

AN OBSERVER FOR PIPELINE FLOW WITH HYDROGEN BLENDING IN GAS NETWORKS: EXPONENTIAL SYNCHRONIZATION *

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Abstract. We consider a state estimation problem for gas flows in pipeline networks where hydrogen is blended into the natural gas. The flow is modeled by the quasi-linear isothermal Euler equations coupled to an advection equation on a graph. The flow through the vertices where the pipes are connected is governed by algebraic node conditions. The state is approximated by an observer system that uses nodal measurements. We prove that the state of the observer system converges to the original system state exponentially fast in the L^2 -norm if the measurements are exact. If measurement errors are present we show that the observer state approximates the original system state up to an error that is proportional to the maximal measurement error. The proof of the synchronization result uses Lyapunov functions with exponential weights.

Key words. Network, nodal observation, node conditions, gas transportation network, exponential synchronization, Lyapunov function, exponential weights, networked hyperbolic system, quasilinear hyperbolic PDE, general graph, synchronization of solutions to PDEs

MSC codes. 35L04, 35Q49

Introduction. Hydrogen is a promising candidate as an energy carrier in a decarbonized economy. In order to use existing infrastructure for natural gas transport hydrogen blending is of interest. We present an observer system for the flow of gas through networks of pipelines that is governed by the isothermal Euler equations with an additional conservation law to allow for hydrogen blending. We consider classical solutions that allow to investigate the isothermal Euler equations in quasilinear form. At the nodes of the gas pipeline network the solutions for the adjacent pipes are coupled by algebraic node conditions that require the conservation of mass and the continuity of the pressure. Moreover, we assume perfect mixing of the blended hydrogen and natural gas.

We present a nodal observer system for the blended flow through the network where the coupling to the original system is governed at each vertex v of the graph by a real parameter $\mu^v \in [0, 1]$. We show that if $\mu^v \geq 0$ is sufficiently small, the observer system yields an approximation of the full state in the original system where in the absence of measurement errors the L^2 -norm of the observer error decays exponentially fast with time if the C^1 -norm of the initial state is sufficiently small and generates a semi-global classical solution. This assumption makes sense since in the operation of gas pipelines, the velocity of the gas flow is usually quite small compared with the speed of sound. If there is a pointwise upper bound for the measurement error, the error in the state estimation obtained from the observer decays exponentially fast up to the level of the measurement error.

Hence the proposed observer yields a reliable approximation for the complete

*Submitted to the editors 04/05/2023.

Funding: This work was funded by DFG in the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Project C03 and C05. The authors thank the Bundesministerium für Bildung und Forschung (BMBF) for funding under DAAD grant 57654073 'Uncertain data in control of PDE systems'.

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state in the original system exponentially fast. Similarly as in [8] the proof of the exponential synchronization uses Lyapunov functions with exponential weights.

For semi-linear hyperbolic equations observers based on distributed measurements have been presented in [4, 5] and for quasi-linear hyperbolic equations in [1, 11]. A boundary observer for semi-linear hyperbolic problems based on the backstepping method is studied [16]. In [13] an observer for semi-linear hyperbolic equations based upon nodal measurements is presented. The proofs in [13] are based upon a suitable observability inequality. The recovery of an unknown initial state using an observer is studied in [17]. The design of boundary observers for hyperbolic PDEs coupled with ODEs is studied in [10].

The novelty of our contribution is the construction of an observer for a system that is governed by networked quasilinear PDEs and uses observations that are given on isolated points in space only. In addition, we include the effect of measurement errors in our analysis. Motivated by current discussions in decarbonization, our model includes the case of blended flow with a small concentration of hydrogen.

This paper has the following structure. In Section 1 we introduce the isothermal Euler equations with an additional conservation law as a model for hydrogen blending. In Section 1.1 we present the corresponding Riemann invariants and transform the system in diagonal form. In Section 2 we present the node conditions that model the flow of the gas mixture through the junctions in a gas pipeline network. In Section 3, we state the observed system **(S)**, the observer system **(R)** and the system **(Diff)** that is satisfied by the observation error. For our analysis, we use the framework of semi-global classical solutions that are defined through a fixed point iteration along the characteristic curves. A well-posedness result is presented in Section 4. It is of a similar type as the results in [22].

In Section 5 the decay of the L^2 -error of the observer for the density and the flow rate of the mixture is studied under smallness assumptions for the initial state. We show that if there is no measurement error it converges to zero exponentially fast. In the proof a quadratic L^2 -Lyapunov function with exponential weights is used. Also in this analysis, possible measurement error is included. It is shown in the presence of measurement error the observer error converges exponentially fast to a level that is of the order of the measurement error. The proof of this statement is based upon an appropriate version of Gronwall's Lemma. In Section 6 we show the exponential decay of the observer error for hydrogen concentration. The result is of a similar type as in Section 5. Note that in the proof of the result in Section 6 the result from Section 5 is also used.

1. The isothermal Euler equations. The one-dimensional isothermal Euler equations are a well-known model for the flow of gas through pipelines, see for example [2]. A pipeline network corresponds to a finite graph $G = (V, E)$ where V denotes the set of vertices and E the set of edges. Each pipe of length $L^e > 0$ is parameterized by the interval $[0, L^e]$ and corresponds to an edge $e \in E$. The diameter of the pipe is denoted by $D^e > 0$ and the friction coefficient by $\lambda_{fric}^e > 0$. Let $\theta^e := \frac{\lambda_{fric}^e}{D^e}$. For the density of the gas mixture that flows through pipe e we use the notation $\hat{\rho}^e$, for the pressure \hat{p}^e and for the mass flow rate \hat{q}^e . The isothermal Euler equations are hyperbolic provided that the pressure $\hat{p} = p(\hat{\rho})$ is given as a monotone increasing function of the density. We assume that this monotonicity is strict. Examples are the isentropic gas law $p(\hat{\rho}) = a(\hat{\rho})^\gamma$ with $a > 0$, $\gamma > 1$ and the model of the American Gas Association (AGA), see [15], $p(\hat{\rho}) = \frac{R_s \mathcal{T} \hat{\rho}}{1 - \tilde{\alpha} \hat{\rho}}$ where \mathcal{T} denotes the temperature, R_s is the gas constant and $\tilde{\alpha} \leq 0$. For $\tilde{\alpha} = 0$ the AGA model reduces to the ideal gas law

$p(\hat{\rho}) = R_s \mathcal{T} \hat{\rho}$. We study a model that is based upon the 2×2 Euler equations

$$(1.1) \quad \begin{cases} \hat{\rho}_t^e + \hat{q}_x^e = 0, \\ \hat{q}_t^e + \left(\hat{p}^e + \frac{(\hat{q}^e)^2}{\hat{\rho}^e} \right)_x = -\frac{1}{2} \theta^e \frac{\hat{q}^e |\hat{q}^e|}{\hat{\rho}^e} \end{cases}$$

that govern the flow of the mixture through pipe e . Similarly as in [18], let $\rho_{(h)}^e$ denote the density of the hydrogen that is blended in the mixture and $q_{(h)}^e$ the corresponding mass flow rate. We assume that $q_{(h)}^e = W(\hat{q}^e, \hat{\rho}^e) \rho_{(h)}^e$. This means that the velocity $W(\hat{q}^e, \hat{\rho}^e)$ of the hydrogen is determined by the flow of the gas mixture. This assumption is justified if the concentration of the hydrogen is sufficiently low. We assume that $W = \frac{\hat{q}^e}{\hat{\rho}^e + \gamma}$ where $\gamma \in [0, \infty)$. For $\gamma = 0$ this means that the velocity of the hydrogen is equal to the velocity of the gas mixture. A choice of $\gamma > 0$ allows to model a situation where the hydrogen velocity is smaller than the velocity of the mixture. For all $e \in E$, the conservation of mass yields the additional conservation law

$$(1.2) \quad (\rho_{(h)}^e)_t + (q_{(h)}^e)_x = 0.$$

1.1. The system in diagonal form. As stated in [9, Chapter 7.3] for every 2×2 system of hyperbolic conservation laws (for example for (1.1)) we can find a system of Riemann invariants. Although the extended system (1.1), (1.2) is a 3×3 system, it is endowed with Riemann invariants that allow to write the system in diagonal form with the Riemann invariants as new variables. For this purpose we introduce the notation $\tilde{R}(\rho) = \int_1^\rho \frac{\sqrt{p'(r)}}{r} dr$. We obtain the Riemann invariants $R_\pm(\hat{\rho}, \hat{q}) = \tilde{R}(\hat{\rho}) \pm \frac{\hat{q}}{\hat{\rho}}$ and $R_0 = \frac{\rho_{(h)}}{\hat{\rho} + \gamma}$. With the eigenvalues $\lambda_\pm = \frac{\hat{q}}{\hat{\rho}} \pm \sqrt{p'}$ and $\lambda_0 = W(\hat{q}, \hat{\rho})$ we can write the system (1.1), (1.2) in the diagonal form

$$(1.3) \quad \begin{pmatrix} R_+^e \\ R_-^e \\ R_0^e \end{pmatrix}_t + \begin{pmatrix} \lambda_+(R_+^e, R_-^e) & 0 & 0 \\ 0 & \lambda_-(R_+^e, R_-^e) & 0 \\ 0 & 0 & \lambda_0(R_+^e, R_-^e) \end{pmatrix} \begin{pmatrix} R_+^e \\ R_-^e \\ R_0^e \end{pmatrix}_x = \begin{pmatrix} -\frac{\theta^e}{8} |R_+^e - R_-^e| (R_+^e - R_-^e) \\ \frac{\theta^e}{8} |R_+^e - R_-^e| (R_+^e - R_-^e) \\ 0 \end{pmatrix}.$$

Here the eigenvalues are represented as functions of the Riemann invariants, e.g.

$$\lambda_\pm^e = \frac{R_+^e - R_-^e}{2} \pm \sqrt{p' \left(\tilde{R}^{-1} \left(\frac{R_+^e + R_-^e}{2} \right) \right)}.$$

The case for the AGA model for real gas is presented in detail in [15].

