Averaged turnpike property for differential equations with random constant coefficients

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Abstract

This paper studies the integral turnpike and turnpike in average for a class of random ordinary differential equations. We prove that, under suitable assumptions on the matrices that define the system, the optimal solutions for an optimal distributed control tracking problem remain, in an averaged sense, sufficiently close to the associated random stationary optimal solution for the majority of the time horizon.

1 Introduction

In this paper we consider an optimal control distributed tracking-type problem of linear ordinary differential equations with random coefficients. This kind of differential equation is the stochastic counterpart of deterministic differential equations in the most complete sense. The term random differential equations in general refers to differential equations with random coefficients, having either deterministic or random inhomogeneous parts and initial conditions.

Differential equations with random coefficients have been studied and used on a wide variety of problems in engineering and science. This is because the solution of a dynamic system is a function of the values of the parameters which constitute the system. These parameters are experimentally determined and are usually taken to be the mean value of a set of experimental observations. However, in reality, the observations might be measured with errors due the variability of the conditions or because of uncertainties or lack of knowledge. Therefore, a more adequate approach to analysis would be to consider systems with random variables as coefficients. We can mention the earlier work on this area [2] where the author studied the propagation of sound waves of high-frequency in the atmosphere of randomly varying refraction index. We refer to the books [18, 19] for a complete study of this kind of equations and interesting applications in science, engineering, physics and biomedical systems, among others.

On the other hand, in the context of optimal control, in [16] the authors studied the concept of turnpike for the solutions of an optimal control problem subject to ordinary differential equations without randomness. The turnpike property, roughly speaking, describes that the optimal evolutionary solution is made of three arcs, the first and the last being transient short time arcs, and the middle piece being a long-time arc remaining exponentially close to the optimal steady state of the corresponding stationary optimal control problem. This concept was formulated in the earlier work [6], in the context of the econometric field.

Motivated by the previous considerations, we will investigate if any connection exists between the average of an optimal solution of a certain optimal control problem for an ordinary differential equation with random coefficients with the corresponding stationary random problem. Specifically, we will analyze the *turnpike phenomenon* for a class of random differential equations, which

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is important to understand the behavior of solutions to optimal control problems on large time horizons. To the best of our knowledge, it is the first time that the turnpike property is studied for random differential equations.

Stating things more mathematically, in this paper we consider a probability space $(\Omega, \mathcal{F}, \mu)$ and three random matrices $A, C \in C^0(\Omega, \mathcal{L}(\mathbb{R}^n))$ and $B \in C^0(\Omega, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, constant in time, which will represent the random coefficients of the equation, the random observation and the random control, respectively. We assume that the joint probability distribution of matrices A, B and C is specified. We consider the following optimal control problem

$$\min_{u \in L^2(0,T;\mathbb{R}^m)} \left\{ J^T(u) = \frac{1}{2} \left(\int_0^T \|u(t)\|^2_{\mathbb{R}^m} \, dt + \int_0^T \|C(\cdot)x(t,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)} \, dt \right) \right\}, \tag{1.1}$$

subject to x solving the following evolutionary problem

$$\begin{cases} x_t + A(\omega)x = B(\omega)u & t \in (0,T), \\ x(0,\omega) = x_0(\omega), \end{cases} \tag{1.2}$$

where $z \in \mathbb{R}^n$ is a fixed target. Here $x = x(t,\omega) \in \mathbb{R}^n$ represents the state and $u(t) \in \mathbb{R}^m$ the control of the system, respectively.

The first aim of this paper it is prove that when the time-horizon goes to infinity, the optimal pair (u^T, x^T) of (1.1)-(1.2) converges in an averaged sense to $(\overline{u}, \overline{x})$ in $\mathbb{R}^m \times L^2(\Omega, \mathbb{R}^n)$, where $(\overline{u}, \overline{x})$ solves the associated stationary random optimal control problem

$$\min_{u \in \mathbb{R}^m} \left\{ J^s(u) = \frac{1}{2} \Big(\|u\|^2_{\mathbb{R}^m} + \|C(\cdot)x(\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)} \Big) \right\}, \tag{1.3}$$

subject to x solving the problem

$$A(\omega)x(\omega) = B(\omega)u, \tag{1.4}$$

where $z \in \mathbb{R}^n$ is the same target of the problem (1.1). That is, we will analyze the following limits, which are usually called integral turnpike property

$$\frac{1}{T} \int_0^T x^T(t,\cdot)dt \to \overline{x}(\cdot) \quad \text{in } L^2(\Omega,\mathbb{R}^n), \qquad \frac{1}{T} \int_0^T u^T(t)dt \to \overline{u} \quad \text{in } \mathbb{R}^m.$$

The second main result is to show an exponential turnpike property. Namely, we will prove the existence of two positive constants K and δ, independent on the time-horizon T, such that the solutions of the extremal equations (u^T, x^T, φ^T) remains exponentially close to the steady state solution, the so-called turnpike, for the majority of the time. That is,

$$\left\|x^T(t,\cdot) - \overline{x}(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|\varphi^T(t,\cdot) - \overline{\varphi}(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K(e^{-\delta(T-t)} + e^{-\delta t}),$$

for every $t \in (0,T)$. Here, $(\varphi^T, \overline{\varphi})$ represents the characterization of minimizers via the first order necessary optimality conditions (the dual variables). In addition, as a consequence of the previous estimate, we prove an average exponential turnpike, that is

$$\left\|\mathbb{E}(x^T(t,\cdot) - \overline{x})\right\|_{\mathbb{R}^n} + \left\|\mathbb{E}(\varphi^T(t,\cdot) - \overline{\varphi})\right\|_{\mathbb{R}^n} + \left\|u^T(t) - \overline{u}\right\|_{\mathbb{R}^m} \leq K(e^{-\delta(T-t)} + e^{-\delta t}),$$

for every $t \in (0,T)$ and where $\mathbb{E}(x^T)$ denotes the expected value of x^T and is given by

$$\mathbb{E}(x^T) = \int_\Omega x^T d\mu.$$

Let us mention that both results are based on stability assumptions for A, B and C, which are the matrices that define the system. These assumptions are related to the existence of feedback operators in a such way that we can ensure an ellipticity type condition. Besides, these hypotheses allow us to establish relevant observability inequalities which play an important role in the proof of our main results. We refer to Section 2 for a complete discussion on the subject.

There is abundant literature on the turnpike phenomenon in various contexts, see for instance [4, 7, 8, 9, 10, 11, 12, 15, 16, 17, 20, 21, 22, 23, 24, 25, 27]. Let us briefly mention some of them. The present work is an extension of the results presented in [16], where the authors studied the turnpike property in the context of ordinary differential equations. They established the exponential turnpike property using the theory for Algebraic Riccati Equation associated to the optimal control. Here, we show the exponential turnpike based on the boundedness of the operator corresponding to the optimality system. This approach was taken from [10], where the authors proved the exponential turnpike result for linear quadratic optimal control problems with dynamics satisfying a particular kind of stabilizability and detectability assumption. Finally, we mention the work [15], where the authors, motivated by reinforcement learning, studied a similar problem to our first main result in the context of optimal control problems with uncertain dynamics.

The rest of the paper is organized as follows. The main assumptions of this work are included in Section 2, in which we also prove the existence and uniqueness for the optimal control problems. Section 3 is devoted to the statement and proof of the main results, namely Theorems 3.1 and 3.2. Finally, in Section 4 we show some computational examples to illustrate the average turnpike phenomenon.

2 Optimal control problems

In this part of the work, we analyze the optimal control problem presented in the Introduction, both the evolutionary and stationary problem. We prove that, under some assumptions on the matrices A, B and C which define the system, these optimal controls are well-defined and satisfy the first order optimality condition.

Throughout the following, without any mention, we consider the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, with $n, m \in \mathbb{N}$, $n \geq m$. We consider a probability space $(\Omega, \mathcal{F}, \mu)$, where the sample space $\Omega \subset \mathbb{R}^d$ is an arbitrary bounded set, $\mathcal{F}$ represents the σ-algebra, and μ is the probability measure on $\mathcal{F}$ which satisfies $\mu(\Omega) = 1$. We will use the letter K to denote an absolute positive constant which might change even in the same line of text.

We consider the following random ODE

$$\begin{cases} x_t + A(\omega)x = B(\omega)u & t \in (0,T), \\ x(0,\omega) = x_0(\omega), \end{cases}$$

where $\omega \in \Omega$ corresponds the random parameter, $x = x(t,\omega) \in \mathbb{R}^n$ is the state of the system, the $n \times n$ matrix $A \in C^0(\Omega, \mathcal{L}(\mathbb{R}^n))$ governs its free dynamics, $u(t) \in \mathbb{R}^m$ is the control function

which is assumed to be independent of the randomness and acts on the system through the control matrix $B \in C^0(\Omega, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ which is a $m \times n$ parameter dependent matrix. The initial datum $x_0(\omega)$ belongs to the space $L^2(\Omega, \mathbb{R}^n; \mu)$, which is defined below.

Let us define the space

$$L^2(\Omega, \mathbb{R}^n; \mu) := \{\omega \in \Omega \mapsto y(\omega) \in \mathbb{R}^n \text{ measurable } : \|y(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n;\mu)} = \int_\Omega \|y(\omega)\|^2_{\mathbb{R}^n} d\mu(\omega) < \infty\},$$

which is a Hilbert space endowed with the inner product

$$\langle x, y\rangle_{L^2(\Omega,\mathbb{R}^n;\mu)} = \int_\Omega \langle x(\omega), y(\omega)\rangle_{\mathbb{R}^n} d\mu(\omega), \quad \forall x, y \in L^2(\Omega, \mathbb{R}^n; \mu).$$

In what follows, we denote by $L^2(\Omega, \mathbb{R}^n) := L^2(\Omega, \mathbb{R}^n; \mu)$.

