Solving Linear Generalized Nash Games Using an Active Signature Method

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Abstract

We propose a method to solve linear generalized Nash equilibrium problems (LGNEPs). For this purpose, a reformulation of the LGNEPs as piecewise linear problems is considered. This requires the calculation of all vertices for a special kind of unbounded convex polyhedra. Then the active signature method for constrained abs-linear problems can be used to determine the Nash equilibria. We analyse the computational effort for the resulting solution procedure. This includes also the verification of suitable optimality conditions. Finally, we present and analyse numerical results for some test problems.

1 Introduction and problem formulation

The task to solve optimization problems with piecewise linear objective functions arises in many applications. This includes train time tabling [12], the solution of local piecewise linear approximation, see, e.g., [25], and the training of deep neural networks with the Rectified Linear Unit (ReLU) as activation function [15, 32]. In this paper we concentrate on linear generalized Nash equilibrium problems (LGNEP) as an application. LGNEPs arise when players solve linear systems and share at least one constraint, for example, in transportation problems. In [30], the authors presented an approach to reformulate LGNEPs as piecewise linear problems with linear constraints.

So far, there is only a limited number of algorithms to solve such constrained nonsmooth optimization problems available. Possible approaches comprise for example quasi-Newton methods originally designed for smooth tasks [3] or bundle methods with various parameters to determine [26]. For a recent overview of these methods see [1]. They all have in common that they do not exploit the structure that is available in the nonsmooth setting. This is partly due to the lack of computationally tractable optimality respectively stationarity conditions. To overcome these challenges, for the unconstrained case optimality conditions that can be verified in polynomial time for a large set of piecewise smooth functions, the so-called abs-smooth functions, were derived in [19]. In [20, 22], Hegerhorst-Schultchen and Steinbach extended these optimality conditions to the constrained case by reformulating the inequality constraints as equality constraints using slack variables. Furthermore, it has been shown that each abs-smooth nonlinear optimization problem has an equivalent formulation as mathematical program with equilibrium constraints (MPECs) [21]. In the same paper, an equivalence of the corresponding regularity conditions was shown. These papers focus on optimality conditions and no algorithm to solve such problems was proposed.

For unconstrained optimization problems with piecewise linear objective functions, the so-called Active Signature Method (ASM) for determining local minima has been proposed in [16]. This approach explicitly builds on the nonsmooth structure to verify corresponding optimality conditions that can be verified in polynomial time even if the target function is nonsmooth as shown in [19]. An extension of this algorithm, i.e., the Constrained Active Signature Method (CASM), for the piecewise linear constrained case is presented in [24], however, without stating the optimality conditions explicitly. In the present paper, we will derive these optimality conditions directly for the problem class considered here providing an alternative proof for the piecewise linear case in comparison to [20, 22].

The structure of the paper is as follows. The remained of this section is dedicated to a more detailed introduction of GNEPs. In the next section, we introduce the mathematical formulation of LGNEPs and summarize the analysis for reformulating them as piecewise linear optimization problems. For the

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actual solution of these problems, i.e., to describe the corresponding feasible set, one has to determine the vertices of a pointed polyhedron, which is an NP-hard problem. Therefore, we examine the effort to find these vertices using a heuristic version of the Double Description Method [13] in Section 3. The optimality conditions employed by CASM are described in Section 4. Numerical results for the solution of the resulting piecewise linear problems are discussed in Section 5. Finally, a conclusion and an outlook are given in Section 6.

A situation where several competing players try to optimize their objective functions over strategy sets that are independent of the decisions of the remaining players is referred to as (classical) Nash equilibrium problem (NEP). A generalization of this problem formulation is given by the setting when the strategy sets of one player may depend on the decisions of the other players. Such dependencies arise for example when players share a common budget or commonly use specific infrastructure and lead to so-called GNEPs. More detailed information on GNEPs can be found in [8] and [11].

The numerical solution of GNEPs is still challenging. Therefore, a wide variety of approaches was already proposed. One possibility is the use of penalty terms. In [9], Facchnei and Kanzow presented a method in which the GNEP is reduced to single penalized and nonsmooth NEP. The idea of penalty terms was also considered in [10, 14]. Nonsmooth reformulations of GNEP based on a regularized Nikaido-Isoda function were investigated, for example in [5, 6] such that a nonsmooth, possibly constrained optimization problem has to be solved. Furthermore, there are also approaches to solve the GNEP based on KKT optimality conditions [7]. Here, concatenating KKT conditions for the optimization problems of each player are merged into a KKT-like system, where one has to handle the complementarity conditions appropriately. Merit functions and interior-point-based methods are used for this purpose. However, up to now, there is no off-the-shelf solution algorithm for (L)GNEPs. This might be due to the fact that the resulting optimization problems are NP-hard as can be concluded from the reformulation as piecewise linear optimization problem presented below.

2 Formulating LGNEPs as piecewise linear Functions

To propose an alternative solution method, we assume in the present paper that the cost functions and the constraints of all players are linear. An LGNEP consists of *players* $\nu \in \{1, \dots, N\}$ which control *decision vectors* $x^\nu \in \mathbb{R}^{n_\nu}$ for $N, n_\nu \in \mathbb{N}$ and $n := n_1 + \dots + n_N$. Each player ν tries to solve the linear optimization problem

$$\mathcal{Q}_\nu(x^{-\nu}): \quad \min_{x^\nu \in \mathbb{R}^{n_\nu}} \langle a^\nu, x^\nu \rangle \quad \text{s.t. } A^\nu x^\nu + B^\nu x^{-\nu} \leq c^\nu,$$

to find a $x := (x^\nu, x^{-\nu}) \in \mathbb{R}^n$, such that for each $\nu \in \{1, \dots, N\}$, x^ν is a global minimum of $\mathcal{Q}_\nu(x^{-\nu})$, where $x^{-\nu} \in \mathbb{R}^{n-n_\nu}$ are the decisions made by the other players. For each player, the matrices $A^\nu \in \mathbb{R}^{m_\nu \times n_\nu}$, $B^\nu \in \mathbb{R}^{m_\nu \times (n-n_\nu)}$ and vectors $a^\nu \in \mathbb{R}^{n_\nu}$, $c^\nu \in \mathbb{R}^{m_\nu}$ are given, with $m_\nu \in \mathbb{N}$. Each player has a *decision set* defined by

$$\mathrm{X}_\nu(x^{-\nu}) := \{x^\nu \in \mathbb{R}^{n_\nu} | A^\nu x^\nu \leq c^\nu - B^\nu x^{-\nu}\}$$

and

$$\mathrm{dom}\ \mathrm{X}_\nu := \{x^{-\nu} \in \mathbb{R}^{n-n_\nu} | \mathrm{X}_\nu(x^{-\nu}) \neq \emptyset\}.$$

In the whole paper we assume that for any $\nu \in \{1, \dots, N\}$ and $x^{-\nu} \in \mathrm{dom}\ \mathrm{X}_\nu$, $\mathcal{Q}_\nu(x^{-\nu})$ is solvable.

Example 2.1. *We consider an LGNEP with two one-dimensional players x^1, x^2. That is $n_1 = n_2 = 1$ and $N = n = 2$. Both players share the constraints*

$$A^1 x_1 + B^1 x_2 = \begin{pmatrix} -1 \\ 2 \\ -\frac{1}{2} \end{pmatrix} x^1 + \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{4} \\ -1 \end{pmatrix} x^2 \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c^1 = c^2 .$$

Hence, the dimensions are given by $m_1 = m_2 = 3$ and $A^2 = B^1, B^2 = A^1$. The target functions are defined by $a^1 = -2$, $a^2 = 3$.