2. The Node Conditions. In our model, the flow through the nodes of the network is governed by algebraic coupling conditions. For any node $v \in V$ let $E_0(v)$ denote the set of edges in the graph that are incident to v . Let $x^e(v) \in \{0, L^e\}$ denote the end of the interval $[0, L^e]$ that corresponds to the edge $e \in E_0(v)$. Define

$$(2.1) \quad \mathbf{n}(v, e) := \begin{cases} -1 & \text{if } x^e(v) = 0 \text{ and } e \in E_0(v), \\ 1 & \text{if } x^e(v) = L^e \text{ and } e \in E_0(v), \\ 0 & \text{if } e \notin E_0(v). \end{cases}$$

The conservation of mass at the interior nodes (i.e. $v \in V$ with $|E_0(v)| \geq 2$) is represented by the Kirchhoff condition

$$(2.2) \quad \sum_{e \in E_0(v)} \mathbf{n}(v, e) (D^e)^2 \hat{q}^e(x^e(v)) = 0.$$

As a further coupling condition, we require the continuity of the pressure at v . This choice leads to well-posed Riemann problems, see [2]. It means that

$$(2.3) \quad p(\hat{\rho}^e(t, x^e(v))) = p(\hat{\rho}^f(t, x^f(v))) \text{ for all } e, f \in E_0(v).$$

Another possible choice is advocated by [23], namely the continuity of enthalpy:

$$(2.4) \quad F'(\rho^e(t, x^e(v))) + \frac{(q^e(t, x^e(v)))^2}{(\rho^e(t, x^e(v)))^2} = F'(\rho^f(t, x^f(v))) + \frac{(q^f(t, x^e(v)))^2}{(\rho^f(t, x^e(v)))^2}$$

for all $e, f \in E_0(v)$ where $F(\rho)$ is the pressure potential that is defined by $F(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2} dr$. In the small velocity limit $\frac{q}{\rho} \rightarrow 0$ both (2.3) and (2.4) enforce the continuity of densities, since $p(\rho)$ and $F'(\rho)$ are both strictly monotone increasing. Thus both conditions coincide asymptotically in the low velocity limit.

Now we state the node conditions (2.2), (2.3) in terms of the Riemann invariants (R_+, R_-) . Similarly as in [13], define the components of the vectors $R_{in}^v(t), R_{out}^v(t) \in \mathbb{R}^{|E_0(v)|}$ as follows: For $e \in E_0(v)$ we set

$$(2.5) \quad R_{in}^e(t, x^e(v)) := \begin{cases} R_-^e(t, x^e(v)) & \text{if } x^e(v) = 0 \text{ and } e \in E_0(v), \\ R_+^e(t, x^e(v)) & \text{if } x^e(v) = L^e \text{ and } e \in E_0(v). \end{cases}$$

$$(2.6) \quad R_{out}^e(t, x^e(v)) := \begin{cases} R_+^e(t, x^e(v)) & \text{if } x^e(v) = 0 \text{ and } e \in E_0(v), \\ R_-^e(t, x^e(v)) & \text{if } x^e(v) = L^e \text{ and } e \in E_0(v). \end{cases}$$

For $v \in V$ and $e \in E_0(v)$ with $|E_0(v)| \geq 2$ define $\omega_v := \frac{2}{\sum_{f \in E_0(v)} (D^f)^2}$ and

$$K_\sigma^{v,e}(R_+, R_-, t) = -R_{in}^e(t, x^e(v)) + \omega_v \sum_{g \in E_0(v)} (D^g)^2 R_{in}^g(t, x^g(v)).$$

For the convenience of the reader we state the following lemma from [13]:

LEMMA 2.1. *For any node $v \in V$ with $|E_0(v)| \geq 2$ and $e \in E_0(v)$ the node conditions (2.2), (2.3) can be written in terms of the vectors $R_{in}^v(t), R_{out}^v(t)$ as*

$$(2.7) \quad R_{out}^e(t, x^e(v)) = K_\sigma^{v,e}(R_+, R_-, t).$$

The proof of Lemma 2.1 is given in [13].

To complete our model, we also impose a coupling condition for the hydrogen flow. The conservation of mass requires the Kirchhoff condition

$$(2.8) \quad \sum_{e \in E_0(v)} \mathbf{n}(v, e) (D^e)^2 q_{(h)}^e(t, x^e(v)) = 0.$$

For $v \in V$ and $t \geq 0$ define the set

$$E_{in}(v, t) = \{e \in E_0(v) : \mathbf{n}(v, e) \lambda_0(R_+^e(t, x_e(v)), R_-^e(t, x_e(v))) \geq 0\}$$

and require for all $e \in E_0(v)$ with $e \notin E_{in}(v, t)$ the perfect mixing condition (see [6], [7])

$$(2.9) \quad \rho_{(h)}^e(t, x^e(v)) = \sum_{f \in E_{in}(v, t)} \lambda_R^f(t) \rho_{(h)}^f(t, x^f(v))$$

where

$$(2.10) \quad \lambda_R^f(t) = \frac{(D^f)^2 |\lambda_0^f|(t, x^f(v))}{\sum_{g \in E_{in}(v, t)} (D^g)^2 |\lambda_0^g|(t, x^g(v))}$$

and, by misuse of notation, we write $\lambda_0(t, x)$ instead of $\lambda_0(R_+(t, x), R_-(t, x))$. Note that perfect mixing implies (2.8) For $v \in V$ with $|E_0(v)| \geq 2$ let

$$(2.11) \quad K_0^{v, e}(R_0, R_+, R_-, t) = \sum_{f \in E_{in}(v, t)} \lambda_R^f(t) R_0^f(t, x^f(v))$$

if $e \notin E_{in}(v, t)$. For the Riemann invariant R_0 this yields

$$(2.12) \quad R_0^e(t, x^e(v)) = K_0^{v, e}(R_0, R_+, R_-, t) \text{ if } e \notin E_{in}(v, t).$$

For a boundary node $v \in V$ where $|E_0(v)| = 1$ we state the boundary conditions for R_+^e, R_-^e respectively in the form

$$(2.13) \quad R_{out}^e(t, x^e(v)) = (1 - \mu^v) u_\sigma^v(t) + \mu^v R_{in}^e(t, x^e(v))$$

where $\mu^v \in [0, 1]$ is a given number. If $e \notin E_{in}(v, t)$ also a boundary condition for R_0 is required. We state it as

$$(2.14) \quad R_0^e(t, x^e(v)) = u_0^e(t).$$

The regularity requirements for the functions u_σ^e and u_0^e will be stated in the section about the well-posedness of the system.

Remark 2.2. For $\mu^v = 0$, (2.13) is a Dirichlet boundary condition for the incoming Riemann invariant. For $\mu^v = 1$, it is a Dirichlet boundary condition for the velocity.

3. The observed system (S), the observer system (R) and the error system (diff). In this section we state the observed system, we introduce the observer system and state the system that is satisfied by the observation error.

For $e \in E$, let $\nu^e = \frac{1}{8}\theta^e$ be given and define

$$(3.1) \quad \sigma^e(R_+^e, R_-^e) = \nu^e |R_+^e - R_-^e| (R_+^e - R_-^e).$$

Define $\hat{\Delta}^e(R_+, R_-)$ as the diagonal 2×2 matrix that contains the eigenvalues λ_+ and λ_- corresponding to the edge $e \in E$. In terms of the Riemann invariants, the quasilinear system (1.1) has the diagonal form

$$(3.2) \quad \partial_t \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} + \hat{\Delta}^e(R_+, R_-) \partial_x \begin{pmatrix} R_+^e \\ R_-^e \end{pmatrix} = \sigma^e(R_+^e, R_-^e) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This system is independent of R_0 . Hence the 3×3 system (1.3) can be solved by solving the 2×2 system (3.2) first and then in a second step the equation for R_0 with the given values for (R_+, R_-) . This approach is possible since also the node conditions for (R_+, R_-) are independent of R_0 .

