

A PDE-CONSTRAINED GENERALIZED NASH EQUILIBRIUM APPROACH FOR MODELING GAS MARKETS WITH TRANSPORT

VERONIKA GRIMM, MICHAEL HINTERMÜLLER, OLIVIER HUBER,
LARS SCHEWE, MARTIN SCHMIDT, GREGOR ZÖTTL

ABSTRACT. We investigate a class of generalized Nash equilibrium problems (GNEPs) in which the objectives of the individuals are interdependent and the shared constraint consists of a system of partial differential equations. This setup is motivated by the modeling of strategic interactions of competing firms, which explicitly take into account the dynamics of transporting a commodity, such as natural gas, through a network. We establish the existence of a variational equilibrium of the GNEP. In the case of symmetric firms, we identify an equivalent optimization problem. We use this model to numerically explore the impact of linepacking, that is the use of the network as a temporary storage device. In particular, we study the firms' decisions under various linepacking abilities and analyze which market participants benefit from it.

1. INTRODUCTION

The framework of equilibrium problems considerably contributes to the understanding of a wide array of real-world problems, ranging from management and micro-economics applications to a variety of design problems in engineering. Specifically, non-cooperative game theory has turned out to be a useful tool in analyzing the actions of agents (or players) on markets. In such a context, one seeks solutions in which each player's individual objective is maximized subject to individual (or so-called private) constraints, and no player has an incentive to take decisions deviating from a so-called Nash equilibrium (Nash 1950). For classic Nash equilibrium problems (NEPs), the coupling between individuals occurs in the objective functionals only, whereas the feasible sets (or constraints) are independent. If the latter are also interdependent, one faces a so-called generalized Nash equilibrium problem (GNEP).

In this work, we investigate GNEPs in which the dynamics of transporting a given good over a network are described by a system of partial differential equations (PDEs), which constitutes a shared constraint that is taken into account by all players. This is highly relevant, e.g., in the case of liberalized energy markets, where conversion and trading of energy carriers is organized by competing private companies that seek to maximize their profits in a non-cooperative fashion. With a short time horizon, as in an intraday market, it is important to consider a high-fidelity model of the dynamics when transporting goods with slow transients such as natural gas or hydrogen. Moreover, in our setup, the objective functionals are interdependent such that the actions of rivals have a direct impact on the value of the objective of the respective individual. This allows us to consider classic situations of strategically interacting firms; see, e.g., Cournot (1838) for the seminal article in this area. The focus here is on the solution concept called *variational equilibrium* (VE) for which the common multipliers associated with the shared constraints have a clear economic interpretation.

In view of the aforementioned application regarding liberalized gas markets, in this work we first provide conditions for the existence of a solution to the governing system of PDEs, with

Date: March 7, 2023.

2010 Mathematics Subject Classification. 35Axx, 35B65, 35Q91, 49J40, 90C35, 90C48,

Key words and phrases. Generalized Nash Equilibrium Problem, PDE-Constrained Optimization, Energy Markets, Existence, State Constraints.

enough regularity to ensure the existence of multipliers to the shared constraints. Secondly, we show the existence of a VE that solves the underlying jointly convex GNEP. For symmetric setups, we identify a reformulation of the original equilibrium problem as a single optimization problem, which allows us to derive a solution algorithm for finding such a VE. Finally, we apply our algorithm to a suitable test instance, which is motivated by the trading and transporting of natural gas. Our numerical results highlight crucial dynamical aspects in the context of gas markets such as the impact and value of *linepacking*. The latter refers to the action of changing the amount of gas stored in the pipe, the so-called linepack. When such a linepacking occurs, the transportation network acts as a temporary storage device. This allows to take advantage of price variations or to overcome fluctuations in supply and demand.

Our work contributes to several strands of the literature. Indeed, equilibrium problems that are subject to network constraints have received considerable attention. In the context of energy markets, these settings have been studied mainly for the analysis of electricity and gas markets. There, we find many contributions that consider strategic interactions of firms in the presence of shared and static, i.e., stationary, network constraints; see, e.g., Chen and Wang (2014), Schiro et al. (2013), Wei and Smeers (1999), and R. Wilson (2008) as well as, more recently, Holmberg and Philpott (2018). Due to its physical properties, a steady-state approximation of electricity transmission is well accepted for many market applications. Owing to slower speeds of propagation, a stationary approach cannot capture, however, all aspects of the dynamic evolution of gas transported in a network.

Another strand of literature explicitly considers strategic interaction of firms in a dynamic context—but in the absence of shared network constraints. In several contributions, firms, e.g., experience adjustment costs when changing their output; see Caruana and Einav (2008), Jun and Vives (2004), and Wirl (2010). Other contributions have analyzed situations in which strategic firms exploit a private resource stock (Ledvina and Sircar 2011) or accumulate private productive capacities over time (Lambertini and Palestini 2014). Similar to our approach, in these contributions, the interactive nature of the situation analyzed also arises due to firms' profits being influenced by all rivals' actions. However, there are no shared constraints that restrict the choices of the firms.

Several works on dynamic oligopoly models do consider shared constraints. Benckekroun (2003, 2008) studies differential games in which firms jointly exploit a common renewable production asset. More recent articles further elaborate on this; see, e.g., Colombo and Labrecciosa (2015, 2019) and Xin and Sun (2018), who consider different types of strategic behavior among firms. Note that for all these problems, the shared constraint can be represented by a single state variable governed by an ordinary differential equation (ODE). For a proper modeling of the physics underlying the network transportation aspect of our setup, we have to consider a spatially distributed state variable governed by a system of PDEs.

Several recent works analyze GNEPs involving shared constraints governed by PDEs (Dreves and Gwinner 2016; Gugat and Steffensen 2018; Hintermüller and Surowiec 2013; Hintermüller, Surowiec, and Kämmler 2015; Kanzow et al. 2019). While the structural assumptions vary among these contributions, the results and methods are mostly applied to instances in which objective functionals are of tracking type and the shared constraint is a scalar PDE. In our setup, we consider objectives that model market interaction of strategic firms. In addition, the shared constraint comprises a system of PDEs, which allows to appropriately capture the physical phenomena under consideration.

The remainder of this paper is structured as follows. After this introduction, we present the stylized GNEP model coupling markets and transport phenomena in Section 2. Section 3 provides general well-posedness results for the system of PDEs under investigation. In Section 4, we establish the existence of a VE for the jointly convex GNEP under consideration. A reformulation of the GNEP that allows for a computational approach to determine a VE is

given. We illustrate the important role of linepacking in Section 5 and finally conclude in Section 6.

Notations. For a given Banach space X , its dual is given by X^* . If $A: X \rightarrow Y$ is a linear and continuous mapping between Banach spaces, its adjoint is denoted by $A^*: Y^* \rightarrow X^*$. The continuous and dense embedding of a Banach space X into Y is represented by $X \hookrightarrow Y$. With Ω being an open interval of \mathbb{R} , for any integer p (and even for $p = +\infty$), the function space $L^p(\Omega)$ is the standard Lebesgue space. For any real s , $H^s(\Omega)$ is the $L^2(\Omega)$ -based Sobolev space of order s . Let $f: \mathbb{R} \rightarrow X$ and $T > 0$. If $t \mapsto \|f(t)\|_X$ lies in $L^p(0, T)$ and is Bochner measurable, then $f \in L^p(0, T; X)$. The space $C([0, T]; X)$ consists of all X -valued continuous functions on $[0, T]$ equipped with the norm of uniform convergence. For more background on these spaces, see, e.g., Adams and Fournier (2003). As we have only one spatial dimension, denoted by x , the Laplacian operator Δ has the same action as taking second partial derivatives $\partial^2/\partial x^2$. When applied to a vector-valued function, its action is to take second partial derivatives of each component. As a shorthand notation, for any function f depending on x and t , f_t , f_x and f_{xx} denote, respectively, the first-order derivative with respect to t , x , as well as the second-order derivative with respect to x . For a given set S , its closure is denoted by \bar{S} , its interior by $\text{int } S$, and its indicator function by δ_S . The subdifferential of a convex function $g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is represented by $\partial g: X \rightrightarrows X^*$. When S is convex, the normal cone $N_S(\cdot)$ is the subdifferential of $\delta_S(\cdot)$.

2. A GNEP MODEL FOR GAS MARKETS WITH TRANSPORT

For the ease of exposition, we investigate a stylized intraday gas market over a single pipe e . The pipe's diameter $D^e > 0$ is typically much smaller than its length $L^e > 0$. Hence, a 1d model for the flow of gas is appropriate. Let $A^e := (D^e)^2\pi/4$ be the constant cross-sectional area of the pipe. We consider the case of an isothermal flow, i.e., the temperature $T_{\text{emp}} > 0$ of gas is constant, and denote the density by ρ^e , p^e is the pressure, and $c_s = \sqrt{R_s T_{\text{emp}}}$ is the speed of sound for a given specific gas constant R_s . The state equation is given by the ideal gas law $p^e = c_s^2 \rho^e$, and the velocity of gas is assumed to be small with respect to c_s . Given the operational conditions in a pipeline system, the evolution of the gas can be modeled by the system

$$\begin{aligned} p_t^e - \varepsilon p_{xx}^e + \frac{c_s^2}{A^e} q_x^e &= 0, \\ q_t^e - \varepsilon q_{xx}^e + A^e p_x^e &= -\sigma^e A^e \frac{\check{q}^e}{\check{p}^e} \left(q^e - \frac{\check{q}^e}{2\check{p}^e} p^e \right) - \frac{A^e g \sin \varphi^e}{c_s^2} p^e, \end{aligned} \quad (1)$$

where $\varepsilon > 0$, $\sigma^e := 16c_s^2\lambda^e/(\pi^2(D^e)^5)$, $\lambda^e \geq 0$ is the friction coefficient, g is the gravitational acceleration, and φ^e is the constant angle of the pipe with the horizontal plane. The pair (\check{p}, \check{q}) , with $\check{p} > 0$ and $\check{q} > 0$, is a reference state. System (1) comes from the linearization and viscosity regularization of the hyperbolic system known as ‘‘ISO2’’ in the model catalog of Domschke et al. (2021), which presents different models for gas flow in pipelines depending on the operational conditions. The boundary conditions for (1) are of Dirichlet type. Among other gases, flows of natural gas and hydrogen can be modeled by (1).

Alternatively to the single-pipe scenario, a small pipeline network whose graph $(\mathcal{N}, \mathcal{E})$ is a directed tree, i.e., a directed acyclic graph, may be considered. In such a setting, at a node where multiple pipes meet, two relations have to hold: (i) The pressures at the common endpoints of these pipes are identical and (ii) the inflows and outflows of gas are balanced. In our subsequent exposition we will, however, stick to the single-pipe setting for convenience as this allows us to emphasize the GNEP aspect.

