

Existence, Uniqueness, and Algorithms for Equilibria in Competitive Energy Markets

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Abstract

Due to the transition towards climate neutrality, energy markets are rapidly evolving. New technologies are developed that allow electricity from renewable energy sources to be stored or to be converted into other energy commodities. As a consequence, new players enter the markets and existing players gain more importance. Market equilibrium problems are capable of capturing these changes and therefore enable us to answer contemporary research questions with regard to energy market design and climate policy.

This cumulative dissertation is devoted to the study of different market equilibrium problems that address such emerging aspects in liberalized energy markets. In the first part, we review a well-studied competitive equilibrium model for energy commodity markets and extend this model by sector coupling, by temporal coupling, and by a more detailed representation of physical laws and technical requirements. Moreover, we summarize our main contributions of the last years with respect to analyzing the market equilibria of the resulting equilibrium problems.

For the extension regarding sector coupling, we derive sufficient conditions for ensuring uniqueness of the short-run equilibrium a priori and for verifying uniqueness of the long-run equilibrium a posteriori. Furthermore, we present illustrative examples that each of the derived conditions is indeed necessary to guarantee uniqueness in general.

For the extension regarding temporal coupling, we provide sufficient conditions for ensuring uniqueness of demand and production a priori. These conditions also imply uniqueness of the short-run equilibrium in case of a single storage operator. However, in case of multiple storage operators, examples illustrate that charging and discharging decisions are not unique in general. We conclude the equilibrium analysis with an a posteriori criterion for verifying uniqueness of a given short-run equilibrium. Since the computation of equilibria is much more challenging due to the temporal coupling, we shortly review why a tailored parallel and distributed alternating direction method of multipliers enables to efficiently compute market equilibria.

For the extension regarding physical laws and technical requirements, we show that, in nonconvex settings, existence of an equilibrium is not guaranteed and that the fundamental welfare theorems therefore fail to hold. In addition,

we argue that the welfare theorems can be re-established in a market design in which the system operator is committed to a welfare objective. For the case of a profit-maximizing system operator, we propose an algorithm that indicates existence of an equilibrium and that computes an equilibrium in the case of existence. Based on well-known instances from the literature on the gas and electricity sector, we demonstrate the broad applicability of our algorithm. Our computational results suggest that an equilibrium often exists for an application involving nonconvex but continuous stationary gas physics. In turn, integralities introduced due to the switchability of DC lines in DC electricity networks lead to many instances without an equilibrium. Finally, we state sufficient conditions under which the gas application has a unique equilibrium and the line switching application has finitely many.

In the second part, all preprints belonging to this cumulative dissertation are provided. These preprints, as well as two journal articles to which the author of this thesis contributed, are referenced within the extended summary in the first part and contain more details.

Author's Contributions

This cumulative dissertation is based on two peer-reviewed journal articles that have already been published and on two preprints that are currently under review. We report the bibliographic data of the journal articles in the following. The preprints are provided in Part II of this dissertation. Part I consists of an extended summary in which we review the related literature, present the connection between the contributions, and derive the key conclusions. Since all articles are co-authored, we highlight below the contributions made by the author of this dissertation. We like to note that the authors are listed in alphabetical order for all articles.

- [JG1] J. Egerer, V. Grimm, J. Grübel, and G. Zöttl. “Long-Run Market Equilibria in Coupled Energy Sectors: A Study of Uniqueness”. Available online at http://www.optimization-online.org/DB_HTML/2021/03/8312.html. Preprint under review, 2021.

The author of this dissertation contributed key ideas to this article, developed the theory based on previous research, proved all theorems and lemmas, and provided the detailed discussion of the uniqueness conditions together with the illustrative examples. In addition, she primarily authored this article.

- [JG2] J. Grübel, T. Kleinert, V. Krebs, G. Orlinskaya, L. Schewe, M. Schmidt, and J. Thürauf. “On Electricity Market Equilibria with Storage: Modeling, Uniqueness, and a Distributed ADMM”. In: *Computers and Operations Research* 114 (2020). DOI: 10.1016/j.cor.2019.104783.

The initial idea to this article came from Lars Schewe and Martin Schmidt and was further developed by all seven authors in a workshop. The author of this dissertation contributed the proofs of all theorems and lemmas, derived the sufficient conditions that ensure partial uniqueness of an equilibrium, and developed the illustrative examples, which show that uniqueness fails to hold in general. Moreover, she is the primary

author of the third section.

- [JG3] V. Grimm, J. Grübel, L. Schewe, M. Schmidt, and G. Zöttl. “Non-convex Equilibrium Models for Gas Market Analysis: Failure of Standard Techniques and Alternative Modeling Approaches”. In: *European Journal of Operational Research* 273.3 (2019), pp. 1097–1108. DOI: 10.1016/j.ejor.2018.09.016.

The initial idea to this article came from Veronika Grimm, Lars Schewe, Martin Schmidt, and Gregor Zöttl and was further developed in joint discussions of all authors. The author of this dissertation proved all theorems and lemmas and derived all illustrative examples. Additionally, she primarily authored this article.

- [JG4] J. Grübel, O. Huber, L. Hümbes, M. Klimm, M. Schmidt, and A. Schwartz. “Existence of Energy Market Equilibria with Convex and Nonconvex Players”. Available online at http://www.optimization-online.org/DB_HTML/2021/05/8426.html. Preprint under review, 2021.

The author of this dissertation contributed the main ideas to this article, developed the major parts of the theory, and also proved all theoretical results. In addition, she contributed significantly to the implementation. Furthermore, she conducted the instance generation and computational study of the DC line switching application. The instance generation and computational study of the gas flow application were performed by Lukas Hümbes. Finally, the writing of the paper was primarily done by the author of this dissertation.

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Part I.
Extended Summary

1. Introduction

To achieve climate neutrality, many countries are striving for an energy supply that is free of fossil fuels and nuclear power and that has a high share of renewable energy sources such as wind and solar power. As a consequence, new technologies including, e.g., storage devices and renewable synthetic fuels are needed to compensate for fluctuations in renewable production. In addition, the role of energy transportation is becoming increasingly important since the locations of renewable generation are often determined by external factors. Thus, centers of demand diverge from centers of generation. All of this leads to new players entering energy markets while some existing players gain more importance. To assess contemporary research questions with regard to energy market design and climate policy, optimization models are required that cover the behavior of these emerging players and capture the interaction with other market participants.

This cumulative dissertation is dedicated to this topic and analyzes classic energy market equilibrium problems extended by new players or new side-constraints for already existing players. In particular, we consider operators of sector-coupling production plants, operators of storage devices, and network operators facing nonconvex physical laws. In the following, we review the related literature and provide an overview of the key types of optimization models that have been studied so far in the literature on energy market modeling. Afterward, we elaborate in more detail on the mathematical challenges addressed in this dissertation and on the structure of this thesis.

Literature Survey on Energy Market Modeling A major strand in the literature focuses on optimizing the energy system as a whole. The respective models are usually referred to as energy system models. Typically, the objective of these models is to find the cost-minimal operation for the given system, while investment is either fixed or co-optimized. Specific political or technical goals are often imposed as constraints. Political goals might be climate targets such as a certain percentage of renewable production in the system. Technical goals might be stability aspects such as sufficient operating reserve to compensate for possible network or supply outages. The considered time horizon can vary from a few minutes to several decades and the geographic scope can also vary widely,

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ranging from a municipal to an international level. One of the drawbacks of all these models is that neither the market design nor the energy pricing in the respective energy systems is included but, of course, different players interact in the systems' markets and pursue individual objectives that may not be in line with those of the overall system. Therefore, the market outcome is not necessarily related to the system optimum and market equilibria are generally not assessed by energy system models. Since there is an enormous amount of articles in this area, we only mention selected contributions below that are relevant to the applications studied in this dissertation.

Early energy system models focusing on sector coupling are provided by, e.g., Correa-Posada and Sánchez-Martín (2014), Geidl and Andersson (2007), Li et al. (2008), and Unsihuay et al. (2007). For more recent works see, e.g., Deane et al. (2017), Zhang et al. (2015), and Zlotnik et al. (2017) or the review articles by Bloess et al. (2018) and Ramsebner et al. (2021) as well as the references therein. For articles on storage devices that are operated to achieve specific political goals, we refer the reader to, e.g., Cebulla et al. (2017) and Zerrahn, Schill, and Kemfert (2018), while specific technical goals are addressed, e.g., in Banshwar et al. (2019), Gayme and Topcu (2013), Muzhikyan et al. (2016), or Samsatli and Samsatli (2015). A comprehensive overview of further energy system models addressing power storage in systems with high shares of renewable generation can be found in Zerrahn and Schill (2017). Finally, concerning the nonconvex optimization of gas networks we refer the interested reader to Koch et al. (2015) and Ríos-Mercado and Borraz-Sánchez (2015) and concerning transmission line switching models to, e.g., Fisher et al. (2008) and Hedman et al. (2008, 2009) or Bienstock and Muñoz (2015) as well as Brown and Moreno-Centeno (2020) for more recent contributions.

Besides energy system models, another strand of the literature focuses on mixed complementarity problems (MCPs) as a modeling framework. For a general overview of complementarity modeling in energy markets see Gabriel, Conejo, et al. (2013). A specific overview of MCPs in gas markets is given by Gabriel and Smeers (2006) in Seeger (2006). For coupled markets of multiple sectors, Conejo et al. (2020) provide complementarity-based models. An advantage of complementarity models is that the energy pricing, the individual objectives of all players, and their interaction can be fully captured. One of the core focuses of papers in this research area is on the computation of equilibria. In the context of sector coupling, Abrell and Weigt (2012) compute, for example, short-run market equilibria in coupled electricity and gas markets. Abrell and Weigt (2016) extend this approach by additionally considering different time intervals for trading as well as investment in transport and production capacity. More recently, Roach and Meeus (2020) analyze the investment in coupling

capacity between the electricity and gas sector using an iterative procedure based on solving MCPs for the clearing of each sector's market. Chen, Conejo, Sioshansi, et al. (2020b) use an MCP-based approach to model independent operations of the electric power and natural gas system that are coupled due to natural gas-fired power plants. Concerning energy storage applications, Awad et al. (2014) state an MCP to study the impact of energy storage size and location on market prices and players' rents. In contrast, Fomeni et al. (2019) focus on the concept of energy storage operators as service providers that support the overall system and stabilize electricity prices during peak periods.

All of the aforementioned articles on energy market modeling by MCPs have one thing in common: the assumption of perfect competition. We focus on this aspect as perfect competition is also one basic assumption throughout this dissertation. Corresponding market models under perfect competition have also been studied extensively in a more general form in the economic literature, where these models are usually referred to as market equilibrium problems (MEPs). We provide a brief history of MEPs and of the strongly related fundamental welfare theorems in Section 2.1.

For the sake of completeness, we also give a brief link to the literature that analyzes strategic behavior and market power in energy systems based on complementarity models. Again, we only state selected contributions that are relevant with regard to the context of this dissertation. The interested reader is referred, e.g., to Chen, Conejo, Sioshansi, et al. (2020a) and Huppmann and Egging (2014) for sector-coupling applications, to Sioshansi (2014) for an energy storage application, and to Boots et al. (2004), Cremer and Laffont (2002), Egging et al. (2008), Xu et al. (2017), Yang et al. (2016), or Yao et al. (2008) for gas market applications. Among the numerous contributions to gas market applications, we like to highlight especially the work of Gabriel, Kiet, et al. (2005), in which perfect competition is also assumed in almost all modeled markets except for marketer sales to consumers. In particular, Gabriel, Kiet, et al. (2005) conduct a thorough existence and uniqueness analysis for their studied mixed nonlinear complementarity problem.

However, most papers refrain from a detailed equilibrium analysis of the respective MCPs. Exceptions are Grimm, Schewe, et al. (2017), Krebs, Schewe, et al. (2018), and Krebs and Schmidt (2018). In Grimm, Schewe, et al. (2017), the electricity market model of Hobbs and Helman (2004) is studied in continuous time. In this MEP, firms operate in different market regions and face fluctuating demand while deciding on their production, which is constrained by their invested capacity. In addition, a network operator runs the network with the aim of maximizing congestion rents subject to predefined transmission capacities. As a major contribution, Grimm, Schewe, et al. (2017) derive general conditions

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under which uniqueness of the short- and long-run market equilibrium holds. Since we study several extensions of their setup in this dissertation and also partly rely on their theoretical results, we review those in Section 3.3. The setup of Grimm, Schewe, et al. (2017) is enhanced by Krebs and Schmidt (2018), who additionally take into account piecewise linear and symmetric transport costs and prove uniqueness of the resulting short-run equilibrium under mild assumptions. Finally, in Krebs, Schewe, et al. (2018), physical laws are incorporated in the MEP of Hobbs and Helman (2004) by adding the classic direct current (DC) lossless approximation to the system operator’s optimization problem. The authors establish uniqueness of the short-run market equilibrium on tree and cycle networks under certain assumptions. Furthermore, it is shown by counterexamples that uniqueness fails to hold in general and a posteriori criteria for uniqueness are derived. Their setup is reviewed in more detail in Section 6.1 of this dissertation, where we also investigate the inclusion of other physical laws in the framework of Hobbs and Helman (2004).

Finally, we review literature that analyzes the design of energy markets by multilevel computational equilibrium frameworks. Often, an MCP as described above forms a lower level of these multilevel optimization problems. Thus, a thorough uniqueness study is a prerequisite for tackling such problems; see, e.g., Colson et al. (2007), Dempe (2002), and Gabriel, Conejo, et al. (2013) for more information on the importance of lower-level uniqueness in multilevel problems. There are numerous examples from the literature on power markets; see, among others, Ambrosius et al. (2020), Asensio et al. (2017), Baringo and Conejo (2012), Egerer et al. (2021), Grimm, Kleinert, et al. (2019), Grimm, Martin, et al. (2016), Grimm, Rückel, et al. (2021), Hu and Ralph (2007), Huppmann and Egerer (2015), Jenabi et al. (2013), Kleinert and Schmidt (2019), Rider et al. (2013), Ruiz and Conejo (2015), Sauma and Oren (2006), or Zeng et al. (2016). Respective work in the context of sector coupling is provided by, e.g., Chen and Conejo (2020), Cong et al. (2019), and Wang et al. (2018), in the context of power storage by, e.g., Abrell, Rausch, et al. (2019) and Grimm, Grübel, et al. (2020), and in the context of gas markets by Böttger et al. (2021), Grimm, Schewe, et al. (2019), Heitsch et al. (2021), and Schewe et al. (2021).

Mathematical Challenges and Structure As argued at the beginning, many extensions of the above addressed MEP of Hobbs and Helman (2004) are becoming increasingly important due to the transition to an energy supply free of fossil fuels and nuclear power. We have seen that only few of them have been examined in the literature so far. Therefore, this cumulative dissertation is dedicated to close this gap by analyzing the following three extensions: sector

coupling, temporal coupling, and nonconvex physical laws. In particular, we consider the following players with challenging side-constraints:

- (i) operators of sector-coupling production plants (see Chapter 4),
- (ii) operators of storage devices, whose actions couple successive trading periods (see Chapter 5), and
- (iii) a system operator that faces nonconvex physical laws at the spot markets (see Chapter 6).

Similar to Grimm, Schewe, et al. (2017), we conduct an equilibrium analysis for the aforementioned cases with a focus on the following challenging questions:

- (i) Does a market equilibrium exist in the short- and long-run?
- (ii) If so, is this market equilibrium unique or, at least, partly unique? If not, which circumstances lead to multiplicities?
- (iii) How can we compute market equilibria?

Part I of this cumulative dissertation summarizes our main contributions to the stated questions for the three mentioned extensions. In this summary, we refrain from presenting detailed results and proofs. Those can be found in the original journal articles and preprints, the latter forming Part II. The summary in Part I is structured in the following way. First, we provide a brief history of the economic concept of MEPs and of the strongly related fundamental welfare theorems in Chapter 2. Furthermore, we discuss an alternative formulation of the welfare theorems shown by Harks (2020), which is based on Lagrangian duality and forms the foundation for several results derived in this dissertation. In Chapter 3, we introduce the unified notation and basic assumptions used throughout Part I. Moreover, we introduce the MEP of Hobbs and Helman (2004) and review the main results of the equilibrium analysis conducted by Grimm, Schewe, et al. (2017). Afterward, we address the three mentioned extensions of the model of Hobbs and Helman (2004) and summarize our main findings with respect to existence, uniqueness, and the computation of equilibria. We start with sector coupling in Chapter 4, continue with temporal coupling in Chapter 5, and conclude with the extension regarding nonconvex physical laws in Chapter 6. Finally, we highlight the key conclusions of this dissertation and discuss the next research challenges in Chapter 7.

2. Foundations of the Theory of Market Equilibria

This chapter focuses on market equilibrium problems (MEPs) and the fundamental welfare theorems. We start with a brief history in Section 2.1 that follows the historical survey of Aliprantis et al. (2002). We conclude with presenting a special type of MEPs in Section 2.2 that covers all MEPs considered in this dissertation. For this kind of MEPs, we provide a formulation of the fundamental welfare theorems based on Lagrangian duality.

2.1. Brief History

The theory of general equilibrium as known today was founded by Léon Walras in his seminal work “Elements of Pure Economics”, which consists of four successive editions published between the years 1874 and 1900; see, e.g., Walras (1900) for the fourth edition or Walras (1954) for an English translation. In his work, Walras formulated a system of equations, whose solutions constitute equilibria of an entire economy. This Walrasian general equilibrium model covers the demand of consumers, the supply of producers, and the clearing of markets, i.e., that demand equals supply on each market. As it is still commonly done in modern equilibrium theory, Walras made the following basic assumptions: All consumers maximize their utility and all producers maximize their profit, while all markets are perfectly competitive. Walras already observed that an equilibrium to the presented system of equations might in general neither be unique nor exist at all. This observation initiated a research route that many great economists of the 20th century followed: Under which conditions can existence and uniqueness of general market equilibria be obtained?

The first general existence results have been established by Abraham Wald in a series of papers appearing between 1933 and 1936; see Wald (1933–1934, 1934–1935, 1936) or Wald (1951) for the English translation of Wald (1936). However, it took another twenty years until, due to the advances in game theory and the mathematical fields of topology and linear programming, “it was perceived independently by a number of scholars that existence theorems of greater simplicity and generality than Wald’s were now possible.”; see Arrow

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and Hahn (1971, p. 11). Exemplary contributions are Debreu (1962), Gale (1955), and McKenzie (1959). One of the most famous existence results is the one of Arrow and Debreu (1954). The model formulated in this work is seen as the archetype of general equilibrium models and is often referred to as the Arrow–Debreu–McKenzie general equilibrium model. In addition to this seminal contribution on the existence of equilibria, Debreu also made one of the first outstanding contributions on the uniqueness of a general equilibrium; see Debreu (1970). For an overview of earlier contributions under much more restrictive assumptions see Chapter 9 in Arrow and Hahn (1971).

Several generalizations of the Arrow–Debreu–McKenzie model have been studied in the literature. Examples include the case of intransitive and incomplete preferences as in Gale and Mas-Colell (1975), Mas-Colell (1974), and Shafer and Sonnenschein (1975a,b), or the case of an infinite number of commodities as in Debreu (1954) and Radner (1967). All of the contributions mentioned so far have one thing in common: They analyze convex economies. Another strand of the literature extends these results to the case of nonconvex economies. Nonconvexities on the consumer side are studied, e.g., by Aumann (1964) and Hildenbrand (1974). Nonconvexities on the producer side most commonly arise due to indivisibilities, increasing returns to scale, and fixed costs. Examples that tackle such nonconvexities are, e.g., Bikhchandani and Ostroy (2002, 2006), Leonard (1983), and Shapley and Shubik (1971) for assignment problems, Baldwin and Klempner (2019) as well as Bikhchandani and Mamer (1997) for general exchange economies with indivisibilities, Guo et al. (2021) and O’Neill et al. (2005) for discrete markets, Fleiner et al. (2019) and Hatfield et al. (2013, 2019) for trading networks, and Beato (1982), Bonnisseau and Cornet (1988, 1990) as well as Brown, Heal, et al. (1986) for economies with increasing returns to scale.

Besides existence and uniqueness, the economic efficiency of general equilibria has also been intensively studied in the literature. The results of these studies are known today as the first and second welfare theorem. Again, Arrow and Debreu were the first to prove these theorems in reasonable generality; see Arrow (1951) and Debreu (1951, 1954). The assumptions imposed, e.g., in Debreu (1954) are

- (i) the convexity of the consumption sets,
- (ii) the convexity of preferences,
- (iii) the continuity of preferences,
- (iv) the convexity of the aggregate production set, and

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(v) the existence of an interior point of the aggregate production set.

Given these assumptions, the first welfare theorem establishes that every equilibrium is Pareto optimal, while the second welfare theorem states that every Pareto optimal allocation is an equilibrium. Consequently, there is an equivalence of equilibria and Pareto optimal solutions, where the latter are typically referred to as welfare-maximal solutions. In the following, we study in more detail the relation of the fundamental welfare theorems and Lagrangian duality.

2.2. The Relation Between Welfare Theorems and Lagrangian Duality

We consider a special type of MEPs for which we present an alternative formulation of the fundamental welfare theorems based on Lagrangian duality. This connection of the theory of market equilibria to Lagrangian duality in optimization is shown by Harks (2020). Our presentation in this section follows Harks (2020) and [JG4].

Let the set of perfectly competitive players I be finite. The preferences of each player $i \in I$ are described by the utility function $u_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ and the strategy set is denoted by $X_i \subseteq \mathbb{R}^{n_i}$. In addition, the function $h_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_\pi}$ with $x_i \mapsto h_i(x_i)$ maps the player's decision variables to the actual consumption vector. Hence, for a given price vector $\pi \in \mathbb{R}^{n_\pi}$, every player $i \in I$ faces the optimization problem

$$\max_{x_i} u_i(x_i) + \pi^\top h_i(x_i) \quad \text{s.t.} \quad x_i \in X_i. \quad (2.1)$$

To obtain an equilibrium, the best responses of all players have to satisfy predefined market-clearing conditions that depend on the variables of all players. We model these market-clearing conditions as

$$\sum_{i \in I} h_i(x_i) = 0, \quad (2.2)$$

i.e., we assume, in contrast to Harks (2020), that there is no excess demand or supply.

In total, we consider the following MEP:

$$\begin{aligned} \text{Optimization problems of the players:} & \quad (2.1) \text{ for all } i \in I, \\ \text{Market-clearing conditions:} & \quad (2.2). \end{aligned} \quad (\text{MEP})$$

In the following, we use $x := (x_i)_{i \in I} \in \mathbb{R}^{n_x}$ with $n_x := \sum_{i \in I} n_i$ to abbreviate the decision variables of all players. Analogously, $X := \times_{i \in I} X_i$ denotes the Cartesian product of the individual strategy sets.

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Based on this notation, the pair (x, π) is a *market equilibrium* of (MEP) if the decision variables x_i are best responses to the prices π for all players $i \in I$ and if, additionally, the market-clearing conditions are satisfied by these best responses. Prices π are called *market-clearing prices* if decision variables $x \in X$ exist such that (x, π) is a market equilibrium.

The corresponding welfare maximization problem of (MEP) is obtained by jointly maximizing the objectives of all players subject to the constraints of all players and the market-clearing conditions. Hence, the welfare problem reads

$$\max_x \sum_{i \in I} u_i(x_i) \quad \text{s.t.} \quad x \in X, \quad \sum_{i \in I} h_i(x_i) = 0. \quad (\text{WFP})$$

Now, the question that arises is the following. Under which conditions do the fundamental welfare theorems apply in our setting? In other words, when do the market equilibria of (MEP) correspond to the solutions of (WFP)? Note that answering this question is not trivial since no further assumptions are imposed on the utility functions, strategy sets, and decision variables, i.e., nonconvexities are possible. Nevertheless, as Harks (2020) proves, a general answer to the raised question can be found by applying Lagrangian duality. In particular, Harks (2020) shows that the equivalence of welfare maxima and equilibria holds if and only if the duality gap of the welfare problem is zero.

To establish this result formally, let the Lagrangian of the welfare problem (WFP) be defined by

$$L(x, \pi) := \sum_{i \in I} \left(u_i(x_i) + \pi^\top h_i(x_i) \right).$$

The corresponding Lagrangian dual problem thus reads

$$\inf_{\pi \in \mathbb{R}^{n_\pi}} \mu(\pi), \quad (2.3)$$

where $\mu(\pi) := \sup_{x \in X} L(x, \pi)$ holds.

The welfare problem has *zero duality gap* if a global solution x of the primal problem (WFP) and a global solution π of the dual problem (2.3) exist for which the respective objective function values are the same. In the presence of nonconvexities, such primal-dual solution pairs with zero duality gap might not exist. However, if such a pair exists, the following result of Harks (2020) guarantees that (x, π) is then a market equilibrium of (MEP).

Theorem 2.2.1 (Harks (2020, Theorem A.1 Part 1)). *The pair (x^*, π^*) is a market equilibrium of (MEP) if and only if x^* and π^* are solutions of the welfare optimization problem (WFP) and the corresponding dual problem (2.3), respectively, with zero duality gap.*

2.2. The Relation Between Welfare Theorems and Lagrangian Duality

This result has several important implications. If a market equilibrium of (MEP) exists, we can directly conclude that this market equilibrium is economically efficient in the sense that the welfare of the overall system is maximized. Moreover, if we know for given applications that the duality gap of the welfare problem is always zero, we have an equivalence of (MEP) and the welfare maximization problem as in the fundamental welfare theorems. Hence, we can then solve the corresponding welfare maximization problem to obtain a market equilibrium.

To conclude, we consider the special case that the optimization problems (2.1) of all players are convex for all prices π . In this case, the fundamental welfare theorems apply if a constraint qualification as, e.g., Slater's condition holds for the welfare problem since the duality gap of the welfare problem is then always zero. Thus, we obtain the following alternative formulation of the fundamental welfare theorems based on Lagrangian duality.

Corollary 2.2.2 (Harks (2020, Corollary A.4)). *Let the strategy sets X_i of all players $i \in I$ be nonempty convex sets. Furthermore, assume that the utility functions u_i are concave and the functions h_i are affine for all players $i \in I$. In addition, assume that Slater's condition holds for the welfare problem (WFP) and that the welfare problem (WFP) is bounded. Then, an equilibrium (x^*, π^*) of (MEP) exists. Moreover, if $\sum_{i \in I} u_i$ is strictly concave over X , this equilibrium is unique. In addition, each equilibrium of (MEP) corresponds to a welfare-maximal solution of (WFP) and vice versa.*

3. A Market Equilibrium Problem to Model Liberalized Energy Markets

Here, we present the equilibrium model for energy commodity markets as discussed in Hobbs and Helman (2004). This model is an application of the market equilibrium problem (MEP) introduced in the previous chapter and forms the basis for all equilibrium models considered in the subsequent chapters. We start with an overview of our basic notation and main assumptions in Section 3.1. Afterward, we describe the addressed equilibrium model in Section 3.2 and review the results from the literature with respect to existence, uniqueness, and the computation of market equilibria in Section 3.3.

3.1. Basic Notation and Assumptions

In this section, we provide the basic notation that we keep general enough to cover the setup of all articles included in this cumulative dissertation. Moreover, we collect the main assumptions used throughout this thesis.

Basic Notation The transport network of the considered energy commodity is modeled as a directed and weakly connected graph $G_s = (V_s, A_s)$ with node set V_s and arc set A_s . The index s represents the respective energy sector of the energy commodity such as electricity or natural gas. The set of all considered sectors is denoted by S . We further partition the set of nodes into the set V_s^+ of nodes at which producers are located, the set V_s^- of nodes at which consumers are located, and the set V_s^0 of inner network nodes. If we speak, e.g., of producer $v \in V_s^+$, we actually refer to the producer located at this node. Analogously, we refer to all producers of a given sector by the set V_s^+ . Finally, we follow common notation and define the set $\delta_s^{\text{in}}(v)$ of all ingoing arcs and the set $\delta_s^{\text{out}}(v)$ of all outgoing arcs at a node $v \in V_s$ by

$$\delta_s^{\text{in}}(v) := \{a = (w, v) \in A_s \mid w \in V_s\}, \quad \delta_s^{\text{out}}(v) := \{a = (v, w) \in A_s \mid w \in V_s\}.$$

To capture the market structure, we now consider the bidding zones Z_s of the sector $s \in S$. Each bidding zone $z \in Z_s$ is the subset of the producers and

3. A Market Equilibrium Problem to Model Liberalized Energy Markets

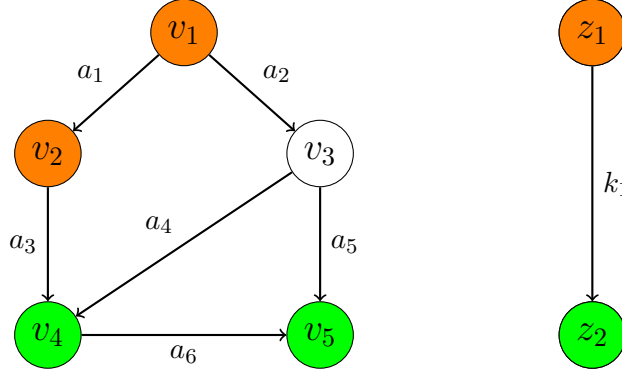


Figure 3.1.: An example for the graph $G_s = (V_s, A_s)$ modeling the network infrastructure (left) and the graph $G'_s = (Z_s, K_s)$ modeling the market structure (right); inner network nodes are colored white while a bidding zone as well as all producers and consumers that trade in this bidding zone are colored alike, i.e., $z(v_1) = z(v_2) = z_1$ and $z(v_4) = z(v_5) = z_2$ hold

consumers $V_s^+ \cup V_s^-$ that trade in this bidding zone. For the ease of presentation, we assume that each producer and each consumer trades in exactly one bidding zone, i.e., Z_s is a partition of $V_s^+ \cup V_s^-$. Given a bidding zone $z \in Z_s$ of the sector $s \in S$, we denote by V_z^+ all producers and by V_z^- all consumers that trade in this bidding zone. For all sectors $s \in S$, we indicate by the function z in which bidding zone the producer or consumer $v \in V_s^+ \cup V_s^-$ trades:

$$z: \bigcup_{s \in S} (V_s^+ \cup V_s^-) \rightarrow \bigcup_{s \in S} Z_s.$$

Besides the graph G_s to model the network infrastructure, we introduce the directed and weakly connected graph $G'_s = (Z_s, K_s)$ to model the market structure of the energy sector $s \in S$. The arc set $K_s \subseteq Z_s \times Z_s$ represents the trading connections between the different bidding zones of this sector. More specifically, if trade between the bidding zone $z \in Z_s$ and another bidding zone $z' \in Z_s$ is possible at the spot market, either (z, z') or (z', z) is an element of the arc set. The level of trade on an arc $k \in K_s$ is limited by the minimum and maximum trading capacity q_k^- and q_k^+ . To account for the possibility that there is no trading between adjacent bidding zones in certain trading periods, we assume that $q_k^- \leq 0$ and $q_k^+ \geq 0$ hold. Again, we use the standard δ -notation as introduced above to define the set $\delta_s^{\text{in}}(z)$ of all ingoing and the set $\delta_s^{\text{out}}(z)$ of all outgoing trading connections of a bidding zone $z \in Z_s$.

Figure 3.1 illustrates the graphs G_s and G'_s modeling the network infrastructure and the market structure for an exemplary instance.

3.1. Basic Notation and Assumptions

For the further economic framework, we consider $T := \{0, 1, \dots, t^e\} \subseteq \mathbb{N}_0$ successive trading periods. The variable production costs of the producer $v \in V_s^+$ are described by the function $c_v^{\text{var}}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. In case of no production, variable costs are zero, i.e., $c_v^{\text{var}}(0) = 0$ holds. Moreover, all functions $c_v^{\text{var}}(\cdot)$ are monotonically increasing, convex, and continuously differentiable. The investment costs of the producer $v \in V_s^+$ linearly increase in the invested capacity by $c_v^{\text{inv}} \in \mathbb{R}_{\geq 0}$. The inverse demand function $P_{t,v}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ models the demand of the consumer $v \in V_s^-$ in trading period $t \in T$. All functions $P_{t,v}(\cdot)$ are continuous and strictly decreasing. Modeling demand by elastic demand functions is a standard approach in economic literature; see, e.g., Mas-Colell et al. (1995).

Main Assumptions Equilibrium problems that result from jointly modeling private investment and spot-market behavior are challenging problems. Additionally considering aspects like temporal and spatial coupling, as done in this dissertation, further complicates the equilibrium analysis. Hence, to keep the analysis tractable, we assume perfect competition at all markets. This is a standard assumption in related contributions; see, e.g., Boucher and Smeers (2001), Daxhelet and Smeers (2007), and Grimm, Martin, et al. (2016). Assuming perfect competition has the two following direct implications. First, all players act as price-takers, i.e., each individual player views the market prices as exogenously given and does not anticipate the impact of the own and the rivals' decisions on these market prices. Second, all players are perfectly informed. Thus, the system operator is, e.g., informed about the capacities of all production facilities in the considered network.

In addition, we assume that each player has perfect foresight. This implies that, e.g., all producers correctly anticipate the outcomes of the spot markets in which they operate when deciding on their investment. Moreover, all storage operators correctly anticipate the future spot-market outcomes when deciding on their level of charging and discharging in the current trading period. More precisely, perfect foresight implies that the spot-market results are not subject to any kind of uncertainty. As discussed in [JG1], the assumption of perfect foresight might be relaxed by using robust or stochastic optimization techniques. Resulting robustified complementarity problems have only recently received increased attention in the literature and still need to be better understood theoretically. For more information on this topic see, e.g., Biefel et al. (2021), Krebs, Müller, et al. (2021), Krebs and Schmidt (2020), and Sierra-Aguilar et al. (2021) or Çelebi et al. (2021) and Kramer et al. (2020) in the context of energy market modeling. For uncertain complementarity problems studied

3. A Market Equilibrium Problem to Model Liberalized Energy Markets

using stochastic optimization we refer the reader to, e.g., Chen and Fukushima (2005), Chen, Wets, et al. (2012), Chen, Zhang, et al. (2009), and Lin and Fukushima (2006).

3.2. Producers, Consumers, System Operators, and Market Clearing

We now continue with the description of the equilibrium model for energy commodity markets as discussed in Hobbs and Helman (2004). The three types of players considered in this model are producers, consumers, and system operators. Before we state the MEP, we present in detail the objectives and strategy sets of the individual players. To simplify notation, we use quantities without an index to denote the vector of indexed quantities. For example, $d_v := (d_{t,v})_{t \in T}$ is the vector of all consumption of the consumer $v \in V_s^-$ and $d := (d_v)_{v \in V_s^-}$ is the vector of overall consumption.

The goal of the producer $v \in V_s^+$ is to maximize the profit from trading, i.e., the revenues from selling the considered energy commodity minus the costs of production and investment. Thus, the optimization problem of the producer reads

$$\max_{y_v, \bar{y}_v} \sum_{t \in T} \left(\pi_{t,z(v)} y_{t,v} - c_v^{\text{var}}(y_{t,v}) \right) - c_v^{\text{inv}} \bar{y}_v \quad (3.1a)$$

$$\text{s.t.} \quad \bar{y}_v \geq y_{t,v} \geq 0, \quad t \in T, \quad (3.1b)$$

where $\pi_{t,z(v)}$ denotes the price of trading period t in the bidding zone $z(v)$ in which the producer is trading. When deciding on the production level $y_{t,v}$, the producer is constrained by the invested production capacity \bar{y}_v .

Consumers aim at choosing the demand levels d_v that maximize their gross consumer surpluses less their purchasing costs. Hence, the consumer $v \in V_s^-$ faces the optimization problem

$$\max_{d_v} \sum_{t \in T} \left(\int_0^{d_{t,v}} P_{t,v}(\mu) d\mu - \pi_{t,z(v)} d_{t,v} \right) \quad (3.2a)$$

$$\text{s.t.} \quad d_{t,v} \geq 0, \quad t \in T. \quad (3.2b)$$

A system operator is responsible for allocating the trading capacities between adjacent bidding zones of the same sector. The goal of the system operator is to maximize the profit from congestion rents, which is achieved by allocating as much trade as the trading capacities allow from low price bidding zones to high

3.2. Producers, Consumers, System Operators, and Market Clearing

price bidding zones. Hence, the optimization problem of the system operator of sector $s \in S$ is given by

$$\max_q \sum_{t \in T} \sum_{k=(z,z') \in K_s} (\pi_{t,z} - \pi_{t,z'}) q_{t,k} \quad (3.3a)$$

$$\text{s.t. } q_k^- \leq q_{t,k} \leq q_k^+, \quad k \in K_s, \quad t \in T. \quad (3.3b)$$

A positive trading value $q_{t,k} > 0$ on $k = (z, z') \in K_s$ implies that the respective amount of the energy commodity is delivered in trading period t from bidding zone z to bidding zone z' . In turn, a negative trading value $q_{t,k} < 0$ implies a delivery from bidding zone z' to bidding zone z .

We like to note that only congestion rents between bidding zones and not within bidding zones are considered. Therefore, the resulting allocation might violate transport constraints. If this is the case, the system operator adjusts production and consumption in a cost-minimal way such that transport becomes feasible. Typically, these adjustments are assumed to be cost-based since the decisions of other players are then not affected by the adjustments; see, e.g., Grimm, Martin, et al. (2016), Grimm, Rückel, et al. (2021), or [JG1] for more information on this topic.

