

OBSERVER-BASED DATA ASSIMILATION FOR BAROTROPIC GAS TRANSPORT USING DISTRIBUTED MEASUREMENTS

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ABSTRACT. We consider a state estimation problem for gas pipeline flow modeled by the one-dimensional barotropic Euler equations. In order to reconstruct the system state, we construct an observer system of Luenberger type based on distributed measurements of one state variable. First, we show the existence of Lipschitz-continuous semi-global solutions of the observer system and of the original system for initial and boundary data satisfying smallness and compatibility conditions for a single pipe and for general networks. Second, based on an extension of the relative energy method we prove that the state of the observer system converges exponentially in the long time limit towards the original system state. We show this for a single pipe and for star-shaped networks.

Keywords: data assimilation, observer systems, barotropic flow, Euler equations, relative energy estimates, networks, well-posedness

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1. INTRODUCTION

The goal of data assimilation is to estimate the current state of some system by combining available measurement data with a physical model of the system [1, 2]. This is important, for example, if control problems are considered that need the current system state as input data. Here, we will consider systems that are described by nonlinear hyperbolic balance laws. In many applications such systems are considered on networks, for a survey on balance laws on networks see [3]. Examples are the Saint-Venant system modeling water flow through open channels, the Aw-Rascle-Zhang model for traffic flow and the barotropic Euler equations modeling gas transport through pipes, where our long-term goal is to study data assimilation for these problems on networks.

There are different approaches for data assimilation, which can mainly be grouped into variational approaches, statistical approaches such as Kalman filtering and observer-based approaches. In this paper we will use the observer-based approach, which means that an observer system is constructed that contains, in comparison to the original system, some additional source terms or boundary conditions accounting for the measurement data. Then, the main goal is to show the synchronization, i.e., the convergence of the observer system towards the original system state. Proving synchronization of observer systems can be viewed as a control problem, where the difference system has to be controlled to zero. A particular difficulty arises from the

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nonlinearity of the balance laws, since for nonlinear systems the difference system does not have the same structure as the original system. To deal with this challenge we construct Lyapunov functions based on relative energy.

Similar to other control problems, we can distinguish between boundary observers based on boundary measurements (cf. [4] for an example for a semilinear model for gas transport), and distributed observers, where distributed measurements are inserted into the observer system through suitable source terms. Synchronization of a boundary observer for linear hyperbolic systems was shown in [5]. An important class of observers for distributed measurements are Luenberger-type observers [6], where the inserted source terms are proportional to the difference between the state of the observer system and the measurements, see e.g. [7] for an application to the Saint-Venant system using a Luenberger-type observer for a kinetic representation. Other works using Luenberger-type observers include [8], where synchronization of an observer for the 2D Bénard convection problem is investigated numerically for coarse-grained measurements of one variable, [9] and [10], where exponential synchronization of an observer for the surface quasi-geostrophic equation and the wave equation, respectively, is shown, and [11], where error estimates for a mixed finite element discretization of an observer system for the Navier-Stokes equation are provided. Luenberger observers for linear hyperbolic systems of second order are investigated in [12] and [13], where in the latter work measurement data are incorporated into the discretization of wave-like equations in order to guarantee bounded approximation errors. Joint state-parameter estimation for a front-tracking problem using a Luenberger observer for state estimation and a Kalman filter for parameter estimation is studied in [14] and in [15] simultaneous state and parameter estimation is investigated for wave equations and exponential synchronization is shown. Luenberger observers can also be designed for boundary measurements, see e.g. [16], where a boundary Luenberger observer is introduced for a linearized model of pipeline flow.

In the following, we will consider observer-based data assimilation for systems modeled by the one-dimensional barotropic Euler equations

$$\partial_t \rho + \partial_x m = 0 \tag{1}$$

$$\partial_t m + \partial_x \left(\frac{m^2}{\rho} + p(\rho) \right) = -\gamma \frac{|m|m}{\rho} \tag{2}$$

for $0 < x < \ell$, $t > 0$ allowing for a nonlinear friction term on the right-hand side. Here, the density ρ and the mass flow rate $m = \rho v$ with the velocity v are the unknowns and the system is complemented by a strictly monotone pressure law $p(\rho)$ and a friction coefficient $\gamma \geq 0$. This system describes gas transport through gas pipes, where the friction term results from the friction at the pipe walls (cf. [17]). Let us note that for a pressure law given by $p(\rho) = \frac{g}{2}\rho^2$ with g the gravity of earth the system (without friction) is equivalent to the Saint-Venant system for flat bathymetry and the system is also similar to the Aw-Rascle-Zhang model (see e.g. [18]) for traffic flow.

In order to investigate the convergence of the state of the observer system towards the original solution, we have to bound the difference between two solutions. There are different frameworks for this in the case of systems of nonlinear hyperbolic balance laws: The L^1 -stability framework by Bressan [19], which needs a small total variation condition, and the relative energy framework, which we will use in the following. The relative energy defined with respect to conservative variables was introduced in [20] and was applied to a variety of thermo-mechanical theories of fluids in [21]. If systems on networks are studied, it turns out, see [22], that it is more suitable to use the relative energy with respect to the non-conservative variables (ρ, v) . Therefore, we

rewrite system (1)-(2) in the Hamiltonian form

$$\partial_t \rho + \partial_x m = 0 \quad (3)$$

$$\partial_t v + \partial_x h = -\gamma |v|v \quad (4)$$

with the specific enthalpy $h(\rho, v) := \frac{1}{2}v^2 + P'(\rho)$, a smooth and strictly convex pressure potential $P = P(\rho)$, connected to the pressure law by $p'(\rho) = \rho P''(\rho)$, and associated energy functional

$$\mathcal{H}(\rho, v) := \int_0^\ell \left(\frac{1}{2}\rho v^2 + P(\rho) \right) dx. \quad (5)$$

In the case of subsonic velocities and smooth solutions, which is the relevant case for gas transport, this formulation allows to use relative energy techniques in order to investigate the stability of the equations (3)-(4) with respect to parameters and initial data, see [22]. The essential idea of our approach is to use a modification of this relative energy method to prove the exponential synchronization of the observer to the original system, i.e., we show the exponential convergence in the long time limit of the state of the observer system to the original system state.

We study the case where distributed partial measurements are available, i.e., measurements of only one state variable on the whole computational domain. Our aim is to estimate the complete system state. A motivation for this scenario is that e.g. for the Saint-Venant system the water height in a network of open channels is accessible more easily than the velocity. For the barotropic Euler system, we assume that we have distributed measurements of one of the fields velocity, density or mass flow. Then, we introduce an observer system for the original system (3)-(4), which reads for the case of velocity measurements

$$\partial_t \hat{\rho} + \partial_x \hat{m} = 0 \quad (6)$$

$$\partial_t \hat{v} + \partial_x \hat{h} = -\gamma |\hat{v}|\hat{v} + \mathcal{L}_v \quad (7)$$

with source term $\mathcal{L}_v = \mu(v - \hat{v})$ of Luenberger type with ‘nudging parameter’ $\mu > 0$. We complement both systems by identical Dirichlet boundary conditions at the left and right end of the pipe. When we consider the equations on networks, we additionally need coupling conditions at inner nodes. Here, we will use the conservation of mass and the continuity of the specific enthalpy, which yields energy conservation as shown in [23].

The two main contributions of this paper are to prove exponential decay of the difference between the state of the observer system and the original system state for partial measurements and to show existence of semi-global Lipschitz-continuous solutions of the observer system. Using only measurements of one of the state variables the other state variable can be reconstructed exponentially fast. Our results are valid for single pipes as well as for star-shaped networks. It turns out that increasing the nudging parameter does not necessarily improve the speed of synchronization. To the best of our knowledge, this is the first time that the synchronization of an observer system is proven for 2×2 -systems of quasilinear hyperbolic balance laws.

This paper is structured as follows. In section 2 we present the general setup, relevant notations and the Luenberger terms for the three cases of measurement of velocity, density or mass flow. In section 3 we show the existence of semi-global Lipschitz-continuous solutions of the observer system (6)-(7) for measurement of velocity or density provided that the original system (3)-(4) admits a unique solution and the solution of the original system as well as the initial and boundary data are sufficiently small when expressed in Riemann invariants and have sufficiently small Lipschitz-constants. For the original system (3)-(4), existence of semi-global solutions was shown in [24] for coupling conditions requiring conservation of mass and continuity of the pressure. This was done by writing the system in terms of Riemann invariants, which yields an integral equation along the characteristic curves. The proof cannot be transferred directly to the observer system due to the form of the observer-terms, but by using a modified

fixed-point iteration, we can show the existence of solutions of the observer system on a single pipe and on general networks with coupling conditions given by the conservation of mass and the continuity of the specific enthalpy. In addition, our proof yields existence of solutions of the original system for these coupling conditions, i.e., for continuity of the specific enthalpy instead of continuity of the pressure as it was used in [24].

In section 4 we show exponential convergence of solutions of the observer system towards the original system state under the assumption that both systems admit classical solutions that are bounded away from vacuum, have sufficiently small velocity and the solution of the original system has small time derivatives. We can show this result for a single pipe and for star-shaped networks for all three cases, i.e., measurement of velocity, density or mass flow. In the convergence proof we use techniques that are similar to those used in [22] to show stability of the barotropic Euler equations, i.e., we will measure the distance between the solution of the original system (3)-(4) and the solution of the observer system (6)-(7) in terms of the relative energy and then estimate the time derivative of the relative energy. This leads to decrease with respect to the variable that is measured, but not with respect to the other state variable. Therefore one of the main ideas of the proof is to use an extension of the relative energy framework. More precisely, we introduce an additional functional whose time derivative yields decrease with respect to the variable that is not measured. This modification of the relative energy is inspired by the extension of the energy that was used in [25] to study convergence of wave equations to steady states, which in turn was based on [26], among others.

2. GENERAL SETUP AND LUENBERGER OBSERVER

In the following we assume that the pressure law $p : \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth and strictly monotone so that for given density bounds $0 < \underline{\rho} < \bar{\rho}$ there exist constants $\underline{C}_{p'}, \bar{C}_{p'} > 0$ such that

$$0 < \underline{C}_{p'} \leq p'(\rho) \leq \bar{C}_{p'} \quad \forall 0 < \rho \leq \bar{\rho}.$$

Since the pressure potential $P : \mathbb{R}_+ \rightarrow \mathbb{R}$ is connected to the pressure law by $p'(\rho) = \rho P''(\rho)$, this implies that the pressure potential is smooth and strongly convex with

$$0 < \underline{C}_{P''} \leq P''(\rho) \leq \bar{C}_{P''}, \quad |P'''(\rho)| \leq C_{P'''} \quad \forall 0 < \rho \leq \bar{\rho}$$

for some constants $\underline{C}_{P''}, \bar{C}_{P''}, C_{P'''} > 0$.

We consider three different cases of available measurements, i.e., we assume that one of the three fields v , ρ or m is measured in the whole interval $[0, \ell]$, while for the other state variable we have no information. Then, we consider the observer system

$$\partial_t \hat{\rho} + \partial_x \hat{m} = \mathcal{L}_\rho \tag{8}$$

$$\partial_t \hat{v} + \partial_x \hat{h} = -\gamma |\hat{v}| \hat{v} + \mathcal{L}_v \tag{9}$$

for $0 < x < \ell$, $t > 0$ with Luenberger observer terms \mathcal{L}_ρ , \mathcal{L}_v depending on the measurements. For velocity measurements we set

$$\mathcal{L}_\rho = 0, \quad \mathcal{L}_v = \mu(v - \hat{v}), \quad \mu > 0 \tag{10}$$

and for density measurements

$$\mathcal{L}_\rho = \mu \frac{c}{\sqrt{p'(\hat{\rho})}} \hat{\rho} (\tilde{P}(\rho) - \tilde{P}(\hat{\rho})), \quad \mathcal{L}_v = 0, \quad \mu > 0 \tag{11}$$

with $c := \sqrt{p'(\rho_{\text{ref}})}$ for a reference density $\rho_{\text{ref}} > 0$ and $\tilde{P}(\rho) := \int_{\rho_{\text{ref}}}^{\rho} \frac{\sqrt{p'(s)}}{cs} ds$. For measurements of the mass flow we will use

$$\mathcal{L}_\rho = 0, \quad \mathcal{L}_v = \mu(m - \hat{m}), \quad \mu > 0. \tag{12}$$

Note that the observer term for density measurements is not exactly in Luenberger form, but for $0 < \underline{\rho} \leq \rho, \hat{\rho} \leq \bar{\rho}$ we have $\mathcal{L}_\rho = \mu \frac{\hat{\rho}}{\bar{\rho}} \frac{\sqrt{p'(\hat{\rho})}}{\sqrt{p'(\bar{\rho})}} (\rho - \hat{\rho})$ with some $\tilde{\rho} \in [\underline{\rho}, \bar{\rho}]$ and due to the monotonicity of p we have

$$0 < \frac{\underline{\rho}}{\bar{\rho}} \frac{\sqrt{C_{p'}}}{\sqrt{C_p}} \leq \frac{\hat{\rho}}{\bar{\rho}} \frac{\sqrt{p'(\hat{\rho})}}{\sqrt{p'(\bar{\rho})}} \leq \frac{\bar{\rho}}{\underline{\rho}} \frac{\sqrt{C_{p'}}}{\sqrt{C_p}}.$$

The reason why we choose this form of the observer term is that this observer term is linear when expressed in Riemann invariants, which will be relevant for the existence proof.

Remark 2.1. If exact measurements of ρ are available one might compute $\partial_t \rho = -\partial_x m$ and, thus, reconstruct the whole system state without recourse to an observer. However, in practical applications, measurements are subject to measurement errors and while these are usually small for the measured field they will be significant for its derivatives. Thus, we provide a framework that uses measurement data of one of the fields v, ρ or m , but does not make use of derivatives of measured data.

Since we will also consider the observer system on networks, we have to introduce some notation. We describe a network by a directed, connected graph with edges $e \in \mathcal{E}$ and vertex set \mathcal{V} consisting of boundary nodes $v \in \mathcal{V}_\partial$ and inner nodes $v \in \mathcal{V} \setminus \mathcal{V}_\partial$. The edge $e \in \mathcal{E}$ is identified with an interval $(0, \ell^e)$ and the set of edges that are incident to some node v is denoted by $\mathcal{E}(v)$. We assume that all pipes have the same diameter. For an edge $e = (v_1, v_2)$ that starts in node v_1 and ends at node v_2 we denote the pipe direction by $s^e(v_1) = -1$ and $s^e(v_2) = 1$. In the following, we will abbreviate

$$\|u - v\|_{L^2(\mathcal{E})}^2 := \sum_{e \in \mathcal{E}} \|u^e - v^e\|_{L^2(0, \ell^e)}^2.$$

Then, we assume that on each edge $e \in \mathcal{E}$ the equations (3)-(4) and (8)-(9), respectively, are satisfied. At the inner nodes we have to prescribe coupling conditions, where we will use the conservation of mass and the continuity of the specific enthalpy, i.e.,

$$\sum_{e \in \mathcal{E}(v)} m^e(v) s^e(v) = 0, \quad h^e(v) = h^f(v) \quad \forall e, f \in \mathcal{E}(v) \quad (13)$$

for every inner node $v \in \mathcal{V} \setminus \mathcal{V}_\partial$. These coupling conditions yield energy conservation, see [23].

3. EXISTENCE OF SOLUTIONS

A prerequisite of our proof of convergence of solutions of the observer system towards the original solution in section 4 is the existence of Lipschitz-continuous solutions of the observer system that have small, subsonic velocities and densities that are bounded away from vacuum. Therefore, in this section we will show existence of solutions of the observer system with suitable bounds on velocity and density for observer terms given by (10) or (11), i.e., for measurement of velocity or density.

For the original system (3)-(4) existence of solutions was shown in [24] for a single pipe and for networks, where in contrast to our coupling conditions conservation of mass and continuity of the pressure is used as coupling condition. If the initial and boundary conditions are sufficiently small when written in terms of Riemann invariants and have sufficiently small Lipschitz constant, existence of semi-global solutions, i.e., existence of solutions on a given time interval, with a priori bounds on velocity and density was established (see Theorem 6.1 in [24]). The main idea in the proof is rewriting the barotropic Euler equations (3)-(4) in terms of Riemann invariants. Then, by integrating along the characteristic curves, the problem is formulated as an integral equation, for which existence of solutions is shown by a fixed-point argument. These techniques have also been used to study semi-global classical solutions, see [27].

