Contact Pressure over Doubly Connected Rectangular Domains and Punch Shape Optimization

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Abstract. Contact problems arise in a variety of industrial processes, engineering and biomechanical systems. 3-D contact problem for a rigid punch with a doubly connected base bounded by the lines close to rectangles is in consideration. An analytic-numerical technique is developed for its solving. The problem contains Fredholm integral equations of the first kind, which are transformed into the second kind by means of regularization. Using the simple layer potential expansion, the kernels of the integrals are presented in the form of expansions in the powers of the polar radius. The difference between the values of the desired function at different points and the subsequent interpolation of the terms are proposed to smooth the kernels and eliminate singularities. The integral equations are reduced to one-dimension and then solved using quadrature formulas. Subsequently a punch shape is taken as a desired function, and as a minimizing functional is considered the root-mean-square deviation of the pressure distribution arising under the punch from some optimal distribution. In this case, the values of the total forces and moments applied to the punch are assumed to be given, which leads to restrictions imposed on the distributions by the equilibrium conditions. The normal displacements are determined which arising under the action of the found contact pressure on the elastic half-space. The desired punch shape is found using the simple layer potential. A solution to the problem is obtained for the punch with the doubly connected base bounded by lines close to rectangles.

INTRODUCTION

The mechanics of contact interaction plays a considerable role in the development of industrial. Each machine is a set of interacting parts, and it is necessary to ensure their contact strength and rigidity exists. Nowadays, contact problems arise constantly, the most frequent and modern example in everyday life is touching the screen of a smartphone by a finger. According to [1-3], one of the main tasks of calculating accuracy in mechanical engineering is the calculation of the loss of accuracy due to force deformations. Contact elastic displacements make up a significant part in the balance of elastic displacements of machines and their units in industry and also in gas and oil equipment. According to the research results, the contact deformations are up to 22% of the total elastic displacements [2, 3]. In real machines, contact deformations have a great influence, so in the spindle units of machine tools contact displacements are up to 50% and in consoles and calipers up to 80-90% [2]. For this reason, considerable effort has been put into the contact phenomena modeling, analysis and numerical simulations. The literature on this field is extensive. The works by Hertz and Boussinesq, published at the end of the 19th century, are generally considered the beginning of classical contact mechanics. Since that, contact interaction has been studied extensively and is still important.

The problems of a rigid punch indented into an elastic half-space are applied to design beams, slabs, foundations lying on an elastic base and so rigid in comparison with the base that their deformations can be neglected when calculating normal stresses. An important problem is the forecasting of subsidence and rolls of operated constructions in the case of difficult engineering and geological conditions. In addition, during operation, the

constructions are exposed to various kinds of local loads, often transmitted through the reinforcing elements (stiffeners, linings, rod systems - frames and trusses). The presence of brackets, branch pipes, various supports and fasteners, as well as other design parts also leads to local loads.

Finding stresses and deformations in the elements of the engineering systems are traditional tasks for contact problems. The solutions of such problems are characterized by significant mathematical difficulties as mentioned in [1]. This leads to simplified and idealized models without taking into account real surface characteristics, initial stresses, friction, etc. The general mathematical apparatus for solving 3-D problems with different conditions and complex geometry of contact still does not exist. Each problem formulation requires the development of its own solution method. The existence of solutions, their uniqueness or non-uniqueness, and stability of solutions should be established. Numerical analysis of the solutions and the construction of reliable and efficient algorithms for their numerical approximations with guaranteed accuracy are necessary.

In 1957 Galin published his treatise in Contact Mechanics, which further was generalized together with others in Galin and Gladwell [2] where the base solution of contact problems with an emphasis on idealized linear elastic problems for classical domains using some analytical methods were presented. A lot of modern publications developed the problems, such as Barber [1]. Alexandrov and Pozharskii [3] gave a systematic approach, based on Green's functions and integral equations, to the analytical and numerical methods and resulted for a great number of 3-D contact problems for elastic bodies.

A mathematical model of contact interaction between two rectangular plates was presented by Krysko et al. [4]. Stress-strain state was analyzed there by the method of variational iterations.

The problem of a ring punch is a relevant one in the contact mechanics. In 1967 Alexandrov constructed the asymptotic solution of the axisymmetric problem of the effect of an annular ring punch on an elastic half-space, which is also described in Alexandrov and Pozharskii [3]. In the same vein, Argatov and Nazarov [5] investigated the linear contact problem for a narrow ring in plan punch by the method of matched asymptotic expansions. For the first time Roitman and Shishkanova [6] obtained an analytical solution to the problem of an annular punch in the form of a double series, in which coefficients were calculated exactly from the simple recurrence relations.

The solution method based on the use of a simple layer potential for the doubly connected punch close to being annular was further developed in [7]-[9].

Babich and Guz [10] considered the problem of the contact interaction of elastic bodies with initial residual stresses disregarding the friction forces. The case of a rigid cylindrical annular punch and an elastic half-space was studied in [11].