We now consider a gas market over one pipe with two nodes labeled in and out, respectively. For the ease of notation, subsequently we drop all superscripts related to the pipe's identifier. Further, we assume that M strategic firms (producers) may inject gas at the in node and sell it at

the out node. Concerning the equilibrium setting, we consider the case of Cournot competition: With the goal of maximizing their respective profit over a finite time period $[0, T]$, $T > 0$, each firm strategically decides on the injected mass flow q_i^{in} at pressure p_i^{in} and the mass flow q_i^{out} at pressure p_i^{out} , which they withdraw from the pipe. Let us state here that below we use the terms “firm”, “agent”, or “player” as synonyms. We denote by $u_i := (p_i^{\text{in}}, q_i^{\text{in}}, p_i^{\text{out}}, q_i^{\text{out}})$ the decisions of firm i and let $u := (u_1, \dots, u_M)$ be the entire decision vector. With respect to notation, below the classic game-theoretic convention is invoked that for any i , $u = (u_i, u_{-i})$, where u_{-i} collects the decisions of all the other firms except u_i . Mathematically, it is supposed that each u_i belongs to $U \subseteq L^2(0, T)^4$. In the pipe, the state variable is $y := (p, q) : (0, L) \times (0, T) \rightarrow \mathbb{R}^2$. Then the aforementioned Dirichlet boundary conditions are given by $y(0, \cdot) = u_\Gamma^0$ and $y(L, \cdot) = u_\Gamma^L$, where the nodal state values $u_\Gamma^0 := (p^{\text{in}}, q^{\text{in}})$ and $u_\Gamma^L := (p^{\text{out}}, q^{\text{out}})$ are related to the firms’ actions by

$$u_\Gamma^0 = \left(\sum_{i=1}^M \frac{p_i^{\text{in}}}{M}, \sum_{i=1}^M q_i^{\text{in}} \right), \quad u_\Gamma^L = \left(\sum_{i=1}^M \frac{p_i^{\text{out}}}{M}, \sum_{i=1}^M q_i^{\text{out}} \right). \quad (2)$$

The relations on the mass flows fit the above stated conditions at a junction. For the pressures, this is also the case if all firms choose the same pressures and we focus on this latter case. Furthermore, we consider lower and upper bound constraints on the pressures and mass flows. For given bounds $\underline{y} := (\underline{p}, \underline{q})$ and $\bar{y} := (\bar{p}, \bar{q})$, these state constraints are represented by

$$y \in K := \{y \in C([0, T]; C([0, L])^2) \mid \underline{y} \leq y(x, t) \leq \bar{y} \text{ for all } (x, t) \in [0, L] \times [0, T]\}, \quad (3)$$

where the space $C([0, T]; C([0, L])^2)$ is chosen in order to guaranteed the existence of associated Lagrangian multipliers for a primal-dual stationarity characterization of optimal solutions resp. equilibria, below. Since firms are fully aware of both the system (1) and the bound constraints (3), they all take into account the same, and thus shared, network constraints. Let S be the so-called control-to-state operator, which gives the solution (state) y of the PDE system for a given (control) u . Then, the inclusion $y = S(u) \in K$ represents the shared constraint of the underlying GNEP.

Now, at any time $t \in [0, T]$, there is a given demand of gas at the consumption node. Further, the price of gas is given by an inverse demand function $P(t, q^{\text{out}}(t))$, where q^{out} is the total amount of gas at the exit node. On the other hand, it is assumed that each firm chooses to inject the amount $q_i^{\text{in}}(t)$ at a cost of $c_i(t) q_i^{\text{in}}(t)$. For our subsequent investigation, we invoke the following assumption on the data.

- Assumption.**
- (i) It holds $P(t, q^{\text{out}}(t)) := \beta(t) - \alpha(t) q^{\text{out}}(t)$, where P is a measurable function in t .
 - (ii) There exist finite numbers α_{\min} and α_{\max} such that $0 < \alpha_{\min} \leq \alpha(t) \leq \alpha_{\max}$ holds for almost all $t \in (0, T)$.
 - (iii) For any firm $i \in \{1, \dots, M\}$, there exists a constant $\bar{c}_i > 0$ such that $|c_i(t)| \leq \bar{c}_i$ for almost all $t \in (0, T)$ and c_i is measurable.

A single agent’s (firm $i \in \{1, \dots, M\}$) maximization problem reads

$$\max_{u_i \in U} \int_0^T \left(P \left(t, \sum_{k=1}^M q_k^{\text{out}}(t) \right) q_i^{\text{out}}(t) - c_i(t) q_i^{\text{in}}(t) \right) dt - \mathcal{R}_i(u_i) \quad (4a)$$

$$\text{subject to (s.t.) } 0 \leq q_i^{\text{in}}(t) \leq \bar{q}_i^{\text{in}} \quad \text{for almost all } t \in (0, T), \quad (4b)$$

$$\int_0^T (q_i^{\text{out}}(t) - q_i^{\text{in}}(t)) dt \leq 0, \quad (4c)$$

$$S(u_i, u_{-i}) \in K, \quad (4d)$$

where \bar{q}_i^{in} is a given production-specific capacity bound. Considering the convex function \mathcal{R}_i in the objective allows for instance to influence, specifically regularize, the state at final time (i.e., at $t = T$) in the pipe. Indeed, the initial state is likely to be rather regular (in space), but

without any additional consideration, the final state is only required to satisfy the constraint (4d). Constraint (4c) ensures that over the entire time period, each firm has injected at least as much gas as it has withdrawn. This is important since at the beginning of the time period, the pipe does contain some gas as we require the initial state $y_0 := (p_0, q_0)$ to fulfill the given upper and lower bounds. Hence, (4c) ensures that there is no free gas for any of the firms. In problem (4), the constraints (4b) and (4c) are private, whereas (4d) is shared among producers. The collection of these M -many optimization problems gives rise to a GNEP, which is studied in more detail in Section 4, below.

As a benchmark for the GNEP solution, we consider the corresponding *welfare maximization problem*. In fact, the overall welfare is given by the total consumer surplus minus the total production cost:

$$\int_0^T \left(\int_0^{q^{\text{out}}(t)=\sum_{i=1}^M q_i^{\text{out}}(t)} P(t, \tau) \, d\tau - \sum_{i=1}^M c_i(t) q_i^{\text{in}}(t) \right) dt.$$

After replacing the innermost integral by its expression and noting that we only need to consider the total amount of gas withdrawn, the associated welfare maximization problem reads:

$$\begin{aligned} \max_{\mathbf{u} \in \mathbf{U}} \quad & \int_0^T \left(\beta(t) q^{\text{out}}(t) - \frac{1}{2} \alpha(t) q^{\text{out}}(t) q^{\text{out}}(t) - \sum_{i=1}^M c_i(t) q_i^{\text{in}}(t) \right) dt - \mathcal{R}(\mathbf{u}) \\ \text{s.t.} \quad & 0 \leq q_i^{\text{in}}(t) \leq \bar{q}_i^{\text{in}} \quad \text{for almost all } t \in (0, T) \text{ and all } i \in \{1, \dots, M\}, \\ & \int_0^T \left(q^{\text{out}}(t) - \sum_{i=1}^M q_i^{\text{in}}(t) \right) dt \leq 0, \\ & S(\mathbf{u}) \in K, \end{aligned} \tag{5}$$

where $\mathbf{u} := (p^{\text{in}}, p^{\text{out}}, (q_1^{\text{in}}, \dots, q_M^{\text{in}}), q^{\text{out}})$, and \mathcal{R} sums up all \mathcal{R}_i , $i = 1, \dots, M$. Note that from any solution u^* of the GNEP with identical pressure variables, i.e., (u_i^*, u_{-i}^*) is a solution to (4) for all $i \in \{1, \dots, M\}$ and $(p_i^{\text{in}}, p_i^{\text{out}}) = (p_j^{\text{in}}, p_j^{\text{out}})$ for all $i, j \in \{1, \dots, M\}$, a feasible point for the welfare maximization problem can be constructed. Therefore, from a solution of the GNEP, a lower bound on the optimal function value in (5) can be obtained.

3. WELL-POSEDNESS OF THE GAS FLOW PDE SYSTEM

Before analyzing our GNEP, we first address the well-posedness of the PDE system (1) over a pipe with a solution of sufficient regularity to ensure the existence of multipliers for the shared constraint. The spatial domain of the pipe is $\Omega := (0, L)$, with $L > 0$, and we set $\Omega_T := \Omega \times (0, T)$ for a time horizon $T > 0$. The system of PDEs modeling the flow of gas is given by

$$p_t - \varepsilon p_{xx} + \frac{c_s^2}{A} q_x = 0, \quad \text{in } \Omega_T, \tag{6a}$$

$$q_t - \varepsilon q_{xx} + A p_x = -\sigma A \frac{\check{q}}{\check{p}} \left(q - \frac{\check{q}}{2\check{p}} p \right) - \frac{Ag \sin \varphi}{c_s^2} p, \quad \text{in } \Omega_T, \tag{6b}$$

$$p(0, \cdot) = p^{\text{in}}, \quad p(L, \cdot) = p^{\text{out}}, \quad q(0, \cdot) = q^{\text{in}}, \quad q(L, \cdot) = q^{\text{out}}, \quad \text{in } (0, T), \tag{6c}$$

$$p(\cdot, 0) = p_0, \quad q(\cdot, 0) = q_0, \quad \text{in } \Omega, \tag{6d}$$

where (p_0, q_0) is the initial state and $(\check{p}, \check{q}) \in C([0, T]; C(\bar{\Omega})^2)$, with \check{p} bounded away from zero, is a given reference state. For our analysis, we transform (6) into a system of PDEs with homogeneous Dirichlet boundary conditions. For this purpose, we require the boundary data $(p^{\text{in}}, p^{\text{out}}, q^{\text{in}}, q^{\text{out}})$ to be in the Sobolev space $H^1(0, T)^4$. Then, for the boundary data $u_{\Gamma}^p := (p^{\text{in}}, p^{\text{out}})$ and $u_{\Gamma}^q := (q^{\text{in}}, q^{\text{out}})$, we construct extensions u_{Ω}^p (resp. u_{Ω}^q) of u_{Γ}^p (resp. u_{Γ}^q) to the entire domain Ω . Specifically, for any $(x, t) \in \Omega_T$, we set $u_{\Omega}^p(x, t)$ and $u_{\Omega}^q(x, t)$ to be the

linear interpolation of the values of u_Γ^p and u_Γ^q at both ends of the pipe. More precisely, we define the operator $\mathcal{E}: H^1(0, T)^2 \rightarrow H^1(0, T; H^s(\Omega))$, with s being an arbitrary positive integer, as

$$\mathcal{E}(v_1, v_2)(t)(x) := \frac{L-x}{L}v_1(t) + \frac{x}{L}v_2(t). \quad (7)$$