Finally, the market-clearing conditions require that all trades are balanced in each trading period and in each bidding zone of each sector $s \in S$:

$$\sum_{k \in \delta_s^{\text{in}}(z)} q_{t,k} - \sum_{k \in \delta_s^{\text{out}}(z)} q_{t,k} = \sum_{v \in V_z^-} d_{t,v} - \sum_{v \in V_z^+} y_{t,v}, \quad z \in Z_s, \quad t \in T. \quad (3.4)$$

In total, we obtain for each sector $s \in S$ the energy market equilibrium problem

$$\begin{aligned} \text{Producers:} & \quad (3.1) \text{ for all } v \in V_s^+, \\ \text{Consumers:} & \quad (3.2) \text{ for all } v \in V_s^-, \\ \text{System operator:} & \quad (3.3), \\ \text{Market-clearing conditions:} & \quad (3.4). \end{aligned} \quad (\text{MEP-E})$$

The corresponding welfare optimization problem reads

$$\begin{aligned} \max_{y, \bar{y}, d, q} & \quad \sum_{t \in T} \left(\sum_{v \in V_s^-} \int_0^{d_{t,v}} P_{t,v}(\mu) \, d\mu - \sum_{v \in V_s^+} c_v^{\text{var}}(y_{t,v}) \right) - \sum_{v \in V_s^+} c_v^{\text{inv}} \bar{y}_v \quad (\text{WFP-E}) \\ \text{s.t.} & \quad (3.1b), \quad v \in V_s^+, \quad (3.2b), \quad v \in V_s^-, \quad (3.3b), \quad (3.4). \end{aligned}$$

3.3. Equilibrium Analysis

In this section, we analyze (MEP-E) with respect to existence, uniqueness, and the computation of equilibria. Grimm, Schewe, et al. (2017) study (MEP-E) in continuous time and conduct a detailed equilibrium analysis. Most of their theoretical results carry over to the case of discrete time studied here and are therefore reviewed in the following.

Existence of an equilibrium is easily obtained for (MEP-E) and follows directly by applying Corollary 2.2.2. The strategy sets of all players are described by linear inequalities and are, by definition, nonempty since the zero vector is a feasible strategy for all players. Furthermore, the utility functions are concave for all players and the functions that map the players' decision variables to their actual consumption vectors are affine for all players. The market-clearing conditions and therefore all constraints of (WFP-E) are linear and satisfied by the zero vector. Hence, if the welfare problem is bounded, a market equilibrium of (MEP-E) exists. In reality, the welfare problem is always bounded due to technical transport restrictions and limited resources.

Concerning uniqueness of the market equilibrium, we first like to note that uniqueness of demand directly follows from Theorem 1a of Mangasarian (1988). This result can also be applied to establish uniqueness of production in case that the variable production costs are strictly convex. Consequently, all players' decisions except for the trading flows are unique. Multiplicities in the trading flows arise if and only if it is possible to modify a given solution with a flow along a cycle while remaining within the trading capacities; see also the discussion before Theorem 3 in Grimm, Schewe, et al. (2017). However, such multiplicities are in line with the optimal decisions of the other players since the market is cleared nonetheless. These multiplicities are thus negligible. Therefore, uniqueness is easily obtained for strictly convex variable costs.

For this reason, we restrict our analysis in the following to linear variable costs and additionally assume that these costs are pairwise distinct.

Assumption 1 (Grimm, Schewe, et al. (2017, Assumption 2)). *The variable costs of producer $v \in V_s^+$ are given in any trading period $t \in T$ by $c_v^{\text{var}} y_{t,v}$, where $c_v^{\text{var}} \in \mathbb{R}_{\geq 0}$ holds. Moreover, the variable production costs are pairwise distinct for the considered sector $s \in S$, i.e., $c_v^{\text{var}} \neq c_w^{\text{var}}$ for all $v \neq w$ with $v, w \in V_s^+$.*

This assumption is sufficient to establish uniqueness of the short-run market equilibrium, i.e., in the case of fixed investment decisions.

Theorem 3.3.1 (Grimm, Schewe, et al. (2017, Theorem 3)). *Let a sector $s \in S$ be given. Suppose Assumption 1 holds. Then, there exists a short-run market equilibrium of (MEP-E) with unique demand and production.*

We refer the interested reader to Krebs and Schmidt (2018) for a generalization of this short-run result to the case of a system operator facing piecewise linear and symmetric transport costs.

When proving the uniqueness of the long-run market equilibrium in the continuous time setting, Grimm, Schewe, et al. (2017) rely on the assumption that sets of scenarios in which specific demand values are realized must have zero measure; see Assumption 3 in Grimm, Schewe, et al. (2017). In the setting of finitely many trading periods, this assumption obviously does not hold in general. However, by using an alternative proof, the author of this dissertation has shown in [JG1] that the theoretical results of Grimm, Schewe, et al. (2017) are nevertheless transferable to the case of discrete time. In order to see this, we need some more notation and therefore introduce the concept of price clusters.

Definition 3.3.1 (Grimm, Schewe, et al. (2017, Definition 1), [JG1, Definition 4.2]). Let x be a market equilibrium of (MEP-E). Furthermore, let $\mathcal{C}_{t,s} = \{C_1, \dots, C_{|\mathcal{C}_{t,s}|}\}$ be a partition of the bidding zones Z_s of the considered sector $s \in S$ in trading period $t \in T$. We call $\mathcal{C}_{t,s}$ a partition into *price clusters* if prices are equal for all bidding zones in each cluster $C \in \mathcal{C}_{t,s}$. To emphasize that price clusters may depend on the considered market equilibrium, we use the notation $\mathcal{C}_{t,s}(x)$.

As shown in Grimm, Schewe, et al. (2017), such a partition into price clusters can be constructed from a given market equilibrium x of (MEP-E) by deleting all arcs with binding trading capacities from the graph G'_s . Then, the remaining weakly connected components of the obtained graph $G''_s = (Z_s, K_{t,s}^o)$ with $K_{t,s}^o := \{k \in K_s \mid q_k^- < q_{t,k} < q_k^+\}$ yield the partition into price clusters for trading period $t \in T$. We call this partition *capacity-induced* since it is induced by the binding trading capacities in the given market equilibrium.

To establish uniqueness of the long-run market equilibrium, we impose a final assumption based on the occurrence of the same capacity-induced partition over multiple trading periods. To this end, let $\gamma_{t,v}^+$ denote the dual variable of the capacity bound in (3.1b) for a given producer $v \in V_s^+$ and a given trading period $t \in T$. For the ease of presentation, we address all producers that operate in the same price cluster by V_C^+ , i.e., $V_C^+ := \bigcup_{z \in C} V_z^+$ holds.

Assumption 2 (Grimm, Schewe, et al. (2017, Assumption 4), [JG1, Assumption 3]). Let x be a market equilibrium of (MEP-E). Then, for the considered sector $s \in S$, there exists a subset of trading periods $\bar{T} \subseteq T$ for which the capacity-induced partitions $\mathcal{C}_{t,s}(x)$, $t \in \bar{T}$, are the same. Moreover, for each price cluster $C \in \mathcal{C}_{t,s}(x)$, there exists a bijective function

$$f_C: \bar{T}_C \rightarrow V_C^+ \quad \text{with} \quad \bar{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|\bar{T}_C|}\} \subseteq \bar{T}$$

3. A Market Equilibrium Problem to Model Liberalized Energy Markets

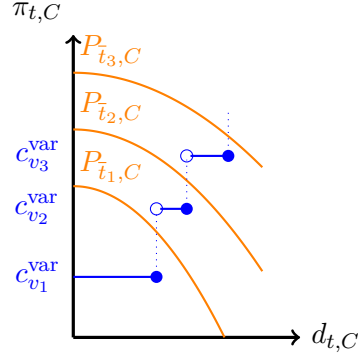


Figure 3.2.: Illustration of Assumption 2 for the case that three producers trade in the cluster (merit-order in blue, aggregated demand functions in orange; adapted from [JG1])

such that the following holds for a given $j \in \{1, \dots, |\overline{T}_C|\}$:

- (i) All producers $f_C(\bar{t}_{j'})$ with $j' \leq j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, produce in trading period \bar{t}_j at maximum capacity, while strict complementarity is satisfied, i.e., $y_{\bar{t}_j, f_C(\bar{t}_{j'})} = \bar{y}_{f_C(\bar{t}_{j'})}^+$ holds with $\gamma_{\bar{t}_j, f_C(\bar{t}_{j'})}^+ > 0$.
- (ii) All producers $f_C(\bar{t}_{j'})$ with $j' > j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, produce in trading period \bar{t}_j zero, i.e., $y_{\bar{t}_j, f_C(\bar{t}_{j'})} = 0$ holds.

In economic terms, Assumption 2 implies that there exists at least one trading period for each producer in which this producer generates the lowest but still positive contribution margin. To explain this relation more precisely, we consider the merit-order $c_{v_1}^{\text{var}} < c_{v_2}^{\text{var}} < \dots < c_{v_n}^{\text{var}}$ of the price cluster C with $n := |V_C^+|$. Assumption 2 imposes that there exists one trading period in which the associated price in the cluster is greater than $c_{v_j}^{\text{var}}$ but smaller than $c_{v_{j+1}}^{\text{var}}$, i.e.,

$$c_{v_1}^{\text{var}} < \pi_{\bar{t}_1, C} < c_{v_2}^{\text{var}} < \pi_{\bar{t}_2, C} < c_{v_3}^{\text{var}} < \dots < c_{v_n}^{\text{var}} < \pi_{\bar{t}_{|\overline{T}_C|}, C}$$

holds. This relation is especially violated if two producers do not produce or always produce at the same time once they are in the same price cluster. However, as soon as producers are sufficiently different and consumer demand is sufficiently fluctuating, such cases should actually not occur and Assumption 2 should be satisfied. In Figure 3.2, the case of sufficiently different economic data is illustrated. For a more detailed discussion of Assumption 2 in the context of a single sector, we refer the interested reader to Grimm, Schewe, et al. (2017).

Based on Assumption 2, we can verify uniqueness of a given long-run market equilibrium of (MEP-E) a posteriori.

Theorem 3.3.2 (Grimm, Schewe, et al. (2017, Theorem 5), [JG1, Theorem 4.20]). *Let a sector $s \in S$ and a long-run market equilibrium x of (MEP-E) be given. Suppose Assumptions 1 and 2 hold. Then, (MEP-E) has a unique solution in demand, production, and investment.*

Finally, we like to comment on how to compute an equilibrium of (MEP-E). As argued above, Corollary 2.2.2 applies for (MEP-E). Consequently, each equilibrium of (MEP-E) corresponds to a welfare-maximal solution of (WFP-E) and vice versa. Hence, state-of-the-art quadratic programming solvers can be applied to the single-level welfare problem (WFP-E) in order to obtain a market equilibrium of (MEP-E). This is done, e.g., in Grimm, Martin, et al. (2016), where the above introduced MEP forms a lower level of a multilevel equilibrium problem. Even for large-scale instances, this approach is easily applicable in the short-run since all trading periods are decoupled over time and can therefore be solved separately. In the long-run, problem-tailored clustering methods can be applied for large-scale instances to reduce the input data size to a manageable number of representative scenarios as done, e.g., in Ambrosius et al. (2020).

4. First Extension: Operators of Sector-Coupling Technologies

In the following, we consider the case of coupled energy sectors as a first extension of the market equilibrium problem (MEP-E) described in Chapter 3. So far, the transport networks and markets of the different energy commodities have not been coupled. Introducing operators of sector-coupling technologies as additional players leads to such coupling aspects in the equilibrium model. As a consequence, the equilibrium analysis becomes more complicated. In [JG1], we show that additional assumptions are necessary to ensure uniqueness of the short- and long-run market equilibrium compared to the single-sector case described in the previous chapter. For a more compact presentation of these results, we refrain from distinguishing between existing and candidate production facilities as in [JG1] and only take candidate facilities into account. Otherwise, the setup and all presented results in this chapter are based entirely on [JG1].

We proceed in this chapter as follows. First, we introduce the extension of (MEP-E) regarding sector coupling in Section 4.1. In Section 4.2, we provide a priori conditions for uniqueness of the short-run equilibrium in coupled markets of multiple energy sectors. Finally, we consider the long-run in Section 4.3 and derive a posteriori conditions for verifying the uniqueness of a given long-run market equilibrium.

4.1. The Market Equilibrium Problem

A sector-coupling technology transforms the energy commodity of one sector s into the energy commodity of another sector s' , where $s \neq s'$ and $s, s' \in S$ hold. Thus, a sector-coupling production facility is characterized by the tuple (v, w) of the network location $v \in V_s$ at which the input factor is withdrawn and the network location $w \in V_{s'}$ at which the production output is injected. We denote the set of all sector-coupling facilities by $A^\pm \subseteq \bigcup_{s, s' \in S: s \neq s'} (V_s \times V_{s'})$. Each facility $a = (v, w) \in A^\pm$ has an efficiency $\eta_a \in (0, 1)$ and investment costs that linearly increase in the invested capacity by $c_a^{\text{inv}} \in \mathbb{R}_{\geq 0}$. Given a bidding zone $z \in Z_s$ of the sector $s \in S$, we denote by $A_z^- \subseteq A^\pm$ all sector-coupling producers that obtain their input factors from this bidding zone. Analogously,

4. First Extension: Operators of Sector-Coupling Technologies

we denote by $A_z^+ \subseteq A^\pm$ all sector-coupling producers that sell their production output in this bidding zone.

An operator of a sector-coupling facility aims at maximizing the profit from trading such as the *sector-specific* producers introduced in the previous chapter. However, the difference between these two types of producers is that, for the sector-specific producers, the energy markets from which they obtain their input factors are not modeled endogenously in our setup, while the input factor markets of the sector-coupling producers are. Hence, the operator of the sector-coupling production facility $a = (v, w) \in A^\pm$ faces the following optimization problem

$$\max_{y_a, \bar{y}_a, d_a} \sum_{t \in T} \left(\pi_{t, z(w)} y_{t,a} - \pi_{t, z(v)} d_{t,a} \right) - c_a^{\text{inv}} \bar{y}_a \quad (4.1a)$$

$$\text{s.t. } \eta_a d_{t,a} = y_{t,a}, \quad t \in T, \quad (4.1b)$$

$$\bar{y}_a \geq y_{t,a} \geq 0, \quad t \in T, \quad (4.1c)$$

$$d_{t,a} \geq 0, \quad t \in T, \quad (4.1d)$$

where $\pi_{t, z(v)}$ denotes the price of the bidding zone in which the input factor is obtained and $\pi_{t, z(w)}$ denotes the price of the bidding zone in which the production output is sold. In accordance with constraints (4.1b), the production output equals the purchased input multiplied by the facility's efficiency. In addition, the production output is limited by the invested capacity.

Finally, we include the operators of sector-coupling technologies in the market-clearing conditions and therefore have

$$\sum_{k \in \delta_s^{\text{in}}(z)} q_{t,k} - \sum_{k \in \delta_s^{\text{out}}(z)} q_{t,k} = \sum_{v \in V_z^-} d_{t,v} - \sum_{v \in V_z^+} y_{t,v} + \sum_{a \in A_z^-} d_{t,a} - \sum_{a \in A_z^+} y_{t,a} \quad (4.2)$$

for all $z \in Z_s$, $s \in S$, and $t \in T$.

In total, we obtain the following market equilibrium problem for coupled markets of multiple energy sectors:

$$\begin{aligned} \text{Sector-specific producers:} & \quad (3.1) \text{ for all } v \in V_s^+, s \in S, \\ \text{Consumers:} & \quad (3.2) \text{ for all } v \in V_s^-, s \in S, \\ \text{System operators:} & \quad (3.3) \text{ for all } s \in S, \\ \text{Sector-coupling producers:} & \quad (4.1) \text{ for all } a \in A^\pm, \\ \text{Market-clearing conditions:} & \quad (4.2). \end{aligned} \quad (\text{MEP-SC})$$

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The corresponding welfare optimization problem reads

$$\begin{aligned}
\max_{y, \bar{y}, d, q} \quad & \sum_{t \in T} \sum_{s \in S} \left(\sum_{v \in V_s^-} \int_0^{d_{t,v}} P_{t,v}(\mu) d\mu - \sum_{v \in V_s^+} c_v^{\text{var}}(y_{t,v}) \right) \\
& - \sum_{s \in S} \sum_{v \in V_s^+} c_v^{\text{inv}} \bar{y}_v - \sum_{a \in A^\pm} c_a^{\text{inv}} \bar{y}_a \quad (\text{WFP-SC}) \\
\text{s.t.} \quad & (3.1b), \quad v \in V_s^+, \quad s \in S, \quad (3.2b), \quad v \in V_s^-, \quad s \in S, \\
& (3.3b), \quad s \in S, \quad (4.1b)-(4.1d), \quad a \in A^\pm, \quad (4.2).
\end{aligned}$$

Existence of an equilibrium is easily obtained for (MEP-SC) by applying Corollary 2.2.2 as in the single-sector case. Moreover, Corollary 2.2.2 implies that each equilibrium of (MEP-SC) corresponds to a welfare-maximal solution of (WFP-SC) and vice versa. Hence, as in the single-sector case, state-of-the-art quadratic programming solvers can be applied to the welfare problem (WFP-SC) to compute a market equilibrium of (MEP-SC) and, if necessary, problem-tailored clustering methods can be used to reduce the input data size to a manageable number of representative scenarios. In contrast to existence and computation, uniqueness of the market equilibrium is not as easy to achieve for multiple coupled sectors, which we discuss below.

4.2. A Priori Conditions for Short-Run Uniqueness

Here, we present conditions that are sufficient for guaranteeing uniqueness of the short-run market equilibrium of (MEP-SC). These conditions are obtained by extending the assumptions required for a single sector to multiple sectors. Before we start with deriving these conditions, we like to point out that all demand is again unique due to Theorem 1a of Mangasarian (1988). Moreover, if the variable cost functions are strictly convex, sector-specific production is also unique due to Mangasarian (1988). Since the additional assumptions for uniqueness of the sector-coupling production are covered by the assumptions for uniqueness of overall production in the case of linear variable costs, we refrain from presenting the special case of strictly convex costs and directly restrict our argumentation to linear costs. To this end, we assume for the rest of this chapter that the variable costs of producer $v \in V_s^+$ are given in any trading period $t \in T$ by $c_v^{\text{var}} y_{t,v}$, where $c_v^{\text{var}} \in \mathbb{R}_{\geq 0}$ holds.

The first condition that is required for establishing short-run uniqueness relates to the sector-specific demand and reads as follows.

Assumption 3 ([JG1, Assumption 1]). *The demand $d_{t,z} := \sum_{v \in V_z^-} d_{t,v}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is positive in all trading periods $t \in T$.*

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As argued in [JG1], this assumption can be justified by the fact that, in real applications, there should always be consumers in a bidding zone who are willing to pay more than the resident operators of sector-coupling facilities.

Assumption 3 allows us to conclude uniqueness of the market prices; see Lemma 4.9 in [JG1]. Since the market prices are unique, we have multiple equilibria if and only if individual players have multiple best responses to these unique market prices. As in the single-sector case, this situation is avoided if the variable production costs are pairwise distinct for all producers. To guarantee this in sector-coupled markets, the following assumption is imposed.

Assumption 4 ([JG1, Assumption 2]). *All variable production costs and all efficiencies of the sector-coupling facilities satisfy the following five properties.*

- (i) *The variable production costs are pairwise distinct in each sector, i.e., $c_v^{\text{var}} \neq c_w^{\text{var}}$ for all $v \neq w$ with $v, w \in V_s^+$, $s \in S$.*
- (ii) *The efficiencies of the sector-coupling facilities are pairwise distinct, i.e., $\eta_a \neq \eta_{a'}$ for all $a \neq a'$ with $a, a' \in A^\pm$.*
- (iii) *Consider the sector-coupling facility $a \in A^\pm$ that produces the commodity of sector $s \in S$ from the commodity of sector $s' \in S$ with $s \neq s'$. Then, the variable production costs across the two sectors are—taking into account the efficiency of the sector-coupling facility—pairwise distinct, i.e., $\eta_a c_v^{\text{var}} \neq c_w^{\text{var}}$ with $v \in V_s^+$ and $w \in V_{s'}^+$.*
- (iv) *It holds*

$$c_v^{\text{var}} \prod_{i=1}^j \eta_{a_i} \neq c_w^{\text{var}} \prod_{i=j+1}^{j'} \eta_{a_i}$$

for all $\{a_1, \dots, a_{j'}\} \subseteq A^\pm$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$ and for all $v \neq w$ with $v \in V_s^+$, $s \in S$, and $w \in V_{s'}^+$, $s' \in S$.

- (v) *It holds*

$$\prod_{i=1}^j \eta_{a_i} \neq \prod_{i=j+1}^{j'} \eta_{a_i}$$

for all $\{a_1, \dots, a_{j'}\} \subseteq A^\pm$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$.

Assumption 4 (i) covers the single-sector short-run assumption of pairwise distinct variable costs for all sector-specific producers.

4.3. A Posteriori Conditions for Long-Run Uniqueness

To explain Assumption 4 in more detail, we now consider the variable production costs of an operator $a = (v, w)$ of a sector-coupling technology. This operator obtains the technology's input factor in bidding zone $z(v)$ and sells the production output in bidding zone $z(w)$. If this operator produces the quantity $\varepsilon > 0$, the respective variable costs of production amount to $\pi_{t,z(v)}\varepsilon/\eta_a$. Thus, Assumption 4 (ii) must hold to avoid that two sector-coupling producers with the same input and output bidding zone operate at the same variable costs. Moreover, if a sector-specific production technology sets the price in the input bidding zone, i.e., $\pi_{t,z(v)} = c_{v'}^{\text{var}}$ holds for some $v' \in V_{z(v)}^+$, Assumption 4 (iii) prevents that any sector-specific producer in the output bidding zone has the same variable costs as the given sector-coupling producer.

The considered operator might sell the technology's production output to another sector-coupling producer $a' \in A^\pm$ that takes this output as input for producing the quantity $\varepsilon' > 0$ of another energy commodity. In this case, the respective variable costs amount to $\pi_{t,z(v)}\varepsilon' / (\eta_a\eta_{a'})$ if $\pi_{t,z(w)} = \pi_{t,z(v)}/\eta_a$ is true. Hence, Assumption 4 (iv) naturally extends Assumption 4 (iii) to chains of successively producing sector-coupling technologies. Finally, Assumption 4 (v) avoids that such chains with the same initial input and final output bidding zone yield the same variable production costs. In summary, Assumption 4 imposes that the variable costs are pairwise distinct for all producers.

Under the stated conditions, we finally obtain uniqueness of the short-run equilibrium in coupled markets of multiple energy sectors.

Theorem 4.2.1 ([JG1, Theorem 4.16]). *Suppose Assumptions 3 and 4 hold. Then, there exists a short-run market equilibrium of (MEP-SC) with unique sector-specific and sector-coupling demand and production.*

Examples B.1–B.5 in [JG1] illustrate that each of the conditions imposed in Assumption 4 is indeed necessary to guarantee uniqueness of demand and production in general. Note that, as in the single-sector case, the trading flows of the system operators can be ambiguous. Using the same arguments as in Section 3.3, it is easy to see that these multiplicities are negligible.

4.3. A Posteriori Conditions for Long-Run Uniqueness

We continue with studying uniqueness of the long-run market equilibrium. The fact that a priori conditions for long-run uniqueness have not yet been derived for a single sector indicates that such conditions are difficult to obtain. In addition, it is even more difficult to obtain a priori criteria for coupled markets of multiple energy sectors than for a single sector. Therefore, we focus only on

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a posteriori conditions for long-run uniqueness in this section. Again, we extend the assumptions required for a single sector to multiple sectors. To this end, we denote by $\gamma_{t,v}^+$ the dual variable of the capacity bound in (3.1b) for a given sector-specific producer $v \in V_s^+$ and a given trading period $t \in T$. Analogously, we denote by $\tau_{t,a}^+$ the dual variable of the capacity bound in (4.1c) for a given sector-coupling production facility $a \in A^\pm$ and a given trading period $t \in T$.

Assumption 5 ([JG1, Assumption 3]). *Let x be an equilibrium of (MEP-SC). Then, for each sector $s \in S$, there exists a subset of trading periods $\bar{T} \subseteq T$ for which the capacity-induced partitions $\mathcal{C}_{t,s}(x)$, $t \in \bar{T}$, are the same. Moreover, for each price cluster $C \in \mathcal{C}_{t,s}(x)$, there exists a bijective function*

$$f_C: \bar{T}_C \rightarrow V_C^+ \cup A_C^+ \cup A_C^- \quad \text{with} \quad \bar{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|\bar{T}_C|}\} \subseteq \bar{T}$$

such that the following holds for a given $j \in \{1, \dots, |\bar{T}_C|\}$:

- (i) *All producers $f_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, produce in trading period \bar{t}_j either zero or at maximum capacity, where in the latter case strict complementarity is satisfied, i.e., either $y_{\bar{t}_j, f_C(\bar{t}_{j'})} = 0$ or*

$$y_{\bar{t}_j, f_C(\bar{t}_{j'})} = \bar{y}_{f_C(\bar{t}_{j'})} \quad \text{with} \quad \begin{cases} \gamma_{\bar{t}_j, f_C(\bar{t}_{j'})}^+ > 0, & \text{if } f_C(\bar{t}_{j'}) \in V_C^+, \\ \tau_{\bar{t}_j, f_C(\bar{t}_{j'})}^+ > 0, & \text{if } f_C(\bar{t}_{j'}) \in A_C^+ \cup A_C^-. \end{cases}$$

- (ii) *The producer $f_C(\bar{t}_j)$ produces in trading period \bar{t}_j at maximum capacity, while strict complementarity is satisfied, i.e.,*

$$y_{\bar{t}_j, f_C(\bar{t}_j)} = \bar{y}_{f_C(\bar{t}_j)} \quad \text{with} \quad \begin{cases} \gamma_{\bar{t}_j, f_C(\bar{t}_j)}^+ > 0, & \text{if } f_C(\bar{t}_j) \in V_C^+, \\ \tau_{\bar{t}_j, f_C(\bar{t}_j)}^+ > 0, & \text{if } f_C(\bar{t}_j) \in A_C^+ \cup A_C^-. \end{cases}$$

- (iii) *All producers $f_C(\bar{t}_{j'})$ with $j' > j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, produce in trading period \bar{t}_j zero, i.e., $y_{\bar{t}_j, f_C(\bar{t}_{j'})} = 0$ holds.*

The above stated long-run uniqueness condition cannot be as easily interpreted via the merit-order as in the case of a single sector. Of course, it must still hold that, given the merit-order $c_{v_1}^{\text{var}} < c_{v_2}^{\text{var}} < \dots < c_{v_n}^{\text{var}}$, $n := |V_C^+|$, of sector-specific production facilities in each cluster, there must exist at least one trading period in which the associated price is greater than $c_{v_j}^{\text{var}}$ but smaller than $c_{v_{j+1}}^{\text{var}}$. However, this condition alone is not sufficient since the sector-coupling production facilities also enter this merit-order but at which positions

4.3. A Posteriori Conditions for Long-Run Uniqueness

is not clear a priori. This depends heavily on the relation of the prices between the coupled sectors. Nevertheless, as soon as sector-specific and sector-coupling producers are sufficiently different and consumer demand is sufficiently fluctuating in all sectors, Assumption 5 should also be satisfied in the context of multiple sectors; see Examples C.1 and C.2 in [JG1] for illustrative instances in which this is not true. For a more detailed discussion of Assumption 5, we refer the interested reader to [JG1].

Finally, given a long-run market equilibrium of (MEP-SC), we can verify a posteriori that this market equilibrium is unique if all stated assumptions are fulfilled.

Theorem 4.3.1 ([JG1, Theorem 4.20]). *Let a long-run market equilibrium x of (MEP-SC) be given. Suppose Assumptions 3, 4, and 5 hold. Then, (MEP-SC) has a unique solution in sector-specific and sector-coupling demand, production, and investment.*

5. Second Extension: Operators of Storage Technologies

In what follows, we consider temporal coupling as a second extension of the market equilibrium problem (MEP-E) described in Chapter 3. Temporal coupling results from, e.g., introducing operators of storage technologies as additional players since their actions couple successive trading periods. This time-dependency complicates the equilibrium analysis. Therefore, we focus on the short-run and on the basic case of a single bidding zone in a single energy sector. To simplify the presentation, we drop the index for the sector and for the bidding zone. The setup and all presented results in this chapter are entirely based on [JG2].

We first introduce the extension of (MEP-E) regarding temporal coupling in Section 5.1. Afterward, we study the corresponding equilibria in Section 5.2. While uniqueness of demand and production is established under additional assumptions, storage operations generally remain ambiguous. However, we show that uniqueness can be verified a posteriori under certain conditions. In Section 5.3, we discuss the computation of equilibria for the problem at hand. We propose a tailored parallel and distributed alternating direction method of multipliers (ADMM) and argue why an ADMM is particularly suitable for the given problem.

5.1. The Market Equilibrium Problem

Let $V^\pm \subseteq V$ denote the set of nodes at which operators of storage technologies are located. Each storage device $v \in V^\pm$ is characterized by the capacity $\bar{\ell}_v > 0$, the maximal charging and discharging $\bar{s}_v > 0$ per trading period, and the charging efficiency $\eta_v \in (0, 1)$. For the ease of presentation, we assume that all losses occur during charging. The initial and final states of charge $\ell_{-1,v}$ and $\ell_{t^e,v}$ are assumed to be zero for all $v \in V^\pm$.

The objective of a storage operator $v \in V^\pm$ is to maximize the profit from trading. Hence, charging $s_{t,v}^c$ of the storage is best carried out in low-price trading periods and discharging $s_{t,v}^d$ in high-price trading periods. While doing so, the state of charge $\ell_{t,v}$ at the end of each trading period is not allowed

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to exceed the capacity of the storage device. Consequently, the optimization problem of the storage operator reads

$$\max_{s_v^c, s_v^d, \ell_v} \sum_{t \in T} \pi_t (s_{t,v}^d - s_{t,v}^c) \quad (5.1a)$$

$$\text{s.t. } \ell_{t,v} = \ell_{t-1,v} - (s_{t,v}^d - \eta_v s_{t,v}^c), \quad t \in T, \quad (5.1b)$$

$$0 \leq \ell_{t,v} \leq \bar{\ell}_v, \quad t \in T, \quad (5.1c)$$

$$0 \leq s_{t,v}^c \leq \bar{s}_v, \quad t \in T, \quad (5.1d)$$

$$0 \leq s_{t,v}^d \leq \bar{s}_v, \quad t \in T, \quad (5.1e)$$

where π_t denotes the price of the respective trading period. The adjusted market-clearing conditions

$$\sum_{v \in V^\pm} (s_{t,v}^d - s_{t,v}^c) = \sum_{v \in V^-} d_{t,v} - \sum_{v \in V^+} y_{t,v}, \quad t \in T, \quad (5.2)$$

complete the time-coupled market equilibrium problem

$$\begin{aligned} \text{Producers:} & \quad (3.1) \text{ for all } v \in V^+, \\ \text{Consumers:} & \quad (3.2) \text{ for all } v \in V^-, \\ \text{Storage operators:} & \quad (5.1) \text{ for all } v \in V^\pm, \\ \text{Market-clearing conditions:} & \quad (5.2). \end{aligned} \quad (\text{MEP-TC})$$

The corresponding welfare optimization problem is given by

$$\begin{aligned} \max_{y, d, s^c, s^d, \ell} & \quad \sum_{t \in T} \left(\sum_{v \in V^-} \int_0^{d_{t,v}} P_{t,v}(\mu) \, d\mu - \sum_{v \in V^+} c_v^{\text{var}}(y_{t,v}) \right) \\ \text{s.t.} & \quad (3.1b), \quad v \in V^+, \quad (3.2b), \quad v \in V^-, \\ & \quad (5.1b) \text{--}(5.1e), \quad v \in V^\pm, \quad (5.2). \end{aligned} \quad (\text{WFP-TC})$$

As for (MEP-E) and (MEP-SC), existence of an equilibrium is easily obtained for (MEP-TC) by applying Corollary 2.2.2. However, we illustrate below that uniqueness and computation of the market equilibrium is not as easy to achieve.

5.2. Uniqueness and Multiplicity of the Short-Run Equilibria

In this section, we discuss the boundaries between uniqueness and multiplicity of the short-run market equilibria of (MEP-TC). We derive sufficient conditions

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that ensure uniqueness of demand and production. However, examples illustrate that storage operations are not unique in general. We conclude with presenting an a posteriori criterion for verifying uniqueness of a given short-run equilibrium.

To establish uniqueness of demand and production, we first impose the following assumption.

Assumption 6 ([JG2, Assumptions 3 and 5]). *The market price π_t and the total demand $\sum_{v \in V^-} d_{t,v}$ are positive in each trading period $t \in T$.*

As argued in [JG2], this assumption is not very restrictive if large-scale storage systems are participating in real-world sport markets. In the presence of negative prices, the storage operators would naturally charge the production surplus, since this automatically leads to positive profit. Hence, non-positive prices are not to be expected. In addition, there should always be consumers who are willing to pay and, more precisely, whose willingness to pay is higher than the willingness to pay of the storage operators.

Again, uniqueness of demand—and of production for strictly convex variable cost functions—follows from Theorem 1a of Mangasarian (1988). Hence, we restrict our analysis in the following to linear variable costs. Moreover, as a prerequisite for deriving uniqueness of production in this case, we proceed with variable costs that are dependent on the trading periods. Thus, we assume for the rest of this chapter that the variable costs of producer $v \in V^+$ are given in trading period $t \in T$ by $c_{t,v}^{\text{var}} y_{t,v}$, where $c_{t,v}^{\text{var}} \in \mathbb{R}_{\geq 0}$ holds.

Next, we extend the classic assumption of pairwise distinct variable costs for all producers to the given setting with storage operators.

Assumption 7 ([JG2, Assumptions 4 and 6]). *All variable production costs and all efficiencies of the storage devices satisfy the following three properties.*

(i) *All variable costs are pairwise distinct, i.e.,*

$$c_{t,v}^{\text{var}} \neq c_{t',w}^{\text{var}} \quad \text{for all } (t,v) \neq (t',w) \in T \times V^+.$$

(ii) *It holds*

$$c_{t,w}^{\text{var}} \prod_{i=1}^j \eta_{v_i} \neq c_{t',w'}^{\text{var}} \prod_{i=j+1}^{j'} \eta_{v_i}$$

for all $(t,w) \neq (t',w') \in T \times V^+$ and $v_1, \dots, v_{j'} \in V^\pm$ with $1 \leq j' \leq |T| - 1$.

(iii) *The efficiencies of the storage devices are pairwise distinct, i.e.,*

$$\eta_v \neq \eta_w \quad \text{for all } v \neq w \in V^\pm.$$

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Assumption 7 (i) covers for $t = t'$ the time-independent short-run assumption of pairwise distinct variable costs for all producers.

To explain Assumption 7 in more detail, we now consider the variable production costs of a storage operator v . If this operator charges the considered energy commodity in trading period t and later discharges the quantity $\varepsilon > 0$ in trading period t' , the respective variable costs of production amount to $\pi_t \varepsilon / \eta_v$. In case that there is another trading period before t' with the same price as in t , the storage operator might be indifferent whether to charge in this trading period or in the trading period t as the resulting variable costs are the same. We cannot assess all prices a priori but we can at least prevent such situations when specific production technologies set the prices. Therefore, we impose Assumption 7 (i). Moreover, we have to ensure that the storage operator's variable costs $\pi_t \varepsilon / \eta_v$ do not meet the variable costs of any producer in the discharging trading period t' . Again, we can only assess situations a priori in which specific production technologies set the prices. These situations are covered by Assumption 7 (ii) for $j' = 1$. Since one storage device might also charge from another storage device or two storage devices might discharge at the same time, the cases $j' > 1$ in Assumption 7 (ii) are of importance. Finally, it remains to avoid the case that two storage operators produce at the same variable costs due to the same efficiency; see Assumption 7 (iii).

Under the stated conditions, we obtain uniqueness of demand and production for a market with producers, consumers, and storage operators.

Theorem 5.2.1 ([JG2, Theorem 3.11]). *Suppose Assumptions 6 and 7 hold. Then, there exists a short-run market equilibrium of (MEP-TC) with unique demand and production. Furthermore, the differences $\sum_{v \in V^\pm} (s_{t,v}^d - s_{t,v}^c)$ of total discharging and charging are unique for all trading periods $t \in T$.*

For the MEPs (MEP-E) and (MEP-SC) studied in the previous chapters, we also only have uniqueness of demand and production, while the trading flows of the system operators remain ambiguous. Nevertheless, these multiplicities are in line with the optimal decisions of the other players since the market is cleared independently of the strategies played by the system operators. This is not true here. If different storage operators play any of their multiple strategies, an infeasible market outcome might result as all traded amounts might no longer be balanced. Hence, none of the equilibria is stable.

We like to note that Theorem 5.2.1 directly implies uniqueness of the short-run market equilibrium if a single storage device is operated in the market; see Corollary 3.12 in [JG2]. However, as soon as multiple storage operators trade in the market, multiplicities might occur; see Examples 3.13–3.16 in [JG2], in which multiplicities are present despite the fact that all of the above stated

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assumptions are fulfilled. This is mainly because we can only capture situations a priori in which specific production technologies set the prices. Nevertheless, we can assess all prices a posteriori and therefore obtain the following a posteriori uniqueness condition.

