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Turnpike properties for (partially) uncontrollable systems

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Abstract

We analyse the turnpike properties for a general, infinite dimensional, linear-quadratic (LQ) optimal control problem, both in the deterministic and in the stochastic case. The novelty of the paper is twofold. Firstly, it obtains positive turnpike results for systems that are (partially) uncontrollable. Secondly, it provides turnpike results for averaged control associated to a family of problems that depend on a random parameter, which is the first turnpike type result in the averaged controllability framework.

Key words: Measure turnpike, Averaged control, LQ optimal control problem, Infinite-time admissibility, Turnpike phenomenon.

1 Introduction

The turnpike property refers to a special phenomenon that occurs in many optimal control problems associated with a time-evolution system, namely the tendency of optimal controls and trajectories to remain nearly stationary most of the time (Figure 1). This allows a time-dependent control problem to be reduced, at least approximately, to the corresponding stationary one. Such a simplification is obviously of great interest, both from an applicational and computational point of view.

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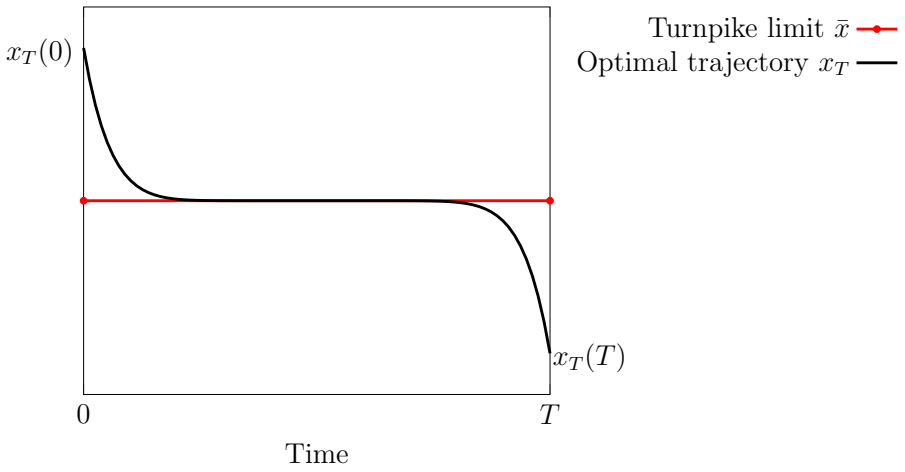

Fig. 1. Optimal trajectory remains close to the steady state $\bar{x}$ most of the time framework.

The term was coined by economists more than half a century ago and introduced in the context of finite-dimensional, discrete-time optimal control problems. However, it remained out of focus of the systems and control community for several decades. Rigorous analysis of the turnpike property started to develop recently in the context of mean field games and model predictive controls [2,12]. A large theory on the topic related to the calculus of variations and optimal control problems has been developed independently by Zaslavski in a series of works (cf. [15,16] and the references therein). Since then, numerous results have been published in this area, both in finite and infinite dimensional context, as well as for time-discrete and time-continuous systems. The notion has been applied in various contexts: shape design problems, residual neural networks, heat conduction etc. For a detailed introduction we refer an interested reader to some recent, extensive survey papers on the topic [5,3]

In most papers on this topic, authors either require the system to be both stabilizable and detectable, or they assume even stronger conditions, such as controllability, observability and dissipativity (cf. Remark 3 for a detailed discussion of these assumptions). The corresponding turnpike results are then obtained by applying sophisticated functional analysis tools to the optimality system and exploring stabilization properties of the corresponding Riccati operator (e.g. [1,5,11]).

In this paper we follow a different approach, based on a rather elegant procedure. By exploring the optimality systems, we derive the key estimate (Theorem 2.1, (13)) for the difference between evolutional and stationary optimal controls and observations in terms of the controllability and observability Gramian operator. It is important to emphasize that this estimate holds in general, without any restrictions on the operators and data defining the problem. From here, the turnpike results follow directly, assuming only infinite-time admissibility of control and observation operator.

In the framework of averaged controllability we follow a similar approach. Here we consider a stochastic system depending on parameters in a random manner. The optimal averaged control provides the minimal value of the cost functional averaged with respect to the parameter, but the control itself is parameter independent. The notion was introduced in [18], and subsequently developed in a series of papers. Links between averaged and simulta-

neous controllability were made in [10] and the notion of long-time averaged controllability was introduced in [8]. We refer the interested reader to [9] for a recent survey on this topic.

It is interesting to note that the same kind of arguments used in the stochastic case allow us to obtain the corresponding turnpike results for an averaged control. To the best of the author's' knowledge, these are the first turnpike type results in the framework of averaged controllability.

The paper is organized as follows. In the next section, we present the method and provide turnpike results for a fixed linear control system. Section 3 analyses stochastic systems and contains turnpike results for averaged controls. An illustrative example supporting the theoretical findings is provided in Section 4. The paper is closed by some concluding remarks and possible directions for further research on this topic.

2 Turnpike properties in a deterministic case

We analyse the control system

$$\begin{aligned} x'(t) + Ax(t) &= Bu(t) \\ x(0) &= x_0, \end{aligned} \tag{1}$$

whose dynamics is governed by a (possibly) unbounded operator A on a Hilbert space X (with the scalar products denoted by $\cdot$). Here $u \in L^2_{\text{loc}}([0,\infty);U)$ denotes the control function, U is a Hilbert space, B is a bounded control operator from $\mathcal{L}(U,X)$, and $x^0 \in H$ denotes the initial state.