Define $\Delta^e(R_+, R_-)$ as the diagonal 3×3 matrix that contains the eigenvalues λ_+ , λ_- and λ_0 corresponding to the edge $e \in E$. The quasilinear model with hydrogen blending has the following form:

$$(\mathbf{S}) \left\{ \begin{array}{l} S_\iota^e(0, x) = y_\iota^e(x), \iota \in \{+, -, 0\}, x \in (0, L^e), e \in E, \\ \\ S_{out}^e(t, x^e(v)) = (1 - \mu^v) u_\sigma^v(t) + \mu^v S_{in}^e(t, x^e(v)), t \in (0, T), \\ \text{if } e \in E_0(v) \text{ and } |E_0(v)| = 1; \\ S_0^e(t, x^e(v)) = u_0^e(t), t \in (0, T), \\ \text{if } e \in E_0(v), |E_0(v)| = 1 \text{ and } e \notin E_{in}(v, t); \\ \\ S_{out}^e(t, x^e(v)) = K_\sigma^{v,e}(S_+, S_-, t), t \in (0, T), \\ \text{if } e \in E_0(v) \text{ and } |E_0(v)| \geq 2; \\ S_0^e(t, x^e(v)) = K_0^{v,e}(S_0, S_+, S_-, t) \\ \text{if } e \in E_0(v), |E_0(v)| \geq 2 \text{ and } e \notin E_{in}(v, t); \\ \\ \partial_t \begin{pmatrix} S_+^e \\ S_-^e \\ S_0^e \end{pmatrix} + \Delta^e(S_+, S_-) \partial_x \begin{pmatrix} S_+^e \\ S_-^e \\ S_0^e \end{pmatrix} = \sigma^e(S_+^e, S_-^e) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \text{on } [0, T] \times [0, L^e], e \in E. \end{array} \right.$$

Now we introduce the observer system (\mathbf{R}) that depends on numbers $\mu^v \in [0, 1]$ that are given for all $v \in V$ and control the flow of information from the original system to the observer system. For an interior node with $\mu^v = 0$, the values at the node v in the observer system are fully determined by the information from the system (\mathbf{S}) . For $\mu^v = 1$, no data from (\mathbf{S}) are used at the node. We study the case that for the data from (\mathbf{S}) we have perturbations that is smoothed to give a C^1 -function of time $Z = (Z_+, Z_-, Z_0)$ for all $v \in V$, $e \in E_0(v)$ at the points $x^e(v)$. The observer (\mathbf{R}) is defined as follows:

$$(\mathbf{R}) \left\{ \begin{array}{l} R_\iota^e(0, x) = z_\iota^e(x), \iota \in \{+, -, 0\}, x \in (0, L^e), e \in E, \\ \\ R_{out}^e(t, x^e(v)) = (1 - \mu^v)[u_\sigma^v(t) + Z_{out}^e(t, x^e(v))] + \mu^v R_{in}^e(t, x^e(v)), \\ t \in (0, T), \text{if } e \in E_0(v) \text{ and } |E_0(v)| = 1; \\ R_0^e(t, x^e(v)) = [u_0^e(t) + Z_0^e(t, x^e(v))], \\ t \in (0, T), \text{if } e \in E_0(v), |E_0(v)| = 1 \text{ and } e \notin E_{in}(v, t); \\ \\ R_{out}^e(t, x^e(v)) = \\ \mu^v K_\sigma^{v,e}(R_+, R_-, t) + (1 - \mu^v)[S_{out}^e(t, x^e(v)) + Z_{out}^e(t, x^e(v))], \\ t \in (0, T), \text{if } e \in E_0(v) \text{ and } |E_0(v)| \geq 2; \\ R_0^e(t, x^e(v)) = \\ \mu^v K_0^{v,e}(R_0, R_+, R_-, t) + (1 - \mu^v)[S_0^e(t, x^e(v)) + Z_0^e(t, x^e(v))], \\ t \in (0, T), \text{if } e \in E_0(v), |E_0(v)| \geq 2 \text{ and } e \notin E_{in}(v, t); \\ \\ \partial_t \begin{pmatrix} R_+^e \\ R_-^e \\ R_0^e \end{pmatrix} + \Delta^e(R_+, R_-) \partial_x \begin{pmatrix} R_+^e \\ R_-^e \\ R_0^e \end{pmatrix} = \sigma^e(R_+^e, R_-^e) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\ \text{on } [0, T] \times [0, L^e], e \in E. \end{array} \right.$$

The initial state (z_+^e, z_-^e, z_0^e) is an approximation of the initial state of the original system. The data from the original system (\mathbf{S}) enters the system state through the

node conditions. Indeed, for $\mu^v = 0$, the Riemann invariants of the observer coincide with the Riemann invariants of the observed system up to the perturbations. In contrast, for $\mu^v = 1$ the observer **(R)** satisfies the same coupling conditions as **(S)** and no measurement information is inserted.

To investigate the exponential synchronization, we consider the difference

$$\delta := R - S$$

between the state R that is generated by the observer **(R)** and the original state S . For the difference δ we have the system

$$\left(\text{Diff} \right) \left\{ \begin{array}{l}
 \delta_\iota^e(0, x) = z_\iota^e(x) - y_\iota^e(x), \iota \in \{+, -, 0\}, x \in (0, L^e), e \in E, \\
 \delta_{out}^e(t, x^e(v)) = (1 - \mu^v) Z_{out}^e(t, x^e(v)) + \mu^v \delta_{in}^e(t, x^e(v)), \\
 t \in (0, T), \text{ if } e \in E_0(v) \text{ and } |E_0(v)| = 1, \\
 \delta_0^e(t, x^e(v)) = Z_0^e(t, x^e(v)), \\
 t \in (0, T), \text{ if } e \in E_0(v), |E_0(v)| = 1 \text{ and } e \notin E_{in}(v, t); \\
 \\
 \delta_{out}^e(t, x^e(v)) = \mu^v K_\sigma^{v,e}(\delta_+, \delta_-, t) + (1 - \mu^v) Z_{out}^e(t, x^e(v)), \\
 t \in (0, T), \text{ if } e \in E_0(v) \text{ and } |E_0(v)| \geq 2, \\
 \delta_0^e(t, x^e(v)) = (1 - \mu^v) Z_0^v(t) \\
 + \mu^v [K_0^v(S_0 + \delta_0, S_+ + \delta_+, S_- + \delta_-, t) - K_0^v(S_0, S_+, S_-, t)] \\
 t \in (0, T), \text{ if } e \in E_0(v), |E_0(v)| \geq 2 \text{ and } e \notin E_{in}(v, t) \\
 \\
 \partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \\ \delta_0^e \end{pmatrix} + \Delta^e(S + \delta) \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \\ \delta_0^e \end{pmatrix} + [\Delta^e(S + \delta) - \Delta^e(S)] \partial_x \begin{pmatrix} S_+^e \\ S_-^e \\ S_0^e \end{pmatrix} \\
 = [\sigma^e(\delta_+^e + S_+^e, \delta_-^e + S_-^e) - \sigma^e(S_+^e, S_-^e)] \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
 \text{on } [0, T] \times [0, L^e], e \in E.
 \end{array} \right.$$

4. A well-posedness result. Our system allows to apply a result about the existence of classical semi-global solutions for quasilinear hyperbolic systems as stated in [20], [22]. The solutions are constructed using integral equations along the characteristic curves whose slopes are given by the eigenvalues. A similar analysis of Lipschitz solutions for the 2×2 system is given in [15]. These results do not only provide the existence of a classical solution on a given finite time interval $[0, T]$ for sufficiently small initial data that are C^1 -compatible with the boundary conditions but also give a priori bounds for the C^1 norm of the solutions that are proportional to the C^1 -norms of the initial data. Solutions of this type have been studied by Ta-Tsien Li and his group in depth, see for example [21].

In the quasilinear model that we consider, the eigenvalues in the diagonal system matrix define three families of characteristics. But it is a particular feature of the system that all the eigenvalues only depend on R_+ and R_- . This allows to assert the well-posedness for the 2×2 system (3.2) since the node conditions for R_+ , R_- are also independent of R_0 . The proof for the 2×2 systems with the node conditions from **(S)**, **(R)** and **(Diff)** is very similar to the analysis presented in [15]. In fact, the only difference are the particular node conditions stated in **(R)** and **(Diff)**. Note however that the construction in [15] does not yield classical solutions but less regular solutions that are Lipschitz continuous with respect to space. Due to the results from

[22], we also obtain the existence of classical solutions for the 2×2 system. This part of the system also completely determines the characteristic curves with slope λ_0 . The Riemann invariant R_0 is constant along these curves.

Under suitable assumptions our initial data generate R_+, R_- such that $|\lambda_0| > 0$ everywhere. Then the sets $E_{in}(v, t)$ do not depend on t and the node conditions for R_0 uniquely determine the values along the characteristic λ_0 -curves. The following result is stated in a C^1 -neighbourhood of a steady reference state that is a classical solution of **(S)**. Steady states in gas networks are discussed in detail in [14], [12].

THEOREM 4.1. *Let $T > 0$, a steady state $(J^e(x))_{e \in E}$ that is a classical solution of **(S)** and a number $M > 0$ be given. Assume that for all $e \in E$ we have*

$$(4.1) \quad \lambda_+(J_+^e, J_-^e) > 0, \lambda_-(J_+^e, J_-^e) < 0$$

and

$$(4.2) \quad |J_+^e - J_-^e| > 0,$$

i.e. there is no point on the network with velocity zero. Then there exists a number $\varepsilon(T, M) > 0$ such that for initial data $y_\iota^e \in C^1(0, L^e)$ ($\iota \in \{+, -, 0\}$, $e \in E$) such that

$$\|y_\pm^e - J_\pm^e\|_{C^1(0, L^e)} \leq \varepsilon(T, M), \|y_0^e - J_0^e\|_{C^1(0, L^e)} \leq \varepsilon(T, M),$$

and control functions $u_\sigma^v, u_0^e \in C^1(0, T)$ ($e \in E$) such that

$$(4.3) \quad \|(1 - \mu^v)u_\sigma^v - J_{out}^e(x^e(v)) + \mu^v J_{in}^e(x^e(v))\|_{C^1(0, T)} \leq \varepsilon(T, M),$$

$$(4.4) \quad \|u_0^e - J_0^e(x^e(v))\|_{C^1(0, T)} \leq \varepsilon(T, M)$$

*that satisfy the C^1 -compatibility conditions for **(S)** at the vertices $v \in V$ for $t = 0$ there exists a unique classical solution of **(S)** such that for all $e \in E$ we have*

$$(4.5) \quad \|S_\pm^e - J_\pm^e\|_{C^1((0, T) \times (0, L^e))} \leq M,$$

$$(4.6) \quad \|S_0^e - J_0^e\|_{C^1((0, T) \times (0, L^e))} \leq M$$

$$(4.7) \quad |S_+^e - S_-^e| > 0.$$

Proof. In order to prove the result, we consider new variables $\alpha_\pm^e = y_\pm^e - J$. Then $\alpha_\pm^e = 0, \alpha_0^e = 0$ yields a constant stationary solution on the network that corresponds to a physical state with zero flow, constant pressure and no hydrogen. Moreover, for $\alpha = 0$ the source term in the PDE in **(S)** vanishes.

