

NODAL STABILIZATION OF THE FLOW IN A NETWORK WITH A CYCLE

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Abstract. In this paper we discuss an approach to the stability analysis for classical solutions of closed loop systems that is based upon the tracing of the evolution of the Riemann invariants along the characteristics. We consider a network where several edges are coupled through node conditions that govern the evolution of the Riemann invariants through the nodes of the network. The analysis of the decay of the Riemann invariants requires to follow backwards all the characteristics that enter such a node and contribute to the evolution. This means that with each nodal reflection/crossing the number of characteristics that contribute to the evolution increases.

We show how for simple networks with a sufficient number of damping nodal controllers it is possible to keep track of this family of characteristics and use this approach to analyze the exponential stability of the system. The analysis is based on an adapted version of Gronwall's lemma that allows us to take into account the possible increase of the Riemann invariants when the characteristic curves cross a node of the network.

Our example is motivated by applications in the control of gas pipeline flow, where the graphs of the networks often contain many cycles.

Key words: Nodal stabilization, classical solutions, networks, gas pipeline, characteristics, cycle, nodal control.

2010 Mathematics Subject Classification: 35L04, 35B35.

Communicated by Prof. P. Kogut

1. Introduction

The boundary stabilization of quasilinear systems has been the subject of numerous investigations, see for example [13], [5]. Also networked hyperbolic systems that are defined on tree-shaped graphs have been studied (see for example [11]). In many applications, the graphs of the networks contain cycles, for example in pipeline networks for gas transportation (see for example [24]). Since the control actions are concentrated in space in such a way that they can be considered as point actions, we do not consider distributed control but the problem to stabilize by controllers that act at certain points in space only. In order to describe such a situation, the term nodal control is used. We show that even if the network contains a cycle, the system can still be exponentially stable if the controls are chosen suitably.

In [22] the boundary control of networks of elastic strings is studied. For networks of vibrating strings, it is natural to study the problem with controllers located at the boundary nodes of the network only. The following statement is made (see also [21]): *Let G be a graph containing a circuit. If the strings*

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constituting that circuit have rationally dependent optical lengths, then approximate controllability fails, regardless of how many controls are imposed on the network. This statement indicates the difficulties that occur in the control of networks with cycles. These difficulties can be avoided, if suitable nodal controllers are included within the cycle. Moreover, the exact type of damping conditions is important. In this paper we consider problems of flow control with linear Riemann dampers that have the property that the maximum norm of the Riemann invariants is decreased as the characteristic curves cross the damping location.

In a network, the scattering effect at the nodes can lead locally to increased maximal values of the states. The nodal control actions have to compensate these augmentation effects. If the combined effects of the amplification by the scattering and the damping effect decrease the maximum norm of the solution, the system can be exponentially stable.

We want to derive results about the exponential stabilization of the system by pointwise control action. Note that while stabilization problems for quasilinear hyperbolic problems have been studied in depth (see for example [6], [18], [19]) in these contributions networked systems with cycles have not been considered. In [1] finite time stabilization for a network of vibrating strings (governed by the linear wave equation) with a tree-shaped graph is shown. It is assumed that the control acts on each vertex of the graph.

This paper can also be seen as an extension of the contribution in [13], where the stabilization of the flow in a sequence of pipes with a single compressor has been analyzed. Lyapunov functions have been used successfully to study this type of problem, for example recently for states with H^2 -regularity (see [18]). However, in order to study the decay of the C^1 -norm that occurs naturally in the context of classical solutions, due to the non-smoothness of this norm Lyapunov functions are not a natural tool. Therefore in this paper we follow a different approach that is based upon the tracing of the characteristic curves along which the values of the Riemann invariants evolve during the process. This type analysis is related to the tracing of optical rays in the analysis of the boundary controllability properties of the wave equation, see [4]. Note that in our analysis, the effect of the source term on the evolution is taken into account since in the model of gas pipeline flow, this effect is essential.

This paper has the following structure: In Section 2 we describe the model for the flow on an edge in the networks that we consider, namely the isothermal Euler equations. For the flow through the nodes, we require the conservation of mass that is guaranteed by the Kirchhoff condition and an additional algebraic condition. In Section 3 we transform the Euler equations using the Riemann invariants. This enables us to define a control action in Section 4, that acts as a Riemann damper. In Section 5 we show the exponential stability for the system. We start with a network consisting of a single edge in 5.1, continue with a star shaped network in 5.2, then consider a network that is a cycle in 5.3 and finally a cycle with two additional edges in 5.4. We show that two controllers suffice for the stabilization of this network. Numerical experiments are presented in 5.5.

2. Networks governed by the isothermal Euler equations

Let a finite directed graph $G = (V, E)$ be given, where each edge $e \in E$ corresponds to an interval $I^e = [0, L^e]$. The flow through the edges is governed by the isentropic Euler equations

$$\partial_t \begin{pmatrix} \rho^e \\ \rho^e v^e \end{pmatrix} + \partial_x \begin{pmatrix} \rho^e v^e \\ (v^e)^2 \rho^e + p^e(\rho^e) \end{pmatrix} = - \begin{pmatrix} 0 \\ f^e \rho^e v^e |v^e| + g \sin(\alpha^e) \rho^e \end{pmatrix} \quad (2.1)$$

for the gas density $\rho^e = \rho^e(t, x)$ and the gas velocity $v^e = v^e(t, x)$. The gas flux is defined by $q^e(t, x) = (\rho^e v^e)(t, x)$ and $t \geq 0, x \in I^e$. The parameters $f^e > 0$ and $\alpha^e = \alpha^e(x)$ are the friction coefficient and the slope of the pipe. The number g denotes the gravitational force and $D^e > 0$ the pipe diameter.

Let $E_0(v)$ denote the set of edges in the graph that are incident to a vertex $v \in V$ and $x^e(v) \in \{0, L^e\}$ denote the end of the interval $[0, L^e]$ that corresponds to the edge e that is adjacent to v . Let $V_0(e)$ denote the set of nodes adjacent to some edge e . Define

$$\mathfrak{s}(v, e) = \begin{cases} -1 & \text{if } x^e(v) = 0 \text{ and } e \in E_0(v), \\ 1 & \text{if } x^e(v) = L^e \text{ and } e \in E_0(v), \\ 0 & \text{if } e \notin E_0(v). \end{cases} \quad (2.2)$$

We impose the Kirchhoff condition

$$\sum_{e \in E_0(v)} \mathfrak{s}(v, e) (D^e)^2 q^e(t, x^e(v)) = 0 \quad (2.3)$$

that expresses conservation of mass at the nodes.

In order to close the system, additional coupling conditions are needed. A typical choice, leading to well-posed Riemann problems (see [3]) is to require the continuity of the pressure at v , which means that for all $e, f \in E_0(v)$ we have

$$p^e(\rho^e(t, x^e(v))) = p^f(\rho^f(t, x^f(v))). \quad (2.4)$$

At one of the the boundary nodes $v \in V$, that is the nodes where $|E_0(v)| = 1$, we impose a boundary condition of the form

$$p^e(\rho^e) = p_{ref}^e \quad (2.5)$$

where $p_{ref}^e > 0$ is a real number that describes a desired reference pressure.

At the other nodes we prescribe a stabilizing feedback law that will be discussed below.

In [12] it has been shown for ideal gas that the boundary condition (2.5) at one of the boundary nodes and boundary conditions of the form

$$q^e = q_{ref}^e$$

at the other boundary nodes (where q_{ref}^e is a real number that describes a desired velocity) and the node conditions that require the continuity of the pressure determine a unique stationary state if for all $e \in E$ we have $f^e > 0$ and $\alpha^e = 0$ that is the pipes are horizontal.

