

# Interpolation Strategy for BT-based Parametric MOR of Gas Pipeline-Networks

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**Abstract** Proceeding from balanced truncation-based parametric reduced order models (BT-pROM) a matrix interpolation strategy is presented that allows the cheap evaluation of reduced order models at new parameter sets. The method extends the framework of model order reduction (MOR) for high-order parameter-dependent linear time invariant systems in descriptor form by Geuss (2013) by treating not only permutations and rotations but also distortions of reduced order basis vectors. The applicability of the interpolation strategy and different variants is shown on BT-pROMs for gas transport in pipeline-networks.

## 1 Introduction

Optimization and control of large transient gas networks require the fast simulation of the underlying parametric partial differential algebraic systems. In this paper we present a surrogate modeling technique that is composed of linearization around stationary states, spatial semi-discretization and model order reduction via balanced truncation (BT). Making use of a matrix interpolation strategy (MIS) in the spirit of [1, 5] we explore its performance for evaluating the BT-pROMs over a wide parameter range of different boundary pressures and temperatures. Our developed variant DTMIS particularly regards possible distortions of the reduced basis vectors.

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## 2 Modeling Approach for Gas Pipeline-Networks

Proceeding from a nonlinear partial differential algebraic gas network model we perform linearization and spatial semi-discretization to obtain a parametric linear time invariant system as basis for MOR.

**Modeling.** The gas dynamics in a horizontal pipe  $e$  can be described by the one-dimensional isothermal Euler equations in terms of pressure  $p_e$  and flow rate  $q_e$  for space parameter  $x \in [x_e^{in}, x_e^{out}]$  and time  $t \in [0, t_{end}]$ ,

$$\partial_t \left( \frac{1}{z} p_e(x, t) \right) + \frac{R_s T}{A_e} \partial_x q_e(x, t) = 0, \quad (1a)$$

$$\partial_t q_e(x, t) + A_e \partial_x p_e(x, t) + R_s T \partial_x \left( z \frac{q_e^2(x, t)}{p_e(x, t)} \right) = - \frac{R_s T}{2A_e D_e} z \lambda \frac{q_e(x, t) |q_e(x, t)|}{p_e(x, t)} \quad (1b)$$

with pipe length  $L_e$ , diameter  $D_e$ , cross-sectional area  $A_e$ , temperature  $T$ , and specific gas constant  $R_s$ . The gas compressibility  $z$  and friction  $\lambda$  are empirically given by AGA and Chen formula, respectively, [4], i.e.,

$$z(p_e, T) = 1 + 0.257 \frac{p_e}{p^*} - 0.533 \frac{p_e T^*}{p^* T}$$

with critical pressure  $p^*$  and temperature  $T^*$  values depending on the gas type, and

$$\frac{1}{\sqrt{\lambda(q_e)}} = -2 \log_{10} \left[ \frac{\kappa_e}{3.707 D_e} - \frac{5.045}{\text{Re}} \log_{10} \left( \frac{1}{2.826} \left( \frac{\kappa_e}{D_e} \right)^{1.110} + \frac{5.851}{\text{Re}^{0.898}} \right) \right]$$

with Reynolds number  $\text{Re}(q_e) = |q_e| D_e / (\eta A_e)$ , dynamic gas viscosity  $\eta$ , and pipe roughness  $\kappa_e$ . A network of pipelines can then be modeled as a directed graph  $\mathcal{G} = (\mathcal{E}, \mathcal{N})$  where the edges are represented by the pipes  $e \in \mathcal{E}$  (with mathematically positive orientation from  $x_e^{in}$  to  $x_e^{out}$ ). The set of nodes  $\mathcal{N}$  consists of sources  $\mathcal{N}_{in}$ , sinks  $\mathcal{N}_{out}$  and branching (neutral) nodes  $\mathcal{N}_{neu}$ . At the branching nodes, mass conservation –known as first Kirchhoff law– and pressure equality in terms of auxiliary variables  $p$  are imposed as coupling conditions, i.e.,

$$\sum_{e \in \delta_v^-} q_e(x_e^{out}, t) = \sum_{e \in \delta_v^+} q_e(x_e^{in}, t), \quad (1c)$$

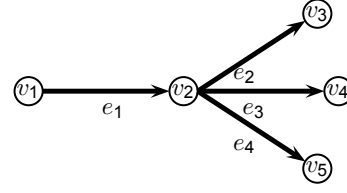
$$p_e(x_e^{in}, t) = p(v, t), \quad e \in \delta_v^+, \quad p_e(x_e^{out}, t) = p(v, t), \quad e \in \delta_v^-, \quad v \in \mathcal{N}_{neu} \quad (1d)$$

where  $\delta_v^-$  and  $\delta_v^+$  denote the sets of ingoing and outgoing arcs at  $v \in \mathcal{N}_{neu}$ , cf. Fig. 1. As boundary conditions we prescribe the pressure profile at the sources and the flow rate at the sinks

$$p(v, t) = f_v(t), \quad v \in \mathcal{N}_{in}, \quad q(v, t) = f_v(t), \quad v \in \mathcal{N}_{out}. \quad (1e)$$

**Fig. 1** Network topology

Fork:  $\mathcal{G} = (\mathcal{E}, \mathcal{N})$  with source  $\mathcal{N}_{\text{in}} = \{v_1\}$ , sinks  $\mathcal{N}_{\text{out}} = \{v_3, v_4, v_5\}$ , as well as  $\mathcal{N}_{\text{neu}} = \{v_2\}$ ,  $\delta_{v_2}^- = \{e_1\}$  and  $\delta_{v_2}^+ = \{e_2, e_3, e_4\}$ .