Most of the primary works deal almost exclusively with idealized contact problems. The emphasis in the work is on the solution of the mixed boundary value problem that arises in various idealized contact problems. Real surfaces are rarely perfectly smooth and have curvature or roughness which often causes the contact areas to be extremely small, and therefore, the resulting contact pressure is usually relatively high [12]. Understanding the interdependence of the contact domain and contact characteristics, such as load, contact stiffness, indentation value, deformations and many others that depend on the morphological properties of roughness, is still considered a challenging problem. Contact mechanics of rough surfaces is a very active area of research today. A great number of works are devoted to it, and the papers [12]-[19] should be mentioned.

Friction forces influence on the contact process has attracted substantial attention from researchers for its paramount importance to address many practical applications in engineering and medicine. During friction phenomena mechanical, electrical, thermal, chemical and vibration processes occur simultaneously. Since contact interaction is always accompanied by the presence of friction, the design of various machine parts should be developed taking into account the friction forces. Trends in understanding the mechanisms of solids contact in tribosystems were generalized by Goryacheva [15] and Guz [20]. Myshkin et al. [21] showed that the development of precision tribosystems requires further advancement in the theory of friction, numerical models, and experimental studies on contact interaction.

The contact problems solutions in the classical theory of elasticity that were featured by Galin and Gladwell [2] certainly enjoy a high level of prominence. Much attention was paid to the study of the friction contact under conditions of complete and incomplete slipping of surfaces in the contact area. The methods for solving plane contact problems of this class were developed, based on the use of the theory of complex variable functions. The solution of 3-D contact problems for a simply connected axisymmetric punch sliding with friction along the boundary of an elastic half-space was proposed.

Goryacheva [15] created a remarkable comprehensive research of contact problems for discrete contact of rough surfaces, the effect of imperfect elasticity and mechanical inhomogeneity of contacting bodies, models of friction and wear, changes in contact characteristics during the wear process, etc. Quasistatic contact problems continue to

be in consideration by many authors [22, 23]. The sliding frictional contact problem between a bidirectional functionally graded material half-plane and a rigid punch with triangular profile was considered in [24]. The solutions were made via two methods: finite element and an expansion-collocation, quadrature integration techniques and a recursive integration method for the Cauchy integral. In connection with the rapid development of computer technology, Finite Element Methods (FEM) and Boundary Element Methods (BEM) are widely used in modern mechanics now [13, 25]. However, an application of these methods are often limited, since contact problems have a singularity in the state equations that impair the stability of the numerical solution when the source point is in the element to be integrated. Treatment of the singularity is necessary in each problem statement. An efficient method of the regularization of singular and hyper-singular kernels for the problem of the BEM's three-dimensional elasto-statical formulation was proposed in [26].

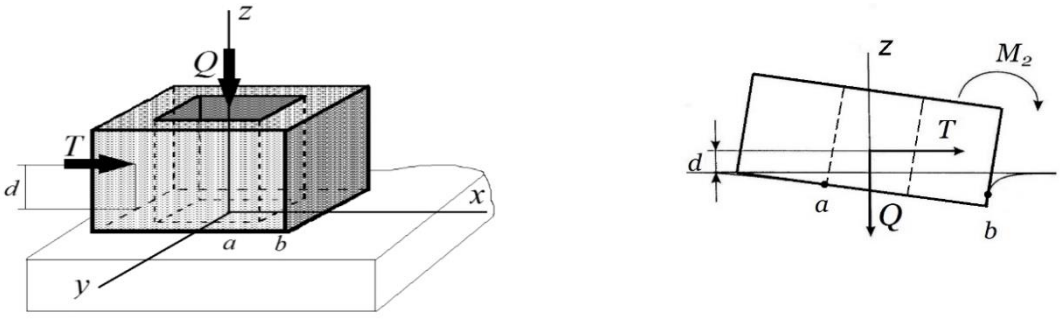
The book by Popov et al. [27] aimed to provide a compendium of exact solutions for rotationally symmetric contact problems, which are suitable for practical applications. Based on a systematic distinction regarding the type of contact, the regime of friction and the contact geometry, adhesion effects, a multitude of technically relevant contact problems from mechanical engineering, the automotive industry and medical engineering were discussed. In addition, contact between transverse-isotropic elastic materials and functionally graded materials was considered. Such problems are a focus of current research especially in the fields of actuator technology and biomechanics. Argatov and Mishuris [28] developed a comprehensive and unifying approach to articular contact mechanics with an emphasis on frictionless contact interaction of thin cartilage layers. Complex contact geometry is also very important in this area of research.

Numerous results concerning the indenter shape optimization were obtained in the theory of elasticity [29-32]. A formulation well suited to optimal design techniques is developed by Banichuk and Ivanova [33]. It derives an effective decomposition method and optimality conditions for contact pressure distribution. As an example, analytical solutions are obtained for a rectangular punch [33, 34].