In [30], the reformulation of an LGNEP as an optimization problem with a piecewise linear target function and linear constraints is presented. For this purpose, the authors introduce a global extension of the so-called gap function as piecewise linear concave function given by

$$\hat{V}(x) \coloneqq \sum_{\nu=1}^{N} \min_{o^\nu \in \mathcal{O}^\nu} \left(\langle a^\nu, x^\nu \rangle + \langle o^\nu, c^\nu - B^\nu x^{-\nu} \rangle \right), \tag{2.1}$$

where O^ν is the finite and nonempty set of all vertices of the pointed polyhedron [30]

$$\mathrm{Z}_\nu \coloneqq \{ o^\nu \in \mathbb{R}^{m_\nu} | a^\nu + (A^\nu)^T o^\nu = 0,\ o^\nu \geq 0 \}. \tag{2.2}$$

As shown in [30], all global minima of $\hat{V}(x)$ that are contained in the closed convex set

$$\begin{aligned} \mathrm{W} &\coloneqq \{ x \in \mathbb{R}^n | x^\nu \in \mathrm{X}_\nu(x^{-\nu}) \ , \forall \nu = 1, \ldots, N \} \\ &= \{ x \in \mathbb{R}^n | A^\nu x^\nu \leq c^\nu - B^\nu x^{-\nu} \ , \forall \nu = 1, \ldots, N \} \end{aligned} \tag{2.3}$$

are exactly the solutions of the LGNEP. Furthermore, it was proven in the same paper that a global minimizer x_* always has the function value $\hat{V}(x_*) = 0$. Since the function $\hat{V}(x)$ is piecewise linear and concave on the set W, all solutions of the minimization problem lie on the boundary of W. Furthermore, it can be deduced from this formulation that the solution of a LGNEP is NP-hard. Thus, one can not expect an efficient solution approach.

Example 2.2. *The polyhedra of Example 2.1 are given by*

$$\begin{aligned} Z_1 &= \{ o^1 \in \mathbb{R}^3 | -2 + \begin{pmatrix} -1 & 2 & -\frac{1}{2} \end{pmatrix} o^1 = 0 \ , \ o^1 \geq 0 \} = \{ o^1 \in \mathbb{R}^3 | A_{Z_2} o^1 \leq b_{Z_2} \}, \\ Z_2 &= \{ o^2 \in \mathbb{R}^3 | 3 + \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & -1 \end{pmatrix} o^2 = 0 \ , \ o^2 \geq 0 \} = \{ o^2 \in \mathbb{R}^3 | A_{Z_2} o^2 \leq b_{Z_2} \} \ . \end{aligned}$$

where

$$A_{Z_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 2 & -\frac{1}{2} \\ 1 & -2 & \frac{1}{2} \end{pmatrix}, \quad b_{Z_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ \frac{1}{2} & -\frac{3}{4} & -1 \\ -\frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix}, \quad \textit{and} \quad b_{Z_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 3 \end{pmatrix}$$

For those polyhedra one can determine the sets of vertices as

$$\mathcal{O}^1 = \{ (0\ 1\ 0)^\top \} \quad \textit{and} \quad \mathcal{O}^2 = \{ (0\ 0\ 3)^\top, (0\ 4\ 0)^\top \} \ .$$

yielding the target function

$$\hat{V}(x) = -2x^1 + \frac{15}{4}x^2 + 4 + \min\{ \frac{3}{2}x^1, 1 - 8x^1 \}.$$

Figure 1 shows the feasible set W and Figure 2 the target function $\hat{V}(x)$ together with the set where $\hat{V}(x) = 0$. The red line indicates the points on the boundary of W, i.e., the set of solutions of the LGNEP given by the line segment $[(\frac{2}{19}, -\frac{20}{19}), (5, 12)]$.

In conclusion one obtains the optimization problem

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \hat{V}(x) \\ &\text{s.t. } x \in \mathrm{W}. \end{aligned} \tag{2.4}$$

with a piecewise linear target function and linear constraints. For the computation of the original LGNEP one faces two challenges. First, one has to calculate all vertices of Z_ν to obtain an explicit formulation of $\hat{V}(x)$. As described in the next section in more detail, this is an NP-hard problem. For this reason, we propose to use a heuristic and analyze its performance in the context of LGNEPs. It should be noted that this procedure does not need any boundedness of the sets Z_ν or X as often assumed in other publications. However, a drawback is the higher effort required to calculate the vertices of Z_ν. Secondly, one has to

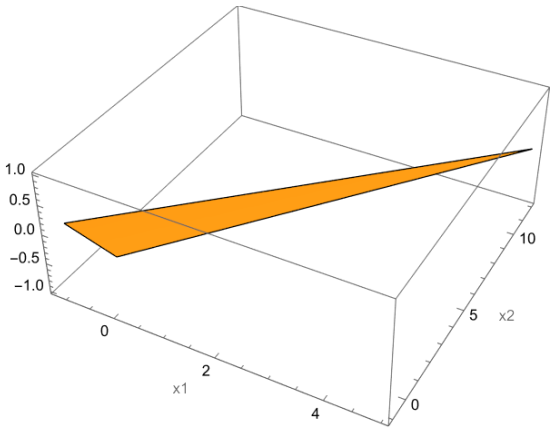


Figure 1: Feasible set W

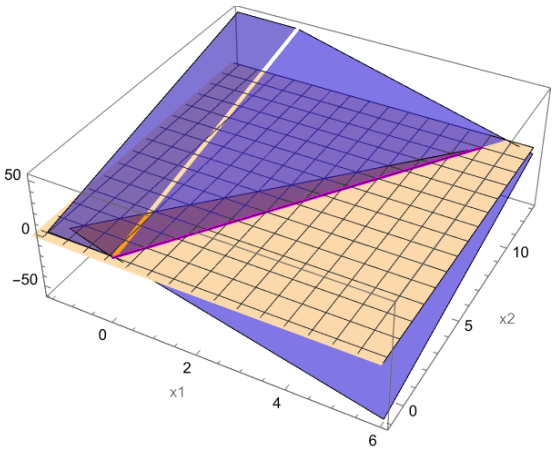


Figure 2: Target function $\hat{V}(x)$

solve the resulting piecewise linear problems with constraints. For this purpose, next we introduce a specific formulation of piecewise linear functions and analyse corresponding optimality conditions for such optimization problems in Section 4.

It follows from [29, Prop. 2.2.2] in combination with the reformulations

$$\max(x,y) = \frac{1}{2}(x+y+|x-y|) \quad \text{and} \quad \min(x,y) = \frac{1}{2}(x+y-|x-y|) \quad \text{for} \quad x,y \in \mathbb{R} \tag{2.5}$$

that every continuous piecewise linear function can be represented in the so-called abs-linear form as introduced in a slightly different form already in, e.g., [16, 17, 18]. Since we want to incorporate constraints in a consistent way, we adapt the abs-linear form slightly and consider the *abs-linear form* given by

$$y = d + a^\top x + b^\top z \,, \tag{2.6a}$$
$$z = c + Zx + Mz + L|z| \,, \tag{2.6b}$$

with $x \in \mathbb{R}^n$ as arguments, $z \in \mathbb{R}^s$ the switching variables, called *switching vector*, and constants $d \in \mathbb{R}$, $a \in \mathbb{R}^n$, $b, c \in \mathbb{R}^s$, $Z \in \mathbb{R}^{s\times n}$, $L, M \in \mathbb{R}^{s\times s}$, where the last two matrices are assumed to be strictly lower triangular requiring that z_i only depends on switching variables z_j with $1 \le j < i$. Here, and throughout, $|z|$ denotes the component-wise modulus of a vector z. Eq. (2.6b) is called *switching system*.