Recall that the mild solution to problem (1) is given by the Duhamel's formula

$$x(t) = S_t x_0 + \int_0^t S_{t-s} B u(s) ds$$

where $(S_t)_{t\geq 0}$ denotes the semigroup generated by $-A$. Based on the above assumptions the solution satisfies $x \in L^2_{\text{loc}}([0,\infty);X) \cap C([0,\infty);X)$.

Given an observation operator $C \in L(X,Z)$, we consider the optimal control problem

$$\min J_T(u) = \min_u \frac{1}{2} \int_0^T \left(|u(t) - u_d|_U^2 + |Cx(t) - z_d|_Z^2 \right) dt + p_d \cdot y(T), \tag{2}$$

where minimization is taken over the space $L^2_{\text{loc}}([0,\infty);U)$, and x is the state determined by control u, i.e. it is the solution to (1). Here u_d and z_d stand for a time independent desirable control and observation, respectively, while $p_d \in X$ determines a linear regularization of the final state. Based on the above assumptions the last integral is well defined. Furthermore, the functional J_T being continuous, coercive and convex admits the unique minimizer that we denote by u_T, while x_T stands for the corresponding optimal state.

The optimality system for the problem (2) reads

$$\begin{aligned} x_T'(t) + Ax_T(t) &= -B(B^* p_T(t) - u_d) & -p_T'(t) + A^* p_T(t) &= C^*(Cx_T(t) - z_d) \\ x_T(0) &= x_0 & p_T(T) &= p_d, \end{aligned} \tag{3}$$

and the optimal control is given by $u_T = -B^* p_T + u_d$.

We also consider the corresponding stationary problem

$$\min_{u \in U} J_s(u) = \min \left\{ \frac{1}{2} \left(|u - u_d|_U^2 + |Cx - z_d|_Z^2 \right) | Ax = Bu \right\}, \tag{4}$$

whose unique solution is given by $\bar{u} = -B^* \bar{p} + u_d$, where $\bar{p}$ satisfies the corresponding stationary optimality system

$$A\bar{x} = -B(B^* \bar{p} - u_d) \qquad A^* \bar{p} = C^*(C\bar{x} - z_d), \tag{5}$$

while $\bar{x}$ is the optimal stationary state.

Remark 1 *In order for the stationary problem* (4) *to be well defined it is necessary that the corresponding state equation $Ax = Bu$ is well posed. Obviously, this holds if A is invertible, but also under significantly weaker conditions (e.g. A injective and its image containing the image of B). Interesting, this issue is even more delicate in a finite dimensional case, than for some unbounded operators (e.g. for A being an elliptic operator).*

It is also remarkable that most of the papers on the turnpike do not comment on the well-posedness of the stationary problem. Probably the most detailed contribution to this issue is the one from [5]. Here we shall not go further into details, instead, we just assume that the problem is well posed.

In the sequel we want to derive estimates on the difference between the solution to the dynamic optimization problem (2) and its stationary counterpart (4), $\|u_T - \bar{u}\|_{L^2(0,T;U)}$, and similarly for the difference of corresponding optimal observations $\|C(x_T - \bar{x})\|_{L^2(0,T;Z)}$.

To this effect let us subtract the two optimality systems (3) and (5). Denoting by $y_T = x_T - \bar{x}$ and $q_T = p_T - \bar{p}$ the corresponding differences of optimal primal and dual states, we get

$$\begin{aligned} y_T'(t) + Ay_T(t) &= -BB^* q_T(t) & -q_T'(t) + A^* q_T(t) &= C^* C y_T(t) \\ y_T(0) &= x_0 - \bar{x} & q_T(T) &= p_d - \bar{p}. \end{aligned}$$

Note that the obtained system is independent of the target data z_d and u_d. In addition, unlike the optimality system (3) it contains non-zero initial datum both for the primal and dual variable. In order to obtain the desired estimates, we split it into two parts. Hereby, the first part preserves the original initial condition for the primal state, while its dual component is run from zero. The reverse situation occurs with the second part of the decomposition.

More precisely, we introduce the decomposition

$$\begin{aligned} y_T &= y_{T,1} + y_{T,2} \\ q_T &= q_{T,1} + q_{T,2}, \end{aligned} \tag{6}$$

where the introduced functions satisfy the systems

$$\begin{aligned} y_{T,1}'(t) + Ay_{T,1}(t) &= -BB^* q_{T,1}(t) & -q_{T,1}'(t) + A^* q_{T,1}(t) &= C^* C y_{T,1}(t) \\ y_{T,1}(0) &= x_0 - \bar{x} & q_{T,1}(T) &= 0. \end{aligned} \tag{7}$$

and

$$\begin{aligned} y'_{T,2}(t) + Ay_{T,2}(t) &= -BB^*q_{T,2}(t) \qquad & -q'_{T,2}(t) + A^*q_{T,2}(t) &= C^*Cy_{T,2}(t) \\ y_{T,2}(0) &= 0 & q_{T,2}(T) &= p_d - \bar{p}. \end{aligned} \tag{8}$$

Obviously, system (7) is the optimality system for the problem

$$\min_{v\in L^2(0,T;U)} J_{T,1}(v) = \min_{v\in L^2(0,T;U)} \frac{1}{2}\int_0^T \left(|v(t)|_U^2 + |Cy(t)|_Z^2\right) dt, \tag{9}$$

where y stands for the solution to

$$\begin{aligned} y'(t) + Ay(t) &= Bv(t) \\ y(0) &= x_0 - \bar{x}. \end{aligned}$$