Based on the current literature review, the problems with a complex contact geometry taking into account friction remain insufficiently studied at present. Analytical solutions of frictionless contact problems for a rigid punch with the base being close to doubly connected rectangles were obtained in [18]. Both cases of smooth and rough surfaces of elastic half-space were treated separately in [8, 18]. Consideration of the similar problem taking into account friction is still relevant. The present work is focused on an analytical-numerical algorithm for solving the problem for a rigid punch with the doubly connected base bounded by lines close to the rectangles taking into account friction forces and shape optimization for the doubly connected punch.

PROBLEM FORMULATION

A rigid punch is indented into an elastic half-space. The punch has the base bounded by the lines close to rectangles. Let a vertical force Q and a horizontal one T act on the punch. The force T is applied at the height d from the base of the punch (see Fig. 1). We assume that the punch moves uniformly, so the problem is quasistatic. The friction force is collinear to the direction of the punch movement.



(a) (b)

FIGURE 1. The loading scheme of the punch: (a) 3-D; (b) cross-section

Let the coordinate system be formed by the z-axis directed perpendicular to the plane of the elastic half-space, x-axis, which is parallel to the direction of the force T, and the origin coincides with the center of the punch symmetry. The line of action of the vertical force Q passes through the origin.

Under the load, the punch indents and rotates. Here, δ denotes the indentation value and γ denotes the projection of the small rotation vector of the punch. The problem is to solve a system of equilibrium equations and a basic integral equation, which under a linear friction law [15] is a Fredholm equation of the first kind containing a weak singularity and has the following form

$$\frac{1-\nu^2}{\pi E}\cdot\iint_S \frac{p(\rho',\theta')}{r}\cdot\left[1+\varepsilon_1\cos(r,x)\right]ds' = g(\rho,\theta), \tag{1}$$

where $\varepsilon_1 = \mu\tilde{\lambda}$, μ is a friction coefficient; $\tilde{\lambda} = \frac{(1-2\nu)}{2\cdot(1-\nu)}$, ν is Poisson's ratio; ρ,θ are polar coordinates; $p(\rho,\theta)$ is the unknown function characterizing the distribution of normal pressure under the punch; S is a doubly connected contact domain bounded by the rectangles, $r^2 = \rho'^2+\rho^2-2\rho\cdot\rho'\cos(\theta-\theta')$, $(\rho',\theta')\in S$, $(\rho,\theta)\in S$; $g(\rho,\theta)$ is a function depending on the shape of the punch surface, the indentation value, the angles of rotation, it can be expressed as $g(\rho,\theta) = \delta+\gamma\rho\cos\theta$ in the case of a flat base of the punch, E is the modulus of elasticity.

The equilibrium equations are

$$Q = \iint_S p(\rho',\theta')ds',\quad M_1 = 0,\quad M_2 = \iint_S p(\rho',\theta')\cdot\rho^2\cos\theta ds'. \tag{2}$$

As follows from (1), the pressure under the moving punch is distributed asymmetrically, this results in an additional moment M_2, Fig. 1 (b). The resultant of the pressure forces is displaced in the direction of the punch movement.

It follows from the equilibrium condition that the force T causing the punch to move must be applied at the height $d = |M_2/(\mu\cdot Q)|$ so that the punch moves without tilting [15]. Otherwise, the punch would have an inclined base. The unknown value γ defining the inclination angle is determined from the condition of equality of the moments of all forces acting on the punch.

METHOD FOR SOLVING THE PROBLEM

Let us write down the equations of the boundaries of the doubly connected domain S in a polar coordinate system $\rho_{\Gamma_1} = a\cdot f(\theta)$; $\rho_{\Gamma_2} = b\cdot f(\theta)$. We present the continuously differentiable function $f(\theta)$ in the following form

$$f(\theta) = 1+\sum_{i=1}^{\infty}\varepsilon^i\ f_i(\theta), \tag{3}$$

where ε is a small parameter, which characterizes the geometry of the stamp base. For example, let $\varepsilon = 0.3583$ [18] and for $i = 1,2,3$ using harmonic analysis one can determine $f_1(\theta) = 0.5\cdot\cos 2\theta + 0.06170$, $f_2(\theta) = -0.4958\cdot\cos 2\theta - \cos 4\theta$, $f_3(\theta) = -1.1952\cos 6\theta + 0.4992\cos 8\theta$.

Equation (1) is an integral equation of the first kind. The problem of solving such equations is ill-posed by J. Hadamard [35]. It is difficult to apply approximation methods to such problems. Instability occurs when solving equations of the first kind by the method of eigenfunctions, iterations, projection methods, the Fourier transform method, and many others. An arbitrary small change in the data can lead to large changes in the solution. This put the expediency of studying ill-posed problems in doubt. The concept of a well-posed problem is due to J. Hadamard, who took the point of view that every mathematical problem corresponding to some physical process must be well-posed. However, a diversity of complex phenomena occurs in real life. The sense of the solution of ill-posed problem was given by A.N. Tikhonov [35]. He also proposed various ways to make it stable under small changes of the original information.