This yields $u_\Omega^p := \mathcal{E}(u_\Gamma^p)$ as well as $u_\Omega^q := \mathcal{E}(u_\Gamma^q)$. From the expression in (7), $(u_\Omega^p)_x$ and $(u_\Omega^q)_x$ are constant with respect to x . Hence, these extensions indeed belong to $H^1(0, T; H^s(\Omega))$. We then consider the shifted state variables $\tilde{p} := p - u_\Omega^p$ and $\tilde{q} := q - u_\Omega^q$, which are solutions to the system

$$\tilde{p}_t - \varepsilon \tilde{p}_{xx} + \frac{c_s^2}{A} \tilde{q}_x = -(u_\Omega^p)_t - \frac{c_s^2}{A} (u_\Omega^q)_x, \quad \text{in } \Omega_T, \quad (8a)$$

$$\begin{aligned} \tilde{q}_t - \varepsilon \tilde{q}_{xx} + A \tilde{p}_x + \sigma A \frac{\tilde{q}}{\tilde{p}} \tilde{q} + \left[\frac{Ag \sin \varphi}{c_s^2} - \sigma A \frac{\tilde{q}^2}{2\tilde{p}^2} \right] \tilde{p} \\ = -(u_\Omega^q)_t - A(u_\Omega^p)_x + \sigma A \frac{\tilde{q}}{\tilde{p}} u_\Omega^q + \left[\frac{Ag \sin \varphi}{c_s^2} - \sigma A \frac{\tilde{q}^2}{2\tilde{p}^2} \right] u_\Omega^p, \quad \text{in } \Omega_T, \quad (8b) \end{aligned}$$

$$\tilde{p}(0, \cdot) = \tilde{p}(L, \cdot) = 0, \quad \tilde{q}(0, \cdot) = \tilde{q}(L, \cdot) = 0, \quad \text{in } (0, T), \quad (8c)$$

$$\tilde{p}(\cdot, 0) = \tilde{p}_0 := p_0 - u_\Omega^p(\cdot, 0), \quad \tilde{q}(\cdot, 0) = \tilde{q}_0 := q_0 - u_\Omega^q(\cdot, 0), \quad \text{in } \Omega. \quad (8d)$$

Remember that the feasible set of each firm includes the constraint $y \in K$ with K given in (3) and that for analytical but also numerical reasons one wishes to characterize a solution of (4) by Lagrange multipliers. This leads us to seek solutions of (6) with higher regularity than the classic parabolic one; cf. (10) below for the latter. In this vein, our subsequent approach provides us, under appropriate assumptions, with a unique solution of (6) in the space

$$Y := \{y \in L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H^1(\Omega)^2) \text{ and } y_t \in L^2(0, T; L^2(\Omega)^2)\}.$$

Then, by the Aubin–Lions Lemma (see Lemma A.1 in Appendix A), we have the (even compact) embedding of Y into $C([0, T]; C(\bar{\Omega})^2)$. The latter space is instrumental in proving the existence of an interior point for the constraint $y \in K$. As a consequence, Slater’s constraint qualification (CQ) is satisfied, which yields the existence of a (bounded) Lagrange multiplier for this constraint.

For the proof of higher regularity we proceed as follows. First, the existence and uniqueness of a solution with classic parabolic regularity is obtained via classic results on abstract parabolic problems. Then, for the additional regularity, we do not wish to follow the usual path of showing temporal regularity and then spatial regularity as terms involving the time derivative of the control are already present in (8a) and (8b). Rather, we only show additional spatial regularity, first for (\tilde{p}, \tilde{q}) and then for $(\tilde{p}_t, \tilde{q}_t)$. Indeed, under appropriate regularity of the shifted initial state and the right-hand side, the solution (\tilde{p}, \tilde{q}) satisfies

$$(\tilde{p}, \tilde{q}) \in L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H_0^1(\Omega)^2),$$

$$(\tilde{p}_t, \tilde{q}_t) \in L^2(0, T; L^2(\Omega)^2),$$

which leads to $(p, q) \in Y$.

Let $\mathcal{V} := H_0^1(\Omega)^2$ and $\mathcal{H} := L^2(\Omega)^2$. We denote any element v of \mathcal{V} , resp. \mathcal{H} , by (v_p, v_q) , where both v_p and v_q belong to $H_0^1(\Omega)$, resp. $L^2(\Omega)$. This specific notation is inspired by the variable names for gas pressure and flow. It is well-known that $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and that the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact. Therefore, we have the Gelfand triple

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*.$$

Let $(\cdot, \cdot)_{\mathcal{H}}$ denote the inner product of \mathcal{H} and let $\|\cdot\|_{\mathcal{H}}$ be the corresponding norm. For any v, w in \mathcal{H} , we have $(v, w)_{\mathcal{H}} = (v_p, w_p)_{L^2(\Omega)} + (v_q, w_q)_{L^2(\Omega)}$ and $\|v\|_{\mathcal{H}}^2 = \|v_p\|_{L^2(\Omega)}^2 + \|v_q\|_{L^2(\Omega)}^2$. The norm of \mathcal{V} is denoted by $\|\cdot\|_{\mathcal{V}}$. In the following, we consider a system of parabolic PDEs with the lower-order terms having generic coefficients. This allows us to streamline the presentation

and also invoke these results for an associated adjoint system in future works. Let the bilinear form $a_\tau: [0, T] \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ be defined as

$$\begin{aligned} a_\tau(t, v, w) &:= -\varepsilon (\nabla v, \nabla w)_{\mathcal{H}} + \theta_p (\nabla v_q, w_p)_{L^2(\Omega)} + \theta_q (\nabla v_p, w_q)_{L^2(\Omega)} \\ &\quad + \sum_{i,j \in \{p,q\}} (\gamma_{ij}(t) v_i, w_j)_{L^2(\Omega)} + \tau (v, w)_{\mathcal{H}}, \end{aligned} \quad (9)$$

where $\varepsilon > 0$, both θ_p and θ_q are non-zero constants, and each γ_{ij} belongs to $C([0, T]; C(\bar{\Omega}))$ with magnitude bounded by $\bar{\gamma}_{ij}$. The parameter τ is fixed later. By applying the Cauchy–Schwarz inequality to each term in (9), we see that the bilinear form a_τ is bounded, i.e., there exists $C > 0$ such that $|a_\tau(t, v, w)| \leq C \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}$ for all $v, w \in \mathcal{V}$ and $t \in [0, T]$. Let $\mathcal{L}_\tau(t)$ be an operator in \mathcal{H} with action

$$\mathcal{L}_\tau(t)v := -\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} \Delta v + \begin{pmatrix} 0 & \theta_p \\ \theta_q & 0 \end{pmatrix} \nabla v + \begin{pmatrix} \gamma_{pp}(t) + \tau & \gamma_{pq}(t) \\ \gamma_{qp}(t) & \gamma_{qq}(t) + \tau \end{pmatrix} v,$$

and domain $D(\mathcal{L}_\tau(t)) \equiv (H^2(\Omega) \cap H_0^1(\Omega))^2$. Recall that $v \in D(\mathcal{L}_\tau(t))$ is equivalent to

$$v \in \mathcal{V}, \quad \mathcal{L}_\tau(t)v \in \mathcal{H}, \quad \text{and} \quad (\mathcal{L}_\tau(t)v, w)_{\mathcal{H}} = a_\tau(t, v, w) \quad \text{for all } w \in \mathcal{V}.$$

We are going to investigate a problem of the following form: Find $v \in L^2(0, T; \mathcal{V})$ with $v_t \in L^2(0, T; \mathcal{V}^*)$ such that given $v_0 \in \mathcal{H}$ and $\ell \in L^2(0, T; \mathcal{V}^*)$, it satisfies $v(0) = v_0$ and for all $w \in \mathcal{V}$, we have

$$\frac{d}{dt} (v(\cdot), w)_{\mathcal{H}} + a_0(\cdot, v(\cdot), w) = (\ell(\cdot), w)_{\mathcal{H}}$$

in the sense of $\mathcal{D}^*(0, T)$, the space of distribution on $(0, T)$. This is referred to as Problem (P). As it is usual in the parabolic case, rather than requiring the coercivity of the bilinear form a_0 , it is sufficient to check that a_0 satisfies a weaker condition, namely Gårding's inequality (also known as weak coercivity).

Lemma 3.1 (Gårding's inequality for a_0). *Consider the bilinear form defined in (9). Then, there exists two constants $\bar{\tau}$, ζ , with $\zeta > 0$, such that for all $v \in \mathcal{V}$ and $t \in [0, T]$, we have*

$$a_0(t, v, v) + \bar{\tau} \|v\|_{\mathcal{H}}^2 \geq \zeta \|v\|_{\mathcal{V}}^2.$$

Proof. Take $t \in [0, T]$. We use classic inequalities to get the following bound on the second term in the right-hand side of (9):

$$(\nabla v_q, v_p)_{L^2(\Omega)} \geq -\|\nabla v_q\|_{L^2(\Omega)} \|v_p\|_{L^2(\Omega)} \geq -\kappa_2 \|\nabla v_q\|_{L^2(\Omega)}^2 - \frac{1}{4\kappa_2} \|v_p\|_{L^2(\Omega)}^2$$

for any $\kappa_2 > 0$. Similarly, for the third term of (9) we have

$$(\nabla v_p, v_q)_{L^2(\Omega)} \geq -\kappa_3 \|\nabla v_p\|_{L^2(\Omega)}^2 - \frac{1}{4\kappa_3} \|v_q\|_{L^2(\Omega)}^2$$

with $\kappa_3 > 0$. Finally, each term in the sum (fourth term) is bounded from below by

$$\begin{aligned} (\gamma_{pp}(t) v_p, v_p)_{L^2(\Omega)} &\geq -\bar{\gamma}_{pp} \|v_p\|_{L^2(\Omega)}^2, & (\gamma_{qq}(t) v_q, v_q)_{L^2(\Omega)} &\geq -\bar{\gamma}_{qq} \|v_q\|_{L^2(\Omega)}^2, \\ (\gamma_{pq}(t) v_p, v_q)_{L^2(\Omega)} &\geq -\kappa_{pq} \bar{\gamma}_{pq} \|v_p\|_{L^2(\Omega)}^2 - \frac{\bar{\gamma}_{pq}}{4\kappa_{pq}} \|v_q\|_{L^2(\Omega)}^2, \\ (\gamma_{qp}(t) v_q, v_p)_{L^2(\Omega)} &\geq -\kappa_{qp} \bar{\gamma}_{pq} \|v_q\|_{L^2(\Omega)}^2 - \frac{\bar{\gamma}_{pq}}{4\kappa_{qp}} \|v_p\|_{L^2(\Omega)}^2, \end{aligned}$$

where $\kappa_{pq} > 0$ and $\kappa_{qp} > 0$ involve $\bar{\gamma}_{pq}$ and $\bar{\gamma}_{qp}$. Now, we set $\kappa_2 := \frac{\varepsilon}{2|\theta_p|}$ and $\kappa_3 := \frac{\varepsilon}{2|\theta_q|}$ and let $\bar{\tau}_p := \bar{\gamma}_{pp} + \bar{\gamma}_{pq}\kappa_{pq} + \bar{\gamma}_{qp}(4\kappa_{qp})^{-1}$ as well as $\bar{\tau}_q := \bar{\gamma}_{qq} + \bar{\gamma}_{qp}\kappa_{qp} + \bar{\gamma}_{pq}(4\kappa_{pq})^{-1}$. Then, we get

$$\begin{aligned} a_0(t, v, v) &\geq \frac{\varepsilon}{2} \|\nabla v\|_{\mathcal{H}}^2 - \left(\frac{|\theta_p|}{2\varepsilon} + \bar{\tau}_p \right) \|v_p\|_{L^2(\Omega)}^2 - \left(\frac{|\theta_q|}{2\varepsilon} + \bar{\tau}_q \right) \|v_q\|_{L^2(\Omega)}^2 \\ &\geq \frac{\varepsilon}{2} \|v\|_{\mathcal{V}}^2 - \left(\frac{\theta_p^2}{2\varepsilon} + \bar{\tau}_p + \frac{\varepsilon}{2} \right) \|v_p\|_{L^2(\Omega)}^2 - \left(\frac{\theta_q^2}{2\varepsilon} + \bar{\tau}_q + \frac{\varepsilon}{2} \right) \|v_q\|_{L^2(\Omega)}^2. \end{aligned}$$

