sets. Let $f : C \rightarrow \mathbb{R}$ be a function. We set

$$C^f := \{(t, f(t, x)) \in \mathbb{R}^n \times \mathbb{R} \mid (t, x) \in C\}$$

and

$$\mathcal{D}^f := \{D^f \mid D \in \mathcal{D}\}.$$

4.10 Remark

The following holds.

- (1) Let $f : C \rightarrow \mathbb{R}$ be definable. Then $D := C^f$ is definable.
- (2) Let $l \in \{1, \dots, r\}$. We define $\mu_l : C \rightarrow \mathbb{R}, (t, x) \mapsto x - \Theta_l(t)$ and inductively for $j \in \{l+1, \dots, r\}$ we define $\mu_j : C \rightarrow \mathbb{R}, (t, x) \mapsto \log(|\mu_{j-1}(t, x)|) - \Theta_j(t)$. Then $\mathcal{Y}_{r-l, D} := (\mu_l, \dots, \mu_r)$ is a well-defined $(r-l)$ -logarithmic scale with center $(\Theta_l, \dots, \Theta_r)$ on $D := C^{\log(|y_{l-1}|)}$.

Proof

(1): Clear

(2): Note that

$$\mathcal{Y}(t, x) = (y_0(t, x), \dots, y_{l-1}(t, x), \mathcal{Y}_{r-l, D}(t, \log(|y_{l-1}(t, x)|)))$$

for every $(t, x) \in C$. So it is straightforward to see with Definition 4.1 that $\mathcal{Y}_{r-l, D}$ is an $(r-l)$ -logarithmic scale on D . ■

4.11 Definition

Let $M \in \mathbb{R}_{>0}$. We set

$$C_{>M}(\mathcal{Y}) := \{(t, x) \in C \mid |y_l(t, x)| > M \text{ for all } l \in \{1, \dots, r\}\},$$

and for every $l \in \{1, \dots, r\}$

$$C_{l, M}(\mathcal{Y}) := \{(t, x) \in C \mid |y_l(t, x)| \leq M\}.$$

We often omit \mathcal{Y} .

4.12 Remark

Let $M \in \mathbb{R}_{>0}$. Then $C_{>M}$ and $C_{1, M}, \dots, C_{r, M}$ are definable.

4.13 Remark

Assume $r \in \mathbb{N}$. The following properties hold.

(1) Let $l \in \{1, \dots, r\}$. We have

$$|y_l| \leq |\log(|y_{l-1}|)|$$

on C .

(2) Let $c, \lambda_1, \dots, \lambda_r \in \mathbb{R}$. Then there is $M \in \mathbb{R}_{>1}$ such that

$$|c + \sum_{k=1}^r \lambda_k \log(|y_k|)| \leq \frac{|y_1|}{2}$$

on $C_{>M}$.

Proof

(1): We find $\epsilon_l \in]0, 1[$ such that $|y_l(t, x)| \leq \epsilon_l |\log(|y_{l-1}(t, x)|)|$ for every $(t, x) \in C$ or $\Theta_l = 0$ by Definition 4.1. Hence $|y_l| \leq |\log(|y_{l-1}|)|$.

(2): Take $M \geq \exp_r(1)$ to obtain with (1)

$$|c + \sum_{k=1}^r \lambda_k \log(|y_k|)| \leq |c| + \sum_{k=1}^r |\lambda_k| \cdot \log_k(|y_1|)$$

on $C_{>M}$. By increasing M if necessary we may assume $|c| \leq \frac{|y_1|}{4}$ and

$$|\lambda_l| \log_l(|y_1|) \leq \frac{|y_1|}{4r}$$

for every $l \in \{1, \dots, r\}$ on $C_{>M}$. This gives the result. ■

4.14 Definition

Let $m \in \mathbb{N}$. Let $D \subset X \subset \mathbb{R}^m$. Let $f, g : X \rightarrow \mathbb{R}$ be functions. We call f **similar to g on D** , written $f \sim_D g$, if there is $\delta \in \mathbb{R}_{>0}$ such that $1/\delta \cdot g < f < \delta \cdot g$ on D .

4.15 Remark

Let $D \subset X \subset \mathbb{R}^m$. Let $f, g : X \rightarrow \mathbb{R}$ be functions. If $f \sim_D g$ then f and g don't have a zero on D .

4.16 Remark

Let $D \subset X \subset \mathbb{R}^m$ and let $f, g : X \rightarrow \mathbb{R}$ be functions.

- (1) It holds $f \sim_D g$ if and only if $\frac{f}{g} \sim_D 1$.
- (2) If $f \sim_D g$ then $\text{sign}(f) = \text{sign}(g)$ on D .
- (3) The relation $f \sim_D g$ is an equivalence relation on the set of all functions on X without a zero on D .
- (4) The set of functions on X which are similar to 1 on D form a divisible group with respect to pointwise multiplication.

4.17 Remark

If $\Theta_0 \neq 0$ then $x \sim_C \Theta_0$. Let $j \in \{1, \dots, r\}$. If $\Theta_j \neq 0$ then $\log(|y_{j-1}|) \sim_C \Theta_j$.

Proof

Assume $\Theta_0 \neq 0$. Then there is $\epsilon \in]0, 1[$ such that $|x - \Theta_0(t)| < \epsilon|x|$ for every $(t, x) \in C$. This gives that $x \neq 0$ and

$$1 - \epsilon < \frac{\Theta_0(t)}{x} < 1 + \epsilon$$

for every $(t, x) \in C$. Set $\delta := \max\{1/(1 - \epsilon), 1 + \epsilon\}$. Then $1/\delta < \frac{\Theta_0(t)}{x} < \delta$ for every $(t, x) \in C$.

Assume $\Theta_j \neq 0$. Then there is $\epsilon_j \in]0, 1[$ such that

$$|\log(|y_{j-1}(t, x)|) - \Theta_j(t)| < \epsilon_j |\log(|y_{j-1}(t, x)|)|$$

for every $(t, x) \in C$. This gives $\log(|y_{j-1}(t, x)|) \neq 0$ for every $(t, x) \in C$. Now proceed in the same way as above. ■

4.18 Proposition

Let $\Psi : \pi(C) \rightarrow \mathbb{R}$ be a function. The following properties hold.

- (1) Let $l \in \{1, \dots, r\}$. If $|y_{l-1}| \sim_{C_{>M}} \Psi$ for some $M > 1$ then there is $N \geq M$ such that $|y_l| \sim_{C_{>N}} |\log(\Psi) - \Theta_l|$.
- (2) Let E be a set of positive definable functions on $\pi(C)$ such that Ψ and $\Theta_1, \dots, \Theta_r$ can be constructed from E . Let $q_1, \dots, q_r \in \mathbb{Q}$.
 - (i) Let $|y_1| \sim_{C_{>M}} \Psi$ for some $M > 1$. There is $N \geq M$ and a function $\mu : \pi(C) \rightarrow \mathbb{R}_{>0}$ which can be constructed from E such that

$$\prod_{j=1}^r |y_j|^{q_j} \sim_{C_{>N}} \mu.$$

(ii) Suppose

$$y_0 \sim_C \Psi \prod_{j=1}^r |y_j|^{q_j}.$$

Then there is $M \in \mathbb{R}_{>1}$ such that $|y_1| \sim_{C_{>M}} |\log(|\Psi|) - \Theta_1|$. Additionally there is $N \geq M$ and a function $\xi : \pi(C) \rightarrow \mathbb{R}$ which can be constructed from E such that $y_0 \sim_{C_{>N}} \xi$.

Proof

(1): Let $\delta > 0$ be with

$$\frac{1}{\delta} |y_{l-1}| < \Psi < \delta |y_{l-1}|$$

on $C_{>M}$. Note that $\delta > 1$. By taking logarithm and subtracting Θ_l we get

$$-\log(\delta) + y_l < \log(\Psi) - \Theta_l < \log(\delta) + y_l$$

on $C_{>M}$. Set $N := \max\{M, 2 \log(\delta)\}$. If $y_l > 0$ on C we obtain on $C_{>N}$ that $-\log(\delta) + y_l > 0$ and therefore

$$|y_l| - \log(\delta) < |\log(\Psi) - \Theta_l| < |y_l| + \log(\delta).$$

If $y_l < 0$ on C we obtain on $C_{>N}$ that $\log(\delta) + y_l < 0$ and therefore

$$|y_l| - \log(\delta) \leq |\log(\delta) + y_l| < |\log(\Psi) - \Theta_l| < |-\log(\delta) + y_l| \leq |y_l| + \log(\delta).$$

In both cases we obtain

$$\frac{|y_l|}{2} < |\log(\Psi) - \Theta_l| < 2|y_l|$$

on $C_{>N}$.

(2),(i): Let $\Psi_1 := \Psi$. With (1) we find inductively for $l \in \{2, \dots, r\}$ a real number $N_l \geq N_{l-1}$ such that

$$|y_l| \sim_{C_{>N_l}} |\log(\Psi_{l-1}) - \Theta_l| := \Psi_l$$

where $N_1 := M$. We see with an easy induction on $l \in \{1, \dots, r\}$ and Proposition 3.11(2) that Ψ_l can be constructed from E for every $l \in \{1, \dots, r\}$. For $N := N_r$ we obtain with Remark 4.16(4)

$$|y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} \sim_{C_{>N}} \Psi_1^{q_1} \cdot \dots \cdot \Psi_r^{q_r} := \mu.$$

Note that $\mu : \pi(C) \rightarrow \mathbb{R}_{>0}$ can be constructed from E by Proposition 3.11(2).

(2),(ii): Let $\delta > 0$ be such that

$$\frac{1}{\delta} |y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} \Psi < y_0 < \delta |y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} \Psi$$

on C . Let $\kappa := \max\{1/\delta, \delta\}$. Taking absolute values, logarithm and subtracting Θ_1 we obtain

$$-\log(\kappa) + L + \log(|\Psi|) - \Theta_1 < y_1 < \log(\kappa) + L + \log(|\Psi|) - \Theta_1$$

on C where $L := \sum_{k=1}^r q_k \log(|y_k|)$. By Remark 4.13(2) we find $M > 1$ such that

$$\log(\kappa) + |L| \leq \frac{|y_1|}{2}$$

on $C_{>M}$. We obtain

$$-\frac{|y_1|}{2} + \log(|\Psi|) - \Theta_1 < y_1 < \frac{|y_1|}{2} + \log(|\Psi|) - \Theta_1$$

and therefore

$$\frac{|y_1|}{2} < |\log(|\Psi|) - \Theta_1| < 2|y_1|$$

on $C_{>M}$. Let $\Gamma := |\log(|\Psi|) - \Theta_1|$. Then $\Gamma : \pi(C) \rightarrow \mathbb{R}_{>0}$ can be constructed from E , and $|y_1| \sim_C \Gamma$. By (2), (i) we find a function $\mu : \pi(C) \rightarrow \mathbb{R}_{>0}$ which can be constructed from E and $N \geq M$ such that

$$|y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} \sim_{C_{>N}} \mu.$$

Therefore by Remark 4.16(1) $\Psi \cdot |y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} \sim_{C_{>N}} \Psi \cdot \mu$ and by Remark 4.16(3) $y_0 \sim_{C_{>N}} \Psi \cdot \mu$. So take $\xi := \Psi \cdot \mu$. Then ξ can be constructed from E . We are done with the proof of Proposition 4.18. ■

4.2 A Preparation Theorem for Log-Analytic Functions

For Definition 4.19 we set the following: For $r \in \mathbb{N}_0$ let $w = (w_0, \dots, w_r)$ range over \mathbb{R}^{r+1} and let $\pi^* : \mathbb{R}^n \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}^n$, $(t, w) \mapsto t$.

4.19 Definition

Let $r \in \mathbb{N}_0$ and let $D \subset \mathbb{R}^n \times \mathbb{R}^{r+1}$ be definable. A function $u : D \rightarrow \mathbb{R}$ is called **r -LA-special unit** on D if $u = v \circ \phi$ where the following holds.

(a) The function ϕ is given by

$$\phi : D \mapsto [-1, 1]^s, (t, w) \mapsto (b_1(t) \prod_{l=0}^r |w_l|^{p_{1l}}, \dots, b_s(t) \prod_{l=0}^r |w_l|^{p_{sl}}),$$

where $s \in \mathbb{N}_0$, $b_1, \dots, b_s : \pi^*(D) \rightarrow \mathbb{R}$ are definable which have no zeros and $p_{1l}, \dots, p_{sl} \in \mathbb{Q}$ for every $l \in \{0, \dots, r\}$.

(b) v is a real power series in s variables which converges absolutely on an open neighbourhood of $[-1, 1]^s$.

(c) It holds $v([-1, 1]^s) \subset \mathbb{R}_{>0}$.

We call $b := (b_1, \dots, b_s)$ a tuple of **base functions** for u and

$$\mathcal{LI} := (s, v, b, P)$$

where

$$P := \begin{pmatrix} p_{10} & \cdot & \cdot & p_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{s0} & \cdot & \cdot & p_{sr} \end{pmatrix} \in M(s \times (r+1), \mathbb{Q})$$

an r -**LA-describing tuple** for u .

4.20 Definition

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be globally subanalytic. Let $f : C \rightarrow \mathbb{R}$ be a function. Then f is called **globally subanalytically prepared in x with center θ** if for every $(t, x) \in C$

$$f(t, x) = a(t) \cdot |x - \theta(t)|^q \cdot u(t, x - \theta(t))$$

where $q \in \mathbb{Q}$, $\theta : \pi(C) \rightarrow \mathbb{R}$ is a globally subanalytic function such that either $x > \theta(t)$ for every $(t, x) \in C$ or $x < \theta(t)$ for every $(t, x) \in C$, a is a globally subanalytic function on $\pi(C)$ which is identically zero or has no zeros, and u is a 0-LA-special unit on

$$C^\theta := \{(t, x - \theta(t)) \mid (t, x) \in C\}$$

with globally subanalytic base functions. Additionally either there is $\epsilon \in]0, 1[$ such that

$$|x - \theta(t)| < \epsilon|x|$$

for every $(t, x) \in C$ or $\theta = 0$.

We see that a globally subanalytic prepared function has roughly speaking the form of a Puiseux series in one variable.

4.21 Fact (Lion/Rolin, [28], Theorem 1)

Let $m \in \mathbb{N}$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and $f_1, \dots, f_m : X \rightarrow \mathbb{R}$ be globally subanalytic. Then there is a globally subanalytic cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is a globally subanalytic $\theta : \pi(C) \rightarrow \mathbb{R}$ such that f_1, \dots, f_m are globally subanalytically prepared in x with center θ on C .

4.22 Remark

There are similar versions of this preparation theorem for reducts of \mathbb{R}_{an} like $\mathbb{R}_{\mathcal{W}}$ where \mathcal{W} is a convergent Weierstrass system (see [30] for the details).

4.23 Definition

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and $f : X \rightarrow \mathbb{R}$ be a function.

- (a) Let $r \in \mathbb{N}_0$. By induction on r we define that f is **log-analytic in x of order at most r** .

Base case: The function f is log-analytic in x of order at most 0 if the following holds: There is a definable cell decomposition \mathcal{C} of X such that $f|_C$ is globally subanalytic in x for $C \in \mathcal{C}$.

Inductive step: The function f is log-analytic in x of order at most r if the following holds: There is a definable cell decomposition \mathcal{C} of X such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, log-analytic functions $g_1, \dots, g_k : C \rightarrow \mathbb{R}, h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ in x of order at most $r - 1$ and a globally subanalytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f|_C = F(g_1, \dots, g_k, \log(h_1), \dots, \log(h_l)).$$

- (b) Let $r \in \mathbb{N}_0$. The function f is **log-analytic in x of order r** if f is log-analytic in x of order at most r but not log-analytic in x of order at most $r - 1$.
- (c) The function f is **log-analytic in x** if there is $r \in \mathbb{N}_0$ such that f is log-analytic in x of order r .

4.24 Remark

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable.

- (1) A log-analytic function in x on X is definable.
- (2) A log-analytic function on X is a log-analytic function in x on X .
- (3) Let $r \in \mathbb{N}_0$. The set of log-analytic functions in x of order at most r on X is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.
- (4) The set of log-analytic functions in x on X is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable.

4.25 Definition

Let $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be a function. We say that g is **r -log-analytically prepared in x with center Θ** if

$$g(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q}u(t, y_0(t, x), \dots, y_r(t, x))$$

for all $(t, x) \in C$ where a is a definable function on $\pi(C)$ which vanishes identically or has no zero, $\mathcal{Y} = (y_0, \dots, y_r)$ is an r -logarithmic scale with center Θ on C , $q \in \mathbb{Q}^{r+1}$ and u is an r -LA-special unit on $C^{\mathcal{Y}}$. We call a **coefficient** of g and the base functions b_1, \dots, b_s of u are called **base functions** of g . We call $\mathcal{LJ} := (r, \mathcal{Y}, a, q, \mathcal{LI})$ where $q \in \mathbb{Q}^{r+1}$ and \mathcal{LI} is an r -LA-describing tuple for u an **LA-preparing tuple** for g .

4.26 Remark

Let $r \in \mathbb{N}_0$. Let $g : C \rightarrow \mathbb{R}$ be a function. If g is r -log-analytically prepared in x then g is log-analytic in x of order at most r .

4.27 Remark

Let $f : C \rightarrow \mathbb{R}$ be a function. Let $f \sim_C g$ where g is r -log-analytically prepared in x with preparing tuple $(r, \mathcal{Y}, a, q, s, v, b, P)$. Then

$$f \sim_C a|\mathcal{Y}|^{\otimes q}.$$

Proof

We show $g \sim_C a|\mathcal{Y}|^{\otimes q}$ and are done with Remark 4.16(3). Note that a does not have a zero. There is $\delta > 1$ such that

$$\frac{1}{\delta} < u(t, y_0(t, x), \dots, y_r(t, x)) < \delta$$

for every $(t, x) \in C$. We see that

$$\frac{1}{\delta} < \frac{g(t, x)}{a(t)|\mathcal{Y}(t, x)|^{\otimes q}} < \delta$$

for every $(t, x) \in C$ and therefore

$$g \sim_C a|\mathcal{Y}|^{\otimes q}.$$

■

4.28 Definition

Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $g_1, \dots, g_m : C \rightarrow \mathbb{R}$ be functions. We say that g_1, \dots, g_m are r -log-analytically prepared in x **in a simultaneous way** if there

is $\Theta : \pi(C) \rightarrow \mathbb{R}^{r+1}$ such that g_1, \dots, g_m are r -log-analytically prepared in x with center Θ .

4.29 Remark

Let $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Let $f_1, \dots, f_m : C \rightarrow \mathbb{R}$ be r -log-analytically prepared in x in a simultaneous way. Then there are preparing tuples for f_1, \dots, f_m which coincide in r, \mathcal{Y}, s, b, P .

Proof

This follows immediately with Definition 4.25 and by redefining the corresponding power series v_1, \dots, v_m . ■

4.30 Fact (Lion/Rolin, [28], Theorem 2)

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $r \in \mathbb{N}_0$ and $m \in \mathbb{N}$. Let $f_1, \dots, f_m : X \rightarrow \mathbb{R}$ be log-analytic functions in x of order at most r . Then there is a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that $f_1|_C, \dots, f_m|_C$ are r -log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$.

4.31 Definition

Let \mathcal{Y} be an r -logarithmic scale on C with center $\Theta = (\Theta_0, \dots, \Theta_r)$. We call \mathcal{Y} **pure** if its center Θ is log-analytic.

4.32 Definition

Let $r \in \mathbb{N}_0$.

- (a) Let $f : C \rightarrow \mathbb{R}$ be a function. We call $f : C \rightarrow \mathbb{R}$ **purely r -log-analytically prepared in x with center Θ** if f is r -log-analytically prepared in x with log-analytic center Θ , log-analytic coefficient and log-analytic base functions. An LA-preparing tuple for g with log-analytic components is then called a **pure LA-preparing tuple** for g .
- (b) Let $f_1, \dots, f_m : C \rightarrow \mathbb{R}$ be functions. We call f_1, \dots, f_m **purely r -log-analytically prepared in x in a simultaneous way** if f_1, \dots, f_m are purely r -log-analytically prepared in x with the same center.

4.33 Remark

Let $r \in \mathbb{N}_0$. If g is r -log-analytically prepared in x then g is definable but not necessarily log-analytic. If g is purely r -log-analytically prepared in x then g is log-analytic.

4.34 Definition

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let $f : X \rightarrow \mathbb{R}$ be a function in x .

- (a) Let $r \in \mathbb{N}_0$. By induction on r we define that f is **strongly log-analytic in x of order at most r** .

Base case: The function f is strongly log-analytic in x of order at most 0 if the following holds: There is a definable cell decomposition \mathcal{C} of X such that $f|_C$ is globally subanalytic in x with log-analytic support function for $C \in \mathcal{C}$.

Inductive step: The function f is strongly log-analytic in x of order at most r if the following holds: There is a definable cell decomposition \mathcal{C} of X such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, strongly log-analytic functions $g_1, \dots, g_k : C \rightarrow \mathbb{R}, h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ in x of order at most $r - 1$, and a globally subanalytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f|_C = F(g_1, \dots, g_k, \log(h_1), \dots, \log(h_l)).$$

- (b) Let $r \in \mathbb{N}_0$. The function f is **strongly log-analytic in x of order r** if f is strongly log-analytic in x of order at most r but not strongly log-analytic in x of order at most $r - 1$.
- (c) The function f is **strongly log-analytic in x** if there is $r \in \mathbb{N}_0$ such that f is strongly log-analytic in x of order r .

4.35 Remark

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. A strongly log-analytic function in x on X is log-analytic.

4.36 Remark

Let $f : C \rightarrow \mathbb{R}$ be a function. Let $r \in \mathbb{N}_0$.

- (1) A log-analytic function $g : C \rightarrow \mathbb{R}$ of order at most r is strongly log-analytic in x of order at most r .
- (2) A purely r -log-analytically prepared function $g : C \rightarrow \mathbb{R}$ in x is strongly log-analytic in x of order at most r .

Here is the promised example that the above Fact 4.30 can in general not be carried out in the log-analytic category.

4.37 Example

Let $\phi :]0, \infty[\rightarrow \mathbb{R}, y \mapsto y/(1+y)$. Consider the log-analytic function

$$f : \mathbb{R}_{>0} \times \mathbb{R}, (t, x) \mapsto -\frac{1}{\log(\phi(x))} - t.$$

Then f is log-analytic of order 1, but does not allow piecewise a pure 1-log-analytic preparation in x .

Proof

Assume that the contrary holds. Let \mathcal{C} be the corresponding cell decomposition. Let $\psi :]0, 1[\rightarrow \mathbb{R}, y \mapsto y/(1-y)$. Then ψ is the compositional inverse of ϕ . Note that $f(t, \psi(e^{-1/t})) = 0$ for all $t \in \mathbb{R}_{>0}$. Let $\alpha : \mathbb{R}_{>0} \rightarrow \mathbb{R}, t \mapsto \psi(e^{-1/t})$. Then α is not log-analytic and $\alpha(t) = \sum_{n=1}^{\infty} e^{-n/t}$ for all $t \in \mathbb{R}_{>0}$. By passing to a finer definable cell decomposition we find a cell C of the form

$$C := \{(t, x) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid 0 < t < \epsilon, \alpha(t) < x < \alpha(t) + \eta(t)\}$$

with some suitable $\epsilon \in \mathbb{R}_{>0}$ and some definable function $\eta :]0, \epsilon[\rightarrow \mathbb{R}_{>0}$ such that f is purely 1-log-analytically prepared in x on C . Let $(1, \mathcal{Y}, a, q, \mathcal{LI})$ be a pure preparing tuple for $f|_C$ and let $\Theta = (\Theta_0, \Theta_1)$ be the center of \mathcal{Y} .

Claim

$\Theta_0 = 0$.

Proof of the claim

Assume that Θ_0 is not the zero function. By the definition of a 1-logarithmic scale we find $\epsilon_0 \in]0, 1[$ such that $|y_0| < \epsilon_0|x|$ on C . This implies $|\alpha(t) - \Theta_0(t)| \leq \epsilon_0\alpha(t)$ for all $0 < t < \epsilon$. But this is not possible since we have $\alpha = o(\Theta_0)$ at 0 by the assumption that Θ_0 is log-analytic and not the zero function. \blacksquare Claim

From

$$f(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} u(t, \mathcal{Y}(t, x))$$

for all $(t, x) \in C$ and $\lim_{x \searrow \alpha(t)} f(t, x) = 0$ for all $t \in]0, \epsilon[$ we get by α -minimality that there is, after shrinking $\epsilon > 0$ if necessary, some $j \in \{0, 1\}$ such that $\lim_{x \searrow \alpha(t)} |y_j(t, x)|^{q_j} = 0$ for all $t \in]0, \epsilon[$. By the Claim the case $j = 0$ is not possible. In the case $j = 1$ we have, again by the Claim, that $q_1 > 0$ and therefore $\Theta_1 = \log(\alpha)$. But this is a contradiction to the assumption that the function Θ_1 is log-analytic, since the function $\log(\alpha)$ on the right is not log-analytic. This can be seen by applying the logarithmic series. We obtain that $\log(\alpha(t)) + 1/t \sim e^{-1/t}$. \blacksquare

Let f, C, α and ψ be as in Example 4.37. In the following we mention that f can be 0-log-analytically prepared on C (after shrinking ϵ and η if necessary) with coefficient, center and base functions which can be constructed from a set E of positive definable functions with the following property: Every $g \in \log(E)$ is a component of the center of a logarithmic scale on C .

For $(w_2, w_3, w_4) \in \mathbb{R}^3$ with $w_2, w_4 > 0$, $w_4/w_2 \in]1/2, 3/2[$ and $1/(1 + w_4) \in]1/2, 3/2[$ we set

$$u(w_2, w_3, w_4) := \log^*\left(\frac{w_4}{w_2}\right) + \log^*\left(\frac{1}{1 + w_4}\right) + w_3$$

where

$$\log^* : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto \begin{cases} \log(y), & y \in [1/2, 3/2], \\ 0, & \text{else.} \end{cases}$$

Consider

$$F : \mathbb{R}^4 \rightarrow \mathbb{R}, (w_1, \dots, w_4) \mapsto \begin{cases} -\frac{1}{u(w_2, w_3, w_4)} - w_1, & w_2, w_4 > 0, w_4/w_2 \in]1/2, 3/2[, 1/(1 + w_4) \in]1/2, 3/2[\\ & \text{and } u(w_2, w_3, w_4) \neq 0, \\ 0, & \text{else.} \end{cases}$$

Note that F is globally subanalytic since \log^* is globally subanalytic. By shrinking ϵ and η if necessary we obtain $x/\alpha(t) \in]1/2, 3/2[, 1/(1 + x) \in]1/2, 3/2[$ and therefore

$$f(t, x) = F(t, \alpha(t), \log(\alpha(t)), x)$$

for $(t, x) \in C$. Note that α can be constructed from $E := \{\pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto e^{-1/t}\}$ (since $\alpha(t) = \psi(e^{-1/t})$ for $t \in \pi(C)$ and ψ is globally subanalytic). Therefore $\log(\alpha)$ can also be constructed from E (since $\pi(C) \rightarrow \mathbb{R}, t \mapsto \log(\psi(t))$, is log-analytic). With Fact 4.21 we find a globally subanalytic cell decomposition \mathcal{D} of \mathbb{R}^4 such that for every $D \in \mathcal{D}$ we have that $F|_D$ is globally subanalytically prepared in x_4 . By shrinking ϵ and η if necessary we may assume that there is $D \in \mathcal{D}$ such that $(t, \alpha(t), \log(\alpha(t)), x) \in D$ for every

$(t, x) \in C$. So we may assume that F is globally subanalytically prepared in w_4 . Consequently f is 0-log-analytically prepared in x with coefficient, center and base functions which can be constructed from E . By further shrinking ϵ and η if necessary we may assume that

$$|\log(x) + 1/t| < 1/2|\log(x)|$$

for $(t, x) \in C$. Let $y_0 := x$ and $y_1 := \log(y_0) + 1/t$. We have that (y_0, y_1) is a 1-logarithmic scale on C with center (Θ_0, Θ_1) where $\Theta_0 := 0$ and $\Theta_1 : \pi(C) \rightarrow \mathbb{R}, t \mapsto -1/t$. So E is a set of one positive definable function on $\pi(C)$ whose logarithm coincides with Θ_1 .

We will see in Chapter 4.3 that functions which can be constructed from a set of positive definable functions whose logarithm coincide with a component of the center of a logarithmic scale play a crucial role in preparing log-analytic functions in one variable.

4.3 A Type of Definable Functions Closed Under Log-Analytic Preparation

There is a piecewise nice preparation for log-analytic functions, but there is no precise statement how the center, the coefficient, and the base functions of such a log-analytic preparation look like on such a piece. So it is not possible to do deep technical proofs on multivariate phenomenons for specific definable functions just by using the existing preparation result. Example 4.37 shows that there are problems even if the underlying function is log-analytic. But as motivated at the end of Chapter 4.2 there is a class of log-analytic functions in x closed under log-analytic preparation which we present in this chapter.

For Section 4.3 we set the following: 'Log-analytically prepared' means always "log-analytically prepared in x ". For $k, l \in \mathbb{N}$ and $s \in \{1, \dots, l\}$ denote by $M_s(k \times l, \mathbb{Q})$ the set of all $k \times l$ -matrices with rational entries where the first s columns are zero. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable.

4.38 Definition

We call $g : \pi(C) \rightarrow \mathbb{R}$ **C -heir** if there is $r \in \mathbb{N}_0$, an r -logarithmic scale \mathcal{Y} with center $(\hat{\Theta}_0, \dots, \hat{\Theta}_r)$ on C , and $l \in \{1, \dots, r\}$ such that $g = \exp(\hat{\Theta}_l)$.

4.39 Remark

Let $D \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $\pi(C) = \pi(D)$. A C -heir is not necessarily a D -heir.

Proof

Let $C := \mathbb{R}^n \times]0, 1[$ and $D := \mathbb{R}^n \times \mathbb{R}_{\neq 0}$. By Remark 4.3 a D -heir does not exist. But it is straightforward to see that (y_0, y_1) with $y_0 := x$ and $y_1 := \log(x)$ is a 1-logarithmic scale with center 0 on C and therefore that $g : \pi(C) \rightarrow \mathbb{R}, x \mapsto 1$, is a C -heir. ■

4.40 Definition

We call $g : \pi(C) \rightarrow \mathbb{R}$ **C -nice** if there is a set E of C -heirs such that g can be constructed from E .

4.41 Example

- (1) A log-analytic function $f : \pi(C) \rightarrow \mathbb{R}$ is C -nice.
- (2) Let $C :=]0, 1[^2$ and let $h :]0, 1[\rightarrow \mathbb{R}, t \mapsto e^{-1/t}$. Then h is not C -nice.

Proof

(1): Note that f can be constructed from $E = \emptyset$. Therefore f is C -nice by Definition 4.40.

(2): Note that $\pi(C) =]0, 1[$. Suppose the contrary. Let E be a set of C -heirs such that h can be constructed from E . Note that C is simple (compare Definition 5.4). The proof of Proposition 5.8 shows that the center of every logarithmic scale on C vanishes. So we see $E = \emptyset$ or $E = 1$.

Claim

The function h is log-analytic.

Proof of the claim

If $E = \emptyset$ then h is clearly log-analytic by Definition 3.5(a). So suppose $E = \{1\}$. Let $e \in \mathbb{N}_0$ be such that h has exponential number at most e with respect to E . We do an induction on e . For $e = 0$ this is clear with Definition 3.5(a). So suppose $e > 0$. Let $u_1, \dots, u_k, v_1, \dots, v_l : \pi(C) \rightarrow \mathbb{R}$ be functions with $\exp(v_1), \dots, \exp(v_l) \in E$ which have exponential number at most $e - 1$ with respect to E and $F : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}$ be log-analytic such that

$$h(t) = F(u_1(t), \dots, u_k(t), \exp(v_1(t)), \dots, \exp(v_l(t)))$$

for $t \in \pi(C)$. Note that u_1, \dots, u_k are log-analytic by the inductive hypothesis and that $\exp(v_1) = \dots = \exp(v_l) = 1$. So we obtain that h is log-analytic.

■ Claim

But we have $h \sim e^{-1/t}$ at zero, a contradiction to the Claim. ■

A C -nice function which is not log-analytic can be found in Example 4.46.

4.42 Remark

Let $r \in \mathbb{N}_0$. Let $(\Theta_0, \dots, \Theta_r)$ be a C -nice center of an r -logarithmic scale \mathcal{Y} on C . Let $f = \exp(\Theta_j)$ for $j \in \{1, \dots, r\}$. Then f is a C -nice C -heir.

Proof

Let E be a set of C -heirs such that $\Theta_1, \dots, \Theta_r$ can be constructed from E . Then $\exp(\Theta_j)$ can be constructed from $E \cup \{\exp(\Theta_j)\}$ for $j \in \{1, \dots, r\}$ by Proposition 3.11(1) and is therefore C -nice, because $\pi(C) \mapsto \mathbb{R}, t \mapsto \exp(\Theta_j(t))$, is a C -heir. ■

4.43 Remark

Let $D \subset C$ be definable.

- (1) Let $g : \pi(C) \rightarrow \mathbb{R}$ be a C -heir. Then $g|_{\pi(D)}$ is a D -heir.
- (2) Let $h : \pi(C) \rightarrow \mathbb{R}$ be C -nice. Then $h|_{\pi(D)}$ is D -nice.

Proof

(1): This follows from the following fact: Let $r \in \mathbb{N}_0$. Let \mathcal{Y} be an r -logarithmic scale on C with center $(\Theta_0, \dots, \Theta_r)$. Then $\mathcal{Y}|_D$ is an r -logarithmic scale with center $(\Theta_0|_{\pi(D)}, \dots, \Theta_r|_{\pi(D)})$ on D .

(2): Let E be a set of C -heirs such that h can be constructed from E . Then by Remark 3.9 $h|_{\pi(D)}$ can be constructed from $E|_{\pi(D)}$ which is a set of $\pi(D)$ -heirs by (1). ■

4.44 Remark

The set of C -nice functions is closed under composition with log-analytic functions, i.e. let $m \in \mathbb{N}$ and $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be log-analytic and $\eta := (\eta_1, \dots, \eta_m) : \pi(C) \rightarrow \mathbb{R}^m$ be C -nice. Then $F \circ \eta : \pi(C) \rightarrow \mathbb{R}$ is C -nice.

Proof

Let E be a set of C -heirs such that η_1, \dots, η_m can be constructed from E . By Proposition 3.11(2) $F(\eta_1, \dots, \eta_m)$ can be constructed from E . ■

4.45 Definition

Let $r \in \mathbb{N}_0$. Let \mathcal{Y} be an r -logarithmic scale on C . Then \mathcal{Y} is **nice** if its center $\Theta := (\Theta_0, \dots, \Theta_r)$ is C -nice.

The next example shows that not every nice logarithmic scale is log-analytic in general.

4.46 Example

Consider the definable cell

$$C := \{(t, x) \in \mathbb{R}^2 \mid 0 < t < 1, \frac{1}{1+t} + e^{-2/t+2e^{-1/t}} < x < \frac{1}{1+t} + e^{-1/t}\}.$$

Then there is a nice logarithmic scale on C which is not log-analytic.

Proof

Note that $\pi(C) =]0, 1[$, because

$$e^{-2/t+2e^{-1/t}} < e^{-1/t}$$

for every $t \in]0, 1[$. Consider

$$\Theta_0 : \pi(C) \rightarrow \mathbb{R}, t \mapsto \frac{1}{1+t},$$

$$\Theta_1 : \pi(C) \rightarrow \mathbb{R}, t \mapsto -1/t,$$

and

$$\hat{\Theta}_1 : \pi(C) \rightarrow \mathbb{R}, t \mapsto -1/t + e^{-1/t}.$$

For $(t, x) \in C$ consider

$$y_0(t, x) := x - \Theta_0(t),$$

$$y_1(t, x) := \log(x - \Theta_0(t)) - \Theta_1(t),$$

and

$$\hat{y}_1(t, x) := \log(x - \Theta_0(t)) - \hat{\Theta}_1(t).$$

Claim

$\mathcal{Y} := (y_0, y_1)$ and $\hat{\mathcal{Y}} := (y_0, \hat{y}_1)$ are 1-logarithmic scales on C .

Proof of the claim

Note that $y_0 > 0$, $y_1 < 0$, and $\hat{y}_1 < 0$ on C . Let $\epsilon_0 := 1/2$ and $\epsilon_1 := 1/2$. Let $(t, x) \in C$. Then $x < \frac{2}{1+t}$, because $e^{-1/t} < \frac{1}{1+t}$. Therefore $|x - \Theta_0(t)| < \epsilon_0|x|$. Note that

$$e^{-2/t+2e^{-1/t}} + \frac{1}{1+t} < x.$$

So an easy calculation shows that

$$|\log(x - \Theta_0(t)) - \Theta_1(t)| < \epsilon_1 \cdot |\log(x - \Theta_0(t))|$$

and

$$|\log(x - \Theta_0(t)) - \hat{\Theta}_1(t)| < \epsilon_1 \cdot |\log(x - \Theta_0(t))|$$

■ Claim

Note that Θ_0 and Θ_1 are log-analytic. Therefore \mathcal{Y} is nice by Example 4.41(1). Note also that $\hat{\Theta}_1$ is not log-analytic. We see that $\hat{\Theta}_1 = G(1/t, e^{-1/t})$ where $G : \mathbb{R}^2 \rightarrow \mathbb{R}, (w_1, w_2) \mapsto -w_1 + w_2$, is log-analytic. So $\hat{\Theta}_1$ has exponential number 1 with respect to $E := \{g\}$ where $g : \pi(C) \rightarrow \mathbb{R}, t \mapsto e^{-1/t}$ (since $t \mapsto -1/t$ is globally subanalytic). So $\hat{\Theta}_1$ can be constructed from E . By the Claim we see that g is a C -heir. Therefore $\hat{\mathcal{Y}}$ is an example for a nice 1-logarithmic scale which is not log-analytic. ■

4.47 Remark

Let $r \in \mathbb{N}_0$. Let $\mathcal{Y} := (y_0, \dots, y_r)$ be a C -nice r -logarithmic scale with center $(\Theta_0, \dots, \Theta_r)$ on C . Let $\Psi : \pi(C) \rightarrow \mathbb{R}$ be C -nice. Assume that

$$y_0 \sim_C \Psi \prod_{j=1}^r |y_j|^{q_j}$$

where $q_1, \dots, q_r \in \mathbb{Q}$. Then there is $M > 1$ and a C -nice $\xi : \pi(C) \rightarrow \mathbb{R}$ such that $y_0 \sim_{C_{>M}} \xi$.

Proof

Let E be a set of C -heirs such that $\Psi, \Theta_0, \dots, \Theta_r$ can be constructed from E . The assertion follows from Proposition 4.18 (2),(ii). ■

4.48 Definition

Let $f : C \rightarrow \mathbb{R}$ be a function. We say that f is **nice** r -**log-analytically prepared with center** Θ if f is r -log-analytically prepared with a nice r -logarithmic scale \mathcal{Y} with center Θ , C -nice coefficient and C -nice base functions. A corresponding LA-preparing tuple for f is then called a **nice LA-preparing tuple** for f .

4.49 Remark

Let $r, m \in \mathbb{N}$ and $k \in \{1, \dots, r\}$. Let $\mathcal{Y}_{k-1} := (y_0, \dots, y_{k-1})$ be a nice $(k-1)$ -logarithmic scale with center $(\Theta_0, \dots, \Theta_{k-1})$ on C . Let $B := C^{\log(|y_{k-1}|)}$. Let $\alpha_1, \dots, \alpha_m : B \rightarrow \mathbb{R}$ be nicely $(r-k)$ -log-analytically prepared with center $(\Theta_k, \dots, \Theta_r)$. For every $j \in \{1, \dots, m\}$ consider

$$\beta_j : C \rightarrow \mathbb{R}, (t, x) \mapsto \alpha_j(t, \log(|y_{k-1}(t, x)|)).$$

Let $\Theta := (\Theta_0, \dots, \Theta_r)$. Then β_1, \dots, β_m are nicely r -log-analytically prepared with center Θ .

Additionally there is a nice LA-preparing tuple $(r, \mathcal{Y}_r, a_j, q_j, s, v_j, b, P)$ for β_j such that $q_j \in \{0\}^k \times \mathbb{Q}^{r+1-k}$ and $P \in M_k(s \times (r+1), \mathbb{Q})$ where \mathcal{Y}_r is the r -logarithmic scale on C with center $\Theta := (\Theta_0, \dots, \Theta_r)$ for $j \in \{1, \dots, m\}$.

Proof

Let

$$(r - k, \hat{\mathcal{Y}}_{r-k, B}, a_j, q_j, s, v_j, b, P)$$

be a nice LA-preparing tuple for α_j where $j \in \{1, \dots, m\}$. (By Remark 4.29 s, b, P are independent from j .) Here $\hat{\mathcal{Y}}_{r-k, B}$ denotes the $(r - k)$ -logarithmic scale with center $(\Theta_k, \dots, \Theta_r)$ on B . We set

$$\mathcal{Y}_r(t, x) := (y_0(t, x), \dots, y_{k-1}(t, x), \hat{\mathcal{Y}}_{r-k, B}(t, \log(|y_{k-1}(t, x)|)))$$

for $(t, x) \in C$. With Definition 4.1 one sees immediately that \mathcal{Y}_r defines an r -logarithmic scale with center $\Theta := (\Theta_0, \dots, \Theta_r)$ on C . Because Θ_j is C -nice for every $j \in \{0, \dots, r\}$ we see that \mathcal{Y}_r is nice. In particular it is

$$|\hat{\mathcal{Y}}_{r-k, B}(t, \log(|y_{k-1}(t, x)|))|^{\otimes q} = |\mathcal{Y}_r(t, x)|^{\otimes q^*}$$

for every $q \in \mathbb{Q}^{r-k+1}$ and $(t, x) \in C$ where $q^* := (0, \dots, 0, q) \in \mathbb{Q}^{r+1}$. This observation gives the desired nice LA-preparing tuple for β_j where $j \in \{1, \dots, m\}$. ■

4.50 Remark

One may replace "nice" by "pure" and "nicely" by "purely" in Remark 4.49.

4.51 Definition

Let $r \in \mathbb{N}_0$. Let $m \in \mathbb{N}$. Let $g_1, \dots, g_m : C \rightarrow \mathbb{R}$ be functions. We call g_1, \dots, g_m nicely r -log-analytically prepared **in a simultaneous way** if there is $\Theta : \pi(C) \rightarrow \mathbb{R}^{r+1}$ such that g_1, \dots, g_m are nicely r -log-analytically prepared with center Θ .

4.52 Proposition

The following properties hold.

- (1) *Let $r \in \mathbb{N}_0$. Let $g : C \rightarrow \mathbb{R}$ be nicely r -log-analytically prepared. Then there is $k \in \mathbb{N}$, a C -nice function $\eta : \pi(C) \rightarrow \mathbb{R}^k$, a globally subanalytic function $G : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$, and a nice r -logarithmic scale \mathcal{Y}_r on C such that*

$$g(t, x) = G(\eta(t), \mathcal{Y}_r(t, x))$$

for all $(t, x) \in C$.

- (2) *Let $r \in \mathbb{N}$. Let $h : C \rightarrow \mathbb{R}_{>0}$ be nicely $(r - 1)$ -log-analytically prepared. Then there is $k \in \mathbb{N}$, a C -nice function $\eta : \pi(C) \rightarrow \mathbb{R}^k$, a globally*

subanalytic function $H : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$, a nice $(r-1)$ -logarithmic scale $\mathcal{Y}_{r-1} := (y_0, \dots, y_{r-1})$ on C such that

$$\log(h(t, x)) = H(\eta(t), \mathcal{Y}_{r-1}(t, x), \log(|y_{r-1}(t, x)|))$$

for all $(t, x) \in C$.

Proof

(1): Let

$$(r, \mathcal{Y}_r, a, q, s, v, b, P)$$

be a nice LA-preparing tuple for g . Take $k := s + 1$,

$$\eta = (\eta_1, \dots, \eta_k) : \pi(C) \rightarrow \mathbb{R}^k, t \mapsto (a(t), b_1(t), \dots, b_s(t)).$$

Then η is C -nice. Let $z := (z_0, \dots, z_s)$ range over \mathbb{R}^{s+1} and $w := (w_0, \dots, w_r)$ range over \mathbb{R}^{r+1} . Set

$$\alpha_0 : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z, w) \mapsto z_0 \prod_{j=0}^r |w_j|^{q_j}.$$

For $i \in \{1, \dots, s\}$ let

$$\alpha_i : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z, w) \mapsto z_i \prod_{j=0}^r |w_j|^{p_{ij}}.$$

Set

$$G : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z_0, \dots, z_s, w_0, \dots, w_r) \mapsto \begin{cases} \alpha_0(z, w)v(\alpha_1(z, w), \dots, \alpha_s(z, w)), & |\alpha_i(z, w)| \leq 1 \text{ for all } i \in \{1, \dots, s\}, \\ 0, & \text{else.} \end{cases}$$

Then G is globally subanalytic and for each $(t, x) \in C$ we have

$$g(t, x) = G(\eta(t), \mathcal{Y}_r(t, x)).$$

(2): Let

$$(r-1, \mathcal{Y}_{r-1}, a, q, s, v, b, P)$$

be a nice LA-preparing tuple for h where $\mathcal{Y}_{r-1} := (y_0, \dots, y_{r-1})$ is a nice $(r-1)$ -logarithmic scale with center $(\Theta_0, \dots, \Theta_{r-1})$ on C . Then $a > 0$ on C . Take $k := s + r + 1$,

$$\eta = (\eta_1, \dots, \eta_k) : \pi(C) \rightarrow \mathbb{R}^k, t \mapsto (\log(a(t)), b_1(t), \dots, b_s(t), \Theta_1(t), \dots, \Theta_{r-1}(t), 0).$$

Then η is C -nice by Remark 4.44. Let $z := (z_0, \dots, z_{s+r})$ range over \mathbb{R}^{s+r+1} and $w := (w_0, \dots, w_r)$ over \mathbb{R}^{r+1} . Set

$$\beta : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z, w) \mapsto z_0 + \sum_{j=0}^{r-1} q_j(w_{j+1} + z_{s+j+1}).$$

For $i \in \{1, \dots, s\}$ let

$$\alpha_i : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z, w) \mapsto z_i \prod_{j=0}^{r-1} |w_j|^{p_{ij}}.$$

Set

$$H : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}, (z_0, \dots, z_{s+r}, w_0, \dots, w_r) \mapsto \begin{cases} \beta(z, w) + \log(v(\alpha_1(z, w), \dots, \alpha_s(z, w))), & |\alpha_i(z, w)| \leq 1 \text{ for all } i \in \{1, \dots, s\}, \\ 0, & \text{else.} \end{cases}$$

Then H is globally subanalytic since $\log(v)$ is globally subanalytic. For every $(t, x) \in C$ we have

$$\log(h(t, x)) = H(\eta(t), \mathcal{Y}_{r-1}(t, x), \log(|y_{r-1}(t, x)|)).$$

■

4.53 Corollary

Let $r, k, l \in \mathbb{N}$. Let $g_1, \dots, g_k : C \rightarrow \mathbb{R}$ and $h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ be nicely $(r-1)$ -log-analytically prepared in a simultaneous way. Let $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ be globally subanalytic. Then there are $m \in \mathbb{N}$, a C -nice $\eta : \pi(C) \rightarrow \mathbb{R}^m$, a globally subanalytic $J : \mathbb{R}^m \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$, and a nice $(r-1)$ -logarithmic scale $\mathcal{Y} := (y_0, \dots, y_{r-1})$ on C such that for all $(t, x) \in C$

$$\begin{aligned} & F(g_1(t, x), \dots, g_k(t, x), \log(h_1(t, x)), \dots, \log(h_l(t, x))) \\ &= J(\eta(t), \mathcal{Y}(t, x), \log(|y_{r-1}(t, x)|)). \end{aligned}$$

4.54 Remark

One may replace "nicely prepared" by "purely prepared", " C -nice" by "log-analytic" and "nice r -logarithmic scale" by "pure r -logarithmic scale" in Proposition 4.52 and Corollary 4.53.

Here is the promised class of log-analytic functions in x which is closed under log-analytic preparation.

4.55 Definition

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let $f : X \rightarrow \mathbb{R}$ be a function.

- (a) Let $r \in \mathbb{N}_0$. By induction on r we define that f is **nicely log-analytic in x of order at most r** .

Base case: The function f is nicely log-analytic in x of order at most 0 if the following holds: There is a definable cell decomposition \mathcal{C} of X such that $f|_C$ is globally subanalytic in x with C -nice support function for $C \in \mathcal{C}$.

Inductive step: The function f is nicely log-analytic in x of order at most r if the following holds: There is a definable cell decomposition \mathcal{C} of X such that for $C \in \mathcal{C}$ there are $k, l \in \mathbb{N}_0$, nicely log-analytic functions $g_1, \dots, g_k : C \rightarrow \mathbb{R}, h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ in x of order at most $r - 1$, and a globally subanalytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f|_C = F(g_1, \dots, g_k, \log(h_1), \dots, \log(h_l)).$$

- (b) Let $r \in \mathbb{N}_0$. The function f is **nicely log-analytic in x of order r** if f is nicely log-analytic in x of order at most r but not nicely log-analytic in x of order at most $r - 1$.
- (c) The function f is **nicely log-analytic in x** if there is $r \in \mathbb{N}_0$ such that f is nicely log-analytic in x of order r .

4.56 Remark

Let $r \in \mathbb{N}_0$. Then the following properties hold.

- (1) Let $f : C \rightarrow \mathbb{R}$ be a function. If f is strongly log-analytic in x of order at most r then it is nicely log-analytic in x of order at most r .
- (2) Let $m \in \mathbb{N}$. Let $f_1, \dots, f_m : C \rightarrow \mathbb{R}$ be nicely log-analytic in x of order at most r and let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be globally subanalytic. Then $F(f_1, \dots, f_m)$ is nicely log-analytic in x of order at most r .
- (3) Let $m \in \mathbb{N}$. Let $f_1, \dots, f_m : C \rightarrow \mathbb{R}$ be nicely log-analytic in x and let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be log-analytic. Then $F(f_1, \dots, f_m)$ is nicely log-analytic in x .

4.57 Proposition

Let $f : C \rightarrow \mathbb{R}$ be a function. Then f is nicely log-analytic in x if and only if there is a definable cell decomposition \mathcal{D} of C such that for every $D \in \mathcal{D}$ there is a set E_D of D -heirs such that $f|_D$ can be constructed from E_D .

Proof

Assume the former. Let f be nicely log-analytic in x of order at most $r \in \mathbb{N}_0$. We do an induction on r .

$r = 0$: There is a definable cell decomposition \mathcal{D} of C such that for every $D \in \mathcal{D}$ there is $m \in \mathbb{N}_0$, a globally subanalytic function $F : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$, and a D -nice function $g : \pi(D) \rightarrow \mathbb{R}^m$ such that $f(t, x) = F(g(t), x)$ for all $(t, x) \in D$. So for $D \in \mathcal{D}$ choose a set E_D of D -heirs such that g can be constructed from E_D . By Proposition 3.11(2) $f|_D$ can be constructed from E_D for every $D \in \mathcal{D}$.

$r - 1 \rightarrow r$: It is enough to consider the following situation: There are $k, l \in \mathbb{N}_0$, a globally subanalytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$, nicely log-analytic functions $g_1, \dots, g_k : C \rightarrow \mathbb{R}$, $h_1, \dots, h_l : C \rightarrow \mathbb{R}_{>0}$ in x of order at most $r - 1$ such that

$$f = F(g_1, \dots, g_k, \log(h_1), \dots, \log(h_l))$$

on C . By the inductive hypothesis there is a definable cell decomposition \mathcal{D} of C such that for every $D \in \mathcal{D}$ there is a set E_D of D -heirs such that $g_1|_D, \dots, g_k|_D, h_1|_D, \dots, h_l|_D$ can be constructed from E_D . By Proposition 3.11(2) $f|_D$ can be constructed from E_D for every $D \in \mathcal{D}$.

Assume the latter. Fix $D \in \mathcal{D}$ and a corresponding $E := E_D$. It is enough to show that $f|_D$ is nicely log-analytic in x . We do an induction on the exponential number $e \in \mathbb{N}_0$ of $f|_D$ with respect to E .

$e = 0$: Then $f|_D$ is log-analytic and the assertion follows.

$e - 1 \rightarrow e$: There are $k, l \in \mathbb{N}_0$, functions $g_1, \dots, g_k, h_1, \dots, h_l : D \rightarrow \mathbb{R}$ with exponential number at most $e - 1$ with respect to E and a log-analytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ with

$$f = F(g_1, \dots, g_k, \exp(h_1), \dots, \exp(h_l))$$

and $\exp(h_1), \dots, \exp(h_l) \in E$. Note that $\exp(h_1), \dots, \exp(h_l)$ are D -nice, depend only on t and are therefore nicely log-analytic in x (since they are globally subanalytic in x with D -nice support function: We have $\exp(h_j(t)) = \pi_2(\exp(h_j(t)), x)$ for $(t, x) \in D$ and $j \in \{1, \dots, l\}$ where $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection on the first coordinate. Clearly π_2 is globally subanalytic). By the inductive hypothesis g_1, \dots, g_k are nicely log-analytic in x and therefore $f|_D$ with Remark 4.56(3). ■

4.58 Remark

A nice logarithmic scale $\mathcal{Y} := (y_0, \dots, y_r)$ is nicely log-analytic. In particular y_j is nicely log-analytic in x of order at most j for $j \in \{0, \dots, r\}$.

Proof

Let $\mathcal{Y} := (y_0, \dots, y_r)$. Let $(\Theta_0, \dots, \Theta_r)$ be the C -nice center of \mathcal{Y} . We show by induction on $j \in \{0, \dots, r\}$ that y_j is nicely log-analytic in x of order at most j .

$j = 0$: We have $y_0(t, x) = x - \Theta_0(t)$ for every $(t, x) \in C$. That y_0 is nicely log-analytic in x of order 0 is clear with Remark 4.56(2).

$j - 1 \rightarrow j$: Note that $y_j(t, x) = \log(|y_{j-1}(t, x)|) - \Theta_j(t)$ for every $(t, x) \in C$. That y_j is nicely log-analytic in x of order at most j is clear with Remark 4.56(2). ■

4.59 Proposition

Let $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Let $F : \mathbb{R}^{k+r+1} \rightarrow \mathbb{R}$ be globally subanalytic and $\eta : \pi(C) \rightarrow \mathbb{R}^k$ be C -nice. Let $\mathcal{Y} := (y_0, \dots, y_r)$ be a nice r -logarithmic scale with center $(\Theta_0, \dots, \Theta_r)$ on C . Let

$$f : C \rightarrow \mathbb{R}, (t, x) \mapsto F(\eta(t), \mathcal{Y}(t, x)).$$

Then the following properties hold.

- (1) f is nicely log-analytic in x of order at most r .
- (2) Assume that there is a C -nice $\xi : \pi(C) \rightarrow \mathbb{R}$ such that $y_0 \sim_C \xi$. Then f is nicely log-analytic in x of order at most $r - 1$.

Proof

(1): By Remark 4.58 y_l is nicely log-analytic in x of order at most l for every $l \in \{0, \dots, r\}$. We are done with Remark 4.56(2).

(2): By Remark 4.16(1) there is $\delta > 1$ such that $1/\delta < \frac{y_0}{\xi} < \delta$ on C . Set

$$\log^* : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto \begin{cases} \log(y), & y \in [1/\delta, \delta], \\ 0, & y \notin [1/\delta, \delta]. \end{cases}$$

Then \log^* is globally subanalytic. We have $y_1 = y_1^*$ on C where

$$y_1^* := \log^*\left(\frac{y_0}{\xi}\right) + \log(|\xi|) - \Theta_1.$$

Then y_1^* is nicely log-analytic in x of order 0, because $\log(|\xi|)$ is C -nice by Remark 4.44. In particular

$$f(t, x) = F(\eta(t), y_0(t, x), y_1^*(t, x), \dots, y_r^*(t, x))$$

for every $(t, x) \in C$ where inductively for $l \in \{2, \dots, r-1\}$

$$y_l^* := \log(|y_{l-1}^*|) - \Theta_l.$$

Note that y_l^* is nicely log-analytic in x of order at most $l-1$ for every $l \in \{2, \dots, r\}$ and therefore $F(\eta, y_0, y_1^*, \dots, y_r^*)$ is nicely log-analytic in x of order at most $r-1$ by Remark 4.56(2). \blacksquare

4.60 Remark

Let $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be a function. If f is nicely r -log-analytically prepared in x then f is nicely log-analytic in x of order at most r .

Proof

By Proposition 4.52(1) there is $k \in \mathbb{N}$, a C -nice function $\eta : \pi(C) \rightarrow \mathbb{R}$, a globally subanalytic function $G : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$, and a nice r -logarithmic scale \mathcal{Y} on C such that

$$g(t, x) = G(\eta(t), \mathcal{Y}(t, x))$$

for all $(t, x) \in C$. By Proposition 4.59(1) g is nicely log-analytic in x of order at most r . \blacksquare

4.61 Proposition

Let $k \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Let $F : \mathbb{R}^{k+r} \rightarrow \mathbb{R}$ be globally subanalytic and $\eta : \pi(C) \rightarrow \mathbb{R}^k$ be C -nice. Let $\mathcal{Y} := (y_0, \dots, y_r)$ be a nice r -logarithmic scale with center $(\Theta_0, \dots, \Theta_r)$ on C . Let

$$f : C \rightarrow \mathbb{R}, (t, x) \mapsto F(\eta(t), y_1(t, x), \dots, y_r(t, x)).$$

Then the following holds.

- (1) There is a nicely log-analytic function $\kappa : C^{\log(|y_0|)} \rightarrow \mathbb{R}$ in x of order at most $r-1$ such that $f(t, x) = \kappa(t, \log(|y_0(t, x)|))$ for every $(t, x) \in C$.
- (2) Assume $r \geq 2$. Let $l \in \{1, \dots, r-1\}$. Let $\xi : \pi(C) \rightarrow \mathbb{R}$ be C -nice such that $y_l \sim_C \xi$. Then there is a nicely log-analytic function $\lambda : C^{\log(|y_0|)} \rightarrow \mathbb{R}$ in x of order at most $r-2$ such that $f(t, x) = \lambda(t, \log(|y_0(t, x)|))$ for every $(t, x) \in C$.

Proof

We set $B := C^{\log(|y_0|)}$. Let $\mu_0 : B \rightarrow \mathbb{R}, (t, x) \mapsto x - \Theta_1(t)$ and inductively for $j \in \{1, \dots, r-1\}$ let $\mu_j : B \rightarrow \mathbb{R}, (t, x) \mapsto \log(|\mu_{j-1}(t, x)|) - \Theta_{j+1}(t)$. Note that

$$y_j(t, x) = \mu_{j-1}(t, \log(|y_0(t, x)|))$$

for every $(t, x) \in C$ and $j \in \{1, \dots, r\}$. With Remark 4.10(2) we obtain that $\mathcal{Y}_{r-1, B} := (\mu_0, \dots, \mu_{r-1})$ is a nice $(r-1)$ -logarithmic scale with center $(\Theta_1, \dots, \Theta_r)$ on B . Then

$$f(t, x) = F(\eta(t), \mathcal{Y}_{r-1, B}(t, \log(|y_0(t, x)|)))$$

for every $(t, x) \in C$.

(1): We set

$$\kappa : B \rightarrow \mathbb{R}, (t, x) \mapsto F(\eta(t), \mathcal{Y}_{r-1, B}(t, x)).$$

By Proposition 4.59(1) κ is a nicely log-analytic function in x of order at most $r-1$ and we obtain

$$f(t, x) = \kappa(t, \log(|y_0(t, x)|))$$

for all $(t, x) \in C$.

(2): Since $y_l \sim_C \xi$ we obtain $\mu_{l-1} \sim_B \xi$ since ξ depends only on t and $\pi(B) = \pi(C)$. Thus by Remark 4.10(1) there is $\delta > 1$ such that $1/\delta < \frac{\mu_{l-1}}{\xi} < \delta$ on B . Consider

$$\log^* : \mathbb{R} \rightarrow \mathbb{R}, y \mapsto \begin{cases} \log(y), & y \in [1/\delta, \delta], \\ 0, & y \notin [1/\delta, \delta]. \end{cases}$$

Then \log^* is globally subanalytic. Set

$$\mu_l^* := \log^*\left(\frac{\mu_{l-1}}{\xi}\right) + \log(|\xi|) - \Theta_{l+1}$$

and inductively for $j \in \{l+1, \dots, r-1\}$

$$\mu_j^* := \log(|\mu_{j-1}^*|) - \Theta_{j+1}.$$

Then by Remark 4.58 and 4.56(2) μ_l^* is a nicely log-analytic function in x of order at most $l-1$, because $\log(|\xi|)$ is C -nice by Remark 4.44. We see similarly as in the proof of Remark 4.58 that μ_j^* is nicely log-analytic in x of order at most $j-1$ for every $j \in \{l, \dots, r-1\}$. We set

$$\lambda : B \rightarrow \mathbb{R}, (t, x) \mapsto F(\eta(t), \mu_0(t, x), \dots, \mu_{l-1}(t, x), \mu_l^*(t, x), \dots, \mu_{r-1}^*(t, x)).$$

By Remark 4.56(2) λ is nicely log-analytic in x of order at most $r-2$. Because $\mu_j = \mu_j^*$ for every $j \in \{l, \dots, r-1\}$ on B we obtain

$$f(t, x) = \lambda(t, \log(|y_0(t, x)|))$$

for every $(t, x) \in C$. ■

An immediate consequence from the globally subanalytic preparation theorem is the following.

4.62 Proposition

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $m \in \mathbb{N}$. Let $f_1, \dots, f_m : X \rightarrow \mathbb{R}$ be nicely log-analytic functions in x of order 0. Then there is a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that $f_1|_C, \dots, f_m|_C$ are nicely 0-log-analytically prepared in a simultaneous way for every $C \in \mathcal{C}$.

Proof

It is enough to consider the following situation: Let $g_1, \dots, g_k : \pi(X) \rightarrow \mathbb{R}$ be X -nice and $F_j : \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ be globally subanalytic such that

$$f_j(t, x) = F_j(g_1(t), \dots, g_k(t), x)$$

for every $(t, x) \in X$ and $j \in \{1, \dots, m\}$. Let $g := (g_1, \dots, g_k)$. Let $z := (z_1, \dots, z_k)$ range over \mathbb{R}^k . Let $\pi^* : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^k$, $(z, x) \mapsto z$. With Fact 4.21 we find a globally subanalytic cell decomposition \mathcal{D} of $\mathbb{R}^k \times \mathbb{R}_{\neq 0}$ such that $F_1|_D, \dots, F_m|_D$ are globally subanalytically prepared in x in a simultaneous way for every $D \in \mathcal{D}$. There is a definable cell decomposition \mathcal{C} of X such that for every $C \in \mathcal{C}$ there is $D_C \in \mathcal{D}$ such that $(g(t), x) \in D_C$ for every $(t, x) \in C$. Fix a $C \in \mathcal{C}$, the globally subanalytic center ϑ , and for $j \in \{1, \dots, m\}$ a preparing tuple $(0, y_0, a_j, q_j, s, v_j, b, P)$ for $F_j|_{D_C}$ where $y_0 := y - \vartheta(z)$ on D_C , $b := (b_1, \dots, b_s)$, $P := (p_1, \dots, p_s)^t$, and $a_1, \dots, a_m, b_1, \dots, b_s$ are globally subanalytic on $\pi^*(D_C)$. We have for every $j \in \{1, \dots, m\}$ and every $(z, x) \in D_C$

$$F_j(z, x) = a_j(z)|x - \vartheta(z)|^{q_j} v_j(b_1(z)|x - \vartheta(z)|^{p_1}, \dots, b_s(z)|x - \vartheta(z)|^{p_s}).$$

Let $h : C \rightarrow \mathbb{R}$, $(t, x) \mapsto x - \vartheta(g(t))$. Then it is immediately seen with Definition 4.20 that h is a 0-logarithmic scale. We obtain

$$f_j(t, x) = a_j(g(t))|h(t, x)|^{q_j} v_j(b_1(g(t))|h(t, x)|^{p_1}, \dots, b_s(g(t))|h(t, x)|^{p_s})$$

for every $(t, x) \in C$ and are done, because $a_1(g), \dots, a_m(g), \vartheta(g), b_1(g), \dots, b_s(g) : \pi(C) \rightarrow \mathbb{R}$ are C -nice by Remark 4.44. \blacksquare

4.63 Remark

One may replace "nicely log-analytic" by "strongly log-analytic" and "nicely 0-log-analytically prepared" by "purely 0-log-analytically prepared" in Proposition 4.62.

4.64 Proposition

Let $m \in \mathbb{N}$ and $r \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f_1, \dots, f_m : X \rightarrow \mathbb{R}$ be nicely log-analytic functions in x of order at most r . Then there is a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that $f_1|_C, \dots, f_m|_C$ are nicely r -log-analytically prepared in a simultaneous way for every $C \in \mathcal{C}$.

Proof

We do an induction on r .

$r = 0$: Then f_1, \dots, f_m are nicely log-analytic in x of order 0 and we are done with Proposition 4.62.

$< r \rightarrow r$: It is enough to consider the following situation: Assume that there are $l, l' \in \mathbb{N}_0$, nicely log-analytic functions $g_1, \dots, g_l : X \rightarrow \mathbb{R}$ and $h_1, \dots, h_{l'} : X \rightarrow \mathbb{R}_{>0}$ in x of order at most $r - 1$ and for every $j \in \{1, \dots, m\}$ there is a globally subanalytic function $F_j : \mathbb{R}^{l+l'} \rightarrow \mathbb{R}$ such that

$$f_j = F_j(g_1, \dots, g_l, \log(h_1), \dots, \log(h_{l'})).$$

Applying the inductive hypothesis to $g_1, \dots, g_l, h_1, \dots, h_{l'}$ and Corollary 4.53 we find a definable cell decomposition \mathcal{U} of $X_{\neq 0}$ such that for every $U \in \mathcal{U}$ there is a nice $(r - 1)$ -logarithmic scale $\mathcal{Y}_{r-1} := (y_0, \dots, y_{r-1})$ on U , a $k \in \mathbb{N}$, a U -nice function $\eta : \pi(U) \rightarrow \mathbb{R}^k$, and for every $j \in \{1, \dots, m\}$ there is a globally subanalytic function $H_j : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ such that

$$f_j(t, x) = H_j(\eta(t), \mathcal{Y}_{r-1}(t, x), \log(|y_{r-1}(t, x)|))$$

for all $(t, x) \in U$. Fix $U \in \mathcal{U}$ and for this U a corresponding \mathcal{Y}_{r-1} , η , and H_j for $j \in \{1, \dots, m\}$. By further decomposing U if necessary we may assume that either $|y_{r-1}| = 1$ or $|y_{r-1}| > 1$ or $|y_{r-1}| < 1$ on U . Assume the former. Then by Proposition 4.59(1) $f|_C$ is nicely log-analytic in x of order at most $(r - 1)$ and we are done with the inductive hypothesis. Assume $|y_{r-1}| > 1$ or $|y_{r-1}| < 1$ on U . Then $\mathcal{Y} := (y_0, \dots, y_r)$ is an r -logarithmic scale on U where $y_r := \log(|y_{r-1}|)$.