Let us use Tikhonov regularization method in relation to an integral equation of the first kind, which consists in transforming the problem to an equation of the second kind, which is well-posed. We introduce a regularizing operator depending on the error in specifying the right-hand side or the kernel of the operator. Regularization of (1) leads to the solution of an equation of the second kind

$$B\cdot p(\rho,\theta)+\frac{1-\nu^2}{\pi E}\cdot\iint_S \frac{p(\rho',\theta')}{r}\cdot\left[1+\varepsilon_1\cos(r,x)\right]ds'=g(\rho,\theta), \tag{4}$$

where B characterizes the deformable properties of the roughness of the half-space surface [15].

Equation (4) can be considered as the main integral equation of the problem of the punch indentation into an elastic rough half-space taking friction into account. We neglect the vertical displacements of microprotrusions due to the action of a tangential force.

After the introduction of the next change in notations, i.e., $B_1=B\pi E/\left(\left(1-\nu^2\right)\cdot b\right)$; $P(\rho,\theta)=\left(1-\nu^2\right)/(\pi E)\cdot p(\rho,\theta)$; for small B_1 the parameter of the equation can be denoted as $B_1/(2\pi)=1-\alpha$ [8]. Thus, (4) is transformed into the next form

$$(1-\alpha)\cdot P(\rho,\theta)+\iint_S \frac{P(\rho',\theta')}{2\pi br}\left[1+\varepsilon_1\frac{\rho\cos\theta-\rho'\cos\theta'}{r}\right]ds'=\frac{g(\rho,\theta)}{2\pi b}. \tag{5}$$

Small parameter expansions of the integrals contained in (5) were obtained. The coefficients of the expansions are integrals of the simple layer potential type over a circular ring. We used the series expansion in ε powers of a simple layer potential by an internal point, developed in [9], analogically it was done for outer one in [7]. The stability and convergence analysis is represented in [9]. In addition, the calculation of the integral with a weak singularity for a circular ring is used [6], [8]. Thus, the problem of indenting the punch with the base bounded by the lines close to rectangles (3) is reduced to a sequence of the problems for a circular ring base Ω bounded by circles $\rho=a$ and $\rho=b$.

The zero approximation $p_0(\rho,\theta)$ is the solution to the integral equation when $\varepsilon_1=0$. We consider solutions $P(\rho,\theta)$ of (5) close to $p_0(\rho,\theta)$ when the parameter ε_1 is small. In this case, we represent the desired function $P(\rho,\theta)$ and the depth of indentation δ as series

$$P(\rho,\theta)=\sum_{i=0}^{\infty}\varepsilon_1{}^i\cdot p_i(\rho,\theta),\quad \delta=\sum_{i=0}^{\infty}\varepsilon_1{}^i\cdot\delta_i. \tag{6}$$

Let us write out the equation for determining the expansion coefficients in powers of ε_1 in the first approximations

$$\begin{gathered}(1-\alpha)p_0(\rho,\theta)+\iint_\Omega\frac{p_0(\rho,\theta)}{2\pi br}d\Omega=\frac{\delta_0}{2\pi b},\\ (1-\alpha)p_1(\rho,\theta)+\iint_\Omega\frac{p_1(\rho',\theta')}{2\pi br}d\Omega'+\iint_\Omega p_0(\rho')\frac{\rho\cos\theta-\rho'\cos\theta'}{2\pi br^2}d\Omega'=\frac{\delta_1+\gamma_1\rho\cos\theta}{2\pi b},\\ (1-\alpha)p_2(\rho,\theta)+\iint_\Omega\frac{p_2(\rho',\theta')}{2\pi br}d\Omega'+\iint_\Omega p_1(\rho',\theta')\frac{\rho\cos\theta-\rho'\cos\theta'}{2\pi br^2}d\Omega'=\frac{\delta_2+\gamma_0\rho\cos\theta}{2\pi b}.\end{gathered} \tag{7}$$

When there is no roughness $\alpha=1$, (5) and (7) are equations of the first kind and in each approximation that corresponds to absolutely smooth contact. When the punch rotations and horizontal displacements are absence, the exact solution, taking into account the first two approximations for a flat circular punch with a linear law of friction, is obtained in the next form

$$p(\rho,\theta)=\frac{Q}{2\pi b^2}\cdot\left[\frac{1}{\sqrt{1-\rho^2/b^2}}+\varepsilon_1\cdot\left(\frac{2}{\pi}+\sum_{k,p=0}^{\infty}\frac{(2k+1)!!}{(2k)!!}\cdot\frac{1}{2k-2p+1}\cdot\frac{1}{2p+1}\left(\frac{\rho}{b}\right)^{2k+1}\right)\cos\theta\right].$$