Additionally, we also assume that the matrices A and B are uniformly bounded with respect to ω, that is, there exists a positive constant $M > 0$ such that

$$\begin{aligned}
&\sup_{\omega\in\Omega} |a_{i,j}(\omega)| \le M, \quad i, j \in \{1, \dots, n\},\\
&\sup_{\omega\in\Omega} |b_{i,j}(\omega)| \le M, \quad i \in \{1, \dots, n\},\ j \in \{1, \dots, m\},\\
&\sup_{\omega\in\Omega} |c_{i,j}(\omega)| \le M, \quad i, j \in \{1, \dots, n\},
\end{aligned}$$

where $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ are the coefficient of matrices A, B and C, respectively. This assumption allows us to ensure the integrability of the solutions of (1.2) with respect to ω.

Concerning the integrability of the solutions for (1.2), we have the next result, which can be consulted in [14].

Theorem 2.1 ([14, Corollary 2.2]). *Assume the map $\omega \mapsto (A(\omega), B(\omega))$ is continuous on Ω. Then, for every $x_0 \in L^2(\Omega, \mathbb{R}^n)$, every $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^m)$, and every $t \ge 0$, the solution x of (1.2) satisfies $x(t, \cdot) \in L^2(\Omega, \mathbb{R}^n)$. In addition, the solution x can be represented by*

$$x(t, \omega) = e^{tA(\omega)}x_0(\omega) + \int_0^T e^{(t-s)A(\omega)}B(\omega)u(s)ds, \quad \forall \omega \in \Omega,\ t \in [0, T].$$

2.1 Evolutionary problem

Let us consider first the optimal control for the evolutionary problem (1.2) with initial datum independent of ω. That is,

$$\min_{u\in L^2(0,T;\mathbb{R}^m)} J^T(u), \tag{2.1}$$

where

$$J^T(u) = \frac{1}{2}\left(\int_0^T \|u(t)\|^2_{\mathbb{R}^m}\, dt + \int_0^T \|C(\cdot)x(t, \cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)}\, dt\right),$$

subject to x solving the following evolutionary problem

$$\begin{cases} x_t + A(\omega)x = B(\omega)u & t \in (0, T),\\ x(0, \omega) = x_0, \end{cases} \tag{2.2}$$

where $z \in \mathbb{R}^n$ is a fixed target. The case of initial condition $x_0 \in L^2(\Omega, \mathbb{R}^n)$ does not lead to any essential new difficulty throughout the following. Thus, for sake of simplicity of the presentation, we only deal with the case where x_0 is independent of ω.

By using the direct method of the calculus of variations, and noting that the solution x of (2.2) depends linearly and continuously in u, we obtain the existence and uniqueness result of the optimal control. Besides, the characterization of the control can be done using the Gateaux derivative of J^T. These results are included in the following theorem.

Theorem 2.2. *There exists a unique solution* $(u^T, x^T) \in L^2(0,T;\mathbb{R}^m) \times C^0([0,T]; L^2(\Omega,\mathbb{R}^n))$ *to the minimization problem* (2.1)-(2.2)*, where* x^T *is the optimal state associated to the control* u^T*. In addition,*

$$u^T(t) = -\int_\Omega B^*(\omega)\varphi^T(t,\omega)d\mu, \tag{2.3}$$

where $\varphi^T \in C^0([0,T]; L^2(\Omega,\mathbb{R}^n))$ *is the solution of the backward problem*

$$\begin{cases} -\varphi_t^T + A^*(\omega)\varphi^T = C^*(\omega)(C(\omega)x^T - z) \quad t \in (0,T), \\ \varphi^T(T,\omega) = 0. \end{cases} \tag{2.4}$$

In what follows, we assume the next two conditions with respect to the dynamics and the cost functional.

Hypothesis 1: For the pair (A, C) we assume the following condition. There exists a feedback operator $K_C \in C^0(\Omega, \mathcal{L}(\mathbb{R}^n))$ uniformly bounded with respect to ω such that

$$\exists \alpha > 0 : \ \langle (A + K_C C)v, v\rangle_{L^2(\Omega,\mathbb{R}^n)} \geq \alpha \left\| v(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)}, \quad \forall v \in L^2(\Omega,\mathbb{R}^n). \tag{2.5}$$

Hypothesis 2: For the pair (A^*, B^*) we consider the next assumption. There exists $\kappa_1 \in \mathbb{R}$ and $\kappa_2 > 0$ such that

$$\langle A^*v, v\rangle_{L^2(\Omega,\mathbb{R}^n)} + \kappa_1 \left\| \int_\Omega B^*(\omega)v(\omega)d\mu \right\|^2_{\mathbb{R}^m} \geq \kappa_2 \left\| v(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)}, \quad \forall v \in L^2(\Omega,\mathbb{R}^n). \tag{2.6}$$

These kinds of assumptions are closely related to exponential stabilizability and exponential detectability, as mentioned in [10] for abstract differential equations. Under the previous assumptions we have the following "observability" estimates for $x^T(T)$ and $\varphi^T(0)$.

Lemma 2.3. *Let us assume that* **Hypothesis 1** *holds. Then, there exists a constant* $K > 0$ *independent of* $T > 0$ *such that, for every* $t \in [0,T]$

$$\left\|x^T(t,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K \left(\int_0^T \left[\left\|B(\cdot)u^T(t)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|C(\cdot)x^T(t,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \right] dt \right.$$
$$\left. + \left\|x_0\right\|^2_{\mathbb{R}^n} \right), \tag{2.7}$$

where (u^T, x^T) *is the optimal pair given by Theorem 2.2.*

Proof. Let $0 < t \leq T$. Testing the equation (2.2) by x^T and integrating over $(0,t)\times\Omega$, we have

$$\frac{1}{2}\|x^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \int_0^t \left(\langle Ax^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)} + \langle K_C C x^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)}\right) ds$$
$$= \int_0^t \left(\langle Bu^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)} + \langle K_C C x^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)}\right) ds + \frac{1}{2}\|x_0\|^2_{\mathbb{R}^n},$$

where K_C is the feedback operator given by **Hypothesis 1**. Using again **Hypothesis 1**, there exists a constant α such that

$$\frac{1}{2}\|x^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \alpha\int_0^t \|x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds$$
$$\leq \int_0^t \left(\langle Bu^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)} + \langle K_C C x^T, x^T\rangle_{L^2(\Omega,\mathbb{R}^n)}\right) ds + \frac{1}{2}\|x_0\|^2_{\mathbb{R}^n}.$$

Now, for every $\varepsilon_1, \varepsilon_2 > 0$ we obtain

$$\frac{1}{2}\|x^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \alpha\int_0^t \|x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds \leq \frac{1}{2\varepsilon_1}\int_0^t \|B(\cdot)u^T(s)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds$$
$$+ \frac{\varepsilon_1}{2}\int_0^t \|x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds + \frac{1}{2\varepsilon_2}\int_0^t \|K_C(\cdot)C(\cdot)x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds$$
$$+ \frac{\varepsilon_2}{2}\int_0^t \|x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds + \frac{1}{2}\|x_0\|^2_{\mathbb{R}^n}.$$

Choosing $\varepsilon_1, \varepsilon_2 > 0$ in a such way $2\alpha - \varepsilon_1 - \varepsilon_2 > 0$, there exists a constant $K > 0$, independent of t, such that

$$\|x^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \int_0^t \|x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds$$
$$\leq K\left(\int_0^t \left[\|B(\cdot)u^T(s)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|C(\cdot)x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right] ds + \|x_0\|^2_{\mathbb{R}^n}\right).$$

Therefore, we obtain

$$\|x^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\Bigg(\int_0^T \left[\|B(\cdot)u^T(s)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|C(\cdot)x^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right] ds$$
$$+ \|x_0\|^2_{\mathbb{R}^n}\Bigg),$$

for every $0 \leq t \leq T$. We have shown (2.7). □

Lemma 2.4. *Let us assume that* **Hypothesis 2** *holds. Then, there exists a constant $K > 0$ independent of $T > 0$ such that for every $t \in [0,T]$*

$$\|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\int_0^T \Bigg[\|C(\cdot)x^T(t,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)}$$
$$+ \left\|\int_\Omega B^*(\omega)\varphi^T(t,\omega)d\mu\right\|^2_{\mathbb{R}^m}\Bigg] dt, \quad (2.8)$$

where φ^T is the solution of (2.4).