We normalize the lengths L^e of all the pipes to 1 without changing the signs of the eigenvalues. Then the state on each pipe corresponds to three components on the interval $[0, 1]$. Thus the state is given by a function with $3|E|$ components defined on $[0, 1]$. After this transformation all the node conditions on the graph G reduce to boundary conditions on the transformed space interval $[0, 1]$. If $\varepsilon(T, M) > 0$ is sufficiently small, we have $\lambda_+(S_+^e, S_-^e) > 0$ and $\lambda_+(S_+^e, S_-^e) < 0$. Hence the boundary conditions for S_+ and S_- give the values for the incoming Riemann invariants as functions of the outgoing Riemann invariants with an additive C^1 -term. Hence we

can apply Theorem 2.5 in [22] (see also Lemma 1.1. in [19]) which yield the existence of a unique semi-global C^1 -solution of the 2×2 system (3.2) that satisfies the a-priori bound (4.5) for S_+ and S_- .

By choosing $\varepsilon(T, M) > 0$ sufficiently small the a priori bound for S_+ and S_- implies that we can even obtain a solution such that

$$|S_+^e - J_+^e| + |S_-^e - J_-^e| < |J_+^e - J_-^e|.$$

Hence due to (4.2) we can obtain a solution of the 2×2 system (3.2) that satisfies

$$|S_+^e - S_-^e| = |S_+^e - J_+^e + J_+^e - J_-^e + J_-^e - S_-^e| > 0$$

and thus (4.7). Then we have $|\lambda_0(S_+, S_-)| > 0$. This implies that for all $t \in [0, T]$ we have $E_{in}(v, t) = E_{in}(v, 0)$. Hence due to our construction the boundary conditions for S_0 give the necessary values for the Riemann invariant R_0 . Hence we can apply Theorem 2.5 in [22] (see also Lemma 1.1. in [19]) to the 3×3 system which yield the existence of a unique semi-global C^1 -solution that satisfies (4.6).

Remark 4.2. We have discussed the well-posedness of **(S)**. A similar result can be stated for the observer system **(R)** with the additional assumption on the functions $Z_\iota^e(t, x_e(v)) \in C^1(0, T)$ ($v \in V$, $e \in E_0(v)$, $\iota \in \{+, -, 0\}$) that should be added to u_σ^v , u_0^v in (4.3) and (4.4) and should also satisfy the C^1 -compatibility conditions at the nodes, i.e. the Z_ι^e and their derivatives need to vanish for $t \searrow 0$. These two results also imply the existence of a unique classical solution of the error system **(Diff)**.

In [15] a result about the existence of solutions with the slightly weaker $W^{1,\infty}$ regularity is shown. The existence result for solutions with higher H^2 -regularity (to be precise in the space $\times_{e \in E} C([0, T], H^2(0, L^e))$) for quasilinear systems given in [3] requires C^2 -regularity of the source term. In our case, the source term only has C^1 -regularity. Therefore, the result from [3] cannot be applied.

5. An L^2 -Lyapunov function for (δ_+, δ_-) on the network. Squared L^2 -norms with exponential weights have been used successfully as Lyapunov functions to show exponential boundary feedback stabilization, see for example [3]. We will use a Lyapunov function of this type to show the exponential synchronization of the observer system. As a first step, we will show the exponential synchronization of (R_+, R_-) and (S_+, S_-) that are governed by the corresponding decoupled 2×2 system. Later, in a second step we will also show the exponential synchronization of R_0 and S_0 with an appropriate Lyapunov function.

For a given real parameter $\psi \geq 0$ and $e \in E$ we introduce the weighting functions

$$(5.1) \quad h_+^e(x) = B_+^e \exp(-\psi x), \quad h_-^e(x) = B_-^e \exp(\psi x)$$

where B_+^e, B_-^e are real numbers in $(0, \infty)$. For $e \in E$, define the Lyapunov function

$$\mathcal{E}_\sigma^e(t) := \int_0^{L^e} h_+^e(x) |\delta_+^e(t, x)|^2 + h_-^e(x) |\delta_-^e(t, x)|^2 dx.$$

We have

$$\mathcal{E}_\sigma^e(t) = \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} |\delta_+^e(t, x)|^2 \\ |\delta_-^e(t, x)|^2 \end{pmatrix} dx.$$

Note that $\mathcal{E}_\sigma^e(t)$ is equivalent to the squared L^2 -norm of δ^e . Assume that on the time-interval $[0, T]$, a semi-global classical solution of **(Diff)** is given. Then for the

time-derivative of $\mathcal{E}_\sigma^e(t)$ we have

$$\frac{d}{dt}\mathcal{E}_\sigma^e(t) = 2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} \delta_+^e(t, x) \partial_t \delta_+^e(t, x) \\ \delta_-^e(t, x) \partial_t \delta_-^e(t, x) \end{pmatrix} dx.$$

Using the partial differential equation from **(Diff)**, we will show in the proof of Theorem 5.2 that the network Lyapunov function

$$\mathcal{E}_\sigma(t) := \sum_{e \in E} \mathcal{E}_\sigma^e(t)$$

satisfies a differential inequality of the form

$$\frac{d}{dt}\mathcal{E}_\sigma(t) \leq -\chi \mathcal{E}_\sigma(t) - Q(t)$$

with a positive real number $\chi > 0$ and

$$(5.2) \quad Q(t) = \sum_{e \in E} [h_+^e(x) \lambda_+^e(S + \delta) |\delta_+^e|^2(t, x) + h_-^e(x) \lambda_-^e(S + \delta) |\delta_-^e|^2(t, x)] \Big|_{x=0}^{L^e}.$$

If we have an inequality of the form $-Q(t) \leq \eta$ with a real number η , this yields the exponential decay of $\mathcal{E}_\sigma(t)$ up to the level of perturbations with the following version of Gronwall's Lemma:

LEMMA 5.1. *Let $\chi > 0$, $\eta \geq 0$ and a function $\mathcal{E}(t)$ that is absolutely continuous on $[0, T]$ be given. Assume that for $t \in [0, T]$ almost everywhere we have*

$$\mathcal{E}'(t) \leq -\chi \mathcal{E}(t) + \eta.$$

Then for all $t \in [0, T]$ we have

$$(5.3) \quad \mathcal{E}(t) \leq \mathcal{E}(0) \exp(-\chi t) + \frac{\eta}{\chi} [1 - \exp(-\chi t)].$$

Proof. Define $\mathcal{H}(t) = \exp(\chi t) \left[\mathcal{E}(t) - \frac{\eta}{\chi} \right]$. The product rule yields

$$\begin{aligned} \mathcal{H}'(t) &= \chi \exp(\chi t) \left[\mathcal{E}(t) - \frac{\eta}{\chi} \right] + \exp(\chi t) \mathcal{E}'(t) \\ &\leq \exp(\chi t) \left[\chi \left[\mathcal{E}(t) - \frac{\eta}{\chi} \right] - \chi \mathcal{E}(t) + \eta \right] \\ &= 0 \end{aligned}$$

for t almost everywhere in $[0, T]$. We get $\mathcal{H}(t) \leq \mathcal{H}(0)$. Hence due to the definition of \mathcal{H} we obtain $\mathcal{E}(t) - \frac{\eta}{\chi} \leq \left[\mathcal{E}(0) - \frac{\eta}{\chi} \right] \exp(-\chi t)$. Thus we have shown Lemma 5.1. \square

In order to study the properties of $Q(t)$ we reorder the sum in terms of the vertices of the graph and obtain

$$(5.4) \quad Q(t) = \sum_{v \in V} \sum_{e \in E_0(v): x^e(v) = L^e} \begin{pmatrix} h_+^e(L^e) \\ h_-^e(L^e) \end{pmatrix}^\top \begin{pmatrix} \lambda_+^e(S + \delta) |\delta_+^e|^2(t, L^e) \\ \lambda_-^e(S + \delta) |\delta_-^e|^2(t, L^e) \end{pmatrix} \\ - \sum_{e \in E_0(v): x^e(v) = 0} \begin{pmatrix} h_+^e(0) \\ h_-^e(0) \end{pmatrix}^\top \begin{pmatrix} \lambda_+^e(S + \delta) |\delta_+^e|^2(t, 0) \\ \lambda_-^e(S + \delta) |\delta_-^e|^2(t, 0) \end{pmatrix}.$$

