

A Turnpike Result for Optimal Boundary Control Problems with the Transport Equation under Uncertainty

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Abstract In this paper we analyze the turnpike phenomenon for optimal boundary control problems with a linear transport equation with source term. The convex objective function depends on the boundary traces of the transport equation and is strictly convex with respect to the boundary control. We show an integral turnpike result for an optimal Dirichlet boundary control problem in the sense that if the time horizon goes to infinity, then the dynamic optimal control converges to the corresponding steady state optimal control.

The novelty of this work is two-sided. On the one hand, even if turnpike results for this kind of optimal boundary control problem already exist, we present a new direct proof without using adjoint calculus that leads to sharper estimates. On the other hand we consider uncertainty in the initial data and/or in the source term. We show that the integral turnpike result also holds considering uncertainty. Throughout the paper we use numerical examples to illustrate the results.

Subject Classification: 49K20, 49K45.

Keywords: Turnpike, Boundary Control, Transport Equation, Random Boundary Data.

1 Introduction

The Turnpike property has been established long time ago in optimal control theory. It has been discussed in mathematical economics by P. A. Samuelson in 1949 (see [3]). Until now the turnpike theory has been analyzed in various contexts, see e.g. [15]. The turnpike theory in the context of differential equations has been developed recently, see for example [9, 14] and [12] for finite-dimensional optimal control problems and [13] for optimal control problems in a Hilbert space setting. The Turnpike phenomenon for the wave equation in a linear-quadratic setting was studied in [6], linear hyperbolic 2x2-systems systems with and without integer constraints have been analyzed in [4, 5].

In this paper we consider optimal boundary control problems with a linear transport equation with and without uncertain data. We first present an integral turnpike result for an optimal boundary control problem s.t. a deterministic linear transport equation is satisfied. Integral turnpike result means, that the L^2 -norm of the difference between optimal dynamic control and corresponding optimal static control converges to 0 if the time horizon T goes to infinity. The proof is based on necessary optimality conditions for the dynamic and the corresponding static optimal control problem. Even if turnpike results for this kind of optimal boundary control problems already exist we present a novel direct proof without using adjoint calculus that leads to sharper estimates and a better time independent right-hand-side constant. Another novelty of this work is that we assume uncertainty in the initial data and/or in the source term of the transport equation. We show that the integral turnpike property also holds for the transport equation with uncertain initial data,

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where the objective function depends on the optimal control and on the expected state. Further we show that an integral turnpike property holds for the transport equation with random source term as well, where the objective function also depends on the optimal control and on the expected state.

This paper is structured as follows: In Section 2 we introduce a deterministic dynamic optimal boundary control problem with the linear transport equation and a corresponding static optimal boundary control problem. The objective function satisfies a property similar to strong convexity and it depends on the optimal control as well as on the corresponding state. We prove an integral turnpike property based on necessary optimality conditions for both optimal control problems.

In Section 3 we consider a dynamic optimal control problem with uncertain initial data (modeled by a Wiener process) and we show that the integral turnpike inequality stated in the deterministic case in Section 2 also holds in this setting.

In Section 4 we consider a dynamic optimal control problem with a linear transport equation in which the source term is multiplied with a random variable. We show that an integral turnpike property also holds for a dynamic optimal control problem with the linear transport equation with random source term and a corresponding static optimal control problem. Further we specify the results for Gaussian and uniformly distributed random variables.

Throughout the paper we use numerical examples to illustrate our results. For the numerical optimization we used the *AMPL* software package¹ with the open-source interior point solver *IPOPT* and the linear sparse systems solver *MUMPS*. For the simulations and the pictures we use *MATLAB*[®] 2019a.

2 A Turnpike result for the deterministic optimal control problem

In this section we consider a deterministic optimal boundary control problem governed by a linear transport equation with source term and the corresponding static optimal problem. We state an integral turnpike result for the optimal controls in the sense that the dynamic optimal control converges to the static optimal control if the time horizon tends to infinity. For $(t, x) \in [0, T] \times [0, L]$, for initial data $r_{\text{ini}}(x) \in L^2(0, L)$, for boundary control $u(t) \in L^2(0, L)$ and for sound speed $c > 0$ consider the linear transport equation with initial and boundary control

$$\begin{cases} r_t(t, x) + cr_x(t, x) = mr(t, x), \\ r(0, x) = r_{\text{ini}}(x), \\ r(t, 0) = u(t), \end{cases} \quad (1)$$

where $r_t(t, x)$ is the time derivative of r and $r_x(t, x)$ is the space derivative of r . If the sound speed c would be negative the information would be transported the other way around and we would have to assume boundary control at $x = L$. The constant m is a real number. Note that the linear transport equation (1) is exactly controllable if $T \geq L/c$. Let convex and differentiable functions

$$f, g : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0},$$

be given. For $u \in L^2(0, T)$ and $w : L^2(0, T) \rightarrow L^2(0, T)$ we define the function

$$J_T : L^2(0, T) \rightarrow \mathbb{R}, \quad J_T(u) = \int_0^T f(u(t)) + g(w(u(t))) dt. \quad (2)$$

Note that J_T is convex due to the convexity of f and g . For $w(u) = r(t, L)$ we consider the dynamic optimal boundary control problem

$$\begin{cases} \min_{u \in L^2(0, T)} & J_T(u) = \int_0^T f(u(t)) + g(r(t, L)) dt \\ \text{s.t.} & r_t(t, x) + cr_x(t, x) = m r(t, x), \\ & r(0, x) = r_{\text{ini}}(x), \\ & r(t, 0) = u(t). \end{cases} \quad (3)$$

¹<https://ampl.com/>

The objective function is chosen such that both control cost and corresponding state are optimal with respect to the choice of f and g . Motivated by applications a natural choice would be minimizing the control cost s.t. the state is close to a desired state $a \in \mathbb{R}$, i.e., $f(x) = x^2$ and $g(x) = (x - a)^2$. Obviously the term $r(t, L)$ also depends on $u(t)$ but we do not explicitly mention this dependence since we will later use the explicit representation of $r(t, L)$.

From the theory of hyperbolic partial differential equations we know that for initial data $r_{\text{ini}} \in L^2(0, L)$ and boundary data $u \in L^2(0, T)$ the linear transport equation (1) has a solution $r \in C([0, T], L^2(0, L))$ in terms of characteristics. In fact the solution of (1) is given by

$$r(t, x) = \begin{cases} \exp(mt) r_{\text{ini}}(x - ct) & \text{for } x > ct, \\ \exp\left(m\frac{x}{c}\right) u\left(t - \frac{x}{c}\right) & \text{for } x \leq ct. \end{cases} \quad (4)$$

If initial and boundary condition are C^1 -regular and satisfy C^1 compatibility then (4) is a classical solution. Using the solution (4) the dynamic optimal boundary control problem (3) is equivalent to

$$\begin{aligned} \min_{u \in L^2(0, T)} J_T(u) &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\exp(mt) r_{\text{ini}}(L - ct)) dt \\ &\quad + \int_{\frac{L}{c}}^T g\left(\exp\left(m\frac{L}{c}\right) u\left(t - \frac{L}{c}\right)\right) dt. \end{aligned}$$

We consider the static optimal control problem corresponding to (3) that is given by

$$\begin{cases} \min_{u \in \mathbb{R}} f(u) + g(r(L)), \\ cr_x(x) = mr(x), \\ r(0) = u. \end{cases} \quad (5)$$

The solution of the static transport equation is given by

$$r(x) = \exp\left(m\frac{x}{c}\right) u, \quad (6)$$

thus the static optimal control problem (5) is equivalent to

$$\min_{u \in \mathbb{R}} f(u) + g\left(\exp\left(m\frac{L}{c}\right) u\right).$$

Before we state the first Turnpike result in this paper we make the following assumptions:

(A1) Let the functions f and g satisfy the property

$$(f'(x_1) - f'(x_2))(x_1 - x_2) + (g'(y_1) - g'(y_2))(y_1 - y_2) \geq \varepsilon \|x_1 - x_2\|_2^2, \quad (7)$$

with constant $\varepsilon > 0$.