The abs-linear form can be generated in an automated fashion using extended algorithmic differentiation (AD) as provided for example by the tool ADOL-C [34]. Hence, it is important to note that it is not necessary to reformulate the global extension of the gap function (2.1) in an abs-linear form to apply the algorithm proposed in this paper. The advantage of the abs-linear form is that it provides a lot of structure to solve piecewise linear optimization problems. Using the signatures of the switching vector (2.6b), it is possible to decompose $\mathbb{R}^n$ into polyhedra [19], using the *signature vector* given by

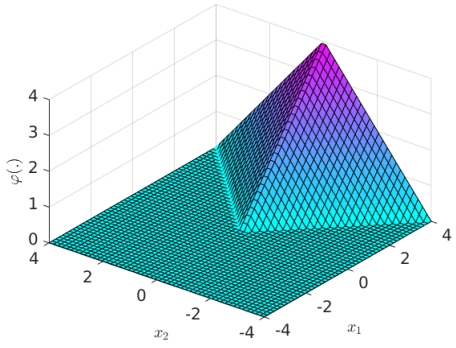
$$\sigma(x) \equiv (\mathbf{sgn}(z_1(x)), \ldots, \mathbf{sgn}(z_s(x))) \in \{-1, 0, 1\}^s \,,$$

where the set on the right-hand side comprises all possible signature vectors. The corresponding *signature matrix* is given by $\Sigma(x) = \mathbf{diag}(\sigma(x))$. A signature vector $\sigma(x)$ is called *definite*, if no component vanishes, i.e., $\sigma(x) \in \{-1, 1\}^s$. This situation is denoted by $0 \notin \sigma(x)$. Otherwise, it is called *indefinite*.

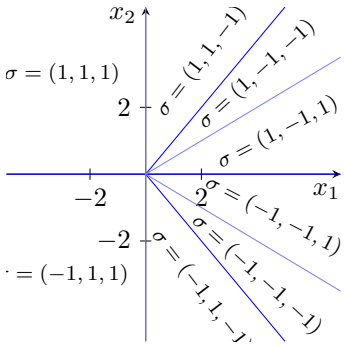
When allowing also piecewise linear constraints as generalization to the linear constraint given by Eq. (2.3) one obtains a constrained abs-linear optimization problem (CALOP) as introduced in [23, 24]

$$\begin{aligned} \min_{x\in\mathbb{R}^n, z\in\mathbb{R}^s} \quad & a^\top x + b^\top z \\ \text{s.t.} \quad & 0 = g + Ax + Bz + C|z| \,, \\ & 0 \ge h + Dx + Ez + F|z| \,, \\ & z = c + Zx + Mz + L|z| \,, \end{aligned} \tag{CALOP}$$

where $g \in \mathbb{R}^m$, $h \in \mathbb{R}^p$, $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{m\times s}$, $D \in \mathbb{R}^{p\times n}$ and $E, F \in \mathbb{R}^{p\times s}$. Since the constant d in the abs-linear formulation (2.6a) does not change the optimal point, it has been dropped. For later use,



The target function $f()$



Polyhedra that belong to the corresponding definite signature vectors

Figure 3: Illustration of the unconstrained case from Example 2.3

we define

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}, & (x,z) &\mapsto a^\top x + b^\top z\,, \\ G &: \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^m, & (x,z,|z|) &\mapsto g + Ax + Bz + C|z|\,, \\ \text{and} \quad H &: \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}^p, & (x,z,|z|) &\mapsto h + Dx + Ez + F|z|\,. \end{aligned} \tag{2.7}$$

For the constrained optimization problem (CALOP), the functions G and H may or may not depend on the value $|z_s|$ as illustrated in the next example. Therefore, we denote the total number of all switching variables z_i that occur as arguments in an evaluation of the absolute value in the target function or in the constraints by $\tilde{s} \leq s$ and assume that they are located in the first $\tilde{s}$ switching variables. If this is not the case, the abs-linear form of (CALOP) can be adapted such that $\tilde{s} \in \{s-1, s\}$.

Example 2.3. *Let the function* $f(x_1, x_2) = \max\{0, x_1 - |x_2|\}$ *be given. This nonsmooth nonconvex function is illustrated on the left-hand side of Fig. 3. Using the reformulation of the* max*-function given in Equation (2.5), we obtain*

$$f(x_1,x_2) = \frac{1}{2}\left(x_1 - |x_2| + \left|-x_1 + |x_2|\right|\right),$$

which can be converted into the following abs-linear form, see Equation (2.6):

$$\begin{aligned} z &= \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + |z_1| \\ -\frac{1}{2}|z_1| + \frac{1}{2}|z_2| \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} |z_1| \\ |z_2| \\ |z_3| \end{pmatrix} \\ y &= 0 + \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = f(x,z)\,. \end{aligned}$$

If we consider this function as objective of an unconstrained optimization problem, the polyhedra resulting from definite signature vectors are shown in Fig. 3, where we used the corresponding signature vectors as labels for the polyhedra. The blue lines mark the arguments that cause the nonsmoothness and the light blue lines correspond to the last component of the switching vector that does not lead to any nonsmoothness in the target function. Now, we add the constraint

$$|z_3| = \left|-\frac{1}{2}|x_2| + \frac{1}{2}\right| - x_1 + |x_2|\Big| \leq 2$$

that can be formulated as

$$H(x,z,|z|) = -2 + \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |z_1| \\ |z_2| \\ |z_3| \end{pmatrix} \leq 0$$

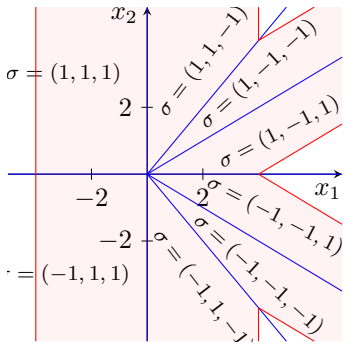


Figure 4: Polyhedra for the definite signature vectors in the constrained case of Ex. 2.3.

to obtain a constrained optimization problem of the form (CALOP). Then, $|z_3|$ contributes explicitly to the evaluation of the abs-linear constraint.

Fig. 4 shows for the constrained situation the polyhedra resulting from the definite signature vectors using the corresponding σ as label. In comparison to the unconstrained case, further kinks are added resulting in more polyhedra. The red area represents the feasible set.

3 Calculation of the Vertices

To derive an explicit formula for $\hat{V}(x)$, first one has to calculate the vertices of the pointed convex polyhedra Z_ν for each $\nu \in \{1, \ldots, N\}$. Since each polyhedron Z_ν is pointed, it must contain at least one vertex [30]. Furthermore, it follows from the finite set of constraints describing the set Z_ν that there can only be a finite number of vertices. Therefore, the task of computing $\mathcal{O}^\nu$ is well posed. The determination of all required vertices may require a high computational effort. On the other hand, it has to be performed only once, since it is independent of the value of the optimization variable x. Hence, this calculation can be done a-priori before the optimization process is started. In this section, we will analyze one possible algorithm to compute the sets $\mathcal{O}^\nu$ and investigate this computational complexity in detail.

So far, there is no method known that can calculate all vertices of an arbitrary unbounded convex polyhedron of the form

$$P = \{x \in \mathbb{R}^d |\ Ax \leq b\} \tag{3.1}$$

in polynomial time. Therefore, we rely on the Double Description Method (DD-method) [27] with improvements proposed in [13]. The general complexity of the DD-method in terms of the sizes of inputs and their properties is difficult to predict. Bremner [2] proved that there is a class of polyhedra for which the DD-method has exponential complexity, a behavior that we also observe in our numerical experiments. This is due to the fact that the intermediate polytopes encountered by the algorithm can become very complex relative to the original polyhedron. Additionally, the algorithm is known to be very sensitive with respect to the permutations of the rows of the matrix A. Therefore, the improvements in [13] are based on rearrangements of the rows of A including heuristic approaches to do so to improve the performance of the DD-Method. These heuristics did also occur in our experiments.