Proof. Let $0 \leq t < T$. Multiplying (2.4) by φ^T, integrating over $(t,T)\times\Omega$, and using the **Hypothesis 2**, we get

$$\frac{1}{2}\|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \kappa_2 \int_t^T \|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}dt \\ \leq \int_t^T \left(\langle Cx^T - z, C\varphi^T\rangle_{L^2(\Omega,\mathbb{R}^n)} + \kappa_1 \left\| \int_\Omega B^*(\omega)\varphi^T(t,\omega)d\mu \right\|^2_{\mathbb{R}^m} \right) dt.$$

Then, for every $\varepsilon > 0$ we obtain

$$\frac{1}{2}\|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left(\kappa_2 - \frac{\varepsilon}{2}\right) \int_t^T \|\varphi^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}ds \\ \leq \int_t^T \left(\frac{1}{2\varepsilon}\|C(\cdot)x^T(s,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)} + \kappa_1 \left\| \int_\Omega B^*(\omega)\varphi^T(s,\omega)d\mu \right\|^2_{\mathbb{R}^m} \right) ds.$$

Taking $\varepsilon > 0$ such that $\varepsilon < 2\kappa_2$, there exists a constant $K > 0$, independent of T, such that

$$\|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \int_t^T \|\varphi^T(s,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}ds \\ \leq K \int_t^T \left[\|C(\cdot)x^T(s,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\| \int_\Omega B^*(\omega)\varphi^T(s,\omega)d\mu \right\|^2_{\mathbb{R}^m} \right] ds.$$

Further,

$$\|\varphi^T(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K \int_0^T \Big[\|C(\cdot)x^T(s,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)} \\ + \left\| \int_\Omega B^*(\omega)\varphi^T(s,\omega)d\mu \right\|^2_{\mathbb{R}^m} \Big] ds,$$

for every $0 \leq t \leq T$ and the proof is finished. □

Remark 1. Let us mention some consequences of the previous considerations. From **Hypothesis 1**, it immediately follows that there exists a constant $K > 0$ such that the following stationary inequality holds:

$$\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\Big(\|Av\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|Cv\|^2_{L^2(\Omega,\mathbb{R}^n)} \Big), \tag{2.9}$$

for every $v \in L^2(\Omega,\mathbb{R}^n)$. Indeed, following the proof of Lemma 2.3, we have

$$\|x(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \\ \leq K \left(\int_0^T \left[\|x_t(t,\cdot) + Ax(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|C(\cdot)x(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \right] dt + \|x_0\|^2_{\mathbb{R}^n} \right). \tag{2.10}$$

Then, for $v \in L^2(\Omega,\mathbb{R}^n)$ we take $x(t,\omega) = tv(\omega)$. Applying the inequality (2.10), we have

$$T^2\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K \int_0^T \left[\|v(\cdot) + tA(\cdot)v(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + t^2\|C(\cdot)v(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \right] dt.$$

Therefore, we get

$$T^2\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\Big(T\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} + \frac{T^3}{3}\|Av\|^2_{L^2(\Omega,\mathbb{R}^n)} + \frac{T^3}{3}\|Cv\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big).$$

Taking $T > K$, there exists $\widetilde{K} > 0$ such that

$$\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq \widetilde{K}\Big(\|Av\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|Cv\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big),$$

and the claim is proven.

2.2 Stationary problem

Now, we continue with the analysis of the stationary optimal control problem (1.3)–(1.4).

Under the **Hypothesis 2** we have the following existence and uniqueness for the optimal pair for problem (1.3)–(1.4).

Theorem 2.5. *Assume that* **Hypothesis 2** *holds true. Then the problem* (1.3)–(1.4) *admits a unique optimal pair* $(\overline{u},\overline{x}) \in \mathbb{R}^m \times L^2(\Omega,\mathbb{R}^n)$*, with* $\overline{x}$ *the optimal state associated to* $\overline{u}$*.*

Proof. We observe that J^s is a strictly convex and continuous functional, therefore, there exists a unique solution $\overline{u} \in \mathbb{R}^m$ of (1.3). In addition, there exists a unique observation $C(\omega)x$ associated to $\overline{u}$.

Let us assume that there exist two optimal states x_1 and x_2 associated to the optimal control $\overline{u}$, that is

$$Ax_1 = B\overline{u} = Ax_2.$$

Since J^s is strictly convex, we obtain that $C(\omega)x_1 = C(\omega)x_2$ in $L^2(\Omega,\mathbb{R}^n)$.

From (2.9) we obtain

$$\|x_1(\cdot) - x_2(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\Big[\|A(\cdot)(x_1(\cdot) - x_2(\cdot))\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|C(\cdot)(x_1(\cdot) - x_2(\cdot))\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big] = 0.$$

Namely, $x_1 = x_2$. □

Now, we define the following set

$$D := \{u \in \mathbb{R}^m \ : \ B(\omega)u \in \mathrm{Ran}(A(\omega)), \text{ for each } \omega \in \Omega\}. \tag{2.11}$$

Theorem 2.6. *Assume that* **Hypothesis 2** *holds true and let* $(\overline{u},\overline{x})$ *be the unique solution of the optimal control* (1.3)–(1.4)*. Then, there exists* $\overline{\varphi} \in L^2(\Omega,\mathbb{R}^n)$ *such that for a.e.* $\omega \in \Omega$ *we have*

$$A^*(\omega)\overline{\varphi} = C^*(\omega)(C(\omega)\overline{x} - z), \tag{2.12}$$

and

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, Bv\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0, \quad \forall v \in D. \tag{2.13}$$

Proof. Since $(\overline{u}, \overline{x})$ is the unique optimal pair, we have

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle C\overline{x} - z, C\varphi\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0, \tag{2.14}$$

for every $v \in D$ and every $\varphi \in L^2(\Omega, \mathbb{R}^n)$ such that $A(\omega)\varphi = B(\omega)v$.

Taking $v = 0$ in the previous expression, then $A(\omega)\varphi(\omega) = 0$. Namely, $\varphi(\omega) \in Ker(A(\omega))$. From (2.14) we obtain

$$\langle C^*(C\overline{x} - z), \varphi\rangle_{L^2(\Omega,\mathbb{R}^n)} = \langle C\overline{x} - z, C\varphi\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0.$$

Therefore, $C^*(C\overline{x} - z) \in Ker(A)^{\perp}$, which implies that $C^*(C\overline{x} - z) \in \mathrm{Ran}(A^*)$. That is, there exists $\overline{\varphi}(\omega) \in Dom(A^*(\omega))$ such that

$$A^*(\omega)\overline{\varphi}(\omega) = C^*(\omega)(C(\omega)\overline{x}(\omega) - z).$$

Finally, replacing in (2.14), we get

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, Bv\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0,$$

for every $v \in D$. □

3 Main results

In this section we state and prove the main results of this work. For this, let us recall the evolutionary and stationary optimality systems. Let (u^T, x^T) be the optimal pair of (2.1)–(2.2), and $(\overline{x}, \overline{u})$ the optimal pair of (1.3)–(1.4) (see Theorems 2.2 and 2.6). In addition, we have that there exist φ^T solution of (2.4), such that the optimal control u^T of (2.1) is given by

$$u^T(t) = -\int_\Omega B^*(\omega)\varphi^T(t,\omega)d\mu, \tag{3.1}$$

and $\overline{u}$ the optimal control associate to (1.3) satisfies

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, B(\omega)v\rangle_{L^2(\Omega,X)} = 0, \quad \forall v \in D, \tag{3.2}$$

where $\overline{\varphi}$ is the solution of (2.12). Besides, the following optimality systems hold:

$$\begin{cases} x_t^T + A(\omega)x^T = B(\omega)u^T \quad t \in (0,T), \\ -\varphi_t^T + A^*(\omega)\varphi^T = C^*(\omega)(C(\omega)x^T - z) \quad t \in (0,T), \\ x^T(0,\omega) = x_0, \quad \varphi^T(T,\omega) = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} A(\omega)\overline{x} = B(\omega)\overline{u}, \\ A^*(\omega)\overline{\varphi} = C^*(\omega)(C(\omega)\overline{x} - z). \end{cases} \tag{3.4}$$

The first main result is concerning to the average convergence of the optimal pair (u^T, x^T) to the corresponding stationary ones $(\overline{u}, \overline{x})$, stated in the following theorem. The proof is based on the result contained in [16].

Theorem 3.1. *Let us assume that* **Hypothesis 1 and 2** *hold. Then,*

$$\frac{1}{T}\int_0^T x^T(t,\cdot)dt \longrightarrow \overline{x}(\cdot) \quad \text{in } L^2(\Omega,\mathbb{R}^n), \quad \frac{1}{T}\int_0^T u^T(t)dt \longrightarrow \overline{u} \quad \text{in } \mathbb{R}^m,$$

as $T \to \infty$*, where* (u^T, x^T) *is the optimal pair of* (2.1)–(2.2)*, and* $(\overline{x}, \overline{u})$ *is the optimal pair of* (1.3)–(1.4)*.*

Proof. We divide the proof into four steps. Since we are interested in the asymptotic behavior when T goes to infinity, we can assume that there exists $\tau > 0$ such that $T > \tau > 0$.