For the initial boundary value problems that describe the evolution of the state on the network, we obtain semi-global classical solutions on a given time interval $[0, T]$, provided that the initial data and the boundary data is sufficiently small in the C^1 -sense and the initial data are compatible with the boundary conditions and the node conditions. The semi-global solutions can be obtained using the methods presented in [23]. The proofs are based upon the method of characteristics, see also [20] as a classical reference or also [17] for a similar construction of Lipschitz-continuous solutions.

3. The system in terms of Riemann invariants

Quasilinear hyperbolic systems can be written in diagonal form in terms of Riemann invariants. Here the flux, its Jacobian and the eigenvectors are given by

$$f(\rho, q) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p \end{pmatrix}, \quad Df(\rho, q) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{\rho^2} + p'(\rho) & 2\frac{q}{\rho} \end{pmatrix}, \quad \ell_{\pm}(\rho, q) = \begin{pmatrix} -1 \\ \frac{q}{\rho} \pm \sqrt{p'(\rho)} \end{pmatrix}$$

In [Chapter 7.3] [9] it is pointed out that every 2×2 system of hyperbolic conservation laws is endowed with a system of Riemann invariants. For the Euler equations with general pressure law they are given by

$$R_{\pm}(\rho, q) = \tilde{R}(\rho) \pm \frac{q}{\rho} \quad (3.1)$$

where \tilde{R} is defined by

$$\tilde{R}(\rho) = \int_1^{\rho} \frac{\sqrt{p'(r)}}{r} dr.$$

The specific forms for the isentropic law $\rho = a\rho^{\gamma}$ can be found in [9, Chapter 7.3]. In the sequel we consider the case of ideal gas ($\gamma = 1$). Now we state the node conditions in terms of Riemann invariants.

Define the vectors $R_{in}^v(t), R_{out}^v(t) \in \mathbb{R}^{|E_0(v)|}$ in the following manner:

If $\mathfrak{s}(v, e) = 1$, $R_+^e(t, x^e(v))$ is a component of $R_{in}^v(t)$ that we refer to as $R_{in}^e(t, x^e(v))$ and if

$\mathfrak{s}(v, e) = -1$, $R_{in}^v(t)$ contains $R_-^e(t, x^e(v))$ as a component that we also refer to as $R_{in}^e(t, x^e(v))$.

Moreover, if $\mathfrak{s}(v, e) = 1$, $R_-^e(t, x^e(v))$ is a component of R_{out}^v that we refer to as $R_{out}^e(t, x^e(v))$ and if $\mathfrak{s}(v, e) = -1$, $R_{out}^v(t)$ contains $R_+^e(t, x^e(v))$ as a component that we also refer to as $R_{out}^e(t, x^e(v))$.

We assume that the components are ordered in such a way that the j -th component of R_{out}^v corresponds to the same edge $e \in E_0(v)$ as the j -th component of R_{in}^v .

For a node $v \in V$ of the graph and $e \in E_0(v)$ the node conditions (2.4), (2.3) can be written in the form of the linear equation

$$R_{out}^e(t, x^e(v)) = -R_{in}^e(t, x^e(v)) + \frac{2}{\sum_{f \in E_0(v)} (Df)^2} \sum_{g \in E_0(v)} (Dg)^2 R_{in}^g(t, x^g(v)). \quad (3.2)$$

This can be seen as follows. Equation (3.2) implies that for all $e \in E_0(v)$, the value of

$$R_+^e(t, x^e(v)) + R_-^e(t, x^e(v)) = (R_{in}^e + R_{out}^e)(t, x^e(v))$$

is the same, which implies that the value of $\tilde{R}^e(\rho^e)$ is independent of e . Since $\rho \mapsto \tilde{R}(\rho)$ is strictly monotone increasing, this implies (2.4). We have

$$\mathfrak{s}(v, e) [R_+^e(t, x^e(v)) - R_-^e(t, x^e(v))] = (R_{in}^e - R_{out}^e)(t, x^e(v)).$$

Moreover, (3.2) implies

$$\sum_{e \in E_0(v)} (D^e)^2 [R_{out}^e(t, x^e(v)) - R_{in}^e(t, x^e(v))] = 0.$$

Due to (2.4) this implies that equation (2.3) holds.

It is important to look at the structure of the scattering at the node v described by (3.2) in detail. First note that in the process of reflection back in the same pipe e the value of the incoming Riemann invariant is multiplied with the number

$$\kappa_R^e = -1 + \frac{2(D^e)^2}{\sum_{f \in E_0(v)} (Df)^2}.$$

Note that $\kappa_R^e \in (-1, 1)$. In particular we have $|\kappa_R^e| < 1$.

4. Control action

For an edge $e \in E$, in terms of Riemann invariants $R^e = (R_+^e, R_-^e)$ our system has the form

$$\partial_t R^e + D(R^e) \partial_x R^e = F^e(R^e) \quad (4.1)$$

where $D(R^e)$ is a diagonal matrix that contains the eigenvalues which depend continuously differentiably on R^e and F^e is the source term that depends continuously differentiably on R^e . We consider subsonic flow, where one eigenvalue is strictly greater than zero and the other eigenvalue is strictly less than zero. We assume that R_+^e corresponds to the positive eigenvalue and R_-^e corresponds to the negative eigenvalue.

Let a desired stationary state \bar{R}^e be given. At a point $x_0^e \in (0, L^e)$, we consider control action in the form

$$\begin{pmatrix} R_+^e(t, x_0^e+) - \bar{R}_+^e(x_0^e) \\ R_-^e(t, x_0^e-) - \bar{R}_-^e(x_0^e) \end{pmatrix} = M^e \begin{pmatrix} R_+^e(t, x_0^e-) - \bar{R}_+^e(x_0^e) \\ R_-^e(t, x_0^e+) - \bar{R}_-^e(x_0^e) \end{pmatrix}. \quad (4.2)$$

Here M^e is a 2×2 matrix with $\|M^e\|_\infty \leq \kappa^e$, where the matrix norm $\|M^e\|_\infty$ is given by

$$\left\| \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \right\|_\infty = \max\{|z_{11}| + |z_{12}|, |z_{21}| + |z_{22}|\}. \quad (4.3)$$

Assume that there exists a real number $\alpha_0 > 0$ such that for all $e \in E$ we have

$$\kappa^e \leq \alpha_0 < 1. \quad (4.4)$$

If $x_0^e = 0$, we have the feedback law

$$R_+^e(t, x_0^e) = \bar{R}_+^e(x_0^e) + \kappa^e [R_-^e(t, x_0^e) - \bar{R}_-^e(x_0^e)]. \quad (4.5)$$

If $x_0^e = L^e$, we have the feedback law

$$R_-^e(t, x_0^e) = \bar{R}_-^e(x_0^e) + \kappa^e [R_+^e(t, x_0^e) - \bar{R}_+^e(x_0^e)]. \quad (4.6)$$

The feedback laws can be stated in terms of the physical variables using (3.1). Note that in terms of the physical variables, linear Riemann feedback becomes nonlinear.

If at each edge $e \in E$, we have a control action, it is possible to ensure exponential decay by choosing κ^e with a sufficiently small absolute value. However, in general, a smaller number of control locations is sufficient. In particular, at the end of the paper we will present a network that contains a cycle that can be stabilized by two controllers that are located at suitable positions in the cycle.