System (1) is supplemented with consistent initial conditions obtained from solving the stationary problem with the boundary conditions (1e) evaluated at time  $t = 0$

**Linearization.** Expanding around a stationary state  $y(x, t) = y^s(x) + \varepsilon y^t(x, t) + \mathcal{O}(\varepsilon^2)$ ,  $y \in \{p_e, q_e\}$ , that is specified by a certain parameter set  $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^d$ , the nonlinear system (1) decomposes in first order into a stationary subsystem

$$\begin{aligned}
 c_{e,1}^s \frac{d}{dx} p_e^s + c_{e,2}^s \frac{1}{p_e^s} &= 0, \quad e \in \mathcal{E}, & \sum_{e \in \delta_v^-} q_e^s(x_e^{\text{out}}) &= \sum_{e \in \delta_v^+} q_e^s(x_e^{\text{in}}) \\
 p_e^s(x_e^{\text{in}}) &= p^s(v), \quad e \in \delta_v^+, & p_e^s(x_e^{\text{out}}) &= p^s(v), \quad e \in \delta_v^-, & v \in \mathcal{N}_{\text{neu}} \\
 p^s(v) &= f_v(0), \quad v \in \mathcal{N}_{\text{in}}, & q^s(v) &= f_v(0), \quad v \in \mathcal{N}_{\text{out}}
 \end{aligned}$$

with

$$c_{e,1}^s = 1 - \frac{R_s T}{A_e^2} z^s \left( \frac{q_e^s}{p_e^s} \right)^2, \quad c_{e,2}^s = \frac{R_s T}{A_e^2} \partial_p z^s (q_e^s)^2 + \frac{R_s T}{2D_e A_e^2} z^s \lambda^s q_e^s |q_e^s|,$$

and a linear transient (correction) subsystem

$$\begin{aligned}
 \partial_t p_e^t + c_{e,1}^t \partial_x q_e^t &= 0 \\
 \partial_t q_e^t + c_{e,2}^t \partial_x q_e^t + c_{e,3}^t \partial_x p_e^t + c_{e,4}^t q_e^t + c_{e,5}^t p_e^t &= 0, \quad e \in \mathcal{E} \\
 \sum_{e \in \delta_v^-} q_e^t(x_e^{\text{out}}, t) &= \sum_{e \in \delta_v^+} q_e^t(x_e^{\text{in}}, t) \\
 p_e^t(x_e^{\text{in}}, t) &= p^t(v, t), \quad e \in \delta_v^+, \quad p_e^t(x_e^{\text{out}}, t) = p^t(v, t), \quad e \in \delta_v^-, \quad v \in \mathcal{N}_{\text{neu}} \\
 p^t(v, t) &= (f_v(t) - f_v(0))/\varepsilon, \quad v \in \mathcal{N}_{\text{in}}, \quad q^t(v, t) = (f_v(t) - f_v(0))/\varepsilon, \quad v \in \mathcal{N}_{\text{out}}
 \end{aligned} \tag{2}$$

with initial conditions  $y^t(x, 0) = 0$ ,  $y \in \{p_e, q_e\}$ , and

$$\begin{aligned}
 c_{e,1}^t &= \frac{R_s T}{A_e} (z^s)^2, & c_{e,2}^t &= \frac{2R_s T}{A_e} z^s \frac{q_e^s}{p_e^s}, & c_{e,3}^t &= A_e - \frac{R_s T}{A_e} \left( \frac{q_e^s}{p_e^s} \right)^2, \\
 c_{e,4}^t &= \frac{R_s T}{A_e} \left( \frac{1}{2D_e} z^s \frac{|q_e^s|}{p_e^s} (2\lambda^s + \partial_q \lambda^s q_e^s) - 2 \frac{q_e^s}{(p_e^s)^2} \frac{d}{dx} p_e^s \right), \\
 c_{e,5}^t &= \frac{R_s T}{A_e} \left( \frac{1}{2D_e} \lambda^s \frac{|q_e^s|}{p_e^s} \left( \partial_p z^s - z^s \frac{q_e^s}{p_e^s} \right) - 2 \left( \frac{q_e^s}{p_e^s} \right)^2 \frac{d}{dx} p_e^s \left( \partial_p z^s + \frac{z^s}{p_e^s} \right) \right).
 \end{aligned}$$

The coefficient functions  $c_{e,j}^t$ ,  $j = 1, \dots, 5$ , depend not only on the stationary state but also on the model parameters of the pipeline-network and the gas flow. Moreover, note that the flow rate should be regularized before the linearization procedure, i.e.,  $|q_e| = (q_e^2 + \alpha^2)^{1/2}$ ,  $\alpha$  small, [7].

**Semi-discretization.** As spatial discretization for (2) we use a conservative first-order finite-volume-like method on a staggered grid to obtain small discretization stencils. Each pipe is distributed in cells of same length where the pressure  $p_e^t$  and the mass balance are evaluated at the cell edges and the flow rate  $q_e^t$  and the momentum balance at the cell midpoints. Sources and sinks are either located on the edges or midpoints of a cell, if pressure or flow rate are given as boundary condition. Neutral nodes are placed at cell boundaries, as suggested in [4]. Function values of  $p_e^t$  at a midpoint and  $q_e^t$  at a cell boundary are interpolated. Note that for readability the indices  $_e^t$  are suppressed in the stated scheme for a pipe interior,

$$\begin{aligned} \frac{d}{dt} p_{i+1/2} &= -\frac{c_{1,i+1/2}}{\Delta x} (q_{i+1} - q_i) \\ \frac{d}{dt} q_i &= -\frac{c_{2,i}}{\Delta x} (q_{i+1/2} - q_{i-1/2}) - \frac{c_{3,i}}{\Delta x} (p_{i+1/2} - p_{i-1/2}) - c_{4,i} q_i - c_{5,i} p_i \end{aligned}$$

with  $p_i = (p_{i+1/2} + p_{i-1/2})/2$  and  $q_{i+1/2} = (q_{i+1} - q_i)/2$  as well as cell size  $\Delta x$ .