For our numerical tests, we used a publicly available Python implementation of the DD-method [33]. Due to the complexity results for the DD-method mentioned above, we explicitly investigate the cost of the DD-method for the LGNEP problem and therefore examine polyhedrons of the form (2.2). Since the DD-method requires a problem formulation of the form (3.1), we reformulated (2.2) into

$$\mathrm{Z}_\nu = \left\{ o^\nu \in \mathbb{R}^{m_\nu} \middle| \begin{pmatrix} -\mathrm{I}_\nu \\ (A^\nu)^\top \\ -(A^\nu)^\top \end{pmatrix} o^\nu \leq \begin{pmatrix} 0 \\ -a^\nu \\ a^\nu \end{pmatrix} \right\},$$

where $\mathrm{I}_\nu \in \mathbb{R}^{m_\nu \times m_\nu}$ is the identity matrix. This reformulation yields $m_\nu + 2n_\nu$ inequalities for each player x^ν. In the remainder of this section, we investigate the compute time of the DD-method for this special case of inequalities and matrices. Hence, we only consider one player of the LGNEP with the suitable polyhedron formulation for the DD-methods.

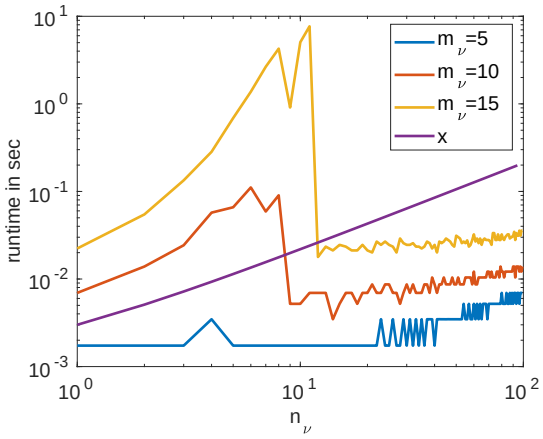


Figure 5: Effort with growing n_ν

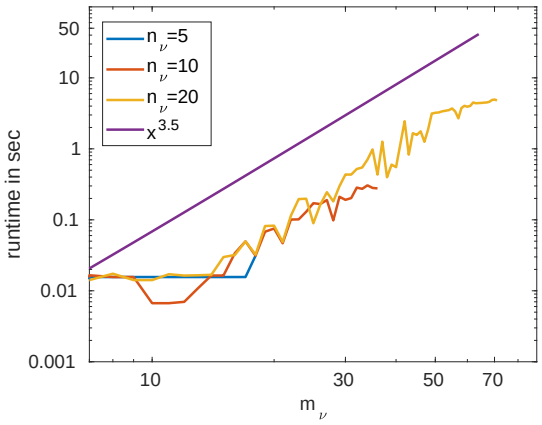


Figure 6: Effort with growing m_ν

For the numerical experiments, the entries of A^ν are chosen to be random numbers in the interval [-1,1]. Due to this construction, the sets Z_ν often appear to be empty for $n_\nu > m_\nu$, since then more possible contradictory inequalities than variables given appear. We distinguish between the number of optimization variables n_ν of player ν and the number of constraints m_ν for player ν.

First, we fix m_ν to be equal to 5, 10 and 15 and increase n_ν, where for the underlying LGNEP, this number n_ν corresponds to the space dimension of the player ν. Due to the heuristic approach in the algorithm, we use the average value from 15 runs for each dimension to obtain an average run time for an increasing number of optimization variables n_ν. As can be seen from Figure 5, there are still some fluctuations visible, but one can clearly see a trend very similar to a linear growth as soon as n_ν gets significantly larger than m_ν. For $n_\nu < m_\nu$ each graph shows an exponential growth with respect to n_ν which corresponds to the theoretical effort of the DD-Method.

Figure 6 shows the computing time depending on m_ν for a fixed value n_ν to be either 5, 10 or 20. Two main observations can be made. First, and most obviously, one observes that the effort grows exponentially with respect to m_ν. This is clearly due to the non emptiness of the sets Z_ν and therefore, finding vertices remains a well posed problem for growing m_ν and the exponential growth of the DD-Method can be verified. Furthermore, we found that the fluctuation of the runtime is not as strong as for varying n_ν. Therefore, it was not necessary to take the average of several values. These fluctuations seam to correspond with the emptiness of the sets Z_ν.

One can see the advantage obtained by using the heuristic approach of the DD-Method in [13] by the fact that the effort for the special case in this paper is significantly lower than expected effort of the nonheuristic DD-Method. But there also accrue some problem formulations in our case which do not benefit from this approach. Hence, there might be an exponentially growing effort to find all vertices. As a conclusion, one finds that for the explicit formulation of the globalized gap function, there is a fast-growing computation effort both in the number of constraints for the players and the number of optimization variables.

4 Necessary and sufficient optimality conditions for constrained abs-linear problems

In this section, we derive optimality conditions for the constrained optimization problem (CALOP) as defined above and show that they can be verified in polynomial time at a given point. This extends directly the optimality conditions derived in [17] for the unconstrained abs-linear case. For the more general class of abs-smooth problems optimality conditions that can be verified in polynomial time were already derived in [20, 22]. However, since we restrict ourselves to the class of abs-linear functions and admit both z and $|z|$ as arguments for all functions, we prove the optimality conditions directly for our formulation considered in this paper instead of showing the equivalence of our problem statement to the problem formulation used in [20, 21, 22]. This approach has the advantage that the optimality conditions derived here can be directly used in CASM and no further adaptation is needed. Furthermore, the more general formulation with z and $|z|$ as arguments for all functions may allow additional structure exploitation like sparsity in the algorithm.

In Section 2, we already illustrated the polyhedra that result from the abs-linear form. Now, we formalize them in the following way:

Definition 4.1 (Feasible (extended) signature domain)**.** *Let an optimization problem of the form* (CALOP) *be given. For a fixed signature vector* $\sigma \in \{-1, 0, 1\}^s$*, we define*

$$\mathcal{F}_\sigma \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} G(x, z(x), \Sigma z(x)) = 0, \\ H(x, z(x), \Sigma z(x)) \leq 0, \\ \mathbf{sgn}(z(x)) = \sigma, \end{array} \right\} \subset \overline{\mathcal{F}}_\sigma \equiv \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} G(x, z(x), |z(x)|) = 0, \\ H(x, z(x), |z(x)|) \leq 0, \\ \Sigma z(x) = |z(x)| \end{array} \right\}.$$

The set $\mathcal{F}_\sigma$ *is called* feasible signature domain *and* $\overline{\mathcal{F}}_\sigma$ *the* feasible extended signature domain*.*

In a similar way as we introduced the signature vector for kinks, we define a vector containing the signs of the inequality constraints.

Definition 4.2 (Signature vector, signature matrix of inequality constraints)**.** *Let a point* $x \in \mathbb{R}^n$ *be given that fulfills the equality and inequality constraints of* (CALOP)*. We define the* signature vector of the inequality constraints *as*

$$\omega(x) \equiv \mathbf{sgn}(H(x, z, |z|)) \in \{-1, 0\}^p \, .$$

The j*th inequality constraint is called* active *if* $\omega_j(x) = 0$ *and* inactive *otherwise. The* signature matrix of the inequality constraints *is denoted by* $\Omega(x) = \mathbf{diag}(\omega(x))$*. Furthermore,* $\mathcal{I} \equiv \mathcal{I}(x)$ *collects the indices of the active inequality constraints at* x*. The projection onto the active components of* $H(x)$ *is defined as* $P_\mathcal{I} \equiv (e_i^\top)_{i \in \mathcal{I}} \in \mathbb{R}^{|\mathcal{I}| \times p}$ *with* e_i *denoting the* i*th unit vector of appropriate size.*

Next, we prepare the formulation of optimality conditions that can be verified in polynomial time. For this purpose, we introduce the following notations:

Definition 4.3 (Active switching variables)**.** *A switching variable* z_i *is called* active *at* x *if* $z_i(x) = 0$*. The* active switching set $\alpha(x)$ *collects all indices of switching variables that are active at* x *occur as arguments of the absolute value, i.e.,*

$$\alpha(x) \equiv \{i \in \{1, \ldots, \tilde{s}\} \mid z_i(x) = 0\} \, .$$

The projection onto the active components of $z(x)$ *is defined as* $P_\alpha \equiv (e_i^\top)_{i \in \alpha} \in \mathbb{R}^{|\alpha| \times s}$ *with* e_i *denoting the* i*th unit vector of appropriate size.*