With $\bar{\tau}$ being the maximum of the coefficients in front of $\|v_p\|_{L^2(\Omega)}^2$ and $\|v_q\|_{L^2(\Omega)}^2$, we obtain

$$a_0(t, v, v) \geq \frac{\varepsilon}{2} \|v\|_{\mathcal{V}}^2 - \bar{\tau} \|v\|_{\mathcal{H}}^2,$$

which completes the proof. \square

Since the time horizon T is finite, we continue our investigation with the coercive bilinear form $a := a_{\bar{\tau}}$. Indeed, if z is the solution of

$$\begin{aligned} \frac{d}{dt} (z, w)_{\mathcal{H}} + a(t, z, w) &= (e^{\bar{\tau}t} \ell, w)_{\mathcal{H}} \quad \text{for all } w \in \mathcal{V}, \\ z(0) &= v_0 \end{aligned}$$

in the sense of $\mathcal{D}^*(0, T)$, then $v = z e^{\bar{\tau}t}$ is a solution of Problem (P). For more details, see Dautray and J.-L. Lions (2000, Remark 2, p. 512), Schwab and Stevenson (2009, Appendix A), or Renardy and Rogers (2004, p. 383). In the following, we set $\mathcal{L} := \mathcal{L}_{\bar{\tau}}$ and we continue to use ℓ to denote the constant terms of the system of PDEs.

Proposition 3.2. *Suppose that $v_0 \in \mathcal{H}$ and $\ell \in L^2(0, T; \mathcal{V}^*)$. Then, there exists a unique solution v to Problem (P) with*

$$v \in L^2(0, T; \mathcal{V}) \quad \text{and} \quad v_t \in L^2(0, T; \mathcal{V}^*). \quad (10)$$

Moreover, there exists $C > 0$ such that the following estimate holds for all $t \in (0, T]$:

$$\|v(t)\|_{\mathcal{H}}^2 + \|v\|_{L^2(0, t; \mathcal{V})}^2 \leq C(\|v_0\|_{\mathcal{H}}^2 + \|\ell\|_{L^2(0, t; \mathcal{V}^*)}^2). \quad (11)$$

Proof. The existence result is classic; see Theorem 1 and 2 in Dautray and J.-L. Lions (2000) (on Pages 512–513), Theorem 1.37 in Hinze et al. (2009), or Theorem 11.3 in Renardy and Rogers (2004). The estimate is provided in Hinze et al. (2009, Theorem 1.35) and Dautray and J.-L. Lions (2000, Page 515). \square

For the additional regularity, we follow the procedure outlined in Dautray and J.-L. Lions (2000, Remark 2, Page 532) for our system. Let us first check that over the domain Ω , the operator \mathcal{L} satisfies the Ladyzhenskaya–Sobolevski inequality.

Lemma 3.3 (Ladyzhenskaya–Sobolevski-type inequality). *There exists $c_1, c_2 > 0$ such that for any fixed $t \in [0, T]$, we have*

$$(\mathcal{L}(t)v, -\Delta v)_{\mathcal{H}} \geq c_1 \|v\|_{H^2(\Omega)^2}^2 - c_2 \|v\|_{\mathcal{H}}^2 \quad \text{for all } v \in D(\mathcal{L}(t)).$$

Proof. We proceed as in P. L. Lions (1981, Proof of Theorem 1.1). Let $v \in D(\mathcal{L}(t)) \equiv (H^2(\Omega) \cap H_0^1(\Omega))^2$ and consider the splitting $v =: (v_p, v_q)$. First, thanks to Lemma A.3, we get that the second-order terms in \mathcal{L} satisfy

$$\varepsilon (\Delta v_p, \Delta v_p)_{L^2(\Omega)} + \varepsilon (\Delta v_q, \Delta v_q)_{L^2(\Omega)} \geq \frac{\varepsilon}{2} \|v\|_{H^2(\Omega)^2}^2 - \varepsilon(C''' + 1) \|v\|_{\mathcal{H}}^2 \quad (12)$$

for some constant $C''' > 0$. Then, moving on to the first-order terms in \mathcal{L} , we have

$$\begin{aligned} (\nabla v_q, \Delta v_p)_{L^2(\Omega)} &\geq -\|\nabla v_q\|_{L^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} \\ &\geq -\kappa_1 \|v_q\|_{H^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} - C_{\kappa_1} \|v_q\|_{L^2(\Omega)} \|\Delta v_p\|_{L^2(\Omega)} \\ &\geq -\frac{\kappa_1}{2} \|v_q\|_{H^2(\Omega)}^2 - C_{\kappa_1} C_{\kappa_1'} \|v_q\|_{L^2(\Omega)}^2 - (\kappa_1' + \frac{\kappa_1}{2}) \|v_p\|_{H^2(\Omega)}^2, \end{aligned} \quad (13)$$

where we used the relation (41) in Lemma A.2 and Cauchy's inequality. Similarly, we have

$$(\nabla v_p, \Delta v_q)_{L^2(\Omega)} \geq -\frac{\kappa_2}{2} \|v_p\|_{H^2(\Omega)}^2 - C_{\kappa_2} C_{\kappa'_2} \|v_p\|_{L^2(\Omega)}^2 - (\kappa'_2 + \frac{\kappa_2}{2}) \|v_q\|_{H^2(\Omega)}^2.$$

Finally, with $\tau_{pp} = \tau_{qq} = \bar{\tau}$ and $\tau_{pq} = \tau_{qp} = 0$, for i, j being p or q we have

$$((\tau_{ij} + \gamma_{ij}(t))v_i, \Delta v_j)_{L^2(\Omega)} \geq (\tau_{ij} + \bar{\gamma}_{ij})(-\kappa_{i,j} \|v_j\|_{H^2(\Omega)}^2 - C_{\kappa_{i,j}} \|v_i\|_{L^2(\Omega)}^2) \quad (14)$$

for all $t \in [0, T]$. Summarizing, we derived lower bounds for $(\mathcal{L}(t)v, \Delta v)_{\mathcal{H}}$ in terms of $\|v_p\|_{H^2(\Omega)}^2, \|v_q\|_{H^2(\Omega)}^2, \|v_p\|_{L^2(\Omega)}^2$, and $\|v_q\|_{L^2(\Omega)}^2$ for all $t \in [0, T]$. Furthermore, the magnitude of the terms $\|v_p\|_{H^2(\Omega)}^2$ and $\|v_q\|_{H^2(\Omega)}^2$ in (13)–(14) can be controlled by choosing $\kappa_1, \kappa'_1, \kappa_2, \kappa'_2$, and $\kappa_{i,j}$. Hence, for any values of $\theta_p, \theta_q, \bar{\tau}$, and $\bar{\gamma}_{ij}$, one can select these quantities such that the contribution of $\|v_p\|_{H^2(\Omega)}^2$ (resp. $\|v_q\|_{H^2(\Omega)}^2$) in (13)–(14) is larger than $-\frac{\varepsilon}{4} \|v_p\|_{H^2(\Omega)}^2$ (resp. $-\frac{\varepsilon}{4} \|v_q\|_{H^2(\Omega)}^2$). Finally, using (12), we obtain

$$(\mathcal{L}(t)v, -\Delta v)_{\mathcal{H}} \geq \frac{\varepsilon}{4} \|v\|_{H^2(\Omega)^2}^2 - C_{\text{total}} \|v\|_{\mathcal{H}}^2,$$

with a suitably chosen $C_{\text{total}} > 0$. □

Now, we are ready to state the desired regularity result.

Proposition 3.4. *Suppose that $v_0 \in \mathcal{V}$ and $\ell \in L^2(0, T; \mathcal{H})$. Then, the solution v to Problem (P) satisfies*

$$v \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega)^2)$$

with the estimate

$$\|v\|_{L^\infty(0, T; \mathcal{V})} + \|v\|_{L^2(0, T; H^2(\Omega)^2)} \leq C(\|v_0\|_{\mathcal{V}} + \|\ell\|_{L^2(0, T; \mathcal{H})})$$

for some $C > 0$.

Proof. We follow Dautray and J.-L. Lions (2000, Remark 2, Page 532). To this end, we consider the following approximation

$$\mathcal{V}_m := \text{span} \{\mathcal{W}_j\}_{j=1}^m$$

of \mathcal{V} , where $\mathcal{W}_j = (e_j, e_j)$ is defined by the (normalized) eigenfunctions e_j of the Laplacian in the Dirichlet problem

$$-\Delta e_j = \nu_j e_j \quad \text{in } \Omega$$

and $e_j(0) = e_j(L) = 0$. These functions form an orthonormal basis of $L^2(\Omega)$ and belong to $H_0^1(\Omega) \cap C^\infty(\Omega)$; see Brezis (2010, Theorem 9.31). Note that the sequence $\{e_j/\sqrt{\nu_j + 1}\}_{j=1}^\infty$ forms an orthonormal basis of $H_0^1(\Omega)$. For the initial state of the finite-dimensional problem over \mathcal{V}_m , we choose $v^{(m)}(0) \in \mathcal{V}_m$ such that $v^{(m)}(0)$ converges to v_0 as $m \rightarrow \infty$ in \mathcal{V} . With $v^{(m)} = \sum_{j=1}^m \zeta_j \mathcal{W}_j$ being the solution to the Galerkin approximation of order m to Problem (P), for almost all $t \in (0, T)$, we have

$$\left(\frac{d}{dt} v^{(m)}(t), w \right)_{\mathcal{H}} + a(t, v^{(m)}(t), w) = (\ell(t), w)_{\mathcal{H}} \quad (15)$$

for all $w \in \mathcal{V}_m \subset \mathcal{H}$. For almost all t in $(0, T)$, we take $w = \nu_j \zeta_j(t) \mathcal{W}_j$ in (15) and sum all the equalities. This yields

$$\left(\frac{d}{dt} v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} + \left(\mathcal{L}(t)v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} = \left(\ell(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}}.$$

After integrating by part the first term and exchanging the partial derivatives, we have

$$\left(\frac{d}{dt} \nabla v^{(m)}(t), \nabla v^{(m)}(t) \right)_{\mathcal{H}} + \left(\mathcal{L}(t)v^{(m)}(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}} = \left(\ell(t), -\Delta v^{(m)}(t) \right)_{\mathcal{H}}.$$

Then, we integrate w.r.t. t and get

$$\begin{aligned} & \frac{1}{2} \|\nabla v^{(m)}(t)\|_{\mathcal{H}}^2 + \int_0^t \left(\mathcal{L}(t)v^{(m)}(s), -\Delta v^{(m)}(s) \right)_{\mathcal{H}} ds \\ &= \int_0^t \left(\ell(s), -\Delta v^{(m)}(s) \right)_{\mathcal{H}} ds + \frac{1}{2} \|\nabla v^{(m)}(0)\|_{\mathcal{H}}^2. \end{aligned}$$

