

AN ALGORITHMIC FRAMEWORK FOR OPTIMAL CONTROL OF HYBRID DYNAMICAL SYSTEM WITH PARABOLIC PDES

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ABSTRACT. We present an algorithmic approach for the computational solution of optimal control problems with hybrid nature governed by linear parabolic PDEs featuring implicit switches. We propose a stepwise reformulation of the original formulation into a more tractable setting via application of methods from disjunctive programming and a time transformation method. After removal of the implicit switching rule at the cost of the introduction of explicit switching variables and vanishing constraints, the connection of the resulting formulation to problems with equilibrium constraints is established and studied. The previous steps in combination with smoothening and a Moreau-Yosida type penalty approach allow the derivation of necessary first order optimality conditions to characterize candidates for optimality to the original system. Following the discussion of each individual reformulation step, we introduce the algorithmic framework founded on a semismooth Newton method. Finally, we report on computational experiments of the proposed framework

1. Introduction

Countless technical and economical processes can be formulated as optimal control problems. Applications involving implicit switching for instance cover safety circuits for heating processes, where further heating is prohibited once a certain averaged temperature threshold has been crossed, c.f. [30]. Another example is posed by bacteria growth within a petri dish, c.f. [31], where the state of the bacteria, active or dormant, is determined by the overall population within the dish. Finally we mention the application to gas networks, c.f. [17], where the transport through the network is optimized.

Apparently the core task lays in an appropriate handling of the implicitly formulated switching rule. For ordinary differential equations (ODEs) one can remove the implicit rule from our formulation by approaches from generalized disjunctive programming and replace it by a combination of explicit switching variables and vanishing constraints (VCs) as proposed by Bock et al., [5]. The authors in [5] proceed to solve the resulting mixed integer optimal control problem (MIOCP) by relaxed partial outer convexification and sum up rounding. A related approach is discussed by Biegler and Baumdrucker in [4], where the combinatorial aspects of the switching rule are formalized as equilibrium constraints (ECs).

For ODEs another approach, that immediately yields a set of necessary optimality conditions, is the application of a suitable hybrid maximum principle. There exist a variety of different formulations, that each consider different combinations of state and control constraints or even the combination of both. For instance, in [10] a version of an hybrid maximum principle is obtained by the application of a transformation method, that reformulates the system such that the (standard)

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maximum principle, as for instance mentioned in [26], is possible. The utilized transformation displays a strong connection to the one applied later in this paper. Furthermore [32] considers non-smooth dynamics and in [33] smooth systems with state constraints have been studied. Although, for the ODE case, a sophisticated level for the application of the maximum principle has been reached, the situation changes drastically, when the dynamics are governed by PDEs. Li et al., [25] have established a maximum principle for elliptic and parabolic systems. Casas has extended this to also include state constraints in [6]. However, a generalized hybrid maximum principle for PDEs is still unknown. A first step in this direction has for instance been performed in [29], where adjoint based representations of the sensitivity of the objective function with respect to the switching times as well as the switching sequence have been derived.

We are aiming to extend the scope of optimal control for systems involving implicit switching to systems governed by linear parabolic partial differential equations (PDEs). We proceed along a certain path established for ODEs and perform numerous non-trivial adaptations to make the approach also available to PDE systems.

We proceed as in [5] to obtain a MIOCP with vanishing constraints. We then employ a time transformation method, which has been introduced by Lee et al., [24] and further developed by Gerdt in [15] to include optimization over mode selection. The contribution by Leyendecker et al., [27] also covers certain mode dependent state constraints. In this framework a new continuous control variable, commonly referred to as time control, is introduced. The purpose of this time control is to select the correct choice for discrete valued controls with the support of the so called integer control function, which itself is a step function and attains only the possible choices for the discrete control variables. After the introduction of the time control and the integer function, the resulting ODE formulation then features only continuous-valued controls with an adapted dynamic. This seemingly small adaption in the dynamics is harmless in the ODE framework, whereas it is troublesome in the PDE framework by severely changing the nature of the involved PDEs. Therefore a new solution concept for the obtained PDE system and a smoothing step are required. Another transformation step is the replacement of the resulting VCs by equilibrium constraints (ECs) and a slack variable. Programs with these type of constraints require special attention even in a finite-dimensional case. We mention the contributions to MPECs by Kanzow et al., in [2], [12], [13] and [14] for the finite dimensional case and refer for instance to [34] on the extension to Banach spaces.

There already exist some results on optimal control problems with explicit switching even for semilinear parabolic systems by Kunisch et al., [9]. However, the problem formulation considered in this work does not allow the inclusion of state constraints, resulting from the switching rule, or equilibrium constraints. But the presented techniques are useful for the discussion of the system, where we ultimately remove the previously mentioned constraints via penalization. Penalization techniques for PDEs with state constraint have been discussed for instance by Hintermüller and Kunisch in [18] and Ito and Kunisch [22], and penalization of equilibrium constraints has also been in the focus of research. We mention the work of Huang et al., [20] and Hu together with Ralph [21]. Altogether, the approach leads to a tractable problem.

This article is organized as follows. In Section 2 the problem formulation together with an example is introduced. Section 3 covers the mentioned reformulations from implicit switching to explicit switching and vanishing constraints before arriving at a formulation with explicit switching and equilibrium constraint. This

section is concluded by the introduction of the penalized formulation and the derivation of the associated optimality conditions. In Section 4 we introduce our algorithm based on a semismooth Newton scheme to numerically solve the necessary optimality conditions derived at the end of Section 3 on a two dimensional rectangular space time domain. In Section 5 we report on the performance of our algorithm in numerical experiments. In Section 6 the core innovations are summarized. Also future research directions are mentioned.

2. Problem formulation

Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ be a domain with smooth boundary. We denote by $Q := \Omega \times (0, T)$ and set $\Sigma := \partial\Omega \times (0, T)$. In this work we want to study the following optimal control problem

$$\begin{aligned}
 (1a) \quad & \min_{u, y} J(u, y) = \frac{1}{2} \|y - y_{dis}\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|u\|_{L^2(Q)}^2 \\
 & \text{s.t.} \\
 (1b) \quad & y_t(x, t) + A(y)(x, t) = f_{d(t)}(x, u(x, t)) & (x, t) \in Q, \\
 (1c) \quad & y(x, t) = 0 & (x, t) \in \Sigma, \\
 (1d) \quad & y(x, 0) = y_0(x) & x \in \Omega, \\
 (1e) \quad & d(t) = C(y(\cdot, t)) & t \in [0, T].
 \end{aligned}$$

Here $\gamma_{Con} > 0$, $y_{dis} \in L^2(Q)$ and $y_0 \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ are fixed. In addition we assume that the partial time derivative of the desired state $(y_{dis})_t \in L^2(Q)$ exists. $C : L^2(\Omega) \rightarrow \{1, \dots, D\}$ denotes the mode function for a fixed number of modes $D \in \mathbb{N}$. Furthermore we suppose that $f_d : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, u) \mapsto f_d(x, u)$ is linear with respect to the control variable, i.e., for all $d \in \{1, \dots, D\}$ it can be written as $f_d(x, u) = \alpha_d(x) \cdot u$ for functions $\alpha_d \in L^\infty(\Omega)$. The operator A is supposed to be a linear, symmetric and uniformly elliptic differential operator of second order in divergence form, i.e.,

$$A(y) = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right) + b(x)y(x),$$

for given coefficient functions $b \in L^\infty(\Omega)$ with $b \geq 0$ and $a_{ij} \in L^\infty(\Omega)$ such that for a constant $c > 0$ it holds

$$(2) \quad \sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \geq c \|\zeta\|^2$$

for a.e. $x \in \mathbb{R}^n$ and all $\zeta \in \mathbb{R}^n$. In the following we will write $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ for the associated continuous and $H_0^1(\Omega)$ coercive bilinear form to A in the weak formulation of (1b).

Example 1. Let $D = 2$ and $\Omega_d \subset \Omega$ for $d \in \{1, 2\}$ be domains such that $\Omega_1 \cap \Omega_2 = \emptyset$. Let χ_A denote the characteristic function for a Lebesgue measurable set $A \subset \mathbb{R}^n$, i.e.,

$$\chi_A : \mathbb{R}^n \rightarrow \mathbb{R}, x \mapsto \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if else.} \end{cases}$$

We set

$$f_d : L^2(Q) \rightarrow L^2(Q), u \mapsto u \cdot \chi_{\Omega_d},$$

$$C : L^2(\Omega) \rightarrow \{1, 2\}, y \mapsto \begin{cases} 1, & \text{if } \int_{\Omega} y(x) dx \leq \delta, \\ 2, & \text{if } \int_{\Omega} y(x) dx > \delta. \end{cases}$$

We fix $A(y) = -\Delta y$ and a threshold $\delta \in \mathbb{R}$. The previous formulation then becomes

$$\begin{aligned} \min_{u, y} J(u, y) &= \frac{1}{2} \|y - y_{dis}\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|u\|_{L^2(Q)}^2 \\ \text{s.t.} & \\ y_t(x, t) - \Delta y(x, t) &= u(x, t) \chi_{\Omega_{d(t)}}(x) && (x, t) \in Q, \\ y(x, t) &= 0 && (x, t) \in \Sigma, \\ y(x, 0) &= y_0(x) && x \in \Omega, \\ d(t) &= \begin{cases} 1, & \text{if } \int_{\Omega} y(x, t) dx \leq \delta, \\ 2, & \text{if } \int_{\Omega} y(x, t) dx > \delta. \end{cases} && t \in [0, T]. \end{aligned}$$

3. Reformulations

In this section we want to specify the reformulations, that will result in a tractable approach to solve (1). We will first establish a transformation, that removes the implicit stated switching rule from the problem formulation. Instead the resulting problem will feature explicit switching points as new variables together with a special type of constraints, namely vanishing constraints, which include the state constraints related to the switching rule. As a result the transformed problem displays aspects of a mathematical problem with vanishing constraints (MPVC).

In a next reformulation step we will further transform these vanishing constraints into equilibrium constraints by the introduction of an additional slack variable. Thereby we obtain a mathematical problem with equilibrium constraints (MPEC). We prefer this approach for MPECs over MPVCs since there exists a variety of well established optimality concepts and algorithms for MPECs. However, the resulting MPEC type reformulation still includes the same state constraints associated with the switching rule as the MPVC formulation, which makes a direct algorithmic approach impossible.

In a last transformation step, we will remove these state constraints together with a complementarity condition, which is a result of the previous two transformations, from the problem formulation via penalization. The resulting problem allows the formulation of necessary optimality conditions and thereafter application of a well established optimization routine, e.g., a semismooth Newton method.

3.1. Transformation from implicit to explicit switching via a time transformation method

In this section we want to discuss the removal of the implicitly stated switching rule from the original problem formulation. We first introduce some assumptions on the switching behavior, c.f. [5]. For that purpose we also further specify the structure of the switching function C . We assume that $C = R \circ S$ for a linear and continuous function $S : L^2(\Omega) \rightarrow \mathbb{R}$, i.e., $S \in \mathcal{L}(L^2(\Omega), \mathbb{R})$ and a piecewise constant function $R : \mathbb{R} \rightarrow \{1, \dots, D\}$ holds. We also postulate that the inverse image of each mode $d \in \{1, \dots, D\}$ is a half closed interval, i.e., there exist real

numbers $a_d < b_d$ such that

$$R^{-1}(d) =]a_d, b_d], \forall d \in \{1, \dots, D\}.$$

Here we use the convention that $b_d := \infty$ if $R^{-1}(d)$ is not bounded from above respectively $a_d := -\infty$ if $R^{-1}(d)$ is not bounded from below.

Assumption 1. Let $t_s \in \mathbb{R}$ be such that

$$(3) \quad S(y(\cdot, t_s)) \in \bigcup_{d \in \{1, \dots, D\}} \{a_d, b_d\}.$$

We set

$$y^-(\cdot, t_s) := \lim_{t \downarrow t_s} y(\cdot, t) \in L^2(\Omega),$$

$$y^+(\cdot, t_s) := \lim_{t \uparrow t_s} y(\cdot, t) \in L^2(\Omega),$$

for the limit of the solution of the state equation (1b) - (1d) from the left and from the right at t_s . Furthermore the derivatives from the left and from the right

$$S'_-(t_s) := \lim_{t \uparrow t_s} \frac{\partial S}{\partial t}(y^-(\cdot, t_s)),$$

$$S'_+(t_s) := \lim_{t \downarrow t_s} \frac{\partial S}{\partial t}(y^+(\cdot, t_s)),$$

are supposed to exist. The problem (1) satisfies the transversality assumption if the evaluation of $S'_+(t_s) \cdot S'_-(t_s) > 0$ for all $t_s \in \mathbb{R}$, that fulfill (3), holds.

Aside from the behavior on the switching manifold, another important aspect of switched processes is the number of switching points, in particular the avoidance of arcs, which display Zeno behavior. For a discussion on that topic for hybrid systems with hyperbolic dynamics we refer to [16] and [28].

Assumption 2. We assume that system (1) possesses only a finite number of switches for each admissible control and state pair (u, y) .

Under Assumption 2, we will utilize the terms switching time, switching time set and switching sequence as introduced in [27]. We will denote the switching sequence by $\bigcup_{m=1}^{|S|} \{S_m\}$ and write $|S|$ for the total number of switches. Let $[0, T]$ be the time interval. We aim to divide this set into smaller units.

Definition 1. Let $A \subset \mathbb{R}$ be a given closed set. We choose open intervals I_i , $i \in \{1, \dots, |I|\} := |I|_{|I|}$, such that

- i) The sets I_i are pairwise disjoint, i.e., $I_k \cap I_l = \emptyset$ for all $k \in |I|_{|I|}$ with $k \neq l$.
- ii) A is perserved, i.e., $A = \bigcup_{i=1}^{|I|} \bar{I}_i$.

We will refer to the elements I_i in the partition of $[0, T]$ as **major intervals**. In a second partition step we now again divide each \bar{I}_i again into open intervals I_i^j for $j \in |I|_{|I_i|} := \{1, \dots, |I_i|\}$. We refer to the elements of this second partition as **minor intervals**. To address the elements in a partition by a single index, we set $m = \sum_{k=1}^{i-1} |I_k| + j$ for an index pair (i, j) , where m then ranges from 1 to $\sum_{k=1}^{|I|} |I_k|$. For simplicity we will make use of the following assumption.

Assumption 3. The number of minor intervals $|I_i|$ is equal for all $i \in |I|_{|I|}$.

We will denote the length of a major interval by ΔI_i respectively the length of a minor interval by ΔI_i^j . We will frequently require to address the boundary of a minor respectively major interval. For convenience we introduce the following notation

$$(4) \quad \begin{aligned} t_{i-1} &= \inf(I_i), & t_i &= \sup(I_i), & i &\in |I|_{|I|}, \\ \tau_{i(j-1)} &= \inf(I_i^j), & \tau_{ij} &= \sup(I_i^j), & (i, j) &\in |I|_{|I|, |I_i^j|}. \end{aligned}$$

with $|I|_{|I|, |I_i^j|} := |I|_{|I|} \times |I|_{|I_i^j|}$. As further simplification we assume

Assumption 4. *The partition into major and minor intervals is chosen equidistant, i.e.,*

$$(5) \quad \begin{aligned} \Delta I_i &= \frac{T}{|I|}, & \forall i &\in |I|_{|I|}, \\ \Delta I_i^j &= \frac{\Delta I_i}{|I_i^j|}, & (i, j) &\in |I|_{|I|, |I_i^j|}. \end{aligned}$$

The combination of the previous properties results in the following definition.

Definition 2. *We call a partition into major and minor intervals, that fulfills $\sup(I_m) = \inf(I_{m+1})$ for all $m \in \{1, \dots, |I| \cdot |I_i|\}$, Assumption 3 and Assumption 4, an ordered partition into major and minor intervals of equal length.*

The main intention behind the introduction of the minor intervals is to fix a certain mode $d \in \{1, \dots, D\}$ on them and as a consequence to be able to resolve switches on the scale of major intervals. In this context we recall the notion of an integer control function, c.f. [27]. To complete the description of the time transformation process, we require to introduce additional terminology, c.f. [27].

Definition 3. *Let $z \in L^\infty([0, T])$. A function, that fulfills*

$$\begin{aligned} z(\tau) &\geq 0 & \text{for a.e. } \tau &\in [0, T], \\ z|_{I_i^j} &\equiv z_{ij} \in \mathbb{R} & \forall (i, j) &\in |I|_{|I|, |I_i^j|}, \\ \Delta I_i &= \int_{I_i} z(s) ds & \forall i &\in |I|_{|I|}, \end{aligned}$$

is called piecewise constant time control.

With that, we can now specify the associated time transformation, c.f. [15], which is a core tool for the subsequent considerations.

Definition 4. *Let z be a piecewise constant time control. We declare the associated time transformation $t_z : [0, T] \rightarrow [0, T]$ by*

$$(6) \quad \tau \mapsto t_z(\tau) := 0 + \int_0^\tau z(s) ds.$$

By definition the time transformation (6) can explicitly be written as

$$t_z(\tau) = t_{i-1} + \sum_{l=1}^{j-1} \Delta I_i^l z_{il} + (\tau - \tau_{i(j-1)}) z_{ij}, \quad \text{for } \tau \in I_i^j.$$

Furthermore the function t_z is in general not injective and therefore not invertible. However as in [15], we can overcome this issue and define a (pseudo) inverse for the back transformation via,

$$\tau_z(t) := \inf\{s \in [0, T] \mid t_z(s) = t\}.$$

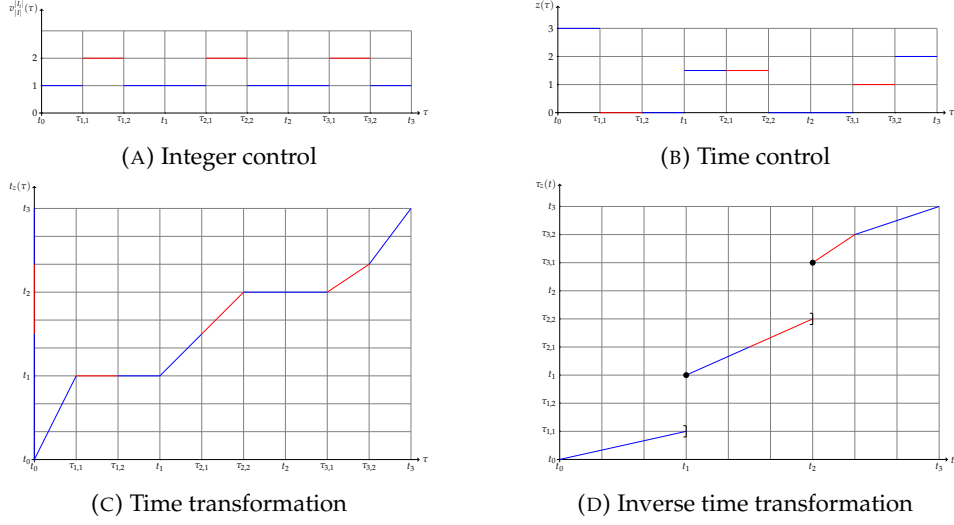


FIGURE 1. Time transformation method

Definition 5. Let z be a piecewise constant time control. We call a minor interval I_i^j active if the corresponding value of the piecewise constant time control satisfies $z_{ij} > 0$.

Now we discuss the resolution of switches. Under Assumption 2 we can w.l.o.g assume that there exists a lower bound on the time between two switches, i.e.,

$$(7) \quad 0 < \Delta S_{\min} := \min_{m \in \{2, \dots, |S|\}} S_m - S_{m-1}.$$

This means the partition of major intervals 1.ii) can be chosen fine enough, such that in each major interval there is at most one switch possible. We require a suitable partition into minor intervals together with an appropriate integer control function to resolve all possible switching combinations. These conditions are combined in the term control consistency for an integer control function, c.f. [27]. It can be deduced that a control consistent integer control function contains at least $2D - 1$ minor intervals. However, this number also suffices as can be observed by the following example.