The minimizer of the functional $J_{T,1}$ is given by the expression $v_{T,1} = B^*q_{T,1}$, which allows us to obtain the following bound

$$\begin{aligned} \|v_{T,1}\|^2_{L^2(0,T;U)} + \|Cy_{T,1}\|^2_{L^2(0,T;Z)} &= 2J_{T,1}(v_{T,1}) \\ &\leq 2J_{T,1}(0) = \|CS_t(x_0 - \bar{x})\|_{L^2(0,T;X)} = Q_T(x_0 - \bar{x}) \cdot (x_0 - \bar{x}), \end{aligned} \tag{10}$$

where Q_T stands for the observability Grammian

$$Q_T = \int_0^T S_t^* C^* C S_t dt.$$

Similar kind of interpretation can be obtained for the second part of the above decomposition, i.e. for the system (8). Indeed, by using the change of variable $s = T - t$ one notice that it coincides with the optimality system for the problem

$$\min J_{T,2}(z) = \min \frac{1}{2}\int_0^T \left(|z(t)|_Z^2 + |B^*q(t)|_X^2\right) dt, \tag{11}$$

where minimization is taken over the space $L^2_{\text{loc}}([0,\infty); Z)$, while q is the solution to the problem

$$\begin{aligned} q'(t) + A^*q(t) &= C^*z(t) \\ q(0) &= p_d - \bar{p}. \end{aligned}$$

Concluding similarly as in (10) we obtain

$$\begin{aligned} \|v_{T,2}\|^2_{L^2(0,T;U)} + \|Cy_{T,2}\|^2_{L^2(0,T;Z)} &= 2J_{T,2}(v_{T,2}) \\ &\leq 2J_{T,2}(0) = \|B^*S_t^*(p_d - \bar{p})\|_{L^2(0,T;X)} = \Lambda_T(p_d - \bar{p}) \cdot (p_d - \bar{p}), \end{aligned} \tag{12}$$

where $v_{T,2} = B^*q_{T,2}$, while Λ_T stands for the controllability Grammian

$$\Lambda_T = \int_0^T S_t B B^* S_t^* dt.$$

Combining the estimates (10) and (12) together we can formulate our first result.

Theorem 2.1 *The difference of solutions to optimal control problems* (2) *and* (4), $v_T = u_T - \bar{u}$, *together with the difference of the corresponding optimal states* $y_T = x_T - \bar{x}$, *satisfies the estimate*

$$\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \|C(x_T - \bar{x})\|^2_{L^2(0,T;Z)} \leq 2\Big(Q_T(x_0 - \bar{x})\cdot(x_0 - \bar{x}) + \Lambda_T(p_d - \bar{p})\cdot(p_d - \bar{p})\Big), \quad (13)$$

where Q_T *is the observability Grammian for the pair* (A, C), *while* Λ_T *stands for the controllability Grammian corresponding to the pair* (A, B).

Remark 2 *It is important to emphasize that the previous theorem does not impose any conditions on the linear operators A, B and C (apart from the well possedness of the stationary problem discussed in Remark 1). In order to obtain the result, we decompose the system obtained by subtracting the time-evolution and the stationary optimality system into two parts. Each of these parts is identified as the optimal system of some auxiliary optimal control problem. This is the crucial observation which allows us to obtain the above estimate with virtually no requirements.*

The last theorem allows us to obtain the first turnpike property.

Theorem 2.2 *Assume that* B *and* C *are an infinite-time admissible control and observation operator for the semigroup generated by* $-A$, *respectively. Then the following result holds.*

a) (Integral turnpike) *For any target data* u_d *and* z_d, *the time averages of optimal solutions to optimal control problems* (3) *converge to the solution of the corresponding stationary problem* (4) *as the time framework goes to infinity, i.e.*

$$\frac{1}{T}\int_0^T u_T \xrightarrow[T\to\infty]{} \bar{u} \qquad \text{strongly in } U,$$

with the convergence rate of $O(1/\sqrt{T})$. *In addition, the same kind of the convergence with the same rate holds for the optimal observations*

$$\frac{1}{T}\int_0^T Cx_T \xrightarrow[T\to\infty]{} C\bar{x} \qquad \text{strongly in } Z.$$

b) (Measure turnpike) *For every* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon > 0$ *(that depends on* $x_0 - \bar{x}$ *and* $p_d - \bar{p}$*) such that for every* $T > 0$ *we have*

$$\mu\Big\{t \in [0,T]\Big|\ |u_T - \bar{u}|^2 + |C(x_T - \bar{x})|^2 \geq \varepsilon\Big\} < C_\varepsilon.$$

In other words, the Lebesgue measure of the set of time instants at which the optimal trajectory and control stay outside an ε *neighborhood of the stationary optimal pair* $(\bar{x}, \bar{u})$ *remains uniformly bounded as the time horizon* T *goes to infinity.*

Proof: a) The infinite-time admissibility assumptions imply that the considered Grammian operators are uniformly bounded (with respect to T) by their infinite-time counterparts $\Lambda_\infty, Q_\infty \in L(X)$. In particular, it follows

$$\|u_T - \bar{u}\|^2_{L^2(0,\infty;U)} + \|C(x_T - \bar{x})\|^2_{L^2(0,\infty;Z)} \leq K\Big(|x_0 - \bar{x}|^2_X + |p_d - \bar{p}|^2_X\Big), \quad (14)$$

with $K = 2(\|\Lambda_\infty\|_{L(X)} + \|Q_\infty\|_{L(X)})$. From here we get

$$\Big| \int_0^T (u_T(t) - \bar{u})dt \Big| \leq \sqrt{T}\|u_T - \bar{u}\|_{L^2(0,T;U)} \leq \sqrt{KT}(|x_0 - \bar{x}|_X + |p_d - \bar{p}|_X). \qquad (15)$$

By dividing the last estimate with T we obtain the first convergence.
In the same way, by repeating the above arguments one easily gets the integral turnpike property for the states, with the same convergence rate.
b) In order to obtain the measure turnpike property, let us assume the contrary. Thus there exists a sequence (T_n) such that for every n.

$$\mu\Big\{t \in [0,T]\Big|\ |u_T - \bar{u}|^2 + |C(x_T - \bar{x})|^2 \geq \varepsilon\Big\} \geq n.$$

From here it follows that

$$\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \|C(x_T - \bar{x})\|^2_{L^2(0,T;Z)} \geq \varepsilon n,$$

which contradicts the estimate (13) and the infinite-time admissibility assumptions.