This result is not directly transferable to the observer system (8)-(9) (since the conditions (5.7) and (6.1) of [24] are not satisfied for the right hand side of the observer term for fixed $\mu > 0$), but with a similar strategy as in [24] we can show existence of semi-global solutions of the observer system. The key idea here is to modify the fixed-point map by using the specific form of the observer terms. Note that by setting the observer terms to zero, i.e., setting $\mu = 0$, our proof also yields existence of solutions of the original system for the coupling conditions (13), i.e., for conservation of mass and continuity of the specific enthalpy.

3.1. Observer system in Riemann invariants. Before we formulate the fixed-point map, we first rewrite the observer system for a single pipe in terms of Riemann invariants in order to diagonalize the advective part of the system. We can write (8)-(9) as

$$\begin{pmatrix} \partial_t \hat{\rho} \\ \partial_t \hat{v} \end{pmatrix} + \begin{pmatrix} \hat{v} & \hat{\rho} \\ P''(\hat{\rho}) & \hat{v} \end{pmatrix} \begin{pmatrix} \partial_t \hat{\rho} \\ \partial_t \hat{v} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_\rho \\ -\gamma|\hat{v}|\hat{v} + \mathcal{L}_v \end{pmatrix}, \quad (14)$$

where the flux Jacobian matrix has the eigenvalues

$$\hat{\lambda}_\pm = \hat{v} \pm \sqrt{p'(\hat{\rho})},$$

where $\sqrt{p'(\hat{\rho})}$ is the speed of sound, and left eigenvectors

$$l_\pm = \begin{pmatrix} \sqrt{p'(\hat{\rho})} \\ c\hat{\rho} \end{pmatrix}, \pm \frac{1}{c}$$

with $c = \sqrt{p'(\rho_{\text{ref}})}$ for a reference density $\rho_{\text{ref}} > 0$. Now, we will write (14) in terms of Riemann invariants S_\pm , which have the property $(\partial_\rho S_\pm, \partial_v S_\pm) = l_\pm$ (for an introduction to Riemann invariants see [28], Chapter 7). Here, the observer system has the Riemann invariants

$$S_\pm = \int_{\rho_{\text{ref}}}^{\hat{\rho}} \frac{\sqrt{p'(s)}}{cs} ds \pm \frac{\hat{v}}{c} =: \tilde{P}(\hat{\rho}) \pm \frac{\hat{v}}{c}. \quad (15)$$

By multiplying (14) from the left by the eigenvectors l_\pm , we see that the observer system can be written in terms of Riemann invariants as the diagonalized system

$$\begin{pmatrix} \partial_t S_+ \\ \partial_t S_- \end{pmatrix} + \begin{pmatrix} \hat{\lambda}_+ & 0 \\ 0 & \hat{\lambda}_- \end{pmatrix} \begin{pmatrix} \partial_x S_+ \\ \partial_x S_- \end{pmatrix} \\ = (\sigma(S_+, S_-) - \frac{1}{c}\mathcal{L}_v(S_+, S_-)) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{\sqrt{p'(\hat{\rho})}}{c\hat{\rho}} \mathcal{L}_\rho(S_+, S_-) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (16)$$

where $\sigma(S_+, S_-) = \frac{\gamma}{c}|\hat{v}|\hat{v} = \gamma\frac{c}{4}|S_+ - S_-|(S_+ - S_-)$ is the friction term and $\mathcal{L}_v(S_+, S_-)$, $\mathcal{L}_\rho(S_+, S_-)$ are the Luenberger terms written in Riemann invariants.

Let us mention some properties of the Riemann invariants and of the eigenvalues of the system that are needed to get existence of characteristic curves. First, the equation for the Riemann invariants implies

$$\hat{v} = \frac{c}{2}(S_+ - S_-), \quad \hat{\rho} = \tilde{P}^{-1}\left(\frac{1}{2}(S_+ + S_-)\right), \quad (17)$$

where \tilde{P}^{-1} is the inverse of the function \tilde{P} introduced in (15), which exists since $\tilde{P}' > 0$ due to the assumption $p' > 0$. Thus, if the Riemann invariants satisfy $|S_\pm| \leq S_{\text{max}}$ for a constant $S_{\text{max}} > 0$, then the corresponding velocity and density satisfy the bounds

$$|\hat{v}| \leq cS_{\text{max}} =: \bar{v}, \quad \underline{\rho} := \tilde{P}^{-1}(-S_{\text{max}}) \leq \hat{\rho} \leq \tilde{P}^{-1}(S_{\text{max}}) =: \bar{\rho}.$$

Since $\tilde{P}^{-1}(0) = \rho_{\text{ref}} > 0$ and \tilde{P}^{-1} is continuous, there exists a constant $S_{\text{max}} > 0$ such that $0 < \underline{\rho} \leq \hat{\rho} \leq \bar{\rho}$. If additionally S_{max} is sufficiently small such that $cS_{\text{max}} \leq \frac{1}{2}\sqrt{C_{p'}}$, then from

the bounds on velocity and density we can deduce the following bounds for the eigenvalues

$$0 < \underline{\Lambda}(S_{\max}) \leq \hat{\lambda}_+ \leq \Lambda(S_{\max}), \quad 0 > -\underline{\Lambda}(S_{\max}) \geq \hat{\lambda}_- \geq -\Lambda(S_{\max}) \quad (18)$$

with $\underline{\Lambda}(S_{\max}) = \frac{1}{2}\sqrt{\bar{C}_{p'}}$ and $\Lambda(S_{\max}) = \frac{3}{2}\sqrt{\bar{C}_{p'}}$. Furthermore, we can show

$$|\hat{\lambda}_{\pm}(S_+, S_-) - \hat{\lambda}_{\pm}(\tilde{S}_+, \tilde{S}_-)| \leq L_{\lambda}(|S_+ - \tilde{S}_+| + |S_- - \tilde{S}_-|),$$

i.e., the eigenvalues are Lipschitz-continuous with Lipschitz constant $L_{\lambda} \leq \frac{c}{2} + \frac{c\bar{\rho}}{4\bar{C}_{p'}}\bar{C}_{p''}$, where $\bar{C}_{p''} := \max_{\rho \leq \bar{\rho}} |p''(\rho)|$.

Using these prerequisites we can show analogously to Lemma 5.1 in [24] that, if $\mathcal{S} = (S_+, S_-) \in C([0, T] \times [0, \ell])^2$ is Lipschitz-continuous with respect to x and satisfies $|S_{\pm}| \leq S_{\max}$ and if $T \in (0, \min_{e \in \mathcal{E}} \frac{\ell^e}{\Lambda(S_{\max})})$, the characteristic curves $\xi_{\pm}^{\mathcal{S}}(s, x, t)$ defined by

$$\xi_{\pm}^{\mathcal{S}}(t, x, t) = x, \quad \partial_s \xi_{\pm}^{\mathcal{S}}(s, x, t) = \lambda_{\pm}(\mathcal{S}(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)))$$

exist locally. They can be extended up to the boundary of $[0, T] \times [0, \ell]$ and are Lipschitz-continuous with respect to x . Now, using the characteristic curves system (16) can be written as

$$\begin{aligned} \partial_s S_{\pm}(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) &= \mp \sigma(S_+, S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) \pm \frac{1}{c} \mathcal{L}_v(S_+, S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) \\ &\quad + \frac{\sqrt{p'(\hat{\rho})}}{c} \frac{1}{\hat{\rho}} \mathcal{L}_{\rho}(S_+, S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)). \end{aligned}$$

In the following we will study the existence of solutions of the observer system for the case of velocity measurement or density measurement, i.e., with observer terms given by (10) and (11), respectively. Therefore, we rewrite the observer terms \mathcal{L}_{ρ} , \mathcal{L}_v in terms of the Riemann invariants $R_{\pm} = \tilde{P}(\rho) \pm \frac{v}{c}$ of the original system and $S_{\pm} = \tilde{P}(\hat{\rho}) \pm \frac{\hat{v}}{c}$ of the observer system. Using (17) we see that the system for velocity measurement, i.e., for $\mathcal{L}_{\rho} = 0$, $\mathcal{L}_v = \mu(v - \hat{v})$, can be written as

$$\begin{aligned} \partial_s S_{\pm}(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) \\ = \mp \sigma(S_+, S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) \pm \frac{\mu}{2} (R_+ - R_- - (S_+ - S_-))(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)). \end{aligned} \quad (19)$$

For measurement of the density we have $\mathcal{L}_v = 0$, $\mathcal{L}_{\rho} = \mu \frac{c}{\sqrt{p'(\hat{\rho})}} \hat{\rho}(\tilde{P}(\rho) - \tilde{P}(\hat{\rho}))$ and therefore

$$\begin{aligned} \partial_s S_{\pm}(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) \\ = \mp \sigma(S_+, S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)) + \frac{\mu}{2} (R_+ + R_- - S_+ - S_-)(s, \xi_{\pm}^{\mathcal{S}}(s, x, t)). \end{aligned} \quad (20)$$

Remark 3.1. In the following, we will show existence of semi-global Lipschitz-continuous solutions of the observer system for velocity measurement with observer terms (10). Since the system (20) for measurement of ρ has the same structure as the system (19) for measurement of v , i.e., both systems can be written as

$$\partial_s S_+ + \frac{\mu}{2} S_+ = -\sigma + b_+(R_+, R_-, S_-), \quad \partial_s S_- + \frac{\mu}{2} S_- = +\sigma + b_-(R_+, R_-, S_+)$$

for suitable functions b_+ , b_- , existence of solutions of (20) can be shown analogously. The crucial point in the proof is that for measurements of velocity or density, the observer terms are linear when written in terms of Riemann invariants. However, this is not the case for measurement of the mass flow such that the same strategy cannot be used to show existence of solutions of the observer system for mass flow measurements.

Before we present the existence proof, we first consider the coupling conditions, since we also show existence of solutions on networks.

3.2. Coupling conditions in Riemann invariants. In order to show existence of solutions of the observer system for networks, we have to rewrite the coupling conditions in terms of Riemann invariants. Using that the mass flow can be written as

$$m = \rho v = \tilde{P}^{-1}\left(\frac{1}{2}(R_+ + R_-)\right) \frac{c}{2}(R_+ - R_-)$$

and the specific enthalpy as

$$h = \frac{1}{2}v^2 + P'(\rho) = \frac{1}{2}\left(\frac{c}{2}(R_+ - R_-)\right)^2 + P'\left(\tilde{P}^{-1}\left(\frac{1}{2}(R_+ + R_-)\right)\right),$$

the coupling conditions in Riemann invariants are given by

$$\sum_{e \in \mathcal{E}(v)} \tilde{P}^{-1}\left(\frac{1}{2}(R_+^e + R_-^e)\right)(R_+^e - R_-^e) = 0, \quad (21)$$

$$\begin{aligned} \frac{c^2}{8}(R_+^e - R_-^e)^2 + P'\left(\tilde{P}^{-1}\left(\frac{1}{2}(R_+^e + R_-^e)\right)\right) = \\ \frac{c^2}{8}(R_+^f - R_-^f)^2 + P'\left(\tilde{P}^{-1}\left(\frac{1}{2}(R_+^f + R_-^f)\right)\right), \quad \forall e, f \in \mathcal{E}(v) \end{aligned} \quad (22)$$

for any inner node $v \in \mathcal{V} \setminus \mathcal{V}_\partial$. For ease of notation we have assumed that the direction of the incident pipes is such that all pipes start in the node v . In order to show existence of solutions also on networks, we need the following lemma:

Lemma 3.2. *Consider an inner node v of a pipe network, where v has n incident edges denoted by $i = 1, \dots, n$. Then there exists some constant $S_{\max} > 0$ such that, if the incoming Riemann invariants $R_- = (R_-^1, \dots, R_-^n)$ satisfy $|R_-^i| \leq S_{\max}$, $i = 1, \dots, n$, then there exist unique outgoing Riemann invariants $R_+ = (R_+^1, \dots, R_+^n)$ satisfying the coupling conditions (21)-(22). In this case there exists a constant $C(n) > 0$ independent of R_- such that the outgoing Riemann invariants are bounded by*

$$|R_+|_\infty \leq C(n)|R_-|_\infty.$$

Proof. We show the assertion by using the implicit function theorem. As a first step, we write (21)-(22) as

$$F(R_-, R_+) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

with

$$F(R_-, R_+) := \begin{pmatrix} \sum_{i=1}^n \tilde{P}^{-1}\left(\frac{1}{2}(R_+^i + R_-^i)\right)(R_+^i - R_-^i) \\ \frac{c^2}{8}(R_+^2 - R_-^2)^2 + P'(\rho_2) - \frac{c^2}{8}(R_+^1 - R_-^1)^2 - P'(\rho_1) \\ \vdots \\ \frac{c^2}{8}(R_+^n - R_-^n)^2 + P'(\rho_n) - \frac{c^2}{8}(R_+^{n-1} - R_-^{n-1})^2 - P'(\rho_{n-1}) \end{pmatrix}$$

with $\rho_i = \tilde{P}^{-1}\left(\frac{1}{2}(R_+^i + R_-^i)\right)$. The entries of the derivative of F with respect to R_+ are given by

$$\begin{aligned} \left(\frac{\partial F}{\partial R_+}\right)_{1,i} &= (\tilde{P}^{-1})'\left(\frac{1}{2}(R_+^i + R_-^i)\right) \frac{1}{2}(R_+^i - R_-^i) + \rho_i, & i \in \{1, \dots, n\}, \\ \left(\frac{\partial F}{\partial R_+}\right)_{i,i} &= \frac{c^2}{4}(R_+^i - R_-^i) + P''(\rho_i)(\tilde{P}^{-1})'\left(\frac{1}{2}(R_+^i + R_-^i)\right) \frac{1}{2}, & i \in \{2, \dots, n\}, \\ \left(\frac{\partial F}{\partial R_+}\right)_{i,i-1} &= -\frac{c^2}{4}(R_+^{i-1} - R_-^{i-1}) - P''(\rho_{i-1})(\tilde{P}^{-1})'\left(\frac{1}{2}(R_+^{i-1} + R_-^{i-1})\right) \frac{1}{2}, & i \in \{2, \dots, n\} \end{aligned}$$

and all other entries of $\frac{\partial F}{\partial R_+}$ are zero. For $R_- = 0, R_+ = 0$, we have $F(0, 0) = 0$ and

$$\frac{\partial F}{\partial R_+}(0, 0) = \begin{pmatrix} \rho_{\text{ref}} & \rho_{\text{ref}} & \rho_{\text{ref}} & \cdots & \rho_{\text{ref}} \\ -a & a & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & -a & a & 0 \\ 0 & \cdots & 0 & -a & a \end{pmatrix},$$

where $a := \frac{1}{2}P''(\rho_{\text{ref}})\rho_{\text{ref}}$ and $a > 0$ since P is strictly convex. We can show by induction that $\det\left(\frac{\partial F}{\partial R_+}(0, 0)\right) > 0$, i.e., $\frac{\partial F}{\partial R_+}(0, 0)$ is invertible. Therefore the implicit function theorem provides the existence of some open neighborhood $W \subset \mathbb{R}^n$ of 0 and a unique, continuous function $g : W \rightarrow \mathbb{R}^n, R_- \mapsto R_+ = g(R_-)$ such that $F(R_-, g(R_-)) = 0$. This means that for incoming Riemann invariants R_- with $|R_-^i| \leq S_{\text{max}}, i = 1, \dots, n$, for some $S_{\text{max}} > 0$ with $B_{S_{\text{max}}}(0) \subset W$, the outgoing Riemann invariants R_+ are given by $R_+ = g(R_-)$.