(A2) Let the derivative of g be Lipschitz continuous with Lipschitz constant L_k , i.e.,

$$\|g'(y_1) - g'(y_2)\|_2 \leq L_k \|y_1 - y_2\|_2. \quad (8)$$

Theorem 1. *Let the assumptions (A1) and (A2) be satisfied. Then there exist optimal controls $u^\delta(t)$ of the dynamic optimal boundary control problem (3) and u^σ of the corresponding static problem (5) that satisfy the integral turnpike property*

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2 dt \leq \mathcal{C}, \quad (9)$$

with a constant $\mathcal{C} > 0$ that is time independent.

Remark 2. The turnpike inequality (9) is equivalent to the normalized turnpike inequality

$$\frac{1}{T} \int_0^T \|u^\delta(t) - u^\sigma\|_2^2 dt \leq \frac{\mathcal{C}}{T}, \quad (10)$$

which implies that for a constant γ the turnpike inequality (10) leads to

$$\frac{1}{T} \int_0^T \gamma^2 dt = \gamma^2.$$

Remark 3. The turnpike inequality (9) implies that the dynamic optimal control converges asymptotically to the static optimal control for $T \rightarrow \infty$, i.e.,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u^\delta(t) dt = u^\sigma.$$

Remark 4. There are no restrictions on the constant m in the source term. Often in turnpike theory for hyperbolic systems the source term is required to be negative in order to avoid a blow-up in the solution of the PDE (see e.g. [4, 7]).

We mention again that the integral turnpike result stated in *Theorem 1* follows e.g. from [4, 5] but we do not consider any restrictions to the source term and we present a direct proof without using adjoint calculus here. Further due to the direct proof leads to sharper estimates.

Proof of Theorem 1. For the reader's convenience we split the proof in three parts. In the first part we compute the derivative of the objective function, in the second part we derive necessary optimality conditions and in the third part we proof the turnpike inequality (9).

Part I: In this part we compute the derivative of the objective function of the dynamic optimal boundary control problem (3). We define the constant

$$k := \exp\left(m \frac{L}{c}\right). \quad (11)$$

For $u \in L^2(0, T)$ we consider a control variation $\tilde{u}(t) = u(t) + h(t)$ with $h \in L^2(0, T)$. We have

$$\begin{aligned} J_T(\tilde{u}) &= \int_0^T f(u(t) + h(t)) dt + \int_0^{\frac{L}{c}} g(\exp(mt) r_{\text{ini}}(L - ct)) dt \\ &\quad + \int_{\frac{L}{c}}^T g\left(k \left(u\left(t - \frac{L}{c}\right) + h\left(t - \frac{L}{c}\right)\right)\right) dt. \end{aligned}$$

Using integration by substitution we have

$$\begin{aligned} J_T(\tilde{u}) &= \int_0^T f(u(t) + h(t)) dt + \int_0^{\frac{L}{c}} g(\exp(mt) r_{\text{ini}}(L - ct)) dt \\ &\quad + \int_0^{T - \frac{L}{c}} g\left(k (u(t) + h(t))\right) dt. \end{aligned}$$

Due to the convexity of f and g we have

$$\begin{aligned} J_T(\tilde{u}) &\geq \int_0^T f(u(t)) + f'(u(t)) h(t) dt + \int_0^{\frac{L}{c}} g(\exp(mt) r_{\text{ini}}(L - ct)) dt \\ &\quad + \int_0^{T - \frac{L}{c}} g(k u(t)) + k g'(k u(t)) h(t) dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} J_T(\tilde{u}) &\geq \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\exp(mt) r_{\text{ini}}(L - ct)) dt + \int_{\frac{L}{c}}^T g\left(k u\left(t - \frac{L}{c}\right)\right) dt \\ &\quad + \int_0^T f'(u(t)) h(t) dt + \int_0^{T - \frac{L}{c}} k g'(k u(t)) h(t) dt. \end{aligned} \quad (12)$$

We define the function

$$\psi : [0, T] \rightarrow \{0, 1\}, \quad \psi : t \mapsto \begin{cases} 1 & 0 \leq t \leq T - \frac{L}{c}, \\ 0 & \text{else.} \end{cases}$$

We have

$$\int_0^{T-\frac{L}{c}} k g'(k u(t)) h(t) dt = \int_0^T k \psi(t) g'(k u(t)) h(t) dt,$$

and thus (12) leads to

$$J_T(\tilde{u}) \geq J_T(u) + \left\langle f'(u(t)) + k \psi(t) g'(k u(t)), h(t) \right\rangle_{L^2(0,T)}.$$

Due to the convexity of J_T this implies that the functional derivative of J_T is given by

$$J'_T(u) = f'(u(t)) + k \psi(t) g'(k u(t)). \quad (13)$$

Note that (13) coincides with the Fréchet derivative.

Part II: In this part we derive necessary optimality conditions for the dynamic optimal boundary control problem (3) and for the corresponding static problem (5). The existence of an optimal control for the dynamic problem (3) follows by assumption (A1) and by applying the Direct Method of the Calculus of Variations (see e.g. [2]). The existence of an optimal solution of the static problem (5) also follows by assumption (A1). Also assumption (A1) guarantees uniqueness of the optimal controls.

Let $u^\delta(t) \in L^2(0, T)$ be the optimal solution of the dynamic optimal control problem (3) with corresponding optimal state $r^\delta(t, x)$ and let $u^\sigma \in \mathbb{R}$ be the optimal solution of the corresponding static optimal control problem (5) with corresponding state $r^\sigma(x)$. Then we have

$$f'(u^\delta(t)) + k \psi(t) g'(k u^\delta(t)) = 0,$$

and

$$f'(u^\sigma) + k g'(k u^\sigma) = 0.$$

Equalizing both equations leads to

$$f'(u^\delta(t)) - f'(u^\sigma) = k g'(k u^\sigma) - k \psi(t) g'(k u^\delta(t)). \quad (14)$$

Part III: Now we can proof the turnpike inequality (9). From assumption (A1) it follows

$$\begin{aligned} \varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &= \int_0^T \varepsilon \|u^\delta(t) - u^\sigma\|_2^2 dt \\ &\leq \int_0^T \left(f'(u^\delta(t)) - f'(u^\sigma) \right) \left(u^\delta(t) - u^\sigma \right) dt \\ &\quad + \int_0^T \left(g'(r^\delta(t, L)) - g'(r^\sigma(L)) \right) \left(r^\delta(t, L) - r^\sigma(L) \right) dt. \end{aligned}$$