Let $(z, w) := (z_1, \dots, z_k, w_0, \dots, w_r)$ range over $\mathbb{R}^k \times \mathbb{R}^{r+1}$. Set $w' := (w_1, \dots, w_r)$. Let $\pi^* : \mathbb{R}^k \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}^{r+k}$ be the projection on $(z_1, \dots, z_k, w_1, \dots, w_r)$. With Fact 4.21 we find a globally subanalytic cell decomposition \mathcal{D} of $\mathbb{R}^k \times \mathbb{R}^r \times \mathbb{R}_{\neq 0}$ such that $H_1|_D, \dots, H_m|_D$ are globally subanalytically prepared in w_0 in a simultaneous way. There is a definable cell decomposition \mathcal{A} of U such that for every $A \in \mathcal{A}$ there is $D_A \in \mathcal{D}$ such that $(\eta(t), \mathcal{Y}_r(t, x)) \in D_A$ for every $(t, x) \in A$. Fix $A \in \mathcal{A}$, the globally subanalytic center ϑ , and for $j \in \{1, \dots, m\}$ an LA-preparing tuple $(0, y, a_j, q_j, s, v_j, b, P)$ for $F_j|_{D_A}$ where $y := w_0 - \vartheta(w', z)$ on D_A , $b := (b_1, \dots, b_s)$, $P := (p_1, \dots, p_s)^t$, and $a_1, \dots, a_m, b_1, \dots, b_s$ are globally subanalytic on $\pi^*(D_A)$. We have for every $j \in \{1, \dots, m\}$

$$H_j|_{D_A} = a_j(z, w') |y(z, w)|^{q_j} v_j(b_1(z, w') |y(z, w)|^{p_1}, \dots, b_s(z, w') |y(z, w)|^{p_s}).$$

From the inductive hypothesis we will derive the following two claims.

Claim 1

Let $d \in \mathbb{N}$ and $p \in \{1, \dots, r\}$. Let $\alpha_1, \dots, \alpha_d : A^{\log(|y_0|)} \rightarrow \mathbb{R}$ be nicely log-analytic in x of order at most $p - 1$. For $j \in \{1, \dots, d\}$ consider

$$\beta_j : A \rightarrow \mathbb{R}, (t, x) \mapsto \alpha_j(t, \log(|y_0(t, x)|)).$$

Then there is a definable cell decomposition \mathcal{Q} of A such that for every $Q \in \mathcal{Q}$ the following holds.

- (1) There are $\hat{\Theta}_1, \dots, \hat{\Theta}_p : \pi(Q) \rightarrow \mathbb{R}$ such that $\beta_1|_Q, \dots, \beta_d|_Q$ are nicely p -log-analytically prepared with center $\hat{\Theta} := (\Theta_0|_{\pi(Q)}, \hat{\Theta}_1, \dots, \hat{\Theta}_p)$.
- (2) Additionally for every $j \in \{1, \dots, d\}$ there is for $\beta_j|_Q$ a nice LA-preparing tuple $(p, \hat{\mathcal{Y}}_p, \hat{a}_j, \hat{q}_j, \hat{s}, \hat{v}_j, \hat{b}, \hat{P})$ such that $\hat{q}_j \in \{0\} \times \mathbb{Q}^p$ and $\hat{P} \in M_1(\hat{s} \times (p+1), \mathbb{Q})$ where $\hat{\mathcal{Y}}_p$ denotes the p -logarithmic scale with center $\hat{\Theta}$ on Q .

Proof of Claim 1

Set $B := A^{\log(|y_0|)}$. Applying the inductive hypothesis to $\alpha_1, \dots, \alpha_d$ we obtain a definable cell decomposition \mathcal{S} of B such that $\alpha_1|_S, \dots, \alpha_d|_S$ are nicely $(p - 1)$ -log-analytically prepared in a simultaneous way for every $S \in \mathcal{S}$. Consider the definable set

$$S^{\log(|y_0|)^*} := \{(t, x) \in A \mid (t, \log(|y_0(t, x)|)) \in S\}$$

for $S \in \mathcal{S}$. We obtain $\bigcup_{S \in \mathcal{S}} S^{\log(|y_0|)^*} = A$. Fix $S \in \mathcal{S}$ and the center $(\hat{\Theta}_1, \dots, \hat{\Theta}_p)$ of $\alpha_1|_S, \dots, \alpha_d|_S$. Let $T := S^{\log(|y_0|)^*}$. Note that $\beta_1|_T, \dots, \beta_d|_T$ are nicely p -log-analytically prepared with center $(\Theta_0|_{\pi(T)}, \hat{\Theta}_1, \dots, \hat{\Theta}_p)$ and for $j \in \{1, \dots, d\}$ there is a nice LA-preparing tuple $(p, \hat{\mathcal{Y}}_{p,T}, \hat{a}_j, \hat{q}_j, \hat{s}, \hat{v}_j, \hat{b}, \hat{P})$ for $\beta_j|_T$ such that $\hat{q}_j \in \{0\} \times \mathbb{Q}^p$ and $\hat{P} \in M_1(\hat{s} \times (p+1), \mathbb{Q})$ by Remark 4.49 where $\hat{\mathcal{Y}}_{p,T}$ denotes the nice p -logarithmic scale with center $(\Theta_0|_{\pi(T)}, \hat{\Theta}_1, \dots, \hat{\Theta}_p)$ on T . With the cell decomposition theorem applied to T we are done (compare with Theorem 2.15). ■ Claim 1

Claim 2

Let $\gamma : \mathbb{R}^k \times \mathbb{R}^r \rightarrow \mathbb{R}$ be globally subanalytic such that

$$y_0 \sim_A \gamma(\eta, y_1, \dots, y_r).$$

Then $f_j|_A$ is nicely log-analytic in x of order at most $r - 1$ for every $j \in \{1, \dots, m\}$.

Proof of Claim 2

By Proposition 4.61(1) there is a nicely log-analytic function $\kappa : A^{\log(|y_0|)} \rightarrow \mathbb{R}$ in x of order at most $r - 1$ such that

$$\gamma(\eta, y_1, \dots, y_r) = \kappa(t, \log(|y_0|))$$

on A . So by Claim 1 there is a definable cell decomposition \mathcal{N} of A such that for every $N \in \mathcal{N}$ the function $\gamma(\eta, y_1, \dots, y_r)|_N$ is nicely r -log-analytically prepared and there is a nice LA-preparing tuple $(r, \tilde{\mathcal{Y}}_r, \Psi, \tilde{q}, \tilde{s}, \tilde{v}, \tilde{b}, \tilde{P})$ for $\gamma(\eta, y_1, \dots, y_r)|_N$ such that $\tilde{q} \in \{0\} \times \mathbb{Q}^{r-1}$ and $\tilde{P} \in M_1(\tilde{s} \times (r+1), \mathbb{Q})$ where $\tilde{\mathcal{Y}}_r$ has a center whose first component is $\Theta_0|_{\pi(N)}$. Fix $N \in \mathcal{N}$.

By Remark 4.27 it is enough to consider the following property $(*)_p$ for $p \in \{0, \dots, r\}$ on N : There is a nice p -logarithmic scale $\tilde{\mathcal{Y}}_p := (y_0, \tilde{y}_1, \dots, \tilde{y}_p)$ with center $(\Theta_0|_{\pi(N)}, \tilde{\Theta}_1, \dots, \tilde{\Theta}_p)$, and an N -nice $\Psi : \pi(N) \rightarrow \mathbb{R}$ such that

$$y_0 \sim_N \Psi \prod_{l=1}^p |\tilde{y}_l|^{\tilde{q}_l}$$

where $\tilde{q}_l \in \mathbb{Q}$ for every $l \in \{1, \dots, p\}$.

If $p = 0$ then $y_0 \sim_N \Psi$. We are done with Proposition 4.59(2) applied to $f_j|_N = H_j(\eta|_N, \mathcal{Y}|_N)$ for every $j \in \{1, \dots, m\}$. Assume $p > 0$. By a suitable induction on p it is enough to establish the following Subclaim.

Subclaim

There is a decomposition \mathcal{K} of N into finitely many definable sets such that the following holds for every $K \in \mathcal{K}$: The function $f_j|_K$ is nicely log-analytic in x of order at most $r - 1$ for every $j \in \{1, \dots, m\}$ or $(*)_{p-1}$ holds on K .

Proof of the Subclaim

For $M > 1$ let $N_{>M} := N_{>M}(\tilde{\mathcal{Y}}_p)$ and for $i \in \{1, \dots, p\}$ let $N_{i,M} := N_{i,M}(\tilde{\mathcal{Y}}_p)$. By Remark 4.47 there is $M > 1$ and an N -nice $\xi : \pi(N) \rightarrow \mathbb{R}$ such that $y_0 \sim_{N_{>M}} \xi$. Fix such a M . Since

$$N = N_{>M} \cup N_{1,M} \cup \dots \cup N_{p,M}$$

it suffices to establish the Subclaim for $N_{>M}$ and $N_{i,M}$ instead for N where $i \in \{1, \dots, p\}$.

$B := N_{>M}$: By Remark 4.43(2) $\xi|_B$ is B -nice. Therefore with Proposition 4.59(2) applied to $f_j|_B = H_j(\eta|_B, \mathcal{Y}|_B)$ we obtain that $f_j|_B$ is a nicely log-analytic function in x of order at most $r - 1$.

$B_i := N_{i,M}$ for $i \in \{1, \dots, p\}$: Set $\tilde{y}_0 := y_0$. We have $|\tilde{y}_i| < M$ on B_i . So we obtain

$$\frac{1}{\delta} < \frac{|\tilde{y}_{i-1}|}{\exp(\tilde{\Theta}_i)} < \delta$$

for $\delta := e^M$ on B_i which gives $\tilde{y}_{i-1} \sim_{B_i} \exp(\tilde{\Theta}_i)|_{\pi(B_i)}$.

Assume $i = 1$. With Remark 4.42 we see that $\exp(\tilde{\Theta}_1)|_{\pi(B_1)}$ is a B_1 -nice B_1 -heir. Again with Proposition 4.59(2) applied to $f_j|_{B_1} = H_j(\eta|_{B_1}, \mathcal{Y}|_{B_1})$ (we have $y_0 \sim_{B_1} \xi$ with $\xi = \exp(\tilde{\Theta}_1)|_{\pi(B_1)}$) we see that $f_j|_{B_1}$ is nicely log-analytic in x of order at most $r - 1$ for every $j \in \{1, \dots, m\}$.

Assume $i > 1$. Then $p > 1$. Let

$$\Xi : N \rightarrow \mathbb{R}, (t, x) \mapsto \Psi(t) \prod_{l=1}^p |\tilde{y}_l(t, x)|^{\tilde{q}_l}.$$

By (the proof of) Proposition 4.52(1) there is a globally subanalytic function $G : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R}$ such that $\Xi = G(\Psi, \tilde{y}_1, \dots, \tilde{y}_p)$ on B_i . By Proposition 4.61(2) there is a nicely log-analytic function $\lambda : B_i^{\log(|y_0|)} \rightarrow \mathbb{R}$ in x of order at most $p - 2$ such that

$$\Xi(t, x) = \lambda(t, \log(|y_0(t, x)|))$$

for every $(t, x) \in B_i$. Note that $0 \leq p - 2 < r$. With Claim 1 applied to Ξ we find a definable cell decomposition \mathcal{K} of B_i such that $\Xi|_K$ is nicely $(p-1)$ -log-analytically prepared with nice LA-preparing tuple $(p-1, \mathcal{Y}_{p-1}, \bar{a}_j, \bar{q}_j, \bar{s}, \bar{v}_j, \bar{b}, \bar{P})$ where $\bar{q}_j \in \{0\} \times \mathbb{Q}^{p-1}$, $\bar{P} \in M_1(\bar{s} \times p, \mathbb{Q})$, and \mathcal{Y}_{p-1} is a nice $(p-1)$ -logarithmic scale with center $\bar{\Theta} := (\Theta_0|_{\pi(K)}, \bar{\Theta}_1, \dots, \bar{\Theta}_{p-1})$. With Remark 4.27 we obtain property $(*)_{p-1}$ applied to every $K \in \mathcal{K}$. ■ Subclaim

■ Claim 2

Case 1: $\vartheta = 0$. Let $a_{m+j} := b_j$ for $j \in \{1, \dots, s\}$. By Proposition 4.61(1) there are nicely log-analytic functions $\alpha_1, \dots, \alpha_{m+s} : A^{\log(|y_0|)} \rightarrow \mathbb{R}$ in x of order at most $r - 1$ such that for every $(t, x) \in A$ it is

$$a_j(\eta(t), y_1(t, x), \dots, y_r(t, x)) = \alpha_j(t, \log(|y_0(t, x)|))$$

for every $j \in \{1, \dots, m + s\}$. With Claim 1 applied to

$$\beta_j : A \rightarrow \mathbb{R}, (t, x) \mapsto \alpha_j(t, \log(|y_0(t, x)|)),$$

for $j \in \{1, \dots, m + s\}$ we obtain the desired preparation using composition of power series.

Case 2: $\vartheta \neq 0$. There is $\epsilon \in]0, 1[$ such that $0 < |w_0 - \vartheta(z, w')| < \epsilon|w_0|$ for $(z, w) \in D_A$. This gives with Remark 4.17 $w_0 \sim_{D_A} \vartheta(z, w')$ and therefore

$$y_0 \sim_A \vartheta(\eta, y_1, \dots, y_r).$$

By Claim 2 $f_j|_A$ is a nicely log-analytic function in x of order at most $r - 1$ for every $j \in \{1, \dots, m\}$. With the inductive hypothesis applied to $f_j|_A$ for $j \in \{1, \dots, m\}$ we obtain the desired preparation.

All in all we are done with the proof of Proposition 4.64. ■

4.65 Corollary

Let $m \in \mathbb{N}$, $r \in \mathbb{N}_0$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $f_1, \dots, f_m : X \rightarrow \mathbb{R}$ be log-analytic functions of order at most r . Then there is a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that $f_1|_C, \dots, f_m|_C$ are nicely r -log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$.

Outgoing from Corollary 4.65 one may ask the following questions:

- (1) Is there a natural bound for the exponential number of the data (the coefficient, the center, and the base functions) of a prepared log-analytic function depending on its log-analytic order with respect to a suitable set of heirs?
- (2) Is there a another kind of pure preparation for log-analytic functions of order > 0 , i.e. a preparation with log-analytic data only?

To investigate question 2 one notes that an interesting example is the class of constructible functions introduced by Cluckers and Miller which is a proper larger class than the globally subanalytic functions (compare with Definition 2.16) but a proper subclass of the class of log-analytic functions of order at most 1 (for a function which is log-analytic of order 1 but not constructible compare with Example 2.22: If f in Example 2.22 would be constructible then the parameterized integral $\int_{\mathbb{R}} f(u, v) dv$ would be definable by Fact 2.20). For constructible functions there is a pure preparation not in terms of units but suitable for questions on integration (compare with [5], [6] respectively [7]).

4.4 A Preparation Theorem for Definable Functions

A Preparation Theorem of Lion and Rolin

Lion and Rolin established the following preparation theorem.

4.66 Fact (Lion/Rolin, [28], Proposition 3)

Let $n \in \mathbb{N}$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ and $f : X \rightarrow \mathbb{R}$ be definable. Then there is a definable cell decomposition \mathcal{C} of X such that for every $C \in \mathcal{C}$ there is a log-analytic function $g : C \rightarrow \mathbb{R}$ in x and a definable function $h : C \rightarrow \mathbb{R}$ such that $f|_C = g \cdot \exp(h)$.

This preparation result shows that there is a nice representation of a definable function in x , because log-analytic functions in x are involved which can be piecewise prepared by Fact 4.30. But there is no information how the function h looks like. Furthermore there is no bound on the number of iterations of the exponential in this preparation.

A Preparation Theorem of Van den Dries and Speissegger

In [16] Van den Dries and Speissegger used the syntax of terms which we will introduce here. We fix distinct formal variables v_1, \dots, v_{n+1} . "Term" means " $\mathcal{L}_{\text{an}}(\exp, \log)$ -term in which no other variable than v_1, \dots, v_{n+1} occurs". Note that such a term τ defines a definable function $\tau : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, $(t, x) \mapsto \tau(t, x)$.

4.67 Definition (Van den Dries/Speissegger, [16], Definition 5.1)

Let τ be a term. The **exponential level** of τ with respect to v_{n+1} is the number $e(\tau) \in \mathbb{N}_0$ defined inductively as follows.

- (a) If τ is a variable or a constant symbol, then $e(\tau) = 0$.
- (b) If $t = g(\tau_1, \dots, \tau_m)$ where g is an m -ary function symbol of \mathcal{L}_{an} , $m \geq 1$, and τ_1, \dots, τ_m are terms, then

$$e(\tau) = \max\{e(\tau_1), \dots, e(\tau_m)\}.$$

- (c) If $\tau = \exp(s)$, where s is a term, then

$$e(\tau) = \begin{cases} e(s) + 1, & \text{if } v_{n+1} \text{ occurs in } s, \\ e(s), & \text{else.} \end{cases}$$

- (d) If $\tau = \log(s)$, where s is a term, then $e(\tau) = e(s)$.

The exponential level of a term t with respect to v_{n+1} is the maximal number of iterations of the exponential depending on v_{n+1} which occur in t . So a term

which has exponential number at most $e \in \mathbb{N}_0$ with respect to a suitable set E of terms (which define positive functions on $\mathbb{R}^n \times \mathbb{R}$) can have exponential level zero with respect to v_{n+1} . Note also that for a log-analytic function $h : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ the following holds: For every $e \in \mathbb{N}_0$ the function h can be piecewise represented by terms which have exponential level e with respect to v_{n+1} .

4.68 Fact (Van den Dries/Speissegger, [16], Theorem 5.4)

Let τ be a term with $e(\tau) > 0$. Then there is a definable cell decomposition \mathcal{C} of $\mathbb{R}^n \times \mathbb{R}$ such that for every $C \in \mathcal{C}$ there are terms σ, λ, u such that $e(\sigma) = 0$, $e(\lambda) < e(\tau)$, and for all $(t, x) \in C$

$$\tau(t, x) = \sigma(t, x) \cdot \exp(\lambda(t, x)) \cdot u(t, x), \quad |u(t, x) - 1| < 1/2.$$

Because terms with exponential level 0 with respect to v_{n+1} define log-analytic functions in x we get the following conclusion with Fact 4.30.

4.69 Corollary

Let $m \in \mathbb{N}$. Let τ_1, \dots, τ_m be terms. Then there is a definable cell decomposition \mathcal{C} of $\mathbb{R}^n \times \mathbb{R}_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is an r -logarithmic scale $\mathcal{Y} = (y_0, \dots, y_r)$ on C , and for every $j \in \{1, \dots, m\}$ there are $q_j := (q_{j0}, \dots, q_{jr}) \in \mathbb{Q}^{r+1}$, a term s_j , a definable $a_j : \pi(C) \rightarrow \mathbb{R}$, and a definable $u_j : C \rightarrow \mathbb{R}$ with $|u_j(t, x) - 1| < 1/2$ for every $(t, x) \in C$ such that

$$\tau_j(t, x) = a_j(t) |\mathcal{Y}(t, x)|^{\otimes q_j} \exp(s_j(t, x)) u_j(t, x)$$

for all $(t, x) \in C$ where $e(s_j) < e(\tau_j)$ if $e(\tau_j) > 0$, and $s_j = 0$ if $e(\tau_j) = 0$.

So we see also that Van den Dries and Speissegger used the exponential level as a natural bound on the number of iterations of the exponential which may occur in its prepared form. This is strong enough to show that $\mathbb{R}_{\text{an,exp}}$ is o-minimal, but not strong enough for our purposes since there is no precise statement how the "unit" looks like. So both known preparation theorems on definable functions are not deep enough to do proofs on multivariate phenomena for specific definable functions. We will prove a similar, but more precise preparation theorem. Furthermore we will give a connection between a set of positive definable functions from which one can construct the underlying definable function and the exponentials which occur in its preparation.

A Preparation Theorem for Definable Functions

We start with some preparations.

Preparations

For this paragraph we fix $m \in \mathbb{N}$, a tuple of variables $v := (v_1, \dots, v_m)$, and a definable set $X \subset \mathbb{R}^m$. Fix $k, l \in \mathbb{N}_0$ and definable functions $g_1, \dots, g_k : X \rightarrow \mathbb{R}$ and $h_1, \dots, h_l : X \rightarrow \mathbb{R}$. Set $\beta := (g, h_l, \exp(h_1), \dots, \exp(h_l))$. (We could also write $\beta = (g, \exp(h_1), \dots, \exp(h_l))$, but this simplifies notation below.) Note that $\beta(X) \subset \mathbb{R}^{k+l} \times \mathbb{R}_{>0}$. So there exists a 0-logarithmic scale on $\beta(X)$, i.e. $\beta(X)$ is 0-admissible.

Fix a log-analytic function $F : \mathbb{R}^{k+l+1} \rightarrow \mathbb{R}$. Let $\alpha : X \rightarrow \mathbb{R}, v \mapsto F(\beta(v))$. Let $y := (y_1, \dots, y_{k+l})$ range over \mathbb{R}^{k+l} . Let z be another single variable such that (y, z) ranges over $\mathbb{R}^{k+l} \times \mathbb{R}$. Let $\pi^* : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}^{k+l}, (y, z) \mapsto y$.

4.70 Proposition

Let $\Theta : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ be log-analytic such that

$$\exp(h_l) \sim_X \Theta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

There is a log-analytic function $G : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$\alpha = G(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$$

on X .

Proof

Let $\kappa := \Theta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$. Note that $\kappa > 0$. There is $\delta > 1$ such that

$$\frac{1}{\delta} < \frac{\exp(h_l)}{\kappa} < \delta$$

on X . By taking logarithm we get with $\lambda := \log(\delta)$

$$-\lambda < h_l - \log(\kappa) < \lambda$$

on X . Set

$$\exp^* : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} \exp(x), & x \in [-\lambda, \lambda], \\ 0, & \text{else.} \end{cases}$$

Then \exp^* is globally subanalytic and we have

$$\alpha = F(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}), \kappa \cdot \exp^*(h_l - \log(\kappa)))$$

on X . Consider

$$G : \mathbb{R}^{k+l} \rightarrow \mathbb{R}, y \mapsto \begin{cases} F(y, \Theta(y) \cdot \exp^*(y_{k+l} - \log(\Theta(y))))), & y \in \pi^*(\beta(X)), \\ 0, & \text{else.} \end{cases}$$

Note that G is well-defined, log-analytic and we obtain

$$\alpha = G(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$$

on X since for $x \in X$ we have

$$(g(x), h_l(x), \exp(h_1(x)), \dots, \exp(h_{l-1}(x))) \in \pi^*(\beta(X)).$$

■

4.71 Proposition

Assume that F is positive and purely 0-log-analytically prepared in z with center 0 on $\beta(X)$. Then there is a strongly log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that

$$\log(F(\beta)) = H(\beta)$$

on X .

Proof

Let

$$(0, \mathcal{Y}, a, q, s, v, b, P)$$

be a purely LA-preparing tuple for f where $b := (b_1, \dots, b_s)$ and $\mathcal{Y} := y$. Note that $a > 0$. Consider

$$\eta := (\eta_0, \dots, \eta_s) : \pi^*(\beta(X)) \rightarrow \mathbb{R}_{>0} \times \mathbb{R}^s, y \mapsto (a(y), b_1(y), \dots, b_s(y)).$$

Note that η is log-analytic. Let $w := (w_0, \dots, w_{s+2})$ range over \mathbb{R}^{s+3} . For $w \in \mathbb{R}_{>0} \times \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}_{\neq 0}$ with $-1 \leq w_i |w_{s+2}|^{p_{0i}} \leq 1$ for every $i \in \{1, \dots, s\}$ let

$$\phi(w) := \log(w_0) + qw_{s+1} + \log(v(w_1 |w_{s+2}|^{p_{01}}, \dots, w_s |w_{s+2}|^{p_{0s}})).$$

Consider

$$G : \mathbb{R}_{>0} \times \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}_{\neq 0} \rightarrow \mathbb{R}, w \mapsto \begin{cases} \phi(w), & -1 \leq w_i |w_{s+2}|^{p_{0i}} \leq 1 \text{ for every } i \in \{1, \dots, s\}, \\ 0, & \text{else.} \end{cases}$$

Then G is strongly log-analytic in w_{s+2} of order 0 since $\log(v)$ is globally subanalytic. Note that

$$\log(F(\beta)) = G(\eta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})), h_l, \exp(h_l))$$

on X . Then

$$H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}, (y, z) \mapsto \begin{cases} G(\eta(y_1, \dots, y_{k+l}), y_{k+1}, z), & (y, z) \in \beta(X), \\ 0, & \text{else,} \end{cases}$$

does the job, because H is strongly log-analytic in z of order 0. ■

4.72 Corollary

Let $c, d \in \mathbb{N}$. Suppose there are functions $\mu_1, \dots, \mu_c : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\nu_1, \dots, \nu_d : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}_{>0}$ which are purely 0-log-analytically prepared in z with center Θ on $\beta(X)$, and a globally subanalytic function $G : \mathbb{R}^{c+d} \rightarrow \mathbb{R}$ such that

$$\alpha = G(\mu_1(\beta), \dots, \mu_c(\beta), \log(\nu_1(\beta)), \dots, \log(\nu_d(\beta))).$$

If $\Theta = 0$ there is a strongly log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that $\alpha = H(\beta)$ on X . If $\Theta \neq 0$ then we have

$$\exp(h_l) \sim_X \Theta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

Proof

Note that $y_0 : \beta(X) \rightarrow \mathbb{R}, (y, z) \mapsto z - \Theta(y)$, is a 0-logarithmic scale with log-analytic center Θ .

Case 1: $\Theta = 0$. Then by Proposition 4.71 there are strongly log-analytic functions $H_1, \dots, H_m : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that $\log(\nu_j(\beta)) = H_j(\beta)$ on X for every $j \in \{1, \dots, m\}$. By Remark 4.36(2) we see that μ_1, \dots, μ_c are strongly log-analytic in z of order 0. Because G is globally subanalytic we are done.

Case 2: $\Theta \neq 0$. By Remark 4.17 $z \sim_{\beta(X)} \Theta$. We obtain the result. \blacksquare

4.73 Corollary

Let $r \in \mathbb{N}_0$. Suppose F is strongly log-analytic in z of order r . Then there is a decomposition \mathcal{C} of X into finitely many definable sets such that for $C \in \mathcal{C}$ the following holds. There is a strongly log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that $\alpha|_C = H(\beta|_C)$ or there is a log-analytic function $\Theta : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$\exp(h_l) \sim_C \Theta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

Proof

We do an induction on r .

$r = 0$: Then F is strongly log-analytic in z of order 0. The assertion follows.

$r - 1 \rightarrow r$: With Definition 4.34 it is enough to consider the following situation. Let $c, d \in \mathbb{N}_0$, a globally subanalytic function $G : \mathbb{R}^{c+d} \rightarrow \mathbb{R}$, strongly log-analytic functions $\rho_1, \dots, \rho_c : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma_1, \dots, \sigma_d : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}_{>0}$ in z of order at most $r - 1$ be such that

$$\alpha = G(\rho_1(\beta), \dots, \rho_c(\beta), \log(\sigma_1(\beta)), \dots, \log(\sigma_d(\beta))).$$

Applying the inductive hypothesis to $\rho_1(\beta), \dots, \rho_c(\beta), \sigma_1(\beta), \dots, \sigma_d(\beta)$ there is a decomposition \mathcal{A} of X into finitely many definable sets such that the following

holds for $A \in \mathcal{A}$. There are functions $\mu_1, \dots, \mu_c : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\nu_1, \dots, \nu_d : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}_{>0}$ which are strongly log-analytic in z of order 0 such that

$$\alpha = G(\mu_1(\beta), \dots, \mu_c(\beta), \log(\nu_1(\beta)), \dots, \log(\nu_d(\beta)))$$

on A (G does not change) or there is a log-analytic function $\tilde{\Theta} : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$\exp(h_l) \sim_A \tilde{\Theta}(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

Fix such an A . If the latter holds we are done. So assume the former. Fix the corresponding $\mu_1, \dots, \mu_c, \nu_1, \dots, \nu_d$. By Remark 4.63 there is a definable cell decomposition \mathcal{D} of $\mathbb{R}^{k+l} \times \mathbb{R}_{\neq 0}$ such that $\mu_1, \dots, \mu_c, \nu_1, \dots, \nu_d$ are purely 0-log-analytically prepared in z in a simultaneous way on every $D \in \mathcal{D}$. There is a definable cell decomposition \mathcal{C} of A such that for every $C \in \mathcal{C}$ there is $D_C \in \mathcal{D}$ with $\beta(C) \subset D_C$. Fix $C \in \mathcal{C}$ and the center Θ of the pure 0-preparation of $\mu_1, \dots, \mu_c, \nu_1, \dots, \nu_d$ on D_C . If $\Theta = 0$ we find with Corollary 4.72 a strongly log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that $\alpha|_C = H(\beta|_C)$. If $\Theta \neq 0$ it is with Remark 4.17 $z \sim_{D_C} \Theta(y)$ and therefore

$$\exp(h_l) \sim_C \Theta(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

■

Formulation and Proof of the Preparation Theorem

For the rest of Section 4.4 let $m \in \mathbb{N}$ and $X \subset \mathbb{R}^m$ be definable.

4.74 Definition

Let $f : X \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on X .

- (1) Let L be a set of log-analytic functions on X . By induction on $e \in \mathbb{N}_0 \cup \{-1\}$ we define that $f : X \rightarrow \mathbb{R}$ is **e -prepared with respect to L and E** .

Base Case: The function f is (-1) -prepared with respect to L and E if f is the zero function.

Inductive step: The function f is e -prepared with respect to L and E if

$$f = a \cdot \exp(c) \cdot u$$

where $a \in L$, $c : X \rightarrow \mathbb{R}$ is $(e-1)$ -prepared with respect to L and E , $\exp(c) \in E$ and $u : X \rightarrow \mathbb{R}$ is a function of the form

$$u = v(b_1 \cdot \exp(d_1), \dots, b_s \cdot \exp(d_s))$$

where $s \in \mathbb{N}_0$, $b_j \in L$ does not have any zero, $d_j : X \rightarrow \mathbb{R}$ is $(e - 1)$ -prepared with respect to L and E and $\exp(d_j) \in E$ for every $j \in \{1, \dots, s\}$.

Additionally v is a real power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$, it holds $b_j(x)\exp(d_j(x)) \in [-1, 1]$ for every $x \in X$ and every $j \in \{1, \dots, s\}$, and $v([-1, 1]^s) \subset \mathbb{R}_{>0}$.

- (2) We say that f is **e -prepared with respect to E** if there is a set L of log-analytic functions on X such that f is e -prepared with respect to L and E .

4.75 Remark

Let E be a set of positive definable functions and L be a set of log-analytic functions on X . Let $e \in \mathbb{N}_0$. Let $f : X \rightarrow \mathbb{R}$ be e -prepared with respect to L and E . There are $k \in \mathbb{N}$, log-analytic functions $h_1, \dots, h_k \in L$, $(e - 1)$ -prepared g_1, \dots, g_k with respect to L and E with $\exp(g_1), \dots, \exp(g_k) \in E$ and a globally subanalytic function $G : \mathbb{R}^{2k} \rightarrow \mathbb{R}$ such that

$$f = G(h_1, \dots, h_k, \exp(g_1), \dots, \exp(g_k)).$$

Proof

There are $s \in \mathbb{N}$, $a, b_1, \dots, b_s \in L$ and $(e - 1)$ -prepared c, d_1, \dots, d_s with respect to L and E such that $\exp(c), \exp(d_1), \dots, \exp(d_s) \in E$ and

$$f = a\exp(c)v(b_1\exp(d_1), \dots, b_s\exp(d_s))$$

where v is a real power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$ and it holds $b_j(x)\exp(d_j(x)) \in [-1, 1]$ for every $j \in \{1, \dots, s\}$ and every $x \in X$. Note that the function

$$H : \mathbb{R}^s \rightarrow \mathbb{R}, (u_1, \dots, u_s) \mapsto \begin{cases} v(u_1, \dots, u_s), & (u_1, \dots, u_s) \in [-1, 1]^s, \\ 0, & \text{else} \end{cases},$$

is globally subanalytic. Choose $k := s + 1$, $h_1 := a$, $h_j := b_{j-1}$ for $j \in \{2, \dots, k\}$, $g_1 := c$ and $g_j := d_{j-1}$ for $j \in \{2, \dots, k\}$. Let $w := (w_1, \dots, w_k)$ and $z := (z_1, \dots, z_k)$ range over \mathbb{R}^k . Consider

$$G : \mathbb{R}^{2k} \rightarrow \mathbb{R}, (w, z) \mapsto w_1 z_1 H(w_2 z_2, \dots, w_k z_k).$$

Then G is globally subanalytic and we have

$$f = G(h_1, \dots, h_k, \exp(g_1), \dots, \exp(g_k)).$$

■

4.76 Remark

- (1) Let $e \geq 0$. If $f : X \rightarrow \mathbb{R}$ is e -prepared with respect to a set E of positive definable functions then $1 \in E$.
- (2) Let $f : X \rightarrow \mathbb{R}$ be log-analytic. Then f is 0-prepared with respect to $L := \{f\}$ and $E := \{1\}$.
- (3) If f is 0-prepared with respect to a set E of definable functions then f is log-analytic.

Proof

(1): We obtain the result immediately with Definition 4.74 and an easy induction on e .

(2): With Definition 4.74 we see immediately that f is 0-prepared with respect to L and E .

(3): Let L be a set of log-analytic functions such that f is 0-prepared with respect to L and E . Note that a function which is (-1) -prepared coincides with the zero function. So by Remark 4.75 there are $k \in \mathbb{N}$, functions $h_1, \dots, h_k \in L$ and a globally subanalytic function $G : \mathbb{R}^k \rightarrow \mathbb{R}$ such that

$$f = G(h_1, \dots, h_k).$$

So it is clear that f is log-analytic. ■

4.77 Remark

Let E be a set of positive definable functions. Let $e \in \mathbb{N}_0$. Let $f : X \rightarrow \mathbb{R}$ be e -prepared with respect to E . Then f has exponential number at most e with respect to E and is therefore definable.

Proof

We do an induction on e . If $e = 0$ then f is log-analytic by Remark 4.76(3). So assume $e > 0$. Let L be a set of log-analytic functions such that f is e -prepared with respect to L and E . By Remark 4.75 there are $k \in \mathbb{N}$, log-analytic functions $h_1, \dots, h_k \in L$, $(e - 1)$ -prepared g_1, \dots, g_k with respect to L and E with $\exp(g_1), \dots, \exp(g_k) \in E$ and a globally subanalytic function $G : \mathbb{R}^{2k} \rightarrow \mathbb{R}$ such that

$$f = G(h_1, \dots, h_k, \exp(g_1), \dots, \exp(g_k)).$$

With the inductive hypothesis and Definition 3.5(a) we are done. ■

4.78 Proposition

Let $e \in \mathbb{N}_0$. Let $f : X \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on X such that f has exponential number at most e with respect to E . Then there is a decomposition \mathcal{C} of X into finitely many definable sets such that for every $C \in \mathcal{C}$ there is a finite set P of positive definable functions on C and a finite set L of log-analytic functions on C such that the following holds.

- (1) $f|_C$ is e -prepared with respect to L and P and for every $g \in \log(P)$ there is $l \in \{-1, \dots, e-1\}$ such that g is l -prepared with respect to L and P .
- (2) P fulfills the following condition $(*_e)$ with respect to $E|_C$: If $g \in \log(P)$ is l -prepared with respect to L and P for $l \in \{0, \dots, e-1\}$ then g is a finite \mathbb{Q} -linear combination of functions from $\log(E)$ restricted to C which have exponential number at most l with respect to E .

Proof

We do an induction on e .

$e = 0$: Then f is log-analytic and we are done by choosing $P = \{1\}$ and $L = \{f\}$.

$e - 1 \rightarrow e$: There are $k, l \in \mathbb{N}$, functions $g_1, \dots, g_k, h_1, \dots, h_l : \mathbb{R}^m \rightarrow \mathbb{R}$ with exponential number at most $e-1$ with respect to E , and a log-analytic function $F : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f = F(g_1, \dots, g_k, \exp(h_1), \dots, \exp(h_l))$$

and $\exp(h_1), \dots, \exp(h_l) \in E$.

By an auxiliary induction on l and involving the inductive hypothesis we may assume that the theorem is proven for functions of the form

$$\kappa = H(g_1, \dots, g_k, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$$

on X where $H : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ is log-analytic. (**)

(If $l = 1$ then (**) holds by the inductive hypothesis: $g = H(g_1, \dots, g_k, h_l)$ has exponential number at most $e-1$ with respect to E by Proposition 3.11(2).)

(This includes also functions of the form

$$\kappa = \hat{H}(g_1, \dots, g_k, \exp(h_1), \dots, \exp(h_{l-1}))$$

on X where $\hat{H} : \mathbb{R}^{k+l-1} \rightarrow \mathbb{R}$ is log-analytic.)

Let $g := (g_1, \dots, g_k)$. Let $y := (y_1, \dots, y_{k+l})$ range over \mathbb{R}^{k+l} . Let z be another single variable such that (y, z) ranges over $\mathbb{R}^{k+l} \times \mathbb{R}$. Let $y' := (y_1, \dots, y_{k+l-1})$.

Let $\pi^+ : \mathbb{R}^{k+l} \rightarrow \mathbb{R}^{k+l-1}, y \mapsto y'$. Let $r \in \mathbb{N}_0$ be such that F is strongly log-analytic in y_{k+l} of order at most r .

Case 1: $r = 0$. By Remark 4.63 we find a definable cell decomposition \mathcal{D} of $\mathbb{R}^{k+l-1} \times \mathbb{R}_{\neq 0}$ such that $F|_D$ is purely 0-log-analytically prepared in y_{k+l} for every $D \in \mathcal{D}$. There is a decomposition \mathcal{A} of X into finitely many definable sets such that for every $A \in \mathcal{A}$ there is $D_A \in \mathcal{D}$ such that for every $x \in A$ we have $(g(x), \exp(h_1(x)), \dots, \exp(h_l(x))) \in D_A$. Fix $A \in \mathcal{A}$ and a purely preparing tuple $(0, \mathcal{Y}, a, q, s, v, b, P)$ for $F|_{D_A}$ where $b := (b_1, \dots, b_s)$ and $P := (p_1, \dots, p_s)^t$. Let Θ be the center of \mathcal{Y} . Then one of the following properties holds.

(1) There is $\epsilon \in]0, 1[$ such that

$$0 < |y_{k+l} - \Theta(y')| < \epsilon |y_{k+l}|$$

for all $y \in D_A$.

(2) $\Theta = 0$ on $\pi^+(D_A)$.

Assume (1). Then by Remark 4.17 we have $y_{k+l} \sim_{D_A} \Theta$. This gives

$$\exp(h_l) \sim_A \Theta(g, \exp(h_1), \dots, \exp(h_{l-1})).$$

By Proposition 4.70 there is a log-analytic function $G : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f = G(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$$

on A . With $(**)$ applied to f we are done.

Assume (2). Let $\beta := (g, \exp(h_1), \dots, \exp(h_{l-1}))$. We have

$$f|_A = a(\beta) \exp(qh_l) v(b_1(\beta) \exp(p_1 h_l), \dots, b_s(\beta) \exp(p_s h_l)).$$

Let $b_0 := a$. Note that we can apply $(**)$ to $b_j(\beta)$ for $j \in \{0, \dots, s\}$ and so there is a decomposition \mathcal{B} of A into finitely many definable sets such that for every $B \in \mathcal{B}$ the following holds: There is a finite set P' of positive definable functions on B and a finite set L' of log-analytic functions on B such that $b_0(\beta)|_B, \dots, b_s(\beta)|_B$ are e -prepared with respect to L' and P' and that P' fulfills property $(*_e)$ with respect to $E|_B$. Fix such a B . Then we have for $j \in \{0, \dots, s\}$

$$b_j(\beta) = \hat{a}_j \exp(\hat{c}_j) \hat{v}_j(\hat{b}_{1j} \exp(\hat{d}_{1j}), \dots, \hat{b}_{s_j j} \exp(\hat{d}_{s_j j}))$$

where $s_j \in \mathbb{N}$, $\hat{a}_j, \hat{b}_{1j}, \dots, \hat{b}_{s_j j}$ are log-analytic and $\hat{c}_j, \hat{d}_{1j}, \dots, \hat{d}_{s_j j}$ are finite \mathbb{Q} -linear combinations of functions from $\log(E)$ restricted to B which have exponential number at most $e - 1$ with respect to E and \hat{v}_j is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^{s_j}$ and

$\hat{v}_j([-1, 1]^{s_j}) \subset \mathbb{R}_{>0}$. Since $h_l \in \log(E)$ has exponential number at most $e - 1$ with respect to E we obtain that $\hat{c}_0 + qh_l$ and $\hat{c}_j + p_j h_l$ for $j \in \{1, \dots, s\}$ are finite \mathbb{Q} -linear combinations of functions from $\log(E)$ restricted to B which have exponential number at most $e - 1$ with respect to E . (Note that h_l is not necessarily $(e - 1)$ -prepared with respect to P' on B .)

Consequently we obtain by composition of power series that there are $\tilde{s} \in \mathbb{N}$, log-analytic functions $\tilde{a}, \tilde{b}_1, \dots, \tilde{b}_{\tilde{s}} : B \rightarrow \mathbb{R}$, and functions $c, d_1, \dots, d_{\tilde{s}} : B \rightarrow \mathbb{R}$ which are finite \mathbb{Q} -linear combinations of elements from $\log(E)$ restricted to B which have exponential number at most $e - 1$ with respect to $E|_B$ such that the following holds.

$$f|_B = \tilde{a} \exp(c) \tilde{v}(\tilde{b}_1 \exp(d_1), \dots, \tilde{b}_{\tilde{s}} \exp(d_{\tilde{s}}))$$

where $\tilde{b}_j(x) \exp(d_j(x)) \in [-1, 1]$ for every $x \in B$ and $j \in \{1, \dots, \tilde{s}\}$, and \tilde{v} is a real power series which converges absolutely on an open neighbourhood of $[-1, 1]^{\tilde{s}}$ and $\tilde{v}([-1, 1]^{\tilde{s}}) \subset \mathbb{R}_{>0}$. Note that c and $d_1, \dots, d_{\tilde{s}}$ have exponential number at most $e - 1$ with respect to E . By the inductive hypothesis there is a further decomposition \mathcal{C}_B of B into finitely many definable sets such that for every $C \in \mathcal{C}_B$ there is a finite set \tilde{P} of positive definable functions on C which fulfills property $(*_{e-1})$ with respect to $E|_C$ and a finite set \tilde{L} of log-analytic functions on C such that the functions $c|_C$ and $d_1|_C, \dots, d_{\tilde{s}}|_C$ are $(e - 1)$ -prepared with respect to \tilde{L} and \tilde{P} . So choose

$$L := \tilde{L} \cup \{\tilde{a}|_C, \tilde{b}_1|_C, \dots, \tilde{b}_{\tilde{s}}|_C\}$$

and

$$P := \tilde{P} \cup \{\exp(c)|_C, \exp(d_1)|_C, \dots, \exp(d_{\tilde{s}})|_C\}.$$

Then P fulfills property $(*_e)$ with respect to $E|_C$. Note that $f|_C$ is e -prepared with respect to L and P and we are done.

Case 2: $r > 0$. By Corollary 4.73 there is a decomposition \mathcal{A} of X into finitely many definable sets such that for every $A \in \mathcal{A}$ one of the following properties holds.

- (1) There is a strongly log-analytic function $H : \mathbb{R}^{k+l} \times \mathbb{R} \rightarrow \mathbb{R}$ in z of order 0 such that

$$f|_A = H(g, h_l, \exp(h_1), \dots, \exp(h_l)).$$

- (2) There is a log-analytic function $\tilde{\Theta} : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$\exp(h_l) \sim_A \tilde{\Theta}(g, h_l, \exp(h_1), \dots, \exp(h_{l-1})).$$

Let $A \in \mathcal{A}$. If (1) holds for A then we are done with Case 1. If (2) holds for A then by Proposition 4.70 there is a log-analytic function $H : \mathbb{R}^{k+l} \rightarrow \mathbb{R}$ such that

$$f = H(g, h_l, \exp(h_1), \dots, \exp(h_{l-1}))$$

on A . We are done with $(**)$ applied to f .

The treatment of the both cases above gives the proof of Proposition 4.78. ■

4.79 Definition

Let $r \in \mathbb{N}_0$. Let $w := (w_0, \dots, w_r)$ range over \mathbb{R}^{r+1} . Let $D \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{r+1}$ be definable. Let $\pi^* : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}^n \times \mathbb{R}$, $(t, x, w) \mapsto (t, x)$ and $\pi^+ : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}^n$, $(t, x, w) \mapsto t$. A function $u : D \rightarrow \mathbb{R}$ is called **r -special unit** if $u = v \circ \phi$ where the following holds:

(a) The function ϕ is given by

$$\phi : D \mapsto [-1, 1]^s, (t, x, w) \mapsto (\phi_1(t, x, w), \dots, \phi_s(t, x, w)),$$

where $s \in \mathbb{N}_0$, and for every $j \in \{1, \dots, s\}$ we have

$$\phi_j(t, x, w) = b_j(t) \exp(d_j(t, x)) \prod_{l=0}^r |w_l|^{p_{jl}}$$

for every $(t, x, w) \in D$ where $d_j : \pi^*(D) \rightarrow \mathbb{R}$ is definable, $b_j : \pi^+(D) \rightarrow \mathbb{R}$ is definable which has no zeros and $p_{jl} \in \mathbb{Q}$ for every $l \in \{0, \dots, r\}$.

(b) v is a real power series in s variables which converges absolutely on an open neighbourhood of $[-1, 1]^s$.

(c) It holds $v([-1, 1]^s) \subset \mathbb{R}_{>0}$.

We call $b := (b_1, \dots, b_s)$ a tuple of **base functions** for u , $e^d := (e^{d_1}, \dots, e^{d_s})$ a tuple of **exponential definable functions** for u , and

$$\mathcal{I} := (s, v, b, e^d, P)$$

where

$$P := \begin{pmatrix} p_{10} & \cdot & \cdot & p_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{s0} & \cdot & \cdot & p_{sr} \end{pmatrix} \in M(s \times (r+1), \mathbb{Q})$$

an **r -describing tuple** for u .

4.80 Remark

Let $r \in \mathbb{N}_0$. Let $D \subset \mathbb{R}^n \times \mathbb{R}^{r+1}$ be definable. Let $u : D \rightarrow \mathbb{R}$ be an r -LA special unit. Then u is an r -special unit considered as function on

$$\tilde{D} := \{(t, x, w) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{r+1} \mid (t, w) \in D\}$$

which does not depend on x .

Proof

Let $\mathcal{LI} = (s, v, b, P)$ be an r -LA-describing tuple for u . Let d_1, \dots, d_s be zero functions on \tilde{D} . Then it is easy to see that u has describing tuple $\mathcal{I} = (s, v, b, \exp(d), P)$ where $\exp(d) := (\exp(d_1), \dots, \exp(d_s))$. Clearly u does not depend on x . \blacksquare

4.81 Definition

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable and \mathcal{Y} be an r -logarithmic scale on C . We set

$$C^{(x, \mathcal{Y})} := \{(t, x, y_0(t, x), \dots, y_r(t, x)) \mid (t, x) \in C\} = \text{graph}(\mathcal{Y}).$$

4.82 Definition

Let $r \in \mathbb{N}_0$ and $e \in \mathbb{N}_0 \cup \{-1\}$. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable and $f : C \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on C .

By induction on e we define that f is (e, r) -**prepared** in x with center Θ with respect to E . To this preparation we assign a **preparing tuple** for f .

$e = -1$: We say that f is $(-1, r)$ -prepared in x with center Θ with respect to E if f is the zero function. A preparing tuple for f is (0) .

$e - 1 \rightarrow e$: We say that f is (e, r) -prepared in x with center Θ with respect to E if

$$f(t, x) = a(t) |\mathcal{Y}(t, x)|^{\otimes q} e^{c(t, x)} u(t, x, y_0(t, x), \dots, y_r(t, x))$$

for every $(t, x) \in C$ where $a : \pi(C) \rightarrow \mathbb{R}$ is C -nice, $\mathcal{Y} := (y_0, \dots, y_r)$ is a nice r -logarithmic scale with center Θ , $q = (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$, $\exp(c) \in E$ where c is $(e-1, r)$ -prepared in x with center Θ with respect to E and u is an r -special unit on $C^{(x, \mathcal{Y})}$ with r -describing tuple $\mathcal{I} = (s, v, b, \exp(d), P)$ where $b = (b_1, \dots, b_s)$ is a tuple of C -nice functions on $\pi(C)$, $\exp(d) = (\exp(d_1), \dots, \exp(d_s)) \in E^s$ is a tuple of functions with $(e-1, r)$ -prepared d_1, \dots, d_s in x with center Θ with respect to E . A preparing tuple for f is then

$$\mathcal{J} := (r, \mathcal{Y}, a, \exp(c), q, \mathcal{I}).$$

4.83 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let E be a set of positive definable functions on C . Let $f : C \rightarrow \mathbb{R}$ be (e, r) -prepared in x with respect to E . If $e = 0$ then f is nicely log-analytically prepared in x .

4.84 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let E be a set of positive definable functions on C . Let $f : C \rightarrow \mathbb{R}$ be (e, r) -prepared in x with respect to E . Then there is a set \mathcal{E} of C -heirs such that f can be constructed from $E \cup \mathcal{E}$.

Proof

We do an induction on e . For $e = -1$ there is nothing to show.

$e - 1 \rightarrow e$: Let

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

be a preparing tuple for f where $b := (b_1, \dots, b_s)$,

$$\exp(d) := (\exp(d_1), \dots, \exp(d_s))$$

and $\exp(c), \exp(d_1), \dots, \exp(d_s) \in E$. By the inductive hypothesis there is a set \mathcal{E}' of C -heirs such that c and d_1, \dots, d_s can be constructed from $E \cup \mathcal{E}'$. By Proposition 3.11(1) we see that $\exp(c), \exp(d_1), \dots, \exp(d_s)$ can be constructed from $E \cup \mathcal{E}'$. Because a, b_1, \dots, b_s and $\Theta_0, \dots, \Theta_r$ are C -nice there is a set $\bar{\mathcal{E}}$ of C -heirs such that a, b_1, \dots, b_s and $\Theta_0, \dots, \Theta_r$ can be constructed from $\bar{\mathcal{E}}$. Set $\mathcal{E} := \bar{\mathcal{E}} \cup \mathcal{E}'$. Then

$$\eta := (a, \exp(c), b_1, \dots, b_s, \exp(d_1), \dots, \exp(d_s))$$

can be constructed from $E \cup \mathcal{E}$. Note also that y_0, \dots, y_r can be constructed from \mathcal{E} (compare the proof of Remark 4.58). Let $k := 2 + 2s + r + 1$. With a similar argument as in the proof of Proposition 4.52(1) or Remark 4.75 we see that there is a globally subanalytic $F : \mathbb{R}^k \rightarrow \mathbb{R}$ such that for every $(t, x) \in C$ it is

$$f(t, x) = F(\eta(t), y_0(t, x), \dots, y_r(t, x)).$$

With Proposition 3.11(2) we are done. ■

Now we give a full formulation for our preparation theorem.

4.85 Proposition

Let $e \in \mathbb{N}_0$. Let $Y \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let E be a set of positive definable functions on Y . Let $f : Y \rightarrow \mathbb{R}$ be a function with exponential number at most e with respect to E . Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $Y_{\neq 0}$ such that for every $C \in \mathcal{C}$ there is a finite set P of positive definable functions on C such that the function $f|_C$ is (e, r) -prepared in x with center Θ with respect to P . Additionally the following holds.

- (1) For every $g \in \log(P)$ there is $l \in \{-1, \dots, e - 1\}$ such that g is (l, r) -prepared in x with center Θ with respect to P .

(2) *The following condition $(+_e)$ is satisfied: If $g \in \log(P)$ is (l, r) -prepared in x with center Θ with respect to P for $l \in \{-1, \dots, e-1\}$ then g is a finite \mathbb{Q} -linear combination of functions from $\log(E)$ restricted to C which have exponential number at most l with respect to E .*

Proof

By Proposition 4.78 we may assume that f is e -prepared with respect to a finite set $L := \{g_1, \dots, g_m\}$ of log-analytic functions and a finite set Q of positive definable functions with the following property:

Every $g \in \log(Q)$ is l -prepared with respect to L and Q for an $l \in \{-1, \dots, e-1\}$. Additionally if $g \in \log(Q)$ is l -prepared for $l \in \{-1, \dots, e-1\}$ with respect to L and Q then g is a finite \mathbb{Q} -linear combination of functions from $\log(E)$ which have exponential number at most l with respect to E . $(*_e)$

Let $r \in \mathbb{N}_0$ be such that g_1, \dots, g_m are log-analytic of order at most r . By Corollary 4.65 there is a definable cell decomposition \mathcal{C} of $Y_{\neq 0}$ such that $g_1|_C, \dots, g_m|_C$ are nicely r -log-analytically prepared in x in a simultaneous way for every $C \in \mathcal{C}$. Fix $C \in \mathcal{C}$ and the corresponding center Θ for the r -logarithmic preparation of $g_1|_C, \dots, g_m|_C$.

Claim

Let $l \in \{-1, \dots, e\}$ and $h \in \log(Q) \cup \{f\}$ be l -prepared with respect to L and Q . Then $h|_C$ is (l, r) -prepared in x with center Θ with respect to $P := Q|_C$.

Proof of the claim

We do an induction on l . If $l = -1$ it is clear. So assume $l \geq 0$. Since $h|_C$ is l -prepared with respect to $\{g_1|_C, \dots, g_m|_C\}$ we have that

$$h|_C = \mu e^\sigma \tilde{v}(\nu_1 e^{\tau_1}, \dots, \nu_k e^{\tau_k})$$

where $k \in \mathbb{N}$, the functions $\mu, \nu_1, \dots, \nu_k \in L|_C$ are nicely r -log-analytically prepared in x with center Θ , the functions $\sigma, \tau_1, \dots, \tau_k \in \log(P)$ are $(l-1)$ -prepared with respect to P and $L|_C$ and $\nu_j(t, x)e^{\tau_j(t, x)} \in [-1, 1]$ for every $(t, x) \in C$ and $j \in \{1, \dots, k\}$. Additionally \tilde{v} is a real power series which converges absolutely on an open neighbourhood of $[-1, 1]^k$ with $\tilde{v}([-1, 1]^k) \subset \mathbb{R}_{>0}$. By the inductive hypothesis we see that σ and τ_1, \dots, τ_k are $(l-1, e)$ -prepared in x with center Θ with respect to P .

With composition of power series we see that there is a C -nice $a : \pi(C) \rightarrow \mathbb{R}$, a nice r -logarithmic scale \mathcal{Y} with center Θ , $q \in \mathbb{Q}^{r+1}$ and an r -special unit u on $C^{(x, \mathcal{Y})}$ with r -describing tuple $\mathcal{I} = (s, v, b, \exp(d), P^*)$ where $b = (b_1, \dots, b_s)$ is a tuple of C -nice functions on $\pi(C)$ and $\exp(d) := (\exp(d_1), \dots, \exp(d_s))$ is a tuple of definable functions on C with $d_j \in \{\tau_1, \dots, \tau_k\} \cup \{0\}$ for every $j \in \{1, \dots, s\}$ such that

$$h(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} e^{\sigma(t, x)} u(t, y_0(t, x), \dots, y_r(t, x))$$

for every $(t, x) \in C$. So we see that h is (l, r) -prepared in x with respect to P . (By Remark 4.76(1) we have $e^0 \in Q$ since $l \geq 0$.) ■ Claim

Because f is e -prepared with respect to Q we see by the Claim that $f|_C$ is (e, r) -prepared in x with center Θ with respect to P . With the Claim we obtain that $g|_C$ is (l, r) -prepared in x with center Θ with respect to P for every $g \in \log(Q)$ which is l -prepared with respect to Q . So with $(*_e)$ we see that $(+_e)$ is fulfilled. ■

For some purposes we need a sharper version of Definition 4.82 when we deal with log-analytic coefficients, base functions and a log-analytic center.

4.86 Definition

Let $r \in \mathbb{N}_0$ and $e \in \mathbb{N}_0 \cup \{-1\}$. Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable and $f : C \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on C .

By induction on e we define that f is **purely (e, r) -prepared** in x with center Θ with respect to E . To this preparation we assign a **pure preparing tuple** for f .

$e = -1$: We say that f is purely $(-1, r)$ -prepared in x with center Θ with respect to E if f is the zero function. A pure preparing tuple for f is (0) .

$e - 1 \rightarrow e$: We say that f is purely (e, r) -prepared in x with center Θ with respect to E if $\Theta := (\Theta_0, \dots, \Theta_r)$ is log-analytic, f is (e, r) -prepared in x with respect to E with preparing tuple

$$\mathcal{J} := (r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

where \mathcal{Y} is an r -logarithmic scale with center Θ , a and $b := (b_1, \dots, b_s)$ are log-analytic and c, d_1, \dots, d_s are purely $(e - 1, r)$ -prepared in x with center Θ with respect to E . We call then \mathcal{J} a pure preparing tuple for f .

4.87 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let E be a set of positive definable functions on C . Suppose that f is purely (e, r) -prepared in x with respect to E . If $E = \{1\}$ or $e = 0$ then f is purely r -log-analytically prepared in x .

4.88 Remark

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let E be a set of positive definable functions on C and let $f : C \rightarrow \mathbb{R}$ be purely (e, r) -prepared in x with respect to E for $e, r \in \mathbb{N}_0$. Then f is (e, r) -prepared in x with respect to E and can be constructed from E .

Proof

We do an induction on e . For $e = -1$ there is nothing to show.

$e - 1 \rightarrow e$: Let

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

be a purely preparing tuple for f where $b := (b_1, \dots, b_s)$,

$$\exp(d) := (\exp(d_1), \dots, \exp(d_s))$$

and $\exp(c), \exp(d_1), \dots, \exp(d_s) \in E$. Note that a, b_1, \dots, b_s and \mathcal{Y} are log-analytic. So we see with the inductive hypothesis and Proposition 3.11(1) that the function

$$\eta := (a, \exp(c), b_1, \dots, b_s, \exp(d_1), \dots, \exp(d_s), y_0, \dots, y_r)$$

can be constructed from E . Now adapt the proof of Remark 4.84. ■

The next example shows that not every (e, r) -prepared function in x with respect to a set E of positive definable functions is also purely (e, r) -prepared in x with respect to E .

4.89 Example

Consider

$$C := \{(u, x) \in \mathbb{R}^2 \mid 0 < u < 1, \frac{1}{1+u} + e^{-2/u+2e^{-1/u}} < x < \frac{1}{1+u} + e^{-1/u}\}.$$

Let

$$f : C \rightarrow \mathbb{R}, (u, x) \mapsto e^{e^{-1/u}x}$$

and $E := \{1, f\}$. Then the following holds.

- (1) f is $(1, 0)$ -prepared in x with respect to E .
- (2) There is no $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$ such that f is purely (e, r) -prepared in x with respect to E .

Proof

(1): In Example 4.46 we have shown that $C \rightarrow \mathbb{R}, u \mapsto e^{-1/u}$, is a C -heir and therefore we obtain with Definition 4.82 that f is $(1, 0)$ -prepared in x with respect to E .

(2): We start with the following Claim.

Claim

There is no $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$ such that the function $h : C \rightarrow \mathbb{R}, (u, x) \mapsto e^{-1/u}x$, is purely (e, r) -prepared in x with respect to E .

Proof of the claim

Since $\pi(C) =]0, 1[$ we have that h is not log-analytic. We do an induction on e .

$e = -1$: There is no $r \in \mathbb{N}_0$ such that h is purely $(-1, r)$ -prepared in x with respect to E since h is not the zero function.

$e = 0$: There is no $r \in \mathbb{N}_0$ such that h is purely $(0, r)$ -prepared in x with respect to E since otherwise h would be log-analytic by Remark 4.87 and Remark 4.33.

$e - 1 \rightarrow e$: We may assume that $e \geq 1$. Assume the contrary. Let

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

be a purely preparing tuple for h where

$$\exp(d) := (\exp(d_1), \dots, \exp(d_s)).$$

Note that h is not purely (e, r) -prepared in x with respect to $\{1\}$ since otherwise h would be log-analytic by Remark 4.87 and Remark 4.33. By Definition 4.86 we have $\exp(c) = f$ or there is $j \in \{1, \dots, s\}$ such that $\exp(d_j) = f$. With Definition 4.86 we obtain that $\log(f)$ is purely $(e - 1, r)$ -prepared in x with respect to E . But $\log(f) = h$. So we obtain a contradiction to the inductive hypothesis. ■ Claim

Assume the contrary. Fix $e, r \in \mathbb{N}_0$ such that f is purely (e, r) -prepared in x with respect to E . Similarly as in the proof of the claim we obtain with Definition 4.86 that $h = \log(f)$ is purely $(e - 1, r)$ -prepared in x with respect to E , a contradiction to the Claim. ■

4.5 A Preparation Theorem for Restricted Log-Exp-Analytic Functions

For Section 4.5 we set the following: Let $m, l \in \mathbb{N}_0$ be with $n = l + m$. Let $w := (w_1, \dots, w_l)$ range over \mathbb{R}^l and $u := (u_1, \dots, u_m)$ over \mathbb{R}^m . Let $C, X \subset \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$ be definable sets with $C \subset X$. Assume that X_w is open for every $w \in \mathbb{R}^l$. Let $\pi_l : \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^l, (w, u) \mapsto w$.

4.90 Remark

Assume that C is a cell such that C_w is open for every $w \in \mathbb{R}^l$. A nicely log-analytically prepared function $f : C \rightarrow \mathbb{R}$ in x is restricted log-exp-analytic in (u, x) .

Proof

By Proposition 4.52(1) and the proof of Remark 4.58 we find a set \mathcal{E} of C -heirs such that f can be constructed from \mathcal{E} . Let $h \in \mathcal{E}$. Note that h depends on

(w, u) and that there is $r \in \mathbb{N}$ and an r -logarithmic scale \mathcal{Y} on C with center $(\Theta_0, \dots, \Theta_r)$ such that $h = \exp(\Theta_j)$ for $j \in \{1, \dots, r\}$ (compare Definition 4.38). We show that Θ_j is locally bounded in (u, x) (considered as function on C). Let $w \in \mathbb{R}^l$, $(u_0, x_0) \in C_w$ and let

$$\gamma :]0, 1[\rightarrow C_w, y \mapsto (\gamma_u(y), \gamma_x(y)) := (\gamma_{u_1}(y), \dots, \gamma_{u_m}(y), \gamma_x(y)),$$

be a definable curve with $\lim_{y \searrow 0} \gamma(y) = (u_0, x_0)$. Then $\lim_{y \searrow 0} \Theta_l(w, \gamma_u(y)) \in \mathbb{R}$ by (I_l) in Proposition 6.28 in Chapter 6 applied to the definable curve $\hat{\gamma} := (w, \gamma)$ (since $\hat{\gamma}$ is compatible with C , compare Definition 6.26). ■

4.91 Definition

Let $e \in \mathbb{N}_0 \cup \{-1\}$. Let $f : C \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, be a function. We call f $(m + 1, X)$ -**restricted e -prepared** if f is e -prepared with respect to a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X .

4.92 Remark

Let $f : C \rightarrow \mathbb{R}$ be a function. Let $e \in \mathbb{N}_0 \cup \{-1\}$. If f is $(m + 1, X)$ -restricted e -prepared then f is restricted log-exp-analytic in (u, x) of order at most e with reference set X .

Proof

There is a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X and f is e -prepared with respect to E . By Remark 4.77 f can be constructed from E . ■

4.93 Proposition

Let $e \in \mathbb{N}_0 \cup \{-1\}$. Let $f : X \rightarrow \mathbb{R}$ be a restricted log-exp-analytic function in (u, x) of order e . Then there is a decomposition \mathcal{C} of X into finitely many definable sets such that $f|_C$ is $(m + 1, X)$ -restricted e -prepared for every $C \in \mathcal{C}$.

Proof

Let E be a set of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) and f has exponential number at most e with respect to E . By Proposition 4.78 there is a decomposition \mathcal{C} of X into finitely many definable sets such that the following holds for every $C \in \mathcal{C}$. The function $f|_C$ is e -prepared with respect to a finite set P of positive definable functions such that every $g \in \log(P)$ is l -prepared with respect to P for $l \in \{0, \dots, e - 1\}$ and if $g \in \log(P)$ is l -prepared with respect to P then g is a finite \mathbb{Q} -linear combination of functions from $\log(E)$ with exponential number at most l with respect to E restricted to C for $l \in \{0, \dots, e - 1\}$. Thus by Remark 3.27 every $g \in \log(P)$ is locally bounded in (u, x) with reference set X and we are done. ■

4.94 Definition

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. We call $f : X \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, $(m + 1, X)$ -**restricted** (e, r) -**prepared in** x if f is (e, r) -prepared in x with respect to a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X .

4.95 Corollary

Let $e \in \mathbb{N}_0 \cup \{-1\}$. Let $f : X \rightarrow \mathbb{R}$ be a restricted log-exp-analytic function in (u, x) of order at most e . Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that $f|_C$ is $(m + 1, X)$ -restricted (e, r) -prepared in x for every $C \in \mathcal{C}$.

Proof

This follows from Proposition 4.93 and Proposition 4.85. ■

4.96 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be $(m + 1, X)$ -restricted (e, r) -prepared in x . Then f is not necessarily a restricted log-exp-analytic function in (u, x) of order at most e with reference set X . The assertion holds if $C = X$.

Proof

Suppose that $w = 0$ and $m = 1$. Consider

$$C := \{(u, x) \in \mathbb{R}^2 \mid 0 < u < 1, \frac{1}{1+u} + e^{-2/u+2e^{-1/u}} < x < \frac{1}{1+u} + e^{-1/u}\}.$$

Let $X \subset \mathbb{R}^2$ be open with $C \subset X$ and $0 \in X$. Let

$$f : C \rightarrow \mathbb{R}, (u, x) \mapsto e^{-1/u}x.$$

We see that f is nicely 0-log-analytically prepared in x , because in Example 4.46 we have shown that $C \rightarrow \mathbb{R}, u \mapsto e^{-1/u}$, is a C -heir. We see also that the function

$$f^* : X \rightarrow \mathbb{R}, (u, x) \mapsto \begin{cases} e^{-1/u}x, & (u, x) \in C, \\ 0, & \text{else,} \end{cases}$$

is flat at $(0, 0)$. But in Chapter 5 we will show that nonzero restricted log-exp-analytic functions do not exhibit this property. So f is not restricted log-exp-analytic in (u, x) with reference set X .

Now suppose that $C = X$. One sees that C_w is open for every $w \in \mathbb{R}^l$. By Definition 4.94 and Remark 4.84 there is a set \mathcal{E} of C -heirs and a set E of positive definable functions where every $g \in \log(E)$ is locally bounded in (u, x) with reference set X such that f can be constructed from $E \cup \mathcal{E}$. By the proof of Remark 4.90 we see that the center of every r -logarithmic scale \mathcal{Y} on C is

locally bounded in (u, x) with reference set X . So every $g \in \log(\mathcal{E})$ is locally bounded in (u, x) with reference set X . So f is restricted log-exp-analytic in (u, x) with reference set X . ■

4.97 Definition

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be a function. We say that f is **purely $(m + 1, X)$ -restricted (e, r) -prepared in x** if f is purely (e, r) -prepared in x with respect to a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X .

4.98 Remark

Let $f : C \rightarrow \mathbb{R}$ be a function. Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. If f is purely $(m + 1, X)$ -restricted (e, r) -prepared in x then f is restricted log-exp-analytic in (u, x) with reference set X .