□

Remark 3 *Several remarks are in order.*

- *The last theorem can be generalized to a larger class of systems, for which the operators B and C do not satisfy infinite-time admissibility assumptions. In that case the results holds assuming that $(x_0 - \bar{x})$ consists of either stable or unobservable modes of the system, while $(p_d - \bar{p})$ contains only stable or uncontrollable modes of the system, which is enough to ensure uniform boundedness of the right hand side of* (13) *with respect to T.*
- *In some papers on this topic, authors derive turnpike results by assuming controllability and observability assumptions for the pairs (A, B) and (A, C), respectively (cf. [11,19]). However, these requirements are quite strong and are usually replaced by weaker assumptions of stabilizability and detectability in more recent works (e.g. [1,7]). The latter ones require that all unstable modes of the dynamical system under consideration are both controllable and observable. It is significant that these assumptions are almost opposite to those required by Theorem 2.2. Namely, the assumed infinite time admissibilities require all unstable modes of the system to be both uncontrollable and unobservable. This raises an interesting question on turnpike behaviour of a dynamics that contains both controllable and uncontrollable unstable modes (and also both observable and unobservable). We shall return to this question in the Conclusion section.*

  For a stable matrix A, of course, both kind of assumptions are directly satisfied.
- *In some works, the measure turnpike is obtained by assuming strict dissipativity at some stationary point with a special choice of storage function (e.g. [13]). This notion is stronger than dissipativity in the sense of Willems [14] and requires the existence of a special point along which the optimal solutions concentrate in a uniform sense with respect to the time frame T. Actually, strict dissipativity at a particular point is a rather strong assumption. For general nonlinear optimal control problems it can be shown to be equivalent to the measure turnpike (with respect to the same point) under some additional conditions [4]. For*

LQ regularization problems, the equivalence can be obtained without additional assumptions in both discrete [6] and continuous time setting [7]. Of course, the question remains open under which conditions strict dissipativity holds and at which point. A step forward in this direction was recently achieved for LQ regularization problems in [6,7], where, under the assumption of stabilizability, it was proved that strict dissipativity is equivalent to detectability of the system under consideration. This characterizes turnpike and strict dissipativity of the problem in terms of checkable, system theoretic properties. However, even if one is able to verify them, the question remains at which point the dissipativity holds, i.e. what is going to be the turnpike limit. In [7] it was shown that the limit is the solution to an approximate stationary problem that does not coincide with the original one. On the contrary, in this paper we not only determine verifiable conditions under which the problem exhibits the turnpike properties, but we also provide a complete characterization of the turnpike limit - it is the solution to the corresponding stationary optimal control problem.

- *In the existing literature the measure-turnpike is often defined in terms of the state only (e.g. [7]). The property stated above takes into account both the state and control, similar as in [13].*
- *The proof of the* **b)** *part of the above theorem actually follows directly from the well established functional analysis results. In particular, it is related to the fact that L^p convergence implies convergence in measure. As it is quite short we provide it explicitly for the sake of completeness.*

The next result provides a similar kind of convergence for the (normalized) minimal values of functionals under consideration.

Theorem 2.3 *Under the assumptions of Theorem 2.2 it holds*

$$\frac{1}{T}\min J_T \xrightarrow[T\to\infty]{} \min J_s,$$

with the convergence rate of order $1/\sqrt{T}$.

Proof: In order to obtain the required result, we analyse the difference

$$\Big|\frac{1}{T}\min J_T - \min J_s\Big| = \frac{1}{T}\int_0^T \Big(|u_T(t)-u_d|_U^2 - |\bar{u}-u_d|_U^2 + |Cx(t)-z_d|_Z^2 - |C\bar{x}-z_d|_Z^2\Big)dt. \tag{16}$$

The first two terms in the above integrand we estimate as follows

$$\begin{aligned}\frac{1}{T}\int_0^T \Big(|u_T(t)-u_d|_U^2 - |\bar{u}-u_d|_U^2\Big)dt &\le \frac{1}{T}\int_0^T |u_T(t)-u_d|_U\Big(|u_T(t)-\bar{u}|_U + 2|\bar{u}-u_d|_U\Big)dt \\ &\le \frac{1}{T}\Big(\|u_T-\bar{u}\|^2_{L^2(0,T;U)} + 2\sqrt{T}\|u_T-\bar{u}\|_{L^2(0,T;U)}|\bar{u}-u_d|_U\Big)\end{aligned} \tag{17}$$

Performing the same analysis for the last two terms in (16), combining it with the last

estimate and the result of Theorem 2.1 we obtain

$$\Big|\frac{1}{T}\min J_T - \min J_s\Big| \leq \frac{K}{T}\Big(|x_0-\bar{x}|_X^2 + |p_d-\bar{p}|_X^2\Big) + \frac{2\sqrt{K}}{\sqrt{T}}\Big(|x_0-\bar{x}|_X + |p_d-\bar{p}|_X\Big)\Big(|\bar{u}-u_d|_U + |C\bar{x}-z_d|_Z\Big).$$