Example 2. Let D be a fixed number of modes and $|I_i| = 2D - 1$, then we obtain a control consistent integer control function via the following construction, c.f. [27].

$$(8) \quad v_{|I|}^{|I|}(\tau) = \begin{cases} j, & \text{if } \tau \in I_i^j \text{ and } 1 \leq j \leq D, 1 \leq i \leq |I|, \\ 2D - j, & \text{if } \tau \in I_i^j \text{ and } D < j \leq 2D - 1, 1 \leq i \leq |I|. \end{cases}$$

We now have all tools to transform (1) into a problem with explicit switches and vanishing constraints. Therefore we study the transformed control $\tilde{u}(x, \tau) := u(x, t_z(\tau))$, together with transformed state $\tilde{y}(x, \tau) := y(x, t_z(\tau))$ and transformed desired state $\tilde{y}_{dis}(x, \tau) := y_{dis}(x, t_z(\tau))$. We consider the following problem for a new unknown piecewise constant time control z , where we fix a control consistent integer control function $v_{|I|}^{|I|}$. In order to keep the notation for the state in the different reformulations clear, we will write $y^{(1)}$ instead of \tilde{y} for the state variable,

although we can identify \tilde{y} with $y^{(1)}$. Thus (1) becomes

$$(9a) \quad \min_{z, \tilde{u}, y^{(1)}} \tilde{J}(z, \tilde{u}, y^{(1)}) = \frac{1}{2} \|\sqrt{z} \cdot (y^{(1)} - \tilde{y}_{dis})\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|\sqrt{z} \cdot \tilde{u}\|_{L^2(Q)}^2$$

s.t.

$$(9b) \quad y_\tau^{(1)}(x, \tau) + z(\tau)A(y^{(1)})(x, \tau) = z(\tau)\alpha_{v_{|I_i|}(\tau)}(x)\tilde{u}(x, \tau) \quad (x, \tau) \in Q,$$

$$(9c) \quad y^{(1)}(x, \tau) = 0 \quad (x, \tau) \in \Sigma,$$

$$(9d) \quad y^{(1)}(x, 0) = y_0(x) \quad x \in \Omega,$$

$$(9e) \quad z(\tau) \geq 0 \quad \tau \in [0, T],$$

$$(9f) \quad z|_{I_i} = z_{ij} \quad (i, j) \in |I|_{|I_i|, |I_j|},$$

$$(9g) \quad \int_{I_i} z(s) ds = \Delta I_i \quad \forall i \in |I|_{|I_i|},$$

$$(9h) \quad z(\tau)(S(y^{(1)}(\cdot, \tau)) - b_{v_{|I_i|}(\tau)}) \leq 0 \quad \tau \in [0, T],$$

$$(9i) \quad z(\tau)(a_{v_{|I_i|}(\tau)} - S(y^{(1)}(\cdot, \tau))) \leq 0 \quad \tau \in [0, T].$$

If $a_{v_{|I_i|}(\tau)} = -\infty$ respectively $b_{v_{|I_i|}(\tau)} = \infty$ holds, we discard the corresponding inequality. The additional appearance of $z(\tau)$ in the state equation and the cost function is justified by the following observations. We have

$$(10) \quad \frac{d}{d\tau}\tilde{y}(x, \tau) = \frac{d}{d\tau}y(x, t_z(\tau)) = y_t(x, t_z(\tau)) \cdot z(\tau).$$

With regard to the factor \sqrt{z} in the cost function, we notice after a change of variables

$$\begin{aligned} & \tilde{J}(z, \tilde{u}, \tilde{y}) \\ &= \frac{1}{2} \int_0^T \int_\Omega z(\tau) |\tilde{y}(x, \tau) - \tilde{y}_{dis}(x, \tau)|^2 dx d\tau + \frac{\gamma_{Con}}{2} \int_0^T \int_\Omega z(\tau) |\tilde{u}(x, \tau)|^2 dx d\tau \\ &= \frac{1}{2} \int_0^T \int_\Omega z(\tau) (|y(x, t_z(\tau)) - y_{dis}(x, t_z(\tau))|^2 + \gamma_{Con}|u(x, t_z(\tau))|^2) dx d\tau \\ &= \frac{1}{2} \sum_{m=1, z_m > 0}^{|I| \cdot |I_i|} \int_{I_m} \int_\Omega z_m (|y(x, t_z(\tau)) - y_{dis}(x, t_z(\tau))|^2 + \gamma_{Con}|u(x, t_z(\tau))|^2) dx d\tau \\ &= \frac{1}{2} \sum_{m=1, z_m > 0}^{|I| \cdot |I_i|} \int_{t_z(I_m)} \int_\Omega |y(x, t) - y_{dis}(x, t)|^2 dx dt + \gamma_{Con} \int_{t_z(I_m)} \int_\Omega |u(x, t)|^2 dx dt \\ &= \frac{1}{2} \int_0^T \int_\Omega |y(x, t) - y_{dis}(x, t)|^2 dx dt + \frac{\gamma_{Con}}{2} \int_0^T \int_\Omega |u(x, t)|^2 dx dt = J(u, y). \end{aligned}$$

This means the time transformation preserves the values of the cost function. Last we want to comment on the two additional inequalities, (9h) and (9i), in (9). By definition of the switching rule, the state constraints connected to each mode have to be fulfilled. However, this has to be valid only if the corresponding minor interval is actually active. Therefore it appears reasonable to replace the state constraints associated with a specific mode in the following fashion

$$(S(\tilde{y}(\cdot, \tau)) - b_{v_{|I_i|}(\tau)}) \leq 0, \tau \in [0, T] \Rightarrow z(\tau)(S(\tilde{y}(\cdot, \tau)) - b_{v_{|I_i|}(\tau)}) \leq 0, \tau \in [0, T],$$

$$(a_{v_{|I_i|}(\tau)} - S(\tilde{y}(\cdot, \tau))) \leq 0, \tau \in [0, T] \Rightarrow z(\tau)(a_{v_{|I_i|}(\tau)} - S(\tilde{y}(\cdot, \tau))) \leq 0, \tau \in [0, T].$$

Remark 1. We note that in (9) a state is also feasible for an active minor interval, if it attains the value of $a_{v_{|I_i|}(\tau)}$. This is not in accordance with the definition of the switching function C , which requires $S(\tilde{y}(\cdot, \tau)) \in]a_{v_{|I_i|}(\tau)}, b_{v_{|I_i|}(\tau)}]$. However, due to Assumption 1, this is only possible on a set of measure 0. Therefore the constraints stated in (9) have to hold almost everywhere.

The constraints in (9h) and (9i) are considered to be vanishing constraints or constraints of vanishing type, c.f. [2].

Definition 6. Let $f, g_i, h_k, G_l, H_l : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable functions.

$$\begin{aligned}
(11a) \quad & \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \\
(11b) \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, p, \\
(11c) \quad & h_k(x) = 0 \quad \forall k = 1, \dots, q, \\
(11d) \quad & H_l(x) \geq 0 \quad \forall l = 1, \dots, r, \\
(11e) \quad & G_l(x)H_l(x) \leq 0 \quad \forall l = 1, \dots, r.
\end{aligned}$$

is called a mathematical program with vanishing constraints (MPVC).

The core issue with MPVC is that they in general fail to fulfill standard constraint qualifications (CQs). As a result minimizers of a MPVC are not necessarily required to be a KKT point, c.f. [2]. This lead to the development of problem tailored constraint qualifications [1].

Moreover, there is a major drawback associated with the approach in (9) concerning the solvability of involved state equation.

$$\begin{aligned}
(12a) \quad & y_\tau^*(x, \tau) + z(\tau)A(y^*)(x, \tau) = z(\tau)\alpha_{v_{|I_i|}(\tau)}(x)\tilde{u}(x, \tau) \quad (x, \tau) \in Q, \\
(12b) \quad & y^*(x, \tau) = 0 \quad (x, \tau) \in \Sigma, \\
(12c) \quad & y^*(x, 0) = y_0(x) \quad x \in \Omega.
\end{aligned}$$

The posed constraints (9e) - (9g) on z only guarantee z to be a piecewise constant time control and therefore will not prevent z from containing inactive minor intervals. Since we assume control consistency for the involved integer control function $v_{|I_i|}^{I_i}$, there will by design exist several inactive minor intervals I_i^j . As a result we can no longer guarantee the existence of a uniform positive coercivity constant for the resulting differential operator $\tilde{A}(y) := zA(y)$ and therefore cannot expect \tilde{A} to be uniformly elliptic, c.f. (2). This means the discussion of (12), requires a different notion of solution, c.f. [11]. For that purpose we introduce the following space for a fixed integer control $v_{|I_i|}^{I_i}$ and piecewise constant time control z

$$V_{v_{|I_i|}, z} := \{y \in C^0([0, T]; H_0^1(\Omega)) \cap C^0(\bar{Q}) \mid y|_{I_i^j} \in Y(I_i^j), \forall I_i^j \text{ with } z_{ij} > 0\}.$$

Definition 7. Let $\cup_{i=1}^{|I|} \cup_{j=1}^{|I_i|} I_i^j$ be an ordered partition of $[0, T]$ into major and minor intervals of equal length. Denote by $v_{|I_i|}^{I_i}$ a fixed control consistent integer control function. We call a function $y^* \in V_{v_{|I_i|}, z}$ forward solution to (12) for a piecewise constant time control $z \in L^\infty([0, T])$, control $\tilde{u} \in L^2(Q)$ and initial value $y_0 \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$ if

$$\begin{aligned}
y^*(x, \tau) &= y^*(x, \tau_{i(j-1)}) \quad \forall (x, \tau) \in \overline{\Omega \times I_i^j} \text{ with } z_{ij} = 0, \\
y^*(x, \tau) &\text{ solves (12) in a weak sense restricted to } \overline{I_i^j} \quad \forall (x, \tau) \in \overline{\Omega \times I_i^j} \text{ with } z_{ij} > 0.
\end{aligned}$$

The core idea behind a forward solution is to continuously connect solutions along active I_i^j with constant continuations along inactive I_i^j . This should be possible at least with respect to $C^0([0, T]; H_0^1(\Omega))$ and $C^0(\overline{Q})$. Before we can state a first lemma on the existence of forward solutions, we recall the main theorems connected to the existence and regularity of solutions to parabolic equations, c.f. [9] and [23]. Let $y_0, v \in L^\infty([0, T])$, such that there exists $c \in \mathbb{R}$ with the property $0 < c < v(t)$ f.a.e. $t \in [0, T]$, together with $\alpha \in L^\infty(\Omega)$ and $u \in L^2(Q)$ be given. We consider the following system

$$(13a) \quad y_t(x, t) + v(t)A(y)(x, t) = \alpha(x)u(x, t) \quad (x, t) \in Q,$$

$$(13b) \quad y(x, t) = 0 \quad (x, t) \in \Sigma,$$

$$(13c) \quad y(x, 0) = y_0(x) \quad x \in \Omega.$$

Then, c.f. [23].

- i) For $y_0 \in L^2(\Omega)$ there exists a unique solution $y \in W(0, T)$ to (13), that satisfies the estimate

$$(14) \quad \|y\|_{W(0, T)} \leq \|\alpha\|_{L^\infty(\Omega)} \cdot \|u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}.$$

- ii) For $y_0 \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ there exists a unique solution $y \in Y(0, T)$ to (13), which fulfills the estimate

$$(15) \quad \|y\|_{Y(0, T)} \leq \|\alpha\|_{L^\infty(\Omega)} \cdot \|u\|_{L^2(Q)} + \|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})}.$$

Here the spaces $W(0, T)$ and $Y(0, T)$ are given by

$$W(0, T) = \{y \in L^2((0, T); H_0^1(\Omega)) \mid \dot{y} \in L^2((0, T); H^{-1}(\Omega))\},$$

$$Y(0, T) = \{y \in L^2((0, T); H^2(\Omega) \cap H_0^1(\Omega)) \mid \dot{y} \in L^2((0, T); L^2(\Omega))\} \cap C^0(\overline{Q}).$$

Both are Banach spaces equipped with their associated norms

$$\begin{aligned} \|y\|_{W(0, T)} &= \|y\|_{L^2(Q)} + \sum_{k=1}^n \left\| \frac{\partial y}{\partial x_k} \right\|_{L^2(Q)} + \|\dot{y}\|_{L^2((0, T); H^{-1}(\Omega))}, \\ \|y\|_{Y(0, T)} &= \|y\|_{L^2(Q)} + \|\dot{y}\|_{L^2(Q)} \\ &\quad + \sum_{k=1}^n \left\| \frac{\partial y}{\partial x_k} \right\|_{L^2(Q)} + \sum_{k=1}^n \sum_{l=1}^n \left\| \frac{\partial^2 y}{\partial x_k \partial x_l} \right\|_{L^2(Q)} + \max_{(x, t) \in \overline{Q}} |y(x, t)|. \end{aligned}$$

At this point we briefly want to discuss the regularity of the state variable y in our implicit switched system (1b) – (1d). By assumption on f_d , $d \in \{1, \dots, D\}$, for all $t \in [0, T]$ the right side of (1b) is given by $f_{d(t)}(\cdot, u(\cdot, t)) = \alpha_{d(t)}(\cdot)u(\cdot, t) \in L^2(\Omega)$ with fixed functions $\alpha_{d(t)} \in L^\infty(\Omega)$. But the set of coefficient functions α_d is still finite and therefore uniformly bounded. Hence the right side of (1b) is in $L^2(Q)$. But then regularity estimate (15) immediately yields $y \in Y(0, T)$.

Lemma 1. Let $\cup_{i=1}^{|I|} \cup_{j=1}^{|I_i|} I_i^j$ be an ordered partition of $[0, T]$ into major and minor intervals of equal length. Furthermore denote by $v|_{|I|}$ a fixed control consistent integer control function. Let $z \in L^\infty([0, T])$ be a piecewise constant time control and $u \in L^2(Q)$. Then there exists a unique forward solution to (12).

Proof. We construct the solution y^* on each $\overline{I_i^j}$ for $(i, j) \in |I|_{|I_i|, |I_i|}$. We start with I_1^1 and consider the two possible options for the value of the time control z_{11} .

Let $z_{11} = 0$. Then we set $y^*(x, \tau) = y_0(x)$ for all $(x, \tau) \in \overline{\Omega} \times \overline{I_1^1}$.

Let $z_{11} > 0$. Then we have to solve

$$(16a) \quad y_\tau(x, \tau) + z_{11}A(y)(x, \tau) = z_{11}\alpha_{\nu|I_1^1}(\tau)(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_1^1,$$

$$(16b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_1^1,$$

$$(16c) \quad y(x, 0) = y_0(x) \quad x \in \Omega.$$

But this is now a linear parabolic equation with an uniformly elliptic differential operator, i.e., it fits into the setting presented in (13). Hence there exists a unique solution $y_{11} \in Y(I_1^1)$ for (16) according to (15). But this immediately yields $y_{11} \in C^0(\overline{I_1^1}; H_0^1(\Omega)) \cap C^0(\overline{I_1^1} \times \overline{\Omega})$ since $Y(I_1^1)$ can be embedded continuously into $C^0(\overline{I_1^1}; H_0^1(\Omega))$, c.f. [11]. We set $y^*(x, \tau) = y_{11}(x, \tau)$.

This construction fulfills the requirements for a forward solution on $\overline{I_1^1}$ in both cases and is uniquely determined.

Let now I_i^j be arbitrary. We assume there has already been constructed a unique forward solution y^* on $I_{i(j-1)}^\cup := \cup_{k=1}^j \cup_{l=1}^{k-1} \overline{I_k^l}$. Since $\cup_{i,j} I_i^j$ is an ordered partition $\sup(I_{i(j-1)}^\cup) = \inf(I_i^j)$ holds. The supremum in $I_{i(j-1)}^\cup$ is attained, as the set $I_{i(j-1)}^\cup$ is compact by construction. We want to extend y^* along $\overline{I_i^j}$ via the following continuation. We distinguish between z_{ij} greater or equal to zero.

Let $z_{ij} = 0$. We extend y^* along I_i^j via $y^*(x, \tau) = y^*(x, \tau_{i(j-1)}) \quad \forall (x, \tau) \in \overline{\Omega \times I_i^j}$.

From this construction it is clear that $y^* \in C^0((I_{ij}^\cup \cup \overline{I_i^j}); H_0^1(\Omega)) \cap C^0(\overline{\Omega} \times (I_{ij}^\cup \cup \overline{I_i^j}))$ and y^* is uniquely determined.

Let $z_{ij} > 0$. We need to find a continuous extension for our solution y^* with respect to $C^0(I_{ij}^\cup; H_0^1(\Omega)) \cap C^0(\overline{\Omega} \times I_{ij}^\cup)$ and guarantee that $y^* \in Y(I_i^j)$. We consider the following state equation on I_i^j

$$(17a) \quad y_\tau(x, \tau) + z_{ij}A(y)(x, \tau) = z_{ij}\alpha_{\nu|I_i^j}(\tau)(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

$$(17b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

$$(17c) \quad y(x, \tau_{i(j-1)}) = y^*(x, \max(I_{ij}^\cup)) \quad x \in \Omega.$$

Since $y^*(x, \tau_{i(j-1)}) \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ and $z_{ij} > 0$ we can apply the regularity theory for parabolic problems with an uniformly elliptic differential operator. Therefore there exists a unique weak solution $y_{ij} \in Y(I_i^j)$ to (17). We extend y^* with the help of the obtained y_{ij} via $y^*(x, \tau) = y_{ij}(x, \tau) \quad \forall (x, \tau) \in \overline{\Omega \times I_i^j}$. This extension is well posed. The extension is also continuous for $y^*(\cdot, \tau)$ in the sense of $H_0^1(\Omega)$ since $y^* \in C^0(I_{i(j-1)}^\cup; H_0^1(\Omega))$ according to assumption and $y^*|_{\overline{I_i^j}} = y_{ij} \in C^0(\overline{I_i^j}; H_0^1(\Omega))$ with $y_{ij}(\cdot, \tau_{i(j-1)}) = y^*(\cdot, \tau_{i(j-1)})$. This reasoning can be repeated to argue that y^* in $C^0(\overline{\Omega} \times (I_{ij}^\cup \cup \overline{I_i^j}))$, since we have assumed $y^*|_{(\overline{\Omega} \times I_{i(j-1)}^\cup)} \in C^0(\overline{\Omega} \times I_{i(j-1)}^\cup)$ and deduced $y^*|_{\overline{\Omega \times I_i^j}} = y_{ij} \in C^0(\overline{\Omega} \times I_i^j)$. We also obtain the desired regularity for y^* since $y^*|_{I_i^j} = y_{ij} \in Y(I_i^j)$.

We continue this construction along $[0, T]$ and obtain a unique function y^* that fulfills the requirements on a forward solution according to Definition 7. \square

We want to end this first section with a lemma on the connection between the formulations (1) and (9).

Lemma 2. *Let $(u, y) \in L^2(Q) \times Y(0, T)$ be an admissible control to state pair for (1) and denote by $v_{|I|}^{|I_i|}$ a control consistent integer function associated with an ordered partition of $[0, T] = \cup_{i=1}^{|I|} \cup_{j=1}^{|I_i|} \bar{I}_i^j$ into major and minor intervals of equal length. Then there exists a piecewise constant time control z such that the triple $(z, \tilde{u}, \tilde{y})$ is admissible for (9) and \tilde{y} solves the involved state equation (12) in the sense of a forward solution, i.e., $\tilde{y} \in V_{v_{|I|}^{|I_i|}, z}$. Vice versa from an admissible triple $(z, \tilde{u}, \tilde{y})$ to (9) involving a forward solution $\tilde{y} \in V_{v_{|I|}^{|I_i|}, z}$ to (12), we can construct a feasible pair (u, y) for (1), where y is a solution in $W_{v_{|I|}^{|I_i|}, z} := \{y \in C^0([0, T]; H_0^1(\Omega)) \cap C^0(\bar{Q}) \mid y|_{t_z(I_i^j)} \in Y(t_z(I_i^j)), \forall I_i^j \text{ with } z_{ij} > 0\}$.*

Proof. Let $|I|$ be chosen such that $\frac{T}{|I|} < \Delta S_{min}$ holds. Furthermore denote by $M : [0, T] \rightarrow \{1, \dots, D\}, t \mapsto C(y(\cdot, t))$ the mode function. W.l.o.g. we select the control consistent integer control $v_{|I|}^{|I_i|}$ as in (8). We divide the proof into two main parts.

We assume an admissible pair (u, y) for (1). We are first required to construct a suitable time control z , such that $(z, \tilde{u}, \tilde{y})$ is feasible for (9). According to the assumption on $|I|$, there is now at most one switch of M possible in each major interval I_i , so we will construct the time control z on each I_i individually. We distinguish between the switching and non switching case.

