

max- p optimal boundary control of gas flow

Martin Gugat* Michael Schuster**

* *Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU),
Department of Data Science, Lehrstuhl für Dynamics, Control and
Numerics (Alexander von Humboldt-Professur), Cauerstr. 11, 91058
Erlangen, Germany (e-mail: martin.gugat@fau.de).*

** *Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU),
Department of Data Science, Lehrstuhl für Dynamics, Control and
Numerics (Alexander von Humboldt-Professur), Cauerstr. 11, 91058
Erlangen, Germany (e-mail: michi.schuster@fau.de).*

Abstract: In the transition to renewable energy sources, hydrogen will potentially play an important role for energy storage. The efficient transport of this gas is possible via pipelines. An understanding of the possibilities to control the gas flow in pipelines is one of the main building blocks towards the optimal use of gas.

For the operation of gas transport networks it is important to take into account the randomness of the consumers' demand, where often information on the probability distribution is available. Hence in an efficient optimal control model the corresponding probability should be included and the optimal control should be such that the state that is generated by the optimal control satisfies given state constraints with large probability. We comment on the modelling of gas pipeline flow and the problems of optimal nodal control with random demand, where the aim of the optimization is to determine controls that generate states that satisfy given pressure bounds with large probability. We include the H^2 norm of the control as control cost, since this avoids large pressure fluctuations which are harmful in the transport of hydrogen since they can cause embrittlement of the pipeline metal.

Keywords: gas pipeline flow, nodal control, boundary control, optimal control, hyperbolic differential equation, random demand, state constraints, pressure bound, classical solutions

1. INTRODUCTION

The isothermal Euler equations (see e.g. Banda et al. (2006), Gugat and Herty (2022))

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{1}{2}\theta \frac{q|q|}{\rho} \end{cases} \quad (1)$$

are a well-established model for gas pipeline flow, where ρ denotes the gas density, p the pressure, q the mass flow rate and $\theta \geq 0$ is a friction parameter. At the end $x = 0$ the flow rate that is desired by the consumers is given by a random variable, so we have $q(t, 0) = q_r(\omega)$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Here we assume that $q_r(\omega) \in C^1([0, T])$ for all $\omega \in \Omega$. Due to the influence of the random boundary term, also the pde solution becomes a random variable. At the end $x = L$ of the pipe, the pressure is controlled, $p(t, L) = u(t)$. We consider controls $u \in H^2([0, T])$. For the deterministic case, in Gugat and Sokolowski (2022), a similar optimal control problem for gas networks is considered and the existence of an optimal control is shown. See Göttlich and Schillinger (2021) for a related study for linear systems.

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2. THE SYSTEM

Let a time horizon $T > 0$ be given. For the case of ideal gas where $p = a^2 \rho$ with the sound speed $a > 0$, our system is governed by the initial boundary value problem

$$(\mathbf{S}^\omega) \begin{cases} q(0, x) = q_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in (0, L), \\ q(t, 0) = q_r(\omega), \quad p(t, L) = u(t), \quad t \in (0, T), \\ \begin{pmatrix} \rho \\ q \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ a^2 - \frac{q^2}{\rho^2} & 2\frac{q}{\rho} \end{pmatrix} \begin{pmatrix} \rho \\ q \end{pmatrix}_x = \begin{pmatrix} 0 \\ -\frac{\theta}{2} \frac{q|q|}{\rho} \end{pmatrix}. \end{cases}$$

Let $R_0 > 0$ denote a constant reference density. Due to the theory of semi-global solutions (see Li (2010), Li et al. (2016)) for any given time horizon $T > 0$ there exist numbers $\varepsilon(T) > 0$ and $C_1(T) > 0$ such that for all $R \geq R_0$ and all initial states that satisfy

$$\max\{\|q_0\|_{C^1([0, L])}, \|\rho_0 - R\|_{C^1([0, L])}\} \leq \varepsilon(T) \quad (2)$$