Using Lemma 3.3, there exists strictly positive constants c_1 and c_2 such that

$$\begin{aligned} & \frac{1}{2} \|\nabla v^{(m)}(t)\|_{\mathcal{H}}^2 + c_1 \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds \\ & \leq \frac{1}{2} \|\nabla v^{(m)}(0)\|_{\mathcal{H}}^2 + \int_0^t \|\ell(s)\|_{\mathcal{H}} \|\Delta v^{(m)}(s)\|_{\mathcal{H}} ds + c_2 \int_0^t \|v^{(m)}(s)\|_{\mathcal{H}}^2 ds, \end{aligned} \quad (16)$$

where we used that $\ell \in L^2(0, T; \mathcal{H})$. Now, there exists $C_{\frac{c_1}{2}} > 0$ such that

$$\int_0^t \|\ell(s)\|_{\mathcal{H}} \|\Delta v^{(m)}(s)\|_{\mathcal{H}} ds \leq C_{\frac{c_1}{2}} \int_0^t \|\ell(s)\|_{\mathcal{H}}^2 ds + \frac{c_1}{2} \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds.$$

Using the estimate in (11), we know that there exists a positive constant C_{c_2} such that

$$c_2 \int_0^t \|v^{(m)}(s)\|_{\mathcal{H}}^2 ds \leq C_{c_2} \left[\int_0^t \|\ell(s)\|_{\mathcal{V}^*}^2 ds \right] + \frac{1}{2} \|v^{(m)}(0)\|_{\mathcal{H}}^2 - \frac{1}{2} \|v^{(m)}(t)\|_{\mathcal{H}}^2$$

holds. To get an upper bound independent of m , we use the last two relations in (16) and the boundedness of the sequence $v^{(m)}(0) \in \mathcal{V}_m$ as it converges to v_0 as $m \rightarrow \infty$ in \mathcal{V} . Thus, we get

$$\frac{1}{2} \|v^{(m)}(t)\|_{\mathcal{V}}^2 + \frac{c_1}{2} \int_0^t \|v^{(m)}(s)\|_{H^2(\Omega)^2}^2 ds \leq C_0 \|v_0\|_{\mathcal{V}}^2 + C_{\text{rhs}} \int_0^t \|\ell(s)\|_{\mathcal{H}}^2 ds. \quad (17)$$

Hence, $v^{(m)} \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; H^2(\Omega)^2)$. Then, using classic arguments, at the limit $m \rightarrow \infty$, the solution v also fulfills (17). The estimate is then obtained based on this inequality. \square

Let us now turn our attention to the characterization of the control-to-state mapping. We start with the following relation between ℓ and the boundary values.

Lemma 3.5. *Let $u_\Gamma \in H^1(0, T)^4$ and ℓ be the right-hand side of the equations (8a)–(8b). Then, there exists $C > 0$ such that*

$$\|\ell\|_{L^2(0, T; \mathcal{H})} \leq C \|u_\Gamma\|_{H^1(0, T)^4}.$$

Proof. In this proof, C is a positive constant that may take different values depending on the context. We show that the inequality $\|\ell\|_{L^2(0, T; \mathcal{H})}^2 \leq C \|u_\Gamma\|_{H^1(0, T)^4}^2$ holds. First, note that for any $(x, t) \in \Omega_T$, we have

$$(u_\Omega^p)_t(x, t) = \frac{L-x}{L} (p^{\text{in}})_t(t) + \frac{x}{L} (p^{\text{out}})_t(t), \quad (u_\Omega^p)_x(x, t) = \frac{p^{\text{out}}(t) - p^{\text{in}}(t)}{L},$$

and similar relations hold for $(u_\Omega^q)_t$ and $(u_\Omega^q)_x$. With $\ell =: (\ell^p, \ell^q)$, for all $s \in (0, T)$, we have

$$\|\ell^p(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left(\|(u_\Omega^p)_t(\cdot, s)\|_{L^2(\Omega)}^2 + \|(u_\Omega^q)_x(\cdot, s)\|_{L^2(\Omega)}^2 \right).$$

Using the expressions above for $(u_\Omega^p)_t$ and $(u_\Omega^q)_x$, it holds that

$$\|\ell^p(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left((p^{\text{in}}(s))^2 + ((p^{\text{out}})_t(s))^2 + (q^{\text{in}}(s))^2 + (q^{\text{out}}(s))^2 \right).$$

Now for ℓ^q , using similar arguments, we have

$$\begin{aligned} & \|\ell^q(\cdot, s)\|_{L^2(\Omega)}^2 \\ & \leq C \left(\|(u_\Omega^q)_t(\cdot, s)\|_{L^2(\Omega)}^2 + \|(u_\Omega^p)_x(\cdot, s)\|_{L^2(\Omega)}^2 + \|u_\Omega^q(\cdot, s)\|_{L^2(\Omega)}^2 + \|u_\Omega^p(\cdot, s)\|_{L^2(\Omega)}^2 \right) \\ & \leq C \left((q^{\text{in}}(s))^2 + ((q^{\text{out}})_t(s))^2 + (p^{\text{in}}(s))^2 + ((p^{\text{out}})_t(s))^2 \right. \\ & \quad \left. + (p^{\text{in}}(s))^2 + (p^{\text{out}}(s))^2 + (q^{\text{in}}(s))^2 + (q^{\text{out}}(s))^2 \right). \end{aligned}$$

Hence, it holds that

$$\begin{aligned} \|\ell\|_{L^2(0,T;\mathcal{H})}^2 &= \|\ell^p\|_{L^2(0,T;L^2(\Omega))}^2 + \|\ell^q\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\leq C \left[\|p^{\text{in}}\|_{L^2(0,T)}^2 + \|p^{\text{out}}\|_{L^2(0,T)}^2 + \|q^{\text{in}}\|_{L^2(0,T)}^2 + \|q^{\text{out}}\|_{L^2(0,T)}^2 \right. \\ & \quad \left. + \|(p^{\text{in}})_t\|_{L^2(0,T)}^2 + \|(p^{\text{out}})_t\|_{L^2(0,T)}^2 + \|(q^{\text{in}})_t\|_{L^2(0,T)}^2 + \|(q^{\text{out}})_t\|_{L^2(0,T)}^2 \right]. \end{aligned}$$

This concludes the proof. \square

Finally, the desired regularity result for the solution of the system can be obtained. As the shifted initial state has to be in $H_0^1(\Omega)^2$, there are compatibility conditions between the initial state and the value of the controls at $t = 0$. To capture these, we define

$$\begin{aligned} Z := \{ (y_0, u_\Gamma) \in H^1(\Omega)^2 \times H^1(0, T)^4 \mid & p_0(0) = p^{\text{in}}(0), \quad p_0(L) = p^{\text{out}}(0), \\ & q_0(0) = q^{\text{in}}(0), \quad q_0(L) = q^{\text{out}}(0) \}. \end{aligned}$$

Lemma 3.6. *Given $y_0 \in H^1(\Omega)^2$, assume that the boundary control u_Γ satisfies $(y_0, u_\Gamma) \in Z$. Then, the unique solution $y = (p, q)$ of (6) has regularity*

$$y \in L^\infty(0, T; H^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \quad \text{and} \quad y_t \in L^2(0, T; \mathcal{H}),$$

and there is a linear and continuous mapping $S_{\text{PDE}}: Z \rightarrow Y$ such that $y = S_{\text{PDE}}(y_0, u_\Gamma)$.

Proof. The hypothesis allows us to invoke Lemma 3.5 and Proposition 3.4. Given data $(y_0, u_\Gamma) \in Z$, let v be the unique solution of Problem (P) with initial state $v_0 = y_0 - (\mathcal{E}(u_\Gamma^p), \mathcal{E}(u_\Gamma^q))^\top(\cdot, 0)$. From Proposition 3.4, we have

$$v \in L^\infty(0, T; H_0^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2) \subset L^2(0, T; (H^2(\Omega) \cap H_0^1(\Omega))^2), \quad (18)$$

and then $v \in D(\mathcal{L}(t))$ for all $t \in [0, T]$. Moreover, Problem (P) is equivalent to $v_t = \ell - \mathcal{L}v$ in $L^2(0, T; \mathcal{V}^*)$ plus initial and boundary conditions; see Remark 2, Page 512 in Dautray and J.-L. Lions (2000). As $\ell - \mathcal{L}v \in L^2(0, T; \mathcal{H}) \subset L^2(0, T; \mathcal{V}^*)$, we have $v_t \in L^2(0, T; \mathcal{H})$. Furthermore, with v satisfying (18), the estimate

$$\begin{aligned} \|v_t\|_{L^2(0,T;\mathcal{H})} &\leq \|\ell\|_{L^2(0,T;\mathcal{H})} + \|\mathcal{L}v\|_{L^2(0,T;\mathcal{H})} \\ &\leq \|\ell\|_{L^2(0,T;\mathcal{H})} + C_{\mathcal{L}}\|v\|_{L^2(0,T;H^2(\Omega)^2)} \end{aligned}$$

holds due to the boundedness of \mathcal{L} . In combination with the estimates from Proposition 3.4 and Lemma 3.5, we get

$$\|v\|_{L^\infty(0,T;\mathcal{V})} + \|v_t\|_{L^2(0,T;\mathcal{H})} + \|v\|_{L^2(0,T;H^2(\Omega)^2)} \leq C(\|v_0\|_{\mathcal{V}} + \|u_\Gamma\|_{H^1(0,T)^4}).$$

From v , the unique solution y of (6) is given by

$$y = v + \begin{pmatrix} \mathcal{E}(u_\Gamma^p) \\ \mathcal{E}(u_\Gamma^q) \end{pmatrix}.$$

Recall that $\mathcal{E}: H^1(0, T)^2 \rightarrow H^1(0, L; H^2(\Omega))$ is linear and bounded. Therefore, we get the boundedness of S_{PDE} . \square

Before moving on to the analysis of the GNEP, let us define the control-to-state mapping S . The compatibility condition $(y_0, u_\Gamma) \in Z$ restricts the solution operator to be well-defined on a subset of $H^1(0, T)^4$. More precisely, with $u_\Gamma^{y_0} := (p_0(0), p_0(L), q_0(0), q_0(L))$, the set of boundary controls compatible with y_0 is $U_{y_0} := \{u_\Gamma \in H^1(0, T)^4 \mid u_\Gamma(0) = u_\Gamma^{y_0}\}$. With $\hat{u}^{y_0} \in U_{y_0}$ the constant function $\hat{u}^{y_0} \equiv u_\Gamma^{y_0}$, the relation between any boundary control $u_\Gamma \in U_{y_0}$ and the state can be written as

$$y = S_{\text{PDE}}(y_0, u_\Gamma) = S_{\text{PDE}}(y_0, \hat{u}^{y_0}) + S_{\text{PDE}}(0, u_\Gamma - \hat{u}^{y_0}),$$

where

$$u_\Gamma - \hat{u}^{y_0} \in \hat{U} := \{\hat{u} \in H^1(0, T)^4 \mid \hat{u}(0) = 0\} \quad (19)$$

is the Banach space (with the norm of $H^1(0, T)^4$) of controls compatible with the initial state being 0. With $\hat{S}_0: \hat{U} \rightarrow Y$ given by $\hat{u} \mapsto S_{\text{PDE}}(0, \hat{u})$, the control-to-state operator $S: U_{y_0} \rightarrow Y$ is given by

$$S(u_\Gamma) := S_{\text{PDE}}(y_0, \hat{u}^{y_0}) + \hat{S}_0(u_\Gamma - \hat{u}^{y_0}) = y. \quad (20)$$