Proof

There is a set E of positive definable functions such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X and f is purely (e, r) -prepared in x with respect to E . By Remark 4.88 f can be constructed from E . ■

4.99 Remark

Let $C \subset X$ be definable. Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. A function $f : C \rightarrow \mathbb{R}$ which is $(m + 1, X)$ -restricted (e, r) -prepared in x is not purely $(m + 1, X)$ -restricted (e, r) -prepared in x in general.

Proof

This is easily seen with Remark 4.96 and Remark 4.98. ■

5 Differentiability Properties of Restricted Log-Exp-Analytic Functions

Log-analytic functions belong to a class of definable functions which avoid the exponential function. They do not exhibit properties of the function $e^{-1/x}$ as flatness or infinite differentiability but not real analyticity.

Theorem (Kaiser/Opris, Theorem C in [22])

Let $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $(t, x) \mapsto f(t, x)$, be log-analytic. Then there is $N \in \mathbb{N}$ such that the following holds for every $(t, x) \in \mathbb{R}^n \times \mathbb{R}^m$. If $f(t, -)$ is C^N at x then $f(t, -)$ is real analytic at x .

(See also [14] for a formulation of this theorem in the globally subanalytic setting and [40] for Tamm's original version.) Kaiser used such results to enable a deep technical induction on the number of variables for proving global complexification for globally subanalytic functions. So our main goal for this chapter is to extend Tamm's theorem to the class of restricted log-exp-analytic functions beyond the class of log-analytic ones. Because we deal with compositions of log-analytic functions and exponentials whose arguments are locally bounded features like flatness but no real analyticity should also not occur. To be able to use the preparation theorems our initial result is the key observation that a restricted log-exp-analytic function can be log-analytically prepared on certain 'pieces' which we call simple. It turns out that on such a piece the coefficient, the base functions and the center of such a preparation is also restricted log-exp-analytic. An immediate consequence is that the class of restricted log-exp-analytic functions is closed under taking derivatives and exhibits strong quasianalyticity (see Miller in [31] for this result in polynomially bounded o-minimal structures). With this both observations we will prove Theorem A which implies that real analyticity of restricted log-exp-analytic functions is a definable property. This shows that this class of functions shares its properties from the viewpoint of analysis and of o-minimality with log-analytic and even with globally subanalytic ones. But this is not true for definable functions in general as remarked in [22] and in the end of [14].

Example

Consider the function

$$f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, (t, x) \mapsto \begin{cases} |x|^{|2t|}, & x \neq 0, \\ 0, & x = 0, \end{cases} \text{ if}$$

which is definable in \mathbb{R}_{exp} . Then the set of all $t \in \mathbb{R}$ such that $f(t, -)$ is real-analytic at 0 is the set of integers.

This chapter is organized as follows. At first we give basic properties for logarithmic scales and prove a preparation theorem for restricted log-exp-analytic functions on simple sets. Then we prove that they are closed under differentiation and exhibit strong quasianalyticity. Finally we derive Theorem A.

For the whole chapter let $n \in \mathbb{N}_0$, t range over \mathbb{R}^n and x over \mathbb{R} . Definable means definable in $\mathbb{R}_{\text{an,exp}}$ if not otherwise mentioned.

5.1 Simple Sets and Simple Preparation

For the following see also [22].

Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be definable. Let $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, (t, x) \mapsto t$, be the projection on the first n coordinates.

Let $r \in \mathbb{N}_0$.

5.1 Definition

An r -logarithmic scale on C is called **elementary** if its center is vanishing.

5.2 Remark

Let $\mathcal{Y} := (y_0, \dots, y_r)$ be a tuple of functions on C . Then \mathcal{Y} is an elementary r -logarithmic scale on C if and only if the following holds.

- (1) We have $y_j < 0$ or $y_j > 0$ for $j \in \{0, \dots, r\}$.
- (2) For $(t, x) \in C$ we have $y_0(t, x) = y$ and $y_j(t, x) = \log(|y_{j-1}(t, x)|)$ for every $j \in \{1, \dots, r\}$.

Notice that an elementary r -logarithmic scale may not exist on C . If it exists it is uniquely determined and log-analytic.

5.3 Definition

If C has an elementary r -logarithmic scale then we call C **r -elementary**. The elementary r -logarithmic scale on C is then denoted by $\mathcal{Y}_r^{\text{el}} = \mathcal{Y}_{r,C}^{\text{el}}$.

For the next definition compare with the setting of [14], Section 4.

5.4 Definition

We call C **simple** if for every $t \in \pi(C)$ we have $C_t =]0, d_t[$ for some $d_t \in \mathbb{R}_{>0} \cup \{+\infty\}$.

5.5 Remark

Let $V := \{(t, x) \in C \mid 0 \text{ is interior point of } C_t\}$. Then V is definable. Let \mathcal{D} be a definable cell decomposition of $V_{\neq 0}$. Then

$$\pi(V) = \bigcup \{\pi(D) \mid D \in \mathcal{D} \text{ simple}\}.$$

We set $e_0 := 0$ and $e_r := \exp(e_{r-1})$ for $r \in \mathbb{N}$. In the following let $1/0 := \infty$.

5.6 Proposition

Let C be simple and r -elementary and let $\mathcal{Y}_{r,C}^{\text{el}} = (y_0, \dots, y_r)$. Then the following holds.

- (1) $\sup(C_t) \leq 1/e_r$ for all $t \in \pi(C)$.
- (2) $y_0 = x, y_1 = \log(x), y_j = \log_{j-1}(-\log(x))$ for $j \in \{2, \dots, r\}$.
- (3) $\text{sign}(\mathcal{Y}_{r,C}^{\text{el}}) = (1, -1, 1, \dots, 1) \in \mathbb{R}^{r+1}$.

Proof

For $t \in \pi(C)$ let $d_t := \sup(C_t)$. Let $(\sigma_0, \dots, \sigma_r)$ be the sign of $\mathcal{Y}_{r,C}^{\text{el}}$. We show by induction on $k \in \{0, \dots, r\}$ that $d_t \leq e_k$ for all $t \in \pi(C)$, that $y_0 = y$, $y_1 = \log(y)$, $y_j = \log_{j-1}(-\log(y))$ for all $j \in \{2, \dots, k\}$ and that $(\sigma_0, \dots, \sigma_k) = (1, -1, 1, \dots, 1) \in \mathbb{R}^{k+1}$.

$k = 0$: We have $y_0 = x$ by Definition 4.1 and Definition 5.1. This gives $\sigma_0 = 1$. That $d_t \leq \infty = 1/e_0$ for all $t \in \pi(C)$ is clear.

$k = 1$: Since $y_0 = x$ and $\sigma_0 = 1$ by above we obtain according to Definition 4.1 and Definition 5.1 that $y_1 = \log(x)$ and that $d_t \leq 1 = 1/e_1$ for all $t \in \pi(C)$. This gives $\sigma_1 = -1$.

$k = 2$: Since $\sigma_1 = -1$ we have $y_1 < 0$. According to Definition 4.1 and 5.1 we get that $y_2 = \log(-y_1) = \log(-\log(x))$ and therefore that $d_t \leq 1/\exp(1) = 1/e_2$ and $\sigma_2 = 1$.

$k \rightarrow k + 1$: We can assume $k \geq 2$. By the inductive hypothesis we have $y_k = \log_{k-1}(-\log(x)) > 0$ and $\sigma_k = 1$. According to Definition 5.1 we obtain that $y_{k+1} = \log_k(-\log(x))$ and that $d_t \leq 1/\exp(e_k) = 1/e_{k+1}$ for all $t \in \pi(C)$. This gives also $\sigma_{k+1} = 1$. ■

5.7 Definition

We call C **r -simple** if it is simple and r -admissible.

5.8 Proposition

Let C be r -simple. Then C is r -elementary and r -unique, i.e. there is a unique logarithmic scale \mathcal{Y} on C whose center vanishes.

Proof

Let $\mathcal{Y} = (y_0, \dots, y_r)$ be an r -logarithmic scale on C . We show that \mathcal{Y} is elementary and are done by Remark 5.2. Let $\Theta = (\Theta_0, \dots, \Theta_r)$ be the center of \mathcal{Y} . We show by induction on $k \in \{0, \dots, r\}$ that $\Theta_0 = \dots = \Theta_k = 0$.

$k = 0$: Assume that $\Theta_0 \neq 0$. Then by Definition 4.1 there is $\epsilon_0 \in]0, 1[$ such that

$$|x - \Theta_0(t)| < \epsilon_0 |x|$$

for all $(t, x) \in C$. Let $t \in \pi(C)$ be such that $\Theta_0(t) \neq 0$. Then we obtain

$$+\infty = \lim_{x \searrow 0} \left| 1 - \frac{\Theta_0(t)}{x} \right| \leq \epsilon_0$$

which is a contradiction.

$k = 1$: Assume that $\Theta_1 \neq 0$. By the case $k = 0$ and Proposition 5.6(2) we have $y_0 = x$. According to Definition 4.1 there is $\epsilon_1 \in]0, 1[$ such that

$$|\log(x) - \Theta_1(t)| < \epsilon_1 |\log(x)|$$

for all $(t, x) \in C$. Therefore

$$1 = \lim_{x \searrow 0} \left| 1 - \frac{\Theta_1(t)}{\log(x)} \right| \leq \epsilon_1$$

for $t \in \pi(C)$ which is a contradiction.

$k = 2$: Assume that $\Theta_2 \neq 0$. By the case $k = 1$ and Proposition 5.6(2) we have $y_1 = \log(x)$. Note that $y_1 < 0$. According to Definition 4.1 there is $\epsilon_2 \in]0, 1[$ such that

$$|\log(-y_1) - \Theta_2(t)| < \epsilon_2 |\log(-y_1)|$$

for all $(t, x) \in C$. Therefore

$$1 = \lim_{x \searrow 0} \left| 1 - \frac{\Theta_2(t)}{\log(-y_1)} \right| \leq \epsilon_2$$

for $t \in \pi(C)$ which is a contradiction.

$k \rightarrow k + 1$: We may assume that $k \geq 2$. Assume that $\Theta_{k+1} \neq 0$. By the inductive hypothesis and Proposition 5.6(2) we have $y_k = \log_{k-1}(-\log(x))$. Note that $y_k > 0$. According to Definition 4.1 there is $\epsilon_{k+1} \in]0, 1[$ such that

$$|\log(y_k) - \Theta_{k+1}(t)| < \epsilon_{k+1} |\log(y_k)|$$

for all $(t, x) \in C$. Therefore

$$1 = \lim_{x \searrow 0} \left| 1 - \frac{\Theta_{k+1}(t)}{\log(y_k)} \right| \leq \epsilon_{k+1}$$

for $t \in \pi(C)$ which is a contradiction. ■

5.9 Corollary

Let C be simple. Then the following are equivalent:

- (1) C is r -simple.
- (2) $\sup C_t \leq 1/e_r$ for every $t \in \pi(C)$.

Proof

(1) \Rightarrow (2): If C is r -simple then C is r -elementary by Proposition 5.8. By Proposition 5.6(1) we obtain (2).

(2) \Rightarrow (1): Let $y_0 = x$, $y_1 = \log(x)$, $y_j = \log_{j-1}(-\log(x))$ for $j \in \{2, \dots, r\}$. Then it is easy to see that \mathcal{Y} is a well-defined elementary r -logarithmic scale on C . ■

5.10 Definition

Let $q = (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$ with $q \neq 0$. We set $j(q) := \min\{j \mid q_j \neq 0\}$ and $\sigma(q) := \text{sign}(q_{j(q)}) \in \{\pm 1\}$. Moreover, let

$$q_{\text{diff}} := (q_0 - 1, \dots, q_{j(q)} - 1, q_{j(q)+1}, \dots, q_r).$$

5.11 Remark

Let C be r -simple. Let $q := (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$ with $q \neq 0$. Then

$$\lim_{x \searrow 0} |\mathcal{Y}_{r,C}^{\text{el}}(x)|^{\otimes q} = \begin{cases} 0, & j(q) = 0, \sigma(q) = +1, \\ +\infty, & j(q) = 0, \sigma(q) = -1, \\ +\infty, & j(q) > 0, \sigma(q) = +1, \\ 0, & j(q) > 0, \sigma(q) = -1. \end{cases}$$

Proof

Note that for $(t, x) \in C$

$$|\mathcal{Y}_{r,C}^{\text{el}}(x)|^{\otimes q} = |x|^{q_0} \cdot |\log(x)|^{q_1} \cdot |\log(-\log(x))|^{q_2} \cdot \dots \cdot |\log_{r-1}(-\log(x))|^{q_r}.$$

So the assertion follows from the growth properties of the iterated logarithm. ■

5.12 Proposition

Let C be r -simple. Let $q \in \mathbb{Q}^{r+1}$ with $q \neq 0$. Then

$$\lim_{x \searrow 0} \left| \frac{\frac{d}{dx} |\mathcal{Y}_{r,C}^{\text{el}}(x)|^{\otimes q}}{|\mathcal{Y}_{r,C}^{\text{el}}(x)|^{\otimes q_{\text{diff}}}} \right| = q_j(q).$$

Proof

Let $\mathcal{Y}_{r,C}^{\text{el}} = (y_0, \dots, y_r)$. We get by Proposition 5.6(2) that $d|y_0|/dx = 1$ and that for $j \in \{1, \dots, r\}$

$$\frac{d|y_j|}{dx} = -\frac{1}{\prod_{j=0}^{k-1} |y_j|}.$$

This gives

$$\begin{aligned} \frac{d}{dx} \prod_{j=0}^r |y_j|^{q_j} &= q_0 |y_0|^{q_0-1} \cdot |y_1|^{q_1} \cdot \dots \cdot |y_r|^{q_r} - \sum_{j=1}^r q_j |y_j|^{q_j-1} \frac{1}{\prod_{i=0}^{j-1} |y_i|} \prod_{i \neq j} |y_i|^{q_i} \\ &= q_0 |y_0|^{q_0-1} \prod_{i>0} |y_i|^{q_i} - \sum_{j=1}^r q_j \prod_{i \leq j} |y_i|^{q_i-1} \prod_{i>j} |y_i|^{q_i}. \end{aligned}$$

We obtain the assertion by the growth properties of the iterated logarithms. ■

5.13 Proposition

Let C be simple and let $g : \pi(C) \rightarrow \mathbb{R}$ be C -nice. Then g is log-analytic.

Proof

Let E be a set of C -heirs such that g can be constructed from E . Let $h \in E$. There is $r \in \mathbb{N}_0$, an r -logarithmic scale \mathcal{Y} with center $(\Theta_0, \dots, \Theta_r)$ on C and $l \in \{1, \dots, r\}$ such that $h = \exp(\Theta_l)$. Note that C is r -simple. With Proposition 5.8 we have $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$. Therefore $h = 1$. So we obtain $E = \emptyset$ or $E = \{1\}$. With the proof of the claim in Example 4.41 one sees that g is log-analytic. ■

5.14 Corollary

Let C be simple and let $g : C \rightarrow \mathbb{R}$ be nicely r -log-analytically prepared in x . Then g is purely r -log-analytically prepared in x .

Proof

Since C is simple and since g is r -log-analytically prepared in x we have that C is r -simple. Let

$$\mathcal{J} := (r, \mathcal{Y}, a, q, s, v, b, P)$$

be a nice LA-preparing tuple for g . We have $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$ by Proposition 5.8. Note that a and b are C -nice. With Proposition 5.13 we get that the coefficient a and the base functions b_1, \dots, b_s are log-analytic. Since the center of $\mathcal{Y}_{r,C}^{\text{el}}$ vanishes we are done. ■

5.15 Proposition

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable and let $g : X \rightarrow \mathbb{R}$ be nicely log-analytic in x of order at most r . Then there is a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the cell C is r -simple and $g|_C$ is purely r -log-analytically prepared in x .

Proof

By Proposition 4.64 there is a definable cell decomposition \mathcal{C} of X such that $g|_C$ is nicely r -log-analytically prepared in x for every $C \in \mathcal{C}$. Fix a simple $C \in \mathcal{C}$. Then C is r -simple by Proposition 5.6(1). With Corollary 5.14 applied to $g|_C$ we are done. ■

So we see that log-analytic functions can be prepared on simple sets (see Definition 5.4 for the notion of a simple set) with log-analytic data only. (See [22], Proposition 2.29 and Theorem 2.30 for another, but shorter proof than Proposition 4.64. This proof is formulated explicitly for log-analytic functions on simple sets.)

5.2 A Preparation Theorem for Restricted Log-Exp-Analytic Functions on Simple Sets

For Section 5.2 we set the following: Let $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, (t, x) \mapsto t$. Let $l, m \in \mathbb{N}_0$ be with $n = l + m$. Let w range over \mathbb{R}^l and u over \mathbb{R}^m . Let $\pi_l : \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^l, (w, u) \mapsto w$. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable such that X_w is open in $\mathbb{R}^m \times \mathbb{R}$ for every $w \in \mathbb{R}^l$. Note that $\pi(X)_w$ is open in \mathbb{R}^m for $w \in \mathbb{R}^l$. Assume that 0 is an interior point of X_t for every $t \in \pi(X)$.

The goal for this section is to show that a restricted log-exp-analytic function $f : X \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, in (u, x) can be log-analytically prepared in x with restricted log-exp-analytic coefficient and base functions in u with reference set $\pi(X)$ on simple sets.

5.16 Proposition

Let $f : X \rightarrow \mathbb{R}$ be a restricted log-exp-analytic function in (u, x) . Then there are $r \in \mathbb{N}_0, e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the cell C is r -simple and $f|_C$ is purely $(m+1, X)$ -restricted (e, r) -prepared in x .

Proof

By Corollary 4.95 there are $r \in \mathbb{N}_0$, $e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every $C \in \mathcal{C}$ the function $f|_C$ is $(m+1, X)$ -restricted (e, r) -prepared in x . Fix such a simple $C \in \mathcal{C}$ and a set E of locally bounded functions in (u, x) with reference set X such that $f|_C$ is (e, r) -prepared in x with respect to E . Note that C is r -simple. We show by induction on $l \in \{-1, \dots, e\}$ that $g \in \log(E) \cup \{f\}$ is purely (l, r) -prepared in x with respect to E . For $l = -1$ there is nothing to show.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

be a preparing tuple for g . Note that a and b are C -nice and that $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$. Therefore $\Theta = 0$. Additionally we have that a and b are log-analytic by Proposition 5.13. Therefore g is purely (e, r) -prepared in x with respect to E by the inductive hypothesis and we are done. \blacksquare

5.17 Proposition

Let $f : X \rightarrow \mathbb{R}$ be restricted log-exp-analytic in (u, x) . Then there is $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the following holds. The cell C is r -simple and $f|_C$ is r -log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$.

Proof

By Proposition 5.16 there are $r \in \mathbb{N}_0$, $e \in \mathbb{N}_0 \cup \{-1\}$ and a definable cell decomposition \mathcal{Q} of X such that for every simple $Q \in \mathcal{Q}$ the cell Q is r -simple and we have that $f|_Q$ is purely $(m+1, X)$ -restricted (e, r) -prepared in x . Fix such a simple $Q \in \mathcal{Q}$. Let $d_y := \sup(Q_y)$ for $y \in \pi(Q)$. Let E be a set of positive definable functions on Q such that every $g \in \log(E)$ is locally bounded in (u, x) with reference set X and $f|_Q$ is purely (e, r) -prepared in x with respect to E . We need the following Claim.

Claim

Let $h \in \log(E)$ be r -log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$. Then there is a definable simple set $D \subset Q$ with $\pi(D) = \pi(Q)$ such that $h = h_1 + h_2$ where

- (1) $h_1 : \pi(D) \rightarrow \mathbb{R}$ is a function such that $\exp(h_1) : \pi(D) \rightarrow \mathbb{R}$ is restricted log-exp-analytic in u with reference set $\pi(X)$ and
- (2) $h_2 : D \rightarrow \mathbb{R}$ is a bounded function such that $\exp(h_2)$ is r -log-analytically prepared in x with coefficient 1 and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$.

Proof of the claim

Let

$$(r, \mathcal{Y}, a, q, s, v, b, P)$$

be a corresponding LA-preparing tuple for h where $b := (b_1, \dots, b_s)$ and $P := (p_1, \dots, p_s)^t$. Note that $\mathcal{Y} = \mathcal{Y}_{r,Q}^{\text{el}}$. We have

$$h(t, x) = a(t)|\mathcal{Y}(x)|^{\otimes q} v(b_1(t)|\mathcal{Y}(x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(x)|^{\otimes p_s})$$

for every $(t, x) \in Q$. We may assume that $a \neq 0$. Note that $q_0 > 0$ or if $q_0 = 0$ we see $q_{j(q)} \leq 0$. (Otherwise we have $\lim_{x \searrow 0} |h(w, u, x)| = \infty$ for every $(w, u) \in \pi(Q)$ by Remark 5.11, but for every $(w, u) \in \pi(Q)$ we have $(u, 0) \in X_w$ and therefore there is an open neighbourhood U of $(u, 0)$ in X_w such that $h_w|_{U \cap Q_w}$ is bounded.)

We also have $p_{i0} > 0$ or if $p_{i0} = 0$ then $p_{ij(p_i)} \leq 0$ for every $i \in \{1, \dots, s\}$ by Remark 5.11. (*)

Case 1: $q \neq 0$.

For $t \in \pi(Q)$ let

$$\hat{d}_t := \sup\{x \in]0, d_t/2[\mid |a(t)||\mathcal{Y}(c)|^{\otimes q} \leq 1 \text{ for every } c \in]0, x[\}$$

and $D := \{(t, x) \in \pi(Q) \times \mathbb{R} \mid x \in]0, \hat{d}_t[\}$. Note that D is definable and that $\pi(Q) = \pi(D)$. Set $h_1 := 0$ and $h_2 := f|_D$. Note that h_2 is bounded. By the construction of D , using the exponential series and composition of power series we see that $\exp(h_2)$ is r -log-analytically prepared in x as desired.

Case 2: $q = 0$.

Subclaim 1

The coefficient a is locally bounded in u with reference set $\pi(X)$.

Proof of Subclaim 1

Let $\hat{\pi} : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^m, (u, x) \mapsto u$. Let $(w, u) \in \pi(X)$. Note that $(w, u, 0) \in X$. Take an open ball U around $(u, 0)$ in X_w such that either $U \cap Q_w = \emptyset$ or $h_w|_{U \cap Q_w}$ is bounded. With Definition 4.19 we see that either $U \cap Q_w = \emptyset$ or $a_w|_{U \cap Q_w}$ is bounded. Let $B := \pi(Q)$. Note that $\hat{\pi}(Q_w) = B_w$.

If the former holds then $\hat{\pi}(U) \cap B_w = \hat{\pi}(U) \cap \hat{\pi}(Q_w) = \emptyset$. (Let $u \in \hat{\pi}(U) \cap \hat{\pi}(Q_w)$. Since U is an open ball in X_w there is $\epsilon > 0$ such that $]0, \epsilon[\subset U_u$. So $]0, \min\{\epsilon, d_{(w,u)}\}[\subset U_u \cap (Q_w)_u$ and therefore $U \cap Q_w \neq \emptyset$. This shows also that $\hat{\pi}(U \cap Q_w) = \hat{\pi}(U) \cap \hat{\pi}(Q_w) = \hat{\pi}(U) \cap B_w$.)

If the latter holds then $a_w|_{\hat{\pi}(U \cap Q_w)}$ is bounded since a depends only on (w, u) . Since $\hat{\pi}(U \cap Q_w) = \hat{\pi}(U) \cap B_w$ we see that $a_w|_{\hat{\pi}(U) \cap B_w}$ is bounded. So take $\hat{\pi}(U) \subset \pi(X)_w$ which is an open neighbourhood of u in $\pi(X)_w$. ■ Subclaim 1

Let $\sum_{\alpha \in \mathbb{N}_0^s} c_\alpha X^\alpha$ be the power series expansion of v . Let

$$\Gamma_1 := \{\alpha \in \mathbb{N}_0^s \mid {}^t P\alpha = 0\}$$

and $\Gamma_2 := \mathbb{N}_0^s \setminus \Gamma_1$. Set $v_1 := \sum_{\alpha \in \Gamma_1} c_\alpha X^\alpha$ and $v_2 := \sum_{\alpha \in \Gamma_2} c_\alpha X^\alpha$. For $l \in \{1, 2\}$ let

$$g_l : Q \rightarrow \mathbb{R}, (t, x) \mapsto a(t)v_l(b_1(t)|\mathcal{Y}(x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(x)|^{\otimes p_s}).$$

Let

$$\mathcal{S} := \{j \in \{1, \dots, s\} \mid p_j \neq 0\}.$$

Subclaim 2

There is a restricted log-exp-analytic function $\Psi : \pi(Q) \rightarrow \mathbb{R}, (w, u) \mapsto \Psi(w, u)$, in u with reference set $\pi(X)$ which is locally bounded in u with reference set $\pi(X)$ such that $g_1(w, u, x) = \Psi(w, u)$ for every $(w, u, x) \in Q$.

Proof of Subclaim 2

For $j \in \mathcal{S}$ we have that $\alpha_j = 0$ for every $\alpha \in \Gamma_1$ by definition of Γ_1 . Let $\{1, \dots, s\} \setminus \mathcal{S} = \{l_1, \dots, l_\lambda\}$ where $\lambda \in \{0, \dots, s\}$ and $l_1, \dots, l_\lambda \in \{0, \dots, s\}$ (if $\lambda = 0$ then $\mathcal{S} = \{1, \dots, s\}$ and if $\lambda = s$ then $\mathcal{S} = \emptyset$). Note that there is a power series \hat{v} which converges absolutely on an open neighbourhood of $[-1, 1]^\lambda$ such that

$$v_1(b_1(t)|\mathcal{Y}(x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(x)|^{\otimes p_s}) = \hat{v}(b_{l_1}(t), \dots, b_{l_\lambda}(t))$$

for $(t, x) \in Q$. Choose

$$\Psi : \pi(Q) \rightarrow \mathbb{R}, t \mapsto a(t)\hat{v}(b_{l_1}(t), \dots, b_{l_\lambda}(t)).$$

By Remark 3.32 Ψ is restricted log-exp-analytic in u with reference set $\pi(X)$ (since \hat{v} defines a globally subanalytic function on $[-1, 1]^\lambda$). Note that Ψ has the desired properties since a is locally bounded in u with reference set $\pi(X)$.

■ Subclaim 2

Let Ψ be as in Subclaim 2. If $\mathcal{S} = \emptyset$ then $g_2 = 0$ and we are done by taking $D = Q$, $h_1 := \Psi$ and $h_2 := 0$. So assume $\mathcal{S} \neq \emptyset$. Since v is a power series which converges absolutely on an open neighbourhood of $[-1, 1]^s$ there is $L \in \mathbb{R}_{>0}$ such that $\sum_{\alpha \in \mathbb{N}_0^s} |c_\alpha| < L$. Fix such an L . For $t \in \pi(Q)$ and $j \in \mathcal{S}$ let

$$\hat{d}_{j,t} := \sup\{x \in]0, d_t/2[\mid |\mathcal{Y}(c)|^{\otimes p_j} < \frac{1}{L|a(t)b_j(t)} \text{ for every } c \in]0, x[\}.$$

For $t \in \pi(Q)$ set $\hat{d}_t := \min\{\hat{d}_{j,t} \mid j \in \mathcal{S}\}$. Consider

$$D := \{(t, x) \in Q \mid x \in]0, \hat{d}_t[\}.$$

Note that D is a simple definable set with $\pi(Q) = \pi(D)$.

Subclaim 3

Note that $|g_2(t, x)| \leq 1$ for every $(t, x) \in D$ and $\exp(g_2|_D)$ is r -log-analytically prepared in x with coefficient 1 and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$.

Proof of Subclaim 3

For $j \in \{1, \dots, s\}$ and $(t, x) \in D$ let $\phi_j(t, x) := b_j(t)|\mathcal{Y}(x)|^{\otimes p_j}$. For every $\alpha := (\alpha_1, \dots, \alpha_s) \in \Gamma_2$ fix $i_\alpha \in \{1, \dots, s\}$ with $i_\alpha \in \mathcal{S}$ and $\alpha_{i_\alpha} \neq 0$. (By definition of Γ_2 such an i_α exists.) We have

$$|a(t)||\phi_{i_\alpha}(t, x)| \leq 1/L$$

for $(t, x) \in D$ and $\alpha \in \Gamma_2$. For $(t, x) \in D$ we obtain

$$\begin{aligned} g_2(t, x) &= a(t)V_2(\phi_1(t, x), \dots, \phi_s(t, x)) \\ &= a(t) \sum_{\alpha \in \Gamma_2} c_\alpha \phi_{i_\alpha}(t, x)^{\alpha_{i_\alpha}} \prod_{j \neq i_\alpha} \phi_j(t, x)^{\alpha_j} \\ &= \sum_{\alpha \in \Gamma_2} c_\alpha a(t) \phi_{i_\alpha}(t, x) \phi_{i_\alpha}(t, x)^{\alpha_{i_\alpha} - 1} \prod_{j \neq i_\alpha} \phi_j(t, x)^{\alpha_j} \end{aligned}$$

and therefore $|g_2(t, x)| \leq 1/L \sum_{\alpha \in \Gamma_2} |c_\alpha| = 1$.

Let $\mathcal{S} := \{j_1, \dots, j_k\}$ where $k \in \mathbb{N}$. Note that $i_\alpha \in \{j_1, \dots, j_k\}$ is unique for $\alpha \in \mathbb{N}_0^s$. Let $(z_{j_1}, \dots, z_{j_k})$ be a new tuple of variables. Consider for $l \in \{1, \dots, k\}$

$$\tilde{\phi}_l : D \rightarrow [-1, 1], (t, x) \mapsto a(t)b_{j_l}(t)|\mathcal{Y}(x)|^{\otimes p_{j_l}}.$$

Consider

$$\hat{v} : [-1, 1]^{s+k} \rightarrow \mathbb{R}, (x_1, \dots, x_s, z_{j_1}, \dots, z_{j_k}) \mapsto \sum_{\alpha \in \Gamma_2} c_\alpha z_{i_\alpha} x_{i_\alpha}^{\alpha_{i_\alpha} - 1} \prod_{j \neq i_\alpha} x_j^{\alpha_j}.$$

Note that \hat{v} is a well-defined globally subanalytic function since \hat{v} defines a power series which converges absolutely on an open neighbourhood of $[-1, 1]^{s+k}$. We have for $(t, x) \in D$

$$g_2(t, x) = \hat{v}(\phi_1(t, x), \dots, \phi_s(t, x), \tilde{\phi}_1(t, x), \dots, \tilde{\phi}_k(t, x))$$

and therefore

$$\exp(g_2(t, x)) = \exp^*(\hat{v}(\phi_1(t, x), \dots, \phi_s(t, x), \tilde{\phi}_1(t, x), \dots, \tilde{\phi}_k(t, x)))$$

where

$$\exp^* : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} \exp(x), & x \in [-L, L], \\ 0, & \text{else.} \end{cases}$$

is globally subanalytic. By using the exponential series and composition of power series we see that $\exp(g_2)$ has the desired properties (since a and b_j are restricted log-exp-analytic in u with reference set $\pi(X)$). ■Subclaim 3

So take $h_1 := \Psi$ where Ψ is as in Subclaim 2 and $h_2 := g_2|_D$. ■Claim

Let $h \in \log(E) \cup \{f\}$ be purely (l, r) -prepared in x with respect to E where $l \in \{-1, \dots, e\}$. We show by induction on l that there is a simple definable $A \subset Q$ with $\pi(A) = \pi(Q)$ such that $h|_A$ is r -log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$. For $l = -1$ it is clear by choosing $A := Q$.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, e^d, q, s, v, b, e^c, P)$$

be a purely preparing tuple for h where $b := (b_1, \dots, b_s)$, $e^c := (e^{c_1}, \dots, e^{c_s})$, and $P := (p_1, \dots, p_s)^t$. Note that $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$, that a, b_1, \dots, b_s are log-analytic and that d, c_1, \dots, c_s are purely $(l - 1, e)$ -prepared in x with respect to E . We have

$$h(t, x) = a(t)|\mathcal{Y}(x)|^{\otimes q} e^{d(t,x)} v(b_1(t)|\mathcal{Y}(x)|^{\otimes p_1} e^{c_1(t,x)}, \dots, b_s(t)|\mathcal{Y}(x)|^{\otimes p_s} e^{c_s(t,x)})$$

for every $(t, x) \in Q$. By the inductive hypothesis and the Claim we find a simple definable set $A \subset Q$ with $\pi(A) = \pi(Q)$ and functions $d_1, c_{11}, \dots, c_{1s} : \pi(A) \rightarrow \mathbb{R}$ and $d_2, c_{21}, \dots, c_{2s} : A \rightarrow \mathbb{R}$ with the following properties:

- (1) The functions $\exp(d_1)$ and $\exp(c_{11}), \dots, \exp(c_{1s})$ are restricted log-exp-analytic in u with reference set $\pi(X)$,
- (2) the functions $\exp(d_2), \exp(c_{21}), \dots, \exp(c_{2s})$ are r -log-analytically prepared in x with coefficient 1 and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$,
- (3) we have $d|_A = d_1 + d_2$ and $c_j|_A = c_{1j} + c_{2j}$ for $j \in \{1, \dots, s\}$.

Since a and b_1, \dots, b_s are log-analytic we see that the functions

$$\hat{a} : \pi(A) \rightarrow \mathbb{R}, (w, u) \mapsto a(w, u) \exp(d_1(w, u)),$$

and

$$\hat{b}_j : \pi(A) \rightarrow \mathbb{R}, (w, u) \mapsto b_j(w, u) \exp(c_{1j}(w, u)),$$

for $j \in \{1, \dots, s\}$ are restricted log-exp-analytic in u with reference set $\pi(X)$. For $(w, u, x) \in A$ we have

$$h(w, u, x) = \hat{a}(w, u)|\mathcal{Y}(x)|^{\otimes q} e^{d_2(w,u,x)} v(\hat{\phi}_1(w, u, x), \dots, \hat{\phi}_s(w, u, x))$$

where $\hat{\phi}_j(w, u, x) := \hat{b}_j(w, u)|\mathcal{Y}(x)|^{\otimes p_j} e^{c_{2j}(w,u,x)}$ for $(w, u, x) \in A$ and $j \in \{1, \dots, s\}$. By composition of power series we obtain the desired r -log-analytical preparation for h in x .

So we find a simple definable set $\hat{C} \subset Q$ with $\pi(\hat{C}) = \pi(Q)$ such that $f|_{\hat{C}}$ is r -log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$. With the cell decomposition theorem applied to every such \hat{C} (compare with Theorem 2.15) we are done with the proof of Proposition 5.17. \blacksquare

An immediate consequence from this observation is the following.

5.18 Proposition

Let $f : X \rightarrow \mathbb{R}$, $(w, u, x) \mapsto f(w, u, x)$, be restricted log-exp-analytic in (u, x) . Assume that $\lim_{x \searrow 0} f(t, x) \in \mathbb{R}$ for every $t \in \pi(X)$. Then

$$h : \pi(X) \rightarrow \mathbb{R}, (w, u) \mapsto \lim_{x \searrow 0} f(w, u, x),$$

is restricted log-exp-analytic in u .

Proof

By Proposition 5.17 there is $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the cell C is r -simple, and $f|_C$ is r -log-analytically prepared in x with coefficient and base functions which are restricted log-exp-analytic in u with reference set $\pi(X)$. Let $C \in \mathcal{C}$ be such a simple cell. Set $g := f|_C$ and let

$$(r, \mathcal{Y}, a, q, s, v, b, P)$$

be a corresponding LA-preparing tuple for g . Note that $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$. Then

$$g(w, u, x) = a(w, u)|\mathcal{Y}(x)|^{\otimes q} v(b_1(w, u)|\mathcal{Y}(x)|^{\otimes p_1}, \dots, b_s(w, u)|\mathcal{Y}(x)|^{\otimes p_s})$$

for $(w, u, x) \in C$. By the assumption, Remark 5.11 and Definition 4.19 we see that

$$A : \pi(C) \rightarrow \mathbb{R}, (w, u) \mapsto \lim_{x \searrow 0} a(w, u)|\mathcal{Y}(x)|^{\otimes q},$$

and, for $j \in \{1, \dots, s\}$, that

$$B_j : \pi(C) \rightarrow [-1, 1], (w, u) \mapsto \lim_{x \searrow 0} b_j(w, u)|\mathcal{Y}(x)|^{\otimes p_j},$$

are well-defined restricted log-exp-analytic functions in u with reference set $\pi(X)$. We obtain for $(w, u) \in \pi(C)$

$$h(w, u) = A(w, u)v(B_1(w, u), \dots, B_s(w, u)).$$

Hence $h|_{\pi(C)}$ is restricted log-exp-analytic in u with reference set $\pi(X)$ by Remark 3.32. So by Remark 3.33 we obtain that h is restricted log-exp-analytic in u . \blacksquare

This gives the fact that restricted log-exp-analytic functions are closed under taking derivatives.

5.19 Proposition

Let $y := (y_1, \dots, y_{m+1})$ range over \mathbb{R}^{m+1} . Let $f : X \rightarrow \mathbb{R}$ be restricted log-exp-analytic in y . Let $j \in \{1, \dots, m+1\}$ be such that f_w is differentiable with respect to y_j on X_w for every $w \in \pi_l(X)$. Then $\partial f / \partial y_j$ is restricted log-exp-analytic in y .

Proof

Let $\pi^+ : \mathbb{R}^l \times \mathbb{R}^{m+1} \times \mathbb{R} \rightarrow \mathbb{R}^l \times \mathbb{R}^{m+1}$, $(w, y, x) \mapsto (w, y)$. We may assume that f is differentiable with respect to the last variable y_{m+1} . We have to show that $\partial f / \partial y_{m+1}$ is restricted log-exp-analytic in y . Let $\mathbf{e}_{m+1} := (0, \dots, 0, 1) \in \mathbb{R}^{m+1}$ be the $(m+1)^{\text{th}}$ unit vector. We define

$$V := \{(w, y, x) \in X \times \mathbb{R} \mid (w, y + x\mathbf{e}_{m+1}) \in X\}.$$

Note that V_w is open and that 0 is an interior point of $(V_w)_y$ for every $(w, y) \in \pi^+(V)$. Let

$$F : V \rightarrow \mathbb{R}, (w, y, x) \mapsto \begin{cases} \frac{f(w, y + x\mathbf{e}_{m+1}) - f(w, y)}{x}, & x \neq 0, \\ 0, & \text{else.} \end{cases}$$

By Remark 3.35 we see that $G_1 : V \rightarrow \mathbb{R}$, $(w, y, x) \mapsto f(w, y + x\mathbf{e}_{m+1})$, and $G_2 : V \rightarrow \mathbb{R}$, $(w, y, x) \mapsto f(w, y)$, are restricted log-exp-analytic in (y, x) . So by Remark 3.32 we see that F is restricted log-exp-analytic in (y, x) . Since

$$\frac{\partial f}{\partial y_{m+1}}(w, y) = \lim_{x \searrow 0} F(w, y, x)$$

for $(w, y) \in X$ we obtain by Proposition 5.18 that $\frac{\partial f}{\partial y_{m+1}}$ is a restricted log-exp-analytic function in y with reference set $\pi^+(V) = X$. ■

5.3 Strong Quasianalyticity and Real Analyticity

The Univariate Case

We start our considerations with restricted log-exp-analytic functions in one variable and transfer our results into the multivariate case in the next paragraph.

For this paragraph we consider the following. Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable such that X_t is open and 0 is an interior point of X_t for every $t \in \pi(X)$. Let $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, (t, x) \mapsto t$.

5.20 Definition

Let $N \in \mathbb{N}$. Let $U \subset \mathbb{R}^n$ be open and let $g : U \rightarrow \mathbb{R}$ be a function. Let $a \in U$. The function g is called **N -flat** at a if g is C^N at a and all partial derivatives of g of order at most N vanish in a . The function g is called **flat** at a if g is C^∞ at a and all partial derivatives of g vanish in a .

The fact that a restricted log-exp-analytic function in x can be log-analytically prepared in x on simple definable sets implies the following.

5.21 Proposition

Let $f : X \rightarrow \mathbb{R}$ be restricted log-exp-analytic in x . Then there is $N \in \mathbb{N}$ such that the following holds for every $t \in \pi(X)$. If $f(t, -)$ is N -flat at $x = 0$ then $f(t, -)$ vanishes identically on an open interval around $0 \in \mathbb{R}$.

Proof

By also considering the function $f(t, -x)$ it is enough to show that the following holds: There is $N \in \mathbb{N}$ such that for every $t \in \pi(X)$ with $f(t, -)$ is N -flat at $x = 0$ we have $f(t, x) = 0$ for all $x \in]0, \varepsilon_t[$ for some $\varepsilon_t \in \mathbb{R}_{>0}$. By Proposition 5.17 we find $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the following holds: C is r -simple and $f|_C$ is r -log-analytically prepared in x . Fix a simple $C \in \mathcal{C}$ and let

$$(r, \mathcal{Y}, a, q, s, v, b, P)$$

be an LA-preparing tuple for $f|_C$ where $q := (q_0, \dots, q_r)$. Note that $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$ since the center of \mathcal{Y} vanishes by Proposition 5.8. Choose $N_C \in \mathbb{N}$ such that $N_C > q_0$. Let $t \in \pi(C)$. If $f(t, -)$ is N_C -flat at $x = 0$ then $f(t, -) = o(x^{N_C})$ at $x = 0$ by Taylor's theorem. But $|\mathcal{Y}(x)|^{\otimes q} / x^{N_C} \neq o(1)$ by Remark 5.11. Therefore we obtain $a(t) = 0$. By Remark 5.5 we are done by taking

$$N := \max\{N_C \mid C \in \mathcal{C} \text{ simple}\}.$$

■

5.22 Remark

Proposition 5.21 does not hold in $\mathbb{R}_{\text{an,exp}}$ in general. Consider

$$f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \begin{cases} e^{-\frac{1}{x}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then f is flat at 0, but not the zero function. Note also that f is C^∞ at 0, but not real analytic.

Another consequence from Proposition 5.17 is the following statement about real analyticity of restricted log-exp-analytic functions in x at $x = 0$.

5.23 Proposition

Let $f : X \rightarrow \mathbb{R}$ be a restricted log-exp-analytic in x . Then there is $N \in \mathbb{N}$ such that the following holds: If $f(t, -)$ is C^N at 0 then $f(t, -)$ is real analytic at 0.

Proof

By Proposition 5.17 we find $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C} of $X_{\neq 0}$ such that for every simple $C \in \mathcal{C}$ the cell C is r -simple and $f|_C$ is r -log-analytically prepared in x . Fix a simple cell $C \in \mathcal{C}$ and set $\eta_t := \sup C_t$ for $t \in \pi(C)$. Let

$$(r, \mathcal{Y}, a, q, s, v, b, P)$$

be an LA-preparing tuple for $g := f|_C$. Note that $\mathcal{Y} = \mathcal{Y}_{r,C}^{\text{el}}$. Let $\sum_{\alpha \in \mathbb{N}_0^s} c_\alpha X^\alpha$ be the power series expansion of v . Let

$$\Gamma_1 := \{ \alpha \in \mathbb{N}_0^s \mid {}^t P\alpha + q \in \mathbb{N}_0 \times \{0\}^r \}$$

and $\Gamma_2 := \mathbb{N}_0^s \setminus \Gamma_1$. Set $v_1 := \sum_{\alpha \in \Gamma_1} c_\alpha X^\alpha$ and $v_2 := \sum_{\alpha \in \Gamma_2} c_\alpha X^\alpha$. For $l \in \{1, 2\}$ let

$$g_l : C \rightarrow \mathbb{R}, (t, x) \mapsto a(t) |\mathcal{Y}(x)|^{\otimes q} v_l(b_1(t) |\mathcal{Y}(x)|^{\otimes p_1}, \dots, b_s(t) |\mathcal{Y}(x)|^{\otimes p_s}).$$

Then g_1, g_2 are log-analytic in x . We have $g = g_1 + g_2$. For $k \in \mathbb{N}_0$ let

$$\Gamma_{1,k} := \{ \alpha \in \mathbb{N}_0^s \mid {}^t P\alpha + q = (k, 0, \dots, 0) \} \subset \Gamma_1$$

and

$$d_k : \pi(C) \rightarrow \mathbb{R}, t \mapsto a(t) \sum_{\alpha \in \Gamma_{1,k}} c_\alpha \prod_{i=1}^s b_i(t)^{\alpha_i}.$$

Then

$$g_1(t, x) = \sum_{k=0}^{\infty} d_k(t) x^k$$

for $(t, x) \in C$. We see that for a fixed $t \in \pi(C)$ the series to the right has radius of convergence at least $\sup(C_t)$ and therefore this series converges absolutely on C . So g_1 extends to a well-defined extension

$$\hat{g}_1 : \hat{C} := \{(t, x) \in \mathbb{R}^{n+1} \mid t \in \pi(C), -\eta_t < x < \eta_t\} \rightarrow \mathbb{R},$$

$$(t, x) \mapsto \sum_{k=0}^{\infty} d_k(t)x^k,$$

such that $\hat{g}_1(t, -)$ is real analytic at 0 for every $t \in \pi(C)$. Note that \hat{C} is definable. By shrinking η_t for $t \in \pi(X)$ if necessary we may assume that $\hat{C} \subset X$.

Claim 1

The function \hat{g}_1 is restricted log-exp-analytic in x with reference set X .

Proof of Claim 1

Note that $\hat{g}_1|_{\hat{C} \cap (\mathbb{R}^n \times \{0\})} = d_0$ is restricted log-exp-analytic in x with reference set X (since d_0 depends only on t). Let $D := \hat{C} \cap (\mathbb{R}^n \times \mathbb{R}_{<0})$. We show that $\hat{g}_1|_D$ is restricted log-exp-analytic in x with reference set X and are done with Remark 3.33. Let $\Gamma_{1,e} := \bigcup_{k \text{ even}} \Gamma_{1,k}$ and $\Gamma_{1,o} := \bigcup_{k \text{ odd}} \Gamma_{1,k}$. Set $v_{1,e} := \sum_{\alpha \in \Gamma_{1,e}} c_\alpha X^\alpha$ and $v_{1,o} := \sum_{\alpha \in \Gamma_{1,o}} c_\alpha X^\alpha$. Then for $(t, x) \in \hat{C}$ with $x < 0$ we have

$$\hat{g}_1(t, x) = a(t)|\mathcal{Y}(-x)|^{\otimes q} \left(v_{1,e}(b_1(t)|\mathcal{Y}(-x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(-x)|^{\otimes p_s}) - v_{1,o}(b_1(t)|\mathcal{Y}(-x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(-x)|^{\otimes p_s}) \right)$$

which implies the desired assertion since \hat{g}_1 is log-analytic in x (so we are done with Remark 3.32 since \hat{g}_1 can be constructed from a set E of positive definable functions which depend only on t). ■ Claim 1

Set

$$\hat{g}_2 : \hat{C} \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x) - \hat{g}_1(t, x).$$

Then \hat{g}_2 is restricted log-exp-analytic in x with reference set X by Claim 1. Note that $\hat{g}_2|_C = g_2$. Let

$$\Lambda := \{ {}^t P\alpha + q \mid \alpha \in \Gamma_2 \}.$$

Then $\Lambda \subset \mathbb{Q}^{r+1} \setminus (\mathbb{N}_0 \times \{0\}^r)$. Fix $t^* \in \pi(C)$. For $\lambda \in \Lambda$, let

$$\Gamma_{2,\lambda} := \{ \alpha \in \mathbb{N}_0^{r+1} \mid {}^t P\alpha + q = \lambda \}$$

and

$$e_{t^*,\lambda} := a(t^*) \sum_{\alpha \in \Gamma_{2,\lambda}} c_\alpha \prod_{i=0}^s b_i(t^*)^{\alpha_i}.$$

Then

$$\hat{g}_2(t^*, -) = \sum_{\lambda \in \Lambda} e_{t^*, \lambda} |\mathcal{Y}|^{\otimes \lambda}$$

on $]0, \eta_t[$. Let

$$\Lambda_{t^*} := \{\lambda \in \Lambda \mid e_{t^*, \lambda} \neq 0\}.$$

If $\Lambda_{t^*} = \emptyset$ then $\hat{g}_2(t^*, -) = 0$ on $]0, \eta_t[$. If $\Lambda_{t^*} \neq \emptyset$ there is $\mu_{t^*} = (\mu_{t^*, 0}, \dots, \mu_{t^*, r}) \in \Lambda_{t^*}$ such that $|\mathcal{Y}|^{\otimes \lambda} = o(|\mathcal{Y}|^{\otimes \mu_{t^*}})$ for all $\lambda \in \Lambda_{t^*}$ with $\lambda \neq \mu_{t^*}$.

Claim 2

Let $M \in \mathbb{N}$ be such that $f(t^*, -)$ is C^M at 0. Then $\mu_{t^*, 0} \geq M$.

Proof of Claim 2

Assume that $\mu_{t^*, 0} < M$.

Case 1: $\mu_{t^*, 0} \in \mathbb{N}_0$. Then $m := \mu_{t^*, 0} + 1 \leq M$. Differentiating g_2 m -times with respect to x we see with Proposition 5.12 that there is $\beta = (-1, \beta_1, \dots, \beta_r) \in \mathbb{Q}^{r+1}$ such that

$$\lim_{x \searrow 0} \frac{\partial^m g_2 / \partial x^m(t^*, x)}{|\mathcal{Y}|^{\otimes \beta}} \in \mathbb{R}^*.$$

Since $\hat{g}_2(t^*, -) = f(t^*, -) - \hat{g}_1(t^*, -)$ is C^M at 0 we obtain that

$$\lim_{x \searrow 0} \frac{\partial^m \hat{g}_2}{\partial x^m}(t^*, x) = \frac{\partial^m \hat{g}_2}{\partial x^m}(t^*, 0) \in \mathbb{R}$$

which contradicts that $\hat{g}_2(t^*, -)$ extends $g_2(t^*, -)$.

Case 2: $\mu_{t^*, 0} \notin \mathbb{N}_0$. Then $m := \lceil \mu_{t^*, 0} \rceil \leq M$. Differentiating g_2 m -times with respect to x we see with Proposition 5.12 (note that $\mu_{t^*} \neq 0$) that there is $\beta = (\beta_0, \beta_1, \dots, \beta_r) \in \mathbb{Q}^{r+1}$ with $\beta_0 < 0$ such that

$$\lim_{x \searrow 0} \frac{\partial^m g_2 / \partial x^m(t^*, x)}{|\mathcal{Y}|^{\otimes \beta}} \in \mathbb{R}^*.$$

But $\hat{g}_2(t^*, -) = f(t^*, -) - \hat{g}_1(t^*, -)$ is C^M at 0. We get the same contradiction as in Case 1. ■ Claim 2

Claim 3

Let $M \in \mathbb{N}$ be such that $f(t^*, -)$ is C^M at 0. Then $\hat{g}_2(t^*, -)$ is $(M - 1)$ -flat at 0.

Proof of Claim 3

Case 1: $\Lambda_{t^*} = \emptyset$. Then $\hat{g}_2(t^*, -) = 0$ on $]0, \eta_t[$ and we are clearly done.

Case 2: $\Lambda_{t^*} \neq \emptyset$. By Claim 1 we obtain that $\mu_{t^*, 0} \geq M$. Hence we obtain by Proposition 5.12 and Remark 5.11 that

$$\lim_{x \searrow 0} \frac{\partial^m g_2}{\partial x^m}(t^*, x) = 0$$

for all $m \in \{0, \dots, M-1\}$. Since $\hat{g}_2(t^*, -) = f(t^*, -) - \hat{g}_1(t^*, -)$ is C^M at 0 and since $\hat{g}_2(t^*, -)$ extends $g_2(t^*, -)$ we are done. ■ Claim 3

Since the function \hat{g}_2 is restricted log-exp-analytic in x with reference set X we find by Proposition 5.21 some $K_C \in \mathbb{N}$ such that the following holds for every $t \in \pi(C)$: If $\hat{g}_2(t, -)$ is K_C -flat at $x = 0$ then $\hat{g}_2(t, -)$ vanishes identically on some open interval around 0. Set $N_C := K_C + 1$. Assume that $f(t, -)$ is C^{N_C} at 0. Then by Claim 3 $\hat{g}_2(t, -)$ is K_C -flat and hence by the above that $f(t, -) = \hat{g}_1(t, -)$ on some open interval around 0. Since $\hat{g}_1(t, -)$ is real analytic at 0 we get that $f(t, -)$ is real analytic at 0. By Remark 5.5 we are done with the proof of Proposition 5.23 by taking

$$N := \max\{N_C \mid C \in \mathcal{C} \text{ simple}\}.$$

■

5.24 Corollary

Let $f : X \rightarrow \mathbb{R}$ be restricted log-exp-analytic in x . Assume that $f(t, -)$ is real analytic at 0 for every $t \in \pi(X)$. Then there is a definable cell decomposition \mathcal{B} of $\pi(X)$ such that $B \rightarrow \mathbb{R}, t \mapsto d^k/dx^k f(t, 0)$, is real analytic for every $B \in \mathcal{B}$ and all $k \in \mathbb{N}_0$.

Proof

Using the notation of the previous proof we have $f(t, x) = \hat{g}_1(t, x)$ for all $(t, x) \in C$ where C is a simple cell of the constructed definable cell decomposition \mathcal{C} of $X_{\neq 0}$. Since functions definable in $\mathbb{R}_{\text{an,exp}}$ are piecewise real analytic we find a definable cell decomposition \mathcal{D} of $\pi(C) \subset \mathbb{R}^n$ such that the coefficient a and the base functions b_1, \dots, b_s are real analytic on every $D \in \mathcal{D}$. Hence on each $D \in \mathcal{D}$ the coefficients d_k of \hat{g}_1 are real analytic for every $k \in \mathbb{N}_0$. Since $d^k/dx^k f(t, 0) = k!d_k(t)$ we are done by Remark 5.5. ■

The Multivariate Case

For this paragraph we set the following. Let $m \in \mathbb{N}$. Let $u := (u_1, \dots, u_m)$ and $v := (v_1, \dots, v_m)$ range over \mathbb{R}^m and x over \mathbb{R} . Let $u' := (u_1, \dots, u_{m-1})$. Let $\pi_n : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $(t, u) \mapsto t$ and $\pi : \mathbb{R}^n \times \mathbb{R}^{m-1} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^{m-1}$, $(t, u', u_m) \mapsto (t, u')$. Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable such that X_t is open for every $t \in \mathbb{R}^n$.

With Proposition 5.21 and familiar connectivity arguments we can easily prove the following result which gives strong quarianalyticity of a restricted log-exp-analytic function.

5.25 Proposition

Assume that X_t is open and connected for every $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}$, $(t, u) \mapsto f(t, u)$, be restricted log-exp-analytic in u . Then there is $N \in \mathbb{N}$ with the following property: Let $t \in \pi_n(X)$. If $f(t, -)$ is C^N and if there is $a \in X_t$ such that $f(t, -)$ is N -flat at a then $f(t, -)$ vanishes identically.

Proof

We set $U_\epsilon(a) := \{t \in \mathbb{R}^n \mid |t - a| < \epsilon\}$ where $a \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}_{>0}$. Let y range over \mathbb{R} . Let

$$\pi^* : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}, (t, u, v, y, x) \mapsto (t, u, v, y),$$

be the projection on the first $(n + 2m + 1)$ coordinates. Let

$$\tilde{V} := \{(t, u, v, y, x) \in X \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \mid (t, u + (x - y)v) \in X\},$$

and

$$G : \tilde{V} \rightarrow \mathbb{R}, (t, u, v, y, x) \mapsto f(t, u + (x - y)v).$$

By Remark 3.35 one sees that G is restricted log-exp-analytic in (u, v, y, x) . Furthermore let

$$V := \{(t, u, v, y, x) \in \tilde{V} \mid 0 \in \tilde{V}_{(t, u, v, y)}\}$$

and $F := G|_V$. Then F is restricted log-exp-analytic in (u, v, y, x) with reference set V . Since $V_{(t, u, v, y)}$ is open for $(t, u, v, y) \in X \times \mathbb{R}^m \times \mathbb{R}$ we see that F is also restricted log-exp-analytic in x . By Proposition 5.21 there is $N \in \mathbb{N}$ such that the following holds for every $(t, u, v, y) \in \pi^*(V)$: If $F(t, u, v, y, -)$ is N -flat at $x = 0$ then $F(t, u, v, y, -)$ vanishes identically on an open interval around $0 \in \mathbb{R}$. Let $t \in \pi_n(X)$ be such that $f(t, -)$ is C^N on X_t . Note that then $F(t, u, v, y, -)$ is C^N on $V_{(t, u, v, y)}$ for $(t, u, v, y) \in \pi^*(V)$. Let $a \in X_t$ be such that $f(t, -)$ is N -flat at a . We show that this implies that $f(t, -)$ vanishes identically and are done. We start with the following

Claim

There is $r \in \mathbb{R}_{>0}$ such that $f(t, -)$ vanishes identically on $U_r(a)$.

Proof of the claim

Let $r \in \mathbb{R}_{>0}$ be such that $U_{4r}(a) \subset X_t$. Then

$$W := U_r(a) \times U_1(a) \times]-r, r[\times]-r, r[\subset V_t.$$

So for $v \in U_1(a)$ we have that $F(t, a, v, 0, -)$ is N -flat at $x = 0$. Then by the above $F(t, a, v, 0, -)$ vanishes on some open interval around 0. Fix such a $v \in U_1(a)$ and let A_v be the set of all $y \in]-r, r[$ such that $F(t, a, v, y, -)$ vanishes identically on some open interval around 0. Then $A_v \neq \emptyset$ since $0 \in A_v$ by the above. Then A_v is open. Let $y \in \overline{A_v} \cap]-r, r[$. Then $F(t, a, v, y, -)$ is N -flat at $x = 0$. Hence by the above $F(t, a, v, y, -)$ vanishes identically on some open interval around 0. Therefore A_v is closed in $]-r, r[$. Since intervals are connected we obtain that $A_v =]-r, r[$ and hence that $F(t, a, v, -, -)$ vanishes identically on $]-r, r[\times]-r, r[$. Since $v \in U_1(a)$ is arbitrary we get that $f(t, -)$ vanishes identically on $U_r(a)$. ■ Claim

Let W_t be the set of all $z \in X_t$ such that f vanishes identically on some open ball around z . Then $W_t \neq \emptyset$ since $a \in W_t$ by the Claim. Clearly W_t is open. Let $\hat{u} \in \overline{W_t} \cap X_t$. Then $f(t, -)$ is N -flat at \hat{u} . Again by the Claim we get that $\hat{u} \in W_t$. Therefore W_t is closed in X_t . Since X_t is connected we obtain that $W_t = X_t$ and hence that $f(t, -)$ vanishes identically on X_t . This finishes the proof of Proposition 5.25. ■

With our results on restricted log-exp-analytic functions above we can establish a parametric version of Tamm's theorem for this class of functions. For this we adapt the reasoning of Van den Dries and Miller as in [14], Section 5 to our setting in the same way as in [22], Section 3.3.

5.26 Definition

Let $U \subset \mathbb{R}^n$ be open and let $g : U \rightarrow \mathbb{R}$ be a function. Let $k \in \mathbb{N}$. Then g is called **k -times Gateaux-differentiable** or G^k at a if $x \mapsto g(a + xu)$ is C^k at $x = 0$ for every $u \in \mathbb{R}^n$ and $(u \mapsto d^k g(a + xu)/dx^k)(0)$ is given by a homogeneous polynomial in u of degree k . The function g is called G^∞ at a if g is G^k at a for every $k \in \mathbb{N}$.

The following holds.

5.27 Fact (Van den Dries/Miller, [14], Section 2)

Let $U \subset \mathbb{R}^n$ be open and let $g : U \rightarrow \mathbb{R}$ be a function. Let $a \in U$. Then the following are equivalent:

- (i) The function g is real analytic at a .
- (ii) The function g is G^∞ at a and there is $\epsilon \in \mathbb{R}_{>0}$ such that for every $v \in \mathbb{R}^n$ with $|v| \leq 1$ the function $x \mapsto g(a + xv)$ is defined and real analytic on $] - \epsilon, \epsilon[$.

5.28 Proposition

Let $f : X \rightarrow \mathbb{R}, (t, u) \mapsto f(t, u)$, be a restricted log-exp-analytic function in u . Then there is $N \in \mathbb{N}$ such that the following holds for every $(t, u) \in X$: If $f(t, -)$ is G^N at u then $f(t, -)$ is G^∞ at u .

Proof

Let $v := (v_1, \dots, v_m)$. Let

$$V := \{(t, u, v, x) \in X \times \mathbb{R}^m \times \mathbb{R} \mid (t, u + xv) \in X\}.$$

Note that $V_t \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ is open and that 0 is an interior point of $V_{(t,u,v)}$ for every $(t, u, v) \in X \times \mathbb{R}^m$. By Remark 3.35 we have that

$$g : V \rightarrow \mathbb{R}, (t, u, v, x) \mapsto f(t, u + xv),$$

is restricted log-exp-analytic in (u, v, x) . By Proposition 5.23 there is $K \in \mathbb{N}$ such that the following holds for every $(t, u, v) \in X \times \mathbb{R}^m$: If $x \mapsto f(t, u + xv)$ is C^K at 0 then $x \mapsto f(t, u + xv)$ is real analytic at 0. For $k \in \mathbb{N}$ let W_k be the set of all $(t, u) \in X$ such that $x \mapsto f(t, u + xv)$ is C^k at 0 for every $v \in \mathbb{R}^m$. We define

$$\Phi_k : X \times \mathbb{R}^m \rightarrow \mathbb{R}, (t, u, v) \mapsto \begin{cases} \frac{d^k f(t, u + xv)}{dx^k}(0), & (t, u) \in W_k, \\ 1, & \text{if } (t, u) \notin W_k. \end{cases}$$

Then we have $W_k = W_K$ for all $k \geq K$. By Corollary 5.24 we find a definable cell decomposition \mathcal{D} of $X \times \mathbb{R}^m$ such that $\Phi_k|_D$ is real analytic for every $D \in \mathcal{D}$ and all $k \in \mathbb{N}$. Let $\pi_X : \mathbb{R}^{n+m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m}, (t, u, v) \mapsto (t, u)$. For a definable cell decomposition \mathcal{P} of $X \times \mathbb{R}^m$ we set

$$\pi_X(\mathcal{P}) := \{\pi_X(P) \mid P \in \mathcal{P}\}$$

which is a definable cell decomposition of X .

Claim 1

There is a definable cell decomposition \mathcal{C} of $X \times \mathbb{R}^m$ compatible with \mathcal{D} such that for every $B \in \pi_X(\mathcal{C})$ there is a non-empty open ball U in \mathbb{R}^m and $C^o \in \mathcal{C}$ with $\pi_X(C^o) = B$ such that for every $z \in B$ we have $U \subset (C^o)_z$.

Proof of Claim 1

Let z range over \mathbb{R}^{n+m} and let $\hat{\pi}_X : \mathbb{R}^{n+m} \times \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{n+m}, (z, u') \mapsto z$ and $\pi^* : \mathbb{R}^{n+m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n+m} \times \mathbb{R}^{m-1}, (z, u) \mapsto (z, u')$. We do an induction on m .

$m = 1$: Let $Q \in \pi_X(\mathcal{D})$. It is enough to find a definable cell decomposition $\hat{\mathcal{C}}$ of Q such that for every $\hat{C} \in \hat{\mathcal{C}}$ there is $A \in \mathcal{D}$ with $\hat{C} \subset \pi_X(A)$ and a non-empty open interval I in \mathbb{R} such that $I \subset A_z$ for every $z \in \hat{C}$. Of course this induces the desired cell decomposition \mathcal{C} of $Q \times \mathbb{R}$. Let

$$\mathcal{U}(\mathcal{D}_Q) := \{(z, u) \in Q \times \mathbb{R} \mid \text{there is } A \in \mathcal{D} \text{ such that } (z, u) \in A \text{ and } A_z \text{ is open}\}.$$

By uniform finiteness there are $k \in \mathbb{N}_0$ and definable continuous functions $\sigma_1, \dots, \sigma_k : Q \rightarrow \mathbb{R}$ such that

$$\mathcal{U}(\mathcal{D}_Q) = \{(z, u) \in Q \times \mathbb{R} \mid u \neq \sigma_j(z) \text{ for } j \in \{1, \dots, k\}\}.$$

Let $\epsilon > 0$. Pick $w_1, \dots, w_{k+1} \in \mathbb{R}$ such that $|w_i - w_j| > \epsilon$ for $i \neq j$. For $l \in \{1, \dots, k+1\}$ let $U_l :=]w_l - \epsilon, w_l + \epsilon[$ and let

$$Q_l := \{z \in Q \mid \sigma_j(z) \notin U_l \text{ for every } j \in \{1, \dots, k\}\}.$$

Note that Q_l is definable, that $U_l \subset \mathcal{U}(\mathcal{D}_Q)_z$ for every $z \in Q_l$ and that $Q = \bigcup_{l=1}^{k+1} Q_l$. For $l \in \{1, \dots, k+1\}$ there is $D^o \in \mathcal{D}$ with $\pi_X(D^o) = Q$ such that $U_l \subset (D^o)_z$ for every $z \in Q_l$ since U_l is connected. By choosing $\hat{\mathcal{C}}$ in this way that for every $\hat{C} \in \hat{\mathcal{C}}$ there is $l \in \{1, \dots, k+1\}$ such that $\hat{C} \subset Q_l$ we are done.

$m - 1 \rightarrow m$: Note that $\pi(\mathcal{D}) := \{\pi(D) \mid D \in \mathcal{D}\}$ is a definable cell decomposition of $X \times \mathbb{R}^{m-1}$. With the base case applied to every $\hat{D} \in \pi(\mathcal{D})$ we find a definable cell decomposition \mathcal{T} of $X \times \mathbb{R}^{m-1}$ compatible with $\pi(\mathcal{D})$ such that for every $T \in \mathcal{T}$ there is a non-empty open interval I_T in \mathbb{R} and $A \in \mathcal{D}$ with $T \subset \pi(A)$ such that $I_T \subset A_{(z, u')}$ for every $(z, u') \in T$. By the inductive hypothesis there is a definable cell decomposition \mathcal{K} of $X \times \mathbb{R}^{m-1}$ compatible with \mathcal{T} such that for every $B_K \in \hat{\pi}_X(\mathcal{K}) := \{\hat{\pi}_X(K) \mid K \in \mathcal{K}\}$ there is $K^o \in \mathcal{K}$ with $\hat{\pi}_X(K^o) = B_K$ and a non-empty open ball U_K in \mathbb{R}^{m-1} such that $U_K \subset (K^o)_z$ for all $z \in B_K$. For $D \in \mathcal{D}$ and $K \in \mathcal{K}$ with $K \subset \pi(D)$ let

$$C_{K,D} := \{(z, u', u_m) \in D \mid (z, u') \in K\}.$$

Fix $K \in \mathcal{K}$ with a corresponding K^o as above. For $T^* \in \mathcal{T}$ with $K^o \subset T^*$ fix a corresponding I_{T^*} and $A \in \mathcal{D}$ with $I_{T^*} \subset A_{(z, u')}$ for $(z, u') \in T^*$. Note

that $C_{K^o,A}$ is a definable cell with $\hat{\pi}_X(K) = \pi_X(C_{K,D}) = \pi_X(C_{K^o,A})$. Consider $U := U_K \times I_{T^*}$. Let $z \in \pi_X(C_{K,D})$. By construction we have $U \subset (C_{K^o,A})_z$. Note that

$$\mathcal{C} := \{C_{K,D} \mid K \in \mathcal{K}, D \in \mathcal{D} \text{ with } K \subset \pi(D)\}$$

is a definable cell decomposition of $X \times \mathbb{R}^m$ with the desired properties. \blacksquare _{Claim 1}

Let \mathcal{C} be as in Claim 1. Let $\mathcal{B} := \pi_X(\mathcal{C})$ and for $B \in \mathcal{B}$ let $\mathcal{D}_B := \{C \in \mathcal{C} \mid \pi_X(C) = B\}$. Since \mathcal{C} refines \mathcal{D} we have that $\Phi_k|_C$ is real analytic for every $k \in \mathbb{N}$ and $C \in \mathcal{C}$.