5. Exponential Stability

First we want to provide a framework that allows us to show exponential stability with respect to maximum norms for hyperbolic systems. Due to the finite speed of propagation of information, our proof requires a special line of arguments with Gronwall's Lemma. In order to prepare the analysis of the hyperbolic system, we start with the corresponding lemma. In this lemma, the damping effects can be modelled with a parameter $\alpha_0 \in [0, 1)$ whereas possible amplification effects can be represented by a parameter $\alpha_1 > 1$.

Lemma 5.1. *Let $T_r > 0$, $a_{\max} \geq 0$, $\alpha_0 \in [0, 1)$ and $\alpha_1 \geq 1$ be given. Let a continuous real function $U(t)$ be given such that $U(t) \geq 0$ for all $t \geq 0$.*

Assume that for all $k \in \{0, 1, 2, 3, \dots\}$ and for all $\tau \in [0, T_r]$ we have the inequality

$$U(kT_r + \tau) \leq \alpha_1 U(kT_r) + \int_{kT_r}^{kT_r + \tau} \alpha_1 a_{\max} U(s) ds. \quad (5.1)$$

Moreover, assume that for all $k \in \{0, 1, 2, 3, \dots\}$ and for all $\tau \in [0, T_r]$ we have the inequality

$$U((k+1)T_r) \leq \alpha_0 U(kT_r) + \int_{kT_r}^{(k+1)T_r} \alpha_1 a_{\max} U(s) ds. \quad (5.2)$$

Define

$$\mu = \alpha_0 + \alpha_1 [\exp(\alpha_1 a_{\max} T_r) - 1]. \quad (5.3)$$

Then for all $n \in \{0, 1, 2, 3, \dots\}$ we have

$$U(n T_r) \leq \mu^n U(0). \quad (5.4)$$

In particular, if $\mu < 1$, this implies that U decays exponentially fast.

Proof. With Gronwall's Lemma, for all $\tau \in [0, T_r]$, inequality (5.1) implies

$$U(k T_r + \tau) \leq \alpha_1 U(k T_r) \exp(\alpha_1 a_{\max} \tau). \quad (5.5)$$

Inequality (5.2) implies

$$U((k+1) T_r) \leq \alpha_0 U(k T_r) + \int_0^{T_r} \alpha_1 a_{\max} U(k T_r + \tau) d\tau.$$

Inserting (5.5) yields

$$\begin{aligned} U((k+1) T_r) &\leq \alpha_0 U(k T_r) + \int_0^{T_r} \alpha_1 \alpha_1 a_{\max} U(k T_r) \exp(\alpha_1 a_{\max} \tau) d\tau \\ &= U(k T_r) [\alpha_0 + \alpha_1 (\exp(\alpha_1 a_{\max} T_r) - 1)] = \mu U(k T_r) \end{aligned}$$

with μ as defined in (5.3). By induction, this implies (5.4). Thus we have proved Lemma 5.1.

5.1. Exponential stability for a single pipe

In order to apply Lemma 5.1 to our hyperbolic system, we assume that it is in diagonal form, that is we consider the evolution of the Riemann invariants along the characteristic curves. We study classical solutions that are defined by integral equations along characteristics as studied for example in [20], [23]. In our context of closed loop system the existence results for semi-global classical solutions state that:

For all $T > 0$ there exists a number $\varepsilon(T) > 0$ such that for all initial data for which the maximal C^1 -norm is less than $\varepsilon(T)$ and that are C^1 -compatible with the node conditions and with the feedback laws there exists a unique classical solution on the time-interval $[0, T]$. A classical solution is a continuously differentiable function that satisfies the initial conditions, the node conditions the feedback laws and the partial differential equations.

Moreover, we have an a priori bound for the solution in the sense that there exists a constant $C_0(T) > 0$ such that the C^1 -norm of the solution is bounded a priori by the product of $C_0(T)$ and the C^1 -norm of the initial data and the boundary data (see [23] and the references therein).

For each edge $e \in E$, The eigenvalues of the system matrix that govern the slopes of the characteristic curves are $\lambda_+^e = c + v^e$ and $\lambda_-^e = -c + v^e$, where c

denotes the sound speed in the gas and v^e the velocity of the gas flow in edge e . If throughout the process, the maximum norm of the velocity of the flow is bounded above by $\frac{c}{2}$, for the eigenvalues we have $\lambda_+^e \in [\frac{c}{2}, \frac{3}{2}c]$ and $\lambda_-^e \in [-\frac{c}{2}, -\frac{3}{2}c]$. Actually, in the practical operation of gas networks $|v^e|$ is much smaller than $c/2$. Then the time that a characteristic curve needs to pass the controller in a pipe of length L^e is bounded above by

$$T_0 = 4 \max_{e \in E} \frac{L^e}{c}. \quad (5.6)$$

Note that in the case of a single edge, we need not take into account any scattering effect (bifurcation respectively) of the characteristic curves at an interior node. At the boundary nodes, the characteristic curves are merely reflected, that is a λ_+ characteristic is continued as a λ_- characteristic and vice versa. We assume that the boundary conditions at $x = 0$ and $x = L^e$ are such that the absolute value of the corresponding component of R^e is not increased in the process. Let R denote the vector of Riemann invariants. Let a desired stationary state \bar{R} be given (that is a system state that satisfies the node conditions and the partial differential equations but does not depend on time). In order to analyze the exponential decay we consider the system for the difference

$$r = R - \bar{R}. \quad (5.7)$$

This system still has diagonal form with the same eigenvalues as the original system, but in the source term different terms appear. More precisely, if we write the system in the form $R_t^e + D(R^e) R_x^e = F^e(R^e)$ (see (4.1)), we have $D(\bar{R}^e) \bar{R}_x^e = F^e(\bar{R}^e)$, since $\bar{R}_t^e = 0$. This yields

$$r_t^e + D(\bar{R}^e + r^e) r_x^e = G^e(r^e, \bar{R}^e),$$

with

$$\begin{aligned} G^e(r^e, \bar{R}^e) &= [F^e(\bar{R}^e + r^e) - F^e(\bar{R}^e)] + [D(\bar{R}^e) - D(\bar{R}^e + r^e)] D^{-1}(\bar{R}^e) F^e(\bar{R}^e) \\ &= F^e(\bar{R}^e + r^e) - D(\bar{R}^e + r^e) D^{-1}(\bar{R}^e) F^e(\bar{R}^e) \end{aligned}$$

and the same diagonal matrix $D(R^e) = D(\bar{R}^e + r^e)$. Due to the linearity of the node conditions (3.2), they are also satisfied by r^e . Note that G^e is continuously differentiable in r^e and $G^e(0, \bar{R}^e) = 0$. Let $a_{\max} > 0$ denote a constant such that for all sufficiently small values of $\|r\|_\infty$ the absolute value of the continuous source term component G_* satisfies the inequality

$$|G_*(r, \bar{R})| \leq a_{\max} \|r\|_\infty, \quad (5.8)$$

where $\|r\|_\infty$ denotes the maximum norm of r :

$$\|r\|_\infty = \max_{e \in E} \{|r_+^e|, |r_-^e|\}.$$

By differentiating the system (4.1), we attain an equation of the form

$$\partial_t r_*^e(\tau, \xi_*^e(\tau)) = \partial_t r_*^e(\tau_0, \xi_*^e(\tau_0)) + \int_{\tau_0}^{\tau} \tilde{G}_*^e(r^e(s, \xi_*^e(s)), \partial_t r^e(s, \xi_*^e(s)), \bar{R}(\xi_*^e(s))) ds$$

with a continuous function \tilde{G}^e

$$\tilde{G}(r, r_t, \bar{R}) = G_r(r, \bar{R})r_t - D_r(r + \bar{R})r_t r_x$$

We assume that for all sufficiently small values of $\|v_0\|_\infty$ and $\|v_1\|_\infty$ we also have the inequality

$$\|\tilde{G}(v_0, v_1, v_2)\|_\infty \leq a_{\max} \max\{\|v_0\|_\infty, \|v_1\|_\infty\}. \quad (5.9)$$