Using the necessary optimality conditions (14) leads to

$$\begin{aligned} \varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \int_0^T \left(k g'(k u^\sigma) - k \psi(t) g'(k u^\delta(t)) \right) \left(u^\delta(t) - u^\sigma \right) dt \\ &\quad + \int_0^T \left(g'(r^\delta(t, L)) - g'(r^\sigma(L)) \right) \left(r^\delta(t, L) - r^\sigma(L) \right) dt. \end{aligned}$$

Using the definition of ψ and the solution of the transport equation (4) resp. (6) we have

$$\begin{aligned}
\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \int_0^{T-\frac{L}{c}} \left(k g'(k u^\sigma) - k g'(k u^\delta(t)) \right) \left(u^\delta(t) - u^\sigma \right) dt \\
&\quad + \int_{T-\frac{L}{c}}^T k g'(k u^\sigma) \left(u^\delta(t) - u^\sigma \right) dt \\
&\quad + \int_0^{\frac{L}{c}} \left(g'(\exp(mt) r_{\text{ini}}(L-ct)) - g'(k u^\sigma) \right) \\
&\quad \quad \cdot \left(\exp(mt) r_{\text{ini}}(L-ct) - k u^\sigma \right) dt \\
&\quad + \int_{\frac{L}{c}}^T \left(g'(k u^\delta(t - \frac{L}{c})) - g'(k u^\sigma) \right) \left(k u^\delta(t - \frac{L}{c}) - k u^\sigma \right) dt.
\end{aligned}$$

We use integration by substitution to get

$$\begin{aligned}
&\int_{\frac{L}{c}}^T \left(g'(k u^\delta(t - \frac{L}{c})) - g'(k u^\sigma) \right) \left(k u^\delta(t - \frac{L}{c}) - k u^\sigma \right) dt \\
&= \int_0^{T-\frac{L}{c}} \left(g'(k u^\delta(t)) - g'(k u^\sigma) \right) \left(k u^\delta(t) - k u^\sigma \right) dt,
\end{aligned}$$

and thus we have

$$\begin{aligned}
\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \int_{T-\frac{L}{c}}^T k g'(k u^\sigma) \left(u^\delta(t) - u^\sigma \right) dt \\
&\quad + \int_0^{\frac{L}{c}} \left(g'(\exp(mt) r_{\text{ini}}(L-ct)) - g'(k u^\sigma) \right) \\
&\quad \quad \cdot \left(\exp(mt) r_{\text{ini}}(L-ct) - k u^\sigma \right) dt.
\end{aligned} \tag{15}$$

Since both terms are L^2 scalar products we can apply *Cauchy-Schwarz* inequality. This leads to

$$\begin{aligned}
\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \|k g'(k u^\sigma)\|_{L^2(T-L/c,T)} \|u^\delta(t) - u^\sigma\|_{L^2(T-L/c,T)} \\
&\quad + \|g'(\exp(mt) r_{\text{ini}}(L-ct)) - g'(k u^\sigma)\|_{L^2(0,L/c)} \\
&\quad \quad \cdot \|\exp(mt) r_{\text{ini}}(L-ct) - k u^\sigma\|_{L^2(0,L/c)}.
\end{aligned}$$

We now apply the Lipschitz continuity stated in assumption (A2). This implies

$$\begin{aligned}
\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \|k g'(k u^\sigma)\|_{L^2(T-L/c,T)} \|u^\delta(t) - u^\sigma\|_{L^2(T-L/c,T)} \\
&\quad + L_k \|\exp(mt) r_{\text{ini}}(L-ct) - k u^\sigma\|_{L^2(0,L/c)}^2.
\end{aligned}$$

Since the term $k g'(k u^\sigma)$ does not depend on the time we have

$$z_1 := \|k g'(k u^\sigma)\|_{L^2(T-L/c,T)} = \sqrt{\frac{L}{c}} \|g'(k u^\sigma)\| > 0. \tag{16}$$

With the triangle inequality it follows

$$\begin{aligned}
\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq z_1 \|u^\delta(t) - u^\sigma\|_{L^2(T-L/c,T)} \\
&\quad + L_k \|\exp(mt) r_{\text{ini}}(L-ct)\|_{L^2(0,L/c)}^2 + \|k u^\sigma\|_{L^2(0,L/c)}^2.
\end{aligned}$$

Depending on the sign of m the term $\exp(mt)$ can be estimated by

$$0 \leq \exp(mt) \leq z_2 := \begin{cases} 1 & m \leq 0, \\ \exp\left(m\frac{L}{c}\right) & m > 0. \end{cases} \tag{17}$$

Thus we have

$$\begin{aligned} \varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq z_1 \|u^\delta(t) - u^\sigma\|_{L^2(T-L/c,T)} \\ &\quad + L_k z_2^2 \|r_{\text{ini}}(L-ct)\|_{L^2(0,L/c)}^2 + \|k u^\sigma\|_{L^2(0,L/c)}^2. \end{aligned}$$

We define the constant

$$z_3 := \frac{L}{c} |k u^\sigma|^2 = \|k u^\sigma\|_{L^2(0,L/c)}^2 \geq 0. \quad (18)$$

With

$$\|u^\delta(t) - u^\sigma\|_{L^2(T-L/c,T)} \leq \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} \quad \text{and} \quad \|r_{\text{ini}}(L-ct)\|_{L^2(0,L/c)}^2 = \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2, \quad (19)$$

we have

$$\varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 \leq z_1 \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} + L_k z_2^2 \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3,$$

which for $\hat{u} := \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}$ is equivalent to the quadratic polynomial

$$\varepsilon \hat{u}^2 - z_1 \hat{u} - \left(L_k z_2^2 \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3 \right) \leq 0. \quad (20)$$

Since the term $L_k z_2^2 \|r_{\text{ini}}\|_{L^2(0,L)}^2 + z_3$ is larger than or equal to zero, since ε is positive and since equation (20) holds for $\hat{u} = 0$ we can find an interval $[0, \sqrt{\mathcal{C}}]$, such that

$$\hat{u} = \|u^\delta(t) - u^\sigma\|_{L^2(0,T)} \leq \sqrt{\mathcal{C}},$$

and thus the theorem is proven. \square

Remark 5. The constant \mathcal{C} is time independent and it is given by

$$\mathcal{C} = \left[\frac{z_1 + \sqrt{z_1^2 + 4 \varepsilon \left(L_k z_2^2 \|r_{\text{ini}}(x)\|_{L^2(0,L)}^2 + z_3 \right)}}{2 \varepsilon} \right]^2,$$

where k is defined in (11), z_1 is defined in (16), z_2 is defined in (17) and z_3 is defined in (18).

The assumption (A1) is essential for the existence of optimal controls in terms of the Direct Method of Calculus of Variations. The assumption holds e.g. if f is strongly convex. Further in (15) in the proof one can see that the constant \mathcal{C} is completely defined by the behaviour of the transport equation in the time intervals $[0, L/c]$ and $[T - L/c, T]$. This behaviour can also be seen in by following the characteristic curves as it is shown in *Figure 1*.

