NONOVERLAPPING DOMAIN DECOMPOSITION FOR INSTANTANEOUS OPTIMAL CONTROL OF FRICTION DOMINATED FLOW IN A GAS-NETWORK

GÜNTER LEUGERING

ABSTRACT. We consider a non-overlapping domain decomposition method for an optimal control problem related to the flow of gas in a pipe network. The equations of motions are taken to be represented by a friction dominated model derived from a semi-linear approximation of the fully nonlinear isothermal Euler gas equations. This involves a p-Laplace-type problem on the graph with $p = \frac{3}{2}$. We continue the work [11] where such a problem has been discussed in the context of an instantaneous control strategy. We provide a non-overlapping domain decomposition in the spirit of P.L. Lions for elliptic problems and extend the method to the first order optimality system.

1. INTRODUCTION

1.1. **Modeling of gas flow in a single pipe.** The Euler equations are given by a system of nonlinear hyperbolic partial differential equations (PDEs) which represent the motion of a compressible non-viscous fluid or a gas. They consist of the continuity equation, the balance of moments and the energy equation. The full set of equations is given by (see [5, 12, 13, 16]). Let ρ denote the density, v the velocity of the gas and p the pressure. We further denote g the gravitational constant, λ the friction coefficient of the pipe, D the diameter, a the area of the cross section. The state variables of the system are ρ, the flux $q = \rho v$. We also denote c the the speed of sound, i.e. $c^2 = \frac{\partial p}{\partial \rho}$ (for constant entropy). For natural gas we have 340 $\frac{\mathrm{m}}{\mathrm{s}}$. In particular, in the subsonic case ($|v| < c$), the one which we consider in the sequel, two boundary conditions have to be imposed on the left end and one at the right end of the pipe. We consider here the isothermal case only. Thus, for horizontal pipes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) &= -\frac{\lambda}{2D}\rho v\,|v|\,. \end{aligned} \tag{1.1}$$

In the particular case, where the we have a constant speed of sound $c = \sqrt{\frac{p}{\rho}}$, for small velocities $|v| \ll c$, we arrive at the semi-linear model

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) &= 0 \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial p}{\partial x} &= -\frac{\lambda}{2D}\rho v\,|v|\,. \end{aligned} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35J70,49J20,49J45,93C73,65M55,65N55.

Key words and phrases. Optimal control, gas networks, p-Laplace problem on a graph, optimality system, domain decomposition.

The author was supported by the DFG-TRR 154 "Modellierung Simulation und Optimierung am Beispiel von Gasnetzwerken" (A05) .

If we further neglect the inertia with respect to the flux and introduce $q = \rho v a$, we arrive at

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{c^2}{a}\frac{\partial}{\partial x} q &= 0 \\ \frac{\partial p^2}{\partial x} &= -\frac{\lambda c^2}{D a^2} q\,|q| =: -\gamma^2 q\,|q|\,. \end{aligned} \tag{1.3}$$

We now set $y := p^2$ and obtain from the second equation in (1.3)

$$q = -\frac{1}{\gamma}\frac{\frac{\partial y}{\partial x}}{\sqrt{\left|\frac{\partial y}{\partial x}\right|}}.$$

With $\alpha := \frac{\gamma a}{c}$ we obtain

$$\alpha\frac{\partial}{\partial t}\frac{y}{\sqrt{|y|}} - \frac{\partial}{\partial x}\frac{\frac{\partial y}{\partial x}}{\sqrt{\left|\frac{\partial y}{\partial x}\right|}} = 0. \tag{1.4}$$

We introduce the monotone function $\beta(s) := \frac{s}{\sqrt{|s|}}$. With this (1.4) reads as

$$\alpha\frac{\partial}{\partial t}\beta(y) - \frac{\partial}{\partial x}\beta(\frac{\partial y}{\partial x}) = 0. \tag{1.5}$$

It is also possible to write this down in the p-Laplace format: (1.4) reads as

$$\alpha\frac{\partial}{\partial t}\left(|y|^{p-2}y\right) - \frac{\partial}{\partial x}\left(|\frac{\partial y}{\partial x}|^{p-2}\frac{\partial y}{\partial x}\right) = 0, \tag{1.6}$$

where $p = \frac{3}{2}$. Equation (1.6) has come to be known as doubly nonlinear parabolic equation of p-Laplace type. See e.g. [14], [4]. Notice that $p < 2$ and that the system is, therefore, singular for $\frac{\partial}{\partial x}y(x) = 0$. For $p > 2$ such equations exhibit instead degeneration. Equations similar to (1.5) have been considered in the literature, see e.g. [2, 3]. In this contribution, we aim at a discussion of such equations together with optimal control problems on networks. A more recent study of doubly nonlinear parabolic equations in the context of friction dominated flow has been provided in [1]. Equations of the type (1.5) are known to exhibit positive solutions and satisfy a maximum principle. As a matter of fact, to the best knowledge of the authors, there are no studies on optimal control of such systems on general graphs available from the literature besides [11]. We note that the doubly nonlinear parabolic problem associated with the friction-dominated flow on a network has been considered in the thesis [15]. Optimal control problems for the p-Laplace operator have been studied since the 1980ies, see e.g. [6]. Moreover, in [17] an optimal control problem for the p-Laplace equation $p \geq 2$ has been recently considered. See also [8] for a problem of optimal control in the coefficient for the p-Laplace equation, again for $p \geq 2$.

The plan of this article is to first recall the network modelling from [11] (also presented in [15]), formulate the optimal control problem together with its instantaneous (rolling horizon) approximation via a time discretization. The resulting static optimal control for each time step is then turned into the corresponding first order optimality system, which, in turn, is then handled via the proposed non-overlapping domain decomposition. We note that such non-overlapping domain decompositions have not been considered in the literature so far with the exception of [9], where, for $p \geq 2$, a similar problem without control has been considered, however, with a different updating rule that is more related to a Gauß-Seidel-type iteration based on a Peacman-Rachford scheme and, hence, is not completely parallel.

1.2. **Network modeling.** Let $G = (V, E)$ denote the graph of the gas network with vertices (nodes) $V = \{n_1, n_2, \dots, n_{|V|}\} = \{n_j | j \in \mathcal{J}$ an edges $E = \{e_1, e_2, \dots, e_{|E|}\} = \{e_i | i \in \mathcal{I}\}$. We associate to each edge a direction.

$$d_{ij} = \begin{cases} -1, & \text{if node } n_j \text{ if the the edge } e_i \text{ starts at node } n_j e_i, \\ +1, & \text{if node } n_j \text{ if the edges } e_i \text{ end at node } n_j e_i, \\ 0, & \text{else.} \end{cases}$$

The pressure variables $y_i(n_j)$ coincide for all $i \in \mathcal{I}_j := \{i \in 1, \dots E | d_{ij} \neq 0\}$. We express the transmission conditions at the nodes in the following way. We introduce the edge degree $d_j := |\mathcal{I}_j|$. Then the continuity conditions read as follows

$$y_i(n_j, t) = y_k(n_j, t), \ \forall i, k \in \mathcal{I}_j, \ d_j > 1. \tag{1.7}$$

The nodal balance equation for the fluxes can be written as the classical Kirchhoff-type condition

$$\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j, t)) = 0, \ d_j > 1. \tag{1.8}$$

We use the d_j in order to decompose the index set for nodes $\mathcal{J}$ into $\mathcal{J} = \mathcal{J}^M \cup \mathcal{J}^S$, where $\mathcal{J}^M = \{j \in \mathcal{J} | d_j > 1\}$ represents the multiple nodes and $\mathcal{J}^S = \{j \in \mathcal{J} | d_j = 1\}$ the simple nodes. According to Dirichlet or Neumann boundary conditions a the simple nodes, we further decompose $\mathcal{J}^S = \mathcal{J}_D^S \cup \mathcal{J}_N^S\}$. We summarize the equations as follows:

$$\begin{aligned} &\alpha_i \partial_t \beta(y_i(x,t)) - \partial_x \left(\beta(\partial_x y_i(x,t))\right) = u_i, \ i \in \mathcal{I}, \ x \in (0, \ell_i), \ t \in (0,T) \\ &y_i(n_j, t) = y_k(n_j, t), \ \forall i, k \in \mathcal{I}_j, \ j \in \mathcal{J}^M, \ t \in (0,T) \\ &\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i(n_j, t)) = 0, \ j \in \mathcal{J}^M, \ t \in (0,T) \\ &y_i(n_j, t) = 0, i \in \mathcal{I}_j, j \in \mathcal{J}_D^S, \ t \in (0,T) \\ &d_{ij} \beta(\partial_x y_i(n_j, t) = u_j(t), \ \ i \in \mathcal{I}_j, j \in \mathcal{J}_N^S, \ t \in (0,T) \\ &p_i(x, 0) = p_{i,0}(x), \ q_i(x, 0) = q_{i0}(x), \ x \in (0, \ell_i), \end{aligned} \tag{1.9}$$

where the functions $u_i, i \in \mathcal{I}, u_j, j \in\in \mathcal{I}_j, j \in \mathcal{J}_N^S$ serve as controls.