For each fixed signature vector $\sigma \in \{-1, 0, 1\}^s$, we obtain from (CALOP) the smooth optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} \quad & a^\top x + b^\top z && \text{(4.1a)} \\ \text{s.t.} \quad & 0 = g + Ax + Bz + C\Sigma z \, , && \text{(4.1b)} \\ & 0 \geq h + Dx + Ez + F\Sigma z \, , && \text{(4.1c)} \\ & z = c + Zx + Mz + L\Sigma z \, , && \text{(4.1d)} \\ & 0 = (I_s - |\Sigma|)z \, , \quad 0 \leq \Sigma z \, . && \text{(4.1e)} \end{aligned}$$

Next, we consider an optimal point of this smooth optimization problem:

Definition 4.4 (Feasible signature optimal point)**.** *Let an optimization problem of the form* (CALOP) *be given. Consider a fixed signature vector* $\sigma \in \{-1, 0, 1\}^s$*. A minimizer* $x_\sigma \in \mathcal{F}_\sigma$ *of the optimization problem* (4.1) *is called* feasible signature optimal point *of the constrained optimization problem* (CALOP)*.*

To derive the optimality conditions, we define

$$\tilde{Z} = (I_s - M - L\Sigma)^{-1} Z \qquad \text{and} \qquad \tilde{c} = (I_s - M - L\Sigma)^{-1} c \, . \tag{4.2}$$

Then one can combine Eqs. (4.1d) and the left equality of (4.1e) to one equality constraint and obtains the following optimization problem that is equivalent to the one stated in Eq. (4.1)

$$\begin{aligned} \min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} \quad & a^\top x + b^\top |\Sigma| z && \text{(4.3a)} \\ \text{s.t.} \quad & 0 = g + Ax + B|\Sigma|z + C\Sigma z \, , && \text{(4.3b)} \\ & 0 \geq h + Dx + E|\Sigma|z + F\Sigma z \, , && \text{(4.3c)} \\ & 0 = |\Sigma|z - \tilde{c} - \tilde{Z}x \, , \quad 0 \leq \Sigma z \, . && \text{(4.3d)} \end{aligned}$$

Since we consider only linear constraints, one has for the optimization problem (4.3) that the set of feasible directions at x coincides with the tangent cone at x, see [28, Lem. 12.7]. In this case, no further constraint qualification is needed to ensure the existence of Lagrange multipliers, but their uniqueness is not guaranteed. Our goal is to derive optimality conditions that can be verified in polynomial time. Hence, any dependence on the signature vectors that would lead to a combinatorial complexity in 2^s in the worst case must be avoided. Therefore, we have to ensure that the Lagrange multipliers are unique, see also [18]. For this reason, we adapt the kink qualification LIKQ introduced in [19] for the unconstrained case and in [22] for general constrained nonsmooth nonlinear optimization problems. Since we focus in this paper on the piecewise linear case, LIKQ can be specified in a matrix-based representation.

In the unconstrained case, LIKQ requires the full rank of $P_\alpha \tilde{Z}$, i.e., the active Jacobian of the reformulated switching system. To derive a similar result for the constrained case, we analyze the optimization problem (4.3) for a fix signature vector σ and a corresponding feasible x_σ in more detail. Due to the continuity of all involved functions and the relation $\Sigma z = |z|$, the components z_i, $i \notin \alpha$, of the vector z as determined by Eq. (4.3d) will not drop to zero in an open neighborhood $U(x_\sigma)$ of x_σ. In combination with the identity $\Sigma z = \Sigma|\Sigma|z$, in $U(x_\sigma)$ the optimization problem (4.3) is then equivalent to

$$\min_{x \in U(x_\sigma)} \quad a^\top x + b^\top |\Sigma|(\tilde{c} + \tilde{Z}x) \tag{4.4a}$$

$$\text{s.t.} \quad 0 = g + Ax + B|\Sigma|(\tilde{c} + \tilde{Z}x) + C\Sigma(\tilde{c} + \tilde{Z}x)\ , \tag{4.4b}$$

$$0 \geq h + Dx + E|\Sigma|(\tilde{c} + \tilde{Z}x) + F\Sigma(\tilde{c} + \tilde{Z}x)\ , \tag{4.4c}$$

$$0 = P_\alpha(\tilde{c} + \tilde{Z}x)\ . \tag{4.4d}$$

Definition 4.5 (Active Jacobian)**.** *Consider for the constrained optimization problem* (CALOP) *and a given signature vector $\sigma \in \{-1, 0, 1\}^s$ a point x_σ that is feasible for the problem given by Eq.* (4.4)*. The* active Jacobian *at x_σ is given by*

$$\mathcal{J}_\sigma \equiv \begin{pmatrix} A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z} \\ P_{\mathcal{I}}(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) \\ P_\alpha \tilde{Z} \end{pmatrix} \in \mathbb{R}^{(m+|\mathcal{I}|+|\alpha|)\times n}\ .$$

Now, the required kink qualification can be stated for the setting considered here:

Definition 4.6 (LIKQ (constrained case))**.** *Let a constrained optimization problem of the form* (CALOP) *and a signature vector $\sigma \in \{-1, 0, 1\}^s$ be given. We say that the* Linear Independence Kink Qualification (LIKQ) *holds at a feasible point x_σ if the active Jacobian $\mathcal{J}_\sigma$ at x_σ has full row rank $m + |\mathcal{I}| + |\alpha|$.*

Next, we show that the optimality of a feasible signature optimal point can be verified in polynomial time extending the results given in [17] to the constrained case.

Theorem 4.6.1 (Necessary and sufficient optimality conditions)**.** *Let a constrained optimization problem of the form* (CALOP) *and a signature vector $\sigma \in \{-1, 0, 1\}^s$ be given. Assume that x_σ is feasible signature optimal for* (CALOP) *and that LIKQ holds at x_σ. Then x_σ is a local minimizer of* (CALOP) *if and only if there exist unique Lagrange multipliers $\delta \in \mathbb{R}^m$, $0 \leq \nu \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^s$, such that*

$$0 = a^\top + b^\top|\Sigma|\tilde{Z} + \delta^\top(A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) - \lambda^\top P_\alpha^\top P_\alpha \tilde{Z}\ , \tag{4.5}$$

$$0 = b^\top|\Sigma| + \delta^\top\left(B|\Sigma| + C\Sigma\right) + \nu^\top\left(E|\Sigma| + F\Sigma\right) + \lambda^\top|\Sigma| \tag{4.6}$$

and

$$|P_\alpha(b + B^\top\delta + E^\top\nu + \lambda)| \leq P_\alpha(C^\top\delta + F^\top\nu - \tilde{L}^\top\lambda) \tag{4.7}$$

with $\tilde{L}$ given by

$$\tilde{L} = (I_s - M - L\Sigma)^{-1}L\ .$$

Proof. First, let x_σ be a local minimizer of (CALOP). Since x_σ is feasible signature optimal for the signature vector σ, x_σ is a minimizer of the optimization problem (4.4). Then, we obtain from KKT theory that there exist unique Lagrange multipliers $\delta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^p_+$ and $\check{\lambda} \in \mathbb{R}^{|\alpha|}$ associated with the equality constraint (4.4b), the inequality constraint (4.4c) and the reformulated switching system (4.4d) such that

$$0 = a^\top + b^\top|\Sigma|\tilde{Z} + \delta^\top(A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) + \check{\lambda}^\top P_\alpha \tilde{Z}\ .$$

Hence, together with $\delta \in \mathbb{R}^m$ and $0 \leq \nu \in \mathbb{R}^p$, each vector $\lambda \in \mathbb{R}^s$ such that $\check{\lambda} = -P_\alpha \lambda$ fulfills Eq. (4.5).