Define $I_L(v) = \{e \in E_0(v) : x^e(v) = L^e\}$ and $I_0(v) = \{e \in E_0(v) : x^e(v) = 0\}$. Using the term for the nodal coupling in the definition of **(Diff)** with the parameter μ^v this yields

$$(5.5) \quad Q(t) = \sum_{v \in V} \left[\sum_{e \in I_L(v)} \left[h_+^e(L^e) \lambda_+^e(S + \delta)(t, L^e) |\delta_{in}^e(t, L^e)|^2 \right. \right. \\ \left. \left. + h_-^e(L^e) \lambda_-^e(S + \delta)(t, L^e) |\mu^v K_\sigma^{v,e}(\delta_+, \delta_-, t) + (1 - \mu^v) Z_{out}^e(t, L^e)|^2 \right] \right. \\ \left. - \sum_{e \in I_0(v)} \left[h_+^e(0) \lambda_+^e(S + \delta)(t, 0) |\mu^v K_\sigma^{v,e}(\delta_+, \delta_-, t) + (1 - \mu^v) Z_{out}^e(t, 0)|^2 \right. \right. \\ \left. \left. + h_-^e(0) \lambda_-^e(S + \delta)(t, 0) |\delta_{in}^e(t, 0)|^2 \right] \right].$$

We define

$$\underline{C}^v(t) := \min \left\{ \min_{e \in I_L(v)} h_+^e(L^e) \lambda_+^e(S + \delta)(t, L^e), \min_{e \in I_0(v)} h_-^e(0) |\lambda_-^e(S + \delta)(t, 0)| \right\} \\ \overline{C}^v(t) := \max \left\{ \max_{e \in I_L(v)} h_-^e(L^e) |\lambda_-^e(S + \delta)(t, L^e)|, \max_{e \in I_0(v)} h_+^e(0) \lambda_+^e(S + \delta)(t, 0) \right\}.$$

and observe that $\underline{C}^v(t) > 0$. Note that $K_\sigma^{v,e}(\cdot, \cdot, t)$ is a linear map and there exist a constant \hat{C}^v such that

$$\|K_\sigma^{v,e}(\delta_+, \delta_-, t)\| \leq \hat{C}^v \|\delta_{in}^v(t)\|.$$

Then we have

$$Q(t) \geq \sum_{v \in V} \left[\underline{C}^v(t) - 2\overline{C}^v(t) \hat{C}^v |\mu^v|^2 \right] \|\delta_{in}^v(t)\|^2 - 2|1 - \mu^v|^2 \|Z_{out}^v(t)\|^2.$$

If for a given value of ψ (which influences the values of $h_\pm^e(0)$ and $h_\pm^e(L^e)$) the nodal parameter μ^v is chosen in such a way that $|\mu^v|$ is sufficiently small, we have

$$(5.6) \quad \underline{C}^v(t) - 2\overline{C}^v(t) \hat{C}^v |\mu^v|^2 \geq 0$$

and thus

$$(5.7) \quad Q(t) \geq \sum_{v \in V} -2|1 - \mu^v|^2 \|Z_{out}^v(t)\|^2.$$

For all $e \in E$, define the number

$$(5.8) \quad \kappa^e := \max_{z \in [0, L^e]} \left\{ \frac{h_+^e(z)}{h_-^e(z)}, \frac{h_-^e(z)}{h_+^e(z)} \right\} \geq 1.$$

The following Theorem contains our result about the exponential synchronization of (R_+, R_-) and (S_+, S_-) . Similar to Theorem 4.1 it is stated in some neighbourhood of a steady reference state that is a classical solution of the first two components of **(S)**. The results in [14], [12] imply that for sufficiently small gas velocities the norms of the derivative in the steady state become arbitrarily small. To be precise, we can choose the steady state in such a way that (J_+^e, J_-^e) is arbitrarily close to a constant function of the form (C, C) for all $e \in E$.

THEOREM 5.2. *Let $M > 0$ and a classical steady state $(J_+^e, J_-^e)_{e \in E}$ of (\mathbf{S}) that satisfies the conditions in Theorem 4.1 be given. Assume that for all $e \in E$ we have*

$$(5.9) \quad |J_+^e - J_-^e| \leq M.$$

Let $T > 0$ be given. For all $e \in E$, let initial states $y_+^e, y_-^e, z_+^e, z_-^e \in C^1(0, L^e)$ be given such that at all nodes $v \in V$ the C^1 -compatibility conditions are satisfied. Assume that the first two components of (\mathbf{S}) and (\mathbf{R}) have classical solutions on $[0, T]$ such that (4.5) and the following a-priori bounds (5.10), (5.11) and (5.12) hold:

There exists numbers $\hat{M} > 0, c > 0$ and $\varepsilon_0 > 0$ such that

$$(5.10) \quad |\partial_x S_{\pm}^e| \leq \hat{M},$$

$$(5.11) \quad \frac{3}{4}c \leq \lambda_+^e(R) \leq \frac{5}{4}c, \quad -\frac{5}{4}c \leq \lambda_-^e(R) \leq -\frac{3}{4}c,$$

$$(5.12) \quad |\partial_x (\lambda_{\pm}^e(R))| \leq \varepsilon_0.$$

Assume that we have a nonempty compact convex set $\mathcal{U} \subset \mathbb{R}^2$ that contains the function values of the solutions S^e and R^e and a number $\beta > 0$ such that

$$(5.13) \quad \max_{\tilde{R} \in \mathcal{U}} \left\{ \left| \partial_{R_+} \lambda_{\pm}^e(\tilde{R}) \right|, \left| \partial_{R_-} \lambda_{\pm}^e(\tilde{R}) \right| \right\} \leq \beta.$$

Then on $[0, T]$ there exists a classical solution (δ_+, δ_-) of the first two components of system (\mathbf{Diff}) .

Let $\psi \in (0, \infty)$ be given such that for all $e \in E$ we have (with κ^e as defined in (5.8))

$$(5.14) \quad \varepsilon_0 + (12\nu^e M + 4\beta\hat{M})\kappa^e < \frac{3}{4}\psi c.$$

Assume that for each node $v \in V$ the number $\mu^v \in [0, 1]$ is sufficiently small in the sense that for all $t \in [0, T]$ inequality (5.6) holds. Assume that the perturbations are bounded from above by η_σ , in the sense that for all $t \in [0, T]$ we have

$$(5.15) \quad \sum_{v \in V} \frac{5}{2}c \exp(\psi L^e) |Z_{out}^v(t)|^2 \leq \eta_\sigma.$$

Then the solution of (\mathbf{Diff}) decays exponentially fast up to the perturbation level in the sense that there exist constants $C_1 > 0, \chi > 0$ such that for all $t \in [0, T]$ we have

$$(5.16) \quad \sum_{e \in E} \int_0^{L^e} |\delta_+^e(t, x)|^2 + |\delta_-^e(t, x)|^2 dx \\ \leq C_1 \exp(-\chi t) \sum_{e \in E} \int_0^{L^e} |\delta_+^e(0, x)|^2 + |\delta_-^e(0, x)|^2 dx + C_1 \frac{\eta_\sigma}{\chi}.$$

Hence the L^2 -norm of the error δ between the state R of the observer and the state S of the original system decays exponentially fast up to the perturbation level.

Remark 5.3. We want to emphasize that the inequalities (5.9) (5.10), (5.11), (5.12) hold on the time interval $[0, T]$ for all $e \in E$ on the intervals $[0, L^e]$.

Inequality (5.10) is satisfied if the steady reference state satisfies $|\partial_x J_{\pm}^e| \leq \hat{M}/2$ and $M \leq \hat{M}/2$ which yields $|\partial_x(S_{\pm}^e - J_{\pm}^e)| \leq \hat{M}/2$. Analogously, we can find $\varepsilon_0 > 0$ such that (5.12) holds. Due to (4.1), we can find $c > 0$ such that (5.11) is satisfied.

Inequality (5.14) is satisfied if ε_0 , M and \hat{M} are sufficiently small. This can be achieved in a neighbourhood of the steady state (J_+^e, J_-^e) if it is sufficiently close to a constant function. This is the case if the values of $\nu^e > 0$ are sufficiently small or if the flow rates are sufficiently small.

For the synchronization result for (δ_+, δ_-) it is not necessary that the steady reference state satisfies (4.2). The assertion of Theorem 5.2 also holds for steady states of the form $(J_+^e, J_-^e) = (C, C)$ for all $e \in E$. This means that the 2×2 observer system for (δ_+, δ_-) can also determine the direction of the flow. Equation (4.2) is only necessary for the analysis of the synchronization of δ_0 that is presented below.