Optimal control problems and outline. We are now in the position to formulate optimal control problems on the level of the gas networks. We first describe the general format for an optimal control problem. This involves a cost function that assigns to each admissible pair (y, u) a 'cost' $I(y, u)$, which is represented on each individual edge by a contribution on the state $I_i(y)$ and the controls acting at simple nodes. The typical example, the one that we will use in the sequel is given by

$$I_i(y_i)(x) := \frac{1}{r} |y_i(x) - y_i^d(x)|^r, \ x \in (0, \ell_i), r \in \{\frac{3}{2}, 2\}.$$

$$\begin{aligned} \min_{(y,u) \in \Xi} I(y, u) := &\sum_{i \in \mathcal{I}} \int_0^T \int_0^{\ell_i} I_i(y_i) dx dt + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} \int_0^T |u_j(t)|^2 dt \\ &s.t. \\ &(y, u) \text{ satisfies (1.9)}, \end{aligned} \tag{1.10}$$

$$\Xi := \{(y,u) : \underline{y}_i \le y_i \le \overline{y}_i,\ i \in \mathcal{I}, \underline{u_j} \le u_j \le \overline{u_j}, j \in \mathcal{J}_N^S\}. \tag{1.11}$$

In (1.11), the quantities $\underline{y}_i, \overline{y}_i$ are given constants that determine the feasible pressures and flows in the pipe i, while $\underline{u}_i, \overline{u}_i$ describe control constraints. In the continuous-time case the inequalities are considered as being satisfied for all times and everywhere along the pipes. To the best knowledge of the authors, for problem (1.10) - in particular with the additional constraints (1.11)- no published result on well-posedness and characterization of optimal controls seems to be available in the literature. As in [11], we penalize the control costs using $\nu > 0$ and restrict ourselves to time discretizations of (1.10) and, in fact, to the instantaneous control regime that has come to be known also as rolling horizon problem.

1.3. **Time discretization.** We, therefore, consider the time discretization (1.9) such that $[0,T]$ is decomposed into break points $t_0 = 0 < t_1 < \cdots < t_N = T$ with widths ${}^{\Delta}t_n := t_{n+1} - t_n, n = 0, \ldots, N-1$ (we use $N+1$ as the number of break points which is not related to N as indicating Neumann conditions). Accordingly, we denote $y_i(x,t_n) := y_{i,n}(x),, n = 0, \ldots, N-1$. We consider a semi-implicit Euler scheme.

$$\begin{aligned}
\frac{1}{\Delta t}\beta(y_{i,n+1})(x) - \partial_x\left(\beta(\partial_x y_{i,n+1}(x))\right) &= \frac{1}{\Delta t}\beta(y_{i,n})(x),\ i \in \mathcal{I},\ x \in (0,\ell_i)\\
y_{i,n+1}(n_j) &= y_{k,n+1}(n_j),\ \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M\\
\sum_{i\in\mathcal{I}_j} d_{ij}\beta(\partial_x y_{i,n+1})(n_j) &= 0,\ j \in \mathcal{J}^M\\
\beta(\partial_x y_{i,n+1})(n_j) &= u_{j,n+1},\ d_j = 1,\ i \in \mathcal{I}_j,\ j \in \mathcal{J}_N^S\\
y_{i,n+1}(n_j) &= 0, i \in \mathcal{I}_j,\ j \in \mathcal{J}_D^S\\
y_{i,0}(x) &= y_{i,0}(x), i \in \mathcal{I},\ x \in (0,\ell_i),\ n = 1,\ldots,N-1.
\end{aligned} \tag{1.12}$$

We then obtain the optimal control problem on the time-discrete level:

$$\begin{aligned}
&\min_{(y,u)} I(y,u) := \sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\int_0^{\ell_i} I_i(y_{i,n})dx + \frac{\nu}{2}\sum_{j\in\mathcal{J}_N^S}\sum_{n=1}^{N}|u_j(n)|^2\\
&s.t.\\
&(y,u) \text{ satisfies (1.12)}
\end{aligned} \tag{1.13}$$

It is clear that (1.13) involves all time steps in the cost functional. We would like to reduce the complexity of the problem even further. To this aim, we consider the so-called *instantaneous control* regime. This amounts to reducing the sums in the cost function of (1.13) to the time-level t_{n+1}. This strategy has also come to be known as *rolling horizon* approach, the simplest case of the *moving horizon* paradigm. Thus, for each $n = 1, \ldots, N-1$ and given $y_{i,n}$, we consider the problems

$$\begin{aligned}
&\min_{(y,u)} I(y,u) := \sum_{i\in\mathcal{I}}\int_0^{\ell_i} I_i(y_i)dx + \frac{\nu}{2}\sum_{j\in\mathcal{J}_N^S}|u_j|^2\\
&s.t.\\
&(y,u) \text{ satisfies (1.12) at time level } n+1.
\end{aligned} \tag{1.14}$$

It is now convenient to discard the actual time level $n+1$ and redefine the states at the former time as input data. To this end, we replace $\alpha_i := \frac{1}{\Delta t}$, $f_i^1 := \alpha_i \beta(y_{i,n})$, rewrite (1.12) as

$$
\begin{aligned}
\alpha_i \beta(y_i)(x) - \partial_x \left(\beta(\partial_x y_i(x))\right) &= f_i^1(x),\ i \in \mathcal{I},\ x \in (0, \ell_i) \\
y_i(n_j) &= y_k(n_j),\ \forall i,k \in \mathcal{I}_j,\ j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x y_i)(n_j) &= 0,\ j \in \mathcal{J}^M \\
\beta(\partial_x y_i)(n_j) &= u_j,\ d_j = 1,\ i \in \mathcal{I}_j,\ j \in \mathcal{J}_N^S \\
y_i(n_j) &= 0, i \in \mathcal{I}_j,\ j \in \mathcal{J}_D^S
\end{aligned}
\tag{1.15}
$$

We now consider in the rest of the paper the following optimal control problem:

$$
\begin{aligned}
&\min_{(y,u)} I(y,u) := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} I_i(y_i) dx + \frac{\nu}{2} \sum_{j \in \mathcal{J}_N^S} |u_j|^2 \\
&s.t. \\
&(y,u) \text{ satisfies (1.15).}
\end{aligned}
\tag{1.16}
$$

Wellposedness of (1.15) has been discussed in [11]. We recall the results as follows.

Theorem 1.1 ([11]). *For $f \in \Pi_{i \in \mathcal{I}} L^3(0, \ell_i)$, $u \in \mathbb{R}^{|\mathcal{J}_N^S|}$, problem* (1.15) *admits a unique weak solution $y \in V$.*

Even though, the mapping $\beta(\cdot)$ is differentiable in $\mathbb{R} \setminus \{0\}$, the control-to-state-mapping $u \to y^u$ is not Gâteaux differentiable for $p < 2$. This has already been observed in [7]. However, the control-to-state-map is continuous. By the continuity of $\beta(\cdot)$ and the strong convergence of y^k to y^0 in V, we obtain

Theorem 1.2 ([11]). *The mapping $u \to y^u$, where y^u solves* (1.15) *is continuous between $\mathbb{R}^{|\mathcal{J}_N^S|}$ and V.*

2. Optimal control

As for the existence of optimal pairs (y,u) for the optimal control problem (1.16), we cite

Theorem 2.1 ([11]). *The optimal control problem* (1.16) *admits a unique solution $(\bar{y}, \bar{u}) \in V \times \mathbb{R}^{|\mathcal{J}_N^S|}$.*