□

Remark 4 *If the desirable control and observation pair (u_d, z_d) coincide with the corresponding optimal stationary values $(\bar{u}, C\bar{x})$, or if the pair (x_0, p_d) equals the solution of the stationary optimality system* (5)*, then the above convergence is of order $1/T$.*

3 Turnpike for averaged controls

In this section we consider a family of the control systems

$$\begin{aligned} x'(\omega,t) + A(\omega)x(\omega,t) &= B(\omega)u(t) \\ x(\omega,0) &= x_0(\omega), \end{aligned} \tag{18}$$

each accompanied by an optimal control problem of the form

$$\min J_T(u) = \min_u \frac{1}{2}\int_0^T \Big(|u(t)-u_d|_U^2 + \Big|\int_\Omega \Big(C(\omega)x(\omega,t) - z_d(\omega)\Big)d\omega\Big|_Z^2\Big)\,dt + \int_\Omega p_d(\omega)\cdot x(\omega,T)d\omega. \tag{19}$$

Here the dynamics, control and observation operator, as well as initial datum and the desired observation depend on a parameter ω in a measurable manner. The parameter is assumed to be a random variable living in a space Ω and following some probability measure $d\omega$. We additionally assume that A, B, C, x_0 and z_d are uniformly bounded with respect to ω in order to ensure the integrability of solutions (by the Lebesgue dominated convergence theorem) with respect to the parameter.

However, the optimal control problem and its solution, that we keep denoting u_T, is parameter independent. This corresponds to the problem of finding a parameter-independent control that on average performs well for particular realizations of the parameter under consideration. This leads to the notion of the averaged control, as introduced in [18]. Additionally, the target control u_d is not just time-, but also parameter-independent.

The goal of this section to check deviation of the optimal averaged control from the solution of the corresponding stationary problem, which in this setting reads as

$$\min_{u\in U} J_s(u) = \min\Big\{\frac{1}{2}\Big(|u-u_d|_U^2 + \Big|\int_\Omega \Big(C(\omega)x(\omega) - z_d(\omega)\Big)\,d\omega\Big|_Z^2\Big)\,\Big|A(\omega)x = B(\omega)u\Big\}. \tag{20}$$

In order to obtain the desired estimates, let us write the optimality system corresponding for the non-stationary problem (19). By using the Lagrangian approach this one reads as

$$
\begin{aligned}
x_T'(\omega,t) + A(\omega)x_T(\omega,t) &= -B(\omega)\int_\Omega (B(\nu)^* p_T(\nu,t) - u_d)d\nu \\
x_T(\omega,0) &= x_0(\omega) \\
-p_T'(\omega,t) + A(\omega)^* p_T(\omega,t) &= C(\omega)^*\int_\Omega (C(\nu)x_T(\nu,t) - z_d(\nu))d\nu \\
p_T(\omega,T) &= p_d(\omega),
\end{aligned}
\tag{21}
$$

and the optimal control is given by $u_T = -\int_\Omega B(\omega)^* p_T(\omega)d\omega + u_d$.

Similarly, the optimality system of the stationary problem (20) reads

$$
\begin{aligned}
A(\omega)\bar{x}(\omega,t) &= -B(\omega)\int_\Omega (B(\nu)^*\bar{p}(\nu,t) - u_d)d\nu \\
A(\omega)^* p_T(\omega,t) &= C(\omega)^*\int_\Omega (C(\nu)x_T(\nu,t) - z_d)d\nu
\end{aligned}
\tag{22}
$$

and the optimal stationary control is $\bar{u} = -\int_\Omega B(\omega)^*\bar{p}(\omega)d\omega + u_d$.

We proceed in the manner similar to the one from the previous section. More precisely, we introduce the variables $y_T = x_T - \bar{x}$ and $q_T = p_T - \bar{p}$ which satisfy

$$
\begin{aligned}
y_T'(\omega,t) + A(\omega)y_T(\omega,t) &= -B(\omega)\int_\Omega B(\nu)^* q_T(\nu,t)d\nu \\
y_T(0) &= (x_0 - \bar{x})(\omega) \\
-q_{T,}'(\omega,t) + A(\omega)^* q_{Tu}(\omega,t) &= C(\omega)^*\int_\Omega C(\nu)y_T(\nu,t)d\nu \\
q_T(\omega,T) &= (p_T - \bar{p})(\omega).
\end{aligned}
$$

It is important to notice the identical structures of the equations satisfied by y_T and q_T. This will be decisive for splitting the system into two parts which can both be analysed as optimality system of a same kind of functional. Also note that the last system is independent of target data u_d, z_d. All this allows us to perform same kind of analysis as in the deterministic case. More precisely, we introduce the decomposition

$$
\begin{aligned}
y_T(\omega) &= y_{T,1}(\omega) + y_{T,2}(\omega) \\
q_T(\omega) &= q_{T,1}(\omega) + q_{T,2(\omega)},
\end{aligned}
\tag{23}
$$

where the introduced functions satisfy the systems

$$
\begin{aligned}
y_{T,1}'(\omega,t) + A(\omega)y_{T,1}(\omega,t) &= -B(\omega)\int_\Omega B^*(\nu)q_{T,1}(\nu,t)d\nu \\
y_{T,1}(\omega,0) &= (x_0 - \bar{x})(\omega) \\
-q_{T,1}'(\omega,t) + A(\omega)^* q_{T,1}(\omega,t) &= C^*(\omega)\int_\Omega C(\nu)y_{T,1}(\nu,t)d\nu \\
q_{T,1}(\omega,T) &= 0
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
y'_{T,2}(\omega,t) + A(\omega)y_{T,2}(\omega,t) &= -B(\omega)\int_\Omega B^*(\nu)q_{T,2}(\nu,t)d\nu \\
y_{T,2}(\omega,0) &= 0 \\
-q'_{T,2}(\omega,t) + A(\omega)^* q_{T,2}(\omega,t) &= C^*(\omega)\int_\Omega C(\nu)y_{T,2}(\nu,t)d\nu \\
q_{T,2}(\omega,T) &= (p_T - \bar{p})(\omega).
\end{aligned}
\tag{25}
$$