The resulting linear time invariant system (LTIS) of differential algebraic equations (DAE) for the pipeline-network is parameter-dependent,  $\Sigma(\mathbf{p})$ ,  $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^d$ ,

$$\Sigma(\mathbf{p}) : \quad \mathbf{E}(\mathbf{p}) \frac{d}{dt} \mathbf{x}(t) = \mathbf{A}(\mathbf{p}) \mathbf{x}(t) + \mathbf{B}(\mathbf{p}) \mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}(\mathbf{p}) \mathbf{x}(t), \quad (3)$$

with system matrices  $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{B} \in \mathbb{R}^{n \times m}$  and  $\mathbf{C} \in \mathbb{R}^{p \times n}$ . The states, inputs and outputs are denoted by  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{u} \in \mathbb{R}^m$  and  $\mathbf{y} \in \mathbb{R}^p$ . The inputs are certainly the boundary conditions, the outputs are taken here as the flow rates at the sources and the pressures at the sinks. As in optimization and control the variation of boundary value profiles (1e) is often of interest, we consider a parameter dependence on the boundary pressure  $p(v, 0)$ ,  $v \in \mathcal{N}_{\text{in}}$  and the temperature  $T$ , i.e.,  $\mathbf{p} \subset \mathbb{R}^2$ . In the following we refer to a sample of  $N_p$  different parameter settings and denote the local LTIS associated with  $\mathbf{p}_k$  by  $\Sigma_k$ ,  $k = 1, \dots, N_p$ . It is assumed that  $\Sigma_k$  is stable with the regular pencil  $\mathbf{A}_k - \lambda \mathbf{E}_k$ . The stability depends, among others, on the applied discretization and is ensured for the discretized gas network under consideration. Note that, whenever possible, we suppress the parameter index  $k$  in the explanations to facilitate the readability.

### 3 BT-MOR for LTIS in Descriptor Form

In the classical method of balanced truncation for ordinary differential equations [2, 12], the original model of order  $n$  is first transformed into a balanced form, where the controllability and observability Gramians are diagonally equal. Then, a

BT-ROM of order  $r$ ,  $r \ll n$  is obtained by truncating the  $(n-r)$  states that are related to the  $(n-r)$  smallest Hankel singular values, i.e., diagonal entries of the Gramians.

Considering the full order model (FOM) of DAEs  $\Sigma$  in (3), a QZ-decomposition leads to a pencil  $\mathbf{A} - \lambda\mathbf{E}$  in the generalized real Schur form. By applying a block-diagonalization [6],  $\Sigma$  can be decoupled into proper and improper subsystems. The spectra of the proper and improper subsystems are the same as the finite and infinite ones of the whole system. Afterwards, the proper and improper subsystems are separately transformed into the balanced form. Whereas the standard BT procedure can be applied to obtain a proper ROM, truncation for the improper subsystem can not be performed in general. If states related to non-zero small HSVs are neglected, the improper ROM may have a finite spectrum with non-negative real parts, which leads to a non-stable inaccurate approximation [8]. In addition, algebraic constraints of the systems might be violated. For example, in case of the gas networks, some of the coupling conditions (Kirchhoff's laws and the pressure equivalence at neutral nodes) may not hold true which implies physically meaningless results. However, states related to zero HSVs can be neglected without affecting the system [11].

Thus, a BT-ROM of order  $r = r_f + r_\infty$  is given by

$$\begin{aligned} \Sigma_r : \quad \mathbf{E}_r \frac{d}{dt} \mathbf{x}_r(t) &= \mathbf{A}_r \mathbf{x}_r(t) + \mathbf{B}_r \mathbf{u}(t), \quad \mathbf{y}_r(t) = \mathbf{C}_r \mathbf{x}_r(t) \quad (4a) \\ \mathbf{E}_r &= \mathbf{W}^T \mathbf{E} \mathbf{V} = \begin{bmatrix} \mathbf{I}_{r_f} & \\ & \mathbf{E}_{r_\infty} \end{bmatrix}, \quad \mathbf{B}_r = \mathbf{W}^T \mathbf{B} = \begin{bmatrix} \mathbf{B}_{r_f} \\ \mathbf{B}_{r_\infty} \end{bmatrix}, \quad \mathbf{W} = [\mathbf{W}_{r_f} \quad \mathbf{W}_{r_\infty}] \\ \mathbf{A}_r &= \mathbf{W}^T \mathbf{A} \mathbf{V} = \begin{bmatrix} \mathbf{A}_{r_f} & \\ & \mathbf{I}_{r_\infty} \end{bmatrix}, \quad \mathbf{C}_r = \mathbf{C} \mathbf{V} = [\mathbf{C}_{r_f} \quad \mathbf{C}_{r_\infty}], \quad \mathbf{V} = [\mathbf{V}_{r_f} \quad \mathbf{V}_{r_\infty}] \end{aligned}$$

with its proper and improper subsystems

$$\Sigma_r^{\text{prop}} : \quad \frac{d}{dt} \mathbf{x}_{r_f}(t) = \mathbf{A}_{r_f} \mathbf{x}_{r_f}(t) + \mathbf{B}_{r_f} \mathbf{u}(t), \quad \mathbf{y}_{r_f}(t) = \mathbf{C}_{r_f} \mathbf{x}_{r_f}(t) \quad (4b)$$

$$\Sigma_r^{\text{improp}} : \quad \mathbf{E}_{r_\infty} \frac{d}{dt} \mathbf{x}_{r_\infty}(t) = \mathbf{x}_{r_\infty}(t) + \mathbf{B}_{r_\infty} \mathbf{u}(t), \quad \mathbf{y}_{r_\infty}(t) = \mathbf{C}_{r_\infty} \mathbf{x}_{r_\infty}(t). \quad (4c)$$

The applied projections  $\mathbf{W}$ ,  $\mathbf{V}$  are obviously parameter-dependent, but not orthonormal. They build bases of the (parameter-dependent) rank- $r$  subspaces  $\mathcal{W}$ ,  $\mathcal{V}$  in  $\mathbb{R}^n$ . Analogously,  $\mathbf{W}_{r_f}$ ,  $\mathbf{V}_{r_f}$  and  $\mathbf{W}_{r_\infty}$ ,  $\mathbf{V}_{r_\infty}$  form bases of rank- $r_f$  subspaces  $\mathcal{W}_{r_f}$ ,  $\mathcal{V}_{r_f}$  and rank- $r_\infty$  subspaces  $\mathcal{W}_{r_\infty}$ ,  $\mathcal{V}_{r_\infty}$ , respectively. The BT-ROM  $\Sigma_r$  is stable as long as the FOM  $\Sigma$  of (3) is stable, [12]. Moreover, since only states related to the improper zero-HSVs are truncated, the DAE-index is preserved, [11].