Theorem 2.2 ([11]). *There exists $\bar{p} \in V$ satisfying together with the optimal pair $(\bar{y}, \bar{u})$ the first order optimality condition.*

$$
\begin{aligned}
\alpha_i \beta(\bar{y}_i) - \partial_x \left(\beta(\partial_x \bar{y}_i)\right) &= -\frac{1}{\nu} p_i,\ i \in \mathcal{I},\ x \in (0, \ell_i) \\
\alpha_i \beta'(\bar{y}_i)\bar{p}_i - \partial_x \left(\beta'(\partial_x \bar{y}_i \partial_x \bar{p}_i)\right) &= \kappa(|\bar{y}_i - y_i^d|^{p-2}(\bar{y}_i - y_i^d),\ i \in \mathcal{I},\ x \in (0, \ell_i) \\
\bar{y}_i(n_j) = \bar{y}_k(n_j),\ \forall i, k \in \mathcal{I}_j,\ \bar{p}_i(n_j) &= \bar{p}_k(n_j),\ \forall i, k \in \mathcal{I}_j,\ j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} \beta(\partial_x \bar{y}_i)(n_j) &= 0,\ j \in \mathcal{J}^M \\
\sum_{i \in \mathcal{I}_j} d_{ij} \beta'(\partial_x \bar{y}_i)(n_j) \partial_x \bar{p}_i(n_j) &= 0,\ j \in \mathcal{J}^M \\
\beta(\partial_x \bar{y}_i)(n_j) = \bar{u}_j,\ d_j = 1,\ i \in \mathcal{I}_j,\ j &\in \mathcal{J}_N^S \\
\beta'(\partial_x \bar{y}_i)(n_j) \partial_x \bar{p}_i(n_j) = 0,\ d_j = 1,\ i \in \mathcal{I}_j,\ j &\in \mathcal{J}_N^S \\
\bar{y}_i(n_j) = 0, \bar{p}_i(n_j) = 0\ i \in \mathcal{I}_j,\ j &\in \mathcal{J}_D^S \\
\bar{p}_i^{\epsilon}(n_j) + \nu \bar{u}_j = 0,\ i \in \mathcal{I}_j, j &\in \mathcal{J}_N^S.
\end{aligned} \tag{2.1}
$$

3. Domain decomposition

We now embark on our domain decomposition method. To fix ideas and for the sake of simplicity, we take a serial network consisting of two links first and discuss cutting out stars from a network afterwards. Moreover, also for the sake of simplicity, we discuss in these notes distributed controls only. Boundary controls will be covered in a forthcoming publication.

We introduce the two consecutive intervals $I_1 := (0, 1), I_2 := (1, 2)$ and look into the optimal control problem

$$
\begin{aligned}
&\min_{(u,y)} \frac{\kappa}{r} \sum_{i=1}^{2} \int_0^{\ell_i} |y_i - y_i^d|^r dx + \frac{\nu}{2} \sum_{i=1}^{2} \int_0^{\ell_i} |u_i|^2 dx \\
&\text{s.t.} \\
&\beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i) = u_i, i = 1, 2,\ x \in I_i \\
&y_1(0) = 0,\ y_2(2) = 0, \\
&y_1(1) = y_2(1), \beta_1(\partial_x y_1)(1) = \beta_2(\partial_x y_2)(1).
\end{aligned} \tag{3.1}
$$

We easily derive the corresponding optimality system.

$$
\begin{aligned}
\beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i)) &= -\frac{1}{\nu} p_i, i = 1, 2,\ x \in I_i \\
\beta_i'(y_i) p_i - \partial_x(\beta_i'(\partial_x y_i)\partial_x p_i) &= \kappa(|y_i - y_i^d|^{r-2}(y_i - y_i^d), i = 1, 2,\ x \in I_i \\
y_1(0) = 0,\ y_2(2) &= 0, \\
p_1(0) = 0,\ p_2(2) &= 0, \\
y_1(1) = y_2(1), \beta_1(\partial_x y_1)(1) &= \beta_2(\partial_x y_2)(1), \\
p_1(1) = p_2(1), \beta_1'(\partial_x y_1)(1)\partial_x p_1(1) &= \beta_2'(\partial_x y_2(1))\partial_x p_2(1).
\end{aligned} \tag{3.2}
$$

We propose the following iterative domain decomposition method.

$$
\begin{aligned}
&\beta_i(y_i^{n+1}) - \partial_x(\beta_i(\partial_x y_i^{n+1})) = -\frac{1}{\nu} p_i^{n+1}, i = 1,2,\ x \in I_i \\
&\beta_i'(y_i^{n+1}) p_i^{n+1} - \partial_x(\beta_i'(\partial_x y_i^{n+1})\partial_x p_i^{n+1}) = \kappa(|y_i^{n+1} - y_i^d|^{r-2}(y_i^{n+1} - y_i^d), i = 1,2,\ x \in I_i \\
&y_1^{n+1}(0) = 0,\ y_2^{n+1}(2) = 0, \\
&p_1^{n+1}(0) = 0,\ p_2^{n+1}(2) = 0,
\end{aligned} \tag{3.3}
$$

$$
d_{ij}\beta_i(\partial_x y_i^{n+1}(1)) + \sigma y_i^{n+1}(1) + \mu p_i^{n+1}(1) = -d_{ij}\beta_j(\partial_x y_2^n(1)) + \sigma y_j^n(1) + \mu p_j^n(1) := \lambda_i^n \tag{3.4}
$$

$$
d_{ij}\beta_i'(\partial_x y_i^{n+1}(1))\partial_x p_i^{n+1} + \sigma p_i^{n+1}(1) - \mu y_i^{n+1}(1) = -d_{ij}\beta_j'(\partial_x y_j^n(1))\partial_x p_j^{n+1}(1) + \sigma p_j^n(1) - \mu y_j^n(1) =: \rho_i^n \tag{3.5}
$$

with parameters $\sigma, \mu \geq 0$. The iteration (3.3), (3.4), (3.5) is started at $n = 0$. We notice that the decomposed optimality system (3.3), (3.4), (3.5) can be seen as the optimality system for the virtual control problem on each edge, with solution y, p updated at the iteration index $n + 1$.

$$
\begin{aligned}
&\min_{u,g,y} \left\{ J(u,y) + \frac{1}{2\mu} \sum_{i=1,2} \left[|g_i|^2 + |\mu y_i + \rho_i^n|^2 \right] \right\} \\
&\text{s. t.} \\
&\beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i)) = u_i, i = 1,2,\ x \in I_i \\
&d_{ij}\beta_i(\partial_x y_i(1)) + \sigma y_i(1) = \lambda_i^n + g_i,\ i = 1,2,\ y_1(0) = 0, y_2(2) = 0.
\end{aligned} \tag{3.6}
$$

The proof of this very important feature is left to the reader. It is analogous to the linear result in [10]. We wish to show the convergence of this iteration, when $n \to \infty$. To this end, we introduce the errors

$$
\tilde{y}_i^n := y_i^n - y_i, \tilde{p}_i^n := p_i^n - p_i, i = 1,2,\ n = 1,2,3....., \tag{3.7}
$$

We subtract the state equations and the adjoint equations and obtain the system in terms of the errors.

$$
\begin{aligned}
&\beta_i(\tilde{y}_i^{n+1} + y_i) - \beta_i(y_i) - \partial_x(\beta_i(\partial_x \tilde{y}_i^{n+1} + \partial_x y_i)) - \partial_x(\beta_i(\partial_x y_i)) = -\frac{1}{\nu}\tilde{p}_i^{n+1}\ i = 1,2,\ x \in I_i \\
&\beta_i'(\tilde{y}_i^{n+1} + y_i)\tilde{p}_i^{n+1} - \partial_x(\beta_i'(\partial_x y_i^{n+1} + \partial_x y_i)\partial_x \tilde{p}_i^{n+1}) \\
&\quad = -(\beta_i'(\tilde{y}_i^{n+1} + y_i) - \beta_i'(y_i))p_i + (\beta_i'(\partial_x \tilde{y}_i^{n+1} + \partial_x y_i) - \beta_i'(\partial_x y_i))\partial_x p_i) \\
&\qquad + \kappa((|\tilde{y}_i^{n+1} + y_i - y_i^d|^{r-2}(\tilde{y}_i^{n+1} + y_i - y_i^d) - (|y_i - y_i^d|^{r-2}(y_i - y_i^d)) =: g_i^n, i = 1,2,\ x \in I_i \\
&\tilde{y}_1^{n+1}(0) = 0,\ \tilde{y}_2^{n+1}(2) = 0, \\
&\tilde{p}_1^{n+1}(0) = 0,\ \tilde{p}_2^{n+1}(2) = 0,
\end{aligned} \tag{3.8}
$$