Obviously, system (24) is the optimality system for the problem

$$
\min_{v\in L^2(0,T;U)} J_{T,1}(v) = \min_{v\in L^2(0,T;U)} \frac{1}{2}\int_0^T \left(|v(t)|_U^2 + \Big|\int_\Omega C(\omega)y(\omega,t)d\omega\Big|_Z^2\right)dt, \tag{26}
$$

where y stands for the solution to

$$
\begin{aligned}
y'(\omega,t) + A(\omega)y(\omega,t) &= B(\omega)v(t) \\
y(\omega,0) &= (x_0 - \bar{x})(\omega),
\end{aligned}
$$

while the minimizer of $J_{T,1}$ is given by $v_1 = -\int_\Omega B^*(\omega)q_{T,1}(\omega,t)d\omega$.

Thus we obtain that

$$
\begin{aligned}
&\|v_{T,1}\|^2_{L^2(0,T;U)} + \|\int_\Omega C(\omega)y_{T,1}(\omega,t)d\omega\|^2_{L^2(0,T;U)} = 2J_{T,1}(v_{T,1}) \le 2J_{T,1}(0) \\
&= \|\int_\Omega C(\omega)S_t(\omega)(x_0-\bar{x})(\omega)d\omega\|_{L^2(0,T;Z)} \le \int_\Omega Q_T(\omega)(x_0-\bar{x})(\omega)\cdot(x_0-\bar{x})(\omega)d\omega,
\end{aligned}
\tag{27}
$$

where $Q_T(\omega)$ stands for the observability Grammian associated to the parameter value ω.

Similarly, system (25) is the optimality system for the problem

$$
\min_{z\in L^2(0,T;U)} J_{T,2}(z) = \min_{z\in L^2(0,T;Z)} \frac{1}{2}\int_0^T \left(|z(t)|_U^2 + \Big|\int_\Omega B^*(\omega)q(\omega,t)d\omega\Big|_Z^2\right)dt, \tag{28}
$$

where q is the solution to the problem

$$
\begin{aligned}
q'(\omega,t) + A^*(\omega)q(\omega,t) &= C^*(\omega)z(\omega,t) \\
q(\omega,0) &= (p_T - \bar{p})(\omega).
\end{aligned}
$$

Concluding similarly as in (27) we obtain

$$
\|v_{T,2}\|^2_{L^2(0,T;U)} + \|\int_\Omega C(\omega)y_{T,2}(\omega,t)d\omega\|^2_{L^2(0,T;U)} \le \int_\Omega \Lambda_T(\omega)(p_T-\bar{p})(\omega)\cdot(p_T-\bar{p})(\omega)d\omega, \tag{29}
$$

where $\Lambda_T(\omega)$ stands for the controllability Grammian associated to the parameter value ω.

Combining the last two estimates we obtain the averaged control analogue of Theorem 2.1:

$$
\begin{aligned}
&\|u_T - \bar{u}\|^2_{L^2(0,T;U)} + \|\int_\Omega C(\omega)\left(x_T(\omega,\cdot) - \bar{x}(\omega)\right)d\omega\|^2_{L^2(0,T;U)} \\
&\quad\le \int_\Omega \Big(Q_T(\omega)(x_0-\bar{x})(\omega)\cdot(x_0-\bar{x})(\omega) + \Lambda_T(\omega)(p_T-\bar{p})(\omega)\cdot(p_T-\bar{p})(\omega)\Big)d\omega.
\end{aligned}
\tag{30}
$$

The obtained result coincides with the key estimate (13) in the deterministic case, with the only difference that all the terms are now averaged with respect to the random parameter. As the optimal controls are parameter independent, the first term is same as in the deterministic case, while for the observations we have an estimate on the average of their deviations from optimal stationary ones.

Using similar kind of arguments as in the previous section, we are able to formulate the associated turnpike result.

Theorem 3.1 *Assume that for almost every $\omega \in \Omega$ we have that $B(\omega)$ and $C(\omega)$ are an infinite-time admissible control and observation operator for the semigroup generated by $-A(\omega)$, respectively. Then the following results hold.*

a) (Integral turnpike) *For any target data u_d and $z_d(\omega)$, we have*

$$\frac{1}{T}\int_0^T u_T = \bar{u} + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty,$$

where u_T and $\bar{u}$ are optimal averaged controls for a time-dependent and stationary optimal control problems (19) *and* (20)*, respectively. Similarly,*

$$\frac{1}{T}\int_0^T \int_\Omega C(\omega)x_T(\omega)d\omega = \int_\Omega C(\omega)\bar{x}(\omega)d\omega + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty.$$

b) (Measure turnpike) *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ (that depends on $\|x_0 - \bar{x}\|_{L^2(\Omega;X)}$ and $\|p_d - \bar{p}\|_{L^2(\Omega;X)}$) such that for every $T > 0$ we have*

$$\mu\Big\{t \in [0,T]\Big|\ |u_T - \bar{u}|^2 + \Big|\int_\Omega C(\omega)(x_T(\omega) - \bar{x}(\omega))d\omega\Big|^2 \geq \varepsilon\Big\} < C_\varepsilon.$$

c) (Convergence of the optimal value functions) *The time average of minimal values of evolutional functionals converge to the optimal value of the steady state problem.*

$$\frac{1}{T}\min J_T = \min J_s + \mathcal{O}(\frac{1}{\sqrt{T}}) \qquad \text{as } T \to \infty.$$

The proof follows the same steps as the proofs of theorems 2.2 and 2.3, and we omit it here.

