Joint model of probabilistic-robust (probust) constraints with application to gas network optimization

D. Adelhütte · D. Aßmann · T. González Grandòn · M. Gugat · H. Heitsch · R. Henrion · F. Liers · S. Nitsche · R. Schultz · M. Stingl · D. Wintergerst

Received: date / Accepted: date

Abstract Optimization problems under uncertain conditions abound in many real-life applications. While solution approaches for probabilistic constraints are often developed in case the uncertainties can be assumed to follow a certain probability distribution, robust approaches are usually applied in case solutions are sought that are feasible for all realizations of uncertainties within some predefined uncertainty set. As many applications contain different types of uncertainties that require robust as well as probabilistic treatments, we

Sabrina Nitsche · Rüdiger Schultz
Faculty of Mathematics
University of Duisburg-Essen, Campus Essen
Thea-Leymann-Str. 9
D-45127 Essen, Germany
E-mail: sabrina.nitsche@uni-due.de, ruediger.schultz@uni-due.de

Holger Heitsch · René Henrion
Weierstrass Institute Berlin
Mohrenstraße 39
D-10117 Berlin, Germany
holger.heitsch@wias-berlin.de, henrion@wias-berlin.de

T. González Grandón
Humboldt University Berlin
Rudower Chausse 25
D-12489 Berlin, Germany
gonzalez@hu-berlin.de

Dennis Adelhütte · Denis Aßmann · Martin Gugat · Frauke Liers · Michael Stingl · David Wintergerst
Friedrich-Alexander Universität Erlangen-Nürnberg
Cauerstraße 11, D-91058 Erlangen, Germany
dennis.adelhuette@fau.de, denis.assmann@fau.de, martin.gugat@fau.de, frauke.liers@fau.de, michael.stingl@fau.de, d.wintergerst@fau.de
introduce a class of joint probabilistic/robust constraints. Focusing on complex uncertain gas network optimization problems, we show the relevance of this class of problems for the task of maximizing free booked capacities in an algebraic model for a stationary gas network. We furthermore present approaches for finding their solution. Finally, we study the problem of controlling a transient system that is governed by the wave equation. The task consists in determining controls such that a certain robustness measure remains below some given upper bound with high probability.

**Keywords** stabilization, wave equation, feedback, robust optimization, probabilistic constraints, probust, Karhunen-Loève

**Mathematics Subject Classification (2000)** 90B15, 90C15, 90C26, 93D15, 93D21

1 Introduction

1.1 Joint probabilistic/robust constraints

Decision making problems are more than often affected by parameter uncertainty. An optimization problem dealing with uncertain variables has the typical form

\[
\begin{align*}
\min_x & \quad g_0(x) \\
\text{subject to} & \quad g_i(x, z) \geq 0 \quad (i = 1, \ldots, k)
\end{align*}
\]  

(1)

Here \( x \in \mathbb{R}^n \) is a decision vector, \( z \in \mathbb{R}^m \) is an uncertain parameter, \( g_0 : \mathbb{R}^n \to \mathbb{R} \) is the objective function and \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \) is the constraint mapping. The decision support schemes with non-deterministic parameters have to take into account the nature and source of uncertainty while balancing the objective and the constraints of the problem. There are two main approaches for dealing in a robust way with uncertainty in the constraints of an optimization problem: the first one is the use of probabilistic constraints. This approach is based on the assumption that historical data is available to estimate the probability distribution of the uncertain parameter and thus considering it as a random vector \( \xi \). Then (1) may be given the form

\[
\begin{align*}
\min_x & \quad g_0(x) \\
\text{subject to} & \quad \mathbb{P}(g(x, \xi) \geq 0) \geq p \in (0, 1]
\end{align*}
\]  

(2)

(note that the first ‘\( \geq \)’ sign is to be understood component-wise). Here, a decision \( x \) is declared to be feasible if and only if the original random inequality system (1) is satisfied with a prescribed probability \( p \), a level usually chosen close to but not identical to one. Of course, higher values of \( p \) lead to smaller set of feasible decisions \( x \) in (2), hence to optimal solutions at higher costs. For a standard reference on optimization problems with probabilistic constraints we refer to the monograph [20].
An alternative approach is given by robust optimization. It applies when the uncertain parameter \( u \) is completely unknown or non-stochastic due to a lack of available data. Then, satisfaction of the uncertain inequality system (1) is required for every realization of the uncertainty parameter within some uncertainty set \( U \subseteq \mathbb{R}^m \), so that one arrives at the following optimization problem:

\[
\min_x g_0(x) \\
\text{subject to } g(x, u) \geq 0 \quad \forall u \in U
\]  

(3)

For a basic monograph on robust optimization, we refer to [3].

We notice that both optimization models with probabilistic and robust constraints are deterministic reformulations of (1), since they depend only on the decision vector \( x \) but no longer on the outcome of the uncertain parameter \( z \).

A central issue in robust optimization is the appropriate choice of the uncertainty set \( U \). Simple-shaped sets like polyhedra or ellipsoids induce computationally tractability and allow one to deal with much larger dimensions than in the case of probabilistic constraints. Moreover, when choosing \( U \) such that \( \mathbb{P}(\xi \in U) = p \), then the feasible set of decision variables \( x \) of (3) is contained in that of probabilistic programming (2), so that an optimal solution to (3) is a feasible solution to (2). For these two reasons, robust optimization is not only preferred in the absence of statistical data, but also as a conservative approximation of the probabilistic programming setting. This conservatism, however, may be significant up to the point of ending up at very small or even empty feasible sets possibly coming at much higher costs than under a probabilistic constraints. This tradeoff propels the use of probabilistic constraints in the presence of statistical information at least in moderate dimension.

Although these approaches, probabilistic programming and robust optimization are often dealt with separately, in many applications, one is faced with uncertain variables of both mentioned types. This leads us naturally to the consideration of uncertain inequalities (2) in which the uncertain variable has a stochastic and a non-stochastic part, i.e., \( z = (\xi, u) \). A typical example is optimization of gas transport in the presence of uncertain loads, for which historical data are available in many situations (stochastic uncertainty) and of uncertain roughness coefficients in pipes, for which observations can hardly be accessed (non-stochastic uncertainty). Therefore, it seems natural, to combine the originally separate models (2) and (3). An appropriate way to do so is to formulate a probabilistic constraint (w.r.t. \( \xi \)) involving a robustified (w.r.t. \( u \)) infinite inequality system:

\[
\mathbb{P}(g(x, \xi, u) \geq 0 \quad \forall u \in U) \geq p.
\]  

(4)

This class of constraints has been recently investigated in [23] under the name of hybrid robust/chance-constraints and in [11] under the name of probabilistic/robust constraints. For easier reference, we shall be using in the following the natural acronym of probust constraints. We note that even the more complex
situation of the uncertainty set depending on the decision and random variable plays an increasing role in applications. Here, the constraint becomes

$$\mathbb{P}(g(x, \xi, u) \geq 0 \ \forall u \in U(x, \xi)) \geq p$$

(5)

where the inner part resembles constraint sets arising in generalized semi-infinite optimization [24].

We note that yet another form of combining the probabilistic and robust parts of the inequality system would result from interchanging their arrangements in (4):

$$\mathbb{P}(g(x, \xi, u) \geq 0) \geq p \ \forall u \in U$$

In this way one does not arrive at a probabilistic constraint involving infinitely many random inequalities as in (4) but rather at an infinite system of probabilistic constraints. This setting is related to (robust) first-order stochastic dominance constraints [7] and to distributionally robust probabilistic constraints [25]. We won’t deal with this model here but rather focus our considerations on (4) and (5), respectively.

The aim of this paper is to illustrate the application of this new class of robust constraints to optimization problems in gas transport under uncertainty. For a recent monograph on gas transport optimization we refer to [19]. We will consider first the problem of maximizing free booked capacities in an algebraic model for a stationary gas network. The corresponding model is presented in Section 2. This overall problem is too complex, however, to be dealt within this paper. Therefore we will split it into two subproblems, namely capacity maximization at exits (consumer side) which is discussed in Section 3 and capacity maximization at entries (provider side) which is analyzed in Section 4. Without loss of generality, we follow the convention of entry loads being non positive and exit loads being non negative.

In contrast to the first two parts of the paper, where stationary problems are studied, in the second application in Section 5, we consider a transient system. The transient system is governed by the wave equation under probabilistic Dirichlet boundary data at one end of the space interval and Neumann velocity feedback at the other end of the space interval. For the system a desired stationary state is given. The robustness of the system is measured by the $L^\infty$-norm of the difference between the actual state $\tilde{v}$ and the desired state $\bar{v}$. Due to the definition of the $L^\infty$-norm (as the essential supremum of the absolute value), this approach gives information about the maximal pointwise distance in space and time. Since our solutions are in fact continuous for appropriate initial and boundary data, the $L^\infty$-norm is equal to the maximum norm. The robustness requirement is that this pointwise distance remains below a given upper bound $v_{\text{max}}$. In our system, the state depends on a uncertain boundary input with a given distribution. The probabilistic constraint requires that the probability that the robustness requirement is satisfied is sufficiently large, i.e. greater or equal to a given value $p \in (0, 1]$. Our probabilistic constraint can easily be written in the notation of (4); for details see Section 5.
2 Maximization of free capacities in a stationary network

We consider a passive stationary gas network given by a directed graph $G = (V, E)$ with a set $V$ of vertices and a set $E$ of edges. We shall assume that the set of nodes decomposes into a set $V_+$ of entries, where gas is injected and a set $V_-$ of exits, where gas is withdrawn. Hence, $V = V_+ \cup V_-$ and $V_+ \cap V_- = \emptyset$. Without loss of generality we label the nodes in a way that entries come first and exits last. The gas transport along the network is triggered by bilateral delivery contracts between traders who inject gas at entries and traders covering customer demands by withdrawing gas at exits. An amount of gas injected into or withdrawn from the network at certain nodes is called a nomination. We shall refer to $b$ and $\xi$ as to the vectors of nominations at entries and exits, respectively.