together with the iteration of the transmission conditions

$$
\begin{aligned}
&\beta_1(\partial_x\tilde{y}_1^{n+1}(1)+\partial_x y_1(1))-\beta_1(\partial_x y_1(1))+\sigma\tilde{y}_1^{n+1}(1)+\mu\tilde{p}_1^{n+1}(1)\\
&\qquad=\beta_2(\partial_x\tilde{y}_2^{n}(1)+\partial_x y_2(1))-\beta_2(\partial_x y_2(1))+\sigma\tilde{y}_2^{n}(1)+\mu\tilde{p}_2^{n}(1)\\
&-(\beta_2(\partial_x\tilde{y}_2^{n+1}(1)+\partial_x y_2(1))-\beta_2(\partial_x y_2(1)))+\sigma\tilde{y}_2^{n+1}(1)+\mu\tilde{p}_2^{n+1}(1)\\
&\qquad=-(\beta_1(\partial_x\tilde{y}_1^{n}(1)+\partial_x y_1(1))-\beta_1(\partial_x y_1(1)))+\sigma\tilde{y}_1^{n}(1)+\mu\tilde{p}_1^{n}(1)
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&\beta_1'(\partial_x\tilde{y}_1^{n+1}(1)+\partial_x y_1(1))(\partial_x\tilde{p}_1^{n+1}(1)+\partial_x p_1(1))-\beta_1'(\partial_x y_1)\partial_x p_1(1)+\sigma\tilde{p}_1^{n+1}(1)-\mu\tilde{y}_1^{n+1}(1)\\
&\qquad=\beta_2'(\partial_x\tilde{y}_2^{n}(1)+\partial_x y_2(1))(\partial_x\tilde{p}_2^{n+1}(1)+\partial_x p_2(1))-\beta_2'(\partial_x y_2(1))\partial_x p_2(1))+\sigma\tilde{p}_2^{n}(1)-\mu\tilde{y}_2^{n}(1)\\
&-(\beta_2'(\partial_x\tilde{y}_2^{n+1}(1)+\partial_x y_2(1))(\partial_x\tilde{p}_2^{n+1}(1)+\partial_x p_2(1))-\beta_2'(\partial_x y_2(1))\partial_x p_2(1)))+\sigma\tilde{p}_2^{n+1}(1)-\mu\tilde{y}_2^{n+1}(1)\\
&\qquad=-(\beta_1'(\partial_x\tilde{y}_1^{n}(1)+\partial_x y_1(1))(\partial_x\tilde{p}_1^{n}+\partial_x p_1(1))-\beta_1'(\partial_x y_1(1))\partial_x p_1(1)))+\sigma\tilde{p}_1^{n}(1)-\mu\tilde{y}_1^{n}(1).
\end{aligned}
\tag{3.10}
$$

We introduce

$$
\begin{aligned}
\mathcal{X}^n :=&(\beta_1(\partial_x y_1^n(1))-\beta_1(\partial_x y_1(1))+\sigma\tilde{y}_1^n(1)+\mu\tilde{p}_1^n(1),\\
&-(\beta_2(\partial_x y_2^n(1))-\beta_2(\partial_x y_2(1)))+\sigma\tilde{y}_2^{n+1}(1)+\mu\tilde{p}_2^{n+1}(1),\\
&\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1)-\beta_1'(\partial_x y_1)\partial_x p_1(1)+\sigma\tilde{p}_1^n(1)-\mu\tilde{y}_1^n(1),\\
&-(\beta_2'(\partial_x y_2^n(1))\partial_x p_2^n(1)-\beta_2'(\partial_x y_2(1))\partial_x p_2(1)))+\sigma\tilde{p}_2^n(1)-\mu\tilde{y}_2^n(1))
\end{aligned}
\tag{3.11}
$$

We also introduce

$$
\begin{aligned}
\mathcal{T}\mathcal{X}^n :=& (\beta_2(\partial_x y_2^n(1))-\beta_2(\partial_x y_2(1))+\sigma\tilde{y}_2^n(1)+\mu\tilde{p}_2^n(1)\\
&-(\beta_1(\partial_x y_1^n(1))-\beta_1(\partial_x y_1(1)))+\sigma\tilde{y}_1^n(1)+\mu\tilde{p}_1^n(1)\\
&\beta_2'(\partial_x y_2^n(1))\partial_x\tilde{p}_2^n(1)-\beta_2'(\partial_x y_2(1))\partial_x p_2(1))+\sigma\tilde{p}_2^n(1)-\mu\tilde{y}_2^n(1)\\
&-(\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1)-\beta_1'(\partial_x y_1(1))\partial_x p_1(1)))+\sigma\tilde{p}_1^n(1)-\mu\tilde{y}_1^n(1).
\end{aligned}
\tag{3.12}
$$

With (3.11), (3.12), the iteration (3.9), (3.10) can be written as a fixed point iteration

$$
\mathcal{X}^{n+1}=\mathcal{T}\mathcal{X}^n,\ n=1,2,.....
\tag{3.13}
$$

We are going to prove that the iteration (3.13) converges. Moreover, in the spirit of [10], we may also introduce the relaxed fixed point iteration for a given relaxation parameter $\epsilon\in[0,1)$.

$$
\mathcal{X}^{n+1}=(1-\epsilon)\mathcal{T}\mathcal{X}^n+\epsilon\mathcal{X}^n,\ n=1,2,.....
\tag{3.14}
$$

To begin with, we look at the norm of $\mathcal{X}^n$. For this, we introduce

$$
\begin{aligned}
\mathcal{F}^n :=&\\
&2\sigma\left\{(\beta_1(\partial_x y_1^n(1))-\beta_1(\partial_x y_1(1)))\tilde{y}_1^n(1)-(\beta_2(\partial_x y_2^n(1))-\beta_2(\partial_x y_2(1)))\tilde{y}_2^n(1)\right.\\
&\quad(\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1)-\beta_1'(\partial_x y_1(1))\partial_x p_1(1))\tilde{p}_1^n(1)\\
&\quad\left.-(\beta_2'(\partial_x y_2^n(1))\partial_x p_2^n(1)-\beta_2'(\partial_x y_2(1))\partial_x p_2(1)))\tilde{p}_2^n(1)\right\}\\
&+2\mu\left\{(\beta_1(\partial_x y_1^n(1)-\beta_1(\partial_x y_1(1)))\tilde{p}_1^n(1)-(\beta_2(\partial_x y_2^n(1))-\beta_2(\partial_x y_2(1)))\tilde{p}_2^n(1)\right.\\
&\quad-(\beta_1'(\partial_x y_1^n(1))\partial_x p_1^n(1)-\beta_1'(\partial_x y_1(1))\partial_x y_1(1))\tilde{y}_1^n(1)\\
&\quad\left.+(\beta_2'(\partial_x y_2^n(1))\partial_x p_2^n(1)-\beta_2'(\partial_x y_2(1))\partial_x p_2(1))\tilde{y}_2^n(1)\right\}
\end{aligned}
\tag{3.15}
$$

and

$$\mathcal{E}^n := \sum_{i=1}^{2} \left\{ (\beta_i(\partial_x y_i^n(1) - \beta_i(\partial_x y_i(1))^2 + (\beta_i'(\partial_x y_i^n(1))\partial_x p_i^n(1) - \beta_i'(\partial_x y_i(1))\partial_x p_i(1))^2 \right. \tag{3.16}$$

$$\left. +(\sigma^2 + \mu^2)(\tilde{y}_i^n(1)^2 + \tilde{p}_i^n(1)^2) \right\} \tag{3.17}$$

After some elementary calculations, we arrive at

$$\|\mathcal{X}^n\|^2 = \mathcal{E}^n + \mathcal{F}^n. \tag{3.18}$$

Similarly, we find

$$\|\mathcal{T}\mathcal{X}^n\|^2 = \mathcal{E}^n - \mathcal{F}^n. \tag{3.19}$$

We proceed to show that $\mathcal{F}^n$ is positive definite with respect to the errors. To this end, we multiply below the state equation and the adjoint equation by $\tilde{y}_i^n$, $\tilde{p}_i^n$, respectively. This gives

$$\begin{aligned} 0 &= \int_{I_i} \left(\beta_i(y_i^n) - \beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i)) + \frac{1}{\nu}\tilde{p}_i^n \right) \tilde{y}_i^n dx \\ &= \int_{I_i} \left((\beta_i(y_i^n) - \beta_i(y_i))(y_i^n - y_i) + (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i))\partial_x(y_i^n - y_i) + \frac{1}{\nu}\tilde{p}_i^n \tilde{y}_i^n) \right) dx \\ &\quad - (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i))\tilde{y}_i^n|_{\partial I_i} := \int_{I_i} H_{i1}^n dx - (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i)|_{\partial I_i}. \end{aligned} \tag{3.20}$$