Claim 2

Let $k \in \mathbb{N}$. There is a definable function $w_k : X \times \mathbb{R}^m \rightarrow \mathbb{R}, (t, u, v) \mapsto w_k(t, u, v)$, such that $w_k|_C$ is real analytic for every $C \in \mathcal{C}$ and the following is equivalent for every $(t, u) \in X$.

- (i) The function $f(t, -)$ is G^k at u .
- (ii) $w_k(t, u, v) = 0$ for every $v \in \mathbb{R}^m$.

Proof of Claim 2

Let $\nu(k)$ be the dimension of the real vector space of homogeneous real polynomials of degree k in the variables $V := (V_1, \dots, V_m)$ and let $M_1(V), \dots, M_{\nu(k)}(V)$ be the homogeneous monomials of degree k in V . For $p_1, \dots, p_{\nu(k)} \in \mathbb{R}^m$ let

$$A(p_1, \dots, p_{\nu(k)}) := \begin{pmatrix} M_1(p_1) & \cdot & \cdot & M_{\nu(k)}(p_1) \\ \vdots & & & \vdots \\ M_1(p_{\nu(k)}) & \cdot & \cdot & M_{\nu(k)}(p_{\nu(k)}) \end{pmatrix} \in M(\nu(k) \times \nu(k), \mathbb{R}).$$

Note that for $p_1, \dots, p_{\nu(k)} \in \mathbb{R}^m$ and all $s := (s_1, \dots, s_{\nu(k)})$ the linear system of equations

$$\sum_{j=1}^{\nu(k)} \kappa_j M_j(p_l) = s_l$$

where $l \in \{1, \dots, \nu(k)\}$ has a unique solution $(\kappa_1, \dots, \kappa_{\nu(k)})$ if

$$\det(A(p_1, \dots, p_{\nu(k)})) \neq 0.$$

We have that

$$T := \{(p_1, \dots, p_{\nu(k)}) \in (\mathbb{R}^m)^{\nu(k)} \mid \det(A(p_1, \dots, p_{\nu(k)})) = 0\}$$

is an algebraic set. So $(\mathbb{R}^m)^{\nu(k)} \setminus T$ is Zariski open and therefore dense in $(\mathbb{R}^m)^{\nu(k)}$. For $B \in \mathcal{B}$ consider the following: Fix $C^o \in \mathcal{D}_B$ and a non-empty open ball

U_B such that $U_B \subset (C^o)_z$ for every $z \in B$. Note that $(U_B)^{\nu(k)} \not\subseteq T$. Therefore there are points $p_{k,1}, \dots, p_{k,\nu(k)} \in U_B$ such that for all $(x, u) \in B$ and all $s := (s_1, \dots, s_{\nu(k)})$

$$P_{k,B}(s, V) := \sum_{j=1}^{\nu(k)} a_j(s) M_j(V) \in \mathbb{R}[V]$$

is the unique homogeneous polynomial of degree k with $P_{k,B}(s, p_{k,i}) = s_i$ for all $i \in \{1, \dots, \nu(k)\}$. We have $p_{k,j} \in (C^o)_z$ for every $z \in B$. Set

$$\hat{w}_{k,B} : B \times \mathbb{R}^m \rightarrow \mathbb{R}, (t, u, v) \mapsto P_{k,B}(\Phi_k(t, u, p_{k,1}), \dots, \Phi_k(t, u, p_{k,\nu(k)}), v)$$

and

$$w_{k,B} := \hat{w}_{k,B} - \Phi_k|_{B \times \mathbb{R}^m}.$$

We have by the choice of $p_{k,1}, \dots, p_{k,\nu(k)}$ that $B \rightarrow \mathbb{R}, z \mapsto \Phi_k(z, p_{k,j})$, is real analytic for $j \in \{1, \dots, \nu(k)\}$ and therefore that $w_{k,B}|_C$ is real analytic for $C \in \mathcal{D}_B$.

Letting $w_k : X \rightarrow \mathbb{R}$ be the function defined as $w_k(z) = w_{k,B}(z)$ if $z \in B$. Note that w_k is well-defined, definable and that $w_k|_C$ is real analytic for every $C \in \mathcal{C}$. We show that w_k fulfills the remaining requirements for $k \in \mathbb{N}$.

$i) \Rightarrow ii)$: Let $(t, u) \in X$ be such that $f(t, -)$ is G^k at u . Then $(t, u) \in W_k$ and $v \mapsto \Phi_k(t, u, v)$ is a homogeneous polynomial of degree k . Let $B \in \mathcal{B}$ be with corresponding points $p_{k,1}, \dots, p_{k,\nu(k)} \in \mathbb{R}^m$ such that $(t, u) \in B$. By the definition of $\hat{w}_{k,B}$ we have $\hat{w}_{k,B}(t, u, p_{k,j}) = \Phi_k(t, u, p_{k,j})$ for all $j \in \{1, \dots, \nu(k)\}$. By the uniqueness of $P_{k,B}$ we obtain that

$$\hat{w}_{k,B}(t, u, v) = P_{k,B}(\Phi_k(t, u, p_{k,1}), \dots, \Phi_k(t, u, p_{k,\nu(k)}), v) = \Phi_k(t, u, v)$$

and therefore $w_{k,B}(t, u, v) = 0$ for all $v \in \mathbb{R}^m$.

$ii) \Rightarrow i)$: Let $(t, u) \in X$ be such that $w_k(t, u, v) = 0$ for all $v \in \mathbb{R}^m$. Let $B \in \mathcal{B}$ be such that $(t, u) \in B$. Then $\Phi_k(t, u, v) = \hat{w}_{k,B}(t, u, v)$ for all $v \in \mathbb{R}^m$ and therefore $v \mapsto \Phi_k(t, u, v)$ is a homogeneous polynomial of degree k . Since $k \geq 1$ it is not constant. Hence we get that $(x, u) \in W_k$ and consequently f is G^k at u . ■ Claim 2

Let w_k be as in Claim 2 for $k \in \mathbb{N}$. Since a real analytic function is quasianalytic and $\mathbb{R}_{\text{an,exp}}$ admits C^∞ cell decomposition we find for every $C \in \mathcal{C}$ some $N_C \in \mathbb{N}$ such that

$$\bigcap_{k \in \mathbb{N}} \{(t, u, v) \in C \mid w_k(t, u, v) = 0\} = \bigcap_{k \leq N_C} \{(t, u, v) \in C \mid w_k(t, u, v) = 0\}.$$

(Compare with the proof of Proposition 1.6 in [14], Section 1). Let $N := \max\{N_C \mid C \in \mathcal{C}\}$. Then

$$\bigcap_{k \in \mathbb{N}} \{(t, u, v) \in X \times \mathbb{R}^m \mid w_k(t, u, v) = 0\} =$$

$$= \bigcap_{k \leq N} \{(t, u, v) \in X \times \mathbb{R}^m \mid w_k(t, u, v) = 0\}.$$

Hence for every $(t, u) \in X$ we have that $f(t, -)$ is G^∞ at u if and only if $w_k(t, u, v) = 0$ for all $k \in \mathbb{N}$ and all $v \in \mathbb{R}^m$ if and only if $w_k(t, u, v) = 0$ for all $k \leq N$ and every $v \in \mathbb{R}^m$ if and only if $f(t, -)$ is G^N at u . This finishes the proof of Proposition 5.28. \blacksquare

Finally we obtain Theorem A.

5.29 Proposition

Let $f : X \rightarrow \mathbb{R}, (t, u) \mapsto f(t, u)$, be a restricted log-exp-analytic function in u . Then there is $N \in \mathbb{N}$ such that for all $t \in \mathbb{R}^n$ if $f(t, -)$ is C^N at u then $f(t, -)$ is real analytic at u .

Proof

By Proposition 5.28 there is $N_1 \in \mathbb{N}$ such that for every $(t, u) \in X$ if $f(t, -)$ is G^{N_1} at u then $f(t, -)$ is G^∞ at u . Let

$$V := \{(t, u, v, x) \in X \times \mathbb{R}^m \times \mathbb{R} \mid (t, u + xv) \in X\}$$

and

$$F : V \rightarrow \mathbb{R}, (t, u, v, x) \mapsto f(t, u + xv).$$

Since X_t is open we see that V_t is open for every $t \in \mathbb{R}^n$. By Remark 3.35 F is restricted log-exp-analytic in (u, v, x) . By Proposition 5.23 there is $N_2 \in \mathbb{N}$ such that if $F(t, u, v, -)$ is C^{N_2} at 0 then $F(t, u, v, -)$ is real analytic at 0. Taking $N := \max\{N_1, N_2\}$ we are done with Fact 5.27. \blacksquare

5.30 Corollary

Let $f : X \rightarrow \mathbb{R}, (t, u) \mapsto f(t, u)$, be restricted log-exp-analytic in u . Then the set of all $(t, u) \in X$ such that $f(t, -)$ is real analytic at u is definable.

6 Global Complexification in $\mathbb{R}_{\text{an,exp}}$

The goal for this chapter is to prove that a real analytic restricted log-exp-analytic function has a global complexification which is again restricted log-exp-analytic. Of course this answers the question "Does $\mathbb{R}_{\text{an,exp}}$ have global complexification?" not completely, but may be satisfactory since it may be that every real analytic definable function is restricted log-exp-analytic (since if the global exponential function comes not locally bounded into the game features like flatness occur (compare with Example 3.24)). But neither real analytic functions nor restricted log-exp-analytic functions are flat.

The first step is to construct a definable holomorphic extension of a prepared restricted log-exp-analytic function in one variable on a single cell C . For this we need a rather sophisticated set-up: Of course it is not difficult to construct a definable holomorphic extension of a prepared restricted log-exp-analytic function in one variable, but the problem is to obtain an extension which is big enough to enable the induction on the number of variables. On the one hand we have to deal with iterations of logarithmic scales, exponentials and special units which occur in a prepared restricted log-exp-analytic function and on the other hand we need also some definability results on integration (compare also Kaiser in [19] for the ideas). We will handle all this appropriately by introducing persistent and non-persistent functions which define "suitable definable subsets" of the domain of such an extension. The key feature is that a holomorphic extension of a prepared restricted log-exp-analytic function $f(t, z)$ in z is complex log-analytically prepared in z on a "suitable definable subset", i.e. of the form $f(t, z) = a(t)(\sigma_0 z_0(t, z))^{q_0} \cdot \dots \cdot (\sigma_r z_r(t, z))^{q_r} U(t, z)$ where $z_0 = z - \Theta_0(t)$, $z_1 = \log(\sigma_0 z_0) - \Theta_1(t), \dots$, where $\sigma_j \in \{-1, 1\}$, the q_j 's are rational exponents and $U(t, z)$ is a function of a special form. It turns out that the data of this log-analytical preparation belongs to a class of definable functions which are an efficient generalization of nice functions. We will call them regular functions. This helps enormously to give the desired result on integration outgoing from considering logarithmic scales.

In the second step we consider a restricted log-exp-analytic function. We use the preparation theorem and construct definable holomorphic extensions of prepared restricted log-exp-analytic functions in one variable on every single cell, glue them together and construct a unary high parametric global complexification which is again restricted log-exp-analytic.

In the third step we derive Theorem B and C by a technical induction on the number of variables (compare also Kaiser in [19] for the ideas).

This chapter is organized as follows. Chapter 6.1 is about constructing the definable holomorphic extension of a prepared restricted log-exp-analytic function in one variable on a single cell: At first we describe geometrical properties and definable holomorphic extensions of logarithmic scales in one variable

(Chapter 6.1.1). Before we answer the question on integration for logarithmic scales suitably for our purposes we define regular, persistent and non-persistent functions and give important properties (Chapter 6.1.2). Finally we study definable holomorphic extensions of prepared restricted log-exp-analytic functions in one variable (Chapter 6.1.3). In Chapter 6.2 we construct the unary high parametric global complexification of a restricted log-exp-analytic function which is again restricted log-exp-analytic (Chapter 6.2.1) and then we prove Theorem B and C (Chapter 6.2.2).

Further Notation:

Let $n \in \mathbb{N}_0$. Let t range over \mathbb{R}^n and x over \mathbb{R} if not otherwise mentioned. Let $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n, (t, x) \mapsto t$. For a definable set $X \subset \mathbb{R}^n \times \mathbb{R}$ we consider the following:

For a function $g : \pi(X) \rightarrow \mathbb{R}$ we say that $g > X$ if $g(t) > x$ for every $t \in \pi(X)$ and $x \in X_t$ and $g < X$ if $g(t) < x$ for every $t \in \pi(X)$ and $x \in X_t$. For a function $f : X \rightarrow \mathbb{R}$ we say that f is continuous in x if f_t is continuous on X_t for every $t \in \pi(X)$, (strictly) monotone increasing in x if f_t is (strictly) monotone increasing for all $t \in \pi(X)$ and (strictly) monotone decreasing in x if f_t is (strictly) monotone decreasing for all $t \in \pi(X)$ respectively. A **subcurve** of a definable curve $\gamma :]0, 1[\rightarrow X$ is a definable curve $\hat{\gamma} :]0, 1[\rightarrow X$ of the following form: There is $\delta > 1$ such that $\hat{\gamma}(y) = \gamma(y/\delta)$ for $y \in]0, 1[$.

Let z range over \mathbb{C} . Let $\pi^* : \mathbb{R}^n \times \mathbb{C} \rightarrow \mathbb{R}^n, (t, z) \mapsto t$, be the projection on the first n real coordinates. For $Z \subset \mathbb{R}^n \times \mathbb{C}$ such that Z_t is open for every $t \in \mathbb{R}^n$ we say that a function $F : Z \rightarrow \mathbb{C}$ is holomorphic in z if F_t is holomorphic for every $t \in \pi^*(Z)$. We set

$$\log^* : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto \begin{cases} \log(z), & |z| \in [1/2, 3/2] \setminus \mathbb{R}_{\leq 0}, \\ 0, & \text{if} \\ & \text{else.} \end{cases}$$

Note that \log^* is globally subanalytic.

Conventions:

Definable means definable in $\mathbb{R}_{\text{an,exp}}$ if not otherwise mentioned. Let $m \in \mathbb{N}$ and let $X, Y \subset \mathbb{R}^m$ be definable with $Y \subset X$ and let $f : Y \rightarrow \mathbb{R}$ be a function. Let E be a set of positive definable functions on X . Then we say that f can be constructed from E if f can be constructed from $E|_Y$. We identify \mathbb{C} with \mathbb{R}^2 via $x + iy \mapsto (x, y)$. So "definable in \mathbb{C}^m " means "definable in \mathbb{R}^{2m} " for $m \in \mathbb{N}$.

6.1 Preparations

For Chapter 6.1 we fix a non-empty definable cell $C \subset \mathbb{R}^n \times \mathbb{R}_{\neq 0}$ and an r -logarithmic scale $\mathcal{Y} := (y_0, \dots, y_r)$ on C with center $\Theta := (\Theta_0, \dots, \Theta_r)$. We fix $\text{sign}(\mathcal{Y}) := \sigma := (\sigma_0, \dots, \sigma_r) \in \{-1, 1\}^{r+1}$.

6.1 Definition

- (a) For $t \in \pi(C)$ define the **length of C** with respect to x as $L_C(t) := \sup(C_t) - \inf(C_t) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.
- (b) We say that C is **fat with respect to x** if $L_C(t) > 0$ for every $t \in \pi(C)$.

6.1.1 Geometrical Properties and Holomorphic Extensions of Logarithmic Scales

For the Chapter 6.1.1 we set $D_0 := \pi(C) \times \mathbb{R}$ and

$$g_0 : D_0 \rightarrow \mathbb{R}, (t, x) \mapsto x - \Theta_0(t).$$

For $l \in \{1, \dots, r\}$ we set inductively

$$D_l := \{(t, x) \in D_{l-1} \mid \sigma_{l-1} g_{l-1}(t, x) > 0\}$$

and

$$g_l : D_l \rightarrow \mathbb{R}, (t, x) \mapsto \log(\sigma_{l-1} g_{l-1}(t, x)) - \Theta_l(t).$$

Set

$$D := \{(t, x) \in D_r \mid \sigma_r g_r(t, x) > 0\}.$$

Geometrical Properties

6.2 Remark

We have $C \subset D \subset D_r \subset \dots \subset D_1$ and for $l \in \{0, \dots, r\}$ we have that $g_l|_C = y_l$, and $(g_l)_t : (D_l)_t \rightarrow \mathbb{R}$ is injective and real analytic for every $t \in \pi(C)$.

6.3 Definition

Let $l \in \{0, \dots, r\}$. Let $\mu_{0,l} : \pi(C) \rightarrow \mathbb{R}, t \mapsto \Theta_l(t)$, and inductively for $j \in \{1, \dots, l\}$ we set

$$\mu_{j,l} : \pi(C) \rightarrow \mathbb{R}, t \mapsto \Theta_{l-j}(t) + \sigma_{l-j} e^{\mu_{j-1,l}(t)}.$$

Set $\mu_l := \mu_{l,l}$.

6.4 Remark

Let $l \in \{0, \dots, r\}$. The functions μ_l and $\mu_{j,l}$ for $j \in \{0, \dots, l-1\}$ are definable.

It turns out that the graph of μ_l is disjoint from C and exactly the zero set of g_l for $l \in \{0, \dots, r\}$. Additionally the sign of \mathcal{Y} defines uniquely the position of μ_l with respect to C and with respect to μ_j for $j \neq l$. This facts are proven in Proposition 6.5 and Corollary 6.6. (The functions $\mu_{j,l}$ for $0 \leq j < l$ are needed for technical reasons in this context.)

6.5 Proposition

Let $t \in \pi(C)$. Let $l \in \{0, \dots, r\}$ and $j \in \{0, \dots, l\}$. We have $(t, \mu_l(t)) \in D_j$ and

$$g_j(t, \mu_l(t)) = \sigma_j e^{\mu_{l-j,l}(t)}$$

if $j < l$.

Proof

That $(t, \mu_l(t)) \in D_0$ is clear. So suppose that $l > 0$. We show the statement by induction on $j \in \{0, \dots, l\}$.

$j = 0$: With Definition 6.3 we obtain

$$g_0(t, \mu_l(t)) = \mu_l(t) - \Theta_0(t) = \sigma_0 e^{\mu_{l-1,l}(t)}.$$

$j - 1 \rightarrow j$: By the inductive hypothesis $(t, \mu_l(t)) \in D_{j-1}$ and

$$g_{j-1}(t, \mu_l(t)) = \sigma_{j-1} e^{\mu_{l-j,l}(t)}.$$

So we see that $(t, \mu_l(t)) \in D_j$. If $j < l$ we obtain

$$\begin{aligned} g_j(t, \mu_l(t)) &= \log(\sigma_{j-1} g_{j-1}(t, \mu_l(t))) - \Theta_j(t) \\ &= \mu_{l-j,l}(t) - \Theta_j(t) = \sigma_j e^{\mu_{l-j-1,l}(t)}. \end{aligned}$$

■

6.6 Corollary

For $l \in \{0, \dots, r\}$ and $t \in \pi(C)$ we have $g_l(t, \mu_l(t)) = 0$ and

$$Z(g_l) := \{(t, x) \in D_l \mid g_l(t, x) = 0\} = \{(t, x) \in D_l \mid x = \mu_l(t)\}.$$

Proof

For $l = 0$ this is clear. So assume $l > 0$. With Proposition 6.5 we obtain $(t, \mu_l(t)) \in D_l$,

$$g_{l-1}(t, \mu_l(t)) = \sigma_{l-1} e^{\Theta_l(t)},$$

and therefore

$$g_l(t, \mu_l(t)) = \log(\sigma_{l-1} g_{l-1}(t, \mu_l(t))) - \Theta_l(t) = 0$$

for every $t \in \pi(C)$. With Remark 6.2 we are done. ■

6.7 Proposition

Let $l \in \{0, \dots, r\}$. The following properties hold.

- (1) The function g_l is strictly monotone in x . It is strictly monotone increasing in x if and only if $\prod_{j=0}^{l-1} \sigma_j = 1$.
- (2) We have $\mu_l < C$ or $\mu_l > C$. We have $\mu_l < C$ if and only if $\prod_{j=0}^l \sigma_j = 1$.
- (3) Let $k \in \{l+1, \dots, r\}$. We have $\mu_l < \mu_k$ or $\mu_k < \mu_l$. We have $\mu_l < \mu_k$ if and only if $\prod_{j=0}^l \sigma_j = 1$.

Proof

(1): We do an induction on $l \in \{0, \dots, r\}$. For $l = 0$ it is clear.

$l-1 \rightarrow l$: We assume $\sigma_{l-1} = 1$. The case " $\sigma_{l-1} = -1$ " is handled completely similar. Then g_l is strictly monotone increasing in x if and only if g_{l-1} is strictly monotone increasing in x . By the inductive hypothesis we have that g_{l-1} is strictly monotone increasing in x if and only if $\prod_{j=0}^{l-2} \sigma_j = 1$. This gives the result.

(2): Because $g_l|_C = y_l$ we obtain with Definition 4.1 $g_l|_C > 0$ or $g_l|_C < 0$. We assume $g_l|_C > 0$, i.e. $\sigma_l = 1$. The case " $g_l|_C < 0$ " is handled completely similar. By (1) we have that g_l is either strictly monotone increasing in x or strictly monotone decreasing in x . If the former holds then $\mu_l < C$ by Corollary 6.6. Additionally $\prod_{j=0}^{l-1} \sigma_j = 1$ by (1) and therefore $\prod_{j=0}^l \sigma_j = 1$. If the latter holds then $\mu_l > C$ by Corollary 6.6. Additionally $\prod_{j=0}^{l-1} \sigma_j = -1$ by (1) and therefore $\prod_{j=0}^l \sigma_j = -1$.

(3): With Proposition 6.5 we obtain

$$g_l(t, \mu_k(t)) = \sigma_l e^{\mu_{k-l-1, k}(t)}$$

for every $t \in \pi(C)$. We assume that g_l is strictly monotone increasing in x . The case " g_l is strictly monotone decreasing" is handled completely similar. By (1) $\prod_{j=0}^{l-1} \sigma_j = 1$. If $\sigma_l = 1$ then $g_l(t, \mu_k(t)) = e^{\mu_{k-l-1, k}(t)} > 0$ for every $t \in \pi(C)$ and consequently $\mu_l < \mu_k$ by Corollary 6.6. If $\sigma_l = -1$ then $g_l(t, \mu_k(t)) = -e^{\mu_{k-l-1, k}(t)} < 0$ for every $t \in \pi(C)$ and consequently $\mu_k < \mu_l$ by Corollary 6.6. ■

6.8 Definition

We define the **change index** k^{ch} for \mathcal{Y} as follows: If there is $l \in \{0, \dots, r\}$ with $\sigma_l = -1$ set

$$k^{\text{ch}} := \max\{l \in \{0, \dots, r\} \mid \sigma_l = -1\} - 1.$$

Otherwise set $k^{\text{ch}} := -2$.

6.9 Remark

If $k^{\text{ch}} = -2$ then $\sigma_0 = \dots = \sigma_r = 1$. If $k^{\text{ch}} = -1$ then $\sigma_0 = -1$ and $\sigma_1 = \dots = \sigma_r = 1$.

The next two Remarks show the geometrical meaning of the change index k^{ch} of \mathcal{Y} if it is nonnegative.

6.10 Remark

Let $k := k^{\text{ch}}$. Assume $k \geq 0$.

- (1) We have $\prod_{j=0}^k \sigma_j = -1$ if and only if $\prod_{j=0}^r \sigma_j = 1$.
- (2) Let $\prod_{j=0}^r \sigma_j = 1$. If $l \in \{0, \dots, r\} \setminus \{k\}$ and $\prod_{j=0}^l \sigma_j = -1$ then $\mu_k < \mu_l$.
- (3) Let $\prod_{j=0}^r \sigma_j = -1$. If $l \in \{0, \dots, r\} \setminus \{k\}$ and $\prod_{j=0}^l \sigma_j = 1$ then $\mu_l < \mu_k$.

Proof

(1): Follows directly from Definition 6.8.

(2): Let $l \in \{0, \dots, r\} \setminus \{k\}$ and suppose $\prod_{j=0}^l \sigma_j = -1$. Note that $\prod_{j=0}^a \sigma_j = 1$ if $a \in \{k+1, \dots, r\}$. So $l < k$. With Proposition 6.7(3) we obtain $\mu_k < \mu_l$.

(3): This is proven similarly as (2). ■

6.11 Remark

Let $k := k^{\text{ch}}$. The following holds.

- (1) If $\prod_{j=0}^r \sigma_j = 1$ and $k < 0$ then

$$D = \{(t, x) \in \pi(C) \times \mathbb{R} \mid x > \mu_r(t)\}.$$

- (2) If $\prod_{j=0}^r \sigma_j = -1$ and $k < 0$ then

$$D = \{(t, x) \in \pi(C) \times \mathbb{R} \mid x < \mu_r(t)\}.$$

- (3) If $\prod_{j=0}^r \sigma_j = 1$ and $k \geq 0$ then

$$D = \{(t, x) \in \pi(C) \times \mathbb{R} \mid \mu_r(t) < x < \mu_k(t)\}.$$

- (4) If $\prod_{j=0}^r \sigma_j = -1$ and $k \geq 0$ then

$$D = \{(t, x) \in \pi(C) \times \mathbb{R} \mid \mu_k(t) < x < \mu_r(t)\}.$$

Proof

We show property (3). The rest is proven similarly. Note that $\prod_{j=0}^k \sigma_j = -1$, $\mu_r < C < \mu_k$ and $\sigma_\kappa g_\kappa(t, x) = \sigma_\kappa y_\kappa(t, x) > 0$ for $(t, x) \in C$ and $\kappa \in \{k, r\}$.

Let $(t, x) \in \pi(C) \times \mathbb{R}$ be with $\mu_r(t) < x < \mu_k(t)$. If $\prod_{j=0}^l \sigma_j = 1$ then $\mu_l < \mu_r$ if $l \neq r$ by Proposition 6.7(3). If $\prod_{j=0}^l \sigma_j = -1$ then $\mu_k < \mu_l$ if $l \neq k$ (since then $l < k$). So we obtain with the intermediate value theorem, Definition 4.1 and Corollary 6.6 that $\sigma_j g_j(t, x) > 0$ for every $j \in \{0, \dots, r\}$ since $C \subset D$ and $(g_j)_t$ is continuous. Therefore $(t, x) \in D$.

Let $(t, x) \in D$. Then we have $\sigma_\kappa g_\kappa(t, x) > 0$ for $\kappa \in \{k, r\}$. Again with Proposition 6.7(2), Corollary 6.6, the continuity of g_κ in x and the monotony property of g_κ for $\kappa \in \{k, r\}$ we obtain the result. \blacksquare

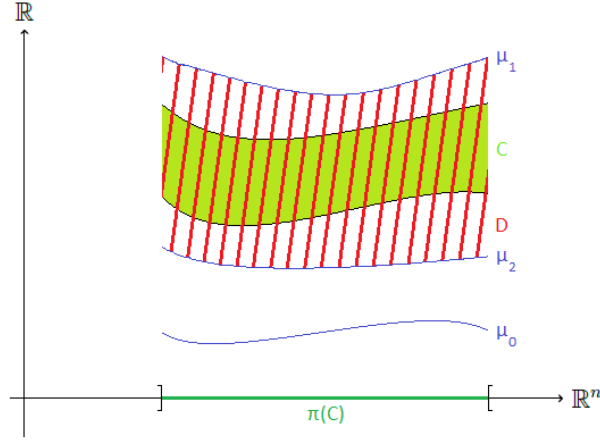


Figure 6.1 The set D and the functions μ_j for a 2-logarithmic scale \mathcal{Y} on C with $\text{sign}(\mathcal{Y}) = (1, -1, -1)$ and continuous center.

6.12 Example

Let $n = 1$ and consider

$$C := \{(t, x) \in \mathbb{R} \times \mathbb{R} \mid t \in]0, 1[, \frac{1}{1+t} + e^{-t-1/t} < x < \frac{1}{1+t} + e^{-1/t}\}.$$

Let $\Theta_0 : \pi(C) \rightarrow \mathbb{R}, t \mapsto \frac{1}{1+t}$, $\Theta_1 : \pi(C) \rightarrow \mathbb{R}, t \mapsto -\frac{1}{t}$ and $\Theta_2 = 0$. Let $y_0 : C \rightarrow \mathbb{R}, (t, x) \mapsto x - \Theta_0(t)$, and inductively for $j \in \{1, 2\}$ let $y_j : C \rightarrow \mathbb{R}, (t, x) \mapsto \log(|y_{j-1}(t, x)|) - \Theta_j(t)$. Then $\mathcal{Y} := (y_0, y_1, y_2)$ is a 2-logarithmic scale with center $(\Theta_0, \Theta_1, \Theta_2)$ and $\text{sign}(\mathcal{Y}) := (\sigma_0, \sigma_1, \sigma_2) = (1, -1, -1) \in \{-1, 1\}^3$. Therefore $k^{\text{ch}} = 1$, $\mu_0 = \Theta_0$, $\mu_1 = \Theta_0 + e^{\Theta_1}$, $\mu_2 = \Theta_0 + e^{\Theta_1-1}$, $\mu_{0,1} = \Theta_1$, $\mu_{0,2} = 0$ and $\mu_{1,2} = \Theta_1 - 1$.

Proof

Compare with Example 4.2 ■

Now we merge into the complex setting.

6.13 Definition

(a) Consider $H_0 := \pi(C) \times \mathbb{C}$ and

$$z_0 : H_0 \rightarrow \mathbb{C}, (t, z) \mapsto z - \Theta_0(t).$$

Inductively for $l \in \{1, \dots, r\}$ we set

$$H_l := \{(t, z) \in H_{l-1} \mid \sigma_{l-1} z_{l-1}(t, z) \in \mathbb{C}^-\}$$

and

$$z_l : H_l \rightarrow \mathbb{C}, (t, z) \mapsto \log(\sigma_{l-1} z_{l-1}(t, z)) - \Theta_l(t).$$

(b) Set

$$H := \{(t, z) \in H_r \mid \sigma_r z_r(t, z) \in \mathbb{C}^-\}.$$

6.14 Remark

The following holds for $l \in \{0, \dots, r\}$.

- (1) It holds $C \subset D_l \subset H_l$, $H_l \subset H_{l-1}$ if $l > 0$ and $(H_l)_t$ is open for every $t \in \pi(C)$. Additionally z_l is well-defined and holomorphic in z .
- (2) It holds $z_l|_{D_l} = g_l$ and $z_l|_C = y_l$.
- (3) Let E be a set of positive definable functions such that $\Theta_0, \dots, \Theta_r$ can be constructed from E . Then z_l can be constructed from E .

Proof

(1): This follows by an easy induction on l and the fact that the complex logarithm function is holomorphic on \mathbb{C}^- .

(2): This follows immediately by an easy induction on l and from Definition 6.13(a).

(3): We do an induction on l . We have $z_0(t, z) = z - \Theta_0(t)$ for every $(t, z) \in H_0$ and are done with Proposition 3.16(2).

$l - 1 \rightarrow l$: We have $z_l(t, z) = \log(\sigma_{l-1} z_{l-1}(t, z)) - \Theta_l(t)$ for every $(t, z) \in H_l$ and are done with the inductive hypothesis and Proposition 3.16(2). ■

We often consider z_l as a function on H for every $l \in \{0, \dots, r\}$.

6.15 Remark

Let $k := k^{\text{ch}}$.

If $\prod_{j=0}^r \sigma_j = 1$ and $k < 0$ then

$$H = \{(t, z) \in \pi(C) \times \mathbb{C} \mid z \in \mathbb{C}_{\mu_r(t)}^-\}.$$

If $\prod_{j=0}^r \sigma_j = -1$ and $k < 0$ then

$$H = \{(t, z) \in \pi(C) \times \mathbb{C} \mid z \in \mathbb{C}_{\mu_r(t)}^+\}.$$

If $\prod_{j=0}^r \sigma_j = 1$ and $k \geq 0$ then

$$H = \{(t, z) \in \pi(C) \times \mathbb{C} \mid z \in \mathbb{C}_{\mu_r(t)}^- \cap \mathbb{C}_{\mu_k(t)}^+\}.$$

If $\prod_{j=0}^r \sigma_j = -1$ and $k \geq 0$ then

$$H = \{(t, z) \in \pi(C) \times \mathbb{C} \mid z \in \mathbb{C}_{\mu_r(t)}^+ \cap \mathbb{C}_{\mu_k(t)}^-\}.$$

Proof

We have

$$H = D \cup (\pi(C) \times \mathbb{C} \setminus \mathbb{R}),$$

because for $(t, z) \in \pi(C) \times \mathbb{C} \setminus \mathbb{R}$ we obtain $\sigma_l z_l(t, z) \in \mathbb{C} \setminus \mathbb{R}$ for every $l \in \{0, \dots, r\}$ by the definition of the complex logarithm function. We are done with Remark 6.14(2) and Remark 6.11. \blacksquare

6.16 Definition

Let $q := (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$. We set

$$(\sigma \mathcal{Z})^{\otimes q} : H \rightarrow \mathbb{C}, (t, z) \mapsto \prod_{j=0}^r (\sigma_j z_j(t, z))^{q_j}.$$

6.17 Remark

Let $q \in \mathbb{Q}^{r+1}$. The following holds.

- (1) H_t is open for every $t \in \pi(C)$ and $(\sigma \mathcal{Z})^{\otimes q}$ is well-defined and holomorphic in z . In particular it is

$$(\sigma \mathcal{Z})^{\otimes q}|_C = |\mathcal{Y}|^{\otimes q}.$$

- (2) Let E be a set of positive definable functions such that $\Theta_0, \dots, \Theta_r$ can be constructed from E . Then $(\sigma \mathcal{Z})^{\otimes q}$ can be constructed from E .

Proof

Note that $\sigma_l z_l(t, z) \in \mathbb{C}^-$ for every $(t, z) \in H$ and $l \in \{0, \dots, r\}$ and that for $p \in \mathbb{Q}$ the function $\mathbb{C}^- \rightarrow \mathbb{C}, z \mapsto z^p = e^{p \log(z)}$, is holomorphic and globally subanalytic. So property (1) follows from Remark 6.14(1) and Remark 6.14(2). Property (2) follows immediately with Remark 6.14(3) and Proposition 3.16(2). \blacksquare

For the rest of Section 6.1.2 we fix $\kappa = r$ or $\kappa \in \{k^{\text{ch}}, r\}$ if $k^{\text{ch}} \geq 0$.

6.18 Definition

Let $H_{0,\kappa}^* := \pi(C) \times \mathbb{C}$ and $\mathcal{P}_{0,\kappa} : H_{0,\kappa}^* \rightarrow \mathbb{C}, (t, z) \mapsto z - \mu_\kappa(t)$, and inductively for $l \in \{1, \dots, \kappa\}$ let

$$H_{l,\kappa}^* := \{(t, z) \in H_{l-1,\kappa}^* \mid 1 + \frac{\sigma_{l-1} \mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}} \in \mathbb{C}^-\}$$

and

$$\mathcal{P}_{l,\kappa} : H_{l,\kappa}^* \rightarrow \mathbb{C}, (t, z) \mapsto \log\left(1 + \frac{\sigma_{l-1} \mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}}\right).$$

Set $\mathcal{P}_\kappa := \mathcal{P}_{\kappa, \kappa}$.

6.19 Remark

Let $l \in \{0, \dots, \kappa\}$. The function $\mathcal{P}_{l, \kappa}$ is well-defined and definable. Furthermore we have $(t, \mu_\kappa(t)) \in H_l^*$ and $\mathcal{P}_{l, \kappa}(t, \mu_\kappa(t)) = 0$ for $t \in \pi(C)$.

We will see that these functions describe z_l as a function of $z - \mu_\kappa$ for $l \in \{0, \dots, \kappa\}$. So they are very helpful for technical proofs for example to show that there is a definable set $Y \subset H$ big enough for our purposes such that $z_l|_Y$ is globally subanalytic in z for $l \in \{0, \dots, \kappa\}$ by using the logarithmic series. (Compare also the second part of Chapter 6.1.2 regarding questions on integration of logarithmic scales in the complex setting).

6.20 Proposition

Let $l \in \{0, \dots, \kappa\}$. We have $H_{l, \kappa}^* = H_l$ and if $l \neq \kappa$ then

$$z_l(t, z) = \mathcal{P}_{l, \kappa}(t, z) + \sigma_l e^{\mu_{\kappa-l-1, \kappa}(t)}$$

for $(t, z) \in H_l$.

Proof

By definition of $H_{0, \kappa}^*$ we have $H_{0, \kappa}^* = H_0$. If $\kappa = 0$ we are done. So assume that $\kappa > 0$. We do an induction on l . For $(t, z) \in H_{0, \kappa}^*$ we obtain

$$\begin{aligned} z_0(t, z) &= z - \Theta_0 - \sigma_0 e^{\mu_{\kappa-1, \kappa}(t)} + \sigma_0 e^{\mu_{\kappa-1, \kappa}(t)} \\ &= z - \mu_\kappa(t) + \sigma_0 e^{\mu_{\kappa-1, \kappa}(t)} = \mathcal{P}_{0, \kappa}(t, z) + \sigma_0 e^{\mu_{\kappa-1, \kappa}(t)}. \end{aligned}$$

$l-1 \rightarrow l$: Note that $H_{l-1} = H_{l-1, \kappa}^*$. Let $(t, z) \in H_{l-1}$. We get with the inductive hypothesis (since $l-1 \neq \kappa$)

$$\begin{aligned} \sigma_{l-1} z_{l-1}(t, z) \in \mathbb{C}^- &\Leftrightarrow \left(1 + \frac{\sigma_{l-1}(z_{l-1}(t, z) - \sigma_{l-1} e^{\mu_{\kappa-l, \kappa}(t)})}{e^{\mu_{\kappa-l, \kappa}(t)}}\right) \in \mathbb{C}^- \\ &\Leftrightarrow \left(1 + \frac{\sigma_{l-1} \mathcal{P}_{l-1, \kappa}(t, z)}{e^{\mu_{\kappa-l, \kappa}(t)}}\right) \in \mathbb{C}^-. \end{aligned}$$

So we obtain $H_l = H_{l, \kappa}^*$. Suppose $l \neq \kappa$. Let $(t, z) \in H_l$. We obtain with the

inductive hypothesis

$$\begin{aligned}
z_l(t, z) &= \log(\sigma_{l-1}z_{l-1}(t, z)) - \Theta_l(t) \\
&= \log(\sigma_{l-1}(\mathcal{P}_{l-1, \kappa}(t, z) + \sigma_{l-1}e^{\mu_{\kappa-l, \kappa}(t)})) - \Theta_l(t) \\
&= \log(\sigma_{l-1}\mathcal{P}_{l-1, \kappa}(t, z) + e^{\mu_{\kappa-l, \kappa}(t)}) - \Theta_l(t) \\
&= \log\left(1 + \frac{\sigma_{l-1}\mathcal{P}_{l-1, \kappa}(t, z)}{e^{\mu_{\kappa-l, \kappa}(t)}}\right) + \mu_{\kappa-l, \kappa}(t) - \Theta_l(t) \\
&= \mathcal{P}_{l, \kappa}(t, z) + \mu_{\kappa-l, \kappa}(t) - \Theta_l(t) \\
&= \mathcal{P}_{l, \kappa}(t, z) + \Theta_l(t) + \sigma_l e^{\mu_{\kappa-l-1, \kappa}(t)} - \Theta_l(t) \\
&= \mathcal{P}_{l, \kappa}(t, z) + \sigma_l e^{\mu_{\kappa-l-1, \kappa}(t)}.
\end{aligned}$$

■

6.21 Corollary

For $(t, z) \in H_\kappa = H_{\kappa, \kappa}^*$ we have $z_\kappa(t, z) = \mathcal{P}_\kappa(t, z)$.

Proof

Let $\kappa = 0$. Then $\Theta_\kappa = \mu_\kappa$ and therefore

$$z_\kappa(t, z) = z - \mu_\kappa(t) = \mathcal{P}_\kappa(t, z)$$

for $(t, z) \in H_\kappa$. So let $\kappa > 0$. With Proposition 6.20 we have for $(t, z) \in H_\kappa$

$$\begin{aligned}
z_\kappa(t, z) &= \log(\sigma_{\kappa-1}z_{\kappa-1}(t, z)) - \Theta_\kappa(t) \\
&= \log(\sigma_{\kappa-1}(\mathcal{P}_{\kappa-1, \kappa}(t, z) + \sigma_{\kappa-1}e^{\mu_{0, \kappa}(t)})) - \Theta_\kappa(t) \\
&= \log(\sigma_{\kappa-1}(\mathcal{P}_{\kappa-1, \kappa}(t, z) + \sigma_{\kappa-1}e^{\Theta_\kappa(t)})) - \Theta_\kappa(t) \\
&= \log(\sigma_{\kappa-1}\mathcal{P}_{\kappa-1, \kappa}(t, z) + e^{\Theta_\kappa(t)}) - \Theta_\kappa(t) \\
&= \log\left(1 + \frac{\sigma_{\kappa-1}\mathcal{P}_{\kappa-1, \kappa}(t, z)}{e^{\Theta_\kappa(t)}}\right) = \mathcal{P}_\kappa(t, z).
\end{aligned}$$

■

We often consider $\mathcal{P}_{l, \kappa}$ as a function on H for $l \in \{0, \dots, \kappa\}$.

6.22 Example

Let $n = 1$ and consider

$$C := \{(t, x) \in \mathbb{R} \times \mathbb{R} \mid t \in]0, 1[, \frac{1}{1+t} + e^{-t-1/t} < x < \frac{1}{1+t} + e^{-1/t}\}.$$

Let $\mathcal{Y} := (y_0, y_1, y_2)$ and $\Theta := (\Theta_0, \Theta_1, \Theta_2)$ be as in Example 6.12. We have

$$H = \{(t, z) \in \pi(C) \times \mathbb{C} \mid \frac{1}{1+t} + e^{-1-1/t} < z < \frac{1}{1+t} + e^{-1/t} \text{ if } z \in \mathbb{R}\}.$$

For $(t, z) \in H$ we have

$$\begin{aligned} z_0(t, z) &= z - \Theta_0(t), \quad z_1(t, z) = \log(z_0(t, z)) - \Theta_1(t), \\ z_2(t, z) &= \log(-z_1(t, z)) - \Theta_2(t) \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_{0,1}(t, z) &= z - \mu_1(t), \quad \mathcal{P}_{1,1}(t, z) = \log\left(1 + \frac{z - \mu_1(t)}{e^{\mu_{0,1}(t)}}\right), \\ \mathcal{P}_{0,2}(t, z) &= z - \mu_2(t), \quad \mathcal{P}_{1,2}(t, z) = \log\left(1 + \frac{z - \mu_2(t)}{e^{\mu_{1,2}(t)}}\right), \\ \mathcal{P}_{2,2}(t, z) &= \log\left(1 - \frac{\mathcal{P}_{1,2}(t, z)}{e^{\mu_{0,2}(t)}}\right) \end{aligned}$$

where $\mu_0, \mu_1, \mu_2, \mu_{0,1}, \mu_{0,2}$ and $\mu_{1,2}$ are as in Example 6.12.

For Remark 6.23 and Remark 6.24 we fix a set E of positive definable functions such that Θ can be constructed from E and we set

$$E_\kappa := E \cup \{\pi(C) \rightarrow \mathbb{R}, t \mapsto e^{\mu_{l,\kappa}(t)} \mid l \in \{0, \dots, \kappa - 1\}\}.$$

6.23 Remark

Let $l \in \{0, \dots, \kappa\}$. The function $\mu_{l,\kappa}$ can be constructed from E_κ .

Proof

We do an induction on l . If $l = 0$ this is clear, because $\mu_{0,\kappa} = \Theta_\kappa$.

$l - 1 \rightarrow l$: We have $\mu_{l,\kappa}(t) = \Theta_{\kappa-l}(t) + \sigma_{\kappa-l} e^{\mu_{l-1,\kappa}(t)}$ for every $t \in \pi(C)$ by Definition 6.3. We are done with Proposition 3.11(2), because $e^{\mu_{l-1,\kappa}}$ can be constructed from E_κ by the inductive hypothesis and Proposition 3.11(1) (since $e^{\mu_{l-1,\kappa}} \in E_\kappa$). ■

6.24 Remark

Let $l \in \{0, \dots, \kappa\}$. Then the following holds.

- (1) The function $\mathcal{P}_{l,\kappa}$ is holomorphic in z and can be constructed from E_κ .
- (2) Let $(t, z) \in H$. Then $(\prod_{j=l}^{\kappa} \sigma_j) \mathcal{P}_{l,\kappa}(t, z) \in \mathbb{C}^-$ and $(\prod_{j=l}^{\kappa} \sigma_j) \mathcal{P}_{l,\kappa}(t, z) \in \mathbb{R}_{>0}$ if $z \in \mathbb{R}$.

Proof

(1): We do an induction on l . If $l = 0$ we have that $\mathcal{P}_0(t, z) = z - \mu_\kappa(t)$ for every $(t, z) \in H$. We obtain the result with Proposition 3.16(2) and Remark 6.23. If $l > 0$ we have that

$$\mathcal{P}_{l,\kappa}(t, z) = \log\left(1 - \frac{\sigma_{l-1} \mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}}\right)$$

for every $(t, z) \in H$. We obtain the result with Proposition 3.16(2) and the proof of Remark 6.23.

(2): If $l = \kappa$ we obtain with Corollary 6.21

$$\sigma_\kappa \mathcal{P}_\kappa(t, z) = \sigma_\kappa z_\kappa(t, z) \in \mathbb{C}^-$$

and if $z \in \mathbb{R}$ then $(t, z) \in D$ (compare with Remark 6.15 and Remark 6.11) and therefore $\sigma_\kappa \mathcal{P}_\kappa(t, z) = \sigma_\kappa g_\kappa(t, z) \in \mathbb{R}_{>0}$ by construction of D and Remark 6.14(2). So assume $l \neq \kappa$. Note that with Proposition 6.20

$$\left(\prod_{j=l}^{\kappa} \sigma_j \right) \mathcal{P}_{l,\kappa}(t, z) = \left(\prod_{j=l}^{\kappa} \sigma_j \right) z_l(t, z) - \left(\prod_{j=l}^{\kappa} \sigma_j \right) \sigma_l e^{\mu_{\kappa-l-1,\kappa}(t)}.$$

Since $z_l(t, z) \in \mathbb{C} \setminus \mathbb{R}$ for $(t, z) \in \mathbb{C} \setminus \mathbb{R}$ we may assume $z \in \mathbb{R}$. Therefore $(t, z) \in D$. We show

$$\left(\prod_{j=l}^{\kappa} \sigma_j \right) \mathcal{P}_{l,\kappa}(t, z) \in \mathbb{R}_{>0}$$

by induction on $l \in \{0, \dots, \kappa - 1\}$. For $l = 0$ we obtain

$$\left(\prod_{j=0}^{\kappa} \sigma_j \right) \mathcal{P}_{0,\kappa}(t, z) = \left(\prod_{j=0}^{\kappa} \sigma_j \right) (z - \mu_\kappa(t)).$$

The following is easy to see with Remark 6.15: If $\prod_{j=0}^{\kappa} \sigma_j = 1$ then $\mu_\kappa(t) < z$ and therefore $z - \mu_\kappa(t) \in \mathbb{R}_{>0}$. If $\prod_{j=0}^{\kappa} \sigma_j = -1$ then $z < \mu_\kappa(t)$ and therefore $\mu_\kappa(t) - z \in \mathbb{R}_{>0}$.

$l - 1 \rightarrow l$: With the inductive hypothesis and Proposition 6.20 we obtain

$$\left(\prod_{j=l-1}^{\kappa} \sigma_j \right) \mathcal{P}_{l-1,\kappa}(t, z) = \left(\prod_{j=l-1}^{\kappa} \sigma_j \right) z_{l-1}(t, z) - \left(\prod_{j=l-1}^{\kappa} \sigma_j \right) \sigma_{l-1} e^{\mu_{\kappa-l,\kappa}(t)} \in \mathbb{R}_{>0}.$$

Assume $\prod_{j=l}^{\kappa} \sigma_j = 1$. The case " $\prod_{j=l}^{\kappa} \sigma_j = -1$ " is handled completely similar. We obtain

$$\sigma_{l-1} z_{l-1}(t, z) > e^{\mu_{\kappa-l,\kappa}(t)}$$

and by taking logarithm

$$\log(\sigma_{l-1} z_{l-1}(t, z)) > \mu_{\kappa-l,\kappa}(t).$$

This gives

$$z_l(t, z) > \mu_{\kappa-l,\kappa}(t) - \Theta_l(t) = \sigma_l e^{\mu_{\kappa-l-1,\kappa}(t)}$$

and therefore

$$\left(\prod_{j=l}^{\kappa} \sigma_j \right) \mathcal{P}_{l,\kappa}(t, z) = \left(\prod_{j=l}^{\kappa} \sigma_j \right) z_l(t, z) - \left(\prod_{j=l}^{\kappa} \sigma_j \right) \sigma_l e^{\mu_{\kappa-l-1,\kappa}(t)} \in \mathbb{R}_{>0}.$$

This finishes the proof. ■

From now on we assume that C is fat with respect to x . For the rest of Chapter 6.1 we consider the following: Let y range over \mathbb{R} . For a definable curve $\gamma :]0, 1[\rightarrow H$ with $\gamma := (\gamma_1, \dots, \gamma_{n+1})$ we set $\gamma_t := (\gamma_1, \dots, \gamma_n)$ for the first n real components, $\gamma_z := \gamma_{n+1}$ and if $\gamma_{n+1}(y) \in \mathbb{R}$ for every $y \in]0, 1[$ we also write γ_x instead of γ_z .

6.25 Remark

Let $\gamma :]0, 1[\rightarrow H$ be a definable curve. Then $\lim_{y \searrow 0} L_C(\gamma_t(y)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

6.26 Definition

Let $\gamma :]0, 1[\rightarrow H$ be a definable curve. We say that γ is **compatible with C** if $\lim_{y \searrow 0} \gamma(y) \in \mathbb{C}$, $\lim_{y \searrow 0} L_C(\gamma_t(y)) > 0$ and there is $\epsilon \in]0, 1[$ such that

$$\{\inf(C_{\gamma_t(y)} \mid y \in]0, \epsilon[)\}$$

is bounded from above and

$$\{\sup(C_{\gamma_t(y)} \mid y \in]0, \epsilon[)\}$$

is bounded from below.

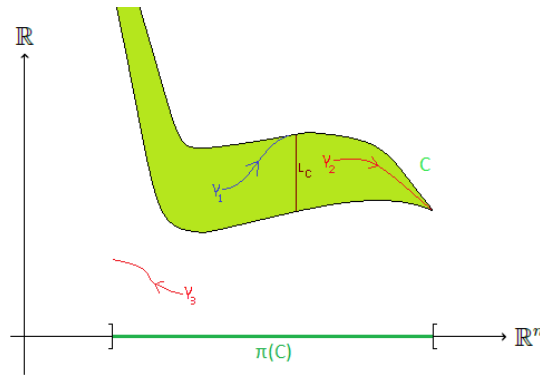


Figure 6.2 The curve γ_1 is compatible with C while the curves γ_2 and γ_3 aren't. (For $j \in \{1, 2, 3\}$ the arrows show $\gamma_j(y)$ for decreasing $y \in]0, 1[$.)

6.27 Remark

Let $\gamma :]0, 1[\rightarrow H$ be a definable curve compatible with C . The following holds.

- (1) There is a continuous definable function $\tilde{\gamma}_x :]0, 1[\rightarrow \mathbb{R}$ such that $\tilde{\gamma} := (\gamma_t, \tilde{\gamma}_x)$ is a definable curve in C compatible with C and

$$\lim_{y \searrow 0} \text{dist}(\tilde{\gamma}_x(y), \mathbb{R} \setminus C_{\gamma_t(y)}) > 0.$$

So for every $l \in \{0, \dots, r\}$ we have

$$\lim_{y \searrow 0} |\tilde{\gamma}_x(y) - \mu_l(\gamma_t(y))| > 0.$$

- (2) Let $\lim_{y \searrow 0} \gamma(y) := (t_0, x_0)$ where $x_0 \in \mathbb{R}$. Suppose that

$$\lim_{y \searrow 0} \text{dist}(\text{Re}(\gamma_z(y)), C_{\gamma_t(y)}) = 0.$$

Then there is a definable function $\hat{\gamma}_x :]0, 1[\rightarrow \mathbb{R}$ such that $\hat{\gamma} := (\gamma_t, \hat{\gamma}_x)$ is a definable curve in C compatible with C and $\lim_{y \searrow 0} (\gamma_t(y), \hat{\gamma}_x(y)) = (t_0, x_0)$.

Proof

We assume that

$$C = \{(t, x) \in \pi(C) \times \mathbb{R} \mid f_1(t) < x < f_2(t)\}$$

where $f_1, f_2 : \pi(C) \rightarrow \mathbb{R}$ are definable and continuous with $f_1 < f_2$. The other cases are handled completely similar (compare Definition 2.13 iii) - v) for the other types of a definable cell). Note that $\lim_{y \searrow 0} (f_2(\gamma_t(y)) - f_1(\gamma_t(y))) > 0$.

- (1): Since γ is compatible with C we find continuous definable functions $h_1 : \pi(C) \rightarrow \mathbb{R}$ and $h_2 : \pi(C) \rightarrow \mathbb{R}$ with $h_1 < h_2$ such that

$$\{(t, x) \in \pi(C) \times \mathbb{R} \mid h_1(t) < x < h_2(t)\} \subset C,$$

$\lim_{y \searrow 0} h_j(\gamma_t(y)) \in \mathbb{R}$ for $j \in \{1, 2\}$ and $\lim_{y \searrow 0} h_2(\gamma_t(y)) - h_1(\gamma_t(y)) \in \mathbb{R}_{>0}$. (Choose $h_1 = f_1$ and $h_2 = \min\{f_1 + 1, f_2\}$ if $\lim_{y \searrow 0} f_1(\gamma_t(y)) \neq -\infty$, $h_1 = \max\{f_1, f_2 - 1\}$ and $h_2 = f_2$ if $\lim_{y \searrow 0} f_2(\gamma_t(y)) \neq \infty$ and otherwise $h_1 = \max\{f_1, 0\}$ and $h_2 = \min\{f_2, 1\}$.)

Consider

$$\tilde{\gamma}_x :]0, 1[\rightarrow \mathbb{R}, y \mapsto \frac{h_1(\gamma_t(y)) + h_2(\gamma_t(y))}{2}.$$

By o-minimality we obtain that $]0, 1[\rightarrow C, y \mapsto (\gamma_t(y), \tilde{\gamma}_x(y))$, is a definable curve compatible with C with

$$\lim_{y \searrow 0} \text{dist}(\tilde{\gamma}_x(y), \mathbb{R} \setminus C_{\gamma_t(y)}) \geq \lim_{y \searrow 0} \frac{h_2(\gamma_t(y)) - h_1(\gamma_t(y))}{2} > 0.$$

We see that $\lim_{y \searrow 0} |\tilde{\gamma}_x(y) - \mu_j(\gamma_t(y))| > 0$ since $\mu_j(\gamma_t(y)) \notin C_{\gamma_t(y)}$ for every $y \in]0, 1[$ and $j \in \{0, \dots, r\}$.

(2): There is $0 < \epsilon < 1$ such that $\text{Re}(\gamma_z(y)) \in C_{\gamma_t(y)}$ for every $y \in]0, \epsilon[$ or $\text{Re}(\gamma_z(y)) \notin C_{\gamma_t(y)}$ for every $y \in]0, \epsilon[$. Assume the latter. The former is handled completely similar.

Suppose that $\text{Re}(\gamma_z(y)) > C_{\gamma_t(y)}$ for $y \in]0, \epsilon[$. The case " $\text{Re}(\gamma_z(y)) < C_{\gamma_t(y)}$ for $y \in]0, \epsilon[$ " is handled completely similar. Note that $\lim_{y \searrow 0} f_2(\gamma_t(y)) = x_0$. Fix a continuous definable function \hat{f}_1 with $f_1 < \hat{f}_1 < f_2$ such that $\lim_{y \searrow 0} \hat{f}_1(\gamma_t(y)) \in \mathbb{R}$. Consider

$$\hat{\gamma}_x :]0, 1[\rightarrow \mathbb{R}, y \mapsto f_2(\gamma_t(y)) + y(\hat{f}_1(\gamma_t(y)) - f_2(\gamma_t(y))).$$

Note that $\hat{\gamma}_x(y) \in C_{\gamma_t(y)}$ for $y \in]0, 1[$. Since $\lim_{y \searrow 0} \hat{\gamma}_x(y) = x_0$ by construction we see that $\hat{\gamma}_x$ does the job. ■

6.28 Proposition

Let $l \in \{0, \dots, r\}$. Let $\gamma :]0, 1[\rightarrow H$ be a definable curve compatible with C . Then the following holds.

(I_l) We have $\lim_{y \searrow 0} \Theta_l(\gamma_t(y)) \in \mathbb{R}$.

(II_l) We have $\lim_{y \searrow 0} |z_l(\gamma(y))| = 0$ if and only if $\lim_{y \searrow 0} (\gamma_z(y) - \mu_l(\gamma_t(y))) = 0$. We have $\lim_{y \searrow 0} |z_l(\gamma(y))| = \infty$ if and only if there is $j \in \{0, \dots, l-1\}$ such that $\lim_{y \searrow 0} (\gamma_z(y) - \mu_j(\gamma_t(y))) = 0$.

Proof

Fix a definable curve $\tilde{\gamma}$ which fulfills the properties from Remark 6.27(1). By o-minimality we have that $\lim_{y \searrow 0} \Theta_l(\gamma_t(y)) \in \mathbb{R} \cup \{\pm\infty\}$. We show both statements by induction on l simultaneously.

$l = 0$: Assume $\lim_{y \searrow 0} |\Theta_0(\gamma_t(y))| = \infty$. So $\Theta_0 \neq 0$. By Definition 4.1 there is $\epsilon_0 \in]0, 1[$ such that

$$|x - \Theta_0(t)| < \epsilon_0|x|$$

for all $(t, x) \in C$. Since $\lim_{y \searrow 0} \tilde{\gamma}_x(y)$ exists we get

$$+\infty = \lim_{y \searrow 0} \left| 1 - \frac{\Theta_0(\gamma_t(y))}{\tilde{\gamma}_x(y)} \right| \leq \epsilon_0.$$

This is a contradiction. This gives (I_0) . Since

$$z_0(\gamma_z(y)) = \gamma_z(y) - \Theta_0(\gamma_t(y)) = \gamma_z(y) - \mu_0(\gamma_t(y))$$

for $y \in]0, 1[$ we obtain (II_0) .

$l - 1 \rightarrow l$: Assume $\lim_{y \searrow 0} |\Theta_l(\gamma_t(y))| = \infty$. So $\Theta_l \neq 0$. So there is $\epsilon_l \in]0, 1[$ such that

$$|\log(\sigma_{l-1}y_{l-1}(t, x)) - \Theta_l(t)| < \epsilon_l |\log(\sigma_{l-1}y_{l-1}(t, x))|$$

for all $(t, x) \in C$. Therefore

$$\lim_{y \searrow 0} \left| 1 - \frac{\Theta_l(\gamma_t(y))}{\log(\sigma_{l-1}y_{l-1}(\tilde{\gamma}(y)))} \right| \leq \epsilon_l.$$

This gives $\lim_{y \searrow 0} \sigma_{l-1}y_{l-1}(\tilde{\gamma}(y)) \in \{0, \infty\}$. With (II_{l-1}) we find $j \in \{0, \dots, l - 1\}$ such that

$$\lim_{y \searrow 0} (\tilde{\gamma}_x(y) - \mu_j(\gamma_t(y))) = 0.$$

But this is a contradiction to the choice of $\tilde{\gamma}$. We obtain (I_l) .

We show (II_l) . We have $\lim_{y \searrow 0} |z_l(\gamma(y))| = \infty$ if and only if $\lim_{y \searrow 0} |z_{l-1}(\gamma(y))| \in \{0, \infty\}$ by (I_l) if and only if there is $j \in \{0, \dots, l - 1\}$ such that $\lim_{y \searrow 0} (\gamma_z(y) - \mu_j(\gamma_t(y))) = 0$ by (II_{l-1}) . We see with (I_a) for $a \in \{1, \dots, l\}$ that

$$\lim_{y \searrow 0} z_{l-j}(\gamma(y)) = \lim_{y \searrow 0} \sigma_{l-j} e^{\mu_{j-1, l}(\gamma_t(y))}$$

if and only if

$$\lim_{y \searrow 0} z_{l-j-1}(\gamma(y)) = \lim_{y \searrow 0} \sigma_{l-j-1} e^{\mu_{j, l}(\gamma_t(y))}$$

by an easy induction on $j \in \{1, \dots, l - 1\}$ (and the continuity of the global exponential function and the global logarithm on the positive real line). So we obtain $\lim_{y \searrow 0} z_l(\gamma(y)) = 0$ if and only if $\lim_{y \searrow 0} z_{l-1}(\gamma(y)) = \sigma_{l-1} e^{\mu_{0, l}(\gamma_t(y))}$ by (I_l) if and only if

$$\lim_{y \searrow 0} z_0(\gamma(y)) = \lim_{y \searrow 0} \sigma_0 e^{\mu_{l-1, l}(\gamma_t(y))}$$

if and only if

$$\lim_{y \searrow 0} \gamma_z(y) = \lim_{y \searrow 0} (\Theta_0(\gamma_t(y)) + \sigma_0 e^{\mu_{l-1, l}(\gamma_t(y))}) = \lim_{y \searrow 0} \mu_l(\gamma_t(y))$$

with (I_0) . ■

We see that the center Θ of \mathcal{Y} has the following property: If for $t_0 \in \mathbb{R}^n$ there is an open neighbourhood U of t_0 in \mathbb{R}^n , $\epsilon > 0$ and $c_1, c_2 \in \mathbb{R}$ such that $L_C(t) > \epsilon$, $\inf(C_t) < c_1$ and $\sup(C_t) > c_2$ for every $t \in \pi(C) \cap U$ then Θ_j is bounded at t_0

for every $j \in \{0, \dots, r\}$ (i.e. there is $V \subset U$ open in \mathbb{R}^n with $t_0 \in V$ such that $\Theta_j|_{\pi(C) \cap V}$ is bounded).

6.29 Corollary

Let $\gamma :]0, 1[\rightarrow H$ be a definable curve compatible with C . The following holds.

(1) Let $l \in \{0, \dots, r\}$.

(i) We have $\lim_{y \searrow 0} \mu_{j,l}(\gamma_t(y)) \in \mathbb{R}$ for every $j \in \{0, \dots, l\}$.

(ii) It is

$$\lim_{y \searrow 0} |\mu_l(\gamma_t(y)) - \mu_j(\gamma_t(y))| > 0$$

for every $j \in \{0, \dots, r\}$ with $j \neq l$.

(2) Let $q \in \mathbb{Q}^{r+1}$. Let $\lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| \in \{0, \infty\}$. Then there is $l \in \{0, \dots, r\}$ with $\lim_{y \searrow 0} \gamma_z(y) - \mu_l(\gamma_t(y)) = 0$.

Proof

(1),(i): This is clear with the continuity of the global exponential function, (I_j) in Proposition 6.28 and the definition of $\mu_{j,l}$ for $j \in \{0, \dots, l\}$.

(1),(ii): Let $j \in \{0, \dots, l-1\}$. With Definition 6.3 and (I_a) in Proposition 6.28 for $a \in \{0, \dots, j-1\}$ we obtain that

$$\lim_{y \searrow 0} |\mu_{l-b}(\gamma_t(y)) - \mu_{j-b}(\gamma_t(y))| > 0$$

if and only if

$$\lim_{y \searrow 0} |\mu_{l-b-1,l}(\gamma_t(y)) - \mu_{j-b-1,j}(\gamma_t(y))| > 0$$

by an easy induction on $b \in \{0, \dots, j-1\}$. So we have

$$\lim_{y \searrow 0} |\mu_l(\gamma_t(y)) - \mu_j(\gamma_t(y))| > 0$$

if and only if

$$\lim_{y \searrow 0} |\mu_{l-j,l}(\gamma_t(y)) - \mu_{0,j}(\gamma_t(y))| = \lim_{y \searrow 0} e^{\mu_{l-j-1,l}(\gamma_t(y))} > 0$$

with (1),(i).

(2): This is clear with $(II)_l$ in Proposition 6.28 for $l \in \{0, \dots, r\}$. ■

The following example shows that Proposition 6.28 and Corollary 6.29 do not hold in general if one dispenses with the property " $\lim_{y \searrow 0} (L_C(\gamma_t(y))) > 0$ " even if the cell is bounded.

6.30 Example

Let C , y_0, y_1 and Θ_0, Θ_1 be as in Example 6.12. Note that (y_0, y_1) is a 1-logarithmic scale with center (Θ_0, Θ_1) . On the other hand we have $\lim_{t \searrow 0} \Theta_1(t) = -\infty$, $\lim_{t \searrow 0} \mu_1(t) - \mu_0(t) = 0$, and $\lim_{t \searrow 0} L_C(t) = 0$.

6.31 Corollary

Let $\gamma :]0, 1[\rightarrow H$ be a definable curve compatible with C . Assume $\lim_{y \searrow 0} \gamma_z(y) - \mu_\kappa(\gamma_t(y)) = 0$. Then

$$\lim_{y \searrow 0} \mathcal{P}_{l,\kappa}(\gamma(y)) = 0$$

for every $l \in \{0, \dots, \kappa\}$.

Proof

We do an induction on l . If $l = 0$ then this is clear with Definition 6.18.

$l - 1 \rightarrow l$: By Definition 6.18 we obtain

$$\lim_{y \searrow 0} \mathcal{P}_{l,\kappa}(\gamma(y)) = \lim_{y \searrow 0} \log\left(1 + \frac{\sigma_{l-1} \mathcal{P}_{l-1,\kappa}(\gamma(y))}{e^{\mu_{\kappa-l,\kappa}(\gamma_t(y))}}\right) = 0$$

by the inductive hypothesis, Corollary 6.29(1),(i) and the continuity of the global logarithm at 1. ■

6.32 Definition

Let $q \in \mathbb{Q}^{r+1}$ we with $(q_\kappa, \dots, q_r) \neq 0$. Assume $k := k^{\text{ch}} \geq 0$. Set

$$l_{\min}(q) := \min\{j \in \{k+1, \dots, r\} \mid q_j \neq 0\}.$$

6.33 Definition

Let $q \in \mathbb{Q}^{r+1}$. Let $l := l_{\min}(q)$ if $k := k^{\text{ch}} \geq 0$.

- a) We call q **r -positive** if $q_r > 0$.
- b) Suppose $k \geq 0$. We call q **k -positive** if $q_k > 0$ or if $q_k = 0$ then $q_l < 0$.
- c) We call q **r -negative** if $q_r < 0$.
- d) Suppose $k \geq 0$. We call q **k -negative** if $q_k < 0$ or if $q_k = 0$ then $q_l > 0$.

6.34 Proposition

Let $q := (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$. Let $\gamma :]0, 1[\rightarrow H$ be a definable curve compatible with C . Suppose $\lim_{y \searrow 0} \gamma_z(y) - \mu_\kappa(\gamma_t(y)) = 0$. Then the following holds.

- (1) $\lim_{y \searrow 0} |z_l(\gamma(y))| \in \mathbb{R}_{>0}$ for $l < \kappa$.
- (2) If q is κ -positive then $\lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| = 0$. If q is κ -negative then $\lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| = \infty$.