Theorem 5.2.2 ([JG2, Theorem 3.17]). *Let a short-run market equilibrium x of (MEP-TC) be given. Suppose Assumptions 6 and 7 hold. Furthermore, assume that $\pi_t \neq \pi_{t'}$ and $\pi_t \neq \pi_{t'}\eta_v$ hold for all $t, t' \in T$, $t \neq t'$, and $v \in V^\pm$. Then, (MEP-TC) has a unique solution in demand, production, and storage operations.*

In this dissertation, we refrain from studying uniqueness of the long-run market equilibrium of (MEP-TC) due to the following two reasons. First, since the outcomes of the future spot markets are ambiguous, a storage operator might not foresee the future profit correctly. Deciding on the basis of our short-run model with multiple equilibria could therefore lead to undesirable investment. Second, and closely connected to the first point, the optimization problems of the producers and storage operators can also be formulated as bilevel problems in the long-run, where investment decisions are made on the upper level and spot-market decisions on the lower level. Moreover, (MEP-TC) might even be a lower-level problem in a more general multilevel computational equilibrium framework; see Ambrosius et al. (2020), Grimm, Martin, et al. (2016), Grimm, Schewe, et al. (2019), and Kleinert and Schmidt (2019) among others for similar frameworks. As already mentioned before, such multilevel problems can only be tackled meaningfully if lower-level problems have a unique solution. As we have shown, this is not the case here. For more information on the importance of lower-level uniqueness in multilevel problems see, e.g., Colson et al. (2007), Dempe (2002), and Gabriel, Conejo, et al. (2013).

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To conclude, we address the computation of equilibria for (MEP-TC). As Corollary 2.2.2 applies for (MEP-TC), each equilibrium of (MEP-TC) corresponds to a welfare-maximal solution of (WFP-TC) and vice versa. Hence, to compute an equilibrium of (MEP-TC), it is sufficient to compute a solution of the single-level welfare problem (WFP-TC). However, due to the coupling of trading periods over time, this is much more challenging compared to the classic setting without storage. In the classic setting, each trading period can be solved separately for short-run applications, which is not possible here. Moreover, problem-tailored clustering methods can be applied in the classic setting to reduce the input data size to a manageable number of representative

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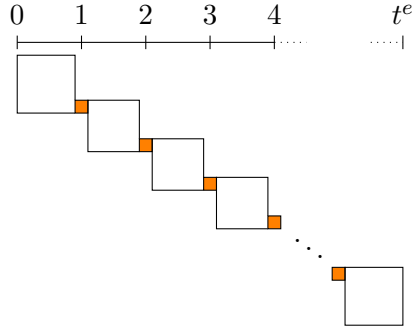


Figure 5.1.: Advantageous almost time-separable block structure of the matrix that describes the feasible set of (WFP-TC)

scenarios. However, for the setup considered in this chapter, it is not straightforward to use such methods due to the time-dependency of storage operations. We therefore propose a tailored parallel and distributed alternating direction method of multipliers (ADMM) in [JG2] and show that this ADMM allows for efficiently solving large-scale instances. Since the author of this dissertation did not contribute to the algorithmic part and the numerical study in [JG2], we refrain from presenting the ADMM and the numerical results here. We refer the interested reader to Sections 4 and 5 in [JG2]. General information on ADMMs can be found in the survey of Boyd et al. (2011) and specific information on the applied ADMM in Chapter 7 of this survey.

We like to close this section with a review on why an ADMM is particularly suitable for the problem at hand; see also the discussion at the end of Section 4 in [JG2]. The basic idea of ADMMs is to decompose the initial problem into smaller problems that can be solved easier, while penalty terms in the objective functions of the smaller problems guarantee that a solution of the initial problem is finally obtained. Taking a closer look at the welfare problem (WFP-TC) reveals that this optimization problem is almost separable. There are several constraints for each trading period but only few ones that couple each trading period with the preceding one, namely the constraints for tracking the states of charge of the storage devices. This underlying structure of the matrix that describes the feasible set of (WFP-TC) is depicted in Figure 5.1 and allows for easily decomposing the initial problem. It is well known that, in practice, ADMMs perform well on problems with this special structure; see, e.g., Geißler et al. (2015, 2018) for some energy-specific applications. We demonstrate in [JG2] that this is also the case for large-scale instances of the energy application studied in this chapter.

6. Third Extension: Operators of Transmission Networks

As the last extension of the market equilibrium problem (MEP-E) presented in Chapter 3, we consider a system operator facing additional constraints besides the allocation of trading capacities. The additional constraints studied in this chapter mainly stem from a more detailed representation of physical laws and technical requirements and therefore also involve nonlinearities or integralities. Since such aspects are challenging already standalone, we restrict our analysis to the short-run and to a single sector. For the ease of notation, we thus drop the index for the trading period and for the sector. In addition, we focus on spot markets in which the network infrastructure and the spatial distribution of demand and production are one-to-one represented. For this reason, we refrain from distinguishing between bidding zones and network nodes in the following and base our notation on the graph G that models the network infrastructure.

In this chapter, we summarize the main contributions of the two articles [JG3] and [JG4]. Since the MEP analyzed in [JG3] is a special application of the MEP studied in [JG4], we mainly follow the setup of [JG4]. Whenever the setup and the results of [JG3] are addressed, we particularly emphasize this. Hence, unless otherwise stated, all presented results are based on [JG4].

We structure this chapter as follows. In Section 6.1, we state in a general form the extension of (MEP-E) by physical laws and technical requirements. Afterward, we discuss three specific applications of this extension. The first application has already been studied thoroughly in the literature and is briefly reviewed due to the similarity with the MEPs analyzed in this thesis. In contrast to the first application, the second and third application include nonconvexities. Existence of an equilibrium is therefore not guaranteed, which we illustrate by an example. Motivated by this example, we propose in Section 6.2 an algorithm that decides whether an equilibrium exists and that computes an equilibrium in the case of existence. By applying this algorithm to well-known instances from the literature on the gas and electricity sector, we identify in Section 6.3 how often and under which conditions an equilibrium exists for the two introduced nonconvex applications. To conclude, we address uniqueness of the short-run market equilibrium for these two applications in Section 6.4.

6.1. The Market Equilibrium Problem

As in (MEP-E), the system operator maximizes the profit from congestion rents but faces additional transport costs $c^{\text{tran}}(q, r)$ that depend on network flows q and possibly on other decision variables r . In total, the system operator faces the following optimization problem

$$\max_{q, r} \sum_{v \in V^- \cup V^+} \pi_v q_v - c^{\text{tran}}(q, r) \quad (6.1a)$$

$$\text{s.t. } F(q, r) \geq 0, \quad q_v = \sum_{a \in \delta^{\text{in}}(v)} q_a - \sum_{a \in \delta^{\text{out}}(v)} q_a, \quad v \in V, \quad (6.1b)$$

$$0 \geq q_v \geq -\bar{y}_v, \quad v \in V^+, \quad q_v \geq 0, \quad v \in V^-, \quad q_v = 0, \quad v \in V^0. \quad (6.1c)$$

Here, $F(q, r) \geq 0$ summarizes the, potentially nonconvex, network-related physical and technical constraints. The constraints (6.1c) ensure that the net flow at consumers can only be positive, at producers negative and within the production capacity, and at inner network nodes equal to zero. The system operator is thus perfectly informed about which type of player is located at which node and, additionally, about the capacities of all production facilities. This is consistent with our assumption of perfect competition. Note that the nodal price π_v is assumed to be zero at all inner network nodes $v \in V^0$, i.e., at all nodes where no producer and no consumer is located.

The new market-clearing conditions are given by

$$\sum_{a \in \delta^{\text{in}}(v)} q_a - \sum_{a \in \delta^{\text{out}}(v)} q_a = -y_v, \quad v \in V^+, \quad (6.2a)$$

$$\sum_{a \in \delta^{\text{in}}(v)} q_a - \sum_{a \in \delta^{\text{out}}(v)} q_a = d_v, \quad v \in V^-, \quad (6.2b)$$

$$\sum_{a \in \delta^{\text{in}}(v)} q_a - \sum_{a \in \delta^{\text{out}}(v)} q_a = 0, \quad v \in V^0. \quad (6.2c)$$

The complete market equilibrium problem that includes a more detailed representation of physical laws and technical requirements reads

$$\begin{aligned} \text{Producers:} & \quad (3.1) \text{ for all } v \in V^+, \\ \text{Consumers:} & \quad (3.2) \text{ for all } v \in V^-, \\ \text{System operator:} & \quad (6.1), \\ \text{Market-clearing conditions:} & \quad (6.2). \end{aligned} \quad (\text{MEP-PL})$$

6.1. The Market Equilibrium Problem

Furthermore, we have the corresponding welfare optimization problem

$$\begin{aligned} \max_{y,d,q,r} \quad & \sum_{v \in V^-} \int_0^{d_v} P_v(\mu) \, d\mu - \sum_{v \in V^+} c_v^{\text{var}}(y_v) - c^{\text{tran}}(q, r) \\ \text{s.t.} \quad & (3.1b), \quad v \in V^+, \quad (3.2b), \quad v \in V^-, \\ & F(q, r) \geq 0, \quad (6.2). \end{aligned} \tag{WFP-PL}$$

Next, we discuss different applications of (MEP-PL). The first application is taken from the literature and is briefly reviewed due to the similarity with the energy MEPs analyzed in this dissertation. The second and third application are presented since these are the main nonconvex applications studied in [JG4].

DC Load Flow As a first application of (MEP-PL), we consider a system operator of a direct current (DC) electricity network facing linear load flow restrictions. Hence, the optimization problems of all players remain convex. This setup has been investigated without transport costs and without constraints (6.1c) by Krebs, Schewe, et al. (2018). The network-related constraints are given by flow bounds for each DC line and by the standard lossless DC load flow approximation as described, e.g., in Wood et al. (2013):

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A, \tag{6.3a}$$

$$q_a = B_a(\theta_v - \theta_w), \quad a = (v, w) \in A, \tag{6.3b}$$

where $B_a > 0$ denotes the DC line's susceptance and θ_v the nodal phase angle. In addition, $q_a^- = -q_a^+$ is assumed. We refer the interested reader to Krebs, Schewe, et al. (2018) for a detailed equilibrium analysis of this DC load flow application of (MEP-PL).

DC Line Switching In this application of (MEP-PL), we again consider a system operator of a DC electricity network. The difference from the previously presented application is that the system operator has the additional possibility to switch on and off certain DC lines in the network. Consequently, integralities enter the system operator's optimization problem.

Let X_a be the DC line's reactance, $A^s \subseteq A$ the set of switchable DC lines, and z_a the binary decision variable indicating whether the respective DC line is switched on ($z_a = 1$) or off ($z_a = 0$). We use the DC load flow approximation from Zimmerman and Murillo-Sánchez (2020b) and therefore introduce a phase shift angle θ_a^{shift} . If a DC line $a \in A^s$ is switched on, the flow on this line follows the DC load flow approximation and is additionally bounded by the line's capacities. Conversely, if a DC line $a \in A^s$ is switched off, no physical laws

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are imposed and the line's flow equals zero. In total, we obtain the following network constraints of the system operator

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A \setminus A^s, \quad (6.4a)$$

$$\theta_v - \theta_w - \theta_a^{\text{shift}} = X_a q_a, \quad a = (v, w) \in A \setminus A^s, \quad (6.4b)$$

$$M_a^-(1 - z_a) \leq \theta_v - \theta_w - \theta_a^{\text{shift}} - X_a q_a, \quad a = (v, w) \in A^s, \quad (6.4c)$$

$$\theta_v - \theta_w - \theta_a^{\text{shift}} - X_a q_a \leq M_a^+(1 - z_a), \quad a = (v, w) \in A^s, \quad (6.4d)$$

$$q_a^- z_a \leq q_a \leq q_a^+ z_a, \quad a \in A^s, \quad (6.4e)$$

$$z_a \in \{0, 1\}, \quad a \in A^s. \quad (6.4f)$$

The transport costs are assumed to quadratically increase in the flow. Moreover, a fixed fee has to be paid if a line is switched on. In total, we have the transport costs

$$c^{\text{tran}}(q, z) = \sum_{a \in A} \alpha q_a^2 + \sum_{a \in A^s} \beta z_a$$

with $\alpha, \beta \in \mathbb{R}_{\geq 0}$.

Later in this chapter, we will see that there exist well-known instances from the literature, in which the considered application of (MEP-PL) does not possess an equilibrium due to the integrality restrictions.

Nonlinear Gas Flow As a last application, we examine a system operator facing nonlinear stationary gas flow equations, which are given by

$$p_v^2 - p_w^2 = \Lambda_a q_a |q_a|, \quad a = (v, w) \in A, \quad (6.5a)$$

$$p_v^- \leq p_v \leq p_v^+, \quad v \in V, \quad (6.5b)$$

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A. \quad (6.5c)$$

The constraints (6.5a) describe the gas flow through the pipes and are called the Weymouth equations; see, e.g., the chapter by Fügenschuh et al. (2015) in Koch et al. (2015) for more information on their derivation. These equations link in a nonlinear way the drop in quadratic pressures ($p_v^2 - p_w^2$) over a pipe to the flow q_a on this pipe. In addition, nodal pressure and flow bounds are imposed to ensure contractual or technical requirements. The transport costs are again assumed to quadratically increase in the flow, i.e., $c^{\text{tran}}(q) = \sum_{a \in A} \alpha q_a^2$ holds for some $\alpha \in \mathbb{R}_{\geq 0}$.

We study this application without transport costs and constraints (6.1c) in detail in [JG3]. As one major contribution, we analyze the relation of the gas flow MEP and the mixed nonlinear complementarity problem (MNCP) that

6.1. The Market Equilibrium Problem

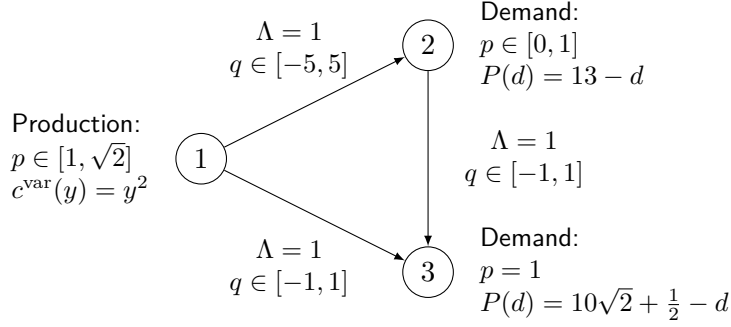


Figure 6.1.: Three-node network with all physical and economic data considered in Example 6.1.1 (adapted from [JG3])

results by joining the KKT conditions of all players with the market-clearing conditions. Both problems are equivalent if and only if a constraint qualification and a second-order sufficient condition hold for the system operator’s problem. We show in [JG3] that the linear independence constraint qualification (LICQ) applies under mild assumptions, i.e., we establish the relation “MEP \Rightarrow MNCP”. However, as we further illustrate, there exist solutions to the MNCP that do not correspond to market equilibria, i.e., the relation “MNCP \Rightarrow MEP” is not true. Hence, in general, we cannot obtain equilibria of the MEP by solving the related complementarity problem as done in similar energy market applications in the literature; see, e.g., Abrell and Weigt (2012, 2016), Awad et al. (2014), Chen, Conejo, Sioshansi, et al. (2020b) as well as Fomeni et al. (2019).

Besides the relation of the MEP and the MNCP, we study the relation of the MEP and the corresponding welfare problem in [JG3]. In particular, we prove that, under the quite restrictive assumption of no network congestion, market equilibria correspond to welfare maxima and vice versa. Thus, the fundamental welfare theorems hold under this assumption. However, we show by the following example that the fundamental welfare theorems do not apply in general; see also Theorem 4.4 in [JG3]. No transport costs are considered in this exemplary instance of the gas flow application of (MEP-PL). Examples with transport costs are examined later in Section 6.3.

Example 6.1.1 ([JG3, Theorem 4.4], [JG4, Example 5.1]). *For the instance depicted in Figure 6.1, exactly two welfare maximal solutions x and \tilde{x} exist, namely*

$$d_2 = d_3 = 1, \quad y_1 = 2, \quad q_{1,2} = q_{1,3} = 1, \quad q_{2,3} = 0, \quad p_1 = \sqrt{2}, \quad p_2 = p_3 = 1,$$

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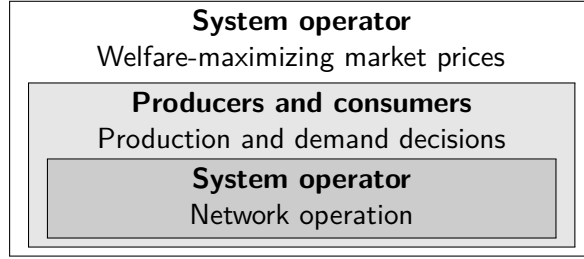


Figure 6.2.: Structure of the multilevel energy market model in which market equilibria correspond to welfare maxima and vice versa (adapted from [JG3])

and

$$\begin{aligned} \tilde{d}_2 &= 1 + \sqrt{2}, \quad \tilde{d}_3 = 0, \quad \tilde{y}_1 = 1 + \sqrt{2}, \quad \tilde{q}_{1,2} = \sqrt{2}, \quad \tilde{q}_{1,3} = 1, \quad \tilde{q}_{2,3} = -1, \\ \tilde{p}_1 &= \sqrt{2}, \quad \tilde{p}_2 = 0, \quad \tilde{p}_3 = 1. \end{aligned}$$

Furthermore, there are no market prices π such that (x, π) or (\tilde{x}, π) is an equilibrium of the gas flow application of (MEP-PL). Hence, by Theorem 2.2.1, no market equilibrium exists for the given instance.

Example 6.1.1 illustrates that the fundamental welfare theorems do not apply in general for the setting considered in this chapter. To conclude this section, we like to comment on one possibility to re-establish the welfare theorems.

Remark 6.1.2. So far, we focused on a system operator moving simultaneously with all other players as a profit-maximizer. However, by appropriate regulations, the system operator can be committed to maximize welfare instead. If, in addition, a hierarchical market structure is adopted, we prove for the described gas flow application in [JG3] that all market solutions correspond to welfare-maximal solutions and vice versa, i.e., the fundamental welfare theorems are re-established under perfect competition and a welfare-maximizing system operator. This result can be easily transferred to other applications. An overview of the resulting multilevel equilibrium problem is provided in Figure 6.2. The system operator decides at the first level on welfare-maximizing market prices. These market prices are then forwarded to the producers and consumers, who still decide simultaneously on the quantities that they produce or consume. The resulting MEP forms the second level. Finally, at the third level, the system operator controls the network in accordance with all physical and technical constraints $F(q, r) \geq 0$.

Nevertheless, the focus of this dissertation is on simultaneous move games. Therefore, we continue with studying (MEP-PL).

6.2. An Algorithm to Indicate the Existence of Equilibria

Due to our results in [JG3], we know that an equilibrium might not exist in nonconvex applications of (MEP-PL). Now, the questions that arise are the following:

- (i) How often does an equilibrium of (MEP-PL) exist in nonconvex settings?
- (ii) Under which conditions does it become more likely that an equilibrium exists?

To answer these questions for specific applications, we choose an algorithmic approach motivated by Theorem 2.2.1. This theorem implies that a market equilibrium of (MEP-PL) exists if and only if the duality gap of the welfare problem (WFP-PL) is zero. The author of this dissertation has proven for a special type of MEPs that, for indicating the existence of an equilibrium, it is sufficient to check for a single primal-dual solution pair whether the welfare problem has zero duality gap or not. We refrain from focusing on the general case and present this result only in the context of energy markets. The interested reader is referred to Section 3 in [JG4].

Theorem 6.2.1 ([JG4, Theorem 4.1]). *Let x^* be a solution of the welfare problem (WFP-PL) and define $\hat{\pi}$ as*

$$\hat{\pi}_v := \begin{cases} P_v(d_v^*), & \text{if } v \in V^-, \\ (c_v^{\text{var}})'(y_v^*), & \text{if } v \in V^+. \end{cases}$$

Then, either $(x^, \hat{\pi})$ is a market equilibrium of (MEP-PL) or there is no market equilibrium.*

Based on this theorem, we state an algorithm that terminates correctly either with a market equilibrium or with the indication that no market equilibrium exists. This algorithm proceeds as follows. First of all, a global solution x^* of the welfare problem (WFP-PL) is computed. Afterward, the price vector $\hat{\pi}$ is determined as in Theorem 6.2.1. Hence, it remains to check if $(x^*, \hat{\pi})$ is a market equilibrium or not. For this purpose, we examine whether the players' strategies as defined by the welfare solution x^* are best responses to the prices $\hat{\pi}$ for all players. By definition of $\hat{\pi}$, this is true for all producers and consumers, which becomes obvious when considering the necessary and sufficient KKT conditions of their optimization problems (3.1) and (3.2); see the proof of Theorem 4.1 in [JG4]. Consequently, it is sufficient to test if the system operator is aligned with the welfare solution x^* for the given prices $\hat{\pi}$. To this end, the optimization

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problem (6.1) of the system operator is solved to global optimality for the fixed prices $\hat{\pi}$. If the objective function values of the obtained optimal strategy and of the strategy as defined by the welfare solution x^* coincide, then the system operator is aligned with the welfare solution. The algorithm returns the market equilibrium $(x^*, \hat{\pi})$. If not, no market equilibrium exists by Theorem 6.2.1.

Algorithm 1: Deciding the existence of an equilibrium of (MEP-PL) and computing an equilibrium in case of existence (adapted from [JG4])

Input: Market equilibrium problem (MEP-PL)

- 1 Compute a global solution x^* of the welfare problem (WFP-PL).
- 2 **if** *the welfare problem (WFP-PL) does not have a solution* **then**
- 3 | **return** *No market equilibrium exists.*
- 4 **else**
- 5 | Define the price vector $\hat{\pi}$ as in Theorem 6.2.1.
- 6 | **if** *the individual players' decisions according to x^* are best responses to the price vector $\hat{\pi}$ for all players* **then**
- 7 | | **return** *$(x^*, \hat{\pi})$ is a market equilibrium.*
- 8 | **else**
- 9 | | **return** *No market equilibrium exists.*
- 10 | **end**
- 11 **end**

Theorem 6.2.2 ([JG4, Theorem 3.8]). *Algorithm 1 terminates correctly either with a market equilibrium of (MEP-PL) or with the information that such an equilibrium does not exist.*

So far, we have developed an algorithm that indicates the existence of an equilibrium of (MEP-PL) for nonconvex applications. Moreover, this algorithm directly computes an equilibrium in the case of existence. Thus, we are ready to answer the questions raised at the beginning of this section.

6.3. Computational Study

We now identify how often and under which conditions a market equilibrium exists for the DC line switching and nonlinear gas flow application of (MEP-PL) introduced in Section 6.1. To this end, we apply the proposed algorithm to well-known instances from the literature on the electricity and gas sector. In total, 102 electricity instances and 96 gas instances are considered. The electricity market instances are based on the technical instances provided by the Software

MATPOWER 7.0; see Zimmerman and Murillo-Sánchez (2020a) and Zimmerman, Murillo-Sánchez, and Thomas (2011). The gas market instances are taken from Heitsch et al. (2021) as well as Schewe et al. (2021). For more information on the instances and the general setup of the computational study see Section 5.2 in [JG4].

Our main findings are as follows. For the gas flow application, an equilibrium exists in all of the 84 instances (87.5 %) that are solved within the set time limit of 1 hour. Thus, instances without an equilibrium as the handcrafted one in Example 6.1.1 seem to hardly occur in practical instances. In turn, 42 instances (41.2 %) do not possess an equilibrium in the DC line switching application. One of the main circumstances leading to non-existence of an equilibrium is that the system operator's profit gains do not outweigh the losses induced by a switching decision while the welfare gains do. We give several examples of this in Section 5.4 in [JG4]. Furthermore, we illustrate by examples in [JG4] that the existence of an equilibrium might be ensured by

- (i) very low switching costs,
- (ii) very high switching costs, or
- (iii) a high increase of the transport costs in the flows.

The interested reader is referred to Sections 5.3 and 5.4 in [JG4] for the detailed results of the computational study.

In summary, our computational study suggests that an equilibrium frequently exists in the nonconvex but continuous gas flow application. In contrast, an equilibrium often does not exist in the DC line switching application due to the integrality restrictions.

6.4. Uniqueness in the Short-Run

So far, we have covered the topics of existence, computation, and economic efficiency of market equilibria of (MEP-PL). By this, we have summarized the main contributions of the two articles [JG3] and [JG4] that we have focused on in this chapter. For the sake of completeness, we like to close the equilibrium analysis with a brief discussion of uniqueness. To this end, we review sufficient conditions for uniqueness that can be established based on existing results from the literature. These results only cover the case of strictly convex variable costs.

Assumption 8. *The variable cost functions c_v^{var} , $v \in V^+$, are strictly convex.*

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Based on this assumption, we obtain uniqueness of demand and production in the case of existence.

Theorem 6.4.1 ([JG4, Corollary 3.3]). *Let (x, π) and $(\tilde{x}, \tilde{\pi})$ be two market equilibria of (MEP-PL). Suppose Assumption 8 holds. Then, $d_v = \tilde{d}_v$ holds for all $v \in V^-$ and $y_v = \tilde{y}_v$ for all $v \in V^+$.*

In the following, we focus on the two nonconvex applications of (MEP-PL) introduced above: DC line switching and nonlinear gas flow. Similar to Assumption 3 in Krebs, Schewe, et al. (2018), we fix a potential at an arbitrary node in the respective network.

Assumption 9. *The phase angle θ_v at an arbitrary node $v \in V$ is fixed and the phase shift angle θ_a^{shift} is equal to zero for all $a \in A$.*

Assumption 10. *The pressure p_v at an arbitrary node $v \in V$ is fixed.*

Now, by applying Theorems 1 and 2 of Ríos-Mercado, Wu, et al. (2002), we can establish that, in the case of existence, the DC line switching application has finitely many equilibria, while the gas flow application has a unique market equilibrium; similar to Theorems 2.1 and 3.1 in Krebs, Schewe, et al. (2018).

Theorem 6.4.2. *Let the network constraints $F(q, r) \geq 0$ be given by the DC line switching equations (6.4). Moreover, let (x, π) and $(\tilde{x}, \tilde{\pi})$ be two market equilibria of (MEP-PL) with $z_a = \tilde{z}_a$ for all $a \in A^s$. Suppose Assumptions 8 and 9 hold. Then, $q_a = \tilde{q}_a$ holds for all $a \in A$ and $\theta_v = \tilde{\theta}_v$ for all $v \in V$. Thus, in the case of existence, (MEP-PL) possesses finitely many equilibria.*

Analogously, we have the following result for the gas flow application.

Theorem 6.4.3. *Let the network constraints $F(q, r) \geq 0$ be given by the nonlinear gas flow equations (6.5). Moreover, let (x, π) and $(\tilde{x}, \tilde{\pi})$ be two market equilibria of (MEP-PL). Suppose Assumptions 8 and 10 hold. Then, $q_a = \tilde{q}_a$ holds for all $a \in A$ and $p_v = \tilde{p}_v$ for all $v \in V$. Thus, in the case of existence, the market equilibrium of (MEP-PL) is unique.*

7. Conclusion

In the following, we summarize the most important findings of this dissertation and provide an outlook on possible next challenges.

Key Conclusions In this thesis, we presented classic energy MEPs extended by new players or new side-constraints for already existing players who have an important role in modern energy systems. We analyzed these problems with a focus on the following three challenging questions:

- (i) Does a market equilibrium exist in the short- and long-run?
- (ii) If so, is this market equilibrium unique or, at least, partly unique? If not, which circumstances lead to multiplicities?
- (iii) How can we compute market equilibria?

Concerning the existence of market equilibria, we found that existence is easy to prove in the case of sector and temporal coupling since the optimization problems of all players remain convex. This result remains true if system operators are considered that face convex technical requirements or physical laws such as the lossless DC load flow approximation in electricity networks. However, not all applications allow for a convex representation of the network restrictions. In this dissertation, we presented two examples for such nonconvex applications: a system operator that operates a gas network under nonlinear stationary gas flow equations and a system operator that switches DC lines in an electricity network. As a main contribution, we developed an algorithm that indicates existence of a short-run equilibrium for energy MEPs including system operators that face any type of nonconvex network restrictions. We tested this algorithm on the two presented applications. Our computational results suggest that a market equilibrium often exists in the nonconvex but continuous gas flow setting, while this is not true in the DC line switching application due to the integrality constraints.

Concerning the uniqueness of market equilibria, we extended the single-sector setting analyzed in Grimm, Schewe, et al. (2017) to a discretized trading horizon and to coupled markets of multiple energy sectors. For this extension,

7. Conclusion

we provided sufficient conditions for ensuring uniqueness of the short-run equilibrium a priori and for verifying uniqueness of the long-run equilibrium a posteriori. For the extension regarding temporal coupling, we identified sufficient conditions that guarantee uniqueness of demand and production as well as uniqueness of storage operations in the case of a single storage operator. To also cover the case of multiple storage operators, we derived an a posteriori criterion for proving uniqueness of a given short-run equilibrium. Finally, we considered the two presented nonconvex applications: DC line switching and gas flow. We studied uniqueness under the assumption of strictly convex variable production costs. Our results show that the short-run equilibrium is unique in the gas flow application and, for fixed switching decisions, in the DC line switching application. Hence, in the latter application, at most finitely many short-run equilibria exist.

Concerning the computation of market equilibria, the equivalence of market equilibria and welfare maxima in the case of existence directly suggests to compute candidates for equilibria via the welfare problem. For the convex case, it is well known that the solutions obtained are indeed equilibria. Hence, for the convex extension of sector coupling, large-scale instances can be tackled by problem-tailored clustering methods to reduce the input data size to a manageable number of representative scenarios. State-of-the-art quadratic programming solvers can then be applied to the resulting welfare problem to compute equilibria. However, for the convex case of temporal coupling, it is not straightforward to use such methods due to the time-dependency of storage operations. Therefore, we reviewed a parallel and distributed ADMM that allows for efficiently solving large-scale instances without the need to reduce the input data size. Finally, we examined the extension by system operators facing nonconvex network restrictions. Here, we computed market equilibria with the developed algorithm to indicate existence of an equilibrium since, by construction, this algorithm also returns an equilibrium in the case of existence.

An overview of these key conclusions is provided in Table 7.1. A check mark implies that the respective results presented in this dissertation cover this topic. In turn, a tilde stands for extension of existing results from the literature while some questions remain open for future research. Finally, the hyphen denotes that the respective topic is not addressed in this thesis. Thus, the given overview points directly to the following next research challenges.

Next Challenges As far as the energy MEPs analyzed in this dissertation are concerned, a deeper understanding of uniqueness would be desirable. In particular, the question remains open whether a priori conditions exist that

Table 7.1.: Main contributions of this dissertation

	Sector coupling	Temporal coupling	Physical laws
Existence	✓	✓	✓
Short-run uniqueness	✓	~	~
Long-run uniqueness	~	-	-
Algorithmic approach	✓	✓	✓

ensure uniqueness of the long-run market equilibrium both for a single energy sector as well as for coupled markets of multiple energy sectors. Moreover, we focused mainly on conditions to guarantee uniqueness. However, sufficient conditions for non-existence of uniqueness would also be of interest.

For the presented energy MEPs with nonconvexities, a potential next challenge might be to identify sufficient conditions that ensure existence of an equilibrium. This leads to the following research question: Under which conditions do the corresponding welfare problems have a zero duality gap?

Another interesting research direction is to analyze how results change theoretically and algorithmically when our main assumptions of perfect foresight and perfect competition are weakened. First, if players do not perfectly foresee future spot-market outcomes, their optimization problems are subject to uncertainties. Hence, energy MEPs under uncertainty arise. Second, the assumption of perfect competition might be relaxed by taking into account strategic interaction between the producers. Here, classic Nash–Cournot models could be used as a modeling approach. For the equilibrium models that result in both cases, the questions of whether equilibria exist, are unique, and how to compute them can again be addressed.

Last but not least, another research direction would be to consider alternative equilibrium concepts such as approximate equilibria. It would be interesting to test whether ε -relaxed solutions to the players’ problems allow to establish existence of an equilibrium in settings in which classic equilibria do not exist. Moreover, to develop algorithms that are particularly suitable for computing approximate equilibria in the energy context would be valuable.

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Part II.
Preprints

Preprint 1

Long-Run Market Equilibria in Coupled Energy Sectors: A Study of Uniqueness

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Preprint under review

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LONG-RUN MARKET EQUILIBRIA IN COUPLED ENERGY SECTORS: A STUDY OF UNIQUENESS

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ABSTRACT. We propose an equilibrium model for coupled markets of multiple energy sectors. The agents in our model are operators of sector-specific production and sector-coupling technologies, as well as price-sensitive consumers with varying demand. We analyze long-run investment in production capacity in each sector and investment in coupling capacity between sectors, as well as production decisions determined at repeated spot markets. We show that in our multi-sector model, multiplicity of equilibria may occur, even if all assumptions hold that would be sufficient for uniqueness in a single-sector model. We then contribute to the literature by deriving sufficient conditions for the uniqueness of short- and long-run market equilibrium in coupled markets of multiple energy sectors. We illustrate via simple examples that these conditions are indeed required to guarantee uniqueness in general. The uniqueness result is an important step to be able to incorporate the proposed market equilibrium problem in more complex computational multilevel equilibrium models, in which uniqueness of lower levels is a prerequisite for obtaining meaningful solutions. Our analysis also paves the way to understand and analyze more complex sector coupling models in the future.

Key words. Energy Markets, Sector Coupling, Regional Pricing, Uniqueness, Short- and Long-Run Market Equilibrium.

1. INTRODUCTION

The goal of climate neutrality in 2050 in Europe and other regions worldwide means that the entire economy must decarbonize or defossilize. On the one hand, renewable electricity will be used for the direct electrification of the heat and transport sectors as well as industry. Where this is not possible, the path is via climate-neutrally generated hydrogen and synthetic energy sources based on it. Hydrogen and renewable synthetic fuels are also needed to compensate for fluctuations of renewable energies in the electricity sector. Sector coupling will thus play an increasingly important role in both directions. In the medium run, during the time of transition, also fossil gas will play a role in the energy system. Overall, the transition towards a climate neutral world will lead to a higher degree of integration between the markets for electricity, fossil gas, and hydrogen. This integration must be taken into account in market designs for energy sectors.

For the assessment of market designs it is a common approach to develop (multilevel) equilibrium models that examine the decisions of stakeholders in energy markets and the corresponding market outcomes. In this paper, we extend approaches from the literature to coupled energy sectors and pave the way for a convincing equilibrium analysis of those more complex environments. For the case of a single energy commodity, Grimm, Schewe,

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et al. (2017) analyze a framework of peak-load pricing in electricity markets, i.e., they consider investment and operation decisions of firms when interacting in a network context with flow constraints. Krebs, Schewe, et al. (2018) extend this analysis to the case of DC power flow in electricity markets and Grübel et al. (2020) analyze uniqueness and multiplicity in a similar setting taking into account storage facilities (without a network). The extension of modeling approaches to the representation of several coupled markets for energy commodities is important to assess the impact of sector-specific market designs on investment and operation decisions in integrated energy systems. In particular, further research is needed to develop a better understanding of the impact that the interactions between different energy markets have on the uniqueness of market outcomes in order to derive necessary model assumptions. This is the main focus of our contribution. First, we provide sufficient conditions for uniqueness of equilibrium in a sector-coupled energy market model. Second, our analysis paves the way to build on the framework to analyze particular issues that arise in coupled energy markets in practice.

There are different strands of literature on the transition of energy markets with several energy commodities and a spatial market representation. Many articles focus on co-optimizing system cost for different energy commodities, which are usually referred to as energy system models. These focus exclusively on determining the cost optimal configuration and operation of an energy system that consists of several energy commodities, potentially subject to specific policy goals (e.g., climate targets or emission pricing). It is not in the nature of energy system models to take into account market design and energy pricing, however. Consequently, the optimization of system cost is not in line with the outcome of market interaction given incentives at markets for individual participants, and market equilibria in coupled energy markets are not assessed. Earlier contributions include Geidl and Andersson (2007), Martínez-Mares and Fuerte-Esquivel (2012), Clegg and Mancarella (2015b), Zhang et al. (2015) and Clegg and Mancarella (2015a), recent work is provided by, e.g., Zlotnik et al. (2017) or Felten (2020).

Another strand of literature, which is more closely related to the present work, explicitly considers the behavior of all market participants and the resulting market equilibria in coupled energy markets. In this context, compare for example Abrell and Weigt (2012) on short-run market equilibria in electricity and gas markets and Abrell and Weigt (2016) for the case of a long-run analysis including firms' investment decisions in production facilities. Similarly, Huppmann and Egging (2014) and Gil et al. (2015) provide frameworks to model the behavior of market participants in fuel markets and the electricity market subject to network constraints resulting both from electricity and fuel transport. Chen et al. (2019) build on this work and explicitly analyze strategic bidding of firms under the anticipation of nodal pricing both in the electricity and the gas market in a bilevel approach. Several other very recent contributions highlight further interesting aspects arising in the context of market interaction in coupled energy markets. Ordoudis et al. (2019) analyze different degrees of coupling day-ahead and real-time electricity and gas markets under uncertainty. And most recently, Roach and Meeus (2020) provide an iterative simulation procedure to analyze the impact of long-term gas contracts on market outcomes in coupled electricity and gas markets. All those studies provide highly valuable insights on the different aspects regarding market interaction in coupled energy markets. Whereas they determine one of the resulting market equilibria for their test instances considered, they do not analyze whether there are many different equilibria yielding potentially different policy conclusions. Moreover, they do not provide conditions when their setup indeed delivers unique market outcomes.

This paper discusses the extension of market equilibrium models that capture the behavior of individual market participants to multiple coupled markets for energy commodities. As compared to single-commodity models, the implementation of market dynamics between integrated energy markets in a joint market equilibrium model requires additional considerations. We propose a model for coupled energy markets that captures long-term investment decisions in transport and production capacity and short-term market decisions. For this

kind of model, we determine sufficient conditions that ensure unique market outcomes. Our contribution is, to the best of our knowledge, the first to fill this gap for the case of coupled energy markets.