It remains to show that we have a bound of the form $|R_+|_{\infty} \leq C|R_-|_{\infty}$ for some constant $C > 0$. In order to show this note that the implicit function theorem implies

$$\frac{\partial g}{\partial R_-}(R_-) = -\left(\frac{\partial F}{\partial R_+}(R_-, g(R_-))\right)^{-1} \frac{\partial F}{\partial R_-}(R_-, g(R_-)).$$

Since all components of F are continuously differentiable, $\left(\frac{\partial F}{\partial R_+}(R_-, g(R_-))\right)^{-1}$ and $\frac{\partial F}{\partial R_-}(R_-, g(R_-))$ depend continuously on R_- , i.e., for $(R_-, g(R_-))$ in a neighbourhood of $(0, 0)$ also the derivatives $\left(\frac{\partial F}{\partial R_+}(R_-, g(R_-))\right)^{-1}$ and $\frac{\partial F}{\partial R_-}(R_-, g(R_-))$ are close to $\left(\frac{\partial F}{\partial R_+}(0, 0)\right)^{-1}$ and $\frac{\partial F}{\partial R_-}(0, 0)$, respectively. Therefore, there exists a constant $C(n)$ such that, if we choose S_{max} sufficiently small, then

$$\left|\frac{\partial g}{\partial R_-}(R_-)\right|_{\infty} \leq \left|\left(\frac{\partial F}{\partial R_+}(R_-, g(R_-))\right)^{-1}\right|_{\infty} \left|\frac{\partial F}{\partial R_-}(R_-, g(R_-))\right|_{\infty} \leq C(n)$$

for all incoming Riemann invariants R_- that satisfy $|R_-|_{\infty} \leq S_{\text{max}}$. \square

3.3. Existence of solutions of the observer system for measurement of v . By formulating the observer system (19) for velocity measurements as a fixed-point iteration along the characteristic curves we can show the following existence theorem:

Theorem 3.3. *Let some $\mu \geq 0$ be given. Consider a pipe network, where any inner node has at most n incident edges. Let some $T \in (0, \min_{e \in \mathcal{E}} \frac{\ell^e}{\Lambda(S_{\text{max}})})$ such that $1 - e^{-\frac{\mu}{2}T} \leq \frac{1}{12} \min\{1, \frac{1}{1+C(n)} \frac{4}{9} \frac{\Lambda(S_{\text{max}})}{\Lambda(S_{\text{max}})}\}$ with the constant $C(n)$ from Lemma 3.2 be given and consider the system (19), i.e., the observer system for measurement of v , together with the initial and boundary conditions*

$$S_{\pm, e}(0, x) = y_{\pm}^e(x), \quad e \in \mathcal{E}, x \in (0, \ell^e), \quad (23)$$

$$S_{\pm, e}(t, v) = u_{\pm}^e(t), \quad v \in \mathcal{V}_{\partial}, e \in \mathcal{E}(v), t \in (0, T). \quad (24)$$

Assume that the boundary and initial data are C^0 -compatible, i.e., $y_{\pm}^e(v) = u_{\pm}^e(0)$ for $v \in \mathcal{V}_{\partial}, e \in \mathcal{E}(v)$, the initial data is compatible with the coupling conditions (21)-(22) and the boundary and initial data are Lipschitz-continuous with Lipschitz-constant L_I . Let B_{max} denote the bound on the initial and boundary data, i.e.,

$$B_{\text{max}} = \max_{e \in \mathcal{E}} \sup_{(t, x) \in [0, T] \times [0, \ell^e]} (|u_{\pm}^e(t)|, |y_{\pm}^e(x)|).$$

For $\mu > 0$, assume in addition that the original system (3)-(4) (written in Riemann invariants, i.e., as (16) with $\mathcal{L}_v = 0$, $\mathcal{L}_\rho = 0$) together with the boundary data

$$R_{\pm,e}(t, v) = u_{\pm}^e(t), \quad v \in \mathcal{V}_\partial, e \in \mathcal{E}(v), t \in (0, T) \quad (25)$$

and the coupling conditions (21)-(22) has a unique solution $\mathcal{R} = ((R_{+,e}, R_{-,e}))_{e \in \mathcal{E}}$, $(R_{+,e}, R_{-,e}) \in C^0([0, T] \times [0, \ell^e])$ that satisfies $\|R_{\pm,e}\|_{L^\infty([0, T] \times [0, \ell])} \leq S_{\max}$ and is Lipschitz-continuous with respect to x with Lipschitz-constant bounded by L_R .

Then, if B_{\max} , L_I , S_{\max} and L_R are sufficiently small, the system (19) together with the initial and boundary conditions (23)-(24) and the coupling conditions (21)-(22) admits a unique solution in

$$\begin{aligned} M(S_{\max}, L_R) := \{ & \mathcal{S} = ((S_{+,e}, S_{-,e}))_{e \in \mathcal{E}}, (S_{+,e}, S_{-,e}) \in C^0([0, T] \times [0, \ell^e]) : \\ & |S_{\pm,e}| \leq S_{\max} \quad \forall e \in \mathcal{E} \text{ and all } S_{\pm,e} \text{ are Lipschitz-continuous} \\ & \text{with respect to } x \text{ with Lipschitz-constant } L_R \}. \end{aligned}$$

Remark 3.4. Theorem 3.3 provides the existence of Lipschitz-continuous solutions satisfying the bounds $|S_{\pm}| \leq S_{\max}$. This implies the bounds

$$|\hat{v}| \leq cS_{\max} =: \bar{v}, \quad \rho := \tilde{P}^{-1}(-S_{\max}) \leq \hat{\rho} \leq \tilde{P}^{-1}(S_{\max}) =: \bar{\rho}$$

for the velocity and density with $\rho > 0$ for S_{\max} sufficiently small. Thus, the bound $|S_{\pm}| \leq S_{\max}$ on the Riemann invariants corresponds to bounds for $\hat{\rho}$, \hat{v} that are of the same form as the bounds that will be used in the convergence proof, cf. assumptions (A1) and (A3) in section 4.

Remark 3.5. Note that Theorem 3.3 also applies to the case $\mu = 0$, which means that the theorem asserts the existence of Lipschitz-continuous solutions of the original system for the coupling conditions (13). This was not known before, since [24] uses other coupling conditions.

Proof of Theorem 3.3. The main idea of the proof is to write (19) as a suitable fixed-point mapping and then apply the Banach fixed-point theorem in order to show existence of a unique fixed-point in $M(S_{\max}, L_R)$, where we use on $M(S_{\max}, L_R)$ the norm

$$\|\mathcal{S}_e\|_M := \max_{(t,x) \in [0, T] \times [0, \ell^e]} |S_{+,e}(t, x)| + |S_{-,e}(t, x)|.$$

The strategy of the proof is similar to the strategy of the proof of Theorem 5.1 in [24], but we use a modification of the fixed-point iteration that is used there in order to deal with the observer terms. More precisely, we denote $\mathcal{S}^i = ((S_{+,e}^i, S_{-,e}^i))_{e \in \mathcal{E}}$ and define the mapping

$$\Phi : M(S_{\max}, L_R) \rightarrow M(S_{\max}, L_R), \quad \mathcal{S}^i \mapsto \mathcal{S}^{i+1},$$

where $\mathcal{S}^{i+1} = ((S_{+,e}^{i+1}, S_{-,e}^{i+1}))_{e \in \mathcal{E}}$ is the solution of the differential equations

$$\begin{aligned} & \partial_s S_{+,e}^{i+1}(s, \xi_{+,e}^{\mathcal{S}^i}(s, x, t)) + \frac{\mu}{2} S_{+,e}^{i+1}(s, \xi_{+,e}^{\mathcal{S}^i}(s, x, t)) \\ & = \frac{\mu}{2} (R_{+,e} - R_{-,e} + S_{-,e}^i)(s, \xi_{+,e}^{\mathcal{S}^i}(s, x, t)) - \sigma(\mathcal{S}^i)(s, \xi_{+,e}^{\mathcal{S}^i}(s, x, t)), \quad e \in \mathcal{E}, \end{aligned} \quad (26)$$

along the ξ_+ -characteristics and

$$\begin{aligned} & \partial_s S_{-,e}^{i+1}(s, \xi_{-,e}^{\mathcal{S}^i}(s, x, t)) + \frac{\mu}{2} S_{-,e}^{i+1}(s, \xi_{-,e}^{\mathcal{S}^i}(s, x, t)) \\ & = -\frac{\mu}{2} (R_{+,e} - R_{-,e} - S_{+,e}^i)(s, \xi_{-,e}^{\mathcal{S}^i}(s, x, t)) + \sigma(\mathcal{S}^i)(s, \xi_{-,e}^{\mathcal{S}^i}(s, x, t)), \quad e \in \mathcal{E}, \end{aligned} \quad (27)$$

along the ξ_- -characteristics complemented with the initial and boundary conditions (23)-(24) and the coupling conditions (21)-(22). Note that on the left hand side there is not only the derivative term $\partial_s S_{\pm}^{i+1}$, but also some part of the observer term. This has the advantage that we get some extra damping from this term, which will be relevant in the estimates that are needed to show that Φ is a contraction and a self-mapping.

For fixed x and t , equations (26) and (27) are ordinary differential equations in the variable s that can be solved by the "variation of constants"- formula. For the equation (26) for S_+ this yields

$$S_{+,e}^{i+1}(s, \xi_+^{\mathcal{S}_e^i}(s, x, t)) = S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), \xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t))e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x, t))} + \int_{t_+^{\mathcal{S}_e^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2}(R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) - \sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) \right) dr \quad (28)$$

on every pipe $e \in \mathcal{E}$ for $s \geq t_+^{\mathcal{S}_e^i}(x, t)$, where $t_+^{\mathcal{S}_e^i}(x, t)$ is the time when the characteristic $\xi_+^{\mathcal{S}_e^i}(s, x, t)$ hits the boundary of the interval $[0, \ell^e]$ or the initial time, i.e., $t_+^{\mathcal{S}_e^i}(x, t) = 0$, if $\xi_+^{\mathcal{S}_e^i}(s, x, t) > 0$ for all $s \in [0, T]$, and $t_+^{\mathcal{S}_e^i}(x, t)$ is defined by $\xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) = 0$ otherwise. In particular, $S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), \xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t))$ is known from the initial data, boundary data or from the coupling conditions. This means that the fixed-point iteration can be written as

$$S_{+,e}^{i+1}(s, \xi_+^{\mathcal{S}_e^i}(s, x, t)) = \int_{t_+^{\mathcal{S}_e^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2}(R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) - \sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) \right) dr + \begin{cases} y_{+,e}(\xi_+^{\mathcal{S}_e^i}(0, x, t))e^{-\frac{\mu}{2}s} & \text{if } t_+^{\mathcal{S}_e^i}(x, t) = 0, \\ u_{+,e}(t_+^{\mathcal{S}_e^i}(x, t))e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x, t))} & \text{if } \xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) \text{ is a boundary node,} \\ S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v)e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x, t))} & \text{if } \xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) = v \text{ is an inner node} \end{cases}$$

on each edge $e \in \mathcal{E}$. Now, if we choose S_{\max} sufficiently small such that (18) holds, from the bound $T \leq \min_{e \in \mathcal{E}} \frac{\ell^e}{\Lambda(S_{\max})}$ we deduce that the value of $S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v)$ in the last case can be followed back to the initial state and we obtain

$$S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v) = g_e(S_{-,f}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v)),$$

where g_e is the e -th component of the function g that represents the coupling conditions, see the proof of Lemma 3.2, and the components $S_{-,f}^{i+1}$, $f \in \mathcal{E}(v)$, of S_-^{i+1} are given by

$$S_{-,f}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v) = \int_0^{t_+^{\mathcal{S}_e^i}(x, t)} e^{-\frac{\mu}{2}(t_+^{\mathcal{S}_e^i}(x, t)-r)} \left(-\frac{\mu}{2}(R_{+,f} - R_{-,f} - S_{+,f}^i)(r, \xi_-^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t))) + \sigma(\mathcal{S}_f^i)(r, \xi_-^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t))) \right) dr + S_{-,f}^{i+1}(0, \xi_-^{\mathcal{S}_f^i}(0, v, t_+^{\mathcal{S}_e^i}(x, t)))e^{-\frac{\mu}{2}t_+^{\mathcal{S}_e^i}(x, t)} \quad (29)$$

with $S_{-,f}^{i+1}(0, \xi_-^{\mathcal{S}_f^i}(0, v, t_+^{\mathcal{S}_e^i}(x, t)))$ given by the initial data. For ease of presentation we assume here that all pipes that are incident to the inner node v are oriented such that they start in v .

In the first part of the proof we show that the fixed-point mapping Φ is well-defined, i.e., $\Phi(\mathcal{S}) \in M(S_{\max}, L_R)$ for $\mathcal{S} \in M(S_{\max}, L_R)$. Before we start, let us note that

$$\sigma(S_+, S_-) = \frac{\gamma}{c} |\hat{v}| \hat{v} = \gamma \frac{c}{4} |S_+ - S_-| (S_+ - S_-) \leq \gamma c (S_{\max})^2 =: \sigma_{\max}$$

and $\sigma(S_+, S_-)$ is Lipschitz-continuous with Lipschitz-constant L_σ , i.e.,

$$|\sigma(R_+, R_-) - \sigma(\tilde{R}_+, \tilde{R}_-)| \leq L_\sigma \left(|R_+ - \tilde{R}_+| + |R_- - \tilde{R}_-| \right) \quad (30)$$

with $L_\sigma \leq \gamma c S_{\max}$ for $\mathcal{S} \in M(S_{\max}, L_R)$. In the following, let us choose S_{\max} sufficiently small such that $\hat{v}(S_+, S_-) = \frac{c}{2} (S_+ - S_-) \leq \frac{1}{2} \sqrt{C_{p'}}$ and the bounds (18) on λ_\pm are satisfied.

Let $\mathcal{S}^i \in M(S_{\max}, L_R)$. Then

$$\begin{aligned}
|S_{+,e}^{i+1}(s, \xi_{+}^{\mathcal{S}_e^i}(s, x, t))| &\leq |S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))e^{-\frac{\mu}{2}(s-t_{+}^{\mathcal{S}_e^i}(x, t))}| \\
&\quad + \int_{t_{+}^{\mathcal{S}_e^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2} |R_{+,e} - R_{-,e} + S_{-,e}^i|(r, \xi_{+}^{\mathcal{S}_e^i}(r, x, t)) + |\sigma(\mathcal{S}_e^i)|(r, \xi_{+}^{\mathcal{S}_e^i}(r, x, t)) \right) dr \\
&\leq |S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))| + \int_{t_{+}^{\mathcal{S}_e^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2} 3S_{\max} + \sigma_{\max} \right) dr \\
&= |S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))| + \left(1 - e^{-\frac{\mu}{2}(s-t_{+}^{\mathcal{S}_e^i}(x, t))} \right) 3S_{\max} + T\sigma_{\max} \\
&\leq |S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))| + \frac{1}{12} \frac{1}{1+C(n)} 3S_{\max} + T\sigma_{\max}
\end{aligned}$$

for all $(x, t) \in [0, \ell] \times [0, T]$, $t_{+}^{\mathcal{S}_e^i}(x, t) \leq s \leq T$, since $0 \leq s - t_{+}^{\mathcal{S}_e^i}(x, t) \leq T$ and $1 - e^{-\frac{\mu}{2}T} \leq \frac{1}{12} \frac{1}{1+C(n)}$. If $S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))$ is given by the initial or boundary data, then $|S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))| \leq B_{\max}$. If $\xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t)$ is an inner node $v \in \mathcal{V} \setminus \mathcal{V}_{\partial}$, then $S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), v) = g_e(S_{-,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), v))$ with

$$\begin{aligned}
|S_{-,f}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), v)| &\leq |S_{-,f}^{i+1}(0, \xi_{+}^{\mathcal{S}_f^i}(0, v, t_{+}^{\mathcal{S}_e^i}(x, t)))e^{-\frac{\mu}{2}t_{+}^{\mathcal{S}_e^i}(x, t)}| \\
&\quad + \frac{3}{2}\mu S_{\max} \int_0^{t_{+}^{\mathcal{S}_e^i}(x, t)} e^{-\frac{\mu}{2}(t_{+}^{\mathcal{S}_e^i}(x, t)-r)} dr + T\sigma_{\max} \\
&\leq B_{\max} + 3S_{\max} \frac{1}{12} \frac{1}{1+C(n)} + T\sigma_{\max} \quad \forall f \in \mathcal{E}(v).
\end{aligned} \tag{31}$$

Therefore

$$\begin{aligned}
|S_{+,e}^{i+1}(s, \xi_{+}^{\mathcal{S}_e^i}(s, x, t))| &\leq 3S_{\max} \frac{1}{12} \frac{1}{1+C(n)} + T\sigma_{\max} + \max \left\{ B_{\max}, C(n) \left(B_{\max} + 3S_{\max} \frac{1}{12} \frac{1}{1+C(n)} + T\sigma_{\max} \right) \right\} \\
&\leq (1 + C(n))B_{\max} + \frac{1}{4}S_{\max} + (1 + C(n))T\sigma_{\max}.
\end{aligned}$$

Now, if $B_{\max} \leq \frac{1}{2} \frac{1}{1+C(n)} S_{\max}$ and S_{\max} is sufficiently small such that $(1 + C(n))T\sigma_{\max} \leq \frac{1}{4}S_{\max}$, then

$$|S_{+,e}^{i+1}(s, \xi_{+}^{\mathcal{S}_e^i}(s, x, t))| \leq S_{\max}.$$

Analogously, we can show $|S_{-,e}^{i+1}| \leq S_{\max}$. Thus, $|\Phi(\mathcal{S}^i)| \leq S_{\max}$ for $\mathcal{S}^i \in M(S_{\max}, L_R)$, if S_{\max} and B_{\max} are sufficiently small.