The method of successive approximations is going to be applied to transformed (5), (7) into one-dimensional equations, and we obtain the next equation to determine $p_0(\rho,\theta)$

$$(1-\alpha)\varphi(\rho)+\int_a^b \mathrm{K}(\rho,\rho')\cdot\left(\varphi(\rho')-\varphi(\rho)\right)\frac{d\rho'}{b}+\varphi(\rho)\cdot\int_a^b \mathrm{K}(\rho,\rho')\cdot\frac{d\rho'}{b}=\frac{\delta_0+\gamma_0\rho\cos\theta}{2\pi b}, \tag{8}$$

where the integrand in the first integral is correct, since the difference $\left(\varphi(\rho')-\varphi(\rho\ )\right)$ vanishes on the diagonal when $\rho'=\rho$, and the integral $\int_a^b \mathrm{K}(\rho\ ,\rho')\frac{d\rho'}{b}$ is calculated without the desired functions in an explicit form

$$\int_a^b \mathrm{K}(\rho\ ,\rho')\frac{d\rho'}{b} \quad = \frac{2}{\pi}\left[E\left(\frac{\rho}{b}\right)+\frac{\rho}{b}\left(\left(1-\frac{a^2}{\rho^2}\right)\cdot K\left(\frac{a}{\rho}\right)-E\left(\frac{a}{\rho}\right)\right)\right],$$

where $K(z)$, $E(z)$ are elliptic integrals of the first kind.

The kernel in (8) has the form

$$\mathrm{K}(\rho,\rho') = \begin{cases} \sum_{n=0}^{\infty}\xi_n\cdot\left(\frac{\rho'}{\rho}\right)^{2n+1}, & \rho'<\rho; \\ \sum_{n=0}^{\infty}\xi_n\cdot\left(\frac{\rho}{\rho'}\right)^{2n}, & \rho'>\rho, \end{cases},$$

where $\xi_n = \left(\frac{(2n-1)!!}{(2n)!!}\right)^2$.

When $\alpha<1$, the equation (8) can be solved by the method of successive approximations. The unknown constant δ_0 is determined from the equilibrium (2), also presented as expansions in the small parameter.

Similarly, the next equation is obtained for $p_1(\rho,\theta)=\psi(\rho)\cos\theta$

$$\psi(t) = \alpha\psi(t)-\int_a^b \frac{\widetilde{K}(t,l)}{b}\cdot[\psi(l)-\psi(t)]\,dl-\frac{\psi(t)}{b}\int_a^b \widetilde{K}(t,l)dl-\int_a^t \frac{l}{b\cdot t}\cdot\psi(l)\,dl,$$

$$\widetilde{K}(t,l) = \begin{cases} \sum_{n=1}^{\infty}\xi_n\cdot\left[1-\frac{1}{2n+2}\right]\left(\frac{l}{t}\right)^{2n+2}, & l<t; \\ \sum_{n=1}^{\infty}\xi_n\cdot\left[1-\frac{1}{2n+2}\right]\left(\frac{t}{l}\right)^{2n+1}, & l>t. \end{cases}$$

We continue in the same way, transforming the next approximation to receive the required accuracy.

Replacing the integrals in the obtained integral equations with quadrature formulas of approximate calculation, we arrive at a system of linear algebraic equations, which can be solved by the method of successive approximations for $\alpha<1$, or $B_1>4K(a/b)$, where $K(a/b)$ is a complete elliptic integral of the first kind. The error resulting from replacing an integral equation with the system of linear algebraic equations depends on the choice of the quadrature formula.

PUNCH SHAPE OPTIMIZATION

Using the classical calculus of variations [34], the problem of optimizing pressure under a rigid punch with a doubly connected rectangular base is studied. The shape of the punch is taken as the desired function, and the role of the minimizing functional is played by the root-mean-square deviation of the pressure distribution under the punch. In this case, the values of the total forces and moments applied to the punch are assumed to be given, which leads to restrictions imposed on the pressure distribution by the equilibrium conditions.

It is assumed that the desired punch shape $g(\rho,\theta)$ is a continuous and smooth function of coordinates. When the function is non-smooth, the problem becomes more complicated and it is necessary to use methods of non-smooth optimization [36], which is supposed to be done in further studies. The normal pressure should be positive $p(\rho,\theta)\geq 0$, and the stresses outside the contact domain and at its boundaries are equal to zero. For the simplification, we take the case when the punch moves without tilting. As an optimizing functional, we consider an integral of the following form [37]

$$J(p(g)) = \iint_S (p_g - p(g))^2 ds',$$

which characterizes the mismatch between the pressure distribution $p(\rho,\theta)$ corresponding to a certain shape of the punch and a given target pressure $p_g(\rho,\theta)$. For example, it is possible to specify the function $p_g(\rho,\theta)$ in the form of a constant (average) or linear pressure distribution [34].