Nominations have to satisfy three conditions:

1. At each node (entry or exit) of the network, nominations must not exceed the capacity booked for that node by the respective trader.
2. Nominations must be balanced over the whole network, i.e., the sum of nominations at entries equals the sum of nominations at exits.
3. Nominations must be technically feasible in the sense that there exist pressures within given bounds at the nodes and a flow through the network such that the nominations at the exits can be served by the nominations at the entries.

The first condition has to be satisfied by the traders. Referring to $C_+$ and $C_-$ as to the vectors of booked capacities at entries and exits, respectively, it can be written as

$$b \in [-C_+, 0], \; \xi \in [0, C_-],$$

where the intervals are to be understood in a multidimensional sense. The second condition is an automatic consequence of the ensemble of bilateral delivery contracts between entries and exits and can be written as

$$1^T_+ b + 1^T_- \xi = 0,$$

where $1_+$ and $1_-$ are vectors filled with entries 1 of the respective dimension of entries and exits.

The third condition of technical feasibility of some joint nomination vector $(b, \xi)$ can be characterized by the existence of vectors $q$ of flows along the edges of the network and $\pi$ of pressure squares at the nodes satisfying the conditions

$$\mathcal{A}q = \begin{pmatrix} b \\ \xi \end{pmatrix}; \; \mathcal{A}^T \pi = -\Phi q \; |q|; \; \pi \in [\pi^*, \pi^*].$$

Here, $\mathcal{A}$ is the incidence matrix of the network graph, $\Phi := \text{diag}((\Phi_e))_{e \in E}$ is a diagonal matrix of roughness coefficients and the modulus sign for a vector has to be understood componentwise. The first two equations in (8) correspond to the first two Kirchhoff’s Laws (mass conservation and pressure drop), whereas
the interval condition imposes bounds on the pressure. It is actually these bounds that constrain the feasibility of nominations \( b, \xi \), see, e.g. [19].

It is the network owners responsibility to make sure - without knowledge of concrete bilateral delivery contracts between entries and exits - that condition 3. is satisfied for all nominations fulfilling conditions 1. and 2. This requirement clearly imposes a constraint on the booked capacities \( C_+, C_- \) via (6). It can be formally written as:

\[
\forall (b, \xi) : (6), (7) \exists (q, \pi) : (8) \tag{9}
\]

Satisfying (9) in a rigorous way would yield (too) small booking capacities at the nodes of the network. Here, the network owner can benefit from the fact that nominations at the exits (gas withdrawn for consumption) are endowed with a typically large historical data basis so that they can be modeled as random vectors obeying some appropriate multivariate distribution. This offers the possibility to relax the 'for all' condition on \( \xi \) in a probabilistic sense as to satisfy (9) with sufficiently high probability \( p \). In this way, by choosing \( p \) close to one, it is possible to keep a robust satisfaction of technical feasibility while allowing for considerably larger booked capacities. A similar probabilistic modeling of entry nominations would not be justified (although historical data might be available here too) because their outcome is market driven and thus not a genuine random vector.

This motivates the network owner to relax the worst case condition in a probabilistic sense on the side of exits but keeping it on the side of entries. He then arrives at the following probabilistic formulation of feasible booked capacities \( C_+, C_- \):

\[
P (\xi \in [0, C_-], \forall b \in [-C_+, 0] : \mathbf{1}_+^T b + \mathbf{1}_-^T \xi = 0 \ \exists (q, \pi) : (8)) \geq p. \tag{10}
\]

Here, \( P \) refers to a probability measure related with the random vector \( \xi \) and \( p \in (0, 1] \) is a desired probability level chosen by the network owner. The expression on the left-hand side of this inequality provides the probability that a random exit nomination (within booked capacity) combined with an arbitrary entry nomination (within booked capacity and in balance with the exit nomination) is technically feasible.

Now, for a given capacity vector \((C_+, C_-)\) it may turn out that the associated probability on the left-hand side of (10) is larger than the desired minimum probability \( p \), e.g. 0.96 vs. 0.9. This would give the network owner the opportunity to offer larger capacities while still keeping the desired probability \( p \). Therefore, he might be led to determine the largest possible additional capacities \((x_+, x_-)\) he could offer for booking by new clients. This would lead
to the following optimization problem:
\[
\max w_+^T x_+ + w_-^T x_- \quad (11)
\]
\[
\mathbb{P}
\left( \begin{array}{c}
\xi \in [0,C_-] \quad \forall y \in [0,x_-], \quad \forall b \in [-C_+ - x_+,0] : 1^T b + 1^T \xi + 1^T y = 0 \\
\exists (q,\pi) : Aq = \begin{pmatrix} b \\
\xi + y \end{pmatrix} ; \quad A^T \pi = -\Phi q \mid q ; \quad \pi \in [\pi^*,\pi^*] \\
\end{array} \right) \geq p
\]
Here, the weight vector \(w\) in the objective reflects some preferences the network owner could have in order to offer new booking capacities at different nodes. In the absence of preferences, he could simply choose \(w := 1\). Note, that the nomination vector at exits has been split into \(\xi\) and \(y\), where \(\xi\) refers to the nominations of already existing clients (thus endowed with historical data and amenable to stochastic modeling) while \(y\) refers to nominations of potentially new clients without nomination history. As these can therefore not be treated stochastically, they are considered with a 'for all' requirement similar to entry nominations. No such splitting is necessary on the side of entries because nominations have to be considered there with a 'for all' requirement anyway as they cannot be modeled stochastically in a straightforward manner. In the following section, we shall address in detail the capacity maximization problem for exits only, a restriction which allows us numerically to solve the arising entire optimization problem subject to robust constraints. In contrast, Section 4 will focus on entries only and discuss essential issues related with the solution of this alternatively restricted optimization problem.

3 Maximization of booked capacities for exits in a stationary gas network

As mentioned in the introduction, the overall problem of capacity maximization (11) is too complex to be dealt with here. Therefore, we shall focus in a first step on maximizing capacities at exits.

3.1 The capacity maximization problem for several exits and one entry

In the following we will make the assumption that the network is a tree and that there exists only one entry point serving \(m\) exits (for an example see Fig 1 below). The presence of cycles in the network would significantly complicate the numerical solution of the problem. Nonetheless, in Section 3.4, we sketch a possible methodology in the presence of cycles. The restriction to a single entry is made here, in order not to deal with the robust uncertainty related with the splitting of nominations within several entry nodes (see ‘\(\forall b \in \ldots\)’ condition in (11)) which will be considered later in Section 4 separately. Without loss of generality, we define the entry to be the root of the network labeled by index
For simplicity, we assume moreover that the booked capacity $C_+$ of the entry is large enough to meet the maximum possible load by all exits as well as possible extensions thereof after adding additional capacity at the exits as a result of an optimization problem. As a consequence of this constellation our general capacity maximization problem (11) reduces to an exit capacity maximization problem of the form

$$\max w^T x_+$$

$$\mathbb{P}(\xi \in [0, C_-] \forall y \in [0, x_-] \exists (q, \pi) :$$

$$Aq = \begin{pmatrix} -1^T \xi - 1^T y \\ \xi + y \end{pmatrix} ; A^T \pi = -\phi q |q| ; \pi \in [\pi^*, \pi^*]$$

$$\geq p \) .$$

Here, the remaining decision variables are just the extensions of exit capacities. Since no capacity extension for the single entry is intended and since its existing capacity is not constraining by our assumption, the corresponding constraint disappears as well as the balance equation which is just substituted in the description of technical feasibility. The resulting optimization problem does no longer contain entry nominations at all but only random exit nominations $\xi$ and deterministic exit nominations $y$ of new clients along with the additionally allocated booking capacities $x_-$. 

Clearly, the probabilistic constraint in (12) is not yet in the explicit form of the probust constraint (4). This can be achieved in our case thanks to the network being a tree having the single entry as its root. According to [10], a vector $z$ of exit loads in this configuration is technically feasible, if and only if in the notation introduced above, the inequality system

$$g_{k,l}(z) := h_k(z) + \pi_k - h_l(z) - \pi_{*,l} \geq 0 \quad (k, l = 0, \ldots, m)$$

(13)

is satisfied, where

$$h_k(z) := \begin{cases} \sum_{e \in H(k)} \phi_e \left( \sum_{e' \geq H(e)} z_{e'} \right)^2 & \text{if } k \geq 1 \\ 0 & \text{if } k = 0 \end{cases}.$$  

(14)

In order to explain the definition of functions $h_k$ above, we denote $k \geq l$ for $k, l \in V$ if the unique directed path from the root to $k$, denoted $H(k)$, passes through $l$. $H(e)$ refers to the head of the (directed) arc $e \in E$.

With these specifications, which are fully explicit in terms of the initial data of the problem, we reformulate problem (12) with the aid of inequalities (13) as

$$\max w^T x_-$$

$$\mathbb{P}(\xi \in [0, C_-] \forall y \in [0, x_-] ; \forall k, l = 0, \ldots, m) \geq p \) .$$

(15)
is satisfied with that \( x \). The meaning of this constraint is as follows: The capacity extension \( x_- \) is feasible if and only if with probability larger than \( p \in (0, 1] \) the sum \( \xi + y \) of the original random nomination vector and of a new nomination vector can be technically realized for every such new nomination vector in the limits \([0, x_-]\). Clearly, the probust constraint (15) is of the form (5), with \( u := y, x := x_- \) and the uncertainty set \( U(x) := [0, x] \).

In [17] it is shown, that the infinite random inequality system
\[
g_{k,l}(\xi + y) \geq 0 \quad \forall y \in [0, x]; \quad \forall k, l = 0, \ldots, m
\]
inside (15) can be reduced - using (13) and (14) - to the following finite one
\[
\sum_{e \in \Pi(k) \setminus \Pi(l)} \Phi_e \left( \sum_{t \geq H(e)} \xi_t \right)^2 - \sum_{e \in \Pi(l) \setminus \Pi(k)} \Phi_e \left( \sum_{t \geq H(e)} \xi_t + (x_-)_t \right)^2 \geq \\
\pi_{*,l} - \pi_{*,k}; \quad \forall k, l = 0, \ldots, m
\]
(16)

For the random vector \( \xi \) of stochastic exit nominations we will suppose that it follows a truncated multivariate Gaussian distribution:
\[
\xi \sim \mathcal{T}\mathcal{N}(\mu, \Sigma, [0, C_-]).
\]

More precisely, the distribution of \( \xi \) is obtained by truncating an \( m \)-dimensional Gaussian distribution with mean \( \mu \) and covariance matrix \( \Sigma \) to an \( m \)-dimensional rectangle \([0, C_-]\) representing the (historical) booked capacity at exit node \( i \).