and all $q_r(\omega)$ with

$$\|q_r(\omega)\|_{C^1([0, T])} \leq \varepsilon(T) \quad (3)$$

and all controls with

$$\|u - a^2 R\|_{C^1([0, T])} \leq \varepsilon(T) \quad (4)$$

that are C^1 -compatible with the initial state (q_0, ρ_0) there exists a classical solution (q^ω, ρ^ω) of (\mathbf{S}^ω) on $[0, T]$ that satisfies the a priori estimate

$$\max\{\|q^\omega\|_{C^1([0, T] \times [0, L])}, \|\rho^\omega - R\|_{C^1([0, T] \times [0, L])}\} \leq C_1(T) \\ \max\{\|q_0\|_{C^1([0, L])}, \|\rho_0 - R\|_{C^1([0, L])}, \|q_r(\omega)\|_{C^1([0, T])}, \|u -$$

$a^2 R|_{C^1([0, T])}$. Moreover, the state depends continuously in $(C^1([0, T] \times [0, L]))^2$ on the control $u \in C^1([0, T])$.

To guarantee that a regular solution exists, in the optimal control problem the control constraint (4) and the C^1 -compatibility conditions with the initial state are prescribed.

3. THE OPTIMAL CONTROL PROBLEM

Let a lower pressure bound $p_{\min} > 0$ be given. For a control $u \in H^2(0, T)$ define the objective function

$$J(u, R) = \|u - R\|_{H^2(0, T)} \quad (5)$$

$$- \ln(\mathbb{P}(\|(p_{\min} - p^\omega)_+\|_{C([0, T] \times [0, L])} = 0)).$$

The optimal control problem $\mathbf{P}_{\text{dyn}}(T)$ is to minimize $J(u, R)$ subject to the constraints $R \geq R_0$, (4) and the C^1 -compatibility conditions for u , where (p_ω, q_ω) solves (\mathbf{S}^ω) .

The H^2 -term in the objective function helps to avoid large pressure fluctuations in the pipe that can be harmful if the gas contains hydrogen due to the danger of embrittlement, see Guy et al. (2021). The optimal control of gas transportation systems is a classical topic in process engineering, see for example Osiadacz and Swierczewski (1994).

3.1 Existence of solutions

Theorem 1. Let $(q_0, \rho_0) \in (C^1([0, T]))^2$ be given such that $a^2 \rho_0 > p_{\min}$ and (2) holds. Assume that (3) and the C^1 -compatibility conditions of $q_r(\omega)$ and the initial data hold almost surely. Then an optimal control that solves $\mathbf{P}_{\text{dyn}}(T)$ does exist in $(0, \infty) \times H^2([0, T])$.

Proof. The set of admissible controls is non-empty, since for all $R > 0$ there exists a control $\hat{u} \in H^2(0, T)$ that is compatible with (q_0, ρ_0) and satisfies (4). The a priori estimate implies that if R is sufficiently large, we have almost surely $p^\omega = a^2 R + a^2(\rho^\omega - R) \geq a^2 R - a^2 C_1(T) \varepsilon(T) \geq p_{\min}$. Hence if R is sufficiently large, there exists a control where the objective function attains a finite value. (Note that this does not require that $p^\omega \geq p_{\min}$ almost surely, but only that $p^\omega \geq p_{\min}$ has a nonzero probability.)

The H^2 -norm is a weakly sequentially lower semi-continuous functional in $H^2(0, T)$. Results from Farshbaf-Shaker et al. (2018) imply that the probabilistic part of the objective function is also weakly sequentially lower semi-continuous in $H^2(0, T)$. This can be seen as follows. A sequence that converges weakly in $H^2(0, T)$ converges strongly in $C^1([0, T])$ to a limit point $u^* \in H^2(0, T)$. Due to the theory of semi-global solutions, (2), (3) and (4) imply that the controls generate classical solutions of (\mathbf{S}^ω) almost surely and the strong convergence in $C^1([0, T])$ of the controls implies that also the corresponding subsequence of generated states given by the classical solutions of (\mathbf{S}^ω) converges strongly in $(C^1([0, T] \times [0, L]))^2$ to the solution that is generated by the limit point u^* . Then Lemma 2 in Farshbaf-Shaker et al. (2018) implies that the probability is weakly sequentially upper semi-continuous in $H^2(0, T)$.