Next, we want to show that the Lipschitz-constant of \mathcal{S}^i with respect to x is uniformly bounded in i , i.e., if \mathcal{S}^i has the Lipschitz-constant L_R with respect to x , then we have to show that \mathcal{S}^{i+1} has the same Lipschitz-constant with respect to x . From (28) we know that

$$\begin{aligned}
S_{+,e}^{i+1}(t, x) &= S_{+,e}^{i+1}(t_{+}^{\mathcal{S}_e^i}(x, t), \xi_{+}^{\mathcal{S}_e^i}(t_{+}^{\mathcal{S}_e^i}(x, t), x, t))e^{-\frac{\mu}{2}(t-t_{+}^{\mathcal{S}_e^i}(x, t))} \\
&\quad + \int_{t_{+}^{\mathcal{S}_e^i}(x, t)}^t e^{-\frac{\mu}{2}(t-r)} \left(\frac{\mu}{2} (R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_{+}^{\mathcal{S}_e^i}(r, x, t)) - \sigma(\mathcal{S}_e^i)(r, \xi_{+}^{\mathcal{S}_e^i}(r, x, t)) \right) dr \\
&=: f_{1,e}(t, x) + f_{2,e}(t, x)
\end{aligned}$$

for all $e \in \mathcal{E}$. Let $x_1, x_2 \in [0, \ell^e]$ with $t_{+}^{\mathcal{S}_e^i}(x_1, t) \leq t_{+}^{\mathcal{S}_e^i}(x_2, t)$. Then

$$|S_{+,e}^{i+1}(t, x_1) - S_{+,e}^{i+1}(t, x_2)| \leq |f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| + |f_{2,e}(t, x_1) - f_{2,e}(t, x_2)|$$

with

$$\begin{aligned}
& |f_{2,e}(t, x_1) - f_{2,e}(t, x_2)| \\
& \leq \int_{t_+^{\mathcal{S}_e^i}(x_2, t)}^t e^{-\frac{\mu}{2}(t-r)} \left| \frac{\mu}{2}(R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_1, t)) \right. \\
& \quad - \frac{\mu}{2}(R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_2, t)) \\
& \quad \left. - (\sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_1, t)) - \sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_2, t))) \right| dr \\
& \quad + \int_{t_+^{\mathcal{S}_e^i}(x_1, t)}^{t_+^{\mathcal{S}_e^i}(x_2, t)} e^{-\frac{\mu}{2}(t-r)} \left| \frac{\mu}{2}(R_{+,e} - R_{-,e} + S_{-,e}^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_1, t)) - \sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x_1, t)) \right| dr \\
& \leq \int_{t_+^{\mathcal{S}_e^i}(x_2, t)}^t e^{-\frac{\mu}{2}(t-r)} \left(\frac{\mu}{2} 3L_R + L_\sigma 2L_R \right) |\xi_+^{\mathcal{S}_e^i}(r, x_1, t) - \xi_+^{\mathcal{S}_e^i}(r, x_2, t)| dr \\
& \quad + \int_{t_+^{\mathcal{S}_e^i}(x_1, t)}^{t_+^{\mathcal{S}_e^i}(x_2, t)} e^{-\frac{\mu}{2}(t-r)} \left(\frac{\mu}{2} 3S_{\max} + \sigma_{\max} \right) dr,
\end{aligned}$$

where L_σ is the Lipschitz-constant of the friction term σ with respect to the Riemann invariants, see (30). Now, we want to estimate $|\xi_+^{\mathcal{S}_e^i}(r, x_1, t) - \xi_+^{\mathcal{S}_e^i}(r, x_2, t)|$. Analogously to Lemma 5.1 in [24] we can show that the characteristics ξ_\pm are Lipschitz-continuous with respect to x with Lipschitz-constant $L_x = \exp(2L_R T L_\lambda)$, i.e.,

$$|\xi_+^{\mathcal{S}_e^i}(r, x_1, t) - \xi_+^{\mathcal{S}_e^i}(r, x_2, t)| \leq L_x |x_1 - x_2|.$$

In addition, as in [24, Lemma 5.1] we can show that $t_+^{\mathcal{S}_e^i}(x, t)$ is Lipschitz-continuous with respect to x with the Lipschitz-constant

$$L_t = \frac{1}{\Lambda(S_{\max})} \exp(2L_R T L_\lambda) = \frac{1}{\Lambda(S_{\max})} L_x.$$

This implies

$$\begin{aligned}
|f_{2,e}(t, x_1) - f_{2,e}(t, x_2)| & \leq \left(\frac{1}{12} 3L_R + 2T L_\sigma L_R \right) L_x |x_1 - x_2| + L_t |x_1 - x_2| \left(\frac{\mu}{2} 3S_{\max} + \sigma_{\max} \right) \\
& \leq \left(\frac{1}{4} L_R L_x + 2T L_\sigma L_R L_x + L_t \frac{\mu}{2} 3S_{\max} + L_t \sigma_{\max} \right) |x_1 - x_2| \\
& =: L_2 |x_1 - x_2|.
\end{aligned}$$

Thus, if we choose S_{\max} and L_R sufficiently small such that $L_x \leq \frac{3}{2}$, $3T L_\sigma \leq \frac{1}{24}$, $L_t \frac{\mu}{2} 3S_{\max} \leq \frac{1}{24} L_R$ and $L_t \sigma_{\max} \leq \frac{1}{24} L_R$, then

$$L_2 \leq \frac{1}{2} L_R. \tag{32}$$

Now, we will estimate the Lipschitz-constant of

$$f_{1,e}(t, x) = S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), \xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t)) e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_e^i}(x, t))}.$$

Here, we distinguish between three cases.

Case 1: $t_+^{\mathcal{S}_+^i}(x_1, t) = 0 = t_+^{\mathcal{S}_+^i}(x_2, t)$.

Then $f_{1,e}(t, x_1)$ and $f_{1,e}(t, x_2)$ are given by the initial condition, i.e.,

$$\begin{aligned} |f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| &= |S_{+,e}^{i+1}(0, \xi_+^{\mathcal{S}_+^i}(0, x_1, t))e^{-\frac{\mu}{2}t} - S_{+,e}^{i+1}(0, \xi_+^{\mathcal{S}_+^i}(0, x_2, t))e^{-\frac{\mu}{2}t}| \\ &\leq |e^{-\frac{\mu}{2}t} |y_+^e(\xi_+^{\mathcal{S}_+^i}(0, x_1, t)) - y_+^e(\xi_+^{\mathcal{S}_+^i}(0, x_2, t))| \\ &\leq L_I |\xi_+^{\mathcal{S}_+^i}(0, x_1, t) - \xi_+^{\mathcal{S}_+^i}(0, x_2, t)| \\ &\leq L_I L_x |x_1 - x_2| \end{aligned}$$

with $L_I L_x \leq \frac{1}{2} L_R$ for L_I sufficiently small.

Case 2: $t_+^{\mathcal{S}_+^i}(x_1, t) > 0$ and $t_+^{\mathcal{S}_+^i}(x_2, t) > 0$.

Then $\xi_+^{\mathcal{S}_+^i}(t_+^{\mathcal{S}_+^i}(x_1, t), x_1, t) = \xi_+^{\mathcal{S}_+^i}(t_+^{\mathcal{S}_+^i}(x_2, t), x_2, t)$ is either a boundary node or an inner node of the network, i.e., $f_{1,e}(t, x_1)$ and $f_{1,e}(t, x_2)$ are given by the boundary condition or the coupling conditions.

Case 2a: If $f_{1,e}(t, x_1)$ and $f_{1,e}(t, x_2)$ are given by the boundary conditions, we have

$$\begin{aligned} &|f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| \\ &= |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), 0)e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_1, t))} - S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_2, t), 0)e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_2, t))}| \\ &\leq |u_+^e(t_+^{\mathcal{S}_+^i}(x_1, t)) - u_+^e(t_+^{\mathcal{S}_+^i}(x_2, t))| |e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_1, t))}| \\ &\quad + |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_2, t), 0)| |e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_1, t))} - e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_2, t))}| \\ &\leq L_I |t_+^{\mathcal{S}_+^i}(x_1, t) - t_+^{\mathcal{S}_+^i}(x_2, t)| + B_{\max} \frac{\mu}{2} |t_+^{\mathcal{S}_+^i}(x_1, t) - t_+^{\mathcal{S}_+^i}(x_2, t)| \\ &\leq (L_I L_t + B_{\max} \frac{\mu}{2} L_t) |x_1 - x_2|. \end{aligned}$$

Case 2b: If $\xi_+^{\mathcal{S}_+^i}(t_+^{\mathcal{S}_+^i}(x_1, t), x_1, t) = \xi_+^{\mathcal{S}_+^i}(t_+^{\mathcal{S}_+^i}(x_2, t), x_2, t)$ is an inner node $v \in \mathcal{V} \setminus \mathcal{V}_\partial$, then we can estimate

$$\begin{aligned} &|f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| \\ &= |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v)e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_1, t))} - S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_2, t), v)e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_2, t))}| \\ &\leq |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v)| |e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_1, t))} - e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_2, t))}| \\ &\quad + |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v) - S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_2, t), v)| |e^{-\frac{\mu}{2}(t-t_+^{\mathcal{S}_+^i}(x_2, t))}| \\ &\leq |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v)| \frac{\mu}{2} L_t |x_1 - x_2| + C(n) |S_{-,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v) - S_{-,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_2, t), v)|_\infty \end{aligned}$$

with

$$\begin{aligned} |S_{+,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v)| &\leq C(n) |S_{-,e}^{i+1}(t_+^{\mathcal{S}_+^i}(x_1, t), v)|_\infty \\ &\leq C(n) \left(B_{\max} + 3S_{\max} \frac{1}{12} \frac{1}{1+C(n)} + T\sigma_{\max} \right) \\ &\leq C(n) B_{\max} + \frac{1}{4} S_{\max} + C(n) T\sigma_{\max} \end{aligned}$$

and

$$\begin{aligned}
& |S_{-,f}^{i+1}(t_+^{S_e^i}(x_1, t), v) - S_{-,f}^{i+1}(t_+^{S_e^i}(x_2, t), v)| \\
& \leq |S_{-,f}^{i+1}(0, \xi_-^{S_f^i}(0, v, t_+^{S_e^i}(x_1, t)))e^{-\frac{\mu}{2}t_+^{S_e^i}(x_1, t)} - S_{-,f}^{i+1}(0, \xi_-^{S_f^i}(0, v, t_+^{S_e^i}(x_2, t)))e^{-\frac{\mu}{2}t_+^{S_e^i}(x_2, t)}| \\
& \quad + \left| \int_0^{t_+^{S_e^i}(x_1, t)} e^{-\frac{\mu}{2}(t_+^{S_e^i}(x_1, t)-r)} \left(-\frac{\mu}{2}(R_{+,f} - R_{-,f} - S_{+,f}^i)(r, \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_1, t))) \right. \right. \\
& \quad \quad \quad \left. \left. + \sigma(S_e^i)(r, \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_1, t))) \right) dr \right. \\
& \quad \left. - \int_0^{t_+^{S_e^i}(x_2, t)} e^{-\frac{\mu}{2}(t_+^{S_e^i}(x_2, t)-r)} \left(-\frac{\mu}{2}(R_{+,f} - R_{-,f} - S_{+,f}^i)(r, \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_2, t))) \right. \right. \\
& \quad \quad \quad \left. \left. + \sigma(S_e^i)(r, \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_2, t))) \right) dr \right| \\
& \leq B_{\max} \frac{\mu}{2} L_t |x_1 - x_2| + L_I |\xi_-^{S_f^i}(0, v, t_+^{S_e^i}(x_1, t)) - \xi_-^{S_f^i}(0, v, t_+^{S_e^i}(x_2, t))| \\
& \quad + \left(\frac{1}{12} \frac{1}{1+C(n)} \frac{4}{9} \frac{\Lambda(S_{\max})}{\Lambda(S_{\max})} 3L_R + 2TL_\sigma L_R \right) \\
& \quad \max_{r \in [0, t_+^{S_e^i}(x_1, t)]} |\xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_1, t)) - \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_2, t))| \\
& \quad + \left(\frac{3}{2} \mu S_{\max} + \sigma_{\max} \right) T \frac{\mu}{2} L_t |x_1 - x_2| + L_t |x_1 - x_2| \left(\frac{3}{2} \mu S_{\max} + \sigma_{\max} \right).
\end{aligned}$$

Using that for all $r \in [0, t_+^{S_e^i}(x_1, t)]$

$$\begin{aligned}
& |\xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_1, t)) - \xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_2, t))| \\
& = |\xi_-^{S_f^i}(r, v, t_+^{S_e^i}(x_1, t)) - \xi_-^{S_f^i}(r, \xi_-^{S_f^i}(t_+^{S_e^i}(x_1, t), v, t_+^{S_e^i}(x_2, t)), t_+^{S_e^i}(x_1, t))| \\
& \leq L_x |v - \xi_-^{S_f^i}(t_+^{S_e^i}(x_1, t), v, t_+^{S_e^i}(x_2, t))| \leq L_x L_s L_t |x_1 - x_2|,
\end{aligned}$$

where $L_s \leq \Lambda(S_{\max})$ is the Lipschitz-constant of ξ_{\pm} with respect to s , this implies

$$\begin{aligned}
& |f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| \\
& \leq \left((C(n)B_{\max} + \frac{1}{4}S_{\max} + C(n)T\sigma_{\max}) \frac{\mu}{2} \right. \\
& \quad \left. + C(n) \left(B_{\max} \frac{\mu}{2} + L_I L_x L_s + \left(\frac{1}{12} \frac{1}{1+C(n)} \frac{4}{9} \frac{\Lambda(S_{\max})}{\Lambda(S_{\max})} 3L_R + 2TL_\sigma L_R \right) L_x L_s \right. \right. \\
& \quad \quad \left. \left. + \left(\frac{3}{2} \mu S_{\max} + \sigma_{\max} \right) \left(1 + \frac{\mu}{2} T \right) \right) L_t |x_1 - x_2| \right. \\
& \leq (C(n)B_{\max} \mu + \frac{\mu}{8} S_{\max} + C(n)T \frac{\mu}{2} \sigma_{\max} + C(n)L_x L_s L_I \\
& \quad + C(n) \left(\frac{3}{2} \mu S_{\max} + \sigma_{\max} \right) \left(1 + \frac{\mu}{2} T \right) L_t |x_1 - x_2| \\
& \quad \left. + L_t L_x L_s \left(\frac{4}{9} \frac{\Lambda(S_{\max})}{\Lambda(S_{\max})} \frac{1}{4} + C(n)2TL_\sigma \right) L_R |x_1 - x_2|. \right.
\end{aligned}$$

Since the bounds from above imply that $L_t L_x L_s \leq \frac{9}{4} \frac{\Lambda(S_{\max})}{\Lambda(S_{\max})}$ for L_R sufficiently small, we can choose B_{\max} , S_{\max} , L_I and L_R sufficiently small such that

$$|f_{1,e}(t, x_1) - f_{1,e}(t, x_2)| \leq \frac{1}{2} L_R |x_1 - x_2|.$$

Case 3: $t_+^{S_e^i}(x_1, t) = 0 < t_+^{S_e^i}(x_2, t)$.

Then $\xi_+^{S_e^i}(t_+^{S_e^i}(x_2, t), x_2, t) = 0$ and $f_{1,e}(t, x_1)$ is given by the initial condition, while $f_{1,e}(t, x_2)$ is given by the boundary condition or the coupling conditions. This case can be treated similarly

to case 2. Therefore we do not present the details here. Note that in the case that $f_{1,e}(t, x_2)$ is given by the coupling conditions the compatibility of the initial data with the coupling conditions has to be used.

Summing up the cases 1–3 shows that the Lipschitz-constant of $f_{1,e}$ is bounded by $\frac{1}{2}L_R$, if B_{\max} , S_{\max} , L_I and L_R are sufficiently small. Together with (32) this shows that the Lipschitz-constant of $S_{+,e}^{i+1}$ is bounded by L_R for every $e \in \mathcal{E}$. Analogously it can be shown that the Lipschitz-constant of S_-^{i+1} is bounded by L_R . In other words, we have shown that the Lipschitz-constant of \mathcal{S} with respect to x remains bounded under the fixed-point mapping. Therefore $\Phi(\mathcal{S}) \in M(S_{\max}, L_R)$ for $\mathcal{S} \in M(S_{\max}, L_R)$. In particular, the fixed-point mapping is well-defined.