First we assume that there exists no switch in the investigated I_i . This means $M(I_i) \equiv d \in \{1, \dots, D\}$. As a result we define $z|_{I_i}$ via

$$z(\tau) = \begin{cases} |I_i|, & \text{if } \tau \in I_i^d, \\ 0, & \text{if else.} \end{cases}$$

The evaluation of the corresponding time transformation t_z is given by

$$t_z(\tau) = \begin{cases} t_{i-1}, & \text{if } \tau \in [t_{i-1}, \tau_{i,(d-1)}], \\ t_{i-1} + (\tau - \tau_{i,(d-1)}) \cdot |I_i|, & \text{if } \tau \in [\tau_{i,(d-1)}, \tau_{i,d}], \\ t_i, & \text{if } \tau \in [\tau_{i,d}, t_i]. \end{cases}$$

We now assume that there is a switch in the investigated I_i from $d_1 \in \{1, \dots, D\}$ to $d_2 \in \{1, \dots, D\}$. Let $t_s \in I_i$ denote the switching point. As a consequence we declare $z|_{I_i}$ by

$$z(\tau) = \begin{cases} \frac{t_s - t_{i-1}}{t_i - t_{i-1}} \cdot |I_i|, & \text{if } \tau \in I_i^{d_1}, \\ \frac{t_i - t_s}{t_i - t_{i-1}} \cdot |I_i|, & \text{if } \tau \in I_i^{(|I_i| - d_2) + 1}, \\ 0, & \text{if else.} \end{cases}$$

Evaluation of the associated time transformation t_z yields

$$t_z(\tau) = \begin{cases} t_{i-1}, & \text{if } \tau \in [t_{i-1}, \tau_{i,(d_1-1)}], \\ t_{i-1} + (\tau - \tau_{i,(d_1-1)}) \cdot \frac{t_s - t_{i-1}}{t_i - t_{i-1}} \cdot |I_i|, & \text{if } \tau \in [\tau_{i,(d_1-1)}, \tau_{i,d_1}], \\ t_s, & \text{if } \tau \in [\tau_{i,d_1}, \tau_{i,(|I_i| - d_2)}], \\ t_s + (\tau - \tau_{i,(|I_i| - d_2)}) \cdot \frac{t_i - t_s}{t_i - t_{i-1}} \cdot |I_i|, & \text{if } \tau \in [\tau_{i,(|I_i| - d_2)}, \tau_{i,(|I_i| - d_2) + 1}], \\ t_i, & \text{if } \tau \in [\tau_{i,(|I_i| - d_2) + 1}, t_i]. \end{cases}$$

As a result the remaining aspect left to demonstrate for the constructed time control z is the selection of the correct mode, i.e.,

$$(18) \quad M(t_z(\tau)) = v_{|I|}^{|I_i|}(\tau) \quad \text{f.a.e. } \tau \in \{\tilde{\tau} \in [0, T] \mid z(\tilde{\tau}) > 0\}.$$

We briefly address the properties of a piecewise constant time control. First $z|_{I_i^j} = z_{ij} \in \mathbb{R}$ holds for all $j \in |I|_{|I_i|}$. Furthermore the non negativity of z and the preservation of major intervals can be immediately deduced from the stated construction of z and the evaluation of t_z . We split the proof of (18) into two parts. We first show that

$$(19) \quad M(t_z(\tau)) = v_{|I|}^{|I_i|}(\tau) \quad \forall \tau \in \{\tilde{\tau} \in [0, T] \mid z(\tilde{\tau}) > 0, t_z(\tilde{\tau}) \notin \{S_1, \dots, S_{|S|}\}\}.$$

Here we again distinguish between the switch and no switch option.

In the no switch case $d \equiv M(t) \forall t \in I_i$ and a fixed $d \in \{1, \dots, D\}$ is satisfied. The only minor interval, where $z(\tau) > 0$ is fulfilled, is given by I_i^d . By construction $v_{|I|}^{|I_i|}(\tau) = d, \forall \tau \in I_i^d$ holds. Therefore (19) is valid.

In the switch case

$$M(t) = \begin{cases} d_1, & \text{if } t \in [t_i, t_s[, \\ d_2, & \text{if } t \in]t_s, t_{i+1}], \end{cases}$$

with the switching time t_s holds. We again consider the set, where $z(\tau) > 0$ is fulfilled. According to the stated construction of z this yields $\tau \in I_i^{d_1} \cup I_i^{|I_i|-d_2}$. Evaluation of the corresponding time transformation t_z indeed shows

$$M(t_z(\tau)) = v_{|I|}^{|I_i|}(\tau)$$

for all τ , that satisfy (19), since $t_s \in \{S_1, \dots, S_{|S|}\}$.

To conclude the proof of (18), we want to show that the set

$$\tilde{S} = \{\tilde{\tau} \in [0, T] \mid z(\tilde{\tau}) > 0, t_z(\tilde{\tau}) \in \{S_1, \dots, S_{|S|}\}\}.$$

has measure 0. In this case it is only necessary to investigate the switching case, since for the no switch option the set \tilde{S} is already empty. We attempt to calculate an upper bound on the set \tilde{S} . We receive the following estimate from the definition of z and t_z

$$\begin{aligned} \tilde{S} &\subseteq \left[\left([\tau_{i,(d_1-1)}, \tau_{i,d_1}] \cup [\tau_{i,(|I_i|-d_2)}, \tau_{i,(|I_i|-d_2+1)}] \right) \right. \\ &\quad \left. \cap \left([t_{i-1}, \tau_{i,(d_1-1)}] \cup [\tau_{i,d_1}, \tau_{i,(|I_i|-d_2)}] \cup [\tau_{i,(|I_i|-d_2+1)}, t_i] \right) \right] \\ &= \{\tau_{i,(d_1-1)}\} \cup \{\tau_{i,d_1}\} \cup \{\tau_{i,(|I_i|-d_2)}\} \cup \{\tau_{i,(|I_i|-d_2+1)}\}. \end{aligned}$$

Hence \tilde{S} contains a finite number of elements and therefore it is a set with measure 0, which yields (18). We are now left to demonstrate that $\tilde{y}(x, \tau) = y(x, t_z(\tau)) \in V_{v_{|I|}^{|I_i|}, z}$. We first address the property $\tilde{y} \in C^0([0, T]; H_0^1(\Omega)) \cap C^0(\overline{Q})$. This is a consequence of $y \in Y(0, T)$ and the continuous embedding of $Y(0, T)$ into the space $C^0([0, T]; H_0^1(\Omega))$. The time transformation t_z is continuous and preserves $[0, T]$. Hence $\tilde{y} : [0, T] \rightarrow H_0^1(\Omega), \tau \mapsto y(\cdot, t_z(\tau))$ is well defined and continuous. This reasoning can be repeated to deduce $\tilde{y} \in C^0(\overline{Q})$. After establishing the continuity as the first aspect of a forward solution, we have to check the claimed regularity on minor intervals. Let I_i^j with $z_{ij} = 0$ be given. From the definition of the time transformation t_z , we deduce $t_z(I_i^j) = \tilde{\tau} \in [0, T]$. Therefore $\tilde{y}(x, \tau) = y(x, \tilde{\tau})$ holds

for all $(x, \tau) \in \Omega \times \overline{I_i^j}$. This means \tilde{y} fulfills the requirements for a forward solution, in case $z_{ij} = 0$. Let now a minor interval I_i^j with $z_{ij} > 0$ be given. We have to show that \tilde{y} solves

(20a)

$$y_\tau^*(x, \tau) + z_{ij}A(y^*)(x, \tau) = z_{ij}\alpha_{v_{|I_i^j|}(\tau)}(x)\tilde{u}(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

(20b)

$$y^*(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

(20c)

$$y^*(x, \tau_{i(j-1)}) = y(x, t_z(\tau_{i(j-1)})) = \tilde{y}(x, \tau_{i(j-1)}) \quad x \in \Omega,$$

and $\tilde{y} \in Y(I_i^j)$. We will denote by $\tilde{d} \in \{1, \dots, D\}$ the fixed value of $v_{|I_i^j|}$ on I_i^j , especially \tilde{d} is the mode choice for the solution y of (1b) – (1d) in $t_z(I_i^j)$ according to (18). The associated weak formulation is given by

$$(21) \quad \begin{aligned} & \int_{I_i^j} \langle y_\tau^*(\tau), \varphi(\tau) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} d\tau + \int_{I_i^j} z_{ij}a(y^*(\tau), \varphi(\tau)) d\tau \\ & = \int_{I_i^j} z_{ij} \langle \alpha_{\tilde{d}}\tilde{u}(\tau), \varphi(\tau) \rangle_{L^2(\Omega)} d\tau \quad \forall \varphi \in L^2(I_i^j, H_0^1(\Omega)). \end{aligned}$$

According to assumption $z_{ij} > 0$, this means $t_z : I_i^j \rightarrow t_z(I_i^j)$ is a bijection. It is even a diffeomorphism, since $\tau_z : t_z(I_i^j) \rightarrow I_i^j$ is also continuously differentiable. This means we apply the chain rule for weak derivatives, c.f. [3], and calculate for $\tau \in I_i^j$

$$\tilde{y}_\tau(x, \tau) = y_t(x, t_z(\tau)) \cdot z_{ij} = \tilde{y}_t(x, \tau) \cdot z_{ij}.$$

Let now $\varphi \in L^2(I_i^j; H_0^1(\Omega))$ be arbitrary. We check, whether \tilde{y} fulfills (21). We plug \tilde{y} into (21) and obtain after a change of variables

$$(22) \quad \begin{aligned} & \int_{I_i^j} z_{ij} \langle \tilde{y}_t(\tau), \varphi(\tau) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} d\tau + \int_{I_i^j} z_{ij}a(\tilde{y}(\tau), \varphi(\tau)) d\tau \\ & = \int_{I_i^j} z_{ij} \langle \alpha_{\tilde{d}}\tilde{u}(\tau), \varphi(\tau) \rangle_{L^2(\Omega)} d\tau, \\ & \int_{t_z(I_i^j)} \langle y_t(t), \varphi(\tau_z(t)) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt + \int_{t_z(I_i^j)} a(y(t), \varphi(\tau_z(t))) dt \\ & = \int_{t_z(I_i^j)} \langle \alpha_{\tilde{d}}u(\tau), \varphi(\tau_z(t)) \rangle_{L^2(\Omega)} dt. \end{aligned}$$

with $\varphi(\cdot, \tau_z(\cdot)) \in L^2(t_z(I_i^j); H_0^1(\Omega))$. But (22) indeed holds true for any $\psi \in L^2(t_z(I_i^j); H_0^1(\Omega))$ since y solves the weak formulation of (1b) – (1c). Furthermore also the initial conditions coincide as $\tilde{y}(\cdot, \tau_{i(j-1)}) = y(\cdot, t_z(\tau_{i(j-1)}))$. Since $y \in Y(0, T) \subset Y(t_z(I_i^j))$ we conclude $\tilde{y} \in Y(I_i^j)$. But this means \tilde{y} is a forward solution.

Let now $\tilde{y} \in V_{v_{|I_i^j|}, z}$ be a forward solution of (12) in the sense of Definition 7. We

set $y(x, t) = \tilde{y}(x, \tau_z(t))$ and $u(x, t) = \tilde{u}(x, \tau_z(t)) \forall (x, t) \in \overline{\Omega \times (0, T)}$. Since \tilde{y} is assumed to be only a forward solution, we generally cannot expect y to solve (1b) – (1d) with the regularity $y \in Y(0, T)$. We first want to discuss the continuity aspects of y and demonstrate that $y \in C^0([0, T]; H^1(\Omega)) \cap C^0(\overline{\Omega})$. This property for y does not follow immediately from the corresponding property of \tilde{y} since the (pseudo) inverse time transformation τ_z is in general not continuous. Let $t \in [0, T]$

be fixed. Denote by $\tau = \tau_z(t)$ the evaluation of the inverse time transformation. We now distinguish between two cases.

Let $\tau \in \text{Int}(I_m)$ for a $m \in \{1, \dots, |I| \cdot |I_i|\} := |I|_{|I| \cdot |I_i|}$. We claim that $z(\tau) > 0$ holds. We assume for a contradiction that $z(\tau) = 0$. This means that $z_m = 0$, since τ is an interior point of I_m . This yields $t_z(\overline{I_m}) = t$. But this contradicts $\tau_z(t) = \tau$, as $\tau_{m-1} = \inf(I_m) < \tau$. This means $t_z|_{I_m}$ is a continuously differentiable bijection. By the inverse function theorem the same holds true for $\tau_z|_{t_z(I_m)}$. Therefore τ_z is continuous at t since $t = t_z(\tau)$ holds.

Let $\tau \in \partial I_m$ for an index $m \in |I|_{|I| \cdot |I_i|}$. Since $(I_m)_{m \in |I|_{|I| \cdot |I_i|}}$ is an ordered partition of $[0, T]$ we assume w.l.o.g. $\tau = \inf(I_m)$. We claim that either $\tau = 0$ or $z_{m-1} > 0$ holds. We assume for a contradiction that $z_{m-1} = 0$ and $\tau \neq 0$. Our assertions ensure that I_{m-1} is well defined. But the definition of t_z together with $z_{m-1} = 0$ yields $t_z(\overline{I_{m-1}}) = t$, which contradicts $\tau = \tau_z(t)$. In particular we deduce that τ_z is left-continuous.

With the previous considerations we have established an important property of τ_z . For any given $t \in [0, T]$ the function τ_z is either continuous at t , or the evaluation $\tau = \tau_z(t)$ is either located at 0 or at the infimum of a I_m with $z_{m-1} > 0$. We will utilize this in the continuity discussion of y in $C^0([0, T]; H_0^1(\Omega)) \cap C^0(\overline{Q})$. The continuity of y at t with $\tau \in \text{Int}(I_m)$ is then established via concatenation of the continuous functions τ_z at t and \tilde{y} at $\tau_z(t)$. Let now be $t \in [0, T]$ such that $\tau \in \partial I_m$. We have already established that τ_z is still left-continuous in this case. Therefore we only consider sequences such that $(t_k)_{k \in \mathbb{N}} \downarrow t$. We denote by $(\tau_k)_{k \in \mathbb{N}}$ the corresponding evaluations of $\tau_z(t_k)$. Since τ_z is monotonous increasing, the sequence $(\tau_k)_{k \in \mathbb{N}}$ is monotonous decreasing. Furthermore $(\tau_k)_{k \in \mathbb{N}}$ is bounded. Hence $(\tau_k)_{k \in \mathbb{N}}$ converges to an element $\bar{\tau}$. We evaluate $t_z(\bar{\tau})$.

$$t_z(\bar{\tau}) = \lim_{k \rightarrow \infty} t_z(\tau_k) = \lim_{k \rightarrow \infty} t_k = t.$$

This means, although we cannot expect $(\tau_k)_{k \in \mathbb{N}}$ to converge to τ , we can at least ensure convergence to an element $\bar{\tau}$, whose evaluation of the time transformation coincides with that of τ . We deduce from the definition of τ that $\tau \leq \bar{\tau}$ is valid. We now remain to compare the evaluations of $\tilde{y}(x, \tau)$ and $\tilde{y}(x, \bar{\tau})$, where $t_z(\tau) = t_z(\bar{\tau})$ holds. We conclude that $z(\hat{\tau}) = 0$ for $\hat{\tau} \in (\tau, \bar{\tau})$ holds, since

$$0 = t_z(\bar{\tau}) - t_z(\tau) = \int_{[\tau, \bar{\tau}]} z(s) ds.$$

This indeed yields $\tilde{y}(x, \tau) = \tilde{y}(x, \bar{\tau})$ since the forward solution \tilde{y} continues constant along sets / minor intervals I_m with $z_m = 0$. This ultimately completes the continuity discussion of y . With respect to the regularity, we claim that $y \in W_{v|_{I_i}|, z}$

holds. The property that $y \in Y(t_z(I_i^j))$ for all I_i^j with $z_{ij} > 0$ follows from the fact that $\tilde{y}|_{I_i^j} \in Y(I_i^j)$ for all such z_{ij} and $\tau_z|_{t_z(I_i^j)}$ defines a continuously differentiable diffeomorphism. The expressions for the weak derivatives of y can then be calculated via the chain rule. The only open task is the validity of the state equation. We again denote by \bar{d} the value of $v|_{I_i}|_{I_i^j}$. We are required to show that y fulfills

$$\begin{aligned} y_t^*(x, t) + A(y^*)(x, t) &= \alpha_{\bar{d}}(x)u(x, t) & (x, t) \in \Omega \times t_z(I_i^j), \\ y^*(x, t) &= 0 & (x, t) \in \partial\Omega \times t_z(I_i^j), \\ y^*(x, \inf(t_z(I_i^j))) &= \bar{y}(x, \inf(t_z(I_i^j))) & x \in \Omega, \end{aligned}$$

for any I_i^j with $z_{ij} > 0$. Here $\bar{y} \in Y(0, T)$ is the solution to (1b) – (1d) for our constructed $u \in L^2(Q)$. The associated weak formulation is given by

$$(23) \quad \begin{aligned} & \int_{t_z(I_i^j)} \langle y_t^*(t), \varphi(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt + \int_{t_z(I_i^j)} a(y^*(t), \varphi(t)) dt \\ & = \int_{t_z(I_i^j)} \int_{\Omega} \langle \alpha_{\bar{d}} u(t), \varphi(t) \rangle_{L^2(\Omega)} dt \quad \forall \varphi \in L^2(t_z(I_i^j), H_0^1(\Omega)). \end{aligned}$$

Since $z_{ij} > 0$, $\tau_z : t_z(I_i^j) \rightarrow I_i^j$ is a bijection and even a C^∞ -diffeomorphism, we are able to calculate y_t for $t \in t_z(I_i^j)$ via the chain rule.

$$y_t(x, t) = \tilde{y}_\tau(x, \tau_z(t)) \cdot \frac{\partial \tau_z}{\partial t}(t) = \tilde{y}_\tau(x, \tau_z(t)) \cdot \frac{1}{z_{ij}}.$$

Note that $y \in Y(t_z(I_i^j))$ and therefore $y_t \in L^2(t_z(I_i^j), L^2(\Omega))$ holds. We evaluate for an arbitrary $\varphi \in L^2(t_z(I_i^j), H_0^1(\Omega))$ the following expressions and remember that \tilde{y} solves (21).

$$(24) \quad \begin{aligned} & \int_{t_z(I_i^j)} \langle y_t(t), \varphi(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt = \int_{t_z(I_i^j)} \frac{1}{z_{ij}} \cdot \langle \tilde{y}_\tau(\tau_z(t)), \varphi(t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt \\ & = \int_{I_i^j} \langle \tilde{y}_\tau(s), \varphi(t_z(s)) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} ds \\ & = - \int_{I_i^j} z_{ij} a(\tilde{y}(s), \varphi(t_z(s))) ds + \int_{I_i^j} z_{ij} \langle \alpha_{\bar{d}} \tilde{u}(s), \varphi(t_z(s)) \rangle_{L^2(\Omega)} ds \\ & = - \int_{t_z(I_i^j)} a(\tilde{y}(\tau_z(t)), \varphi(t)) dt + \int_{t_z(I_i^j)} \langle \alpha_{\bar{d}} \tilde{u}(\tau_z(t)), \varphi(t) \rangle_{L^2(\Omega)} dt \\ & = - \int_{t_z(I_i^j)} a(y(t), \varphi(t)) dt + \int_{t_z(I_i^j)} \langle \alpha_{\bar{d}} u(t), \varphi(t) \rangle_{L^2(\Omega)} dt \end{aligned}$$

But this yields that $y(x, t) = \tilde{y}(x, \tau_z(t))$ indeed solves (23). One open task is the accordance of the initial values, i.e., $y(x, \inf(t_z(I_i^j))) = \bar{y}(x, \inf(t_z(I_i^j)))$ for each minor interval I_{ij} with $z_{ij} > 0$, where \bar{y} solves (1b) – (1c) for u .