An error estimate for the system's transfer function  $\mathbf{G}$  in the frequency domain is related to the (decreasingly sorted) proper HSVs  $\sigma_i$ ,  $i = 1, \dots, n_f$ , [11],

$$\|\mathbf{G} - \mathbf{G}_r\|_{\mathbb{H}_\infty} = \|\mathbf{G}^{\text{prop}} - \mathbf{G}_r^{\text{prop}}\|_{\mathbb{H}_\infty} \leq 2 \sum_{i=r_f+1}^{n_f} \sigma_i$$

with the  $\mathbb{H}_\infty$ -norm defined as  $\|\mathbf{G}\|_{\mathbb{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|\mathbf{G}(i\omega)\|_2$ . Hereby,  $\mathbf{G}^{\text{prop}}$  and  $\mathbf{G}_r^{\text{prop}}$  denote the strictly proper part of  $\mathbf{G}$  and  $\mathbf{G}_r$ , respectively. Due to the Paley-Wiener Theorem, this error estimate also holds in the time domain [2] where the  $\mathbb{H}_\infty$ -norm is regarded as the 2-induced operator norm,

$$\|\mathbf{y}(t) - \mathbf{y}_r(t)\|_2 \leq \|\mathbf{G} - \mathbf{G}_r\|_{\mathbb{H}_\infty} \|\mathbf{u}(t)\|_2.$$

## 4 Interpolation for BT-pROMs

BT-MOR requires the determination of the generalized Schur form and the solving of generalized time-continuous/-discrete Lyapunov equations. Each requires  $\mathcal{O}(n^3)$  flops and has  $\mathcal{O}(n^2)$  memory complexity, [2]. Obviously, the computational effort is so extremely high for a large-scale LTIS that this procedure is not suitable for online parameter variations. Therefore, we suggest an interpolation strategy in the spirit of [1, 5].

Once BT-pROMs  $\Sigma_{r,k}$  are computed for different parameter settings  $k = 1, \dots, N_p$ , one could think of efficiently approaching a reduced order model at a new parameter  $\mathbf{p}$  by means of matrix interpolation

$$\Sigma_r(\mathbf{p}) : \quad \mathbf{E}_r(\mathbf{p}) \frac{d}{dt} \mathbf{x}_r(t) = \mathbf{A}_r(\mathbf{p}) \mathbf{x}_r(t) + \mathbf{B}_r(\mathbf{p}) \mathbf{u}(t), \quad \mathbf{y}_r(t) = \mathbf{C}_r(\mathbf{p}) \mathbf{x}_r(t) \quad (5a)$$

$$\mathbf{M}_r(\mathbf{p}) = \sum_{k=1}^{N_p} \alpha_k(\mathbf{p}) \mathbf{M}_{r,k}, \quad \mathbf{M} \in \{\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}\} \quad (5b)$$

where the weighting functions  $\alpha_k$  are determined by the selected interpolation method. However, note that in BT, the states of  $\Sigma_k$  are recombined during the decoupling phase in order to be separated with respect to the finite and infinite spectra. In the MOR phase the states are again recombined such that they can be rearranged according to the HSVs in decreasing order. States related to the small proper HSVs and to the zero-valued improper HSVs are truncated until the local reduced systems  $\Sigma_{r,k}$  have the same order  $r$ . Thus, the projections  $\mathbf{W}_k, \mathbf{V}_k$  usually span different rank- $r$  subspaces  $\mathcal{W}_k, \mathcal{V}_k$  in  $\mathbb{R}^n$ . Consequently, the reduced states  $\mathbf{x}_{r,k}$  have in general no common physical interpretation, which implies that a interpolation of type (5) might not be meaningful.

**Generalized rank- $r$  subspace and respective transformation.** To make sense of the interpolation, all local reduced states  $\mathbf{x}_{r,k}$  are transformed in a generalized rank- $r$  subspace  $\mathcal{V}$ . Choosing its basis  $\tilde{\mathbf{V}}$  requires in general a priori knowledge about the dynamics of the local ROMs. Different strategies are discussed in literature. For example, one of the local bases might act as generalized basis  $\tilde{\mathbf{V}} = \mathbf{V}_{k_0}$ ,  $k_0 \in \{1, \dots, N_p\}$ , [1]. This is suitable, if all local reduced states lie in the same subspace. In case that the local bases are very different, the generalized basis must catch the most important characteristics of all local ROMs. For this purpose, a Proper Orthogonal Decomposition (POD) [10] can be employed [9], i.e.,

$$[\bar{\mathbf{V}} \mathbf{U}] \Sigma [\bar{\mathbf{V}} \mathbf{U}]^T = [\mathbf{V}_1 \cdots \mathbf{V}_{N_p}] [\mathbf{V}_1 \cdots \mathbf{V}_{N_p}]^T.$$

The state transformation  $\mathbf{T}_{V,k}$  maps  $\mathbf{x}_{r,k}$  in  $\mathcal{V}$ , i.e.,  $\bar{\mathbf{x}}_r = \mathbf{T}_{V,k} \mathbf{x}_{r,k}$ . A transformation proposed in [9]

$$\mathbf{T}_{V,k} = (\bar{\mathbf{V}}^T \mathbf{V}_k)^{-1} \quad (6)$$

describes permutations, rotations and length distortions of the basis vectors. Furthermore, it maximizes correlations between each  $i$ -th base vectors,  $i = 1, \dots, r$ , but minimizes correlations between the  $i$ -th and  $i'$ -th base vectors,  $i \neq i'$  in  $\bar{\mathbf{V}}$  and  $\mathbf{V}_k$ , where the correlations are defined according to the Modal Assurance Criterion (MAC) [1]

$$MAC(\bar{\mathbf{V}}_{i'}, \mathbf{V}_{k,i}) = \frac{|\langle \bar{\mathbf{V}}_{i'}, \mathbf{V}_{k,i} \rangle|^2}{\langle \bar{\mathbf{V}}_{i'}, \bar{\mathbf{V}}_{i'} \rangle \langle \mathbf{V}_{k,i}, \mathbf{V}_{k,i} \rangle}.$$

However,  $\mathbf{T}_{V,k}$  of (6) can be singular, if  $\bar{\mathbf{V}}$  is orthogonal to  $\mathbf{V}_k$ . To avoid this crucial weakness, one seeks state transformations such that the sum of the correlations of all  $i$ -th base vectors in  $\bar{\mathbf{V}}$  and  $\mathbf{V}_k$  is maximized.