In order to apply the Banach fixed-point theorem, it remains to show the contraction property. We will only show the estimation for S_+ , the estimation for S_- can be done similarly. Let $\mathcal{S}^i, \tilde{\mathcal{S}}^i \in M(S_{\max}, L_R)$ and denote $\mathcal{S}^{i+1} := \Phi(\mathcal{S}^i)$, $\tilde{\mathcal{S}}^{i+1} := \Phi(\tilde{\mathcal{S}}^i)$. Let $e \in \mathcal{E}$ and assume without loss of generality $t_+^{\tilde{\mathcal{S}}^i}(x, t) \leq t_+^{\mathcal{S}^i}(x, t)$. Then

$$\begin{aligned}
& S_{+,e}^{i+1}(s, \xi_+^{\mathcal{S}^i}(s, x, t)) - \tilde{S}_{+,e}^{i+1}(s, \xi_+^{\tilde{\mathcal{S}}^i}(s, x, t)) \\
&= \int_{t_+^{\tilde{\mathcal{S}}^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2} ((R_{+,e} - R_{-,e})(r, \xi_+^{\mathcal{S}^i}(r, x, t)) - (R_{+,e} - R_{-,e})(r, \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t))) \right. \\
&\quad + \frac{\mu}{2} (S_{-,e}^i(r, \xi_+^{\mathcal{S}^i}(r, x, t)) - \tilde{S}_{-,e}^i(r, \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t))) \\
&\quad \left. - (\sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}^i}(r, x, t)) - \sigma(\tilde{\mathcal{S}}_e^i)(r, \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t))) \right) dr \\
&\quad - \int_{t_+^{\tilde{\mathcal{S}}^i}(x, t)}^{t_+^{\mathcal{S}^i}(x, t)} e^{-\frac{\mu}{2}(s-r)} \left(\frac{\mu}{2} (R_{+,e} - R_{-,e} + \tilde{S}_{-,e}^i)(r, \xi_+^{\mathcal{S}^i}(r, x, t)) - \sigma(\tilde{\mathcal{S}}_e^i)(r, \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t)) \right) dr \\
&\quad + (S_{+,e}^{i+1}(t_+^{\mathcal{S}^i}(x, t), \xi_+^{\mathcal{S}^i}(t_+^{\mathcal{S}^i}(x, t), x, t)) e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}^i}(x, t))} \\
&\quad - \tilde{S}_{+,e}^{i+1}(t_+^{\tilde{\mathcal{S}}^i}(x, t), \xi_+^{\tilde{\mathcal{S}}^i}(t_+^{\tilde{\mathcal{S}}^i}(x, t), x, t)) e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}^i}(x, t))}) \\
&=: \int_{t_+^{\tilde{\mathcal{S}}^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left((F_a) + (F_b) + (F_c) \right) dr + (F_d) + (F_e).
\end{aligned}$$

Using the Lipschitz-constant of R_{\pm} with respect to x , we can estimate (F_a) by

$$\begin{aligned}
|(F_a)| &= \frac{\mu}{2} |(R_{+,e} - R_{-,e})(r, \xi_+^{\mathcal{S}^i}(r, x, t)) - (R_{+,e} - R_{-,e})(r, \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t))| \\
&\leq \frac{\mu}{2} 2 L_R |\xi_+^{\mathcal{S}^i}(r, x, t) - \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t)|.
\end{aligned}$$

Analogously to Lemma 5.1 in [24] we can show that

$$|\xi_+^{\mathcal{S}^i}(r, x, t) - \xi_+^{\tilde{\mathcal{S}}^i}(r, x, t)| \leq TL_{\lambda} \exp(2L_R TL_{\lambda}) \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M = L_{\lambda} TL_x \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \quad (33)$$

This implies

$$|(F_a)| \leq \mu L_R L_{\lambda} TL_x \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M.$$

For (F_b) , we can estimate

$$\begin{aligned} |(F_b)| &\leq \frac{\mu}{2} (|\mathcal{S}_{-,e}^i(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) - \tilde{\mathcal{S}}_{-,e}^i(r, \xi_+^{\mathcal{S}_e^i}(r, x, t))| + |\tilde{\mathcal{S}}_{-,e}^i(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) - \tilde{\mathcal{S}}_{-,e}^i(r, \xi_+^{\tilde{\mathcal{S}}_e^i}(r, x, t))|) \\ &\leq \frac{\mu}{2} \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M + \frac{\mu}{2} L_R |\xi_+^{\mathcal{S}_e^i}(r, x, t) - \xi_+^{\tilde{\mathcal{S}}_e^i}(r, x, t)| \\ &\leq \frac{\mu}{2} (1 + L_R L_\lambda T L_x) \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \end{aligned}$$

The friction term (F_c) can be bounded by

$$\begin{aligned} |(F_c)| &= |\sigma(\mathcal{S}_e^i)(r, \xi_+^{\mathcal{S}_e^i}(r, x, t)) - \sigma(\tilde{\mathcal{S}}_e^i)(r, \xi_+^{\tilde{\mathcal{S}}_e^i}(r, x, t))| \\ &\leq L_\sigma 2L_R |\xi_+^{\mathcal{S}_e^i}(r, x, t) - \xi_+^{\tilde{\mathcal{S}}_e^i}(r, x, t)| + L_\sigma \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M \\ &\leq L_\sigma (1 + 2L_R L_\lambda T L_x) \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \end{aligned}$$

The integral (F_d) can be estimated by

$$|(F_d)| \leq \left(\frac{3\mu}{2} S_{\max} + \sigma_{\max}\right) |t_+^{\tilde{\mathcal{S}}_e^i}(x, t) - t_+^{\mathcal{S}_e^i}(x, t)| \leq \left(\frac{3\mu}{2} S_{\max} + \sigma_{\max}\right) \frac{L_\lambda}{\Delta(S_{\max})} T L_x \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M,$$

where we have used in the last step that $|t_+^{\tilde{\mathcal{S}}_e^i}(x, t) - t_+^{\mathcal{S}_e^i}(x, t)|$ can be estimated as in Lemma 5.1 of [24]. Therefore we can summarize

$$\begin{aligned} &\left| \int_{t_+^{\mathcal{S}_e^i}(x, t)}^s e^{-\frac{\mu}{2}(s-r)} \left((F_a) + (F_b) + (F_c) \right) dr + (F_d) \right| \\ &\leq \left(\frac{1}{12} \frac{1}{1+C(n)} (1 + 3L_\lambda L_R T L_x) + T L_\sigma (1 + 2L_\lambda L_R T L_x) \right. \\ &\quad \left. + \left(\frac{3\mu}{2} S_{\max} + \sigma_{\max}\right) \frac{L_\lambda}{\Delta(S_{\max})} T L_x \right) \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M \\ &\leq \left(\frac{1}{12} + \left(\frac{1}{4} L_R + \left(\frac{3\mu}{2} S_{\max} + \sigma_{\max}\right) \frac{1}{\Delta(S_{\max})}\right) L_\lambda T L_x + T L_\sigma (1 + 2L_\lambda L_R T L_x) \right) \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \end{aligned}$$

Now, we choose $S_{\max} > 0$ and $L_R > 0$ sufficiently small such that

$$\frac{1}{12} + \left(\frac{1}{4} L_R + \left(\frac{3\mu}{2} S_{\max} + \sigma_{\max}\right) \frac{1}{\Delta(S_{\max})}\right) L_\lambda T L_x + T L_\sigma (1 + 2L_\lambda L_R T L_x) \leq \frac{2}{3},$$

i.e., the contraction constant of the terms $(F_a) - (F_d)$ is bounded by $\frac{2}{3}$.

For the estimation of term (F_e) we distinguish three cases.

Case 1: $t_+^{\mathcal{S}_e^i}(x, t) = 0 = t_+^{\tilde{\mathcal{S}}_e^i}(x, t)$.

In this case we have

$$\begin{aligned} |(F_e)| &= |S_{+,e}^{i+1}(0, \xi_+^{\mathcal{S}_e^i}(0, x, t)) e^{-\frac{\mu}{2}s} - \tilde{S}_{+,e}^{i+1}(0, \xi_+^{\tilde{\mathcal{S}}_e^i}(0, x, t)) e^{-\frac{\mu}{2}s}| \\ &\leq |y_+^e(\xi_+^{\mathcal{S}_e^i}(0, x, t)) - y_+^e(\xi_+^{\tilde{\mathcal{S}}_e^i}(0, x, t))| e^{-\frac{\mu}{2}s} \\ &\leq L_I |\xi_+^{\mathcal{S}_e^i}(0, x, t) - \xi_+^{\tilde{\mathcal{S}}_e^i}(0, x, t)| \leq L_I L_\lambda T L_x \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \end{aligned}$$

Case 2: $t_+^{\mathcal{S}_e^i}(x, t) > 0$ and $t_+^{\tilde{\mathcal{S}}_e^i}(x, t) > 0$.

Then $\xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) = \xi_+^{\tilde{\mathcal{S}}_e^i}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), x, t)$ is a boundary node or an inner node.

Case 2a: If $\xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) = \xi_+^{\tilde{\mathcal{S}}_e^i}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), x, t)$ is a boundary node, we have

$$\begin{aligned} |(F_e)| &= |S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), 0)e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x,t))} - \tilde{S}_{+,e}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), 0)e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))}| \\ &= |u_+^e(t_+^{\mathcal{S}_e^i}(x, t)) - u_+^e(t_+^{\tilde{\mathcal{S}}_e^i}(x, t))|e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x,t))} \\ &\quad + |u_+^e(t_+^{\tilde{\mathcal{S}}_e^i}(x, t))|e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))} - e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))}| \\ &\leq L_I |t_+^{\mathcal{S}_e^i}(x, t) - t_+^{\tilde{\mathcal{S}}_e^i}(x, t)| + B_{\max} \frac{\mu}{2} |t_+^{\mathcal{S}_e^i}(x, t) - t_+^{\tilde{\mathcal{S}}_e^i}(x, t)| \\ &\leq (L_I + B_{\max} \frac{\mu}{2}) \frac{L_\lambda}{\Lambda(S_{\max})} TL_x \| \mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i \|_M. \end{aligned}$$

Case 2b: If $\xi_+^{\mathcal{S}_e^i}(t_+^{\mathcal{S}_e^i}(x, t), x, t) = \xi_+^{\tilde{\mathcal{S}}_e^i}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), x, t)$ is an inner node v of the network, then

$$\begin{aligned} |(F_e)| &= |S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v)e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x,t))} - \tilde{S}_{+,e}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v)e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))}| \\ &\leq |S_{+,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v)|e^{-\frac{\mu}{2}(s-t_+^{\mathcal{S}_e^i}(x,t))} - e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))}| \\ &\quad + |S_{+,e}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v) - \tilde{S}_{+,e}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v)|e^{-\frac{\mu}{2}(s-t_+^{\tilde{\mathcal{S}}_e^i}(x,t))}| \\ &\leq S_{\max} \frac{\mu}{2} \frac{L_\lambda}{\Lambda(S_{\max})} TL_x \| \mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i \|_M \\ &\quad + C(n) |S_{-,e}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v) - \tilde{S}_{-,e}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v)|_\infty \end{aligned}$$

with

$$\begin{aligned} &|S_{-,f}^{i+1}(t_+^{\mathcal{S}_e^i}(x, t), v) - \tilde{S}_{-,f}^{i+1}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v)| \\ &\leq |S_{-,f}^{i+1}(0, \xi_{-,f}^{\mathcal{S}_f^i}(0, v, t_+^{\mathcal{S}_e^i}(x, t)))e^{-\frac{\mu}{2}t_+^{\mathcal{S}_e^i}(x,t)} - \tilde{S}_{-,f}^{i+1}(0, \xi_{-,f}^{\tilde{\mathcal{S}}_f^i}(0, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t)))e^{-\frac{\mu}{2}t_+^{\tilde{\mathcal{S}}_e^i}(x,t)}| \\ &\quad + \left| \int_0^{t_+^{\mathcal{S}_e^i}(x,t)} e^{-\frac{\mu}{2}(t_+^{\mathcal{S}_e^i}(x,t)-r)} \left(-\frac{\mu}{2}(R_{+,f} - R_{-,f} - S_{+,f}^i)(r, \xi_{-,f}^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t))) \right. \right. \\ &\quad \quad \left. \left. + \sigma(\mathcal{S}_f^i)(r, \xi_{-,f}^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t))) \right) dr \right. \\ &\quad \left. - \int_0^{t_+^{\tilde{\mathcal{S}}_e^i}(x,t)} e^{-\frac{\mu}{2}(t_+^{\tilde{\mathcal{S}}_e^i}(x,t)-r)} \left(-\frac{\mu}{2}(R_{+,f} - R_{-,f} - \tilde{S}_{+,f}^i)(r, \xi_{-,f}^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))) \right. \right. \\ &\quad \quad \left. \left. + \sigma(\tilde{\mathcal{S}}_f^i)(r, \xi_{-,f}^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))) \right) dr \right| \\ &\leq L_I |\xi_{-,f}^{\mathcal{S}_f^i}(0, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_{-,f}^{\tilde{\mathcal{S}}_f^i}(0, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))| + B_{\max} \frac{\mu}{2} \frac{L_\lambda}{\Lambda(S_{\max})} TL_x \| \mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i \|_M \\ &\quad + \left(\frac{1}{12} \frac{1}{1+C(n)} 3L_R + 2TL_\sigma L_R \right) \max_{r \in [0, t_+^{\tilde{\mathcal{S}}_e^i}(x,t)]} |\xi_{-,f}^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_{-,f}^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))| \\ &\quad + \left(\frac{1}{12} \frac{1}{1+C(n)} + 2TL_\sigma \right) \| \mathcal{S}_f^i - \tilde{\mathcal{S}}_f^i \|_M + (3\frac{\mu}{2} S_{\max} + \sigma_{\max}) (1 + \frac{\mu}{2} T) \frac{L_\lambda}{\Lambda(S_{\max})} TL_x \| \mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i \|_M, \end{aligned}$$

where we used in the last step the estimation of $|t_+^{\mathcal{S}_e^i}(x, t) - t_+^{\tilde{\mathcal{S}}_e^i}(x, t)|$ as in (F_d) and estimated the integral terms similar to (31). For the characteristic curves we can estimate for $r \in [0, t_+^{\mathcal{S}_e^i}(x, t)]$

$$\begin{aligned} & |\xi_-^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))| \\ & \leq |\xi_-^{\mathcal{S}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t))| + |\xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t))| \\ & \stackrel{(33)}{\leq} L_\lambda T L_x \|\mathcal{S}_f^i - \tilde{\mathcal{S}}_f^i\|_M + |\xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\mathcal{S}_e^i}(x, t)) - \xi_-^{\tilde{\mathcal{S}}_f^i}(r, \xi_-^{\tilde{\mathcal{S}}_f^i}(t_+^{\mathcal{S}_e^i}(x, t), v, t_+^{\mathcal{S}_e^i}(x, t)), t_+^{\mathcal{S}_e^i}(x, t))| \\ & \leq L_\lambda T L_x \|\mathcal{S}_f^i - \tilde{\mathcal{S}}_f^i\|_M + L_x L_s \frac{L_\lambda}{\Lambda(S_{\max})} T L_x \|\mathcal{S}_e^i - \tilde{\mathcal{S}}_e^i\|_M. \end{aligned}$$

In the second step we used $\xi_-^{\tilde{\mathcal{S}}_f^i}(r, v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t)) = \xi_-^{\tilde{\mathcal{S}}_f^i}(r, \xi_-^{\tilde{\mathcal{S}}_f^i}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), v, t_+^{\tilde{\mathcal{S}}_e^i}(x, t)), t_+^{\tilde{\mathcal{S}}_e^i}(x, t))$. This yields

$$\begin{aligned} |(F_e)| & \leq \left(S_{\max} \frac{\mu}{2} + C(n) L_I (\Lambda(S_{\max}) + L_s L_x) + C(n) B_{\max} \frac{\mu}{2} \right. \\ & \quad \left. + \left(\frac{1}{4} L_R + 2TC(n) L_\sigma L_R \right) (\Lambda(S_{\max}) + L_s L_x) + C(n) (3 \frac{\mu}{2} S_{\max} + \sigma_{\max}) (1 + \frac{\mu}{2} T) \right) \\ & \quad \cdot \frac{L_\lambda}{\Lambda(S_{\max})} T L_x \|\mathcal{S}^i - \tilde{\mathcal{S}}^i\|_{C([0, T] \times \mathcal{E})^2} + \left(\frac{1}{12} + 2TC(n) L_\sigma \right) \|\mathcal{S}^i - \tilde{\mathcal{S}}^i\|_{C([0, T] \times \mathcal{E})^2}. \end{aligned}$$

Case 3: $t_+^{\mathcal{S}_e^i}(x, t) = 0$, $t_+^{\tilde{\mathcal{S}}_e^i}(x, t) > 0$.