Proof

(1): Assume there is $l \in \{0, \dots, \kappa - 1\}$ such that $\lim_{y \searrow 0} |z_l(\gamma(y))| \in \{0, \infty\}$. Then with (II_l) in Proposition 6.28 we find $j \in \{0, \dots, l\}$ such that $\lim_{y \searrow 0} (\gamma_z(y) - \mu_j(\gamma_t(y))) = 0$ and therefore

$$\lim_{y \searrow 0} (\mu_\kappa(\gamma_t(y)) - \mu_j(\gamma_t(y))) = 0,$$

a contradiction to Corollary 6.29(1),(ii). We obtain the assertion with o-minimality.

(2): If $\kappa = r$ we obtain with $(II)_r$ in Proposition 6.28 $\lim_{y \searrow 0} z_r(\gamma(y)) = 0$. With (1) we are done. So let $0 \leq k := k^{\text{ch}}$ and suppose $\kappa = k$. Let

$$W := \{(t, z) \in H \mid |z_k(t, z)| < 1/\exp_{r-k}(1)\},$$

$$L_1 : W \rightarrow \mathbb{C}, (t, z) \mapsto \log(|z_k(t, z)|),$$

and for $l \in \{2, \dots, r - k\}$

$$L_l : W \rightarrow \mathbb{C}, (t, z) \mapsto \log_{l-1}(-\log(|z_k(t, z)|)).$$

Note that L_l is well-defined and definable for $l \in \{1, \dots, r - k\}$. By o-minimality and considering a suitable subcurve of γ if necessary we may assume that $\gamma(]0, 1[) \subset W$.

Claim

Let $l \in \{1, \dots, r - k\}$. There is a definable function $h_l : W \rightarrow \mathbb{C}$ with $\lim_{y \searrow 0} h_l(\gamma(y)) = 0$ such that

$$z_{k+l}(t, z) = L_l(t, z)(1 + h_l(t, z))$$

for every $(t, z) \in W$.

Proof of the claim

We do an induction on l . Suppose $l = 1$. We have for $(t, z) \in W$

$$z_{k+1}(t, z) = \log(|z_k(t, z)|) \left(1 - \frac{\Theta_{k+1}(t) - i \arg(\sigma_k z_k(t, z))}{\log(|z_k(t, z)|)} \right).$$

So choose

$$h_1 : W \rightarrow \mathbb{C}, (t, z) \mapsto -\frac{\Theta_{k+1}(t) - i \arg(\sigma_k z_k(t, z))}{\log(|z_k(t, z)|)}.$$

We have $\lim_{y \searrow 0} h_1(\gamma(y)) = 0$ by (I_{k+1}) and (II_k) in Proposition 6.28.

$l = 2$: Note that $\sigma_{k+1} = -1$. We have for $(t, z) \in W$

$$\begin{aligned} z_{k+2}(t, z) &= \log(\sigma_{k+1} L_1(t, z)(1 + h_1(t, z))) - \Theta_{k+2}(t) \\ &= \log(-L_1(t, z)) \left(1 + \frac{\log(1 + h_1(t, z)) - \Theta_{k+2}(t)}{\log(-L_1(t, z))} \right). \end{aligned}$$

Choose

$$h_2 : W \rightarrow \mathbb{C}, (t, z) \mapsto \frac{\log(1 + h_1(t, z)) - \Theta_{k+2}(t)}{\log(-L_1(t, z))}.$$

Then h_2 is definable and by (I_{k+2}) in Proposition 6.28 we get $\lim_{y \searrow 0} h_2(\gamma(y)) = 0$.

$l - 1 \rightarrow l$: We may assume that $l > 2$. Note that $\sigma_{k+l-1} = 1$. With the inductive hypothesis we obtain

$$\begin{aligned} z_{k+l}(t, z) &= \log(\sigma_{k+l-1} L_{l-1}(t, z)(1 + h_{l-1}(t, z))) - \Theta_{k+l}(t) \\ &= \log(L_{l-1}(t, z)) \left(1 + \frac{\log(1 + h_{l-1}(t, z)) - \Theta_{k+l}(t)}{\log(L_{l-1}(t, z))} \right) \end{aligned}$$

for every $(t, z) \in W$. Choose

$$h_l : H \rightarrow \mathbb{C}, (t, z) \mapsto \frac{\log(1 + h_{l-1}(t, z)) - \Theta_{k+l}(t)}{\log(L_{l-1}(t, z))}.$$

Then h_l is definable and by (I_{k+l}) in Proposition 6.28 we get $\lim_{y \searrow 0} h_l(\gamma(y)) = 0$. ■ Claim

Let h_l and L_l be as in the Claim for $l \in \{1, \dots, r - k\}$. Then we obtain for $y \in]0, 1[$

$$|(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| = d(\gamma(y)) |z_k(\gamma(y))|^{q_k} \prod_{l=1}^{r-k} L_l(\gamma(y))^{q_{\kappa+l}} (1 + h_l(\gamma(y)))^{q_{\kappa+l}}$$

where

$$d(\gamma(y)) := \prod_{l=0}^{k-1} |z_l(\gamma(y))|^{q_l}.$$

Note that $\lim_{y \searrow 0} |z_k(\gamma(y))| = 0$ by (II_k) in Proposition 6.28. So with the growth properties of the iterated logarithm we obtain

$$\lim_{y \searrow 0} |z_k(\gamma(y))|^{q_k} \prod_{l=1}^{r-k} |L_l(\gamma(y))|^{q_{\kappa+l}} = 0$$

if q is k -positive and

$$\lim_{y \searrow 0} |z_k(\gamma(y))|^{q_k} \prod_{l=1}^{r-k} L_l(\gamma(y))^{q_{\kappa+l}} = \infty$$

if q is k -negative. So one sees with (1) that $\lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| = 0$ if q is k -positive and $\lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(\gamma(y))| = \infty$ if q is k -negative. This finishes the proof of Proposition 6.34. \blacksquare

6.35 Corollary

Let $q \in \mathbb{Q}^{r+1}$ with $(q_\kappa, \dots, q_r) \neq 0$. Let $j_\kappa(q) := \min\{j \in \{\kappa, \dots, r\} \mid q_j \neq 0\}$ and

$$q_{\kappa, \text{diff}} := (0, \dots, 0, q_\kappa - 1, \dots, q_{j_\kappa(q)} - 1, q_{j_\kappa(q)+1}, \dots, q_r).$$

For $t \in \pi(C)$ it is

$$\lim_{z \rightarrow \mu_\kappa(t)} \left| \frac{\frac{d}{dz}(\sigma \mathcal{Z})^{\otimes q}(t, z)}{(\sigma \mathcal{Z})^{\otimes q_{\kappa, \text{diff}}}(t, z)} \right| \in \mathbb{R}^*.$$

(Compare Definition 5.10 and Proposition 5.12 for the case that C is simple and $\kappa \neq r$: On a simple cell we have $k = 0$.)

Proof

Let $t \in \pi(C)$. We have

$$\frac{d}{dz} \sigma_l z_l = \frac{\sigma_l}{\prod_{j=0}^{l-1} z_j}.$$

We obtain

$$\frac{d}{dz} (\sigma \mathcal{Z})^{\otimes q} = \sum_{j=0}^r q_j (\sigma_j z_j)^{q_j-1} \frac{1}{\prod_{i=0}^{j-1} z_i} \prod_{i \neq j} (\sigma_i z_i)^{q_i}.$$

Note also that $\lim_{z \rightarrow \mu_\kappa(t)} |\mathcal{P}_{j,\kappa}(t, z)| = 0$ and therefore with Proposition 6.20

$$\lim_{z \rightarrow \mu_\kappa(t)} |z_j(t, z)| = \lim_{z \rightarrow \mu_\kappa(t)} |\mathcal{P}_{j,\kappa}(t, z) + \sigma_j e^{\mu_{\kappa-j-1,\kappa}(t)}| = e^{\mu_{\kappa-j-1,\kappa}(t)} \in \mathbb{R}^*$$

for $j \in \{0, \dots, \kappa - 1\}$. We get the assertion with the growth properties of the complex logarithm (compare the proof of Proposition 6.34(2)). ■

6.36 Definition

We call C **near with respect to** μ_κ if there is a definable curve $\gamma :]0, 1[\rightarrow C$ compatible with C such that $\lim_{y \searrow 0} (\mu_\kappa(\gamma_t(y)) - \gamma_x(y)) = 0$. Otherwise we call C **far with respect to** μ_κ .

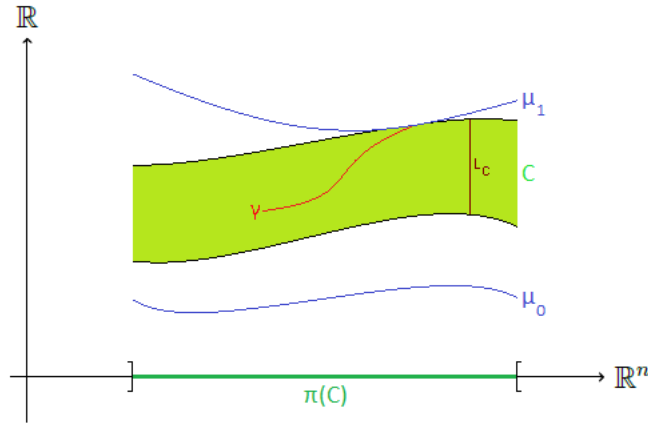


Figure 6.3 The functions μ_j for a 1-logarithmic scale \mathcal{Y} with $\text{sign}(\mathcal{Y}) = (1, -1)$ where μ_0 is far and μ_1 is near with respect to C .

6.37 Proposition

Let $k := k^{\text{ch}} \geq 0$ and let C be near with respect to μ_k . Then $\Theta_l = 0$ for every $l \in \{k + 1, \dots, r\}$.

Proof

Let $\gamma :]0, 1[\rightarrow C$ be a definable curve compatible with C and let

$$\lim_{y \searrow 0} (\mu_k(\gamma_t(y)) - \gamma_x(y)) = 0.$$

Then $\lim_{y \searrow 0} y_k(\gamma(y)) = 0$ by (II_k) in Proposition 6.28. We do an induction on $l \in \{k + 1, \dots, r\}$.

$l = k + 1$: Assume $\Theta_{k+1} \neq 0$. Then there is $\epsilon_{k+1} \in]0, 1[$ such that

$$|\log(\sigma_k y_k(t, x)) - \Theta_{k+1}(t)| < \epsilon_{k+1} |\log(\sigma_k y_k(t, x))|$$

for all $(t, x) \in C$. We obtain

$$\lim_{y \searrow 0} \left| 1 - \frac{\Theta_{k+1}(\gamma_t(y))}{\log(\sigma_k y_k(\gamma(y)))} \right| \leq \epsilon_{k+1}$$

and therefore $\lim_{y \searrow 0} |\Theta_{k+1}(\gamma_t(y))| = \infty$, a contradiction to (I_{k+1}) in Proposition 6.28.

$l \rightarrow l + 1$: We have $\sigma_{k+1} = -1$ and $\sigma_{k+2} = \dots = \sigma_l = 1$. So we obtain

$$\log(\sigma_l y_l) = \log_{l-k}(-\log(\sigma_k y_k))$$

on C by the inductive hypothesis. Assume $\Theta_{l+1} \neq 0$. Then there is $\epsilon_{l+1} \in]0, 1[$ such that

$$|\log_{l-k}(-\log(\sigma_k y_k(t, x))) - \Theta_{l+1}(t)| < \epsilon_{l+1} |\log_{l-k}(-\log(\sigma_k y_k(t, x)))|$$

for all $(t, x) \in C$. We obtain

$$\lim_{y \searrow 0} \left| 1 - \frac{\Theta_{l+1}(\gamma_t(y))}{\log_{l-k}(-\log(\sigma_k y_k(\gamma(y))))} \right| \leq \epsilon_{l+1}$$

and therefore $\lim_{y \searrow 0} |\Theta_{l+1}(\gamma_t(y))| = \infty$, a contradiction to (I_{l+1}) in Proposition 6.28. ■

6.1.2 Regularity, Persistence and Integrability of Logarithmic Scales in the Complex Setting

For this section we set the following: Fix $l, m \in \mathbb{N}_0$ such that $l + m = n$. Let $w := (w_1, \dots, w_l)$ range over \mathbb{R}^l and $u := (u_1, \dots, u_m)$ over \mathbb{R}^m . Let $C \subset \mathbb{R}^n \times \mathbb{R}$ be a definable fat cell with respect to x and $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $C \subset X$. Assume that X_w is open for every $w \in \mathbb{R}^l$. Let $\pi_l : \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^l, (w, u, x) \mapsto w$. For $w \in \pi_l(C)$ and a definable curve $\gamma :]0, 1[\rightarrow H_w = \{(u, z) \in \mathbb{R}^m \times \mathbb{C} \mid (w, u, z) \in H\}$ we set $\gamma_u := (\gamma_1, \dots, \gamma_m)$ for the first m real components and $\gamma_z := \gamma_{m+1}$ for the last complex component and if $\gamma_{m+1}(y) \in \mathbb{R}$ for every $y \in]0, 1[$ we also write γ_x instead of γ_z . For $w \in \pi_l(C)$ we say that a definable curve $\gamma :]0, 1[\rightarrow H_w$ is compatible with C_w if the curve $\hat{\gamma} :]0, 1[\rightarrow H, y \mapsto (w, \gamma(y))$, is compatible with C . Fix $\kappa = r$ or $\kappa \in \{k^{\text{ch}}, r\}$ if $k^{\text{ch}} \geq 0$.

6.38 Definition

Let $g : \pi(C) \rightarrow \mathbb{R}$ be definable. We call $g : \pi(C) \rightarrow \mathbb{R}$ **C -consistent in u with respect to X** if the following holds. Let $w \in \pi_l(C)$. Then for every definable curve $\gamma :]0, 1[\rightarrow C_w$ compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ it is

$$\lim_{y \searrow 0} g(w, \gamma_u(y)) \in \mathbb{R}.$$

So a C -consistent function $g : \pi(C) \rightarrow \mathbb{R}$ in u with respect to X has the following property: If for $(w_0, u_0) \in \pi(X)$ there is an open neighbourhood U of u_0 in $\pi(X)_{w_0}$ and $\epsilon > 0$ such that $L_C(w_0, u) > \epsilon$ for every $u \in \pi(C)_{w_0} \cap U$ then g_{w_0} is bounded at u_0 , i.e. there is $V \subset U$ open with $u_0 \in V$ such that $g_{w_0}|_{\pi(C)_{w_0} \cap V}$ is bounded. If $C = X$ then g is locally bounded in u .

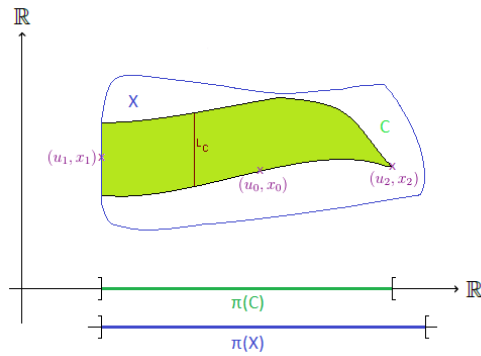


Figure 6.4 Let $m = 1$ and $l = 0$. A C -consistent function $f : C \rightarrow \mathbb{R}$ in u with respect to X is bounded at u_0 , but not necessarily bounded at u_1 respectively u_2 .

6.39 Remark

Let $Y \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $X \subset Y$ such that Y_w is open for every $w \in \mathbb{R}^l$. A function $g : \pi(C) \rightarrow \mathbb{R}$ which is C -consistent in u with respect to Y is also C -consistent in u with respect to X .

6.40 Remark

- (1) The set of C -consistent functions in u with respect to X on $\pi(C)$ is an \mathbb{R} -algebra with respect to pointwise addition and multiplication.
- (2) The set of positive C -consistent functions in u with respect to X is a divisible monoid with respect to pointwise multiplication.
- (3) Let $q \in \mathbb{Q}$. Let $f : \pi(C) \rightarrow \mathbb{R}$ be C -consistent in u with respect to X . Then $\exp(q \cdot f)$ is C -consistent in u with respect to X .

6.41 Example

- (1) The functions $\Theta_0, \dots, \Theta_r$ are C -consistent in u with respect to $\mathbb{R}^n \times \mathbb{R}$.
- (2) Let $l \in \{0, \dots, \kappa\}$. The function $\pi(C) \mapsto \mathbb{R}, t \mapsto \mu_{j,l}(t)$, is C -consistent in u with respect to $\mathbb{R}^n \times \mathbb{R}$ for $j \in \{0, \dots, l\}$.
- (3) The logarithm of a C -heir is C -consistent in u with respect to $\mathbb{R}^n \times \mathbb{R}$.

Proof

- (1): Follows from (I_j) in Proposition 6.28 for $j \in \{0, \dots, r\}$.
- (2): Follows from Corollary 6.29(1),(i).
- (3): Let $g : \pi(C) \rightarrow \mathbb{R}_{>0}$ be a C -heir. Note that there is $l \in \mathbb{N}$, an l -logarithmic scale $\hat{\mathcal{Y}}$ with center $(\hat{\Theta}_0, \dots, \hat{\Theta}_l)$ such that $g = \exp(\hat{\Theta}_j)$ for $j \in \{1, \dots, l\}$. So this property follows from (I_j) in Proposition 6.28 applied to $\hat{\mathcal{Y}}$ for $j \in \{1, \dots, l\}$. ■

6.42 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, be $(m+1, X)$ -restricted (e, r) -preparing in x with preparing tuple

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

with $b = (b_1, \dots, b_s)$, $\exp(d) = (\exp(d_1), \dots, \exp(d_s))$ and $P = (p_1, \dots, p_s)^t$. Then the following holds.

- (1) Let $j \in \{1, \dots, s\}$. The function b_j is C -consistent in u with respect to X .

- (2) Assume that f is locally bounded in (u, x) with reference set X . The function a is C -consistent in u with respect to X .

Proof

For $(t, x) \in C$ let

$$V(t, x) := v(b_1(t)|\mathcal{Y}(t, x)|^{\otimes p_1} e^{d_1(t, x)}, \dots, b_s(t)|\mathcal{Y}(t, x)|^{\otimes p_s} e^{d_s(t, x)}).$$

For $(t, x) \in C$ we have

$$f(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} e^{c(t, x)} V(t, x).$$

Let $w \in \pi_l(C)$ and let $\gamma :]0, 1[\rightarrow C_w$ be a definable curve compatible with C_w with $(u_0, x_0) := \lim_{y \searrow 0} \gamma(y) \in X_w$. We find $\delta > 0$ and $B > 0$ such that $U := Q^m(u_0, \delta) \times]x_0 - \delta, x_0 + \delta[\subset X_w$, $|c(w, u, x)| < B$ and $|d_j(w, u, x)| < B$ for $j \in \{1, \dots, s\}$ and every $(u, x) \in U$. By passing to a suitable subcurve we may assume that $\gamma(y) \in Q^m(u_0, \delta) \times]x_0 - \delta/2, x_0 + \delta/2[$ for every $y \in]0, 1[$. Similarly as in the proof of Remark 6.27(1) we find a continuous definable function $\tilde{\gamma}_x :]0, 1[\rightarrow \mathbb{R}$ such that $\tilde{\gamma} := (\gamma_u, \tilde{\gamma}_x)$ is a definable curve in $C_w \cap U$ compatible with C_w and

$$\lim_{y \searrow 0} |\tilde{\gamma}_x(y) - \mu_l(w, \gamma_u(y))| > 0$$

for every $l \in \{0, \dots, r\}$.

(1): Let $j \in \{1, \dots, s\}$. Note that

$$\lim_{y \searrow 0} |\mathcal{Y}(w, \tilde{\gamma}(y))|^{\otimes p_j} \in \mathbb{R} \setminus \{0\}$$

by Corollary 6.29(2). Since

$$b_j(t)|\mathcal{Y}(t, x)|^{\otimes p_j} e^{d_j(t, x)} \in [-1, 1]$$

for every $(t, x) \in C$ we obtain $\lim_{y \searrow 0} b_j(w, \gamma_u(y)) \in \mathbb{R}$.

(2): Note that there is $\rho > 1$ such that $V(t, x) \in]1/\rho, \rho[$ for every $(t, x) \in C$ and that

$$\lim_{y \searrow 0} |\mathcal{Y}(w, \tilde{\gamma}(y))|^{\otimes q} \in \mathbb{R} \setminus \{0\}$$

by Corollary 6.29(2). Since $\lim_{y \searrow 0} f(w, \tilde{\gamma}(y)) \in \mathbb{R}$ we obtain

$$\lim_{y \searrow 0} a(w, \gamma_u(y)) \in \mathbb{R}.$$

■

6.43 Definition

Let $D \subset \pi(C)$ be definable. A function $f : D \rightarrow \mathbb{R}$ is called **C -regular in u with respect to X** if there is a set E of positive definable functions on $\pi(C)$ such that the following holds. The set $\log(E)$ consists only of C -consistent functions in u with respect to X and f can be constructed from E .

6.44 Remark

- (1) Let $D \subset \pi(C)$ be definable and $f : D \rightarrow \mathbb{R}$ be C -regular and C -consistent in u with respect to X . Then $\exp(f)$ is C -regular and C -consistent in u with respect to X .
- (2) Let $D \subset \pi(C)$ be definable and $f_1, \dots, f_m : D \rightarrow \mathbb{R}$ be C -regular in u with respect to X . Let $G : \mathbb{R}^m \rightarrow \mathbb{R}$ be log-analytic. Then $G(f_1, \dots, f_m)$ is C -regular in u with respect to X .

Proof

Let E be a set of positive definable functions on $\pi(C)$ such that $\log(E)$ consists only of C -consistent functions in u with respect to X and f can be constructed from E .

(1): With Remark 6.40(3) we see that $\exp(f)$ is C -consistent in u with respect to X . So with Proposition 3.11(1) we see that f can be constructed from $E \cup \{\exp(f)\}$ and is therefore C -regular.

(2): Follows with Proposition 3.11(2). ■

6.45 Example

- (1) A C -nice function on $\pi(C)$ is C -regular in u with respect to X .
- (2) Let $q_0, \dots, q_{\kappa-1} \in \mathbb{Q}$. Suppose that Θ is C -regular. Then the function $\pi(C) \rightarrow \mathbb{R}, t \mapsto \prod_{j=0}^{\kappa-1} e^{q_j \mu_{j, \kappa}(t)}$, is C -regular and C -consistent in u with respect to X .
- (3) Suppose that $X \subset \mathbb{R}^2$ is open with $0 \in X$. Let $C :=]0, 1]^2$ and let $h :]0, 1[\rightarrow \mathbb{R}, u \mapsto e^{-1/u}$. Then h is not C -regular in u with respect to X .

Proof

(1): Follows directly with Example 6.41(1), Definition 4.40 and Definition 6.43.

(2): This follows from Example 6.41(2), Remark 6.40(3) and Remark 6.44.

(3): Suppose the contrary. Let E be a set of C -regular functions with respect to X such that f can be constructed from E . Note that for every $g \in \log(E)$ there is an open neighbourhood U of 0 in $\pi(X)$ such that $g|_{U \cap \pi(C)}$ is bounded. Therefore $h \sim \alpha$ for a log-analytic $\alpha : \pi(C) \rightarrow \mathbb{R}$ at zero. But we have $h \sim e^{-1/u}$ at zero, a contradiction. ■

For the rest of Chapter 6.1 we assume that Θ is C -regular.

6.46 Definition

Let $g : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be a function.

- (a) We say that g is (C, \mathcal{Y}, κ) -**persistent in u with respect to X** if g is definable and the following holds. Let $w \in \pi_l(C)$. Then for every definable curve $\gamma :]0, 1[\rightarrow C_w$ compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ and

$$\lim_{y \searrow 0} (\gamma_x(y) - \mu_\kappa(w, \gamma_u(y))) = 0$$

it holds $\lim_{y \searrow 0} g(w, \gamma_u(y)) > 0$.

- (b) We say that g is (C, \mathcal{Y}, κ) -**non-persistent in u with respect to X** if g is definable and the following holds. Let $w \in \pi_l(C)$. Then for every definable curve $\gamma :]0, 1[\rightarrow C_w$ compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ and

$$\lim_{y \searrow 0} (\gamma_x(y) - \mu_\kappa(w, \gamma_u(y))) = 0$$

it holds $\lim_{y \searrow 0} g(w, \gamma_u(y)) = 0$.

6.47 Remark

Let $g : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be a function. Let $Y \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $X \subset Y$ such that Y_w is open for every $w \in \mathbb{R}^l$. If g is (C, \mathcal{Y}, κ) -persistent ((C, \mathcal{Y}, κ) -non-persistent) in u with respect to Y then g is (C, \mathcal{Y}, κ) -persistent ((C, \mathcal{Y}, κ) -non-persistent) in u with respect to X .

Note that a function which is not (C, \mathcal{Y}, κ) -persistent in u with respect to X is not automatically (C, \mathcal{Y}, κ) -non-persistent in u with respect to X in general (compare with Remark 6.49 or Example 6.52).

Since C , \mathcal{Y} and X are fixed we say "κ-persistent" respectively "κ-non-persistent" instead of " (C, \mathcal{Y}, κ) -persistent in u with respect to X " respectively " (C, \mathcal{Y}, κ) -non-persistent in u with respect to X " in Chapter 6.1 if not otherwise mentioned. We also say "consistent" respectively "regular" instead of " C -consistent in u with respect to X " respectively " C -regular in u with respect to X " in Chapter 6.1 if not otherwise mentioned.

6.48 Remark

- (1) The set of all κ -persistent (κ -non-persistent) functions on $\pi(C)$ is closed under pointwise addition and multiplication and closed under taking positive roots.
- (2) Let $f : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be κ -persistent (κ -non-persistent). Let $q \in \mathbb{Q}$. Then $\pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \sqrt[q]{f(t)}$, is κ -persistent (κ -non-persistent).
- (3) Let $f_1, f_2 : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be κ -persistent (κ -non-persistent). Then

$$\pi(C) \mapsto \mathbb{R}_{\geq 0}, t \mapsto \min\{f_1(t), f_2(t)\},$$

and

$$\pi(C) \mapsto \mathbb{R}_{\geq 0}, t \mapsto \max\{f_1(t), f_2(t)\},$$

are κ -persistent (κ -non-persistent).

6.49 Remark

If C is far with respect to μ_κ then every definable function $\pi(C) \rightarrow \mathbb{R}_{\geq 0}$ is κ -persistent and κ -non-persistent.

6.50 Remark

Let $g : \pi(C) \rightarrow \mathbb{R}_{> 0}$ be a consistent function. Then $\pi(C) \rightarrow \mathbb{R}_{> 0}, t \mapsto 1/g(t)$, is κ -persistent.

6.51 Remark

Let $g : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be a definable function. If for every $w \in \pi_l(C)$ and every definable curve $\gamma :]0, 1[\rightarrow C_w$ compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ and

$$\lim_{y \searrow 0} \text{dist}(\mu_\kappa(w, \gamma_u(y)), (C_w)_{\gamma_u(y)}) = 0$$

we have $\lim_{y \searrow 0} g(w, \gamma_u(y)) > 0$ ($\lim_{y \searrow 0} g(w, \gamma_u(y)) = 0$) then g is κ -persistent (κ -non-persistent).

6.52 Example

Let $w = 0$ and $u = 1$. Let $X = \mathbb{R}^2$ and $l = 0$. Let C and $\mathcal{Y} = (y_0, y_1, y_2)$ be as in Example 6.12 (i.e. $r = 2$ and $k^{\text{ch}} = 1$). Then $g_1 :]0, 1[\rightarrow \mathbb{R}_{> 0}, u \mapsto u$, is 2-persistent and $g_2 :]0, 1[\rightarrow \mathbb{R}_{> 0}, u \mapsto 1 - u$, is 2-non-persistent. Additionally $g_3 :]0, 1[\rightarrow \mathbb{R}_{\geq 0}, u \mapsto (u - 1/2)^2$, is neither 1-persistent nor 1-non-persistent and there is only one 1-non-persistent function, namely the zero function. We have that $g_4 :]0, 1[\rightarrow \mathbb{R}_{\geq 0}, u \mapsto u + 1$, is 1-persistent and 2-persistent.

Proof

We have

$$C = \{(u, x) \in \mathbb{R} \times \mathbb{R} \mid u \in]0, 1[, \frac{1}{1+u} + e^{-u-1/u} < x < \frac{1}{1+u} + e^{-1/u}\}.$$

For $u \in]0, 1[$ we have $L_C(u) = e^{-1/u}(1 - e^{-u})$. Note that $L_C(u) > 0$ for $u \in]0, 1[$, that $\lim_{u \searrow 0} L_C(u) = 0$ and that $\lim_{u \nearrow 1} L_C(u) > 0$. For $u \in]0, 1[$ we have $\mu_2(u) = \frac{1}{1+u} + e^{-1-1/u}$. We see that

$$\lim_{u \nearrow 1} \text{dist}(\mu_2(u), C_u) = \lim_{u \nearrow 1} (e^{-u-1/u} - e^{-1-1/u}) = 0$$

and if $u \in]0, 1[$ we see that $\text{dist}(\mu_2(u), C_u) > 0$. So with Remark 6.51 it is enough to show that $\lim_{u \nearrow 1} g_1(u) > 0$ and $\lim_{u \nearrow 1} g_2(u) = 0$. But this is clear.

For $u \in]0, 1[$ we have $\mu_1(u) = \frac{1}{1+u} + e^{-1/u}$ and we see that $\text{dist}(\mu_1(u), C_u) = 0$ for every $u \in]0, 1[$. So the zero function is the only 1-non-persistent function on $\pi(C)$. (For every $u \in]0, 1[$ there is a definable function $f_u :]0, 1[\rightarrow C_u$ with $\lim_{y \searrow 0} f_u(y) - \mu_1(u) = 0$. So $\gamma :]0, 1[\rightarrow C, y \mapsto (u, f_u(y))$, is a definable curve compatible with C with $\lim_{y \searrow 0} \gamma_x(y) - \mu_1(u) = 0$. Definition 6.46(b) gives $g(u) = 0$.) Similarly one sees that if a function on $\pi(C)$ is 1-persistent it does not have any zero. So g_3 is neither 1-non-persistent nor 1-persistent. Since $g_4(u) \geq 1$ for every $u \in]0, 1[$ the assertion for g_4 follows. ■

6.53 Example

Let $X = \mathbb{R}^n \times \mathbb{R}$. Let $l \in \{1, \dots, r\}$ and $j \in \{0, \dots, l - 1\}$. Then

$$\pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto e^{\mu_{j,l}(t)},$$

and

$$\pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto |\mu_l(t) - \mu_j(t)|,$$

are κ -persistent.

Proof

Let $w \in \pi_l(C)$. Then for every definable curve $\gamma :]0, 1[\rightarrow C_w$ compatible with C_w we have $\lim_{y \searrow 0} e^{\mu_{j,l}(w, \gamma_u(y))} > 0$ by Corollary 6.29(1),(i) and

$$\lim_{y \searrow 0} |\mu_l(w, \gamma_u(y)) - \mu_j(w, \gamma_u(y))| > 0$$

if $j \neq l$ by Corollary 6.29(1),(ii). ■

6.54 Remark

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, be $(m+1, X)$ -restricted (e, r) -prepared in x with preparing tuple

$$(r, \mathcal{Y}, a, \exp(c), q, s, v, b, \exp(d), P)$$

with $b = (b_1, \dots, b_s)$, $\exp(d) = (\exp(d_1), \dots, \exp(d_s))$ and $P = (p_1, \dots, p_s)^t$. Then the following holds.

- (1) Let $j \in \{1, \dots, s\}$. The function $\pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto |1/b_j(t)|$, is κ -persistent. Additionally if p_j is κ -negative then $|b_j|$ is κ -non-persistent.
- (2) Suppose that $a \neq 0$ and that f is locally bounded in (u, x) with reference set X . Then the function $\pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto |1/a(t)|$, is κ -persistent. Additionally if q is κ -negative then $|a|$ is κ -non-persistent.

Proof

For $(t, x) \in C$ let

$$V(t, x) := v(b_1(t)|\mathcal{Y}(t, x)|^{\otimes p_1} e^{d_1(t, x)}, \dots, b_s(t)|\mathcal{Y}(t, x)|^{\otimes p_s} e^{d_s(t, x)}).$$

For $(t, x) \in C$ we have

$$f(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} e^{c(t, x)} V(t, x).$$

Let $w \in \pi_l(C)$ and $\gamma :]0, 1[\rightarrow C_w$ be a definable curve compatible with C_w such that $\lim_{y \searrow 0} \gamma_u(y) \in X_w$ and

$$\lim_{y \searrow 0} |\gamma_x(y) - \mu_\kappa(w, \gamma_u(y))| = 0.$$

We have $\lim_{y \searrow 0} (e^{c(w, \gamma(y))}) \in \mathbb{R}_{>0}$ and $\lim_{y \searrow 0} (e^{d_i(w, \gamma(y))}) \in \mathbb{R}_{>0}$ for every $i \in \{1, \dots, s\}$.

(1): The first part follows immediately with Remark 6.42(1) and Remark 6.50. Assume that p_j is κ -negative. Then by Proposition 6.34(2) we have $\lim_{y \searrow 0} |\mathcal{Y}(w, \gamma(y))|^{\otimes p_j} = \infty$ and therefore

$$\lim_{y \searrow 0} b_j(w, \gamma_u(y)) = 0$$

since

$$b_j(t)|\mathcal{Y}(t, x)|^{\otimes p_j} e^{d_j(t, x)} \in [-1, 1]$$

for every $(t, x) \in C$.

(2): The first part follows immediately with Remark 6.42(2) and Remark 6.50. Note that there is $\rho > 1$ such that $V(t, x) \in]1/\rho, \rho[$ for every $(t, x) \in C$. Assume that q is κ -negative. Then by Proposition 6.34(2) and Remark 6.17(1) we have

$$\lim_{y \searrow 0} |\mathcal{Y}(w, \gamma(y))|^{\otimes q} = \infty$$

and therefore $\lim_{y \searrow 0} a(w, \gamma_u(y)) = 0$ since $\lim_{y \searrow 0} f(w, \gamma_u(y)) \in \mathbb{R}$. ■

6.55 Definition

For definable functions $U, W : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a definable function $G : H \rightarrow \mathbb{C}$ we set

$$\mathcal{B}(G, U) := \{(t, z) \in H \mid |G(t, z)| < U(t)\}$$

and

$$\mathcal{A}(G, W, U) := \{(t, z) \in H \mid W(t) < |G(t, z)| < U(t)\}.$$

6.56 Remark

Let $U, W : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be definable. For a definable $G : H \rightarrow \mathbb{C}$ the sets $\mathcal{B}(G, U) \subset \mathbb{R}^n \times \mathbb{C}$ and $\mathcal{A}(G, W, U) \subset \mathbb{R}^n \times \mathbb{C}$ are definable. We have

$$\mathcal{B}(\mathcal{P}_{0, \kappa}, U) = \{(t, z) \in H \mid z \in B(\mu_\kappa(t), U(t))\}$$

and

$$\mathcal{A}(\mathcal{P}_{0, \kappa}, W, U) = \{(t, z) \in H \mid z \in A(\mu_\kappa(t), W(t), U(t))\}.$$

Outgoing from Definition 6.46 and Definition 6.55 we form "suitable definable subsets" of H which we need later in Chapter 6.1.3 to give a suitable result on integration for holomorphic extensions of prepared restricted log-exp-analytic function in one variable.

From Proposition 6.57 to Corollary 6.58 we fix regular functions $U : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and $W : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$. Suppose that U is κ -persistent.

6.57 Proposition

The following properties hold.

- (1) *Let $0 < \delta \leq 1/2$. There is a regular κ -persistent $\zeta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $K := \mathcal{B}(\mathcal{P}_{0, \kappa}, \zeta) \subset \mathcal{B}(\mathcal{P}_{l, \kappa}, U)$ for $l \in \{0, \dots, \kappa\}$ and*

$$\left| \frac{\mathcal{P}_{j-1, \kappa}(t, z)}{e^{\mu_{\kappa-j, \kappa}(t)}} \right| < \delta$$

for every $(t, z) \in K$ and every $j \in \{1, \dots, \kappa\}$. If $U > 0$ then $\zeta > 0$.

- (2) *Let W be κ -non-persistent. Then there is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0, \kappa}, N, M) \subset \mathcal{A}(z_\kappa, W, U)$.*

Proof

We start with the following claim.

Claim

Let $0 < \delta \leq 1/2$. There are globally subanalytic functions $\lambda_1, \lambda_2 : \mathbb{R}_{\geq 0} \rightarrow [0, \delta]$ such that the following holds.

- (i) λ_j is monotone increasing for $j \in \{1, 2\}$.
- (ii) For $r \in \mathbb{R}_{\geq 0}$ and $z \in \mathbb{C}$ we have $|\log(1+z)| < r$ if $|z| < \lambda_1(r)$.
- (iii) For $r \in \mathbb{R}_{\geq 0}$ and $z \in \mathbb{C}$ we have $r < |\log(1+z)|$ if $\lambda_2(r) < |z| < \delta$ for $z \in \mathbb{C}$.
- (iv) We have $\lambda_2(0) = 0$, $\lim_{r \searrow 0} \lambda_2(r) = 0$ and $\lim_{r \nearrow r_0} \lambda_j(r) > 0$ for $r_0 \in \mathbb{R}_{> 0}$ and $j \in \{1, 2\}$.

Proof of the claim

Define

$$\lambda_1 : \mathbb{R}_{\geq 0} \rightarrow [0, \delta], r \mapsto \sup\{c \in [0, \delta[\mid \text{for every } z \in \mathbb{C} \text{ with } |z| < c \text{ it holds } |\log^*(1+z)| < r\}.$$

Note that λ_1 is well-defined, monotone increasing and we have $\lambda_1(0) = 0$. For $r \in \mathbb{R}_{\geq 0}$ set

$$s(r) := \inf\{c \in [0, \delta[\mid \text{for every } z \in \mathbb{C} \text{ with } c < |z| < \delta \text{ it holds } r < |\log^*(1+z)|\}.$$

Note that $s(r_1) \leq s(r_2)$ for $0 \leq r_1 \leq r_2$. Set

$$\lambda_2 : \mathbb{R}_{\geq 0} \rightarrow [0, \delta], r \mapsto \begin{cases} s(r), & s(r) \neq \infty, \\ \delta, & \text{else.} \end{cases}$$

Note that λ_j is globally subanalytic since \log^* is globally subanalytic and fulfills the desired properties for $j \in \{1, 2\}$. ■ Claim

(1): Let λ_1 be as in the Claim. Let $l \in \{0, \dots, \kappa\}$. Let $\zeta_{l,l} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto U(t)$. Define by descending induction on $j \in \{1, \dots, l\}$

$$\zeta_{j-1,l} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \lambda_1(\zeta_{j,l}(t))e^{\mu_{\kappa-j,\kappa}(t)}.$$

By an easy descending induction on $j \in \{0, \dots, l\}$ one sees with the Claim that $\zeta_{j,l}$ is a well-defined regular κ -persistent function (compare with Remark 6.48(1) and Example 6.53). If $|\mathcal{P}_{j-1,\kappa}(t, z)| < \zeta_{j-1,l}(t)$ then

$$\left| \frac{\mathcal{P}_{j-1,\kappa}(t, z)}{e^{\mu_{\kappa-j,\kappa}(t)}} \right| < \lambda_1(\zeta_{j,l}(t)) \leq \delta$$

and therefore

$$|\mathcal{P}_{j,\kappa}(t, z)| = \left| \log \left(1 + \frac{\mathcal{P}_{j-1,\kappa}(t, z)}{e^{\mu_{\kappa-j,\kappa}(t)}} \right) \right| < \zeta_{j,l}(t)$$

for $j \in \{1, \dots, \kappa\}$ and $(t, z) \in H$ by definition of λ_1 . So we take

$$\zeta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{\zeta_{0,0}(t), \dots, \zeta_{0,\kappa}(t)\}.$$

(2): Let $0 < \delta \leq 1/2$. Let λ_2 be as in the Claim. By (1) there is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $K := \mathcal{B}(\mathcal{P}_{0,\kappa}, M) \subset \mathcal{B}(\mathcal{P}_\kappa, U) = \mathcal{B}(z_\kappa, U)$ and we have

$$\left| \frac{\mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}} \right| < \delta$$

for every $(t, z) \in K$ and every $l \in \{1, \dots, \kappa\}$. Let $N_\kappa := W$. Define by descending induction on $l \in \{1, \dots, \kappa\}$

$$N_{l-1} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \lambda_2(N_l(t)) e^{\mu_{\kappa-l,\kappa}(t)}.$$

By an easy descending induction on $l \in \{0, \dots, \kappa\}$ one sees that N_l is a well-defined regular κ -non-persistent function. This gives the following: If $N_{l-1}(t) < |\mathcal{P}_{l-1,\kappa}(t, z)|$ then

$$\lambda_2(N_l(t)) < \left| \frac{\mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}} \right| < \delta$$

and therefore

$$N_l(t) < \left| \log \left(1 + \frac{\mathcal{P}_{l-1,\kappa}(t, z)}{e^{\mu_{\kappa-l,\kappa}(t)}} \right) \right| = |\mathcal{P}_{l,\kappa}(t, z)|$$

for $l \in \{1, \dots, \kappa\}$ and $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, M)$ by definition of λ_2 . So set $N := N_0$. We obtain $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{A}(z_\kappa, W, U)$. This finishes the proof of Proposition 6.57 \blacksquare

For Proposition 6.58 and Corollary 6.59 we set for a definable function $V : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$

$$V_{\text{up}} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} 1/V(t), & V(t) \neq 0, \\ 0, & \text{else,} \end{cases}$$

and

$$V_{\text{down}} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} 1/V(t), & V(t) \neq 0, \\ 1, & \text{else.} \end{cases}$$

Note that V_{up} and V_{down} are regular if V is regular. Furthermore let $k := k^{\text{ch}}$.

6.58 Proposition

Let $q \in \mathbb{Q}^{r+1}$ with $(q_\kappa, \dots, q_r) \neq 0$. The following holds.

- (1) Let $q_\kappa > 0$. Then there is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0,\kappa}, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.
- (2) Let $k \geq 0$. Suppose that $\kappa = k$ and let q be k -positive with $q_k = 0$. Suppose that U_{up} is consistent. Then there is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0,\kappa}, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.
- (3) Suppose that U_{down} is κ -non-persistent and $q_\kappa < 0$. Then there is a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.

Proof

We may assume that C is near with respect to μ_κ . Otherwise we are done with Remark 6.49 by choosing $M = 0$ in (1),(2) and $N = M = 0$ in (3). Note that if $\kappa = k$ then $\sigma_{k+1} = -1$ and $\sigma_{k+2} = \dots = \sigma_r = 1$ and $\Theta_{\kappa+1} = \dots = \Theta_r = 0$ by Proposition 6.37 and therefore by definition of H

$$\log_l(1/\sigma_k z_k(t, z)) = \log_{l-1}(-\log(\sigma_k z_k(t, z))) \in \mathbb{C}^-$$

for $(t, z) \in H$ and $l \in \{1, \dots, r - k\}$. By Proposition 6.20 we have

$$\prod_{j=0}^r |z_j(t, z)|^{q_j} = \prod_{j=0}^{\kappa-1} |\sigma_j \mathcal{P}_{j,\kappa}(t, z) + e^{\mu_{\kappa-j-1,\kappa}(t)}|^{q_j} \prod_{l=\kappa}^r |z_l(t, z)|^{q_l}$$

for $(t, z) \in H$. We have $(t, z) \in \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$ if

$$\prod_{l=\kappa}^r |z_l(t, z)|^{q_l} < U(t) \prod_{j=0}^{\kappa-1} \frac{e^{-q_j \mu_{\kappa-j-1,\kappa}(t)}}{\left| \frac{\sigma_j \mathcal{P}_{j,\kappa}(t, z)}{e^{\mu_{\kappa-j-1,\kappa}(t)}} + 1 \right|^{q_j}}$$

for $(t, z) \in H$. By Proposition 6.57(1) there is a regular κ -persistent $M^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that for every $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, M^+)$ and $j \in \{0, \dots, \kappa - 1\}$ we have

$$\left| \frac{\sigma_j \mathcal{P}_{j,\kappa}(t, z)}{e^{\mu_{\kappa-j-1,\kappa}(t)}} \right| < 1/2$$

So we have $(t, z) \in \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$ if

$$\prod_{l=\kappa}^r |z_l(t, z)|^{q_l} < \rho U(t) \prod_{j=0}^{\kappa-1} e^{-q_j \mu_{\kappa-j-1,\kappa}(t)}$$

for $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, M^+)$ where $\rho := \prod_{j=0}^{\kappa-1} (1/3)^{|q_j|}$. (+)

Set

$$\phi : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \rho U(t) \prod_{j=0}^{\kappa-1} e^{-q_j \mu_{\kappa-j-1, \kappa}(t)}.$$

Note that ϕ is regular by Example 6.45(2) and Remark 6.44(2) and κ -persistent by Remark 6.48(1) (compare also with Example 6.53).

Claim 1

Let $l \in \mathbb{N}$, $m \in \mathbb{N}$ and $z \in \mathbb{C}^-$ with $\exp_l(m^2\pi) < |z|$. Then $|\log_l(z)| < \sqrt[2^l]{|z|}$.

Proof of Claim 1

For $p \in \mathbb{N}$ we set $c_p := \exp(p^2\pi)$. We do an induction on l .

$l = 1$: Assume $c_m < |z|$. We show

$$\sqrt{|z|^{-1/m}(\log^2(|z|) + \arg^2(z))} < 1.$$

One sees that

$$[c_m, \infty[\rightarrow \mathbb{R}_{>0}, x \mapsto \log^2(x)x^{-1/m},$$

is strictly monotone decreasing and that the function

$$\mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}, x \mapsto x^4\pi^2/\exp(x\pi),$$

takes its values on $]0, 1/2[$. Therefore

$$\log^2(c_m)c_m^{-1/m} = \log^2(c_m)/\exp(m\pi) = m^4\pi^2/\exp(m\pi) < 1/2$$

and consequently

$$\log^2(|z|)|z|^{-1/m} < 1/2.$$

Clearly

$$\arg^2(z)|z|^{-1/m} < 1/2.$$

$l - 1 \rightarrow l$: Assume $\exp_l(m^2\pi) < |z|$. We have by the definition of the complex logarithm

$$c_1 \leq c_m < \log_{l-1}(|z|) \leq |\log_{l-1}(z)|.$$

Therefore by the base case for $m = 1$ applied to $\log_{l-1}(z)$

$$|\log_l(z)| = |\log(\log_{l-1}(z))| < \sqrt{|\log_{l-1}(z)|}$$

and by the inductive hypothesis

$$\sqrt{|\log_{l-1}(z)|} < \sqrt{\sqrt[2^{l-1}]{|z|}} = \sqrt[2^l]{|z|}.$$

■ Claim 1

If $k \geq 0$ we set

$$m := \max\{\lceil |q_j/q_b| \rceil \mid j \in \{b+1, \dots, r\}\}$$

where $b = k$ if $q_k \neq 0$ or $b = l_{\min}(q)$ if $q_k = 0$.

(1): We do two cases.

Case 1:

Assume $\kappa = r$ or if $\kappa = k$ (i.e. $k \geq 0$) then $q_{k+1} = \dots = q_r = 0$. Let

$$\hat{M} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \sqrt[q_k]{\phi(t)}.$$

Then \hat{M} is κ -persistent by Remark 6.48(2) and regular by Remark 6.44(2). If $|z_\kappa(t, z)| < \hat{M}(t)$ then $|z_\kappa(t, z)|^{q_\kappa} < \phi(t)$ for $(t, z) \in H$ (so the inequality in (+) is fulfilled for such a (t, z)). By Proposition 6.57(1) there is a regular κ -persistent $M^* : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0, \kappa}, M^*) \subset \mathcal{B}(z_\kappa, \hat{M})$. So take $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M^*(t), M^+(t)\}$. Clearly M is regular by Remark 6.44(2) and κ -persistent by Remark 6.48(3). With (+) we have $\mathcal{B}(\mathcal{P}_{0, \kappa}, M) \subset \mathcal{B}((\sigma \mathcal{Z})^{\otimes q}, U)$.

Case 2:

Assume $\kappa = k$ (i.e. $k \geq 0$), $q_k > 0$ and $(q_{k+1}, \dots, q_r) \neq 0$. Set

$$\hat{M} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{\sqrt[s]{\phi(t)}, 1/\exp_{r-k}(m^2\pi)\},$$

where $s := \frac{q_k}{2^{r-k}}$. Then \hat{M} is regular and k -persistent. Let $(t, z) \in \mathcal{B}(z_k, \hat{M})$. We show that the estimation in (+) holds. We have

$$|z_k(t, z)| \prod_{j=1}^{r-k} |z_k(t, z)|^{-\frac{1}{2^j}} < \sqrt[q_k]{\phi(t)}$$

and therefore

$$|z_k(t, z)| \prod_{j=1}^{r-k} (1/|z_k(t, z)|)^{\frac{1}{2^j}} < \sqrt[q_k]{\phi(t)}.$$

We obtain

$$|z_k(t, z)| \prod_{j=1}^{r-k} (1/|z_k(t, z)|)^{\left| \frac{q_{k+j}}{2^j m q_k} \right|} < \sqrt[q_k]{\phi(t)}$$

since $1/|z_k(t, z)| > \exp_{r-k}(m^2\pi)$ and $\left| \frac{q_{k+j}}{m q_k} \right| \leq 1$. With Claim 1 applied to $1/\sigma_k z_k(t, z)$ we obtain

$$|\log_j(1/\sigma_k z_k(t, z))| < 2^j \sqrt[m]{1/|z_k(t, z)|}$$

for $j \in \{1, \dots, r - k\}$ and therefore

$$|z_k(t, z)| \prod_{j=1}^{r-k} |\log_j(1/\sigma_k z_k(t, z))|^{q_k+j/q_k} < \sqrt[q_k]{\phi(t)}.$$

Because $|\log_j(1/\sigma_k z_k(t, z))| > 1$ for every $j \in \{1, \dots, r - k\}$ we obtain

$$|z_k(t, z)| \prod_{j=1}^{r-k} |\log_j(1/\sigma_k z_k(t, z))|^{q_k+j/q_k} < \sqrt[q_k]{\phi(t)}.$$

Now define M outgoing from \hat{M} in the same way as in case 1. With (+) we have $\mathcal{B}(\mathcal{P}_{0,k}, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.

(2): We have that ϕ_{up} is consistent by Example 6.41(2), Remark 6.40(3) and Remark 6.40(1) (since U_{up} is consistent). Let $l := l_{\min}(q)$. Note that $q_l < 0$. Set

$$\hat{M} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \min\{1/\exp_{l-k}(\sqrt[s]{\phi_{\text{up}}(t)}), 1/\exp_{r-k}(m^2\pi)\}, & \phi_{\text{up}}(t) \neq 0, \\ 0, & \text{else,} \end{cases}$$

where $s := -(1/2)^{r-l}q_l$. Since ϕ_{up} is consistent we obtain that $\sqrt[s]{\phi_{\text{up}}}$ is consistent by Remark 6.40(2). By Remark 6.40(3) we see that $\exp_j(\sqrt[s]{\phi_{\text{up}}})$ is also consistent for $j \in \{1, \dots, l - k\}$. So we see that \hat{M} is regular by Remark 6.44(2).

Claim 2

\hat{M} is κ -persistent.

Proof of Claim 2

Let $\gamma :]0, 1[\rightarrow C_w$ be a definable curve compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ and

$$\lim_{y \searrow 0} (\gamma_x(y) - \mu_{\kappa}(w, \gamma_u(y))) = 0.$$

Since ϕ is κ -persistent we see $\lim_{y \searrow 0} \phi(w, \gamma_u(y)) > 0$ and therefore

$$\lim_{y \searrow 0} \phi_{\text{up}}(w, \gamma_u(y)) \in \mathbb{R}_{>0}.$$

■ Claim 2

Let $(t, z) \in \mathcal{B}(z_k, \hat{M})$. Then $\phi_{\text{up}}(t) \neq 0$. We show that the inequality in (+) is satisfied for this (t, z) . Let $y := \log_{l-k}(1/\sigma_k z_k(t, z))$. Note that

$$\max\{\sqrt[s]{\phi_{\text{up}}(t)}, \exp_{r-l}(m^2\pi)\} < |y|.$$

We obtain with $\phi_{\text{up}}(t) = 1/\phi(t)$

$$|y| \prod_{j=l+1}^r |y|^{-\frac{1}{2^{j-l}}} > \sqrt[q]{\phi(t)}$$

and therefore

$$|y| \prod_{j=l+1}^r (1/|y|)^{\frac{1}{2^{j-l}}} > \sqrt[q]{\phi(t)}.$$

This gives

$$|y| \prod_{j=l+1}^r (1/|y|)^{\left|\frac{q_j}{2^{j-l}mq_l}\right|} > \sqrt[q]{\phi(t)}$$

since $|y| > 1$ and $\left|\frac{q_j}{mq_l}\right| \leq 1$. Since $\exp_{r-l}(m^2\pi) < |y|$ we obtain with Claim 1 applied to y

$$|\log_{j-l}(y)| < 2^{j-l}m\sqrt{|y|}$$

for $j \in \{l+1, \dots, r\}$ and therefore

$$|y| \prod_{j=l+1}^r |\log_{j-l}(y)|^{-|q_j/q_l|} > \sqrt[q]{\phi(t)}.$$

Because $|\log_{j-l}(y)| > 1$ for every $j \in \{l+1, \dots, r\}$ we obtain

$$\prod_{j=l}^r |\log_{j-k}(1/\sigma_k z_k(t, z))|^{q_j/q_l} > \sqrt[q]{\phi(t)}$$

and therefore

$$\prod_{j=l}^r |\log_{j-k}(1/\sigma_k z_k(t, z))|^{q_j} < \phi(t).$$

Now define M outgoing from \hat{M} in the same way as in (1) in case 1. With (+) we have $\mathcal{B}(\mathcal{P}_{0,k}, M) \subset \mathcal{B}((\sigma \mathcal{Z})^{\otimes q}, U)$.

(3): Since U_{down} is κ -non-persistent we see with Remark 6.40(3) that ϕ_{down} is also κ -non-persistent.

Case 1:

Assume $\kappa = r$ or if $\kappa = k$ then $q_{k+1} = \dots = q_r = 0$. Let

$$\hat{N} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \sqrt[-q\kappa]{\phi_{\text{down}}(t)}.$$

Then \hat{N} is regular and κ -non-persistent. For $(t, z) \in H$ with $\hat{N}(t) < |z_\kappa(t, z)| < 1$ we see that the inequality in (+) is satisfied (since $\phi(t) \neq 0$). By Proposition 6.57(2) there is a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M^* : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M^*) \subset \mathcal{A}(z_\kappa, \hat{N}, 1)$.

So take $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M^*(t), M^+(t)\}$. With (+) we see that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.

Case 2:

Assume $\kappa = k$ and $(q_{k+1}, \dots, q_r) \neq 0$. Let $s := -(2 - (1/2)^{r-k})q_k$. Set

$$\hat{N} : \pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto \sqrt[s]{\phi_{\text{down}}(t)},$$

and

$$\hat{M} : \pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto 1/\exp_{r-k}(m^2\pi).$$

Then \hat{N} is k -non-persistent and regular. Let $(t, z) \in \mathcal{A}(z_\kappa, \hat{N}, \hat{M})$. Then $\phi(t) \neq 0$. We show that the inequality in (+) is satisfied for this (t, z) . We obtain

$$|z_k(t, z)| \prod_{j=1}^{r-k} |z_k(t, z)|^{\frac{1}{2^j}} > {}^{q_k}\sqrt{\phi(t)}$$

and therefore

$$|z_k(t, z)| \prod_{j=1}^{r-k} (1/|z_k(t, z)|)^{-\frac{1}{2^j}} > {}^{q_k}\sqrt{\phi(t)}.$$

We obtain

$$|z_k(t, z)| \prod_{j=1}^{r-k} (1/|z_k(t, z)|)^{-\left|\frac{q_{k+j}}{2^j m q_k}\right|} > {}^{q_k}\sqrt{\phi(t)}$$

since $1/|z_k(t, z)| > \exp_{r-k}(m^2\pi)$ and $\left|\frac{q_{k+j}}{m q_k}\right| \leq 1$. With Claim 1 applied to $1/\sigma_k z_k(t, z)$ we obtain

$$|\log_j(1/\sigma_k z_k(t, z))| < {}^{2^j m}\sqrt{1/|z_k(t, z)|}$$

for $j \in \{1, \dots, r-k\}$ and therefore

$$|z_k(t, z)| \prod_{j=1}^{r-k} |\log_j(1/\sigma_k z_k(t, z))|^{-|q_{k+j}/q_k|} > {}^{q_k}\sqrt{\phi(t)}.$$

Because $|\log_j(1/\sigma_k z_k(t, z))| > 1$ for every $j \in \{1, \dots, r-k\}$ we obtain

$$|z_k(t, z)| \prod_{j=1}^{r-k} |\log_j(1/\sigma_k z_k(t, z))|^{q_{k+j}/q_k} > {}^{q_k}\sqrt{\phi(t)}.$$

Now define N and M analogously as in case 1 (outgoing from \hat{N} and \hat{M} instead from \hat{N} and 1). With (+) we have $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$.

This finishes the proof of Proposition 6.58. ■

6.59 Corollary

Let U_{down} be κ -non-persistent. Let $q \in \mathbb{Q}^{r+1}$ be κ -negative. Then there is a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U).$$

Proof

Note that \sqrt{U} is κ -persistent and that $\sqrt{U_{\text{down}}}$ is κ -non-persistent. Let $q_\kappa^* := |q_\kappa| + 1$. Let

$$\hat{\mathcal{Z}} := (\sigma_\kappa z_\kappa)^{q_\kappa - q_\kappa^*} \prod_{j \neq \kappa} (\sigma_j z_j)^{q_j}.$$

Then we have $\mathcal{B}((\sigma_\kappa z_\kappa)^{q_\kappa^*}, \sqrt{U}) \cap \mathcal{B}(\hat{\mathcal{Z}}, \sqrt{U}) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$. With Proposition 6.58(1) respectively Proposition 6.58(3) we find regular κ -persistent functions $M_1, M_2 : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent function $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0,\kappa}, M_1) \subset \mathcal{B}((\sigma_\kappa z_\kappa)^{q_\kappa^*}, \sqrt{U})$ and $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M_2) \subset \mathcal{B}(\hat{\mathcal{Z}}, \sqrt{U})$. Set $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M_1(t), M_2(t)\}$. We see that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes q}, U)$. \blacksquare

Integrability of Logarithmic Scales in the Complex Setting

In this paragraph we give some results on integration of logarithmic scales in the complex setting "on suitable subsets" which we will need to investigate integrability of holomorphic extensions of prepared restricted log-exp-analytic functions in one variable (compare with Chapter 6.1.3).

For this paragraph we set the following: Let s and τ range over \mathbb{R} . Let $\pi^+ : \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n \times \mathbb{R}, (t, s, z) \mapsto (t, s)$, be the projection on the first $(n+1)$ real coordinates. For $(t, s) \in \pi(C) \times \mathbb{R}_{>0}$ we parameterize the circle $\partial B(\mu_\kappa(t), s) \subset \mathbb{C}$ with the counterclockwise oriented curve

$$\rho : [-\pi, \pi] \rightarrow \mathbb{R}, \tau \mapsto \mu_\kappa(t) + \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau}.$$

For a function $f : X \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x)$, we say that f is bounded in x if f_t is bounded on X_t for every $t \in \pi(X)$.

6.60 Definition

Let $D \subset \pi(C) \times \mathbb{C}$ be definable. We set

$$D(\mu_\kappa) := \left\{ (t, s, \tau) \in \pi(C) \times \mathbb{R}_{>0} \times [-\pi, \pi] \mid \mu_\kappa(t) + \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau} \in D_t \right\}.$$

6.61 Remark

Let $M, N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be definable functions. Suppose that $M < |\mu_r - \mu_k|$ if $k \geq 0$. Let $D := \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M)$. Then

$$D(\mu_\kappa) = \{(t, s) \in \pi(C) \times \mathbb{R}_{>0} \mid N(t) < s < M(t)\} \times] - \pi, \pi[.$$

Proof

Let

$$B := \{(t, s) \in \pi(C) \times \mathbb{R}_{>0} \mid N(t) < s < M(t)\} \times] - \pi, \pi[.$$

Suppose $\kappa = r$. Let $\prod_{j=0}^r \sigma_j = 1$. The case " $\prod_{j=0}^r \sigma_j = -1$ " is treated completely similar. For $t \in \pi(C)$, $s \in \mathbb{R}_{>0}$ and $\tau \in [-\pi, \pi]$ we set $\delta(t, s, \tau) := \mu_r(t) + s e^{i\tau}$. Note that we have $\delta(t, s, \tau) \in \mathbb{C}_{\mu_r(t)}^-$ if $k < 0$ respectively $\delta(t, s, \tau) \in \mathbb{C}_{\mu_r(t)}^- \cap \mathbb{C}_{\mu_k(t)}^+$ if $k \geq 0$ for $(t, s, \tau) \in B$. Remark 6.15 gives $B \subset D(\mu_\kappa)$. Let $(t, s, \tau) \in D(\mu_\kappa)$. Then $t \in \pi(C)$, $N(t) < s < M(t)$ and since $\delta(t, s, \tau) \in D_t$ we have $\tau \in] - \pi, \pi[$. (If $\tau \in \{-\pi, \pi\}$ we obtain $\delta(t, s, \tau) = \mu_r(t) - s < \mu_r(t)$, i.e. $\delta(t, s, \tau) \notin \mathbb{C}_{\mu_r}^-$ and therefore $\delta(t, s, \tau) \notin D_t$ by Remark 6.15.) We obtain $(t, s, \tau) \in B$. The case " $\kappa = k$ " is treated completely similar. ■

6.62 Proposition

Let $M, N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ be definable functions. Suppose that $M < |\mu_r - \mu_k|$ if $k \geq 0$. Let $D := \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M)$. Let $F : D \rightarrow \mathbb{C}$ be definable such that

$$F^* : D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto F(t, s + \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau}),$$

is bounded in τ . Suppose that F^* is globally subanalytic in τ with support function $g : \pi^+(D(\mu_\kappa)) \rightarrow \mathbb{R}^l$ for an $l \in \mathbb{N}_0$. Let

$$\Delta := \{(t, s, z) \in \pi(C) \times \mathbb{R}_{>0} \times \mathbb{C} \mid N(t) < s < M(t), z \in B(\mu_\kappa(t), s)\}.$$

Then the function

$$\Psi : \Delta \rightarrow \mathbb{C}, (t, s, z) \mapsto \frac{1}{2\pi i} \int_{\partial B(\mu_\kappa(t), s)} \frac{F(t, \xi)}{\xi - z} d\xi,$$

is well-defined and constructible in z with support function

$$h : \pi^+(\Delta) \rightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}, (t, s) \mapsto (g(t, s), \mu_\kappa(t), s).$$

Proof

Let $G : \mathbb{R}^l \times \mathbb{R} \rightarrow \mathbb{R}$ be globally subanalytic such that for $(t, s, \tau) \in D(\mu_\kappa)$

$$F^*(t, s, \tau) = G(g(t, s), \tau).$$

For $(t, s, z) \in \Delta$ we have

$$\begin{aligned} \int_{\partial B(\mu_\kappa(t), s)} \frac{F(t, \xi)}{\xi - z} d\xi &= \int_{-\pi}^{\pi} \frac{F^*(t, s, \tau)}{\mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau} - z} i \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau} d\tau \\ &= \int_{-\pi}^{\pi} \frac{G(g(t, s), \tau)}{\mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau} - z} i \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau} d\tau. \end{aligned}$$

Note that

$$\Delta \times] - \pi, \pi[\rightarrow \mathbb{C}, (t, s, z, \tau) \mapsto \frac{(\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau}}{\mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau} - z},$$

is bounded in τ . So Ψ is well-defined.

Let $v := (v_1, \dots, v_l)$ range over \mathbb{R}^l and y over \mathbb{R} . Set

$$Q := \{(v, x, y, z, \tau) \in \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{C} \times] - \pi, \pi[\mid |x - z| < y\}$$

and

$$\begin{aligned} \Omega : \mathbb{R}^l \times \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{R} &\rightarrow \mathbb{C}, (v, x, y, z, \tau) \mapsto \\ \left\{ \begin{array}{ll} \frac{G(v, \tau)}{x + (\prod_{j=0}^{\kappa} \sigma_j) y e^{i\tau} - z} i \left(\prod_{j=0}^{\kappa} \sigma_j \right) y e^{i\tau}, & (v, x, y, z, \tau) \in Q, \\ 0, & \text{else.} \end{array} \right. \end{aligned}$$

Then Ω is globally subanalytic since we have that

$$] - \pi, \pi[\rightarrow \mathbb{C}, \tau \mapsto e^{i\tau} = \cos(\tau) + i \sin(\tau),$$

is globally subanalytic. Then for $(t, s, z, \tau) \in \Delta \times] - \pi, \pi[$ we have $|\mu_\kappa(t) - z| < s$ and therefore $(h(t, s), z, \tau) \in Q$. So we obtain

$$\frac{G(g(t, s), \tau)}{\mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau} - z} i \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau} = \Omega(h(t, s), z, \tau)$$

for every $(t, s, z, \tau) \in \Delta \times] - \pi, \pi[$. By Remark 3.19 there is a constructible $G : \mathbb{R}^l \times \mathbb{R} \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{C}$ such that $\Psi(t, s, z) = G(h(t, s), z)$ for $(t, s, z) \in \Delta$. So we see that Ψ is constructible in z with support function h . \blacksquare

In the construction of the unary high parametric global complexification of a real analytic restricted log-exp-analytic function in Chapter 6.2 the parameter s is "replaced" by a regular function $\alpha : \pi(C) \rightarrow \mathbb{R}$. So every component of

the support function g of F^* in Proposition 6.62 should also be log-analytic in s with regular data.

6.63 Definition

Let $D \subset \pi(C) \times \mathbb{C}$ be definable. We call a function $f : D(\mu_\kappa) \rightarrow \mathbb{C}$ **free-regular in s** if there is $l \in \mathbb{N}_0$ such that f is globally subanalytic in τ with support function $g := (g_1, \dots, g_l) : \pi^+(D(\mu_\kappa)) \rightarrow \mathbb{R}^l$ and the following holds for g . There is $p \in \mathbb{N}_0$, a regular function $\beta : \pi(C) \rightarrow \mathbb{R}^p$ and for every $j \in \{1, \dots, l\}$ a log-analytic $h_j : \mathbb{R}^{p+1} \rightarrow \mathbb{R}$ such that $g_j(t, s) = h_j(\beta(t), s)$ for every $(t, s) \in \pi^+(D(\mu_\kappa))$.

6.64 Remark

Let $D \subset \pi(C) \times \mathbb{C}$ be definable.

- (1) Let $f : D(\mu_\kappa) \rightarrow \mathbb{C}$ be globally subanalytic. Then f is free-regular in s .
- (2) Let $f : D(\mu_\kappa) \rightarrow \mathbb{C}$ be globally subanalytic in τ with log-analytic support function. Then f is free-regular in s .

6.65 Remark

Let $D \subset \pi(C) \times \mathbb{C}$ be definable. Let $f : D(\mu_\kappa) \rightarrow \mathbb{C}$ be free-regular in s . The following holds.