Proof. Since $R := S + \delta$, (5.11) and (5.12) can also be seen as conditions on δ . Define

$$(5.17) \quad F^e(S, \delta) := (\sigma^e(\delta_+^e + S_+^e, \delta_-^e + S_-^e) - \sigma^e(S_+^e, S_-^e)) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - [\hat{\Delta}^e(S + \delta) - \hat{\Delta}^e(S)] \begin{pmatrix} \partial_x S_+^e \\ \partial_x S_-^e \end{pmatrix}.$$

The first two components of the partial differential equation in **(Diff)** can be stated as

$$\partial_t \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} + \begin{pmatrix} \lambda_+(S_+^e + \delta_+^e, S_-^e + \delta_-^e) & 0 \\ 0 & \lambda_-(S_+^e + \delta_+^e, S_-^e + \delta_-^e) \end{pmatrix} \partial_x \begin{pmatrix} \delta_+^e \\ \delta_-^e \end{pmatrix} = \begin{pmatrix} F_+^e(S, \delta) \\ F_-^e(S, \delta) \end{pmatrix}.$$

Since the PDE is in diagonal form this yields

$$(5.18) \quad \frac{d}{dt} \mathcal{E}_\sigma^e(t) = 2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \left[- \begin{pmatrix} \lambda_+(S + \delta) \delta_+^e \partial_x \delta_+^e \\ \lambda_-(S + \delta) \delta_-^e \partial_x \delta_-^e \end{pmatrix} + \begin{pmatrix} \delta_+^e F_+^e(S, \delta) \\ \delta_-^e F_-^e(S, \delta) \end{pmatrix} \right] dx \\ = \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \left[- \begin{pmatrix} \lambda_+(S + \delta) \partial_x (\delta_+^e)^2 \\ \lambda_-(S + \delta) \partial_x (\delta_-^e)^2 \end{pmatrix} + 2 \begin{pmatrix} \delta_+^e F_+^e(S, \delta) \\ \delta_-^e F_-^e(S, \delta) \end{pmatrix} \right] dx.$$

Using integration by parts we obtain

$$(5.19) \quad \frac{d}{dt} \mathcal{E}_\sigma^e(t) = \int_0^{L^e} \begin{pmatrix} \partial_x [h_+^e(x) \lambda_+(S + \delta)] \\ \partial_x [h_-^e(x) \lambda_-(S + \delta)] \end{pmatrix}^\top \begin{pmatrix} (\delta_+^e)^2 \\ (\delta_-^e)^2 \end{pmatrix} dx \\ - \begin{pmatrix} h_+^e(x) \lambda_+(S + \delta) \\ h_-^e(x) \lambda_-(S + \delta) \end{pmatrix}^\top \begin{pmatrix} (\delta_+^e)^2 \\ (\delta_-^e)^2 \end{pmatrix} \Big|_{x=0}^{L^e} + 2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} \delta_+^e F_+^e(S, \delta) \\ \delta_-^e F_-^e(S, \delta) \end{pmatrix} dx.$$

We have

$$\partial_x (h_{\pm}^e(x) \lambda_{\pm}^e(S + \delta)) = h_{\pm}^e(x) (\mp \psi \lambda_{\pm}^e(S + \delta) + \partial_x (\lambda_{\pm}^e(S + \delta))).$$

Due to (5.11) and (5.12) we have

$$\partial_x (h_{\pm}^e(x) \lambda_{\pm}^e(S + \delta)) \leq -h_{\pm}^e(x) \left(\psi \frac{3}{4} c - \varepsilon_0 \right).$$

This yields the inequality

$$(5.20) \quad \int_0^{L^e} \partial_x \begin{pmatrix} h_+^e(x) \lambda_+^e(S + \delta) \\ h_-^e(x) \lambda_-^e(S + \delta) \end{pmatrix}^\top \begin{pmatrix} |\delta_+^e|^2 \\ |\delta_-^e|^2 \end{pmatrix} dx \leq - \left(\frac{3}{4} \psi c - \varepsilon_0 \right) \mathcal{E}_\sigma^e(t).$$

Since the right-hand side of the inequality is negative, it is a suitable start to derive a differential inequality that can be used to apply Gronwall's lemma. It remains to derive bounds for the other terms that appear in $\frac{d}{dt} \mathcal{E}_\sigma^e(t)$. We have

$$(5.21) \quad \begin{aligned} & 2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} \delta_+^e F_+^e(S, \delta) \\ \delta_-^e F_-^e(S, \delta) \end{pmatrix} dx \\ &= 2 \int_0^{L^e} [\sigma^e(\delta_+^e + S_+^e, \delta_-^e + S_-^e) - \sigma^e(S_+^e, S_-^e)] \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} -\delta_+^e \\ \delta_-^e \end{pmatrix} dx \\ &\quad - 2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \delta_+^e \\ h_-^e(x) \delta_-^e \end{pmatrix}^\top [\hat{\Delta}^e(S + \delta) - \hat{\Delta}^e(S)] \begin{pmatrix} \partial_x S_+^e \\ \partial_x S_-^e \end{pmatrix} dx. \end{aligned}$$

Let us define the term that is connected with the source term that models the friction at the pipe walls

$$I_1^e := 2 \int_0^{L^e} [\sigma^e(\delta_+^e + S_+^e, \delta_-^e + S_-^e) - \sigma^e(S_+^e, S_-^e)] \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} -\delta_+^e \\ \delta_-^e \end{pmatrix} dx.$$

Moreover, we define

$$I_2^e := -2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \delta_+^e \\ h_-^e(x) \delta_-^e \end{pmatrix}^\top [\hat{\Delta}^e(S + \delta) - \hat{\Delta}^e(S)] \begin{pmatrix} \partial_x S_+^e \\ \partial_x S_-^e \end{pmatrix} dx.$$

Now we derive an upper bound for $|I_1^e|$. Since the function $z \mapsto \nu^e z |z|$ is differentiable with the derivative $z \mapsto 2\nu^e |z|$, definition (3.1) of the function σ^e , the mean value theorem, (5.9) and (4.5) yield

$$|\sigma^e(\delta_+^e + S_+^e, \delta_-^e + S_-^e) - \sigma^e(S_+^e, S_-^e)| \leq 2\nu^e (3M) |\delta_+^e - \delta_-^e|.$$

Hence we have

$$(5.22) \quad \begin{aligned} |I_1^e| &\leq 6\nu^e M \int_0^{L^e} [h_+^e(x) |\delta_+^e| + h_-^e(x) |\delta_-^e|] |\delta_+^e - \delta_-^e| dx \\ &\leq 6\nu^e M \int_0^{L^e} [h_+^e(x) |\delta_+^e| + h_-^e(x) |\delta_-^e|] [|\delta_+^e| + |\delta_-^e|] dx \\ &= 6\nu^e M \int_0^{L^e} [h_+^e(x) |\delta_+^e|^2 + h_-^e(x) |\delta_-^e|^2] + [h_+^e(x) + h_-^e(x)] |\delta_+^e \delta_-^e| dx \\ &\leq 6\nu^e M \int_0^{L^e} \left[\frac{3}{2} h_+^e(x) + \frac{1}{2} h_-^e(x) \right] |\delta_+^e|^2 + \left[\frac{1}{2} h_+^e(x) + \frac{3}{2} h_-^e(x) \right] |\delta_-^e|^2 dx. \end{aligned}$$

Since $h_\pm^e(x) = \frac{h_\pm^e(x)}{h_\mp^e(x)} h_\mp^e(x)$, we have $h_\pm^e(x) \leq \kappa^e h_\mp^e(x)$. Thus we have the bound

$$|I_1^e| \leq 6\nu^e M 2\kappa^e \mathcal{E}_\sigma^e(t).$$

It remains to derive a bound for the term $|I_2^e|$ that comes from the variations of the eigenvalues of the system matrix. Due to (5.10) and (5.13) we have

$$(5.23) \quad |I_2^e| \leq 2 \int_0^{L^e} [h_+^e(x) |\delta_+^e| + h_-^e(x) |\delta_-^e|] \beta [|\delta_+^e| + |\delta_-^e|] \hat{M} dx \\ = 2\beta \hat{M} \int_0^{L^e} h_+^e(x) |\delta_+^e|^2 + h_-^e(x) |\delta_-^e|^2 + [h_+^e(x) + h_-^e(x)] |\delta_+^e \delta_-^e| dx.$$

Similarly as for I_1^e we obtain the inequality

$$(5.24) \quad |I_2^e| \leq 2\beta \hat{M} \int_0^{L^e} \left[\frac{3}{2} h_+^e(x) + \frac{1}{2} h_-^e(x) \right] |\delta_+^e|^2 \\ + \left[\frac{1}{2} h_+^e(x) + \frac{3}{2} h_-^e(x) \right] |\delta_-^e|^2 dx \leq 2\beta \hat{M} 2\kappa^e \mathcal{E}_\sigma^e(t).$$

We have $2 \int_0^{L^e} \begin{pmatrix} h_+^e(x) \\ h_-^e(x) \end{pmatrix}^\top \begin{pmatrix} \delta_+^e F_+^e(S, \delta) \\ \delta_-^e F_-^e(S, \delta) \end{pmatrix} dx = I_1^e + I_2^e$.

Hence (5.20) and the bounds for $|I_1^e|$ and $|I_2^e|$ yield the differential inequality

$$\frac{d}{dt} \mathcal{E}_\sigma^e(t) \leq \left[-\frac{3}{4} \psi c + \varepsilon_0 + (6\nu^e M + 2\beta \hat{M}) 2\kappa^e \right] \mathcal{E}_\sigma^e(t) \\ - [h_+^e(x) \lambda_+^e(S + \delta) |\delta_+^e(t, x)|^2 + h_-^e(x) \lambda_-^e(S + \delta) |\delta_-^e(t, x)|^2] \Big|_{x=0}^{L^e}.$$

We define the number

$$\chi := \min_{e \in E} \frac{3}{4} \psi c - \varepsilon_0 - (6\nu^e M + 2\beta \hat{M}) 2\kappa^e. \quad \square$$

Then (5.14) implies $\chi > 0$. For the network Lyapunov function we obtain

$$\frac{d}{dt} \mathcal{E}_\sigma(t) \leq -\chi \mathcal{E}_\sigma(t) - Q(t)$$

with $Q(t)$ as defined in (5.2). Since (5.6) holds, due to (5.15) we obtain

$$\frac{d}{dt} \mathcal{E}_\sigma(t) \leq -\chi \mathcal{E}_\sigma(t) + \eta_\sigma.$$

Now Lemma 5.1 implies (5.3). Since $\mathcal{E}_\sigma(t)$ is equivalent to the L^2 -norm, we obtain (5.16).