This implies that the objective functional is a weakly sequentially lower semi-continuous functional in $(0, \infty) \times H^2(0, T)$.

We consider a minimizing sequence of feasible controls (R_k, u_k) . Due to the H^2 -term and (4), this sequence is bounded in $\mathbb{R} \times H^2(0, T)$. Hence it contains a subsequence that converges weakly in $\mathbb{R} \times H^2(0, T)$ and thus converges strongly in $\mathbb{R} \times C^1([0, T])$ to a limit point $(R^*, u^*) \in \mathbb{R} \times H^2(0, T)$. Moreover, this also implies that u^* satisfies (4) and that the values of the objective function $J(u^*)$ is minimal.

3.2 Numerical approaches

For the numerical solution, a kernel density estimator should be used to obtain a differentiable approximation of the objective function similar to the approach in Schuster et al. (2021). The controls are represented as Fourier series, $u(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(j \frac{2\pi}{T} t) + b_j \sin(j \frac{2\pi}{T} t)$. Then we have $\frac{2}{T} \|u\|_{H^2(0, T)}^2$

$= \frac{a_0^2}{2} + \sum_{j=1}^{\infty} \left(1 + (j \frac{2\pi}{T})^2 + (j \frac{2\pi}{T})^4\right) (|a_j|^2 + |b_j|^2)$. Truncation of the Fourier series after a finite number of modes leads to a semi-infinite optimization problem, for a survey see Stein (2012).

4. A NUMERICAL EXAMPLE

Gas network optimization has been of interest for decades, see e.g., Herty and Sachers (2007); Zlotnik et al. (2015) for a semilinear hyperbolic gas transport model and Mak et al. (2019) for a parabolic gas transport model. But gas network optimization with H^2 control and probabilistic terms in the objective function was not considered yet. We present a numerical example on a single edge for both, a probabilistic objective function with a L^2 control term and a probabilistic objective function with an H^2 control term.

We consider the isothermal Euler equations for ideal gases, i.e., for $(t, x) \in [0, T] \times [0, L]$ we have

$$p(t, x) = a^2 \rho(t, x),$$

where a denotes the speed of sound in the gas. Due to the proportionality of pressure and density we consider density control at $x = L$ instead of pressure control. All values and constants are given in Table 1.

Letter	Value	Unit
T	12	h
L	30	km
a	343	m/s
θ	0.2	
R	46.3	kg/m ³
ρ_{\min}	40.4	kg/m ³

Table 1. Values for the numerical example.

At the end $x = 0$ we assume random gas outflow. Therefore we define a deterministic function

$$q_D(t) = -\frac{16}{\pi} \sin\left(\frac{\pi}{12 \cdot 60^2} t\right) \cdot \frac{16}{\pi} \cos\left(\frac{\pi}{8 \cdot 60^2} t\right) + 140.$$

Let

$$\xi \sim \mathcal{N}\left(1, \sqrt{0.1}\right),$$

be a Gaussian distributed random variable on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Similar as in Schuster et al. (2021) we write $q_D(t)$ as Fourier series and multiply every Fourier term with a random number $\xi(\omega)$, $\omega \in \Omega$. For the implementation we cut the Fourier series after 10 terms. A sample of 20 random boundary functions $q_r(\omega)$ and the corresponding deterministic function $q_D(t)$ are shown in *Figure 1*. For the implementation we use the negative flow values since gas is transported from the end of the pipe ($x = L$) to its beginning ($x = 0$).