Our work lays the foundation for addressing timely research questions on energy market coupling with regard to, e.g., climate policy, regulation of networks, energy pricing and its effects on infrastructure planning, investment incentives, and market outcomes. A substantive analysis of the long-run interaction of the different agents requires techno-economic multilevel modeling of coupled energy markets similar to the sector-specific models of Grimm, Martin, et al. (2016) and Ambrosius et al. (2020) for electricity markets and Grimm, Schewe, et al. (2019) for gas markets. In those multilevel energy market models, uniqueness of the market outcome at the second level typically is key in order to enable a coherent analysis of decision makers' investments in transport capacities at the first level. In the existing literature, uniqueness for multilevel market models has only been addressed for one energy sector and additional aspects like storage and load-flow representation. Our results show that multi-sector market models, with a representation of the agents active in different markets, require additional conditions to guarantee the uniqueness of short- and long-term market equilibrium. Beyond our general analysis, we illustrate in small examples. We also discuss the current limits for ensuring uniqueness in model applications and open questions for further research.

The remainder of the paper is organized as follows. Section 2 introduces the general problem setting with timeline, model assumptions, and basic notation. Section 3 states the model formulation, Sect. 4 provides the analysis of uniqueness, and Sect. 5 concludes.

2. SETUP, BASIC MODEL ASSUMPTIONS, AND NOTATION

In liberalized energy markets, various agents with different objectives interact. If several coupled energy sectors are taken into account, some of these agents participate in multiple commodity markets. In this paper we explicitly consider network-based energy sectors such as electricity, fossil gas, and hydrogen, which have a regulated and centrally operated transmission system, or could have one in the future. The types of agents considered are sector-specific consumers and producers, sector-coupling producers, as well as sector-specific transmission companies and system operators. Our model covers their individual decisions on investment, spot market trading, and adjustments of spot market allocations necessary for technical operation of transmission systems. At the spot market stage, network constraints in energy pricing are partly considered, which requires a spatial definition of market zones and a pricing regime for transmission capacity. The private agents seek to maximize their own objectives on the spot market. While consumers decide on the level of their consumption by maximizing their consumer surpluses, producers—with the aim of maximizing profits—decide on investment in their production technologies and on the operation of established facilities. The difference between sector-specific and sector-coupling production technologies is that for sector-coupling technologies, the energy sector from which the input factors are obtained is also modeled endogenously. This is not the case for sector-specific production technologies.

In the literature that analyses semi-liberalized energy markets in multilevel equilibrium models (see, e.g., Sauma and Oren (2006), Baringo and Conejo (2012), Jenabi et al. (2013), Grimm, Martin, et al. (2016), Grimm, Schewe, et al. (2019), or Ambrosius et al. (2020)), all the above-mentioned market-driven decisions of private agents are typically captured by one level of a multilevel equilibrium model. In this paper we focus on a particular, very important aspect of this level, namely the uniqueness of the equilibrium at the "market level", for the case of multiple coupled energy markets. Only precise knowledge about the set of equilibria and the conditions under which a unique market equilibrium at the lower level obtains allows a meaningful analysis of the entire model. It is evident that uniqueness requires simplifying technical assumptions, which might seem too restrictive for many applications of interest. However, it is important to understand under which assumptions it is possible to derive coherent results and under which assumptions, due to multiplicity of equilibria, one has to

be very careful in interpreting the results of more complex models. In order to understand precisely under which assumptions one can actually expect uniqueness of equilibrium, a very detailed understanding of the underlying multilevel problem is essential. For this reason we start with a description of the timeline of the overall model. We then present the basic model assumptions with regard to the economic environment. Finally, the notation is introduced.

2.1. Setup. Our market model for coupled energy sectors has the following multilevel structure, which is illustrated in Figure 1 for two sectors. First, network companies in the different sectors decide on network expansion, anticipating private investment and the resulting spot market outcomes, as well as subsequent adjustments of the market outcomes to ensure technical feasibility in the transmission systems. Next, the private agents observe network investment and the implied trade restrictions and decide on their investment in production capacities, both in sector-specific and sector-coupling technologies. These investment decisions depend on their expectations on spot market outcomes in all sectors. Finally, trading on the spot market takes place over several time periods, and possibly subsequent adjustments are made in the individual sectors in case of violated transport constraints. This is relevant if transmission capacity constraints are not reflected in spot market prices and therefore accounted for in the market outcome, i.e., under zonal pricing. If these adjustments are made cost-based, there is no interference with trading on the spot market since the private agents cannot realize additional profits. In this case, anticipation of decisions at the adjustment stage does not affect investment and operating decisions.

Under the assumption of perfect competition, the investment and spot market trading levels can be subsumed in one level that contains all market-driven decisions. The multilevel structure described above can thus be reduced to a techno-economic trilevel problem with the following underlying structure: network investment on the first level, private investment in production capacity as well as spot market trading on the second level, and adjustments of market outcomes on the third level.

This paper focuses on the uniqueness analysis of the second level, i.e., of investment in sector-specific and sector-coupling production capacity as well as spot market trading in all sectors considered. Uniqueness of the second level is a prerequisite for uniqueness of the overall equilibrium solution. The question of uniqueness of the first and third level is not addressed in detail in this paper. At the first level, multiplicities occur if and only if different investment decisions yield the same maximal welfare, given cost minimal adjustment decisions at the third level. While this kind of multiplicities might occur, they are much easier to handle upon applications of multilevel models to specific scenarios.

2.2. Basic model assumptions. Modeling private investment and spot market behavior in multiple sectors jointly results in a large market equilibrium problem. In order to keep the analysis tractable, we assume—as it is standard in many related contributions—perfect competition at all markets, i.e., upon investment and spot market decisions.

Second, we assume that each involved agent has perfect foresight, i.e., producers correctly anticipate the spot market outcomes of the sectors in which they operate when they make their investment decisions. In particular, perfect foresight implies that spot market results are not subject to uncertainty on final production and demand levels due to forecasting errors. This assumption may be relaxed by stochastic or robust optimization techniques. In this case, complementarity problems under uncertainty arise. However, this kind of equilibrium problems have only recently received increased attention in literature and are theoretically not yet well understood. In particular, the existence of a single-level counterpart as in the case of perfect foresight is not generally guaranteed. For more information on uncertain linear complementarity problems see, e.g., Krebs and Schmidt (2020) and Krebs, Müller, et al. (2019).

As third assumption, operators of storage devices are excluded as possible agents in our context since Grübel et al. (2020) already showed that uniqueness fails to hold in general as

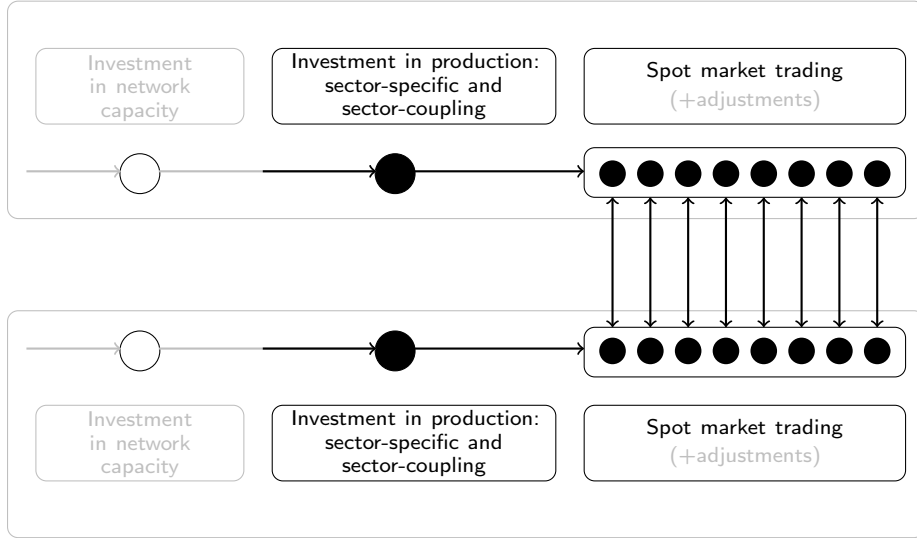


FIGURE 1. Multilevel structure of investment into network and production capacity and subsequent trade in and across multiple sectors—the uniqueness analysis in this paper focuses on the levels highlighted in black

soon as several storage operators are considered in a market equilibrium problem similar to the one analyzed here.

Fourth, we consider trade capacity between adjacent bidding zones at the spot market while we abstract from additional constraints on trade capacity related to potential-based network flows. For some applications, in particular in network-based energy sectors, it might be of interest to incorporate those constraints at the spot market; e.g., if nodal pricing or zonal flow-based pricing systems are modeled. In this paper, we refrain from taking potential-based trade constraints into account since, even in case of linear models, multiplicities may easily arise; see the results for the electricity market equilibrium problems in Krebs, Schewe, et al. (2018) and Krebs and Schmidt (2018) that both include as trade constraints the linear direct current (DC) lossless setup to approximate Kirchhoff’s voltage law once without and once with transportation costs. Moreover, if nonconvex transportation constraints are considered in a market equilibrium problem, the equivalence between market equilibria and welfare maxima may no longer hold; see Grimm, Grübel, et al. (2019) who address the case of nonconvex gas physics.

Finally, we assume that the time horizon and the time intervals for trading the products (i.e., electricity or gas) are the same on spot markets across all sectors. If this is not the case, technologies from sectors where products are traded more frequently will face equal conditions in the coupled sector for a number of subsequent trading periods, which might obviously lead to multiplicities. This issue is left for future research.

2.3. Notation. In the following, we introduce the basic notation. We start with the time horizon and market structure, continue with the sector-specific demand and supply, and conclude with the sector-coupling preliminaries. For the sake of completeness, an overview of all sets, parameters, and variables used throughout the paper is presented in Tables 1, 2, and 3 in App. A.

Time horizon. The time horizon T for all considered energy markets is assumed to be equidistantly discretized, i.e., in time periods $T = \{t_1, \dots, t_{|T|}\}$ with the duration $\tau = t_i - t_{i-1}$ for $i = 2, \dots, |T|$. This time horizon typically resembles all trading periods of, e.g., one year. The related data is based either on historical or representative data, using the net present value approach for investment.

Bidding zones. In each sector, the spot market represents the network infrastructure and the spatial distribution of demand and supply within one or multiple bidding zones and with trade products between the bidding zones. We denote the set of bidding zones per sector $s \in S$ by $Z_s = \{z_1, \dots, z_{|Z_s|}\}$. In case of a market with a single bidding zone, $|Z_s| = 1$ holds. In case of $|Z_s| > 1$, the trade capacity between adjacent bidding zones is specified and accounted for at the spot market. In our setting, the set $K_s \subseteq Z_s \times Z_s$ represents all bidding zones of the same sector $s \in S$ between which a positive trade capacity is available. Each element $k \in K_s$ is characterized by its maximum and minimum trade capacity f_k^+ , f_k^- . A positive trade value $f_{t,k}$ on $k = (z, z') \in K_s$ in time period $t \in T$ implies that the respective amount of the sector commodity is traded from bidding zone z to z' , while a negative trade value represents trading in the opposite direction. Finally, we follow common notation and define the set of all ingoing and outgoing trade capacities of a bidding zone $z \in Z_s$ by

$$\begin{aligned}\delta_s^{\text{in}}(z) &:= \{k = (z', z) \in K_s \mid z' \in Z_s\}, \\ \delta_s^{\text{out}}(z) &:= \{k = (z, z') \in K_s \mid z' \in Z_s\}.\end{aligned}$$

Sector-specific demand. The demand in each sector varies over time and is assumed to be price-elastic. For the ease of notation, it is furthermore assumed that there is one (possibly aggregated) demand function per bidding zone. The inverse demand function $P_{t,z} : [0, \infty) \rightarrow \mathbb{R}^+$ of bidding zone $z \in Z_s$ in sector $s \in S$ and in time period $t \in T$ is continuously differentiable and strictly decreasing.

Sector-specific supply. The demand is met on the one hand by production technologies that operate only within one sector and on the other hand by technologies that couple the different sectors. The difference between the two types of technologies is that the sector-specific production technologies do not rely on input from the other sectors in our model scope. All existing production facilities in bidding zone $z \in Z_s$ that are operated sector-specific form the set G_z^{ex} . In order to account for the possibility of further investment in sector-specific production technologies, we introduce the set G_z^{new} of candidate production facilities. We refer to the set of all existing and candidate sector-specific production facilities by G_z^{all} , i.e., $G_z^{\text{all}} := G_z^{\text{ex}} \cup G_z^{\text{new}}$. Each facility $g \in G_z^{\text{all}}$ is characterized by its variable costs of production c_g^{var} . Moreover, existing facilities have a given capacity y_g^{ex} , while the capacity y_g^{new} of candidate facilities is variable with associated investment costs of c_g^{inv} . Finally, we introduce the facility-specific availability $\alpha_{t,g}$. In the case of renewable energy sources, like wind and solar, $\alpha_{t,g}$ describes variability in physical availability whereas in the other cases it may be interpreted as planned availability, which includes shut-down times due to maintenance.

Sector coupling. Each sector-coupling technology transforms the commodity from one sector $s \in S$ into the commodity of another sector $s' \in S$ with $s \neq s'$. We introduce the set X^{ex} for all existing production facilities with sector-coupling technologies. An element (o, i) of this set represents the respective withdrawal point o and the respective injection point i of the facility. Moreover, we introduce the set X^{new} for all candidate sector-coupling facilities and refer to the set of all sector-coupling facilities by X^{all} , i.e., $X^{\text{all}} := X^{\text{ex}} \cup X^{\text{new}}$. Each sector-coupling facility $x \in X^{\text{all}}$ is characterized by its efficiency $\eta_x \in (0; 1)$. Note that an efficiency is not specified for the sector-specific production facilities because the respective efficiency is captured by the variable production costs.

The set O_z^{ex} represents all existing sector-coupling facilities that withdraw the commodity from sector $s \in S$ in bidding zone $z \in Z_s$, while the set O_z^{new} represents the candidate facilities. The set of all facilities is denoted by O_z^{all} , i.e., $O_z^{\text{all}} := O_z^{\text{ex}} \cup O_z^{\text{new}}$.

Analogously, the set I_z^{ex} represents all existing sector-coupling facilities that inject the commodity from sector $s \in S$ in bidding zone $z \in Z_s$. As before, we also consider the respective sets of candidate facilities I_z^{new} and all facilities I_z^{all} . All existing sector-coupling facilities have a given capacity y_i^{ex} , while the capacity y_i^{new} of candidate facilities is variable with associated investment costs of c_i^{inv} . Note that the capacity restriction needs only to

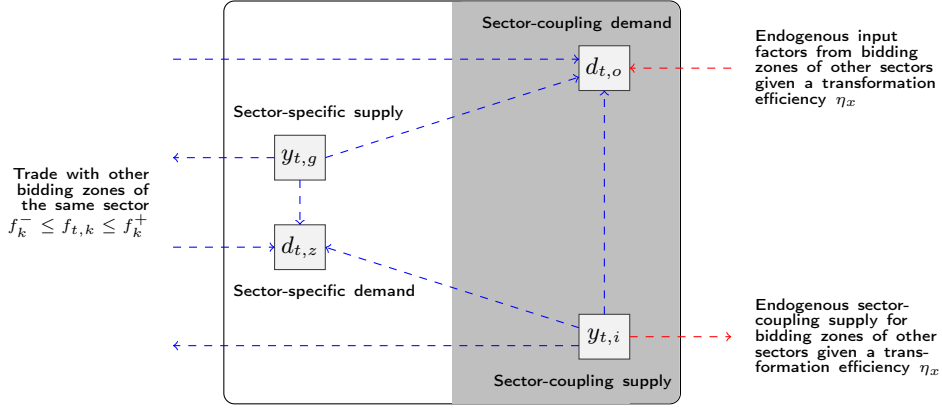


FIGURE 2. Dependencies of the smallest component of our model in one time period: One bidding zone in one sector (endogenous trade of the sector's commodity in blue and endogenous trade of sector-coupled commodities in red)

be defined for the injection point because by this, the capacity at the withdrawal point is also restricted given the efficiency of the underlying facility. The equivalent availability of sector-coupling facilities is denoted by $\alpha_{t,i}$.

Figure 2 illustrates the trading dependencies of one bidding zone in one sector, i.e., of the smallest component of our model. First of all, the sector's commodity is traded between the agents within this bidding zone. For this purpose, the sector-specific production facilities purchase their input factors at an exogenously determined price. In turn, the purchase of input factors of sector-coupling facilities is endogenously modeled. For the transformation of one commodity into another, a transfer efficiency is presumed. Hence, depending on the relation of market prices between the considered bidding zone and bidding zones of other sectors, both trade for transforming the considered commodity into others or vice versa is possible. Finally, the sector's commodity can be sold to or bought from agents of other bidding zones of the same sector. However, the trade between the considered bidding zone and an adjacent one is limited by the respective trade capacity.

3. AN EQUILIBRIUM MODEL FOR COUPLED ENERGY MARKETS AND ITS SINGLE-LEVEL COUNTERPART

In this section, we proceed as follows. First, we propose a market equilibrium problem that covers the private investment in sector-specific and sector-coupling production technologies together with the subsequent spot market trading in the different coupled energy sectors. Afterward, we present a welfare maximization problem suitably chosen to form a single-level counterpart to the proposed market equilibrium problem, i.e., each welfare maximal solution corresponds to a market equilibrium and vice versa. We close with the proof of the latter relation.

To simplify notation, we denote, e.g., by y_t the vector of sector-specific and sector-coupling production given a fixed time period $t \in T$. Furthermore, when we speak, e.g., of the producer $g \in G_z^{\text{all}}$ in the following, we are actually referring to the operator of the production facility g .

3.1. The market equilibrium problem. On the spot markets of different sectors, multiple agents interact who all pursue their own objectives. The main agents are the consumers, the operators of sector-specific production as well as sector-coupling technologies. We continue

with the description of the individual optimization problems of each agent type and start with the consumers.

Consumers. Consumers maximize their gross consumer surpluses less their purchasing costs. Hence, the consumers located in bidding zone $z \in Z_s$ of sector $s \in S$ face the following optimization problem

$$\max \sum_{t \in T} \left(\int_0^{d_{t,z}} P_{t,z}(\mu) d\mu - p_{t,z} d_{t,z} \right) \quad (1a)$$

$$\text{s.t. } 0 \leq d_{t,z}, \quad \text{for all } t \in T, \quad [\gamma_{t,z}] \quad (1b)$$

where $p_{t,z}$ denotes the price of the respective bidding zone in the considered time period. Due to the assumption of perfect competition, this price is—from the point of view of the individual consumers—exogenously given, i.e., effects of own or others' decisions on the price are not taken into account.

The Greek letters behind constraints denote the associated dual variables. Since the inverse demand function $P_{t,z}$ is assumed to be continuously differentiable and strictly decreasing, Problem (1) is a concave maximization problem with linear constraints. Thus, the corresponding Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for global optimality. Consequently, all solutions of the stated problem are characterized by

$$P_{t,z}(d_{t,z}) - p_{t,z} + \gamma_{t,z} = 0, \quad \text{for all } t \in T, \quad (2a)$$

$$0 \leq \gamma_{t,z} \perp d_{t,z} \geq 0, \quad \text{for all } t \in T. \quad (2b)$$

Sector-specific producers. Each operator of an existing sector-specific production technology maximizes profits from trading, i.e., the revenues from trading minus the variable production costs. Thus, the producer $g \in G_z^{\text{ex}}$ located in bidding zone $z \in Z_s$ of sector $s \in S$ has the following maximization problem

$$\max \sum_{t \in T} (p_{t,z} - c_g^{\text{var}}) y_{t,g} \quad (3a)$$

$$\text{s.t. } 0 \leq y_{t,g} \leq \alpha_{t,g} y_g^{\text{ex}}, \quad \text{for all } t \in T. \quad [\beta_{t,g}^{\pm}] \quad (3b)$$

The latter inequality implies that the production must remain within the real-time available capacities. The associated KKT conditions are necessary and sufficient for global optimality since Problem (3) is a linear maximization problem.

$$p_{t,z} - c_g^{\text{var}} + \beta_{t,g}^- - \beta_{t,g}^+ = 0, \quad \text{for all } t \in T, \quad (4a)$$

$$0 \leq \beta_{t,g}^- \perp y_{t,g} \geq 0, \quad \text{for all } t \in T, \quad (4b)$$

$$0 \leq \beta_{t,g}^+ \perp \alpha_{t,g} y_g^{\text{ex}} - y_{t,g} \geq 0, \quad \text{for all } t \in T. \quad (4c)$$

The operators of candidate sector-specific production technologies additionally consider their investment costs. Therefore, such an operator $g \in G_z^{\text{new}}$ located in bidding zone $z \in Z_s$ of sector $s \in S$ optimizes the following maximization problem

$$\max \sum_{t \in T} (p_{t,z} - c_g^{\text{var}}) y_{t,g} - c_g^{\text{inv}} y_g^{\text{new}} \quad (5a)$$

$$\text{s.t. } 0 \leq y_{t,g} \leq \alpha_{t,g} y_g^{\text{new}}, \quad \text{for all } t \in T. \quad [\delta_{t,g}^{\pm}] \quad (5b)$$

The respective (necessary and sufficient) KKT conditions read

$$-c_g^{\text{inv}} + \sum_{t \in T} \alpha_{t,g} \delta_{t,g}^+ = 0, \quad (6a)$$

$$p_{t,z} - c_g^{\text{var}} + \delta_{t,g}^- - \delta_{t,g}^+ = 0, \quad \text{for all } t \in T, \quad (6b)$$

$$0 \leq \delta_{t,g}^- \perp y_{t,g} \geq 0, \quad \text{for all } t \in T, \quad (6c)$$

$$0 \leq \delta_{t,g}^+ \perp \alpha_{t,g} y_g^{\text{new}} - y_{t,g} \geq 0, \quad \text{for all } t \in T. \quad (6d)$$

Note that the specified setup does not exclude the case where an operator owns more than one production facility. Since we assume perfect competition and consequently that the operators cannot influence market prices by their decisions, optimizing jointly over all owned facilities is equivalent to optimizing individually each facility (see also Theorem 3.3 below for the proof of this assertion).

Sector-coupling producers. The objective of an operator of a sector-coupling technology is to maximize profits from trading. As a consequence, the operator of the existing sector-coupling facility $x = (o, i) \in X^{\text{ex}}$ faces the following optimization problem

$$\max \sum_{t \in T} (p_{t,z} y_{t,i} - p_{t,z'} d_{t,o}) \quad (7a)$$

$$\text{s.t. } \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } t \in T, \quad [\zeta_{t,x}] \quad (7b)$$

$$0 \leq y_{t,i} \leq \alpha_{t,i} y_i^{\text{ex}}, \quad \text{for all } t \in T, \quad [\nu_{t,x}^{\pm}] \quad (7c)$$

$$0 \leq d_{t,o}, \quad \text{for all } t \in T, \quad [\theta_{t,x}] \quad (7d)$$

where the injection point $i \in I_z^{\text{ex}}$ of the sector-coupling facility is located in bidding zone $z \in Z_s$ of sector $s \in S$ and the withdrawal point $o \in O_{z'}^{\text{ex}}$ is located in bidding zone $z' \in Z_{s'}$ of sector $s' \in S$ with $s \neq s'$.

Here, the KKT conditions are—due to the linearity of Problem (7)—again necessary and sufficient for global optimality

$$p_{t,z} + \nu_{t,x}^- - \nu_{t,x}^+ - \zeta_{t,x} = 0, \quad \text{for all } t \in T, \quad (8a)$$

$$-p_{t,z'} + \theta_{t,x} + \eta_x \zeta_{t,x} = 0, \quad \text{for all } t \in T, \quad (8b)$$

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } t \in T, \quad (8c)$$

$$0 \leq \nu_{t,x}^- \perp y_{t,i} \geq 0, \quad \text{for all } t \in T, \quad (8d)$$

$$0 \leq \nu_{t,x}^+ \perp \alpha_{t,i} y_i^{\text{ex}} - y_{t,i} \geq 0, \quad \text{for all } t \in T, \quad (8e)$$

$$0 \leq \theta_{t,x} \perp d_{t,o} \geq 0, \quad \text{for all } t \in T. \quad (8f)$$

The operator of the candidate sector-coupling facility $x = (o, i) \in X^{\text{new}}$ additionally considers required investment costs and thus plans in accordance with the following maximization problem

$$\max \sum_{t \in T} (p_{t,z} y_{t,i} - p_{t,z'} d_{t,o}) - c_i^{\text{inv}} y_i^{\text{new}} \quad (9a)$$

$$\text{s.t. } \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } t \in T, \quad [\xi_{t,x}] \quad (9b)$$

$$0 \leq y_{t,i} \leq \alpha_{t,i} y_i^{\text{new}}, \quad \text{for all } t \in T, \quad [\tau_{t,x}^{\pm}] \quad (9c)$$

$$0 \leq d_{t,o}, \quad \text{for all } t \in T. \quad [\rho_{t,x}] \quad (9d)$$

The respective (necessary and sufficient) KKT conditions read

$$-c_i^{\text{inv}} + \sum_{t \in T} \alpha_{t,i} \tau_{t,x}^+ = 0, \quad (10a)$$

$$p_{t,z} + \tau_{t,x}^- - \tau_{t,x}^+ - \xi_{t,x} = 0, \quad \text{for all } t \in T, \quad (10b)$$

$$-p_{t,z'} + \rho_{t,x} + \eta_x \xi_{t,x} = 0, \quad \text{for all } t \in T, \quad (10c)$$

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } t \in T, \quad (10d)$$

$$0 \leq \tau_{t,x}^- \perp y_{t,i} \geq 0, \quad \text{for all } t \in T, \quad (10e)$$

$$0 \leq \tau_{t,x}^+ \perp \alpha_{t,i} y_i^{\text{new}} - y_{t,i} \geq 0, \quad \text{for all } t \in T, \quad (10f)$$

$$0 \leq \rho_{t,x} \perp d_{t,o} \geq 0, \quad \text{for all } t \in T. \quad (10g)$$

System operators. For the market allocation of the predetermined trade capacity between bidding zones an independent system operator or, alternatively, an exchange might be responsible. The objective is to exploit price differences between adjacent bidding zones to maximize profits from congestion rents. This is done by allocating as much flow as the trade capacities allow from low price bidding zones to high price bidding zones, i.e., the following optimization problem is faced

$$\max \sum_{k=(z,z') \in K_s} \sum_{t \in T} (p_{t,z} - p_{t,z'}) f_{t,k} \quad (11a)$$

$$\text{s.t. } f_k^- \leq f_{t,k} \leq f_k^+, \quad \text{for all } k \in K_s, t \in T, \quad [\kappa_{t,k}^\pm] \quad (11b)$$

by the responsible system operator of sector $s \in S$. The modeling mainly follows Hobbs and Helman (2004).

The KKT conditions are again necessary and sufficient for global optimality due to the linearity of the considered optimization problem

$$p_{t,z} - p_{t,z'} + \kappa_{t,k}^- - \kappa_{t,k}^+ = 0, \quad \text{for all } k = (z, z') \in K_s, t \in T, \quad (12a)$$

together with

$$0 \leq \kappa_{t,k}^- \perp f_{t,k} - f_k^- \geq 0, \quad \text{for all } k \in K_s, t \in T, \quad (12b)$$

$$0 \leq \kappa_{t,k}^+ \perp f_k^+ - f_{t,k} \geq 0, \quad \text{for all } k \in K_s, t \in T. \quad (12c)$$

Market clearing. Finally, we add the flow balance equation for each bidding zone $z \in Z_s$ in each sector $s \in S$ and time period $t \in T$

$$d_{t,z} - \sum_{g \in G_z^{\text{all}}} y_{t,g} + \sum_{o \in O_z^{\text{all}}} d_{t,o} - \sum_{i \in I_z^{\text{all}}} y_{t,i} = \sum_{k \in \delta_s^{\text{in}}(z)} f_{t,k} - \sum_{k \in \delta_s^{\text{out}}(z)} f_{t,k}, \quad (13)$$

i.e., the inflows minus the outflows of a bidding zone must equal the sector-specific and sector-coupling demand minus the sector-specific and sector-coupling production of this bidding zone.

The game. Joining all KKT conditions (2), (4), (6), (8), (10), and (12) together with the market clearing (13) yields the following mixed complementarity problem (MCP) that models the spot market trading in different coupled energy markets:

$$\begin{aligned} & \text{Consumers : (2), for all } s \in S, z \in Z_s, \\ & \text{Sector-specific producers} \\ & \quad \text{without investment : (4), for all } s \in S, z \in Z_s, g \in G_z^{\text{ex}}, \\ & \text{Sector-specific producers} \\ & \quad \text{with investment : (6), for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, \\ & \text{Sector-coupling producers} \quad (MCP) \\ & \quad \text{without investment : (8), for all } x = (o, i) \in X^{\text{ex}}, \\ & \text{Sector-coupling producers} \\ & \quad \text{with investment : (10), for all } x = (o, i) \in X^{\text{new}}, \\ & \text{System operators : (12), for all } s \in S, \\ & \text{Market clearing : (13), for all } s \in S, z \in Z_s, t \in T. \end{aligned}$$

3.2. Equivalence to a single-level counterpart. Next, we prove that all market equilibria, i.e., all solutions of the above stated (MCP), correspond to solutions of a suitably chosen welfare maximization problem, and vice versa. This directly implies the economic efficiency of the market equilibria. The welfare maximization problem reads

$$\max \sum_{s \in S} \sum_{z \in Z_s} \sum_{t \in T} \left(\int_0^{d_{t,z}} P_{t,z}(\mu) d\mu - \sum_{g \in G_z^{\text{all}}} c_g^{\text{var}} y_{t,g} \right)$$

$$-\sum_{s \in S} \sum_{z \in Z_s} \left(\sum_{g \in G_z^{\text{new}}} c_g^{\text{inv}} y_g^{\text{new}} + \sum_{i \in I_z^{\text{new}}} c_i^{\text{inv}} y_i^{\text{new}} \right) \quad (15a)$$

$$\text{s.t. } 0 \leq d_{t,z}, \quad \text{for all } s \in S, z \in Z_s, t \in T, \quad [\gamma_{t,z}] \quad (15b)$$

$$0 \leq y_{t,g} \leq \alpha_{t,g} y_g^{\text{ex}}, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{ex}}, t \in T, \quad [\beta_{t,g}^{\pm}] \quad (15c)$$

$$0 \leq y_{t,g} \leq \alpha_{t,g} y_g^{\text{new}}, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, t \in T, \quad [\delta_{t,g}^{\pm}] \quad (15d)$$

$$\eta_x d_{t,o} = y_{t,i}, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad [\zeta_{t,x}] \quad (15e)$$

$$0 \leq y_{t,i} \leq \alpha_{t,i} y_i^{\text{ex}}, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad [\nu_{t,x}^{\pm}] \quad (15f)$$

$$0 \leq d_{t,o}, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad [\theta_{t,x}] \quad (15g)$$

$$\eta_x d_{t,o} = y_{t,i}, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad [\xi_{t,x}] \quad (15h)$$

$$0 \leq y_{t,i} \leq \alpha_{t,i} y_i^{\text{new}}, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad [\tau_{t,x}^{\pm}] \quad (15i)$$

$$0 \leq d_{t,o}, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad [\rho_{t,x}] \quad (15j)$$

$$f_k^- \leq f_{t,k} \leq f_k^+, \quad \text{for all } s \in S, k \in K_s, t \in T, \quad [\kappa_{t,k}^{\pm}] \quad (15k)$$

$$(13), \quad \text{for all } s \in S, z \in Z_s, t \in T, \quad [\lambda_{t,z}] \quad (15l)$$

together with its KKT conditions: first stationarity

$$P_{t,z}(d_{t,z}) - \lambda_{t,z} + \gamma_{t,z} = 0, \quad \text{for all } s \in S, z \in Z_s, t \in T, \quad (16a)$$

$$\lambda_{t,z} - c_g^{\text{var}} + \beta_{t,g}^- - \beta_{t,g}^+ = 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{ex}}, t \in T, \quad (16b)$$

$$-c_g^{\text{inv}} + \sum_{t \in T} \alpha_{t,g} \delta_{t,g}^+ = 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, \quad (16c)$$

$$\lambda_{t,z} - c_g^{\text{var}} + \delta_{t,g}^- - \delta_{t,g}^+ = 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, t \in T, \quad (16d)$$

$$\lambda_{t,z} + \nu_{t,x}^- - \nu_{t,x}^+ - \zeta_{t,x} = 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (16e)$$

$$-\lambda_{t,z'} + \theta_{t,x} + \eta_x \zeta_{t,x} = 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (16f)$$

$$-c_i^{\text{inv}} + \sum_{t \in T} \alpha_{t,i} \tau_{t,x}^+ = 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, \quad (16g)$$

$$\lambda_{t,z} + \tau_{t,x}^- - \tau_{t,x}^+ - \xi_{t,x} = 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (16h)$$

$$-\lambda_{t,z'} + \rho_{t,x} + \eta_x \xi_{t,x} = 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (16i)$$

$$\lambda_{t,z} - \lambda_{t,z'} + \kappa_{t,k}^- - \kappa_{t,k}^+ = 0, \quad \text{for all } s \in S, k = (z, z') \in K_s, t \in T, \quad (16j)$$

subsequent primal feasibility

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (17a)$$

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (17b)$$

and finally complementarity

$$0 \leq \gamma_{t,z} \perp d_{t,z} \geq 0, \quad \text{for all } s \in S, z \in Z_s, t \in T, \quad (18a)$$

$$0 \leq \beta_{t,g}^- \perp y_{t,g} \geq 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{ex}}, t \in T, \quad (18b)$$

$$0 \leq \beta_{t,g}^+ \perp \alpha_{t,g} y_g^{\text{ex}} - y_{t,g} \geq 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{ex}}, t \in T, \quad (18c)$$

$$0 \leq \delta_{t,g}^- \perp y_{t,g} \geq 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, t \in T, \quad (18d)$$

$$0 \leq \delta_{t,g}^+ \perp \alpha_{t,g} y_g^{\text{new}} - y_{t,g} \geq 0, \quad \text{for all } s \in S, z \in Z_s, g \in G_z^{\text{new}}, t \in T, \quad (18e)$$

$$0 \leq \nu_{t,x}^- \perp y_{t,i} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (18f)$$

$$0 \leq \nu_{t,x}^+ \perp \alpha_{t,i} y_i^{\text{ex}} - y_{t,i} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (18g)$$

$$0 \leq \theta_{t,x} \perp d_{t,o} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{ex}}, t \in T, \quad (18h)$$

$$0 \leq \tau_{t,x}^- \perp y_{t,i} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (18i)$$

$$0 \leq \tau_{t,x}^+ \perp \alpha_{t,i} y_i^{\text{new}} - y_{t,i} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (18j)$$

$$0 \leq \rho_{t,x} \perp d_{t,o} \geq 0, \quad \text{for all } x = (o, i) \in X^{\text{new}}, t \in T, \quad (18k)$$

$$0 \leq \kappa_{t,k}^- \perp f_{t,k} - f_k^- \geq 0, \quad \text{for all } s \in S, k \in K_s, t \in T, \quad (18l)$$

$$0 \leq \kappa_{t,k}^+ \perp f_k^+ - f_{t,k} \geq 0, \quad \text{for all } s \in S, k \in K_s, t \in T. \quad (18m)$$

Since all inverse demand functions are assumed to be continuously differentiable and strictly decreasing, the welfare maximization problem is a concave maximization problem with linear constraints. Consequently, its KKT conditions (16)–(18) are necessary and sufficient for global optimality. Therefore, it is sufficient to show the 1-1 correspondence of the solutions of (MCP) and the stated KKT conditions in order to prove the equivalence of market equilibria and welfare maxima. Thus, we start with the implication that each solution of the KKT conditions yields a market equilibrium.

Lemma 3.1. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, \lambda)$ be a solution of the KKT conditions (16)–(18). Then, w corresponds to a solution of (MCP).*

Proof. Comparison of (16)–(18) to (MCP) shows that all equations coincide if $p_{t,z} := \lambda_{t,z}$ holds for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$. Hence, $(d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ is a solution of (MCP). \square

Next, we continue with the implication that each market equilibrium yields a solution of the KKT conditions.

Lemma 3.2. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of (MCP). Then, w corresponds to a solution of the KKT conditions (16)–(18).*

Proof. Again, comparison of (MCP) to (16)–(18) shows that all equations coincide if $\lambda_{t,z} := p_{t,z}$ holds for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$. Thus, $(d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, \lambda)$ is a solution of the KKT conditions (16)–(18). \square

Finally, we are able to prove the equivalence of market equilibria and welfare maxima.

Theorem 3.3. *There is a 1-1 correspondence of the market equilibrium problem (MCP) and the welfare maximization problem (15).*

Proof. The assertion follows from Lemmata 3.1 and 3.2. \square

So, we have shown that the presented welfare maximization problem is indeed a single-level counterpart to the proposed market equilibrium problem, i.e., each welfare maximal solution corresponds to a market equilibrium and vice versa. This 1-1 correspondence is exploited in the next section during the study of existence and uniqueness of the underlying market equilibrium.

4. EXISTENCE AND UNIQUENESS

In the following, we study existence and uniqueness of short- and long-run market equilibrium of the proposed market equilibrium problem (MCP) that models the private investment and subsequent spot market trading in different coupled energy sectors. Since we established the equivalence of (MCP) to a single-level counterpart in the previous section, it is now easy to see existence of equilibrium by applying the classical theorem of Weierstraß to the single-level counterpart. However, uniqueness is not so easily achieved. For this purpose, additional conditions have to be fulfilled. Before we address these conditions, we show in Sect. 4.1 that if multiple equilibria exist, at least two of them share some structural properties. The obtained insights are later exploited to deduce contradictions when showing uniqueness under further assumptions. In Sect. 4.2, we derive conditions that we then prove to be sufficient for guaranteeing uniqueness of equilibrium in the short-run, i.e., for the case of fixed investment decisions. Afterward, in Sect. 4.3, additional sufficient conditions for long-run uniqueness are presented. We conclude with a detailed discussion of all conditions for uniqueness in Sect. 4.4.