We prove this by induction. Let I_m be the first minor interval such that $z_m > 0$. Then $\inf(t_z(I_m)) = 0$ and $\tau_z(0) = 0$ holds. Also $y(x, \inf(t_z(I_m))) = y(x, 0) = \tilde{y}(x, \tau_z(0)) = \tilde{y}(x, 0) = y_0(x) = \bar{y}(x, \inf(t_z(I_m)))$ is deduced. Here we utilized that \tilde{y} is a forward solution and this concludes that the initial values indeed match. We conclude that $\tilde{y}(x, \tau_z(\sup(t_z(I_m)))) = y(x, \sup(t_z(I_m))) = \bar{y}(x, \sup(t_z(I_m)))$ as we discovered $y|_{\overline{t_z(I_m)}} = \bar{y}|_{\overline{t_z(I_m)}}$ from the calculation in (24). In this case we obtain $\tilde{y}(x, \tau_z(\sup(t_z(I_m)))) = \tilde{y}(x, \sup(I_m))$ as $t_z|_{\overline{I_m}}$ is a bijection from $\overline{I_m} \rightarrow \overline{t_z(I_m)}$. Since $(I_m)_{m \in |I|, |I_i|}$ is an ordered partition, we conclude that $\tilde{y}(x, \sup(t_z(I_m))) = \bar{y}(x, \inf(t_z(I_{\alpha_m})))$ with $\alpha_m := \min\{k \in \mathbb{N} \mid k > m \text{ and } z_k > 0\}$ holds. From the definition of α_m and τ_z , we deduce $\tilde{y}(x, \sup(I_m)) = \tilde{y}(x, \tau_z(\inf(t_z(I_{\alpha_m}))))$. By combining the arguments mentioned in this paragraph, we obtain

$$\begin{aligned} y(x, \inf(t_z(I_{\alpha_m}))) &= \tilde{y}(x, \tau_z(\inf(t_z(I_{\alpha_m})))) = \tilde{y}(x, \sup(I_m)) \\ &= \tilde{y}(x, \tau_z(\sup(t_z(I_m)))) = y(x, \sup(t_z(I_m))) \\ &= \bar{y}(x, \sup(t_z(I_m))) = \bar{y}(x, \inf(t_z(I_{\alpha_m}))). \end{aligned}$$

Let now I_m with $z_m > 0$ be fixed. We assume that the initial values $y(x, \inf(t_z(I_m)))$ and $\bar{y}(x, \inf(t_z(I_m)))$ coincide. We assert that m does not equal $\beta = \max\{k \in \mathbb{N} \mid z_k > 0\}$, because in this case there would be nothing left to prove. We have to demonstrate the claim for $y(x, \inf(t_z(I_{\alpha_m})))$, where α_m is defined in the

same fashion as in the previous paragraph. From the calculation (24) the property $y|_{\overline{t_z(I_m)}} = \tilde{y}|_{\overline{t_z(I_m)}}$ is inferred. In particular $y(x, \sup(t_z(I_m))) = \tilde{y}(x, \sup(t_z(I_m)))$ holds. As in the initial step, we obtain $\tilde{y}(x, \sup(t_z(I_m))) = \tilde{y}(x, \inf(t_z(I_{\alpha_m})))$ and $\sup(I_m) = \tau_z(\sup(t_z(I_m)))$. According to the definition of α_m , $z_k = 0$ for all $m < k < \alpha_m$ is valid. Therefore this implies $\sup(I_m) = \tau_z(\inf(t_z(I_{\alpha_m})))$. By combination of the previous steps we again obtain

$$\begin{aligned} y(x, \inf(t_z(I_{\alpha_m}))) &= \tilde{y}(x, \tau_z(\inf(t_z(I_{\alpha_m})))) = \tilde{y}(x, \sup(I_m)) \\ &= \tilde{y}(x, \tau_z(\sup(t_z(I_m)))) = y(x, \sup(t_z(I_m))) \\ &= \tilde{y}(x, \sup(t_z(I_m))) = \tilde{y}(x, \inf(t_z(I_{\alpha_m}))). \end{aligned}$$

Last we want to comment on the choice of the mode $\tilde{d} \in \{1, \dots, D\}$ on $t_z(I_m)$ for each I_m with $z_m > 0$. Since \tilde{y} is feasible for (9), the obtained state $y(x, t) = \tilde{y}(x, \tau_z(t))$ fulfills

$$\begin{aligned} (S(y(\cdot, t)) - b_{\tilde{d}}) &\leq 0, \\ (a_{\tilde{d}} - S(y(\cdot, t))) &\leq 0, \end{aligned}$$

for all $t \in t_z(I_m)$ and by construction $[0, T] = \cup_{m, z_m > 0} \overline{t_z(I_m)}$. This yields that y also fulfills the required state constraints. This completes the construction of an admissible pair $(u, y) \in L^2(Q) \times W_{v_{|i|}, z}$ for (1). \square

3.2. Transformation of vanishing constraints to equilibrium constraints and regularisation

In the previous section we have transformed the problem formulation with implicit switches (1) into a formulation with explicit switches (9). The drawback of (9) is the inclusion of state and vanishing type constraints on each minor interval. However, vanishing constraints and the resulting programs are closely related to so called mathematical programs with equilibrium constraints (MPECs). These programs have been studied intensively in the recent decades and satisfactory knowledge has been achieved in the process. We proceed along this established path and begin this section with a short overview on MPECs and sketch their relation to MPVCs.

Definition 8. Let $f, g_i, h_k, G_l, H_l : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuously differentiable functions. The optimization problem

$$\begin{aligned} (25a) \quad & \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \\ (25b) \quad & g_i(x) \leq 0 \quad \forall i = 1, \dots, p, \\ (25c) \quad & h_k(x) = 0 \quad \forall k = 1, \dots, q, \\ (25d) \quad & G_l(x) \geq 0 \quad \forall l = 1, \dots, r, \\ (25e) \quad & H_l(x) \geq 0 \quad \forall l = 1, \dots, r, \\ (25f) \quad & G_l(x)H_l(x) = 0 \quad \forall l = 1, \dots, r, \end{aligned}$$

is called mathematical program with equilibrium constraints (MPEC).

As announced earlier, we want to establish a connection between MPECs and MPVCs, c.f. [2]. Therefore we introduce slack variables s_l , $l \in \{1, \dots, r\}$. Let an instance of a MPVC in the form of (11) be given. We denote the input argument for

this problem by $x \in \mathbb{R}^n$. We consider the associated MPEC in $z = (x, s) \in \mathbb{R}^{n+r}$.

$$\begin{aligned}
(26a) \quad & \min_{(x,s)} f(x) \quad \text{s.t.} \\
(26b) \quad & g_i(x) \leq 0 \quad \forall i \in \{1, \dots, p\} \\
(26c) \quad & h_k(x) = 0 \quad \forall k \in \{1, \dots, q\} \\
(26d) \quad & G_l(x) - s_l \leq 0 \quad \forall l \in \{1, \dots, r\} \\
(26e) \quad & H_l(x) \geq 0 \quad \forall l \in \{1, \dots, r\} \\
(26f) \quad & s_l \geq 0 \quad \forall l \in \{1, \dots, r\} \\
(26g) \quad & H_l(x)s_l = 0 \quad \forall l \in \{1, \dots, r\}
\end{aligned}$$

There exist direct connections between local minima of (11) and (26), c.f. [2]. We now apply the transformation from (11) into (26) to (9). This results in the subsequent formulation. As explained in the previous section, we utilize a different notation for the state variable in each major reformulation.

$$\begin{aligned}
(27a) \quad & \min_{z, \tilde{u}, y^{(2)}, s} \tilde{J}(z, \tilde{u}, y^{(2)}) = \frac{1}{2} \|\sqrt{z} \cdot (y^{(2)} - \tilde{y}_{dis})\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|\sqrt{z} \cdot \tilde{u}\|_{L^2(Q)}^2 \\
& \quad \text{s.t.} \\
(27b) \quad & y_\tau^{(2)}(x, \tau) + z(\tau)A(y^{(2)})(x, \tau) = z(\tau)\alpha_{v_{|I|}^{(1)}(\tau)}(x)\tilde{u}(x, \tau) \quad (x, \tau) \in Q, \\
(27c) \quad & y^{(2)}(x, \tau) = 0 \quad (x, \tau) \in \Sigma, \\
(27d) \quad & y^{(2)}(x, 0) = y_0(x) \quad x \in \Omega, \\
(27e) \quad & z(\tau) \geq 0 \quad \tau \in [0, T], \\
(27f) \quad & z|_{I_i^j} \equiv z_{ij} \quad (i, j) \in |I|_{|I|, |I_i|}, \\
(27g) \quad & \int_{I_i} z(s) ds = \Delta I_i \quad i \in |I|_{|I|}, \\
(27h) \quad & (S(y^{(2)}(\cdot, \tau)) - b_{v_{|I|}^{(1)}(\tau)}) \leq s_{ij, UB} \quad \tau \in I_i^j, (i, j) \in |I|_{|I|, |I_i|}, \\
(27i) \quad & (a_{v_{|I|}^{(1)}(\tau)} - S(y^{(2)}(\cdot, \tau))) \leq s_{ij, LB} \quad \tau \in I_i^j, (i, j) \in |I|_{|I|, |I_i|}, \\
(27j) \quad & s_{ij, k} \geq 0 \quad (i, j, k) \in |I|_{|I|, |I_i|, k}, \\
(27k) \quad & s_{ij, k} \cdot z_{ij} = 0 \quad (i, j, k) \in |I|_{|I|, |I_i|, k}.
\end{aligned}$$

Here we set $|I|_{|I|, |I_i|, k} := |I|_{|I|} \times |I|_{|I_i|} \times \{UB, LB\}$.

Lemma 3 (Feasibility perservation). *We can establish the following connection between admissible points for (9) and (27).*

i) *Let $(z, \tilde{u}, y^{(1)})$ be feasible for (9) then $(z, \tilde{u}, y^{(1)}, s)$ is admissible (27), where we initialize $\begin{pmatrix} s_{LB} \\ s_{UB} \end{pmatrix} = s \in \mathbb{R}^{2 \cdot |I| \cdot |I_i|}$ as*

$$\begin{aligned}
s_{ij, LB} &= \begin{cases} 0, & \text{if } z_{ij} > 0, \\ \sup\{0, \max_{\tau \in I_i^j} a_{v_{|I|}^{(1)}(\tau)} - S(y^{(1)}(\cdot, \tau))\}, & \text{if else.} \end{cases} \\
s_{ij, UB} &= \begin{cases} 0, & \text{if } z_{ij} > 0, \\ \sup\{0, \max_{\tau \in I_i^j} S(y^{(1)}(\cdot, \tau)) - b_{v_{|I|}^{(1)}(\tau)}\}, & \text{if else.} \end{cases}
\end{aligned}$$

ii) If $(z, \tilde{u}, y^{(2)}, s)$ is admissible to (27), then $(z, \tilde{u}, y^{(2)})$ is feasible for (9).

Proof. We proof each statement individually.

Let $(z, \tilde{u}, y^{(1)})$ be feasible for (9). We first consider the state equation (9b) – (9d). According to assumption, there exists a well posed control to state operator $S^O : L^\infty([0, T]) \times L^2(Q) \rightarrow V_{v_{|I|}, z}^{|I|} : (z, \tilde{u}) \mapsto y^*$ for (12). This means $y^{(1)}$ also fulfills the state equation (27b) – (27d) in terms of a forward solution. The set of constraints in (27e) – (27g), which guarantees z to be a piecewise constant time control, can also be discovered in (9e) – (9g). Next we discuss the constraints (27h) – (27k). The postulate (27j) on $s_{ij,k}$ as well as the validity of (27k) in combination with z_{ij} can be immediately deduced from the construction of s . In order to check the validity of (27h) and (27i), we fix a $\tau \in I_i^j$. We distinguish between the case z_{ij} greater or equal to zero.

Let $z_{ij} > 0$. By construction of s , $s_{ij,k} = 0$ holds. Since $y^{(1)}$ is feasible for (9), we conclude that (9h) and (9i) are fulfilled. Due to our assumption on z_{ij} , we obtain

$$\begin{aligned} S(y^{(1)}(\cdot, \tau)) - b_{v_{|I|}(\tau)} &\leq 0, \\ a_{v_{|I|}(\tau)} - S(y^{(1)}(\cdot, \tau)) &\leq 0. \end{aligned}$$

But this equals exactly (27h) and (27i).

Let $z_{ij} = 0$. Then the conditions (9h) and (9i) pose no further restrictions. Since $y^{(1)}$ solves (12), the mapping $S_\tau : [0, T] \rightarrow H_0^1(\Omega), \tau \mapsto y^{(1)}(\cdot, \tau)$ is continuous. This means the mapping $S \circ S_\tau$ is a continuous mapping from $[0, T] \rightarrow \mathbb{R}$. Therefore it attains its maximum. We cannot immediately apply these arguments for $|S \circ S_\tau - b_{v_{|I|}(\tau)}|$ respectively $|a_{v_{|I|}(\tau)} - S \circ S_\tau|$ on the compact set \bar{I}_i^j , as the integer control function $v_{|I|}(\tau)$ is discontinuous on \bar{I}_i^j . However replacing the max by a sup expression resolves this minor issue. Eventually this construction guarantees that the formulated entries of $s_{ij, LB}$ and $s_{ij, UB}$ are actually well posed and finite. The validity of (27h) and (27i) is imminent from the definition of the entries for s . This establishes the admissibility of $(z, \tilde{u}, y^{(2)}, s)$ for (27). In particular we have discovered the superfluous distinction between $y^{(1)}$ and $y^{(2)}$.

Let $(z, \tilde{u}, y^{(2)}, s)$ be admissible to (27). The discussion of the state equation (9b) – (9d) and the requirements on z to be a piecewise constant time control (9e) – (9g) is performed in the same fashion as in the first part of this proof. We fix $\tau \in I_i^j$ and remain with the study of (9h) and (9i). In fact we only need to study the case where z_{ij} is greater than 0 since otherwise (9h) and (9i) pose no restriction. Therefore let w.l.o.g. $z_{ij} > 0$. From (27k) we receive that $s_{ij,k} = 0$ for $k \in \{UB, LB\}$. But then the (27h) and (27i) just read as (9h) and (9i). Ultimately this yields the feasibility of $(z, \tilde{u}, y^{(2)})$ for (9). \square

Remark 2. We still keep the conservation condition of the involved time control on each major interval I_i as an integral formulation in the previous problem (re)formulation. However, in the numerical implementation we will exploit the equivalent discrete formulation via

$$\sum_{l=1}^{|I_i|} z_{il} = |I_i|.$$

Of course this formulation heavily relies on the stated properties of the time control z as well as on the assumption of an ordered partition into major and minor intervals of equal length.

As a direct consequence of Lemma 3, we obtain the following connection between global solutions of (9) and (27).

Corollary 1. *Let $(z, \tilde{u}, y^{(2)}, s)$ be a global optimal solution of (27), then $(z, \tilde{u}, y^{(1)})$ is a global optimal solution of (9). Vice versa we obtain a global solution $(z, \tilde{u}, y^{(2)}, s)$ to (27) from a global solution $(z, \tilde{u}, y^{(1)})$ to (9) and initialization of s as presented in Lemma 3. In particular we have $y^{(2)} = y^{(1)}$.*

Proof. This is true to the fact that the cost functions in (9) and (27) coincide. The required feasibility discussion has been performed in Lemma 3. In particular the connection between the involved states has been explained. \square

Despite the portrayed relationship between the MPVC and MPEC formulation we still perform a last minor adaption to the cost function \tilde{J} by the inclusion of the term $\frac{\gamma_s}{2} \|s\|^2$ with a parameter $\gamma_s \geq 0$. The intention behind this additional contribution is to cope with the ambiguity in the choice of the slack variable and to enforce a unique choice for the slack variable, [2].

3.3. A differentiable penalty approach

In this section we will discuss a differentiable penalty approach since the formulation (27) still possesses some major drawbacks. The first one concerns the involved state equation (12), which can, as explained in subsection 3.1, only be solved in terms of a forward solution. Lemma 4 will allow us the approximation of forward solutions by regularized states (28), which can again be treated with standard parabolic theory. Eventually we will replace the state equation in (27b) – (27d) by its regularization (28a) – (28c). For a fixed $\varepsilon > 0$, we investigate for a given piecewise constant time control z and $u \in L^2(Q)$ the subsequent problem on Q

(28a)

$$y_\tau(x, \tau) + (z_{ij} + \varepsilon)A(y)(x, \tau) = (z_{ij} + \varepsilon)\alpha_{\frac{v_{|i|}}{v_{|i|}(\tau)}}(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

(28b)

$$y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

(28c)

$$y(x, 0) = y_0(x) \quad x \in \Omega.$$

Thereby the conditions (28a) and (28b) have to be fulfilled for all $(i, j) \in |I|_{|I|, |I_i|}$. The values of z_{ij} in the definition of (28) correspond to $z|_{I_i^j}$ for a given piecewise constant time control z . We obtain from (14) respectively (15) the following regularity estimates for the state y in (28).

$$(29) \quad \|y\|_{W((0, T))} \leq C((\varepsilon + \|z\|_\infty)\|u\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}),$$

$$(30) \quad \|y\|_{Y(0, T)} \leq C((\varepsilon + \|z\|_\infty)\|u\|_{L^2(Q)} + \|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})}).$$

The connection between the states obtained from (12) respectively (28) is formulated in the subsequent lemma.

Lemma 4. *Let an initial condition $y_0 \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$, a piecewise constant time control $z \in L^\infty([0, T])$ together with a distributed control $u \in L^2(Q)$ be given. Furthermore denote by $y^\varepsilon \in Y(0, T)$ the solution to (28) for a fixed $\varepsilon > 0$ and let $y^f \in V_{\frac{v_{|i|}}{v_{|i|}z}}$ be the*

forward solution to (12). Then

$$\lim_{\varepsilon \downarrow 0} \|y^\varepsilon - y^f\|_{C^0([0,T];H_0^1(\Omega))} + \|y^\varepsilon - y^f\|_{C^0(\overline{\Omega})} = 0.$$

Proof. Before we discuss the core part of the proof, we consider two intermediate results. We select a fixed minor interval I_i^j and assume $z_{ij} > 0$ together with a fixed $\varepsilon > 0$. We investigate the following two systems

$$(31a) \quad y_\tau(x, \tau) - (z_{ij} + \varepsilon)A(y)(x, \tau) = (z_{ij} + \varepsilon)\alpha_{\frac{|I_i^j|}{v|I_i^j|}}(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

$$(31b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

$$(31c) \quad y(x, \tau_{i,(j-1)}) = y_0^\varepsilon(x) \quad x \in \Omega,$$

and

$$(32a) \quad y_\tau(x, \tau) - z_{ij}A(y)(x, \tau) = z_{ij}\alpha_{\frac{|I_i^j|}{v|I_i^j|}}(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

$$(32b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

$$(32c) \quad y(x, \tau_{i,(j-1)}) = y_0(x) \quad x \in \Omega.$$

We assume the initial values satisfy $y_0 \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ and $y_0^\varepsilon \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$, $\forall \varepsilon > 0$, such that $y_0^\varepsilon \rightarrow y_0$ in $H_0^1(\Omega) \cap C^0(\overline{\Omega})$. We denote the solution to (31) for a given right side u by \tilde{y}_ε . Similarly we write \tilde{y} for the solution of (32) associated with the same distributed control u . We set $y_\varepsilon^* := \tilde{y}_\varepsilon - \tilde{y}$. By construction y_ε^* then solves the system

$$(33a) \quad y_\tau(x, \tau) - z_{ij}A(y)(x, \tau) = \varepsilon A(\tilde{y}_\varepsilon)(x, \tau) + \varepsilon \alpha_{\frac{|I_i^j|}{v|I_i^j|}}(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

$$(33b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

$$(33c) \quad y(x, \tau_{i,(j-1)}) = y_0^\varepsilon(x) - y_0(x) \quad x \in \Omega.$$

From (15) we receive the estimate

$$(34) \quad \begin{aligned} \|y_\varepsilon^*\|_{Y(I_i^j)} &\leq \varepsilon \cdot C \cdot (\|A(\tilde{y}_\varepsilon)\|_{L^2(\Omega \times I_i^j)} + \|u\|_{L^2(\Omega \times I_i^j)}) \\ &\quad + C \cdot (\|y_0^\varepsilon - y_0\|_{H^1(\Omega)} + \|y_0^\varepsilon - y_0\|_{C^0(\overline{\Omega})}). \end{aligned}$$

By the assumption on $y_0^\varepsilon - y_0$ and since $\|A(\tilde{y}_\varepsilon)\|_{L^2(\Omega \times I_i^j)}$ is uniformly bounded for small ε , we conclude

$$\lim_{\varepsilon \downarrow 0} \|y_\varepsilon^*\|_{Y(I_i^j)} = 0.$$

Since $Y(I_i^j)$ is continuously embedded into $C^0(\overline{I_i^j}; H_0^1(\Omega))$, we deduce for $\varepsilon \downarrow 0$

$$\lim_{\varepsilon \downarrow 0} \|y_\varepsilon^*\|_{C^0(\overline{\Omega \times I_i^j})} + \|y_\varepsilon^*\|_{C^0(\overline{I_i^j}, H_0^1(\Omega))} = 0.$$

Next we consider the following problem for a set of initial values $y_0^\varepsilon \rightarrow 0$ in $H_0^1(\Omega) \cap C^0(\overline{\Omega})$. We write y_ε^{ij} for the solution of

$$(35a) \quad y_\tau(x, \tau) - \varepsilon A(y)(x, \tau) = \varepsilon \alpha_{\frac{|I_i^j|}{v|I_i^j|}}(x)u(x, \tau) \quad (x, \tau) \in \Omega \times I_i^j,$$

$$(35b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_i^j,$$

$$(35c) \quad y(x, \tau_{i,(j-1)}) = y_0^\varepsilon(x) \quad x \in \Omega.$$

Since for each fixed $\varepsilon > 0$ the estimate (15) holds, we can deduce

$$(36) \quad \|y_\varepsilon^{ij}\|_{Y(I_i^j)} \leq C \cdot \left(\varepsilon \cdot \|u\|_{L^2(\Omega \times I_i^j)} + \|y_0^\varepsilon\|_{H^1(\Omega)} + \|y_0^\varepsilon\|_{C^0(\bar{\Omega})} \right).$$

We obtain for the limit $\varepsilon \downarrow 0$

$$\lim_{\varepsilon \downarrow 0} \|y_\varepsilon^{ij}\|_{Y(I_i^j)} \leq \lim_{\varepsilon \downarrow 0} C \cdot \left(\varepsilon \cdot \|u\|_{L^2(\Omega \times I_i^j)} + \|y_0^\varepsilon\|_{H^1(\Omega)} + \|y_0^\varepsilon\|_{C^0(\bar{\Omega})} \right) = 0.$$

Let us now discuss the main part of the proof. We begin with the element I_1^1 and distinguish between z_{11} greater or equal to zero.