4. ANALYSIS OF THE JOINTLY CONVEX GNEP

Given the well-posedness of the PDE governing the gas flow, we now study the underlying GNEP. The space of pressures (resp. mass flows) at the pipe's endpoints is $P := H^1(0, T)^2$ (resp. $Q := H^1(0, T)^2$). Each firm's decision variable $u_i := (p_i, q_i)$, where p_i and q_i are the pressure and mass flow variables, is an element of $U := P \times Q$ and we define $\mathcal{U} := U^M$ as we consider M -many firms. The set of initial states compatible with the constraint $y \in K$ is given by:

$$Y_0^{\text{ad}} := \{y_0 \in H^1(\Omega)^2 \mid \underline{y} \leq y_0(x) \leq \bar{y} \text{ for all } x \in [0, L]\},$$

where $\underline{y}, \bar{y} \in \mathbb{R}$ with $\underline{y} \leq \bar{y}$ are fixed. The mapping in (2) between the players' decisions and the boundary values is $\mathcal{L}: \mathcal{U} \rightarrow U$ with

$$\mathcal{L} := (\mathcal{L}^p, \mathcal{L}^q) \quad \text{with} \quad \mathcal{L}^p(u) := \frac{1}{M} \sum_{i=1}^M p_i \quad \text{and} \quad \mathcal{L}^q(u) := \sum_{i=1}^M q_i.$$

While this does not impose the continuity of the pressure, we shall see that this particular choice of \mathcal{L}^p allows us to show existence of solutions to the game such that the continuity of pressure holds.

In our analysis below we invoke the following notation. As previously discussed, the control-to-state operator S is only well-defined on a subset of \mathcal{U} that lacks the vector space structure. To facilitate the handling of the compatibility conditions, we perform a change of variables. Given $y_0 = (p_0, q_0)$, consider the constant functions $\hat{p}^0(t) := (p_0(0), p_0(L))$ and $\hat{q}^0(t) := (q_0(0)/M, q_0(L)/M)$ for all $t \in [0, T]$. With $\hat{u}_i^{y_0} := (\hat{p}^0, \hat{q}^0)$, the shifted decisions for player i are given by $\hat{u}_i = u_i - \hat{u}_i^{y_0}$, and these quantities are collected in \hat{u} and \hat{u}^{y_0} . With $\hat{P} := \{\hat{p} \in P \mid \hat{p}(0) = 0\}$ and $\hat{Q} := \{\hat{q} \in Q \mid \hat{q}(0) = 0\}$, the assumptions stated in (23) and (24) below allow us to define the feasible sets for the pressures and mass flows as $\hat{P}^{\text{ad}} := \{\hat{p} \in \hat{P} \mid \hat{p} + \hat{p}^0 \in P^{\text{ad}}\}$ and $\hat{Q}_i^{\text{ad}} := \{\hat{q} \in \hat{Q} \mid \hat{q} + \hat{q}^0 \in Q_i^{\text{ad}}\}$. In the light of the definition of \hat{U} in (19), we note that $\hat{U} = \hat{P} \times \hat{Q}$. Also, as closed subspaces of $H^1(0, T)^2$, equipped with the norm of $H^1(0, T)^2$, both \hat{P} and \hat{Q} are Banach spaces. Furthermore, from the inclusions $H_0^1(0, T)^2 \subset \hat{P} \subset L^2(0, T)^2$ and $\hat{P}^* \subset H^{-1}(0, T)^2$, we infer that \hat{P} is dense in $L^2(0, T)^2$. Coupled with the continuous injection of $H^1(0, T)$ in $L^2(0, T)$, we deduce that we have the Gelfand triples $\hat{P} \hookrightarrow L^2(0, T)^2 \hookrightarrow \hat{P}^*$ and $\hat{Q} \hookrightarrow L^2(0, T)^2 \hookrightarrow \hat{Q}^*$. The space $\mathcal{Z} := (\hat{U}^*)^M$ is isomorphic to \hat{U}^* , denoted by $\mathcal{Z} \simeq \hat{U}^*$. We can thus identify any element of \mathcal{Z} with a unique element of \hat{U}^* and vice-versa. The duality pairing between \hat{U} and \mathcal{Z} is given by

$\langle z, u \rangle_{\mathcal{Z}, \hat{U}} = \sum_{i=1}^M \langle z_i, u_i \rangle_{\hat{U}^*, \hat{U}}$. Then, the feasible set for \hat{u}_i is a subset of the Banach space \hat{U} . The affine mapping $\hat{S}: \hat{U} \rightarrow Y$, from the players' actions to the state, is given by

$$y = \hat{S}(\hat{u}) = S_{\text{PDE}}(y_0, \hat{u}^{y_0}) + \hat{S}_0(\mathcal{L}(\hat{u})).$$

With $\mathcal{L}_i: \hat{P} \times \hat{Q} \rightarrow \hat{P} \times \hat{Q}$ given by $\mathcal{L}_i(\hat{u}_i) := (M^{-1}\hat{q}_i, \hat{q}_i)$ for all $i \in \{1, \dots, M\}$, we define $\hat{S}_i(\hat{u}_i) := \hat{S}_0(\mathcal{L}_i(\hat{u}_i))$. Then, for a given \hat{u}_{-i} , the mapping from each player's control \hat{u}_i to the state y reads

$$\hat{S}(\hat{u}_i, \hat{u}_{-i}) = \hat{S}_i(\hat{u}_i) + S_{\text{PDE}}(y_0, \hat{u}^{y_0}) + \hat{S}_0(\mathcal{L}(0, \hat{u}_{-i})).$$

Note that $\hat{S}_i: \hat{U} \rightarrow Y$ is well defined and linear. The feasible set for each agent is given by the set-valued map $\hat{C}_i: \hat{U}^{M-1} \rightrightarrows \hat{U}$ defined as

$$\hat{C}_i(\hat{u}_{-i}) := \{\hat{u}_i \in \hat{U}_i^{\text{ad}} \mid \hat{S}(\hat{u}_i, \hat{u}_{-i}) \in K\}, \quad (21)$$

where $\hat{U}_i^{\text{ad}} := \hat{P}^{\text{ad}} \times \hat{Q}_i^{\text{ad}}$. As both \hat{U}_i^{ad} and K are non-empty, closed, and convex, the fact that \hat{S} is affine ensures that the images of $\hat{C}_i(\hat{u}_{-i})$ are closed and convex (if non-empty). The part of the objective functional that intertwines the players' decision variables is the revenue functional

$$r_i(q_i^{\text{out}}, q_{-i}^{\text{out}}) := \int_0^T \left(\alpha(t) \sum_{k=1}^M q_k^{\text{out}}(t) - \beta(t) \right) q_i^{\text{out}}(t) dt.$$

With $\hat{u}_i := (\hat{p}_i^{\text{in}}, \hat{p}_{-i}^{\text{out}}, \hat{q}_i^{\text{in}}, \hat{q}_{-i}^{\text{out}})$, in the shifted variables, we get

$$\begin{aligned} \hat{r}_i(\hat{q}_i^{\text{out}}, \hat{q}_{-i}^{\text{out}}) &:= r_i(\hat{q}_i^{\text{out}} + q_0(L)/M, \hat{q}_{-i}^{\text{out}} + q_0(L)/M) \\ &= \int_0^T \left[\alpha(t) \sum_{k=1}^M \hat{q}_k^{\text{out}}(t) - (\beta(t) - \alpha(t)q_0(L)) \right] [\hat{q}_i^{\text{out}}(t) + q_0(L)/M] dt. \end{aligned}$$

Note that for all $\hat{q}_{-i}^{\text{out}} \in L^2(0, T)^{M-1}$, the mapping $\hat{q}_i^{\text{out}} \mapsto \hat{r}_i(\hat{q}_i^{\text{out}}, \hat{q}_{-i}^{\text{out}})$ is convex and continuously Fréchet differentiable. The expression of the gradient w.r.t. the first variable reads

$$\nabla_{\hat{q}_i^{\text{out}}} \hat{r}_i(\hat{q}_i^{\text{out}}, \hat{q}_{-i}^{\text{out}}) = \alpha \sum_{k=1}^M \hat{q}_k^{\text{out}} + \alpha \hat{q}_i^{\text{out}} - (\beta - \hat{\alpha}q_0(L))$$

with $\hat{\alpha} := \alpha(1 + M)/M$. Let $e \in \hat{U}^*$, $d_1: \hat{U} \times \hat{U}^{M-1} \rightarrow \hat{U}^*$, and $d_2: \hat{U} \rightarrow \hat{U}^*$ be such that for any $w_i = (w_i^{p_0}, w_i^{p_L}, w_i^{q_0}, w_i^{q_L}) \in \hat{U}$ the following holds:

$$\begin{aligned} \langle d_1(\hat{u}_i, \hat{u}_{-i}), w_i \rangle_{\hat{U}^*, \hat{U}} &= \left(\alpha \sum_{k=1}^M \hat{q}_k^{\text{out}}, w_i^{qL} \right)_{L^2(0, T)}, \\ \langle d_2(\hat{u}_i), w_i \rangle_{\hat{U}^*, \hat{U}} &= \left(\alpha \hat{q}_i^{\text{out}}, w_i^{qL} \right)_{L^2(0, T)}, \\ \langle e, w_i \rangle_{\hat{U}^*, \hat{U}} &= \left(\beta - \hat{\alpha}q_0(L), w_i^{qL} \right)_{L^2(0, T)}. \end{aligned}$$

Finally, we define the functionals $\hat{g}^p(\hat{p}_i) := g^p(\hat{p}_i + \hat{p}^0)$ and $\hat{g}_i^q(\hat{q}_i) := g_i^q(\hat{q}_i + \hat{q}^0)$.

We are now ready to state our main result.