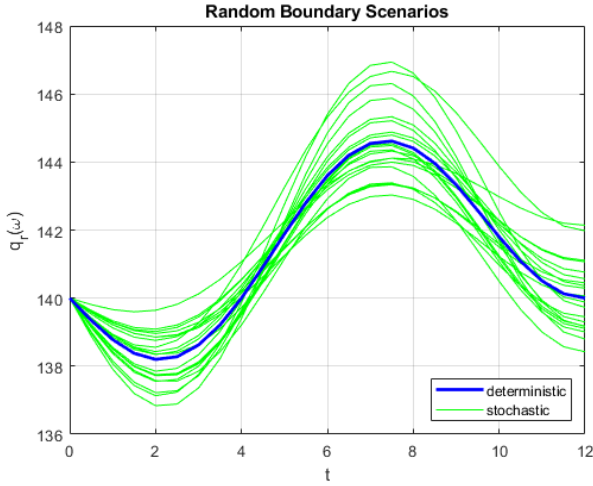


Fig. 1. Sample of 20 random boundary functions $q_r(\omega)$.

For the initial state we solve the stationary isothermal Euler equations

$$\begin{cases} q_x^\sigma = 0, \\ \left(a^2 \rho^\sigma + \frac{(q^\sigma)^2}{\rho^\sigma} \right)_x = -\frac{1}{2} \theta \frac{q^\sigma |q^\sigma|}{\rho^\sigma}, \end{cases} \quad (6)$$

with the boundary conditions $q^\sigma(0) = q_D(0)$ and $\rho^\sigma(L) = 46.3 \text{ kg/m}^3$. The solution $(\rho_{\text{init}}^\sigma, q_{\text{init}}^\sigma)$ of (6) serves as initial state for the dynamic problem.

The probabilistic term in the objective function is computed with a kernel density estimator approach (see Schuster et al. (2021)). Due to the friction along the pipe and due to the choice of initial states, for every time the density is minimal at $x = 0$. Thus we have $\rho^\omega(t, x) \geq \rho_{\text{min}}$ iff $\rho^\omega(t, 0) \geq \rho_{\text{min}}$. We discretize the time interval using $n_T + 1$ equidistant points $0 = t_0 < \dots < t_{n_T} = T$ and we use a multivariate kernel density estimator approach with Gaussian product kernels to approximate the probabilistic term. For

$$\mathcal{P}_{\text{min}} := \otimes_{i=1}^{n_T} [\rho_{\text{min}}, \infty),$$

we have

$$\begin{aligned} & \mathbb{P}(\rho^\omega(t, 0) \geq \rho_{\text{min}} \quad \forall t \in [0, T]) \approx \\ & \int_{\mathcal{P}_{\text{min}}} \frac{1}{N \sqrt{\det H}} \sum_{i=1}^N \prod_{j=1}^{n_T} \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\frac{z_j - \rho_i(t_j)}{\sqrt{2H_{j,j}}}\right)^2\right) \\ & = \frac{1}{N} \frac{1}{2^{n_T}} \sum_{i=1}^N \prod_{j=1}^{n_T} \left[1 - \text{erf}\left(\frac{\rho_{\text{min}} - \rho_i(t_j)}{\sqrt{2H_{j,j}}}\right) \right]. \end{aligned}$$

Here N is the number of samples, $\rho_i(t_j)$ is the density for the i -th sample at $(t, x) = (t_j, 0)$ and H is a diagonal

positive definite bandwidth matrix.

We define objective functions

$$J_{L^2}(u, R) = w_1 \|u - R\|_{L^2(0, T)} - \ln\left(\mathbb{P}(\rho^\omega(t, 0) \geq \rho_{\text{min}} \quad \forall t \in [0, T])\right),$$

and

$$J_{H^2}(u, R) = w_1 \|u - R\|_{L^2(0, T)} + w_2 \|u'\|_{L^2(0, T)} + w_3 \|u''\|_{L^2(0, T)} - \ln\left(\mathbb{P}(\rho^\omega(t, 0) \geq \rho_{\text{min}} \quad \forall t \in [0, T])\right),$$

with weights

$$w_1 = 2 \cdot 10^{-3}, \quad w_2 = 1 \cdot 10^5, \quad w_3 = 1 \cdot 10^{12}.$$

The optimal controls for both objective functions ($N = 20$ and $n_T = 25$) are shown in *Figure 2*. The blue line shows the optimal density control for $J_{L^2}(u, R)$ and the red line in shows the optimal density control for $J_{H^2}(u, R)$. The results can be interpreted as follows: For an objective function without probabilistic term the optimal solution would obviously be $u \equiv R$. The density at $x = 0$ is only lower than ρ_{min} for the peak around 7 hours (cf. *Figure 1*). Thus the control only needs to be active in this time span. As it was expected the H^2 control is smoother than the L^2 control.