4.1. Structural properties of multiple equilibria. Grimm, Schewe, et al. (2017) have shown in the context of a single sector that if multiple equilibria exist, there also exist two equilibria that share some structural properties. By applying Lemma 1 of Grimm, Schewe, et al. (2017), we obtain the same result for the case of multiple sectors.

Lemma 4.1. *Exactly one of the following two cases occurs:*

- (i) *There exist a demand vector d^* , a production vector y^* , and an investment vector $(y^{\text{new}})^*$ such that the primal variables of every market equilibrium of (MCP) are of the form $(d^*, y^*, f, (y^{\text{new}})^*)$ for some flow f .*
- (ii) *There exist two market equilibria w and \tilde{w} of (MCP) with $(d, y, y^{\text{new}}) \neq (\tilde{d}, \tilde{y}, \tilde{y}^{\text{new}})$ and*

$$\begin{aligned} \{k \in K_s : f_{t,k} = f_k^-\} &= \{k \in K_s : \tilde{f}_{t,k} = f_k^-\}, \\ \{k \in K_s : f_{t,k} = f_k^+\} &= \{k \in K_s : \tilde{f}_{t,k} = f_k^+\}, \\ \{g \in G_z^{\text{all}} : y_{t,g} = 0\} &= \{g \in G_z^{\text{all}} : \tilde{y}_{t,g} = 0\}, \\ \{g \in G_z^{\text{ex}} : y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\} &= \{g \in G_z^{\text{ex}} : \tilde{y}_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\}, \\ \{i \in I_z^{\text{all}} : y_{t,i} = 0\} &= \{i \in I_z^{\text{all}} : \tilde{y}_{t,i} = 0\}, \\ \{i \in I_z^{\text{ex}} : y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\} &= \{i \in I_z^{\text{ex}} : \tilde{y}_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\}, \end{aligned}$$

for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.¹

Proof. This follows from Proposition 1 in Grimm, Schewe, et al. (2017). \square

We establish uniqueness of short- and long-run equilibrium in Sect. 4.2 and 4.3 by showing that, under appropriate conditions, only Case (i) of Lemma 4.1 occurs. To this end, it is advantageous to introduce the concept of price clusters as in Grimm, Schewe, et al. (2017). Later, we will state conditions on the appearance of these price clusters in order to ensure long-run uniqueness. For now, we use the concept to reformulate Lemma 4.1 suitably.

Definition 4.2. Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of the market equilibrium problem (MCP). Furthermore, let $\mathcal{C}_{t,s} = \{C_1, \dots, C_{|\mathcal{C}_{t,s}|}\}$ be a partition of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$. We call $\mathcal{C}_{t,s}$ a partition into *price clusters* if prices are equal for all bidding zones in a cluster $C \in \mathcal{C}_{t,s}$. To emphasize that price clusters may depend on the considered solution, we also use the notation $\mathcal{C}_{t,s}(w)$. The adjacent bidding zones $k = (z, z') \in K_s$ are called *intercluster adjacent bidding zones* if $z \in C_i$ and $z' \in C_j$ with $i \neq j$ holds.

To simplify notation in the following, we denote, e.g., by G_C^{new} the set of all candidate sector-specific production facilities in price cluster $C \in \mathcal{C}_{t,s}$ or by $\delta_s^{\text{in}}(C)$ the set of all ingoing trade capacities of the price cluster.

In case of a uniform pricing system in sector s , $\mathcal{C}_{t,s} = \{Z_s\}$ is a partition into price clusters for all time periods. The same is true if there is never congestion between bidding zones, i.e., if inequalities (11b) are never binding over the considered time horizon. This relation is shown in the next lemma and the subsequent corollary.

Lemma 4.3. *Let adjacent bidding zones $k = (z, z') \in K_s$ of sector $s \in S$ be given. For time periods $t \in T$ with $p_{t,z} \neq p_{t,z'}$, i.e., with different prices in the adjacent bidding zones, $f_{t,k} = f_k^+$ or $f_{t,k} = f_k^-$ holds.*

Proof. For $k = (z, z') \in K_s$ and $t \in T$ with $p_{t,z} < p_{t,z'}$, $\kappa_{t,k}^+ > 0$ follows from (12a) and therefore $f_{t,k} = f_k^+$ holds by (12c). For $k = (z, z') \in K_s$ and $t \in T$ with $p_{t,z} > p_{t,z'}$, $\kappa_{t,k}^- > 0$ follows from (12a) and therefore $f_{t,k} = f_k^-$ holds by (12b). \square

¹If this case applies, it follows directly that infinitely many market equilibria exist. As shown in the previous section, the two distinct market equilibria correspond to two distinct welfare maximal solutions. All convex combinations of this two welfare maxima also form welfare maxima since the welfare maximization problem (15) is a concave maximization problem with linear constraints. However, all these welfare maxima in turn correspond to market equilibria.

Lemma 4.3 directly implies that the prices of adjacent bidding zones are the same if the trade capacities between these bidding zones are not binding.

Corollary 4.4. *Let adjacent bidding zones $k = (z, z') \in K_s$ of sector $s \in S$ be given. For time periods $t \in T$ with $f_{t,k} \neq f_k^+$ and $f_{t,k} \neq f_k^-$, i.e., with non-binding trade capacities, $p_{t,z} = p_{t,z'}$ holds.*

In particular, this guarantees the following: if all adjacent bidding zones with one binding trade capacity are disregarded, then, remaining adjacent bidding zones have the same price. Based on this insight, a partition into price clusters of the bidding zones can be constructively derived (see Theorem 2 in Grimm, Schewe, et al. (2017) for the same result in the context of one sector).

Lemma 4.5. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of the market equilibrium problem (MCP). Consider the set $K_{t,s}^o$ of adjacent bidding zones with non-binding trade capacities in time period $t \in T$ and sector $s \in S$, i.e., $K_{t,s}^o := \{k \in K_s : f_k^- < f_{t,k} < f_k^+\}$. Furthermore, consider the graph $\mathcal{G} := (Z_s, K_{t,s}^o)$. Let $\mathcal{C}_{t,s} = \{C_1, \dots, C_{|\mathcal{C}_{t,s}|}\}$ be the set of connected components of the graph \mathcal{G} . Then, $\mathcal{C}_{t,s}$ is a partition into price clusters of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$.*

Proof. The assertion follows directly from Corollary 4.4. \square

We call the latter partition capacity-induced because it is induced by the binding trade capacities in the given market equilibrium.

Definition 4.6. The partition $\mathcal{C}_{t,s}$ described in Lemma 4.5 is called the *capacity-induced* partition into price clusters of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$.

Now, we are able to reformulate Lemma 4.1 suitably.

Lemma 4.7. *Exactly one of the following two cases occurs:*

- (i) *There exist a demand vector d^* , a production vector y^* , and an investment vector $(y^{\text{new}})^*$ such that the primal variables of every market equilibrium of (MCP) are of the form $(d^*, y^*, f, (y^{\text{new}})^*)$ for some flow f .*
- (ii) *There exist two market equilibria w and \tilde{w} of (MCP) with $(d, y, y^{\text{new}}) \neq (\tilde{d}, \tilde{y}, \tilde{y}^{\text{new}})$ such that*

- *the capacity-induced partitions $\mathcal{C}_{t,s}(w)$ and $\mathcal{C}_{t,s}(\tilde{w})$ into price clusters of the bidding zones Z_s are the same for all sectors and all time periods, i.e., $\mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ holds for all $s \in S, t \in T$,*
- *the total in- or outflow of each price cluster $C \in \mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ is unique for all $s \in S, t \in T$:*

$$f_C := \sum_{k \in \delta_s^{\text{in}}(C)} f_{t,k} - \sum_{k \in \delta_s^{\text{out}}(C)} f_{t,k} = \sum_{k \in \tilde{\delta}_s^{\text{in}}(C)} \tilde{f}_{t,k} - \sum_{k \in \tilde{\delta}_s^{\text{out}}(C)} \tilde{f}_{t,k} =: \tilde{f}_C,$$

- *any operator of a sector-specific or sector-coupling facility who does not produce in one market equilibrium does also not produce in the other one, i.e.,*

$$\begin{aligned} \{g \in G_z^{\text{all}} : y_{t,g} = 0\} &= \{g \in G_z^{\text{all}} : \tilde{y}_{t,g} = 0\}, \\ \{i \in I_z^{\text{all}} : y_{t,i} = 0\} &= \{i \in I_z^{\text{all}} : \tilde{y}_{t,i} = 0\}, \end{aligned}$$

for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$,

- *any operator of an existing sector-specific or sector-coupling facility who does produce at maximum available capacity in one market equilibrium does also produce at maximum available capacity in the other one, i.e.,*

$$\begin{aligned} \{g \in G_z^{\text{ex}} : y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\} &= \{g \in G_z^{\text{ex}} : \tilde{y}_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\}, \\ \{i \in I_z^{\text{ex}} : y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\} &= \{i \in I_z^{\text{ex}} : \tilde{y}_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\}, \end{aligned}$$

for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.

Proof. The lemma follows directly from Lemmata 4.1 and 4.5 with the following observation. By definition of the capacity-induced partition, $f_{t,k} = f_k^+$ or $f_{t,k} = f_k^-$ holds for all intercluster adjacent bidding zones $k \in K_s$ and $f_{t,k} \neq f_k^+$ and $f_{t,k} \neq f_k^-$ for all other adjacent bidding zones $k \in K_s$. Since binding patterns are assumed to coincide in w and \tilde{w} (cf. Lemma 4.1 (ii)), $\mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ and $f_C = \tilde{f}_C$ for $C \in \mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ follows for all $s \in S$, $t \in T$. \square

So far, we have shown that if multiple equilibria exist, there necessarily exist two different equilibria that share the structural properties listed in Lemma 4.7 (ii). Next, we show gradually that, under additional assumptions, two such market equilibria are identical in the short-run (Sect. 4.2) as well as in the long-run (Sect. 4.3), i.e., finally only Case (i) of Lemma 4.7 occurs under all additional assumptions. This implies uniqueness of equilibrium of the proposed market equilibrium problem (MCP).

4.2. Uniqueness of equilibrium in the short-run. For the moment, we only consider the short-run, i.e., all investment decisions have already been made and are therefore fixed. We begin our short-run uniqueness analysis with the important observation that at least all consumer demand is unique.

Theorem 4.8. *Let w and \tilde{w} be two market equilibria of (MCP). Then, the sector-specific demand is unique, i.e., $d_{t,z} = \tilde{d}_{t,z}$ holds for each sector $s \in S$, bidding zone $z \in Z_s$, and time period $t \in T$.*

Proof. The assertion follows from applying Theorem 1a in Mangasarian (1988) to the single-level counterpart (15) under the assumption of continuously differentiable and strictly decreasing inverse demand functions. \square

However, to obtain uniqueness of the other primal decision variables and therefore of the short-run market equilibrium, additional assumptions are necessary. The first one relates to the sector-specific demand and is similar to Assumption 5 in Grübel et al. (2020).

Assumption 1. *The sector-specific demand $d_{t,z}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is positive in all time periods $t \in T$.*

This assumption can be justified by the fact that, in real applications, there should always be consumers in a bidding zone who are willing to pay enough and especially more than the resident operators of sector-coupling facilities. The benefit of this assumption is that it enables us to prove uniqueness of market prices (cf. Lemma 3.8 in Grübel et al. (2020) for the same result in the context of a single sector).

Lemma 4.9. *Suppose Assumption 1 holds. Then, the zonal price $p_{t,z}$ is unique for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.*

Proof. Since $d_{t,z} > 0$ holds for all $s \in S$, $z \in Z_s$, and $t \in T$ by Assumption 1, $\gamma_{t,z} = 0$ follows from (2b). Moreover, $P_{t,z}(d_{t,z}) = p_{t,z}$ follows from (2a). Since all sector-specific demand is unique (Theorem 4.8), uniqueness of zonal prices is given due to the assumption of continuously differentiable and strictly decreasing inverse demand functions. \square

Since we assume perfect competition, all players act as price takers, i.e., view market prices as exogenously given. This means that each player does not take the effects of his own or others' decisions on the prices into account when reacting to the market prices. Hence, due to the uniqueness of market prices, multiplicity of equilibria directly implies multiplicity of primal decisions of at least one individual player, i.e., players with multiple best responses to the unique market prices exist. For this reason, we next consider each player type independently and analyze under which circumstances these types are indeed indifferent about their actions and under which circumstances this is not the case.

First, we consider the operators of sector-specific production facilities. In accordance to Lemma 3.9 in Grübel et al. (2020), we prove that the production of a sector-specific facility is unique if the zonal market price does not meet its variable costs of production.

Lemma 4.10. *Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,g}$ of producer $g \in G_z^{\text{ex}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is unique if $p_{t,z} \neq c_g^{\text{var}}$ holds.*

Proof. For $g \in G_z^{\text{ex}}$ with $c_g^{\text{var}} < p_{t,z}$, $\beta_{t,g}^+ > 0$ follows from (4a) and therefore $y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}$ holds by (4c). For $g \in G_z^{\text{ex}}$ with $c_g^{\text{var}} > p_{t,z}$, $\beta_{t,g}^- > 0$ follows from (4a) and therefore $y_{t,g} = 0$ holds by (4b). \square

Analogously, the following applies for candidate sector-specific production facilities.

Lemma 4.11. *Given fixed investment decisions y^{new} . Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,g}$ of producer $g \in G_z^{\text{new}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is unique if $p_{t,z} \neq c_g^{\text{var}}$ holds.*

Proof. For $g \in G_z^{\text{new}}$ with $c_g^{\text{var}} < p_{t,z}$, $\delta_{t,g}^+ > 0$ follows from (6b) and therefore $y_{t,g} = \alpha_{t,g} y_g^{\text{new}}$ holds by (6d). For $g \in G_z^{\text{new}}$ with $c_g^{\text{var}} > p_{t,z}$, $\delta_{t,g}^- > 0$ follows from (6b) and therefore $y_{t,g} = 0$ holds by (6c). \square

Next, we examine the operators of sector-coupling facilities and show that those are only indifferent about their actions if the price of the withdrawing zone z' matches the price of the injection zone z multiplied by the facility's efficiency.

Lemma 4.12. *Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,i}$ and the demand $d_{t,o}$ of the sector-coupling facility $x = (o, i) \in X^{\text{ex}}$ are unique if $p_{t,z'} \neq \eta_x p_{t,z}$ holds.*

Proof. From equations (8a) and (8b), $-p_{t,z'} + \theta_{t,x} + \eta_x(p_{t,z} + \nu_{t,x}^- - \nu_{t,x}^+) = 0$ follows for all time periods $t \in T$. For $x = (o, i) \in X^{\text{ex}}$ with $p_{t,z'} < \eta_x p_{t,z}$, $\eta_x \nu_{t,x}^+ > 0$ follows from the latter equation and therefore $y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}$ holds by (8e). For $x = (o, i) \in X^{\text{ex}}$ with $p_{t,z'} > \eta_x p_{t,z}$, $\theta_{t,x} + \eta_x \nu_{t,x}^- > 0$ follows and therefore $d_{t,o} = y_{t,i} = 0$ holds by (8c), (8d), and (8f). \square

Analogously, the following applies for candidate sector-coupling facilities.

Lemma 4.13. *Given fixed investment decisions y^{new} . Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,i}$ and the demand $d_{t,o}$ of the sector-coupling facility $x = (o, i) \in X^{\text{new}}$ are unique if $p_{t,z'} \neq \eta_x p_{t,z}$ holds.*

Proof. From equations (10b) and (10c), $-p_{t,z'} + \rho_{t,x} + \eta_x(p_{t,z} + \tau_{t,x}^- - \tau_{t,x}^+) = 0$ follows for all time periods $t \in T$. For $x = (o, i) \in X^{\text{new}}$ with $p_{t,z'} < \eta_x p_{t,z}$, $\eta_x \tau_{t,x}^+ > 0$ follows from the latter equation and therefore $y_{t,i} = \alpha_{t,i} y_i^{\text{new}}$ holds by (10f). For $x = (o, i) \in X^{\text{new}}$ with $p_{t,z'} > \eta_x p_{t,z}$, $\rho_{t,x} + \eta_x \tau_{t,x}^- > 0$ follows and therefore $d_{t,o} = y_{t,i} = 0$ holds by (10d), (10e), and (10g). \square

Before we finally prove uniqueness of equilibrium in the short-run, we formulate additional sufficient conditions required in order to carry out the proof.

Assumption 2. *Here and in what follows, we assume that the below stated properties are satisfied by the variable production costs and the efficiencies of the sector-coupling facilities.*

- (a) *The variable production costs are pairwise distinct in each sector, i.e., $c_g^{\text{var}} \neq c_{g'}^{\text{var}}$ for $g \neq g'$ with $g \in G_z^{\text{all}}$ and $g' \in G_{z'}^{\text{all}}$ for all $z, z' \in Z_s$ and $s \in S$.*
- (b) *The efficiencies of all sector-coupling facilities are pairwise distinct, i.e., $\eta_x \neq \eta_{x'}$ for $x \neq x'$ with $x, x' \in X^{\text{all}}$.*
- (c) *Given the sector-coupling facility $x = (o, i) \in X^{\text{all}}$ that produces the commodity of sector $s \in S$ from the commodity of sector $s' \in S$ with $s' \neq s$. Then, the variable production costs across the sectors are—taking into account the sector-coupling facility's efficiency—pairwise distinct, i.e., $\eta_x c_g^{\text{var}} \neq c_{g'}^{\text{var}}$ with $g \in G_z^{\text{all}}$ for all $z \in Z_s$ and with $g' \in G_{z'}^{\text{all}}$ for all $z' \in Z_{s'}$.*

(d) Moreover, the following is true

$$c_g^{\text{var}} \prod_{i=1}^j \eta_{x_i} \neq c_{g'}^{\text{var}} \prod_{i=j+1}^{j'} \eta_{x_i}$$

for all $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$ and for $g \neq g'$ with $g \in G_z^{\text{all}}$ for all $z \in Z_s$, $s \in S$, and with $g' \in G_{z'}^{\text{all}}$ for all $z' \in Z_{s'}$, $s' \in S$.

(e) Furthermore, it holds that

$$\prod_{i=1}^j \eta_{x_i} \neq \prod_{i=j+1}^{j'} \eta_{x_i}$$

for all $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$.

In principle, each condition prevents facilities from producing at identical variable production costs in the same sector. For example, Assumption 2 (b) guarantees that two sector-coupling facilities with the same input and target sector have distinct variable production costs. Assumption 2 (e) ensures, e.g., that one sector-coupling facility does not produce at the same costs as sequentially producing sector-coupling facilities with the same primary input and final target sector. In Sect. 4.4, we elaborate in more detail on this topic and also show with illustrative examples that as soon as one of the above conditions is removed, then, these conditions are no longer sufficient for guaranteeing uniqueness in the short-run.

In addition, we like to observe that Assumption 2 (a) is the direct extension of the classical uniqueness assumption for one sector (see, e.g., Grimm, Schewe, et al. (2017)) to the case of multiple sectors. Moreover, it directly follows from Assumption 2 (a) and Lemmata 4.10 and 4.11 that the sector-specific production in a price cluster is not unique for at most one player. Analogously, it directly follows from Assumption 2 (b) and Lemmata 4.12 and 4.13 that the sector-coupling production and demand between two price clusters of different sectors is not unique for at most one player. With Assumption 2 (c), the subsequent case is avoided: indifference about the sector-specific production in one sector leads to indifference about the sector-coupling demand and production between two sectors that in turn triggers indifference about the sector-specific production in the other sector. However, indifference about a sector-coupling demand and production between the other and another sector may also be triggered. To avoid such paths and cycles of indifference, Assumptions 2 (d) and (e) are required. Given these assumptions, we are finally able to prove short-run uniqueness. The proof is split into two parts. First, we show that all sector-specific production is unique. In a second step, we establish uniqueness of sector-coupling demand and production.

Lemma 4.14. *Given fixed investment decisions y^{new} . Suppose that Assumptions 1 and 2 hold. Let w and \tilde{w} be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-specific production is unique, i.e., $y_{t,g} = \tilde{y}_{t,g}$ holds for each sector $s \in S$, bidding zone $z \in Z_s$, producer $g \in G_z^{\text{all}}$, and time period $t \in T$.*

Proof. In the following, we denote for sector $s \in S$ and time period $t \in T$ by $\mathcal{C}_{t,s}$ the identical capacity-induced partitions $\mathcal{C}_{t,s}(w)$ and $\mathcal{C}_{t,s}(\tilde{w})$ into price clusters of the bidding zones Z_s . Let a price cluster $C \in \mathcal{C}_{t,s}$ be given. Moreover, let $p_{t,C}$ be the price of this price cluster. If $p_{t,C} \neq c_g^{\text{var}}$ holds for all producers $g \in G_C^{\text{all}}$, all sector-specific production in this cluster is unique (follows from Lemmata 4.10 and 4.11). In turn, if $p_{t,C} = c_g^{\text{var}}$ holds for a producer $g \in G_C^{\text{all}}$, the respective sector-specific production may not be unique. It remains to show that this is not the case. Hence, for the sake of contradiction, we assume that the production of producer g is not unique, i.e., $y_{t,g} \neq \tilde{y}_{t,g}$ holds for some $t \in T$. First, we like to observe that the production is then unique for all other sector-specific facilities in this cluster (follows from Lemmata 4.10 and 4.11 under Assumption 2 (a)). Since, in addition, all sector-specific demand (Theorem 4.8) and the total in- and outflow of this cluster (Lemma 4.7 (ii)) are unique, it directly follows from the market clearing conditions (13) for this cluster that $d_{t,o}$ or $y_{t,i}$ is not unique for at least one $o \in O_C^{\text{all}}$ or $i \in I_C^{\text{all}}$. This, in turn, implies by (7b) or

(9b) that the sector-coupling demand or production in at least one price cluster $C' \in \mathcal{C}_{t,s'}$ in another sector $s' \in S$ is not unique. Again, we like to observe that the sector-coupling demand and production is then unique for all other sector-coupling facilities that operate between price clusters C and C' (follows from Lemmata 4.12 and 4.13 under Assumption 2 (b)). Moreover, Lemmata 4.12 and 4.13 establish as a prerequisite for multiplicities that

$$p_{t,C} = c_g^{\text{var}} = \eta_x p_{t,C'} \quad \text{or} \quad p_{t,C} = c_g^{\text{var}} = (1/\eta_x) p_{t,C'}$$

holds for the respective sector-coupling facility $x \in X^{\text{all}}$. By Assumption 2 (c), it directly follows that $p_{t,C'} \neq c_{g'}^{\text{var}}$ for all $g' \in G_{C'}^{\text{all}}$. Hence, all sector-specific production is unique in this price cluster. The same argumentation as above implies that $d_{t,o}$ or $y_{t,i}$ is not unique for at least another $o \in O_{C'}^{\text{all}}$ or $i \in I_{C'}^{\text{all}}$. This, in turn, implies by (7b) or (9b) that the sector-coupling demand or production in at least one price cluster of another sector is not unique. The same argumentation can be applied again and again until either the non-unique sector-coupling demand or production is met by a non-unique sector-specific production of producer $g' \neq g$, i.e., the term

$$c_g^{\text{var}} \prod_{i=1}^j \eta_{x_i} = c_{g'}^{\text{var}} \prod_{i=j+1}^{j'} \eta_{x_i}$$

is—in accordance with the prerequisite for multiplicities—true for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$, or, since there are at most $\sum_{s \in S} |Z_s|$ price clusters, a price cluster C'' is again reached, i.e., the term

$$p_{t,C''} \prod_{i=1}^j \eta_{x_i} = p_{t,C''} \prod_{i=j+1}^{j'} \eta_{x_i} \Leftrightarrow \prod_{i=1}^j \eta_{x_i} = \prod_{i=j+1}^{j'} \eta_{x_i}$$

holds—in accordance with the prerequisite for multiplicities—for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$. Both cases are forbidden by Assumptions 2 (d) and (e). Therefore, $y_{t,g} = \tilde{y}_{t,g}$ holds. \square

Next, we show the short-run uniqueness of sector-coupling demand and production under Assumptions 1 and 2.

Lemma 4.15. *Given fixed investment decisions y^{new} . Suppose that Assumptions 1 and 2 hold. Let w and \tilde{w} be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-coupling demand and production are unique, i.e., $d_{t,o} = \tilde{d}_{t,o}$ and $y_{t,i} = \tilde{y}_{t,i}$ hold for each sector $s \in S$, bidding zone $z \in Z_s$, withdrawing facility $o \in O_z^{\text{all}}$, injecting facility $i \in I_z^{\text{all}}$, and time period $t \in T$.*

Proof. Since all sector-specific demand and production (Theorem 4.8 and Lemma 4.14) and the total in- and outflow of each cluster (Lemma 4.7 (ii)) are unique, non-unique demand or production of one sector-coupling facility $x \in X^{\text{all}}$ in a cluster directly implies by the market clearing conditions (13) for this cluster non-unique demand or production of another sector-coupling facility $x' \in X^{\text{all}}$ in this cluster. Furthermore, since at most one sector-coupling demand and production is not unique between two price clusters of different sectors (follows from Lemmata 4.12 and 4.13 under Assumption 2 (b)), this non-uniqueness continues until, due to at most $\sum_{s \in S} |Z_s|$ price clusters, the same price cluster C is again reached, i.e., the term

$$p_{t,C} \prod_{i=1}^j \eta_{x_i} = p_{t,C} \prod_{i=j+1}^{j'} \eta_{x_i} \Leftrightarrow \prod_{i=1}^j \eta_{x_i} = \prod_{i=j+1}^{j'} \eta_{x_i}$$

holds for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$. This contradicts Assumption 2 (e) and, consequently, there is uniqueness with regard to sector-coupling demand and production.² \square

So, we finally obtain uniqueness of short-run market equilibrium in coupled markets of multiple energy sectors.

Theorem 4.16. *Suppose that Assumptions 1 and 2 hold. Then, the market equilibrium of (MCP) is unique in the short-run, i.e., in the case of fixed investment decisions.*

Proof. The assertion follows directly from Theorem 4.8 and Lemmata 4.14 and 4.15. \square

4.3. Uniqueness of equilibrium in the long-run. We now continue our study of uniqueness with the long-run, i.e., investment decisions are no longer fixed. It is common knowledge that investment decisions are mainly driven by the generated contribution margins over time. What is special in our context is that the time periods with positive contribution margins are uniquely determined for each facility due to the uniqueness of zonal prices (Lemma 4.9).

Lemma 4.17. *Suppose that Assumption 1 holds. Let w be a market equilibrium of (MCP). Moreover, let T_g for producer $g \in G_z^{\text{new}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ be the time periods in which this producer obtains positive contribution margins, i.e., $T_g := \{t \in T : \delta_{t,g}^+ > 0\}$ holds. Then, T_g is unique.*

Proof. Since the zonal price $p_{t,z}$ is unique for all time periods $t \in T$ by Lemma 4.9, the time periods with $p_{t,z} > c_g^{\text{var}}$ are also unique but $p_{t,z} > c_g^{\text{var}}$ holds in a time period t if and only if $\delta_{t,g}^+ > 0$ holds due to (6b)–(6d); see also the argumentation in the proof of Lemma 4.11. \square

Analogously, the same applies for candidate sector-coupling facilities.

Lemma 4.18. *Suppose that Assumption 1 holds. Let w be a market equilibrium of (MCP). Moreover, let T_x for producer $x \in X^{\text{new}}$ be the time periods in which this producer generates positive contribution margins, i.e., $T_x := \{t \in T : \tau_{t,x}^+ > 0\}$ holds. Then, T_x is unique.*

Proof. Since the zonal price $p_{t,z}$ of the bidding zone $z \in Z_s$ of sector $s \in S$, in which the injection point of facility x is located, and the zonal price $p_{t,z'}$ of the bidding zone $z' \in Z_{s'}$ of sector $s' \in S$ with $s \neq s'$, in which the withdrawal point of facility x is located, are unique for all time periods $t \in T$ by Lemma 4.9, the time periods with $p_{t,z'} < \eta_x p_{t,z}$ are also unique but $p_{t,z'} < \eta_x p_{t,z}$ holds in a time period t if and only if $\tau_{t,x}^+ > 0$ holds due to (10b), (10c), (10e), and (10f); see also the argumentation in the proof of Lemma 4.13. \square

Nevertheless, in order to achieve uniqueness of the long-run equilibrium, further assumptions are required. These are obtained by naturally extending Assumptions 3 and 4 of Grimm, Schewe, et al. (2017) to the case of multiple sectors.

Assumption 3. *Let w be a market equilibrium of (MCP). Here and in what follows, we assume that, for each sector $s \in S$, there exists a subset of time periods $\bar{T} \subseteq T$ for which the capacity-induced partitions $\mathcal{C}_{t,s}(w)$, $t \in \bar{T}$, are the same. Moreover, for each price cluster $C \in \mathcal{C}_{t,s}(w)$, there exists a bijective function $h_C : \bar{T}_C \rightarrow G_C^{\text{new}} \cup I_C^{\text{new}} \cup O_C^{\text{new}}$ with $\bar{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|\bar{T}_C|}\} \subseteq \bar{T}$ such that the following holds for a given $j \in \{1, \dots, |\bar{T}_C|\}$:*

- (i) *all producers $h_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, produce in time period \bar{t}_j either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied in Equation (6d) or (10f), i.e., either*

$$y_{\bar{t}_j, h_C(\bar{t}_{j'})} = 0 \quad \text{or} \quad y_{\bar{t}_j, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_j, h_C(\bar{t}_{j'})} y_{h_C(\bar{t}_{j'})}^{\text{new}}$$

²In order to increase the understanding of the latter expression and therefore of the contradiction, we like to point out again that Lemmata 4.12 and 4.13 establish as a prerequisite for multiplicities that

$$p_{t,C} = \eta_x p_{t,C'} \quad \text{or} \quad p_{t,C} = (1/\eta_x) p_{t,C'}$$

holds for a sector-coupling facility $x \in X^{\text{all}}$ that operates between the price clusters C and C' .

with

$$\begin{aligned} \delta_{\bar{t}_j, h_C(\bar{t}_{j'})}^+ &> 0, & \text{if } h_C(\bar{t}_{j'}) \in G_C^{\text{new}}, \text{ and} \\ \tau_{\bar{t}_j, x}^+ &> 0, & \text{if } x = (o, h_C(\bar{t}_{j'})) \in X^{\text{new}}. \end{aligned}$$

- (ii) all sector-coupling consumers $h_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, demand in time period \bar{t}_j either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied in Equation (10f), i.e., either

$$d_{\bar{t}_j, h_C(\bar{t}_{j'})} = 0 \quad \text{or} \quad \eta_x d_{\bar{t}_j, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_j, i} y_i^{\text{new}} \quad \text{with} \quad \tau_{\bar{t}_j, x}^+ > 0,$$

where $x = (h_C(\bar{t}_{j'}), i) \in X^{\text{new}}$.

- (iii) if $h_C(\bar{t}_j)$ is a producer, then, this producer produces in time period \bar{t}_j at maximum available capacity while strict complementarity is satisfied in Equation (6d) or (10f), i.e.,

$$y_{\bar{t}_j, h_C(\bar{t}_j)} = \alpha_{\bar{t}_j, h_C(\bar{t}_j)} y_{h_C(\bar{t}_j)}^{\text{new}}$$

with

$$\begin{aligned} \delta_{\bar{t}_j, h_C(\bar{t}_j)}^+ &> 0, & \text{if } h_C(\bar{t}_j) \in G_C^{\text{new}}, \text{ and} \\ \tau_{\bar{t}_j, x}^+ &> 0, & \text{if } x = (o, h_C(\bar{t}_j)) \in X^{\text{new}}. \end{aligned}$$

- (iv) if $h_C(\bar{t}_j)$ is a sector-coupling consumer, then, this consumer demands in time period \bar{t}_j at maximum available capacity while strict complementarity is satisfied in Equation (10f), i.e.,

$$\eta_x d_{\bar{t}_j, h_C(\bar{t}_j)} = \alpha_{\bar{t}_j, i} y_i^{\text{new}} \quad \text{with} \quad \tau_{\bar{t}_j, x}^+ > 0,$$

where $x = (h_C(\bar{t}_j), i) \in X^{\text{new}}$.

- (v) all producers $h_C(\bar{t}_{j'})$ with $j' > j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, produce in time period \bar{t}_j zero, i.e., $y_{\bar{t}_j, h_C(\bar{t}_{j'})} = 0$,
- (vi) all sector-coupling consumers $h_C(\bar{t}_{j'})$ with $j' > j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, demand in time period \bar{t}_j zero, i.e., $d_{\bar{t}_j, h_C(\bar{t}_{j'})} = 0$,
- (vii) all existing sector-specific producers produce in time period \bar{t}_j either zero or at maximum available capacity, i.e., $y_{\bar{t}_j, g} = 0$ or $y_{\bar{t}_j, g} = \alpha_{\bar{t}_j, g} y_g^{\text{ex}}$ for all $g \in G_C^{\text{ex}}$,
- (viii) all existing sector-coupling producers produce in time period \bar{t}_j either zero or at maximum available capacity, i.e., $y_{\bar{t}_j, i} = 0$ or $y_{\bar{t}_j, i} = \alpha_{\bar{t}_j, i} y_i^{\text{ex}}$ for all $i \in I_C^{\text{ex}}$, and
- (ix) all existing sector-coupling consumers demand in time period \bar{t}_j either zero or at maximum available capacity, i.e., $d_{\bar{t}_j, o} = 0$ or $\eta_x d_{\bar{t}_j, o} = \alpha_{\bar{t}_j, i} y_i^{\text{ex}}$ for all $x = (o, i) \in X^{\text{ex}}$ with $o \in O_C^{\text{ex}}$.

In the following, we explain why Assumption 3 prevents multiplicities in the long-run and how likely it is that Assumption 3 is fulfilled in general. Therefore, let us assume that the long-run investment of a given facility is not unique, e.g., $y_g^{\text{new}} + \varepsilon_g = \tilde{y}_g^{\text{new}}$ holds for two market equilibria w and \tilde{w} of (MCP) as described in Lemma 4.7 (ii) with $\varepsilon_g > 0$.³ By Lemmata 4.17 and 4.18, it directly follows that also the production is not unique, i.e.,

$$\tilde{y}_{t, g} = \alpha_{t, g} \tilde{y}_g^{\text{new}} = \alpha_{t, g} (y_g^{\text{new}} + \varepsilon_g) = y_{t, g} + \alpha_{t, g} \varepsilon_g \quad (19)$$

holds for all time periods $t \in T_g$ due to $\delta_{t, g}^+ > 0$, $\tilde{\delta}_{t, g}^+ > 0$, and (6d). However, since the sector-specific demand is unique (Theorem 4.8) and must be served in each time period, all these changes in production have to be compensated by changes in production of other facilities of the cluster. Now, this is the point where Assumption 3 comes in. Assumption 3 (i)–(vi) involves that at least one time period exists for each candidate facility in which this facility and all other producing candidate facilities generate positive contribution margins. Moreover,

³The same argumentation can be applied for the sector-coupling facility $x \in X^{\text{new}}$.

in this time period, the production of all existing facilities coincides in w and \tilde{w} due to Assumption 3 (vii)–(ix) and Lemma 4.7 (ii). Hence, one of the compensating facilities in this time period is a candidate facility that produces with a positive contribution margin. Thus, also the investment of this facility is not unique and resulting changes in production of this facility have to be compensated by changes in production of other facilities of the cluster. At this point, Assumption 3 again intervenes and prevents that these two facilities entirely compensate each other, i.e., the sets of time periods with positive contribution margins of facility g and the mentioned compensating candidate facility are different.⁴ Furthermore, Assumption 3 imposes that there exists a time period in the union of the set differences in which all conditions of Assumption 3 are again fulfilled. Therefore, applying the same argument as above leads to multiplicities in investment of a third candidate facility. Assumption 3 allows us to apply this argument until a time period is reached in which only one candidate facility from this cluster produces while the production of all existing facilities coincides in this time period in w and \tilde{w} due to Assumption 3 (vii)–(ix) and Lemma 4.7 (ii).⁵ Since no compensation is possible in this time period, the initial assumption that the long-run investment of facility g is not unique must have been wrong.

In what follows, we want to elaborate on how likely it is that Assumption 3 is violated in general. In the case of one sector, Assumption 3 (iii) and (iv) involve that at least one time period exists for each candidate facility in which this facility generates the lowest but still positive contribution margin. This implies that—given the merit order of all candidate facilities in a cluster C , i.e., $c_1^{\text{var}} < c_2^{\text{var}} < \dots < c_n^{\text{var}}$ with $n := |G_C^{\text{new}}|$ —there exists at least one time period \bar{t}_1 such that the relation $c_1^{\text{var}} < p_{\bar{t}_1, C} < c_2^{\text{var}}$ is true, at least one time period \bar{t}_2 such that $c_2^{\text{var}} < p_{\bar{t}_2, C} < c_3^{\text{var}}$ holds, and so on. This situation is illustrated for the case of three candidate and no existing production facility in Figure 3. In particular, the above mentioned condition implies the following: over time, there runs at least one aggregated cluster demand function through each dashed area depicted in Figure 3 (exemplary inverse demand functions are indicated). Thus, the wider the gap between two subsequent facilities in the merit order, the more likely it is that the stated assumption is satisfied. We like to observe that this result is in line with the classical peak-load pricing setting for a single sector with one bidding zone, where significant gaps between variable costs are also a prerequisite. For more information on the case of a single sector see Grimm, Schewe, et al. (2017).