- (1) f is globally subanalytic in τ with support function $g : \pi^+(D(\mu_\kappa)) \rightarrow \mathbb{R}^l$ which can be constructed from a set E of positive definable functions such that every $h \in \log(E)$ is consistent.
- (2) Let $q \in \mathbb{N}$ and let $G : \mathbb{C}^q \rightarrow \mathbb{C}$ be globally subanalytic. Let $J_1, \dots, J_q : D(\mu_\kappa) \rightarrow \mathbb{C}$ be functions which are free-regular in s . Then $H := G(J_1, \dots, J_q) : D(\mu_\kappa) \rightarrow \mathbb{C}$ is free-regular in s .

Proof

(1): Let $g := (g_1, \dots, g_l)$. Definition 6.63, Definition 6.43 and Remark 6.44(2) show immediately that g_j can be constructed from a set E of positive definable functions such that every $h \in \log(E)$ is consistent for $j \in \{1, \dots, l\}$.

(2): Follows directly with Remark 3.18(2). ■

6.66 Proposition

There is a regular κ -persistent function $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $M < |\mu_r - \mu_k|$ if $k \geq 0$ such that for every $l \in \{0, \dots, \kappa\}$ the following holds where $D := \mathcal{B}(\mathcal{P}_{0,\kappa}, M)$.

(1) The function

$$\mathcal{P}_{l,\kappa}^* : D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto \mathcal{P}_{l,\kappa}(t, \mu_\kappa(t)) + \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau},$$

is free-regular in s .

(2) The function

$$z_l^* : D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_l(t, \mu_\kappa(t)) + \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau},$$

is free-regular in s .

Proof

By Proposition 6.57(1) there is a regular κ -persistent $T : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\left| \frac{\mathcal{P}_{j-1,\kappa}(t, z)}{e^{\mu_{\kappa-j,\kappa}(t)}} \right| < 1/2$$

for $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, T)$ for every $j \in \{1, \dots, \kappa\}$. Let $M := T$ and $D = \mathcal{B}(\mathcal{P}_{0,\kappa}, M)$. Note that a regular function $h : \pi(C) \rightarrow \mathbb{R}$ considered as a function on $D(\mu_\kappa)$ is free-regular in s .

(1): We do an induction on l . Note that $\mathcal{P}_{0,\kappa}^*$ is globally subanalytic, because

$$\mathcal{P}_{0,\kappa}^*(t, s, \tau) = \left(\prod_{j=0}^{\kappa} \sigma_j \right) s e^{i\tau} = \left(\prod_{j=0}^{\kappa} \sigma_j \right) s (\cos(\tau) + i \sin(\tau))$$

for $(t, s, \tau) \in D(\mu_\kappa)$. So $\mathcal{P}_{0,\kappa}^*$ is free-regular in s by Remark 6.64(1).

$l-1 \rightarrow l$: For $(t, s, \tau) \in D(\mu_\kappa)$ we obtain

$$\mathcal{P}_{l,\kappa}^*(t, s, \tau) = \log^* \left(1 + \frac{\mathcal{P}_{l-1,\kappa}^*(t, s, \tau)}{e^{\mu_{\kappa-l,\kappa}(t)}} \right).$$

Because \log^* is globally subanalytic we see with Remark 6.65(2) that $\mathcal{P}_{l,\kappa}^*$ is free-regular in s since $e^{\mu_{\kappa-l,\kappa}}$ is regular (and therefore $e^{\mu_{\kappa-l,\kappa}}$ is also free-regular in s considered as a function on $D(\mu_\kappa)$ since coordinate projections are globally subanalytic).

(2): For $(t, s, \tau) \in D(\mu_\kappa)$ we obtain with Proposition 6.20

$$z_l^*(t, s, \tau) = \mathcal{P}_{l,\kappa}^*(t, s, \tau) + \sigma_l e^{\mu_{\kappa-l-1,\kappa}(t)}$$

for $l \in \{0, \dots, \kappa - 1\}$ and with Corollary 6.21

$$z_\kappa^*(t, s, \tau) = \mathcal{P}_{\kappa,\kappa}^*(t, s, \tau).$$

So we obtain that z_l^* is free-regular in s for every $l \in \{0, \dots, \kappa\}$ with (1) if $l = \kappa$ respectively (1) and Remark 6.65(2) if $l \in \{0, \dots, \kappa - 1\}$ since $e^{\mu_{\kappa-l-1,\kappa}}$ is regular. \blacksquare

6.67 Corollary

Let $\kappa = r$. Let $q := (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$. There is a regular r -persistent function $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $M < |\mu_r - \mu_k|$ if $k \geq 0$ such that for $D := \mathcal{B}(\mathcal{P}_{0,r}, M)$ the function

$$\mathcal{Z}_{\mu_r}^{\otimes q} : D(\mu_r) \mapsto \mathbb{C}, (t, s, \tau) \mapsto (\sigma \mathcal{Z})^{\otimes q}(t, \mu_r(t) + \left(\prod_{j=0}^r \sigma_j\right) s e^{i\tau}),$$

is bounded in τ and free-regular in s .

Proof

By Corollary 6.66(2) we find a regular r -persistent $T : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$z_l^* : \hat{D}(\mu_r) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_l(t, \mu_r(t) + \left(\prod_{j=0}^r \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for $l \in \{0, \dots, r\}$ where $\hat{D} := \mathcal{B}(\mathcal{P}_{0,r}, T)$. Then $\sigma_l z_l^*(t, s, \tau) \in \mathbb{C}^-$ for every $(t, s, \tau) \in \hat{D}(\mu_r)$ and $l \in \{0, \dots, r\}$ by Definition 6.13(a) since $\mu_r(t) + \left(\prod_{j=0}^r \sigma_j\right) s e^{i\tau} \in H_t$ for $(t, s, \tau) \in \hat{D}(\mu_r)$. Let

$$\hat{\mathcal{Z}}_{\mu_r}^{\otimes q} : \hat{D}(\mu_r) \mapsto \mathbb{C}, (t, s, \tau) \mapsto (\sigma \mathcal{Z})^{\otimes q}(t, \mu_r(t) + \left(\prod_{j=0}^r \sigma_j\right) s e^{i\tau}),$$

and

$$G : (\mathbb{C}^-)^{r+1} \rightarrow \mathbb{C}, (w_0, \dots, w_r) \mapsto \prod_{j=0}^r (w_j)^{q_j}.$$

Note that G is globally subanalytic. We obtain

$$\hat{\mathcal{Z}}_{\mu_r}^{\otimes q}(t, s, \tau) = G(\sigma_0 z_0^*(t, s, \tau), \dots, \sigma_r z_r^*(t, s, \tau))$$

for every $(t, s, \tau) \in \hat{D}(\mu_r)$. Therefore $\hat{\mathcal{Z}}_{\mu_r}^{\otimes q}$ is free-regular in s by Remark 6.65(2).

Let

$$d : \pi(C) \rightarrow \mathbb{R}, t \mapsto \min\{|\mu_r(t) - \mu_j(t)| \mid j \in \{0, \dots, r-1\}\}.$$

Note that d is regular (by Example 6.45(1) and Remark 6.44(2)) and r -persistent (by Example 6.53 and Remark 6.48(3)). Choose $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{T(t), d(t)\}$. Let $D := \mathcal{B}(\mathcal{P}_{0,r}, M)$. The following claim finishes the proof of Corollary 6.67.

Claim

The function $\mathcal{Z}_{\mu_r}^{\otimes q} = \hat{\mathcal{Z}}_{\mu_r}^{\otimes q}|_{D(\mu_r)}$ is bounded in τ .

Proof of the claim

Assume the contrary. Then there is $(t, s) \in \pi^+(D(\mu_r))$ and a definable curve $\gamma_\tau :]0, 1[\rightarrow]-\pi, \pi[$ such that $(t, s, \gamma_\tau(y)) \in D(\mu_r)$ for $y \in]0, 1[$ and

$$\lim_{y \searrow 0} |\mathcal{Z}_{\mu_r}^{\otimes q}(t, s, \gamma_\tau(y))| = \lim_{y \searrow 0} |(\sigma \mathcal{Z})^{\otimes q}(t, \mu_r(t) + (\prod_{j=0}^r \sigma_j) s e^{i\gamma_\tau(y)})| = \infty.$$

Note that $0 < s < d$. Consider $\gamma :]0, 1[\rightarrow H, y \mapsto (\gamma_t(y), \gamma_z(y))$ with $\gamma_t(y) = t$ and $\gamma_z(y) = \mu_r(t) + (\prod_{j=0}^r \sigma_j) s e^{i\gamma_\tau(y)}$ for $y \in]0, 1[$. Note that this curve is definable and compatible with C . Since $0 < s < d$ we have

$$\lim_{y \searrow 0} \gamma_z(y) - \mu_l(t) = \lim_{y \searrow 0} \mu_r(t) + (\prod_{j=0}^r \sigma_j) s e^{i\gamma_\tau(y)} - \mu_l(t) \neq 0$$

for $l \in \{0, \dots, r\}$ which is a contradiction to Corollary 6.29(2). ■

Before we investigate the case " $\kappa = k$ " we need the following three statements.

6.68 Proposition

Let $D \subset H$ be definable such that D_t is connected and $\mathbb{R} \cap D_t \neq \emptyset$ for $t \in \pi^*(D)$. Let $f, g : D \rightarrow \mathbb{C}^-$ be functions with the following properties.

- (1) f and g are continuous in z .
- (2) We have $f(t, z) \in \mathbb{R}_{>0}$ if $(t, z) \in D \cap (\pi(C) \times \mathbb{R})$.
- (3) It holds $f(t, z)g(t, z) \in \mathbb{C}^-$ for every $(t, z) \in D$.

Then for every $(t, z) \in D$ it is

$$\log(f(t, z)g(t, z)) = \log(f(t, z)) + \log(g(t, z)).$$

Proof

Suppose there is $(t, z^*) \in D$ such that

$$\log(f(t, z^*)g(t, z^*)) \neq \log(f(t, z^*)) + \log(g(t, z^*)).$$

So we have

$$\log(f(t, z^*)g(t, z^*)) = \log(f(t, z^*)) + \log(g(t, z^*)) \pm 2\pi i$$

and therefore

$$|\arg(f(t, z^*)) + \arg(g(t, z^*))| \geq \pi.$$

Note that

$$D_t \rightarrow \mathbb{R}, z \mapsto \arg(f(t, z)) + \arg(g(t, z)),$$

is continuous and that $|\arg(f(t, z)) + \arg(g(t, z))| < \pi$ if $z \in \mathbb{R}$ for $t \in \pi(C)$. By the intermediate value theorem (by taking a definable curve which connects z^* and a point $a \in D_t \cap \mathbb{R}$) there is $z' \in D_t$ such that

$$|\arg(f(t, z')) + \arg(g(t, z'))| = \pi.$$

This gives $\arg(f(t, z')g(t, z')) = \pi$. But then $f(t, z')g(t, z') \in \mathbb{R}_{<0}$, a contradiction. ■

6.69 Remark

Let $m \in \mathbb{N}_0$. Let $D := \{z \in \mathbb{C}^- \mid |z| < 1/\exp_m(1)\}$ and

$$L_m : D \rightarrow \mathbb{C}, z \mapsto \log_m(-\log(z)).$$

Then L_m is well-defined, definable and holomorphic. We have $\log_m(-\log(z)) \in \mathbb{C}^-$ and $\log_m(-\log(z)) \in \mathbb{R}_{>0}$ if $z \in \mathbb{R}_{>0}$ for $z \in D$.

Proof

Let $z \in D$. If $z \notin \mathbb{R}_{>0}$ it is

$$\log(z) = \log(|z|) + i\arg(z) \in \mathbb{C} \setminus \mathbb{R}$$

since $\arg(z) \neq 0$ and therefore $\log_m(-\log(z)) \in \mathbb{C}^-$. So assume $z \in \mathbb{R}_{>0}$. We obtain $\exp_{m-1}(1) < -\log(z)$ and therefore

$$\exp_{m-1-l}(1) < \log_l(-\log(z))$$

for every $l \in \{1, \dots, m-1\}$ (where $\exp_0 := 1$). We obtain $\log_m(-\log(z)) > 0$. ■

6.70 Proposition

Let $m \in \mathbb{N}$. Let

$$c := \frac{1}{\exp_m(2\sqrt{\log^2(1.5) + \pi^2})}$$

and $D :=]0, c[\times] - \pi, \pi[$. The function

$$T_l : D \rightarrow \mathbb{C}^-, (s, \tau) \mapsto \log_l(-\log(se^{i\tau})),$$

is a well-defined globally subanalytic function in τ with log-analytic support function for every $l \in \{0, \dots, m\}$.

Proof

Note that $se^{i\tau} \in \mathbb{C}^-$ for every $(s, \tau) \in \mathbb{R}_{>0} \times] - \pi, \pi[$. So we obtain well-definability with Remark 6.69. Set

$$S_0 : D \rightarrow \mathbb{C}, (s, \tau) \mapsto \frac{i\tau}{\log(s)},$$

and inductively for $l \in \{1, \dots, m-1\}$

$$S_l : D \rightarrow \mathbb{C}, (s, \tau) \mapsto \frac{\log(1 + S_{l-1}(s, \tau))}{\log_l(-\log(s))}.$$

Claim

Let $l \in \{0, \dots, m-1\}$. Then S_l is a well-defined globally subanalytic function in τ with log-analytic support function. For every $(s, \tau) \in D$ we have $|S_l(s, \tau)| < 1/2$ and

$$T_l = \log_l(-\log(s))(1 + S_l(s, \tau)).$$

Proof of the claim

Let $(s, \tau) \in D$. We obtain

$$2\sqrt{\log^2(1.5) + \pi^2} < \log_j(-\log(s)) \quad (*)$$

for every $j \in \{0, \dots, m-1\}$. We do an induction on l . The case $l = 0$ is clear.

$l-1 \rightarrow l$: With the inductive hypothesis, $(*)$, and the definition of the complex logarithm we see that S_l is well-defined and that $|S_l(s, \tau)| < 1/2$ for every $(s, \tau) \in D$. We have for $(s, \tau) \in D$

$$S_l(s, \tau) = \frac{\log^*(1 + S_{l-1}(s, \tau))}{\log_l(-\log(s))}.$$

With the inductive hypothesis and Remark 3.18(2) it follows that S_l is a globally subanalytic function in τ with log-analytic support function. Additionally we obtain for $(s, \tau) \in D$ with the inductive hypothesis

$$\begin{aligned} \log_l(-\log(s) - i\tau) &= \log(\log_{l-1}(-\log(s) - i\tau)) \\ &= \log(\log_{l-1}(-\log(s))(1 + S_{l-1}(s, \tau))) \\ &= \log_l(-\log(s)) + \log(1 + S_{l-1}(s, \tau)) \\ &= \log_l(-\log(s)) \left(1 + \frac{\log(1 + S_{l-1}(s, \tau))}{\log_l(-\log(s))}\right) \\ &= \log_l(-\log(s))(1 + S_l(s, \tau)). \end{aligned}$$

■ Claim

Note that by the Claim

$$\begin{aligned} T_m(s, \tau) &= \log(T_{m-1}(s, \tau)) \\ &= \log(\log_{m-1}(-\log(s))(1 + S_{m-1}(s, \tau))) \\ &= \log(\log_{m-1}(-\log(s))) + \log(1 + S_{m-1}(s, \tau)) \\ &= \log_m(-\log(s)) + \log^*(1 + S_{m-1}(s, \tau)). \end{aligned}$$

By the Claim and Remark 3.18(2) we obtain immediately that T_m is globally subanalytic in τ with log-analytic support function. ■

For the rest of Subchapter 6.1.2 we assume $k := k^{\text{ch}} \geq 0$ and that $\kappa = k$. For $j \in \{0, \dots, k\}$ we set $\mathcal{Q}_j := \mathcal{P}_{j,k}$.

6.71 Proposition

Let $k = 0$. There is a regular 0-persistent function $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $M < |\mu_r - \mu_0|$ such that for $D := \mathcal{B}(\mathcal{Q}_0, M)$ and every $l \in \{0, \dots, r\}$ the function

$$z_l^* : D(\mu_0) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_l(\mu_0(t) + \sigma_0 s e^{i\tau}),$$

is globally subanalytic in τ with log-analytic support function.

Proof

We may assume that C is near with respect to $\mu_0 = \Theta_0$. Otherwise we are done with Remark 6.49 by choosing $M = 0$. Note that $\sigma_1 = -1$, $\sigma_2 = \dots = \sigma_r = 1$ and that $\Theta_1 = \dots = \Theta_r = 0$ by Proposition 6.37. By Definition 6.3 we have $\mu_0 = \Theta_0$, $\mu_1 = \Theta_0 + \sigma_0$ and for $j \in \{2, \dots, r\}$ we have $\mu_j = \Theta_0 + \sigma_0 e^{-e_{j-1}}$ where $e_0 := 0$ and $e_m := \exp(e_{m-1})$ for $m \in \mathbb{N}$. Note that for $(t, z) \in H$

$$z_1(t, z) = \log(\sigma_0 z_0(t, z))$$

and

$$z_l(t, z) = \log_{l-1}(-\log(\sigma_0 z_0(t, z)))$$

for $l \in \{2, \dots, r\}$. Let

$$c := \frac{1}{\exp_{r-1}(2\sqrt{\log^2(1.5) + \pi^2})}.$$

Then $c < 1/\exp_{r-1}(1) = 1/e_r = |\mu_r(t) - \mu_0(t)|$ for $t \in \pi(C)$. Consider $M := c$. With Remark 6.61 we see that

$$D(\mu_0) = \mathcal{B}(\mathcal{Q}_0, c) = \pi(C) \times]0, c[\times] - \pi, \pi[.$$

(Since $D(\mu_0) = \mathcal{A}(\mathcal{Q}_0, 0, c)$.) Note that $z_0^*(t, s, \tau) = \sigma_0 s e^{i\tau} = \sigma_0 s (\cos(\tau) + i \sin(\tau))$ and $z_1^*(t, s, \tau) = \log(s) + i\tau$ for $(t, s, \tau) \in D(\mu_0)$. So z_0^* and z_1^* fulfill the desired properties since the restriction of the global sine and cosine function on $] - \pi, \pi[$ are globally subanalytic. We obtain for $l \in \{1, \dots, r-1\}$

$$\begin{aligned} z_{l+1}^*(t, s, \tau) &= \log_l(-\log(\sigma_0 z_0(t, \Theta_0(t) + \sigma_0 s e^{i\tau}))) \\ &= \log_l(-\log(s e^{i\tau})) \end{aligned}$$

for every $(t, s, \tau) \in D(\mu_0)$. With Proposition 6.70 we see that

$$]0, c[\times] - \pi, \pi[\rightarrow \mathbb{C}, (s, \tau) \mapsto \log_l(-\log(s e^{i\tau})),$$

is a well-defined globally subanalytic function in τ with log-analytic support function for $l \in \{0, \dots, r-1\}$ and we are done. \blacksquare

6.72 Proposition

There is a regular k -persistent function $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $M < |\mu_r - \mu_k|$ such that for $D := \mathcal{B}(\mathcal{Q}_0, M)$ and every $l \in \{0, \dots, r\}$ the function

$$z_l^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_l(\mu_k(t) + (\prod_{j=0}^k \sigma_j) s e^{i\tau}),$$

is free-regular in s .

Proof

We may assume that C is near with respect to μ_k . Otherwise we are done with Remark 6.49 by choosing $M = 0$. Note that $\sigma_{k+1} = -1, \sigma_{k+2} = \dots = \sigma_r = 1$ and that $\Theta_{k+1} = \dots = \Theta_r = 0$ by Proposition 6.37. So for $(t, z) \in H$ we have with Corollary 6.21

$$z_{k+1}(t, z) = \log(\sigma_k z_k(t, z)) = \log(\sigma_k \mathcal{Q}_k(t, z))$$

and

$$z_l(t, z) = \log_{l-k-1}(-\log(\sigma_k z_k(t, z))) = \log_{l-k-1}(-\log(\sigma_k \mathcal{Q}_k(t, z)))$$

for $l \in \{k+2, \dots, r\}$.

If $k = 0$ we are done with Proposition 6.71. So assume $k > 0$.

Fix $\lambda \in]0, 1/2[$ such that

$$\left| \frac{\log(1+z) - z}{z} \right| < 1/2$$

for $z \in \mathbb{C} \setminus \{0\}$ with $|z| < \lambda$. For $l \in \{1, \dots, k\}$ set

$$\zeta_l : \pi(C) \rightarrow \mathbb{R}_{>0}, t \mapsto \min \left\{ \frac{1}{e^{2|\mu_{k-l,k}(t)|}}, \frac{1}{\exp_{r-k-1}(2\sqrt{\log^2(1.5)+\pi^2})}, \lambda e^{\mu_{k-l,k}(t)}, \frac{e^{\mu_{k-l,k}(t)}}{\exp_{r-k-1}(2\sqrt{\log^2(1.5)+\pi^2})} \right\}.$$

We see that ζ_l is regular and k -persistent for every $l \in \{1, \dots, k\}$. For $l \in \{1, \dots, k\}$ set

$$\Omega_l := \{(t, z) \in H \mid |\mathcal{Q}_{l-1}(t, z)| < \zeta_l(t)\}.$$

By Proposition 6.57(1) there is a regular k -persistent $M_l : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{Q}_0, M_l) \subset \Omega_l$ for $l \in \{1, \dots, k\}$. Consider

$$M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{|\mu_r(t) - \mu_k(t)|, M_1(t), \dots, M_l(t)\}.$$

Then M is k -persistent and regular. We have $D := \mathcal{B}(\mathcal{Q}_0, M) \subset \bigcap_{l=1}^k \Omega_l$. Clearly D_t is connected and $D_t \cap \mathbb{R} \neq \emptyset$ for $t \in \pi^*(D)$. Let for $l \in \{1, \dots, k\}$

$$c_l : D \rightarrow \mathbb{C}, (t, z) \mapsto \frac{\sigma_{l-1} \mathcal{Q}_{l-1}(t, z)}{e^{\mu_{k-l,k}(t)}}.$$

Then for every $(t, z) \in D$ and $l \in \{1, \dots, k\}$ it is

$$\mathcal{Q}_l(t, z) = \log(1 + c_l(t, z)).$$

Let $\mathcal{V} \in \{c, \mathcal{Q}\}$. By Remark 6.24(2) we have

$$\left(\prod_{j=l}^k \sigma_j\right) \mathcal{V}_l(t, z) \in \mathbb{C}^-,$$

and

$$\left(\prod_{j=l}^k \sigma_j\right) \mathcal{V}_l(t, z) \in \mathbb{R}_{>0}$$

if $z \in \mathbb{R}$ for every $(t, z) \in D$ and $l \in \{1, \dots, k\}$. $(+\nu)$

Note also that

$$|\mathcal{V}_l(t, z)| < \frac{1}{\exp_{r-k-1}(2\sqrt{\log^2(1.5) + \pi^2})}$$

for every $(t, z) \in D$ and $l \in \{1, \dots, k\}$. $(++\nu)$

By shrinking M if necessary we may assume with Proposition 6.66(1) that

$$\mathcal{V}_l^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto \mathcal{V}_l(t, \mu_k(t) + \left(\prod_{j=0}^k \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for every $l \in \{0, \dots, k\}$. (Therefore

$$z_l^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_l(t, s + \left(\prod_{j=0}^k \sigma_j\right) s e^{i\tau}),$$

is also free-regular in s .) $(+++ \nu)$

For $l \in \{0, \dots, k\}$ and $m \in \{0, \dots, r - k - 1\}$ we define

$$R_{m,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto \log_m(-\log(\left(\prod_{j=l}^k \sigma_j\right) \mathcal{Q}_l(t, z))).$$

By Remark 6.69, $(+\mathcal{Q})$ and $(++\mathcal{Q})$ we obtain that the function $R_{m,l}$ is well-defined, continuous in z , and for $(t, z) \in D$ we have $R_{m,l}(t, z) \in \mathbb{C}^-$ and if $z \in \mathbb{R}$ then $R_{m,l}(t, z) \in \mathbb{R}_{>0}$. $(+R)$

For $l \in \{1, \dots, k\}$ we define

$$S_{0,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto \mu_{k-l,k}(t),$$

and inductively for $m \in \{1, \dots, r - k - 1\}$

$$S_{m,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto \log\left(1 + \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)}\right).$$

Claim 1

Let $l \in \{1, \dots, k\}$. The function $S_{m-1,l}$ is well-defined, continuous in z and for $(t, z) \in D$ we have

$$\left| \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)} \right| < 1/2$$

for every $m \in \{1, \dots, r - k - 1\}$.

Proof of Claim 1

Note that for $(t, z) \in D$

$$|\mathcal{Q}_{l-1}(t, z)| < \min\left\{ \frac{1}{e^{2|\mu_{k-l,k}(t)|}}, \frac{1}{\exp_{r-k-1}(2\sqrt{\log(1.5)^2 + \pi^2})} \right\}. \quad (*)$$

We show Claim 1 by induction on $m \in \{1, \dots, r - k - 1\}$. Fix $(t, z) \in D$.

$m = 1$: That $S_{0,l}$ is well-defined and continuous in z is clear. We obtain with (*)

$$|\mu_{k-l,k}(t)| < -1/2 \log(|\mathcal{Q}_{l-1}(t, z)|)$$

and with the definition of the complex logarithm

$$|\mu_{k-l,k}(t)| < 1/2 |\log((\prod_{j=l-1}^k \sigma_j) \mathcal{Q}_{l-1}(t, z))|.$$

This gives

$$\left| \frac{S_{0,l}(t, z)}{R_{0,l-1}(t, z)} \right| < 1/2.$$

$m - 1 \rightarrow m$: We obtain well-definability of $S_{m-1,l}$ by the inductive hypothesis. We get

$$\begin{aligned} |S_{m-1,l}(t, z)| &= \left| \log\left(1 + \frac{S_{m-2,l}(t, z)}{R_{m-2,l-1}(t, z)}\right) \right| \\ &< \sqrt{\log^2(1.5) + \pi^2} \end{aligned}$$

by the inductive hypothesis and definition of the complex logarithm. So we obtain by (*)

$$|\mathcal{Q}_{l-1}(t, z)| < \frac{1}{\exp_m(2|S_{m-1,l}(t, z)|)},$$

i.e.

$$2|S_{m-1,l}(t, z)| < \log_{m-1}(-\log(|\mathcal{Q}_{l-1}(t, z)|)).$$

By the definition of the complex logarithm we obtain that

$$\left| \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)} \right| < 1/2$$

and we are done since $R_{m-1,l-1}$ is continuous in z . ■ Claim 1

By Claim 1 and $(+)_R$ we have that $S_{m,l}$ is well-defined, continuous in z and

$$1 + \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)} \in \mathbb{C}^-$$

for $(t, z) \in D$, $l \in \{1, \dots, k\}$ and $m \in \{1, \dots, r - k - 1\}$. $(+)_S$

Let for $l \in \{1, \dots, k\}$ and $m \in \{0, \dots, r - k - 1\}$

$$T_{m,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto \log_m(-\log(\prod_{j=l}^k \sigma_j c_l(t, z))).$$

By Remark 6.69, $(+_c)$ and $(++_c)$ we obtain that the function $T_{m,l}$ is well-defined, continuous in z , and for $(t, z) \in D$ we have $T_{m,l}(t, z) \in \mathbb{C}^-$ and if $z \in \mathbb{R}$ then $T_{m,l}(t, z) \in \mathbb{R}_{>0}$. $(+)_T$

Let for $l \in \{1, \dots, k\}$

$$b_l : D \rightarrow \mathbb{C}, (t, z) \mapsto \frac{\log(1 + c_l(t, z)) - c_l(t, z)}{c_l(t, z)},$$

$$U_{0,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto -\log(1 + b_l(t, z)),$$

and inductively for $m \in \{1, \dots, r - k - 1\}$

$$U_{m,l} : D \rightarrow \mathbb{C}, (t, z) \mapsto \log\left(1 + \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)}\right).$$

Note that $|c_l(t, z)| < \lambda$ since $|\mathcal{Q}_{l-1}(t, z)| < \lambda e^{\mu_{k-l,k}(t)}$ for $(t, z) \in D$. So we see that b_l is well-defined, continuous in z , $|b_l(t, z)| < 1/2$ by the choice of λ and $1 + b_l(t, z) \in \mathbb{C}^-$ for $(t, z) \in D$. $(+_b)$

Claim 2

Let $l \in \{1, \dots, k\}$. The function $U_{m-1,l}$ is well-defined, continuous in z and for $(t, z) \in D$ we have

$$\left| \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)} \right| < 1/2$$

for every $m \in \{1, \dots, r - k - 1\}$.

Proof of Claim 2

Fix $(t, z) \in D$. We have

$$|\mathcal{Q}_{l-1}(t, z)| < \frac{e^{\mu_{k-l,k}(t)}}{\exp_{r-k-1}(2\sqrt{\log^2(1.5) + \pi^2})}$$

and by (+_b)

$$|U_{0,l}(t, z)| < \sqrt{\log^2(1.5) + \pi^2}. (**)$$

We do an induction on m .

$m = 1$: With (+_b) we obtain that $U_{0,l}$ is well-defined and continuous in z . With (**) we obtain

$$|\mathcal{Q}_{l-1}(t, z)| < \frac{e^{\mu_{k-l,k}(t)}}{\exp(2\sqrt{\log^2(1.5) + \pi^2})}$$

and therefore

$$|\mathcal{Q}_{l-1}(t, z)| < \frac{e^{\mu_{k-l,k}(t)}}{\exp(2|U_{0,l}(t, z)|)}.$$

This gives

$$|U_{0,l}(t, z)| < -1/2 \log(|c_l(t, z)|)$$

and with the definition of the complex logarithm

$$|U_{0,l}(t, z)| < 1/2 |\log((\prod_{j=l}^k \sigma_j) c_l(t, z))|.$$

We obtain

$$\left| \frac{U_{0,l}(t, z)}{T_{0,l}(t, z)} \right| < 1/2.$$

$m - 1 \rightarrow m$: We obtain well-definability of $U_{m-1,l}$ by the inductive hypothesis.

We get

$$\begin{aligned} |U_{m-1,l}(t, z)| &= \left| \log\left(1 + \frac{U_{m-2,l}(t, z)}{T_{m-2,l}(t, z)}\right) \right| \\ &< \sqrt{\log^2(1.5) + \pi^2} \end{aligned}$$

by the inductive hypothesis. This gives with (**)

$$|\mathcal{Q}_{l-1}(t, z)| < \frac{e^{\mu_{k-l,k}(t)}}{\exp_m(2|U_{m-1,l}(t, z)|)}.$$

This implies

$$2|U_{m-1,l}(t, z)| < \log_{m-1}(-\log(|c_l(t, z)|))$$

and with the definition of the complex logarithm we obtain

$$\left| \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)} \right| < 1/2$$

and are done. (We obtain the continuity of $U_{m-1,l}$ in z since $T_{m-2,l}$ is continuous in z .) ■ Claim 2

By Claim 2 we have that $U_{m,l}$ is well-defined, continuous in z , and we have

$$1 + \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)} \in \mathbb{C}^-$$

for $(t, z) \in D$, $l \in \{1, \dots, k\}$ and $m \in \{1, \dots, r - k - 1\}$. (+ U)

Claim 3

Let $(t, z) \in D$. Let $l \in \{1, \dots, k\}$. For every $m \in \{0, \dots, r - k - 1\}$ it is

$$T_{m,l}(t, z) = R_{m,l-1}(t, z) + S_{m,l}(t, z).$$

Proof of Claim 3

We do an induction on m .

$m = 0$: We obtain

$$\begin{aligned} T_{0,l}(t, z) &= -\log\left(\prod_{j=l}^k \sigma_j c_l(t, z)\right) \\ &= -\log\left(\prod_{j=l-1}^k \sigma_j \mathcal{Q}_{l-1}(t, z)\right) + \mu_{k-l,k}(t) \\ &= R_{0,l-1}(t, z) + S_{0,l}(t, z). \end{aligned}$$

$m - 1 \rightarrow m$: By Proposition 6.68, (+ R), (+ S), (+ T) we obtain with the inductive hypothesis

$$\begin{aligned} \log(T_{m-1,l}(t, z)) &= \log(R_{m-1,l-1}(t, z) + S_{m-1,l}(t, z)) \\ &= \log(R_{m-1,l-1}(t, z)) + \log\left(1 + \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)}\right) \end{aligned}$$

and therefore

$$T_{m,l}(t, z) = \log(T_{m-1,l}(t, z)) = R_{m,l-1}(t, z) + S_{m,l}(t, z).$$

Claim 4

Let $(t, z) \in D$. Let $l \in \{1, \dots, k\}$. For every $m \in \{0, \dots, r - k - 1\}$ it is

$$R_{m,l}(t, z) = T_{m,l}(t, z) + U_{m,l}(t, z).$$

Proof of Claim 4

We do an induction on m .

$m = 0$: We obtain

$$\begin{aligned} R_{0,l} &= -\log\left(\left(\prod_{j=l}^k \sigma_j\right) \mathcal{Q}_l(t, z)\right) = -\log\left(\left(\prod_{j=l}^k \sigma_j\right) \log(1 + c_l(t, z))\right) \\ &= -\log\left(\left(\prod_{j=l}^k \sigma_j\right) (c_l(t, z) + \log(1 + c_l(t, z)) - c_l(t, z))\right) \\ &= -\log\left(\left(\prod_{j=l}^k \sigma_j\right) c_l(t, z) (1 + b_l(t, z))\right). \end{aligned}$$

With $(+_b)$, $(+_c)$, $(+_Q)$ and Proposition 6.68 we obtain

$$-\log\left(\left(\prod_{j=l}^k \sigma_j\right) c_l(t, z) (1 + b_l(t, z))\right) = -\log\left(\left(\prod_{j=l}^k \sigma_j\right) c_l(t, z)\right) - \log(1 + b_l(t, z)).$$

Therefore

$$R_{0,l} = T_{0,l}(t, z) + U_{0,l}(t, z).$$

$m - 1 \rightarrow m$: By Proposition 6.68, $(+_R)$, $(+_T)$, $(+_U)$ and the inductive hypothesis we obtain

$$\begin{aligned} \log(R_{m-1,l}(t, z)) &= \log(T_{m-1,l}(t, z) + U_{m-1,l}(t, z)) \\ &= \log(T_{m-1,l}(t, z)) + \log\left(1 + \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)}\right) \end{aligned}$$

and therefore

$$R_{m,l}(t, z) = \log(R_{m-1,l}(t, z)) = T_{m,l}(t, z) + U_{m,l}(t, z).$$

Claim 5

(I_l) for $l \in \{1, \dots, k\}$: The function

$$T_{m,l}^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto T_{m,l}(t, \mu_k(t) + \left(\prod_{j=0}^k \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for every $m \in \{0, \dots, r - k - 1\}$.

(II_l) for $l \in \{0, \dots, k\}$: The function

$$R_{m,l}^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto R_{m,l}(t, \mu_k(t) + \left(\prod_{j=0}^k \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for every $m \in \{0, \dots, r - k - 1\}$.

Proof of Claim 5

We start with (II₀). Let $(t, s, \tau) \in D(\mu_k)$. We have

$$0 < s < \frac{1}{\exp_{r-k-1}(2\sqrt{\log(1.5)^2 + \pi^2})}$$

since

$$|z - \mu_k(t)| = |\mathcal{Q}_0(t, z)| < \frac{1}{\exp_{r-k-1}(2\sqrt{\log(1.5)^2 + \pi^2})}$$

for all $(t, z) \in D$. Note that

$$R_{m,0}^*(t, s, \tau) = \log_m(-\log(se^{i\tau}))$$

for $m \in \{0, \dots, r - k - 1\}$. Therefore with Proposition 6.70 and Remark 6.64(2) we are done.

Assume that (I_{l-1}) and (II_{l-1}) have already been shown for $l \in \{1, \dots, k\}$ where (I₀) is any true statement. We show (I_l). By Claim 3 we have $T_{m,l} = R_{m,l-1} + S_{m,l}$ on D for every $m \in \{0, \dots, r - k - 1\}$. With (II_{l-1}) and Remark 6.65(2) it suffices to show that

$$S_{m,l}^* : D(\mu_k) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto S_{m,l}(t, \mu_k(t) + \left(\prod_{j=0}^k \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for every $m \in \{0, \dots, r - k - 1\}$. We do an induction on m . For $m = 0$ this is clear since $S_{0,m}^*$ is regular considered as a function on $\pi^*(D)$ and with Definition 6.63 one sees immediately that $S_{0,m}^*$ is free-regular in s . Assume $m > 0$. By Claim 1 we obtain that

$$\left| \frac{S_{m-1,l}(t, z)}{R_{m-1,l-1}(t, z)} \right| < 1/2$$

for every $(t, z) \in D$. So

$$S_{m,l} = \log^* \left(1 + \frac{S_{m-1,l}}{R_{m-1,l-1}} \right).$$

With (II_{l-1}) , the inductive hypothesis and Remark 6.65(2) we are done.

We show (II_l) . By Claim 4 we have $R_{m,l} = T_{m,l} + U_{m,l}$ on D for every $m \in \{0, \dots, r - k - 1\}$. With (I_l) and Remark 6.65(2) it suffices to show that

$$U_{m,l}^* : D(\mu_k) \rightarrow \mathbb{R}, (t, s, \tau) \mapsto U_{m,l}(t, \mu_k(t) + \left(\prod_{j=0}^k \sigma_j \right) s e^{i\tau}),$$

is free-regular in s for every $m \in \{0, \dots, r - k - 1\}$. We do an induction on m .

$m = 0$: We have $U_{0,l}(t, z) = \log^*(1 + b_l(t, z))$ and

$$b_l(t, z) = \frac{\log^*(1 + c_l(t, z)) - c_l(t, z)}{c_l(t, z)}$$

for every $(t, z) \in D$ (compare $(+_b)$, $(++_c)$). With $(+++_c)$ and Remark 6.65(2) we are done.

$m - 1 \rightarrow m$: By Claim 4 we obtain that

$$\left| \frac{U_{m-1,l}(t, z)}{T_{m-1,l}(t, z)} \right| < 1/2$$

for every $(t, z) \in D$. So

$$U_{m,l} = \log^* \left(1 + \frac{U_{m-1,l}}{T_{m-1,l}} \right).$$

With (I_l) , the inductive hypothesis and Remark 6.65(2) we are done. \blacksquare Claim 5

We have

$$\sigma_{k+m+1} z_{k+m+1} |_D = R_{m,k}$$

for every $m \in \{0, \dots, r - k - 1\}$. Therefore we obtain with (II_k) in Claim 5 that z_{k+m+1}^* is free-regular in s for every $m \in \{0, \dots, r - k - 1\}$. By $(+++_{\mathcal{Q}})$ we also have that z_l^* is free-regular in s for $l \in \{0, \dots, k\}$ and we are done with the proof of Proposition 6.72. \blacksquare

6.73 Corollary

Let $q := (q_0, \dots, q_r) \in \mathbb{Q}^{r+1}$. There is a k -persistent regular function $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $M < |\mu_r - \mu_k|$ such that

$$\mathcal{Z}_{\mu_k}^{\otimes q} : D(\mu_k) \mapsto \mathbb{C}, (t, s, \tau) \mapsto (\sigma \mathcal{Z})^{\otimes q}(t, \mu_k(t) + (\prod_{j=0}^k \sigma_j) s e^{i\tau}),$$

is bounded in τ and free-regular in s where $D := \mathcal{B}(\mathcal{Q}_0, M)$.

Proof

The proof for this Corollary is the same as the proof of Corollary 6.67. ■

6.1.3 Holomorphic Extensions of Prepared Restricted Log-Exp-Analytic Functions

Fix $\kappa = r$ or $\kappa \in \{k^{\text{ch}}, r\}$ if $k^{\text{ch}} \geq 0$.

Complex Log-Analytically Prepared Functions

For the rest of Chapter 6.1 we set the following: For $m, p \in \mathbb{N}$ and $l \in \mathbb{N}$ with $l \leq p$ we denote by $M_l(m \times p, \mathbb{Q})$ the set of all $m \times p$ -matrices over \mathbb{Q} where the first l columns are zero.

6.74 Definition

Let $Y \subset H$ be definable. Let $r \in \mathbb{N}_0$. Let $F : Y \rightarrow \mathbb{C}$ be a function. We say that F is **complex r -log-analytically prepared in z with center Θ** if

$$F(t, z) = a(t)(\sigma\mathcal{Z})^{\otimes q}(t, z)U(t, z)$$

for all $(t, z) \in Y$ where a is a definable function on $\pi^*(Y)$ which vanishes identically or has no zero, $q \in \mathbb{Q}^{r+1}$, and we have $U = V \circ \phi$ where the following holds.

- (a) The function ϕ is given by

$$\phi : Y \rightarrow \overline{D^s(0, 1)}, (t, z) \mapsto (b_1(t) \prod_{l=0}^r (\sigma_l z_l(t, z))^{p_{1l}}, \dots, b_s(t) \prod_{l=0}^r (\sigma_l z_l(t, z))^{p_{sl}}),$$

where $s \in \mathbb{N}_0$, $b_1, \dots, b_s : \pi^*(Y) \rightarrow \mathbb{R}$ are definable which have no zeros and $p_{1l}, \dots, p_{sl} \in \mathbb{Q}$ for every $l \in \{0, \dots, r\}$.

- (b) V is a power series which converges absolutely on an open neighbourhood of $\overline{D^s(0, 1)}$.

We call a **coefficient** of F and b_1, \dots, b_s are called **base functions** of F . We call $\mathcal{L}\mathcal{A}_{\mathbb{C}} := (r, \mathcal{Z}, a, q, s, V, b, P)$ where $\mathcal{Z} := (z_0, \dots, z_r)$ and $b := (b_1, \dots, b_s)$ and

$$P := \begin{pmatrix} p_{10} & \cdot & \cdot & p_{1r} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ p_{s0} & \cdot & \cdot & p_{sr} \end{pmatrix} \in M(s \times (r+1), \mathbb{Q})$$

a **complex LA-preparing tuple** for F .

6.75 Remark

Let $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be r -log-analytically prepared in x . Let $\mathcal{LA} := (r, \mathcal{Y}, a, q, s, v, b, P)$ be an LA-preparing tuple for f with $b := (b_1, \dots, b_s)$ and $P := (p_1, \dots, p_s)^t$. (So we have

$$f(t, x) = a(t)|\mathcal{Y}(t, x)|^{\otimes q} v(b_1(t)|\mathcal{Y}(t, x)|^{\otimes p_1}, \dots, b_s(t)|\mathcal{Y}(t, x)|^{\otimes p_s}).$$

Let $R > 1$ be such that v converges absolutely on an open neighbourhood of $\overline{Q^s(0, R)}$. For $l \in \{1, \dots, s\}$ set

$$\Gamma_l := \{(t, z) \in H \mid |b_l(t)(\sigma \mathcal{Z})^{\otimes p_l}(t, z)| < R\}.$$

Set

$$\Gamma := \bigcap_{l=1}^s \Gamma_l$$

and consider

$$F : \Gamma \rightarrow \mathbb{C}, (t, z) \mapsto a(t)(\sigma \mathcal{Z})^{\otimes q}(t, z) \hat{V}(b_1(t)(\sigma \mathcal{Z})^{\otimes p_1}(t, z), \dots, b_s(t)(\sigma \mathcal{Z})^{\otimes p_s}(t, z))$$

where \hat{V} is a complex power series in s variables which converges absolutely on an open neighbourhood of $\overline{D^s(0, R)}$ with $\hat{V}|_{Q^s(0,1)} = v$. Then the function F is well-defined, definable and holomorphic in z . Additionally we have $C \subset \Gamma$ and $F|_C = f$. Therefore F is an extension for f which is holomorphic in z . Additionally F is complex r -log-analytically prepared in z with preparing tuple $\mathcal{LA}_{\mathbb{C}} := (r, \mathcal{Z}, a, q, s, V, 1/R \cdot b, P)$ where $V(z_1, \dots, z_s) := \hat{V}(Rz_1, \dots, Rz_s)$.

6.76 Proposition

Let $\nu := (\nu_0, \dots, \nu_{\kappa-1}) \in \mathbb{Q}^{\kappa}$. There is a regular κ -persistent $M_{\nu} : \pi(C) \rightarrow \mathbb{R}_{>0}$ such that the following holds where $D_{\nu} := \mathcal{B}(\mathcal{P}_{0,\kappa}, M_{\nu})$.

$\prod_{j=0}^{\kappa-1} (\sigma_j z_j)^{\nu_j}|_{D_{\nu}}$ is complex r -log-analytically prepared in z with complex preparing tuple

$$(r, \mathcal{Z}, a_{\nu}, q_{\nu}, s_{\nu}, V_{\nu}, b_{\nu}, P_{\nu})$$

such that $q_{\nu} = 0$, $P_{\nu} = (p_{i(j-1)})_{i,j}$ with $p_{ij} = 0$ for $j \in \{0, \dots, r\} \setminus \{\kappa\}$ and $i \in \{1, \dots, s_{\nu}\}$, $a_{\nu} \neq 0$, $1/|a_{\nu}|$ is κ -persistent and a_{ν} respectively b_{ν} are regular. Additionally $|V_{\nu}(\overline{D^{s_{\nu}}(0, 1)})| \subset \mathbb{R}_{>0}$.

Proof

Let $0 < \delta \leq 1/2$ be such that $1/2 < \prod_{j=0}^{\kappa-1} |1 + z|^{\nu_j} < 2$ for $z \in B(0, \delta)$. By Proposition 6.57(1) there is a regular κ -persistent $M_\delta : \pi(C) \rightarrow \mathbb{R}_{>0}$ such that

$$\left| \frac{\mathcal{P}_{j,\kappa}(t, z)}{e^{\mu_{\kappa-j-1,\kappa}(t)}} \right| < \delta$$

for every $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, M_\delta)$ and every $j \in \{0, \dots, \kappa - 1\}$. We have

$$\prod_{j=0}^{\kappa-1} (\sigma_j z_j(t, z))^{\nu_j} = \prod_{j=0}^{\kappa-1} e^{\nu_j \mu_{\kappa-j-1,\kappa}(t)} \left(1 + \frac{\sigma_j \mathcal{P}_{j,\kappa}(t, z)}{e^{\mu_{\kappa-j-1,\kappa}(t)}}\right)^{\nu_j} \quad (+)$$

for $(t, z) \in \mathcal{B}(\mathcal{P}_{0,\kappa}, M_\delta)$ (compare with Proposition 6.20).

We say that a function $g : Y \rightarrow \mathbb{C}$ where $Y \subset \mathcal{B}(\mathcal{P}_{0,\kappa}, M_\delta)$ is definable with $\pi^*(Y) = \pi(C)$ fulfills property $(*)_p$ if the following holds:

g is complex r -log-analytically prepared in z with complex LA-preparing tuple

$$(r, \mathcal{Z}, a_g, q_g, s_g, V_g, b_g, P_g)$$

with $q_g := ((q_g)_0, \dots, (q_g)_r)$ such that $(q_g)_\kappa = 1$ and $(q_g)_j = 0$ for $j \in \{0, \dots, r\} \setminus \{\kappa\}$, $P_g = (\hat{p}_{i(j-1)})_{i,j}$ with $\hat{p}_{i(j-1)} = 0$ for $j \in \{1, \dots, r+1\} \setminus \{\kappa+1\}$ and $i \in \{1, \dots, s_g\}$, $a_g \neq 0$, $1/|a_g|$ is κ -persistent and a_g respectively b_g are regular.

Let $\Omega := M_\delta$.

Claim 1

By shrinking Ω if necessary we have that $\mathcal{P}_{0,\kappa}|_D$ fulfills property $(*)_p$ (i.e. there is a regular κ -persistent $\hat{\Omega} : \pi(C) \rightarrow \mathbb{R}_{>0}$ with $\hat{\Omega} \leq \Omega$ such that $\mathcal{P}_{0,\kappa}|_{\mathcal{B}(\mathcal{P}_{0,\kappa}, \hat{\Omega})}$ fulfills property $(*)_p$.)

Proof of Claim 1

Let $E_0 := z_\kappa|_D$. Inductively for $l \in \{1, \dots, \kappa\}$ let

$$E_l : D \rightarrow \mathbb{C}, (t, z) \mapsto \sigma_{\kappa-l} e^{\mu_{l-1,\kappa}(t)} (e^{E_{l-1}(t,z)} - 1).$$

We see by an easy induction on $l \in \{0, \dots, \kappa\}$ that $E_l(t, z) = \mathcal{P}_{\kappa-l,\kappa}(t, z)$ for $(t, z) \in D$. We show by induction on $l \in \{0, \dots, \kappa\}$ that $E_l|_D$ fulfills property $(*)_p$ by shrinking Ω if necessary. For $l = 0$ this is clear.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Z}, a, q, s, V, b, P)$$

be a corresponding complex preparing tuple for $E_{l-1}|_D$ where $P := (p_{i(j-1)})_{i,j}$. We have for $(t, z) \in D$

$$E_l(t, z) = \sigma_{\kappa-l} e^{\mu_{l-1,\kappa}(t)} (\exp(a(t) \sigma_\kappa z_\kappa(t, z) Q(t, z)) - 1)$$

where for $(t, z) \in D$

$$Q(t, z) := V(b_1(t)(\sigma_\kappa z_\kappa(t, z))^{p_{1\kappa}}, \dots, b_s(t)(\sigma_\kappa z_\kappa(t, z))^{p_{s\kappa}}).$$

We see for $(t, z) \in D$

$$E_l(t, z) = \sigma_{\kappa-l} e^{\mu_{l-1, \kappa}(t)} a(t) \sigma_\kappa z_\kappa(t, z) \frac{\exp(a(t) \sigma_\kappa z_\kappa(t, z) Q(t, z)) - 1}{a(t) \sigma_\kappa z_\kappa(t, z)}.$$

Since $1/|a|$ is κ -persistent and regular we may assume that $|z_\kappa(t, z)| < 1/|a(t)|$ for $(t, z) \in D$ after shrinking Ω if necessary (by Proposition 6.58(1) there is a regular κ -persistent $\hat{M} : \pi(C) \rightarrow \mathbb{R}_{>0}$ such that $\mathcal{B}(\mathcal{P}_{0, \kappa}, \hat{M}) \subset \mathcal{B}(z_\kappa, 1/|a|)$). Since $\sigma_{\kappa-l} e^{\mu_{l-1, \kappa}} a$ is regular and $1/(|e^{\mu_{l-1, \kappa}} a|)$ is κ -persistent we see that E_l fulfills property $(*)_p$ by using the exponential series and composition of power series. ■ Claim 1

Claim 2

By shrinking Ω if necessary we have that $\mathcal{P}_{l, \kappa}|_D$ fulfills property $(*)_p$ for $l \in \{0, \dots, \kappa - 1\}$.

Proof of Claim 2

We do an induction on l . By Claim 1 we get the assertion for $l = 0$.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Z}, a, q, s, V, b, P)$$

be a corresponding complex preparing tuple for $\mathcal{P}_{l-1, \kappa}|_D$ where $P := (p_{i(j-1)})_{i, j}$. Let $\hat{a} := \frac{a}{e^{\mu_{\kappa-l, \kappa}}}$. We have for $(t, z) \in D$

$$\begin{aligned} \mathcal{P}_{l, \kappa}(t, z) &= \log^*(1 + \frac{\mathcal{P}_{l-1, \kappa}(t, z)}{e^{\mu_{\kappa-l, \kappa}(t)}}) \\ &= \hat{a}(t) \sigma_\kappa z_\kappa(t, z) \frac{\log^*(1 + \hat{a}(t) \sigma_\kappa z_\kappa(t, z) Q(t, z))}{\hat{a}(t) \sigma_\kappa z_\kappa(t, z)} \end{aligned}$$

where for $(t, z) \in D$

$$Q(t, z) := V(b_1(t)(\sigma_\kappa z_\kappa(t, z))^{p_{1\kappa}}, \dots, b_s(t)(\sigma_\kappa z_\kappa(t, z))^{p_{s\kappa}}).$$

Note that \hat{a} is regular and $1/|\hat{a}|$ is κ -persistent since $e^{\mu_{\kappa-l, \kappa}}$ is regular and κ -persistent. So by shrinking Ω if necessary we may assume that $|z_\kappa(t, z)| < 1/|\hat{a}(t)|$ for $(t, z) \in D$. We see by using the logarithmic series and composition of power series that $\mathcal{P}_{l, \kappa}|_D$ fulfills $(*)_p$. ■ Claim 2

So let for $j \in \{0, \dots, \kappa - 1\}$

$$(r, \mathcal{Z}, a_j, q_j, s_j, V_j, b_j, P_j)$$

be a complex preparing tuple for $\mathcal{P}_{j,\kappa}|_D$ according to property $(*)_p$. Note that $a_j(t)/e^{\mu_{\kappa-j-1,\kappa}(t)}$ is regular and the absolute value of its reciprocal is κ -persistent. So by shrinking Ω if necessary we may assume that $|z_\kappa(t, z)| < |a_j(t)/e^{\mu_{\kappa-j-1,\kappa}(t)}$ for $(t, z) \in D$ and $j \in \{0, \dots, \kappa - 1\}$. So with (+) and composition of power series we obtain the desired preparation for $\prod_{j=0}^{\kappa-1} (\sigma_j z_j)^{\nu_j}|_D$ since $\prod_{j=0}^{\kappa-1} e^{\nu_j \mu_{\kappa-j-1,\kappa}}$ is regular and its reciprocal is κ -persistent (by writing $\prod_{j=0}^{\kappa-1} (1+z)^{\nu_j}$ as a power series for $z \in B(0, 1/2)$ which converges absolutely on an open neighbourhood of $\overline{B(0, 1/2)}$). This finishes the proof of Proposition 6.76. \blacksquare

For the rest of this paragraph we fix a definable $Y \subset H$. Fix a complex r -log-analytically prepared function $F : Y \rightarrow \mathbb{C}$ in z with complex LA-preparing tuple $(r, \mathcal{Z}, a, q, s, V, b, P)$.

6.77 Remark

Let $\eta := (a, b_1, \dots, b_s)$. Then there is a globally subanalytic $G : \mathbb{C}^{s+1} \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}$ such that for every $(t, z) \in Y$ we have

$$F(t, z) = G(\eta(t), \mathcal{Z}(t, z)).$$

So F is definable.

Proof

Let $u := (u_0, \dots, u_s)$ range over \mathbb{C}^{s+1} and $w := (w_0, \dots, w_r)$ over \mathbb{C}^{r+1} . Set

$$\alpha_0 : \mathbb{C}^{s+1} \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}, (u, w) \mapsto \begin{cases} u_0 \prod_{j=0}^r (\sigma_j w_j)^{q_j}, & \sigma_j w_j \in \mathbb{C}^- \text{ for every } j \in \{0, \dots, r\}, \\ 0, & \text{else.} \end{cases}$$

For $l \in \{1, \dots, s\}$ let

$$\alpha_l : \mathbb{C}^{s+1} \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}, (u, w) \mapsto \begin{cases} u_l \prod_{j=0}^r (\sigma_j w_j)^{p_{lj}}, & \sigma_j w_j \in \mathbb{C}^- \text{ for every } j \in \{0, \dots, r\}, \\ 0, & \text{else.} \end{cases}$$

Set

$$G : \mathbb{C}^{s+1} \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}, (u_0, \dots, u_s, w_0, \dots, w_r) \mapsto \begin{cases} \alpha_0(u, w) V(\alpha_1(u, w), \dots, \alpha_s(u, w)), & |\alpha_i(u, w)| \leq 1 \text{ for all } i \in \{1, \dots, s\}, \\ 0, & \text{else.} \end{cases}$$

Then G is globally subanalytic. Note that $\sigma_l z_l(t, z) \in \mathbb{C}^-$ for every $l \in \{0, \dots, r\}$ and $(t, z) \in H$. So we have

$$F(t, z) = G(\eta(t), \mathcal{Z}(t, z))$$

for every $(t, z) \in Y$. ■

For the rest of this paragraph let $Y := \mathcal{A}(\mathcal{P}_{0,\kappa}, N, T)$ where $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ is regular and κ -non-persistent and $T : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ is regular and κ -persistent. So we may also assume that C is near with respect to μ_κ and consequently $\Theta_{k+1} = \dots = \Theta_r = 0$ if $\kappa = k$. Let $\pi^+ : \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n \times \mathbb{R}$, $(t, s, z) \mapsto (t, s)$, be the projection on the first $n + 1$ real coordinates.

6.78 Proposition

There is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that the following holds. Let

$$\Delta := \{(t, s, z) \in \pi(C) \times \mathbb{R}_{>0} \times \mathbb{C} \mid N(t) < s < M(t), z \in B(\mu_\kappa(t), s)\}.$$

Then the function

$$\Psi : \Delta \rightarrow \mathbb{C}, (t, s, z) \mapsto \frac{1}{2\pi i} \int_{\partial B(\mu_\kappa(t), s)} \frac{F(t, \xi)}{\xi - z} d\xi,$$

is well-defined. Additionally there is $l \in \mathbb{N}$ and a definable function $g = (g_1, \dots, g_l) : \pi^+(\Delta) \rightarrow \mathbb{R}^l$ such that Ψ is constructible in z with support function

$$\pi^+(\Delta) \rightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}, (t, s) \mapsto (g(t, s), \mu_\kappa(t), s),$$

where the following holds for g_j where $j \in \{1, \dots, l\}$: There is $p \in \mathbb{N}_0$, a regular function $\beta : \pi(C) \rightarrow \mathbb{R}^p$ and a log-analytic $h_j : \mathbb{R}^{p+1} \rightarrow \mathbb{R}$ such that $g_j(t, s) = h_j(\beta(t), s)$ for every $(t, s) \in \pi^+(\Delta)$.

Proof

There is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ (with $M < |\mu_r - \mu_k|$ if $k \geq 0$) such that for $D := \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M)$ we have $D \subset Y$ and

$$z_\alpha^* : D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto z_\alpha(t, \mu_\kappa(t) + \left(\prod_{j=0}^{\kappa} \sigma_j\right) s e^{i\tau}),$$

is free-regular in s for $\alpha \in \{0, \dots, r\}$ where

$$D(\mu_\kappa) = \{(t, s, \tau) \in \pi(C) \times \mathbb{R}_{>0} \times \mathbb{C} \mid N(t) < s < M(t)\}.$$

(By Proposition 6.66(2) if $\kappa = r$ respectively Proposition 6.72 if $\kappa = k$.) Let $\eta := (a, b_1, \dots, b_s)$. Note that η is regular. Let $l := s + 1$. By Remark 6.77 there is a globally subanalytic function $G : \mathbb{C}^l \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}$ such that

$$F(t, z) = G(\eta(t), z_0(t, z), \dots, z_r(t, z))$$

for every $(t, z) \in D$. With Remark 6.65(2) we have that

$$F^* : D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto F(t, \mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau}),$$

is free-regular in s (since the components of η considered as functions on $D(\mu_\kappa)$ are free-regular in s). By shrinking M if necessary we may assume that F^* is bounded in τ . (By Corollary 6.67 if $\kappa = r$ respectively Corollary 6.73 if $\kappa = k$ we obtain after shrinking M if necessary that

$$D(\mu_\kappa) \rightarrow \mathbb{C}, (t, s, \tau) \mapsto (\sigma \mathcal{Z})^{\otimes q}(t, \mu_\kappa(t) + (\prod_{j=0}^{\kappa} \sigma_j) s e^{i\tau}),$$

is bounded in τ .) Let $l \in \mathbb{N}$ and $g := (g_1, \dots, g_l) : \pi^+(D(\mu_\kappa)) \rightarrow \mathbb{R}^l$ be a corresponding support function for F^* . With Proposition 6.62 we see that the function

$$\Psi : \Delta \rightarrow \mathbb{C}, (t, s, z) \mapsto \frac{1}{2\pi i} \int_{\partial B(\mu_\kappa(t), s)} \frac{F(t, \xi)}{\xi - z} d\xi,$$

is well-defined and constructible in z with support function

$$\pi^+(\Delta) \rightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}, (t, s) \mapsto (g(t, s), \mu_\kappa(t), s).$$

Since $\pi^+(D(\mu_\kappa)) = \pi^+(\Delta)$ we see with Definition 6.63 that g_j fulfills the desired properties for $j \in \{1, \dots, l\}$. ■

6.79 Proposition

There is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$D := \mathcal{A}(\mathcal{P}_{0, \kappa}, N, M) \subset Y$$

and the following holds:

(1) For $(t, z) \in D$ and $j \in \{0, \dots, \kappa - 1\}$ we have that

$$\left| \frac{\mathcal{P}_{j, \kappa}(t, z)}{e^{\mu_{\kappa-j-1, \kappa}(t)}} \right| < 1/2.$$

(2) $F|_D$ is complex r -log-analytically prepared in z with preparing tuple

$$(r, \mathcal{Z}, \hat{a}, \hat{q}, \hat{s}, \hat{V}, \hat{b}, \hat{P})$$

where $\hat{q} := (\hat{q}_0, \dots, \hat{q}_r)$ with $\hat{q}_0 = \dots = \hat{q}_{\kappa-1} = 0$ and $\hat{P} \in M_\kappa(s \times (r+1), \mathbb{Q})$. Additionally \hat{a} and \hat{b} are regular.

Proof

For $q \in \mathbb{Q}^{r+1}$ we set $q^* := (q_0, \dots, q_{\kappa-1})$. By Proposition 6.76 there is a regular κ -persistent $\hat{M} : \pi(C) \rightarrow \mathbb{R}_{>0}$ such that for $\nu := (\nu_0, \dots, \nu_{\kappa-1}) \in \{q^*, p_1^*, \dots, p_s^*\}$ the following holds. The function $\prod_{j=0}^{\kappa-1} (\sigma_j z_j)^{\nu_j} |_{\mathcal{B}(\mathcal{P}_{0,\kappa}, \hat{M})}$ is complex r -log-analytically prepared in z with preparing tuple

$$(r, \mathcal{Z}, a_\nu, q_\nu, s_\nu, V_\nu, b_\nu, P_\nu)$$

such that $q_\nu = 0$, $P_\nu = ((p_\nu)_{i(j-1)})_{i,j}$ with $(p_\nu)_{i(j-1)} = 0$ for $j \in \{1, \dots, r+1\} \setminus \{\kappa+1\}$ and $i \in \{1, \dots, s_\nu\}$, a_ν respectively $b_\nu := ((b_\nu)_1, \dots, (b_\nu)_{s_\nu})$ are regular and we have $|V_\nu(D^{s_\nu}(0,1))| \subset \mathbb{R}_{>0}$. By shrinking \hat{M} if necessary we obtain property (1) with Proposition 6.57(1). For $\nu \in \{q^*, p_1^*, \dots, p_s^*\}$ we set

$$S_\nu : \mathcal{B}(\mathcal{P}_{0,\kappa}, \hat{M}) \rightarrow \mathbb{C}, (t, z) \mapsto$$

$$V_\nu((b_\nu)_1(t)(\sigma_\kappa z_\kappa(t, z))^{(p_\nu)_{1\kappa}}, \dots, (b_\nu)_{s_\nu}(t)(\sigma_\kappa z_\kappa(t, z))^{(p_\nu)_{s_\nu \kappa}}).$$

So we obtain for $(t, z) \in \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M)$ where

$$M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{T(t), \hat{M}(t)\},$$

$$F(t, z) = a(t)a_{q^*}(t)S_{q^*}(t, z) \prod_{j=\kappa}^r (\sigma_j z_j(t, z))^{q_j}$$

$$V(b_1(t)a_{p_1^*}(t)S_{p_1^*}(t, z) \prod_{j=\kappa}^r (\sigma_j z_j(t, z))^{p_{1j}}, \dots, b_s(t)\alpha_{p_s^*}(t)S_{p_s^*}(t, z) \prod_{j=\kappa}^r (\sigma_j z_j(t, z))^{p_{sj}}).$$

By composition of power series we get property (2). ■

For Proposition 6.80 fix M and D from Proposition 6.79. Suppose that $F|_D$ is complex r -log-analytically prepared in z as in Proposition 6.79.

6.80 Proposition

Let $t \in \pi^*(D)$. Assume that $N(t) = 0$, $M(t) > 0$ and that $F(t, -)$ has a holomorphic extension at $\mu := \mu_\kappa(t)$ (i.e. there is an open neighbourhood W of μ in $B(\mu, M(t))$ and a holomorphic $J : W \rightarrow \mathbb{C}$ such that $J|_{W \cap D_t} = F(t, -)$). Then $F(t, -)$ has a holomorphic extension on $B(\mu, M(t))$ (i.e. there is a holomorphic function $h : B(\mu, M(t)) \rightarrow \mathbb{C}$ such that $h|_{D_t} = F(t, -)$).

Proof

We have

$$F(t, -) = a(t)(\sigma \mathcal{Z})^{\otimes q}(t, z)V(b_1(t)(\sigma \mathcal{Z})^{\otimes p_1}(t, z), \dots, b_s(t)(\sigma \mathcal{Z})^{\otimes p_s}(t, z)).$$

Since $z_\kappa(t, z) = \mathcal{P}_\kappa(t, z)$ we obtain by successively using the logarithmic series

$$z_\kappa(t, z) = L(z - \mu) (*)$$

for $z \in D_t$ where L is a power series around zero which converges absolutely on $B(0, M(t))$. We set $Z := (Z_1, \dots, Z_s)$. Let $\sum_{\alpha \in \mathbb{N}_0^s} c_\alpha Z^\alpha$ be the power series expansion of V .

We may assume that $a(t) \neq 0$. Note that $(p_{j\kappa}, \dots, p_{jr}) = 0$ or p_j is κ -positive for $j \in \{1, \dots, s\}$. Let

$$\Gamma_1 := \{\alpha \in \mathbb{N}_0^s \mid {}^t P\alpha + q \in \{0\}^\kappa \times \mathbb{N}_0 \times \{0\}^{r-\kappa}\}$$

and $\Gamma_2 := \mathbb{N}_0^s \setminus \Gamma_1$. Set $V_1 := \sum_{\alpha \in \Gamma_1} c_\alpha Z^\alpha$ and $V_2 := \sum_{\alpha \in \Gamma_2} c_\alpha Z^\alpha$. For $l \in \{1, 2\}$ let

$$G_l : D_t \rightarrow \mathbb{C}, z \mapsto a(t)(\sigma Z)^{\otimes q}(t, z)V_l(b_1(t)(\sigma Z)^{\otimes p_1}(t, z), \dots, b_s(t)(\sigma Z)^{\otimes p_s}(t, z)).$$

So we see by composition of power series and $(*)$ that

$$G_1(z) = \sum_{j=0}^{\infty} d_j (z - \mu)^j$$

for $z \in D_t$ where $d_j \in \mathbb{C}$ and the series to the right converges absolutely on $B(\mu, M(t))$. So G_1 has a holomorphic extension \hat{G}_1 on $B(\mu, M(t))$. Let $0 < \epsilon < M(t)$ be such that $F|_{B(\mu, \epsilon) \cap D_t}$ has a holomorphic extension \hat{F} on $W := B(\mu, \epsilon)$. Set

$$\hat{G}_2 : W \rightarrow \mathbb{C}, z \mapsto \hat{F}(z) - \hat{G}_1(z).$$

Note that \hat{G}_2 is holomorphic and that $\hat{G}_2|_{D_t \cap W} = G_2$. We show that $\hat{G}_2 = 0$ and we are done by the identity theorem (since F_t coincides then with G_1 on $W \cap D_t$). Let

$$\begin{aligned} \Omega := \{(\omega_\kappa, \dots, \omega_r) \in \mathbb{Q}^{r-\kappa+1} \mid \text{there is } \alpha \in \Gamma_2 \text{ such that} \\ ({}^t P\alpha + q)_j = \omega_j \text{ for } j \in \{\kappa, \dots, r\}\}. \end{aligned}$$

Then $\Omega \subset \mathbb{Q}^{r-\kappa+1} \setminus (\mathbb{N}_0 \times \{0\}^{r-\kappa})$. For $\omega \in \Omega$ let

$$\Gamma_{2, \omega} := \{\alpha \in \mathbb{N}_0^s \mid ({}^t P\alpha + q)_j = \omega_j \text{ for } j \in \{\kappa, \dots, r\}\}.$$

We have

$$\hat{G}_2(z) = \sum_{\omega \in \Omega} e_\omega \prod_{j=\kappa}^r (\sigma_j z_j(t, z))^{\omega_j}$$

where

$$e_\omega = a(t) \sum_{\alpha \in \Gamma_{2, \omega}} c_\alpha \prod_{j=1}^s b_j(t)^{\alpha_j}$$

for $z \in W$. Let

$$\Omega^* := \{\omega \in \Omega \mid e_\omega \neq 0\}.$$

If $\Omega^* = \emptyset$ we are done. So assume $\Omega^* \neq \emptyset$. There is $\xi := (\xi_\kappa, \dots, \xi_r) \in \Omega$ such that $\prod_{j=\kappa}^r |z_j|^{\omega_j} = o(\prod_{j=\kappa}^r |z_j|^{\xi_j})$ for all $\omega \in \Omega$ with $\omega \neq \xi$. If $\xi_\kappa \in \mathbb{N}_0$ we differentiate \hat{G}_2 m -times with respect to z where $m := \xi_\kappa + 1$ and we see with Corollary 6.35 that there is $\beta = (-1, \beta_{\kappa+1}, \dots, \beta_r) \in \mathbb{Q}^{r-\kappa+1}$ such that

$$\lim_{z \rightarrow \mu} \left| \frac{d^m \hat{G}_2 / dz^m(t, z)}{\prod_{j=\kappa}^r z_j^{\beta_j}} \right| \in \mathbb{R}^*.$$

But we obtain

$$\lim_{z \rightarrow \mu} \frac{d^m \hat{G}_2}{dz^m}(t, z) = \frac{d^m \hat{G}_2}{dz^m}(t, \mu) \in \mathbb{C},$$

a contradiction. If $\xi_\kappa \notin \mathbb{N}_0$ we get a similar contradiction by taking $m := \lceil \xi_\kappa \rceil$. ■

Holomorphic Extensions of Prepared Restricted Log-Exp-Analytic Functions

In the following we investigate definable holomorphic extensions of restricted log-exp-analytic functions in one variable, show that such an extension is complex log-analytically prepared on "suitable definable subsets" and give the desired result on integration.