6. An L^2 -Lyapunov function for δ_0 on the network. In order to complete our analysis for the case of hydrogen blending in this section we discuss a Lyapunov function for δ_0 . Note that in order to show the exponential decay of δ_0 , we use the exponential decay of (δ_+, δ_-) that we have shown in the previous section. For our analysis, we use the following version of Gronwall's Lemma:

LEMMA 6.1. *Let $\chi_0 > 0$, $\chi > 0$, $\eta_0 \geq 0$, $\eta \geq 0$, $D_0 > 0$ and a function $\mathcal{E}(t)$ that is absolutely continuous on $[0, T]$ be given. Assume that*

$$(6.1) \quad \mathcal{E}'(t) \leq -\chi_0 \mathcal{E}(t) + D_0 \left[\sqrt{\mathcal{E}(t)} \sqrt{\exp(-\chi t) + \eta} + \exp(-\chi t) + \eta \right] + \eta_0$$

for $t \in [0, T]$ almost everywhere. Then for all $\zeta < \chi_0$, $\zeta \neq \chi$ and $t \in [0, T]$ we have (6.2)

$$\mathcal{E}(t) \leq \mathcal{E}(0) e^{-\zeta t} + \frac{\eta_0}{\zeta} + \left[\frac{D_0^2}{4(\chi_0 - \zeta)} \frac{1}{(\zeta - \chi)} + D_0 \right] \left[\frac{e^{-\chi t} - e^{-\zeta t}}{\zeta - \chi} + \frac{\eta}{\zeta} (1 - e^{-\zeta t}) \right].$$

Proof. Let $\zeta < \chi_0$ be given. Young's inequality implies

$$\sqrt{\mathcal{E}(t)} \frac{\sqrt{2(\chi_0 - \zeta)}}{\sqrt{2(\chi_0 - \zeta)}} D_0 \sqrt{\exp(-\chi t) + \eta} \leq \mathcal{E}(t)(\chi_0 - \zeta) + \frac{D_0^2}{4(\chi_0 - \zeta)} [\exp(-\chi t) + \eta].$$

Thus we have $\mathcal{E}'(t) \leq -\zeta \mathcal{E}(t) + \left[\frac{D_0^2}{4(\chi_0 - \zeta)} + D_0 \right] [\exp(-\chi t) + \eta] + \eta_0$.

Define $\mathcal{H}(t) = \exp(\zeta t) \left[\mathcal{E}(t) - \frac{\eta_0}{\zeta} \right]$. The product rule yields

$$\begin{aligned} \mathcal{H}'(t) &= \zeta \exp(\zeta t) \left[\mathcal{E}(t) - \frac{\eta_0}{\zeta} \right] + \exp(\zeta t) \mathcal{E}'(t) \\ &\leq \exp(\zeta t) \left[\zeta \left[\mathcal{E}(t) - \frac{\eta_0}{\zeta} \right] - \zeta \mathcal{E}(t) + \eta_0 + \left[\frac{D_0^2}{4(\chi_0 - \zeta)} + D_0 \right] [\exp(-\chi t) + \eta] \right] \\ &= \left[\frac{D_0^2}{4(\chi_0 - \zeta)} + D_0 \right] [\exp((\zeta - \chi)t) + \eta \exp(\zeta t)] \end{aligned}$$

for $t \in [0, T]$ almost everywhere. This yields

$$\mathcal{H}(t) \leq \mathcal{H}(0) + \left[\frac{D_0^2}{4(\chi_0 - \zeta)} + D_0 \right] \left[\frac{\exp((\zeta - \chi)t) - 1}{\zeta - \chi} + \frac{\eta}{\zeta} (\exp(\zeta t) - 1) \right].$$

Hence due to the definition of \mathcal{H} we have

$$\mathcal{E}(t) - \frac{\eta}{\zeta} \leq \left[\mathcal{E}(0) - \frac{\eta}{\zeta} \right] e^{-\zeta t} + \left[\frac{D_0^2}{4(\chi_0 - \zeta)} + D_0 \right] \left[\frac{e^{-\chi t} - e^{-\zeta t}}{\zeta - \chi} + \frac{\eta}{\zeta} (1 - e^{-\zeta t}) \right]$$

and (6.2) follows. Thus we have proved Lemma 6.1.

In order to analyze the exponential decay of δ_0 , we assume that the first two components (δ_+, δ_-) of the solution of **(Diff)** are given and that for all $e \in E$, $x \in [0, L^e]$ and $t \in [0, T]$ we have

$$(6.3) \quad |\lambda_0(S_+^e + \delta_+^e, S_-^e + \delta_-^e)(t, x)| > 0.$$

Due to (4.7) in Theorem 4.1 we know that this assumption holds in a C^1 -neighborhood of a steady state that has non-zero velocity everywhere. We use the standard notation $sign(x) = 1$ if $x > 0$, $sign(x) = -1$ if $x < 0$ and $sign(0) = 0$. Due to (6.3) for $e \in E$ the number

$$s^e := sign(\lambda_0(S_+^e + \delta_+^e, S_-^e + \delta_-^e)(t, x))$$

is well-defined.

Let $\psi_0 \geq 0$ denote a real parameter. For $e \in E$ define

$$(6.4) \quad \mathcal{E}_0^e(t) = \int_0^{L^e} e^{-s^e \psi_0 x} |\delta_0^e(t, x)|^2 dx.$$

We will use $\mathcal{E}_0^e(t)$ as a Lyapunov function to show the following theorem about the synchronization of δ_0 :

THEOREM 6.2. *Let $M > 0$ and a classical steady state $(J_+^e, J_-^e, J_0^e)_{e \in E}$ of **(S)** that satisfies the conditions in Theorem 4.1 be given. Assume that for all $e \in E$ we have (5.9). Let $T > 0$ be given. For all $e \in E$, let initial states $y_+^e, y_-^e, y_0^e, z_+^e, z_-^e, z_0^e \in C^1(0, L^e)$ be given such that at all nodes $v \in V$ the C^1 -compatibility conditions are satisfied. Assume that **(S)** and **(R)** have classical solutions on $[0, T]$ such that (4.5) and the following a-priori bounds hold: There exists a numbers $\hat{M} > 0, c > 0$ and $\varepsilon_0 > 0$ such that (5.10), (5.11), (5.12) hold and*

$$(6.5) \quad |\partial_x (\lambda_0^e(R))| \leq \varepsilon_0.$$

Assume that there is a nonempty compact convex set $\mathcal{U} \subset \mathbb{R}^2$ that contains the function values of the solutions (S_+^e, S_-^e) and (R_+^e, R_-^e) and a number $\beta > 0$ such that (5.13) holds and we have

$$(6.6) \quad \max_{\tilde{R} \in \mathcal{U}} \left\{ \left| \partial_{R_+} \lambda_0^e(\tilde{R}) \right|, \left| \partial_{R_-} \lambda_0^e(\tilde{R}) \right| \right\} \leq \beta.$$

Assume that there exist numbers $\mathbf{v} > 0, \bar{\mathbf{v}} > 0$, such that for all $e \in E$

$$(6.7) \quad \mathbf{v} \leq |\lambda_0(R_+^e, R_-^e)| \leq \bar{\mathbf{v}}.$$

*Then on $[0, T]$ there exists a classical solution $(\delta_+, \delta_-, \delta_0)$ of system **(Diff)**. Let $\psi \in (0, \infty)$ be given such that for all $e \in E$ we have (5.14) (with κ^e as defined in (5.8)) and*

Assume that for each node $v \in V$ the number $\mu^v \in [0, 1]$ is sufficiently small in the sense that for all $t \in [0, T]$ inequality (5.6) holds. Assume that the perturbations are bounded above by η_0 , in the sense that for all $t \in [0, T]$ we have (5.15) and

$$(6.8) \quad 3 \sum_{v \in V} \sum_{e \notin E_{in}(v, t)} \exp(-s^e \psi_0 x^e(v)) |\lambda_0^e(S + \delta)|(t, x^e(v)) (1 - \mu^v) |Z_0^v(t)|^2 \leq \eta_0.$$

*Then the solution of **(Diff)** decays exponentially fast up to the perturbation level in the sense that there exist constants $C_0 > 0, \chi_0 > 0$ such that for all $t \in [0, T]$ we have*

$$(6.9) \quad \sum_{e \in E} \int_0^{L^e} |\delta_0^e(t, x)|^2 dx \leq C_0 \exp(-\chi_0 t) \sum_{e \in E} \int_0^{L^e} |\delta_0^e(0, x)|^2 + C_0 [\eta_0 + \eta_\sigma].$$

Hence the L^2 -norm of the error δ_0 between the state R_0 of the observer and the state S_0 of the original system decays exponentially fast up to the perturbation level.

Proof. Due to (4.2) Theorem 4.1 implies that (6.3) holds and we have

$$s^e = \text{sign}(\lambda_0(J_+^e, J_-^e)(x)) = \text{sign}(\lambda_0(S_+^e, S_-^e)(t, x)).$$

In particular, s^e is independent of t and x . We use the notation

$$W^e(S + \delta)(t, x) = \lambda_0(S_+^e + \delta_+^e, S_-^e + \delta_-^e)(t, x).$$

For all $e \in E$ we have $(\delta_0^e)_t + W^e(S + \delta)(\delta_0^e)_x = -[W^e(S + \delta) - W^e(S)] S_x^e$.