This gives the boundary expressions

$$(\beta_1(\partial_x y_1^n) - \beta_1(\partial_x y_1))(1)\tilde{y}_1^n(1) = \int_{I_1} H_{11}^n dx, \tag{3.21}$$

$$(\beta_2(\partial_x y_2^n) - \beta_2(\partial_x y_2))(1)\tilde{y}_2^n(1) = -\int_{I_2} H_{21}^n dx. \tag{3.22}$$

Similarly, after multiplying the state equation by $\tilde{p}_i^n$, we obtain

$$\begin{aligned} 0 &= \int_{I_i} \left(\beta_i(y_i^n) - \beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i)) + \frac{1}{\nu}\tilde{p}_i^n \right) \tilde{p}_i^n dx \\ &= \int_{I_i} \left((\beta_i(y_i^n) - \beta_i(y_i))\tilde{p}_i^n + (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i))\partial_x \tilde{p}_i^n + \frac{1}{\nu}(\tilde{p}_i^n)^2 \right) dx \\ &\quad - (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i))\tilde{p}_i^n|_{\partial I_i} := \int_{I_i} H_{i2}^n dx - (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i)|_{\partial I_i}. \end{aligned} \tag{3.23}$$

$$(\beta_1(\partial_x y_1^n) - \beta_1(\partial_x y_1))(1)\tilde{p}_1^n(1) = \int_{I_1} H_{12}^n dx, \tag{3.24}$$

$$(\beta_2(\partial_x y_2^n) - \beta_2(\partial_x y_2))(1)\tilde{p}_2^n(1) = -\int_{I_2} H_{22}^n dx. \tag{3.25}$$

We now multiply the adjoint equation first by $\tilde{p}_i^n$ and obtain

$$0 = \int_{I_i} \left(\beta_i'(y_i^n)p_i^n - \beta_i'(y_i)p_i - \partial_x(\beta_i'(\partial_x y_i^n)\partial_x p_i^n - \beta_i'(\partial_x y_i)\partial_x p_i) - \kappa g_i^n\right)\tilde{p}_i^n dx \tag{3.26}$$

$$= \int_{I_i} \left((\beta_i'(y_i^n)p_i^n - \beta_i'(y_i)p_i)\tilde{p}_i^n + (\beta_i'(\partial_x y_i^n)\partial_x p_i^n - \beta_i(\partial_x y_i)\partial_x p_i)\partial_x\tilde{p}_i^n - \kappa g_i^n\tilde{p}_i^n\right) dx$$

$$- (\beta_i'(\partial_x y_i^n)p_i^n - \beta_i(\partial_x y_i)p_i)\tilde{p}_i^n|_{\partial I_i} := \int_{I_i} G_{i1}^n dx - (\beta_i'(\partial_x y_i^n)p_i^n - \beta_i'(\partial_x y_i))p_i)\tilde{p}_i^n|_{\partial I_i}.$$

This leads to the boundary expressions

$$(\beta_1'(\partial_x y_1^n)\partial_x p_1^n - \beta_1(\partial_x y_1)\partial_x p_1)(1)\tilde{p}_1^n(1) = \int_{I_1} G_{11}^n dx, \tag{3.27}$$

$$(\beta_2'(\partial_x y_2^n)\partial_x p_2^n - \beta_2(\partial_x y_2)\partial_x p_2)(1)\tilde{p}_2^n(1) = -\int_{I_2} G_{21}^n dx. \tag{3.28}$$

Finally, we multiply the adjoint equation by $\tilde{y}_i^n$ and obtain

$$0 = \int_{I_i} \left(\beta_i'(y_i^n)p_i^n - \beta_i'(y_i)p_i - \partial_x(\beta_i'(\partial_x y_i^n)\partial_x p_i^n - \beta_i'(\partial_x y_i)\partial_x p_i) - \kappa g_i^n\right)\tilde{y}_i^n dx \tag{3.29}$$

$$= \int_{I_i} \left((\beta_i'(y_i^n)p_i^n - \beta_i'(y_i)p_i)\tilde{y}_i^n + (\beta_i'(\partial_x y_i^n)\partial_x p_i^n - \beta_i(\partial_x y_i)\partial_x p_i)\partial_x\tilde{y}_i^n - \kappa g_i^n\tilde{y}_i^n\right) dx$$

$$- (\beta_i'(\partial_x y_i^n)p_i^n - \beta_i'(\partial_x y_i)p_i)\tilde{y}_i^n|_{\partial I_i} := \int_{I_i} G_{i2}^n dx - (\beta_i'(\partial_x y_i^n)p_i^n - \beta_i'(\partial_x y_i))p_i)\tilde{y}_i^n|_{\partial I_i}.$$

From this, we arrive at the boundary terms

$$(\beta_1'(\partial_x y_1^n)\partial_x p_1^n - \beta_1(\partial_x y_1)\partial_x p_1)(1)\tilde{y}_1^n(1) = \int_{I_1} G_{12}^n dx, \tag{3.30}$$

$$(\beta_2'(\partial_x y_2^n)\partial_x p_2^n - \beta_2(\partial_x y_2)\partial_x p_2)(1)\tilde{y}_2^n(1) = -\int_{I_2} G_{22}^n dx. \tag{3.31}$$

Recalling the definition (3.15) of $\mathcal{F}^n$, we obtain

$$\mathcal{F}^n = 2\sigma\sum_{i=1}^{2}\int_{I_i}\left(H_{i1}^n + G_{i1}^n\right)dx + 2\mu\sum_{i=1}^{2}\int_{I_i}\left(H_{i2}^n - G_{i2}^n\right)dx. \tag{3.32}$$

In order to proceed, we employ the monotonicity property of the functions β_i. During these notes, it will be convenient to assume the p-structure $\beta_i(s) = \gamma_i|s|^{p-2}s$. For $1 < p \leq 2$ we have

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq (|a| + |b|)^{p-2}(a-b)^2. \tag{3.33}$$

We first look at the terms with factor σ. We use (3.33) for H^n_{i1} as follows

$$\int_{I_i} H^n_{i1}dx \geq \gamma_i \int_{I_i} \left((|y^n_i| + |y_i|)^{p-2}(\tilde{y}^n_i)^2 + (|\partial_x y^n_i| + |\partial_x y_i|)^{p-2}(\partial_x \tilde{y}^n_i)^2 \right) dx + \frac{1}{\nu}\int_{I_i} \tilde{p}^n_i \tilde{y}^n_i dx. \tag{3.34}$$

For G^n_{12} we have

$$\begin{aligned}\int_{I_i} G^n_{i1}dx =&\gamma_i \int_{I_i} \left(\beta_i'(y^n_i)(\tilde{p}^n_i)^2 + \beta_i'(\partial_x y^n_i)(\partial_x \tilde{p}^n_i)^2 \right) dx \\ &+ \gamma_i \int_{I_i} \left(\beta_i''(y_i + \theta_n(\tilde{y}^n_i))p_i \tilde{y}^n_i \tilde{p}^n_i + \beta_i''(\partial_x y_i + \theta_n(\partial_x \tilde{y}^n_i))\partial_x p_i \partial_x \tilde{y}^n_i \partial_x \tilde{p}^n_i \right) dx + \kappa \int_{I_i} g^n_i \tilde{p}^n_i dx. \end{aligned} \tag{3.35}$$