Let z_{11} be equal to 0. According to the definition of a forward solution, we set $y^f(x, \tau) = y_0(x) \quad \forall (x, \tau) \in \overline{\Omega \times I_1^1}$. With y^ε for $\varepsilon > 0$ we solve

$$(37a) \quad y_\tau(x, \tau) - \varepsilon A(y)(x, \tau) = \varepsilon \alpha_{\frac{|I_1^1|}{|\tau|}}(x) u(x, \tau) \quad (x, \tau) \in \Omega \times I_1^1,$$

$$(37b) \quad y(x, \tau) = 0 \quad (x, \tau) \in \partial\Omega \times I_1^1,$$

$$(37c) \quad y(x, 0) = y_0(x) \quad x \in \Omega.$$

We denote by y^h the solution to (37) with distributed control 0 and initial condition y_0 . From the representation of y^h via a Fourier series, c.f. [11], we obtain the estimate

$$(38) \quad \|y^h - y^f\|_{C^0(\overline{\Omega \times I_1^1})} + \|y^h - y^f\|_{C^0(\bar{I}_1^1; H_0^1(\Omega))} \leq s(\varepsilon) \cdot (\|y_0\|_{C^0(\bar{\Omega})} + \|y_0\|_{H_0^1(\Omega)})$$

with a continuous function $s : \mathbb{R} \rightarrow \mathbb{R}_0^+$, which possesses the property $s(0) = 0$. The function s just depends on T but is independent from Ω . By combining the second intermediate result, (36), and (38) we deduce

$$(39) \quad \begin{aligned} & \|y^\varepsilon - y^f\|_{C^0(\overline{\Omega \times I_1^1})} + \|y^\varepsilon - y^f\|_{C^0(\bar{I}_1^1; H_0^1(\Omega))} \\ &= \|y^\varepsilon - y^h\|_{C^0(\overline{\Omega \times I_1^1})} + \|y^\varepsilon - y^h\|_{C^0(\bar{I}_1^1; H_0^1(\Omega))} \\ &+ \|y^h - y^f\|_{C^0(\overline{\Omega \times I_1^1})} + \|y^h - y^f\|_{C^0(\bar{I}_1^1; H_0^1(\Omega))} \\ &\leq C \cdot \|y^\varepsilon - y^h\|_{Y(I_1^1)} + s(\varepsilon) \cdot (\|y_0\|_{C^0(\bar{\Omega})} + \|y_0\|_{H_0^1(\Omega)}) \\ &\leq C \cdot \varepsilon \cdot \|u\|_{L^2(\Omega \times I_1^1)} + s(\varepsilon) \cdot (\|y_0\|_{C^0(\bar{\Omega})} + \|y_0\|_{H_0^1(\Omega)}) \\ &\leq \tilde{s}(\varepsilon) \cdot (\|u\|_{L^2(\Omega \times I_1^1)} + \|y_0\|_{C^0(\bar{\Omega})} + \|y_0\|_{H_0^1(\Omega)}). \end{aligned}$$

Here \tilde{s} possesses the same properties as s .

Let $z_{11} > 0$. Then we conclude from the first part of our lemma, (34), and the estimate for $\|y^\varepsilon\|_{Y(I_1^1)}$,

$$(40) \quad \begin{aligned} & \|y^\varepsilon - y^f\|_{C^0(\overline{\Omega \times I_1^1})} + \|y^\varepsilon - y^f\|_{C^0(\bar{I}_1^1; H_0^1(\Omega))} \\ &\leq \varepsilon \cdot C \cdot (\|A(y_\varepsilon)\|_{L^2(\Omega \times I_1^1)} + \|u\|_{L^2(\Omega \times I_1^1)}) \\ &\leq \varepsilon \cdot C \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_1^1)}). \end{aligned}$$

This yields the claim for I_1^1 , but most importantly the estimates in (39) and (40) demonstrate that the norm difference of the initial values on the upcoming minor interval does depend continuously on ε . This concludes the initial case. Let now I_i^j be fixed. We again consider the compact set $I_{ij}^U := \cup_{k=1}^i \cup_{l=1}^{j-1} \bar{I}_{kl}$. The forward

solution y^f and regularized solution y^ε are assumed to satisfy

$$(41) \quad \begin{aligned} & \|y^\varepsilon - y^f\|_{C^0(\overline{I_{ij}^\cup \times \Omega})} + \|y^\varepsilon - y^f\|_{C^0(I_{ij}^\cup; H_0^1(\Omega))} \\ & \leq \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^\cup)}). \end{aligned}$$

with a continuous and nonnegative function \tilde{s} satisfying $\tilde{s}(0) = 0$. For an arbitrary I_i^j we again distinguish between z_{ij} greater or equal to zero.

Let $z_{ij} = 0$. Then the forward solution $y^f(x, \tau)$ for $(x, \tau) \in \overline{\Omega \times I_i^j}$ is given as $y^f(x, \tau) = y^f(x, \max(I_{ij}^\cup))$. Let $y_{ij,0}^\varepsilon(x) = y^\varepsilon(x, \max(I_{ij}^\cup)) - y^f(x, \max(I_{ij}^\cup))$ denote the initial condition for system (35). This system covers the difference between y^ε and y^h , where y^h solves (35) with distributed control 0 and initial condition $y^f(x, \max(I_{ij}^\cup))$. This set of initial conditions fulfills $y_{ij,0}^\varepsilon \rightarrow 0$ in $H_0^1(\Omega) \cap C^0(\overline{\Omega})$. We again split the difference into two parts

$$\begin{aligned} & \|y^\varepsilon - y^f\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^\varepsilon - y^f\|_{C^0(I_{ij}; H_0^1(\Omega))} \\ & \leq \|y^\varepsilon - y^h\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^\varepsilon - y^h\|_{C^0(I_{ij}; H_0^1(\Omega))} \\ & \quad + \|y^h - y^f\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^h - y^f\|_{C^0(I_{ij}; H_0^1(\Omega))}. \end{aligned}$$

By definition of $y_{ij,0}^\varepsilon$ and the involved norms

$$(42) \quad \|y_{ij,0}^\varepsilon\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon\|_{C^0(\overline{\Omega})} \leq \|y^\varepsilon - y^f\|_{C^0(\overline{I_{ij}^\cup \times \Omega})} + \|y^\varepsilon - y^f\|_{C^0(I_{ij}^\cup; H_0^1(\Omega))}$$

holds. For the first contribution we apply the estimate (36) in combination with (41) and (42)

$$(43) \quad \begin{aligned} & \|y^\varepsilon - y^h\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^\varepsilon - y^h\|_{C^0(I_{ij}; H_0^1(\Omega))} \\ & \leq C \cdot (\|y_{ij,0}^\varepsilon\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon\|_{C^0(\overline{\Omega})} + \varepsilon \cdot \|u\|_{L^2(\Omega \times I_{ij})}) \\ & \leq C \cdot (\tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^\cup)}) + \varepsilon \|u\|_{L^2(\Omega \times I_{ij})}). \end{aligned}$$

For the second half we argue again with the representation of y_h via its Fourier series and ultimately deduce a similar estimate to (38). Furthermore the evaluations of $\|y_f\|_{C^0(\overline{I_{ij}^\cup \times \Omega})}$ and $\|y_f\|_{C^0(I_{ij}^\cup; H_0^1(\Omega))}$ depend Lipschitz continuous on y_0 and the involved control u , since y_f is either the solution to a parabolic equation on active minor intervals or continued in a constant way along inactive minor intervals. We obtain the estimate

$$(44) \quad \begin{aligned} & \|y^h - y^f\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^h - y^f\|_{C^0(I_{ij}; H_0^1(\Omega))} \\ & \leq s(\varepsilon) \cdot (\|y^f(\cdot, \max(I_{ij}^\cup))\|_{H_0^1(\Omega)} + \|y^f(\cdot, \max(I_{ij}^\cup))\|_{C^0(\overline{\Omega})}) \\ & \leq s(\varepsilon) \cdot C \cdot (\|y_0\|_{H^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^\cup)}). \end{aligned}$$

Finally we combine (43) and (44)

$$\begin{aligned} & \|y^\varepsilon - y^f\|_{C^0(\overline{\Omega \times I_{ij}})} + \|y^\varepsilon - y^f\|_{C^0(I_{ij}; H_0^1(\Omega))} \\ & \leq C \cdot \tilde{s}(\varepsilon) (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^\cup)} + \|u\|_{L^2(\Omega \times I_{ij})}) \\ & \leq \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\overline{\Omega})} + \|u\|_{L^2(\Omega \times (I_{ij}^\cup \cup \overline{I_i^j})})). \end{aligned}$$

The function \tilde{s} is still non negative and continuous. In addition it fulfills $\tilde{s}(0) = 0$.

We assume $z_{ij} > 0$. Then the forward solution y^f solves (32) with the initial value $y_{ij,0} = y^f(x, \max(I_{ij}^U))$. The set of regularized systems solves (31) for a set of initial values $y_{ij,0}^\varepsilon$, which satisfy $y_{ij,0}^\varepsilon \rightarrow y_{ij,0}$ in $H_0^1(\Omega) \cap C^0(\bar{\Omega})$ for $\varepsilon \downarrow 0$, according to (41). From (34) and (41) we obtain the following estimate for $y^\varepsilon - y^f$,

$$\begin{aligned} & \|y^\varepsilon - y^f\|_{Y(I_{ij}^j)} \\ & \leq \varepsilon \cdot C \cdot (\|A(y^\varepsilon)\|_{L^2(\Omega \times I_{ij}^j)} + \|u\|_{L^2(\Omega \times I_{ij}^j)}) \\ & + C \cdot (\|y_{ij,0}^\varepsilon - y_{ij,0}\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon - y_{ij,0}\|_{C^0(\bar{\Omega})}) \\ & \leq \varepsilon \cdot C \cdot (\|A(y^\varepsilon)\|_{L^2(\Omega \times I_{ij}^j)} + \|u\|_{L^2(\Omega \times I_{ij}^j)}) \\ & + \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)}) \end{aligned}$$

The evaluation of $A(y^\varepsilon)$ satisfies $\|A(y^\varepsilon)\|_{L^2(\Omega \times I_{ij}^j)} \leq \|y^\varepsilon\|_{Y(I_{ij}^j)}$ according to the definition of $\|\cdot\|_{Y(I_{ij}^j)}$. Furthermore (30) yields $\|y^\varepsilon\|_{Y(I_{ij}^j)} \leq C \cdot (\|y_{ij,0}^\varepsilon\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^j)})$. Hence we deduce

$$\begin{aligned} \|y^\varepsilon - y^f\|_{Y(I_{ij}^j)} & \leq \varepsilon \cdot C \cdot (\|y_{ij,0}^\varepsilon\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^j)}) + \|u\|_{L^2(\Omega \times I_{ij}^j)} \\ & + \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)}). \end{aligned}$$

The evaluations of $\|y_{ij,0}^\varepsilon\|_{H_0^1(\Omega)} + \|y_{ij,0}^\varepsilon\|_{C^0(\bar{\Omega})}$ are bounded by $\|y^\varepsilon\|_{C^0(\bar{\Omega} \times I_{ij}^U)}$ and $\|y^\varepsilon\|_{C^0(I_{ij}^U; H_0^1(\Omega))}$, which themselves can ultimately be estimated by $\|y^\varepsilon\|_{Y(I_{ij}^U)}$. As a result we obtain the estimate

$$\begin{aligned} \|y^\varepsilon - y^f\|_{Y(I_{ij}^j)} & \leq \varepsilon \cdot C \cdot (\|y^\varepsilon\|_{Y(I_{ij}^U)} + \|u\|_{L^2(\Omega \times I_{ij}^j)}) \\ & + \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)}). \end{aligned}$$

Finally y_ε is the solution to the system (28) on the set $\Omega \times I_{ij}^U$. Therefore an estimate for $\|y_\varepsilon\|_{Y(I_{ij}^U)}$ similar to (30) holds. We conclude

$$\begin{aligned} \|y^\varepsilon - y^f\|_{Y(I_{ij}^j)} & \leq \varepsilon \cdot C \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)}) + \|u\|_{L^2(\Omega \times I_{ij}^j)} \\ & + \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)}) \\ & \leq \tilde{s}(\varepsilon) \cdot (\|y_0\|_{H_0^1(\Omega)} + \|y_0\|_{C^0(\bar{\Omega})} + \|u\|_{L^2(\Omega \times I_{ij}^U)} + \|u\|_{L^2(\Omega \times I_{ij}^j)}) \end{aligned}$$

with a non negative and continuous function \tilde{s} such that $\tilde{s}(0) = 0$. We obtain the claim of the lemma by induction and taking $\varepsilon \rightarrow 0$ since

$$\|y^\varepsilon - y^f\|_{C^0(\bar{\Omega} \times I_{ij}^j)} + \|y^\varepsilon - y^f\|_{C^0(I_{ij}^U; H_0^1(\Omega))} \leq \|y^\varepsilon - y^f\|_{Y(I_{ij}^j)}$$

holds. \square

In the process we also replace z by $z + \varepsilon$ in the cost function of (27). The reasoning behind this substitution can be discovered in the discussion of the algorithmic approach. For now we consider the following problem for $\varepsilon > 0$. We will write $\tilde{z} \in L^\infty([0, T])$ for the time control that is uniquely determined by a vector $z \in \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|}$ via the property $\tilde{z}|_{I_{ij}^i} = z_{ij}$. Also we adapt the notation to $z \in \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|}$ instead of $z \in L^\infty([0, T])$. Finally we set $\tilde{y}_{dis}^z(x, \tau) := y_{dis}(x, t_{\tilde{z}}(\tau))$. This gives

$$\begin{aligned}
(45a) \quad & \min_{z, \tilde{u}, y^{(2,\varepsilon)}, s} \tilde{J}_\varepsilon(z, \tilde{u}, \tilde{y}) = \frac{1}{2} \|\sqrt{\tilde{z} + \varepsilon} \cdot (y^{(2,\varepsilon)} - \tilde{y}_{dis}^z)\|_{L^2(Q)}^2 \\
& + \frac{\gamma_{Con}}{2} \|\sqrt{\tilde{z} + \varepsilon} \cdot \tilde{u}\|_{L^2(Q)}^2 + \frac{\gamma_s}{2} \|s\|^2 \\
& \text{s.t.} \\
(45b) \quad & y_\tau^{(2,\varepsilon)}(x, \tau) + (\tilde{z}(\tau) + \varepsilon) A(y^{(2,\varepsilon)})(x, \tau) = (\tilde{z}(\tau) + \varepsilon) \alpha_{v_{|I_i|}(\tau)}(x) \tilde{u}(x, \tau) \quad (x, \tau) \in Q, \\
(45c) \quad & y^{(2,\varepsilon)}(x, \tau) = 0 \quad (x, \tau) \in \Sigma, \\
(45d) \quad & y^{(2,\varepsilon)}(x, 0) = y_0(x) \quad x \in \Omega, \\
(45e) \quad & \tilde{z}(\tau) \geq 0 \quad \tau \in [0, T], \\
(45f) \quad & \tilde{z}|_{I_i^j} \equiv z_{ij} \quad (i, j) \in |I|_{|I_i|, |I_i|}, \\
(45g) \quad & \int_{I_i} \tilde{z}(s) ds = \Delta I_i \quad i \in |I|_{|I|}, \\
(45h) \quad & (S(y^{(2,\varepsilon)}(\cdot, \tau)) - b_{v_{|I_i|}(\tau)}) \leq s_{ij, UB} \quad \tau \in I_i^j \quad (i, j) \in |I|_{|I_i|, |I_i|}, \\
(45i) \quad & (a_{v_{|I_i|}(\tau)} - S(y^{(2,\varepsilon)}(\cdot, \tau))) \leq s_{ij, LB} \quad \tau \in I_i^j \quad (i, j) \in |I|_{|I_i|, |I_i|}, \\
(45j) \quad & s_{ij, k} \geq 0 \quad (i, j, k) \in |I|_{|I_i|, |I_i|, k}, \\
(45k) \quad & s_{ij, k} \cdot z_{ij} = 0 \quad (i, j, k) \in |I|_{|I_i|, |I_i|, k}.
\end{aligned}$$

This performed ε -regularization simplifies the discussion of the involved state equation. However, the formulation (45) still contains state constraints on each minor interval (45h) and (45i) together with the equilibrium constraints (45k), which require separate attention.

At this stage we want to give a brief explanation and justification for the chosen penalty approach. In [9] optimality conditions for semi linear parabolic systems with explicit switches are discussed. Herein the authors also apply a transformation method with similar properties compared to our time transformation. However, they do not concern themselves with state constraints. Traditionally state constraints require special treatment, as their associated multipliers can be quite irregular. For state constraints the acquainted technique of Moreau-Yosida penalization possesses known convergence results, c.f. [18] and [22]. But also in the realm of finite dimensional MPECs there exist known results about differentiable penalty approaches, c.f. [20] and [21]. With the knowledge about the mentioned results, a penalty approach for the involved state and equilibrium constraints seems promising.