As introduced before, $\omega = \omega(x_\sigma)$ denotes the signature vector of the inequality constraints. Then, it is necessary and sufficient for local minimality that $(x_\sigma, z(x_\sigma))$ is a minimizer of $f(.,.)$ as defined in Eq. (2.7) on all feasible extended signature domains $\overline{\mathcal{F}}_{\tilde{\sigma}}$ with definite $\tilde{\sigma} \succ \sigma$, where $\tilde{\sigma} \succeq \sigma$ holds if

$$\tilde{\sigma}_j \sigma_j \geq \sigma_j^2 \quad \text{for} \quad j = 1, \ldots, s \, .$$

Any such $\tilde{\sigma} \succ \sigma$ can be written as $\tilde{\sigma} = \sigma + \gamma$ with $\gamma \in \{-1, 0, 1\}^s$ structurally orthogonal to σ such that for $\Gamma \equiv \mathbf{diag}(\gamma)$ we have the matrix equations

$$\tilde{\Sigma} = \Sigma + \Gamma \quad \text{and} \quad \Sigma\Gamma = 0 = |\Sigma|\Gamma \, . \tag{4.8}$$

Then we can express $z(x) = z_{\tilde{\sigma}}(x)$ for $x \in \mathcal{P}_{\tilde{\sigma}}$ as

$$z_{\tilde{\sigma}}(x) = z_{\sigma+\gamma}(x) = (I_s - M - L\Sigma - L\Gamma)^{-1}(c + Zx) = (I_s - \tilde{L}\Gamma)^{-1}(\tilde{c} + \tilde{Z}x) \, . \tag{4.9}$$

Since x_σ must be a minimizer of the objective also on $\overline{\mathcal{F}}_{\tilde{\sigma}}$, it solves the smooth optimization problem

$$\begin{aligned}
&\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^s} && a^\top x + b^\top \left(|\Sigma| + |\Gamma|\right) z && \text{(4.10a)}\\
&\text{s.t.} && 0 = g + Ax + B\left(|\Sigma| + |\Gamma|\right) z + C(\Sigma + \Gamma)z \, , && \text{(4.10b)}\\
& && 0 \geq h + Dx + E\left(|\Sigma| + |\Gamma|\right) z + F(\Sigma + \Gamma)z \, , && \text{(4.10c)}\\
& && 0 = (I_s - \tilde{L}\Gamma)z - \tilde{c} - \tilde{Z}x \, , && \text{(4.10d)}\\
& && 0 \leq P_\alpha \Gamma z \, . && \text{(4.10e)}
\end{aligned}$$

Then, we obtain from KKT theory that there exist Lagrange multipliers $\delta \in \mathbb{R}^m$, $0 \leq \nu \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^s$ and $0 \leq \mu \in \mathbb{R}^{|\alpha|}$ associated with the equality constraint, the inequality constraint, the reformulated switching system and the sign conditions such that

$$0 = a^\top + \delta^\top A + \nu^\top D - \lambda^\top \tilde{Z} \quad \text{and} \tag{4.11}$$

$$\begin{aligned}
0 = {} & b^\top \left(|\Sigma| + |\Gamma|\right) + \delta^\top \left(B\left(|\Sigma| + |\Gamma|\right) + C(\Sigma + \Gamma)\right)\\
& + \nu^\top \left(E\left(|\Sigma| + |\Gamma|\right) + F(\Sigma + \Gamma)\right) + \lambda^\top \left(I_s - \tilde{L}\Gamma\right) - \mu^\top P_\alpha \Gamma \, .
\end{aligned} \tag{4.12}$$

Since the optimization problem (4.10) is linear, these conditions together with the feasibility of the variables and the complementarity condition are necessary as well as sufficient for x_σ to be a minimizer. Multiplying the last equation from the right by $|\Sigma|\tilde{Z}$, we obtain with the identity $\Sigma = \Sigma|\Sigma|$ and Eq. (4.8)

$$0 = b^\top |\Sigma| \tilde{Z} + \delta^\top \left(B|\Sigma| + C\Sigma\right) \tilde{Z} + \nu^\top \left(E|\Sigma| + F\Sigma\right) \tilde{Z} + \lambda^\top |\Sigma| \tilde{Z} \, . \tag{4.13}$$

Adding this equality to Equation (4.11) and exploiting

$$I_s = |\Sigma| + P_\alpha^\top P_\alpha \tag{4.14}$$

yields

$$0 = a^\top + b^\top |\Sigma| \tilde{Z} + \delta^\top (A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}) + \nu^\top (D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}) - \lambda^\top P_\alpha^\top P_\alpha \tilde{Z} \, .$$

Hence, it follows that the Lagrange multipliers $\delta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^s$ fulfill Eq. (4.5) with $\check{\lambda} = -P_\alpha \lambda$. Due to the kink qualification LIKQ, one also has that the vectors $\delta \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ as well as the components $P_\alpha \lambda \in \mathbb{R}^{|\alpha|}$ are determined uniquely. The remaining components of $\lambda \in \mathbb{R}^s$ can be obtained by multiplying Eq. (4.12) this time only with $|\Sigma|$ from the right yielding

$$0 = b^\top |\Sigma| + \delta^\top \left(B|\Sigma| + C\Sigma\right) + \nu^\top \left(E|\Sigma| + F\Sigma\right) + \lambda^\top |\Sigma|$$

and thus Eq. (4.6). To derive the third condition (4.7), we multiply Eq. (4.12) from the right by $\Gamma P_\alpha^\top$. Using

$$P_\alpha^\top P_\alpha = \Gamma\Gamma = |\Gamma| \text{ and } P_\alpha P_\alpha^\top = I_{|\alpha|} \tag{4.15}$$

and $\mu \geq 0$, it follows that

$$\begin{aligned}
-(b^\top + \delta^\top B + \nu^\top E + \lambda^\top)\Gamma P_\alpha^\top &= (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L})\Gamma\Gamma P_\alpha^\top - \mu^\top\\
&\leq (\delta^\top C + \nu^\top F - \lambda^\top \tilde{L}) P_\alpha^\top \, .
\end{aligned}$$

Now the key observation is that this condition is linear in Γ and is strongest for the choice

$$\gamma_i = -\mathbf{sgn}(\lambda^\top + b^\top + \delta^\top B + \nu^\top E)_i \quad \text{for} \quad i \in \alpha\,.$$

Hence, one obtains the inequalities

$$|(b + B^\top\delta + E^\top\nu + \lambda)_i| \leq e_i(C^\top\delta + F^\top\nu - \tilde{L}^\top\lambda) \quad \text{for} \quad i \in \alpha$$

showing Eq. (4.7) and therefore the necessary optimality conditions.

Second, we show that these conditions are also sufficient. For this purpose, we consider again all adjacent extended signature domains $\overline{\mathcal{F}}_{\tilde{\sigma}}$. For this purpose, we multiply Eq. (4.12) again from the right by $\Gamma P_\alpha^\top$ and use Eqs. (4.15) and (4.7) to obtain

$$\mu^\top = \left(b^\top + \delta^\top B + \nu^\top E + \lambda^\top\right)\Gamma P_\alpha^\top + \left(\delta^\top C + \nu^\top F - \lambda^\top\tilde{L}\right)P_\alpha^\top \geq 0$$

and thus the feasibility. Exploiting Eq. (4.14), Eq. (4.6) multiplied from the right by $\tilde{Z}$ and Eq. (4.5) yields

$$\begin{aligned}
\lambda^\top\tilde{Z} &= \lambda^\top\left(|\Sigma| + P_\alpha^\top P_\alpha\right)\tilde{Z} = \lambda^\top|\Sigma|\tilde{Z} + \lambda^\top P_\alpha^\top P_\alpha\tilde{Z} \\
&= -b^\top|\Sigma|\tilde{Z} - \delta^\top\left(B|\Sigma| + C\Sigma\right)\tilde{Z} - \nu^\top\left(E|\Sigma| + F\Sigma\right)\tilde{Z} \\
&\quad + a^\top + b^\top|\Sigma|\tilde{Z} + \delta^\top\left(A + B|\Sigma|\tilde{Z} + C\Sigma\tilde{Z}\right) + \nu^\top\left(D + E|\Sigma|\tilde{Z} + F\Sigma\tilde{Z}\right) \\
&= a^\top + \delta^\top A + \nu^\top D
\end{aligned}$$