Step 1. We claim that there exist two constants $\hat{K}$, K, independent of $T > 0$, such that x^T and φ^T satisfy

$$\left\|x^T(T,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq KT, \tag{3.5}$$

$$\left\|\varphi^T(0,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq \hat{K}T. \tag{3.6}$$

Indeed, multiplying the first equation of (3.3) by φ^T and integrating by parts over $(0,T)$ we obtain

$$\begin{aligned}
\left\langle \varphi^T(0,\omega), x_0\right\rangle_{\mathbb{R}^n} &= \int_0^T \left\langle C^*(\omega)(C(\omega)x^T(t,\omega) - z), x^T(t,\omega)\right\rangle_{\mathbb{R}^n} dt \\
&\qquad - \int_0^T \left\langle B(\omega)u^T(t), \varphi^T(t,\omega)\right\rangle_{\mathbb{R}^n} dt \\
&= \int_0^T \left\langle C(\omega)x^T(t,\omega) - z, C(\omega)x^T(t,\omega)\right\rangle_{\mathbb{R}^n} dt \\
&\qquad - \int_0^T \left\langle u^T(t), B^*(\omega)\varphi^T(t,\omega)\right\rangle_{\mathbb{R}^m} dt \\
&= \int_0^T \left(\left\|C(\omega)x^T(t,\omega) - z\right\|^2_{\mathbb{R}^n} + \left\langle C(\omega)x^T(t,\omega) - z, z\right\rangle_{\mathbb{R}^n}\right) dt \\
&\qquad - \int_0^T \left\langle u^T(t), B^*(\omega)\varphi^T(t,\omega)\right\rangle_{\mathbb{R}^m} dt.
\end{aligned} \tag{3.7}$$

Now, integrating over Ω and using the fact that u^T only depends on $t \in (0,T)$ and is given by (3.1), we have

$$\begin{aligned}
\left\langle \varphi^T(0,\cdot), x_0\right\rangle_{L^2(\Omega,\mathbb{R}^n)} &= \int_0^T \left(\left\|C(\cdot)x^T(t,\cdot) - z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\langle Cx^T - z, z\right\rangle_{L^2(\Omega,\mathbb{R}^n)}\right) dt \\
&\quad - \int_0^T \left\langle u^T, \int_\Omega B^*\varphi^T d\mu\right\rangle_{\mathbb{R}^m} dt \\
&= \int_0^T \left(\left\|C(\cdot)x^T(t,\cdot) - z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\langle Cx^T - z, z\right\rangle_{L^2(\Omega,\mathbb{R}^n)} \right. \\
&\quad \left. + \left\|u^T(t)\right\|^2_{\mathbb{R}^m}\right) dt.
\end{aligned}$$

That is,

$$\int_0^T \Big(\left\|C(\cdot)x^T(t,\cdot)-z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t)\right\|^2_{\mathbb{R}^m}\Big)dt$$
$$= \left\langle \varphi^T(0,\omega), x_0\right\rangle_{L^2(\Omega,\mathbb{R}^n)} - \int_0^T \left\langle Cx^T - z, z\right\rangle_{L^2(\Omega,\mathbb{R}^n)} dt. \quad (3.8)$$

Next, by Cauchy–Schwarz and Young inequalities, for every $\varepsilon > 0$, the first term at the right-hand side of (3.8) can be bound as

$$\begin{aligned}\left\langle \varphi^T(0,\omega), x_0\right\rangle_{L^2(\Omega,\mathbb{R}^n)} &\le \frac{\varepsilon \left\|\varphi^T(0,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2} + \frac{\|x_0\|^2_{\mathbb{R}^n}}{2\varepsilon} \\ &\le \frac{K\varepsilon}{2}\Big(\int_0^T \left\|C(\cdot)x^T(t,\cdot)-z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t)\right\|^2_{\mathbb{R}^m} dt\Big) \\ &\quad + \frac{\|x_0\|^2_{\mathbb{R}^n}}{2\varepsilon}, \end{aligned} \quad (3.9)$$

where in the last inequality we used (2.8). For the second term at the right-hand side of (3.8), for every ε_1 we get

$$-\int_0^T \left\langle Cx^T - z, z\right\rangle_{L^2(\Omega,\mathbb{R}^n)} dt$$
$$\le \int_0^T \varepsilon_1 \frac{\left\|C(\cdot)x^T(t,\cdot)-z\right\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2} dt + T\frac{\|z\|^2_{\mathbb{R}^n}}{2\varepsilon_1}. \quad (3.10)$$

Thus, replacing (3.10) and (3.9) into (3.8), we deduce

$$\begin{aligned}&\int_0^T \Big(1 - \frac{K\varepsilon}{2} - \varepsilon_1\Big) \left\|C(\cdot)x^T(t,\cdot)-z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \Big(1-\frac{K\varepsilon}{2}\Big) \left\|u^T(t)\right\|^2_{\mathbb{R}^m}\Big)dt \\ &\qquad \le T\frac{\|z\|^2_{\mathbb{R}^n}}{2\varepsilon_1} + \frac{\|x_0\|^2_{\mathbb{R}^n}}{2\varepsilon} = \frac{T\left\|z\right\|^2_{\mathbb{R}^n}\varepsilon + \varepsilon_1 \left\|x_0\right\|^2_{\mathbb{R}^n}}{2\varepsilon\varepsilon_1} \\ &\qquad \le \max\Big\{\left\|z\right\|^2_{\mathbb{R}^n}, \left\|x_0\right\|^2_{\mathbb{R}^n}\Big\} \frac{T\varepsilon + \varepsilon_1}{2\varepsilon\varepsilon_1}. \end{aligned} \quad (3.11)$$

Then taking $\varepsilon, \varepsilon_1$ in a such way that $1 - \frac{K\varepsilon}{2} > \varepsilon_1$, there exists a constant $\hat{K} > 0$ (independent of T) such that

$$\int_0^T \left\|C(\cdot)x^T(t,\cdot)-z\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t)\right\|^2_{\mathbb{R}^m} dt \le \hat{K}T. \quad (3.12)$$

From (2.8) and (3.12), we get the desired inequality for $\varphi^T(0)$, that is

$$\left\|\varphi^T(0,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \le \hat{K}T. \quad (3.13)$$

The inequality for $x^T(T,\omega)$ is a direct consequence of (2.7) and (3.12).

Step 2. We claim that the terms

$$\frac{1}{T}\int_0^T x^T(t,\cdot)dt, \quad \frac{1}{T}\int_0^T u^T(t)dt,$$

are bounded in $L^2(\Omega,\mathbb{R}^n)$ and $\mathbb{R}^m$, respectively.

Indeed, integrating over $(0,T)$ the first equation of (3.3) we obtain

$$\frac{1}{T}\int_0^T A(\omega)x^T(t,\omega)dt = \frac{1}{T}\int_0^T B(\omega)u^T(t)dt - \left(\frac{x^T(T,\omega)-x_0}{T}\right). \tag{3.14}$$

From (3.5) we obtain that $\frac{x^T(T,\omega)-x_0}{T}$ converges to zero as $T\to\infty$. Namely, it is bounded by a positive constant K, independent of T in $L^2(\Omega,\mathbb{R}^n)$. Therefore, taking $L^2(\Omega,\mathbb{R}^n)$ norm we get

$$\begin{aligned}
\left\|\frac{1}{T}\int_0^T A(\cdot)x^T(t,\cdot)dt\right\|_{L^2(\Omega,\mathbb{R}^n)} &\le \left\|\frac{1}{T}\int_0^T B(\cdot)u^T(t)dt\right\|_{L^2(\Omega,\mathbb{R}^n)} \\
&\quad + \left\|\frac{x^T(T,\omega)-x_0}{T}\right\|_{L^2(\Omega,\mathbb{R}^n)} \\
&\le \frac{1}{T}\int_0^T \left\|B(\cdot)u^T(t)\right\|_{L^2(\Omega,\mathbb{R}^n)} dt + K \\
&\le \frac{1}{T}\left(\int_0^T \left\|B(\cdot)u^T(t)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} dt\right)^{1/2}\left(\int_0^T 1dt\right)^{1/2} \\
&\quad + K \\
&\le \frac{1}{T}\left(\int_0^T \left\|B(\cdot)u^T(t)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} dt\right)^{1/2}\sqrt{T} + K.
\end{aligned}$$

Using (3.12), we obtain that $\frac{1}{T}\int_0^T A(\omega)x^T(t,\omega)dt$ is bounded in $L^2(\Omega,\mathbb{R}^n)$. Finally, from (2.9) we get

$$\begin{aligned}
&\left\|\frac{1}{T}\int_0^T x^T(t,\cdot)\,dt\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \\
&\qquad \le K\left(\left\|\frac{1}{T}\int_0^T A(\cdot)x^T(t,\cdot)\,dt\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|\frac{1}{T}\int_0^T C(\cdot)x^T(t,\cdot)\,dt\right\|^2_{L^2(\Omega,\mathbb{R}^n)}\right).
\end{aligned} \tag{3.15}$$

We observe that

$$\left\|\frac{1}{T}\int_0^T C(\cdot)x^T(t,\cdot)\,dt\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \le \frac{1}{T^2}\int_0^T \left\|C(\cdot)x^T(t,\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} dt$$

and by (3.12) we obtain that the second term in the right-hand side of (3.15) vanishing as T goes to infinity. Therefore, from (3.15) the term $\frac{1}{T}\int_0^T x^T(t,\omega)\,dt$ is bounded in $L^2(\Omega,\mathbb{R}^n)$.

The bound for u^T follow again by (3.12) and Cauchy–Schwarz inequality. Indeed,

$$\begin{aligned}
\left\|\int_0^T u^T(t)\,dt\right\|_{\mathbb{R}^m} &\le \int_0^T \left\|u^T(t)\right\|_{\mathbb{R}^m} dt \le \left(\int_0^T \left\|u^T(t)\right\|^2_{\mathbb{R}^m} dt\right)^{1/2}\left(\int_0^T dt\right)^{1/2} \\
&\le \sqrt{KT}\sqrt{T} = \sqrt{K}T.
\end{aligned}$$

Step 3. In this step, we will prove that

$$\frac{1}{T}\left(\int_0^T \left\|C(\cdot)(x^T(t,\cdot)-\overline{x})\right\|^2_{L^2(\Omega,\mathbb{R}^n)} dt + \int_0^T \left\|u^T(t)-\overline{u}\right\|^2_{\mathbb{R}^m} dt\right) \to 0, \tag{3.16}$$

as $T \to \infty$.