The optimization problem is to find the function $g_*(\rho,\theta)$ delivering a minimum to the mismatch functional

$$J(p(g_*)) = \min_g J(p(g)), \tag{9}$$

when the equilibrium conditions (2) are satisfied and the normal pressures in the contact area are positive

$$g \in \Lambda_g = \left\{ g : p(g) \geq 0, \iint_S p(g)\,ds' = Q, \iint_S p(g)\cdot\rho^2\cos\theta ds' = M_2 \right\}. \tag{10}$$

The solution of the formulated optimization problem (9), (10) admits an effective decomposition [33] and reduces to the subsequent solution of two problems. The first problem is to minimize the functional

$$J(p) = \iint_S (p_g - p)^2 ds' \to \min,\ p \in \Lambda_p = \left\{ p : (p)_S \geq 0, \iint_S p\,ds' = Q, \iint_S p\cdot\rho^2\cos\theta ds' = M_2 \right\},$$

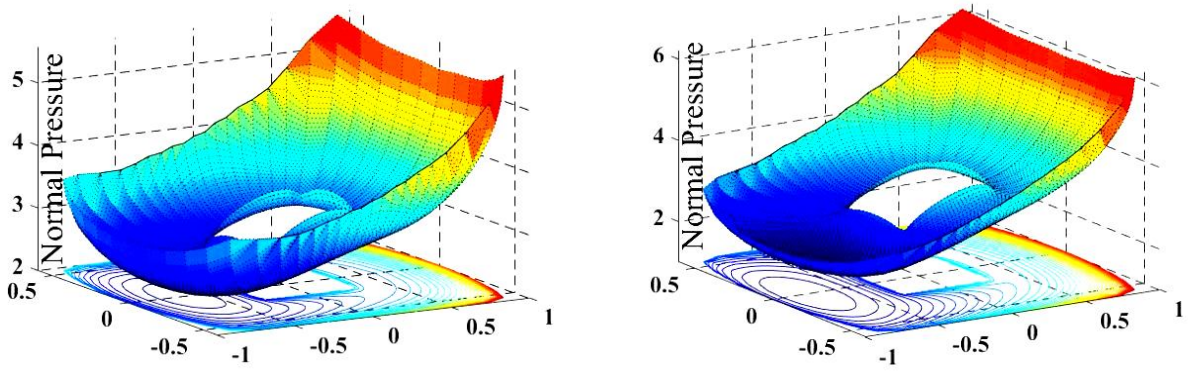
and to find the best admissible pressure distribution $p_*(\rho,\theta)$, the algorithm for solving this optimization problem is described in detail in [34, 37].

The second problem is to find normal displacements $\delta_*(\rho,\theta)$ corresponding to the optimal pressure distribution $p_*(\rho,\theta)$ and determining the optimal shape of the punch $g_*(\rho,\theta)$ in the absence of inclination. To determine the normal displacements, a spatial boundary value problem of the elasticity theory for a half-space is solved, the solution method for which is described in the previous paragraph. In the case of an initially unknown contact domain, the method proposed in [9] is used.

RESULTS AND DISCUSSION

Software based on the proposed method has been developed for engineering design calculations to determine the distribution of normal pressure over the contact area in the form of a doubly connected rectangular ring, find the zones with the highest and lowest pressure, value of indentation depth and punch rotation angles. The input parameters are the geometric dimensions of the punch base, coefficients characterizing the properties of the contacting surfaces, the sliding friction coefficient (if it is zero, the contact is considered without regard to friction forces), the coefficient characterizing the roughness of the elastic half-space, the relative value d/b of the height from the base to the point of horizontal force application. Calculations are carried out in dimensionless relative values, assuming that a unit load $p^* = Q/(2\pi b^2)$ is applied to the punch. Then the relative dimensionless pressure values are determined from the relation $p\ (\rho,\theta)/p^*$. Numerical calculations of specific examples are carried out to study the influence of the roughness coefficient, the friction coefficient and the height of horizontal force application on the distribution of normal pressures under the punch.

The found normal pressure distribution function under the punch with a base in the form of a doubly connected rectangular ring is plotted in Fig. 2 (a), where the chosen values of the parameters are ε_1 =0.057; α =0.7; a/b =0.3 and d/b = 0.5, and Fig. 2 (b) differs only in the value d/b =0.5.



(a) (b)

FIGURE 2. Surfaces of normal pressure distribution: (a) d/b =0.5; (b) d/b =0.8

Figure 3 shows graphs of the normal pressure dependencies p/p^* for different heights of the horizontal force application d/b in the cross-section $\theta = 0;\pi$ for the values of the parameters as in the previous Fig. 2. Lines 1-4 correspond to values d/b =0.2; 0.8; 1.4; 2.0.