Hence,

$$\begin{aligned}\int_{I_i} (H^n_{i1} + G^n_{i1})dx \geq&\gamma_i \int_{I_i} \left((|y^n_i| + |y_i|)^{p-2}(\tilde{y}^n_i)^2 + (|\partial_x y^n_i| + |\partial_x y_i|)^{p-2}(\partial_x \tilde{y}^n_i)^2 \right) dx \\ &+ \gamma_i \int_{I_i} \left(|y^n_i|^{p-2}(\tilde{p}^n_i)^2 + |\partial_x y^n_i|^{p-2}(\partial_x \tilde{p}^n_i)^2 \right) dx \qquad (3.36) \\ &+ \gamma_i \int_{I_i} \left(\beta_i''(y_i + \theta_n(\tilde{y}^n_i))p_i \tilde{y}^n_i \tilde{p}^n_i + \beta_i''(\partial_x y_i + \theta_n(\partial_x \tilde{y}^n_i))\partial_x p_i \partial_x \tilde{y}^n_i \partial_x \tilde{p}^n_i \right) dx \qquad (3.37) \\ &- \frac{1}{2\nu}\int_{I_i} (\tilde{p}^n_i)^2 + (\tilde{y}^n_i)^2 dx - \frac{\kappa}{2}\int_{I_i} (\tilde{p}^n_i)^2 + (g^n_i)^2 dx. \end{aligned}$$

We now look at the terms with factor μ. For H^n_{i2} we have

$$\int_{I_i} H^n_{i2}dx =\gamma_i \int_{I_i} \left(\beta_i(y^n_i) - \beta_i(y_i))(\tilde{p}^n_i) + (\beta_i(\partial_x y^n_i) - \beta_i(\partial_x y_i))(\partial_x \tilde{p}^n_i) \right) dx + \frac{1}{\nu}\int_{I_i} (\tilde{p}^n_i)^2 dx, \tag{3.38}$$

while $-G^n_{i2}$ gives

$$\begin{aligned}-\int_{I_i} G^n_{i2}dx =& -\gamma_i \int_{I_i} \{\beta_i'(y^n_i)\tilde{y}^n_i \tilde{p}^n_i + \beta_i'(\partial_x y^n_i)\partial_x \tilde{y}^n_i \partial_x \tilde{p}^n_i dx \\ &+(\beta_i'(\partial_x y^n_i) - \beta_i'(\partial_x y_i))p_i \tilde{y}^n_i + (\beta_i'(\partial_x y^n_i) - \beta_i'(\partial_x y_i))\partial_x p_i \partial_x \tilde{y}^n_i \} dx + \kappa \int_{I_i} g^n_i \tilde{y}^n_i dx. \end{aligned} \tag{3.39}$$

Therefore,

$$\begin{aligned}\int_{I_i} H^n_{i2} - G^n_{i2}dx =&\gamma_i \int_{I_i} \left((\beta_i(y^n_i) - \beta_i(y_i) - \beta_i'(y^n_i)\tilde{y}^n_i)\tilde{p}^n_i + (\beta_i(\partial_x y^n_i) - \beta_i(\partial_x y_i) - \beta_i'(\partial_x y^n_i)\partial_x \tilde{y}^n_i)(\partial_x \tilde{p}^n_i) \right) dx \\ &+ \gamma_i \int_{I_i} \left((\beta_i'(\partial_x y^n_i) - \beta_i'(\partial_x y_i))p_i \tilde{y}^n_i + (\beta_i'(\partial_x y^n_i) - \beta_i'(\partial_x y_i))\partial_x p_i \partial_x \tilde{y}^n_i \right) dx \qquad (3.40) \\ &+ \kappa \int_{I_i} g^n_i \tilde{y}^n_i dx + \frac{1}{\nu}\int_{I_i} (\tilde{p}^n_i)^2 dx. \end{aligned}$$

We note that

$$\begin{aligned} \kappa g_i^n \tilde{y}_i^n &= \kappa \left\{ (|y_i^n - y_i^d|^{r-2}(y_i^n - y_i^d) - |y_i - y_i^d|(y_i - y_i^d)\tilde{y}_i^n \right\} \\ &\geq \kappa (|y_i^n - y_i^d|^{r-2}(\tilde{y}_i^n)^2. \end{aligned} \tag{3.41}$$

Moreover, we have

$$\begin{aligned} |\beta_i(y_i^n) - \beta_i(y_i) - \beta_i'(y_i^n)\tilde{y}_i^n| &\leq |y_i + \delta\tilde{y}_i^n|^{p-3}|\tilde{y}_i^n|^2 \\ |\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i) - \beta_i'(\partial_x y_i^n)\partial_x\tilde{y}_i^n| &\leq |\partial_x y_i + \delta\partial_x\tilde{y}_i^n|^{p-3}|\partial_x\tilde{y}_i^n|^2. \end{aligned} \tag{3.42}$$

We can, therefore, estimate the mixed terms

$$\begin{aligned} (\beta_i(y_i^n) - \beta_i(y_i) - \beta_i'(y_i^n)\tilde{y}_i^n)\tilde{p}_i^n &\leq \frac{1}{2}|y_i + \delta\tilde{y}_i^n|^{p-3}|\tilde{y}_i^n|(|\tilde{y}_i^n|^2 + |\tilde{p}_i^n|^2) =: L(\tilde{y}_i^n)(|\tilde{y}_i^n|^2 + |\tilde{p}_i^n|^2) \\ (\beta_i(\partial_x y_i^n) - \beta_i(\partial_x y_i) - \beta_i'(\partial_x\tilde{y}_i^n)\partial_x\tilde{y}_i^n)\tilde{p}_i^n &\leq \frac{1}{2}|\partial_x y_i + \delta\partial_x\tilde{y}_i^n|^{p-3}|\partial_x\tilde{y}_i^n|(|\partial_x\tilde{y}_i^n|^2 + |\partial_x\tilde{p}_i^n|^2) \\ &=: L(\partial_x\tilde{y}_i^n)(|\partial_x\tilde{y}_i^n|^2 + |\partial_x\tilde{p}_i^n|^2), \end{aligned} \tag{3.43}$$

where we note that the factors $L(\tilde{y}_i^n), L(\partial_x\tilde{y}_i^n)$ in front of the quadratic terms tend to zero as $\tilde{y}_i^n, \partial_x\tilde{y}_i^n$ tend to zero. The other mixed terms also involve $\beta_i''(s)$, which lead to factors bounded by $C_i(\tilde{y}_i, \partial_x\tilde{y}_i, \tilde{p}_i, \partial_x\tilde{p}_i)$ multiplied the adjoint states $p_i, \partial_x p_i$, which, in turn, we will assume to be small in the sequel. Smallness of $p_i, \partial_x p_i$ can be interpreted as the optimal states $y_i, \partial_x y_i$ being close to the target y_i^d. We also define

$$K_i(\tilde{y}_i^n) := (|\tilde{y}_i^n| + 2|y_i|)^{p-2},\ K_i(\partial_x\tilde{y}_i^n) := (|\partial_x\tilde{y}_i^n| + 2|\partial_x y_i|)^{p-2}.$$

Then

$$\begin{aligned}
\mathcal{F}^n \geq 2\sum_{i=1}^{2}\int_{I_i}\Big[&\sigma\Big\{K_i(\tilde{y}_i^n)((\tilde{y}_i^n)^2+(\tilde{p}_i^n)^2)+K_i(\partial_x\tilde{y}_i^n)((\partial_x\tilde{y}_i^n)^2+(\partial_x\tilde{p}_i^n)^2)\\
&-\frac{1}{2}C_ip_i((\tilde{y}_i^n)^2+(\tilde{p}_i^n)^2)-\frac{1}{2}C_i\partial_xp_i((\partial_x\tilde{y}_i^n)^2+(\partial_x\tilde{p}_i^n)^2)\\
&-\frac{1}{2}(\frac{1}{\nu}+\kappa)((\tilde{y}_i^n)^2+(\tilde{p}_i^n)^2)\Big\}\\
&+\mu\Big\{\kappa(\tilde{y}_i^n)^2+\frac{1}{\nu}(\tilde{p}_i^n)^2\\
&-\frac{1}{2}(C_ip_i+L_i(\tilde{y}_i^n))(y_i^n)^2-\frac{1}{2}(C_i\partial_xp_i+L_i(\partial_x\tilde{y}_i^n))(\partial_x\tilde{y}_i^n)^2\\
&-\frac{1}{2}L_i(\tilde{y}_I^n)(\tilde{p}_i^n)^2-\frac{1}{2}L_i(\partial_x\tilde{y}_i^n)(\partial_x\tilde{p}_i^n)^2\Big\}\Big]dx\\
\geq 2\sum_{i=1}^{2}\int_{I_i}\Big[&\sigma\Big\{(K_i(\tilde{y}_i^n)-\frac{1}{2}(\frac{1}{\nu}+\kappa-C_i|p_i|))((\tilde{y}_i^n)^2+(\tilde{p}_i^n)^2)\\
&+(K_i(\partial_x\tilde{y}_i^n)-\frac{1}{2}C_i|\partial_xp_i|)((\partial_x\tilde{y}_i^n)^2+(\partial_x\tilde{p}_i^n)^2)\Big\}\\
&+\mu\Big\{(\kappa-\frac{1}{2}(C_i|p_i|+L_i(\tilde{y}_i^n)))(\tilde{y}_i^n)^2+(\frac{1}{\nu}-\frac{1}{2}L_i(\tilde{y}_i^n))(\tilde{p}_i^n)^2\\
&-\frac{1}{2}(C_i\partial_xp_i+L_i(\partial_x\tilde{y}_i^n))(\partial_x\tilde{y}_i^n)^2-\frac{1}{2}L_i(\partial_x\tilde{y}_i^n)(\partial_x\tilde{p}_i^n)^2\Big\}\Big]
\end{aligned}\tag{3.44}$$