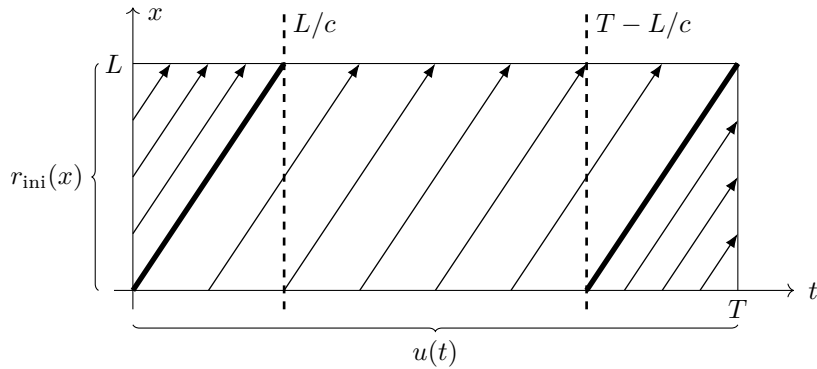


Figure 1: Characteristics of (1) on $[0, T] \times [0, L]$

Example At the end of this section we present an example. For $x \in \mathbb{R}$ we consider the functions

$$f(x) = \omega_1 x^2 \quad \text{and} \quad g(x) = \omega_2 (x - r_D)^2,$$

where r_D is a desired state. We consider the optimal boundary control problem (3) with the constants and variables given in *Table 1*. The initial condition is given by the solution of the stationary problem corresponding to the initial control u_0

$$\begin{cases} c r_x(x) = m r(x), \\ r(0) = u_0. \end{cases}$$

Variable	Letter	Value
speed of propagation	c	$10 [m/s]$
right hand side weight	m	$-5 \cdot 10^{-4} [1/s]$
time horizon	T	$60^2 [s]$
space horizon	L	$10^4 [m]$
weight 1	ω_1	1
weight 2	ω_2	5
desired state	r_D	25
initial control	u_0	25

Table 1: Values for the deterministic example

We assume that the control and state variable are non-dimensionalized. For the time and space discretization we consider 401 equidistant grid points in time and 21 equidistant grid points in space. The transport equation is solved using an upwind scheme in space and an implicit Euler method in time. As mentioned in *Section 1* we solve the optimal control problem using the *AMPL* software with the *IPOPT* and the *MUMPS* solver. The pictures and simulation of the corresponding optimal states are done in *MATLAB*[®] 2019a. The solution is shown in *Figure 2*.

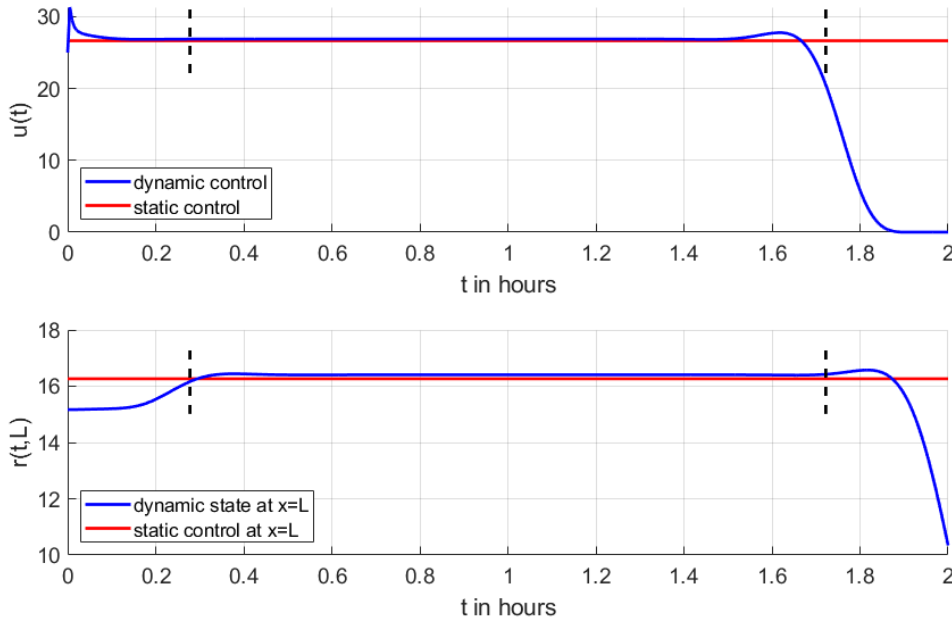


Figure 2: Optimal control (upper picture) and corresponding optimal states at $x = L$ (lower picture)

The results shown in *Figure 2* coincide with the statement of *Theorem 1*, especially with (15) in the proof of *Theorem 1*: The difference between the dynamic and the static solution depends on

the time intervals $[0, L/c]$ and $[0, T - L/c]$. Further for $t > T - L/c$ due to the hyperbolic character of the transport equation the information provided by the control does not reach the $x = L$, so the control is not active anymore.

3 Uncertain Initial Data

In this section we consider an optimal boundary control problem with uncertain initial data given by

$$\left\{ \begin{array}{l} \min_{u \in L^2(0,T)} \quad J_T(u) = \int_0^T f(u(t)) + g(\mathbb{E}[r(t, L)]) dt, \\ \text{s.t.} \quad r(0, x) = r_{\text{ini}}^\omega(x), \\ \quad \quad r_t(t, x) + cr_x(t, x) = mr(t, x), \\ \quad \quad r(t, 0) = u(t), \end{array} \right. \quad (21)$$

where r_{ini}^ω is defined by a space dependent random variable. Motivated by applications the random variable often is not completely unknown than rather given by some expected value. We model the uncertainty here by an expected deterministic state and an additional term given by a *Wiener process* $(W_x)_{x \geq 0}$ (see e.g. [10]) on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$. So we have

$$r_{\text{ini}}^\omega(x) = r_{\text{ini}}(x) + W_x. \quad (22)$$

One might also have a look at [11] which gives an excellent introduction and overview on probability theory and stochastic processes. Since the paths of a Wiener process are \mathbb{P} -almost surely continuous, regularity regarding the existence of solutions for the transport equation with random initial state does not cause problems. A Wiener process can be represented as random Fourier series. Let ξ_1, ξ_2, \dots be independent standard normal distributed random numbers. Then

$$W_x = \sqrt{2} \sum_{k=1}^{\infty} \xi_k \frac{\sin\left(\left(k - \frac{1}{2}\right)\pi x\right)}{\left(k - \frac{1}{2}\right)\pi},$$

represents a Wiener process on $[0, 1]$ and $\sqrt{L} W_{x/L}$ represents a Wiener process on $[0, L]$. For the simulation the sum can be cut after N_F terms. A way of simulating the discretized paths of a Wiener process is a Gaussian random walk. For an equidistant space grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = L$ with $x_k - x_{k-1} = \Delta x$ and independent standard normal distributed random numbers ξ_1, \dots, ξ_n a path of a Wiener process is given by

$$W_0 = 0 \quad \text{and} \quad W_{n\Delta x} = \sqrt{\Delta x} \sum_{k=1}^n \xi_k.$$

Both representations are shown in *Figure 3* for $L = 10$, $n = 101$, $\Delta x = 0.1$ and $N_F = 10$.