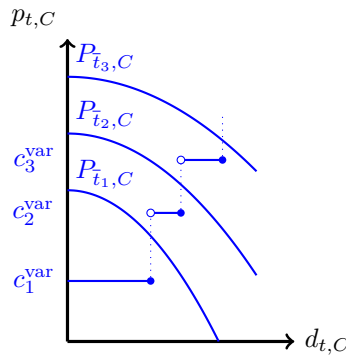


FIGURE 3. Illustration of Assumption 3 for the case of a single sector

⁴In time period \bar{t}_1 , the only producing candidate facility with positive contribution margin is $h_C(\bar{t}_1)$ due to Assumption 3 (i)–(vi). In time period \bar{t}_j with $j \in \{2, \dots, |\bar{T}_C|\}$, the candidate facility $h_C(\bar{t}_j)$ produces with positive contribution margin due to Assumption 3 (iii) and (iv). In addition, at most the candidate facilities $h_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \dots, |\bar{T}_C|\}$, produce due to Assumption 3 (i), (ii), (v), and (vi). And in case of production, these facilities also produce with positive contribution margins. Consequently, different candidate facilities of a cluster never produce always at the same time with positive contribution margins.

⁵This is, e.g., the case in time period \bar{t}_1 .

For the case of multiple sectors, the long-run uniqueness condition cannot be represented as easily as in the case of one sector via the merit order. Of course, it must still be true for the merit order of candidate sector-specific production facilities in each cluster that there exists at least one time period \bar{t}_1 such that the relation $c_1^{\text{var}} < p_{\bar{t}_1, C} < c_2^{\text{var}}$ is true, at least one time period \bar{t}_2 such that $c_2^{\text{var}} < p_{\bar{t}_2, C} < c_3^{\text{var}}$ holds, and so on. Hence, significant gaps between variable costs within clusters still increase the chance of Assumption 3 being valid. Nevertheless, this condition alone is not enough since also the candidate sector-coupling facilities injecting or withdrawing in this cluster have to be considered. However, it is not a priori clear where the sector-coupling facilities enter the respective merit orders. This depends heavily on the underlying demand and supply data of the coupled sectors and the resulting relation of prices between these sectors. Again, the more diverse the data, the more likely it is that Assumption 3 is satisfied.

As a first step in order to formally obtain long-run uniqueness under Assumption 3, we prove that investment decisions are unique under this assumption.

Lemma 4.19. *Suppose that Assumptions 1, 2, and 3 hold. Let w and \tilde{w} be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-specific and sector-coupling investment is unique in each bidding zone $z \in Z_s$ and sector $s \in S$, i.e., $y_g^{\text{new}} = \tilde{y}_g^{\text{new}}$ holds for all candidate sector-specific production facilities $g \in G_z^{\text{new}}$ and $y_i^{\text{new}} = \tilde{y}_i^{\text{new}}$ holds for all candidate sector-coupling facilities $i \in I_z^{\text{new}}$.*

Proof. In what follows, we assume that, for each sector $s \in S$, capacity-induced partitions $\mathcal{C}_{t,s}(w)$, $t \in \bar{T} \subseteq T$, as well as bijective functions $h_C : \bar{T}_C \rightarrow G_C^{\text{new}} \cup I_C^{\text{new}} \cup O_C^{\text{new}}$ with $\bar{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|T_C|}\} \subseteq \bar{T}$ for all price clusters $C \in \mathcal{C}_{t,s}(w)$ are given that fulfill the properties stated in Assumption 3. Now, we prove the assertion by induction over the time periods in \bar{T}_C . We start the induction with period \bar{t}_1 . First of all, we know by Theorem 4.8 that

$$d_{\bar{t}_1, z} = \tilde{d}_{\bar{t}_1, z}, \quad \text{for all } z \in C, \quad (20)$$

holds. In accordance with Assumption 3 (vii)–(ix), all existing producers produce either zero or at maximum available capacity and all existing sector-coupling consumers demand either zero or at maximum available capacity. Thus, it directly follows by Lemma 4.7 (ii) that

$$y_{\bar{t}_1, g} = \tilde{y}_{\bar{t}_1, g}, \quad \text{for all } g \in G_C^{\text{ex}}, \quad (21)$$

$$y_{\bar{t}_1, i} = \tilde{y}_{\bar{t}_1, i}, \quad \text{for all } i \in I_C^{\text{ex}}, \quad (22)$$

$$d_{\bar{t}_1, o} = \tilde{d}_{\bar{t}_1, o}, \quad \text{for all } o \in O_C^{\text{ex}}, \quad (23)$$

is true. Moreover, Assumption 3 (v) and (vi) imply that $y_{\bar{t}_1, g} = y_{\bar{t}_1, i} = d_{\bar{t}_1, o} = 0$ is valid for all $g \in G_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}$, $i \in I_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}$, and $o \in O_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}$. Hence,

$$y_{\bar{t}_1, g} = \tilde{y}_{\bar{t}_1, g} = 0, \quad \text{for all } g \in G_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}, \quad (24)$$

$$y_{\bar{t}_1, i} = \tilde{y}_{\bar{t}_1, i} = 0, \quad \text{for all } i \in I_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}, \quad (25)$$

$$d_{\bar{t}_1, o} = \tilde{d}_{\bar{t}_1, o} = 0, \quad \text{for all } o \in O_C^{\text{new}} \setminus \{h_C(\bar{t}_1)\}, \quad (26)$$

follows from Lemma 4.7 (ii). Furthermore, by Lemma 4.7 (ii), the capacity-induced partitions $\mathcal{C}_{\bar{t}_1, s}(w)$ and $\mathcal{C}_{\bar{t}_1, s}(\tilde{w})$ are the same for each sector $s \in S$ and, in addition, the total in- and outflow of each price cluster $C \in \mathcal{C}_{\bar{t}_1, s}(w) = \mathcal{C}_{\bar{t}_1, s}(\tilde{w})$ is unique. Consequently, by summing up the market clearing conditions (13) for all bidding zones of the price cluster C , we obtain the following relation

$$\begin{aligned} f_C &= \sum_{z \in C} d_{\bar{t}_1, z} - \sum_{g \in G_C^{\text{all}}} y_{\bar{t}_1, g} + \sum_{o \in O_C^{\text{all}}} d_{\bar{t}_1, o} - \sum_{i \in I_C^{\text{all}}} y_{\bar{t}_1, i} \\ &= \sum_{z \in C} \tilde{d}_{\bar{t}_1, z} - \sum_{g \in G_C^{\text{all}}} \tilde{y}_{\bar{t}_1, g} + \sum_{o \in O_C^{\text{all}}} \tilde{d}_{\bar{t}_1, o} - \sum_{i \in I_C^{\text{all}}} \tilde{y}_{\bar{t}_1, i} = \tilde{f}_C, \end{aligned}$$

which is due to (20)–(26) equivalent to

$$\begin{aligned} y_{\bar{t}_1, h_C(\bar{t}_1)} &= \tilde{y}_{\bar{t}_1, h_C(\bar{t}_1)}, & \text{if } h_C(\bar{t}_1) \in (G_C^{\text{new}} \cup I_C^{\text{new}}), \text{ and} \\ d_{\bar{t}_1, h_C(\bar{t}_1)} &= \tilde{d}_{\bar{t}_1, h_C(\bar{t}_1)}, & \text{if } h_C(\bar{t}_1) \in O_C^{\text{new}}. \end{aligned}$$

Therefore, all production and demand decisions coincide in w and \tilde{w} in time period \bar{t}_1 .

From now on, we assume that $h_C(\bar{t}_1) \in G_C^{\text{new}}$ holds and derive the assertion of the induction start. The same argumentation can be applied to the case $h_C(\bar{t}_1) \in I_C^{\text{new}} \cup O_C^{\text{new}}$. By Assumption 3 (iii), $\delta_{\bar{t}_1, h_C(\bar{t}_1)}^+ > 0$ holds. Thus, by Lemma 4.17, $\tilde{\delta}_{\bar{t}_1, h_C(\bar{t}_1)}^+ > 0$ is also true.

Due to (6d), we get

$$y_{\bar{t}_1, h_C(\bar{t}_1)} = \alpha_{\bar{t}_1, h_C(\bar{t}_1)} y_{h_C(\bar{t}_1)}^{\text{new}} \quad \text{and} \quad \tilde{y}_{\bar{t}_1, h_C(\bar{t}_1)} = \alpha_{\bar{t}_1, h_C(\bar{t}_1)} \tilde{y}_{h_C(\bar{t}_1)}^{\text{new}}.$$

Since we have already shown that all production decisions coincide in w and \tilde{w} in time period \bar{t}_1 , the investment in the facility $h_C(\bar{t}_1)$ also coincides by the latter two equations. Hence, we are finally finished with the start of our induction:

$$y_{h_C(\bar{t}_1)}^{\text{new}} = \tilde{y}_{h_C(\bar{t}_1)}^{\text{new}}.$$

Now, for a given time period \bar{t}_{j+1} with $j \in \{1, \dots, |\bar{T}_C| - 1\}$, we assume that the investment in the facilities $h_C(\bar{t}_1), \dots, h_C(\bar{t}_j)$ coincide in the two market equilibria w and \tilde{w} . It remains to show that this is also the case for the facility $h_C(\bar{t}_{j+1})$. The same argumentation as in the start of the induction yields

$$\begin{aligned} d_{\bar{t}_{j+1}, z} &= \tilde{d}_{\bar{t}_{j+1}, z}, & \text{for all } z \in C, \\ y_{\bar{t}_{j+1}, g} &= \tilde{y}_{\bar{t}_{j+1}, g}, & \text{for all } g \in G_C^{\text{ex}}, \\ y_{\bar{t}_{j+1}, i} &= \tilde{y}_{\bar{t}_{j+1}, i}, & \text{for all } i \in I_C^{\text{ex}}, \\ d_{\bar{t}_{j+1}, o} &= \tilde{d}_{\bar{t}_{j+1}, o}, & \text{for all } o \in O_C^{\text{ex}}, \\ y_{\bar{t}_{j+1}, g} &= \tilde{y}_{\bar{t}_{j+1}, g} = 0, & \text{for all } g \in G_C^{\text{new}} \setminus \{h_C(\bar{t}_1), \dots, h_C(\bar{t}_{j+1})\}, \\ y_{\bar{t}_{j+1}, i} &= \tilde{y}_{\bar{t}_{j+1}, i} = 0, & \text{for all } i \in I_C^{\text{new}} \setminus \{h_C(\bar{t}_1), \dots, h_C(\bar{t}_{j+1})\}, \\ d_{\bar{t}_{j+1}, o} &= \tilde{d}_{\bar{t}_{j+1}, o} = 0, & \text{for all } o \in O_C^{\text{new}} \setminus \{h_C(\bar{t}_1), \dots, h_C(\bar{t}_{j+1})\}. \end{aligned}$$

It remains to consider the production and demand decisions of the facilities $h_C(\bar{t}_1), \dots, h_C(\bar{t}_{j+1})$. By Assumption 3 (i) and (ii), the facilities $h_C(\bar{t}_1), \dots, h_C(\bar{t}_j)$ produce or demand in the market equilibrium w in time period \bar{t}_{j+1} either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied. In accordance with Lemma 4.7 (ii), it follows for the case of zero production or demand that there is also zero production or demand in the market equilibrium \tilde{w} . Moreover, for the case of production or demand at maximum available capacity, strict complementarity in w implies strict complementarity in \tilde{w} by Lemmata 4.17 and 4.18. Due to (6d) or (10f), we therefore obtain for $j' \in \{1, \dots, j\}$

$$y_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} y_{h_C(\bar{t}_{j'})}^{\text{new}} \quad \text{and} \quad \tilde{y}_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} \tilde{y}_{h_C(\bar{t}_{j'})}^{\text{new}},$$

if $h_C(\bar{t}_{j'}) \in (G_C^{\text{new}} \cup I_C^{\text{new}})$, and

$$\eta_x d_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1}, i} y_i^{\text{new}} \quad \text{and} \quad \eta_x \tilde{d}_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1}, i} \tilde{y}_i^{\text{new}},$$

if $x = (h_C(\bar{t}_{j'}), i) \in X^{\text{new}}$ with $h_C(\bar{t}_{j'}) \in O_C^{\text{new}}$. Since, in both cases, the right-hand sides are equal by the induction hypothesis, it directly follows for all $j' \in \{1, \dots, j\}$ that

$$\begin{aligned} y_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} &= \tilde{y}_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})}, & \text{if } h_C(\bar{t}_{j'}) \in (G_C^{\text{new}} \cup I_C^{\text{new}}), \text{ and} \\ d_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})} &= \tilde{d}_{\bar{t}_{j+1}, h_C(\bar{t}_{j'})}, & \text{if } h_C(\bar{t}_{j'}) \in O_C^{\text{new}}, \end{aligned}$$

is true. Now, by applying the same argument as in the start of our induction, we obtain that the production or demand associated to $h_C(\bar{t}_{j+1})$ also coincide in w and \tilde{w} in time period \bar{t}_{j+1} . Therefore, all production and demand decisions coincide in w and \tilde{w} in time period \bar{t}_{j+1} .

The same argumentation as in the start of our induction finally yields

$$y_{h_C(\bar{t}_{j+1})}^{\text{new}} = \tilde{y}_{h_C(\bar{t}_{j+1})}^{\text{new}}.$$

□

So, we finally obtain uniqueness of long-run market equilibrium in coupled markets of multiple energy sectors.

Theorem 4.20. *Suppose that Assumptions 1, 2, and 3 hold. Then, the market equilibrium of (MCP) is unique in the long-run.*

Proof. The assertion directly follows from Theorem 4.16 and Lemma 4.19. □

4.4. A discussion of the sufficient conditions for uniqueness. So far, we presented sufficient conditions that ensure uniqueness of equilibrium for coupled markets of multiple energy sectors in the short-run and in the long-run. We discuss those conditions in more detail below in order to provide further insights on when these conditions are violated and how resulting multiplicities might be resolved. In addition, we show that each condition is indeed needed for guaranteeing uniqueness in general.

We start with the short-run and therefore assume for the moment that all investment is fixed. Multiplicities arise in the short-run mainly due to similar cost structures of different production facilities. Following Assumption 2, again five cases can be distinguished

- (1) two sector-specific facilities have the same variable production costs (violates Ass. 2 (a); see Example B.1)
- (2) two sector-coupling facilities with the same input and target sector have the same variable production costs since their efficiencies are not distinct (violates Ass. 2 (b); see Example B.2)
- (3) a sector-coupling facility and a sector-specific facility from its target sector have the same variable production costs (violates Ass. 2 (c); see Example B.3)
- (4) sequentially producing sector-coupling facilities and either a sector-specific facility from their final target sector or other sequentially producing sector-coupling facilities with the same final target sector have the same variable production costs (violates Ass. 2 (d); see Example B.4)
- (5) sequentially producing sector-coupling facilities and other sequentially producing sector-coupling facilities with the same initial and final target sector have the same variable production costs (violates Ass. 2 (e); see Example B.5)

Each mentioned example illustrates how multiplicities arise if the corresponding case occurs. All examples are structured such that exactly one condition of Assumption 2 is violated, e.g., for Case (5) only Assumption 2 (e) is violated in Example B.5 while Conditions (a)–(d) of Assumption 2 hold. This demonstrates in particular that each sufficient condition identified by us is indeed needed for guaranteeing uniqueness in general. For more information on the examples see App. B.

Of course, we are aware that not all of the conditions in Assumption 2 might be fulfilled if real data is analyzed. Several sector-specific production facilities might have the same cost structure, e.g., if several power plants of the same technology are considered. One way to resolve such multiplicities is to artificially perturb same cost structures a priori by small values. Nevertheless, this can lead to undesirable boundary solutions in which, e.g., one player never produces while others with basically the same cost structure always produce. Thus, as an alternative, suitable and transparent tie-breaking rules can be implemented that select a certain equilibrium. To be able to formulate such rules, it is crucial to have a fundamental understanding of the conditions that lead to multiplicities. As explained above,

identical cost structures are mainly responsible for occurring multiplicities in the short-run. Due to same variable production costs, there is no way to distinguish between two facilities. A suitable tie-breaking rule is for example to consider these facilities as a single facility for the purpose of calculating the equilibrium and then to allocate the aggregated production to the individual facilities based on their installed capacities. Consequently, facilities with the same production costs have equal capacity utilization in the resulting equilibrium.

Also in the long-run, multiplicities mainly occur due to similar cost structures of different facilities. In contrast to the short-run, the total costs of production are now the decisive factor. In particular, if all facilities are fully available and two facilities always produce at the same time with positive contribution margins and, additionally, at the same average costs, multiplicities might arise due to the possibility of shifting investment and corresponding production cost-neutrally from one facility to the other. This result is in line with the findings of Grimm, Schewe, et al. (2017) for a single sector. For the sake of completeness, we illustrate in Example C.1 of App. C how multiplicities might arise in a single sector in the long-run. The case of multiple sectors is treated in Example C.2. In both cases, it is not possible due to similar cost structures to distinguish in which facility should best be invested to satisfy the overall demand. A possible remedy to this situation is again to implement a suitable tie-breaking rule. For example, both operators could each invest the same amount. Alternatively, external costs that have not yet been taken into account might determine whether more investment is made at one location or at the other. However, if data with sufficient fluctuations in, e.g., the demand is available, such cases should actually not occur.

5. CONCLUSION

In this paper, we have extended the existing literature by providing a framework to analyze coupled markets of multiple energy sectors, as well as conditions for uniqueness of short- and long-run market equilibrium in such a setup. Our framework lays the foundation for addressing timely research questions on energy market coupling in the context of the transition towards a sustainable energy system. It paves the way to analyze timely questions with regard to, e.g., climate policy, regulation of network and energy pricing, and their effects on infrastructure planning, investment incentives, and market outcomes.

Our results show that multi-sector market models that account for multiple agents with different objectives require additional conditions to guarantee uniqueness of short- and long-run equilibrium compared to the single sector case. For the short-run, we derived sufficient conditions on the cost structures of production facilities that ex ante ensure uniqueness of equilibrium. Moreover, using illustrative examples, we proved that each of these conditions is indeed required to guarantee uniqueness in general. In addition, a straightforward interpretation of the circumstances that lead to multiplicities has been provided: multiplicities on the production level might arise if and only if variable production costs are the same for two distinct facilities. This understanding allows us to formulate suitable and transparent tie-breaking rules that select a certain equilibrium in case the presented conditions for short-run uniqueness are violated, as for instance when several facilities of the same technology are considered in the same bidding zone. One such possibility would be to select the equilibrium in which the facilities with similar variable production costs have equal capacity utilization.

For the long-run, we also derived sufficient conditions that guarantee uniqueness of equilibrium. However, these conditions can only be verified ex post and are fulfilled if and only if enough variability in the data is assured ex ante. It is therefore not recommendable to smooth out variability in data used to analyze specific applications. This result is consistent with the results obtained in similar contexts when sector coupling is not considered; see, e.g., Grimm, Schewe, et al. (2017). The decisive factor for multiplicities in the long-run are the total costs of production. In the case where multiplicities are caused by facilities with similar cost structures, tie-breaking rules might again be a natural choice to select a certain equilibrium, e.g., the one in which the size of these facilities is equal or the one where the

facilities are ideally located given the underlying network. However, if sufficient fluctuations in, e.g., the demand are present, such multiplicities are not to be expected.

Some of the limitations in our analysis could be addressed in future research to allow for implementation of more realistic applications. First of all, one limitation of our analysis is that the frequency of spot market trade is assumed to be the same for all sectors. Of course, the case of different trade frequencies would be of interest since it allows to consider the coupling of markets that trade products on spot markets at different frequencies; in Europe, e.g., spot markets for gas define daily reference prices compared to hourly or even more frequent trade in spot markets for electricity. Second, we mentioned the issue of multiplicities in case potential-based trade constraints are considered or storage operators are modelled explicitly as agents on the spot markets of a single sector. We referred to relevant literature that analyzes uniqueness in these contexts. However, in some timely multi-sector applications, it might be absolutely necessary to model these aspects; e.g., for the production, storage, and utilization of electric fuels or for nodal pricing as benchmark in coupled energy markets. Hence, for sector-coupling applications, a more in-depth understanding of how multiplicities arise in such cases and could be resolved would be desirable.

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APPENDIX A. NOTATION

Here, a complete overview of all sets, parameters, and variables used throughout the paper is provided; see Tables 1, 2, and 3.

TABLE 1. Technical and economic sets

Symbol	Explanation	Unit
T	Set of time periods	-
S	Set of sectors	-
Z_s	Set of bidding zones in sector $s \in S$	-
K_s	Set of all adjacent bidding zones with positive trade capacity in sector $s \in S$	-
G_z^{ex}	Set of existing sector-specific production facilities in bidding zone $z \in Z_s$	-
G_z^{new}	Set of candidate sector-specific production facilities in bidding zone $z \in Z_s$	-
G_z^{all}	Set of all sector-specific production facilities in bidding zone $z \in Z_s$	-
I_z^{ex}	Set of existing sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
I_z^{new}	Set of candidate sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
I_z^{all}	Set of all sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
O_z^{ex}	Set of existing sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
O_z^{new}	Set of candidate sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
O_z^{all}	Set of all sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
X^{ex}	Set of existing sector-coupling facilities	-
X^{new}	Set of candidate sector-coupling facilities	-
X^{all}	Set of all sector-coupling facilities	-

TABLE 2. Technical and economic parameters (*sector-dependent unit)

Symbol	Explanation	Unit
f_k^+	Maximum trade capacity of $k \in K_s$	*
f_k^-	Minimum trade capacity of $k \in K_s$	*
$P_{t,z}(\cdot)$	Inverse demand function of bidding zone $z \in Z_s$ in time period $t \in T$	€/*
$p_{t,z}$	Price of bidding zone $z \in Z_s$ in time period $t \in T$	€/*
c_g^{var}	Variable costs of sector-specific production facility $g \in G_z^{\text{all}}$	€/*
y_g^{ex}	Capacity of existing sector-specific production facility $g \in G_z^{\text{ex}}$	*
c_g^{inv}	Investment costs of candidate sector-specific production facility $g \in G_z^{\text{new}}$	€/*
$\alpha_{t,g}$	Availability of sector-specific production facility $g \in G_z^{\text{all}}$ in time period $t \in T$	-
y_i^{ex}	Capacity of existing sector-coupling facility $i \in I_z^{\text{ex}}$	*
c_i^{inv}	Investment costs of candidate sector-coupling facility $i \in I_z^{\text{new}}$	€/*
$\alpha_{t,i}$	Availability of sector-coupling facility $i \in I_z^{\text{all}}$ in time period $t \in T$	-
η_x	Efficiency of sector-coupling facility $x \in X^{\text{all}}$	-

TABLE 3. Technical and economic variables (*sector-dependent unit)

Symbol	Explanation	Unit
$f_{t,k}$	Trade on $k \in K_s$ in time period $t \in T$	*
$d_{t,z}$	Demand in bidding zone $z \in Z_s$ in time period $t \in T$	*
$y_{t,g}$	Production of sector-specific production facility $g \in G_z^{\text{all}}$ in time period $t \in T$	*
y_g^{new}	Installed capacity of candidate sector-specific production facility $g \in G_z^{\text{new}}$	*
$d_{t,o}$	Demand of sector-coupling facility $o \in O_z^{\text{all}}$ in time period $t \in T$	*
$y_{t,i}$	Production of sector-coupling facility $i \in I_z^{\text{all}}$ in time period $t \in T$	*
y_i^{new}	Installed capacity of candidate sector-coupling facility $i \in I_z^{\text{new}}$	*

TABLE 4. Two equilibria of (MCP) for the scenarios of Example B.1 (left) and Example B.2 (right)

	d_{z_1}	y_{g_1}	y_{g_2}		d_{z_1}	d_{z_2}	y_{g_1}	y_{g_2}	d_{x_1}	d_{x_2}	y_{x_1}	y_{x_2}
1	10	10	0	1	14	12	38	0	4	20	2	10
2	10	5	5	2	14	12	38	0	12	12	6	6

APPENDIX B. ILLUSTRATIVE EXAMPLES FOR THE SHORT-RUN

In the following, we present illustrative examples on how multiplicities arise in the short-run due to similar cost structures of different production facilities. All investment decisions are assumed to be already determined in the short-run. Hence, it is sufficient to consider only existing production facilities. Moreover, since the market equilibrium problem decouples over time for this case, it is sufficient to consider a single time period for demonstrating the underlying effects that cause multiplicities. For the ease of notation, we therefore drop the time index in the following examples. In addition, we denote by d_x and y_x the demand and production of the sector-coupling facility $x \in X^{\text{ex}}$. Furthermore, we assume without loss of generality that all facilities are fully available.

Example B.1 illustrates the case in which two production facilities of the same sector have the same variable production costs, i.e., Assumption 2 (a) is not satisfied. Due to the same cost structures, there is no way to distinguish between the two facilities and thus, multiple equilibria exist.

Example B.1. *Let one sector $S = \{s_1\}$ with one bidding zone $Z_{s_1} = \{z_1\}$ be given. The demand of this bidding zone is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$. Moreover, we assume that two producers $G_{z_1}^{\text{ex}} = \{g_1, g_2\}$ with the same variable production costs $c_{g_1}^{\text{var}} = c_{g_2}^{\text{var}} = 5$ and installed capacities $y_{g_1}^{\text{ex}} = 10$ and $y_{g_2}^{\text{ex}} = 8$ exist. Due to the same variable production costs, no distinction can be made between the two facilities. Therefore, it is not clear which of the producers best meets the demand. As a consequence, multiplicity of market equilibria results. Two of them are exemplarily listed in Table 4 (left).*

In addition to the case where, due to identical cost structures, no distinction can be made between sector-specific production facilities, there might also be the case where there is no way to distinguish between sector-coupling facilities. This occurs in particular if the efficiencies of sector-coupling facilities are not pairwise distinct; see Example B.2 for an illustration. The situation presented there violates Assumption 2 (b) and multiplicities arise from this violation.

Example B.2. *Let two sectors $S = \{s_1, s_2\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$ and $Z_{s_2} = \{z_2\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$ while the demand of bidding zone z_2 is determined by the inverse demand function $P_{z_2}(d_{z_2}) = 26 - 2d_{z_2}$. Moreover, we assume that one sector-specific producer produces in bidding zone z_1 and one in bidding zone z_2 , i.e., $G_{z_1}^{\text{ex}} = \{g_1\}$ and*

TABLE 5. Two equilibria of (MCP) for the scenario of Example B.3

	d_{z_1}	d_{z_2}	y_{g_1}	y_{g_2}	d_{x_1}	y_{x_1}
1	14	11	18	10	4	1
2	14	11	26	8	12	3

$G_{z_2}^{\text{ex}} = \{g_2\}$. The variable production costs are $c_{g_1}^{\text{var}} = 1$ and $c_{g_2}^{\text{var}} = 4$ with installed capacities $y_{g_1}^{\text{ex}} = 40$ and $y_{g_2}^{\text{ex}} = 10$. In addition, two sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2\}$ with the same efficiency $\eta_{x_1} = \eta_{x_2} = 0.5$ exist that both withdraw from sector s_1 and inject in sector s_2 . The capacities of these facilities are $y_{x_1}^{\text{ex}} = 30$ and $y_{x_2}^{\text{ex}} = 20$. Due to the same efficiency of the two sector-coupling facilities, there is no possibility to distinguish which of the operators of the two facilities best meets the demand in sector s_2 . Consequently, there exist multiple market equilibria; for two of them see Table 4 (right).

Moreover, there may exist instances in which it is not possible to distinguish between a sector-specific and a sector-coupling facility. Example B.3 depicts such a case. There, multiplicity of equilibria is caused by the fact that the variable production costs across the sectors are not—taking into account the sector-coupling facility’s efficiency—pairwise distinct, i.e., Assumption 2 (c) is not fulfilled.

Example B.3. Let two sectors $S = \{s_1, s_2\}$ with the same bidding zone structure as well as demand and sector-specific production data as in Example B.2 be given. In addition, there is one sector-coupling facility x_1 with efficiency $\eta_{x_1} = 0.25$ that withdraws from sector s_1 and injects in sector s_2 . The capacity of this facility is $y_{x_1}^{\text{ex}} = 10$. Due to the relation $\eta_{x_1} c_{g_2}^{\text{var}} = c_{g_1}^{\text{var}}$, there is no possibility to distinguish whether the operator of the sector-specific facility g_2 or of the sector-coupling facility x_1 best meets the demand in sector s_2 . Hence, there exist multiple market equilibria. Two of them are depicted in Table 5.

The latter situation of indistinguishable sector-specific and sector-coupling facilities can also arise across several sectors. Example B.4 presents an instance in which multiplicity of equilibria is triggered by the same variable production costs across two sectors taking into account the combined efficiency of two sequentially producing sector-coupling facilities, which contradicts Assumption 2 (d).

Example B.4. Let three sectors $S = \{s_1, s_2, s_3\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$, $Z_{s_2} = \{z_2\}$, and $Z_{s_3} = \{z_3\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$, the one of z_2 by $P_{z_2}(d_{z_2}) = 11.25 - d_{z_2}$, and the one of z_3 by $P_{z_3}(d_{z_3}) = 30.5 - 2d_{z_3}$. Moreover, we assume that one sector-specific producer produces in each bidding zone, i.e., $G_{z_1}^{\text{ex}} = \{g_1\}$, $G_{z_2}^{\text{ex}} = \{g_2\}$, and $G_{z_3}^{\text{ex}} = \{g_3\}$. The variable production costs are $c_{g_1}^{\text{var}} = 1$, $c_{g_2}^{\text{var}} = 0.75$, and $c_{g_3}^{\text{var}} = 2.5$ with installed capacities $y_{g_1}^{\text{ex}} = 50$, $y_{g_2}^{\text{ex}} = 10$, and $y_{g_3}^{\text{ex}} = 5$. In addition, two sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2\}$ with the efficiencies $\eta_{x_1} = 0.8$ and $\eta_{x_2} = 0.5$ exist. The facility x_1 withdraws from sector s_1 and injects in sector s_2 while the facility x_2 transfers the commodity of sector s_2 into the commodity of sector s_3 . The capacities of these facilities are $y_{x_1}^{\text{ex}} = 40$ and $y_{x_2}^{\text{ex}} = 30$. Due to the relation $\eta_{x_1} \eta_{x_2} c_{g_3}^{\text{var}} = c_{g_1}^{\text{var}}$, there is no possibility to decide whether the supply of the sector-coupling or of the sector-specific production facility is the best to meet the demand in sector s_3 . As a consequence, there exist multiple market equilibria; see Table 6 for two of them.

Finally, the case of indistinguishable sector-coupling facilities can also appear due to sequentially producing sector-coupling facilities. In Example B.5, multiplicities occur since the efficiency of the facility that transforms commodity two into commodity three is the same as the combined efficiency of the facilities that transform commodity two into commodity one and commodity one into commodity three. This situation violates Assumption 2 (e), which states that a chain of efficiencies is not allowed to meet another chain of efficiencies.

TABLE 6. Two equilibria of (MCP) for the scenario of Example B.4

	d_{z_1}	d_{z_2}	d_{z_3}	y_{g_1}	y_{g_2}	y_{g_3}	d_{x_1}	d_{x_2}	y_{x_1}	y_{x_2}
1	14	10	14	49	10	0	35	28	28	14
2	14	10	14	39	10	4	25	20	20	10

TABLE 7. Two equilibria of (MCP) for the scenario of Example B.5

	d_{z_1}	d_{z_2}	d_{z_3}	y_{g_1}	y_{g_2}	y_{g_3}	d_{x_1}	d_{x_2}	d_{x_3}	y_{x_1}	y_{x_2}	y_{x_3}
1	7	6	23.5	0	73.5	0	67.5	47	0	54	23.5	0
2	7	6	23.5	0	73.5	0	8.75	0	58.75	7	0	23.5

Example B.5. Let three sectors $S = \{s_1, s_2, s_3\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$, $Z_{s_2} = \{z_2\}$, and $Z_{s_3} = \{z_3\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 8.25 - d_{z_1}$, the one of z_2 by $P_{z_2}(d_{z_2}) = 7 - d_{z_2}$, and the one of z_3 by $P_{z_3}(d_{z_3}) = 26 - d_{z_3}$. Moreover, we assume that one sector-specific producer produces in each bidding zone, i.e., $G_{z_1}^{\text{ex}} = \{g_1\}$, $G_{z_2}^{\text{ex}} = \{g_2\}$, and $G_{z_3}^{\text{ex}} = \{g_3\}$. The variable production costs are $c_{g_1}^{\text{var}} = 8$, $c_{g_2}^{\text{var}} = 1$, and $c_{g_3}^{\text{var}} = 25$ with installed capacities $y_{g_1}^{\text{ex}} = 5$, $y_{g_2}^{\text{ex}} = 80$, and $y_{g_3}^{\text{ex}} = 20$. In addition, three sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2, x_3\}$ with the efficiencies $\eta_{x_1} = 0.8$, $\eta_{x_2} = 0.5$, and $\eta_{x_3} = 0.4$ exist. The facility x_1 withdraws from sector s_2 and injects in sector s_1 while the facility x_2 transfers the commodity of sector s_1 into the commodity of sector s_3 . Finally, the facility x_3 couples sector s_2 with sector s_3 , injecting in the latter sector. The capacities of the sector-coupling facilities are $y_{x_1}^{\text{ex}} = 100$, $y_{x_2}^{\text{ex}} = 120$, and $y_{x_3}^{\text{ex}} = 140$. Due to the relation $\eta_{x_1}\eta_{x_2} = \eta_{x_3}$, there is no possibility to decide which sector-coupling producer is best to meet the demand in sector s_3 . Thus, there exist multiple market equilibria. Two of them are stated in Table 7.

APPENDIX C. ILLUSTRATIVE EXAMPLES FOR THE LONG-RUN

In the following, we introduce examples that illustrate how multiplicities arise in the long-run. Also in the long-run, multiplicities are mainly caused by similar cost structures of different production facilities. Now, the total production costs and not only the variable production costs are decisive. For all examples, we assume without loss of generality that there exist only candidate facilities and that all facilities are fully available. We do not present examples to all possible violations of the nine conditions specified in Assumption 3 but choose two exemplary cases. The first example focuses on a single sector, the second one on multiple sectors.

The single sector case is treated in Example C.1. Since, in this example, both operators of the candidate sector-specific facilities produce in all time periods and additionally at the same average costs, there is no possibility to decide which of the operators should invest best and thus, multiple equilibria exist.

Example C.1. Given two time periods $T = \{t_1, t_2\}$. Moreover, let one sector $S = \{s_1\}$ with one bidding zone $Z_{s_1} = \{z_1\}$ be given. The inverse demand function of bidding zone z_1 in time period t_1 is determined by $P_{t_1, z_1} = 10 - d_{t_1, z_1}$, the one of time period t_2 by $P_{t_2, z_1} = 7.5 - d_{t_2, z_1}$. In addition, we assume that there are two candidate sector-specific facilities in bidding zone z_1 , i.e., $G_{z_1}^{\text{new}} = \{g_1, g_2\}$. The variable production costs are $c_{g_1}^{\text{var}} = 1$ and $c_{g_2}^{\text{var}} = 3$ and the investment costs $c_{g_1}^{\text{inv}} = 6$ and $c_{g_2}^{\text{inv}} = 2$. No sector-coupling facilities exist. Due to the fact that both operators of the sector-specific facilities produce always at the same time and at the same average costs, there is no possibility to distinguish which of the producers should invest best in order to meet the demand in sector s_1 . Hence, multiple market equilibria result; see Table 8 for two of them.

TABLE 8. Two equilibria of (MCP) for the scenario of Example C.1

	d_{t_1, z_1}	d_{t_2, z_1}	y_{t_1, g_1}	y_{t_1, g_2}	y_{t_2, g_1}	y_{t_2, g_2}	$y_{g_1}^{\text{new}}$	$y_{g_2}^{\text{new}}$
1	5	4.5	2.5	2.5	2.5	2	2.5	2.5
2	5	4.5	1	4	1	3.5	1	4

TABLE 9. Two equilibria of (MCP) for the scenario of Example C.2

	d_{t_1, z_1}	d_{t_1, z_2}	d_{t_2, z_1}	d_{t_2, z_2}	y_{t_1, g_1}	y_{t_1, g_2}	y_{t_2, g_1}	y_{t_2, g_2}	$y_{g_1}^{\text{new}}$	$y_{g_2}^{\text{new}}$
1	8	5	7	4.5	8.625	4.5	7.625	4	8.625	4.5
2	8	5	7	4.5	10.5	3	9.5	2.5	10.5	3

	d_{t_1, x_1}	d_{t_2, x_1}	y_{t_1, x_1}	y_{t_2, x_1}	$y_{x_1}^{\text{new}}$
1	0.625	0.625	0.5	0.5	0.5
2	2.5	2.5	2	2	2

For multiple sectors, Example C.2 provides an instance in which multiplicities occur in the long-run. Again, multiplicities result from the fact that both operators of the sector-coupling and the sector-specific facility of sector two produce always at the same time and in addition at the same average costs.

Example C.2. Given two time periods $T = \{t_1, t_2\}$. Furthermore, let two sectors $S = \{s_1, s_2\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$ and $Z_{s_2} = \{z_2\}$, be given. All sector-specific demand is characterized by the inverse demand functions $P_{t_1, z_1} = 10 - d_{t_1, z_1}$, $P_{t_2, z_1} = 8 - d_{t_2, z_1}$, $P_{t_1, z_2} = 10 - d_{t_1, z_2}$, and $P_{t_2, z_2} = 7.5 - d_{t_2, z_2}$. Moreover, we assume that there is one candidate sector-specific facility in each bidding zone, i.e., $G_{z_1}^{\text{new}} = \{g_1\}$ and $G_{z_2}^{\text{new}} = \{g_2\}$. The variable production costs are $c_{g_1}^{\text{var}} = 1$ and $c_{g_2}^{\text{var}} = 3$ and the investment costs are $c_{g_1}^{\text{inv}} = 1$ and $c_{g_2}^{\text{inv}} = 2$. In addition, one candidate sector-coupling facility exists that withdraws from sector s_1 and injects in sector s_2 , i.e., $X^{\text{new}} = \{x_1\}$. The efficiency of this facility is $\eta_{x_1} = 0.8$ and the investment costs are $c_{x_1}^{\text{inv}} = 4.25$. Since the operators of the sector-coupling and the sector-specific facility of sector two produce always at the same time and at the same average costs, it is not clear which of these producers best meets the demand in sector s_2 . Thus, there exist multiple market equilibria. Two of them are depicted in Table 9.