4 An illustrative example

Now we present an example that illustrates that the turnpike phenomenon can also occur for systems with unstable modes, if the unstable modes are not part of the observation. For given real numbers x_0^1, x_0^2 consider the problem

$$\min \frac{1}{2}\int_0^T \left(|u(t)|^2 + |x_1(t) + x_2(t) - 1|^2\right) dt$$

subject to $x_1(0) = x_0^1$, $x_2(0) = x_0^2$, $x_1' = x_1 + u$, $x_2' = x_2 + u$.

The static problem is

$$\min \frac{1}{2}\left(|u|^2 + |x_1 + x_2 - 1|^2\right)$$

subject to $u = -x_1$, $u = -x_2$. This is equivalent to

$$\min \frac{1}{2}\left(|u|^2 + |-2\,u - 1|^2\right)$$

hence the optimal static control is $\bar{u} = -\frac{2}{5}$ and the optimal static state is $(\frac{2}{5}, \frac{2}{5})$.

For the state we have $(x_1 - x_2)' = x_1 - x_2$. Hence $x_1 - x_2$ is independent of the control. Thus in this example, the unstable mode is uncontrollable. We have

$$x_1(t) - x_2(t) = (x_0^1 - x_0^2)\,\mathrm{e}^t.$$

Note that the unstable mode does not influence the observation.

In order to derive the optimal observation $x_{T,1} + x_{T,2}$, define $s_T(t) = x_{T,1}(t) + x_{T,2}(t)$. Then we have $s_T(0) = x_0^1 + x_0^2$. For this example, for the optimal control we have $u_T(T) = 0$ which is equivalent to $s_T'(T) = s_T(T)$. The optimal observation s_T is the minimizer of

$$\frac{1}{2}\int_0^T \left(\frac{1}{4}\,|s'(t) - s(t)|^2 + |s(t) - 1|^2\right) dt,$$

Hence we have $s_T'' = 5\,s_T - 4$. For the optimal observation this yields

$$s_T(t) = x_{T,1}(t) + x_{T,2}(t) = \frac{4}{5}\left(1 + \frac{1}{\sqrt{5} - \tanh(\sqrt{5}\,T)}\,\frac{\sinh(\sqrt{5}\,t)}{\cosh(\sqrt{5}\,T)}\right)$$

$$+\frac{\left(x_0^1 + x_0^2 - \frac{4}{5}\right)}{\sqrt{5} - \tanh(\sqrt{5}\,T)}\left((\sqrt{5} - 1)(1 + \tanh(\sqrt{5}\,T))\right)\,\frac{\mathrm{e}^{-\sqrt{5}\,t}}{2}$$

$$+\frac{\left(x_0^1 + x_0^2 - \frac{4}{5}\right)}{\sqrt{5} - \tanh(\sqrt{5}\,T)}\left(\frac{\sqrt{5} + 1}{(1 + \tanh(\sqrt{5}\,T))}\right)\frac{\mathrm{e}^{\sqrt{5}\,t}}{2\,\cosh^2(\sqrt{5}\,T)}.$$

So we see that for all $t \in [0, T]$ we have the inequality

$$\left|x_{T,1}(t) + x_{T,2}(t) - \frac{4}{5}\right|$$

$$\leq \frac{4}{5}\,\frac{1}{\sqrt{5} - 1}\,\frac{\sinh(\sqrt{5}\,t)}{\cosh(\sqrt{5}\,T)} + \left|x_0^1 + x_0^2 - \frac{4}{5}\right|\left(\mathrm{e}^{-\sqrt{5}\,t} + \frac{\sqrt{5} + 1}{\sqrt{5} - 1}\,\frac{\mathrm{e}^{\sqrt{5}\,t}}{2\,\cosh^2(\sqrt{5}\,T)}\right).$$

This inequality illustrates in particular the integral turnpike phenomenon that is implied by Theorem 2.2. Note that we have

$$\lim_{T\to\infty} x_{T,1}(T) + x_{T,2}(T) - \frac{4}{5} = \frac{4}{5}\,\frac{1}{\sqrt{5} - 1} = 0.6472....$$

The corresponding optimal control is $u(t) = \frac{1}{2}(s_T'(t) - s_T(t))$.

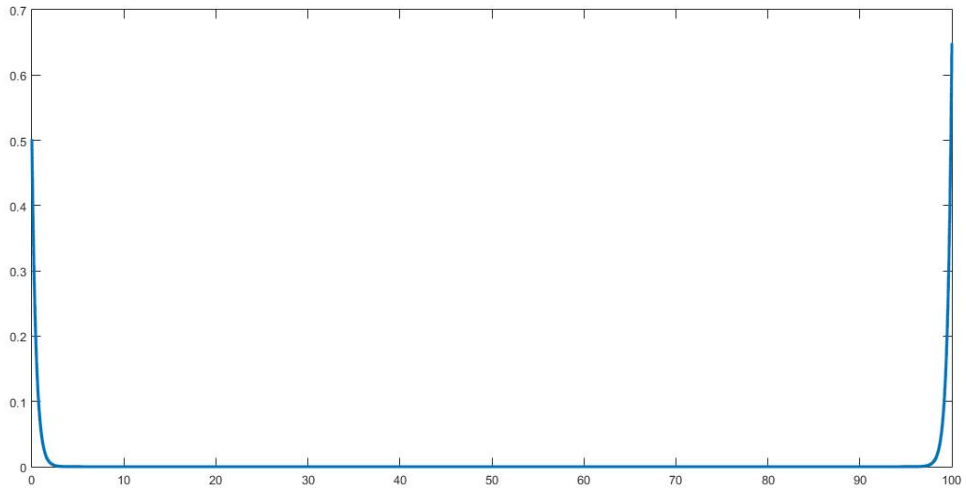

Fig. 2. The graph of $x_{T,1}(t) + x_{T,2}(t) - 0.8$ for $x_0^1 + x_0^2 - 0.8 = 0.5$.