By definition of truncation, this means that there exists a Gaussian random vector \( \tilde{\xi} \sim \mathcal{N}(\mu, \Sigma) \) such that
\[
P(\xi \in A) = \frac{P(\tilde{\xi} \in A \cap [0, C_-])}{P(\tilde{\xi} \in [0, C_-])}
\]
holds true for all Borel measurable subsets \( A \subseteq \mathbb{R}^m \). Hence, in order to determine probabilities under a truncated Gaussian distribution, it is sufficient to be able to determine probabilities under a Gaussian distribution itself. Applying this observation to the probabilistic constraint (15) and combining it with (16) yields the equivalent description
\[
P(\tilde{\xi} \in [0, C_-] : \\
\sum_{e \in \Pi(k) \setminus \Pi(l)} \Phi_e \left( \sum_{t \geq H(e)} \tilde{\xi}_t \right)^2 - \sum_{e \in \Pi(l) \setminus \Pi(k)} \Phi_e \left( \sum_{t \geq H(e)} \tilde{\xi}_t + (x_-)_t \right)^2 \geq \\
\pi_{*,l} - \pi_{*,k}; \quad \forall k, l = 0, \ldots, m) \geq p \cdot P(\tilde{\xi} \in [0, C_-]).
\]
(17)

This is now, in contrast to (15) a conventional probabilistic constraint over a finite inequality system. In order to deal algorithmically with the probabilistic constraint (17), one has evidently to be able to calculate for each fixed decision vector \( x_- \) the probabilities occurring there, as well as their derivatives with respect to \( x_- \). In the following section we briefly sketch the methodology used here.
3.2 Spheric-Radial decomposition of Gaussian random vectors

From the well-known spheric-radial decomposition (see, e.g., [9]) of a Gaussian random vector \( \xi \sim N(\mu, \Sigma) \) it follows that the probability of an arbitrary Borel measurable subset \( M \) of \( \mathbb{R}^m \) may be represented as the following integral over the unit sphere \( S^{m-1} \):

\[
P(\hat{\xi} \in M) = \int_{v \in S^{m-1}} \mu_\chi(E(v)) \, d\mu_\eta(v).
\]

Here, \( \mu_\chi \) refers to the one-dimensional Chi-distribution with \( m \) degrees of freedom, \( \mu_\eta \) is the uniform distribution on \( S^{m-1} \) and

\[
E(v) := \{ r \geq 0 | \mu + rPv \in M \},
\]

where \( P \) is a factor from a decomposition \( \Sigma = PP^T \) of the covariance matrix \( \Sigma \). Following these remarks, the probability on the left-hand side of (17) (depending also on the decision variable \( x_- \)) can be represented as

\[
\int_{v \in S^{m-1}} \mu_\chi(E(v, x_-)) \, d\mu_\eta(v),
\]

where

\[
E(v, x_-) = \{ r \geq 0 | \mu + rPv \in [0, C_-] \} \cap \bigcap_{k,l=0,\ldots,m} E^{k,l}(v, x_-)
\]

and, with \( P_t \) denoting row no. \( t \) of \( P \), for \( k, l = 0, \ldots, m \):

\[
E^{k,l}(v, x_-) := \{ r \in \mathbb{R} | \sum_{e \in \Pi(k) \setminus \Pi(l)} \Phi_e \left( \sum_{t \geq h(e)} \mu_t + rP_tv \right)^2 \geq \pi^*_k - \pi^*_l \}.
\]

In order to evaluate the integrand in (18), one has to be able to characterize (for each given \( v \in S^{m-1} \) and \( x_- \in \mathbb{R}^m \)) the set \( E(v, x_-) \) and to determine its Chi probability. Using some elementary calculus which we omit here, the set \( E(v, x_-) \) can be represented as a finite union of intervals. Then, using available numerically highly precise approximations of the one dimensional Chi distribution function, the desired Chi probability can be determined by summing up the corresponding Chi probabilities of the single intervals. It is important to note that, at the same time, the partial derivatives of the probability with respect to the decision variable \( x_- \) can be calculated as a spherical integral of type (18) again, however with a different integrand which is easily obtained from the partial derivatives of the initial data [22]. In these gradient formulae, the same disjoint union of intervals as in the computation of the probability itself is employed. The spherical integrals can be approximated.
by finite sums using Quasi-Monte Carlo sampling on the sphere (see, e.g., [4]). Then, for each sampled direction \( v \) on the sphere, one may update first the probability itself and then, simultaneously, the gradient of the probability with respect to \( x_\theta \) by using the same disjoint union of intervals in both cases. This approach makes the gradient come almost for free as far as computation time is concerned. Having access to values and gradients of the probabilistic constraint (17), one may set up an appropriate nonlinear optimization solver for solving (15). For the subsequent numerical results, we employed a simple projected gradient method.

3.3 Numerical results for an example

As an illustrating example we considered a network as displayed in Fig. 1 with one entry (filled black circle) and 26 exits. The parameters of the network (i.e., pressure bounds, roughness coefficients, truncated Gaussian distribution for the random nominations at exits) were chosen in nearly realistic quantities. We didn’t assume any preferences in the allocation of new capacities, hence the weight vector in the objective of (15) was chosen as \( w_- := 1_- \). The colored rings around exit points refer to the optimal cumulative capacities (historical+new), i.e. \( C_- + x_- \) after maximization, upon choosing probability levels \( p = 0.95, 0.9, 0.85, 0.8 \). It can be clearly seen, how decreasing of the probability level allows for increasing the allocation of capacity in certain regions of the network.

Fig. 2 illustrates how the computed solution for a probability level \( p = 0.8 \) works for two random exit nomination scenarios \( \xi \) simulated a posteriori according to the chosen truncated multivariate Gaussian distribution. The first scenario is feasible because one could uniformly add a common capacity to every exit (green color) in order to satisfy this scenario. In contrast, the second scenario is infeasible because one would have to (uniformly) reduce the capacities by an amount corresponding to the dark red rings, in order to satisfy this scenario. When simulating a large set of such scenarios, say 1000, it would turn out that according to the probability level \( p = 0.8 \) approximately 800 are feasible, while 200 are infeasible.

3.4 Methodology in the presence of cycles

So far, we have assumed that the gas network is a tree. In general, one is faced, however, with the presence of a certain number of fundamental cycles in the network. In the following, we sketch, how to calculate the probability of feasible nominations in a gas network. When compared to the situation of a tree, the difficulty consists in the presence of an additional equation system, whose solution has to be determined first, before the probability can be calculated.
As mentioned above, the algorithm using spheric-radial decomposition requires the computation of the set

\[ E_i := E(v_i) = \{ r \geq 0 | \mu + rPv_i \in M \} \]

for every sample \( v_1, \ldots, v_s \) on the sphere. Since the set \( E_i \) can be expressed as a finite union of disjoint intervals, \( E_i = \bigcup_{j=1}^{l_i} [a_j, b_j] \), it is sufficient to determine the points where the ray \( rPv_i + \mu \) enters and exits the set of feasible load vectors \( M \).

Let \( A \) decompose into a basis part \( A_B \) and a non-basis part \( A_N \) and let \( q_B \), \( q_N \) and \( \Phi_B \), \( \Phi_N \) be according partitions of \( q \) and \( \Phi \). In [10] it is proven that a
 Optimization problems with robust constraints

Fig. 2: Two scenarios for random exit loads $\xi$ according to the chosen multivariate truncated Gaussian distribution. Left: feasible scenario; Right: infeasible scenario.

given load $(-1^T \xi, \xi)$ is feasible iff there exists a $q_N$ such that

\[
\begin{align*}
A_N^T g(\xi, q_N) &= \Phi_N|q_N|q_N \\
\min_{i=1, \ldots, |V|-1} [\pi_i^* + g_i(\xi, q_N)] &\geq \max_{i=1, \ldots, |V|-1} [\pi_i^* + g_i(\xi, q_N)] \\
\pi_{*,0} &\leq \min_{i=1, \ldots, |V|-1} [\pi_i^* + g_i(\xi, q_N)] \\
\pi_0^* &\geq \max_{i=1, \ldots, |V|-1} [\pi_i^* + g_i(\xi, q_N)]
\end{align*}
\]

(21)\hspace{1cm} (22)\hspace{1cm} (23)\hspace{1cm} (24)

with

\[
g: \mathbb{R}^{|V|-1} \times \mathbb{R}^{|N|} \to \mathbb{R}^{|V|-1}, \quad (s, t) \mapsto (A_B^{-1})^T \Phi_B |A_B^{-1}(s-A_N t)| (A_B^{-1}(s-A_N t)).
\]

Inserting $\xi(r) := rPv_i + \mu$ into this characterization of feasible loads and comparing the terms in the minima and maxima separately, $E_i$ consists of all $r \in \mathbb{R}_{\geq 0}$ for which there is a $q_N$ such that

\[
A_N^T g(rPv_i + \mu, q_N) = \Phi_N|q_N|q_N
\]

\[
(\pi)^* j + g_j(rPv_i + \mu, q_N) \geq (\pi^* k) + g_k(rPv_i + \mu, q_N) \quad \text{for all } j, k = 1, \ldots, |V|-1, j \neq k
\]

\[
\pi_{*,0} \leq \pi^* j + g_j(rPv_i + \mu, q_N) \quad \text{for all } j = 1, \ldots, |V|-1
\]

\[
\pi_0^* \geq \pi^* j + g_j(rPv_i + \mu, q_N) \quad \text{for all } j = 1, \ldots, |V|-1.
\]

Introducing a map $D: \mathbb{R}^{|V|-1} \times \mathbb{R}^{|N|} \to \mathbb{R}^{|V|-1}(|V|-2+2(|V|-1))$ this is equivalent to

\[
A_N^T g(rPv_i + \mu, q_N) = \Phi_N|q_N|q_N
\]

\[
D(rPv_i + \mu, q_N) \leq 0.
\]
and thus, to determine the end points of all of the subintervals of \( E_i \) one has to solve
\[
A_N^T g(rPv_i + \mu, q_N) = \Phi_N|q_N|q_N
\]
\[
D_j (rPv_i + \mu, q_N) = 0
\]
\[
D_k (rPv_i + \mu, q_N) \leq 0 \quad \forall k = 1, \ldots, (|V| - 1)(|V| - 2) + 2(|V| - 1), \ k \neq j
\]
for each \( j = 1, \ldots, (|V| - 1)(|V| - 2) + 2(|V| - 1) \).