**Theorem 1.** [1, Proposition 4.1.] *The optimization problem wrt. the Frobenius norm  $\|\cdot\|_F$*

$$\min_{\mathbf{R}_{V,k} \in \mathcal{O}(r)} \|\bar{\mathbf{V}} - \mathbf{V}_k \mathbf{R}_{V,k}\|_F^2$$

has the unique solution  $\mathbf{R}_{V,k} = \mathbf{U} \mathbf{V}^T$ , where  $\mathbf{U}$  and  $\mathbf{V}$  are the left and right singular vectors of  $\mathbf{V}_k \bar{\mathbf{V}}^T = \mathbf{U} \Sigma \mathbf{V}^T$ .

*Proof.* The first optimality condition together with the uniqueness of the singular value decomposition yields the result.  $\square$

The orthogonal mapping  $\mathbf{R}_{V,k}$  can handle permutations and rotations of the basis vectors in  $\mathbf{V}_k$  wrt.  $\bar{\mathbf{V}}$ , but cannot capture length distortions. To deal also with distortions, we propose a modification on top of the transformation  $\mathbf{R}_{V,k}$ .

**Theorem 2.** *Let  $\bar{\mathbf{V}}_{i'}$  and  $\tilde{\mathbf{V}}_{k,i}$  be the  $i$ -th column vectors in  $\bar{\mathbf{V}}$  and  $\bar{\mathbf{V}}_k = \mathbf{V}_k \mathbf{R}_{V,k}$ . The optimization problem*

$$\min_{\mathcal{W}_{V,i} \geq 0} \|\bar{\mathbf{V}}_{i'} - \mathcal{W}_{V,i} \tilde{\mathbf{V}}_{k,i}\|_F^2$$

has the unique solution

$$\mathcal{W}_{V,i} = \frac{\langle \bar{\mathbf{V}}_{i'}, \tilde{\mathbf{V}}_{k,i} \rangle}{\langle \tilde{\mathbf{V}}_{k,i}, \tilde{\mathbf{V}}_{k,i} \rangle}.$$

*Proof.* The statement follows from the first optimality condition using the fact that  $\|\bar{\mathbf{V}}_{i'} - \mathcal{W}_{V,i} \tilde{\mathbf{V}}_{k,i}\|_F^2 = \|\bar{\mathbf{V}}_{i'} - \mathcal{W}_{V,i} \tilde{\mathbf{V}}_{k,i}\|_2^2$ . The sign  $\mathcal{W}_{V,i} \geq 0$  can be particularly concluded from Theorem 1.  $\square$

Combining Theorem 1 and Theorem 2 we consider the state transformation

$$\mathbf{T}_{V,k} = \mathbf{R}_{V,k} \mathbf{D}_{V,k}, \quad \mathbf{D}_{V,k} = \text{diag}(\mathcal{W}_{V,1}, \dots, \mathcal{W}_{V,r}). \quad (7)$$

Analogously to  $\tilde{\mathcal{V}}$ ,  $\tilde{\mathbf{V}}$  and  $\mathbf{T}_{V,k}$ , we construct  $\tilde{\mathcal{W}}$ ,  $\tilde{\mathbf{W}}$  and  $\mathbf{T}_{W,k}$ . This step is necessary, since the local left projections  $\mathbf{W}_k$  contain the local left Hankel singular vectors related to the local-preserved HSVs in MOR. Similarly to (7), we obtain  $\mathbf{T}_{W,k} = \mathbf{R}_{W,k} \mathbf{D}_{W,k}$  with  $\mathbf{D}_{W,k} = \text{diag}(\gamma_{W,1}, \dots, \gamma_{W,r})$ . Note that in control theory it is often made use of the fact that  $\mathbf{W}_k$  are equal to the right projections  $\mathbf{V}_k^{\text{adj}}$  of the adjoint BT-ROM [5], i.e.,

$$\Sigma_{r,k}^{\text{adj}} : \quad \mathbf{E}_{r,k}^T \frac{d}{dt} \mathbf{x}_{r,k}^{\text{adj}}(t) = \mathbf{A}_{r,k}^T \mathbf{x}_{r,k}^{\text{adj}}(t) + \mathbf{B}_{r,k}^T \mathbf{y}_{r,k}(t), \quad \mathbf{u}(t) = \mathbf{B}_{r,k}^T \mathbf{x}_{r,k}^{\text{adj}}(t).$$

Consequently, the BT-ROM associated with the parameter  $\mathbf{p}_k$ ,  $k = 1, \dots, N_p$ , is given with respect to the generalized rank- $r$  subspaces by

$$\begin{aligned} \hat{\Sigma}_{r,k} : \quad & \hat{\mathbf{E}}_{r,k} \frac{d}{dt} \hat{\mathbf{x}}_{r,k}(t) = \hat{\mathbf{A}}_{r,k} \hat{\mathbf{x}}_{r,k}(t) + \hat{\mathbf{B}}_{r,k} \mathbf{u}(t), \quad \mathbf{y}_{r,k}(t) = \hat{\mathbf{C}}_{r,k} \hat{\mathbf{x}}_{r,k}(t) \quad (8) \\ & \hat{\mathbf{E}}_{r,k} = \mathbf{T}_{W,k}^T \mathbf{E}_{r,k} \mathbf{T}_{V,k}, \quad \hat{\mathbf{A}}_{r,k} = \mathbf{T}_{W,k}^T \mathbf{A}_{r,k} \mathbf{T}_{V,k}, \\ & \hat{\mathbf{B}}_{r,k} = \mathbf{T}_{W,k}^T \mathbf{B}_{r,k}, \quad \hat{\mathbf{C}}_{r,k} = \mathbf{C}_{r,k} \mathbf{T}_{V,k}, \quad \hat{\mathbf{x}}_{r,k} = \mathbf{T}_{V,k} \mathbf{x}_{r,k}. \end{aligned}$$

Note that the basis change has no influence on the input-output properties of the system.