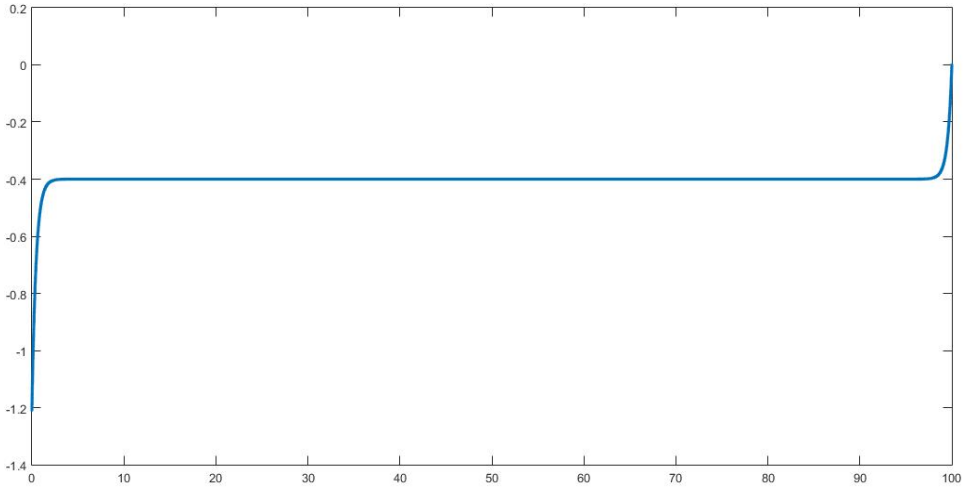

Fig. 3. The graph of the optimal control $u_T(t)$ for $x_0^1 + x_0^2 - 0.8 = 0.5$.

Figure 2 shows the graph of $x_{T,1}(t) + x_{T,2}(t) - \frac{4}{5}$ on $[0,\, T]$ for the case $x_0^1 + x_0^2 - \frac{4}{5} = \frac{1}{2}$ and $T = 100$. In accordance with theoretical findings, we note that most of the time the optimal observation is almost indispensable from its stationary counterpart. The largest deviations occur at initial and terminal point, and decay exponentially as the time approaches the center of the interval.

The evolution of the optimal control $u_T(t)$ for the same case is depicted on Figure 3. Again we observe the largest deviations from the stationary optimal control $\bar{u} = -2/5$ in the boundary layers and their exponential decay as we move away from $t = 0$ and $t = T$.

5 Conclusion

In this paper, we have obtained turnpike results for infinite-dimensional, LQ optimal control problems, either of deterministic or stochastic nature. The results are based on the key estimate (Theorem 2.1, (13)) on the difference between evolutional and stationary optimal controls and observations, which we derive with virtually no assumptions on the operators A, B, and C (cf. Remark 2).

The turnpike properties follow directly by assuming infinite-time admissibility of control and observation operator. In the case of a stochastic system, the assumption has to be satisfied for almost every value of the random parameter. This assumption implies neither controllability nor observability of the system. Instead, it requires all unstable modes of the system to be both uncontrollable and unobservable. It is important to emphasize that the optimal control problem under consideration does not impose any constraint on the final state of the system, and only includes a linear terminal regularization. This allows us to obtain positive results for a (partially) uncontrollable system.

Remarkably, previous works on this topic assume stabilizability and detectability of the system under consideration, implying that all unstable modes of the dynamical system under consideration are both controllable and observable. In the case of a system that contains both controllable and uncontrollable unstable modes (and likewise both observable and unobservable), one could unify the above approaches. Using the Kalman canonical decomposition, one can construct a transformation that splits the system into a controllable and an uncontrollable part (e.g. [17, Section 3.3]). The same procedure can be performed in terms of observability. If two transformations coincide, i.e,. if the controllable part of the system coincides with the observable one, then one can apply two types of turnpike results separately. Those that assume controllability and observability (or more generally, stabilizability and detectability) are to be applied to the first part of the decomposition, and those derived in this paper are to be applied to the second, uncontrollable and unobservable part of the system. Of course, if two canonical transformations do not coincide, this kind of approach is not at disposal, and in general the measure turnpike will not hold. Existence of additional conditions that will ensure the turnpike behaviour in this case remains an interesting open problem.

The results obtained are closely related to those in [1], where the authors prove a stronger, exponential turnpike property. They perform a careful analysis of the optimality system leading to a-priori bounds in weighted Sobolev spaces (with the exponential function as the weight). Hereby, they use a particular decoupling of the optimality system (possible under a special kind of the final state regularization) in the terms of the solution to the Riccati equation. For this purpose, the assumptions of stabilizability and detectability must be satisfied, which, as discussed above, are almost opposite to those considered in this paper. Obtaining an exponential turnpike property for a non-stabilizable system remains another interesting open problem.

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