In each step, a system with the same number of equations and indeterminates has to be solved. If the absolute values in these equations are resolved, the equations become polynomial ones, and hence amenable to the application of techniques from computer algebra.

To this end, consider two sets \( f_1, \ldots, f_s \in \mathbb{k}[x_1, \ldots, x_n] \) and \( g_1, \ldots, g_t \in \mathbb{k}[x_1, \ldots, x_n] \) of polynomials generating the same ideal. Then the sets of solutions of the systems
\[
f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0
\]
and
\[
g_1(x_1, \ldots, x_n) = 0, \ldots, g_t(x_1, \ldots, x_n) = 0
\]
coincide. So the aim is to find a basis for the ideal generated by the polynomials from the original system that is much easier to solve than the original system. Such a basis is, for example, the reduced Gröbner basis w.r.t. the lexicographic monomial ordering. This basis yields a system that is in triangular form so this approach is comparable to the Gauss elimination for systems of linear equations. For an introduction to Gröbner bases see for example [6] or [21]. Moreover, if the ideal generated by the polynomials describing the system fulfills some additional assumptions, that is the ideal is 0-dimensional and radical, and there is an indeterminate in that all complex roots are different, then the reduced Gröbner basis w.r.t. the lexicographic monomial ordering has the structure (Shape Lemma)
\[
x_1 - \text{poly}_1 (x_n)
\]
\[
x_2 - \text{poly}_2 (x_n)
\]
\[
\vdots
\]
\[
x_{n-1} - \text{poly}_{n-1} (x_n)
\]
\[
\text{poly}_n (x_n),
\]
where \( \text{poly}_n \) is a univariate polynomial of degree \( d \) an the remaining polynomials \( \text{poly} \), have degree \( \leq d - 1 \), see, e.g., [2], [21].

In addition to the fact that Gröbner bases yield systems that are much easier to solve than the original system, it is very easy to determine if a system does not have a solution, because then the reduced Gröbner basis consists only of the constant polynomial 1.

With this method, the number of systems to be solved for each sample is \((|V| - 1)(|V| - 2) + 2(|V| - 1)\) times the number of arcs that are contained in at
least one cycle. To reduce this huge number, we used two preprocessings. First, since the resolvings of the absolute values correspond to the flow direction along the pipes, one possibility is to fix the flow direction along as many pipes as possible. The second possibility is to find redundant pressure bounds. These preprocessings are discussed in more detail in [5]. Moreover, [5] contains some numerical results on the computation of the probability of exit loads to be feasible in networks containing cycles.

4 Capacity maximization under uncertain loads and uncertain distribution of entry nominations

After discussing the methodology for the case of uncertain exit loads, we address the case of uncertain entry loads. Instead of solving the full capacity problem, we focus on the probust constraint of Problem (11) and decide whether it is satisfied with a given entry capacity extension. This can be seen as an intermediate step towards the solution of the capacity problem for which also gradient information for the probust constraint would be required, see e.g. Section 3.2.

In the following, this task is formally introduced as a decision problem. Analogously to Section 3, the spheric radial decomposition is applied to check whether the probust constraint is satisfied. Let \( m \) be the number of exit nodes, \( v \in S^{m-1} \) be a sampled vector, \( \mu \in \mathbb{R}^{m-1} \) be the mean of the probability distribution of the exit loads and \( P \) is obtained from the covariance matrix of the probability distribution. As discussed in Section 3.2, we need to determine the length of the intersection of the rays \( \mu + rPv \) with the set of robust feasible loads \( M \), i.e. the set

\[
E(v) := \{ r \in \mathbb{R}_{\geq 0} \mid \mu + rPv \in M \}.
\]

This results in a finite union of intervals \( E(v) = \bigcup_{j=0}^{k} E_j(v) \). We denote \( E_0(v) := [0, E_0(v)^*] \). The Chi probability of \( E(v) \) can be bounded from below by computing a lower bound for \( E_0(v)^* \). In order to compute it, we apply a problem specific binary search which in every iteration decides whether a given number is an element of \( E_0(v) \). In theory, this decision is made by solving a nonlinear optimization problem (NLP) which checks if, for a given realization of (squared) pressure and flow values, all pressures lie within prescribed bounds. In order to make sure that all potential violations of pressure bounds are detected, globally optimal solutions are required. In order to achieve this, we linearize the NLPs by the incremental method. In particular, we apply a piecewise relaxation method as developed in [8] that is known to be very efficient in practice for solving large instances from gas network operation. As binary variables are introduced for the piecewise relaxation, a Mixed-Integer Linear Problem (MIP) is solved instead of the original NLP.

In the remainder of this section, we present the developed algorithm that yields lower bounds on the value of \( E_0(v)^* \). We prove its correctness under the assumption that there exists a node with fixed pressure. This is not a critical
assumption in reality, since gas is injected at some entry node and we can assume that the pressure at this node is known. We conclude this section by the presentation of computational results that show the effectiveness of the method.

4.1 Methodology for general stationary networks

Analogously to Section 3, we assume that the capacities at the exits are large enough to meet the maximum possible loads by all entries. As introduced in Section 2, we assume that the exit loads follow some probability distribution, as enough historical data is available. In constrast, for the entry loads, data is less accessible which means that we cannot rely on a method that is purely probabilistic. Instead, we will develop a robust approach. Furthermore, instead of solving optimization problem (11), we solve the following decision problem: For a given exit load $\xi$, we have to guarantee the feasibility of every possible entry load $b$ with a probability of at least $p$. Since we have to take extensions $x_+$ of the entry capacities $C_+$ into account, the possible entry nominations, i.e., the uncertainty set, is

$$U(\xi, x_+) := \{b \in [-C_+ - x_+, 0] : 1^T_x b + 1^T \xi = 0\}. \quad (25)$$

The robust constraint of problem (11) is then

$$P \left( \xi \geq 0 : \forall b \in U(\xi, x_+) \exists (q, \pi) : Aq = \left( \begin{array}{c} b \\ \xi \end{array} \right), A^T \pi = -\Phi q \right) \geq p. \quad (26)$$

We introduce the following definition:

**Definition 1** Let an exit nomination $\xi \geq 0$ and a capacity extension $x_+ \geq 0$ be given. If for every $b \in U(\xi, x_+)$ there exists a flow $q$ and a pressure $\pi$, such that (8) is satisfied, $(\xi, x_+)$ is called robust feasible.

In other words, in the robust constraint (26) we want for $\xi \geq 0$ and for $x_+ \geq 0$ to ensure robust feasibility of $x_+$ with a probability of at least $p$. As $U(\xi, x_+)$ contains in general infinitely many points, answering robust feasibility for $(\xi, x_+)$ requires the solution of a semiinfinite feasibility problem already for fixed $x_+$. As this is already a challenge on its own, we decide whether $(\xi, x_+)$ is feasible for problem (11), i.e. we check whether constraint (26) is satisfied. This decision problem is denoted by DecProb$(\xi, x_+)$. In order to answer DecProb$(\xi, x_+)$, we develop an optimization problem and apply the spheric radial decomposition analogously to Section 3.2. Let $v \in S^{m-1}$, a capacity extension $x_+ \geq 0$ and $r \geq 0$ be given. Instead of analyzing the entire uncertainty set (25), we analyze the rays which arise in the spheric radial decomposition. Hence set (25) is additionally parametrized by $v$ and $r$ and substituted by

$$U(r, v, x_+) := \{b \in [-C_+ - x_+, 0] : 1^T_x b + 1^T \mu + rPv = 0\}. \quad (27)$$
It has been discussed in Section 3.2 that validating (26) for each ray \( \mu + rPv \) amounts to computing the set

\[
E(v,x+) := \{ r \geq 0 : \forall b \in U(r,v,x+) \exists (\pi,q) : \\
Aq = \begin{pmatrix} b \\ \mu + rPv \end{pmatrix}, \quad A^T \pi = -\Phi|q|q \},
\]

the set which contains all nonnegative \( r \) for which all entry nominations of the uncertainty set \( U(r,v,x+) \) are realizable. As explained in Section 3.2, we can write

\[
E(v,x+) = \bigcup_{j=0}^{t} E_j(v,x+),
\]

where \( E_j(v,x+) \) is an interval for \( j = 0, \ldots, t \).

In the following, we only take the first interval

\[
E_0(v,x+) := \left[0, E_0(v,x+)^* \right],
\]

where \( E_0(v,x+)^* \) denotes the upper bound of \( E_0(v,x+) \), into account. The Chi probability of \( E_0(v,x+)^* \) is a lower bound of the Chi probability of \( E(v,x+) \) which, consequently, results in a lower bound for the probability in (26).

Therefore, our validation of (26) is conservative: the determined lower bounds result in a Chi probability which is at most \( p \).

We note that in the case of \( M \) being star-shaped with respect to \( \mu \), \( E(v,x+) \) is an interval itself. In this case, our validation of (26) is less conservative as we do not neglect intervals and thus derive a lower bound of the Chi probability which is much closer to the probability. In many discussions, the set of robust feasible exit loads is assumed to be star-shaped. We point out that there exist examples of \( M \) not fulfilling this assumption.