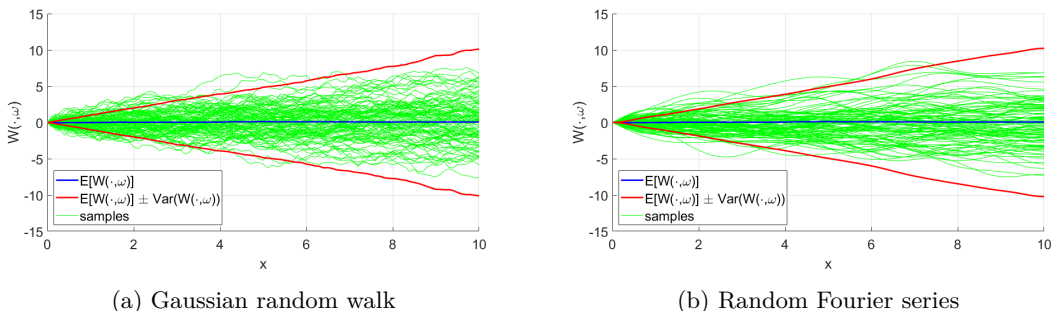


Figure 3: Simulation of 100 paths of a Wiener process by Gaussian random walk (a) and random Fourier series (b)

The fact that $W_0 = 0$ allows us to demand compatibility between initial and boundary condition. Due to the hyperbolic structure of (21) and due to $c > 0$ the information given by the initial

condition is transported through the system and vanishes at the boundary $c = L$ after $t = L/c$ (see *Figure 1*). So we can formulate the following statement:

Theorem 6. *Let the assumptions (A1) and (A2) be satisfied. Then there exist optimal controls $u^\delta(t)$ of the dynamic optimal boundary control problem (21) with uncertain data given by the Wiener process and u^σ of the corresponding static problem (5) that satisfy the integral turnpike property*

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2 dt \leq \mathcal{C}, \quad (23)$$

where the time independent constant \mathcal{C} is given in *Remark 5*.

Proof. For the objective function in (21) we have

$$\begin{aligned} J_T(u) &= \int_0^T f(u(t)) + g(\mathbb{E}(r(t, L))) dt \\ &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\mathbb{E}[\exp(mt)r_{\text{ini}}^\omega(L - ct)]) dt \\ &\quad + \int_{\frac{L}{c}}^T g(\mathbb{E}[\exp\left(m\frac{L}{c}\right)u\left(t - \frac{L}{c}\right)]) dt \end{aligned}$$

Since the last part does not depend on the uncertainty, we have

$$J_T(u) = \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\mathbb{E}[\exp(mt)r_{\text{ini}}^\omega(L - ct)]) dt + \int_{\frac{L}{c}}^T g\left(\exp\left(m\frac{L}{c}\right)u\left(t - \frac{L}{c}\right)\right) dt$$

For the term depending on the uncertainty we have

$$\begin{aligned} \mathbb{E}[\exp(mt)r_{\text{ini}}^\omega(L - ct)] &= \exp(mt) \mathbb{E}[r_{\text{ini}}(L - ct) + W_{L-ct}] \\ &= \exp(mt) \mathbb{E}[r_{\text{ini}}(L - ct)] + \exp(mt) \mathbb{E}[W_{L-ct}] \end{aligned}$$

Since r_{ini} is deterministic and since Wiener processes satisfy $\mathbb{E}[W_x] = 0$ for all $x \geq 0$ we have

$$\mathbb{E}[\exp(mt)r_{\text{ini}}^\omega(L - ct)] = \exp(mt) r_{\text{ini}}(L - ct),$$

and consequently

$$J_T(u) = \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\exp(mt)r_{\text{ini}}(L - ct)) dt + \int_{\frac{L}{c}}^T g\left(\exp\left(m\frac{L}{c}\right)u\left(t - \frac{L}{c}\right)\right) dt.$$

Thus the result of *Theorem 1* can be applied here and the turnpike inequality (23) holds with constant \mathcal{C} given in *Remark 5*. \square

Example: We consider almost the same example as above in *Section 2*. The only difference is that we add a Wiener process simulated with a Gaussian random walk to the deterministic initial data. All values and constants except the initial data are equal to the values and constants in the example of *Section 2*, given in *Table 1*. The results are shown in *Figure 4*.

The result of *Theorem 6* imply that the optimal controls of the dynamic optimal control problems (3) and (21) are equal. Since the initial condition has no influence on the solution at $x = L$ after $t = L/c$, different initial scenarios lead to different states only in $[0, L/c]$ (cf. *Figure 4*).

Since in this setting only the initial data is uncertain, the uncertainty in the system vanishes for times larger than L/c . So another approach would be to exclude the uncertainty part from the objective function and consider the optimal control problem

$$\left\{ \begin{array}{l} \min_{u \in L^2(0, T)} J_T(u) = \int_0^T f(u(t)) dt + \int_{\frac{L}{c}}^T g(r(t, L)) dt, \\ \text{s.t. } r(0, x) = r_{\text{ini}}^\omega(x), \\ r_t(t, x) + cr_x(t, x) = mr(t, x), \\ r(t, 0) = u(t). \end{array} \right. \quad (24)$$

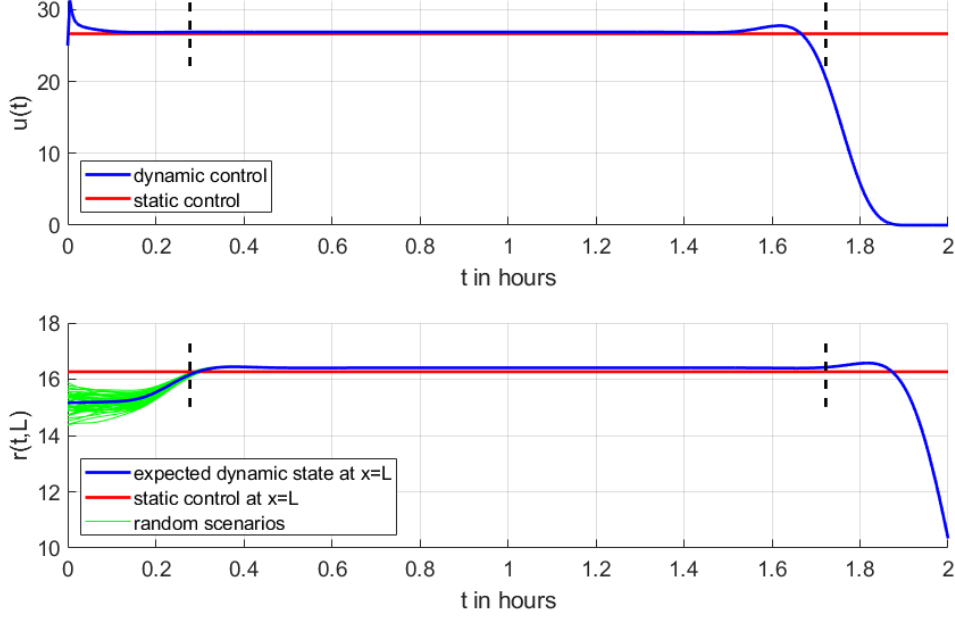


Figure 4: Optimal control (upper picture) and corresponding optimal states at $x = L$ (lower picture)

Then the derivative of J_T is given by (13) with

$$\psi(t) = \begin{cases} 1 & \frac{L}{c} \leq t \leq T - \frac{L}{c}, \\ 0 & \text{else,} \end{cases}$$

and instead of equation (15) we have

$$\begin{aligned} \varepsilon \|u^\delta(t) - u^\sigma\|_{L^2(0,T)}^2 &\leq \int_0^{\frac{L}{c}} k g'(k u^\sigma) (u^\delta(t) - u^\sigma) dt + \int_{T-\frac{L}{c}}^T k g'(k u^\sigma) (u^\delta(t) - u^\sigma) dt \\ &+ \int_0^{\frac{L}{c}} \left(g'(\exp(mt) r_{\text{ini}}(L - ct)) - g'(k u^\sigma) \right) \\ &\quad \cdot \left(\exp(mt) r_{\text{ini}}(L - ct) - k u^\sigma \right) dt. \end{aligned}$$

Due to the estimate (19) this leads to the same constants and the turnpike inequality (23) holds as well.