Theorem 5.1. *Consider a single pipe, that is $E = \{e\}$ with the system governed by (4.1) with a stabilizing control action (4.3) as defined in Section 4 at a point $x_0^e \in (0, L^e)$ and boundary conditions*

$$R_+^e(t, 0) = \bar{R}_+^e(0) + \sigma_0(R_-^e(t, 0) - \bar{R}_-^e(0)) \quad (5.10)$$

$$R_-^e(t, L^e) = \bar{R}_-^e(L^e) + \sigma_L(R_+^e(t, L^e) - \bar{R}_+^e(L^e)) \quad (5.11)$$

with $\sigma_0, \sigma_L \in [-1, 1]$ at $x = 0, x = L^e$ respectively. Define $T_r = T_0$ and with α_0 from (4.4)

$$\mu = \alpha_0 + [\exp(a_{\max} T_r) - 1].$$

Assume that $a_{\max} > 0$ from (5.8), (5.9) is sufficiently small, such that we have $\mu < 1$. Assume furthermore that

$$\frac{2}{c}(1 + a_{\max}) \leq 1. \quad (5.12)$$

Then there exists a number $\varepsilon > 0$ such that for all initial states with C^1 -norm less than ε that are compatible with the boundary and node conditions there exists a global classical solution and the C^1 -norm of the r as defined in (5.7) decays exponentially in the sense that $V(t + T_r) \leq \mu V(t)$ where V is defined by

$$V(t) = \max_{x \in [0, L^e]} \{\|r^e(t, x)\|_\infty, \|\partial_t r^e(t, x)\|_\infty\}.$$

Proof. We have chosen the time T_0 sufficiently large such that after the time T_0 all characteristic curves on the edge e are at least once affected by the node conditions at one of the ends of the edge. Let a natural number N be given such that

$$\mu^N \max\left\{1, \frac{3}{2}c + a_{\max}\right\} \leq 1. \quad (5.13)$$

We assume that for the time interval $[0, N T_0]$, a semi-global classical solution exists, that is we choose

$$T = N T_0$$

and assume that the C^1 -norm of the initial state is less than the corresponding bound $\varepsilon(T)$ from the existence result. Note that this also implies that we have an a priori bound for the solution on $[0, NT_0]$, that tends towards 0 with $\varepsilon \rightarrow 0$.

We want to use Lemma 5.1 to show that the C^1 -norm of $r^e(t)$ decays exponentially. For $* \in \{+, -\}$ let r_*^e denote a component that evolves along the characteristic curve $(s, \xi_*^e(s))$ that satisfies the differential equation $\partial_s \xi_*^e(s) = \lambda_*(R_*^e(s, \xi_*^e(s)))$ where λ_* denotes the eigenvalues corresponding to r_*^e . Along the characteristic curve, with $\tau \in [0, NT_0]$ and $\tau_0 \in [0, \tau]$ as long as the characteristic curve is in the interior of the domain, r_*^e satisfies the integral equation

$$r_*^e(\tau, \xi_*^e(\tau)) = r_*^e(\tau_0, \xi_*^e(\tau_0)) + \int_{\tau_0}^{\tau} G_*(r^e(s, \xi_*^e(s)), \bar{R}(\xi_*^e(s))) ds.$$

Then for the evolution of r_*^e along the characteristic curve $(s, \xi_*^e(s))$ due to the triangle inequality and (5.8) we obtain an estimate of the form

$$|r_*^e(\tau, \xi_*^e(\tau))| \leq |r_*^e(\tau_0, \xi_*^e(\tau_0))| + \int_{\tau_0}^{\tau} a_{\max} \|r^e(s, \xi_*^e(s))\|_{\infty} ds. \quad (5.14)$$

For $t \in [0, NT_0]$, define

$$U(t) = \max_{x \in [0, L^e]} \|r^e(t, x)\|_{\infty}.$$

Then due to (5.14), inequality (5.1) holds for all $k \in \{0, 1, 2, 3, \dots, N-1\}$ with $\alpha_1 = 1$. Now we have chosen the time T_0 sufficiently large, such that at one moment $t_* \in [0, T_0]$, each characteristic curve $(s, \xi_*^e(s))$ reaches the controller at x_0^e where the control action takes place. We have assumed that this control action reduces the absolute value of the ingoing Riemann invariant by the factor α_0 . This means that if $\tau - \tau_0 \geq T_0$ we improve (5.14) by said factor α_0 at least once and obtain

$$|r_*^e(\tau, \xi_*^e(\tau))| \leq \alpha_0 |r_*^e(\tau_0, \xi_*^e(\tau_0))| + \int_{\tau_0}^{\tau} a_{\max} \|r^e(s, \xi_*^e(s))\|_{\infty} ds. \quad (5.15)$$

Since we have inequality (5.15) for both components of r , we obtain (5.2) for all $k \in \{0, 1, 2, \dots, N-1\}$. This implies that (5.2) holds for all $k \in \{0, 1, 2, \dots, N-1\}$. By Lemma 5.1, we obtain the inequality

$$U(kT_0) \leq \mu^k U(0)$$

for all $k \in \{0, 1, 2, \dots, N\}$. With $\mu < 1$ this yields the exponential decay on the finite time interval $[0, NT_0]$.

Now we consider the evolution of $\partial_t r_*^e$ along $(s, \xi_*^e(s))$. With analogous arguments as above we obtain

$$V(kT_0) \leq \mu^k V(0).$$

Due to the partial differential equation for r^e we have

$$r_x^e = D(\bar{R}^w + r^e)^{-1}[G^e(r^e) - r_t^e].$$

This yields an estimate of the form

$$\max_{x \in [0, L^e]} |r_x^e(t, x)| \leq \frac{2}{c} V(t) + \frac{2}{c} a_{\max} U(t) \leq \frac{2}{c} (1 + a_{\max}) V(t).$$

Now (5.12) implies

$$\max_{x \in [0, L^e]} \{|r^e(NT_r, x)|, |r_x^e(NT_r, x)|\} \leq V(NT_r).$$

Moreover we have

$$\max_{x \in [0, L^e]} |r_t^e(t, x)| \leq \left[\frac{3}{2} c + a_{\max} \right] \max_{x \in [0, L^e]} \{|r^e(t, x)|, |r_x^e(t, x)|\}.$$

Since $V(NT_r) \leq \mu^N V(0)$ we obtain

$$\begin{aligned} & \max_{x \in [0, L^e]} \{|r_t^e(NT_r, x)|, |r^e(NT_r, x)|, |r_x^e(NT_r, x)|\} \\ & \leq \mu^N \max \left\{ 1, \left(\frac{3}{2} c + a_{\max} \right) \right\} V(0) \leq \varepsilon. \end{aligned}$$

where the last inequality follows with (5.13).

This implies that at the time $T = NT_r$, the C^1 -norm of the solution is sufficiently small such that we can use it as initial data, to extend our solution to the next time interval $[T, 2T]$. In this way we can obtain a global solution on the time interval $[0, \infty)$ that decays exponentially. \square

5.2. Exponential stability with respect to the C^1 -norm for a classical solution on a star shaped network

In order to develop the analysis for networked structure, it is an important step to consider star-shaped networks. Star-shaped networks of gas-pipelines have been the subject of previous investigations, where the exponential decay with respect to the L^2 -norm has been shown by suitably chosen Lypunov funtions, see [15]. In this contribution we present a different approach, where the decay of the maximum value of the error of Riemann invariants is analyzed by tracing the evolution along the characteristic curves.