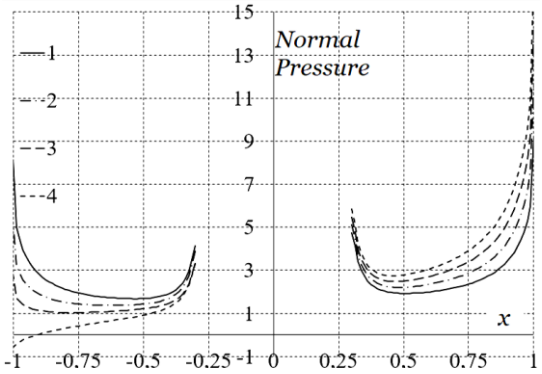
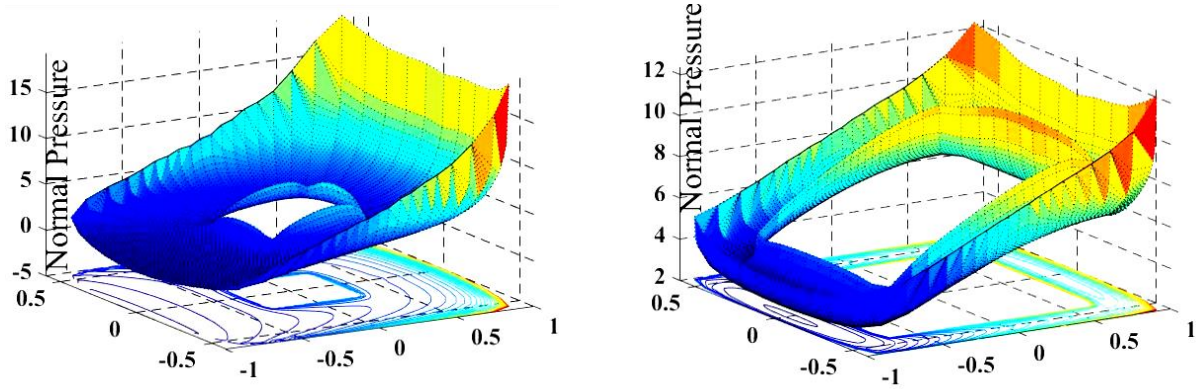


FIGURE 3. Normal pressure dependence for different height values of the horizontal force application.

An increase in the height of application of the horizontal force leads to a greater asymmetry of the pressure distribution. The surfaces of normal pressure distribution give a three-dimensional picture, which also confirms this conclusion in Fig. 2. Curves of equal pressures located under the pressure surfaces show a large asymmetry in the distribution of normal pressure under the punch with a higher point of application of the horizontal force in Fig. 2 (b). Further increase in the height of the force application can lead to the separation of the punch from the surface of the elastic half-space and even subsequent overturning. A zone of negative pressures appears at a value of d/b = 2 for the previously chosen values of the remaining parameters, which corresponds to line 4 in Fig. 3 and the surface in Fig. 4 (a). The normal pressure distribution function under a punch with a narrower base is plotted in Fig. 4 (b), where the chosen values of the parameters are ε_1 =0.08; α =0.7; a/b =0.7 and d/b =0.8.



(a) (b)

FIGURE 4. Surfaces of normal pressure distribution: (a) moment of the punch separation from the elastic half-space d/b =2; a/b =0.3; (b) the narrow base punch a/b =0.7

Comparison of Fig. 2 with Fig. 4 (b) shows that as the ring width decreases, which corresponds to a decrease in the contact area, the contact pressures increase, which makes physical sense. A zone of negative pressures appears at a value of d/b = 1.2 for the considered narrow base punch.

Figure 5 (a) presents the graphs of the change in the normal pressure p/p^* in the cross-section $\theta = 0;\pi$ depending on the influence of the value of the coefficient characterizing the deformation properties of the surface roughness at the parameter values ε_1 = 0.057; d/b = 0.5; a/b = 0.3. Lines 1-4 correspond to values α =0.6; 0.8;

0.9; 0.99. As one can see, from the fact that all graphs in Fig. 5 (a) intersect in a small neighborhood of four points, i.e. for any roughness of the half-space, there are zones that preserve approximately the same pressure. This fact can be used in mathematical modeling of various problems of finding zones with constant pressure, as well as when approximating the solution by curves of such a family [18].

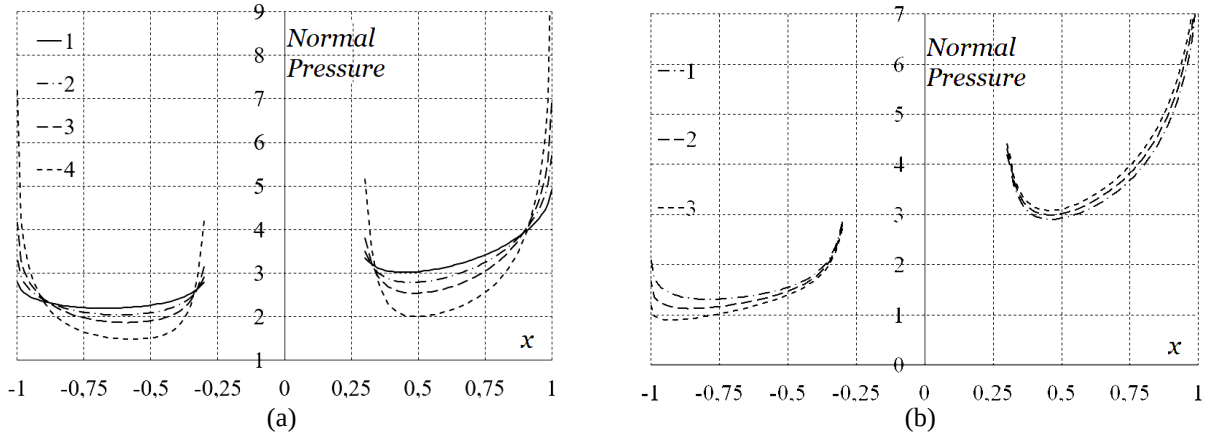


FIGURE 5. Graphs of the contact pressure dependence under the influence of changing: (a) the roughness coefficient; (b) the friction coefficient