Next we proceed with a short overview on the utilized penalty approaches. For the state constraints we apply the following penalization strategy with a penalty parameter $\gamma_{MY} > 0$.

$$(46) \quad g(y(x, t)) \leq 0 \implies \frac{\gamma_{MY}}{2} \max\{0, g(y(x, t))\}^2.$$

We set $g(x)^+ := \max\{0, g(x)\}$. Furthermore we make use of a non-linear complementarity function (NCP function) in form of the established Fischer-Burmeister function

$$\varphi_{FB} : \mathbb{R}^2 \rightarrow \mathbb{R}, (a, b) \mapsto \sqrt{a^2 + b^2} - a - b.$$

Any NCP function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ displays the paramount property

$$(47) \quad \varphi(a, b) \iff a \geq 0, b \geq 0, a \cdot b = 0.$$

However, NCP functions are usually not differentiable. Therefore we have to employ φ_{FB}^2 to ensure a differentiable penalty term with a semismooth derivative, c.f. [7]. Of course the utilization of just φ_{FB} would be possible, but this would result in a non-smooth penalty approach. We obtain the following penalized formulation for (45). We introduce the following abbreviations for penalty functions associated with the equilibrium respectively state constraints

$$\begin{aligned} J_{\gamma EC}(z, s) &:= \frac{\gamma_{EC}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \varphi_{FB}(z_{ij}, s_{ij,UB})^2 + \frac{\gamma_{EC}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \varphi_{FB}(z_{ij}, s_{ij,LB})^2, \\ J_{\gamma MY}(\tilde{y}, s) &:= \frac{\gamma_{MY}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} ((S(\tilde{y}(\cdot, s)) - b_{v|I_i|}(s) - s_{ij,UB})^+)^2 ds \\ &\quad + \frac{\gamma_{MY}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} ((a_{v|I_i|}(s) - S(\tilde{y}(\cdot, s)) - s_{ij,LB})^+)^2 ds. \end{aligned}$$

The penalized version of (45) reads as

$$(48a) \quad \min_{z, \tilde{u}, y^{(3,\varepsilon)}, s} \tilde{J}_\varepsilon^P(z, \tilde{u}, y^{(3,\varepsilon)}, s) = \frac{1}{2} \|\sqrt{\tilde{z}} + \varepsilon \cdot (y^{(3,\varepsilon)} - \tilde{y}_{dis}^z)\|_{L^2(Q)}^2 \\ + \frac{\gamma_{Con}}{2} \|\sqrt{\tilde{z}} + \varepsilon \cdot \tilde{u}\|_{L^2(Q)}^2 + \frac{\gamma_s}{2} \|s\|^2 \\ + J_{\gamma EC}(z, s) + J_{\gamma MY}(y^{(3,\varepsilon)}, s) \\ \text{s.t.}$$

$$(48b) \quad y_\tau^{(3,\varepsilon)}(x, \tau) + (\tilde{z}(\tau) + \varepsilon) A(y^{(3,\varepsilon)})(x, \tau) = (\tilde{z}(\tau) + \varepsilon) \alpha_{v|I_i|(\tau)}(x) \tilde{u}(x, \tau) \quad (x, \tau) \in Q,$$

$$(48c) \quad y^{(3,\varepsilon)}(x, \tau) = 0 \quad (x, \tau) \in \Sigma,$$

$$(48d) \quad y^{(3,\varepsilon)}(x, 0) = y_0(x) \quad x \in \Omega,$$

$$(48e) \quad \tilde{z}|_{I_i^j} \equiv z_{ij} \quad (i, j) \in |I|_{|I_i|, |I_i|},$$

$$(48f) \quad \int_{I_i} \tilde{z}(s) ds = \Delta I_i \quad i \in |I|_{|I|},$$

$$(48g) \quad z_{ij} \geq 0 \quad (i, j) \in |I|_{|I_i|, |I_i|},$$

$$(48h) \quad s_{ij,k} \geq 0 \quad (i, j, k) \in |I|_{|I_i|, |I_i|, k}.$$

Regarding existence of minimizers for (48) we get the following result.

Lemma 5. *Let there be an upper bound, S^+ , on the entries of $s = \begin{pmatrix} s_{UB} \\ s_{LB} \end{pmatrix} \in \mathbb{R}^{2 \cdot |I| \cdot |I_i|}$. Furthermore there exist a function $f \in L^2(Q)$ such that $|y_{dis}(\cdot, t_z(\cdot))| \leq f$ a.e. in Q for all piecewise constant time controls z associated with the partition $\cup_{i=1}^{|I|} \cup_{j=1}^{|I_i|} I_i^j$. Then (48) possesses a solution.*

Proof. Let $((z, \tilde{u}, y^{(3,\varepsilon)}, s)_k)_{k \in \mathbb{N}} \in \mathbb{R}^{|I| \cdot |I_i|} \times L^2(Q) \times Y(0, T) \times \mathbb{R}^{2 \cdot |I| \cdot |I_i|}$ be feasible for (48) and tending to the infimum. Such a choice is possible as the cost function is bounded from below. The function $\tilde{J}_\varepsilon^P(z, \tilde{u}, y^{(3,\varepsilon)}, s)$ is coercive with respect to the control variable. Therefore the sequence $(\tilde{u}_k)_{k \in \mathbb{N}}$ is bounded in $L^2(Q)$. Furthermore the sequences $(z_k)_{k \in \mathbb{N}}$ and $(s_k)_{k \in \mathbb{N}}$ are bounded by construction respectively

assumption. The estimate for the state variable, (30), yields the boundedness of $(y_k^{(3,\varepsilon)})_{k \in \mathbb{N}}$ in $Y(0, T)$. Since the spaces $L^2(Q)$ and $Y(0, T)$ are reflexive, there exist elements $y^* \in Y(0, T)$ respectively $\tilde{u}^* \in L^2(Q)$ such that $(\tilde{u}_k)_{k \in \mathbb{N}} \rightharpoonup \tilde{u}^*$ in $L^2(Q)$ and $(y_k^{(3,\varepsilon)})_{k \in \mathbb{N}} \rightharpoonup y^*$ in $Y(0, T)$ for a suitable subsequence. W.l.o.g. we assume that the entire sequences are weakly convergent. Since z and s are finite dimensional, we can apply Weierstrass's theorem to select convergent (sub)sequences and elements $z^* \in \mathbb{R}^{|I| \cdot |I_i|}$ and $s^* \in \mathbb{R}^{2 \cdot |I| \cdot |I_i|}$ with $z_k \rightarrow z^*$ and $s_k \rightarrow s^*$. We first show that the quadruple $(z^*, \tilde{u}^*, y^*, s^*)$ is feasible for (48). We discuss each component and their involved constraints individually. For the limit of the time control z^* we obtain the following results

$$z_m^* = \lim_{k \rightarrow \infty} z_m^k \geq 0 \quad \forall m \in |I|_{|I| \cdot |I_i|},$$

$$\sum_{l=1}^{|I_i|} z_{il}^* = \lim_{k \rightarrow \infty} \sum_{l=1}^{|I_i|} z_{il}^k = \Delta I_i \quad \forall i \in |I|_{|I|}.$$

As a result z^* is feasible. To check the feasibility of y^* , we require the weak formulation of (28), which is given by

$$(49) \quad \int_0^T \langle y_\tau(\tau), \varphi(\tau) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} d\tau + \int_0^T (\tilde{z}(\tau) + \varepsilon) a(y(\tau), \varphi(\tau)) d\tau =$$

$$\int_0^T (\tilde{z}(\tau) + \varepsilon) \langle \alpha_{v_{|I_i|}(\tau)} u(\tau), \varphi(\tau) \rangle_{L^2(\Omega)} d\tau \quad \forall \varphi \in L^2((0, T); H_0^1(\Omega)).$$

We plug in the triple $(z_k, \tilde{u}_k, y_k^{(3,\varepsilon)})$ for $k \in \mathbb{N}$ and partition $[0, T]$. Then (49) reads as.

$$\sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \langle y_{k,\tau}^{(3,\varepsilon)}(\tau), \varphi(\tau) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} d\tau + \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (z_{ij}^k + \varepsilon) a(y_k^{(3,\varepsilon)}(\tau), \varphi(\tau)) d\tau$$

$$= \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (z_{ij}^k + \varepsilon) \langle \alpha_{v_{|I_i|}(\tau)} \tilde{u}_k(\tau), \varphi(\tau) \rangle_{L^2(\Omega)} d\tau \quad \forall \varphi \in L^2((0, T); H_0^1(\Omega)).$$

The linearity of the integral together with the weak convergence of \tilde{u}_k respectively $y_k^{(3,\varepsilon)}$ and the strong convergence of z_k lead to

$$(50) \quad \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \langle y_\tau^*(\tau), \varphi(\tau) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} d\tau + \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (z_{ij}^* + \varepsilon) a(y^*(\tau), \varphi(\tau)) d\tau$$

$$= \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (z_{ij}^* + \varepsilon) \langle \alpha_{v_{|I_i|}(\tau)} \tilde{u}^*(\tau), \varphi(\tau) \rangle_{L^2(\Omega)} d\tau \quad \forall \varphi \in L^2((0, T); H_0^1(\Omega)).$$

This means y^* fulfills the state equation (28) for \tilde{u}^* . There are no further constraints on the control \tilde{u}^* . Hence (\tilde{u}^*, y^*) is feasible. Finally we investigate the slack variable.

$$s_m^* = \lim_{k \rightarrow \infty} s_m^k \leq S^\dagger \quad \forall m \in \{1, \dots, 2 \cdot |I| \cdot |I_i|\},$$

$$s_m^* = \lim_{k \rightarrow \infty} s_m^k \geq 0 \quad \forall m \in \{1, \dots, 2 \cdot |I| \cdot |I_i|\}.$$

We now want to demonstrate that the feasible quadruple $(z^*, \tilde{u}^*, y^*, s^*)$ is indeed optimal. For that purpose we consider all contributions to the cost function individually

$$(51a) \quad \frac{1}{2} \|\sqrt{\bar{z} + \varepsilon} \cdot (y^{(3,\varepsilon)} - \tilde{y}_{dis}^z)\|_{L^2(Q)}^2 := \psi_1(z, y^{(3,\varepsilon)}),$$

$$(51b) \quad \frac{\gamma_{Con}}{2} \|\sqrt{\bar{z} + \varepsilon} \cdot \tilde{u}\|_{L^2(Q)}^2 := \psi_2(z, \tilde{u}),$$

$$(51c) \quad \frac{\gamma_{EC}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \varphi_{FB}(z_{ij}, s_{ij,UB})^2 := \psi_3(z, s_{UB}),$$

$$(51d) \quad \frac{\gamma_{MY}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} ((S(y^{(3,\varepsilon)}(\cdot, s)) - b_{v_{|I_i|}(s)} - s_{ij,UB})^+)^2 ds := \psi_4(y^{(3,\varepsilon)}, s_{UB}),$$

$$(51e) \quad \frac{\gamma_{EC}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \varphi_{FB}(z_{ij}, s_{ij,LB})^2 := \psi_5(z, s_{LB}),$$

$$(51f) \quad \frac{\gamma_{MY}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} ((a_{v_{|I_i|}(s)} - S(y^{(3,\varepsilon)}(\cdot, s)) - s_{ij,LB})^+)^2 ds := \psi_6(y^{(3,\varepsilon)}, s_{LB}),$$

$$(51g) \quad \frac{\gamma_s}{2} \|s\|^2 := \psi_7(s).$$

We start with the discussion of (51a) and (51b). The corresponding functions ψ_k , $k \in \{1, 2\}$, are convex and continuous with respect to their second argument. Therefore they are weakly lower semicontinuous in this argument. They are also continuous with respect to their first argument. We obtain the following estimate along $(z, y^{(3,\varepsilon)})_k$ respectively $(z, \tilde{u})_k$

$$\begin{aligned} \lim_{k \rightarrow \infty} \psi_1(z_k, y_k^{(3,\varepsilon)}) &= \lim_{k \rightarrow \infty} \psi_1(z_k, y_k^{(3,\varepsilon)}) - \psi_1(z^*, y_k^{(3,\varepsilon)}) + \psi_1(z^*, y_k^{(3,\varepsilon)}) \\ &\geq \lim_{k \rightarrow \infty} \left(\psi_1(z_k, y_k^{(3,\varepsilon)}) - \psi_1(z^*, y_k^{(3,\varepsilon)}) \right) + \psi_1(z^*, y^*), \\ \lim_{k \rightarrow \infty} \psi_2(z_k, \tilde{u}_k) &= \lim_{k \rightarrow \infty} \psi_2(z_k, \tilde{u}_k) - \psi_2(z^*, \tilde{u}_k) + \psi_2(z^*, \tilde{u}_k) \\ &\geq \lim_{k \rightarrow \infty} \left(\psi_2(z_k, \tilde{u}_k) - \psi_2(z^*, \tilde{u}_k) \right) + \psi_2(z^*, u^*). \end{aligned}$$

For the discussion of the remaining limits we utilize the structure of ψ_k for $k \in \{1, 2\}$. According to definition of ψ_1 and the triangle inequality we deduce

$$\begin{aligned} &|\psi_1(z_k, y_k^{(3,\varepsilon)}) - \psi_1(z^*, y_k^{(3,\varepsilon)})| \\ &\leq \left| \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^k (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right. \\ (52) \quad &\left. - \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right| \\ &+ \left| \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right. \end{aligned}$$

$$(53) \quad \left. - \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z^*}(\tau)))^2 dx d\tau \right|.$$

By linearity of the integral we obtain for (52)

$$\begin{aligned}
& \left| \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^k (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right. \\
(54) \quad & \left. - \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right| \\
& \leq \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} |z_{ij}^k - z_{ij}^*| \int_{I_i^j} \int_{\Omega} (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau.
\end{aligned}$$

Expansion of the quadratic expressions in (53) yields

$$\begin{aligned}
& \left| \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_k}(\tau)))^2 dx d\tau \right. \\
& \left. - \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* (y_k^{(3,\varepsilon)}(x, \tau) - y_{dis}(x, t_{z_*}(\tau)))^2 dx d\tau \right| \\
(55) \quad & = \left| \gamma_{Con} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* y_k^{(3,\varepsilon)}(x, \tau) (y_{dis}(x, t_{z_*}(\tau)) - y_{dis}(x, t_{z_k}(\tau))) dx d\tau \right| \\
& + \frac{\gamma_{Con}}{2} \left| \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} z_{ij}^* y_{dis}(x, t_{z_k}(\tau))^2 - y_{dis}(x, t_{z_*}(\tau))^2 dx d\tau \right|.
\end{aligned}$$

Due to weak convergence the sequence $y_k^{(3,\varepsilon)}$ remains bounded in $L^2(Q)$ and the same holds true for $y_{dis}(\cdot, t_{z_k}(\cdot))$ according to the assumption posed in the lemma, whereas the first factor z_{ij}^k tends to z_{ij}^* for each $(i, j) \in |I|_{|I_i|, |I_i|}$. Furthermore the associated time transformation t_{z_k} converges pointwise to t_{z_*} and by dominated convergence $y_{dis}(x, t_{z_k}(\cdot))$ tends to $y_{dis}(x, t_{z_*}(\cdot))$ in $L^2([0, T])$ for almost all $x \in \Omega$, as $y_{dis}(x, \cdot) \in H^1(0, T) \hookrightarrow C^0([0, T])$. This means the expressions (54) and (55) tend to 0 as $k \rightarrow \infty$. Repetition of the previous calculation yields

$$|\psi_2(z_k, \tilde{u}_k) - \psi_2(z^*, \tilde{u}_k)| \leq \frac{\gamma_{Con}}{2} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} |z_{ij}^k - z_{ij}^*| \|\tilde{u}_k\|_{L^2(Q)}^2.$$

From the strong convergence of z_k and s_k together with the continuity of φ_{FB} we immediately deduce for (51c), (51e) and (51g) that

$$\begin{aligned}
\lim_{k \rightarrow \infty} \psi_3(z_k, s_{UB}^k) &= \psi_3(z^*, s_{UB}^*), \\
\lim_{k \rightarrow \infty} \psi_5(z_k, s_{LB}^k) &= \psi_5(z^*, s_{LB}^*), \\
\lim_{k \rightarrow \infty} \psi_7(s_k) &= \psi_7(s^*).
\end{aligned}$$

We are now left with the discussion of the convergence properties of ψ_4 and ψ_6 , (51d) and (51f). We first fix a $\tau \in [0, T]$ and investigate the convergence of $y_k^{(3,\varepsilon)}(\tau)$. We claim that $y_k^{(3,\varepsilon)}(\tau) \rightharpoonup y^*(\tau)$ in $L^2(\Omega)$. We first remind the reader on the continuity of the embedding from $Y(0, T)$ into $C^0([0, T]; L^2(\Omega))$. As a result we deduce that $y_k^{(3,\varepsilon)}(\tau)$ and $y^*(\tau) \in L^2(\Omega)$ for all $\tau \in [0, T]$ and $k \in \mathbb{N}$. Let now be

$\chi \in L^2(\Omega)^*$. By the Riesz representation theorem we can identify χ with a function $f \in L^2(\Omega)$ such that

$$(f, \psi)_{L^2(\Omega)} = \langle \chi, \psi \rangle_{L^2(\Omega)^*, L^2(\Omega)} \quad \forall \psi \in L^2(\Omega).$$

Let $f \in L^2(\Omega)$ be fixed. We now consider $f_\tau : Y(0, T) \rightarrow \mathbb{R}$,

$$y \mapsto (y(\tau), f)_{L^2(\Omega)}.$$

for a fixed $\tau \in [0, T]$. The element f_τ is an element in $Y(0, T)^*$, since the following estimate is valid

$$\begin{aligned} f_\tau(y) &:= (y(\tau), f)_{L^2(\Omega)} \leq \|y(\tau)\|_{L^2(\Omega)} \cdot \|f\|_{L^2(\Omega)} \\ &\leq \max_{\tau \in [0, T]} \|y(\tau)\|_{L^2(\Omega)} \cdot \|f\|_{L^2(\Omega)} \leq \|y\|_{Y(0, T)} \cdot \|f\|_{L^2(\Omega)}. \end{aligned}$$

Due to the weak convergence of $y_k^{(3, \varepsilon)}$ against y^* in $Y(0, T)$, we have by definition

$$\lim_{k \rightarrow \infty} f_\tau(y_k^{(3, \varepsilon)}) = f_\tau(y^*).$$

But this means

$$\lim_{k \rightarrow \infty} (y_k^{(3, \varepsilon)}(\tau), f)_{L^2(\Omega)} = (y^*(\tau), f)_{L^2(\Omega)}.$$

Hence $y_k^{(3, \varepsilon)}(\tau)$ converges weakly to $y^*(\tau)$ in $L^2(\Omega)$ for all $\tau \in [0, T]$. Since the operator S is assumed to be in $\mathcal{L}(L^2(\Omega), \mathbb{R})$, we obtain $S(y_k^{(3, \varepsilon)}(\tau)) \rightarrow S(y^*(\tau))$ for all $\tau \in [0, T]$. By continuity of the Moreau-Yosida penalty term (46) we obtain for any $\tau \in I_i^j$.

$$\begin{aligned} &\lim_{k \rightarrow \infty} ((S(y_k^{(3, \varepsilon)}(\cdot, \tau)) - b_{v_{|I_i^j|}(\tau)} - s_{ij, UB}^k)^+)^2 \\ &= ((S(y^*(\cdot, \tau)) - b_{v_{|I_i^j|}(\tau)} - s_{ij, UB}^*)^+)^2. \\ &\lim_{k \rightarrow \infty} ((a_{v_{|I_i^j|}(\tau)} - S(y_k^{(3, \varepsilon)}(\cdot, \tau)) - s_{ij, LB}^k)^+)^2 \\ &= ((a_{v_{|I_i^j|}(\tau)} - S(y^*(\cdot, \tau)) - s_{ij, LB}^*)^+)^2. \end{aligned}$$

We can now use Fatou's lemma to get an estimate for (51d) and (51f)

$$\begin{aligned} \liminf_{k \rightarrow \infty} \psi_4(y_k^{(3, \varepsilon)}, s_{UB}^k) &\geq \psi_4(y^*, s_{UB}^*). \\ \liminf_{k \rightarrow \infty} \psi_6(y_k^{(3, \varepsilon)}, s_{LB}^k) &\geq \psi_6(y^*, s_{LB}^*). \end{aligned}$$

Combining the estimates for $\psi_k, k \in \{1, \dots, 7\}$, we deduce

$$\liminf_{k \rightarrow \infty} \tilde{J}_\varepsilon^P(z_k, \tilde{u}_k, y_k^{(3, \varepsilon)}, s_k) \geq \tilde{J}_\varepsilon^P(z^*, \tilde{u}^*, y^*, s^*).$$

Since the quadruple (z^*, u^*, y^*, s^*) is feasible, the problem (48) possesses a solution. \square

Remark 3. Although the penalty function (47) would allow us to drop the sign constraint for z_{ij} and $s_{ij, k}$, we still keep them as explicit constraints in (48), as they play an integral part in the discussion on the existence of a minimizer. The existence of S^\dagger is only necessary if $\gamma_s = 0$. Otherwise the coercivity of $\|s\|^2$ already ensures the boundedness of the corresponding entries in the sequence tending to the infimum.