Theorem 4.1. *Given an initial state $(p_0, q_0) \in Y_0^{\text{ad}}$, consider the following jointly convex GNEP with M players, where the optimization problem associated with player $i \in \{1, \dots, M\}$ is given by*

$$\begin{aligned} \min_{u_i \in U} \quad & f_i(u_i, u_{-i}) := \int_0^T \left(\alpha(t) \sum_{k=1}^M q_k^{\text{out}}(t) - \beta(t) \right) q_i^{\text{out}}(t) dt + g_i(u_i) \\ \text{s.t.} \quad & u_i \in P^{\text{ad}} \times Q_i^{\text{ad}}, \quad S(u_i, u_{-i}) \in K, \end{aligned} \quad (22)$$

where $g_i(u_i) = g^p(p_i) + g_i^q(q_i)$ with both g^p and g_i^q being lower semicontinuous and proper convex functions. Assume that for each player, the subdifferential ∂g_i is defined everywhere and that the sets K , $P^{\text{ad}} \subset P$ as well as $Q_i^{\text{ad}} \subset Q$ are closed and convex. In addition, P^{ad} and Q_i^{ad} satisfy the property that

$$p \in P^{\text{ad}} \implies p^{\text{in}}(0) = p_0(0) \text{ and } p^{\text{out}}(0) = p_0(L), \quad (23)$$

$$q \in Q_i^{\text{ad}} \implies Mq_i^{\text{in}}(0) = q_0(0) \text{ and } Mq_i^{\text{out}}(0) = q_0(L). \quad (24)$$

Furthermore, suppose that there exists $\bar{u} \in \prod_{i=1}^M (P^{\text{ad}} \times Q_i^{\text{ad}})$ such that Slater's constraint qualification (CQ)

$$S(\bar{u}) \in \text{int } K \quad (25)$$

holds. Finally, assume that the subdifferential of the sum $Mg^p(p) + \sum_{i=1}^M g_i^q(q_i)$, when restricted to $P^{\text{ad}} \times \prod_{i=1}^M Q_i^{\text{ad}}$, is coercive. Then, there exists a variational equilibrium to the GNEP with all pressures p_i , $i \in \{1, \dots, M\}$, being identical.

Proof. We proceed in three steps: First, we show that there exists a solution to the GNEP for which the first-order conditions for the players' optimization problems exhibit the same multiplier for the shared constraint $S(u) \in K$. Second, we formulate a reduced generalized equation with only one boundary pressure variable and show that its solution set is non-empty. Lastly, from any solution of the latter problem, we construct a solution to an operator equation associated with the first part and therefore to the jointly convex GNEP.

Step #1: We focus on the first-order optimality condition for the minimization problem associated with player i . For this purpose, take any fixed \hat{u}_{-i} such that $\hat{C}_i(\hat{u}_{-i}) \neq \emptyset$ holds. Since for any \hat{u}_{-i} , the partial map $f_i(\cdot, \hat{u}_{-i})$ is lower semicontinuous and convex, for \bar{u}_i to be optimal, it is necessary and sufficient that the inclusion

$$0 \in \partial(f_i(\cdot, \hat{u}_{-i}) + \delta_{\hat{C}_i(\hat{u}_{-i})})(\bar{u}_i) \quad (26)$$

holds, where $\delta_{\hat{C}_i(\hat{u}_{-i})}$ is the indicator function of the set $\hat{C}_i(\hat{u}_{-i})$. As r_i is C^1 and $\partial \hat{g}_i$ has full domain, by applying the sum rule for convex subdifferentials we obtain

$$\partial(f_i(\cdot, \hat{u}_{-i}) + \delta_{\hat{C}_i(\hat{u}_{-i})})(\hat{u}_i) = d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) - e + \partial \hat{g}_i(\hat{u}_i) + \partial(\delta_{\hat{C}_i(\hat{u}_{-i})})(\hat{u}_i).$$

Using the chain rule for convex subdifferentials we get that for any $\hat{u}_i \in \hat{U}_i^{\text{ad}}$ with $\hat{S}(\hat{u}_i, \hat{u}_{-i}) \in K$, the inclusion

$$\partial \delta_{\hat{C}_i(\hat{u}_{-i})}(\hat{u}_i) \supseteq N_{\hat{U}_i^{\text{ad}}}(\hat{u}_i) + \partial(\delta_K(\hat{S}(\hat{u}_i, \hat{u}_{-i}))) \quad (27)$$

holds. Hence, any solution to the generalized equation

$$0 \in d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) - e + \partial \hat{g}_i(\hat{u}_i) + N_{\hat{U}_i^{\text{ad}}}(\hat{u}_i) + \partial(\delta_K(\hat{S}(\hat{u}_i, \hat{u}_{-i}))) \quad (28)$$

satisfies (26) and therefore is optimal for player i . Recall from (20) that $\hat{u}_i \mapsto \hat{S}(\hat{u}_i, \hat{u}_{-i})$ is an affine mapping. Invoking again the chain rule for convex subdifferentials, we have

$$\partial(\delta_K(\hat{S}(\hat{u}_i, \hat{u}_{-i}))) \supseteq \hat{S}_i^* N_K(\hat{S}(\hat{u}_i, \hat{u}_{-i})). \quad (29)$$

With $\mu_i \in N_K(\hat{S}(\hat{u}_i, \hat{u}_{-i}))$, from (28) we get the generalized equation

$$e \in d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) + \partial \hat{g}_i(\hat{u}_i) + N_{\hat{U}_i^{\text{ad}}}(\hat{u}_i) + \hat{S}_i^* \mu_i.$$

Any $\hat{u} \in \hat{\mathcal{U}}$ that satisfies the above inclusion for all $i \in \{1, \dots, M\}$ is a solution of the GNEP. Note that if an appropriate constraint qualification, like Slater's CQ, were to hold for $\hat{C}_i(\hat{u}_{-i})$ as given in (21), then both (27) and (29) would hold with equality. Since the state variable

$y = \hat{S}(\hat{u}_i, \hat{u}_{-i})$ is the same for all players, we can restrict all μ_i to be equal. This leads us to the following operator equations:

$$e \in d_1(\hat{u}_i, \hat{u}_{-i}) + d_2(\hat{u}_i) + \partial \hat{g}_i(\hat{u}_i) + N_{\hat{C}_i^{\text{ad}}}(\hat{u}_i) + \hat{S}_i^* \mu, \quad i \in \{1, \dots, M\}, \quad (30)$$

$$\mu \in N_K(\hat{S}(\hat{u})). \quad (31)$$

Any $\hat{u} \in \hat{\mathcal{U}}$ that satisfies the above relations is a variational equilibrium of the GNEP. To conclude the first step and streamline the remainder of the proof, let us provide an equivalent system of inclusions that characterize a solution to the jointly convex GNEP. Let $\tilde{d}_1: \hat{Q} \times \hat{Q}^{M-1} \rightarrow \hat{Q}^*$ and $\tilde{d}_2: \hat{Q} \rightarrow \hat{Q}^*$ be such that

$$\begin{aligned} \left\langle \tilde{d}_1(\hat{q}_i, \hat{q}_{-i}), w_i \right\rangle_{\hat{Q}^*, \hat{Q}} &= \left(\alpha \sum_{k=1}^M \hat{q}_k^{\text{out}}, w_i^{\text{out}} \right)_{L^2(0,T)}, \\ \left\langle \tilde{d}_2(\hat{q}_i), w_i \right\rangle_{\hat{Q}^*, \hat{Q}} &= (\alpha \hat{q}_i^{\text{out}}, w_i^{\text{out}})_{L^2(0,T)}, \\ \langle \tilde{e}, w_i \rangle_{\hat{Q}^*, \hat{Q}} &= (\beta - \hat{\alpha} q_0(L), w_i^{\text{out}})_{L^2(0,T)}, \end{aligned}$$

for any $w_i = (w_i^{\text{in}}, w_i^{\text{out}}) \in \hat{Q}$. Remembering that $\mathcal{Z} \simeq \hat{\mathcal{U}}^*$, we conclude that the solution set of the system (30)–(31) coincides with the one of

$$0 \in \partial \hat{g}^p(\hat{p}_i) + N_{\hat{P}^{\text{ad}}}(\hat{p}_i) + M^{-1} \eta^p, \quad i \in \{1, \dots, M\}, \quad (32a)$$

$$\tilde{e} \in \tilde{d}_1(\hat{q}_i, \hat{q}_{-i}) + \tilde{d}_2(\hat{q}_i) + \partial \hat{g}_i^q(\hat{q}_i) + N_{\hat{Q}_i^{\text{ad}}}(\hat{q}_i) + \eta^q, \quad i \in \{1, \dots, M\}, \quad (32b)$$

$$\mu \in N_K(\hat{S}(\hat{u})), \quad (32c)$$

$$\begin{pmatrix} \eta^p \\ \eta^q \end{pmatrix} = \hat{S}_0^* \mu. \quad (32d)$$

Step #2: We are interested in the existence of a solution for which the endpoint pressure is the same for all players. For this purpose, we consider the operator equation in the space $\tilde{\mathcal{U}} := \hat{P} \times \hat{Q}^M$. Let $\tilde{\mathcal{L}}: \hat{P} \times \hat{Q}^M$ be given by $\tilde{\mathcal{L}}(\tilde{p}, \hat{q}) = (\tilde{p}, \sum_{i=1}^M \hat{q}_i)$. The relation between the state variable y and the controls $(\tilde{p}, \hat{q}) \in \tilde{\mathcal{U}}$ is now given by $\tilde{S}(\tilde{p}, \hat{q}) := S_{\text{PDE}}(y_0, \hat{u}^{y_0}) + \hat{S}_0(\tilde{\mathcal{L}}(\tilde{p}, \hat{q}))$. The constraint set is $\tilde{\mathcal{C}} := \{(\tilde{p}, \hat{q}) \in \hat{P}^{\text{ad}} \times \hat{Q}^{\text{ad}} \mid \tilde{S}(\tilde{p}, \hat{q}) \in K\}$ with $\hat{Q}^{\text{ad}} := \prod_{i=1}^M \hat{Q}_i^{\text{ad}}$. Thanks to the convexity of \hat{P}^{ad} , Condition (25) ensures the non-emptiness of $\tilde{\mathcal{C}}$. With $\tilde{\mathcal{Z}} := \hat{P}^* \times (\hat{Q}^*)^M$, we associate the operators $\tilde{\mathcal{A}}: \tilde{\mathcal{U}} \rightarrow \tilde{\mathcal{Z}}$ and $\tilde{\mathcal{B}}: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{Z}}$ with

$$\tilde{\mathcal{A}}(\tilde{p}, \hat{q}) = \begin{pmatrix} 0 \\ \tilde{d}_1(\hat{q}_1, \hat{q}_{-1}) + \tilde{d}_2(\hat{q}_1) \\ \vdots \\ \tilde{d}_1(\hat{q}_M, \hat{q}_{-M}) + \tilde{d}_2(\hat{q}_M) \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{B}}(\tilde{p}, \hat{q}) = \begin{pmatrix} \partial M \hat{g}^p(\tilde{p}) \\ \partial \hat{g}_1^q(\hat{q}_1) \\ \vdots \\ \partial \hat{g}_M^q(\hat{q}_M) \end{pmatrix}. \quad (33)$$