Then $\xi_+^{\tilde{\mathcal{S}}_e^i}(t_+^{\tilde{\mathcal{S}}_e^i}(x, t), x, t) = 0$ is either a boundary node or an inner node. Again, this case can be treated similar to case 2.

In all cases 1, 2 and 3 we can choose S_{\max} , B_{\max} , L_I and L_R sufficiently small such that $|(F_e)| \leq \frac{1}{6} \|\mathcal{S}^i - \tilde{\mathcal{S}}^i\|_{C([0, T] \times \mathcal{E})^2}$. Together with the estimates for terms $(F_a) - (F_d)$ above this shows that the fixed-point map Φ is a contraction with contraction constant bounded by $\frac{2}{3} + \frac{1}{6} < 1$, if S_{\max} , B_{\max} , L_I and L_R are sufficiently small. Therefore the Banach fixed-point theorem shows the existence of a unique fixed-point, which solves the observer system (19) together with the initial and boundary conditions (23)-(24) and the coupling conditions (21)-(22). \square

Using Theorem 3.3 we can now show the existence of semi-global solutions on any finite time interval $[0, T]$ (cf. Theorem 6.1 in [24]).

Theorem 3.6. *Let some $T > 0$ be given and assume that the conditions of Theorem 3.3 (apart from the restriction on T) are satisfied. Then there exist a constant $C_T > 0$ and constants $c_1(T)$, $c_2(T)$, $c_3(T)$, $c_4(T) > 0$ such that, if*

$$B_{\max} \leq c_1(T), \quad L_I \leq c_2(T), \quad S_{\max} \leq c_3(T), \quad \text{and } L_R \leq c_4(T),$$

then the observer system (19) for velocity measurements together with the initial and boundary conditions (23)-(24) and the coupling conditions (21)-(22) admits a solution on $[0, T]$ satisfying

$$\|\mathcal{S}_\pm\|_{L^\infty([0, T] \times [0, \ell])} \leq C_T.$$

Proof. Let $T_1 > 0$ such that $T_1 < \min_{e \in \mathcal{E}} \frac{\ell^e}{\Lambda(S_{\max})}$ and $1 - e^{-\frac{\mu}{2} T_1} \leq \frac{1}{12} \min\{1, \frac{1}{1+C(n)} \frac{4}{9} \frac{\Lambda(S_{\max})}{\Lambda(S_{\max})}\}$.

The main idea of the proof of Theorem 3.6 is to apply Theorem 3.3 $\lceil \frac{T}{T_1} \rceil$ -times.

First, let us note that in addition to the bounds on B_{\max} , L_I , S_{\max} and L_R in the proof of Theorem 3.3 we have also bounds on these variables that depend on each other, which can be summarized as

$$B_{\max} \leq \min\{\tilde{c}_1 S_{\max}, \tilde{c}_2 L_R\}, \quad L_I \leq \tilde{c}_3 L_R, \quad S_{\max} \leq \min\{\tilde{c}_4 L_R, \tilde{c}_5 \sqrt{L_R}\}. \quad (34)$$

Now, consider the interval $[0, T_1]$ and let some initial and boundary data be given that are bounded by $B_{\max,1}$ and are Lipschitz-continuous with respect to x with Lipschitz-constant bounded by $L_{I,1}$. Let

$$\begin{aligned} S_{\max,1} &:= \frac{1}{c_1} B_{\max,1} =: f_1(B_{\max,1}), \\ L_{R,1} &:= \max\left\{\frac{1}{c_2} B_{\max,1}, \frac{1}{c_3} L_{I,1}, \frac{1}{c_4} \frac{1}{c_1} B_{\max,1}, \frac{1}{c_2^2} \frac{1}{c_1^2} B_{\max,1}^2\right\} =: f_2(B_{\max,1}, L_{I,1}). \end{aligned}$$

Then, $B_{\max,1}$, $L_{I,1}$, $S_{\max,1}$ and $L_{R,1}$ satisfy the bounds (34) and we can choose $B_{\max,1}$ and $L_{I,1}$ sufficiently small such that $B_{\max,1}$, $L_{I,1}$, $S_{\max,1}$ and $L_{R,1}$ satisfy the bounds of Theorem 3.3, i.e., Theorem 3.3 provides existence of solutions S_{\pm} on the time interval $[0, T_1]$.

Next, we consider the time interval $[T_1, 2T_1]$ and use $S_{\pm}(T_1)$ as initial data, which is bounded by $B_{\max,2} := S_{\max,1}$ and its Lipschitz-constant is bounded by $L_{I,2} \leq L_{R,1}$. Again we define

$$S_{\max,2} := f_1(B_{\max,2}), \quad L_{R,2} := f_2(B_{\max,2}, L_{I,2}).$$

In particular, $S_{\max,2}$ and $L_{R,2}$ can be written as a monotonically increasing function of $B_{\max,1}$ and $L_{I,1}$. Thus, we can choose $B_{\max,1}$ and $L_{I,1}$ sufficiently small such that Theorem 3.3 provides the existence of solutions also on the time interval $[T_1, 2T_1]$. Proceeding in the same way, we can show existence of a solution $\mathcal{S} = (S_+, S_-)$ on the time interval $[0, T]$ satisfying $\|S_{\pm}\|_{L^\infty([0, T] \times [0, \ell])} \leq S_{\max, N} =: C_T$ with $N = \lceil \frac{T}{T_1} \rceil$, if $B_{\max} := B_{\max,1} \leq c_1(T)$, $L_I := L_{I,1} \leq c_2(T)$, $S_{\max} := S_{\max,1} \leq c_3(T)$ and $L_R := L_{R,1} \leq c_4(T)$ with $c_1(T)$, $c_2(T)$, $c_3(T)$, $c_4(T)$ sufficiently small. \square

4. EXPONENTIAL SYNCHRONIZATION

In this section, we show convergence of solutions of the observer system (8)-(9) towards solutions of the original system (3)-(4) for the case of measurement of one of the fields velocity, density or mass flow, i.e., for observer terms given by (10), (11) or (12). In general, (3)-(4) and (8)-(9) have different initial data, since usually only an approximation of the initial data of the exact solution is known. In the first part of this section, we consider the observer system on a single pipe with length ℓ and complement the equations for the original system and the observer system by the boundary conditions

$$m(t, 0) = \hat{m}(t, 0) = m_b(t), \quad h(t, \ell) = \hat{h}(t, \ell) = h_b(t), \quad 0 \leq t \leq T, \quad (35)$$

for the mass flow m at the left end of the pipe and the total specific enthalpy h at the right end of the pipe.

For the proof of exponential synchronization we use the following assumptions:

- (A1) **Bounded state solution of original system:** For the system (3)-(4) together with the boundary conditions (35) there exists a *classical solution*, i.e., a pair of functions

$$(\rho, v) \in (C^1([0, T]; L^\infty(0, \ell)) \cap C^0([0, T]; H^1(0, \ell)))^2$$

that satisfies (3)-(4) in a pointwise sense a.e., such that there exist $\underline{\rho}, \bar{\rho}, \tilde{v} > 0$ so that

$$0 < \underline{\rho} \leq \rho(t, x) \leq \bar{\rho} \quad \text{and} \quad -\tilde{v} \leq v(t, x) \leq \tilde{v}$$

for all $0 \leq x \leq \ell$ and $0 \leq t \leq T$.

- (A2) **Subsonic condition:** The velocity bound \tilde{v} satisfies

$$\rho P''(\rho) \geq 4|\tilde{v}|^2 \quad \forall \underline{\rho} \leq \rho \leq \bar{\rho}.$$

- (A3) **Bounded state solution of observer system:** The observer system (8)-(9) with the boundary conditions (35) admits a classical solution

$$(\hat{\rho}, \hat{v}) \in (C^1([0, T]; L^2(0, \ell)) \cap C^0([0, T]; H^1(0, \ell)))^2$$

that satisfies

$$0 < \underline{\rho} \leq \hat{\rho}(t, x) \leq \bar{\rho} \quad \text{and} \quad -\tilde{v} \leq \hat{v}(t, x) \leq \bar{v}$$

for all $0 \leq x \leq \ell$ and $0 \leq t \leq T$.

(A4) **Smallness-condition:** Assume that both solutions satisfy the following a priori bounds

$$\begin{aligned} \|\partial_t \rho\|_{L^\infty([0, T] \times [0, \ell])} + \|\partial_t v\|_{L^\infty([0, T] \times [0, \ell])} &\leq C_t, \\ \|v\|_{L^\infty([0, T] \times [0, \ell])}, \|\hat{v}\|_{L^\infty([0, T] \times [0, \ell])} &\leq \bar{v}, \end{aligned}$$

for $C_t, \bar{v} > 0$ sufficiently small. This means that both solutions have small velocity and the exact solution has sufficiently small time derivatives.

These assumptions are reasonable in practice, since in the usual operational range of gas pipes the gas velocities are small and only slow changes in the gas flow occur. Moreover, existence of semi-global Lipschitz-continuous solutions of the observer system for measurement of v or ρ and of the original system has been shown in Section 3. These solutions satisfy bounds on density and velocity that are of the same form as the bounds in (A1) and (A3).

Now we can state the main theorem of this section.

Theorem 4.1. *Consider a solution $u = (\rho, v)$ of (3)-(4) and a solution $\hat{u} = (\hat{\rho}, \hat{v})$ of (8)-(9) with observer terms given by (10), (11) or (12) satisfying the boundary conditions (35), the properties stated in (A1)-(A3) and the bounds in (A4) with $C_t, \bar{v} > 0$ sufficiently small.*

Then there exist constants $C_1, C_2 > 0$ such that

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)} \leq C_1 \|u_0 - \hat{u}_0\|_{L^2(0, \ell)} e^{-C_2 t}$$

for all $0 \leq t \leq T$.

Remark 4.2. Note that we have the bound \tilde{v} on the velocity, which asserts that we have subsonic flow, see assumption (A2), and in addition the bound $\bar{v} \leq \tilde{v}$ in Theorem 4.1 ensures that the velocity is sufficiently small in order to control certain terms in our analysis.

In other words, Theorem 4.1 asserts that the state of the observer system converges exponentially towards the exact solution, provided that the time derivative of the exact solution and the velocity in both systems is sufficiently small. The rest of this section is devoted to the proof of this theorem and to the extension to star-shaped networks.

4.1. General strategy of the proof of Theorem 4.1. In order to measure the distance between the exact solution and the solution of the observer system, we will use the *relative energy*, which is defined by

$$\mathcal{H}(\hat{u}|u) := \mathcal{H}(\hat{u}) - \mathcal{H}(u) - \langle \mathcal{H}'(u), \hat{u} - u \rangle,$$

cf. [20, 22]. Here, $u = (\rho, v)$, $\hat{u} = (\hat{\rho}, \hat{v})$ denote the solution of the original system and of the observer system, respectively, $\langle \cdot, \cdot \rangle$ denotes the L^2 -scalar product on $[0, \ell]$ and $\mathcal{H}'(u)$ is the variational derivative of the energy \mathcal{H} , see (5), that is given by

$$\delta_\rho \mathcal{H}(\rho, v) = \frac{1}{2} v^2 + P'(\rho) = h, \quad \delta_v \mathcal{H}(\rho, v) = m. \quad (36)$$

Using these derivatives we can write the system (3)-(4) in the Hamiltonian form

$$\begin{pmatrix} \partial_t \rho \\ \partial_t v \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta_\rho \mathcal{H} \\ \delta_v \mathcal{H} \end{pmatrix} = \begin{pmatrix} 0 \\ -\gamma|v|v \end{pmatrix},$$

which will be crucial in the estimation of the time derivative of the relative energy.

Due to the convexity of the pressure potential P , the relative energy is positive and defines a distance measure on subsonic states. The proof of the following result can be found in [22, Lemma 9].

Lemma 4.3. *Under the assumptions (A1)–(A3) there exist constants $c_0, C_0 > 0$ such that*

$$c_0 \|u_1 - u_2\|_{L^2(0,\ell)}^2 \leq \mathcal{H}(u_1|u_2) \leq C_0 \|u_1 - u_2\|_{L^2(0,\ell)}^2$$

for all $u_1 = (\rho_1, v_1), u_2 = (\rho_2, v_2) \in L^2(0, \ell)$ with

$$0 < \underline{\rho} \leq \rho_1(x), \rho_2(x) \leq \bar{\rho}, \quad -\tilde{v} \leq v_1(x), v_2(x) \leq \tilde{v}, \quad \forall x \in (0, \ell).$$

The main idea of the proof of Theorem 4.1 is to use an extension of the relative energy method, i.e., we estimate $\partial_t \mathcal{H}(\hat{u}|u)$ using a similar strategy as in [22] and then, motivated by an extension of the energy functional used in [25], we add a suitable functional to the relative energy in order to get decrease also with respect to the variables that are not measured. As a first step, we estimate the time derivative of the relative energy.

Lemma 4.4. *Under the assumptions of Theorem 4.1 the time derivative of the relative energy can be bounded by*

$$\begin{aligned} \partial_t \mathcal{H}(\hat{u}|u) &\leq 2\gamma \left(\max\{\|v\|_{L^\infty([0,\ell])}, \|\hat{v}\|_{L^\infty([0,\ell])}\} \right)^3 \frac{1}{\underline{\rho}} \|\rho - \hat{\rho}\|_{L^2([0,\ell])}^2 \\ &\quad + \tilde{C} (\|\partial_t \rho\|_{L^\infty([0,\ell])} + \|\partial_t v\|_{L^\infty([0,\ell])}) \|u - \hat{u}\|_{L^2([0,\ell])}^2 - \langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle \end{aligned} \quad (37)$$

with $\tilde{C} := \max(\frac{1}{2}, C_{P''})$.

Proof. Using the definition of the relative energy, we can compute

$$\begin{aligned} \partial_t \mathcal{H}(\hat{u}|u) &= \langle \mathcal{H}'(\hat{u}), \partial_t \hat{u} \rangle - \langle \mathcal{H}'(u), \partial_t u \rangle - \langle \mathcal{H}''(u) \partial_t u, \hat{u} - u \rangle - \langle \mathcal{H}'(u), \partial_t \hat{u} - \partial_t u \rangle \\ &= \langle \mathcal{H}'(\hat{u}) - \mathcal{H}'(u), \partial_t \hat{u} - \partial_t u \rangle + \langle \mathcal{H}'(\hat{u}) - \mathcal{H}'(u) - \mathcal{H}''(u)(\hat{u} - u), \partial_t u \rangle. \end{aligned}$$

Now we can use that the variational derivatives of the energy \mathcal{H} are given by (36). Together with the equations (3)–(4) and (8)–(9), this leads to

$$\begin{aligned} \partial_t \mathcal{H}(\hat{u}|u) &= \langle h - \hat{h}, \partial_t \rho - \partial_t \hat{\rho} \rangle + \langle m - \hat{m}, \partial_t v - \partial_t \hat{v} \rangle + \langle \mathcal{H}'(\hat{u}) - \mathcal{H}'(u) - \mathcal{H}''(u)(\hat{u} - u), \partial_t u \rangle \\ &= -\langle m - \hat{m}, \gamma|v|v - \gamma|\hat{v}|\hat{v} \rangle + \langle \mathcal{H}'(\hat{u}) - \mathcal{H}'(u) - \mathcal{H}''(u)(\hat{u} - u), \partial_t u \rangle \\ &\quad - \langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle, \end{aligned} \quad (38)$$

where we have used

$$\langle h - \hat{h}, \partial_x \hat{m} - \partial_x m \rangle + \langle \partial_x h - \partial_x \hat{h}, \hat{m} - m \rangle = (h - \hat{h})(\hat{m} - m)|_{x=0}^\ell = 0$$

due to the boundary conditions for m and h . For the first term we note that $|v|v - |\hat{v}|\hat{v} = 2(v - \hat{v}) \int_0^1 |\hat{v} + s(v - \hat{v})| ds$ with

$$\frac{1}{4}(|\hat{v}| + |v|) \leq \int_0^1 |\hat{v} + s(v - \hat{v})| ds \leq \frac{1}{2}(|\hat{v}| + |v|),$$

compare [22]. This implies

$$\begin{aligned} -\langle m - \hat{m}, \gamma|v|v - \gamma|\hat{v}|\hat{v} \rangle &\leq \int_0^\ell \gamma|v| |\rho - \hat{\rho}| |v - \hat{v}| (|v| + |\hat{v}|) dx - \int_0^\ell \frac{1}{2} \gamma \hat{\rho} (v - \hat{v})^2 (|v| + |\hat{v}|) dx \\ &\leq 2\gamma \left(\max\{\|v\|_{L^\infty([0,\ell])}, \|\hat{v}\|_{L^\infty([0,\ell])}\} \right)^3 \frac{1}{\underline{\rho}} \|\rho - \hat{\rho}\|_{L^2([0,\ell])}^2 - \int_0^\ell \frac{1}{4} \gamma \hat{\rho} (|v| + |\hat{v}|) (v - \hat{v})^2 dx. \end{aligned}$$

where the second term in the last line is nonpositive. Since

$$\mathcal{H}'(\hat{u}) - \mathcal{H}'(u) - \mathcal{H}''(u)(\hat{u} - u) = \begin{pmatrix} P'(\hat{\rho}|\rho) + \frac{1}{2}(\hat{v} - v^2) \\ (\hat{\rho} - \rho)(\hat{v} - v) \end{pmatrix},$$

the second term in (38) can be estimated by

$$\begin{aligned} & \langle \mathcal{H}'(\hat{u}) - \mathcal{H}'(u) - \mathcal{H}''(u)(\hat{u} - u), \partial_t u \rangle \\ & \leq \max(\tfrac{1}{2}, C_{P''}) (\|\partial_t \rho\|_{L^\infty([0, \ell])} + \|\partial_t v\|_{L^\infty([0, \ell])}) \|u - \hat{u}\|_{L^2([0, \ell])}^2. \end{aligned} \quad \square$$

We expect that the observer terms $-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle$ in the estimate of $\partial_t \mathcal{H}(\hat{u}|u)$ lead to decrease with respect to the measured variables. In order to get decrease also with respect to the variables that are not measured, we introduce an additional functional \mathcal{G} (cf. the extension of the energy functional in [25]).