In Lemma 5 we have established the existence of a minimizer for (48). Next we want to discuss the differentiability properties of the involved control to state operator $S_O : \mathbb{R}_{0,+}^{|\mathcal{I}| \cdot |\mathcal{I}_i|} \times L^2(Q) \rightarrow Y(0, T)$, $(z, \tilde{u}) \mapsto \tilde{y}$, where \tilde{y} solves (28). We follow the ideas presented in [9].

Lemma 6. *The control to state operator S_O is continuously Fréchet-differentiable (continuously F-differentiable).*

Proof. We are not able to directly apply the result posed in [9, Lemma 4.], since the involved control only possesses the regularity $\tilde{u} \in L^2(Q)$ instead of $L^\infty(Q)$. However, minor adaptations in the presented proof still allow us to derive the claimed result. \square

An integral part in the proof of the previous lemma is played by the system

(56a)

$$w_\tau(x, \tau) + (\hat{z}_{ij} + \varepsilon)A(w)(x, \tau) = \alpha(x, \tau) \quad (x, \tau) \in I_i^j \times \Omega, \quad \forall (i, j) \in |\mathcal{I}|_{|\mathcal{I}_i|},$$

(56b)

$$w(x, \tau) = 0 \quad (x, \tau) \in I_i^j \times \partial\Omega, \quad \forall (i, j) \in |\mathcal{I}|_{|\mathcal{I}_i|},$$

(56c)

$$w(x, 0) = \beta(x) \quad x \in \Omega.$$

After successfully establishing the differentiability of S_O , we require suitable representations of $\frac{\partial S_O}{\partial \tilde{u}}^*$ and $\frac{\partial S_O}{\partial z_{ij}}^*$. We follow the ideas presented in [9] and define the linear mapping $\mathcal{K} : L^2(Q) \rightarrow Y(0, T) \subset L^2(Q)$, $\alpha \mapsto w$, where w solves (56) with $\beta = 0$. This mapping is well-posed and continuous by (30). We now consider the mapping $\mathcal{K}^* : L^2(Q) \rightarrow Y(0, T) \subset L^2(Q)$, $\beta \mapsto q$, where q solves the system

$$\begin{aligned} -q_\tau(x, \tau) + (\hat{z}_{ij} + \varepsilon)A^*(q)(x, \tau) &= \beta(x, \tau) & (x, \tau) \in I_i^j \times \Omega, & \quad \forall (i, j) \in |\mathcal{I}|_{|\mathcal{I}_i|}, \\ q(x, \tau) &= 0 & (x, \tau) \in I_i^j \times \partial\Omega, & \quad \forall (i, j) \in |\mathcal{I}|_{|\mathcal{I}_i|}, \\ q(x, T) &= 0 & x \in \Omega. & \end{aligned}$$

The state q is the solution of a backward linear parabolic equation. Hence the regularity of q is established via the transformation $\varphi : [0, T] \rightarrow [0, T]$, $t \mapsto T - t$ and the estimate (30). We now have that \mathcal{K}^* is in deed the adjoint operator to $\mathcal{K} \in \mathcal{L}(L^2(Q), L^2(Q))$.

Lemma 7. *For all $\alpha \in L^2(Q)$ and $\beta \in L^2(Q)$*

$$\langle \beta, \mathcal{K}(\alpha) \rangle_{L^2(Q)} = \langle \mathcal{K}^*(\beta), \alpha \rangle_{L^2(Q)}$$

holds.

Proof. The proof can be taken from [9, Lemma 6]. \square

By combining the results from Lemma 6 and Lemma 7 for $A(y) = -\Delta y$, we obtain the characterization of $\frac{\partial S_O}{\partial \tilde{u}}(z, \tilde{u})\mathbf{u}$ for $\mathbf{u} \in L^2(Q)$ and $\frac{\partial S_O}{\partial z_{ij}}(z, \tilde{u})\mathbf{z}_{ij}$ for $\mathbf{z}_{ij} \in \mathbb{R}$.

$$\begin{aligned} \frac{\partial S_O}{\partial \tilde{u}}(z, \tilde{u})\mathbf{u} &= \mathcal{K}((\tilde{z} + \varepsilon)\mathbf{u}), \\ \frac{\partial S_O}{\partial z_{ij}}(z, \tilde{u})\mathbf{z}_{ij} &= \mathcal{K}(\chi_{I_i^j}(\Delta \tilde{y} + \tilde{u})\mathbf{z}_{ij}). \end{aligned}$$

Here we again utilized the identification $\tilde{\mathbf{z}} \in L^\infty([0, T])$ with $\tilde{\mathbf{z}}|_{I_i^j} = \mathbf{z}_{ij}$ for the entries of $\mathbf{z} \in \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|}$ and identify $\tilde{\mathbf{y}} = S_O(\mathbf{z}, \tilde{\mathbf{u}})$. For any element $\psi \in L^2(Q)$ holds

$$\begin{aligned} \langle \psi, \frac{\partial S_O}{\partial \tilde{\mathbf{u}}}(\mathbf{z}, \tilde{\mathbf{u}}) \mathbf{u} \rangle_{L^2(Q)} &= \int_{(0, T)} \int_{\Omega} \mathcal{K}^*(\psi)(x, \tau) (\tilde{\mathbf{z}}(\tau) + \varepsilon) \mathbf{u}(x, \tau) dx d\tau, \\ \langle \psi, \frac{\partial S_O}{\partial \mathbf{z}_{ij}}(\mathbf{z}, \tilde{\mathbf{u}}) \mathbf{z}_{ij} \rangle_{L^2(Q)} &= \int_{I_i^j} \int_{\Omega} \mathcal{K}^*(\psi)(x, \tau) (\Delta \tilde{\mathbf{y}}(x, \tau) + \tilde{\mathbf{u}}(x, \tau)) \mathbf{z}_{ij} dx d\tau. \end{aligned}$$

Next we introduce the function $\mathcal{L}_{par} : \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|} \times L^2(Q) \times Y(0, T) \times \mathbb{R}^{2 \cdot |\mathcal{I}| \cdot |\mathcal{I}_i|} \rightarrow \mathbb{R}$, (57)

$$\begin{aligned} (\mathbf{z}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}, s) &\mapsto \mathcal{L}_{par}(\mathbf{z}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}, s) \\ &= \frac{1}{2} \|\sqrt{\tilde{\mathbf{z}} + \varepsilon} \cdot (\tilde{\mathbf{y}} - \tilde{\mathbf{y}}_{dis}^{\tilde{\mathbf{z}}})\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|\sqrt{\tilde{\mathbf{z}} + \varepsilon} \cdot \tilde{\mathbf{u}}\|_{L^2(Q)}^2 + J_{\gamma_{EC}}(\mathbf{z}, s) + J_{\gamma_{MY}}(\tilde{\mathbf{y}}, s) \\ &+ \frac{\gamma_s}{2} \|s\|^2 - \int_{\Omega} p(x, 0) (y(x, 0) - y_0(x)) dx \\ &- \sum_{i=1}^{|\mathcal{I}|} \sum_{j=1}^{|\mathcal{I}_i|} \int_{I_i^j} \int_{\Omega} p(x, \tau) \left(\tilde{\mathbf{y}}_\tau(x, \tau) - (\mathbf{z}_{ij} + \varepsilon) (\Delta \tilde{\mathbf{y}}(x, \tau) + \alpha_{v_{|I_i|}(\tau)}(x) \tilde{\mathbf{u}}(x, \tau)) \right) dx d\tau \end{aligned}$$

associated with (48). Let $(\hat{\mathbf{z}}, \hat{\mathbf{u}}, \hat{\mathbf{y}}, \hat{s}) \in \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|} \times L^2(Q) \times Y(0, T) \times \mathbb{R}^{2 \cdot |\mathcal{I}| \cdot |\mathcal{I}_i|}$ be a feasible point for (48), we define the adjoint state $p \in Y(0, T)$ via

$$(58) \quad \begin{aligned} p &= \mathcal{K}^* \left((\tilde{\mathbf{z}} + \varepsilon) (\hat{\mathbf{y}} - \tilde{\mathbf{y}}_{dis}^{\tilde{\mathbf{z}}}) \right. \\ &\left. + \gamma_{MY} \left((S(\hat{\mathbf{y}}) - b_{v_{|I_i|}} - \hat{s}_{UB})^+ - (a_{v_{|I_i|}} - S(\hat{\mathbf{y}}) - \hat{s}_{LB})^+ \right) R[S'(\hat{\mathbf{y}})] \right). \end{aligned}$$

Here $R : L^2(\Omega)^* \rightarrow L^2(\Omega)$, $v \mapsto R[v]$ is defined via the relation $\langle v, f \rangle_{L^2(\Omega)^*, L^2(\Omega)} = \langle R[v], f \rangle_{L^2(\Omega)}$ for all $f \in L^2(\Omega)$, i.e., the Riesz representation of v . Furthermore an element $y \in L^2(\Omega)$ can be interpreted as an element in $L^2(Q)$, whose evaluation is independent of the time variable. We now calculate the corresponding derivatives at $(\hat{\mathbf{z}}, \hat{\mathbf{u}}, \hat{\mathbf{y}}, \hat{s})$ in the direction of $(\mathbf{z}, \mathbf{u}, \mathbf{y}, s) \in \mathbb{R}^{|\mathcal{I}| \cdot |\mathcal{I}_i|} \times L^2(Q) \times Y(0, T) \times \mathbb{R}^{2 \cdot |\mathcal{I}| \cdot |\mathcal{I}_i|}$.

We begin with the discussion of $D_{\tilde{\mathbf{y}}} \mathcal{L}_{par}(\hat{\mathbf{z}}, \hat{\mathbf{u}}, \hat{\mathbf{y}}, \hat{s})(\mathbf{y})$.

$$(59) \quad \begin{aligned} &D_{\tilde{\mathbf{y}}} \mathcal{L}_{par}(\hat{\mathbf{z}}, \hat{\mathbf{u}}, \hat{\mathbf{y}}, \hat{s})(\mathbf{y}) \\ &= \sum_{i=1}^{|\mathcal{I}|} \sum_{j=1}^{|\mathcal{I}_i|} \int_{I_i^j} \int_{\Omega} (\hat{\mathbf{z}}_{ij} + \varepsilon) (\hat{\mathbf{y}}(x, \tau) - y_{dis}(x, t_{\tilde{\mathbf{z}}}(\tau))) \mathbf{y}(x, \tau) dx d\tau \\ &+ \gamma_{MY} \sum_{i=1}^{|\mathcal{I}|} \sum_{j=1}^{|\mathcal{I}_i|} \int_{I_i^j} (S(\hat{\mathbf{y}}(\cdot, s)) - b_{v_{|I_i|}(s)} - \hat{s}_{ij, UB})^+ \langle S'(\hat{\mathbf{y}}(\cdot, s)), \mathbf{y}(\cdot, s) \rangle_{L^2(\Omega)^*, L^2(\Omega)} ds \\ &- \gamma_{MY} \sum_{i=1}^{|\mathcal{I}|} \sum_{j=1}^{|\mathcal{I}_i|} \int_{I_i^j} (a_{v_{|I_i|}(s)} - S(\hat{\mathbf{y}}(\cdot, s)) - \hat{s}_{ij, LB})^+ \langle S'(\hat{\mathbf{y}}(\cdot, s)), \mathbf{y}(\cdot, s) \rangle_{L^2(\Omega)^*, L^2(\Omega)} ds \\ &- \sum_{i=1}^{|\mathcal{I}|} \sum_{j=1}^{|\mathcal{I}_i|} \int_{I_i^j} \int_{\Omega} p(x, \tau) (\mathbf{y}_\tau(x, \tau) - (\hat{\mathbf{z}}_{ij} + \varepsilon) \Delta \mathbf{y}(x, \tau)) dx d\tau \\ &- \int_{\Omega} p(x, 0) \mathbf{y}(x, 0) dx. \end{aligned}$$

We apply integration by parts with respect to the time variable and conclude with the definition of the adjoint state p

$$\begin{aligned}
&= \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} (\hat{z}_{ij} + \varepsilon) (\hat{y}(x, \tau) - y_{dis}(x, t_{\bar{z}}(\tau))) \mathbf{y}(x, \tau) \, dx \, d\tau \\
&+ \gamma_{MY} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (S(\hat{y}(\cdot, s)) - b_{v_{|I_i|}(s)} - \hat{s}_{ij,UB})^+ \langle R[S'(\hat{y}(\cdot, s))], \mathbf{y}(\cdot, s) \rangle_{L^2(\Omega)} \, ds \\
&- \gamma_{MY} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} (a_{v_{|I_i|}(s)} - S(\hat{y}(\cdot, s)) - \hat{s}_{ij,LB})^+ \langle R[S'(\hat{y}(\cdot, s))], \mathbf{y}(\cdot, s) \rangle_{L^2(\Omega)} \, ds \\
&- \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} \mathbf{y}(x, \tau) (-p_{\tau}(x, \tau) - (\hat{z}_{ij} + \varepsilon) \Delta p(x, \tau)) \, dx \, d\tau \\
&+ \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{\Omega} \mathbf{y}(x, \tau_{ij-1}) p(x, \tau_{ij-1}) \, dx - \int_{\Omega} \mathbf{y}(x, \tau_{ij}) p(x, \tau_{ij}) \, dx \\
&- \int_{\Omega} p(x, 0) \mathbf{y}(x, 0) \, dx = 0.
\end{aligned}$$

We proceed with

$$\begin{aligned}
(60) \quad &D_{\bar{\mathbf{u}}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{u}) \\
&= \gamma_{Con} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} (\hat{z}_{ij} + \varepsilon) \hat{u}(x, \tau) \mathbf{u}(x, \tau) \, dx \, d\tau \\
&+ \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \int_{I_i^j} \int_{\Omega} (\hat{z}_{ij} + \varepsilon) p(x, \tau) \alpha_{v_{|I_i|}(\tau)}(x) \mathbf{u}(x, \tau) \, dx \, d\tau.
\end{aligned}$$

Next we investigate the time derivative in the direction of \mathbf{z}_{ij} for $(i, j) \in |I|_{|I_i|, |I_i|}$. We emphasize that $\hat{y}_{dis}^{\bar{z}}(x, \tau) = y_{dis}(x, t_{\bar{z}}(\tau))$ is also dependent on the choice of \bar{z} and the existence of $(y_{dis})_t$ is postulated.

$$\begin{aligned}
(61) \quad &D_{\mathbf{z}_{ij}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{z}_{ij}) \\
&= \frac{1}{2} \int_{I_i^j} \int_{\Omega} \mathbf{z}_{ij} (\hat{y}(x, \tau) - y_{dis}(x, t_{\bar{z}}(\tau)))^2 \, dx \, d\tau + \frac{\gamma_{Con}}{2} \int_{I_i^j} \int_{\Omega} \mathbf{z}_{ij} (\hat{u}(x, \tau))^2 \, dx \, d\tau \\
&- \sum_{k=1}^{|I|} \sum_{l=1}^{|I_l|} \int_{I_{kl}} \int_{\Omega} (\hat{z}_{kl} + \varepsilon) (\hat{y}(x, \tau) - y_{dis}(x, t_{\bar{z}}(\tau))) \frac{\partial y_{dis}}{\partial t}(x, t_{\bar{z}}(\tau)) \frac{\partial t_{\bar{z}}}{\partial z_{ij}}(\tau) \mathbf{z}_{ij} \, d\tau \\
&+ \gamma_{EC} \varphi_{FB}(\hat{z}_{ij}, \hat{s}_{ij,UB}) \frac{\partial \varphi_{FB}}{\partial z_{ij}}(\hat{z}_{ij}, \hat{s}_{ij,UB}) \mathbf{z}_{ij} + \gamma_{EC} \varphi_{FB}(\hat{z}_{ij}, \hat{s}_{ij,LB}) \frac{\partial \varphi_{FB}}{\partial z_{ij}}(\hat{z}_{ij}, \hat{s}_{ij,LB}) \mathbf{z}_{ij} \\
&+ \int_{I_i^j} \int_{\Omega} \mathbf{z}_{ij} p(x, \tau) (\Delta \hat{y}(x, \tau) + \alpha_{v_{|I_i|}(\tau)}(x) \hat{u}(x, \tau)) \, dx \, d\tau
\end{aligned}$$

We conclude the calculations of the partial derivatives with $D_{s_{ij,k}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s})$ for $k \in \{UB, LB\}$ in the direction of $\mathbf{s}_{ij,k}$

$$(62) \quad D_{s_{ij,UB}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{s}_{ij,UB}) = \gamma_{EC} \varphi_{FB}(\hat{z}_{ij}, \hat{s}_{ij,UB}) \frac{\partial \varphi_{FB}}{\partial s_{ij,UB}}(\hat{z}_{ij}, \hat{s}_{ij,UB}) \mathbf{s}_{ij,UB} \\ - \gamma_{MY} \int_{I_i^j} (S(\hat{y}(\cdot, s)) - b_{\frac{|I_i|}{|I|}} - \hat{s}_{ij,UB})^+ \mathbf{s}_{ij,UB} ds + \gamma_s \hat{s}_{ij,UB} \mathbf{s}_{ij,UB},$$

$$(63) \quad D_{s_{ij,LB}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{s}_{ij,LB}) = \gamma_{EC} \varphi_{FB}(\hat{z}_{ij}, \hat{s}_{ij,LB}) \frac{\partial \varphi_{FB}}{\partial s_{ij,LB}}(\hat{z}_{ij}, \hat{s}_{ij,LB}) \mathbf{s}_{ij,LB} \\ - \gamma_{MY} \int_{I_i^j} (a_{\frac{|I_i|}{|I|}} - S(\hat{y}(\cdot, s)) - \hat{s}_{ij,LB})^+ \mathbf{s}_{ij,LB} ds + \gamma_s \hat{s}_{ij,LB} \mathbf{s}_{ij,LB}.$$

We want to study the relationship between the Lagrangian \mathcal{L}_{par} and the cost function \tilde{J}_ε^p . For an admissible quadruple $(z, \tilde{u}, y^{(3,\varepsilon)}, s)$ to (48) the following connection can be established

$$(64) \quad \tilde{J}_\varepsilon^p(z, \tilde{u}, y^{(3,\varepsilon)}, s) = \mathcal{L}_{par}(z, \tilde{u}, y^{(3,\varepsilon)}, s).$$

The inclusion of the differentiable control to state operator S_O , c.f. Lemma 6, allows us to remove $y^{(3,\varepsilon)} = S_O(z, \tilde{u})$ from the problem formulation. As a result we can state the reduced formulation of (48), which is given by

$$(65) \quad \min_{z, \tilde{u}, s} \tilde{J}_{\varepsilon,r}^p(z, \tilde{u}, s) \\ = \frac{1}{2} \|\sqrt{\tilde{z} + \varepsilon} \cdot (S_O(z, \tilde{u}) - \tilde{y}_{dis}^{\tilde{z}})\|_{L^2(Q)}^2 + \frac{\gamma_{Con}}{2} \|\sqrt{\tilde{z} + \varepsilon} \cdot \tilde{u}\|_{L^2(Q)}^2 \\ + J_{\gamma_{EC}}(z, s) + J_{\gamma_{MY}}(S_O(z, \tilde{u}), s) + \frac{\gamma_s}{2} \|s\|^2 \\ \text{s.t.} \\ \tilde{z}(\tau) \geq 0 \quad \tau \in [0, T], \\ \tilde{z}|_{I_i^j} \equiv z_{ij} \quad (i, j) \in |I|_{|I|, |I_i|}, \\ \int_{I_i} \tilde{z}(s) ds = \Delta I_i \quad i \in |I|_{|I|}, \\ s_{ij,k} \geq 0 \quad (i, j, k) \in |I|_{|I|, |I_i|, k}.$$

The mentioned relationship between \tilde{J}_ε^p and \mathcal{L}_{par} , (64), for an admissible triple to (65) can be translated into

$$(66) \quad \tilde{J}_{\varepsilon,r}^p(z, \tilde{u}, s) = \mathcal{L}_{par}(z, \tilde{u}, S_O(z, \tilde{u}), s).$$