**Manifold for interpolation.** The matrix interpolation (5) performs well, if the dependence of the matrix entries on the parameter  $\mathbf{p} \in \mathcal{P} \subset \mathbb{R}^d$  is accurately described by the interpolants. For example, the interpolants capture the critical points (wrt. first and second derivatives) of the functions that describe the behavior of the matrix entries on  $\mathbf{p}$ . Unfortunately, this requirement is hardly fulfilled by the BT-ROMs (8) as interpolants. Hence, it may be advantageous to map the matrices into a space where the dependencies can be approximated as well as possible, to perform the interpolation there and map then the results back to the original space where the BT-ROMs lie, [1, 3, 5]. In an appropriate space the matrix entries might be regarded as smooth functions of the parameter by applying the concept of a differential Riemannian manifold  $\mathcal{M}$ . There exists a tangent space  $\mathcal{T}_{\mathbf{M}}$  for each matrix  $\mathbf{M} \in \mathcal{M}$ .

Let  $\mathbf{M}_k$  denote a matrix associated to the parameter  $\mathbf{p}_k$ ,  $k = 1, \dots, N_p$  with  $N_p$  sample size. For the interpolation the lifting of regular matrices  $\mathbf{M}_k$  into the tangent space  $\mathcal{T}_{\mathbf{M}_{k_0}}$  of a regular reference matrix  $\mathbf{M}_{k_0}$  can be achieved by the logarithmic mapping which preserves some matrix properties such as symmetric positive definiteness [1]. However, this is only possible if all  $\mathbf{M}_k$  lie in the neighborhood of  $\mathbf{M}_{k_0}$  and  $\mathbf{M}_k$  have an unique and real logarithm at  $\mathbf{M}_{k_0}$ . The last condition is fulfilled if  $\mathbf{M}_k \mathbf{M}_{k_0}^{-1}$  have a nonnegative spectrum [5]. The inverse at  $\mathbf{M}_{k_0}$  is obtained by the exponential mapping. The corresponding interpolation can be read as

$$\mathbf{M}(\mathbf{p}) = \exp \left( \sum_{k=1}^{N_p} \omega(\mathbf{p}) \ln \left( \mathbf{M}_k \mathbf{M}_{k_0}^{-1} \right) \right) \mathbf{M}_{k_0}$$

with weight function  $\omega$ . In case that  $\mathbf{M}_k$  and/or  $\mathbf{M}_{k_0}$  are singular, the interpolation is performed in the vector space at  $\mathbf{M}_{k_0}$  by



$$\mathbf{M}(\mathbf{p}) = \sum_{k=1}^{N_p} \omega(\mathbf{p}) (\mathbf{M}_k - \mathbf{M}_{k_0}) + \mathbf{M}_{k_0}$$

according to [1].

The choice of an appropriate reference matrix  $\mathbf{M}_{k_0}$  to build the respective tangent space requires in general a priori knowledge about the dependencies of the local BT-ROMs on the parameter, which is hard to analyze. In [3] a heuristic selection criterion is based on the assumption that the entries of the regular  $\mathbf{M}_k$  lifted in the tangent space  $\mathcal{T}_{\mathbf{M}_{k_0}}$  depend almost linearly on  $\mathbf{p}_k = (p_{k,1}, \dots, p_{k,d}) \in \mathbb{R}^d$ . This means that considering  $\mathbf{\Gamma}_{k_0,k} = \ln(\mathbf{M}_k \mathbf{M}_{k_0}^{-1})$  the respective  $(i, j)$ -th matrix entry is approximated by  $\gamma_{k_0,k}^{i,j} \approx \alpha_{k_0,0}^{i,j} + \sum_{\ell=1}^d \alpha_{k_0,\ell}^{i,j} p_{k,\ell}$  with constant coefficients  $\alpha_{k_0,m}^{i,j}$ . Then, the normalized least-squares residual of the sample is used as indicator of the parameter dependence, and the maximal values over all matrix entries are considered as selection criterion for the reference parameter  $\mathbf{p}_{k_0}$ ,

$$k_0^* = \arg \min_{k_0} \mu_{k_0}, \quad \mu_{k_0} = \max_{i,j} \mu_{k_0}^{i,j}, \quad \mu_{k_0}^{i,j} = \frac{\sqrt{\sum_{k=1}^{N_p} (\alpha_{k_0,0}^{i,j} + \sum_{\ell=1}^d \alpha_{k_0,\ell}^{i,j} p_{k,\ell} - \gamma_{k_0,k}^{i,j})^2}}{\max_k \gamma_{k_0,k}^{i,j} - \min_k \gamma_{k_0,k}^{i,j}}.$$

Alternatively, one could consider the normalized least-squares residual in the original manifold

$$\theta = \max_{i,j} \theta^{i,j}, \quad \theta^{i,j} = \frac{\sqrt{\sum_{k=1}^{N_p} (\alpha_0^{i,j} + \sum_{\ell=1}^d \alpha_\ell^{i,j} p_{k,\ell} - m_k^{i,j})^2}}{\max_k m_k^{i,j} - \min_k m_k^{i,j}}$$

where  $m_k^{i,j}$  denotes the matrix entries of  $\mathbf{M}_k$ . Comparing  $\theta$  and  $\mu_{k_0}$ , the interpolation is performed in the respective tangent manifold if  $\mu_{k_0} \leq \theta$ . A similar procedure can be also done for the case of  $\mathbf{M}_k$  singular.

Note that in our application of the gas network, the system matrices  $\hat{\mathbf{A}}_{r,k}$  are regular while  $\hat{\mathbf{E}}_{r,k}$ ,  $\hat{\mathbf{B}}_{r,k}$  and  $\hat{\mathbf{C}}_{r,k}$  are singular.

**Interpolation of decoupled system.** The BT-ROM  $\hat{\Sigma}_{r,k}$  of (8) is in general not decoupled in proper and improper subsystems (cf. (4)) any more. Hence, any interpolated reduced order model is also not decoupled, as the matrix interpolation (5) preserves the structure of the matrices due to the element-wise performance. If the algebraic subsystem of the FOM  $\Sigma$  is parameter-invariant, then there is no interchange between the proper and improper BT-ROMs wrt. the parameter (involving a decoupled form of  $\hat{\Sigma}_r$ ). Hence, the subsystems can be adjusted and interpolated separately. In case of decoupled  $\hat{\Sigma}_r$ , only  $(r_f^2 + r_\infty^2)$  elementary operations are needed to approximate the matrix pencil  $\mathbf{A} - \lambda \mathbf{E}$ , instead of  $2(r_f + r_\infty)^2$  operation for the coupled system. This is more amenable to real-time applications. Note that in the gas networks under consideration the assumption on the FOM holds true, i.e., the algebraic coupling conditions are parameter-independent.