Indeed, let us consider the variables $\psi = x^T - \overline{x}$ and $p = \varphi^T - \overline{\varphi}$. Then, ψ and p solve the following problem

$$\begin{cases} \psi_t + A(\omega)\psi = B(\omega)(u^T - \overline{u}), & t \in (0,T), \\ -p_t + A^*(\omega)p = C^*(\omega)C(\omega)\psi, & t \in (0,T), \\ \psi(0,\cdot) = x_0 - \overline{x}, \quad p(T,\cdot) = -\overline{\varphi}. \end{cases} \tag{3.17}$$

Multiplying by p the first equation of (3.17) and integrating by parts over $(0,T)$, we get for each $\omega \in \Omega$

$$\begin{aligned} \langle p(0,\omega), \psi(0,\omega)\rangle_{\mathbb{R}^n} - \langle p(T,\omega), \psi(T,\omega)\rangle_{\mathbb{R}^n} = \int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{\mathbb{R}^n}\, dt \\ - \int_0^T \left\langle u^T - \overline{u}, B^*\varphi^T - B^*\overline{\varphi}\right\rangle_{\mathbb{R}^m} dt. \end{aligned} \tag{3.18}$$

Integrating over Ω, we can obtain the following

$$\begin{aligned} &\langle p(0,\cdot), \psi(0,\cdot)\rangle_{L^2(\Omega,\mathbb{R}^n)} - \langle p(T,\cdot), \psi(T,\cdot)\rangle_{L^2(\Omega,\mathbb{R}^n)} \\ &= \int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\, dt - \int_0^T \left\langle u^T - \overline{u}, \int_\Omega B^*\varphi^T d\mu - \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} dt \\ &= \int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\, dt + \int_0^T \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} dt \\ &\quad + \int_0^T \left\langle u^T, \overline{u} + \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} dt - \int_0^T \left(\langle \overline{u}, \overline{u}\rangle_{\mathbb{R}^m} + \langle B\overline{u}, \overline{\varphi}\rangle_{L^2(\Omega,\mathbb{R}^n)} \right) dt. \end{aligned}$$

We observe that $\overline{u}$ belongs to D, where we recall that D is given by (2.11). Therefore, from (3.2) we have that $\langle \overline{u}, \overline{u}\rangle_{\mathbb{R}^m} + \langle B\overline{u}, \overline{\varphi}\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0$. Thus,

$$\begin{aligned} &\int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\, dt + \int_0^T \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} dt \\ &\qquad \le \|p(0,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \|\psi(0,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \|\psi(T,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \\ &\qquad\qquad - \int_0^T \left\langle u^T, \overline{u} + \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} dt. \end{aligned} \tag{3.19}$$

Now, by the previous step we have that, up to subsequences, there exists y, v such that

$$\frac{1}{T}\int_0^T x^T(t,\cdot)dt \to y(\cdot), \quad \frac{1}{T}\int_0^T u^T(t)dt \to v.$$

From (3.14) and since the term $\frac{x^T(T,\omega)-x_0}{T}$ converges to zero as $T \to \infty$ (see (3.5)), we deduce

$$A(\omega)\left(\frac{1}{T}\int_0^T x^T(t,\cdot)\, dt\right) = B(\omega)\left(\frac{1}{T}\int_0^T u^T(t)\, dt\right) - \left(\frac{x^T(T,\cdot) - x_0}{T}\right) \to B(\omega)v,$$

in $L^2(\Omega,\mathbb{R}^n)$. Namely, we have that $v \in D$ and by Theorem 2.6

$$\langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, Bv\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0,$$

which implies

$$\frac{1}{T}\int_0^T \left\langle u^T, \overline{u} + \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} dt \to \left\langle v, \overline{u} + \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} = \langle \overline{u}, v\rangle_{\mathbb{R}^m} + \langle \overline{\varphi}, Bv\rangle_{L^2(\Omega,\mathbb{R}^n)} = 0.$$

On the other hand, we observe that by (3.13)

$$\begin{aligned}\|p(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} &= \|\varphi^T(0,\cdot) - \overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \\ &\le 2\|\varphi^T(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + 2\|\overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \\ &\le \hat{K}T + 2\|\overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}.\end{aligned}$$

Analogously,

$$\|\psi(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \le KT + 2\|\overline{x}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}.$$

Therefore, the first and second term at the right-hand side of (3.19) can be estimated as follows

$$\begin{aligned}&\|p(0,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)}\|\psi(0,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)}\|\psi(T,\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \\ &\qquad \le \left(\hat{K}T + 2\|\overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right)^{1/2}\|x_0 - \overline{x}(\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \\ &\qquad\qquad + \left(KT + 2\|\overline{x}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right)^{1/2}\|\overline{\varphi}(\cdot)\|_{L^2(\Omega,\mathbb{R}^n)}.\end{aligned}$$

That is, we obtain

$$\begin{aligned}&\frac{1}{T}\int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\,dt + \frac{1}{T}\int_0^T \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} dt \\ &\qquad \le \frac{1}{T}\left(\hat{K}T + 2\|\overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right)^{1/2}\|x_0 - \overline{x}(\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \\ &\qquad\quad + \frac{1}{T}\left(KT + 2\|\overline{x}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\right)^{1/2}\|\overline{\varphi}(\cdot)\|_{L^2(\Omega,\mathbb{R}^n)} \\ &\qquad\qquad - \frac{1}{T}\int_0^T \left\langle u^T, \overline{u} + \int_\Omega B^*\overline{\varphi} d\mu \right\rangle_{\mathbb{R}^m} dt \to 0,\end{aligned}$$

as $T \to \infty$.

Step 4. Finally, we show that

$$\frac{1}{T}\int_0^T x^T(t,\cdot)dt \quad \text{and} \quad \frac{1}{T}\int_0^T u^T(t)dt$$

converge to $\overline{x}(\cdot)$ and $\overline{u}$ in $L^2(\Omega,\mathbb{R}^n)$ and $\mathbb{R}^m$, respectively.

Indeed, from step 3, we have that

$$\left\|\frac{1}{T}\int_0^T (u^T(t) - \overline{u})\,dt\right\|^2_{\mathbb{R}^m} \le \frac{1}{T}\int_0^T \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} dt \to 0.$$

In a similar way,

$$\left\| \frac{1}{T} \int_0^T C(\cdot)\psi(t,\cdot)\, dt \right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq \frac{1}{T} \int_0^T \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\, dt \to 0.$$

Therefore,

$$\frac{1}{T} \int_0^T u^T(t)dt \to \overline{u}, \quad \frac{1}{T} \int_0^T C(\cdot)\psi(t,\cdot)dt \to 0, \tag{3.20}$$

in $\mathbb{R}^m$ and $L^2(\Omega,\mathbb{R}^n)$, respectively. From (3.15) and (2.9), and the previous convergence, we have

$$\frac{1}{T} \int_0^T (x^T(t,\cdot) - \overline{x}(\cdot))\, dt = \frac{1}{T} \int_0^T \psi(t,\cdot)\, dt \to 0, \text{ as } T \to \infty,$$

and the proof is finished. □

Corollary 1. *Let us assume that* **Hypothesis 1 and 2** *hold. Then, there exists a unique* $\overline{\varphi}$ *solution of* (2.12). *In addition, the stationary optimal control* $\overline{u}$ *is given by*

$$\overline{u} = -\int_\Omega B^* \overline{\varphi} d\mu.$$

Proof. Repeating the argument made for x^T in the previous theorem, it is possible to prove that the term

$$\frac{1}{T} \int_0^T \varphi^T(t,\cdot)dt$$

is bounded in $L^2(\Omega,\mathbb{R}^n)$. Then, up to subsequences, there exists some $\overline{\varphi} \in L^2(\Omega,\mathbb{R}^n)$ such that

$$\frac{1}{T} \int_0^T \varphi^T(t,\cdot)dt \ \to \ \overline{\varphi}(\cdot) \quad \text{ in } L^2(\Omega,\mathbb{R}^n).$$

Next, integrating over $(0,T)$ and multiplying by $\frac{1}{T}$ the second equation of (3.3), we obtain

$$\frac{1}{T} \int_0^T A^*(\cdot)\varphi^T(t,\cdot)dt = \frac{1}{T} \int_0^T C^*(Cx^T - z)dt - \frac{\varphi^T(0)}{T}. \tag{3.21}$$

From the average convergence of φ^T and x^T, the estimate (3.6) and the uniqueness of the limit, we deduce that $\overline{\varphi}$ is a solution of (3.4).

Now, since the optimal control u^T is given by $u^T = -\int_\Omega B^*\varphi^T dt$, using the average converge of u^T, we get that $\overline{u} = -\int_\Omega B^*\overline{\varphi}$. The uniqueness of $\overline{\varphi}$ follow by contradiction and the next stationary inequality which follows directly by **Hypothesis 2**

$$\|v\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\left(\left\| \int_\Omega B^* v d\mu \right\|^2_{\mathbb{R}^m} + \|A^* v\|^2_{L^2(\Omega,\mathbb{R}^n)} \right), \tag{3.22}$$

for every $v \in L^2(\Omega,\mathbb{R}^n)$. □

Our second main result, which is the following theorem, shows the average exponential turnpike property. The proof is inspired by the results obtained in [10].

Theorem 3.2. *Let us assume that* **Hypothesis 1 and 2** *hold. Let $\delta \geq 0$ be a small enough nonnegative constant. Let (u^T, x^T, φ^T) be the solution of (3.3) and $(\overline{u}, \overline{x}, \overline{\varphi})$ the corresponding stationary solution of (3.4). Then, there exists a positive constant $K = K(\delta) > 0$ (independent of T) such that*

$$\left\|x^T(t,\cdot) - \overline{x}(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|\varphi^T(t,\cdot) - \overline{\varphi}(\cdot)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K(e^{-\delta(T-t)} + e^{-\delta t}), \tag{3.23}$$

for every $t \in (0,T)$. In particular, we obtain an averaged exponential turnpike as follows

$$\left\|\mathbb{E}(x^T - \overline{x})\right\|_{\mathbb{R}^n} + \left\|\mathbb{E}(\varphi^T - \overline{\varphi})\right\|_{\mathbb{R}^n} + \left\|u^T - \overline{u}\right\|_{\mathbb{R}^m} \leq K(e^{-\delta(T-t)} + e^{-\delta t}),$$

for every $t \in (0,T)$.