Let again a desired stationary state \bar{R} on a star-shaped network $G = (V, E)$ be given that satisfies the partial differential equation and the node conditions but does not depend on time. On each edge $e \in E$, the corresponding state is denoted by \bar{R}^e and satisfies (2.1). Moreover, \bar{R} satisfies the coupling conditions (3.2). We consider the system for the difference r defined in (5.7).

E is a finite set of edges and the graph has exactly one interior node v . For all edges $e \in E$, the end $x_e = 0$ of the interval $[0, L^e]$ is adjacent to v . At the ends L^e , for all $e \in E$ we have boundary nodes of the network. We assume that at each of these boundary nodes, control action occurs, that is we have $x_0^e = L^e$. Define

$$\kappa = \max_{e \in E} |\kappa^e|.$$

The scattering process that occurs at the central node v can be described by a scattering matrix

$$A^v = (a_{(e,f)}) \quad (5.16)$$

where $(e, f) \in E_0(v) \times E_0(v)$. According to (3.2) the diagonal elements that describe how much of the ingoing Riemann invariants is reflected back into the pipe from where it entered have the form

$$a_{(e,e)} = -1 + \frac{2(D^e)^2}{\sum_{f \in E_0(v)} (Df)^2} = \frac{(D^e)^2 - \sum_{g \in E_0(v), g \neq e} (Dg)^2}{\sum_{f \in E_0(v)} (Df)^2} \in (-1, 1). \quad (5.17)$$

The elements that describe how a part of the Riemann invariant that enters v from an edge $f \neq e$ is diverted into the pipe e are given by

$$a_{(e,f)} = \frac{2(D^f)^2}{\sum_{g \in E_0(v)} (Dg)^2}. \quad (5.18)$$

Clearly we have $|a_{(e,e)}| < 1$. Moreover, we have

$$\|A^v\|_\infty = \sum_{g \in E_0(v)} |a_{(e,g)}| < 3, \quad (5.19)$$

where $\|A^v\|_\infty$ denotes the maximum norm of A^v :

$$\|A^v\|_\infty = \max_{e \in E_0(v)} \sum_{g \in E_0(v)} |a_{(e,g)}|.$$

We use

$$\nu_v := \|A^v\|_\infty \quad (5.20)$$

as a short hand notation. Since ν_v is the operator norm of the scattering matrix A^v with respect to $\|\cdot\|_\infty$, inequality (5.19) implies that the crossing through the interior vertex can have for the maximal absolute value of the involved Riemann invariants an amplification effect of at most multiplication with $\nu_v < 3$.

For the analysis of the exponential decay in the network, it is essential that this amplification effect in the interior node can be compensated by the damping effect at the boundary nodes at L^e . The feedback law (4.6) leads to a decrease of the size of the absolute value of the reflected Riemann invariants by multiplication of the Riemann invariants that enter the boundary node by a factor α_0 .

Note that $\partial_t r_{\pm}^e$ satisfies a node condition and boundary conditions similar as r_{\pm}^e . At v we have

$$\partial_t r_{out}^e(t, x^e(v)) = -\partial_t r_{in}^e(t, x^e(v)) + \frac{2}{\sum_{f \in E_0(v)} (Df)^2} \sum_{g \in E_0(v)} (Dg)^2 \partial_t R_{in}^g(t, x^g(v)). \quad (5.21)$$

Thus we can use the same matrix A^v from (5.16) to describe the scattering process for $\partial_t r_{\pm}^e$.

Theorem 5.2. *Consider a star-shaped network with one interior node v , that is governed by (4.1), the coupling condition (3.2) at the interior node (that corresponds to $x_e = 0$) and stabilizing control actions (4.6) at the boundary nodes $x_e = L^e$ as defined in Section 4. Define $T_r = T_0$ with T_0 as in (5.6) and*

$$\mu = \alpha_0 + [\exp(a_{\max} T_r) - 1],$$

where $\alpha_0 = \kappa \nu_v^2$. Assume that $a_{\max} > 0$ is sufficiently small, such that we have $\mu < 1$. Assume furthermore that

$$\frac{2}{c}(1 + a_{\max}) \leq 1. \quad (5.22)$$

Then there exists a number $\varepsilon > 0$ such that for all initial states with C^1 -norm less than ε that are compatible with the boundary node conditions there exists a global classical solution and the C^1 -norm of r decays exponentially in the sense that $V(t + T_r) \leq \mu V(t)$ where V is defined by

$$V(t) = \max_{e \in E} \max_{x \in [0, L^e]} \{ \|r^e(t, x)\|_{\infty}, \|\partial_t r^e(t, x)\|_{\infty} \}. \quad (5.23)$$

First we discuss the case with zero source term, that is with $f^e = 0$ and $\alpha^e = 0$. We assume that

$$\kappa < \frac{1}{9}$$

such that

$$\alpha_0 < 1.$$

Thus one damping effect at a boundary node can compensate the effect of two crossings through the interior node. This is sufficient to prove exponential decay. Which can be seen as follows. The time T_r is greater than two times the longest running time through one of the edges. Starting from some point in the network at a given time $t \geq T_r$ we follow all the characteristics connected with that point backwards in time. Then we have at least one crossing through the interior node v . The Amplification of this crossing is bounded by ν_v . Moreover, in each connected characteristic curve, either before or after this crossing there is a reflection at a boundary node where the control leads to a damping by the factor κ .

Furthermore, it may happen that we have more reflections on short edges, but the number of amplifications at the interior node v is at most 1 larger than the number of damping actions. Hence for

$$U(t) = \max_{e \in E} \max_{x \in [0, L^e]} |r_{\pm}^e(t, x)| \quad (5.24)$$

we obtain the inequality

$$U(t + T_r) \leq \kappa \nu_v^2 U(t) = \alpha_0 U(t).$$

Due to (5.21) we obtain

$$V(t + T_r) \leq \alpha_0 V(t)$$

completely analogously. Assume that the semi-global solution exists on an interval $[0, T]$ with $T = NT_r$. Since $\alpha_0 < 1$, similarly as in Lemma 2 from [16], this implies exponential decay of $V(t)$ on $[0, T]$.

Now we consider the case with non-zero source term, that is where $f \geq 0$ or $\alpha^e \neq 0$ and give a more detailed proof using a similar approach as in the previous section. On each edge $e \in E$, we have the integral equations

$$r_{\pm}^e(\tau, \xi_{\pm}^e(\tau)) = r_{\pm}^e(\tau_0, \xi_{\pm}^e(\tau_0)) + \int_{\tau_0}^{\tau} G_{\pm}^e(r^e(s, \xi_{\pm}^e(s))) ds.$$

Assume once again that for all sufficiently small values of $\|r\|_{\infty}$ the absolute value of the source term G_{\pm}^e satisfies the inequality (5.8), where $a_{\max} > 0$ is an a priori known constant.

Then for the evolution of r_{\pm}^e along the characteristic curve $(s, \xi_{\pm}^e(s))$ we obtain an estimate of the form

$$|r_{\pm}^e(\tau, \xi_{\pm}^e(\tau))| \leq \nu_v |r_{\pm}^e(\tau_0, \xi_{\pm}^e(\tau_0))| + \int_{\tau_0}^{\tau} \nu_v a_{\max} \|r^e(s, \xi_{\pm}^e(s))\|_{\infty} ds. \quad (5.25)$$

Here the factor ν_v appears due to (5.19) and takes into account the effect of a possible scattering at the interior node v .