**Theorem 3.** Assume that the algebraic part of the FOM (3) is parameter-invariant. Then, the proper and improper systems  $\Sigma_{r,k}^{\text{prop}}$  and  $\Sigma_{r,k}^{\text{improp}}$  of the BT-ROMs (4),  $k = 1, \dots, N_p$ , can be separately transformed into generalized subspaces  $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_{r_f} \oplus \tilde{\mathcal{W}}_{r_\infty}$  and  $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{r_f} \oplus \tilde{\mathcal{V}}_{r_\infty}$ , which are spanned by  $\tilde{\mathbf{W}}_{r_f}$ ,  $\tilde{\mathbf{W}}_{r_\infty}$  and  $\tilde{\mathbf{V}}_{r_f}$ ,  $\tilde{\mathbf{V}}_{r_\infty}$  respectively. Furthermore, the transformation only requires the mapping of  $\Sigma_{r,k}^{\text{prop}}$  and  $\Sigma_{r,k}^{\text{improp}}$  into  $\tilde{\mathcal{V}}$ .

*Proof.* To facilitate the readability we suppress the parameter index  $k$ . Assume that  $\mathbf{T}_{V_{r_f}}$ ,  $\mathbf{T}_{W_{r_f}}$  are the transformations associated with the proper BT-ROM. According to (8), the transformed system  $\hat{\Sigma}_r^{\text{prop}}$  is given by

$$\begin{aligned} \hat{\mathbf{E}}_{r_f} \frac{d}{dt} \hat{\mathbf{x}}_{r_f}(t) &= \hat{\mathbf{A}}_{r_f} \hat{\mathbf{x}}_{r_f}(t) + \hat{\mathbf{B}}_{r_f} \mathbf{u}(t), & \mathbf{y}_{r_f}(t) &= \hat{\mathbf{C}}_{r_f} \hat{\mathbf{x}}_{r_f}(t) \\ \hat{\mathbf{E}}_{r_f} &= \mathbf{T}_{W_{r_f}}^T \mathbf{E}_{r_f} \mathbf{T}_{V_{r_f}}, & \hat{\mathbf{A}}_{r_f} &= \mathbf{T}_{W_{r_f}}^T \mathbf{A}_{r_f} \mathbf{T}_{V_{r_f}}, & \mathbf{E}_{r_f} &= \mathbf{I}_{r_f}, \\ \hat{\mathbf{B}}_{r_f} &= \mathbf{T}_{W_{r_f}}^T \mathbf{B}_{r_f}, & \hat{\mathbf{C}}_{r_f} &= \mathbf{C}_{r_f} \mathbf{T}_{V_{r_f}}, & \hat{\mathbf{x}}_{r_f} &= \mathbf{T}_{V_{r_f}} \mathbf{x}_{r_f}. \end{aligned}$$

Since  $\hat{\mathbf{E}}_{r_f}$  is regular,

$$\frac{d}{dt} \hat{\mathbf{x}}_{r_f}(t) = \hat{\mathbf{E}}_{r_f}^{-1} \hat{\mathbf{A}}_{r_f} \hat{\mathbf{x}}_{r_f}(t) + \hat{\mathbf{E}}_{r_f}^{-1} \hat{\mathbf{B}}_{r_f} \mathbf{u}(t)$$

leads to

$$\mathbf{T}_{V_{r_f}} \frac{d}{dt} \mathbf{x}_{r_f}(t) = \mathbf{T}_{V_{r_f}}^{-1} \mathbf{A}_{r_f} \mathbf{T}_{V_{r_f}} \mathbf{x}_{r_f}(t) + \mathbf{T}_{V_{r_f}}^{-1} \mathbf{B}_{r_f} \mathbf{u}(t).$$

The same can be shown for the improper BT-ROM in an analogue manner.  $\square$

## 5 Results and Discussion

Proceeding from a sample of BT-pROMs for gas pipeline-networks we demonstrate the applicability of the matrix interpolation strategy (MIS) for an efficient model order reduction. In particular, we compare different variants (with and without distortion treatment, with original and tangent manifold), regarding the outputs of the interpolated systems.

As test scenario we consider exemplarily the network topology *Fork* visualized in Fig. 1 over the time horizon  $[0, t_{\text{end}}]$ ,  $t_{\text{end}} = 48$  [h]. Although it is a rather small network consisting only of four pipes, the results are representative for the application. The pipes  $e_1, \dots, e_4$  have different lengths  $L_{e_1, \dots, 4} = (16, 45, 7, 38)$  [km], but same diameter  $D_e = 1$  [m] and roughness parameter  $\kappa_e = 5 \cdot 10^{-5}$  [m]. The last enters with the dynamical gas viscosity  $\mu = 10^{-5}$  [kg/(ms)] in the Chen formula for the friction  $\lambda$ . The specific gas constant is  $R_s = 448$  [J/(kg K)]. The boundary conditions (1e) of the gas network given by  $p(v, t) = p_0 + 0.5(1.05p_0 - p_0)(1 - \cos \pi t / t_{\text{end}})$  [bar] at  $v \in \mathcal{N}_{\text{in}}$  and  $q(v, t) = 200$  [kg/s] at  $v \in \mathcal{N}_{\text{out}}$  act as inputs, whereas the pres-