4 Random Source Term

In this section we consider uncertainty in the source term of the PDE. Consider a random variable ξ with absolutely continuous probability distribution and probability density function ϱ_ξ on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that ξ is integrable w.r.t. the Lebesgue-measure λ , i.e., the expected value of ξ exists. Let $m^\omega = \xi(\omega)$ be a realization of the random variable. We consider the linear transport equation with uncertain source term

$$\begin{cases} r_t^\omega(t, x) + c r_x^\omega(t, x) = m^\omega r(t, x), & \omega \in \Omega \\ r^\omega(0, x) = r_{\text{ini}}(x), \\ r^\omega(t, 0) = u(t). \end{cases} \quad (25)$$

In contrast to *Section 3* the system (25) is uncertain for all $(t, x) \in [0, T] \times [0, L]$. Since $m^\omega \in \mathbb{R}$, for $r_{\text{ini}} \in L^2(0, L)$ and $u \in L^2(0, T)$ a solution of (25) exists \mathbb{P} -almost surely in $C([0, T], L^2(0, L))$. The solution is given by

$$r(t, x) = \begin{cases} \exp(m^\omega t) r_{\text{ini}}(x - ct) & \text{for } x > ct, \\ \exp\left(m^\omega \frac{x}{c}\right) u\left(t - \frac{x}{c}\right) & \text{for } x \leq ct. \end{cases} \quad (26)$$

For J_T as in (2) we consider the dynamic optimal boundary control problem

$$\left\{ \begin{array}{l} \min_{u \in L^2(0, T)} J_T(u) = \int_0^T f(u(t)) + g(\mathbb{E}[r(t, L)]) dt \\ \text{s.t. } r_t(t, x) + cr_x(t, x) = m^\omega r(t, x), \\ r(0, x) = r_{\text{ini}}(x), \\ r(t, 0) = u(t), \end{array} \right. \quad (27)$$

and the corresponding static optimal control problem

$$\left\{ \begin{array}{l} \min_{u \in \mathbb{R}} f(u) + g(\mathbb{E}[r(L)]), \\ cr_x(x) = m^\omega r(x), \\ r(0) = u. \end{array} \right. \quad (28)$$

Due to the solution of the transport equation (26) the dynamic optimal control problem (27) is equivalent to

$$\begin{aligned} \min_{u \in L^2(0, T)} J_T(u) &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g\left(\mathbb{E}\left[\exp(m^\omega t) r_{\text{ini}}(L - ct)\right]\right) dt \\ &\quad + \int_{\frac{L}{c}}^T g\left(\mathbb{E}\left[\exp\left(m^\omega \frac{L}{c}\right) u\left(t - \frac{L}{c}\right)\right]\right) dt. \end{aligned}$$

The solution of the static transport equation is given by

$$r(x) = \exp\left(m^\omega \frac{x}{c}\right) u, \quad (29)$$

thus the static optimal control problem (28) is equivalent to

$$\min_{u \in \mathbb{R}} f(u) + g\left(\mathbb{E}\left[\exp\left(m^\omega \frac{L}{c}\right) u\right]\right).$$

Before we state a turnpike result for the optimal boundary control problem (27) and (28) we need to make another assumption. Define the function

$$e_0(t) : [0, T] \rightarrow \mathbb{R} \cup \{\pm\infty\}, \quad t \mapsto \int_{-\infty}^{\infty} \exp(zt) \varrho_\xi(z) dz, \quad (30)$$

where ϱ_ξ is the probability density function of the random variable ξ , and define the number

$$e_1 := \int_{-\infty}^{\infty} \exp\left(z \frac{L}{c}\right) \varrho_\xi(z) dz. \quad (31)$$

Both e_0 and e_1 are obviously positive.

(A3) We assume that

$$e_0(t) \leq e_2 \in \mathbb{R} < \infty \quad \forall t \in [0, L/c], \quad (32)$$

(A4) We assume that

$$e_1 < \infty. \quad (33)$$

Theorem 7. *Let the assumptions (A1) and (A2) be satisfied. Then there \mathbb{P} -almost surely exist optimal controls $u^\delta(t)$ of the dynamic optimal boundary control problem (27) and u^σ of the corresponding static problem (28) that satisfy the integral turnpike property*

$$\int_0^T \|u^\delta(t) - u^\sigma\|_2^2 dt \leq \mathcal{C}_2, \quad (34)$$

with a constant $\mathcal{C}_2 > 0$ that is time independent.

Proof. Consider the objective function $J_T(u)$. We have

$$\begin{aligned} J_T(u) &= \int_0^T f(u(t)) + g(\mathbb{E}[r(t, L)]) dt \\ &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(\mathbb{E}[r(t, L)]) dt + \int_{\frac{L}{c}}^T g(\mathbb{E}[r(t, L)]) dt. \end{aligned}$$

For a random variable X and a function h we have (see [8])

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(z) \varrho_X(z) dz,$$

where ϱ_X is the probability density function of the random variable X . Thus it follows

$$\begin{aligned} J_T(u) &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g\left(\int_{-\infty}^{\infty} \exp(zt) r_{\text{ini}}(L - ct) \varrho_\xi(z) dz\right) dt \\ &\quad + \int_{\frac{L}{c}}^T g\left(\int_{-\infty}^{\infty} \exp\left(z\frac{L}{c}\right) u\left(t - \frac{L}{c}\right) \varrho_\xi(z) dz\right) dt \\ &= \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g\left(r_{\text{ini}}(L - ct) \int_{-\infty}^{\infty} \exp(zt) \varrho_\xi(z) dz\right) dt \\ &\quad + \int_{\frac{L}{c}}^T g\left(u\left(t - \frac{L}{c}\right) \int_{-\infty}^{\infty} \exp\left(z\frac{L}{c}\right) \varrho_\xi(z) dz\right) dt, \end{aligned}$$

and with the definitions of $e_0(t)$ and e_1 given in (30) and (31) we have

$$J_T(u) = \int_0^T f(u(t)) dt + \int_0^{\frac{L}{c}} g(e_0(t) r_{\text{ini}}(L - ct)) dt + \int_{\frac{L}{c}}^T g\left(e_1 u\left(t - \frac{L}{c}\right)\right) dt.$$

Then the proof is analogous to the proof of *Theorem 1* except the constants. We have e_1 instead of k here ($e_1 < \infty$ due to assumption (A4)) and we define

$$\begin{aligned} \tilde{z}_1 &:= \|e_1 g'(e_1 u^\sigma)\|_{L^2(T-L/c, T)} = \sqrt{\frac{L}{c}} e_1 |g'(e_1 u^\sigma)| > 0, \\ \tilde{z}_2 &:= e_2, \\ \tilde{z}_3 &:= \frac{L}{c} |e_1 u^\sigma|^2 = \|e_1 u^\sigma\|_{L^2(0, L/c)}^2 \geq 0. \end{aligned}$$

Note that \tilde{z}_2 exists due to assumption (A3). Thus there exists a time independent constant \mathcal{C}_2 given by

$$\left[\frac{\tilde{z}_1 + \sqrt{\tilde{z}_1^2 + 4 \varepsilon \left(L_k \tilde{z}_2^2 \|r_{\text{ini}}(x)\|_{L^2(0, L)}^2 + \tilde{z}_3 \right)}}{2 \varepsilon} \right]^2,$$

s.t. the turnpike inequality (34) holds and the proof is complete. \square

The assumptions (A3) and (A4) might look strong at first glance but they basically guarantee that the expected values even exist, as it is the case for most of the common distributions. In the following part we will specify the constants for a Gaussian distributed and for a uniform distributed random variable. For this we first proof an auxiliary lemma.