Our goal is to bound \( E_0(v,x+)^* \) from below. To this end, we exploit that DecProb(\( \mu + rPv, x_+ \)) is always answered positively for a gas network if the pressure is unbounded at all nodes ([19], Theorem 7.1). The respective solutions \((\pi,q)\) are labeled pressure flow solutions. As in real gas network operations the pressures are bounded, we introduce so-called penalty functions for every \( u \in V \), in formulas,

\[
f_u : \mathbb{R} \rightarrow \mathbb{R}^+, \quad \pi_u \mapsto \max\{0, \pi^*_u - \pi_u, \pi_u - \pi^*_u\},
\]

If \( f_u(\pi_u) > 0 \) for a node \( u \in V \), \( \pi_u \) lies outside its bounds. Consequently, \( \pi \in [\pi^*_u, \pi^*_u] \) if and only if

\[
\sum_{u \in V} f_u(\pi_u) = 0. \tag{28}
\]

Therefore, for some \( R \in \mathbb{R}_{\geq 0} \) the statement \([0,R] \subset E_0(v,x+) \) is true, if, for every \( r \in [0,R] \) and for every \( b \in U(r,v,x+) \), the pressure flow solutions \((\pi,q)\) fulfill (28). Since \( 0 \in E_0(v,x+) \) and since \( E_0(v,x+) \) is an interval, \([0,R] \subset E_0(v,x+) \) if and only if \( R \in E_0(v,x+) \). Hence we only need to check \( R \in E_0(v,x+) \).

To decide whether (28) is true for some \((\pi,q)\) we introduce the optimization
problem
\[
\max \sum_{u \in V} f_u(\pi_u),
\]
\[
s.t. \quad Aq = \begin{pmatrix} b \\ \mu + rPv \end{pmatrix},
\]
\[
A^T \pi = -\Phi|q|,
\]
\[
1^T (\mu + rPv) + 1^T b = 0,
\]
\[
b \in [-C_+ - x_+, 0],
\]
\[
0 \leq r \leq R.
\]

Under the assumption of a fixed pressure at one node, we will show in the following lemma that \( R \in E_0(v, x_+) \) if and only if the optimal value of problem \( \text{Pen}(R, v, x_+) \) is zero.

**Theorem 1** Let \( v \in S^{n-1} \) and let \( x^+ \geq 0 \) be a capacity extension at the entries. Assume that the pressure at one node is fixed. Then \( R \in E_0(v, x_+) \) if and only if problem \( \text{Pen}(R, v, x_+) \) is solvable with optimal value 0.

**Proof** Since \( f_u(\pi_u) \) is nonnegative for all nodes \( u \in V \), the optimal value of \( \text{Pen}(R, v, x_+) \) is at least zero.

Assume on the one hand that \( R \in E_0 \). Hence, for an arbitrary choice of \( r \in [0, R] \) and for any entry nomination \( b \in U(r, v, x_+) \), there exist a flow vector \( q \) and a squared pressure vector \( \pi \) such that the nomination \( (b, \mu + rPv) \) is feasible. Consequently, \( \pi \in [\pi_*, \pi^*] \) and the objective value for \( (r, b, \pi, q) \) is 0. Due to the fact that the pressure is fixed at one node, the pressure flow solution is unique, see Theorem 7.2. in [19]. Hence the objective value is zero for all feasible solutions of \( \text{Pen}(R, v, x_+) \).

On the other hand, assume that the optimal value is zero. Therefore, for all \( b \in U(r, v, x_+) \) there exist a flow \( q \) and a squared pressure \( \pi \), such that \( (r, b, \pi, q) \) is feasible for \( \text{Pen}(R, v, x_+) \). Since the optimal value is zero, \( \pi \in [\pi_*, \pi^*] \). Hence the pressures lie within their demanded bounds and \( R \in E_0(v, x_+) \).

We note that the assumption of a fixed pressure is crucial in the proof of Theorem 1. Unless this assumption is not satisfied, the pressure values are not necessarily unique and the optimal value of problem \( \text{Pen}(R, v, x_+) \) can be infinity although \( R \in E_0(v, x_+) \).

Problem \( \text{Pen}(R, v, x_+) \) can be used to decide whether \( R \in E_0(v, x_+) \). Since \( E_0(v, x_+) \) is an interval, we can check whether constraint (26) is satisfied using a standard binary search which solves the subproblem \( \text{Pen}(R, v, x_+) \) in every iteration.

A binary search algorithm starts with a lower and an upper bound. Since \( 0 \in E(v, x_+) \subset \mathbb{R}_{\geq 0} \), we choose zero as a lower bound. A trivial upper bound for \( E(v, x_+) \) is given by the exit capacities

\[
0 \leq \mu + rPv \leq C_-.
\]
A tighter bound is given as follows. Due to (8) for an edge \((i, j) \in E\), the pressure drop constraint

\[
\pi_i - \pi_j = \Phi_{i,j} |q_{i,j}|
\]

holds. Since the right hand side of (29) is invertible and the pressures are bounded, we can derive flow bounds for every edge which do not depend on the actual nomination. In the following, these flow bounds, which are called *implicit flow bounds* and are denoted by \(q^*\) and \(q^+\), are exploited to determine an improved upper bound for \(E(v, x_+):\)

**Lemma 1** Let \(v \in \mathbb{S}^n-1\) and \(x_+ \geq 0\) be a capacity extension at the entry nodes. For a node \(u \in V\), let \(\delta^+(u)\) denote the set of incoming arcs and let \(\delta^-(u)\) denote the set of outgoing arcs. Then an upper bound for \(E(v, x_+):\)

\[
\text{max } r, \quad \text{s.t.} \quad 0 \leq \mu + rPv \leq C_-, \quad \sum_{e \in \delta^+(u)} q^+_e - \sum_{e \in \delta^-(u)} q^+_e \geq \mu_u + r(Pv)_u \quad \forall u \in V_-, \quad (UB(v, x_+))
\]

\[
\sum_{e \in \delta^+(u)} q^+_e - \sum_{e \in \delta^-(u)} q^+_e \leq \mu_u + r(Pv)_u \quad \forall u \in V_-,
\]

\[r \geq 0.\]

**Proof** Since we are interested in an upper bound for \(E(v, x_+):\), \(0 \leq \mu + rPv \leq C_-\) and \(r \geq 0\) are satisfied. Furthermore, Kirchoff’s first law demands

\[
\sum_{e \in \delta^+(u)} q_e - \sum_{e \in \delta^-(u)} q_e = \mu_u + r(Pv)_u \quad \forall u \in V_.
\]

for a flow \(q\). Relaxing this equation by substituting the flow variables by the implicit flow bounds \(q^*\) and \(q^+\) results in

\[
\sum_{e \in \delta^+(u)} q^+_e - \sum_{e \in \delta^-(u)} q^+_e \geq \mu_u + r(Pv)_u \quad \forall u \in V_-
\]

and

\[
\sum_{e \in \delta^+(u)} q^+_e - \sum_{e \in \delta^-(u)} q^+_e \leq \mu_u + r(Pv)_u \quad \forall u \in V_-
\]

This concludes the proof.

Algorithm 1 summarizes the procedure for bounding \(E_0(v, x_+)^*\) from below. In every iteration of Algorithm 1, \((\text{Pen}(R, v, x_+))\) is solved for a given \(R\). With a given tolerance \(\text{tol}\), Algorithm 1 bounds \(E_0(v, x_+)^*\) from below with an error of at most \(\text{tol}\). Due to Theorem 1 and Lemma 1, Algorithm 1 terminates with a correct lower bound. Since the algorithm applies binary search, its number of iterations is \(\lfloor \log_2(\frac{\text{tol}}{0.1}) \rfloor \).
Algorithm 1 Finding a lower bound for $E_0(v, x_+)^*$ through Bisection

Input: Sphere vector $v \in \mathbb{R}^{m-1}$, capacity $x_+ \geq 0$, tolerance tol > 0
Output: $R \in \mathbb{R}$, such that $E_0(v, x_+)^* - R < \text{tol}$

$l \leftarrow 0$

$z \leftarrow \text{Optimal value of (UB}(v, x_+))$.

while $|z - l| > \text{tol}$ do

$R \leftarrow \frac{z + l}{2}$

Solve (Pen$(R, v, x_+))$

if (Pen$(R, v, x_+))$ is infeasible then

$z \leftarrow R$

else

let $z^*$ be the optimal value of (Pen$(R, v, x_+))$

if $z^* = 0$ then

$l \leftarrow R$

else

$z \leftarrow R$

end if

end if

end while

$R \leftarrow l$

return $R$

In every iteration of Algorithm 1, $l$ denotes a lower bound for $E_0(v, x_+)^*$. Therefore, the lower bound for the variable $r$ in (Pen$(R, v, x_+))$ is set to $l$, since we do not have to check if $R \in E_0(v, x_+)$ twice.

In practice, Algorithm 1 is applied for every vector $v$ that is sampled in the spheric radial decomposition. Afterwards, the lower bounds for $E_0(v, x_+)^*$ for every $v$ are used to check the validity of (26). However, (Pen$(R, v, x_+))$ is a NLP that has to be optimized globally. In order to circumvent this problem in practice, we use piecewise relaxations of the nonlinear function

$$\Phi_{i,j}|q_{i,j}|q_{i,j}$$

for every $(i, j) \in E$. We apply the incremental method and substitute the functions (30) with linear splines. The applied procedure has been developed in [8]. It has been shown that this approach leads to a very effective method for gas network optimization problems that is able to solve very large instances in practice, see [19]. For a given linearization error $\epsilon > 0$, we construct a linear spline $s_{i,j}(q_{i,j})$, such that

$$s_{i,j}(q_{i,j}) - \epsilon \leq \Phi_{i,j}|q_{i,j}|q_{i,j} \leq s_{i,j}(q_{i,j}) + \epsilon$$

for every $e \in E$. Therefore, we substitute (29) with

$$s_{i,j}(q_{i,j}) - \epsilon \leq \pi_i - \pi_j \leq s_{i,j}(q_{i,j}) + \epsilon$$

for all $(i, j) \in E$. An example can be seen in Figure 3.