Exactly as in the case with zero source term, we can argue that for the connected characteristics the number of amplifications at the interior node v is at most 1 larger than the number of damping actions. Since $\nu_v^2 \kappa < 1$, with inequality (5.25) for U as defined in (5.24) this yields (5.1) with $\alpha_1 = \nu_v$ for all $k \in \{0, 1, 2, 3, \dots, N-1\}$.

Now we have chosen the time T_r sufficiently large such that at least at one moment in

$$[kT_r, (k+1)T_r]$$

each characteristic curve is reflected at a boundary node where the damping control action takes place. Moreover, the number of crossings through the central

note v is at most 1 larger than the number of damping actions. Then due to (5.25), we obtain

$$|r_{\pm}^e((k+1)T_r, \xi_{\pm}^e((k+1)T_r))| \leq \alpha_0 |r_{\pm}^e(kT_r, \xi_{\pm}^e(kT_r))| + \int_{kT_r}^{(k+1)T_r} \nu_v a_{\max} \|r^e(s, \xi_{\pm}^e(s))\|_{\infty} ds.$$

This implies that (5.2) holds for all $k \in \{0, 1, 2, \dots, N-1\}$ with α_0 . As in the proof of Lemma 5.1, we obtain the inequality

$$U(kT_0) \leq \mu^k U(0)$$

for all $k \in \{0, 1, 2, \dots, N\}$ with

$$\mu = \alpha_0 + \alpha_1 [\exp(\alpha_1 a_{\max} T_r) - 1].$$

If $\mu < 1$ this yields exponential decay on the finite time interval $[0, NT_r]$.

Now we consider the evolution of $\partial_t r_{\pm}^e$ along $(s, \xi_{\pm}^e(s))$. Once again we can make use of the fact that $\partial_t r_{\pm}^e$ satisfies a node condition (5.21) and boundary conditions similar as r_{\pm}^e . At $x = L^e$ we have

$$\partial_t r_{-}^e(t, x_0^e) = \kappa^e \partial_t r_{+}^e(t, x_0^e). \quad (5.26)$$

We have an integral equation of the form

$$\partial_t r_{\pm}^e(\tau, \xi_{\pm}^e(\tau)) = \partial_t r_{\pm}^e(\tau_0, \xi_{\pm}^e(\tau_0)) + \int_{\tau_0}^{\tau} \tilde{G}_{\pm}^e(r^e(s, \xi_{\pm}^e(s)), \partial_t r^e(s, \xi_{\pm}^e(s)), \bar{R}(\xi_{\pm}^e(s))) ds$$

with a continuous function \tilde{G}_{\pm}^e and (5.9). With analogous arguments as above we obtain

$$V(kT_r) \leq \mu^k V(0).$$

This implies that at the time $T = NT_r$, the C^1 -norm of the solution is sufficiently small such that we can use it as initial data, to extend our solution to the next time interval $[NT_0, (N+1)T_0]$. In this way we can obtain a global solution on the time interval $[0, \infty]$ that decays exponentially.

5.3. Exponential stability for a network that is a cycle

In order to illustrate the damping effect of our control action, consider a network that is a cycle, that is we have a single edge e with the boundary conditions

$$r_{+}^e(t, 0) = r_{+}^e(t, L^e), \quad r_{-}^e(t, L^e) = r_{-}^e(t, 0).$$

We assume that in this interval a control action as defined in Section 4 at a point $x_0^e \in (0, L^e)$ takes place. For the sake of simplicity we consider the case where the source term is zero, that is the Riemann invariants are constant along the characteristics. Define

$$U^e(t) = \max_{x \in [0, L^e]} \{|r_{+}^e(t, x)|, |r_{-}^e(t, x)|\}.$$

For each value $r_+^e(t, x)$ we can follow the corresponding ξ_+ characteristic that transports the value backwards in time. In this system, no bifurcations and reflections occur. Therefore it is easy to see that if we follow the characteristic curve backwards in time for a time interval of length T_0 , the characteristic curve crosses the point x_0^e where the controller is located. Hence for $t \geq T_0$ we have the inequality

$$|r_+^e(t, x)| \leq |\kappa^e| U^e(t - T_0).$$

Analogous arguments yield the corresponding inequality for $r_-^e(t, x)$. Hence we have

$$U^e(t) \leq |\kappa^e| U^e(t - T_0).$$

If $|\kappa^e| < 1$, this implies the exponential decay of $U^e(t)$ (see Lemma 2 from [16]).

For the case with non-zero source terms, we have to adapt the arguments as in the previous section. In particular, again we obtain the decay of the C^1 norm on sufficiently large time intervals, which yields the global existence of a classical solution.

Of course in practice, circular networks are only interesting if there are additional entry and exit pipes. We consider an example of this type in the next section.

5.4. Exponential stability for a network with a cycle and two additional edges.

We consider a network with the four edges $\alpha, \beta, \gamma, \delta$ and two interior nodes v_0 and v_1 , see Figure 5.1. Let the lengths of the pipes for β and γ be given with $L^\beta > 0$ and $L^\gamma > 0$. We assume that $L^\alpha = L^\delta > L^\beta$ and that $L^\alpha > L^\gamma$. The ends 0 of the edge α and the end L^δ of the edge δ are boundary nodes. At the node v_0 , the edge α is connected with β and γ . The adjacent ends are L^α for the edge α and 0 for the other two edges. At v_1 , the edges β and γ are connected with the edge δ . The adjacent ends are L^β for the edge β , L^γ for the edge γ and 0 for the edge δ .

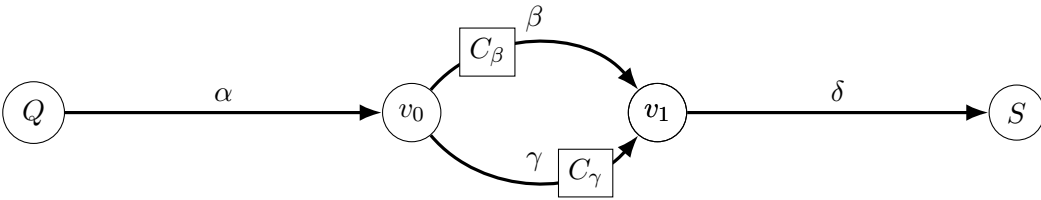


Fig. 5.1. Circle with two border nodes and two locations for control action

Moreover, we assume that there is a control action (4.2) in the two edges β and γ represented by the nodes C_β and C_γ . We assume that the reflections at the boundary nodes Q and S do not increase the absolute value of the corresponding Riemann invariants by the boundary conditions.

At Q , with a number $\sigma^\alpha \in [-1, 1]$ we have the feedback law

$$R_+^\alpha(t, 0) = \bar{R}_+^\alpha(0) + \sigma^\alpha [R_-^\alpha(t, 0) - \bar{R}_-^\alpha(0)]. \quad (5.27)$$

At S , with $\sigma^\delta \in [-1, 1]$ we have the feedback law

$$R_-^\delta(t, L^\delta) = \bar{R}_-^\delta(L^\delta) + \sigma^\delta [R_+^\delta(t, L^\delta) - \bar{R}_+^\delta(L^\delta)]. \quad (5.28)$$

The network can be thought of as a compressor station where two compressors work in parallel. For all $e \in E = \{\alpha, \beta, \gamma, \delta\}$ define $T_e = 2\frac{L^e}{c}$.