With growth of the roughness of the elastic half-space, which corresponds to a decrease in the coefficient α, the pressure distribution becomes more uniform over the contact area, that is, the minimum pressure increases and the maximum pressure decreases, which is similar to the results in [8, 18]. When $\alpha \approx 1$, we obtain an approximate solution to the problem of smooth contact, which coincides with an accuracy of hundredths with the analytical solution determined for the case of the absence of friction $\varepsilon_1 = 0$ and without taking into account the roughness in [18]. However, the pressure increases at the boundaries, but remains finite in our case.

The effect of friction is estimated by changing its coefficient. Figure 5 (b) shows graphs of the contact pressure p/p^* dependence in the cross-section $\theta = 0; \pi$ under the punch for different values of the coefficient ε_1 characterizing the sliding friction, for the values of the other parameters α =0.9; d/b =0.8 and a/b =0.3.

Lines 1-3 correspond to values ε_1 = 0.08; 0.09; 0.1. The effect of friction is similar to the effect of roughness, which is obvious, since these are interrelated concepts. With an increase in the friction coefficient, the pressure distribution also becomes more uniform over the contact area which corresponds to the results in [9, 15].

The optimal punch shape is shown in Fig. 6, in the case when the average value is taken as the optimal pressure value, the punch moves without inclination, and the other constants are respectively equal to ε_1 =0.057; α =0.7; a/b =0.3 Fig.6 (a), a/b =0.7 Fig.6 (b).

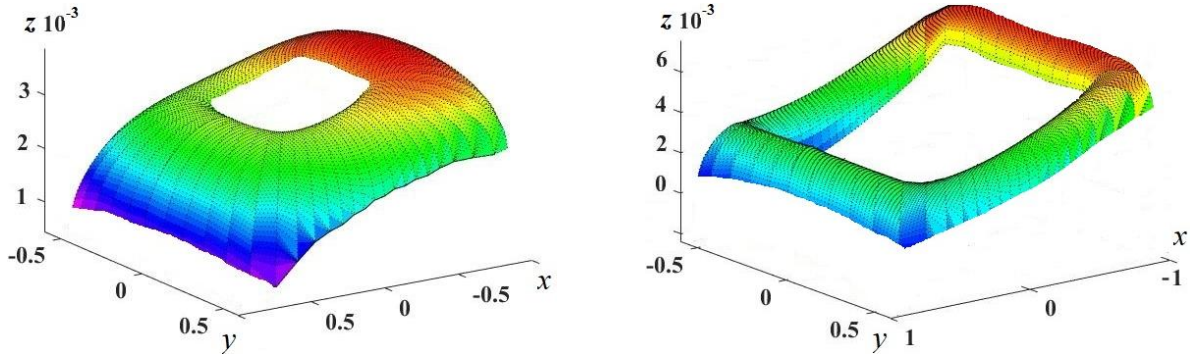


FIGURE 6. Optimal surfaces of the punches: (a) a/b =0.3; (b) a/b =0.7

CONCLUSION

The analytic-numerical method is developed for solving contact problems of a punch having the base bounded by a rectangular ring, taking into account friction and roughness of an elastic half-space. The expansion of the simple layer potential in a small parameter, regularization of the integral equation, the introduction of the difference between the values of the desired function at different points, the subsequent interpolation of the terms to eliminate the singularities and the method of successive approximations are combined in the method.

Numerical examples are given, which confirm the effectiveness of the proposed method. It is shown that with an increase in roughness the maximum pressure decreases and the minimum pressure increases. The pressure at the boundaries of the contact area increases sharply, but remains finite, in contrast to the idealized smooth contact, when it goes to infinity at the boundaries. Analysis of patterns of contact pressure distribution surfaces and curves of equal pressure allows monitoring stress concentrations. An increase in the height of the application of the horizontal force leads to a greater asymmetry of the pressure distribution, which can make the separation of the punch from the surface of the elastic half-space.

With a larger value of the friction coefficient, the pressure distribution also becomes more uniform over the contact area. The geometrical dimensions of the punch also has an influence on the pressure distribution. A decrease in the contact area leads to an increase in the maximum pressure value and a more uneven distribution.

A punch shape was taken as a desired function, and the role of a minimizing functional is played by deviation of the pressure distribution from the optimal one. The optimization problem was transformed into two successively solved problems, according to [34]. Optimal shapes were found using the simple layer potential. Examples were considered for the punch with the doubly connected base bounded by lines close to rectangles.

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