Lemma 4.5. *Assume that the assumptions of Theorem 4.1 are satisfied and assume that there are observer terms $\mathcal{L}_\rho, \mathcal{L}_v$ and a functional \mathcal{G} so that*

$$\frac{c_0}{2} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)}^2 \leq \mathcal{H}(\hat{u}|u)(t) + \mathcal{G}(\hat{u}|u)(t) \leq \frac{3}{2} C_0 \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)}^2 \quad (39)$$

and there exists a constant $\bar{c} > 0$ so that

$$\begin{aligned} & -\langle \mathcal{L}_\rho(t, \cdot), h(t, \cdot) - \hat{h}(t, \cdot) \rangle - \langle \mathcal{L}_v(t, \cdot), m(t, \cdot) - \hat{m}(t, \cdot) \rangle + \partial_t \mathcal{G}(\hat{u}|u)(t) \\ & \leq -\bar{c} (\mathcal{H}(\hat{u}|u)(t) + \mathcal{G}(\hat{u}|u)(t)) \end{aligned} \quad (40)$$

for all $0 \leq t \leq T$.

Then there exist constants $C_1, C_2 > 0$ such that

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)} \leq C_1 \|u_0 - \hat{u}_0\|_{L^2(0, \ell)} e^{-C_2 t}.$$

Proof. Combining (40) with Lemma 4.4 yields

$$\begin{aligned} & \partial_t \mathcal{H}(\hat{u}|u)(t) + \partial_t \mathcal{G}(\hat{u}|u)(t) \\ & \leq 2\gamma \frac{\bar{v}^3}{\rho} \|\rho(t, \cdot) - \hat{\rho}(t, \cdot)\|_{L^2}^2 + \tilde{C} C_t \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2}^2 - \bar{c} (\mathcal{H}(\hat{u}|u)(t) + \mathcal{G}(\hat{u}|u)(t)). \end{aligned}$$

By Lemma 4.3, the first and second term can be absorbed into the last term for sufficiently small $C_t, \bar{v} > 0$. This yields

$$\partial_t \mathcal{H}(\hat{u}|u)(t) + \partial_t \mathcal{G}(\hat{u}|u)(t) \leq -C (\mathcal{H}(\hat{u}|u)(t) + \mathcal{G}(\hat{u}|u)(t))$$

for some $C > 0$. Then, using (39) and the Gronwall Lemma shows the assertion. \square

In the following, we state possible functionals \mathcal{G} that allow to verify the conditions (39) and (40) for the observer terms (10), (11) and (12) corresponding to measurements of m, v or ρ , respectively. For velocity measurements we present the proof in detail, while for density and mass flow measurements we just present the main idea of the proof. Once it is shown that the conditions (39) and (40) are satisfied, Theorem 4.1 follows directly from Lemma 4.5.

4.2. Proof for measurement of v . First, we show exponential convergence of the state of the observer system towards the original system state for measurement of the velocity. In this case, we use the observer terms

$$\mathcal{L}_\rho = 0, \quad \mathcal{L}_v = \mu(v - \hat{v})$$

with $\mu > 0$. Under the assumption that $|v|, |\hat{v}| \leq \bar{v}$, we can estimate

$$\begin{aligned} & -\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle \leq -\mu \rho \|v - \hat{v}\|_{L^2}^2 - \langle \mu(v - \hat{v}), \hat{v}(\rho - \hat{\rho}) \rangle \\ & \leq -\frac{1}{2} \mu \rho \|v - \hat{v}\|_{L^2}^2 + \frac{1}{2} \mu \frac{\bar{v}^2}{\rho} \|\rho - \hat{\rho}\|_{L^2}^2. \end{aligned} \quad (41)$$

In order to show decrease with respect to $\|\rho - \hat{\rho}\|_{L^2}^2$, we use the following additional functional, where the construction of \mathcal{F}_1 is inspired by the construction of the additional functional used for a linear problem in [25].

Definition 4.6. Let

$$\mathcal{F}_1(\hat{u}|u)(t) := \int_0^\ell (M(x, t) - \hat{M}(x, t))(v(x, t) - \hat{v}(x, t)) dx$$

with

$$M(x, t) := \int_0^t m(x, t') dt' - \int_0^x \rho_0(x') dx'.$$

The strategy for the proof of Theorem 4.1 in the case of measurement of v is that we will estimate $\partial_t \mathcal{H}(\hat{u}|u) + \delta \partial_t \mathcal{F}_1(\hat{u}|u)$ with a constant $\delta > 0$ and show that $\partial_t \mathcal{F}_1(\hat{u}|u)$ guarantees decrease with respect to $\|\rho - \hat{\rho}\|_{L^2(0, \ell)}^2$, i.e., we will use Lemma 4.5 and verify the conditions (39) and (40) for $\mathcal{G} := \delta \mathcal{F}_1$. As a first step, let us note that for δ sufficiently small, the term $\mathcal{H}(\hat{u}|u) + \delta \mathcal{F}_1(\hat{u}|u)$ is equivalent to $\|u - \hat{u}\|_{L^2(0, \ell)}^2$.

Lemma 4.7. *Suppose that the assumptions (A1)–(A3) hold and*

$$\delta \leq \frac{c_0}{C_{Poin} \ell}$$

with the constant C_{Poin} from the Poincaré inequality. Then

$$\frac{c_0}{2} \|\hat{u}(t, \cdot) - u(t, \cdot)\|_{L^2(0, \ell)}^2 \leq \mathcal{H}(\hat{u}|u)(t) + \delta \mathcal{F}_1(\hat{u}|u)(t) \leq \frac{3}{2} C_0 \|\hat{u}(t, \cdot) - u(t, \cdot)\|_{L^2(0, \ell)}^2$$

for all $0 \leq t \leq T$.

Proof. Since

$$M(0, t) = \int_0^t m(0, t') dt' = \int_0^t m_b(t') dt' = \hat{M}(0, t),$$

we can apply the Poincaré inequality to $\|M - \hat{M}\|_{L^2}^2$. Therefore we can compute

$$\begin{aligned} |\mathcal{F}_1(\hat{u}|u)| &\leq \|M - \hat{M}\|_{L^2} \|v - \hat{v}\|_{L^2} \leq C_{Poin} \ell \|\partial_x M - \partial_x \hat{M}\|_{L^2} \|v - \hat{v}\|_{L^2} \\ &\leq \frac{1}{2} C_{Poin} \ell \|\rho - \hat{\rho}\|_{L^2}^2 + \frac{1}{2} C_{Poin} \ell \|v - \hat{v}\|_{L^2}^2, \end{aligned}$$

where we have used the fact that $\partial_x M(x) = \int_0^t \partial_x m(x, t') dt' - \rho_0(x) = -\rho(x)$ and Young's inequality in the last step. If $\delta \leq \frac{c_0}{C_{Poin} \ell}$, we have

$$\delta |\mathcal{F}_1(\hat{u}|u)| \leq \frac{c_0}{2} \|u - \hat{u}\|_{L^2}^2.$$

Applying Lemma 4.3 yields the assertion. \square

Remark 4.8. Lemma 4.7 can be generalized to functions $u_1 = (\rho_1, v_1)$, $u_2 = (\rho_2, v_2) \in L^2(0, \ell)$ with

$$0 < \underline{\rho} \leq \rho_1(x), \rho_2(x) \leq \bar{\rho}, \quad -\tilde{v} \leq v_1(x), v_2(x) \leq \tilde{v}, \quad \forall x \in (0, \ell),$$

and $M_1(\ell, t) = M_2(\ell, t)$ instead of $M_1(0, t) = M_2(0, t)$. In this case the definition of M has to be modified, i.e., the integral $\int_0^x \rho_0(x') dx'$ has to be replaced by $\int_\ell^x \rho_0(x') dx'$ and boundary conditions for m have to be prescribed at the right end of the pipe.

Thus, we have shown that condition (39) is satisfied. It remains to verify condition (40).

Lemma 4.9. *Consider a solution $u = (\rho, v)$ of (3)–(4) and a solution $\hat{u} = (\hat{\rho}, \hat{v})$ of (8)–(9) with observer terms given by (10) satisfying the boundary conditions (35), the properties stated in (A1)–(A3) and the bounds in (A4) with $C_t, \bar{v} > 0$ sufficiently small.*

Then there exist constants $\delta, \bar{c} > 0$ with $\delta \leq \frac{c_0}{C_{Poin} \ell}$ such that

$$-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_1(\hat{u}|u) \leq -\bar{c} (\mathcal{H}(\hat{u}|u) + \delta \mathcal{F}_1(\hat{u}|u)).$$

Proof. We start by computing the time derivative of \mathcal{F}_1 , which is given by

$$\begin{aligned}\partial_t \mathcal{F}_1(\hat{u}|u) &= \int_0^\ell (\partial_t M - \partial_t \hat{M})(v - \hat{v}) dx + \int_0^\ell (M - \hat{M})(\partial_t v - \partial_t \hat{v}) dx \\ &= \int_0^\ell (m - \hat{m})(v - \hat{v}) dx + \int_0^\ell (M - \hat{M})(\partial_x \hat{h} - \partial_x h) dx \\ &\quad + \int_0^\ell (M - \hat{M})(-\mathcal{L}_v) dx + \int_0^\ell (M - \hat{M})(-\gamma|v|v + \gamma|\hat{v}|\hat{v}) dx \\ &= (E_1) + (E_2) + (E_3) + (E_4),\end{aligned}$$

where the first term can be estimated by

$$\begin{aligned}(E_1) &= \int_0^\ell (m - \hat{m})(v - \hat{v}) dx = \int_0^\ell (\rho(v - \hat{v})^2 + \hat{v}(\rho - \hat{\rho})(v - \hat{v})) dx \\ &\leq \frac{1}{2} \frac{\bar{v}^2}{\bar{\rho}} \|\rho - \hat{\rho}\|_{L^2}^2 + \frac{3}{2} \bar{\rho} \|v - \hat{v}\|_{L^2}^2.\end{aligned}$$

Using integration by parts and the boundary conditions (35) we can estimate

$$\begin{aligned}(E_2) &= \int_0^\ell (\partial_x M - \partial_x \hat{M})(h - \hat{h}) dx = \int_0^\ell (\hat{\rho} - \rho) \left(\frac{1}{2}(v^2 - \hat{v}^2) + P'(\rho) - P'(\hat{\rho}) \right) dx \\ &\leq -C_{P''} \|\rho - \hat{\rho}\|_{L^2}^2 + \int_0^\ell \frac{1}{2} (v + \hat{v})(v - \hat{v})(\hat{\rho} - \rho) dx \\ &\leq -\left(C_{P''} - \frac{\bar{v}^2}{2\rho} \right) \|\rho - \hat{\rho}\|_{L^2}^2 + \frac{1}{2} \underline{\rho} \|v - \hat{v}\|_{L^2}^2.\end{aligned}$$

Applying the Poincaré inequality to (E_3) yields

$$(E_3) \leq C_{Poin} \ell \|\rho - \hat{\rho}\|_{L^2} \mu \|v - \hat{v}\|_{L^2} \leq \frac{1}{4} C_{P''} \|\rho - \hat{\rho}\|_{L^2}^2 + C_{Poin}^2 \ell^2 \mu^2 \frac{1}{C_{P''}} \|v - \hat{v}\|_{L^2}^2.$$

Finally, the last term can be bounded by

$$\begin{aligned}(E_4) &= \int_0^\ell (M - \hat{M})(-\gamma|v|v + \gamma|\hat{v}|\hat{v}) dx \leq C_{Poin} \ell \|\rho - \hat{\rho}\|_{L^2} 2\gamma \bar{v} \|v - \hat{v}\|_{L^2} \\ &\leq \frac{1}{4} C_{P''} \|\rho - \hat{\rho}\|_{L^2}^2 + 4\gamma^2 C_{Poin}^2 \ell^2 \frac{\bar{v}^2}{C_{P''}} \|v - \hat{v}\|_{L^2}^2.\end{aligned}$$

Assume that $\bar{v}^2 \leq \frac{1}{4} \underline{\rho} C_{P''}$. Then summing the contributions from (E_1) – (E_4) yields

$$\partial_t \mathcal{F}_1(\hat{u}|u) \leq -\frac{1}{4} C_{P''} \|\rho - \hat{\rho}\|_{L^2}^2 + (2\bar{\rho} + C_{Poin}^2 \ell^2 \mu^2 \frac{1}{C_{P''}} + C_{Poin}^2 \ell^2 \gamma^2 \underline{\rho}) \|v - \hat{v}\|_{L^2}^2.$$

Combining this with (41) leads to

$$\begin{aligned}& -\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_1(\hat{u}|u) \\ & \leq \left(-\frac{1}{4} \delta C_{P''} + \frac{1}{2} \mu \frac{\bar{v}^2}{\underline{\rho}} \right) \|\rho - \hat{\rho}\|_{L^2}^2 \\ & \quad + \left(-\frac{1}{2} \mu \underline{\rho} + \delta \left(2\bar{\rho} + C_{Poin}^2 \ell^2 \mu^2 \frac{1}{C_{P''}} + C_{Poin}^2 \ell^2 \gamma^2 \underline{\rho} \right) \right) \|v - \hat{v}\|_{L^2}^2.\end{aligned}$$

Now, we choose δ sufficiently small such that

$$\delta \left(2\bar{\rho} + C_{Poin}^2 \ell^2 \mu^2 \frac{1}{C_{P''}} + C_{Poin}^2 \ell^2 \gamma^2 \underline{\rho} \right) \leq \frac{1}{4} \mu \underline{\rho} \quad (42)$$

and the bound $\delta \leq \frac{c_0}{C_{Poin}^\ell}$ from Lemma 4.7 is satisfied. Then, for this δ we choose the velocity bound \bar{v} sufficiently small such that

$$\frac{1}{2}\mu \frac{\bar{v}^2}{\rho} \leq \frac{1}{8}\delta C_{P''}.$$

This yields

$$-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_1(\hat{u}|u) \leq -\frac{1}{8}\delta C_{P''} \|\rho - \hat{\rho}\|_{L^2}^2 - \frac{1}{4}\mu \rho \|v - \hat{v}\|_{L^2}^2,$$

which together with Lemma 4.7 implies the statement of the lemma. \square

Applying Lemma 4.5 finishes the proof of Theorem 4.1 for the case of velocity measurement.

Remark 4.10. Increasing the parameter μ in the observer term does not necessarily improve the speed of convergence of the state of the observer system towards the original system state: In the proof of Lemma (4.9) we see that $-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_1(\hat{u}|u)$ has the decrease rate $\min\{\frac{1}{8}\delta C_{P''}, \frac{1}{4}\mu \rho\}$ and due to the bound (42) on δ a large μ enforces a small δ .