The following theorem gives sufficient conditions for the exponential stability of the system.

Theorem 5.3. *Consider a network consisting of a cycle with locations for control action in the edges β and γ and two additional edges as defined above. Define $T_r = 2T_\alpha + \max\{T_\beta, T_\gamma\}$. Let $\alpha_0, a_{\max} > 0$ be given. Define*

$$Z_0 = \max\{|a(\alpha, \alpha)| + \alpha_0 \nu_v [|a(\alpha, \beta)| + |a(\alpha, \gamma)|], \nu_v^2 \alpha_0\} \quad (5.29)$$

(where $a(\alpha, \alpha)$, $a(\alpha, \beta)$ and $a(\alpha, \gamma)$ are the entries of the corresponding scattering matrix, see (5.16) and ν_v as defined in (5.20) denotes the corresponding norm, see (5.19)) and for a given a_{\max} define

$$\mu = Z_0 + \nu_v [\exp(\nu_v a_{\max} T_r) - 1].$$

Assume that $\mu < 1$ and that for the control actions we have $|\kappa^e| \leq \alpha_0$. Moreover assume that

$$\frac{2}{c}(1 + a_{\max}) \leq 1. \quad (5.30)$$

Then there exists a number $\varepsilon > 0$ such that for all initial states with C^1 -norm less than ε that are compatible with the node conditions and the boundary conditions there exists a global classical solution and the C^1 -norm decays exponentially in the sense that $V(t + T_r) \leq \mu V(t)$ where V is as defined in (5.23).

Proof. Let a natural number N be given such that

$$\mu^N \max\{1, \frac{3}{2}c + a_{\max}\} \leq 1. \quad (5.31)$$

The theory of semi-global solutions implies that for $T = NT_r$ there exists a number $\varepsilon = \varepsilon(T) > 0$ such that for all initial states with C^1 -norm less than $\varepsilon(T)$ that are compatible with the node conditions and the boundary conditions there exists a classical solution on the interval $[0, NT_r]$. Moreover, this solution satisfies an a priori inequality, hence by choosing $\varepsilon > 0$ sufficiently small it can also be guaranteed that the C^1 -norm of the state is sufficiently small.

First we analyze the evolution of $|r_\pm^\alpha|$. Due to the definition of T_r , if we trace back any characteristic ξ_\pm^α , it reaches the interior node v_0 , possibly after a reflection at $x^\alpha = 0$ that does not increase the size of the Riemann invariant.

Let us say the node v_0 is reached at the time $t^*(v_0) \in (0, T_r]$. Then we have for all $\tau \in [t^*(v_0), T_r]$

$$|r_{\pm}^{\alpha}(\tau, \xi_{\pm}^{\alpha}(\tau))| \leq |r_{\pm}^{\alpha}(t^*(v_0), \xi_{\pm}^{\alpha}(t^*(v_0)))| + \int_{t^*(v_0)}^{\tau} a_{\max} \|r^{\alpha}(s, \xi_{\pm}^{\alpha}(s))\|_{\infty} ds \quad (5.32)$$

where the ξ_{+}^{α} characteristic is continued as a ξ_{-}^{α} characteristic after the possible reflection at $x^{\alpha} = 0$. The definition of T_r implies that the time $t^*(v_0) > \max\{T_{\beta}, T_{\gamma}\}$ and thus the two characteristic curves coming from β and γ have already crossed the damper in the corresponding edge. Hence the size of the corresponding Riemann invariants has already been reduced by a factor α_0 . Moreover, due to the structure of the graph in any case in the edges β and γ there can be at most one interaction more with a node than with the damper. Hence for $t > t^*(v_0)$ before the next interaction with v_0 we have

$$|r_{\pm}^{\alpha}(t, \xi_{\pm}^{\alpha}(t))| \leq [|a(\alpha, \alpha)| + \alpha_0 \nu_v [|a(\alpha, \beta)| + |a(\alpha, \gamma)|]] U(0) + \int_0^t a_{\max} U(s) ds \quad (5.33)$$

where U is as defined in (5.24). Moreover, our definition of T_r implies that for the reflected characteristic in the edge α the number of scattering interactions at v_0 can be at most one higher than the number of damping effects in the time interval $[0, T_r]$. Hence we have

$$|r_{\pm}^{\alpha}(T_r, \xi_{\pm}^{\alpha}(T_r))| \leq [|a(\alpha, \alpha)| + \alpha_0 \nu_v [|a(\alpha, \beta)| + |a(\alpha, \gamma)|]] U(0) + \int_0^{T_r} a_{\max} U(s) ds. \quad (5.34)$$

Moreover, for all $\tau \in [0, T_r]$ we have

$$|r_{\pm}^{\alpha}(\tau, \xi_{\pm}^{\alpha}(\tau))| \leq U(0) + \int_0^{\tau} a_{\max} U(s) ds. \quad (5.35)$$

For the edge δ , we obtain completely analogous estimates.

Now we consider the evolution of the absolute value of the Riemann invariants on the edge β where the damping control action is located. For $* \in \{+, -\}$ we consider the evolution of r_{*}^{β} along the corresponding characteristic curve ξ_{*}^{β} . Then after a possible control action at x_0^{β} , due to the definition of T_r there is a first time $t^* \in [0, T_{\beta}] \subset [0, T_r]$ where the characteristic curve reaches the central node v_0 in the minus-case or the central node v_1 in the plus-case.

Then we have for all $\tau \in [0, t^*]$

$$|r_{\pm}^{\beta}(\tau, \xi_{\pm}^{\beta}(\tau))| \leq U(0) + \int_0^{\tau} a_{\max} U(s) ds. \quad (5.36)$$

After the reflection at the central node v_1 , the ξ_{+}^{β} characteristic is continued as a ξ_{-}^{β} characteristic and vice versa after the reflection at v_0 . Since the scattering is

governed by the matrices A^{v_0} and A^{v_1} respectively, for $\tau > t^*$ due to (5.19) we obtain

$$|r_{\pm}^{\beta}(\tau, \xi_{\pm}^{\beta}(\tau))| \leq \nu_v U(0) + \int_0^{\tau} \nu_v a_{\max} U(s) ds. \quad (5.37)$$

Since $T^{\alpha} > T^{\beta}$ we have $3T_{\alpha} > T_r > 3T_{\beta}$. Hence before the first interaction with a central node, for the reflected characteristic that is continued backwards in time in the edge β , there is at least one interaction with the damping control at x_0^{β} before the next scattering. In any case the number of scattering interaction at an interior node is at most one larger than the number of times that of the characteristic curve crosses the controller. Since $\nu_v \alpha_0 < 1$, we obtain the inequality

$$|r_{\pm}^{\beta}(T_r, \xi_{\pm}^{\alpha}(T_r))| \leq \nu_v^2 \alpha_0 U(0) + \int_0^{\tau} \nu_v^2 \alpha_0 a_{\max} U(s) ds. \quad (5.38)$$

Moreover, for all $\tau \in [0, T_r]$ we have

$$|r_{\pm}^{\beta}(\tau, \xi_{\pm}^{\alpha}(\tau))| \leq \nu_v U(0) + \int_0^{\tau} \nu_v a_{\max} U(s) ds \quad (5.39)$$

and the same holds for the edge γ . With (5.39) and (5.35) for all $\tau \in [0, T_r]$ this yields the inequality

$$U(\tau) \leq \nu_v U(0) + \int_0^{\tau} \nu_v a_{\max} U(s) ds. \quad (5.40)$$

Define

$$Z_0 = \max\{|a(\alpha, \alpha)| + \alpha_0 \nu_v [|a(\alpha, \beta)| + |a(\alpha, \gamma)|], \nu_v^2 \alpha_0\}$$

Then from (5.34) and (5.38) we obtain

$$U(T_r) \leq Z_0 U(0) + \int_0^{\tau} a_{\max} U(s) ds.$$

Then by Lemma 5.1 with $\alpha_1 = \nu_v$ we have $U(T_r) \leq \mu U(0) \leq \varepsilon$. By induction, we obtain $U(NT_r) \leq \mu^N U(0) \leq \varepsilon$.