and hence Eq. (4.11). Using Eq. (4.14), Eq. (4.12) holds if and only if

$$\begin{aligned}
0 = {} & b^\top\left(|\Sigma| + |\Gamma|\right) + \delta^\top\left(B\left(|\Sigma| + |\Gamma|\right) + C(\Sigma + \Gamma)\right) \\
& + \nu^\top\left(E\left(|\Sigma| + |\Gamma|\right) + F(\Sigma + \Gamma)\right) + \lambda^\top\left(|\Sigma| + P_\alpha^\top P_\alpha - \tilde{L}\Gamma\right) - \mu^\top P_\alpha\Gamma
\end{aligned}$$

is valid. Using Eq. (4.6) the last equation is equivalent to

$$0 = b^\top|\Gamma| + \delta^\top\left(B|\Gamma| + C\Gamma\right) + \nu^\top\left(E|\Gamma| + F\Gamma\right) + \lambda^\top\left(P_\alpha^\top P_\alpha - \tilde{L}\Gamma\right) - \mu^\top P_\alpha\Gamma\,.$$

Multiplying the last equation from the right by $\Gamma P_\alpha^\top$ and using Eq. (4.15), we obtain

$$\mu^\top = -\lambda^\top\tilde{L}P_\alpha^\top + \left(b^\top + \delta^\top B + \nu^\top E + \lambda^\top\right)\Gamma P_\alpha^\top + \left(\delta^\top C + \nu^\top F\right)P_\alpha^\top\,.$$

Thus, defining the Lagrange multiplier μ as given above, it satisfies Eq. (4.12) since

$$\begin{aligned}
& b^\top\left(|\Sigma| + |\Gamma|\right) + \delta^\top\left(B\left(|\Sigma| + |\Gamma|\right) + C\left(\Sigma + \Gamma\right)\right) + \nu^\top\left(E\left(|\Sigma| + |\Gamma|\right) + F\left(\Sigma + \Gamma\right)\right) + \lambda^\top\left(I_s - \tilde{L}\Gamma\right) \\
& \quad - \left(-\lambda^\top\tilde{L}P_\alpha^\top + \left(b^\top + \delta^\top B + \nu^\top E + \lambda^\top\right)\Gamma P_\alpha^\top + \left(\delta^\top C + \nu^\top F\right)P_\alpha^\top\right)P_\alpha\Gamma \\
= {} & b^\top\left(|\Sigma| + |\Gamma|\right) + \delta^\top\left(B\left(|\Sigma| + |\Gamma|\right) + C\left(\Sigma + \Gamma\right)\right) + \nu^\top\left(E\left(|\Sigma| + |\Gamma|\right) + F\left(\Sigma + \Gamma\right)\right) + \lambda^\top\left(I_s - \tilde{L}\Gamma\right) \\
& \quad + \lambda^\top\tilde{L}\Gamma - \left(b^\top|\Gamma| + \delta^\top B|\Gamma| + \nu^\top E|\Gamma| + \lambda^\top|\Gamma| + \delta^\top C\Gamma + \nu^\top F\Gamma\right) \\
= {} & -\lambda^\top|\Sigma| + \lambda^\top\tilde{L}\Gamma - \lambda^\top|\Gamma| + \lambda^\top I_s - \lambda^\top\tilde{L}\Gamma \\
= {} & -\lambda^\top|\Sigma| - \lambda^\top|\Gamma| + \lambda^\top|\Sigma| + \lambda^\top P_\alpha^\top P_\alpha = -\lambda^\top|\Gamma| + \lambda^\top\Gamma\Gamma = -\lambda^\top|\Gamma| + \lambda^\top|\Gamma| = 0
\end{aligned}$$

using Eq. (4.6) and (4.14). Due to the fact that the optimization problem (4.10) is linear, the KKT conditions are necessary and sufficient for the minimality of x_σ. Thus, we have shown that x_σ satisfies the KKT conditions for all adjacent extended signature domains. Therefore, x_σ is also a minimizer of (4.10) and hence of (CALOP).

For the uniqueness of the Lagrange multipliers, the first paragraph of this proof states that the Lagrange multipliers δ and ν as well as the components λ_i belonging to the index set $\alpha(x_\sigma)$ are unique. Finally, for the remaining $i \in \alpha^C$, the complement of α, the components λ_i can be uniquely determined by Eq. (4.6). □

It is important to note that for given Lagrange multipliers δ, ν, and λ, it can be verified in polynomial time whether the conditions (4.5)–(4.7) hold. Hence, this optimality test at a feasible signature optimal point is independent from the combinatorial complexity caused by all the possible values of Γ.

Furthermore, for the unconstrained case, i.e., $A = 0$, $B = 0$, $C = 0$, $D = 0$, $E = 0$, $F = 0$ in the appropriate dimensions, one rediscovers the conditions

$$0 = a^\top + b^\top |\Sigma| \tilde{Z} + \lambda^\top P_\alpha \tilde{Z} \qquad \text{and} \qquad |P_\alpha(b + \lambda)| \leq P_\alpha(-\tilde{L}^\top \lambda) \,,$$

i.e., tangential stationarity and normal growth as introduced in [19].

These optimality conditions ensure just local minimality. However, as stated in Section 2, only global minima of the globalized gap function $\hat{V}(x)$ are the solutions of the LGNEP. Hence, one has to employ the additional criterion $\hat{V}(x) = 0$ to verify that a point that fulfills the derived necessary and sufficient optimality conditions is indeed a global minimizer. If this is not the case, one could start the optimization again using a starting point that lies in a different polyhedron than the first one. In the worst case, then one would have to visit all definite polyhedra to determine global minimizer. The corresponding computational effort fits to the fact that the concave piecewise linear problem is NP hard to solve.

As illustrated in the next section, the solution approach proposed here finds a global minimizer for the examples considered here. So far, we do not have a theoretical justification for this behaviour. Hence, it is subject of future research.

5 Numerical Examples

The active signature method proposed in [16] was extended in [23, 24] to optimization problems with piecewise linear constraints of the form (CALOP). The resulting algorithm, i.e., CASM relies heavily on the optimality conditions considered in the previous section. So far, a MATLAB implementation of CASM is available such that the abs-linear form as stated above can not be generated in an automated fashion. Currently, we are working on an C/C++-based implementation with the corresponding interface to ADOL-C. Then, the abs-linear form is also available for larger problem instances. Hence, corresponding tests are the subject of future work.

We present results for three small LGNEPs illustrating the numerical solution proposed in this paper.

Example 5.1. *We start with an example taken from [30], recap the calculation for the piecewise linear form and apply CASM to calculate a solution of this LGNEP. One considers a game with two one-dimensional players, given as x^1, x^2. That is $n_1 = n_2 = 1$ and $n = 2$. The two players share the constraints*

$$A^1 x^1 + B^1 x^2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{pmatrix} x^1 + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} x^2 \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = c^1 = c^2,$$

i.e., $m_1 = m_2 = 3$ and $A^2 = B^1, B^2 = A^1$. The target functions are defined by $a^1 = -1$, $a^2 = 1$. This yields the polyhedra

$$Z_1 = \{o^1 \in \mathbb{R}^3 | -1 + (-0,5\ 1\ -1)\, o^1 = 0 \ , \ o^1 \geq 0\} = \left\{ o^1 \in \mathbb{R}^3 \middle| \begin{pmatrix} -I_1 \\ (A^1)^\top \\ -(A^1)^\top \end{pmatrix} o^1 \leq \begin{pmatrix} 0 \\ -a^1 \\ a^1 \end{pmatrix} \right\},$$

$$Z_2 = \{o^2 \in \mathbb{R}^3 | 1 + (1\ -1\ -1)\, o^2 = 0 \ , \ o^2 \geq 0\} = \left\{ o^2 \in \mathbb{R}^3 \middle| \begin{pmatrix} -I_2 \\ (A^2)^\top \\ -(A^2)^\top \end{pmatrix} o^1 \leq \begin{pmatrix} 0 \\ -a^2 \\ a^2 \end{pmatrix} \right\} .$$