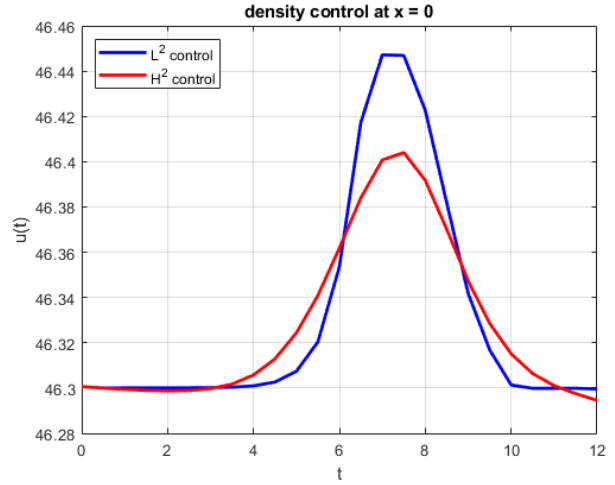


Fig. 2. Optimal control for $J_{L^2}(u, R)$ and $J_{H^2}(u, R)$.

We have

$$\mathbb{P}_{L^2}(\rho^\omega(t, 0) \geq \rho_{\text{min}} \quad \forall t \in [0, T]) \approx 74\%,$$

and

$$\mathbb{P}_{H^2}(\rho^\omega(t, 0) \geq \rho_{\text{min}} \quad \forall t \in [0, T]) \approx 66\%.$$

Thus a slight decrease of the probability leads to a smoother density control and less density fluctuations. This can also be seen in *Figure 3* and *Figure 4*. The peaks around 7 hours are smoother in *Figure 4* than in *Figure 3*. The blue line shows ρ_{min} . The optimal density control and the corresponding densities at $x = 0$ would be even smoother if we would increase the weights w_2, w_3 for the L^2 -Norm of the control derivatives.

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REFERENCES

- Banda, M.K., Herty, M., and Klar, A. (2006). Coupling conditions for gas networks governed by the isothermal Euler equations. *Netw. Heterog. Media*, 1(2), 295–314.
- Farshbaf-Shaker, H., M.H., Henrion, R., and Hömberg, D. (2018). Properties of chance constraints in infinite dimensions with an application to pde constrained optimization. *Set-Valued Var. Anal*, 26, 821–841.
- Gugat, M. and Sokolowski, J. (2022). On problems of dynamic optimal nodal control for gas networks. *Pure and Applied Functional Analysis*, x, x–x.
- Gugat, M. and Herty, M. (2022). Modeling, control, and numerics of gas networks. In E. Trelat and E. Zuazua (eds.), *Handbook of Numerical Analysis*. doi: 10.1016/bs.hna.2021.12.002.
- Guy, P., Laszlo, T., and Julien, C. (2021). Effects of hydrogen addition on design, maintenance and surveillance of gas networks. *Processes*, 9(7). doi:10.3390/pr9071219.
- Göttlich, S. and Schillinger, T. (2021). Control strategies for transport networks under demand uncertainty. ArXiv preprint arXiv:2111.09674.
- Herty, M. and Sachers, V. (2007). Adjoint calculus for optimization of gas networks. *Networks and Heterogeneous Media*, 2(4), 733–750.
- Li, T. (2010). *Controllability and observability for quasilinear hyperbolic systems*, volume 3 of *AIMS Series on Applied Mathematics*. American Institute of Mathematical Sciences (AIMS), Springfield, MO; Higher Education Press, Beijing.
- Li, T., Wang, K., and Gu, Q. (2016). *Exact boundary controllability of nodal profile for quasilinear hyperbolic systems*. SpringerBriefs in Mathematics. Springer, Singapore. doi:10.1007/978-981-10-2842-7.
- Mak, T., Hentenryck, P.V., Zlotnik, A., and Bent, R. (2019). Dynamic compressor optimization in natural gas pipeline systems. *Inform Journal on Computing*, 31, 40–65.
- Osiadacz and Swierczewski (1994). Optimal control of gas transportation systems. In *1994 Proceedings of IEEE International Conference on Control and Applications*, 795–796 vol.2. doi:10.1109/CCA.1994.381219.
- Schuster, M., Strauch, E., Gugat, M., and Lang, J. (2021). Probabilistic constrained optimization on flow networks. *Optimization and Engineering*. doi:10.1007/s11081-021-09619-x.
- Stein, O. (2012). How to solve a semi-infinite optimization problem. *Eur. J. Oper. Res.*, 223(2), 312–320. doi: 10.1016/j.ejor.2012.06.009.
- Zlotnik, A., Chertkov, M., and Backhaus, S.N. (2015). Optimal control of transient flow in natural gas networks. *2015 54th IEEE Conference on Decision and Control (CDC)*, 4563–4570.