We would like to see under which condition (3.44) can be made positive. This is indeed possible if $L_i(\tilde{y}_i^n), L_i(\partial_x\tilde{y}_i^n)$ are small which, in turn, is true if $\tilde{y}_i^n, \partial_x\tilde{y}_i^n$ are small. This is an assumption on the initial neighbourhood of $y_i, \partial_x y_i$. Moreover, we can assume, as mentioned above, from the beginning that $|p_i|, |\partial_x p_i|$ are small. If the smallness assumption on the original adjoint problem holds and we have an integer n such that $L_i(\tilde{y}_i^n), L_i(\partial_x\tilde{y}_i^n)$ are small enough to make (3.44) positive, then according to $\|\mathcal{X}^{n+1}\|^2 = \|\mathcal{T}\mathcal{X}^n\|^2 = \|\mathcal{X}^n\|^2 - \mathcal{F}^n < \|\mathcal{X}\|^n$, also the next iteration will stay in these bounds. Notice that for $p < 2$, as in our case here, the coefficients $K_i(\tilde{y}_i^n)), K_i(\partial_x\tilde{y}_i^n)$ increase when $|\tilde{y}_i^n|, |\partial_x\tilde{y}_i^n|$ decrease. Thus, if we start close enough to to the original solution and if we assume that the norms of the adjoint problem are small, then we can conclude that by choosing σ, μ, κ, ν appropriately, we can make (3.44) positive definite with respect to the quadratic terms $(\tilde{y}_i^n)^2, (\partial_x\tilde{y}_i^n)^2, (\tilde{p}_i^n)^2, (\partial_x\tilde{p}_i^n)^2$.

Theorem 3.1. *Let the $(y_i, p_i) \in H^1(I_i)$ and initial errors $\mathcal{X}^0$ be sufficiently small. Then there are parameters σ, μ, κ, ν such that the iteration $\mathcal{X}^{n+1} = \mathcal{T}\mathcal{X}^n$ converges. Moreover, we have*

$$\begin{aligned}
&\tilde{y}_i^n, \tilde{p}_i^n \to 0,\ n\to\infty,\ \text{in } H^1(I_i),\ i=1,2,\\
&\tilde{y}_i^n(1), \partial_x\tilde{y}_i^n(1), \tilde{p}_i^n(1), \partial_x\tilde{p}_i^n(1) \to 0,\ n\to 0,\ i=1,2.
\end{aligned}\tag{3.45}$$

Proof. We iterate the equation

$$\|\mathcal{X}^{n+1}\|^2 = \|\mathcal{T}\mathcal{X}^{n+1}\|^2 = \|\mathcal{X}^n\| - 2\mathcal{F}^n$$

down to zero. This gives

$$\|\mathcal{X}^{n+1}\|^2 \leq \|\mathcal{X}^0\|^2 - \sum_{l=1}^{n}\mathcal{F}^l,\ \forall n\in\mathbb{N}.$$

This implies that $\mathcal{F}^l \to 0$ as $l \to \infty$. From the positive definiteness of $\mathcal{F}$ provided by (3.44) and our assumption on the smallness of the data, we arrive at the conclusion. □

Remark 3.2. We remark that when we take $p = 2$, then the iterative domain decomposition procedure above reduces to the well-known iteration from [10].

3.1. **Global convergence and separate domain decomposition.** The assumption of theorem 3.45 require starting points close to the true solution. As it becomes obvious from (3.44), the difficulty is with the mixed terms in $\tilde{y}_i^n, \tilde{p}_i^n$ and $\partial_x \tilde{y}_i^n, \partial_x \tilde{p}_i^n$ that have to be compensated by the corresponding quadratic terms. The point is that we can use the monotonicity essentially with respect to the state $\tilde{y}_i^n$, while the fact that we work with the system in terms of the errors forced us to handle differences in the adjoint equation that, in turn, could not be treated by monotonicity. Therefore, we may want to decouple the state equation from the adjoint equation using another fixed point iteration.

To this end, we introduce the common algorithm

Algorithm

(1) Choose u^0
(2) For $i = 1$ until satisfied
(3) (a) Solve for y_i

$$\begin{cases} \beta_i(y_i) - \partial_x(\beta_i(\partial_x y_i) = u_i, i = 1, 2, \ x \in I_i \\ y_1(0) = 0, \ y_2(2) = 0, \\ y_1(1) = y_2(1), \beta_1(\partial_x y_1)(1) = \beta_2(\partial_x y_2)(1). \end{cases}$$

(b) Solve for p_i

$$\begin{cases} \beta_i'(y_i)p_i - \partial_x(\beta_i'(\partial_x y_i)\partial_x p_i) = \kappa(|y_i - y_i^d|^{r-2}(y_i - y_i^d), i = 1, 2, \ x \in I_i \\ p_1(0) = 0, \ p_2(2) = 0, \\ p_1(1) = p_2(1), \beta_1'(\partial_x y_1)(1)\partial_x p_1(1) = \beta_2'(\partial_x y_2(1))\partial_x p_2(1). \end{cases}$$

(c) Set $u = \frac{1}{\nu} p$
(4) Terminate with the (approximate) fixed point u, y, p.

This algorithm is classic for linear problems. For the p-Laplace operator, this needs further arguments. For the sake of brevity, we leave the convergence analysis to a forthcoming paper. See, however [17].

The point of interest in these note is that, giving up the concept of decomposing the optimialtiy system at once, we can concentrate on the two systems in step 3. iteratively. That is to say, we first apply the domain decomposition procedure to the state system (a) and then to the adjoint system (b). Of course, the adjoint system is linear, given the state y_i. Hence, the classical P. Lions type non-overlapping domain decompositions applies. See e.g. [10] for the details. Thus, the problem is with the state system. To the best knowledge of the author, not even for this p-Laplace type problem results are available in the literature. The only exception is a very recent article [9] where the p-Laplace problem problem is treated, however, with a slightly but importantly different iteration which, in turn, is reminiscent of a GaußSeidel type version of the P.L. Lions method, whereas the method of consideration here is more a Jacobi-type iteration and is, therefore, completely parallel.

In these notes, we just outline the procedure by reducing the one for the optimal control problem in setting the adjoint variable as well as the parameter μ to zero. For a given control u, we look into the iteration

$$
\begin{aligned}
\beta_i(y_i)^{n+1} - \partial_x(\beta_i(\partial_x y_i)^{n+1} &= u_i, i = 1, 2,\ x \in I_i \\
y_1^{n+1}(0) = 0,\ y_2^{n+1}(2) &= 0, \\
\beta_1(\partial_x y_1^{n+1})(1) + \sigma y_1^{n+1}(1) &= \beta_2(\partial_x y_2^n(1)) + \sigma y_2^n(1) \\
-\beta_2(\partial_x y_2^{n+1}(1)) + \sigma y_2^{n+1}(1) &= -\beta_1(\partial_x y_1^n(1)) + \sigma y_1^n(1).
\end{aligned} \tag{3.46}
$$

In this case the variable $\mathcal{X}^n$ as well as the iteration operator $\mathcal{T}$ reduce to

$$
\begin{aligned}
\mathcal{X}^n :=& (\beta_1(\partial_x y_1^n(1)) - \beta_1(\partial_x y_1(1)) + \sigma\tilde{y}_1^n(1) + \mu\tilde{p}_1^n(1), \\
&- (\beta_2(\partial_x y_2^n(1)) - \beta_2(\partial_x y_2(1))) + \sigma\tilde{y}_2^{n+1}(1) + \mu\tilde{p}_2^{n+1}(1))
\end{aligned} \tag{3.47}
$$