Preprint 2

Existence of Energy Market Equilibria with Convex and Nonconvex Players

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EXISTENCE OF ENERGY MARKET EQUILIBRIA WITH CONVEX AND NONCONVEX PLAYERS

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ABSTRACT. Motivated by examples from the energy sector, we consider a special type of market equilibrium problems (MEPs) involving players both with convex and nonconvex strategy spaces or objective functions. Such problems naturally occur in settings in which energy trading is combined with aspects of the actual energy transport through networks. We propose an algorithm that determines if an equilibrium of a given MEP exists and that computes a market equilibrium in case of existence. Moreover, we provide both a uniqueness and a non-existence result for MEPs that include players with a unique best response. Finally, we test the proposed algorithm on well-known energy market instances from the power and gas literature. There, nonconvexities—either due to integrality conditions or nonlinearities—mainly arise from considering the transmission system operator as an additional player who, e.g., switches lines or faces nonlinear physical laws. Our numerical results indicate that an equilibrium often exists, especially for the case of continuous nonconvexities in the context of gas market problems.

1. INTRODUCTION

Market equilibrium problems are an important mathematical tool to model many practically relevant applications such as energy markets for power or gas, auctions, or transport network planning. Usually, these problems consist of a number of rational players that compete for a set of goods, which they want to purchase to maximize their utility. In such situations one asks whether there exists a price for these goods so that the market clears and so that no player can improve her utility by unilaterally changing her decisions. Mathematically, rationality is typically modeled via optimization problems and a market equilibrium price then clears the market while all players choose an optimal solution. The main economic or mathematical questions are (i) whether such an equilibrium exists, (ii) whether it is unique, and (iii) how to compute it. In this paper, we address these three topics but focus on the algorithmic aspects.

The classic results ensure existence of market equilibria under suitable convexity assumptions; see, e.g., Wald (1951), Arrow and Debreu (1954), Gale (1955), McKenzie (1959), or Debreu (1962). Unfortunately, many real-world market equilibrium problems do not satisfy these assumptions. Our study is mainly motivated by practically relevant aspects of market equilibrium problems in energy—namely gas and power market equilibrium models on networks. In these settings, the set of players includes producers and consumers that are located at the nodes of the energy network as well as the transmission system operator (TSO), who acts as an arbitrageur and who also controls the network itself; see, e.g., Hobbs and Helman (2004), Gabriel and Smeers (2006) in Seeger (2006), Gabriel, Conejo, et al. (2012),

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Grimm, Schewe, et al. (2017), or Krebs et al. (2018). The main challenge regarding these equilibrium problems is when either the energy flow model is nonlinear or when controlling the network includes deciding on discrete switching variables. Both aspects lead to nonconvex player problems and, thus, the classic existence theory is not applicable anymore. See Scarf (1994) for the special case of integralities in a power market or Grimm, Grübel, et al. (2019) for the special case of gas market interaction on a network with nonconvex flow models.

There are different parts of the applied literature that tackle such nonconvex situations; see, e.g., Shapley and Shubik (1971), Leonard (1983), Bikhchandani, Ostroy, et al. (2002), and Bikhchandani and Ostroy (2006) for assignment problems, Bikhchandani and Mamer (1997) and Baldwin and Klemperer (2019) for general exchange economies with indivisibilities, O’Neill et al. (2005) and Guo et al. (2021) for discrete markets, Hatfield et al. (2013), Fleiner et al. (2019), and Hatfield et al. (2019) for trading networks, and Beato (1982), Brown et al. (1986), Bonnisseau and Cornet (1988), and Bonnisseau and Cornet (1990) for economies with increasing returns to scale. Very recently, Harks (2020) presented a unifying framework for many (possibly nonconvex) equilibrium problems including network tolls for transportation networks, indivisible item auctions, bilateral trade, or congestion control. This framework is based on Lagrangian duality and enables to characterize the existence of solutions to (possibly) nonconvex equilibrium problems by checking if a suitably chosen optimization problem (e.g., the overall welfare maximization problem in economic settings) has a zero duality gap.

The first main contribution of this paper is that we, based on the results of Harks (2020), derive an algorithm to decide the existence of a solution of the market equilibrium problem with convex and nonconvex player problems. If such an equilibrium exists, our algorithm computes it—otherwise, it indicates that no such equilibrium exists. To implement this algorithm, however, we need some specific knowledge about potential candidates for equilibrium prices. These equilibrium price candidates may be challenging to analyze in general but can be derived for specific instances. Hence, the second main contribution is that we detail this knowledge for two practically relevant equilibrium problems in energy: (i) a power market problem in which the TSO controls the underlying DC network by switching on or off DC power lines and (ii) a gas market problem in which the TSO’s model is nonconvex due to the inherent nonlinearity of gas flow models. By doing so, we consider two very different settings of nonconvex market equilibrium problems: one in which the nonconvexity is continuous but nonlinear and one in which the nonconvexity is due to the presence of integer variables. Thus, these studies nicely illustrate the broad applicability of our methods. We present a detailed computational study for both problems and discuss the reasons why equilibria exist or why not. One main conclusion of this computational study is that in the case in which the nonconvexity is due to integrality constraints, we show that no equilibrium exists for a large number of instances. On the other hand, we confirm the existence of an equilibrium for all solvable instances in the nonconvex but continuous case.

The remainder of the paper is organized as follows. In Section 2 we present the abstract model of the market equilibrium problem with convex and nonconvex players. Then, the existence and uniqueness of equilibria is studied in Section 3. The beginning of this section is based on the results by Harks (2020) and extends this work to obtain the algorithm that computes equilibria or proves that no such equilibria exist. Moreover, we provide a uniqueness result for the special case of players with unique best responses. In Section 4 we present the networked power and gas market equilibrium problems and derive theoretical results that are used in Section 5, where the developed algorithm is applied to these two cases. The paper

closes with some concluding remarks and some further topics of future research in Section 6.

2. THE MARKET EQUILIBRIUM PROBLEM

We consider a special type of market equilibrium problems with a finite set of players $i \in I$. We assume perfect competition, i.e., the players act as price takers and do not anticipate the impact of their own actions on prices. Formally, for an exogenously given price vectors $\pi \in \mathbb{R}^{n_\pi}$, every player $i \in I$ solves an optimization problem of the form

$$\min_{y_i} f_i(y_i, \pi) := c_i(y_i) + \pi^\top h_i(y_i) \quad \text{s.t.} \quad y_i \in Y_i, \quad (1)$$

where $y_i \in \mathbb{R}^{n_i}$ are the decision variables of player i , $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_\pi} \rightarrow \mathbb{R}$ is the objective function consisting of $c_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ and $h_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_\pi}$. Moreover, $Y_i \subseteq \mathbb{R}^{n_i}$ is the non-empty feasible set of player i .

Solving the market equilibrium problem in this context means to find market-clearing prices π , i.e., prices to which the best responses of all players exist and satisfy predefined market-clearing conditions that depend on the variables of all players. We model these market-clearing conditions as

$$\sum_{i \in I} h_i(y_i) = 0. \quad (2)$$

Thus, we consider the following market equilibrium problem:

$$\begin{aligned} \text{optimization problems of the players:} & \quad (1) \text{ for all } i \in I, \\ \text{market-clearing conditions:} & \quad (2). \end{aligned} \quad (\text{MEP})$$

A broad range of market equilibrium problems can be modeled this way. For instance, many economic applications, in which a large number of price-taking players purchase and sell goods at certain prices, fall into this category. In this context, $\pi^\top h_i(y_i)$ describes the part of the players' money either spent or gained by trading a specific amount of the goods. Moreover, the market-clearing conditions (2) ensure that the traded amounts are balanced, i.e., there is no excess demand or supply. Problems of this kind naturally arise in energy market modeling; see, e.g., Hobbs and Helman (2004) or Gabriel, Conejo, et al. (2012) and the references therein. For two examples from the field of energy markets see also Section 4, where we discuss them thoroughly. A detailed overview of applications from other fields can be found in, e.g., Harks (2020).

Although the existence of an equilibrium is not guaranteed in general if nonconvex players are part of the (MEP), it is well-known that in the case of existence, market equilibria correspond to welfare optima; cf. Part 1 of Theorem 2.3 in Harks (2020). Formally, the welfare problem is the optimization problem in which the sum of all players' objectives is minimized subject to the constraints of all players and the market-clearing conditions. This welfare optimization problem thus reads

$$\min_y \sum_{i \in I} c_i(y_i) \quad \text{s.t.} \quad y \in Y, \quad \sum_{i \in I} h_i(y_i) = 0. \quad (\text{WFP})$$

Here, we use $y := (y_i)_{i \in I} \in \mathbb{R}^{n_y}$ with $n_y := \sum_{i \in I} n_i$ as an abbreviation for the decision variables of all players $i \in I$. Analogously, let $Y := \times_{i \in I} Y_i$ denote the Cartesian product of the individual feasible sets.

Even if a solution to the welfare optimization problem (WFP) exists, this solution does not necessarily constitute a market equilibrium if nonconvexities are present in the optimization problems (1). However, if the optimization problems (1) are convex for all π , the existence of market equilibria is well-understood. In particular, there exists a market equilibrium of (MEP) if there exists a solution to the welfare

optimization problem (WFP). To see this note that, for each player i , the convexity of the objective function over the convex set Y_i yields

$$f_i(\lambda\tilde{y}_i + (1-\lambda)\hat{y}_i, \pi) \leq \lambda f_i(\tilde{y}_i, \pi) + (1-\lambda)f_i(\hat{y}_i, \pi)$$

for all $\tilde{y}_i, \hat{y}_i \in Y_i$, $\lambda \in [0, 1]$, and all $\pi \in \mathbb{R}^{n_\pi}$. Inserting the definition of the objective function and rearranging then yields

$$\begin{aligned} & c_i(\lambda\tilde{y}_i + (1-\lambda)\hat{y}_i) - \lambda c_i(\tilde{y}_i) - (1-\lambda)c_i(\hat{y}_i) \\ & \leq \pi^\top (\lambda h_i(\tilde{y}_i) + (1-\lambda)h_i(\hat{y}_i) - h_i(\lambda\tilde{y}_i + (1-\lambda)\hat{y}_i)). \end{aligned} \quad (3)$$

Note that the left-hand side is independent of π , and the latter can take any value in \mathbb{R}^{n_π} . Hence, we conclude that

$$\lambda h_i(\tilde{y}_i) + (1-\lambda)h_i(\hat{y}_i) = h_i(\lambda\tilde{y}_i + (1-\lambda)\hat{y}_i). \quad (4)$$

Since the right-hand side of (3) is zero, we see that c_i has to be convex on Y_i . Let \tilde{y}, \hat{y} be feasible for (WFP) and $\lambda \in [0, 1]$. The point $\lambda\tilde{y} + (1-\lambda)\hat{y}$ is also feasible since the set Y is convex and the relation (4) implies that $\sum_{i \in I} h_i(\lambda\tilde{y} + (1-\lambda)\hat{y}) = 0$ holds. Hence, the optimization problem (WFP) is convex. Furthermore, suppose that Slater's condition holds for (WFP). If there exists a solution to (WFP), then this solution together with the multipliers of the market-clearing conditions as prices is a market equilibrium. In the presence of nonconvex players, the existence of a market equilibrium to (MEP) is not guaranteed in general; see, e.g., Example 5.1 and Example 5.2. However, in many applications there is at least one player—or possibly more—whose optimization problem is not convex.

3. EXISTENCE OF EQUILIBRIA

In this section we analyze how solutions of the welfare optimization problem (WFP) can be used to either find a market equilibrium or to determine that no market equilibria exist. To this end, we build on results from Harks (2020) to observe that market equilibria of (MEP) are equivalent to primal-dual solution pairs of the corresponding welfare problem with zero duality gap and provide some practical consequences of this result. Among those are the fact that players with unique best responses to given market prices also have a unique optimal strategy over all market equilibria (if any exists). The close relation between solutions of the welfare problem and market equilibria motivates to first compute a solution y^* of the welfare problem and then try to find suitable market prices π^* such that (y^*, π^*) is a market equilibrium. We show that under certain technical assumptions, which are satisfied in the applications presented in Section 4, it suffices to check one critical price to either obtain a market equilibrium or to know that none can exist. This result is then formalized in an algorithm, which we use in Section 5 to compute market equilibria of nonconvex energy market models.

If we denote the Lagrangian of the welfare optimization problem (WFP) by

$$L(y, \pi) := \sum_{i \in I} (c_i(y_i) + \pi^\top h_i(y_i)),$$

the corresponding Lagrangian dual problem is given by

$$\sup_{\pi \in \mathbb{R}^{n_\pi}} d(\pi), \quad (5)$$

where $d(\pi) := \inf_{y \in Y} L(y, \pi)$. Due to weak duality we always have the relation

$$\inf_{y \in Y} \left\{ \sum_{i \in I} c_i(y_i) : y \in Y, \sum_{i \in I} h_i(y_i) = 0 \right\} \geq \sup_{\pi \in \mathbb{R}^{n_\pi}} d(\pi).$$

The problem has zero duality gap if there exist globally optimal solutions y^* of the primal problem and π^* of the dual problem with the same objective function value.

However, the duality gap of the welfare problem can be positive in the presence of nonconvexities. On the other hand, Part 1 of Theorem 2.3 in Harks (2020) states that a zero duality gap is equivalent to the existence of a market equilibrium. In our setting, this result reads as follows.

Theorem 3.1 (See Part 1 of Theorem 2.3 in Harks (2020)). *The pair (y^*, π^*) is a market equilibrium of (MEP) if and only if y^* and π^* are solutions of the welfare optimization problem (WFP) and the corresponding dual problem (5), respectively, with zero duality gap.*

This result has several immediate consequences:

Corollary 3.2. (a) *If (y^*, π^*) is a market equilibrium of (MEP), then y^* is a global solution of the welfare problem (WFP).*
 (b) *If (y^*, π^*) is a market equilibrium of (MEP), then (y, π^*) is a market equilibrium of (MEP) for all global solutions y of the welfare problem.*
 (c) *If (y^*, π^*) and $(\hat{y}, \hat{\pi})$ are two market equilibria of (MEP), then so are $(y^*, \hat{\pi})$ and (\hat{y}, π^*) .*
 (d) *If y^* is a global solution of the welfare problem (WFP), for which there exists no π such that (y^*, π) is a market equilibrium of (MEP), then the market equilibrium problem (MEP) has no solution.*

Part (a) of the corollary ensures that only solutions y^* of the welfare problem are candidates for a market equilibrium. Part (d) states that there does not exist any market equilibrium at all, if we find a global solution y^* of the welfare problem, which is not a market equilibrium for all π . In general, neither y^* nor π^* have to be unique in a market equilibrium and Parts (b) and (c) state that we can mix and match different solutions.

In some applications, we know for a subset $S \subseteq I$ of the players that, for all possible π , their optimization problem (1) has at most one solution. This means that these players have a unique best response if it exists at all. This is, e.g., the case for players with convex feasible sets Y_i and strictly convex objective functions $y_i \mapsto f_i(y_i, \pi)$. Those players then also have a unique optimal strategy in all market equilibria and multiplicity in solutions of the welfare problem immediately implies that no market equilibrium exists. For a graphic illustration of this effect see Figure 1.

Corollary 3.3. *Let $S \subseteq I$ be the set of players with unique best responses for all price vectors $\pi \in \mathbb{R}^{n_\pi}$.*

- (a) *If (y^*, π^*) and $(\hat{y}, \hat{\pi})$ are two market equilibria of (MEP), then $y_S^* = \hat{y}_S$.*
- (b) *If y^* and \hat{y} are two solutions of the welfare problem (WFP) with $y_S^* \neq \hat{y}_S$, then the market equilibrium problem (MEP) does not have a solution.*

Proof. (a) By Corollary 3.2 (c) we know that (\hat{y}, π^*) is also a market equilibrium and thus both y_i^* and \hat{y}_i are solutions of the optimization problem (1) of player i with price π^* . For all players $i \in S$ this solution is unique, i.e., $y_S^* = \hat{y}_S$ holds.

(b) If the market equilibrium problem would have a solution for some price vector π , then both (y^*, π) and (\hat{y}, π) would be market equilibria, which contradicts (a). \square

Note, however, that the market equilibrium problem can have more than one solution as long as the solutions differ only in the prices or in the strategies of the players in $I \setminus S$, who do not have unique best responses to given prices.

Going back to Corollary 3.2, Part (a) motivates the following approach to compute market equilibria of (MEP): First, we compute a global solution y^* of the welfare problem (WFP). Then, we find a price vector π^* such that for all players $i \in I$, the

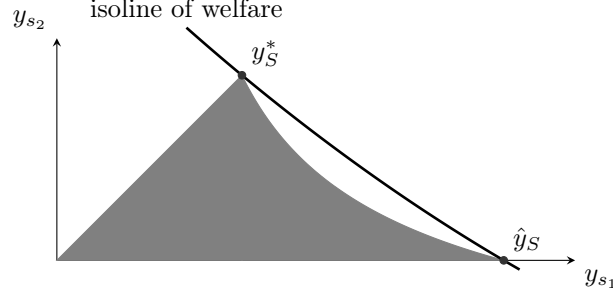


FIGURE 1. Illustration of Corollary 3.3 (based on Figures 2 and 3 of Grimm, Grübel, et al. (2019); see also Example 5.1). The nonconvexity of the feasible set of the welfare problem (WFP) (gray) leads to multiple welfare optima y_S^* and \hat{y}_S that differ in the decisions of the strictly convex players. Thus, no market equilibrium to (MEP) exists.

vector y_i^* is a solution of the player's optimization problem (1) for the given price vector π^* . To determine such an equilibrium price π^* , let $\Pi(y^*) \subseteq \mathbb{R}^{n_\pi}$ be a set that includes all market equilibrium prices, i.e., it has the property

$$(y^*, \pi^*) \text{ is a market equilibrium of (MEP)} \implies \pi^* \in \Pi(y^*). \quad (6)$$

Given a welfare solution y^* and such a candidate set $\Pi(y^*)$, only prices $\pi \in \Pi(y^*)$ can be equilibrium prices. Thus, we can assume that $\Pi(y^*) \neq \emptyset$, since otherwise no market equilibrium exists.

For the forthcoming analysis, we assume that we can compute an enclosing box of $\Pi(y^*)$ which takes the form

$$\{\pi \in \mathbb{R}^{n_\pi} : \pi_k^- \leq \pi_k \leq \pi_k^+ \text{ for all } k \in \{1, \dots, n_\pi\}\},$$

where the upper and lower bounds belong to the extended real line. We now show that it is possible in some instances to reduce the study of the existence of a market equilibrium to checking whether a welfare optimal solution and a particular price constitute a market equilibrium.

Theorem 3.4. *Let y^* be a solution of the welfare problem (WFP) and let $\Pi(y^*) \neq \emptyset$ be a set satisfying Condition (6). Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:*

- (a) $\pi_k^- = \pi_k^+$,
- (b) $\pi_k^+ < \infty$ and $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$,
- (c) $\pi_k^- > -\infty$ and $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$,
- (d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$, and $(h_i(y_i^*))_k = (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$.

Then, there exists a market equilibrium of (MEP) if and only if $(y^, \hat{\pi})$ is a market equilibrium, where the critical price $\hat{\pi}$ is defined as*

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

Proof. If $(y^*, \hat{\pi})$ is a market equilibrium, then obviously one exists. So let us assume that $(y^*, \hat{\pi})$ is not a market equilibrium, but that there exists a market

equilibrium (y^*, π^*) . Then, $\pi^* \in \Pi(y^*)$ and there exists a player $i \in I$ and $\hat{y}_i \in Y_i$ such that

$$f_i(y_i^*, \pi^*) \leq f_i(\hat{y}_i, \pi^*) \quad \text{but} \quad f_i(\hat{y}_i, \hat{\pi}) < f_i(y_i^*, \hat{\pi}).$$

Plugging in the definition of the objective functions yields

$$c_i(y_i^*) - c_i(\hat{y}_i) + (\pi^*)^\top (h_i(y_i^*) - h_i(\hat{y}_i)) \leq 0$$

and

$$c_i(\hat{y}_i) - c_i(y_i^*) + \hat{\pi}^\top (h_i(\hat{y}_i) - h_i(y_i^*)) < 0.$$

Putting both inequalities together and using the assumptions then yields

$$0 > (\pi^* - \hat{\pi})^\top (h_i(y_i^*) - h_i(\hat{y}_i)) = \sum_{k=1}^{n_\pi} (\pi^* - \hat{\pi})_k (h_i(y_i^*) - h_i(\hat{y}_i))_k.$$

Let us show that each term of the sum is non-negative. For all indices k such that either (a) or (d) holds, the respective term is zero. It is easy to see that the sign conditions in (b) and (c) ensure that the product $(\pi^* - \hat{\pi})_k (h_i(y_i^*) - h_i(\hat{y}_i))_k$ is non-negative. Hence, the sum is non-negative and we reach a contradiction. Therefore, either $(y^*, \hat{\pi})$ is a market equilibrium or there is no market equilibrium. \square

Our approach relies on exploiting the interplay between the structural properties of the players' problems and the set of admissible prices $\Pi(y^*)$ for a given solution y^* of the welfare problem. This is best understood when looking at the definition of the critical price $\hat{\pi}$. Looking at Case (b), one sees that if all players contribute in their minimum way to the market-clearing conditions, then it is sufficient to test for the upper bound as an equilibrium price candidate for this component. If, analogously, all players reach their maximum possible contribution for the market-clearing conditions, which is Case (c), then it is sufficient to check whether the lower bound for the component is an equilibrium price. Such upper and lower bounds might implicitly result from (necessary) optimality conditions of the players.

Applications fulfilling the properties stated in Theorem 3.4 arise, e.g., in transportation networks. For an application in the context of energy markets, in which all the stated properties are satisfied, see Section 4. Since these applications are often formulated in terms of maximization problems, we provide the analogue of Theorem 3.4 for maximization problems in Appendix A.

In general, one wants to choose the candidate set $\Pi(y^*)$ as small as possible in order to satisfy the conditions for Theorem 3.4. However, one cannot choose $\Pi(y^*)$ arbitrarily small since Condition (6) needs to be satisfied, i.e., $\Pi(y^*)$ needs to include all market equilibrium prices. A straightforward approach to construct the set $\Pi(y^*)$ is to exploit necessary optimality conditions of each player's optimization problem. If (y^*, π) is a market equilibrium, then for all player $i \in I$, y_i^* is a solution of optimization problem (1) with a price vector π and thus has to satisfy the necessary optimality conditions for (1). These optimality conditions evaluated at y_i^* provide constraints for possible market equilibrium prices π . Since we impose only very few assumptions on the optimization problems (1), different types of necessary optimality conditions might be needed for different classes of players. Fortunately, mixing varying types of necessary optimality conditions is not a problem here.

Remark 3.5. Consider a player $i \in I$. If the feasible set Y_i is given by standard constraints, e.g., $Y_i = \{y_i : g_i(y_i) \leq 0\}$, if all functions c_i , h_i , and g_i are continuously differentiable, and if a constraint qualification for Y_i is satisfied at y_i^* , then the KKT conditions for (1) are necessary. Thus, only prices π , for which there exist multipliers μ_i with

$$0 = \nabla c_i(y_i^*) + \nabla h_i(y_i^*)\pi + \nabla g_i(y_i^*)\mu_i, \quad 0 \leq \mu_i \perp g_i(y_i^*) \leq 0 \quad (7)$$

can be market equilibrium prices. If, additionally, $y_i \mapsto f_i(y_i, \pi)$ and g_i are convex functions, the KKT conditions are not only necessary but also sufficient optimality conditions for (1).

For other players, using the KKT conditions as necessary optimality conditions might not be possible, e.g., because they have discrete decision variables, their optimization problem is not differentiable, or their feasible set does not satisfy a constraint qualification. These players can either be ignored in the definition of $\Pi(y^*)$ or one can use alternative optimality conditions to generate conditions on market equilibrium prices π . For example, the textbooks by Clarke (1990), Luo et al. (1996), Mordukhovich (2018), Rockafellar (1970), and Rockafellar and Wets (1998) provide optimality conditions based on subdifferentials and variational analysis, which can be used in the presence of nondifferentiable or degenerate constraints.

In situations, in which the KKT conditions (7) of certain players are necessary and sufficient and used to obtain the candidate set $\Pi(y^*)$, it is possible to weaken the assumptions of Theorem 3.4. Note that this modification only works in case $\hat{\pi} \in \Pi(y^*)$, which is a nontrivial assumption, because $\hat{\pi}$ is per definition a vertex of an enclosing box of $\Pi(y^*)$. However, in the application considered in Section 4, this assumption will indeed be fulfilled.

Corollary 3.6. *Let y^* be a solution of the welfare problem (WFP). Moreover, let $C \subseteq I$ denote the subset of players, for which the KKT conditions (7) are necessary and sufficient optimality conditions and choose the candidate set $\Pi(y^*)$ such that Condition (6) as well as the KKT conditions of all players $i \in C$ are satisfied, i.e.,*

$$\Pi(y^*) \subseteq \{\pi \in \mathbb{R}^{n_\pi} : \text{for all } i \in C \text{ exists } \mu_i \text{ such that (7) holds}\}.$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

- (a) $\pi_k^- = \pi_k^+$,
- (b) $\pi_k^+ < \infty$ and $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$,
- (c) $\pi_k^- > -\infty$ and $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$,
- (d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ and $(h_i(y_i^*))_k = (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$.

Now, let the critical price $\hat{\pi}$ be defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

If the critical price satisfies $\hat{\pi} \in \Pi(y^*)$, then there exists a market equilibrium of (MEP) if and only if $(y^*, \hat{\pi})$ is a market equilibrium.

Proof. If $(y^*, \hat{\pi})$ is a market equilibrium, then obviously one exists. So let us assume that $(y^*, \hat{\pi})$ is not a market equilibrium. Due to $\hat{\pi} \in \Pi(y^*)$, the KKT conditions of all players $i \in C$ are satisfied, i.e., y_i^* is a best response to $\hat{\pi}$ for all players $i \in C$. Thus, if $(y^*, \hat{\pi})$ is not a market equilibrium, this has to be due to one of the players $i \notin C$. For those players, one obtains a contradiction by applying the same argument as in the proof of Theorem 3.4. \square

Finally, we utilize the previously derived results to formally state Algorithm 1, which terminates either with a market equilibrium or with the information that no market equilibrium exists. The presented algorithm is based on Theorem 3.4. Note that for the situation described in Corollary 3.6, the algorithm remains the same except for Line 6, where it suffices to check whether y_i^* is a best response to the price vector $\hat{\pi}$ for all players $i \in I \setminus C$.

Algorithm 1: Deciding the existence of an equilibrium of (MEP) and computing an equilibrium in case of existence

Input: Market equilibrium problem (MEP)

- 1 Compute a global solution y^* of the welfare optimization problem (WFP).
- 2 **if** the welfare optimization problem (WFP) does not have a solution **then**
- 3 | **return** “No market equilibrium exists.”
- 4 **else**
- 5 | Define the critical price vector $\hat{\pi}$ as in Theorem 3.4.
- 6 | **if** y_i^* is a best response to the price vector $\hat{\pi}$ for all players $i \in I$ **then**
- 7 | | **return** $(y^*, \hat{\pi})$ is a market equilibrium.
- 8 | **else**
- 9 | | **return** “No market equilibrium exists.”
- 10 | **end**
- 11 **end**

- Remark 3.7.**
- (a) In Lines 1 and 6, we assume that it is possible to solve the potentially nonconvex problems (WFP) and (1) to global optimality.
 - (b) If the solver computing the global solution of (WFP) in Line 1 additionally provides dual variables such that strong duality holds, then a market equilibrium exists and the dual variables associated to the market-clearing conditions are market equilibrium prices. However, there is no guarantee that the solver is able to do so; see, e.g., Section 5.3 in the *BARON* manual by Sahinidis (2021).
 - (c) To execute Line 5, Algorithm 1 relies on the existence of a procedure to compute the set $\Pi(y^*)$ for any given welfare optimal solution y^* . Please see Section 4, where we present such procedure for an application.

Theorem 3.8. *If the assumptions of Theorem 3.4 are satisfied, then Algorithm 1 terminates correctly with either a market equilibrium of (MEP) or with the information that such an equilibrium does not exist.*

Proof. Follows from Theorem 3.4 and Corollary 3.2. □

4. APPLICATIONS IN ENERGY MARKET MODELING

In this section, we consider energy networks modeled as graphs $G = (V, A)$ and assume that the graph G is directed and weakly connected. The node set V can further be split in the set $V_- \subset V$ of consumer locations, the set $V_+ \subset V$ of producer locations, and the set $V_0 \subset V$ of so-called inner nodes. For the ease of presentation, we assume that these three sets are disjoint and that $V_- \cup V_+ \cup V_0 = V$ holds. A possible approach to handle nodes, where both a producer and a consumer are located, is discussed in Section 5. In the market model, there are three types of players: producers, consumers, and the transmission system operator (TSO). For the sake of simplicity, we assume that each consumer node can be identified with a single consumer and each producer node with a single producer.

Consider the consumer located at node $u \in V_-$, let π_u be the market price at this node, and let $P_u(\cdot)$ be the consumer’s inverse demand function. The consumer maximizes his surplus by choosing his demand d_u as a solution of

$$\max_{d_u} \int_0^{d_u} P_u(t) dt - \pi_u d_u \quad \text{s.t.} \quad d_u \geq 0. \quad (8)$$

For the producer located at $u \in V_+$, let π_u again be the market price at this node and let $c_u(\cdot)$ denote the variable cost of production. The producer maximizes her

profits by choosing her production level y_u within her production capacity $\bar{y}_u > 0$ as a solution of

$$\max_{y_u} \pi_u y_u - c_u(y_u) \quad \text{s.t.} \quad \bar{y}_u \geq y_u \geq 0. \quad (9)$$

The TSO is responsible for operating the network and his goal is to maximize congestion rents by routing as much of the commodity from low-price to high-price areas. For all arcs $a \in A$, let q_a denote the flow along that arc and let $q := (q_a)_{a \in A}$ be the vector of all flows. Besides the flow q , the TSO can have an additional decision variable x to describe choices such as switching on or off a power line in the power network or activating a compressor station in the gas network. In general, the components of x can be continuous or integer. The optimization problem of the TSO is then given by

$$\max_{q,x} \sum_{u \in V_- \cup V_+} \pi_u \left(\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a \right) - c^t(q, x) \quad (10a)$$

$$\text{s.t.} \quad \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a \geq 0 \quad \text{for all } u \in V_-, \quad (10b)$$

$$\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a \leq 0 \quad \text{for all } u \in V_+, \quad (10c)$$

$$\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a \geq -\bar{y}_u \quad \text{for all } u \in V_+, \quad (10d)$$

$$\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = 0 \quad \text{for all } u \in V_0, \quad (10e)$$

$$F(q, x) \geq 0. \quad (10f)$$

Here, $c^t(q, x)$ describes the transportation costs and $\delta^{\text{in}}(u)$ ($\delta^{\text{out}}(u)$) denote the sets of incoming (outgoing) arcs at node u . Finally, the mapping $F(q, x)$ summarizes the, potentially nonconvex, network-related physical and technical constraints. The first three constraints make sure that the net flow can only be positive at consumer nodes, negative at producer nodes, and that it has to be zero at inner nodes. Constraints (10d) ensure that at each supply node, the TSO does not obtain more of the respective energy carrier than the nodal capacity allows for, i.e., the TSO is informed about the capacities of all production facilities. Note that this is consistent with the assumption of perfect competition.

The model is completed by the nodal market-clearing conditions

$$\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = d_u \quad \text{for all } u \in V_-, \quad (11a)$$

$$\sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = -y_u \quad \text{for all } u \in V_+, \quad (11b)$$

which ensure that production, consumption, and the in- and outgoing flows are balanced at every node of the network.

The complete energy market equilibrium problem is thus given by

$$\begin{aligned} \text{consumers:} & \quad (8) \text{ for all } u \in V_-, \\ \text{producers:} & \quad (9) \text{ for all } u \in V_+, \\ \text{TSO:} & \quad (10), \\ \text{market-clearing conditions:} & \quad (11). \end{aligned} \quad (\text{MEP-E})$$

Since it has the form discussed in Sections 2 and 3, we know that market equilibria $(d^*, y^*, q^*, x^*, \pi^*)$ are related to solutions of the welfare maximization problem

$$\begin{aligned}
 \max_{d, y, q, x} \quad & \sum_{u \in V_-} \int_0^{d_u} P_u(t) dt - \sum_{u \in V_+} c_u(y_u) - c^t(q, x) \\
 \text{s.t.} \quad & \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = 0 \quad \text{for all } u \in V_0, \\
 & \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = d_u \quad \text{for all } u \in V_-, \\
 & \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = -y_u \quad \text{for all } u \in V_+, \\
 & F(q, x) \geq 0, \quad d \geq 0, \quad \bar{y} \geq y \geq 0.
 \end{aligned} \tag{WFP-E}$$

In the remainder of this section, we show that the conditions from Corollary A.2 are satisfied under standard assumptions on the market equilibrium problem (MEP-E). Thus, we can use Algorithm 1 to decide on the existence of equilibria and to compute an equilibrium if one exists. The imposed standard assumptions read as follows.

- Assumption 1.** (1) *The inverse demand functions $P_u(\cdot)$ are continuous and strictly decreasing for all $u \in V_-$.*
 (2) *The variable cost functions $c_u(\cdot)$ are monotonically increasing with $c_u(0) = 0$, convex, and continuously differentiable for all $u \in V_+$.*

These assumptions ensure that the producers and consumers have concave maximization problems subject to linear constraints. Consequently, any solution of (8) and (9) is characterized by the respective KKT conditions. Exploiting this observation, we can prove that Corollary A.2 is applicable.

Theorem 4.1. *Suppose Assumption 1 holds. Let (d^*, y^*, q^*, x^*) be a solution of the welfare problem (WFP-E) and define $\hat{\pi}$ as*

$$\hat{\pi}_u := \begin{cases} P_u(d_u^*), & \text{if } u \in V_-, \\ c'_u(y_u^*), & \text{if } u \in V_+. \end{cases}$$

Then, either $(d^, y^*, q^*, x^*, \hat{\pi})$ is a market equilibrium of (MEP-E), or there is no market equilibrium.*

Proof. We use the KKT conditions of the producers and consumers to define the set $\Pi(d^*, y^*, q^*, x^*)$. For a consumer located at $u \in V_-$, these KKT conditions can be reduced to

$$0 \leq d_u^* \perp \pi_u \geq P_u(d_u^*).$$

For a producer located at $u \in V_+$, the KKT conditions read

$$\pi_u - c'_u(y_u^*) + \beta_u^- - \beta_u^+ = 0, \quad 0 \leq \beta_u^- \perp y_u^* \geq 0, \quad 0 \leq \beta_u^+ \perp \bar{y}_u - y_u^* \geq 0,$$

where β_u^- and β_u^+ denote the corresponding dual variables.

All candidates for market equilibrium prices are thus elements of the set $\Pi(d^*, y^*, q^*, x^*)$ defined by

$$\left\{ \pi \in \mathbb{R}^{|V_+ \cup V_-|} : \pi_u \in \begin{cases} \{P_u(d_u^*)\}, & \text{if } u \in V_-, d_u^* > 0, \\ [P_u(d_u^*), \infty), & \text{if } u \in V_-, d_u^* = 0, \\ \{c'_u(y_u^*)\}, & \text{if } u \in V_+, \bar{y}_u > y_u^* > 0, \\ (-\infty, c'_u(y_u^*)], & \text{if } u \in V_+, y_u^* = 0, \\ [c'_u(y_u^*), +\infty), & \text{if } u \in V_+, y_u^* = \bar{y}_u, \end{cases} \right\}.$$

Due to the simple structure of $\Pi(d^*, y^*, q^*, x^*)$, the critical price $\hat{\pi}$ can be stated explicitly and satisfies $\hat{\pi} \in \Pi(d^*, y^*, q^*, x^*)$. By definition of $\Pi(d^*, y^*, q^*, x^*)$, d^* thus is the consumers' best response to $\hat{\pi}$ and y^* is the producers' best response to $\hat{\pi}$. Consequently, it remains to show that Cases (a)–(d) of Corollary A.2 are fulfilled for the TSO.

For all production nodes $u \in V_+$ with positive but not binding production level and for all consumption nodes $u \in V_-$ with positive demand, there is exactly one candidate for an equilibrium price in $\Pi(d^*, y^*, q^*, x^*)$, namely $\hat{\pi}_u$. Hence, we are in Case (a) of Corollary A.2.

For all demand nodes $u \in V_-$ with zero demand, we have

$$\begin{aligned} (h_{\text{TSO}}(q^*))_u &= \sum_{a \in \delta^{\text{in}}(u)} q_a^* - \sum_{a \in \delta^{\text{out}}(u)} q_a^* \\ &= d_u^* = 0 \leq \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = (h_{\text{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite minimum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (c) of Corollary A.2 are satisfied.

For all supply nodes $u \in V_+$ with zero production, we have

$$\begin{aligned} (h_{\text{TSO}}(q^*))_u &= \sum_{a \in \delta^{\text{in}}(u)} q_a^* - \sum_{a \in \delta^{\text{out}}(u)} q_a^* \\ &= -y_u^* = 0 \geq \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = (h_{\text{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite maximum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (b) of Corollary A.2 are satisfied.