Proof. We divide the proof into several steps.

Step 1. Let $\psi = x^T - \overline{x}$ and $p = \varphi^T - \overline{\varphi}$ be the solutions of the system

$$\begin{cases} \psi_t + A(\omega)\psi + B(\omega)\Big(\displaystyle\int_\Omega B^*(\omega)p(t,\omega)\,d\mu\Big) = 0, \\ -p_t + A^*(\omega)p - C^*(\omega)C(\omega)\psi = 0, \\ \psi(0,\cdot) = x_0 - \overline{x}, \quad p(T,\cdot) = -\overline{\varphi}. \end{cases} \tag{3.24}$$

We claim that there exist $K > 0$, independent of T, such that

$$\|\psi\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} + \|p\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} \leq K\Big(\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big).$$

Indeed, applying (2.7) to ψ, noting that $u^T(t) = -\int_\Omega B^*(\Omega)\varphi^T(t,\omega)d\omega$ and $\overline{u} = -\int_\Omega B^*(\Omega)\overline{\varphi}(\omega)d\omega$ and that B^* is a uniformly bounded matrix, we get

$$\|\psi\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} \leq K\Big(\int_0^T \Big(\left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} + \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big)dt + \|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big). \tag{3.25}$$

For an estimate for p, we proceed as follow. Applying the inequality (2.8) to p, noting that p solves a differential equation with right-hand side $C^*(\omega)C(\omega)\psi$ and recalling that

$$\left\|\int_\Omega B^*(\omega)p(t,\omega)d\mu\right\|^2_{\mathbb{R}^m} = \left\|\int_\Omega B^*(\omega)(\varphi^T(t,\omega) - \overline{\varphi}(\omega))d\mu\right\|^2_{\mathbb{R}^m} = \|u^T(t) - \overline{u}\|^2_{\mathbb{R}^m},$$

we deduce

$$\|p\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} \leq K\Big(\int_0^T \Big(\left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} + \|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big)dt + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}\Big). \tag{3.26}$$

Now, let us give an estimate for the two first terms at the right hand side of (3.25). From Corollary 1 we have that $\overline{u} = -\int_\Omega B^*\overline{\varphi}$. Therefore, using the identity (3.18) we get for every

$\varepsilon_1, \varepsilon_2 > 0$

$$\int_0^T \left(\|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} \right) dt \leq \frac{\varepsilon_1 \|p(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2} + \frac{\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2\varepsilon_1} + \frac{\|\psi(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2\varepsilon_2} + \frac{\varepsilon_2 \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}}{2}.$$

Using the estimates given in (3.25), (3.26) at $t = T$ and $t = 0$, respectively, we obtain

$$\begin{aligned}\int_0^T \left(\|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} \right) dt \\ \leq K \left[\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \left(\frac{1}{2\varepsilon_1} + \frac{1}{2\varepsilon_2} \right) + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \frac{(\varepsilon_1 + \varepsilon_2)}{2} \right. \\ \left. + K \left(\frac{\varepsilon_1}{2} + \frac{1}{2\varepsilon_2} \right) \int_0^T \left(\|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} \right) dt \right] \quad (3.27)\end{aligned}$$

We can take $\varepsilon_1, \varepsilon_2 > 0$ such that the last term in the right-hand side of (3.27) can be absorbed by the left-hand side. Therefore, we deduce that there exists a constant $K > 0$, independent of T, such that

$$\begin{aligned}\int_0^T \left(\|C(\cdot)\psi(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \left\|u^T(t) - \overline{u}\right\|^2_{\mathbb{R}^m} \right) dt \\ \leq K \left(\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \right). \quad (3.28)\end{aligned}$$

Therefore, from (3.25) and (3.28), there exists a positive constant $K > 0$, independent of T, such that

$$\begin{aligned}\|\psi\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} + \|p\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))} \\ \leq K \left(\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \right). \quad (3.29)\end{aligned}$$

Step 2. On the other hand, we observe that the system (3.24) with arbitrary right-hand side $f_1, f_2 \in L^2(0,T;L^2(\Omega,\mathbb{R}^n))$, can be rewrite as follows

$$\begin{pmatrix} -C^*C & -\frac{d}{dt} + A^* \\ 0 & E_T \\ \frac{d}{dt} + A & B\left(\int_\Omega B^* \cdot d\mu\right) \\ E_0 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ p \end{pmatrix} = \begin{pmatrix} f_2 \\ p(T) \\ f_1 \\ \psi(0) \end{pmatrix}, \quad (3.30)$$

where $E_0\psi := \psi(0)$ and $E_T p := p(T)$.

Denoting by Λ the matrix

$$\Lambda := \begin{pmatrix} -C^*C & -\frac{d}{dt} + A^* \\ 0 & E_T \\ \frac{d}{dt} + A & B\left(\int_\Omega B^* \cdot d\mu\right) \\ E_0 & 0 \end{pmatrix},$$

we have that the operator Λ corresponds to the two homogeneous evolution equations, that is, the adjoint equation in the first two rows and the state equation in the last two rows. Consecuently the operador Λ maps the solution to their respective initial data and source terms. Now observe that the operator denoted by Λ^{-1}, maps initial data and source terms to the corresponding state and adjoint solutions. Therefore the operator Λ^{-1} is the solution operator. Since the system (3.24) has a unique solution (by Theorem 2.2 and Corollary 1), it implies that operator Λ^{-1} is well defined as a mapping from $(L^2(0,T;L^2(\Omega,\mathbb{R}^n))\times L^2(\Omega,\mathbb{R}^n))^2$ to $(C([0,T];L^2(\Omega,\mathbb{R}^n))^2$.

According to what was done to prove (3.29), there exists a constant $K>0$, independent of T, such that

$$\|\psi\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))}+\|p\|^2_{C([0,T];L^2(\Omega,\mathbb{R}^n))}\leq K\Big(\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}+\|f_1\|^2_{L^2(0,T;L^2(\Omega),\mathbb{R}^n)}+\|f_1\|^2_{L^2(0,T;L^2(\Omega),\mathbb{R}^n)}\Big). \tag{3.31}$$

From (3.31) we obtain that there exists a constant $K>0$, independent of T, such that

$$\|\Lambda^{-1}\|_{\mathcal{L}((L^2(0,T;L^2(\Omega,\mathbb{R}^n))\times L^2(\Omega,\mathbb{R}^n))^2,(C([0,T];L^2(\Omega,\mathbb{R}^n))^2)}\leq K. \tag{3.32}$$

In the following denote $\mathcal{W}=(L^2(0,T;L^2(\Omega,\mathbb{R}^n))\times L^2(\Omega,\mathbb{R}^n))^2$ and $\mathcal{X}=C([0,T];L^2(\Omega,\mathbb{R}^n))$.

Step 3. In this step we will prove the turnpike property (3.23). Indeed, let us consider the following variables

$$\begin{cases}\widehat{\psi}=\dfrac{\psi}{e^{-\delta(T-t)}+e^{-\delta t}},\\ \widehat{p}=\dfrac{p}{e^{-\delta(T-t)}+e^{-\delta t}},\end{cases}$$

where $\delta>0$. It is immediate that $\widehat{\psi}$ and $\widehat{p}$ solves the system

$$\begin{cases}\widehat{\psi}_t+A(\omega)\widehat{\psi}=\dfrac{B(\omega)(u^T-\overline{u})}{e^{-\delta(T-t)}+e^{-\delta t}}+\widehat{\psi}\delta h(t), & t\in(0,T)\\ -\widehat{p}_t+A^*(\omega)\widehat{p}=C^*(\omega)C(\omega)\widehat{\psi}-\widehat{p}\delta h(t), & t\in(0,T)\\ \widehat{\psi}(0,\cdot)=\dfrac{\psi(0,\cdot)}{1+e^{-\delta T}},\quad \widehat{p}(T,\cdot)=\dfrac{p(T,\cdot)}{1+e^{-\delta T}},\end{cases} \tag{3.33}$$

where $h(t)$ denotes

$$h(t)=\frac{e^{-\delta t}-e^{-\delta(T-t)}}{e^{-\delta t}+e^{-\delta(T-t)}}.$$

Using the definition of Λ, we can rewrite the respective equations as

$$\left[\Lambda-\delta\begin{pmatrix}h(t)&0\\0&0\\0&-h(t)\\0&0\end{pmatrix}\right]\begin{pmatrix}\widehat{\psi}\\ \widehat{p}\end{pmatrix}=\frac{1}{1+e^{-\delta T}}\begin{pmatrix}0\\p(T)\\0\\\psi(0)\end{pmatrix}. \tag{3.34}$$

We observe that $\|\mathcal{P}\|_{\mathcal{L}(\mathcal{X}^2)}\leq 1$, where

$$\mathcal{P}=\begin{pmatrix}h(t)&0\\0&0\\0&-h(t)\\0&0\end{pmatrix}.$$

Now, letting

$$\mathcal{Z} := \begin{pmatrix} \widehat{\psi} \\ \widehat{p} \end{pmatrix}, \quad \mathcal{Y} := \frac{1}{1+e^{-\delta T}} \begin{pmatrix} 0 \\ p(T) \\ 0 \\ \psi(0) \end{pmatrix},$$

system (3.34) can be written as

$$(I - \delta\Lambda^{-1}\mathcal{P})\mathcal{Z} = \Lambda^{-1}\mathcal{Y}. \tag{3.35}$$