For the rest of Chapter 6.1 we set the following.

Let $e \in \mathbb{N}_0 \cup \{-1\}$ and $r \in \mathbb{N}_0$. Let $f : C \rightarrow \mathbb{R}$ be an (e, r) -prepared function in x with center Θ with respect to a finite set E of positive definable functions such that every $g \in \log(E)$ is (l, r) -prepared in x with respect to E for a $l \in \{-1, \dots, e - 1\}$. For $g \in \log(E) \cup \{f\}$ we fix a corresponding preparing tuple $(r, \mathcal{Y}, a_g, \exp(d_{0,g}), q_g, s, v_g, b_g, \exp(d_g), P_g)$ where $b_g := (b_{1,g}, \dots, b_{s,g})$, $\exp(d_g) := (\exp(d_{1,g}), \dots, \exp(d_{s,g}))$ (compare with Definition 4.82). Fix a set \mathcal{E} of C -heirs such that Θ and a_g, b_g can be constructed from \mathcal{E} for $g \in \log(E)$.

Fix $R > 1$ such that v_g converges absolutely on $[-R, R]^s$ and a complex power series V_g which converges absolutely on $D^s(0, R)$ with $V_g|_{] -1, 1[^s} = v_g$ for every $g \in \log(E) \cup \{f\}$. By redefining a_g if necessary we may suppose that $|V_g| < \pi$ (since V_g is bounded. So we have also $|v_g| < \pi$).

6.81 Definition

For an (l, r) -prepared function $g \in \log(E) \cup \{f\}$ we define by induction on $l \in \{-1, \dots, e\}$ a set $\Lambda_g \subset \pi(C) \times \mathbb{C}$ and a function $\Phi_g : \Lambda_g \rightarrow \mathbb{C}$ which is **complex (l, r) -prepared in z** and a **complex preparing tuple $\mathcal{J}_{\mathbb{C}}$** for Φ_g . To this preparation we assign a set E_{Φ_g} of **complex definable functions** on Λ_g .

Base Case: $l = -1$: We set $\Lambda_g = \pi(C) \times \mathbb{C}$, $\Phi_g = 0$, $E_{\Phi_g} = \emptyset$ and $\mathcal{J}_{\mathbb{C}} = (0)$.

Inductive Step: Let

$$(r, \mathcal{Y}, a, e^{d_0}, q, s, v, b, e^d, P) := (r, \mathcal{Y}, a_g, e^{d_0, g}, q_g, s, v_g, b_g, e^{d_g}, P_g)$$

where $b_g := (b_1, \dots, b_s)$, $e^{d_g} := (e^{d_1}, \dots, e^{d_s})$, $P_g := (p_1, \dots, p_s)^t$ and for d_i the corresponding Λ_{d_i} , the complex $(l-1, r)$ -prepared Φ_{d_i} in z and the corresponding set $E_{\Phi_{d_i}}$ have already been defined for $i \in \{0, \dots, s\}$. Set

$$\Lambda_{0, g} := \{(t, z) \in \Lambda_{d_0} \cap H \mid |\operatorname{Im}(\Phi_{d_0}(t, z))| < \pi\}$$

and for $i \in \{1, \dots, s\}$ set

$$\Lambda_{i, g} := \{(t, z) \in \Lambda_{d_i} \cap H \mid |b_i(t)(\sigma \mathcal{Z})^{\otimes p_i}(t, z) \exp(\Phi_{d_i}(t, z))| < R, |\operatorname{Im}(\Phi_{d_i}(t, z))| < \pi\}.$$

Set $\Lambda_g = \bigcap_{i=0}^s \Lambda_{i, g}$ and

$$\begin{aligned} \Phi_g : \Lambda_g \rightarrow \mathbb{C}, (t, z) &\mapsto a(t)(\sigma \mathcal{Z})^{\otimes q}(t, z) e^{\Phi_{d_0}(t, z)} \\ &V(b_1(t)(\sigma \mathcal{Z})^{\otimes p_1}(t, z) e^{\Phi_{d_1}(t, z)}, \dots, b_s(t)(\sigma \mathcal{Z})^{\otimes p_s}(t, z) e^{\Phi_{d_s}(t, z)}). \end{aligned}$$

We set

$$E_{\Phi_g} = (E_{\Phi_{d_0}} \cup E_{\Phi_{d_1}} \cup \dots \cup E_{\Phi_{d_s}} \cup \{e^{\Phi_{d_0}}, \dots, e^{\Phi_{d_s}}\})|_{\Lambda_g}$$

and

$$\mathcal{J}_{\mathbb{C}} = (r, \mathcal{Z}, a, \exp(\Phi_{d_0}), q, s, V, b, \exp(\Phi_d), P)$$

for Φ_g where $\exp(\Phi_d) := (\exp(\Phi_{d_1}), \dots, \exp(\Phi_{d_s}))$.

We often write F instead for Φ_g and Λ instead for Λ_g if $g = f$.

6.82 Remark

Let $g \in \log(E) \cup \{f\}$ be (l, r) -prepared in x with respect to E for $l \in \{-1, \dots, e\}$. We have that Λ_g is definable, $(\Lambda_g)_t$ is open for $t \in \pi(C)$ and Φ_g is holomorphic in z . Additionally we have $\Phi_g|_C = g$ and Φ_g can be constructed from $E_F \cup \mathcal{E}$.

Proof

We show that Φ_g can be constructed from $E_{\Phi_g} \cup \mathcal{E}$. The other properties are clear. We do an induction on l . For $l = -1$ there is nothing to show.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, e^{d_0}, q, s, v, b, e^d, P) := (r, \mathcal{Y}, a_g, e^{d_0, g}, q_g, s, v_g, b_g, e^{d_g}, P_g)$$

where $b_g := (b_1, \dots, b_s)$, $\exp(d_g) := (\exp(d_1), \dots, \exp(d_s))$ and $P_g := (p_1, \dots, p_s)^t$. Note that a and b can be constructed from \mathcal{E} . Let

$$\mathcal{J}_{\mathbb{C}} := (r, \mathcal{Z}, a, \exp(D_0), q, s, V, b, \exp(D), P)$$

be a complex preparing tuple for Φ_g where $D_j := \Phi_{d_j}$ for $j \in \{0, \dots, s\}$ and

$$\exp(D) := (\exp(D_1), \dots, \exp(D_s)).$$

Note that z_0, \dots, z_r can be constructed from \mathcal{E} by Remark 6.14(3) since $\Theta_0, \dots, \Theta_r$ can be constructed from \mathcal{E} . By the inductive hypothesis we have that the functions D_0, \dots, D_s can be constructed from $E_{D_0} \cup \dots \cup E_{D_s} \cup \mathcal{E}$. So we obtain that $D_0|_{\Lambda_g}, \dots, D_s|_{\Lambda_g}$ can be constructed from $(E_{D_0} \cup \dots \cup E_{D_s})|_{\Lambda_g} \cup \mathcal{E}$ and we see with Proposition 3.16(1) that the functions $e^{D_0}|_{\Lambda_g}, \dots, e^{D_s}|_{\Lambda_g}$ can be constructed from $E_{\phi_g} \cup \mathcal{E}$ since the imaginary part of D_j is bounded and $e^{D_j}|_{\Lambda_g} \in E_{\Phi_g}$ for $j \in \{0, \dots, s\}$. Let

$$\eta : \Lambda_g \rightarrow \mathbb{C} \times \mathbb{C}^s \times \mathbb{C}^{s+1}, (t, z) \mapsto (a(t), b_1(t), \dots, b_s(t), e^{D_0(t, z)}, \dots, e^{D_s(t, z)}).$$

Then η can be constructed from $E_{\Phi_g} \cup \mathcal{E}$. Similarly as in Remark 6.77 one can show that there is a globally subanalytic $J : \mathbb{C} \times \mathbb{C}^s \times \mathbb{C}^{s+1} \times \mathbb{C}^{r+1} \rightarrow \mathbb{C}$ such that

$$\Phi_g(t, z) = J(\eta(t, z), z_0(t, z), \dots, z_r(t, z))$$

for every $(t, z) \in \Lambda_g$. With Proposition 3.16(2) we are done. ■

For the rest of Chapter 6.1 we set the following. Let $l, m \in \mathbb{N}_0$ be with $n = l + m$. Let w range over \mathbb{R}^l , u over \mathbb{R}^m and x over \mathbb{R} . Let $X \subset \mathbb{R}^n \times \mathbb{R}$ be definable with $C \subset X$ such that X_w is open for $w \in \mathbb{R}^l$. For the rest of this paragraph we suppose that $\log(E)$ consists only of restricted log-exp-analytic functions in (u, x) with reference set X which are locally bounded in (u, x) with reference set X . So $f : C \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, is $(m + 1, X)$ -restricted (e, r) -prepared in x and every $g \in \log(E)$ is $(m + 1, X)$ -restricted (l, r) -prepared in x for $l \in \{0, \dots, e - 1\}$.

It may happen that there is an unbounded $g \in \log(E)$ such that Φ_g has bounded imaginary part or there can be a bounded $g \in \log(E)$ such that Φ_g is unbounded and has a bounded imaginary part.

6.83 Example

Let $w = 0$. Let $X = C :=]0, 1[^2$. Let $E := \{C \rightarrow \mathbb{R}, (u, x) \mapsto \exp(u \log(x))\}$. Then E is a set of one positive definable function with $\log(E) = \{g\}$ where $g : C \rightarrow \mathbb{R}, (u, x) \mapsto u \log(x)$, is locally bounded in (u, x) with reference set X , but not bounded. Consider

$$f : C \rightarrow \mathbb{R}, (u, x) \mapsto \exp(u \log(x)).$$

Then f is $(2, X)$ -restricted $(1, 1)$ -prepared in x with center 0 (since it is $(1, 1)$ -prepared in x with respect to E , the underlying logarithmic scale is $\mathcal{Y} = (x, \log(x))$ and $v_f = v_g = 1$). Note that $\mu_0 = 0$ and $\mu_1 = 1$. Therefore $H =]0, 1[\times (\mathbb{C}_0^- \cap \mathbb{C}_1^+)$. We have $\Lambda = \Lambda_g = H$ (since $\text{Im}(u \log(z)) < \pi$ for $(u, z) \in H$), $\Phi_g : \Lambda \rightarrow \mathbb{C}, (u, z) \mapsto u \log(z)$, and

$$F : \Lambda \rightarrow \mathbb{C}, (u, z) \mapsto \exp(u \log(z)).$$

Note that Φ_g has bounded imaginary part, but is unbounded at zero. Let

$$D := \{(u, x) \in]0, 1[\times \mathbb{R}_{>0} \mid e^{-1/u} < x < 1\}$$

and consider $h := f|_D$. Then h is $(2, X)$ -restricted $(1, 1)$ -prepared in x since it is $(1, 1)$ -prepared in x with respect to $E|_D$. Note that $\log(E|_D) = \{g|_D\}$ is a set of one bounded function. We have $\Lambda_h = \Lambda_{g|_D} = \Lambda$ and $\Phi_h = F$.

For the next two Propositions let H_{small} be defined as follows.

- If $\prod_{j=0}^r \sigma_j = 1$ and $k < 0$ we set

$$H_{\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]\mu_r(t), \infty[\}.$$

- If $\prod_{j=0}^r \sigma_j = -1$ and $k < 0$ we set

$$H_{\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]-\infty, \mu_r(t)[\}.$$

- If $\prod_{j=0}^r \sigma_j = 1$ and $k \geq 0$ we set

$$H_{\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]\mu_r(t), \mu_k(t)[\}.$$

- If $\prod_{j=0}^r \sigma_j = -1$ and $k \geq 0$ we set

$$H_{\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]\mu_k(t), \mu_r(t)[\}.$$

Note that H_{small} is definable and that $C \subset H_{\text{small}} \subset H$.

For a definable curve $\gamma :]0, 1[\rightarrow H_w$ with $\gamma := (\gamma_u, \gamma_z)$ we say that γ is real if $\text{Im}(\gamma_z) = 0$. We write then γ_x instead of γ_z .

6.84 Proposition

Let $g \in \log(E)$. Let $w \in \mathbb{R}^l$. Let $\gamma, \hat{\gamma} :]0, 1[\rightarrow (\Lambda_g)_w \cap (H_{\text{small}})_w$ be definable curves compatible with C_w where $\hat{\gamma}$ is real and $\gamma_u = \hat{\gamma}_u$. Suppose that

- $\lim_{y \searrow 0} (\gamma_z(y) - \hat{\gamma}_z(y)) = 0$,
- $\lim_{y \searrow 0} |\gamma_z(y) - \mu_\kappa(w, \gamma_u(y))| > 0$ for $\kappa = r$ or $\kappa \in \{k, r\}$ if $k \geq 0$ and
- $\lim_{y \searrow 0} \gamma(y) \in X_w$.

Then we have

$$\lim_{y \searrow 0} \Phi_g(w, \gamma(y)) = \lim_{y \searrow 0} \Phi_g(w, \hat{\gamma}(y)) \in \mathbb{R}.$$

Proof

Note that g is (l, r) -prepared in x with respect to E for $l \in \{-1, \dots, e-1\}$. We do an induction on l . For $l = -1$ the statement is clear.

$l-1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, e^{d_0}, q, s, v, b, e^d, P) := (r, \mathcal{Y}, a_g, e^{d_0, g}, q_g, s, v_g, b_g, e^{d_g}, P_g)$$

where $b_g := (b_1, \dots, b_s)$, $\exp(d_g) := (\exp(d_1), \dots, \exp(d_s))$ and $P_g := (p_1, \dots, p_s)^t$. Note that $\lim_{y \searrow 0} b_i(w, \gamma_u(y)) \in \mathbb{R}$ for $i \in \{1, \dots, s\}$ by Remark 6.42(1) and Definition 6.38 and $\lim_{y \searrow 0} a(w, \gamma_u(y)) \in \mathbb{R}$ by Remark 6.42(2) (since g is locally bounded in (u, x) with reference set X). Let

$$\mathcal{J}_{\mathbb{C}} := (r, \mathcal{Z}, a, \exp(D_0), q, s, V, b, \exp(D), P)$$

be a complex preparing tuple for Φ_g where $D_j := \Phi_{d_j}$ for $j \in \{0, \dots, s\}$ and $\exp(D) := (\exp(D_1), \dots, \exp(D_s))$. By the inductive hypothesis we have

$$\lim_{y \searrow 0} \exp(D_j(w, \gamma(y))) = \lim_{y \searrow 0} \exp(D_j(w, \hat{\gamma}(y))) \in \mathbb{R}_{>0}$$

for $j \in \{0, \dots, s\}$.

Claim

Let $l \in \{0, \dots, r\}$. Then we have

$$\lim_{y \searrow 0} \sigma_l z_l(w, \gamma(y)) = \lim_{y \searrow 0} \sigma_l z_l(w, \hat{\gamma}(y)) \in \mathbb{R}_{>0}.$$

Proof of the claim

Note that $\lim_{y \searrow 0} |\hat{\gamma}_x(y) - \mu_j(w, \gamma_u(y))| > 0$ for $j \in \{0, \dots, l\}$. Since $\sigma_l z_l(w, \hat{\gamma}_u(y)) > 0$ for every $y \in]0, 1[$ we obtain

$$\lim_{y \searrow 0} \sigma_l z_l(w, \hat{\gamma}_u(y)) \in \mathbb{R}_{>0}$$

with (II_l) in Proposition 6.28.

We show the statement by induction on l . For $l = 0$ we have with (I_0) in Proposition 6.28

$$\begin{aligned} \lim_{y \searrow 0} \sigma_0 z_0(w, \gamma(y)) &= \lim_{y \searrow 0} \sigma_0(\gamma_z(y) - \Theta_0(w, \gamma_u(y))) \\ &= \lim_{y \searrow 0} \sigma_0(\hat{\gamma}_z(y) - \Theta_0(w, \gamma_u(y))) \in \mathbb{R}_{>0}. \end{aligned}$$

$l - 1 \rightarrow l$: We have with the inductive hypothesis, (I_l) in Proposition 6.28 and the continuity of the global logarithm

$$\begin{aligned} \lim_{y \searrow 0} \sigma_l z_l(w, \gamma(y)) &= \lim_{y \searrow 0} (\sigma_l(\log(\sigma_{l-1} z_{l-1}(w, \gamma(y))) - \Theta_l(w, \gamma_u(y)))) \\ &= \lim_{y \searrow 0} (\sigma_l(\log(\sigma_{l-1} z_{l-1}(w, \hat{\gamma}(y))) - \Theta_l(w, \gamma_u(y)))) \\ &= \lim_{y \searrow 0} \sigma_l z_l(w, \hat{\gamma}(y)) \in \mathbb{R}_{>0}. \end{aligned}$$

■ Claim

By the Claim we have

$$\lim_{y \searrow 0} (\sigma \mathcal{Z})^{\otimes \nu}(\gamma(y)) = \lim_{y \searrow 0} (\sigma \mathcal{Z})^{\otimes \nu}(\hat{\gamma}(y)) \in \mathbb{R}_{>0}$$

for every $\nu \in \{q, p_1, \dots, p_s\}$ and we are done with the proof of Proposition 6.84 since V is a continuous function on $\overline{D^s(0, R)}$. ■

For a definable $Z \subset \mathbb{R}^n \times \mathbb{C}$ and $(t, x) \in \mathbb{R}^n \times \mathbb{R}$ we set

$$\zeta_Z(t, x) := \sup\{y \in \mathbb{R}_{\geq 0} \mid x + iQ(0, y) \subset Z_t\}.$$

6.85 Proposition

Let $g \in \log(E) \cup \{f\}$. Let $w \in \mathbb{R}^l$. Let $\gamma :]0, 1[\rightarrow (H_{\text{small}})_w$ be a real definable curve compatible with C_w with

- $\lim_{y \searrow 0} \gamma(y) \in X_w$,
- $\lim_{y \searrow 0} \text{dist}(\gamma_x(y), C_{(w, \gamma_u(y))}) = 0$ and
- $\lim_{y \searrow 0} \gamma_x(y) - \mu_\kappa(w, \gamma_u(y)) > 0$ for $\kappa = r$ or $\kappa \in \{k^{\text{ch}}, r\}$ if $k^{\text{ch}} \geq 0$.

Then we have $\lim_{y \searrow 0} \zeta_{\Lambda_g}(w, \gamma(y)) > 0$.

Proof

Let $(u_0, x_0) := \lim_{y \searrow 0} \gamma(y)$. Assume that g is (l, r) -prepared in x with respect to E for $l \in \{-1, \dots, e\}$. We do an induction on l . The assertion is clear for $l = -1$.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, e^{d_0}, q, s, v, b, e^d, P) := (r, \mathcal{Y}, a_g, e^{d_0, g}, q_g, s, v_g, b_g, e^{d_g}, P_g)$$

where $b := (b_1, \dots, b_s)$, $\exp(d_g) := (\exp(d_{g_1}), \dots, \exp(d_{g_s}))$ and $P := (p_1, \dots, p_s)^t$. Let

$$\mathcal{J}_{\mathbb{C}} = (r, \mathcal{Z}, a, \exp(D_0), q, s, V, b, \exp(D), P)$$

be a complex preparing tuple for Φ_g where $D_j := \phi_{d_j}$ for $j \in \{0, \dots, s\}$ and $\exp(D) := (\exp(D_1), \dots, \exp(D_s))$. We set for $j \in \{1, \dots, s\}$ and $(t, z) \in \Lambda_g$

$$\phi_j(t, z) := b_j(t)(\sigma \mathcal{Z})^{\otimes p_j}(t, z)e^{D_j(t, z)}.$$

Note that $|\phi_j(t, z)| \leq 1$ if $(t, z) \in C$. By the inductive hypothesis and passing to a suitable subcurve of γ if necessary we may assume that $(w, \gamma(y)) \in \Lambda_{d_i}$ for $i \in \{0, \dots, s\}$ and every $y \in]0, 1[$.

Assume $\lim_{y \searrow 0} \zeta_{\Lambda_g}(w, \gamma(y)) = 0$. Then there is a definable curve $\hat{\gamma} :]0, 1[\rightarrow (H_{\text{small}})_w$ with $\hat{\gamma}_u = \gamma_u$, $\text{Re}(\hat{\gamma}) = \gamma_x$, $\lim_{y \searrow 0} \hat{\gamma}_z(y) = x_0$ and $0 < \epsilon < 1$ such that $\hat{\gamma}_z(y) \notin (\Lambda_g)_{(w, \gamma_u(y))}$ for $y \in]0, \epsilon[$.

By shrinking ϵ if necessary we may assume that $\hat{\gamma}(y) \in (\Lambda_{d_i})_w$ for every $y \in]0, \epsilon[$ and $i \in \{0, \dots, s\}$. By further shrinking ϵ if necessary we find $j^* \in \{0, \dots, s\}$ such that

$$|\text{Im}(D_{j^*}(w, \hat{\gamma}(y)))| \geq \pi$$

for every $y \in]0, \epsilon[$ or $j^+ \in \{1, \dots, s\}$ such that $|\phi_{j^+}(w, \hat{\gamma}(y))| \geq R$ for every $y \in]0, \epsilon[$. By Remark 6.27(2) there is a definable curve $\tilde{\gamma} :]0, \epsilon[\rightarrow C_w$ compatible with C_w with $\tilde{\gamma}_u = \hat{\gamma}_u = \gamma_u$ and $\lim_{y \searrow 0} \tilde{\gamma}(y) = (u_0, x_0)$. Since $\tilde{\gamma}$ runs through C_w we have $|\phi_{j^+}(w, \tilde{\gamma}(y))| \leq 1$ for $y \in]0, \epsilon[$ and therefore

$\lim_{y \searrow 0} |\phi_{j^+}(w, \tilde{\gamma}(y))| \leq 1$. So the latter cannot hold by Proposition 6.84 (since $R > 1$). Since $\tilde{\gamma}$ runs through C_w we obtain

$$\text{Im}(D_{j^*}(w, \tilde{\gamma}(y))) = \text{Im}(d_{j^*}(w, \tilde{\gamma}(y))) = 0$$

for $y \in]0, \epsilon[$. So the former cannot hold by Proposition 6.84, a contradiction. \blacksquare

For Proposition 6.86 we introduce the following notations. For a definable function $V : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ we set

$$V_{\text{up}} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} 1/V(t), & V(t) \neq 0, \\ 0, & \text{else,} \end{cases}$$

and

$$V_{\text{down}} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} 1/V(t), & V(t) \neq 0, \\ 1, & \text{else.} \end{cases}$$

Note that V_{up} and V_{down} are regular if V is regular. For a definable function $W : Y \rightarrow \mathbb{R}_{\geq 0}$ where $Y \subset \pi(C)$ is definable we set

$$W^C : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} W(t), & t \in Y, \\ 0, & \text{else.} \end{cases}$$

Note that W^C is regular if W is regular. Fix $\kappa = r$ or $\kappa \in \{k^{\text{ch}}, r\}$ if $k^{\text{ch}} \geq 0$. Let $k := k^{\text{ch}}$

6.86 Proposition

Let $g \in \log(E) \cup \{f\}$. There is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $B := \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M) \subset \Lambda_g$ and $G := \Phi_g|_B$ is complex r -log-analytically prepared in z with a complex LA-preparing tuple

$$(r, \mathcal{Z}, a_G, q_G, s_G, V_G, b_G, P_G)$$

where $q_G := ((q_G)_0, \dots, (q_G)_r)$, $b_G := (b_{1,G}, \dots, b_{s_G,G})$ and the following holds.

- (1) We have $(q_G)_0, \dots, (q_G)_{\kappa-1} = 0$ and $P_G \in M_\kappa(s \times (r+1), \mathbb{Q})$.
- (2) a_G is regular. If $g \in \log(E)$ then the following holds: a_G is consistent and if q_G is κ -negative then $1/((|a_G|^C)_{\text{down}})$ is κ -non-persistent.
- (3) b_G is regular, consistent and if $p_{i,G}$ is κ -negative for $i \in \{1, \dots, s_G\}$ then $1/((|b_{i,G}|^C)_{\text{down}})$ is κ -non-persistent where $P_G := (p_{1,G}, \dots, p_{s_G,G})^t$.

Proof

Let g be (l, r) -prepared in x with respect to E for $l \in \{-1, \dots, e\}$. We do an induction on l . For $l = -1$ there is nothing to show.

$l - 1 \rightarrow l$: Let

$$(r, \mathcal{Y}, a, e^{d_0}, q, s, v, b, e^d, P) := (r, \mathcal{Y}, a_g, e^{d_0, g}, q_g, s, v_g, b_g, e^{d_g}, P_g)$$

where $b_g := (b_1, \dots, b_s)$, $\exp(d_g) := (\exp(d_1), \dots, \exp(d_s))$ and $P_g := (p_1, \dots, p_s)^t$. We may assume that $a \neq 0$. Otherwise the statement is clear. Note that d_i is locally bounded in (u, x) with reference set X for every $i \in \{0, \dots, s\}$. For $i \in \{0, \dots, s\}$ let $\Lambda_i := \Lambda_{d_i}$. Let

$$\mathcal{J}_C = (r, \mathcal{Z}, a, \exp(\tilde{D}_0), q, s, V, b, \exp(\tilde{D}), P)$$

be a complex preparing tuple for Φ_g where $\tilde{D}_j := \Phi_{d_j}$ for $j \in \{0, \dots, s\}$ and $\exp(\tilde{D}) := (\exp(\tilde{D}_1), \dots, \exp(\tilde{D}_s))$.

By the inductive hypothesis there is a regular κ -non-persistent $N^* : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M^* : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that for $i \in \{0, \dots, s\}$

$$B^* := \mathcal{A}(\mathcal{P}_{0, \kappa}, N^*, M^*) \subset \Lambda_i$$

and $D_i := \Phi_{d_i}|_{B^*}$ is complex r -log-analytically prepared in z with complex LA-preparing tuple

$$(r, \mathcal{Z}, a_{D_i}, q_{D_i}, s_{D_i}, V_{D_i}, b_{D_i}, P_{D_i})$$

where $q_{D_i} := ((q_{D_i})_0, \dots, (q_{D_i})_r)$, $P_{D_i} := (p_{1, D_i}, \dots, p_{s_{D_i}, D_i})^t$, $b_{D_i} := (b_{1, D_i}, \dots, b_{s_{D_i}, D_i})$ and the following holds for $i \in \{0, \dots, s\}$:

- (1) $(q_{D_i})_0 = \dots = (q_{D_i})_{\kappa-1} = 0$ and $P_{D_i} \in M_\kappa(s_{D_i} \times (r+1), \mathbb{Q})$.
- (2) a_{D_i} is regular. If $g \in \log(E)$ the following holds: a_{D_i} is consistent and if q_{D_i} is κ -negative then $1/((|a_{D_i}|^C)_{\text{down}})$ is κ -non-persistent.
- (3) b_{D_i} is regular, consistent and if p_{j, D_i} is κ -negative then $1/((|b_{j, D_i}|^C)_{\text{down}})$ is κ -non-persistent for $j \in \{1, \dots, s_{D_i}\}$.

At first we deal with $\exp(D_0), \dots, \exp(D_s)$ and start with the following

Claim 1

Let $0 < \delta < 1/2$. Then there is a regular κ -non-persistent $S_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $T_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ with $Y_\delta := \mathcal{A}(\mathcal{P}_{0, \kappa}, S_\delta, T_\delta) \subset B^*$ and for $i \in \{0, \dots, s\}$ definable functions $D_{i,1} : \pi^*(Y_\delta) \rightarrow \mathbb{R}$ and $D_{i,2} : Y_\delta \rightarrow \mathbb{C}$ with the following properties.

- (1)* We have $D_i|_{Y_\delta} = D_{i,1} + D_{i,2}$.
- (2)* $D_{i,1}$ is regular and consistent.
- (3)* $D_{i,2}$ is bounded by δ . Additionally $\exp(D_{i,2})$ is complex r -log-analytically prepared in z with complex LA-preparing tuple $(r, \mathcal{Z}, 1, 0, \hat{s}, \hat{V}, \hat{b}, \hat{P})$ where $\hat{b} := (\hat{b}_1, \dots, \hat{b}_{\hat{s}})$ and \hat{b}_i is as in (3) for $i \in \{1, \dots, \hat{s}\}$.

Proof of Claim 1

Let $K \in \{D_0, \dots, D_s\}$. It suffices to find a regular κ -non-persistent $\hat{S}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $\hat{T}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and corresponding functions $K_1 : \pi^*(W) \rightarrow \mathbb{R}$ and $K_2 : W \rightarrow \mathbb{C}$ which fulfill properties (1)* - (3)* where $W := \mathcal{A}(\mathcal{P}_{0,\kappa}, \hat{S}_\delta, \hat{T}_\delta)$. Let

$$(r, \mathcal{Z}, \mathbf{a}, \mathbf{q}, \mathbf{s}, \mathfrak{V}, \mathbf{b}, \mathfrak{P}) := (r, \mathcal{Z}, a_K, q_K, s_K, V_K, b_K, P_K).$$

Suppose $\mathbf{a} \neq 0$. Otherwise the claim follows. Fix $R > 1$ such that \mathfrak{V} converges absolutely on $\overline{D^s(0, R)}$. Let $\sum_{\alpha \in \mathbb{N}_0^s} c_\alpha Z^\alpha$ be the power series expansion of \mathfrak{V} where $Z := (Z_1, \dots, Z_s)$. Fix $L \in \mathbb{R}$ with $\delta < L$ such that $\sum_{\alpha \in \mathbb{N}^s} |c_\alpha w^\alpha| < L$ for $w \in D^s(0, R)$.

1. Case: $(q_\kappa, \dots, q_r) \neq 0$.

Consider the regular function

$$T_\delta^* : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \min\left\{\frac{\delta}{L|a(t)|}, \frac{1}{|a(t)|}\right\}, & t \in \pi^*(B^*), \\ 0, & \text{else.} \end{cases}$$

Subclaim 1

T_δ^* is κ -persistent.

Proof of Subclaim 1

Let $w \in \mathbb{R}^l$. Let $\gamma :]0, 1[\rightarrow C_w$ be a definable curve compatible with C_w with $\lim_{y \searrow 0} \gamma(y) \in X_w$ and $\lim_{y \searrow 0} |\gamma_x(y) - \mu_\kappa(w, \gamma_u(y))| = 0$. Note that there is $0 < \epsilon \leq 1$ such that $(w, \gamma_u(y)) \in \pi^*(B^*)$ for every $y \in]0, \epsilon[$ (since N^* is κ -non-persistent and M^* is κ -persistent). Since \mathbf{a} is consistent we see that $\lim_{y \searrow 0} T_\delta^*(w, \gamma_u(y)) \in \mathbb{R}_{>0}$. ■ Subclaim 1

Note that if $|(\sigma \mathcal{Z})^{\otimes \mathbf{q}}(t, z)| < T_\delta^*(t)$ then $|K(t, z)| < \delta$ for $(t, z) \in B^*$.

1.1 Case: \mathbf{q} is κ -positive.

By Proposition 6.58(1) if $q_\kappa > 0$ respectively Proposition 6.58(2) if $\kappa = k \geq 0$ and $q_k = 0$ (since $(T_\delta^*)_{\text{up}}$ is consistent, because \mathbf{a} is consistent) there is a regular

κ -persistent $M_\delta^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0,\kappa}, M_\delta^+) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes \mathfrak{q}}, T_\delta^*)$. Take $\hat{S}_\delta := N^*$,

$$\hat{T}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M_\delta^+(t), M^*(t)\},$$

$K_1 := 0$ and $K_2 := K|_W$ where $W := \mathcal{A}(\mathcal{P}_{0,\kappa}, \hat{S}_\delta, \hat{T}_\delta)$. Note that $\exp(K_2) = \exp^*(K_2)$ where

$$\exp^* : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto \begin{cases} \exp(z), & |z| \leq \delta, \\ 0, & \text{else.} \end{cases}$$

Note that \exp^* is globally subanalytic. Additionally we obtain $|\mathfrak{a}(\sigma\mathcal{Z})^{\otimes \mathfrak{q}}(t, z)| < 1$ for $(t, z) \in W$. By composition of power series we get the desired preparation for $\exp(K_2)$.

1.2 Case: \mathfrak{q} is κ -negative.

Similarly as in the proof of Subclaim 1 we see that $(T_\delta^*)_{\text{down}}$ is κ -non-persistent (since $1/((|\mathfrak{a}|^C)_{\text{down}})$ is κ -non-persistent). By Corollary 6.59 there is a regular κ -non-persistent $N_\delta^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M_\delta^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N_\delta^+, M_\delta^+) \subset \mathcal{B}((\sigma\mathcal{Z})^{\otimes \mathfrak{q}}, T_\delta^*)$. Take $\hat{S}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \max\{N_\delta^+(t), N^*(t)\}$, $\hat{T}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M_\delta^+(t), M^*(t)\}$, $K_1 := 0$ and $K_2 := K|_W$ where $W = \mathcal{A}(\mathcal{P}_{0,\kappa}, \hat{S}_\delta, \hat{T}_\delta)$. We are done in a completely similar way as in case 1.1.

2. Case: $(\mathfrak{q}_\kappa, \dots, \mathfrak{q}_r) = 0$.

Let

$$\mathcal{S} := \{i \in \{1, \dots, \mathfrak{s}\} \mid (\mathfrak{p}_{i\kappa}, \dots, \mathfrak{p}_{ir}) \neq 0\}.$$

If $\mathcal{S} = \emptyset$ then $K = \mathfrak{a}\mathfrak{Y}(\mathfrak{b}_1, \dots, \mathfrak{b}_\mathfrak{s})$ is a regular and consistent function considered as function on $\pi^*(B^*)$. So we are done by taking $\hat{S}_\delta := N^*$, $\hat{T}_\delta := M^*$, $K_1 := 0$ and $K_2 := K$. So suppose $\mathcal{S} \neq \emptyset$. Let

$$\Gamma_1 := \{\alpha \in \mathbb{N}_0^\mathfrak{s} \mid {}^t P\alpha = 0\}$$

and $\Gamma_2 := \mathbb{N}_0^\mathfrak{s} \setminus \Gamma_1$. Note that $\Gamma_2 \neq \emptyset$. Set $\mathfrak{Y}_1 := \sum_{\alpha \in \Gamma_1} c_\alpha Z^\alpha$ and $\mathfrak{Y}_2 := \sum_{\alpha \in \Gamma_2} c_\alpha Z^\alpha$. For $l \in \{1, 2\}$ let

$$J_l := \mathfrak{a}\mathfrak{Y}_l(\mathfrak{b}_1(\sigma\mathcal{Z})^{\otimes \mathfrak{p}_1}, \dots, \mathfrak{b}_\mathfrak{s}(\sigma\mathcal{Z})^{\otimes \mathfrak{p}_\mathfrak{s}}).$$

Note that $K = J_1 + J_2$. Let $\phi_i := \mathfrak{b}_i(\sigma\mathcal{Z})^{\otimes \mathfrak{p}_i}$ for $i \in \{1, \dots, \mathfrak{s}\}$. So suppose $\mathcal{S} \neq \emptyset$. For $j \in \mathcal{S}$ we define the regular function

$$U_{j,\delta} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \left| \frac{\delta}{L\mathfrak{a}(t)\mathfrak{b}_j(t)} \right|, & t \in \pi^*(B^*), \\ 0, & \text{else.} \end{cases}$$

Similarly as in the proof of Subclaim 1 one sees for $j \in \mathcal{S}$ that $U_{j,\delta}$ is κ -persistent (since \mathfrak{b}_j and \mathfrak{a} are consistent) and that if \mathfrak{p}_j is κ -negative then

$(U_{j,\delta})_{\text{down}}$ is κ -non-persistent (since \mathbf{a} is consistent and $1/((|\mathbf{b}_j|^C)_{\text{down}})$ is κ -non-persistent). Additionally $(U_{j,\delta})_{\text{up}}$ is consistent for $j \in \mathcal{S}$.

We see that for $j \in \mathcal{S}$ there is a regular κ -non-persistent $\hat{S}_{j,\delta} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $\hat{T}_{j,\delta} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\mathcal{A}(\mathcal{P}_{0,\kappa}, \hat{S}_{j,\delta}, \hat{T}_{j,\delta}) \subset \mathcal{B}((\sigma \mathcal{Z})^{\mathbf{p}_j}, U_{j,\delta}).$$

(By Proposition 6.58(1) if \mathbf{p}_j is κ -positive and $\mathbf{p}_{j\kappa} \neq 0$ respectively Proposition 6.58(2) if \mathbf{p}_j is κ -positive, $\kappa = k \geq 0$ and $\mathbf{p}_{jk} = 0$ since $(U_{j,\delta})_{\text{up}}$ is consistent respectively Corollary 6.59 if \mathbf{p}_j is κ -negative. If \mathbf{p}_j is κ -positive take $\hat{S}_{j,\delta} = 0$.) Of course we may assume that $N^* \leq \hat{S}_{j,\delta}$ and $\hat{T}_{j,\delta} \leq M^*$ for $j \in \mathcal{S}$. Consider $\hat{S}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \max\{\hat{S}_{j,\delta}(t) \mid j \in \mathcal{S}\}$ and $\hat{T}_\delta : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{\hat{T}_{j,\delta}(t) \mid j \in \mathcal{S}\}$. Then \hat{S}_δ is regular and κ -non-persistent and \hat{T}_δ is regular and κ -persistent. Consider $W := \mathcal{A}(\mathcal{P}_{0,\kappa}, \hat{S}_\delta, \hat{T}_\delta)$.

To obtain Claim 1 it remains to show that $\exp(J_2)|_W$ is complex r -log-analytically prepared in z as desired (see Subclaim 2) and that $J_1|_W$ coincides with a regular and consistent function on $\pi(W)$.

Subclaim 2

For $(t, z) \in W$ we have $|J_2(t, z)| < \delta$. The function $\exp(J_2)|_W$ is complex r -log-analytically prepared in z as required in property (3)*.

Proof of Subclaim 2

Set $\phi := (\phi_1, \dots, \phi_{\mathfrak{s}})$. For every $\alpha := (\alpha_1, \dots, \alpha_{\mathfrak{s}}) \in \Gamma_2$ fix $i_\alpha \in \{1, \dots, \mathfrak{s}\}$ with $i_\alpha \in \mathcal{S}$ and $\alpha_{i_\alpha} \neq 0$. Therefore we have for $(t, z) \in W$

$$|\mathbf{a}(t)\phi_{i_\alpha}(t, z)| < \delta/L.$$

We obtain

$$\begin{aligned} |J_2(t, z)| &= |\mathbf{a}(t)\mathfrak{B}_2(\phi(t, z))| \\ &= |\mathbf{a}(t) \sum_{\alpha \in \Gamma_1} c_\alpha \phi_{i_\alpha}(t, z)^{\alpha_{i_\alpha}} \prod_{j \neq i_\alpha} \phi_j(t, z)^{\alpha_j}| \\ &= \left| \sum_{\alpha \in \Gamma_1} c_\alpha \mathbf{a}(t) \phi_{i_\alpha}(t, z) \phi_{i_\alpha}(t, z)^{\alpha_{i_\alpha} - 1} \prod_{j \neq i_\alpha} \phi_j(t, z)^{\alpha_j} \right| \\ &< \delta/L \sum_{\alpha \in \Gamma_1} |c_\alpha \phi_{i_\alpha}(t, z)^{\alpha_{i_\alpha} - 1} \prod_{j \neq i_\alpha} \phi_j(t, z)^{\alpha_j}| \\ &< \delta. \end{aligned}$$

Let $\mathcal{S} := \{j_1, \dots, j_\beta\}$ where $\beta \in \mathbb{N}$. Note that $i_\alpha \in \{j_1, \dots, j_\beta\}$ is unique for $\alpha \in \mathbb{N}_0^{\mathfrak{s}}$. Let $(w_{j_1}, \dots, w_{j_\beta})$ be a new tuple of complex variables. Consider for

$\gamma \in \{1, \dots, \beta\}$

$$\tilde{\phi}_\gamma : W \rightarrow \overline{B(0, 1)}, (t, z) \mapsto a(t)b_{j_\gamma}(t)(\sigma\mathcal{Z})^{\otimes p_{j_\gamma}}(t, z).$$

Consider

$$\hat{V} : \overline{D^{s+\beta}(0, 1)} \rightarrow \mathbb{C}, (z_1, \dots, z_s, w_{j_1}, \dots, w_{j_\beta}) \mapsto \sum_{\alpha \in \Gamma_2} c_\alpha w_{i_\alpha} z_{i_\alpha}^{\alpha_{i_\alpha} - 1} \prod_{j \neq i_\alpha} z_j^{\alpha_j}.$$

Note that \hat{V} defines a power series which converges absolutely on an open neighbourhood of $\overline{D^{s+\beta}(0, 1)}$ (and is therefore globally subanalytic). We have for $(t, z) \in W$

$$J_2(t, z) = \hat{V}(\phi_1(t, z), \dots, \phi_s(t, z), \tilde{\phi}_1(t, z), \dots, \tilde{\phi}_\beta(t, z))$$

and therefore

$$\exp(J_2(t, z)) = \exp^*(\hat{V}(\phi_1(t, z), \dots, \phi_s(t, z), \tilde{\phi}_1(t, z), \dots, \tilde{\phi}_\beta(t, z)))$$

where

$$\exp^* : \mathbb{C} \rightarrow \mathbb{C}, z \mapsto \begin{cases} \exp(z), & |z| \leq \delta, \\ 0, & \text{else.} \end{cases}$$

By using the exponential series and composition of power series we see that $\exp(J_2)$ has the desired properties. ■ Subclaim 2

Set

$$\Xi : \pi^*(W) \rightarrow \mathbb{R}, t \mapsto \sum_{\alpha \in \Gamma_1} \mathbf{a}(t) c_\alpha \prod_{j=1}^s \mathbf{b}_j(t)^{\alpha_j}.$$

Then we obtain $J_1(t, z) = \Xi(t)$ for $(t, z) \in W$. We show that Ξ is consistent and regular and are done with Claim 1. Note that

$$\Xi(t) = \mathbf{a}(t) \mathfrak{Y}_1(\mathbf{b}_1(t)(\sigma\mathcal{Z})^{\otimes p_1}(t, \lambda(t)), \dots, \mathbf{b}_s(t)(\sigma\mathcal{Z})^{\otimes p_s}(t, \lambda(t)))$$

for $t \in \pi^*(W)$ where

$$\lambda : \pi^*(W) \rightarrow \mathbb{R}, t \mapsto \mu_\kappa(t) + \left(\prod_{j=0}^{\kappa} \sigma_j \right) \frac{\hat{S}_\delta(t) + \hat{T}_\delta(t)}{2},$$

is regular. (Note also that $(t, \lambda(t)) \in W$ for $t \in \pi(W)$.) So we see that Ξ is consistent since \mathbf{a} is consistent.

Subclaim 3

The function Ξ is regular.

Proof of Subclaim 3

Similarly as in Remark 6.77 we find a globally subanalytic function $G : \mathbb{C}^{s+1} \times \mathbb{C}^{r-\kappa+1} \rightarrow \mathbb{C}$ such that for $t \in \pi^*(W)$

$$\Xi(t) = G(\eta(t), \mathcal{Z}_\kappa(t, \lambda(t)))$$

where $\mathcal{Z}_\kappa := (z_\kappa, \dots, z_r)$ and $\eta : \pi^*(W) \rightarrow \mathbb{R}^{s+1}, t \mapsto (\mathbf{a}(t), \mathbf{b}_1(t), \dots, \mathbf{b}_s(t))$. Note that η and Θ are regular. By Remark 6.44(2) it is enough to show that $\pi(C) \rightarrow \mathbb{C}, t \mapsto z_l(t, \lambda(t))$, is regular. We do an induction on $l \in \{0, \dots, r\}$.

$l = 0$: We have $z_0(t, \lambda(t)) = \lambda(t) - \Theta_0(t)$ for $t \in \pi^*(W)$. Remark 6.44(2) gives the result since Θ_0 is regular.

$l - 1 \rightarrow l$: We have $z_l(t, \lambda(t)) = \log(\sigma_{l-1} z_{l-1}(t, \lambda(t))) - \Theta_l(t)$ for $t \in \pi^*(W)$. Remark 6.44(2) and the inductive hypothesis give the result since Θ_l is regular.

■ Subclaim 3

So we set $K_1 := \Xi$ and $K_2 := J_2|_W$.

All in all we obtain Claim 1.

■ Claim 1

Fix $0 < \delta < \min\{\log(\sqrt[4]{R}), \pi\}$ such that

$$1/\sqrt[4]{R} < \prod_{j=0}^{\kappa-1} (1 + \delta)^{p_{ij}} < \sqrt[4]{R}$$

for every $i \in \{1, \dots, s\}$. To this δ fix the corresponding regular κ -non-persistent S_δ and the regular κ -persistent T_δ from Claim 1. Let $Y := Y_\delta = \mathcal{A}(\mathcal{P}_{0,\kappa}, S_\delta, T_\delta)$ and let $D_{i,1}$ and $D_{i,2}$ be as in Claim 1 for $i \in \{0, \dots, s\}$. (Note that $1/\sqrt[4]{R} < |\exp(D_{i,2}(t, z))| < \sqrt[4]{R}$ for $(t, z) \in Y$ and $i \in \{0, \dots, s\}$.) (+)

For $(t, z) \in \Lambda_g \cap Y$ we have

$$\begin{aligned} \Phi_g(t, z) &= a(t) \exp(D_{0,1}(t)) (\sigma \mathcal{Z})^{\otimes q}(t, z) \exp(D_{0,2}(t, z)) \\ &V(b_1(t) \exp(D_{1,1}(t)) (\sigma \mathcal{Z})^{\otimes p_1}(t, z) \exp(D_{1,2}(t, z)), \dots, \\ &b_s(t) \exp(D_{s,1}(t)) (\sigma \mathcal{Z})^{\otimes p_s}(t, z) \exp(D_{s,2}(t, z))). \end{aligned}$$

Note that $|\operatorname{Im}(D_{i,2}(t, z))| < \delta < \pi$ for $(t, z) \in Y$ and $i \in \{0, \dots, s\}$. So we have

$$\begin{aligned} \Lambda_g \cap Y &= \{(t, z) \in Y \mid |b_i(t) \exp(D_{i,1}(t)) (\sigma \mathcal{Z})^{\otimes p_i}(t, z) \exp(D_{i,2}(t, z))| < R \\ &\text{for every } i \in \{1, \dots, s\}\}. \end{aligned}$$

Claim 2

There is a regular κ -persistent $M^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent $N^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N^+, M^+) \subset \Lambda_g \cap Y$.

Proof of Claim 2

For $i \in \{1, \dots, s\}$ let

$$\Phi_i : Y \rightarrow \mathbb{C}, (t, z) \mapsto b_i(t) \exp(D_{i,1}(t)) (\sigma \mathcal{Z})^{\otimes p_i}(t, z) \exp(D_{i,2}(t, z)).$$

It is enough to show that there is a regular κ -persistent $M_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent $N_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N_i^+, M_i^+) \subset \mathcal{B}(\Phi_i, R) \cap Y$. (Take then $M^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{M_1(t), \dots, M_s(t)\}$ and $N^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \max\{N_1(t), \dots, N_s(t)\}$.)

Case 1:

Let $(p_{i\kappa}, \dots, p_{ir}) \neq 0$. Let

$$U : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \frac{\sqrt{R}}{|b_i(t)| \exp(D_{i1}(t))}, & t \in \pi^*(Y), \\ 0, & \text{else.} \end{cases}$$

By the proof of Subclaim 1 in Claim 1 we see that U is κ -persistent since $\pi^*(Y) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto |b_i(t)| \exp(D_{i1}(t))$, is consistent by Remark 6.42(1), (2)* in Claim 1 and Remark 6.40(3). So U_{up} is also consistent. For $(t, z) \in \mathcal{B}((\sigma \mathcal{Z})^{\otimes p_i}, U) \cap Y$ we have $|\Phi_i(t, z)| < R$ since $|\exp(D_{i,2}(t, z))| < \sqrt[4]{R}$.

Suppose that p_i is κ -positive. By Proposition 6.58(1) if $p_{i\kappa} \neq 0$ respectively Proposition 6.58(2) if $\kappa = k \geq 0$ and $p_{i\kappa} = 0$ (since U_{up} is consistent) there is a regular κ -persistent $\hat{M}_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{B}(\mathcal{P}_{0,\kappa}, \hat{M}_i^+) \subset \mathcal{B}((\sigma \mathcal{Z})^{\otimes p_i}, U)$. Choose $N_i^+ := S_\delta$ and

$$M_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \min\{\hat{M}_i^+(t), T_\delta(t)\}.$$

Then $\mathcal{A}(\mathcal{P}_{0,\kappa}, N_i^+, M_i^+) \subset \mathcal{B}((\sigma \mathcal{Z})^{\otimes p_i}, U) \cap Y$.

Suppose that p_i is κ -negative. Note that U_{down} is κ -non-persistent (since $|b_i|$ is κ -non-persistent by Remark 6.54(1), compare also the proof of Subclaim 1 in Claim 1). With Corollary 6.59 we find a regular κ -non-persistent $N_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $\mathcal{A}(\mathcal{P}_{0,\kappa}, N_i^+, M_i^+) \subset \mathcal{B}((\sigma \mathcal{Z})^{\otimes p_i}, U)$. The rest of the proof is similar as in the " κ -positive"-case.

Case 2:

Let $(p_{i\kappa}, \dots, p_{ir}) = 0$. Take $N_i^+ := S_\delta$. By Proposition 6.20 we get

$$(\sigma \mathcal{Z})^{\otimes p_i}(t, z) = \prod_{j=0}^{\kappa-1} (\sigma_j \mathcal{P}_{j,\kappa}(t, z) + e^{\mu_{\kappa-j-1,\kappa}(t)})^{p_{ij}}$$

for $(t, z) \in Y$. For $t \in \pi^*(Y)$ let

$$Q(t) := b_i(t) \exp(D_{i,1}(t)) \prod_{j=0}^{\kappa-1} e^{p_{ij} \mu_{\kappa-j-1, \kappa}(t)}.$$

Note that

$$b_i(t) \exp(D_{i,1}(t)) (\sigma \mathcal{Z})^{\otimes p_i}(t, z) = Q(t) \prod_{j=0}^{\kappa-1} \left(1 + \frac{\sigma_j \mathcal{P}_{j, \kappa}(t, z)}{e^{\mu_{\kappa-j-1, \kappa}(t)}}\right)^{p_{ij}}$$

for $(t, z) \in Y$. By Proposition 6.57(1) we find a regular κ -persistent $M^{**} : \pi(C) \rightarrow \mathbb{R}_{>0}$ such that

$$\left| \frac{\sigma_j \mathcal{P}_{j, \kappa}(t, z)}{e^{\mu_{\kappa-j-1, \kappa}(t)}} \right| < \delta$$

for $(t, z) \in \mathcal{B}(\mathcal{P}_{0, \kappa}, M^{**})$ and $j \in \{0, \dots, \kappa-1\}$. Note that if $|Q(t)| < \sqrt{R}$ then

$$|\Phi_i(t, z)| = |Q(t) \prod_{j=0}^{\kappa-1} \left(1 + \frac{\sigma_j \mathcal{P}_{j, \kappa}(t, z)}{e^{\mu_{\kappa-j-1, \kappa}(t)}}\right)^{p_{ij}} \exp(D_{i,2}(t, z))| < \sqrt{R} \cdot \sqrt[4]{R} \cdot \sqrt[4]{R} = R$$

for $(t, z) \in Y \cap \mathcal{B}(\mathcal{P}_{0, \kappa}, M^{**})$ by (+). Let

$$M_i^+ : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \min\{M^{**}(t), T_\delta(t)\}, & t \in \pi^*(Y), |Q(t)| < \sqrt{R}, \\ 0, & \text{else.} \end{cases}$$

Note that M_i^+ is regular and by construction that $\mathcal{A}(\mathcal{P}_{0, \kappa}, N_i^+, M_i^+) \subset \mathcal{B}(\Phi_i, R) \cap Y$. We show that M_i^+ is κ -persistent and are done.

Note that

$$|b_i(t) \exp(D_{i,1}(t)) (\sigma \mathcal{Z})^{\otimes p_i}(t, z) \exp(D_{i,2}(t, z))| \leq 1$$

for every $(t, z) \in Y \cap C$. Let $w \in \pi_l(C)$. Let $\gamma :]0, 1[\rightarrow C_w$ be a definable curve compatible with C_w with $\lim_{y \searrow 0} (\gamma(y)) \in X_w$ and

$$\lim_{y \searrow 0} (\gamma_x(y) - \mu_\kappa(w, \gamma_u(y))) = 0.$$

By passing to a suitable subcurve of γ if necessary we may assume that $(w, \gamma_u(y)) \in \pi^*(Y)$ for $y \in]0, 1[$ (compare the proof of Subclaim 1 in Claim 1) and that there is a definable continuous $\hat{\gamma}_x :]0, 1[\rightarrow \mathbb{R}$ with $(\gamma_u(y), \hat{\gamma}_x(y)) \in C_w \cap Y_w$ for $y \in]0, 1[$. We have

$$|b_i(w, \gamma_u(y)) e^{D_{i1}(w, \gamma_u(y))} (\sigma \mathcal{Z})^{\otimes p_i}(w, \gamma_u(y), \gamma_x(y)) e^{D_{i2}(w, \gamma_u(y), \gamma_x(y))}| \leq 1$$

and therefore

$$\begin{aligned} |Q(w, \gamma_u(y))| &\leq \frac{1}{|\exp(D_{i2}(w, \gamma_u(y), \gamma_x(y))) \prod_{j=0}^{\kappa-1} \left(1 + \frac{\sigma_j \mathcal{P}_{j, \kappa}(w, \gamma_u(y), \gamma_x(y))}{e^{\mu_{\kappa-j-1, \kappa}(w, \gamma_u(y))}}\right)^{p_{ij}}|} \\ &< \sqrt[4]{R} \cdot \sqrt[4]{R} = \sqrt{R} \end{aligned}$$

for $y \in]0, 1[$. So we have

$$\lim_{y \searrow 0} M_i^+(w, \gamma_u(y)) = \lim_{y \searrow 0} \min\{M^{**}(w, \gamma_u(y)), T_\delta(w, \gamma_u(y))\} > 0.$$

■ Claim 2

Let N^+ and M^+ be as in Claim 2. Choose $N := N^+$ and $M := M^+$. Let $B = \mathcal{A}(\mathcal{P}_{0,\kappa}, N, M)$. Define

$$\hat{a} : \pi^*(B) \rightarrow \mathbb{R}, t \mapsto a(t)\exp(D_{0,1}(t))$$

and for $i \in \{1, \dots, s\}$

$$\hat{b}_i : \pi^*(B) \rightarrow \mathbb{R}, t \mapsto b_i(t)\exp(D_{i,1}(t)).$$

Then by (2)* in Claim 1 and Remark 6.44(1) we see that $\exp(D_{i,1})|_{\pi^*(B)}$ is regular and consistent for $i \in \{0, \dots, s\}$. So \hat{a} is regular and if $g \in \log(E)$ then \hat{a} is consistent (by Remark 6.42(2)) and $1/((\hat{a}|^C)_{\text{down}})$ is κ -non-persistent if g is κ -negative (by Remark 6.54(2) and the proof of Subclaim 1). Note that $\hat{b}_1, \dots, \hat{b}_s$ are regular and consistent by (2*) in Claim 1 and Remark 6.40(3). If p_i is κ -negative for $i \in \{1, \dots, s\}$ then $1/((\hat{b}_i|^C)_{\text{down}})$ is κ -non-persistent (by Remark 6.54(1) and the proof of Subclaim 1). For $(t, z) \in B$ we set for $\nu := (\nu_0, \dots, \nu_r) \in \mathbb{Q}^{r+1}$

$$L_\nu := L_\nu(t, z) := \prod_{j=0}^{\kappa-1} (\sigma_j z_j(t, z))^{\nu_j} = \prod_{j=0}^{\kappa-1} e^{\nu_j \mu_{\kappa-j-1, \kappa}(t)} \prod_{j=0}^{\kappa-1} \left(1 + \frac{\sigma_j \mathcal{P}_{j, \kappa}(t, z)}{e^{\mu_{\kappa-j-1, \kappa}(t)}}\right)^{\nu_j}$$

and $\mathcal{Z}_\nu := \prod_{j=\kappa}^r (\sigma_j z_j(t, z))^{\nu_j}$. So for $(t, z) \in B$ we obtain

$$\Phi_g(t, z) = \hat{a}(t) \mathcal{Z}_q L_q e^{D_{0,2}(t,z)} V(\hat{b}_1(t) \mathcal{Z}_{p_1} L_{p_1} e^{D_{1,2}(t,z)}, \dots, \hat{b}_s(t) \mathcal{Z}_{p_s} L_{p_s} e^{D_{s,2}(t,z)}).$$

By shrinking M if necessary we may assume that $B \rightarrow \mathbb{C}, (t, z) \mapsto L_\nu(t, z)$, is complex r -log-analytically prepared in z as in Proposition 6.76 for $\nu \in \{q, p_1, \dots, p_s\}$. By composition of power series we obtain the desired preparation since $B \rightarrow \mathbb{C}, (t, z) \mapsto e^{D_{j,2}(t,z)}$, fulfills property (3)* from Claim 1 for $j \in \{0, \dots, s\}$. This finishes the proof of Proposition 6.86. ■

Let $\pi^+ : \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n \times \mathbb{R}, (t, s, z) \mapsto (t, s)$, be the projection on the first $n + 1$ real coordinates.

6.87 Corollary

There is a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that the following holds. Let

$$\Delta := \{(t, s, z) \in \pi(C) \times \mathbb{R}_{> 0} \times \mathbb{C} \mid N(t) < s < M(t), z \in B(\mu_\kappa(t), s)\}.$$

Then the function

$$\Psi : \Delta \rightarrow \mathbb{C}, (t, s, z) \mapsto \frac{1}{2\pi i} \int_{\partial B(\mu_\kappa(t), s)} \frac{F(t, \xi)}{\xi - z} d\xi,$$

is well-defined. Additionally there is $l \in \mathbb{N}$ and a definable function $g = (g_1, \dots, g_l) : \pi^+(\Delta) \rightarrow \mathbb{R}^l$ such that Ψ is constructible in z with support function

$$\pi^+(\Delta) \rightarrow \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}, (t, s) \mapsto (g(t, s), \mu_\kappa(t), s),$$

where the following holds for g_j where $j \in \{1, \dots, l\}$: There is $p \in \mathbb{N}_0$, a regular function $\beta : \pi(C) \rightarrow \mathbb{R}^p$ and a log-analytic $h_j : \mathbb{R}^{p+1} \rightarrow \mathbb{R}$ such that $g_j(t, s) = h_j(\beta(t), s)$ for every $(t, s) \in \pi^+(\Delta)$.

Proof

By Proposition 6.86 there is a regular κ -persistent $M : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a regular κ -non-persistent $N : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that $F|_B$ is complex r -log-analytically prepared in z with center Θ , regular coefficient and base functions where $B := \mathcal{A}(\mathcal{P}_{0, \kappa}, N, M)$. By shrinking M if necessary we are done with Proposition 6.78. ■

6.2 Global Complexification of Restricted Log-Exp-Analytic Functions

Now we are able to show that a real analytic restricted log-exp-analytic function has a global complexification which is again restricted log-exp-analytic. We start with the following result from topology.

6.88 Proposition

Let Y be a non-empty topological space. Let $g : Y \rightarrow \mathbb{R}$ be a function that is bounded from below. Let

$$f : Y \rightarrow \mathbb{R}, \quad x \mapsto \liminf_{y \rightarrow x} g(y).$$

Then f is lower semi-continuous, i.e. for every $x \in Y$ and every $a < f(x)$ there is an open neighbourhood V of x such that $a < f(y)$ for all $y \in V$.

Proof

See for example Bourbaki ([4], Chapter IV, §6, Section 2, Proposition 4). ■

6.2.1 The One Dimensional Case

In this section we show that a real analytic restricted log-exp-analytic function has a unary high parametric global complexification which is again restricted log-exp-analytic. (See also [19] for the proof in the globally subanalytic setting.)

For this section let $l, m \in \mathbb{N}_0$ be with $n = l + m$. Let w range over \mathbb{R}^l and u over \mathbb{R}^m . Let $\pi_l : \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^l, (w, u, x) \mapsto w$. For a definable set $Z \subset \mathbb{R}^n \times \mathbb{C}$ and $(t, x) \in \mathbb{R}^n \times \mathbb{R}$ we set

$$\zeta_Z := \sup\{r \geq 0 \mid x + iQ(0, r) \subset Z_t\}.$$

Let $\pi^+ : \mathbb{R}^n \times \mathbb{R} \times \mathbb{C} \rightarrow \mathbb{R}^n \times \mathbb{R}, (t, s, z) \mapsto (t, s)$, be the projection on the first $n + 1$ real coordinates.

6.89 Proposition

Let $X \subset \mathbb{R}^{n+1}$ be definable such that X_w is open for every $w \in \pi_l(X)$. Let $f : X \rightarrow \mathbb{R}, (w, u, x) \mapsto f(w, u, x)$, be restricted log-exp-analytic in (u, x) such that f_w is real analytic for all $w \in \mathbb{R}^l$. Then f has a unary high parametric global complexification $F : Z \rightarrow \mathbb{C}, (w, u, z) \mapsto F(w, u, z)$, with respect to (u, x) which is restricted log-exp-analytic in (u, z) .

Proof

For a definable cell $C \subset X$ which is fat with respect to x we say the following: The expression " C -consistent" means always " C -consistent in u with respect to X " and " C -regular" means always " C -regular in u with respect to X ".

Let $e \in \mathbb{N}_0$ be such that f is restricted log-exp-analytic in (u, x) of order at most e . By Corollary 4.95 there is $r \in \mathbb{N}_0$ and a definable cell decomposition \mathcal{C}' of $X_{\neq 0}$ such that $f|_{\mathcal{C}'}$ is $(m+1, X)$ -restricted (e, r) -prepared in x . Let $\mathcal{C} \subset \mathcal{C}'$ be the set of fat cells with respect to x .

For every $C \in \mathcal{C}$ we consider the following.

Fix a finite set E_C of locally bounded functions in (u, x) with reference set X such that every $g \in \log(E_C) \cup \{f|_C\}$ is (l, r) -prepared in x with respect to E_C for $l \in \{-1, \dots, e\}$. For $g \in \log(E_C) \cup \{f|_C\}$ fix a corresponding preparing tuple

$$(r, \mathcal{Y}_C, a_{C,g}, \exp(d_{C,0,g}), q_{C,g}, s, v_{C,g}, b_{C,g}, \exp(d_{C,g}), P_{C,g})$$

where $b_{C,g} := (b_{C,1,g}, \dots, b_{C,s,g})$ and $\exp(d_{C,g}) := (\exp(d_{C,1,g}), \dots, \exp(d_{C,s,g}))$ (without loss of generality s is independent of C and of the function $g \in \log(E_C)$). Note that the center $\Theta_C := (\Theta_{C,0}, \dots, \Theta_{C,r})$ of \mathcal{Y}_C and for every $g \in \log(E_C) \cup \{f|_C\}$ the components of $b_{C,g}$ are C -consistent (compare Example 6.41(1) and Remark 6.42(1)) and C -regular (since they are C -nice and every C -nice function is C -regular by Example 6.45(1)). For every $g \in \log(E_C) \cup \{f|_C\}$ we have that a_g is C -regular since it is C -nice and if $g \in \log(E_C)$ then a_g is C -consistent in addition (by Remark 6.42(2)).

For every $C \in \mathcal{C}$ fix the sign $\sigma_C := (\sigma_{C,0}, \dots, \sigma_{C,r})$ of \mathcal{Y}_C , its change index

$$k_C := k_C^{\text{ch}} := \min\{l \in \{0, \dots, r\} \mid \sigma_{C,l} = -1\} - 1,$$

its corresponding functions $\mu_{C,r}$ and μ_{C,k_C} if $k_C \geq 0$ from Definition 6.3 which are C -regular by Example 6.45(2).

Step 1: Constructing a suitable holomorphic extension of $f|_C$ in the last variable x for $C \in \mathcal{C}$:

For $C \in \mathcal{C}$ we construct a holomorphic extension of $f|_C$ such that, after gluing the single extensions together, we obtain a unary **high** parametric global complexification with respect to (u, x) which is **restricted log-exp-analytic** in (u, z) . A very complex and detailed construction is necessary to obtain this both properties at once.

Let H_C be from Definition 6.13(b). For every $g \in \log(E_C)$ fix $\Phi_{C,g}$ from Definition 6.81 and for $\Phi_{C,g}$ a complex preparing tuple

$$(r, \mathcal{Z}_C, a_{C,g}, \exp(D_{C,0,g}), q_{C,g}, s, V_{C,g}, b_{C,g}, \exp(D_{C,g}), P_{C,g})$$

where

$$\exp(D_{C,g}) := (\exp(D_{C,1,g}), \dots, \exp(D_{C,s,g})).$$

Let $\Lambda_C := \Lambda_{C,f}$, $\Phi_C := \Phi_{C,f}$ and $E_{\Phi_C} := E_{\Phi_{C,f}}$. The expression " (C, κ) -(non-)persistent" means always " $(C, \mathcal{Y}_C, \kappa)$ -(non-)persistent in u with respect to X ".