We also introduce

$$
\begin{aligned}
\mathcal{T}\mathcal{X}^n :=& (\beta_2(\partial_x y_2^n(1)) - \beta_2(\partial_x y_2(1)) + \sigma\tilde{y}_2^n(1) + \mu\tilde{p}_2^n(1) \\
&- (\beta_1(\partial_x y_1^n(1)) - \beta_1(\partial_x y_1(1))) + \sigma\tilde{y}_1^n(1) + \mu\tilde{p}_1^n(1)).
\end{aligned} \tag{3.48}
$$

and $\mathcal{F}$ to

$$
\mathcal{F}^n := 2\sigma\left\{(\beta_1(\partial_x y_1^n(1)) - \beta_1(\partial_x y_1(1)))\tilde{y}_1^n(1) - (\beta_2(\partial_x y_2^n(1)) - \beta_2(\partial_x y_2(1)))\tilde{y}_2^n(1)\right\}. \tag{3.49}
$$

From this, we retrieve

$$
\mathcal{F}^n \geq 2\sum_{i=1}^{2}\sigma\int_{I_i}\left\{K_i(\tilde{y}_i^n)(\tilde{y}_i^n)^2 + K_i(\partial_x\tilde{y}_i^n)(\partial_x\tilde{y}_i^n)^2\right\}dx. \tag{3.50}
$$

Then, by the same arguments as above, we obtain convergence $y_i^n \to y_i$ strongly in $H^1(I_i)$ together with the correct transmission conditions for the limit y. As mentioned above, once the state y is established, it is inserted into (3.46) which provides a linear problem in p. This can be decomposed by the standard method, provided in [10]. These decompositions work without a smallness assumption and can thus be applied until the neighbourhood required in Theorem 3.45 is reached. Due to space limitations, we provide the details along with numerical simulations in a forthcoming publication.

3.2. **General graphs.** In order to apply the domain decomposition to a general graph, we first note that it is sufficient to treat a star-graph with a center node at $x = 0$ with, say, m edges emerging from this node. We couple these edges, say at $x = 1$ with another m edges which, in turn stretch from $x = 1$ to $x = 2$. This graph fits with the description in (2.1). The iteration applied to the corresponding optimality system is then identical to one in (3.3), (3.4), (3.5) with the exception that it is applied to each coupling point and that for $i = 1, \ldots, m$ the solutions y_i^n have to be solved for the star graph. The complete picture along with numerical evidence will be published in a forthcoming article. We confine ourselves to just two examples.

3.3. **Some numerical experiments.** In this section, we consider the static optimal control problem for a two-link model (3.1) and the corresponding optimality system (3.2). The iterative decomposition is based on the system (3.3), (3.4), (3.4). We choose in particular a constant target function $y_d = 1$, constant Dirichlet conditions $y(x,t) = 1$ at the outer ends $x = 0, x = 2$ and parameters $\sigma = 10, \mu = .001, \kappa = 100, \nu = .1$. We perform two experiments, one for $p = 3/2$, the case of interest in gas flow, and $p = 1.1$, which is important in e.g. imaging, where $1 < p \leq 2$ and the limiting case $p = 1$ are considered. We solve the boundary value problems arising on the full domain and the two subdomains using the routine **bvpc4** from Matlab with a resolution of $1e-14$. We have chosen a constant target because it is critical, as vanishing y and $\partial_x y$ play a special role for the p-Laplacian. For $p > 2$, we, therefore, needed a regularization. On the left of figure 3.3, where $p = 3/2$, we show on the top left the solution of the optimality system, on the top right the result of the iteration on each subdomain stitched together at $x = 1$. On the bottom

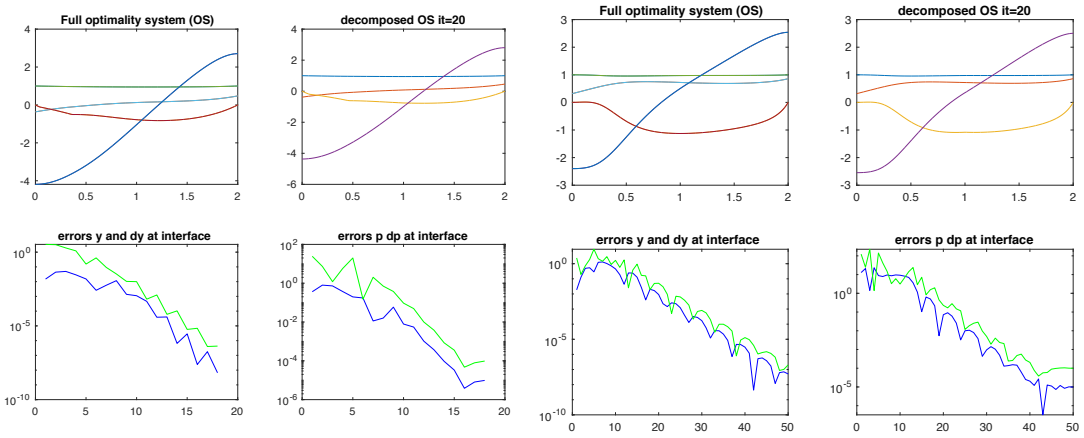

FIGURE 1. Left figure: $p = 1.5, \sigma = 10, \mu = .001, \kappa = 100, \nu = .1$, right figure: $p = 1.1$, same parameters

left we display the errors for the primal variables and their derivatives at the interface $x = 1$ and on the right the corresponding dual variables. The same configuration is used for $p = 1.1$ in the right part of figure 3.3. For more experiments and a more detailed description, we refer to a forthcoming publication.

REFERENCES

[1] R. Alonso, M. Santillana, and C. Dawson. On the diffusive wave approximation of the shallow water equations. *European J. Appl. Math.*, 19(5):575–606, 2008.

[2] A. Bamberger. étude d'une équation doublement non linéaire. *J. Functional Analysis*, 24(2):148–155, 1977.

[3] A. Bamberger, M. Sorine, and J. P. Yvon. Analyse et contrôle d'un réseau de transport de gaz. In *Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977), II*, volume 91 of *Lecture Notes in Phys.*, pages 347–359. Springer, Berlin-New York, 1979.

[4] V. Bögelein, F. Duzaar, P. Marcellini, and C. Scheven. A variational approach to doubly nonlinear equations. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 29(4):739–772, 2018.

[5] J. Brouwer, I. Gasser, and M. Herty. Gas pipeline models revisited: Model hierarchies, nonisothermal models, and simulations of networks. 9(2):601–623.

[6] E. Casas and L. A. Fernández. Optimal control of quasilinear elliptic equations. In *Control of partial differential equations (Santiago de Compostela, 1987)*, volume 114 of *Lect. Notes Control Inf. Sci.*, pages 92–99. Springer, Berlin, 1989.

[7] E. Casas and L. A. Fernández. Optimal control of quasilinear elliptic equations with nondifferentiable coefficients at the origin. *Rev. Mat. Univ. Complut. Madrid*, 4(2-3):227–250, 1991.

[8] E. Casas, P. I. Kogut, and G. Leugering. Approximation of optimal control problems in the coefficient for the p-Laplace equation. I. Convergence result. *SIAM J. Control Optim.*, 54(3):1406–1422, 2016.
[9] E. Engström and E. Hansen. Convergence analyis of the nonoverlapping robin-robin method for nonlinear elliptic equations, 2021.
[10] J. E. Lagnese and G. Leugering. *Domain decomposition methods in optimal control of partial differential equations*, volume 148 of *International Series of Numerical Mathematics*. Birkhäuser Verlag.
[11] G. Leugering and G. Mophou. Instantaneous optimal control of friction dominated flow in a gas-network. In *Shape optimization, homogenization and optimal control*, volume 169 of *Internat. Ser. Numer. Math.*, pages 75–88. Birkhäuser/Springer, Cham, 2018.
[12] R J. Le Veque.
[13] R. J. Le Veque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press.
[14] T. Roubí ček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2013.
[15] L. Schöbel-Krön. *Analysis and Numerical Approximation of Nonlinear Evolution Euqations on Network Structures*. PhD thesis, TU-Darmstadt, 2020.
[16] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*, volume 258 of *Grundlehren der mathematischen Wissenschaften*. Springer Verlag.
[17] W. Wollner. An optimal control problem for equations with p-structure and its finite element discretization, 2021.

Department of Data Science, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU) Cauerstr. 11, D-91058 Erlangen, Germany,

Email address: guenter.leugering@fau.de