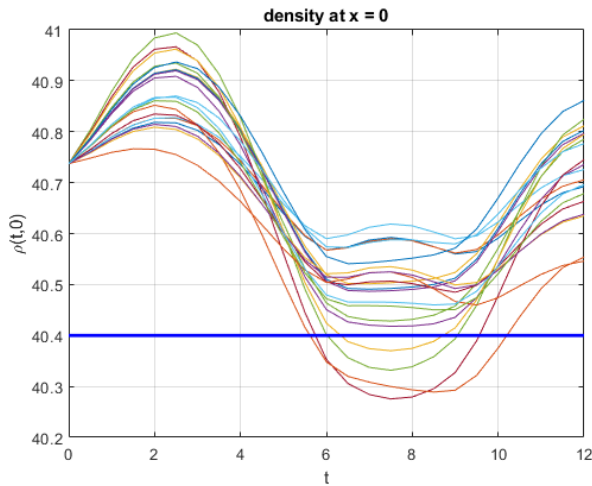


Fig. 3. Scenarios at $x = 0$ for the optimal density control of $J_{L^2}(u, R)$

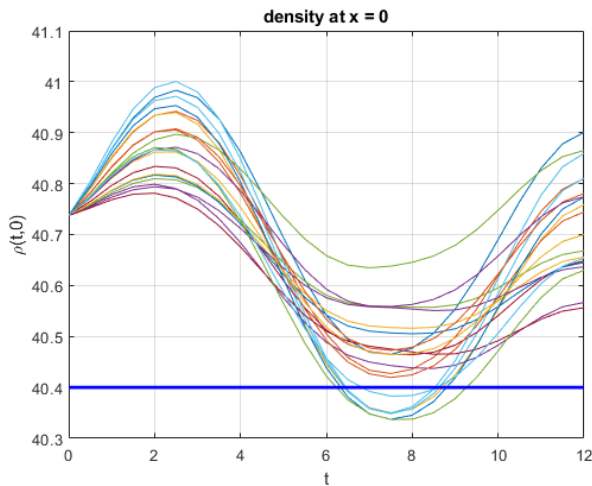


Fig. 4. Scenarios at $x = 0$ for the optimal density control of $J_{H^2}(u, R)$

5. CONCLUSION

In optimal control problems, it is important to take into account the uncertainty of the problem data in order to obtain controls that work sufficiently well in the set of data that is expected. Since in many applications information on the probability distribution of the data is available, this information should be used in an optimal control model. In our contribution we choose the probability that state constraints are satisfied as a part of the objective function. In this way, it is ensured that the optimization generates controls that are robust in the sense that the pressure bounds are satisfied with a high probability. We include an H^2 control cost in the objective functional, which is of particular interest in the context of hydrogen transport. It also serves as a Tychonov regularization term that is important for the proof of the existence of optimal controls.