By Proposition 6.86 and Corollary 6.87 there is for $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$ a C -regular (C, κ) -persistent $\hat{M}_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ and a C -regular (C, κ) -non-persistent $N_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}$ such that

$$K_{C,\kappa} := \{(t, z) \in H_C \mid N_{C,\kappa}(t) < |z - \mu_{C,\kappa}(t)| < \hat{M}_{C,\kappa}(t)\} \subset \Lambda_C,$$

$\Phi_C|_{K_{C,\kappa}}$ is complex r -log-analytically prepared in z as in Proposition 6.86 with center Θ_C and for

$$\hat{\Delta}_{C,\kappa} := \{(t, s, z) \in \pi(C) \times \mathbb{R}_{>0} \times \mathbb{C} \mid N_{C,\kappa}(t) < s < \hat{M}_{C,\kappa}(t), z \in B(\mu_{C,\kappa}(t), s)\}$$

that

$$\Psi_{C,\kappa} : \hat{\Delta}_{C,\kappa} \rightarrow \mathbb{C}, (t, s, z) \mapsto \frac{1}{2\pi i} \int_{\partial B(\mu_{C,\kappa}(t), s)} \frac{\Phi_C(t, \xi)}{\xi - z} d\xi,$$

is well-defined. Additionally there is $l_C \in \mathbb{N}_0$ and a definable function $g_{C,\kappa}^* := ((g_{C,\kappa}^*)_1, \dots, (g_{C,\kappa}^*)_{l_C}) : \pi^+(\hat{\Delta}_{C,\kappa}) \rightarrow \mathbb{R}^{l_C}$ such that $\Psi_{C,\kappa}$ is constructible in z with support function

$$\pi^+(\hat{\Delta}_{C,\kappa}) \rightarrow \mathbb{R}^{l_C} \times \mathbb{R} \times \mathbb{R}, (t, s) \mapsto (g_{C,\kappa}^*(t, s), \mu_{C,\kappa}(t, s), s),$$

where the following holds for $(g_{C,\kappa}^*)_j$ where $j \in \{1, \dots, l_C\}$: There is $p_{C,\kappa} \in \mathbb{N}_0$, a regular function $\beta_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}^{p_{C,\kappa}}$ and a log-analytic $h_{C,\kappa,j} : \mathbb{R}^{p_{C,\kappa}+1} \rightarrow \mathbb{R}$ such that $(g_{C,\kappa}^*)_j(t, s) = h_{C,\kappa,j}(\beta_{C,\kappa}(t), s)$ for every $(t, s) \in \pi^+(\hat{\Delta}_{C,\kappa})$. Fix the C -regular coefficient $\hat{a}_{C,\kappa}$ and a tuple of C -regular base functions $\hat{b}_{C,\kappa}$ for the complex r -log-analytical preparation of $\Phi_C|_{K_{C,\kappa}}$. So we see by shrinking $\hat{M}_{C,\kappa}$ if necessary that $(\Phi_C|_{K_{C,\kappa}})_t$ has a holomorphic extension on $B(\mu_{C,\kappa}(t), \hat{M}_{C,\kappa}(t))$ if $N_{C,\kappa}(t) = 0$ and $(\Phi_C|_{K_{C,\kappa}})_t$ has a holomorphic extension at $\mu_{C,\kappa}(t)$ by Proposition 6.80 (here we have to shrink $\hat{M}_{C,\kappa}$ in general since one needs that z_κ can be written as a convergent power series on $B(\mu_{C,\kappa}(t), \hat{M}_{C,\kappa}(t))$). This can be done with Proposition 6.79(1) and Proposition 6.20). (+)

Let

$$\alpha_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} 2N_{C,\kappa}(t), & N_{C,\kappa}(t) \neq 0, \\ \hat{M}_{C,\kappa}(t)/2, & N_{C,\kappa}(t) = 0. \end{cases}$$

Note that $\alpha_{C,\kappa}$ is C -regular.

Fix a finite set \mathcal{E}_C of positive definable functions on $\pi(C)$ such that every $g \in \log(\mathcal{E}_C)$ is C -consistent and the following holds.

- (1) The functions $\Theta_{C,0}, \dots, \Theta_{C,r}$ can be constructed from \mathcal{E}_C and we have $\exp(\Theta_{C,0}), \dots, \exp(\Theta_{C,r}) \in \mathcal{E}_C$.
- (2) For every $g \in \log(E_C) \cup \{f\}$ we have that $a_{C,g}, b_{C,1,g}, \dots, b_{C,s,g}$ can be constructed from \mathcal{E}_C and that $\exp(a_{C,g}) \in \mathcal{E}_C$ if $g \in \log(E_C)$.
- (3) The functions $N_{C,\kappa}, \hat{M}_{C,\kappa}, \beta_{C,\kappa}, \hat{a}_{C,\kappa}, \hat{b}_{C,\kappa}$ and $\mu_{C,\kappa}$ can be constructed from \mathcal{E}_C for $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$.

Since $\Phi_C|_{K_{C,\kappa}}$ is complex r -log-analytically prepared in z with coefficient, center and base functions which can be constructed from \mathcal{E}_C we have that $\Phi_C|_{K_{C,\kappa}}$ can be constructed from \mathcal{E}_C for $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$ (compare with Remark 6.77 and Proposition 3.16(2)). (β)

Note also that Φ_C can be constructed from $\mathcal{E}_C \cup E_{\phi_C}$ by Remark 6.82. (ββ)

Fix $l_C^* \in \mathbb{N}$ and positive definable functions $e_{C,1}, \dots, e_{C,l_C^*}$ on $\pi(C)$ such that $\mathcal{E}_C = \{e_{C,1}, \dots, e_{C,l_C^*}\}$. For $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$ let

$$\tilde{M}_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto$$

$$\min\{L_C(t), \hat{M}_{C,\kappa}(t), \exp(-|\log(e_{C,1}(t))|), \dots, \exp(-|\log(e_{C,l_C^*}(t))|)\}$$

where L_C denotes the length of C with respect to x . Consider

$$M_{C,\kappa} : \pi(C) \rightarrow \mathbb{R}_{\geq 0}, t \mapsto \begin{cases} \tilde{M}_{C,\kappa}(t), & C_t \cap B(\mu_{C,\kappa}(t), \tilde{M}_{C,\kappa}(t)) \neq \emptyset, \\ 0, & \text{else.} \end{cases}$$

Note that $\tilde{M}_{C,\kappa}$ is (C, κ) -persistent by Remark 6.50 and Remark 6.48(2) (since $\exp(-|\log(e_{C,j})|)$ is C -consistent), but not necessarily C -regular (since it depends on L_C). So one sees also that $M_{C,\kappa}$ is κ -persistent. By shrinking $M_{C,\kappa}$ if necessary we may assume that $M_{C,\kappa} < 1/2|\mu_{C,r} - \mu_{C,k_C}|$ if $k_C \geq 0$.

Let

$$\Delta_{C,\kappa} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid N_{C,\kappa}(t) < \hat{M}_{C,\kappa}(t), z \in B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t))\}.$$

We have that

$$\Delta_{C,\kappa} \rightarrow \mathbb{C}, (t, z) \mapsto \Psi_{C,\kappa}(t, \alpha_{C,\kappa}(t), z),$$

can be constructed from \mathcal{E}_C (since $\mu_{C,\kappa}, \alpha_{C,\kappa}, \beta_{C,\kappa}$ and therefore also $g_{C,\kappa}^*(t, \alpha_{C,\kappa})$ can be constructed from \mathcal{E}_C by construction of $g_{C,\kappa}^*$. Note that a constructible function is log-analytic). Set

$$\hat{\Lambda}_C := \{(t, z) \in \Lambda_C \mid |\text{Im}(z)| <$$

$$\min\{L_C(t), \exp(-|\log(e_{C,1}(t))|), \dots, \exp(-|\log(e_{C,l_C^*}(t))|)\}$$

and

$$\tilde{\Gamma}_C := \{(t, z) \in \hat{\Lambda}_C \mid z \text{ is in the same connected component of } (\hat{\Lambda}_C)_t \text{ as } C_t\}.$$

Note that $\tilde{\Gamma}_C$ is definable with $C \subset \tilde{\Gamma}_C$ and $(\tilde{\Gamma}_C)_t$ is open and connected for $t \in \pi(C)$. Additionally we have $\mathcal{A}(\mu_\kappa, N_{C,\kappa}, M_{C,\kappa}) \subset \tilde{\Gamma}_C$ for $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$.

Let $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$. We set

$$B_{C,\kappa} := \{t \in \pi(C) \mid \alpha_{C,\kappa}(t) < \hat{M}_{C,\kappa}(t) \text{ and } B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t)) \rightarrow \mathbb{C}, \\ z \mapsto \Psi_{C,\kappa}(t, \alpha_{C,\kappa}(t), z), \text{ is holomorphic}\}.$$

Then $B_{C,\kappa}$ is definable since $\Psi_{C,\kappa}$ is definable, $\alpha_{C,\kappa}$ is definable and since the property of being holomorphic is definable by the Cauchy-Riemann equations. We set

$$\mathcal{G}_{C,\kappa} := \{(t, z) \in H_C \mid t \in B_{C,\kappa} \text{ and } z \in B(\mu_{C,\kappa}(t), \min\{\alpha_{C,\kappa}(t), M_{C,\kappa}(t)\})\}.$$

Note that $\mathcal{G}_{C,\kappa} \subset \Delta_{C,\kappa}$. Consider

$$\chi_{C,\kappa} : \mathcal{G}_{C,\kappa} \rightarrow \mathbb{C}, (t, z) \mapsto \Psi_{C,\kappa}(t, \alpha_{C,\kappa}(t), z).$$

Note that $\chi_{C,\kappa}$ can be constructed from \mathcal{E}_C . (BBB)

Let

$$\hat{B}_{C,\kappa} := \{t \in B_{C,\kappa} \mid (\Phi_C)_t = (\chi_{C,\kappa})_t \text{ on } (\tilde{\Gamma}_C)_t \cap (\mathcal{G}_{C,\kappa})_t\}$$

and

$$\mathcal{T}_{C,\kappa} := \{(t, z) \in H_C \mid t \in \hat{B}_{C,\kappa} \text{ and } z \in B(\mu_{C,\kappa}(t), M_{C,\kappa}(t))\}.$$

Note that $\mathcal{T}_{C,\kappa} \subset \tilde{\Gamma}_C \cup \mathcal{G}_{C,\kappa}$ and if $t \in \hat{B}_{C,\kappa}$ then $(\mathcal{T}_{C,\kappa})_t = (K_{C,\kappa})_t \cup (\mathcal{G}_{C,\kappa})_t$. Further let $\mathfrak{d}_{C,\kappa}(t) := \text{dist}(\mu_{C,\kappa}(t), C_t)/2$ for $t \in \pi(C)$. Let $H_{C,\text{small}}$ be defined in the following way.

- If $\prod_{j=0}^r \sigma_{C,j} = 1$ and $k_C < 0$ we set

$$H_{C,\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]\mu_{C,r}(t) + \mathfrak{d}_{C,r}(t), \infty[\}.$$

- If $\prod_{j=0}^r \sigma_{C,j} = -1$ and $k_C < 0$ we set

$$H_{C,\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]-\infty, \mu_{C,r}(t) - \mathfrak{d}_{C,r}(t)[\}.$$

- If $\prod_{j=0}^r \sigma_{C,j} = 1$ and $k_C \geq 0$ we set

$$H_{C,\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \text{Re}(z) \in]\mu_{C,r}(t) + \mathfrak{d}_{C,r}(t), \mu_{C,k_C}(t) - \mathfrak{d}_{C,k_C}(t)[\}.$$

- If $\prod_{j=0}^r \sigma_{C,j} = -1$ and $k_C \geq 0$ we set

$$H_{C,\text{small}} := \{(t, z) \in \pi(C) \times \mathbb{C} \mid \operatorname{Re}(z) \in]\mu_{C,k_C}(t) + \mathfrak{d}_{C,k_C}(t), \mu_{C,r}(t) - \mathfrak{d}_{C,r}(t)[\}$$

Note that $H_{C,\text{small}} \subset H_C$ and that $H_{C,\text{small}}$ is definable. Consider $\Gamma_C := H_{C,\text{small}} \cap \tilde{\Gamma}_C$.

Suppose that $k := k_C < 0$. We set

$$(\hat{\Xi}_C)_t := \begin{cases} (\Gamma_C)_t \cup (\mathcal{T}_{C,r})_t, & t \in \hat{B}_{C,r}, \\ (\Gamma_C)_t, & t \in \pi(C) \setminus \hat{B}_{C,r}, \\ \emptyset, & t \notin \pi(C). \end{cases}$$

We define $G_C : \hat{\Xi}_C \rightarrow \mathbb{C}$ by letting

$$(G_C)_t(z) := \begin{cases} (\Phi_C)_t(z) \text{ if } z \in (\Gamma_C)_t \cup (\mathcal{T}_{C,r})_t \text{ and} \\ (\chi_{C,r})_t(z) \text{ if } z \in (\mathcal{G}_{C,r})_t, & t \in \hat{B}_{C,r}, \\ (\Phi_C)_t(z), & t \in \pi(C) \setminus \hat{B}_{C,r}. \end{cases}$$

Note that $(\hat{\Xi}_C)_t$ is open for every $t \in \pi(C)$ and that G_C is a well-defined unary parametric global complexification of $f|_C : C \rightarrow \mathbb{R}$.

Suppose that $k := k_C \geq 0$. Let for $t \in \mathbb{R}^n$

$$(\hat{\Xi}_C)_t := \begin{cases} (\Gamma_C)_t \cup (\mathcal{T}_{C,k})_t \cup (\mathcal{T}_{C,r})_t, & t \in \hat{B}_{C,k} \cap \hat{B}_{C,r}, \\ (\Gamma_C)_t \cup (\mathcal{T}_{C,k})_t, & t \in \hat{B}_{C,k} \setminus \hat{B}_{C,r}, \\ (\Gamma_C)_t \cup (\mathcal{T}_{C,r})_t, & t \in \hat{B}_{C,r} \setminus \hat{B}_{C,k}, \\ (\Gamma_C)_t, & t \in \pi(C) \setminus (\hat{B}_{C,k} \cup \hat{B}_{C,r}), \\ \emptyset, & t \notin \pi(C). \end{cases}$$

Finally define $G_C : \hat{\Xi}_C \rightarrow \mathbb{C}$ by letting

$$(G_C)_t(z) := \begin{cases} (\Phi_C)_t(z) \text{ if } z \in (\Gamma_C)_t \cup (\mathcal{T}_{C,k})_t \cup (\mathcal{T}_{C,r})_t \text{ and } (\chi_{C,r})_t(z) \\ \text{if } z \in (\mathcal{G}_{C,r})_t \text{ and } (\chi_{C,k})_t(z) \text{ if } z \in (\mathcal{G}_{C,k})_t, & t \in \hat{B}_{C,k} \cap \hat{B}_{C,r}, \\ (\Phi_C)_t(z) \text{ if } z \in (\Gamma_C)_t \cup (\mathcal{T}_{C,k})_t \text{ and } (\chi_{C,k})_t(z) \text{ if } z \in (\mathcal{G}_{C,k})_t, & t \in \hat{B}_{C,k} \setminus \hat{B}_{C,r}, \\ (\Phi_C)_t(z) \text{ if } z \in (\Gamma_C)_t \cup (\mathcal{T}_{C,r})_t \text{ and } (\chi_{C,r})_t(z) \text{ if } z \in (\mathcal{G}_{C,r})_t, & t \in \hat{B}_{C,r} \setminus \hat{B}_{C,k}, \\ (\Phi_C)_t(z), & \text{else.} \end{cases}$$

Note that $(\hat{\Xi}_C)_t$ is open for every $t \in \pi(C)$ and that G_C is a well-defined unary parametric global complexification of $f|_C : C \rightarrow \mathbb{R}$.

Claim 1

Let $C \in \mathcal{C}$. Let $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$. Let $t \in \pi(C)$ be with $N_{C,\kappa}(t) = 0$ and $0 < M_{C,\kappa}(t)$. Suppose there is $\epsilon > 0$ such that $Q(\mu_{C,\kappa}(t), \epsilon) \subset X_t$ and $f_t(x) = p(x - \mu_{C,\kappa}(t))$ for $x \in Q(\mu_{C,\kappa}(t), \epsilon)$ where p is a convergent power series on $] -\epsilon, \epsilon[$. Suppose that $C_t \cap B(\mu_{C,\kappa}(t), \epsilon) \neq \emptyset$. Then $t \in \hat{B}_{C,\kappa}$ and therefore $B(\mu_{C,\kappa}(t), M_{C,\kappa}(t)) \cap (H_C)_t \subset (\hat{\Xi}_C)_t$.

Proof of Claim 1

By (+) and the identity theorem we see that $(\Phi_C|_{K_C})_t$ has a holomorphic extension h on $B(\mu_{C,\kappa}(t), \hat{M}_{C,\kappa}(t))$. Since $\alpha_{C,\kappa}(t) = \hat{M}_{C,\kappa}(t)/2$ we get by Cauchy's integral formula that

$$h(z) = \frac{1}{2\pi i} \int_{\partial B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t))} \frac{(\Phi_C)_t(\xi)}{\xi - z} d\xi$$

for all $z \in B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t))$. So we get that

$$B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t)) \rightarrow \mathbb{C}, z \mapsto \Psi_{C,\kappa}(t, \alpha_{C,\kappa}(t), z),$$

coincides with h and is therefore holomorphic. Hence $t \in \hat{B}_{C,\kappa}$. This gives

$$B(\mu_{C,\kappa}(t), M_{C,\kappa}(t)) \cap (H_C)_t \subset (\hat{\Xi}_C)_t.$$

■ Claim 1

Claim 2

Let $C \in \mathcal{C}$. Let $\kappa = r$ or $\kappa \in \{k_C, r\}$ if $k_C \geq 0$. Let $t \in \pi(C)$ be with $N_{C,\kappa}(t) \neq 0$. Suppose there is $\epsilon > 0$ such that $Q(\mu_{C,\kappa}(t), \epsilon) \subset X_t$ and $f_t(x) = p(x - \mu_{C,\kappa}(t))$ for $x \in Q(\mu_{C,\kappa}(t), \epsilon)$ where p is a convergent power series on $] -\epsilon, \epsilon[$. Suppose that $\alpha_{C,\kappa}(t) < \min\{\hat{M}_{C,\kappa}(t), \epsilon\}$ and that $C_t \cap B(\mu_{C,\kappa}(t), \epsilon) \neq \emptyset$. Then $t \in \hat{B}_{C,\kappa}$ and therefore $B(\mu_{C,\kappa}(t), M_{C,\kappa}(t)) \cap (H_C)_t \subset (\hat{\Xi}_C)_t$.

Proof of Claim 2

Note that $\alpha_{C,\kappa}(t) = 2N_{C,\kappa}(t)$. By power series expansion there is a holomorphic extension $h : B(\mu_{C,\kappa}(t), \epsilon) \rightarrow \mathbb{C}$ of $f_t : Q(\mu_{C,\kappa}(t), \epsilon) \rightarrow \mathbb{R}$. We have $h = (\Phi_C)_t$ on $B(\mu_{C,\kappa}(t), \epsilon) \cap (\tilde{\Gamma}_C)_t$ by the identity theorem (since $B(\mu_{C,\kappa}(t), \epsilon) \cap (\tilde{\Gamma}_C)_t$ is connected). By Cauchy's integral formula we get for all $z \in B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t))$ that

$$h(z) = \frac{1}{2\pi i} \int_{\partial B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t))} \frac{(\Phi_C)_t(\xi)}{\xi - z} d\xi$$

since $N_{C,\kappa}(t) < \alpha_{C,\kappa}(t) < \min\{\hat{M}_{C,\kappa}(t), \epsilon\}$. So we get that

$$B(\mu_{C,\kappa}(t), \alpha_{C,\kappa}(t)) \rightarrow \mathbb{C}, z \mapsto \Psi_{C,\kappa}(t, \alpha_{C,\kappa}(t), z),$$

coincides with h and is therefore holomorphic. Hence $t \in \hat{B}_{C,\kappa}$. We obtain

$$B(\mu_{C,\kappa}(t), M_{C,\kappa}(t)) \cap (H_C)_t \subset (\hat{\Xi}_C)_t.$$

■ Claim 2

Step 2: Constructing a unary high parametric global complexification $G : Z \rightarrow \mathbb{R}$ of f with respect to (u, x) .

Now we are ready to construct a unary high parametric global complexification $G : Z \rightarrow \mathbb{C}$ of f with respect to (u, x) by gluing the single functions G_C together. In step 3 below we show that G can be "redefined" to a function F which is indeed restricted log-exp-analytic in (u, z) (i.e. there is a restricted log-exp-analytic function $F : Z \rightarrow \mathbb{R}$ in (u, z) which coincides with G).

We say that $(t, z) \in \hat{\Xi}_C$ fulfills property (ω) if the following holds.

(ω_1) We have $(t, \operatorname{Re}(z)) \in X$.

(ω_2) There is an open neighbourhood U of z in $(\hat{\Xi}_C)_t$ such that for every $v_1 + iv_2 \in U$ we have that $v_1 + i]0, v_2[\subset (\hat{\Xi}_C)_t$ if $\operatorname{Re}(v_2) > 0$ or $v_1 + i]v_2, 0[\subset (\hat{\Xi}_C)_t$ if $\operatorname{Re}(v_2) < 0$.

(ω_3) The point z is in the same connected component of $(\hat{\Xi}_C)_t$ as C_t .

Let

$$\Xi_C := \{(t, z) \in \hat{\Xi}_C \mid (t, z) \text{ fulfills property } (\omega)\}.$$

Then Ξ_C is definable, $C_t \subset (\Xi_C)_t$ and $(\Xi_C)_t$ is open for $t \in \pi(C)$. Let

$$\Omega := X \cup \bigcup_{C \in \mathcal{C}} \Xi_C.$$

Then Ω is definable and it is easy to check with Proposition 6.85 and Claim 1 that Ω_t is open for every $t \in \pi(C)$. We define

$$G : \Omega \rightarrow \mathbb{C}, (t, z) \mapsto \begin{cases} G_C(t, z), & (t, z) \in \Xi_C, \\ f(t, z), & (t, z) \in X. \end{cases}$$

By definition of Ω and the identity theorem we see that $G : \Omega \rightarrow \mathbb{C}$ is a well-defined unary parametric global complexification of $f : X \rightarrow \mathbb{R}$.

We show that G is a unary high parametric global complexification of $f : X \rightarrow \mathbb{R}$ with respect to (u, x) . This is done with the following

Claim 3

Let $w \in \mathbb{R}^l$ and $(u_0, x_0) \in X_w$. Then

$$\liminf_{(s,v) \rightarrow (u_0,x_0)} \zeta_\Omega(w, s, v) > 0.$$

Proof of Claim 3

We find $\epsilon > 0$ such that $W^* := Q^m(u_0, \epsilon) \times]x_0 - \epsilon, x_0 + \epsilon[\subset X_w$ and $f_w(u, x) = p(u - u_0, x - x_0)$ for $(u, x) \in W^*$ for a convergent power series p on $Q^{m+1}(0, \epsilon)$. We assume

$$\liminf_{(s,v) \rightarrow (u_0, x_0)} \zeta_\Omega(w, s, v) = 0.$$

Then there is a definable curve $\gamma :]0, 1[\rightarrow Q^m(u_0, \epsilon) \times]x_0 - \epsilon/2, x_0 + \epsilon/2[$, $y \mapsto (\gamma_u(y), \gamma_x(y))$, with $\lim_{y \searrow 0} \gamma(y) = (u_0, x_0)$ such that $\lim_{y \searrow 0} \zeta_\Omega(w, \gamma(y)) = 0$. For $y \in]0, 1[$ let

$$I(y) := \{C \in \mathcal{C} \mid C_{(w, \gamma_u(y))} \cap]x_0 - \epsilon/2, x_0 + \epsilon/2[\neq \emptyset\}.$$

By passing to a suitable subcurve of γ if necessary we may assume that $I(y)$ is independent of y . We set

$$I_{\text{big}} := \{C \in I : \lim_{y \searrow 0} L_C(w, \gamma_u(y)) > 0\}$$

and

$$I_{\text{small}} := \{C \in I : \lim_{y \searrow 0} L_C(w, \gamma_u(y)) = 0\}.$$

Because $\sum_{C \in I} L_C(w, \gamma_u(y)) \geq \epsilon$ for all $y \in]0, 1[$ we see that $I_{\text{big}} \neq \emptyset$. For $y \in]0, 1[$ we choose $C_y \in \mathcal{C}$ such that $(w, \gamma(y)) \in \overline{C_y}$. We may assume that $C^* := C_y$ is independent of y by passing to a suitable subcurve of γ if necessary. We also have $(u_0, x_0) \in \overline{(C^*)_w}$. Let $k^* := k_{C^*}$. We consider two cases.

Case 1: $C^* \in I_{\text{big}}$. Note that γ is compatible with $(C^*)_w$.

Subcase 1.1:

Assume $\lim_{y \searrow 0} |\gamma_x(y) - \mu_{C^*, \kappa}(w, \gamma_u(y))| > 0$ for $\kappa = r$ or $\kappa \in \{k^*, r\}$ if $k^* \geq 0$. By passing to a suitable subcurve of γ if necessary we may assume that $\gamma(y) \in (H_{C^*, \text{small}})_w$ for $y \in]0, 1[$. So by Proposition 6.85 we obtain

$$\lim_{y \searrow 0} \zeta_{\Lambda_{C^*}}(w, \gamma(y)) > 0.$$

Note also that

$$\lim_{y \searrow 0} \exp(-|\log(e_{C^*, j}(w, \gamma_u(y)))|) > 0$$

for $j \in \{1, \dots, l_{C^*}^*\}$ since $(u_0, x_0) \in X_w$, $e_{C^*, j} \in \mathcal{E}_{C^*}$ (and so $e_{C^*, j}$ is C^* -consistent) and γ is compatible with $(C^*)_w$. Since $C^* \in I_{\text{big}}$ and $\Gamma_{C^*} = \hat{\Gamma}_{C^*} \cap H_{C^*, \text{small}}$ we obtain

$$\lim_{y \searrow 0} \zeta_{\Gamma_{C^*}}(w, \gamma(y)) > 0.$$

Therefore

$$\lim_{y \searrow 0} \zeta_\Omega(w, \gamma(y)) \geq \lim_{y \searrow 0} \zeta_{\Gamma_{C^*}}(w, \gamma(y)) > 0$$

by construction of Ω , a contradiction.

Subcase 1.2:

Suppose that

$$\lim_{y \searrow 0} |\gamma_x(y) - \mu_{C^*, \kappa}(w, \gamma_u(y))| = 0$$

for $\kappa = r$ or $\kappa \in \{k^*, r\}$ if $k^* \geq 0$. Fix this κ . Since $M_{C^*, \kappa}$ is (C^*, κ) -persistent respectively $N_{C^*, \kappa}$ is (C^*, κ) -non-persistent we have with Definition 6.46 that $\lim_{y \searrow 0} M_{C^*, \kappa}(w, \gamma_u(y)) > 0$ and $\lim_{y \searrow 0} N_{C^*, \kappa}(w, \gamma_u(y)) = 0$ (by considering a definable curve $\hat{\gamma}$ in C_w^* converging to (u_0, x_0) with $\hat{\gamma}_u = \gamma_u$. Such a definable curve is compatible with C_w^* and exists by Remark 6.27(2) since $\lim_{y \searrow 0} \text{dist}(\gamma_x(y), (C^*)_{(w, \gamma_u(y))}) = 0$). For $y \in]0, 1[$ set

$$\mathcal{S}(y) := \min\{M_{C^*, \kappa}(w, \gamma_u(y)), \epsilon/2\}.$$

By passing to a suitable subcurve of γ if necessary we may assume that

$$|\gamma_x(y) - \mu_{C^*, \kappa}(w, \gamma_u(y))| < 1/2\mathcal{S}(y), \quad (*)$$

that

$$B(\mu_{C^*, \kappa}(w, \gamma_u(y)), \mathcal{S}(y)) \cap C_{(w, \gamma_u(y))}^* \neq \emptyset \quad (**)$$

and that $2N_{C^*, \kappa}(w, \gamma_u(y)) < 1/2\mathcal{S}(y)$ for $y \in]0, 1[$. Additionally we may suppose that either $N_{C^*, \kappa}(w, \gamma_u(y)) > 0$ for all $y \in]0, 1[$ or $N_{C^*, \kappa}(w, \gamma_u(y)) = 0$ for all $y \in]0, 1[$.

Assume the former. Then we have $\alpha_{C^*, \kappa}(w, \gamma_u(y)) = 2N_{C^*, \kappa}(w, \gamma_u(y))$ and therefore $\alpha_{C^*, \kappa}(w, \gamma_u(y)) < 1/2\mathcal{S}(y)$ for every $y \in]0, 1[$. By Claim 2 and $(**)$ it is

$$B(\mu_{C^*, \kappa}(w, \gamma_u(y)), M_{C^*, \kappa}(w, \gamma_u(y))) \cap (H_{C^*})_{(w, \gamma_u(y))} \subset (\hat{\Xi}_{C^*})_{(w, \gamma_u(y))}$$

for all $y \in]0, 1[$. We have with the construction of Ξ_{C^*} and $M_{C^*, \kappa}$

$$B(\mu_{C^*, \kappa}(w, \gamma_u(y)), \mathcal{S}(y)) \cap (H_{C^*})_{(w, \gamma_u(y))} \subset (\Xi_{C^*})_{(w, \gamma_u(y))}$$

and therefore

$$B(\mu_{C^*, \kappa}(w, \gamma_u(y)), \mathcal{S}(y)) \subset \Omega_{(w, \gamma_u(y))}$$

for all $y \in]0, 1[$. This gives with $(*)$ and Pythagoras

$$\{\gamma_x(y) + iv \in \mathbb{C} \mid v \in]0, \frac{\sqrt{3}}{2}\mathcal{S}(y)[\} \subset \Omega_{(w, \gamma_u(y))}$$

for all $y \in]0, 1[$. So we have

$$\lim_{y \searrow 0} \zeta_{\Omega}(w, \gamma(y)) \geq \lim_{y \searrow 0} \frac{\sqrt{3}}{2}\mathcal{S}(y) = \lim_{y \searrow 0} \frac{\sqrt{3}}{2} \min\{M_{C^*, \kappa}(w, \gamma_u(y)), \epsilon/2\} > 0,$$

since $M_{C^*, \kappa}$ is (C^*, κ) -persistent, a contradiction. The latter case is handled completely similar.

Case 2: $C^* \in I_{\text{small}}$.

Note that there is $C_{\mathcal{L}} \in I_{\text{big}}$ with $C_{\mathcal{L}} < C^*$ such that

$$\lim_{y \searrow 0} \text{dist}(\gamma_x(y), (C_{\mathcal{L}})_{(w, \gamma_u(y))}) = 0.$$

So $(u_0, x_0) \in \overline{(C_{\mathcal{L}})_w}$.

Subcase 2.1:

Suppose that

$$\lim_{y \searrow 0} |\gamma_x(y) - \mu_{C_{\mathcal{L}}, \kappa}(w, \gamma_u(y))| > 0$$

for $\kappa = r$ or $\kappa \in \{k_{C_{\mathcal{L}}}, r\}$ if $k_{C_{\mathcal{L}}} \geq 0$. By passing to a suitable subcurve of γ if necessary we may assume that γ runs through $(H_{C_{\mathcal{L}}, \text{small}})_w$ since for $\mu_{C_{\mathcal{L}}, \kappa} \geq C_{\mathcal{L}}$

$$\lim_{y \searrow 0} \text{dist}(\mu_{C_{\mathcal{L}}, \kappa}(w, \gamma_u(y)), (C_{\mathcal{L}})_{(w, \gamma_u(y))}) > 0$$

for $\kappa = r$ or $\kappa \in \{k_{C_{\mathcal{L}}}, r\}$ if $k_{C_{\mathcal{L}}} \geq 0$. So γ is compatible with $(C_{\mathcal{L}})_w$. By Proposition 6.85 we obtain

$$\lim_{y \searrow 0} \zeta_{\Lambda_{C_{\mathcal{L}}}}(w, \gamma_u(y)) > 0.$$

A completely similar argument as in Subcase 1.1 applied to $C_{\mathcal{L}}$ gives the desired contradiction.

Subcase 2.2:

Suppose

$$\lim_{y \searrow 0} |\gamma_x(y) - \mu_{C_{\mathcal{L}}, \kappa}(w, \gamma_u(y))| = 0$$

for $\kappa = r$ or a $\kappa \in \{k_{C_{\mathcal{L}}}, r\}$ if $k_{C_{\mathcal{L}}} \geq 0$. Fix such a κ . Then

$$\lim_{y \searrow 0} N_{C_{\mathcal{L}}, \kappa}(w, \gamma_u(y)) = 0$$

and $\lim_{y \searrow 0} M_{C_{\mathcal{L}}, \kappa}(w, \gamma_u(y)) > 0$ by Definition 6.46 (by considering a suitable definable curve $\hat{\gamma}$ in $(C_{\mathcal{L}})_w$ converging to (u_0, x_0) with $\hat{\gamma}_u = \gamma_u$. Such a curve is compatible with $(C_{\mathcal{L}})_w$ and exists by Remark 6.27(2) since

$$\lim_{y \searrow 0} \text{dist}(\gamma_x(y), (C_{\mathcal{L}})_{(w, \gamma_u(y))}) = 0.)$$

A completely similar argument as in subcase 1.2 applied to $C_{\mathcal{L}}$ gives the desired contradiction. ■ Claim 3

We set

$$Z := \{(w, u, z) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{C} \mid (w, u, \operatorname{Re}(z)) \in X \text{ and} \\ \operatorname{Im}(z) \in Q(0, \rho_\Omega(w, u, \operatorname{Re}(z)))\}$$

where for $(w, u, x) \in \mathbb{R}^l \times \mathbb{R}^m \times \mathbb{R}$

$$\rho_\Omega(w, u, x) := \liminf_{(s,v) \rightarrow (u,x)} \zeta_\Omega(w, s, v).$$

Note that Z is definable with $X \subset Z$ by Claim 3.

Claim 4

Z_w is open for every $w \in \mathbb{R}^l$ (considered as a subset of $\mathbb{R}^m \times \mathbb{R}^2$).

Proof of Claim 4

Let $w \in \mathbb{R}^l$ and $(u_0, z_0) \in Z_w$. Let $z_0 := x_0 + iy_0$ for $x_0, y_0 \in \mathbb{R}$. Then there is $r > 0$ such that $y_0 < r < \rho_\Omega(w, u_0, x_0)$. Since the function

$$\rho_\Omega : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}, (u, x) \mapsto \liminf_{(s,v) \rightarrow (u,x)} \zeta_\Omega(w, s, v),$$

is lower semi-continuous by Proposition 6.88 we find an open neighbourhood U of (u_0, x_0) contained in the open set X_w such that $\rho_\Omega(w, u, x) > r$ for every $(u, x) \in U$. This shows that $V := U + iQ(0, r) \subset Z_w$ and we have $(u_0, z_0) \in V$.

■ Claim 4

Claim 4 shows that $F|_Z$ is a unary high parametric global complexification of f with respect to (u, x) .

Step 3: Constructing a unary high parametric global complexification $F : Z \rightarrow \mathbb{R}$ of f with respect to (u, x) which is restricted log-exp-analytic in (u, z) .

Now we are able to redefine G suitably in order to obtain the desired function F .

Let

$$\pi^* : \mathbb{R}^n \times \mathbb{C} \rightarrow \mathbb{R}^n, (t, z) \mapsto t.$$

Let $\mathcal{B} := \{\pi(C) \mid C \in \mathcal{C}\}$. Then \mathcal{B} is a definable cell decomposition of $\pi(X)$. For $B \in \mathcal{B}$ consider the following.

Set

$$\mathcal{C}_B := \{C \in \mathcal{C} \mid \pi(C) = B\}$$

and let $X_B := \{(t, x) \in X \mid t \in B\}$ and $Z_B := \{(t, z) \in Z \mid t \in B\}$. Then $X = \bigcup_{B \in \mathcal{B}} X_B$ and $Z = \bigcup_{B \in \mathcal{B}} Z_B$. Let u_B be the cardinality of \mathcal{C}_B . Let $C_{B,1}, \dots, C_{B,u_B} \in \mathcal{C}_B$ be the unique cells with $C_{B,1} < \dots < C_{B,u_B}$. Note

that $C_{B,j}$ is fat with respect to x for $j \in \{1, \dots, u_B\}$. For $j \in \{1, \dots, u_B\}$ we set $\Gamma_{B,j} := \Gamma_{C_{B,j}}$, $\Xi_{B,j} := \Xi_{C_{B,j}}$, $\mathcal{E}_{B,j} := \mathcal{E}_{C_{B,j}}$, $\Phi_{B,j} := \Phi_{C_{B,j}}$, $k_{B,j} := k_{C_{B,j}}$, $G_{B,j} := G_{C_{B,j}}$ and for $\kappa = r$ or $\kappa \in \{k_{B,j}, r\}$ if $k_{B,j} \geq 0$ we set $\mathcal{T}_{B,j,\kappa} := \mathcal{T}_{C_{B,j},\kappa}$. From now on we consider every $g \in \mathcal{E}_{B,j}$ as a function on $\Xi_{B,j}$.

Let $Y_{B,j} := (X_B \cup \Xi_{B,j}) \cap Z$ for $j \in \{1, \dots, u_B\}$. We define

$$Z_{B,1} := \{(t, z) \in Y_{B,1} \mid \zeta_{Y_{B,1}}(t, \operatorname{Re}(z)) \geq \zeta_{Y_{B,j}}(t, \operatorname{Re}(z))\}$$

$$\text{for every } j \in \{2, \dots, u_B\}$$

and inductively for $l \in \{2, \dots, u_B\}$

$$Z_{B,l} := \{(t, z) \in Y_{B,l} \mid \zeta_{Y_{B,l}}(t, \operatorname{Re}(z)) \geq \zeta_{Y_{B,j}}(t, \operatorname{Re}(z)) \text{ for every}$$

$$j \in \{l+1, \dots, u_B\} \text{ and } (t, z) \notin Z_{B,1} \cup \dots \cup Z_{B,l-1}\}.$$

Note that $Z_{B,j}$ is definable for $j \in \{1, \dots, u_B\}$ and that $Z_B = \dot{\bigcup}_{j=1}^{u_B} Z_{B,j}$. Note also that for $(t, x) \in X_B$ there is a unique $j \in \{1, \dots, l\}$ such that $(t, x + iy) \in Z_{B,j}$ for every $y \in \mathbb{R}$ with $(t, x + iy) \in Z_B$. For $j \in \{1, \dots, u_B\}$ set

$$X_{B,j} := \{(t, x) \in X_B \mid (t, x) \in Z_{B,j}\}.$$

Note that $X_B = \dot{\bigcup}_{j=1}^{u_B} X_{B,j}$. Fix $j \in \{1, \dots, u_B\}$. We have

$$\lim_{y \searrow 0} \zeta_{Z_{B,j}}(w, \gamma_u(y), \gamma_x(y)) > 0$$

for $(w, u_0, x_0) \in X$ and a real definable curve $\gamma :]0, 1[\rightarrow (X_{B,j})_w$ converging to (u_0, x_0) . (++)

(Otherwise

$$\liminf_{(u,x) \rightarrow (u_0, x_0)} \zeta_Z(w, u, x) = 0$$

by construction of $Z_{B,j}$ which is a contradiction to the fact that F is a unary high parametric global complexification of f which respect to (u, x) .)

If $k_{B,j} < 0$ we set $Z_{B,j,\mathcal{T}} := \mathcal{T}_{B,j,r} \cap Z_{B,j}$. If $k_{B,j} \geq 0$ we set $Z_{B,j,\mathcal{T}} := (\mathcal{T}_{B,j,k_{B,j}} \dot{\cup} \mathcal{T}_{B,j,r}) \cap Z_{B,j}$. Set $Z_{B,j,\Gamma} := (\Gamma_{B,j} \cap Z_{B,j}) \setminus Z_{B,j,\mathcal{T}}$. Then $Z_{B,j} = Z_{B,j,\mathcal{T}} \dot{\cup} Z_{B,j,\Gamma}$.

For $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$ let

$$E_{B,j,\mathcal{T}} := \mathcal{E}_{B,j} \big|_{Z_{B,j,\mathcal{T}} \setminus X}$$

and

$$E_{B,j,\Gamma} := (\mathcal{E}_{B,j} \cup E_{\Phi_{B,j}}) \big|_{Z_{B,j,\Gamma} \setminus X}.$$

Additionally consider

$$F_{B,j,\mathcal{T}} : Z_{B,j,\mathcal{T}} \setminus X \rightarrow \mathbb{C}, (t, z) \mapsto G_{B,j}(t, z),$$

and

$$F_{B,j,\Gamma} : Z_{B,j,\Gamma} \setminus X \rightarrow \mathbb{C}, (t, z) \mapsto \Phi_{B,j}(t, z).$$

Note that $F_{B,j,\Gamma}$ can be constructed from $E_{B,j,\Gamma}$ by (ββ).

Claim 5

$F_{B,j,\mathcal{T}}$ can be constructed from $E_{B,j,\mathcal{T}}$.

Proof of Claim 5

For $\kappa = r$ or $\kappa \in \{k_{B,j}, r\}$ if $k_{B,j} \geq 0$ we set $\mathcal{W}_\kappa := G_{B,j}|_{\mathcal{T}_{B,j,\kappa}}$. Then we have for $(t, z) \in \mathcal{T}_{B,j,\kappa}$

$$\mathcal{W}_\kappa(t, z) = \begin{cases} \Phi_{B,j}(t, z), & (t, z) \in K_{C_{B,j},\kappa}, \\ \chi_{C_{B,j},\kappa}(t, z), & (t, z) \in \mathcal{G}_{C_{B,j},\kappa}. \end{cases}$$

We have that $\Phi_{B,j}|_{K_{C_{B,j},\kappa}}$ can be constructed from $\mathcal{E}_{B,j}$ by (β) and that $\chi_{C_{B,j},\kappa}$ can be constructed from $\mathcal{E}_{B,j}$ by (βββ) for $\kappa = r$ or $\kappa \in \{k_{B,j}, r\}$ if $k_{B,j} \geq 0$. So with Remark 3.8 we see that \mathcal{W}_κ can be constructed from $\mathcal{E}_{B,j}$ since every $g \in \mathcal{E}_{B,j}$ is a positive definable function which depends only on t .

If $k_{B,j} < 0$ we have that $F_{B,j,\mathcal{T}} = \mathcal{W}_r|_{Z_{B,j,\mathcal{T}} \setminus X}$. So $F_{B,j,\mathcal{T}}$ can be constructed from $E_{B,j,\mathcal{T}}$ and we are done. So suppose $k_{B,j} \geq 0$. Since for $\kappa \in \{k_{B,j}, r\}$

$$F_{B,j,\mathcal{T}}|_{(\mathcal{T}_{B,j,\kappa} \cap Z_{B,j}) \setminus X} = \mathcal{W}_\kappa|_{(\mathcal{T}_{B,j,\kappa} \cap Z_{B,j}) \setminus X}$$

we have that $F_{B,j,\mathcal{T}}|_{(\mathcal{T}_{B,j,\kappa} \cap Z_{B,j}) \setminus X}$ can be constructed from $E_{B,j,\mathcal{T}}|_{(\mathcal{T}_{B,j,\kappa} \cap Z_{B,j}) \setminus X}$. By Remark 3.8 we see that $F_{B,j,\mathcal{T}}$ can be constructed from $E_{B,j,\mathcal{T}}$ since every $g \in E_{B,j,\mathcal{T}}$ is a positive definable function which depends only on t . ■_{Claim 5}

Finally we define $F : Z \rightarrow \mathbb{C}$ in the following way.

If $(t, z) \in Z_{B,j,\mathcal{T}} \setminus X$ for $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$ let $F(t, z) = F_{B,j,\mathcal{T}}(t, z)$. If $(t, z) \in Z_{B,j,\Gamma} \setminus X$ for $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$ let $F(t, z) = F_{B,j,\Gamma}(t, z)$. If $(t, z) \in X$ set $F(t, z) = f(t, z)$.

Then $F : Z \rightarrow \mathbb{C}, (w, u, z) \mapsto F(w, u, z)$, is a unary high parametric global complexification of f with respect to (u, x) since we have $F(t, z) = G(t, z)$ for every $(t, z) \in Z$. Let

$$E := \{g \mid g : Z \rightarrow \mathbb{C} \text{ is a function with } g|_X \in E_f, g|_{Z_{B,j,\mathcal{T}} \setminus X} \in E_{B,j,\mathcal{T}},$$

$$g|_{Z_{B,j,\Gamma} \setminus X} \in E_{B,j,\Gamma} \text{ for } B \in \mathcal{B}, j \in \{1, \dots, u_B\}\}$$

where E_f is a set of positive definable functions on X such that every $g \in \log(E_f)$ is locally bounded in (u, x) and f can be constructed from E_f .

By Remark 3.8 we see that F can be constructed from E (compare also with Definition 3.14).

Claim 6

Let $h \in E_f$ and $t \in \pi^*(Z)$. Then the function $\log(h)$ is locally bounded in (u, z) with respect to Z .

Proof of Claim 6

Note that h is defined on X . Let $w \in \mathbb{R}^l$. Let $(u_0, z_0) \in Z_w$. We show that $\log(h_w)$ is bounded at (u_0, z_0) and are done.

If $\text{Im}(z_0) \neq 0$ we are clearly done by taking an open neighbourhood of (u_0, z_0) in Z_w disjoint from X . So assume $\text{Im}(z_0) = 0$. Then $(u_0, z_0) \in X_w$ and we find an open neighbourhood U of (u_0, z_0) in X_w such that $\log(g)|_U$ is bounded. So take an open neighbourhood V of (u_0, z_0) in Z_w with $V \cap X \subset U$. ■_{Claim 6}

For $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$ we set $\hat{E}_{B,j,\Gamma} := \{\exp(\text{Re}(g)) \mid g \in \log(E_{B,j,\Gamma})\}$.

Claim 7

Let $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$. Let $h \in \hat{E}_{B,j,\Gamma}$. The function $\log(h)$ is locally bounded in (u, z) with respect to Z .

Proof of Claim 7

Let $C := C_{B,j}$, $\mathcal{K} := Z_{B,j,\Gamma} \setminus X$, $w \in \mathbb{R}^l$ and $(u_0, z_0) \in Z_w$. If $(u_0, z_0) \notin \overline{\mathcal{K}_w}$ we are clearly done. So assume $(u_0, z_0) \in \overline{\mathcal{K}_w}$. We show that $\log(h_w)$ is bounded at (u_0, z_0) .

Case 1:

Let $h \in \mathcal{E}_{B,j}|_{\mathcal{K}}$. Suppose that $\log(h_w)$ is not bounded at (u_0, z_0) (i.e. at u_0). Then there is a definable curve $\gamma :]0, 1[\rightarrow \mathcal{K}_w$ such that $\lim_{y \searrow 0} \gamma(y) = (u_0, z_0)$ and $\lim_{y \searrow 0} |\log(h(w, \gamma_u(y)))| = \infty$. Note that $(\gamma_u, \text{Re}(\gamma_z))$ runs through $X_{B,j}$. By construction of $Z_{B,j}$ (since $\zeta_{Z_{B,j}}$ is bounded by $\exp(-|\log(h)|)$) we have $\lim_{y \searrow 0} \text{Im}(\gamma_z(y)) = 0$ and thus $z_0 \in \mathbb{R}$. So we see

$$\lim_{y \searrow 0} \zeta_{Z_{B,j}}(w, \gamma_u(y), \text{Re}(\gamma_z(y))) = 0,$$

a contradiction to the property $(++)$.

Case 2:

Assume that $h \in E_{\Phi_C}|_{\mathcal{K}}$. Then $h = \exp(\Phi_g)|_{\mathcal{K}}$ for a $g \in \log(E_C)$ where Φ_g is complex (l, r) -prepared in z for $r \in \mathbb{N}_0$ and $l \in \{-1, \dots, e-1\}$. We show the statement by induction on l . If $l = -1$ then $g = 0$ and the assertion follows.

$l-1 \rightarrow l$: Let

$$(r, \mathcal{Z}, a, \exp(D_0), q, s, V, b, \exp(D), P) := (r, \mathcal{Z}_{C,g}|_{\mathcal{K}}, a_{C,g}|_{\mathcal{K}}, \exp(D_{C,0,g})|_{\mathcal{K}}, q_{C,g}, s, V_{C,g}, b_{C,g}|_{\mathcal{K}}, \exp(D_{C,g})|_{\mathcal{K}}, P_{C,g}).$$

Let $b := (b_1, \dots, b_s)$, $\exp(D) := (\exp(D_1), \dots, \exp(D_s))$ and $P := (p_1, \dots, p_s)^t$. So we have

$$h(t, z) = a(t) \mathcal{Z}^{\otimes q}(t, z) \exp(D_0(t, z))$$

$$V(b_1(t) \mathcal{Z}^{\otimes p_1}(t, z) \exp(D_1(t, z)), \dots, b_s(t) \mathcal{Z}^{\otimes p_s}(t, z) \exp(D_s(t, z)))$$

for $(t, z) \in \mathcal{K}$ where $t := (w, u)$. Let $U \subset \mathbb{R}^m \times \mathbb{C}$ be an open neighbourhood of (u_0, z_0) such that $(D_0)_w, \dots, (D_s)_w$ are bounded on $\hat{U} := U \cap \mathcal{K}_w$. We may also assume that a_w and $(\Theta_0)_w, \dots, (\Theta_r)_w$ are bounded on \hat{U} . (Since we have that $\exp(a) \in \mathcal{E}_C$ and $\exp(\Theta_j) \in \mathcal{E}_C$ for $j \in \{0, \dots, r\}$. Otherwise we derive the same contradiction as in case 1.) Assume that there is an open neighbourhood $W \subset U$ of (u_0, x_0) such that h_w is not bounded on $\hat{W} := W \cap \mathcal{K}_w$. Then there is a definable curve $\gamma :]0, 1[\rightarrow \hat{W}$ such that $\lim_{y \searrow 0} |\mathcal{Z}^{\otimes q}(w, \gamma(y))| = \infty$. Note that γ runs through $(\Gamma_C)_w \setminus (\mathcal{T}_{C,r})_w$ respectively $(\Gamma_C)_w \setminus ((\mathcal{T}_{C,r})_w \cup (\mathcal{T}_{C,k_C})_w)$ if $k_C \geq 0$. Therefore γ runs also through $(H_{C,\text{small}})_w$. By the proof of (II_l) in Proposition 6.28 for $l \in \{0, \dots, r\}$ (since $(\Theta_0)_w, \dots, (\Theta_r)_w$ are bounded on \hat{W}_w) and by construction of Γ_C we have that $\lim_{y \searrow 0} |\gamma_z(y) - \mu_{C,\kappa}(w, \gamma_u(y))| = 0$ for $\kappa = r$ or a $\kappa \in \{k_C, r\}$ if $k_C \geq 0$ (since $\lim_{y \searrow 0} \mu_{C,\kappa}(w, \gamma_u(y)) \in \mathbb{R}$, compare also Corollary 6.29).

Fix such a κ . We see $(u_0, z_0) \in X_w$. Let $x_0 := z_0$. By definition of $(H_{C,\text{small}})_w$ we see that

$$\lim_{y \searrow 0} \mathfrak{d}_{C,\kappa}(w, \gamma_u(y)) = \lim_{y \searrow 0} \text{dist}(\mu_{C,\kappa}(w, \gamma_u(y)), C_{(w, \gamma_u(y))}) = 0.$$

So $(u_0, x_0) \in \overline{C_w}$. Fix $\epsilon > 0$ such that $Q^{m+1}((u_0, x_0), \epsilon) \subset X_w$ and $f_w(u, x) = p((u, x) - (u_0, x_0))$ for $(u, x) \in Q^{m+1}((u_0, x_0), \epsilon)$ where p is a convergent power series on $Q^{m+1}(0, \epsilon)$.

We have $\lim_{y \searrow 0} L_C(w, \gamma_u(y)) > 0$ since otherwise we see that

$$\lim_{y \searrow 0} \zeta_{Z_{B,j}}(w, \gamma_u(y), \text{Re}(\gamma_z(y))) = 0$$

(since $(Z_{B,j})_t \subset \mathbb{R} + iQ(0, L_C(t))$ for $t \in \pi(C)$). But this is a contradiction to property $(++)$.

So we see that γ is compatible with C_w . We have $\lim_{y \searrow 0} M_{C,\kappa}(w, \gamma_u(y)) > 0$ and $\lim_{y \searrow 0} N_{C,\kappa}(w, \gamma_u(y)) = 0$. (Since $M_{C,\kappa}$ is (C, κ) -persistent and $N_{C,\kappa}$ is (C, κ) -non-persistent: Consider a suitable definable curve $\hat{\gamma}$ in C_w converging to (u_0, x_0) with $\hat{\gamma}_u = \gamma_u$. Such a curve is compatible with C_w and exists by Remark 6.27(2) since

$$\lim_{y \searrow 0} \text{dist}(\gamma_x(y), C_{(w, \gamma_u(y))}) = 0.)$$

By passing to a suitable subcurve of γ if necessary we may suppose the following.

- (1) We have $\gamma_u(y) \in Q^m(u_0, \epsilon/2)$ for every $y \in]0, 1[$.
- (2) We have $|\mu_{C,\kappa}(w, \gamma_u(y)) - \gamma_z(y)| < 1/2 \min\{M_{C,\kappa}(w, \gamma_u(y)), \epsilon/2\}$ and $C_{(w, \gamma_u(y))} \cap B(\mu_{C,\kappa}(w, \gamma_u(y)), \epsilon) \neq \emptyset$ for every $y \in]0, 1[$.
- (3) We have either $N_{C,\kappa}(w, \gamma_u(y)) > 0$ for every $y \in]0, 1[$ or $N_{C,\kappa}(w, \gamma_u(y)) = 0$ for every $y \in]0, 1[$.

Assume the former in (3). The latter is handled completely similar. Then $\alpha_{C,\kappa}(w, \gamma_u(y)) = 2N_{C,\kappa}(w, \gamma_u(y))$ for $y \in]0, 1[$. Therefore

$$\alpha_{C,\kappa}(w, \gamma_u(y)) < 1/2 \min\{M_{C,\kappa}(w, \gamma_u(y)), \epsilon/2\}$$

for every $y \in]0, 1[$ after passing to a suitable subcurve of γ if necessary. We obtain with Claim 2 that $(w, \gamma_u(y)) \in \hat{B}_{C,\kappa}$ and therefore

$$B(\mu_{C,\kappa}(w, \gamma_u(y)), M_{C,\kappa}(w, \gamma_u(y))) \setminus X_{(w, \gamma_u(y))} \subset (\mathcal{T}_{C,r})_{(w, \gamma_u(y))} \setminus X_{(w, \gamma_u(y))}$$

respectively

$$B(\mu_{C,\kappa}(w, \gamma_u(y)), M_{C,\kappa}(w, \gamma_u(y))) \setminus X_{(w, \gamma_u(y))} \subset (\mathcal{T}_{C,r} \cup \mathcal{T}_{C,k_C})_{(w, \gamma_u(y))} \setminus X_{(w, \gamma_u(y))}$$

if $k_C \geq 0$ for every $y \in]0, 1[$. This gives $(w, \gamma(y)) \in \mathcal{T}_{C,r} \setminus X$ respectively $(w, \gamma(y)) \in (\mathcal{T}_{C,r} \cup \mathcal{T}_{C,k_C}) \setminus X$ for every $y \in]0, 1[$ if $k_C \geq 0$ with property (2). But this is a contradiction since γ runs through $(\Gamma_C)_w \setminus (\mathcal{T}_{C,r})_w$ respectively through $(\Gamma_C)_w \setminus ((\mathcal{T}_{C,r})_w \cup (\mathcal{T}_{C,k_C})_w)$ if $k_C \geq 0$. ■_{Claim 7}

Similarly as in Claim 7 one sees that $\log(h)$ is locally bounded in (u, z) with respect to Z if $h \in E_{B,j,\mathcal{T}}$ for $B \in \mathcal{B}$ and $j \in \{1, \dots, u_B\}$. By Remark 3.28 one sees that E is a set of locally bounded functions in (u, z) and we are done with the proof of Proposition 6.89. ■

6.2.2 The Higher Dimensional Case

We show by induction on the number of variables that the class of restricted log-exp-analytic functions exhibits parametric global complexification. The "highness" from the unary case above is necessary to enable the induction. We adapt Kaiser's arguments from [19] to our situation and establish Theorem C.

6.90 Proposition

Let $m \in \mathbb{N}$. Let w over \mathbb{R}^m and v over \mathbb{C}^m . Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable. Assume that X_t is open for all $t \in \mathbb{R}^n$. Let $f : X \rightarrow \mathbb{R}$, $(t, w) \mapsto f(t, w)$, be restricted log-exp-analytic in w such that f_t is real analytic for every $t \in \mathbb{R}^n$. Then f has an m -ary parametric global complexification $F : Z \rightarrow \mathbb{C}$, $(t, v) \mapsto F(t, v)$, which is restricted log-exp-analytic in v .

Proof

For $l \in \{1, \dots, m\}$ let u range over \mathbb{R}^{m-l} and x over \mathbb{R}^l . Let $\pi_n : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$, $(t, u, x) \rightarrow t$, be the projection on the first n real coordinates. We show by induction on $l \in \{1, \dots, m\}$ that $f : X \rightarrow \mathbb{R}$, $(t, u, x) \mapsto f(t, u, x)$, has an l -ary high parametric global complexification $F : Z \rightarrow \mathbb{C}$, $(t, u, z) \mapsto F(t, u, z)$, with respect to (u, x) where F is restricted log-exp-analytic in (u, z) .

$l = 1$: This follows from Proposition 6.89.

$l - 1 \rightarrow l$: Given $x = (x_1, \dots, x_l) \in \mathbb{R}^l$ we set $x'' := (x_2, \dots, x_l)$. The set

$$X_{(t,u,x_1)} = \{x'' \in \mathbb{R}^{l-1} \mid (t, u, x_1, x'') \in X\} \subset \mathbb{R}^{l-1}$$

is open and the function $f_{(t,u,x_1)} : X_{(t,u,x_1)} \rightarrow \mathbb{R}$ is real analytic since it coincides with $(f_t)_{(u,x_1)}$ for $(t, u, x_1) \in \mathbb{R}^n \times \mathbb{R}^{m-l} \times \mathbb{R}$. Let $\mathbb{R}^{n+m-l+1}$ be the parameter space and write \hat{X} if we consider X as a subset of $\mathbb{R}^{n+m-l+1} \times \mathbb{R}^{l-1}$. Moreover we write \hat{f} if we consider f as a function on \hat{X} . By the inductive hypothesis we find a definable set $\hat{Y} \subset \mathbb{R}^{n+m-l+1} \times \mathbb{C}^{l-1}$ such that $\hat{Y}_t \subset \mathbb{R}^{m-l+1} \times \mathbb{C}^{l-1}$ is open for $t \in \mathbb{R}^n$ and an $(l-1)$ -ary high parametric global complexification $\hat{G} : \hat{Y} \rightarrow \mathbb{C}$, $(t, u, x_1, z'') \mapsto \hat{G}(t, u, x_1, z'')$, of f with respect to (u, x) which is restricted log-exp-analytic in (u, x_1, z'') . We set $G_1 := \text{Re}(\hat{G}) : \hat{Y} \rightarrow \mathbb{R}$ and $G_2 := \text{Im}(\hat{G}) : \hat{Y} \rightarrow \mathbb{R}$. Note that $G_1|_X = f$ and $G_2|_X = 0$. By Definition 3.37 G_1 and G_2 are restricted log-exp-analytic in (u, x_1, z'') . Let

$$Y := \{(t, u, x_1, z'') \in \hat{Y} \mid (G_j)_t \text{ is real analytic at } (u, x_1, z'') \text{ for } j \in \{1, 2\}\}.$$

Then Y is definable by Theorem A (respectively by Corollary 6.30). We have $X \subset Y \subset \hat{Y}$. Since real analyticity is an open property we have that Y_t is open in $\mathbb{R}^{m-l+1} \times \mathbb{C}^{l-1}$ for all $t \in \mathbb{R}^n$.

Considering z'' as a tuple of $2(l-1)$ real variables we find with Proposition 6.89 a unary high parametric global complexification

$$F_j : Z_j \rightarrow \mathbb{C}, (t, u, \hat{z}, z'') \mapsto F_j(t, u, \hat{z}, z''),$$

for $G_j|_Y$ with respect to (u, x_1, z'') which is restricted log-exp-analytic in (u, \hat{z}, z'') where $j \in \{1, 2\}$. (Here \hat{z} is an additional variable which ranges over \mathbb{C} . Note also that $(Z_j)_t$ is open in $\mathbb{R}^{m-l} \times \mathbb{C}^l$ for $t \in \mathbb{R}^n$ with $X \subset Z_j$ for $j \in \{1, 2\}$.)
(*)

Let

$$\tilde{Z} := \{(t, u, z) \in Z_1 \cap Z_2 \mid (F_1 + iF_2)_{(t,u)} \text{ is holomorphic at } z\}.$$

Then \tilde{Z} is definable and $F := (F_1 + iF_2)|_{\tilde{Z}} : \tilde{Z} \rightarrow \mathbb{C}$ is restricted log-exp-analytic in (u, z) with reference set $Z_1 \cap Z_2$ by Remark 3.32 and Definition 3.37.

Let

$$\zeta_{\tilde{Z}}(t, u, x) := \sup\{r \in \mathbb{R}_{\geq 0} \mid x + iQ^l(0, r) \subset \tilde{Z}_{(t,u)}\}.$$

Claim

Let $(t, u, x) \in X$. Then

$$\liminf_{(s,y) \rightarrow (u,x)} \zeta_{\tilde{Z}}(t, s, y) > 0.$$

Proof of the claim

We find some $0 < r$ and an open neighbourhood V of (u, x) in X_t such that $f_t(s, y) = p((s, y) - (u, x))$ for $(s, y) \in V$ where p is a convergent power series on $Q^m(0, r)$. We find some $0 < r' < r/2$ such that

$$U := Q^{m-l}(u, r') \times]x_1 - r', x_1 + r'[\times (Q^{l-1}(x'', r') + iQ^{l-1}(0, r')) \subset \hat{Y}_t$$

and that $\tilde{G}_t(s, y_1, z'') = p((s, y_1, z'') - (u, x))$ for all $(s, y_1, z'') \in U$. So for $(s, y) \in Q^m((u, x), r')$ we see that

$$y'' + iQ^{l-1}(0, r') \subset Y_{(t,s,y_1)}.$$

So by (*) we find some $0 < \epsilon < r'/2$ and an open neighbourhood W of (u, x) in X_t with $W \subset Q^m((u, x), r')$ such that for $(s, y) \in W$ we have the following. We have $y + iQ^l(0, \epsilon) \subset (Z_j)_{(t,s)}$ for $j \in \{1, 2\}$ and $F_{(t,s)}(z) = p_{s-u}(z - x)$ for all $z \in D^l(y, \epsilon)$. So by construction $y + iQ^l(0, \epsilon) \subset \tilde{Z}_{(t,s)}$ for all $(s, y) \in W$. Hence

$$\liminf_{(s,y) \rightarrow (u,x)} \zeta_{\tilde{Z}}(t, s, y) \geq \epsilon > 0.$$

■ Claim

Let

$$Z := \{(t, u, z) \in \mathbb{R}^n \times \mathbb{R}^{m-l} \times \mathbb{C}^l \mid (t, u, \operatorname{Re}(z)) \in X \\ \text{and } \operatorname{Im}(z) \in Q^l(0, \rho(t, u, \operatorname{Re}(z)))\}$$

where

$$\rho(t, u, x) := \liminf_{(s,y) \rightarrow (u,x)} \zeta_{\tilde{Z}}(t, s, y)$$

for $(t, u, x) \in \mathbb{R}^n \times \mathbb{R}^{m-l} \times \mathbb{R}^l$. Note that Z_t is open for all $t \in \mathbb{R}^n$. (Compare with the proof of Claim 4 in Proposition 6.89: The function $\mathbb{R}^{m-l} \times \mathbb{R}^l, (u, x) \mapsto \rho(t, u, x)$, defines a lower semi-continuous function for every $t \in \mathbb{R}^n$.) By the Claim we have $X \subset Z$. So we see that $F|_Z$ is an l -ary high parametric global complexification of f with respect to (u, x) . By the above we see that F is restricted log-exp-analytic in (u, z) . This finishes the proof of Proposition 6.90. ■

So Theorem *B* is also established.

6.91 Corollary

Let $U \subset \mathbb{R}^n$ be definable and open. Let $f : U \rightarrow \mathbb{R}$ be a real analytic restricted log-exp-analytic function. Then there is an open definable $V \subset \mathbb{C}^n$ with $U \subset V$ and a restricted log-exp-analytic $F : V \rightarrow \mathbb{C}$ such that $F|_U = f$.

Proof

This follows directly from Proposition 6.90. ■

6.92 Corollary

(1) Let $X \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable such that X_t is open for every $t \in \mathbb{R}^n$ and let $f : X \rightarrow \mathbb{R}, (t, x) \mapsto f(t, x)$, be log-analytic. Suppose that f_t is real analytic for every $t \in \mathbb{R}^n$. Then there is a definable $V \subset \mathbb{R}^n \times \mathbb{C}^m$ with $U \subset V$ such that V_t is open for every $t \in \mathbb{R}^n$ and a restricted log-exp-analytic $F : V \rightarrow \mathbb{C}, (t, z) \mapsto F(t, z)$, in z such that F_t is holomorphic for every $t \in \mathbb{R}^n$ and $F|_U = f$.

(2) Let $U \subset \mathbb{R}^n$ be open. Let $g : U \rightarrow \mathbb{R}$ be a real analytic log-analytic function. Then there is an open definable $V \subset \mathbb{C}^n$ with $U \subset V$ and a restricted log-exp-analytic $G : V \rightarrow \mathbb{C}$ such that $G|_U = g$.

Proof

(1): This follows directly from Proposition 6.90 since f is restricted log-exp-analytic in x .

(2): This follows directly from Theorem B (resp. Corollary 6.91) since g is restricted log-exp-analytic. ■

7 Conclusion and Outlook

We conclude this thesis with a short summary of the shown results. Motivated by the research of Kaiser, Lion, Rolin, Van den Dries and Miller we found a big non-trivial class of definable functions in $\mathbb{R}_{\text{an,exp}}$ which show the same behaviour as the globally subanalytic ones from the viewpoint of analysis. These are functions which are compositions of log-analytic functions and exponentials whose arguments are locally bounded. We called them restricted log-exp-analytic. Our first step was to establish a preparation theorem for this class of functions. Then we could prove a parametric version of Tamm's theorem by considering an easier form of this preparation theorem on so called simple sets. Finally we could show that a restricted log-exp-analytic function has a global complexification which is again restricted log-exp-analytic, the main result of this thesis. We also gave a version for parameters.

Now we briefly want to discuss open questions for further research in this context outgoing from Theorem B and Theorem C.

- (1) Does a real analytic log-analytic function have a global complexification which is again log-analytic?
- (2) Has the structure $\mathbb{R}_{\text{an,exp}}$ global complexification?

We have the vague assumption that question (1) is true simply by the observation that a log-analytic function extends piecewise to a holomorphic log-analytic function (by considering the single $\mathcal{L}_{\text{an}}(-1, (\sqrt{\dots})_{n=2,3,\dots}, \log)$ -terms). But to enable the induction on the number of variables one needs a preparation theorem for log-analytic functions with log-analytic data only. So one has to get better control on the exponentials which are involved in the construction of the coefficient, the center and the base functions of such a preparation. But such questions are pretty unsolved at present. An idea would be to adapt somehow the preparation theorem of Cluckers and Miller for constructible functions from [5], [6] respectively [7] on the log-analytic case.

With our results above Question 2 can be reformulated in the following way: Is every real analytic definable function restricted log-exp-analytic? This may be true from the point of analysis: If the global exponential function comes not locally bounded into the game features like flatness may occur (compare with Example 3.24). But neither real analytic functions nor restricted log-exp-analytic functions are flat.

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