At $(\hat{z}, \hat{u}, \hat{y}, \hat{s})$ with $\hat{y} = S_O(\hat{z}, \hat{u})$, we obtain in particular the following identities for the involved derivatives of $\tilde{J}_{\varepsilon,r}^p$ in the direction of $(z, \mathbf{u}, \mathbf{s}) \in \mathbb{R}^{|I| \cdot |I_i|} \times L^2(Q) \times \mathbb{R}^{2 \cdot |I| \cdot |I_i|}$

$$\begin{aligned}
(67) \quad \frac{\partial \tilde{J}_{\varepsilon,r}^p}{\partial \mathbf{z}_{ij}}(\hat{z}, \hat{u}, \hat{s})(\mathbf{z}_{ij}) &= \frac{\partial \mathcal{L}_{par}}{\partial \mathbf{z}_{ij}}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{z}_{ij}) + \underbrace{\frac{\partial \mathcal{L}_{par}}{\partial \mathbf{y}}(\hat{z}, \hat{u}, \hat{y}, \hat{s}) \cdot \frac{\partial S_O}{\partial \mathbf{z}_{ij}}(\hat{z}, \hat{u})(\mathbf{z}_{ij})}_0, \\
&= \frac{\partial \mathcal{L}_{par}}{\partial \mathbf{z}_{ij}}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{z}_{ij}), \\
(68) \quad \frac{\partial \tilde{J}_{\varepsilon,r}^p}{\partial \tilde{u}}(\hat{z}, \hat{u}, \hat{s})(\mathbf{u}) &= \frac{\partial \mathcal{L}_{par}}{\partial \tilde{u}}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{u}) + \underbrace{\frac{\partial \mathcal{L}_{par}}{\partial \mathbf{y}}(\hat{z}, \hat{u}, \hat{y}, \hat{s}) \cdot \frac{\partial S_O}{\partial \tilde{u}}(\hat{z}, \hat{u})(\mathbf{u})}_0, \\
&= \frac{\partial \mathcal{L}_{par}}{\partial \tilde{u}}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{u}), \\
(69) \quad \frac{\partial \tilde{J}_{\varepsilon,r}^p}{\partial s_{ij,k}}(\hat{z}, \hat{u}, \hat{s})(\mathbf{s}_{ij,k}) &= \frac{\partial \mathcal{L}_{par}}{\partial s_{ij,k}}(\hat{z}, \hat{u}, \hat{y}, \hat{s})(\mathbf{s}_{ij,k}).
\end{aligned}$$

Since the remaining constraints in (65) are of a simple structure, we consider the following Lagrange function $\mathcal{L} : \mathbb{R}^{|I| \cdot |I_i|} \times L^2(Q) \times \mathbb{R}^{2 \cdot |I| \cdot |I_i|} \times \mathbb{R}^{|I|} \times \mathbb{R}^{|I| \cdot |I_i|} \times \mathbb{R}^{2 \cdot |I| \cdot |I_i|} \times \mathbb{R}^{2 \cdot |I| \cdot |I_i|} \rightarrow \mathbb{R}$, where we also include the postulate on the existence of an upper bound, S^\dagger , for the entries in s . We also replace the integral constraint on the time control by its equivalent sum formulation

$$\begin{aligned}
(70) \quad \mathcal{L} : (z, \tilde{u}, s, \lambda, \mu^z, \mu^s, \zeta^s) &\mapsto \\
&J_{\varepsilon,r}^p(z, \tilde{u}, s) + \sum_{i=1}^{|I|} \lambda_i \left(\sum_{j=1}^{|I_i|} z_{ij} - |I_i| \right) + \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \mu_{ij}^z (-z_{ij}) \\
&+ \sum_{k \in \{LB, UB\}} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \mu_{ij,k}^s (-s_{ij,k}) + \sum_{k \in \{LB, UB\}} \sum_{i=1}^{|I|} \sum_{j=1}^{|I_i|} \zeta_{ij,k}^s (s_{ij,k} - S^\dagger).
\end{aligned}$$

With the aid of the Lagrangian \mathcal{L} we are able to characterize candidates for optimality $(\hat{z}, \hat{u}, \hat{s})$ if a suitable constraint qualification is fulfilled. With the definition of the adjoint state, (58), and the identities (67) – (69), we are looking for points $(\hat{z}, \hat{u}, \hat{s})$, that satisfy the subsequent KKT-system.

$$(71a) \quad \hat{y} = S_O(\hat{z}, \hat{u}),$$

$$(71b) \quad \hat{p} = \mathcal{K}^* \left((\hat{z} + \varepsilon)(\hat{y} - \hat{y}_{dis}^z) + \gamma_{MY} \left((S(\hat{y}) - b_{v_{|I_i|}} - \hat{s}_{UB})^+ - (a_{v_{|I_i|}} - S(\hat{y}) - \hat{s}_{LB})^+ \right) R[S'(\hat{y})] \right).$$

$$(71c) \quad 0 = D_{z_{ij}} \mathcal{L}(\hat{z}, \hat{u}, \hat{s}, \lambda, \mu^z, \mu^s, \zeta^s) = D_{z_{ij}} \tilde{J}_{\varepsilon,r}^p(\hat{z}, \hat{u}, \hat{s}) + \lambda_i - \mu_{ij}^z, \\ = D_{z_{ij}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s}) + \lambda_i - \mu_{ij}^z,$$

$$(71d) \quad 0 = D_{\hat{u}} \mathcal{L}(\hat{z}, \hat{u}, \hat{s}, \lambda, \mu^z, \mu^s, \zeta^s) = D_{\hat{u}} \tilde{J}_{\varepsilon,r}^p(\hat{z}, \hat{u}, \hat{s}) = D_{\hat{u}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s}),$$

$$(71e) \quad 0 = D_{s_{ij,k}} \mathcal{L}(\hat{z}, \hat{u}, \hat{s}, \lambda, \mu^z, \mu^s, \zeta^s) = D_{s_{ij,k}} \tilde{J}_{\varepsilon,r}^p(\hat{z}, \hat{u}, \hat{s}) - \mu_{ij,k}^s + \zeta_{ij,k}^s, \\ = D_{s_{ij,k}} \mathcal{L}_{par}(\hat{z}, \hat{u}, \hat{y}, \hat{s}) - \mu_{ij,k}^s + \zeta_{ij,k}^s,$$

$$(71f) \quad 0 = \sum_{j=1}^{|I_i|} \hat{z}_{ij} - |I_i|,$$

$$(71g) \quad 0 \geq -\hat{z}_{ij}, \quad \mu_{ij}^z \geq 0, \quad 0 = \hat{z}_{ij} \cdot \mu_{ij}^z,$$

$$(71h) \quad 0 \geq -\hat{s}_{ij,k}, \quad \mu_{ij,k}^s \geq 0, \quad 0 = \hat{s}_{ij,k} \cdot \mu_{ij,k}^s,$$

$$(71i) \quad 0 \geq \hat{s}_{ij,k} - S^+, \quad \zeta_{ij,k}^s \geq 0, \quad 0 = (\hat{s}_{ij,k} - S^+) \cdot \zeta_{ij,k}^s.$$

4. Algorithm

In this section we propose an algorithmic approach for the computation of solutions to (1). Revising the arguments in Section 3 we have discovered the importance of minimizers for (9) respectively (27). Furthermore the approximation property of solutions to (45) for $\varepsilon > 0$ with respect to feasible states of (27) has motivated to solve the relaxed formulation (48) or the associated reduced formulation (65) by utilizing first order necessary optimality conditions (71a) – (71i).

We want to apply a semismooth Newton method, c.f. [19], to solve the conditions posed in (71a) – (71i). Therefore we reformulate the equilibrium conditions, (71g) – (71i), on $z_{ij}, s_{ij,k}$ and their associated multipliers $\mu_{ij}^z, \mu_{ij,k}^s, \zeta_{ij,k}^s$ for $k \in \{UB, LB\}$ via a NCP function, namely φ_{FB} .

We now propose the following algorithm. Let $M : X \rightarrow Y$ for

$$X := Y(0, T)^2 \times \mathbb{R}^z \times L^2(Q) \times \mathbb{R}^{s+\lambda+\mu^z+\mu^s+\zeta^s}, \\ Y := L^2(Q)^2 \times \mathbb{R}^z \times L^2(Q) \times \mathbb{R}^{s+\lambda+\mu^z+\mu^s+\zeta^s},$$

be the non linear mapping, which presents the previously discussed conditions, (71a) – (71i), and variables in chronological appearance, i.e., solutions of the optimality system are exactly the roots of M . The dimension of the spaces $\mathbb{R}^z, \mathbb{R}^s$, etc. corresponds to the dimension of the associated variables z, s , etc. in the definition of \mathcal{L} , (70). We define M via

$$M : X \rightarrow Y, \quad (\tilde{y}, p, z, \tilde{u}, s, \lambda, \mu^z, \mu^s, \zeta^s)^T \mapsto M(\tilde{y}, p, z, \tilde{u}, s, \lambda, \mu^z, \mu^s, \zeta^s).$$

We denote by ∂M the evaluation of the generalized gradient according to Clarke, c.f. [8].

Algorithm 1. Let $x = (\tilde{y}, p, z, \tilde{u}, s, \lambda, \mu^z, \mu^s, \zeta^s)^T \in X$. Initialize $x_0 \in X$ together with γ_{EC}, γ_{MY} and set $k = 0$. Choose $Tol_{Newton} > 0, Tol_{EC} > 0, Tol_{MY} > 0$ together with $\sigma_{EC} > 1$ and $\sigma_{MY} > 1$.

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while  $\|M(x_k)\| \geq Tol_{Newton}$  or  $J_{\gamma_{EC}}(z_k, s_k) \geq Tol_{EC}$  or  $J_{\gamma_{MY}}(\tilde{y}_k, s_k) \geq Tol_{MY}$  do
  while  $\|M(x_k)\| \geq Tol_{Newton}$  do
    Select an element  $N \in \partial M(x_k)$  and solve  $Nd_k = -M(x_k)$ ;
    Update  $x_{k+1} = x_k + d_k$ ;
  end
  if  $J_{\gamma_{EC}}(z_{k+1}, s_{k+1}) \geq J_{\gamma_{MY}}(\tilde{y}_{k+1}, s_{k+1})$  then
     $\gamma_{EC} = \gamma_{EC} \cdot \sigma_{EC}$ ;
  if  $J_{\gamma_{EC}}(z_{k+1}, s_{k+1}) \geq J_{\gamma_{MY}}(\tilde{y}_{k+1}, s_{k+1})$  then
     $\gamma_{MY} = \gamma_{MY} \cdot \sigma_{MY}$ ;
end

```

5. Numerical results

In the following section we report on numerical results for Algorithm 1. on selected instances. As a benchmark, we attempt to numerically recover distinguished input parameters to our system.

For that purpose let Ω from now on be chosen as an one-dimensional domain and represented by an open interval, i.e., $\Omega = (0, L)$ with an $L > 0$. Hence the domain under consideration for our problem is formulated by $Q = (0, L) \times (0, T)$. For our given instance let the space time domain then be realized by the open unit square in two dimensions, i.e., $Q = (0, 1) \times (0, 1)$.

Proceeding with the "first optimize then discretize" paradigm, we solve the involved state equation, (71a), respectively adjoint equation, (71b), via a Θ - finite difference scheme for $\Theta \in [0, 1]$. With the choice $\Theta = 0$ an explicit method is obtained, $\Theta = 1$ corresponds to an implicit method, and for $\Theta = \frac{1}{2}$ the well known Crank-Nicolson method is received.

We apply an equidistant grid with $P^* = 72$ space and $Q^* = 72$ time intervals. Let $M_{dis}: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the non-linear mapping in $m = (P^* + 1) \cdot (Q^* + 1) + (P^* + 1) \cdot (Q^* + 1) + |I| \cdot |I_i| + (P^* - 1) \cdot (Q^* + 1) + 2 \cdot |I| \cdot |I_i| + |I| + |I| \cdot |I_i| + 2 \cdot |I| \cdot |I_i| + 2 \cdot |I| \cdot |I_i|$ variables, which presents discretizations of the previously discussed conditions, (71a) – (71i), and variables in chronological appearance, i.e., solutions of the discretized optimality system are exactly the roots of M_{dis} similar to M in the previous section.

We consider $D = 2$ modes, whose associated areas of effect are given by $\Omega_1 = (\frac{1}{6}, \frac{2}{6})$ and $\Omega_2 = (\frac{4}{6}, \frac{5}{6})$. The switching rule is formulated via

$$C: L^2(\Omega) \rightarrow \{1, 2\}, y \mapsto \begin{cases} 1, & \text{if } \int_{\Omega} y(x) dx \leq 0.1, \\ 2, & \text{if else } \int_{\Omega} y(x) dx > 0.1. \end{cases}$$

This equals the switching rule formulated in Example 1. by the choice of $\delta = 0.1$ as a switching threshold. We consider $|I| \cdot |I_i| = 6$ intervals, which consist out of $|I| = 2$ major intervals, that are again each formed by $|I_i| = 3$ minor intervals. We plug in the (constant) desired distributed control $u_{des}(x, t) = 7.5$. Carefully note that the actual acting distributed control in the problem takes the area of effect associated with each mode into account and is therefore given by $u_{des,act}(x, t) = u_{des}(x, t) \cdot \chi_{\Omega_{d(t)}}(x)$. Numerical calculation suggests that area of effect for the dynamic changes once from Ω_1 to Ω_2 around $S_1 \approx 0.18$. Hence $|S| = 1$ holds. We consider the control consistent integer control function v_2^3 , whose general form has been established in Example 2. As the switching takes place from

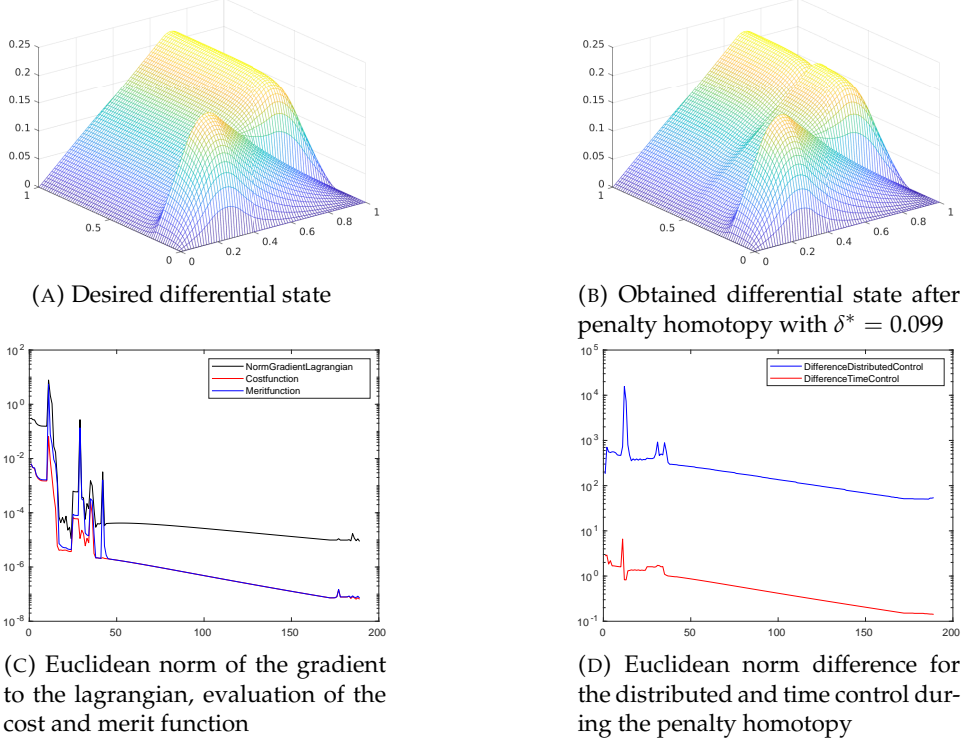


FIGURE 2. Algorithmic results

mode 1 to 2 in the first major interval, we obtain the following desired time control $z_{des} = (1.08 \ 1.92 \ 0 \ 0 \ 3 \ 0)^T$ and transformed desired distributed control $\tilde{u}_{des}(x, \tau) = u_{des}(x, t_{z_{des}}(\tau)) = 7.5$. Consequently the transformed acting desired distributed control is given by $\tilde{u}_{des,act}(x, \tau) = u_{des}(x, t_{z_{des}}(\tau)) \cdot \chi_{\Omega_{v_2^3(\tau)}}(x) = 7.5 \cdot \chi_{\Omega_{v_2^3(\tau)}}(x)$. Therefore the task at hand is the reconstruction of z_{des} together with $\tilde{u}_{des,act}$.

We initialize the system with $z_{start} = \hat{e}$ and $\tilde{u}_{start}(x, \tau) = 1$ for all $(x, \tau) \in Q$. Here \hat{e} corresponds to the vector in $\mathbb{R}^{|I| \cdot |I_i|}$, whose entries are all ones, c.f. Lemma 6. The initial input is completed by $s_{start} = 0_{\mathbb{R}^6}$, $\lambda_{start} = 0_{\mathbb{R}^2}$, $\mu_{start}^z = 0_{\mathbb{R}^6}$, $\mu_{start}^s = 10^{-9} \cdot \hat{e}$ and $\zeta_{start}^s = 0_{\mathbb{R}^6}$. Furthermore we select an upper bound for the slack variable of $S^+ = 10^1$ and a regularization threshold of $\varepsilon = 10^{-1}$. We initialize the respective penalty parameters as $\gamma_{EC} = 10^{-8}$, $\gamma_{MY} = 10^{-8}$, $\gamma_s = 10^{-6}$ and $\gamma_{Con} = 10^{-9}$. The evaluation of the merit function includes, besides the evaluation of the cost function, the primal admissibility of the involved variables. This covers the satisfaction of the discretized state equation, the sum constraint on the time control for each major interval together with the fulfillment of the lower and upper bounds on the entries of the time control respectively slack control. We abort the current Newton iteration of the penalty homotopy once $\|\nabla \mathcal{L}(z_k, \tilde{u}_k, s_k, \lambda_k, \mu_k^z, \mu_k^s, \zeta_k^s)\| < \text{Tol}_{Newton} = 10^{-5}$. Furthermore we set $\text{Tol}_{EC} = \text{Tol}_{MY} = 10^{-8}$ together with $\sigma_{EC} = 10$ and $\sigma_{MY} = 10$.

In our computational experiments Algorithm 1. terminates with the results presented in Figures 2a – 2d. In Figures 2a and 2b the desired state for the original problem together with the obtained state is presented. The (re)constructed state by the algorithm fits visibly well to the desired state. However, note that the utilized

switching threshold δ^* differs from the original choice of δ . In general we observe quite an ill conditioned behavior of the solution with respect to small changes of the switching threshold, i.e. the resulting states differ quite drastically for small perturbations in δ . In Figure 2c. the evolution of the euclidean norm of the gradient to the Lagrangian together with the cost and merit function is displayed. On the one hand we observe an reasonable decay for the cost and merit function, such that after termination of the algorithm there is no visible difference for the two curves, which indicates that the obtained point satisfies primal feasibility. On the other hand the curve of the norm of the gradient to Lagrangian saturates slightly below $\text{ToI}_{\text{Newton}} = 10^{-5}$, but this is to be expect from the termination condition of the semismooth newton step, c.f. Algorithm 1. In Figure 2d. the euclidean norm difference for the distributed respectively time control is displayed. During the iterations of the algorithm both curves decrease, which hints to a successful reconstruction of the input parameters. The numerical testing indicates that the rate of convergence for the complete penalty homotopy is linear for both the distributed and time control.

6. Conclusion

We presented a promising algorithmic approach to solve linear parabolic systems with implicit switching behavior. We introduced and performed several reformulation steps to the original problem setting, which were based on methods established for the ODE framework, e.g., disjunctive programming, a time transformation method and penalization techniques. For a relaxed formulation this approach ultimately allowed us to formulate and algorithmically utilize necessary optimality conditions, which were unavailable for the original setting. In the process we emphasized the required adaptations towards the presented PDE framework. In particular we pointed out the necessity of the newly introduced regularization step. Ultimately a numerical experiment was conducted to underline the promising nature of the demonstrated method by computational results. Together with a first set of theoretical outcomes, the presented technique could pave the path for the proof a hybrid maximum principle with PDE dynamic. We concluded first approximation properties for the relaxed solution, although a rigorous convergence study was clearly out of scope for this paper. Future research projects could feature aspects of the convergence behavior for the surrogate problems together with a detailed regularity analysis of the obtained solutions. In this regard the regularity with respect to the time variable appears paramount. But also further numerical validation of the presented approach and the comparison to other reformulation approaches originating from the ODE framework is crucial.

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