Lemma 8. For constants $a_0, a_1, a_2 \in \mathbb{R}$ we have

$$\int \exp\left(a_0 + a_1 x - a_2 x^2\right) dx = \sqrt{\frac{\pi}{4a_2}} \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \operatorname{erf}\left(\sqrt{a_2} x - \frac{a_1}{2\sqrt{a_2}}\right) + \gamma,$$

where erf is the Gauss error function (see e.g., [1]) and $\gamma \in \mathbb{R}$ is some constant.

Proof. We have

$$\begin{aligned} \frac{d}{dx} \left[\sqrt{\frac{\pi}{4a_2}} \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \operatorname{erf}\left(\sqrt{a_2} x - \frac{a_1}{2\sqrt{a_2}}\right) + \gamma \right] \\ = \sqrt{\frac{\pi}{4a_2}} \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \frac{d}{dx} \left[\operatorname{erf}\left(\sqrt{a_2} x - \frac{a_1}{2\sqrt{a_2}}\right) \right]. \end{aligned}$$

The derivative of the Gauss error function is given by

$$\frac{d}{dx} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \exp(-x^2).$$

Thus it follows

$$\begin{aligned} \frac{d}{dx} \left[\sqrt{\frac{\pi}{4a_2}} \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \operatorname{erf}\left(\sqrt{a_2} x - \frac{a_1}{2\sqrt{a_2}}\right) + \gamma \right] \\ = \sqrt{\frac{\pi}{4a_2}} \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \exp\left(-\left(\sqrt{a_2} x - \frac{a_1}{2\sqrt{a_2}}\right)^2\right) \frac{2}{\sqrt{\pi}} \sqrt{a_2} \\ = \exp\left(a_0 + \frac{a_1^2}{4a_2}\right) + \exp\left(-\left(a_2 x^2 - 2\sqrt{a_2} \frac{a_1}{2\sqrt{a_2}} x + \frac{a_1^2}{4a_2}\right)\right) \\ = \exp\left(a_0 + \frac{a_1^2}{4a_2} - a_2 x^2 + a_1 x - \frac{a_1^2}{4a_2}\right) \\ = \exp\left(a_0 + a_1 x - a_2 x^2\right). \end{aligned}$$

□

Lemma 9. Let ξ be Gaussian distributed with mean value $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$, i.e., $\xi \sim \mathcal{N}(\mu, \sigma^2)$. Then the constant e_2 in assumption (A3) is given by

$$e_2 := \max\left\{1, \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right)\right\},$$

and the constant e_1 in assumption (A4) is given by

$$e_1 := \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right).$$

Proof. The probability density function of the Gaussian distribution with mean value $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$ is given by

$$\varrho_\xi(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right).$$

So for $e_0(t)$ defined in (30) we have

$$\begin{aligned}
e_0(t) &= \int_{-\infty}^{\infty} \exp(z t) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) dz \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \lim_{a \rightarrow \infty} \int_{-a}^a \exp(z t) \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) dz \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \lim_{a \rightarrow \infty} \int_{-a}^a \exp\left(t z - \frac{1}{2\sigma^2} z^2 + \frac{\mu}{\sigma^2} z - \frac{\mu^2}{2\sigma^2}\right) dz \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \lim_{a \rightarrow \infty} \int_{-a}^a \exp\left(b_0 + b_1(t) z - b_2 z^2\right) dz,
\end{aligned}$$

with the values

$$b_0 := -\frac{\mu^2}{2\sigma^2}, \quad b_1(t) := \frac{\mu}{\sigma^2} + t \quad \text{and} \quad b_2 := \frac{1}{2\sigma^2}.$$

An antiderivative is given by *Lemma 8*, so it follows

$$\begin{aligned}
e_0(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \lim_{a \rightarrow \infty} \left[\sqrt{\frac{\pi}{4b_2}} \exp\left(b_0 + \frac{b_1^2(t)}{4b_2}\right) \operatorname{erf}\left(\sqrt{b_2} z - \frac{b_1(t)}{2\sqrt{b_2}}\right) \right]_{z=-a}^{z=a} \\
&= \frac{1}{2\sqrt{2\sigma^2 b_2}} \exp\left(b_0 + \frac{b_1^2(t)}{4b_2}\right) \lim_{a \rightarrow \infty} \left[\underbrace{\operatorname{erf}\left(\sqrt{b_2} a - \frac{b_1(t)}{2\sqrt{b_2}}\right)}_{\rightarrow 1} - \underbrace{\operatorname{erf}\left(-\sqrt{b_2} a - \frac{b_1(t)}{2\sqrt{b_2}}\right)}_{\rightarrow -1} \right] \\
&= \sqrt{\frac{1}{2\sigma^2 b_2}} \exp\left(b_0 + \frac{b_1^2(t)}{4b_2}\right).
\end{aligned}$$

Inserting the values b_0 , $b_1(t)$ and b_2 leads to

$$e_0(t) = \sqrt{\frac{1}{2\sigma^2 \frac{1}{2\sigma^2}}} \exp\left(-\frac{\mu^2}{2\sigma^2} + \frac{\sigma^2}{2}\left(\frac{\mu^2}{4\sigma^4} + 2\frac{\mu}{\sigma^2} t + t^2\right)\right) = \exp\left(\mu t + \frac{\sigma^2}{2} t^2\right).$$

For $t \in [0, L/c]$ this can be estimated by

$$e_0(t) \leq \max\left\{1, \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right)\right\} := e_2,$$

depending on the sign and the value of μ . Equivalently we get

$$e_1 = \int_{-\infty}^{\infty} \exp\left(z \frac{L}{c}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) dz = \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right),$$

and the lemma is proven. \square

From the proof of *Lemma 9* we can explicitly state the expected value of $r(t, L)$ for Gaussian distributed random variables:

Corollary 10. *Let ξ be Gaussian distributed with mean value $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$, i.e., $\xi \sim \mathcal{N}(\mu, \sigma^2)$. Then for the solution $r(t, x)$ of the transport equation at $x = L$ with boundary control $u^\delta(t)$ we have*

$$\mathbb{E}[r(t, L)] = \begin{cases} \exp\left(\mu t + \frac{\sigma^2}{2} t^2\right) r_{ini}(L - ct) & 0 \leq t < \frac{L}{c}, \\ \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right) u^\delta\left(t - \frac{L}{c}\right) & \frac{L}{c} \leq t \leq T, \end{cases}$$

and for the solution $r(x)$ of the static transport equation at $x = L$ with boundary control u^σ we have

$$\mathbb{E}[r(L)] = \exp\left(\mu \frac{L}{c} + \frac{\sigma^2 L^2}{2 c^2}\right) u^\sigma.$$

Lemma 11. Let ξ be uniformly distributed on $[a, b]$, i.e., $\xi \sim \mathcal{U}([a, b])$. Then the constants e_2 and e_1 in assumptions (A3) and (A4) are given by

$$e_1 = e_2 = \frac{c}{L} \frac{1}{b-a} \left[\exp\left(b\frac{L}{c}\right) - \exp\left(a\frac{L}{c}\right) \right].$$