For supply nodes $u \in V_+$ at full capacity, we have

$$\begin{aligned} (h_{\text{TSO}}(q^*))_u &= \sum_{a \in \delta^{\text{in}}(u)} q_a^* - \sum_{a \in \delta^{\text{out}}(u)} q_a^* \\ &= -y_u^* = -\bar{y}_u \leq \sum_{a \in \delta^{\text{in}}(u)} q_a - \sum_{a \in \delta^{\text{out}}(u)} q_a = (h_{\text{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite minimum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (c) of Corollary A.2 are satisfied.

Consequently, all conditions of Corollary A.2 are satisfied and the claim follows. \square

In general, choosing a sufficiently tight superset of all possible equilibrium prices and computing the critical price can be difficult. However, in this application, we get the critical price $\hat{\pi}$ “for free” once a solution (d^*, y^*, q^*, x^*) of the welfare problem (WFP-E) is known.

Additionally, since $\Pi(d^*, y^*, q^*, x^*)$ is defined via the KKT conditions of the consumers and producers, which are necessary and sufficient under Assumption 1, and since $\hat{\pi} \in \Pi(d^*, y^*, q^*, x^*)$, we immediately know that d_u^* and y_u^* are best responses of all consumers $u \in V_-$ and all producers $u \in V_+$ to the prices $\hat{\pi}$. To check if $(d^*, y^*, q^*, x^*, \hat{\pi})$ is a market equilibrium, one thus only has to verify that (q^*, x^*) is a best response of the TSO to the prices $\hat{\pi}$.

5. COMPUTATIONAL STUDY

We now consider in more detail two applications in energy market modeling that fit into the framework described above. We start with a detailed description of the two applications in Section 5.1 and continue with giving information about the computational setup and our test instances in Section 5.2. Afterward, we present the numerical results for Algorithm 1; first for the gas application (Section 5.3) and then for the power application (Section 5.4). The results are discussed and insights are given on the conditions leading to non-existence of equilibria.

5.1. The Optimization Problems of the TSOs. We examine as applications the case of a TSO operating a gas network under nonlinear stationary gas flow equations and the case of a TSO switching DC lines in a DC power network. For each application we provide an instance for which the duality gap of the welfare problem is nonzero, i.e., for which no market equilibrium exists. In order to establish non-existence of an equilibrium for the gas flow instance, we apply Corollary 3.3. For the DC line switching instance, Algorithm 1 is applied to determine that no market equilibrium exists.

We start with describing the optimization problems (10) of the respective TSOs, focusing mostly on the transportation costs $c^t(q, x)$ and the network constraints $F(q, x)$. The optimization problems of the consumers (8) and producers (9) stay the same as in Section 4. An overview of all technical and economic parameters and variables together with the respective units used in this section can be found in Table 3 located in the Appendix B.

5.1.1. Gas Flow. This application is taken from Grimm, Grübel, et al. (2019). We choose this application to study in particular the impact of continuous nonconvexities on the existence of an equilibrium. The network-related physical constraints are given by the following model of stationary gas physics:

$$\begin{aligned} p_u^2 - p_v^2 &= \Lambda_a q_a |q_a|, & a &= (u, v) \in A, \\ p_u^- &\leq p_u \leq p_u^+, & u &\in V, \\ q_a^- &\leq q_a \leq q_a^+, & a &\in A. \end{aligned}$$

Here, the gas flow through the pipes is determined by the so-called Weymouth equation; see, e.g., the chapter by Fügenschuh et al. (2015) in Koch et al. (2015) for more information on this topic. This equation links the flow q_a on an arc to the pressure drop ($p_u^2 - p_v^2$) over this arc in a nonlinear way. Finally, nodal pressure and flow bounds are imposed to, e.g., guarantee technical and contractual requirements. For more information on the general setup see Grimm, Grübel, et al. (2019).

The transportation costs are assumed to increase quadratically with the flow, i.e., we have

$$c^t(q) = \sum_{a \in A} \alpha q_a^2.$$

Next, we present an instance in which no market equilibrium exists for the described gas application of (MEP-E). To simplify the presentation, we set the transportation costs to zero for now.

Example 5.1 (Non-existence of Equilibria). *For the instance depicted in Figure 2, Grimm, Grübel, et al. (2019) show that exactly two welfare maximal solutions exist, namely*

$$\begin{aligned} d_2 &= 1 + \sqrt{2}, & d_3 &= 0, & y_1 &= 1 + \sqrt{2}, & q_{1,2} &= \sqrt{2}, & q_{1,3} &= 1, & q_{2,3} &= -1, \\ p_1 &= \sqrt{2}, & p_2 &= 0, & p_3 &= 1, \end{aligned}$$

and

$$\bar{d}_2 = \bar{d}_3 = 1, \quad \bar{y}_1 = 2, \quad \bar{q}_{1,2} = \bar{q}_{1,3} = 1, \quad \bar{q}_{2,3} = 0, \quad \bar{p}_1 = \sqrt{2}, \quad \bar{p}_2 = \bar{p}_3 = 1,$$

which differ in demand and generation. Hence, by Corollary 3.3, we know that there is no market equilibrium for this instance. For an illustration of the given example see Figure 1, in which the demand at node 2 is plotted on the abscissa and the demand at node 3 on the ordinate.

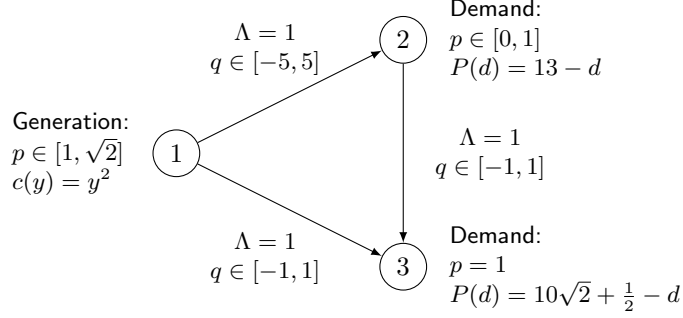


FIGURE 2. 3-node network with all physical and economic data considered in Examples 5.1 and 5.2; based on Figure 1 of Grimm, Grübel, et al. (2019).

5.1.2. *DC Line Switching.* This application addresses the problem introduced in, e.g., Fisher et al. (2008), Hedman et al. (2008), or Hedman et al. (2009) from an economic point of view. We split the set A of arcs into the set of switchable arcs A_+ and non-switchable arcs A_- . As soon as a power line $a \in A_+$ is switched off (indicated by the binary variable z_a being equal to 0), no power flow over this line is possible and no physical laws are imposed for this line. Conversely, as soon as a power line $a \in A_+$ is switched on ($z_a = 1$), the power flow over this line is bounded by its capacities and follows physical laws.

As in Fisher et al. (2008), Hedman et al. (2008), and Hedman et al. (2009), we use the lossless direct current (DC) load flow approximation to model power flow. In particular, we follow the formulation given in Section 3.7 in Zimmerman and Murillo-Sánchez (2021). In the previously presented application, we focus on the effect of continuous nonconvexities on the existence of an equilibrium. Here, we focus on integrality restrictions and therefore choose this linear power flow model. In total, the network-related physical constraints read

$$q_a^- \leq q_a \leq q_a^+, \quad a \in A_-, \quad (12a)$$

$$\theta_u - \theta_v - \theta_a^{\text{shift}} = X_a q_a, \quad a = (u, v) \in A_-, \quad (12b)$$

$$M_a^-(1 - z_a) \leq \theta_u - \theta_v - \theta_a^{\text{shift}} - X_a q_a \leq M_a^+(1 - z_a), \quad a = (u, v) \in A_+, \quad (12c)$$

$$q_a^- z_a \leq q_a \leq q_a^+ z_a, \quad a \in A_+, \quad (12d)$$

$$z_a \in \{0, 1\}, \quad a \in A_+. \quad (12e)$$

First, all flows q_a are bounded from below and above by the capacities of the respective DC lines. In accordance with the DC load flow approximation, the flow on a line multiplied by its reactance X_a has to equal the nodal phase angle change ($\theta_u - \theta_v$) for all non-switchable lines. In addition, for transformer nodes, a phase shift angle θ_a^{shift} is considered. The DC load flow approximation is only

fulfilled for switchable lines if they are switched on. Finally, if a switchable line is switched off, the flow on this line must be zero.

Let us briefly comment on how we choose the big- M values M_a^- and M_a^+ in Equation (12c). On the one hand, it is crucial to find sufficiently large big- M values to obtain a correct linearization. On the other hand, the big- M values should be as tight as possible to avoid numerical problems. While such big- M values are easily obtained for (12d) by the flow bounds, computing big- M values for (12c) is harder. Due to physics, there are bounds on the differences of nodal phase angles. One can derive nodal phase angle bounds by fixing the phase angle at a reference node to zero. Since the path from the reference node to any other node leads over at most $|V| - 1$ arcs, the bounds on the nodal phase angles differences can be transferred to bounds on the nodal phase angles. These bounds could be tightened by solving an all-pairs longest-path problem. However, since this problem is known to be NP-hard, we refrain from using this approach. The big- M values finally result from the DC load flow approximation utilizing the derived phase angle bounds and the flow bounds.

As before, the transportation costs increase quadratically with the flow. In addition, a fee has to be paid for each line that is switched on. Hence, the transportation costs are given by

$$c^\dagger(q, z) = \sum_{a \in A} \alpha q_a^2 + \sum_{a \in A_+} \beta z_a.$$

In the following example, we provide an instance for which no market equilibrium exists for the described application of (MEP-E). For the sake of simplicity, we fix the transportation costs to zero. Transformers are also not taken into account in this example. Later in the computational study, we also provide instances without an equilibrium when transportation costs are nonzero and transformers are included.

Example 5.2 (Non-existence of Equilibria). *The instance considered here is based on the instance considered in Example 5.1. The economic data is the same. In addition, all reactances are 1 and the line (2, 3) is switchable. The welfare maximum computed in Step 1 of Algorithm 1 is the following:*

$$\begin{aligned} d_2 = \frac{11}{3}, \quad d_3 = 1, \quad y_1 = \frac{14}{3}, \quad q_{1,2} = \frac{11}{3}, \quad q_{1,3} = 1, \quad q_{2,3} = 0, \\ \theta_1 = \frac{11}{3}, \quad \theta_2 = 0, \quad \theta_3 = \frac{8}{3}, \quad z_{2,3} = 1. \end{aligned}$$

The next step is to test whether the TSO's best response to the resulting critical price vector

$$\hat{\pi}_1 = \frac{28}{3}, \quad \hat{\pi}_2 = \frac{28}{3}, \quad \hat{\pi}_3 = 10\sqrt{2} - \frac{1}{2},$$

which is defined as in Corollary A.2, coincides with the TSO's strategy in the welfare maximum. However, these prices are not incentive-compatible for the TSO since—given these prices—the TSO's objective is to route as much as possible to node 3, neglecting node 2. This goal is achieved by the strategy

$$\bar{q}_{1,2} = 0.5, \quad \bar{q}_{1,3} = 1, \quad \bar{q}_{2,3} = 0.5, \quad \bar{\theta}_1 = 1, \quad \bar{\theta}_2 = 0.5, \quad \bar{\theta}_3 = 0, \quad \bar{z}_{2,3} = 0$$

and not by (q, θ, z) . Thus, Algorithm 1 terminates with the indication that no market equilibrium exists.

5.2. Computational Setup and Test Instances. We implemented Algorithm 1 in Python 3.8.5 using Pyomo 5.7.3; see Hart et al. (2017). In order to solve the NLPs arising in the gas application, we use ANTIGONE 1.1; see Misener and Floudas (2014). In turn, we solve the MILPs arising in the power application with Gurobi 9.1.1; see Gurobi Optimization (2021). The computations have been carried out on the Woody compute nodes with four Xeon E3-1240 v5 CPUs running at 3.50 GHz with 32 GB of

TABLE 1. Overview of the instances of the gas flow application

Name	$ V $	$ V_- $	$ V_+ $	$ A $	# instances
Gas-134-S	134	45	3	133	60
Gas-11-H	11	3	3	10	36

RAM; for more information on the compute cluster see Regionales Rechenzentrum Erlangen (2021). For each considered instance, we limit the runtime to 1 hour.

5.2.1. *Gas Flow Instances.* For the gas application, we adapted instances from Heitsch et al. (2021) and Schewe et al. (2020), which are both based on networks from Schmidt et al. (2017). An overview of the resulting 96 instances is presented in Table 1.

The 60 instances **Gas-134-S** are identical with those presented in Schewe et al. (2020) but we set fixed transportation costs to 0, choose the transportation cost factor α in $\{0.01, 0.05, 0.1\}$, and “shifts” of the intercepts of the demand functions in $\{-10, -5, 0, 5, 10\}$. We do not include fixed transportation costs here as the existence of a short-run market equilibrium is independent of considering any fixed costs. In addition, we divide all slopes of the demand functions by 10 and multiply all pressure drop coefficients by 8 to obtain network congestion for more instances.¹ This approach is chosen since Grimm, Grübel, et al. (2019) show in Theorem 4.3 that—as long as nodal pressures are not binding and further mild assumptions are fulfilled—an equilibrium exists for the considered market equilibrium problem. Therefore, to avoid too many instances with guaranteed equilibrium in our numerical study, a high percentage of instances with network congestion is desirable. We address this issue further when presenting the numerical results in Section 5.3.

The instances **Gas-11-H** are taken from Heitsch et al. (2021).² Varying the transportation cost factor α within $\{0.01, 0.05, 0.1\}$ yields 36 instances. To increase the percentage of instances with network congestion, all pressure drop coefficients in **Gas-11-H** are again multiplied by the factor 8. For further information on the data, see Heitsch et al. (2021).

5.2.2. *DC Line Switching Instances.* For the power application, we adapted the instances included in the Software MATPOWER 7.0; see Zimmerman and Murillo-Sánchez (2019) or Zimmerman, Murillo-Sánchez, and Thomas (2011) for details. We restrict our analysis to those instances for which the generation cost data is provided as polynomial cost functions.³ In addition, we neglect all instances for which the reported minimum and maximum phase angle difference coincide in all nodes. The reason is that the same minimum and maximum phase angle difference imply the same lower and upper flow bound, i.e., there is a unique solution w.r.t. the flow in the welfare problem (WFP-E) and a matching unique solution of the TSO problem (10). Hence, for these instances, an equilibrium always exists as the TSO has no possibility to deviate from the welfare-maximal solution. After deleting the described instances, 29 instances remain. Since all instances with more than 1000 nodes and more than 1500 arcs cannot be solved to optimality within the time limit, we only report on the remaining 17 instances here. Finally, we vary the

¹An increase of all pressure drop coefficients around the factor 8 corresponds, e.g., to a diameter reduction by 33% for all pipes or the consideration of a hydrogen network instead of a natural gas network (this approximately equals the change in the specific gas constant).

²Since the flow values in Heitsch et al. (2021) are given as volumetric flows under normal conditions, we convert them to mass flow assuming a gas density of 0.87 kg/m^3 .

³For the two instances **case30pwl** and **case_RTS_GMLC**, the generation cost data is given by piecewise linear functions.

TABLE 2. Overview of the instances of the DC line switching application

Instance	$ V $	$ V_- $	$ V_+ $	$ A $	$ A_+ $
Smallest	5	3	4	6	1
Average	56	33	15	97	10
Biggest	300	191	69	411	42

transportation cost factor α and switching costs β as follows: $\alpha \in \{0.01, 0.05, 0.1\}$ and $\beta \in \{20, 50\}$. Thus, the final test set contains 102 instances in total. An overview of this test set is given in Table 2.

The economic data is obtained in the following way. For the calibration of the demand functions, we assume an elasticity of -0.1 . The respective reference price is chosen to be the mean of the suppliers' critical prices given the real power outputs y_u reported by MATPOWER, i.e., the reference price equals

$$\frac{1}{|V_+|} \sum_{u \in V_+} c'_u(y_u).$$

The respective reference quantity is the real power demand reported by MATPOWER. For generators, we utilize the reported generator cost data and generation capacities. Again, fixed costs are not considered as the existence of a short-run market equilibrium is independent of any given fixed costs. If multiple generators are located at one node, we use the average of the coefficients of the reported polynomial cost functions. We further note that there is also the possibility of a consumer and a producer being located at the same node. For these nodes, Case (a) of Corollary A.2 has to be valid in order to apply Algorithm 1. This is always the case.

Since there is not enough information available on switchable arcs, we randomly select 10% of all arcs as switchable.⁴ The reactance X_a and the transformer phase angle shift θ_a^{shift} are chosen as described in Section 3 in Zimmerman and Murillo-Sánchez (2021). The flow bounds are obtained from the minimum and maximum phase angle difference reported by MATPOWER in combination with the DC load flow approximation for all arcs $a \in A$, i.e.,

$$q_a^- = \frac{(\theta_u - \theta_v)^- - \theta_a^{\text{shift}}}{X_a} \quad \text{and} \quad q_a^+ = \frac{(\theta_u - \theta_v)^+ - \theta_a^{\text{shift}}}{X_a}$$

holds if $X_a > 0$, and

$$q_a^- = \frac{(\theta_u - \theta_v)^+ - \theta_a^{\text{shift}}}{X_a} \quad \text{and} \quad q_a^+ = \frac{(\theta_u - \theta_v)^- - \theta_a^{\text{shift}}}{X_a}$$

holds if $X_a < 0$.

Finally, the main questions that arise are the following:

- (1) How often does a market equilibrium exist for the two considered applications of (MEP-E)?
- (2) Under which circumstances does it become more likely that an equilibrium exists for the two considered applications of (MEP-E)?

We answer these questions now by applying Algorithm 1 to the presented instances.

5.3. Numerical Results: Gas Flow. The numerical results of Algorithm 1 applied to the described gas instances are as follows. In total, 84 instances out of the 96 instances (87.5%) are solved within the time limit of 1 hour. The average runtime over all solved instances is 53.1s and the median runtime is 18.7s. To

⁴The number is rounded up to the next integer. The respective random seed to initialize the random number generator of the Python package random equals the number of arcs in the network.

interpret the results, we call an instance congested if at least one of the following cases applies in the welfare solution:

- (1) the flow on an arc is strictly positive and at the arc's upper flow bound, or
- (2) the flow on an arc is strictly negative and at the arc's lower flow bound, or
- (3) the nodal pressure is at an upper bound and another nodal pressure is at a lower bound, while a directed flow path from this node to the other node exists.

We observe network congestion in 56 (66.7%) solved instances. Similar arguments as used by Grimm, Grübel, et al. (2019) reveal that an equilibrium exists for the uncongested 28 solved instances. Our computational results confirm this and show that there also exists a market equilibrium in all solved and congested 56 instances.

Even though no instances without equilibrium exist in our numerical study, we have seen in Example 5.1 that the non-existence of an equilibrium is possible for the considered application. We note that this example was handcrafted with very specific data to ensure that it does not have a market equilibrium. In contrast, our results suggest that non-existence of an equilibrium hardly occurs for practical instances.

5.4. Numerical Results: DC Line Switching. For the 102 instances of the DC line switching application, the average runtime of Algorithm 1 is 0.4 s and the median is 0.3 s. A market equilibrium exists for 60 out of the 102 instances. No market equilibrium exists for the MATPOWER instances `case30`, `case30Q`, `case_ieee30`, `case39`, `case57`, `case145`, and `case300` for all variations of the transportation cost factor and the switching costs. To simplify notation, we use, e.g., `case30-0.1-20` as an abbreviation for `case30` with transportation cost factor $\alpha = 0.1$ and switching costs $\beta = 20$.

In the following, we discuss the circumstances leading to non-existence of an equilibrium based on examples from our computational study. In particular, there does not exist an equilibrium when the welfare gains outweigh the losses induced by a network decision, while the TSO's profit gains do not. This may even result in the TSO incurring losses in the welfare solution as the corresponding gains of the producers and the consumers are greater than the losses of the TSO. Since the TSO's objective function value is bounded from below by zero (zero flows and all lines switched off), no equilibrium exists in these instances. Actually, this situation occurs in 29 out of the 42 instances without equilibrium.

All instances `case30`, `case30Q`, and `case_ieee30` are based on the same network and admit negative TSO profits in the welfare solution. Non-existence of an equilibrium is caused in all instances by a single line being switched on in the welfare solution, which is switched off in the optimal solution of the TSO. In Figure 3, the underlying situation is exemplarily depicted for the instance `case30-0.1-20`. The welfare solution and the TSO solution differ only in the flows on the three depicted lines, of which (10, 20) is the mentioned switchable line. By switching this line off, the TSO separates the consumers located at nodes 18, 19, and 20 from the rest of the network. The reason for this is obvious. While the welfare gains from serving these three consumers clearly outweigh the losses induced by switching on the line (10, 20) in the welfare solution, the profit gains for the TSO in case of switching this line on are by far too low to outweigh the switching costs since price differences are too low in the described subgraph. Consequently, the TSO has no incentive to switch on line (10, 20).

A similar situation occurs for the instance `case57`. Here, non-existence of an equilibrium is caused by two lines being switched on in the welfare solution, which are switched off in the optimal solution of the TSO. In Figure 4, all differences of

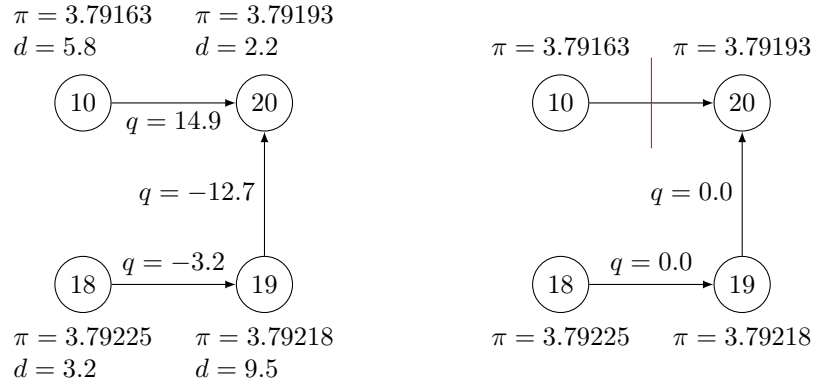


FIGURE 3. Part of the graph on which the welfare solution (left) and the best response of the TSO (right) differ for the instance case30-0.1-20

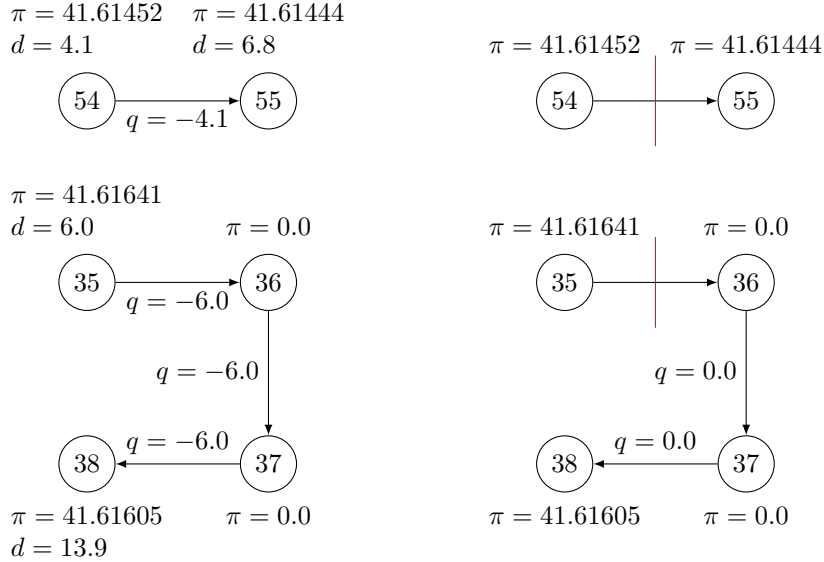


FIGURE 4. Parts of the graph on which the welfare solution (left) and the best response of the TSO (right) differ for the instance case57-0.1-20

the welfare and the TSO solution w.r.t. the flows are presented. The switching decisions differ for the lines (54, 55) and (35, 36). Again, individual consumers are separated from the rest of the network in the TSO solution, namely the consumers located at node 54 and at node 35. The reason for this is the same as before. The profit gains induced by connecting these consumers to the network do not outweigh the related switching costs for the TSO.

In all instances described so far, one or multiple adjacent nodes at which consumers are located have been separated from the rest of the network in the TSO solution but not in the welfare solution. We like to note that, e.g., in the instance case39, the same is true for a node where a generator is located.

There are three main circumstances that might ensure the existence of an equilibrium in the above described situations: (i) lower switching costs, (ii) higher

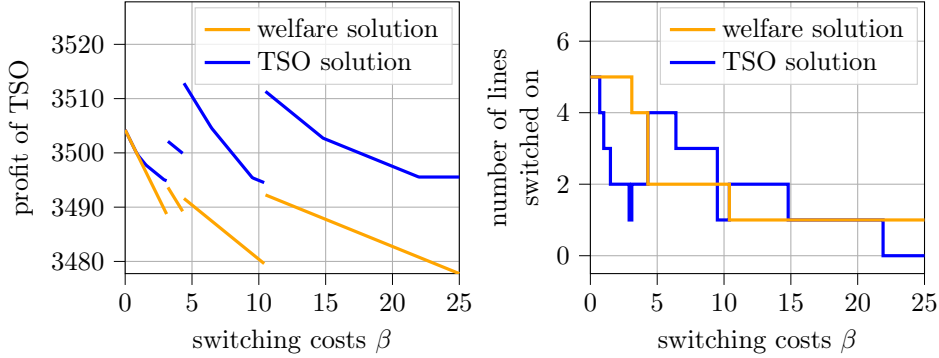


FIGURE 5. Comparison of a welfare optimum (“welfare solution”) and the TSO’s best response to the corresponding critical prices (“TSO solution”) for varying switching costs β and constant transportation cost factor $\alpha = 0.1$. Left: Profits of the TSO. Right: Number of lines switched on in the TSO’s strategies.

switching costs, or (iii) a higher transportation cost factor. If the switching costs are low enough (Case (i)), the TSO’s decisions align with the welfare solution, since then the gained profits indeed outweigh the losses due to switching. If, on the other hand, switching costs are high enough (Case (ii)), the welfare solution matches the TSO’s solution since then the welfare gains as well as the profit gains no longer outweigh the switching costs. Actually, in all instances of our computational study for which a market equilibrium exists, this case applies. Due to relatively high switching costs, all lines are switched off in the welfare and the TSO solution. A higher transportation cost factor (Case (iii)) leads to increased price differences within the network. Resulting profit gains for the TSO might indeed exceed possible switching costs and therefore it becomes more likely that the TSO switches on lines.

To conclude, we study the Cases (i)–(iii) by varying in more detail the switching costs and the transportation cost factor for the MATPOWER case `case39`. The results for varying the switching costs are depicted in Figure 5. Indeed, if the switching costs are low enough, the TSO’s decisions align with the welfare solution. Since the profit gains and the welfare gains outweigh all losses due to switching, all 5 lines are switched on. Now, if the switching costs rise above 0.7, the profit gains of the TSO no longer outweigh all switching costs and the number of switched-on lines in the TSO solution decreases. This continues until the value 3.2 is reached, at which also the welfare gains no longer outweigh all losses due to switching and the number of switched-on lines in the welfare solution reduces to 4. However, this reduction does not lead to the TSO aligning again with the welfare solution. Even the reverse behavior can be observed as, e.g., when the number of switched-on lines in the welfare solution further reduces to 2. There, the TSO exploits the price differences resulting from switching off the two additional lines in the welfare solution by even switching on 4 lines in total. Nevertheless, the incentive to do so decreases with further increasing switching costs. Finally, an equilibrium is obtained again when the switching costs are high enough such that neither the welfare gains nor the profit gains outweigh the losses from switching on any line. This point is reached for significantly larger switching costs of 17 524, which are omitted in the figure for the ease of readability.

Figure 6 shows the results for varying the transportation cost factor for the MATPOWER case `case39`. For a transportation cost factor of 0, no market equilibrium

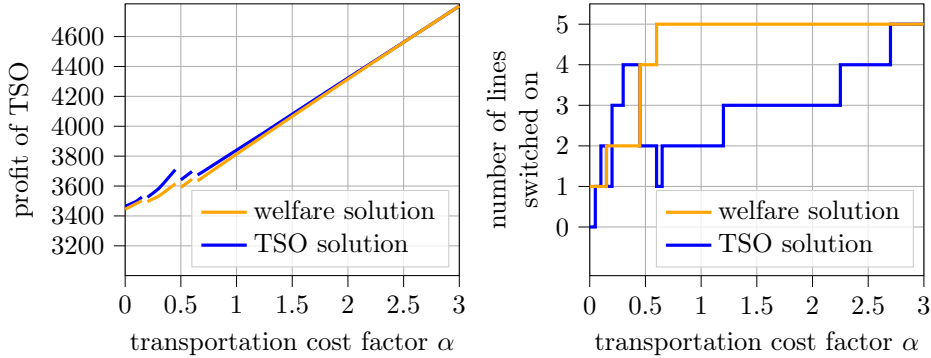


FIGURE 6. Comparison of a welfare optimum (“welfare solution”) and the TSO’s best response to the corresponding critical prices (“TSO solution”) for constant switching costs $\beta = 20$ and varying transportation cost factor α . Left: Profits of the TSO. Right: Number of lines switched on in the TSO’s strategies.

exists. The price differences are not yet high enough for the TSO to outweigh the losses due to switching. In turn, one line is switched on in the welfare solution. With increasing transportation cost factor, the number of switched-on lines in the welfare solution increases monotonically. This is because the additional transport possibilities allow for a cheaper transport despite the additional switching costs. Since an increasing transportation cost factor furthermore leads to increasing price differences, the number of switched-on lines also increases monotonically in the TSO solution for a constant number of switched-on lines in the welfare solution. At the points where additional lines are switched on in the welfare solution, the TSO responds to the corresponding critical prices by reducing the number of switched-on lines. The main reason behind this behavior is as follows. The additional switched-on lines in the welfare solution decrease the price differences. As a result, profit gains due to switching no longer outweigh the arising costs. Consequently, less lines are switched on. Finally, if the transportation cost factor is larger than 2.7, an equilibrium always exists since then all lines are switched on in the TSO and in the welfare solution.

6. CONCLUSION

In this paper we considered market equilibrium problems in which both convex as well as nonconvex player problems appear. This setting is motivated by applications from energy markets, where, e.g., nonconvexities arise in power markets due to integer decisions of certain players or in gas markets due to nonlinear flow models. In the cases studied in this paper, these nonconvexities always appear in the optimization problem of the TSO. Based on the recent results presented in Harks (2020), we derived an algorithm that computes a solution of such nonconvex market equilibrium problems or correctly indicates the non-existence of such an equilibrium. Our computational study reveals interesting aspects. In the continuous but nonlinear and nonconvex market equilibrium problems from the gas sector, all tested instances have an equilibrium. This is different in the power application. Here, integrality restrictions lead to many instances for which no equilibrium exists.

Our results pave the way for some interesting topics of future research. First, one could try to find sufficient conditions under which an equilibrium exists, i.e., under which the resulting welfare problem has a zero duality gap. The discussed

instances from the gas sector indicate that this might be possible. Second, one could consider approximate market equilibria in the nonconvex setting. A potential research question might be whether ε -relaxed optimal solutions to the players' problems give enough freedom to prove the existence of equilibria in settings in which classic equilibria fail to exist. Third, alternative pricing schemes that support an equilibrium could be tested for the DC line switching application as, e.g., the scheme of O'Neill et al. (2005) or the one of Huppmann and Siddiqui (2018). Fourth, one could investigate how the results change in a computational study in which nonlinearities and integralities are combined, as it would be the case for AC line switching models.

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APPENDIX A. EXISTENCE OF EQUILIBRIA FOR MAXIMIZATION GAMES

Instead of the minimization problem (1), each player $i \in I$ now faces the maximization problem

$$\max_{y_i} f_i(y_i, \pi) := c_i(y_i) + \pi^\top h_i(y_i) \quad \text{s.t.} \quad y_i \in Y_i. \quad (13)$$

Thus, the corresponding maximization game reads

$$\begin{aligned} \text{optimization problems of the players:} & \quad (13) \text{ for all } i \in I, \\ \text{market-clearing conditions:} & \quad (2), \end{aligned} \quad (\text{MEP-max})$$

while the corresponding welfare optimization problem now reads

$$\max_y \sum_{i \in I} c_i(y_i) \quad \text{s.t.} \quad y \in Y, \quad \sum_{i \in I} h_i(y_i) = 0. \quad (\text{WFP-max})$$

In this setting, Theorem 3.4 reads as follows.

Corollary A.1. *Let y^* be a solution of the welfare problem (WFP-max) and let $\Pi(y^*) \neq \emptyset$ be a set satisfying the condition*

$$(y^*, \pi^*) \text{ is a market equilibrium of (MEP-max)} \implies \pi^* \in \Pi(y^*).$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

- (a) $\pi_k^- = \pi_k^+$,
- (b) $\pi_k^+ < \infty$ and $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$,
- (c) $\pi_k^- > -\infty$ and $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$,
- (d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ and $(h_i(y_i^*))_k = (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I$.

Then, there exists a market equilibrium of **(MEP-max)** if and only if $(y^*, \hat{\pi})$ is a market equilibrium, in which the critical price $\hat{\pi}$ is defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

Consider again a player $i \in I$ for which the feasible set Y_i is given by standard constraints, e.g., $Y_i = \{y_i : g_i(y_i) \leq 0\}$. If all functions c_i , h_i , and g_i are continuously differentiable, if $y_i \mapsto f_i(y_i, \pi)$ are concave functions and g_i are convex functions, and if a constraint qualification for Y_i is satisfied at y_i^* , then the KKT conditions for (13) are necessary and sufficient for optimality. As a consequence, only prices π for which there exists multipliers μ_i with

$$0 = -\nabla c_i(y_i^*) - \nabla h_i(y_i^*)\pi + \nabla g_i(y_i^*)\mu_i, \quad 0 \leq \mu_i \perp g_i(y_i^*) \leq 0 \quad (14)$$

can be market equilibrium prices. The analogue of Corollary 3.6 for maximization problems thus reads as follows.

Corollary A.2. *Let y^* be a solution of the welfare problem **(WFP-max)**. Moreover, let $C \subseteq I$ denote the subset of players for which the KKT conditions (14) are necessary and sufficient optimality conditions and choose the candidate set $\Pi(y^*)$ such that the condition*

$$(y^*, \pi^*) \text{ is a market equilibrium of } \mathbf{(MEP-max)} \implies \pi^* \in \Pi(y^*)$$

as well as the KKT conditions of all players $i \in C$ are satisfied, i.e.,

$$\Pi(y^*) \subseteq \{\pi \in \mathbb{R}^{n_\pi} : \text{for all } i \in C \text{ exists } \mu_i \text{ such that (14) holds}\}.$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

- (a) $\pi_k^- = \pi_k^+$,
- (b) $\pi_k^+ < \infty$ and $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$,
- (c) $\pi_k^- > -\infty$ and $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$,
- (d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ and $(h_i(y_i^*))_k = (h_i(y_i))_k$ for all $y_i \in Y_i$ and all players $i \in I \setminus C$.

Now, let the critical price $\hat{\pi}$ be defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

If the critical price satisfies $\hat{\pi} \in \Pi(y^*)$, then there exists a market equilibrium of **(MEP)** if and only if $(y^*, \hat{\pi})$ is a market equilibrium.

APPENDIX B. NOTATION

All technical and economic parameters and variables used throughout the computational study in Section 5 are presented together with their respective units in Table 3. We do not use SI units here but the units that are commonly used in the literature of the respective applications, e.g., Nm^3/h denotes volumetric flow under normal conditions. This is in line with the literature from which we adapted our instances; see, e.g., Schewe et al. (2020) and Zimmerman and Murillo-Sánchez (2021).

TABLE 3. Technical and economic parameters (top) and variables (bottom)

Sym.	Explanation	Unit gas	Unit power
π_u	Market price at node $u \in V_- \cup V_+$	€/((1000 Nm ³ /h)	\$/MWh
a_u	Intercept of inverse demand $P_u(\cdot)$ of consumer $u \in V_-$	€/((1000 Nm ³ /h)	\$/MWh
b_u	Slope of inverse demand $P_u(\cdot)$ of consumer $u \in V_-$	€/((1000Nm ³ /h) ²	\$/MW ² h
$c_{u,1}$	Coefficient of linear term of variable cost $c_u(\cdot)$ of producer $u \in V_+$	€/((1000 Nm ³ /h)	\$/MWh
$c_{u,2}$	Coefficient of quadratic term of variable cost $c_u(\cdot)$ of producer $u \in V_+$	€/((1000Nm ³ /h) ²	\$/MW ² h
\bar{y}_u	Capacity of producer $u \in V_+$	1000 Nm ³ /h	MW
α	Transportation cost factor	€/((1000Nm ³ /h) ²	\$/MW ² h
β	Switching costs	—	\$/h
Λ_a	Pressure drop coefficient of arc $a \in A$	(bar) ² /((1000Nm ³ /h) ²	—
p_u^-	Lower pressure bound at node $u \in V$	bar	—
p_u^+	Upper pressure bound at node $u \in V$	bar	—
q_a^-	Lower flow bound of arc $a \in A$	1000 Nm ³ /h	MW
q_a^+	Upper flow bound of arc $a \in A$	1000 Nm ³ /h	MW
X_a	Reactance of arc $a \in A$	—	p. u.
θ_a^{shift}	Transformer phase shift angle ($a \in A$)	—	rad
d_u	Demand of consumer $u \in V_-$	1000 Nm ³ /h	MW
y_u	Production of producer $u \in V_+$	1000 Nm ³ /h	MW
q_a	Flow on arc $a \in A$	1000 Nm ³ /h	MW
p_u	Pressure at node $u \in V$	bar	—
θ_u	Phase angle at node $u \in V$	—	rad
z_a	Switching decision of arc $a \in A_+$	—	—

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