The splines $s_{i,j}(q_{i,j})$ are modeled with binary and continuous variables. Hence, the subproblem (UB$(v, x_+))$ is relaxed and becomes a MIP. This can be solved to global optimality with off-the-shelf software. For applying the incremental method, we require flow bounds. However, we
cannot use the flow bounds we derived from equation (29) since the pressure is not bounded in $\text{Pen}(R,v,x_+)$. Instead, we apply the preprocessing developed in [1] where the pressure bounds are neglected. Furthermore, due to (31) and (32), the optimal value of the linearized problem is an upper bound for the optimal value of the original problem. Therefore, if the optimal value of the linearized problem is zero, the optimal value of Problem $\text{Pen}(R,v,x_+)$ is zero as well. Due to Theorem 1 it follows that $R \in E_0(v,x_+)$. This concludes the presentation of the developed algorithm. In the next subsection, we evaluate Algorithm 1 with respect to running time and quality of the obtained solutions.

4.2 Numerical results

We modify the gas network instance from Section 3.2 by adding a second entry and fixing the pressure at a leaf node. Beyond that, we provide sphere vectors $v$ by sampling a collection of 10000 elements on the unit sphere using a Quasi-Monte Carlo method. Our goal is to calculate the probability of robust feasibility for this network and uncertain entry loads by using a spheric radial decomposition where the ray length is determined by Algorithm 1. The performance of the algorithm is investigated by benchmarking the algorithm on the given instance under a variety of parameter combinations.

All experiments were carried out using GUROBI 7.5 [16] with 4 threads running on machines with Xeon E3-1240 v5 CPUs (4 cores, 3.5 GHz each).

We apply Algorithm 1 to all 10000 rays using all combinations of relaxation parameters $\epsilon \in \{2^{-6}, 2^{-5}, \ldots, 2^4\}$ and bisection termination tolerances $\text{tol} \in \{0.001, 0.01, 0.1\}$. Experiments for smaller tolerances down to $\text{tol} = 10^{-6}$ were carried out as well but are omitted here since they produced almost identical probabilities, when compared to $\text{tol} = 10^{-3}$. The results of this study are
displayed in Figure 4. The probabilities for robust feasibility are displayed in Figure 4a. We determine the overall probability to be between 78% and 78.5%, depending on $\varepsilon$ and $\text{tol}$. As expected, we obtain more conservative solutions for increasing approximation parameters $\varepsilon$. However, the influence of $\varepsilon$ is much smaller than expected, even for large $\varepsilon$. In the same fashion, increasing the bisection termination tolerances $\text{tol}$ leads to more conservative solutions. We note that for both parameters, a combination of $\varepsilon = \frac{1}{2}$ and $\text{tol} = 0.001$ produces solutions that can be improved only very little (within the scope of this study) by decreasing both parameters further. The overall running times for all rays are plotted in Figure 4b. As expected, the running times increase for decreasing parameter $\varepsilon$, as the latter leads to more complex MIP models. A decrease in tolerances $\text{tol}$ leads to a larger number of iterations of the bisection algorithm and thus to longer running times as well.

Fig. 4: Resulting probability and total running time for 10 000 rays using different relaxation qualities $\varepsilon$ and bisection termination tolerances $\text{tol}$.

In the previous experiments, we focused on the influence of the relaxation parameter $\varepsilon$ and the bisection precision tolerance $\text{tol}$ on the algorithm's running time and on the reliability of the obtained probability. Another important impact on the overall running time is the number of rays that needs to be used. Figure 5a shows the resulting probability when only the first $k$ rays of the 10 000 given samples are used for the probability calculation. At a glance, we observe large fluctuations when using only up to about 2500 rays. A considerable decrease in the magnitude of the probability fluctuation can be seen for values of $k \geq 2500$. We further strengthen this observation in Figure 5b by comparing the first graph with a second graph that was obtained from 5000 other random sphere vectors in the same fashion. Since the second graph follows the same pattern, we conclude that for the instance considered here, the number of rays should not be smaller than 2500 if a reliable probability has to be calculated.
Under the assumption that the parameter choice \( k = 2500 \), \( \varepsilon = \frac{1}{2} \), and \( \text{tol} = 0.001 \) is sufficient for a reliable probability calculation, the total running time sums up to about 8 min. This demonstrates the feasibility of using MIP subproblems within the spheric radial decomposition for probability calculations. Moreover, as probability evaluations are possible within minutes, using the presented techniques as part of a projected gradient method like in Section 3 is within practical reach. An actual application as a part of such a method will be subject of research in the future.

![Probability plot](image1)
(a) Using up to 10,000 rays for the probability calculation.

![Probability plot](image2)
(b) Comparison between two collections of 5000 rays each.

This concludes the discussion and presentation of the methodology of stationary gas networks. In the next section, we discuss transient systems which are governed by the wave equation.

5 Stabilization with probabilistic constraints of a system governed by the wave equation

In this section we consider a transient system that is governed by the wave equation. The wave equation occurs as a linear model of the gas flow in gas pipelines for sufficiently small velocities. The state is determined by an initial boundary value problem with Dirichlet boundary data at one end and Neumann boundary feedback at the other end of the space interval. The initial data and the boundary data are given by a stochastic process. We consider the probability to stay near a desired state everywhere in the time space domain. In the introduction, we have pointed out that this problem yields a probabilistic/robust constraint of the form (4).

Let a finite length \( L > 0 \) and a finite time \( T > 0 \) be given. Let \( c > 0 \) denote the sound speed in the gas. Let a stationary velocity field \( \bar{v} \) be given, see [14].
Let \( v = \tilde{v} - \bar{v} \) denote the difference between the velocity and the stationary state. If the norm of \( \bar{v} \) is sufficiently small, the dynamics for \( v \) are governed by the wave equation
\[
v_{tt} = c^2 v_{xx}. \tag{W}
\]
Moreover the gas density \( \rho \) satisfies the wave equation \( \rho_{tt} = c^2 \rho_{xx} \) and the flow rate \( q \) of the gas satisfies the wave equation \( q_{tt} = c^2 q_{xx} \); see [15]. For given uncertain boundary data (that model the uncertain demand) \( \xi \in L^\infty(0, T) \), an initial state \((v_0, v_1) \in L^\infty(0, L) \times L^1(0, L) \) and a feedback parameter \( \eta > 0 \), we consider the closed loop system that is governed by the initial boundary value problem for \((t, x) \in [0, T] \times [0, L] \)
\[
\begin{cases}
  v(0, x) = v_0(x), & v_t(0, x) = v_1(x), \\
  v_x(t, 0) = \eta v_t(t, 0), & v(t, L) = \xi(t), \\
  v_{tt}(t, x) = c^2 v_{xx}(t, x).
\end{cases} \tag{S}
\]
An explicit representation of the generated state in terms of travelling waves (d’Alembert’s solution) is given in [12], [13]. This allows the computation of the system state \( v \in L^\infty((0, T) \times (0, L)) \) without discretization errors. In the operation of pipeline networks, there is a constraint on the magnitude of the flow velocity in the pipe. Let \( v_{\text{max}} > 0 \) be an upper bound for the velocity. We consider the probabilistic constraint for the probability
\[
\mathbb{P} \left( \| v \|_{L^\infty} \leq v_{\text{max}} \right) \tag{33}
\]
where \( v \) solves (S) and \( \| \cdot \|_{L^\infty} \) denotes the norm on \( L^\infty((0, T) \times (0, L)) \). In order to write our probabilistic constraint in the notation similar as in (4), we introduce the notation
\[
g(\tilde{v}, \xi, u) := v_{\text{max}} - |\tilde{v}(u) - \bar{v}(u)|,
\]
with \( \xi = (a, b), \ a = (a_k)_{k=1}^N, b = (b_k)_{k=1}^N, u = (t, x), \ U = [0, T] \times [0, L], \) where \( \tilde{v} \) solves the initial boundary value problem (S) with initial and boundary data that depend on the parameter \((a, b)\) (see (KL-id) and (KL-bd)).

**Theorem 2 (Solution of system (S))** Consider system (S) with \( \xi \in L^\infty((0, T)) \) and \((v_0, v_1) \in L^\infty(0, L) \times L^1(0, L) \) for the optimal feedback parameter \( \eta = \frac{1}{c} \). Define the antiderivative of \( v_1 \) by
\[
V_1(x) := \int_0^x v_1(s) \, ds
\]
and define for
\[
\alpha(s) := \begin{cases} 
  v_0(cs) + \frac{1}{c} V_1(cs) & \text{for } s \in [0, \frac{L}{c}], \\
  2\xi(s - \frac{L}{c}) - \beta(s - \frac{L}{c}) & \text{for } s \in \left[\frac{L}{c}, T + \frac{L}{c}\right].
\end{cases}
\]
and

\[
\beta(s) := \begin{cases} 
  v_0(L - cs) - \frac{1}{c} V_1(L - cs) & \text{for } s \in [0, \frac{L}{c}) \\
  v_0(0) & \text{for } s \in [\frac{L}{c}, T + \frac{L}{c}].
\end{cases}
\]

Then the function

\[
v(t, x) := \frac{1}{2} \alpha \left( t + \frac{c}{2} x \right) + \frac{1}{2} \beta \left( t + \frac{L-c}{c} x \right)
\]

solves system (S) and the solution \( v \) lies in \( L^\infty((0, T) \times (0, L)) \).

**Proof** We show that \( v \) defined in (34) fulfills the pde, the initial conditions and the boundary conditions.

**Wave Equation:**
First we see that \( v \) satisfies the wave equation \( (W) \), because we have

\[
\begin{align*}
  v_{tt} & = \frac{1}{2} \alpha^\prime\prime \left( t + \frac{c}{2} x \right) + \frac{1}{2} \beta^\prime\prime \left( t + \frac{L-c}{c} x \right) \\
  v_{xx} & = \frac{1}{2c^2} \alpha^\prime\prime \left( t + \frac{c}{2} x \right) + \frac{1}{2c^2} \beta^\prime\prime \left( t + \frac{L-c}{c} x \right).
\end{align*}
\]

**Initial Conditions:**
At \( t = 0 \) we have for \( x \in (0, L) \)

\[
v(0, x) = \frac{1}{2} \alpha \left( \frac{c}{2} x \right) + \frac{1}{2} \beta \left( \frac{L-c}{c} x \right) = \frac{1}{2} \left[ v_0(x) + \frac{1}{c} V_1(x) \right] + \frac{1}{2} \left[ v_0(x) - \frac{1}{c} V_1(x) \right] = v_0(x).
\]

For the time derivative at \( t = 0, x \in (0, L) \) we have

\[
v_t(0, x) = \frac{1}{2} \alpha' \left( \frac{c}{2} x \right) + \frac{1}{2} \beta' \left( \frac{L-c}{c} x \right) = \frac{1}{2} \left[ v_0'(x) + v_1(x) \right] - \frac{1}{2} \left[ v_0'(x) - v_1(x) \right] = v_1(x),
\]

where the derivatives are to be understood in the sense of distributions.