Remark 4.11. In the proof of Lemma 4.9 we see that we need at one end of the pipe boundary conditions for m and at the other end boundary conditions for h , i.e., (35) or

$$m(t, \ell) = \hat{m}(t, \ell) = \tilde{m}_b(t), \quad h(t, 0) = \hat{h}(t, 0) = \tilde{h}_b(t), \quad 0 \leq t \leq T.$$

The reason for this is that we want to use the Poincaré inequality for M in the estimation of (E_3) , so that we need $M = \hat{M}$ at at least one point in the interval $(0, \ell)$. Due to the definition of M in Definition 4.6, this is satisfied at $x = 0$, if we choose boundary conditions for m at the left end of the pipe. Then, if we want to have vanishing boundary contributions in the partial integration in the estimation of (E_2) , we need $(M - \hat{M})(\hat{h} - h)(x = \ell) = 0$. Since in general

$$M(\ell, t) = \int_0^t m(\ell, t') dt' - \int_0^\ell \rho_0(x') dx' \neq \hat{M}(\ell, t),$$

we need boundary conditions for h at the other end of the pipe.

4.3. Idea of proof for measurement of ρ or m . In the case of measurements of ρ the nudging terms are given by

$$\mathcal{L}_\rho = \mu \frac{c}{\sqrt{p'(\hat{\rho})}} \hat{\rho} (\tilde{P}(\rho) - \tilde{P}(\hat{\rho})), \quad \mathcal{L}_v = 0, \quad \mu > 0.$$

The proof of Theorem 4.1 for measurements of ρ is similar to the proof for measurements of v . Again, we start by estimating the last two terms of (37), i.e.,

$$\begin{aligned} & -\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle \\ & \leq -\frac{1}{2}\mu \frac{\rho}{\bar{\rho}} C_{P''} \frac{\sqrt{C_{P'}}}{\sqrt{\bar{C}_{P'}}} \|\rho - \hat{\rho}\|_{L^2}^2 + \frac{1}{2}\mu \left(\frac{\bar{\rho}}{\rho}\right)^3 \frac{\bar{v}^2}{C_{P''}} \left(\frac{\bar{C}_{P'}}{C_{P'}}\right)^{3/2} \|v - \hat{v}\|_{L^2}^2, \end{aligned}$$

where we have used the bounds in assumptions (A1)–(A3) as well as the assumption $|v|, |\hat{v}| \leq \bar{v}$ for some constant $\bar{v} > 0$. In order to ensure decrease also with respect to $\|v - \hat{v}\|_{L^2}^2$, we define the functional

$$\mathcal{F}_2(\hat{u}|u) := \int_0^\ell (N - \hat{N})(\rho - \hat{\rho}) dx$$

with

$$N(x, t) := \int_0^t h(x, t') dt' - \int_\ell^x v_0(x') dx' + \int_\ell^x \int_0^t \gamma |v(x', t')| v(x', t') dt' dx'.$$

Using that $(N - \hat{N})(\ell, t) = 0$ and the Poincaré inequality, we can show that

$$\frac{c_0}{2} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)}^2 \leq \mathcal{H}(\hat{u}|u)(t) + \delta \mathcal{F}_2(\hat{u}|u)(t) \leq \frac{3}{2} C_0 \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(0, \ell)}^2 \quad (43)$$

for all $0 \leq t \leq T$ for δ sufficiently small. Finally, it can be shown that, if $|v|, |\hat{v}| \leq \bar{v}$ with \bar{v} sufficiently small, then there exist constants $\delta, \bar{c} > 0$ such that (43) is satisfied and

$$-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_2(\hat{u}|u) \leq -\bar{c}(\mathcal{H}(\hat{u}|u) + \delta \mathcal{F}_2(\hat{u}|u)),$$

i.e., condition (40) is satisfied for $\mathcal{G} := \delta \mathcal{F}_2$. This finishes the proof for measurement of v .

For the case of measurement of the mass flow m we use the nudging terms

$$\mathcal{L}_\rho = 0, \quad \mathcal{L}_v = \mu(m - \hat{m})$$

with $\mu > 0$. This implies

$$-\langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle \leq -\frac{1}{2} \mu \underline{\rho}^2 \|v - \hat{v}\|_{L^2}^2 + 2\mu \bar{v}^2 \frac{\bar{\rho}^2}{\underline{\rho}^2} \|\rho - \hat{\rho}\|_{L^2}^2.$$

Again, the observer terms give decrease only with respect to one state variable. We use the same functional as for the velocity measurement, i.e. the functional \mathcal{F}_1 defined in Definition 4.6, in order to achieve decrease also with respect to the other state variable. Since the equivalence to the L^2 -norm was already shown in Lemma 4.7, it remains to show that the decrease condition (40) is satisfied. This can be shown similar to the proof of Lemma 4.9.

4.4. Extension to star-shaped networks. Now we extend the convergence results that are shown in Theorem 4.1 for a single pipe to star-shaped networks. Such networks have only one inner node, which we denote by v_0 . We assume that the edges are oriented such that they start in the central node v_0 and end at a boundary node. For the case of measurements of v or m , we prescribe at every boundary node the boundary conditions

$$m^e(t, v) = \hat{m}^e(t, v) = m_b^e(t, v) \quad \forall v \in \mathcal{V}_\partial, \quad (44)$$

while for density measurements we prescribe the boundary conditions

$$h^e(t, v_\partial) = \hat{h}^e(t, v_\partial) = h_b^e(t, v_\partial) \quad (45)$$

for one arbitrary, but fixed boundary node $v_\partial \in \mathcal{V}_\partial$ and

$$m^e(t, v) = \hat{m}^e(t, v) = m_b^e(t, v) \quad \forall v \in \mathcal{V}_\partial \setminus \{v_\partial\}. \quad (46)$$

Together with the energy conserving coupling conditions (13) this allows us to extend the results obtained above for a single pipe to star-shaped networks, up to some limitations detailed below. The main idea in the proof is summing the estimates over all edges, where we have to pay attention to the application of the Poincaré inequality in the proof of Lemma 4.7 and of equation (43) as well as to the partial integration used in the verification of condition (40). For measurements of v we present the proof in detail, while for measurements of m or ρ we just present the idea of the proof.

We start with the result for measurements of v or m . In this case, we get exponential synchronization up to some error that is proportional to the difference between the total mass in the original system and the total mass in the observer system at the initial time $t = 0$.

Theorem 4.12. *Let $u^e = (\rho^e, v^e)$ and $\hat{u}^e = (\hat{\rho}^e, \hat{v}^e)$, $e \in \mathcal{E}$, be Lipschitz continuous solutions of (3)-(4) and (8)-(9), respectively, on the network $(\mathcal{V}, \mathcal{E})$ with boundary conditions (44) and source terms $\mathcal{L}_\rho, \mathcal{L}_v$ given by (10) or (12), that satisfy the coupling conditions (13), the properties stated in the assumptions (A1)-(A3) and the bounds in (A4) with $C_t, \bar{v} > 0$ sufficiently small on all pipes $e \in \mathcal{E}$.*

Then there exist constants $C_1, C_2, C_3 > 0$ such that

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(\mathcal{E})}^2 \leq C_1 \|u_0 - \hat{u}_0\|_{L^2(\mathcal{E})}^2 \exp(-C_2 t) + C_3 |\Delta M_0|$$

for all $0 \leq t \leq T$ with mass difference at time $t = 0$

$$\Delta M_0 := \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \rho_0^e(x) dx - \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \hat{\rho}_0^e(x) dx.$$

Remark 4.13. In contrast to the result on a single pipe, in Theorem 4.12 there is the additional term $C_3|\Delta M_0|$ depending on the initial total mass difference. Due to the choice of the boundary conditions for m at each boundary node and the choice $\mathcal{L}_\rho = 0$ for measurement of v or m , there is no chance that this contribution vanishes: For the mass difference at time t

$$\Delta M(t) := \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \rho^e(x, t) dx - \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \hat{\rho}^e(x, t) dx$$

we have

$$\partial_t(\Delta M(t)) = \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (-\partial_x m^e(x, t) + \partial_x \hat{m}^e(x, t)) dx = \sum_{e \in \mathcal{E}} (m^e(v_0) - \hat{m}^e(v_0)) = 0,$$

where we have used the boundary conditions (44) and the coupling conditions (13). This implies that $\Delta M(t) = \Delta M_0$ for all $t \in [0, T]$, i.e. the total mass difference at time t is the same as the total mass difference at the initial time $t = 0$.

Proof of Theorem 4.12. By summing over all edges $e \in \mathcal{E}$, the proof in Section 4.2 for measurement of v transfers almost verbatim to star-shaped networks except for the application of the Poincaré inequality in the proof of Lemma 4.7 and the partial integration used in the proof of Lemma 4.9, which we will detail in the following. In order to transfer Lemma 4.7 to networks, we define analogously to Definition 4.6

$$\mathcal{F}_1(\hat{u}|u)(t) := \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (M^e(x, t) - \hat{M}^e(x, t))(v^e(x, t) - \hat{v}^e(x, t)) dx$$

with

$$M^e(x, t) := \int_0^t m^e(x, t') dt' - \int_{\ell^e}^x \rho_0^e(x') dx', \quad e \in \mathcal{E}.$$

Due to the boundary conditions (44) we have

$$M^e(v) = \int_0^t m^e(v, t') dt' = \hat{M}^e(v) \quad \forall v \in \mathcal{V}_\partial, e \in \mathcal{E}(v).$$

Since the network is star-shaped, every edge $e \in \mathcal{E}$ is incident to some boundary node. Therefore, we can apply the Poincaré inequality to each edge separately and show similarly to Lemma 4.7 that

$$\frac{c_0}{2} \|\hat{u}(t, \cdot) - u(t, \cdot)\|_{L^2(\mathcal{E})}^2 \leq \sum_{e \in \mathcal{E}} \mathcal{H}(\hat{u}^e|u^e)(t) + \delta \mathcal{F}_1(\hat{u}|u)(t) \leq \frac{3}{2} C_0 \|\hat{u}(t, \cdot) - u(t, \cdot)\|_{L^2(\mathcal{E})}^2$$

for all $0 \leq t \leq T$, if $\delta \leq \frac{c_0}{C_{\text{Poin}} \ell_{\max}}$ with $\ell_{\max} := \max_{e \in \mathcal{E}} \ell^e$.

Next, we consider the partial integration that is used in the estimation of term (E_2) in the proof of Lemma 4.9. For star-shaped networks, we get

$$\begin{aligned}
(E_2) &= \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (M^e - \hat{M}^e)(\partial_x \hat{h}^e - \partial_x h^e) dx \\
&= \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (\partial_x M^e - \partial_x \hat{M}^e)(h^e - \hat{h}^e) dx + \sum_{e \in \mathcal{E}} (M^e - \hat{M}^e)(\hat{h}^e - h^e)|_0^{\ell^e} \\
&= \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (\partial_x M^e - \partial_x \hat{M}^e)(h^e - \hat{h}^e) dx + \sum_{v \in \mathcal{V}_\partial} \sum_{e \in \mathcal{E}(v)} (M^e(v) - \hat{M}^e(v))(\hat{h}^e(v) - h^e(v)) \\
&\quad - (\hat{h}(v_0) - h(v_0)) \sum_{e \in \mathcal{E}(v_0)} (M^e(v_0) - \hat{M}^e(v_0)),
\end{aligned}$$

where we have used the coupling conditions (13) in the last step. The sum over the boundary nodes vanishes due to the boundary condition (44) and for the last term we can compute

$$\begin{aligned}
&\sum_{e \in \mathcal{E}(v_0)} (M^e(v_0) - \hat{M}^e(v_0)) \\
&= \int_0^t \left(\sum_{e \in \mathcal{E}(v_0)} m^e(v_0, t') - \sum_{e \in \mathcal{E}(v_0)} \hat{m}^e(v_0, t') \right) dt' + \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \rho_0^e(x') dx' - \sum_{e \in \mathcal{E}} \int_0^{\ell^e} \hat{\rho}_0^e(x') dx',
\end{aligned}$$

where the first term vanishes due to the coupling conditions (13). Using the estimation of (E_2) in the proof of Lemma 4.9 and the fact that by Assumption (A2) the term $\hat{h}(v_0) - h(v_0)$ is bounded by a constant C_h depending only on the bounds in the assumptions, this implies

$$(E_2) \leq - \left(C_{P''} - \frac{\bar{v}^2}{2\rho} \right) \|\rho - \hat{\rho}\|_{L^2(\mathcal{E})}^2 + \frac{1}{2} \rho \|v - \hat{v}\|_{L^2(\mathcal{E})}^2 + C_h |\Delta M_0|.$$

For measurements of v , we can proceed as in the proof of Lemma 4.9. This yields

$$\begin{aligned}
& - \langle \mathcal{L}_\rho, h - \hat{h} \rangle - \langle \mathcal{L}_v, m - \hat{m} \rangle + \delta \partial_t \mathcal{F}_1(\hat{u}|u) \\
& \leq - \frac{1}{8} \delta C_{P''} \|\rho - \hat{\rho}\|_{L^2(\mathcal{E})}^2 - \frac{1}{4} \mu \rho \|v - \hat{v}\|_{L^2(\mathcal{E})}^2 + \delta C_h |\Delta M_0|.
\end{aligned}$$

As in the proof of Lemma 4.5, we can then apply a Gronwall Lemma, which yields the assertion. The statement for measurement of m follows similarly. \square

Now, we want to extend the exponential synchronization for measurement of ρ to star-shaped networks. Here, we do not get an additional term depending on the initial mass difference, but we will use a slightly adapted definition of the functional \mathcal{F}_2 , which makes use of the continuity of the specific enthalpy h included into the coupling conditions.

Theorem 4.14. *Let $u^e = (\rho^e, v^e)$ and $\hat{u}^e = (\hat{\rho}^e, \hat{v}^e)$, $e \in \mathcal{E}$, be Lipschitz continuous solutions of (3)-(4) and (8)-(9), respectively, on the network $(\mathcal{V}, \mathcal{E})$ with boundary conditions (45)-(46) and source terms $\mathcal{L}_\rho, \mathcal{L}_v$ given by (11), that satisfy the coupling conditions (13), the properties stated in the assumptions (A1)-(A3) and the bounds in (A4) with $C_t, \bar{v} > 0$ sufficiently small on all pipes $e \in \mathcal{E}$.*

Then there exist constants $C_1, C_2 > 0$ such that

$$\|u^e(t, \cdot) - \hat{u}^e(t, \cdot)\|_{L^2(\mathcal{E})}^2 \leq C_1 \|u_0^e - \hat{u}_0^e\|_{L^2(\mathcal{E})}^2 \exp(-C_2 t) \quad (47)$$

for all $0 \leq t \leq T$.

Idea of proof. For measurements of ρ , we define similar to the definition in Section 4.3

$$\mathcal{F}_2(\hat{u}|u) := \sum_{e \in \mathcal{E}} \int_0^{\ell^e} (N^e - \hat{N}^e)(\rho^e - \hat{\rho}^e) dx$$

with

$$N^e(x, t) := \int_0^t h^e(x, t') dt' - \int_{v_\partial}^x v_0^e(x') dx' + \int_{v_\partial}^x \int_0^t \gamma |v^e(x', t')| v^e(x', t') dt' dx',$$

where we integrate along the unique path from v_∂ to $x \in e$. It can be shown that

$$\frac{c_0}{2} \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(\mathcal{E})}^2 \leq \sum_{e \in \mathcal{E}} \mathcal{H}(\hat{u}^e|u^e)(t) + \delta \mathcal{F}_2(\hat{u}|u)(t) \leq \frac{3}{2} C_0 \|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^2(\mathcal{E})}^2$$

for all $0 \leq t \leq T$ for δ sufficiently small. Finally, using the coupling conditions (13) we can show that all boundary terms at inner nodes of the network vanish and (40) is satisfied for $\mathcal{G} = \delta \mathcal{F}_2$ for suitable constants $\delta, \bar{v}, \bar{c} > 0$. \square

Remark 4.15. For measurements of v or m we use the boundary conditions (44), i.e., we prescribe the value of the mass flow at every boundary node. Due to this choice of the boundary conditions it is possible to apply the Poincaré inequality in order to show equivalence of $\mathcal{H} + \delta \mathcal{F}_1$ to the L^2 -norm (see the proof of Theorem 4.12), but this boundary conditions also lead to the mass difference ΔM_0 in Theorem (4.12). In contrast to this, for measurements of the density we use the boundary conditions (45)-(46), i.e. at one fixed boundary node we use boundary conditions for h , while at all other boundary nodes we prescribe the value of m . This can be done since the continuity of h is included in the coupling conditions such that the Poincaré inequality, which is needed in order to show the norm equivalence, can be applied. In combination with the coupling conditions this choice of boundary conditions has the advantage that we do not have a term of the form ΔM_0 at the right hand side of (47).

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