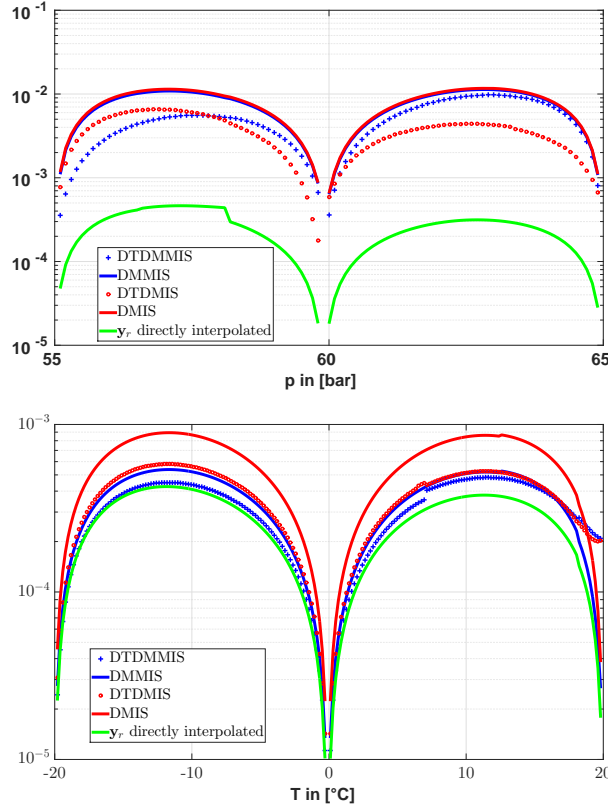
sure  $p(v, t)$  at  $v \in \mathcal{N}_{\text{out}}$  and the flow rate  $q(v, t)$  at  $v \in \mathcal{N}_{\text{in}}$  are considered as outputs for  $t \in [0, t_{\text{end}}]$ . In addition to the boundary pressure  $p_0 \in [55, 65]$  [bar], the temperature  $T \in [-20, 20]$  [°C] is regarded as parameter of the model problem, i.e.,  $\mathbf{p} = (p_0, T) \in \mathcal{P} \subset \mathbb{R}^2$ . Note that typical values  $p^* = 10^6$ ,  $q^* = 10$  and  $t^* = 10^2$  are used to scale pressure, flow rate and time so that the equation system is numerically easier to solve. The stationary problem is determined as follows: using the first Kirchhoff law (1c) and the boundary condition, the stationary flow rates of the pipeline-network are evaluated. Afterwards, the stationary pressure of each pipe is calculated by solving an initial value problem for  $e_1, \dots, e_4$ .

In the following the original FOMs are of order  $n = 35$  due to the spatial discretization with grid size  $\Delta x_{e_1, \dots, 4} = (6.4, 15, 2.33, 12.67)$  [km]. The BT-ROMs are chosen to be of order  $r = 15$ . The BT-ROMs decouple in proper and improper subsystems since the algebraic constraints of the FOM are parameter-independent. We solve them by means of the MATLAB routine `ode15s` (with the default values). Moreover, we use a cubic MIS. Note that the effective choices of the reduced model order and the interpolation order affect quantitatively, but not qualitatively the observed results. Quantitative improvement might be obtained by adapted more sophisticated choices, but this goes beyond the topic of this paper. Focusing on the matrix interpolation we explore here two exemplary model cases that show different parameter-dependent characteristics:

- Case I:  $p_0$  sampled at  $\{55, 59.5, 65\}$  and  $T = 0$ ,  
the generalized bases for the proper and improper BT-ROMs are constructed by using the POD method
- Case II:  $T$  sampled at  $\{-20, -0.49, 20\}$  and  $p_0 = 57.7$ ,  
the local bases at  $T = -0.49$  are chosen as generalized bases for the proper and improper BT-ROMs

We apply four different MIS variants: DMIS and DTMIS operate without and with distortion treatment on the original manifold, DMMIS and DTDMMIS operate without and with distortion treatment on the tangent manifold.

The approximation quality of the interpolation methods is presented in terms of the relative  $\mathcal{L}^2(0, t_{\text{end}})$ -error in Fig. 2, comparing the output of the interpolated system with that of the directly computed BT-ROM. In both model cases our proposed handling of length distortions (Theorem 2) shows a clear improvement. The approximation results are better than the ones achieved with the hitherto existing matrix interpolation strategy by [5]. The influence of the chosen manifold on the results depends on the considered case. Whereas the use of the original manifold seems beneficially in Case I, it is the tangent manifold in Case II. In total, the results concerning Case II are in size an order better than those of Case I which might be explained by less differences in the underlying local rank- $r$  subspaces. The larger the differences of the local rank- $r$  subspaces, the more difficult is the construction of a generalized subspace (to cover the most important dynamics of the system). In our application, the interpolation results are very robust for temperature variations. Changes in pressure, in contrast, might cause instabilities in the interpolated reduced order models, although the underlying sample of ROMs (interpolants) is



**Fig. 2** Comparison of different matrix interpolation strategies, relative  $\mathcal{L}^2(0, t_{end})$ -error between the outputs of the interpolated system and the computed BT-ROM. *Top*: Case I (pressure variations); *bottom*: Case II (temperature variations).

stable. This happens for example outside the considered interval  $[55, 65]$  in Case I. Developing interpolation techniques that preserve stability is hence topic of recent research. Considering the performance, the combination of MOR and an interpolation strategy is superior to computing directly a ROM at a new parameter setting, because the overall computational costs are dominated by the model order reduction technique. The costs due to our additional distortion handling are marginal.

Figure 2 shows additionally the results for directly interpolated outputs  $\mathbf{y}_{r,k}$ . As it is less error-prone, the direct output interpolation is certainly superior to MIS when only the outputs are of interest. However, optimization and control of transient gas networks require the input-output behavior for large input/output variations over a wide range of parameters. For this purpose, knowledge about the system matrices that belong to the different parameter settings is needed to make possible the cheap and fast evaluation of many reduced order models by help of MIS.

## 6 Conclusion

In this paper we proposed an extension of the matrix interpolation strategy by Geuss [5] for parametric MOR, regarding length distortions of the reduced order basis vector. We showed the applicability and especially the improvement of the results for gas transport in pipeline-networks. The combination of MOR and matrix interpolation allows for the efficient computation of parametric reduced order models and makes optimization and control of large transient gas networks possible. Thereby, the underlying model order reduction technique (here balanced truncation) and the interpolation order that are used are replaceable in view of the desired approximation quality. We remark that non-stable interpolated reduced order models might occur, although the sample of interpolants is stable. Thus, the development of stability-preserving interpolation techniques is addressed in future.

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