**Boundary Conditions:**
Now we proof that the Dirichlet boundary condition at \( x = L \) is fulfilled for \( t > 0 \). We have

\[
v(t, L) = \frac{1}{2} \alpha \left( t + \frac{c}{2} L \right) + \frac{1}{2} \beta(t) = \frac{1}{2} [2c(t) - \beta(t)] + \frac{1}{2} \beta(t) = \xi(t).
\]

For the feedback law at \( x = 0 \), we have

\[
v_x(t, 0) = \frac{1}{2c} \alpha'(t) - \frac{1}{2c} \beta'(t + \frac{L}{c}) = \frac{1}{2c} \alpha'(t) - \frac{1}{2c} v_0'(0)
\]

and

\[
v_{tt}(t, 0) = \frac{1}{2c} \alpha''(t) - \frac{1}{2c} \beta''(t + \frac{L}{c}) = \frac{1}{2c} \alpha''(t) - \frac{1}{2c} v_0''(0).
\]

Now we show that \( v \) lies in \( L^\infty((0, T) \times (0, L)) \). By the assumptions, we have \( v_0 \in L^\infty(0, L) \) and \( \xi \in L^\infty(0, L) \). The claim is true if \( V_1 \) is in \( L^\infty((0, L)) \). We know that \( v_1 \) is in \( L^1(0, L) \). This implies

\[
\| V_1 \|_{L^\infty} = \text{ess sup}_{x \in (0, L)} \int_0^x v_1(s) \, ds \leq \text{ess sup}_{x \in (0, L)} \int_0^x |v_1(s)| \, ds \leq \int_0^L |v_1(s)| \, ds = \| v_1 \|_{L^1}.
\]
Theorem 3 (Value of $\|v\|_{L^\infty}$ in terms of initial and boundary data)
Let $v$ be a solution of system (S) under the assumptions of Theorem 2. For $(t, x) \in (0, T) \times (0, L)$, define

$$m_1(t, x) := \frac{1}{2} \left[ v_0(x + ct) + \frac{1}{2} v_1(x + ct) \right] + \frac{1}{2} \left[ v_0(x - ct) - \frac{1}{2} v_1(x - ct) \right],$$

$$m_2(t, x) := \frac{1}{2} \left[ v_0(x + ct) + \frac{1}{2} v_1(x + ct) + v_0(0) \right],$$

$$m_3(t, x) := \xi \left( t + \frac{x - L}{c} \right) + \frac{1}{2} \left[ v_0(0) - V_2(0) - v_0(2L - x - ct) \right],$$

$$m_4(t, x) := \xi \left( t + \frac{x - L}{c} \right) + \frac{1}{2} \left[ \xi \left( t + \frac{x - L}{c} \right) + v_0(0) - v_0(2L - x - ct) \right],$$

$$m_5(t, x) := \xi \left( t + \frac{x - L}{c} \right).$$

and set

$$M_1 := \sup \{ m_1(t, x) : t < \min \left\{ \frac{x}{c}, \frac{L - x}{c} \right\}, \ (t, x) \in (0, T) \times (0, L) \}.$$

$$M_2 := \sup \{ m_2(t, x) : \frac{x}{c} \leq t < \frac{L - x}{c}, \ (t, x) \in (0, T) \times (0, L) \}.$$

$$M_3 := \sup \{ m_3(t, x) : \frac{L - x}{c} \leq t < \frac{x}{c}, \ (t, x) \in (0, T) \times (0, L) \}.$$

$$M_4 := \sup \{ m_4(t, x) : \max \left\{ \frac{L - x}{c}, \frac{x}{c} \right\} \leq t < \frac{L - x}{c}, \ (t, x) \in (0, T) \times (0, L) \}.$$

$$M_5 := \sup \{ m_5(t, x) : t \geq \max \left\{ \frac{L - x}{c}, \frac{x}{c}, \frac{L - x}{c} \right\}, \ (t, x) \in (0, T) \times (0, L) \}.$$

Then the $L^\infty$-norm of the velocity $v$ is given by

$$\|v\|_{L^\infty} = \max \{ M_1, M_2, M_3, M_4, M_5 \}.$$

Proof By Theorem 2 the solution of system (S) is given by

$$v(t, x) := \frac{1}{2} \alpha \left( t + \frac{x}{c} \right) + \frac{1}{2} \beta \left( t + \frac{L - x}{c} \right).$$

By the definition of $\alpha$ and $\beta$, there are four cases to consider. The last case is split into two subcases. The first case $t < \min \left\{ \frac{x}{c}, \frac{L - x}{c} \right\}$ corresponds to being in the first case for both $\alpha$ and $\beta$. Then we have

$$v(t, x) = \frac{1}{2} \left[ v_0(x + ct) + \frac{1}{2} v_1(x + ct) \right] + \frac{1}{2} \left[ v_0(x - ct) - \frac{1}{2} v_1(x - ct) \right].$$

For $\frac{x}{c} \leq t < \frac{L - x}{c}$, we are in the first case for $\alpha$ and in the second case for $\beta$. Note that the interval for $t$ can only be nonempty for $x \in (0, \frac{L}{2})$. We have

$$v(t, x) = \frac{1}{2} \left[ v_0(x + ct) + \frac{1}{2} v_1(x + ct) + v_0(0) \right].$$

For $\frac{L - x}{c} \leq t < \frac{x}{c}$, we are in the second case for $\alpha$ and in the first case for $\beta$. Note that the interval for $t$ can only be nonempty for $x \in (\frac{L}{2}, L)$. We have

$$v(t, x) = \frac{1}{2} \left[ 2\xi \left( t + \frac{x - L}{c} \right) - \beta \left( t + \frac{x - L}{c} \right) + \beta \left( t + \frac{L - x}{c} \right) \right].$$
(by \( t < \frac{2}{\epsilon} < \frac{1}{\epsilon} \) and \( \frac{2}{\epsilon} - \frac{L}{c} < 0 \), we have \( t + \frac{x - L}{c} < \frac{L}{c} \))

\[
\xi(t + \frac{x - L}{c}) - \frac{1}{\epsilon} \left[ v_0(2L - x - ct) - \frac{1}{\epsilon} V_1(2L - x - ct) \right] + \frac{1}{2} \left[ v_0(ct + x) - \frac{1}{\epsilon} V_1(ct + x) \right].
\]

The last case to consider is \( t \geq \max\{\frac{2}{\epsilon}, \frac{x}{c}\} \). It leads to

\[
v(t, x) = \frac{1}{2} \left[ 2\xi(t + \frac{x - L}{c}) - \beta(t + \frac{x - L}{c}) + v_0(0) \right] =
\begin{cases}
\xi(t + \frac{x - L}{c}) + \frac{1}{2} \left[ \frac{1}{c} V_1(2L - x - ct) + v_0(0) - v_0(2L - x - ct) \right] & t < \frac{L}{c} + \frac{L-x}{c}, \\
\xi(t + \frac{x - L}{c}) & t \geq \frac{L}{c} + \frac{L-x}{c}.
\end{cases}
\]

5.1 Boundary data given as cosine with random amplitude, frequency and phase shift

We consider the parametric family

\[
\xi(t) := \lambda \cos(\omega t + \kappa)
\]

with a random variable \((\lambda, \kappa, \omega)\) and the compatible initial data

\[
v_0(x) = \lambda \cos(\kappa), \quad v_1 = 0.
\]

We assume that \((\lambda, \kappa, \omega)\) is normally distributed with expected value \(\mu \in \mathbb{R}^3\) and a positive definite covariance matrix \(\Sigma \in \mathbb{R}^{3 \times 3}\). For the numerical computation of the probability, we use the spheric radial decomposition described in Section 3.2.

**Corollary 1** *(Analytic formula for \(\|v\|_{L^\infty}\)) Let \(v\) be a solution of system \((S)\) under the assumptions of Theorem 2 for Dirichlet boundary data at \(x = L\), given by \((\text{cos-bd})\) and the initial conditions given by \((\text{cos-id})\). Then

\[
\|v\|_{L^\infty} = |\lambda|.
\]

**Proof** With the definitions from Theorem 3 and \((\text{cos-bd})\) as well as \((\text{cos-id})\), we have

\[
\begin{align*}
m_1(t, x) &:= v_0(ct + x) = \lambda \cos(\kappa), \\
m_2(t, x) &:= \frac{1}{2} \left[ v_0(ct + x) + \frac{1}{\epsilon} V_1(ct + x) + v_0(0) \right] = \lambda \cos(\kappa), \\
m_3(t, x) &:= \xi(t + \frac{x - L}{c}) + \frac{1}{2} \left[ v_0(ct + x) - \frac{1}{\epsilon} V_1(ct + x) - v_0(2L - x - ct) + \frac{1}{\epsilon} V_1(2L - x - ct) \right] \cos(\omega t + \kappa) \\
&= \lambda \cos(\omega t + \frac{x - L}{c} + \kappa), \\
m_4(t, x) &:= \xi(t + \frac{x - L}{c}) + \frac{1}{2} \left[ \frac{1}{c} V_1(2L - x - ct) + v_0(0) - v_0(2L - x - ct) \right] \cos(\omega t + \frac{x - L}{c} + \kappa), \\
m_5(t, x) &:= \xi(t + \frac{x - L}{c}) = \lambda \cos(\omega t + \frac{x - L}{c} + \kappa).
\end{align*}
\]

By \(|m_i(t, x)| = |\lambda|\) for \(i = 1, \ldots, 5\) the claim follows.
Fig. 6: The solution $v$ of the wave equation for nine samples $(\lambda, \omega, \kappa)$ on the sphere. The radius $r$ is scaled such that $\|v\|_{L^\infty} = v_{\text{max}}$ holds. The value of the cumulative distribution of the chi distribution evaluated at this radius is given on top of each picture. The probability for the solution to be bounded by $v_{\text{max}}$ is $P(\|v\|_{L^\infty} \leq v_{\text{max}}) \approx 0.7856$ for the data $T = 6$, $L = 2$, $c = 0.5$, $v_{\text{max}} = 1.8$.