Again due to the linearity of the node condition and the control law for the evolution of $\partial_t r$ we obtain completely analogously $V(NT_r) \leq \mu^N V(0) \leq \varepsilon$, where V is as defined in (5.23).

For all $e \in E$, due to the partial differential equation for r^e we have

$$r_x^e = D(\bar{R}^w + r^e)^{-1} [G^e(r^e) - r_t^e].$$

This yields an estimate of the form

$$\max_{e \in E} \max_{x \in [0, L^e]} |r_x^e(t, x)| \leq \frac{2}{c} V(t) + \frac{2}{c} a_{\max} U(t) \leq \frac{2}{c} (1 + a_{\max}) V(t).$$

Now (5.30) implies

$$\max_{e \in E} \max_{x \in [0, L^e]} \{|r^e(NT_r, x)|, |r_x^e(NT_r, x)|\} \leq V(NT_r).$$

Moreover we have

$$\max_{e \in E} \max_{x \in [0, L^e]} |r_t^e(t, x)| \leq \left[\frac{3}{2}c + a_{\max} \right] \max_{e \in E} \max_{x \in [0, L^e]} \{|r^e(t, x)|, |r_x^e(t, x)|\}.$$

Since $V(NT_r) \leq \mu^N V(0)$ we obtain

$$\begin{aligned} & \max_{e \in E} \max_{x \in [0, L^e]} \{|r^e(NT_r, x)|, |r_x^e(NT_r, x)|\} \\ & \leq \mu^N \max\{1, \left(\frac{3}{2}c + a_{\max}\right)\} \max_{e \in E} \max_{x \in [0, L^e]} \{|r^e(0, x)|, |r_x^e(0, x)|\} \leq \varepsilon. \end{aligned}$$

where the last inequality follows with (5.31).

This implies that at the time $T = NT_r$, the C^1 -norm of the solution is sufficiently small such that we can use it as initial data, to extend our solution to the next time interval $[NT_r, 2NT_r]$. In this way we can obtain a global solution on the time interval $[0, \infty]$ that decays exponentially.

Remark 5.1. Note that if there is only one control action in the cycle, say only on β and not on γ , at least in the case without source term (that is with $a_{\max} = 0$) stabilization is not possible. This can be seen as follows. In this case there exist several stationary states on the network with the same state on β , say \bar{R}_1 and \bar{R}_2 . Hence if we choose $r^e = \bar{R}_2 - \bar{R}_1$, we have a stationary state that does not decay.

The stationary states can be chosen with constant states for all $e \in E$. In particular the pressure is constant throughout the network. For both stationary states we can choose the same value for q^β and p^β . Then we have $r^\beta = 0$.

5.5. Exponential stability for non-classical solutions on a star shaped network: Numerical experiments

The proof of our results for classical solutions depends on the description by integral equations along the characteristic curves. It is well known, that for less regular solutions (that is non-classical solutions) such a representation does not exist. In this Section we consider the same system as in Section 5.2 but we consider non-classical solutions as defined in [7] and [8]. We present a numerical study for the behaviour of the system with a piecewise constant initial state and absorbing boundary conditions at the boundary node. The numerical results indicate that also this system is stabilized by the absorbing boundary conditions. However, we do not have a proof for this, so this remains an open question for future research. The asymptotic stabilization of systems of conservation laws by controls acting at a single boundary point has been considered in [2]. However, for the system of balance laws that we consider, a method that allows to take into account the effect of the source term is essential.

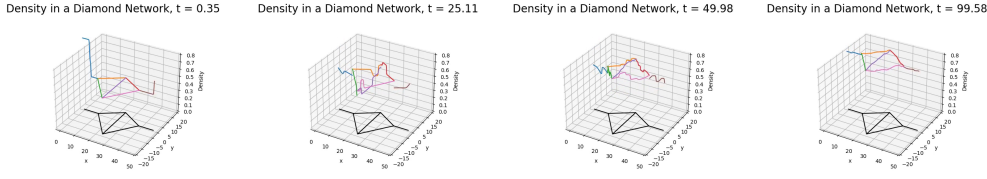


Fig. 5.2. snapshot at $t = 0.35$, $t = 25.11$, $t = 49.98$ and $t = 99.58$

In [14], not only the limits of stabilizability for a semilinear model for gas pipeline flow are studied, but also an example is presented, where the quasilinear flow model becomes unstabilizable for a certain feedback law of NEUMANN type namely $p_x = f_0 p_t$ at $x = 0$ and $v_x = -\frac{1}{c} v_t$ at $x = L$ for all feedback parameters f_0 that are greater than the reciprocal value of the sound speed. However, this does not imply that the situation for DIRICHLET feedback as we consider in this paper is similar.

The snapshots in Figure 5.5 of the evolution of the pressure starting from a piecewise constant initial state illustrate the complexity of the dynamics. Here the aim is to stabilize the system to a state with constant pressure and zero velocity. At the beginning the pressure is also constant, but on a lower level. At the inflow boundary node at $x = 0$ and the outflow boundary node at $x = 50$ the desired values of the Riemann invariants (that is the values of the Riemann invariants corresponding to the desired terminal state) are prescribed by the boundary conditions. The numerical solution approaches the desired state rapidly. However, the question whether the analytical solution shows the same behaviour is open.

6. Conclusion

We show that for a networked quasi-linear system that is defined on a graph with a cycle, the C^1 -norm of the state decays exponentially fast if there is sufficient nodal control action located in the cycle. On account of the fact that the C^1 -norm is non-smooth, Lyapunov functions are not a natural tool to show the exponential decay of classical solutions. Therefore, in our contribution we suggest a different approach that is based upon the study of the evolution of the value of the Riemann invariants along the characteristic curves. In the context of networks, this requires in particular the study of scattering effects at interior nodes, that leads to a branching of the connected characteristic curves. In the analysis, we carefully trace the evolution of the Riemann invariants along the characteristic curves. We look at the evolution with a special focus on the moments when the characteristic curves go through vertices of the graph, both when the characteristic curves are scattered at interior nodes and when they are reflected at boundary nodes.

We study a network with a cycle and show that with a two nodal controllers in the cycle, the system state given by classical solutions can be stabilized exponentially

fast. In particular we show the existence of a global classical solution. Note that this requires weaker regularity assumptions than for the H^2 solution (see for example [19]) that have been studied recently. We expect these results can be extended to more general networks. For future research, it would be desirable to extend the analysis to more general networks and to the case where some model parameters are uncertain, see for example [10].

This work was supported by DFG in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Project C03 and Project C05.

7. Acknowledgment

This work was supported by DFG in the framework of the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Project C03, Projektnummer 239904186 and Project C05 Projektnummer 239904186.

This research was inspired by numerical experiments with the software

<https://caa-avh.nat.fau.eu/pygascontrols-framework>

that has been generously funded by the ALEXANDER VON HUMBOLDT Foundation through E. ZUAZUA, Chair of Dynamics, Control and Numerics вЂ“ ALEXANDER VON HUMBOLDT Professorship.

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Received 25.10.2021