For those polyhedra one can determine the sets of vertices as

$$\mathcal{O}^1 = \{(0\ 1\ 0)^\top\} \quad \text{and} \quad \mathcal{O}^2 = \{(0\ 0\ 1)^\top, (0\ 1\ 0)^\top\} \,.$$

Combining these sets with definition (2.1) and the reformulation Equation (2.5), one obtains the piecewise linear function

$$\hat{V}(x) = -x^1 + 1 + x^2 + \min\{x^2 + 1 + x^1, x^2 + 1 - x^1\} = -x^1 + 2 + 2x^2 - |x^1| \,.$$

An abs-linear form for the target function is given by

$$\begin{aligned} y &= 2 + (1\ 2)\, x + (0\ \ -1)\, z, \\ z &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} |z|. \end{aligned}$$

This yields the

$$\begin{aligned} &\min_{x \in \mathbb{R}^2, z \in \mathbb{R}^2} (1\ 2)\, x + (0\ \ -1)\, z \\ s.t.\ 0 &\geq \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1/2 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix} x, \\ z &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} |z|, \end{aligned}$$

and hence a formulation in (CALOP) form. Applying CASM yields after two iterations the optimal point $x_* = (0,5\ \ -0,5)$ *with the function value* $\hat{V}(x_*) = 0$. *Therefore, this is indeed a global solution of the given LGNEP.*

Example 5.2. *This LGNEP is formulated in [4] and given by the two-player game*

$$\min_{x_1^1, x_2^1} -x_1^1 - 2x_2^1, \qquad \min_{x^2} -x^2, \qquad s.t.\ Ax \leq b\,,$$

with $n_1 = 2$, $n_2 = 1$, $A = \begin{bmatrix} A^1 & A^2 \end{bmatrix} = \begin{bmatrix} B^2 & B^1 \end{bmatrix}$, $b = c^1 = c^2$ *and*

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 14 \\ 30 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $a^1 = (-1\ \ -2)^\top, a^2 = -1$. *Analogous reformulations of* Z_1, Z_2 *as in Example 5.1 yield as the sets of vertices*

$$\begin{aligned} \mathcal{O}^1 &= \{(1\ 0\ 0\ 0\ 0)^\top, (0\ 1\ 2\ 0\ 0)^\top\}\,, \\ \mathcal{O}^2 &= \{(0\ 1\ 0\ 0\ 0)^\top\}\,. \end{aligned}$$

Then the target function is given by

$$\begin{aligned} \hat{V}(x) &= \min\{-x_1^1 - 2x_2^1 + 14 + x^2, -x_1^1 - 2x_2^1 + 30 - x^2\} - x^2 + 30 - 3x_1^1 - 2x_2^1 \\ &= -4x_1^1 - 4x_2^1 - x^2 + 44 + \min\{x^2, 16 - x^2\} \\ &= -4x_1^1 - 4x_2^1 - x^2 + 52 - |x^2 - 8|\,. \end{aligned}$$

The corresponding abs-linear form reads

$$\begin{aligned} y &= 52 + (-4\ \ -4\ \ -1)\, x + (0\ \ -1)\, z, \\ z &= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} |z|, \end{aligned}$$

yielding the (CALOP) formulation

$$\begin{aligned} &\min_{x \in \mathbb{R}^3, z \in \mathbb{R}^2} (-4\ \ -4\ \ -1)\, x + (0\ \ -1)\, z \\ &s.t.\ 0 \geq -b + Ax, \\ &z = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} |z|. \end{aligned}$$

Applying CASM one obtains

$$x_* = \left(\frac{19}{3}\ \frac{14}{3}\ \frac{5}{3}\right)^\top,$$

with the function value $\hat{V}(x_*) = 0$ *within three iterations. This is indeed an optimal point and therefore also included in the set of solutions calculated in [4] using the algorithm proposed in that paper. One has to note that convergence analysis of the algorithm discussed in [4] can only be applied if the feasible set* X_ν *is bounded.*

Example 5.3. *The next example taken from [31] is given by*

$$\min_{x^1\in\mathbb{R}} -x^1, \qquad \min_{x^2\in\mathbb{R}} -x^2, \qquad s.t.\ A^1x^1 + A^2x^2 \le b$$

with

$$A^1 = \left(\tfrac{1}{2}\ 1\ -1\ 0\right)^\top, \quad A^2 = \left(1\ \tfrac{1}{2}\ 0\ -1\right)^\top, \qquad \text{and} \quad c = (1\ 1\ 0\ 0)^\top.$$

Proceeding as in the first two examples, one obtains

$$\mathcal{O}^1 = \{(0\ 1\ 0\ 0)^\top, (2\ 0\ 0\ 0)^\top\}, \quad \mathcal{O}^2 = \{(0\ 2\ 0\ 0)^\top, (1\ 1\ 0\ 0)^T\},$$

and the target function

$$\begin{aligned}\hat{V}(x) &= \min\left\{-x^1 + (0\ 1\ 0\ 0)\left(c - A^2x^2\right), -x^1 + (2\ 0\ 0\ 0)\left(c - A^2x^2\right)\right\}\\ &\quad + \min\left\{-x^2 + (0\ 2\ 0\ 0)\left(c - A^1x^1\right), -x^2 + (1\ 0\ 0\ 0)\left(c - A^1x^1\right)\right\}\\ &= -\frac{9}{4}(x^1 + x^2) + 3 - \frac{1}{4}\left(|3x^2 - 2| + |3x^1 - 2|\right).\end{aligned}$$

This is a piecewise linear function with two absolute value evaluations. Similar to above, one obtains the (CALOP) formulation

$$\begin{aligned}&\min_{x\in\mathbb{R}^2, z\in\mathbb{R}^3} (-9/4\ \ -9/4)\,x + (0\ 0\ \ -1/4)\,z\\ &s.t.\ 0 \ge \begin{pmatrix}-1\\-1\\0\\0\end{pmatrix} + \begin{pmatrix}1/2 & 1\\1 & 1/2\\-1 & 0\\0 & -1\end{pmatrix} x,\\ &z = \begin{pmatrix}-2\\-2\\0\end{pmatrix} + \begin{pmatrix}0 & 3\\3 & 0\\0 & 0\end{pmatrix} x + \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\1 & 1 & 0\end{pmatrix} |z|.\end{aligned}$$

Applying CASM, one obtains $\tilde{x} = (0.6667\ 0.6667)$ *with the function value* $\hat{V}(\tilde{x}) = 4.4409 \cdot 10^{-16}$ *within five iterations. This approximates the point* $x_* = (2/3\ 2/3)$ *with the function value* $\hat{V}(x_*) = 0$ *which therefore is, as also shown in [31] via straightforward calculation, an optimal point.*

6 Conclusion and Outlook

In this paper, we considered linear generalized Nash equilibrium problems as one example for optimization problems with a piecewise linear target function and possibly piecewise linear constraints.

For this class of problems, we prove necessary and sufficient optimality conditions for the formulation of the piecewise linear problems considered here. Similar to the result for an alternative formulation discussed in [20, 22], we can verify the optimality of a given point with polynomial effort. This is in contrast to most optimality conditions available for nonsmooth optimization. Using a few small examples, we illustrate the usage of these optimality conditions in the Constrained Active Signature Method (CASM).

As discussed at the end of Section 4, the presented optimality conditions only ensure local minimality. The globalization of CASM is subject to current research. One possible approach that is currently investigated is the exploitation of the polyhedra decomposition that is given by the signature vectors.

There are some LGNEPS that yield smooth globalized gap functions that are linear. In this case it is much faster to use some linear solver for calculating optimal points. Therefore, it is important to check for such property before using the approach proposed here.

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