5.2 Karhunen-Loève approximation of a Wiener process as initial and boundary data

We consider the Karhunen-Loève representation (see [18]) of a Wiener process on $[0, T]$ with covariance function $\text{Cov}(W_t, W_s) = \min(s, t)$ given by

$$W_t = \sqrt{2T} \sum_{k=1}^{\infty} a_k \frac{\sin(\omega_k \pi t)}{\omega_k \pi}, \quad \omega_k = k - \frac{1}{2},$$

with independently normally distributed random variables $a_k$. It is reasonable to use a finite approximation of it as boundary data, i.e.

$$\xi(t) = \sqrt{2T} \sum_{k=1}^{N} a_k \frac{\sin(\omega_k \pi t)}{\omega_k \pi}, \quad \omega_k = k - \frac{1}{2}, \text{ on } [0, T]. \quad (\text{KL-bd})$$

Analogously, we choose the compatible initial data

$$v_0(x) = \sqrt{2L} \sum_{k=1}^{N} b_k \frac{\sin(\omega_k \pi \frac{x}{L})}{\omega_k \pi}, \quad \omega_k = k - \frac{1}{2}, \text{ on } [0, L], \quad (\text{KL-id})$$
with independently normally distributed random variables $b_k$. We have the compatibility condition $\xi(0) = v_0(L) = 0$. Furthermore, set $v_1 = 0$. Different realizations of the initial and boundary data can be seen in Figures 7 and 8. The solution of the wave equation for different realizations of the initial and boundary data is depicted in Figure 9.

The case is much more involved than that in Section 5.1, since the value of $\|v\|_{L^\infty}$ is not easily expressed as an analytic function of the random variables. This means a sampling scheme based on spheric radial decomposition can not be directly be applied. We use a quasi Monte Carlo method based on a Sobol sequence instead.

If one wants to approximate the $L^\infty$- norm of the velocity by pointwise evaluation on a grid, Lipschitz continuity of the velocity is required.

**Theorem 4 (Lipschitz continuity of the solution)** Assume the boundary data $\xi \in C^{0,1}(0,T)$ and initial data $v_0 \in C^{0,1}(0,L)$ to be Lipschitz continuous and assume that Lipschitz compatibility over the edge holds, i.e. that we have

$$|\xi(t) - v_0(L - x)| \leq K|t - L + x|, \quad \text{for} \quad (t, x) \in [0,T] \times [0,L]$$

with a Lipschitz constant $K > 0$. Furthermore, let $v_1 \in L^\infty(0,L)$.

Then the solution $v$ of system (S) is Lipschitz continuous in time and space, i.e., $v \in C^{0,1}((0,T) \times (0,L))$. 

---

![Fig. 7: Different realizations (21) of the initial data for a Karhunen-Loève sum with 20 standard normally distributed coefficients](image1)

![Fig. 8: Different realizations (21) of the boundary data for a Karhunen-Loève sum with 20 standard normally distributed coefficients](image2)
Stabilization of the wave equation

The solution $v$ of the wave equation with boundary and initial data
given by the functions in (KL-bd) and (KL-id) for nine samples of the standard
normal distributed random vector $(a, b)$ with realizations in $\mathbb{R}^4$, i.e., $N = 20$.
The constants $T = 6$, $L = 2$, $c = 0.5$ were used. The bound $v_{\text{max}} = 5$ was chosen
and $\bar{v} = 0$ was used. The probability of $\|v + \bar{v}\|_{L^\infty} \leq v_{\text{max}}$ is
0.8808 with 10000 samples used. The value of the $L^\infty$-norm is approximated by evaluation on a
$100 \times 100$ grid on $[0, T] \times [0, L]$. The points, where the value of the $L^\infty$- norm
is attained are marked in red.

**Proof** The sum of Lipschitz continuous functions is Lipschitz continuous. It
is therefore sufficient to show the Lipschitz continuity of $\alpha$ and $\beta$ defined
as in Theorem 2. Without loss of generality—by going to the maximum of
the occurring Lipschitz constants—we assume that they are all the same and
denote each of them by $K > 0$. First, we show the Lipschitz continuity of $V_1$.
We have, for $x, y \in [0, L]$,

$$|V(x) - V(y)| = \left| \int_0^x v_1(s) \, ds - \int_0^y v_1(s) \, ds \right| = \left| \int_x^y v_1(s) \, ds \right| \leq |x - y| \, \|v_1\|_{L^\infty} \leq K|x - y|.$$  

The Lipschitz continuity of $\beta$ is clear in the individual integrals $[0, \frac{L}{c}]$ and
$[\frac{L}{c}, T + \frac{L}{c}]$. Consider $s \in [0, \frac{L}{c})$ and $r \in [\frac{L}{c}, T + \frac{L}{c})$. Then, using $V_1(0) = 0$
and $|\frac{L}{c} - s| = \frac{L}{c} - s \leq r - s = |r - s|$, leads to

$$|\beta(s) - \beta(r)| = \frac{1}{c} |cv_0(L - cs) - V_1(L - cs) - cv_0(0) + V_1(0)| \leq \frac{K}{c} |L - cs| \leq K|r - s|.$$
The Lipschitz continuity of $\beta$ of $\bar{\xi}$ imply that $\alpha$ is Lipschitz on $t \geq \frac{L}{c}$ and by the Lipschitz continuity of $v_0$ and $V_1$ it is Lipschitz on $[0, \frac{L}{c})$. Again, the case $s \in [0, \frac{L}{c})$ and $r \geq \frac{L}{c}$ is remaining. We obtain

$$|\alpha(s) - \alpha(r)| = |v_0(cs) + \frac{1}{c}V_1(cs) - 2\bar{\xi}(r - \frac{L}{c}) + \beta(r - \frac{L}{c})|.$$ 

For $\frac{L}{c} \leq r < \frac{2L}{c}$, we have by the definition of $\beta$

$$|\alpha(s) - \alpha(r)| = |v_0(cs) + \frac{1}{c}V_1(cs) - 2\bar{\xi}(r - \frac{L}{c}) + v_0(2L - cr) - \frac{1}{c}V_1(2L - cr)|$$

$$= |v_0(cs) - v_0(L) + \frac{2}{c}(v_0(L) - \xi(r - \frac{L}{c})) + v_0(2L - cr) - v_0(L) + \frac{1}{c}(V_1(cs) - V_1(2L - cr))|$$

(By the triangle inequality and the compatibility $v_0(L) = \bar{\xi}(0)$)

$$\leq |v_0(cs) - v_0(L)| + 2|\xi(0) - \xi(r - \frac{L}{c})|$$

$$+ |v_0(2L - cr) - v_0(L)| + \frac{1}{c}|V_1(cs) - V_1(2L - cr)|$$

$$\leq K|cs - L| + 2K|\bar{\xi}|r + \frac{L}{c} + K|L - cr|$$

$$+ \frac{K}{c}|cs - L| + \frac{K}{c}|L + cr|$$

$$= K[L - cs - 2(r - \frac{L}{c}) + cr - L + \frac{L}{c} - s + r - \frac{L}{c}]$$

(As $-\frac{L}{c} \leq -s$)

$$\leq K[c(r - s) + 2(r - s) + (r - s)] = K(c + 3)|r - s|$$

For $r \geq \frac{2L}{c}$, we have by the definition of $\beta$ and $V_1(0) = 0$

$$|\alpha(s) - \alpha(r)| = |v_0(cs) + \frac{1}{c}V_1(cs) - 2\bar{\xi}(r - \frac{L}{c}) + v_0(0)|$$

$$= |v_0(cs) - v(L) + 2(v_0(L) - \xi(r - \frac{L}{c})) + v_0(0) - v_0(L) + \frac{1}{c}V_1(cs) - \frac{1}{c}V_1(0)|$$

(By the triangle inequality and the compatibility $v_0(L) = \bar{\xi}(0)$)

$$\leq |v_0(cs) - v(L)| + 2|\xi(0) - \xi(r - \frac{L}{c})|$$

$$+ |v_0(0) - v_0(L)| + \frac{1}{c}|V_1(cs) - V_1(0)|$$

$$\leq K|cs - L| + 2K|\bar{\xi}|r + \frac{L}{c} + K|L| + \frac{K}{c}|cs|$$

$$= K[L - cs + 2(r - \frac{L}{c}) + L + s]$$

(Using $2L \leq rc$, $-\frac{L}{c} \leq -s$ and $r - s \geq \frac{2L}{c} - s \geq \frac{L}{c} \geq s$)

$$\leq K[c(r - s) + 2(r - s) + (r - s)] \leq K(c + 3)|r - s|.$$ 

This shows the Lipschitz continuity of $\alpha$ and concludes the proof.

In the application to gas networks, the boundary feedback control is realized by compressor stations. The goal is to stabilize the flow and stay near an equilibrium.
6 Conclusion

In this paper we introduced a joint model of probabilistic and robust constraints, so-called probust constraints and illustrated their importance for gas transport under uncertainty. In particular, we addressed the problem of capacity maximization under uncertainty thereby distinguishing between the cases of uncertain exit and uncertain entry loads. Moreover, we considered a stabilization problem in a transient system governed by the wave equation and subject to probust constraints. By applying the spheric radial decomposition of Gaussian random vectors, we approximated the occurring probabilities and—where possible—their sensitivities with respect to the decision variables in order to numerically solve the resulting optimization problems. There are a lot of remaining challenges for future work, such as efficient incorporation of cycles or active elements in the network. Moreover, a full integration of the methodology outlined in Section 4 for the robust treatment of uncertain entries with the capacity maximization problem described in Section 3 ultimately would allow an application of the probust approach to arbitrary network topologies.

Acknowledgement The authors thank the DFG for their support within Projects B04, B05, B06, C03, and Z01 in CRC TRR 154.

References