With integration by parts this yields $\frac{d}{dt}\mathcal{E}_0^e(t)$

$$\begin{aligned}
&= \int_0^{L^e} 2 \exp(-s^e \psi_0 x) \delta_0^e \partial_t \delta_0^e dx \\
&= - \int_0^{L^e} 2 \exp(-s^e \psi_0 x) \delta_0^e W^e(S + \delta) \partial_x \delta_0^e + 2 \exp(-s^e \psi_0 x) \delta_0^e [W(S^e + \delta^e) - W(S^e)](S^e)_x dx \\
&= - \int_0^{L^e} \exp(-s^e \psi_0 x) W^e(S + \delta) (|\delta_0^e|^2)_x + 2 \exp(-s^e \psi_0 x) \delta_0^e [W(S^e + \delta^e) - W(S^e)](S^e)_x dx \\
&= -e^{-s^e \psi_0 x} W^e(S + \delta) |\delta_0^e(t, x)|^2 \Big|_{x=0}^{L^e} + \int_0^{L^e} [-\psi_0 |W^e(S + \delta)| + W_x^e(S + \delta)] e^{-s^e \psi_0 x} |\delta_0^e|^2 \\
&\quad + 2 \exp(-s^e \psi_0 x) \delta_0^e [W(S^e + \delta^e) - W(S^e)](S^e)_x dx.
\end{aligned}$$

With (6.6) we obtain $|W(S^e + \delta^e) - W(S^e)| \leq \beta (|\delta_+^e| + |\delta_-^e|)$. Using (6.7), (6.5) and (5.10) this yields the inequality

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_0^e(t) &\leq [-\psi_0 \mathbf{v} + \varepsilon_0] \mathcal{E}_0^e(t) + 2\sqrt{2} \exp(\psi L^e) \beta \hat{M} \int_0^{L^e} |\delta_0^0| [(\delta_+^e)^2 + (\delta_-^e)^2]^{1/2} dx \\
&\quad - \exp(-s^e \psi_0 x) W^e(S + \delta)(t, x) |\delta_0^e(t, x)|^2 \Big|_{x=0}^{L^e} \\
&\leq [-\psi_0 \mathbf{v} + \varepsilon_0] \mathcal{E}_0^e(t) + 2\sqrt{2} \exp(\psi L^e) \beta \hat{M} \left[\int_0^{L^e} |\delta_0^0|^2 dx \right]^{\frac{1}{2}} \left[\int_0^{L^e} (\delta_+^e)^2 + (\delta_-^e)^2 dx \right]^{\frac{1}{2}} \\
&\quad - \exp(-s^e \psi_0 x) W^e(S + \delta)(t, x) |\delta_0^e(t, x)|^2 \Big|_{x=0}^{L^e}.
\end{aligned}$$

Define $\chi_0 := \psi_0 \mathbf{v} - \varepsilon_0$, $\mathcal{E}_0(t) := \sum_{e \in E} \mathcal{E}_0^e(t)$ and

$$Q_0(t) := \sum_{e \in E} \exp(-s^e \psi_0 x) W^e(S + \delta)(t, x) |\delta_0^e(t, x)|^2 \Big|_{x=0}^{L^e}.$$

Then with $D_0 = 2\sqrt{2} \exp((\psi + \psi_0) L^e) \beta \hat{M}$ we have

$$(6.10) \quad \frac{d}{dt}\mathcal{E}_0(t) \leq -\chi_0 \mathcal{E}_0(t) + D_0 \sqrt{\mathcal{E}_0(t)} \left[\sum_{e \in E} \int_0^{L^e} (\delta_+^e)^2 + (\delta_-^e)^2 dx \right]^{\frac{1}{2}} + Q_0(t).$$

To finish the proof, a detailed analysis of the nodal term $Q_0(t)$ is necessary. Due to (6.7) for all $v \in V$ the sets $e \in E_{in}(v, t)$ are independent of t . Hence we have $E_{in}(v, t) = E_{in}(v, 0)$. We have

$$(6.11) \quad Q_0(t) = \sum_{v \in V} \left[\sum_{e \in E_{in}(v, 0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) |\delta_0^e(t, x)|^2 \right. \\ \left. - \sum_{e \in E(v) \setminus E_{in}(v, 0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) |\delta_0^e(t, x)|^2 \right]$$

Note that the coupling condition for δ_0 in **(Diff)** can be expressed as

$$(6.12) \quad \delta_0^e(t, x^e(v)) = (1 - \mu^v) Z_0^v(t) + \mu^v K_0^v(\delta_0, S_+ + \delta_+, S_- + \delta_-, t) \\ + \mu^v [K_0^v(S_0, S_+ + \delta_+, S_- + \delta_-, t) - K_0^v(S_0, S_+, S_-, t)]$$

since K_0^v is linear in its first component. Inserting (6.12) into (6.11) yields

$$(6.13) \quad \begin{aligned} Q_0(t) \geq & \sum_{v \in V} \left[\sum_{e \in E_{in}(v,0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) |\delta_0^e(t, x)|^2 \right. \\ & - \sum_{e \in E(v) \setminus E_{in}(v,0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) \\ & \times \left\{ (1 - \mu^v) Z_0^v(t) + \mu^v K_0^v(\delta_0, S_+ + \delta_+, S_- + \delta_-, t) \right. \\ & \left. \left. + \mu^v [K_0^v(S_0, S_+ + \delta_+, S_- + \delta_-, t) - K_0^v(S_0, S_+, S_-, t)] \right\}^2 \right]. \end{aligned}$$

For $v \in V$ we define

$$\begin{aligned} X_1^v &:= 3 \sum_{e \in E(v) \setminus E_{in}(v,0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) \\ & \quad \times |\mu^v K_0^v(\delta_0, S_+ + \delta_+, S_- + \delta_-, t)|^2, \\ X_2^v &:= 3 \sum_{e \in E(v) \setminus E_{in}(v,0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) \\ & \quad \times |\mu^v [K_0^v(S_0, S_+ + \delta_+, S_- + \delta_-, t) - K_0^v(S_0, S_+, S_-, t)]|^2, \\ X_3^v &:= 3 \sum_{e \in E(v) \setminus E_{in}(v,0)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) (1 - \mu^v) |Z_0^v(t)|^2. \end{aligned} \quad \square$$

Since for any $x_1, x_2, x_3 \in \mathbb{R}$ we have $(x_1 + x_2 + x_3)^2 \leq 3x_1^2 + 3x_2^2 + 3x_3^2$ this yields

$$Q_0(t) \geq \sum_{v \in V} \sum_{e \in E_{in}(v,0)} e^{-s^e \psi_0 x^e(v)} |W^e(S + \delta)|(t, x^e(v)) |\delta_0^e(t, x)|^2 - X_1^v - X_2^v - X_3^v.$$

The definition of $K_0^v(\delta_0, S_+ + \delta_+, S_- + \delta_-, t)$ implies that similarly as for Q_σ for μ^v sufficiently small, we have

$$\sum_{v \in V} \sum_{e \in E_{in}(v)} \exp(-s^e \psi_0 x^e(v)) |W^e(S + \delta)|(t, x^e(v)) |\delta_0^e(t, x)|^2 - X_1^v \geq 0.$$

Hence $Q_0(t)$ is bounded below by a term that is determined by $\sum_{v \in V} X_2^v$ and $\sum_{v \in V} X_3^v$.

Due to (6.8) we have $\sum_{v \in V} X_3^v \leq \eta_0$. In Theorem 5.2 it is stated that (δ_+, δ_-) decays exponentially fast and thus can be bounded above by an exponential decaying term and a term of the order $\mathcal{O}(\eta_\sigma)$. The bound (4.7) implies that there is no pipe with zero velocity. Hence due to (2.2), $E_{in}(v, 0)$ is not empty and (6.7) implies

$$\sum_{g \in E_{in}(v,0)} (D^g)^2 |\lambda_0^g(t, x^g(v))| \geq \mathbf{v} \min_{g \in E_0(v)} |D^g|^2.$$

Hence the denominators that appear in $\lambda_R^f(t)$ in the definition (2.11) of K_0^v can be strictly bounded away from zero. Thus we obtain a bound for $\sum_{v \in V} X_2^v$ of the same order $\mathcal{O}(\exp(-\chi t) + \eta_\sigma)$. Hence by further increasing the number $D_0 > 0$ if necessary due to (5.16) and (6.10) we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0(t) &\leq -\chi_0 \mathcal{E}_0(t) + D_0 \sqrt{\mathcal{E}_0(t)} \sqrt{\exp(-\chi t) + \eta_\sigma} - Q_0(t) \\ &\leq -\chi_0 \mathcal{E}_0(t) + D_0 \sqrt{\mathcal{E}_0(t)} \sqrt{\exp(-\chi t) + \eta_\sigma} + D_0(\eta_\sigma + \exp(-\chi t)) + \eta_0. \end{aligned}$$

Therefore (6.1) holds with $\eta = \eta_\sigma$. Then Lemma 6.1 implies the assertion.

7. Conclusion. We have defined an observer system for the gas flow through a pipeline network that is governed by a quasilinear model. The system allows for hydrogen blending in the natural gas flow. As input into the observer system, measurement data that is obtained at the nodes of the network is used. We have allowed for a certain measurement error with the assumption that it is smoothed to have C^1 regularity.

We have shown that under suitable regularity conditions for the solution the observation error decays exponentially fast up to the level of the measurement error. In the proofs of our results appropriately chosen Lyapunov functions with exponential weights play an essential role. Our result requires smallness assumptions on the initial error. To be precise, the initial data of both the original and the observer system need to be close to a given steady reference state. It remains a challenge for future studies to find results that allow for large initial errors.

Acknowledgements: This work was supported by DFG in the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Project C03 and Project C05. The authors thank the Bundesministerium für Bildung und Forschung (BMBF) for support under DAAD grant 57654073 'Uncertain data in control of PDE systems'.

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