Choosing $\gamma := \delta\|\Lambda^{-1}\|_{\mathcal{L}(\mathcal{W},\mathcal{X}^2)}$, which is independent of T since the norm of Λ^{-1} is independent of T (see (3.32)), we obtain that $\delta\Lambda^{-1}\mathcal{P}$ is a contraction. Therefore, the existence and uniqueness of solutions for operator equations as in (3.35) can be established in terms of Neumann series (see Theorem 2.14 of [13]). That is,

$$\left\|(I - \delta\Lambda^{-1}\mathcal{P})^{-1}\right\|_{\mathcal{L}(\mathcal{W},\mathcal{X}^2)} \leq \frac{1}{1-\gamma}.$$

Namely, we obtain

$$\begin{aligned} &\|\widehat{\psi}\|^2_{\mathcal{X}} + \|\widehat{p}\|^2_{\mathcal{X}} \\ &\qquad \leq \frac{\|\Lambda^{-1}\|_{\mathcal{L}(\mathcal{W},\mathcal{X}^2)}}{1-\gamma} \frac{1}{1+e^{-\delta T}} \Big(\|\psi(0,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|p(T,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \Big), \end{aligned}$$

and since $\mathcal{X} = C([0,T];L^2(\Omega,\mathbb{R}^n))$, we get that for every $t \in [0,T]$

$$\|x^T(t,\cdot) - \overline{x}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|\varphi^T(t,\cdot) - \overline{\varphi}(\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)} \leq K\Big(e^{-\delta(T-t)} + e^{-\delta t}\Big), \tag{3.36}$$

where $K > 0$ is a constant independent of T. That is, we obtain the desired estimate (3.23).

Step 4. Finally, from (3.36) we can obtain the averaged exponential turnpike property. Indeed,

$$\begin{aligned} \|\mathbb{E}(x^T) - \mathbb{E}(\overline{x})\|_{\mathbb{R}^n} &= \left\| \int_\Omega x^T d\mu - \int_\Omega \overline{x} d\mu \right\|_{\mathbb{R}^n} \\ &\leq \int_\Omega \|x^T - \overline{x}\|_{\mathbb{R}^n} d\mu \leq \left(\int_\Omega d\mu \right)^{1/2} \left(\int_\Omega \|x^T - \overline{x}\|^2_{\mathbb{R}^n} d\mu \right)^{1/2} \\ &\leq K\Big(e^{-\delta(T-t)} + e^{-\delta t}\Big). \end{aligned}$$

Analogous, for φ^T we have

$$\|\mathbb{E}(\varphi^T) - \mathbb{E}(\overline{\varphi})\|_{\mathbb{R}^n} \leq K\Big(e^{-\delta(T-t)} + e^{-\delta t}\Big).$$

Finally, for the control estimate, we observe that

$$\begin{aligned} \|u^T(t) - \overline{u}\|^2_{\mathbb{R}^m} &= \left\| \int_\Omega B^* p \, d\mu \right\|^2_{\mathbb{R}^m} \\ &\leq K \left\| \int_\Omega p \, d\mu \right\| \leq K \|p(t,\cdot)\|^2_{L^2(\Omega,\mathbb{R}^n)}. \end{aligned}$$

Therefore, from all the previous computations we get the desired result and the proof is finished. □

We conclude this section with the following observations.

Remark 2.

1. It is immediately noted that in Theorems 3.1 and 3.2 one can also consider the case where $x_0, z \in L^2(\Omega, \mathbb{R}^n)$. The proof of both Theorems applies replacing the terms $\|x_0\|_{\mathbb{R}^n}, \|z\|_{\mathbb{R}^n}$ by $\|x_0(\cdot)\|_{L^2(\Omega,\mathbb{R}^n)}, \|z\|_{L^2(\Omega,\mathbb{R}^n)}$, respectively.
2. Our main results, namely Theorems 3.1 and 3.2, are similar to those obtained in [16] in the context of ordinary differential equations without randomness. The authors in [16] proved their results under the assumptions that the dynamics are controllable, and the cost functional is observable. In our setting, the right notion of control will be *average control* and *average observability* conditions, concepts introduced in [26]. However, we do not know if the average exponential turnpike property can be proved only assuming the average control and average observability conditions.
3. Finally, it is interesting to note that our second main result, namely estimate (3.23) in Theorem 3.2, means that the turnpike holds for each parameter separately. This is a strong consequence of our main assumptions Hypothesis 1 and 2. Also, it is interesting that this holds with an optimal control independent of the random parameter. However, it captures at the same time, all the information of the adjoint system (in an average sense).

4 Numerical experiments

In this section, we will perform some numerical experiments to validate the turnpike property in average. We focus our attention on the particular case $A(\omega) = \alpha(\omega)A$ and $B(\omega) = \beta(\omega)B$, with A and B constant matrices and α, β scalar random variables. Besides, the observability matrix will be independent of ω. In addition, we consider a discrete sample space Ω.

4.1 Example 1

Let $\Omega = \mathbb{R}^+$ and β be a random variable with exponential distribution with parameter $\lambda = 7$ i.e. $\beta \sim exp(7)$. We consider the following optimal control problem

$$\min\left\{J^T(u) = \frac{1}{2}\int_0^T \left\{\|u(t)\|^2_{\mathbb{R}^m} + \|Cx(t,\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)}\right\} dt\right\},$$

subject to x solves the system

$$\begin{cases} x_t + Ax = \beta(\omega)Bu \quad t \in (0,T), \\ x(0) = x_0, \end{cases}$$

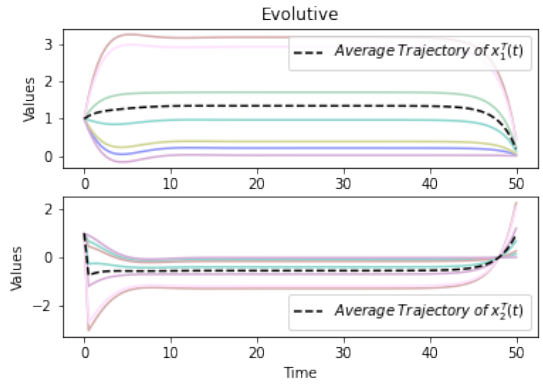
where

$$A = \begin{pmatrix} 0.2 & 0.5 \\ -0.5 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

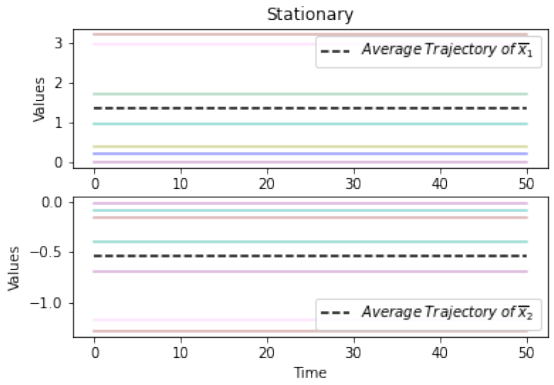
The corresponding stationary optimal control problem is

$$\min\left\{J^s(u) = \frac{1}{2}\left(\|u\|^2_{\mathbb{R}^m} + \|Cx(\cdot) - z\|^2_{L^2(\Omega,\mathbb{R}^n)}\right)\right\},$$

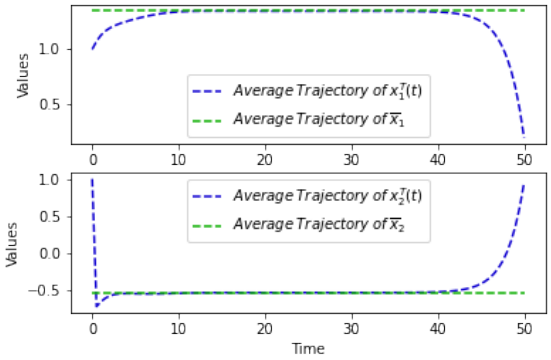
subject to x solves the problem $Ax = \beta(\omega)Bu$.



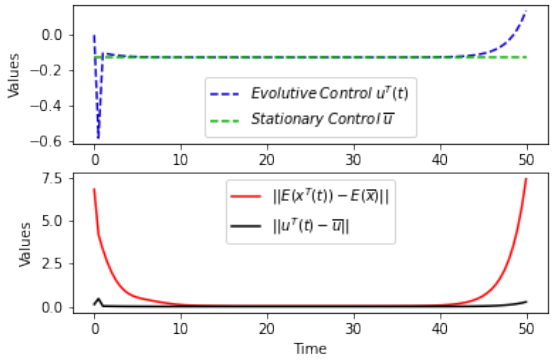
(a) Evolutionary trajectories.



(b) Stationary trajectories.



(a) average trajectories.



(b) Norm difference between evolutionary and stationary solutions and controls.

Figure 1: Evolutionary v/s stationary systems.

It is a straightforward computation that matrices A, B and C satisfy the **Hypothesis 1 and 2**.