Proof. The probability density function of the uniform distribution on $[a, b]$ is given by

$$\varrho_\xi(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \leq b, \\ 0 & \text{else.} \end{cases}$$

Thus we have

$$e_0(t) = \int_{-\infty}^{\infty} \exp(zt) \varrho_\xi(z) dz = \frac{1}{b-a} \int_a^b \exp(zt) dz.$$

Since we have

$$\int_a^b \exp(zt_1) dz < \int_a^b \exp(zt_2) dz \quad \text{for } 0 \leq t_1 < t_2,$$

the function $e_0(t)$ is monotonously increasing on $[0, L/c]$ and we have

$$\begin{aligned} e_0(t) &\leq e_0\left(\frac{L}{c}\right) = \frac{1}{b-a} \int_a^b \exp\left(z\frac{L}{c}\right) dz \\ &= \frac{1}{b-a} \left[\frac{c}{L} \exp\left(z\frac{L}{c}\right) \right]_{z=a}^{z=b} \\ &= \frac{c}{L} \frac{1}{b-a} \left[\exp\left(b\frac{L}{c}\right) - \exp\left(a\frac{L}{c}\right) \right] =: e_2. \end{aligned}$$

Since $e_0(L/c) = e_1$ the same holds for e_1 and the lemma is proven. \square

From the proof of *Lemma 11* we can explicitly state the expected value of $r(t, L)$ for uniformly distributed random variables:

Corollary 12. Let ξ be uniformly distributed on $[a, b]$, i.e., $\xi \sim \mathcal{U}([a, b])$. Then for the solution $r(t, x)$ of the transport equation at $x = L$ with boundary control $u^\delta(t)$ we have

$$\mathbb{E}[r(t, L)] = \begin{cases} \frac{1}{b-a} \left[\exp(bt) - \exp(at) \right] \frac{1}{t} r_{ini}(L-ct) & 0 \leq t \leq \frac{L}{c}, \\ \frac{1}{b-a} \left[\exp\left(b\frac{L}{c}\right) - \exp\left(a\frac{L}{c}\right) \right] \frac{c}{L} u^\delta\left(t - \frac{L}{c}\right) & \frac{L}{c} \leq t \leq T - \frac{L}{c} \end{cases}$$

and for the solution $r(x)$ of the static transport equation at $x = L$ with boundary control u^σ we have

$$\mathbb{E}[r(L)] = \frac{1}{b-a} \left[\exp\left(b\frac{L}{c}\right) - \exp\left(a\frac{L}{c}\right) \right] \frac{c}{L} u^\sigma.$$

Example We consider almost the same example as in *Section 2* except the right-hand-side constant m^ω which is random here. Let a Gaussian distributed random variable ξ be given with

$$\xi \sim \mathcal{N}(-5 \cdot 10^{-4}, 0.001), \quad \text{with } m^\omega = \xi(\omega).$$

All remaining values and constants are given in *Table 1*. The results are shown in *Figure 5*. One can see the turnpike structure of the deterministic control as well as of the expected states at $x = L$ and the scenarios for different realizations of m^ω .

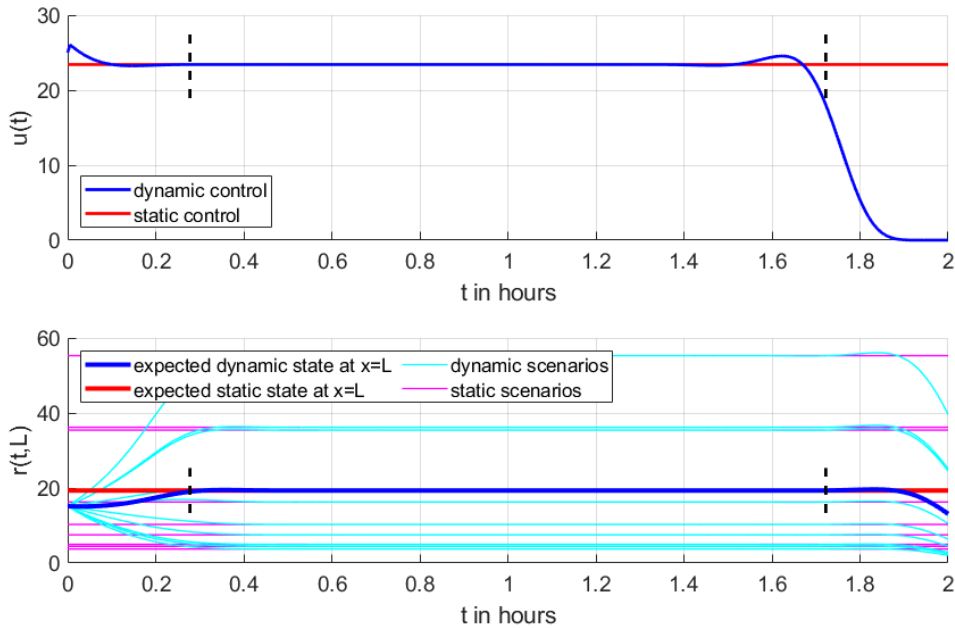


Figure 5: Optimal control (upper picture) and corresponding expected optimal state at $x = L$ (lower picture). The lines in cyan and magenta show 10 dynamic resp. static scenarios for different values of m^ω .

5 Conclusion

In this paper we have proven turnpike results for optimal boundary control problems for the linear transport equation with and without uncertainty. In *Section 2* we have considered a deterministic optimal boundary control problem for the transport equation with linear source term and we have proven an integral turnpike inequality (see *Theorem 1*) exploiting the explicit solution of the transport equation. As it was mentioned several times before, even if this result is not new (see e.g. [4, 5]) we presented a new proof based on less assumptions, that led to sharper estimates for the time independent constant in (1).

In *Section 3* and *Section 4* we have considered uncertainty in the system. First we considered uncertain initial data in which the initial condition was given by a deterministic function that is perturbed by a Wiener process. We have shown in *Theorem 6* that the deterministic turnpike inequality from (23) holds with the same constant as in the deterministic setting. Then we have considered an optimal boundary control problem for the transport equation with random source term. The source term was multiplied with a random variable while the state in the objective function was replaced with the expected state. In *Theorem 7* we have shown that the integral turnpike inequality from the deterministic case holds as well with another constant. Further we have specified the time independent turnpike constant for Gaussian and uniformly distributed random variables.

These results can be generalized in several directions. Instead of the transport equation we can consider any scalar hyperbolic partial differential equation where an explicit solution is known (even nonlinear ones), merely the estimates in the proofs and thus the time independent turnpike constant changes.

For the part in *Section 3* one can assume deterministic initial data that is multiplied with a random variable, as we did for the source term. The other way around one can consider a deterministic source term for the transport equation that is perturbed by a Wiener process, as we did for the initial data.

Even a coupling of transport equations is possible with wisely chosen coupling conditions, but one

has to be aware of the splitting of the time interval for the solution of the transport equation as we did in the proofs of the turnpike theorems. An extension to systems of hyperbolic PDEs is interesting and might be possible but explicit solutions for systems of PDEs are rarely known.

Acknowledgements

This work was supported by Deutsche Forschungsgemeinschaft (DFG) in the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Project C03, Projektnummer 239904186.

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