We compute the optimal solutions (x^T, u^T) in time $T = 50$, and $(\overline{x}, \overline{u})$, by using the Gekko library on Python and considering seven realizations of the random variable β, which was generated using the numpy.random library on Python. Gekko is a library specializing in dynamic optima problems for mixed-integer, nonlinear, and differential algebraic equations (DAE) problems [1]. In the case of the evolution problem, we use the Gekko module to solve optimal control problems (imode 6) with 100 time steps. Its module is based on simultaneous strategies for dynamic optimization [3]. More precisely, in this mode, Gekko uses orthogonal collocation of finite elements [5] this is a form of implicit Runga–Kutta methods.

The trajectory solutions of both problems are collected in Figure 1 (see (A) and (B)). In the figure above, the color lines in (A) and (B) are different realizations of the random variable. We observe that the turnpike property on average can be observed on Figure 1 (see (C)). As expected, except transient initial and final arcs, the expected value of $x^T = (x_1^T, x_2^T)$ remains close to the corresponding expectation of the steady-state $\overline{x} = (\overline{x}_1, \overline{x}_2)$, as in Theorem 3.2. Besides, according to Theorem 3.2, in (D) we can observe that the optimal control u^T remains close to the optimal stationary control $\overline{u}$.

Remark 3. In this example, if we take average to the equation, we obtain the following system

$$\begin{cases} y_t + Ay = \hat{B}u & t \in (0,T), \\ y(0) = x_0. \end{cases}$$

where $y = \mathbb{E}(x)$ and $\hat{B} = \mathbb{E}(B)$. Then if the pair $(A, \hat{B})$ is controllable, and the pair (A, C) is observable, the turnpike property follow directly from [16].

4.2 Example 2

In this example, we consider the same functional J^T as in the previous example. However, x solves the following system

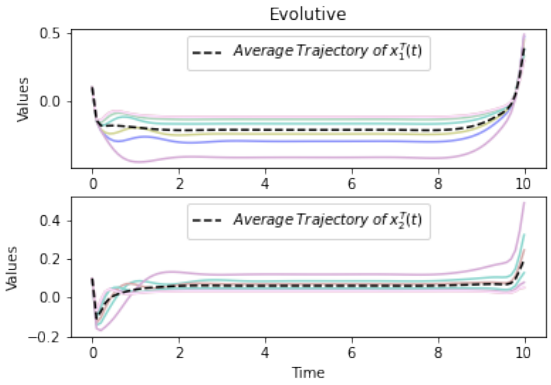
$$\begin{cases} x_t + \alpha(\omega)Ax = Bu & t \in (0,T), \\ x(0) = x_0. \end{cases}$$

The corresponding stationary system is $\alpha(\omega)Ax = Bu$, where in this case

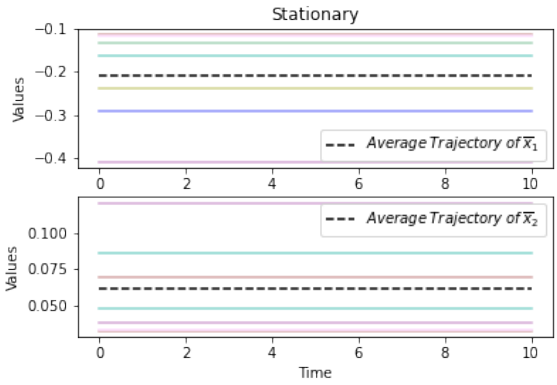
$$A = \begin{pmatrix} 2 & -5 \\ 5 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}.$$

Contrary to the previous case, we consider $\alpha \sim Unif([1/2, 2])$. Once again, the matrices A, B and C satisfy the **Hypothesis 1 and 2**.

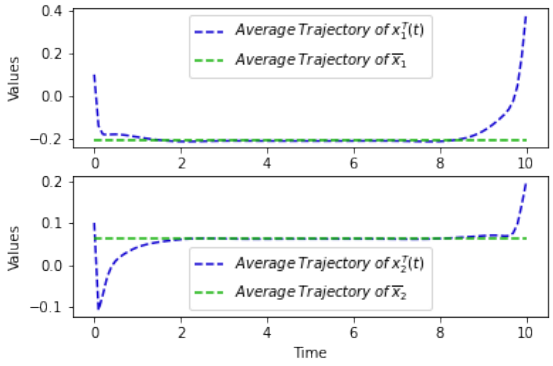
As in the example 4.1, we compute the optimal evolutionary solution $x^T = (x_1^T, x_2^T)$ and stationary solution $\overline{x} = (\overline{x}_1, \overline{x}_2)$ by using Gekko library and we generate seven realizations of the random variable using numpy.random library on Python. However, in this example, we compute the optimal evolutionary solution in time $T = 10$. The result is drawn in Figure 2 (see (A) and (B)).



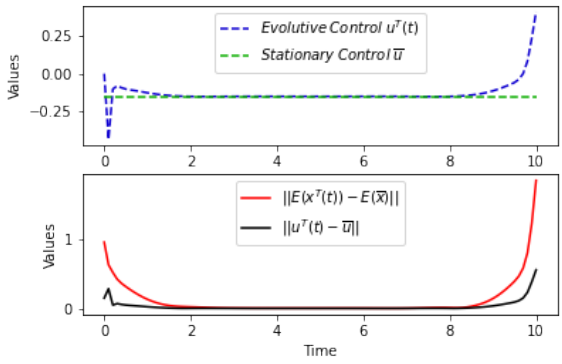
(a) Evolutionary trajectories.



(b) Stationary trajectories.



(c) average trajectories.



(d) Norm difference between evolutionary and stationary solutions and controls.

Figure 2: Evolutionary v/s stationary systems.

As in the previous example, the colored lines in (A) and (B) are different realizations of the random variables. Note that (see Figure 2 (A) and (C)) the average dynamics is more perturbed than in the previous example. However, as expected, except transient initial and final arcs, the optimal evolutionary pair (x^T, u^T) remains close to the steady state $(\overline{x}, \overline{u})$.

4.3 Example 3

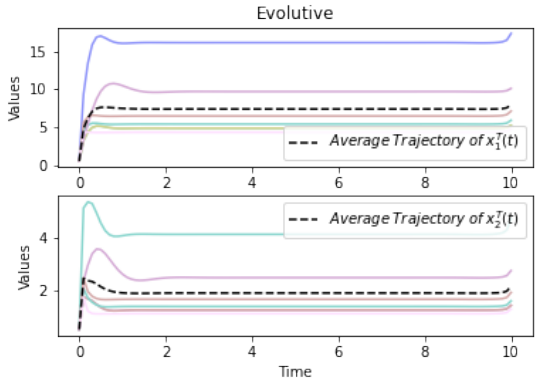
In this final example, we consider the same functional J^T as in example 1, however x solves the problem

$$\begin{cases} x_t + \alpha(\omega)Ax = \beta(\omega)Bu & t \in (0,T), \\ x(0) = x_0. \end{cases}$$

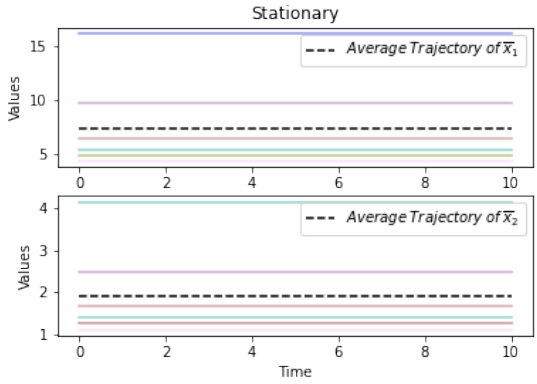
And the corresponding stationary problem $\alpha(\omega)Ax = \beta(\omega)Bu$, where

$$A = \begin{pmatrix} 2 & -5 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 7 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

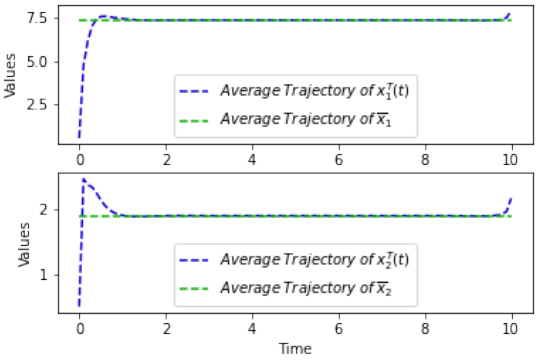
We consider α, $\beta \sim Poisson(5)$. Again computing the optimal solution by using Gekko and generating seven realizations of the random variables using the numpy.random library, we obtain the following trajectories, see Figure 3, where the colored lines, in (A) and (B) are the different realizations of the random variables.



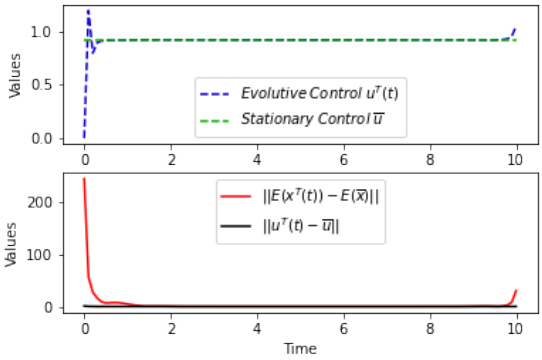
(a) Evolutionary trajectories.



(b) Stationary trajectories.



(c) average trajectories.



(d) Norm difference between evolutionary and stationary solutions and controls.

Figure 3: Evolutionary v/s stationary systems.

We note that the dynamics are more perturbed than in the previous two examples, because in this case, the matrices A and B are both random. Nevertheless, we can see the turnpike property for the average dynamics.

The code used in these numerical examples is free available in the repository https://github.com/Martinshs/Turnpike-property GitHub–Average Turnpike Property.

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