Our goal is to show existence of a solution to the generalized equation

$$\tilde{e} \in \tilde{\mathcal{A}}'(\tilde{p}, \hat{q}) + \tilde{\mathcal{B}}'(\tilde{p}, \hat{q}) + N_{\tilde{\mathcal{C}}}(\tilde{p}, \hat{q}), \quad (34)$$

where $\tilde{e} := (0, \tilde{e}, \dots, \tilde{e})$ and $\tilde{\mathcal{A}}'$ as well as $\tilde{\mathcal{B}}'$ are defined by identifying the sets $\tilde{\mathcal{A}}(\tilde{p}, \hat{q})$ and $\tilde{\mathcal{B}}(\tilde{p}, \hat{q})$ with the corresponding subsets of $\tilde{\mathcal{U}}^*$. Conversely, the operator $\tilde{\mathcal{C}}: \tilde{\mathcal{U}} \rightrightarrows \tilde{\mathcal{Z}}$ is defined by setting $\tilde{\mathcal{C}}(\tilde{p}, \hat{q})$ to the corresponding subset of $N_{\tilde{\mathcal{C}}}(\tilde{p}, \hat{q})$. We show that the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ is maximally monotone and start by establishing that $\tilde{\mathcal{A}}$ is monotone and demicontinuous. The latter property is true whenever for any sequence $\{v_n\} \subset \text{dom } \tilde{\mathcal{A}}$ converging strongly to v in $\tilde{\mathcal{U}}$, we have that $\tilde{\mathcal{A}}(v_n)$ converges weakly to $\tilde{\mathcal{A}}(v)$ in $\tilde{\mathcal{Z}}$; see Barbu and Precupanu (2012,

Definition 1.137). With $z^p \in \hat{P}$, $z_i^q \in \hat{Q}$, $z_i^q =: (z_i^{\text{in}}, z_i^{\text{out}})$, and z^q collecting all the z_i^q , we have

$$\begin{aligned} \left\langle \tilde{\mathcal{A}}(\tilde{p}, \hat{q}), (z^p, z^q) \right\rangle_{\tilde{z}, \tilde{u}} &= \sum_{i=1}^M \left[\left(\alpha \sum_{k=1}^M \hat{q}_k^{\text{out}}, z_i^{\text{out}} \right)_{L^2(0,T)} + (\alpha \hat{q}_i^{\text{out}}, z_i^{\text{out}})_{L^2(0,T)} \right] \\ &= \left(\sum_{k=1}^M \hat{q}_k^{\text{out}}, \sum_{i=1}^M \alpha z_i^{\text{out}} \right)_{L^2(0,T)} + \sum_{i=1}^M (\hat{q}_i^{\text{out}}, \alpha z_i^{\text{out}})_{L^2(0,T)}. \end{aligned}$$

As $\alpha \in L^\infty(0, T)$, we have $\alpha z_i^{\text{out}} \in L^2(0, T)$. Then, for any sequence $\{(\tilde{p}_n, \hat{q}_n)\} \subset \text{dom } \tilde{\mathcal{A}}$ strongly converging to (\tilde{p}, \hat{q}) , we have weak convergence of all the terms in the sum above and $\tilde{\mathcal{A}}(\tilde{p}_n, \hat{q}_n)$ converges weakly to $\tilde{\mathcal{A}}(\tilde{p}, \hat{q})$. For the monotonicity, as $\hat{q} \mapsto \tilde{d}_1(\hat{q}_i, \hat{q}_{-i})$ is linear, the operator $\tilde{\mathcal{A}}$ is a linear. Taking $z^p = \tilde{p}$ and $z^q = \hat{q}$ in the previous relation yields

$$\begin{aligned} \left\langle \tilde{\mathcal{A}}(\tilde{p}, \hat{q}), (\tilde{p}, \hat{q}) \right\rangle_{\tilde{z}, \tilde{u}} &= \sum_{i=1}^M \left[\left(\alpha \sum_{k=1}^M \hat{q}_k^{\text{out}}, \hat{q}_i^{\text{out}} \right)_{L^2(0,T)} + (\alpha \hat{q}_i^{\text{out}}, \hat{q}_i^{\text{out}})_{L^2(0,T)} \right] \\ &= \left(\alpha \sum_{k=1}^M \hat{q}_k^{\text{out}}, \sum_{i=1}^M \hat{q}_i^{\text{out}} \right)_{L^2(0,T)} + \sum_{i=1}^M (\alpha \hat{q}_i^{\text{out}}, \hat{q}_i^{\text{out}})_{L^2(0,T)}. \end{aligned}$$

Since $\alpha(t) \geq \alpha_{\min} > 0$ for almost all $t \in (0, T)$, we infer that

$$\left\langle \tilde{\mathcal{A}}(\tilde{p}, \hat{q}), (\tilde{p}, \hat{q}) \right\rangle_{\tilde{z}, \tilde{u}} \geq \alpha_{\min} \sum_{i=1}^M \|\hat{q}_i^{\text{out}}\|_{L^2(0,T)}^2 + \alpha_{\min} \sum_{i=1}^M \|\hat{q}_i^{\text{out}}\|_{L^2(0,T)}^2.$$

As $\tilde{\mathcal{A}}$ has full domain, is monotone and demicontinuous, it is maximally monotone; see Barbu and Precupanu (2012, Corollary 1.142). The maximal monotonicity of the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ follows from the fact that both $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ have full domain.

For the existence of a solution to (34), we rely on the coercivity of the maximally monotone sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$; see Barbu and Precupanu (2012, Definition 1.138). By the assumption on the subdifferential of $Mg^p(p) + \sum_{i=1}^M g_i^q(q_i)$, namely $\tilde{\mathcal{B}}$, we have the coercivity of the sum $\tilde{\mathcal{B}} + \tilde{\mathcal{C}}$. As $\tilde{\mathcal{A}}$ has full domain and is maximally monotone, the sum $\tilde{\mathcal{A}} + \tilde{\mathcal{B}} + \tilde{\mathcal{C}}$ is coercive. Now under the isomorphism between $\tilde{\mathcal{Z}}$ and $\tilde{\mathcal{U}}^*$, the operator $\tilde{\mathcal{A}}' + \tilde{\mathcal{B}}' + N_{\tilde{\mathcal{C}}}$ is also coercive and maximally monotone. It is well-known that a coercive and maximally monotone operator from a reflexive Banach space to its dual is surjective; see, e.g., Barbu and Precupanu (2012, Theorem 1.143). Therefore, there is a solution of the operator equation

$$\mathbf{e} \in (\tilde{\mathcal{A}}' + \tilde{\mathcal{B}}' + N_{\tilde{\mathcal{C}}})(u). \quad (35)$$

Step #3: Finally, we construct from any solution (p^*, q^*) of this generalized equation a solution u^* of the system (30)–(31) and, thus, of the GNEP. First, since (25) holds at $\tilde{u} = (\tilde{p}, \hat{q})$, we infer that with $p' := (\frac{1}{M} \sum_{i=1}^M \tilde{p}_i) - \hat{p}^0$ and $q' \in \hat{Q}^M$ defined as $q'_i := \tilde{q}_i - \hat{q}^0$, the inclusion

$$\tilde{\mathcal{S}}(p', q') \in \text{int } K$$

holds and $(p', q') \in \tilde{\mathcal{C}}$. Invoking the chain rule for convex subdifferentials, we get that for any $(\tilde{p}, \hat{q}) \in \tilde{\mathcal{C}}$, we have

$$N_{\tilde{\mathcal{C}}}(\tilde{p}, \hat{q}) = N_{\hat{P}^{\text{ad}} \times \hat{Q}^{\text{ad}}}(\tilde{p}, \hat{q}) + \tilde{\mathcal{L}}^* \hat{S}_0^* N_K(\tilde{\mathcal{S}}(\tilde{p}, \hat{q})).$$

Let $\mu^* \in N_K(\tilde{\mathcal{S}}(p^*, q^*))$ be such that the inclusion (35) holds. With $\eta^* := \hat{S}_0^* \mu^*$, we get $(\eta^p)^* \in \hat{P}^*$ and $(\eta^q)^* \in \hat{Q}^*$ from η^* . Then, from (35) we obtain that

$$0 \in \partial M \hat{g}^p(p^*) + N_{\hat{P}^{\text{ad}}}(p^*) + (\eta^p)^*$$

holds. Since $\partial M \hat{g}^p(p^*) = M \partial \hat{g}^p(p^*)$, the following inclusion is satisfied:

$$0 \in \partial \hat{g}^p(p^*) + N_{\hat{P}^{\text{ad}}}(p^*) + M^{-1}(\eta^p)^*. \quad (36)$$

We define $u^* \in \hat{U}^{\text{ad}}$ by setting $u_i^* := (p^*, q_i^*)$ and since $\tilde{S}(p^*, q^*) = S(u^*)$ by construction, we have $\mu^* \in N_K(S(u^*))$. Using the definition of the operators in (33) and of $\hat{\mathcal{L}}$, for all $i \in \{1, \dots, M\}$ we get

$$0 \in \tilde{d}_1(q_i^*, q_{-i}^*) + \tilde{d}_2(q_i^*) + \partial \hat{g}_i^q(q_i^*) + N_{\hat{Q}_i^{\text{ad}}}(q_i^*) + (\eta^q)^*. \quad (37)$$

Since $\mu^* \in N_K(S(u^*))$, $\eta^* = \hat{S}_0^* \mu^*$, and (36)–(37) hold for all u_i^* , we infer that u^* satisfies (32). \square

Corollary 4.2. *Given an initial state $(p_0, q_0) \in Y_0^{\text{ad}}$, consider the GNEP formed by the collection of problems of the form (4). Suppose that for each firm i , the functional $\mathcal{R}_i(p_i, q_i) = \mathcal{R}_i^q(q_i) + \mathcal{R}_i^p(p_i)$ with both $\mathcal{R}_i^q: Q \rightarrow \mathbb{R}$ and $\mathcal{R}_i^p: P \rightarrow \mathbb{R}$ being lower semicontinuous, proper and convex with subdifferentials defined everywhere. We set*

$$Q_i^{\text{ad}} = \{(q^{\text{in}}, q^{\text{out}}) \in Q \mid q_0(0) = Mq^{\text{in}}(0), q_0(L) = Mq^{\text{out}}(0), \int_0^T q^{\text{out}}(t) - q^{\text{in}}(t) dt \leq 0, \\ 0 \leq q^{\text{in}} \leq \bar{q}_i^{\text{in}}, \quad 0 \leq q^{\text{out}} \leq \bar{q} \text{ a.e. in } (0, T)\}.$$

With $P^{\text{ad}} = \{(p^{\text{in}}, p^{\text{out}}) \in P \mid p \leq p^{\text{in}}, p^{\text{out}} \leq \bar{p}, p_0(0) = p^{\text{in}}(0) \text{ and } p_0(L) = p^{\text{out}}(0)\}$, assume there exists $\bar{u} \in \prod_{i=1}^M (P^{\text{ad}} \times Q_i^{\text{ad}})$ such that $S(\bar{u}) \in \text{int } K$ holds. Suppose that the subdifferential of the sum $M\mathcal{R}^p(p) + \sum_{i=1}^M \mathcal{R}_i^q(q_i)$, when restricted to $P^{\text{ad}} \times \prod_{i=1}^M Q_i^{\text{ad}}$, is coercive. Then, there exists a variational equilibrium to the GNEP with all pressures p_i being identical.

Proof. To fit into the format of (22), we let

$$g_i(u_i) = \int_0^T c_i(t) q_i^{\text{in}}(t) dt + \mathcal{R}_i(u_i).$$

and note that all the hypothesis are satisfied. \square

Remark 4.3. *The existence of feasible controls satisfying (25) is automatically given whenever the initial state y_0 is a stationary solution satisfying the bound constraints and there exists feasible controls to satisfy the injection requirements given in (24). The coercivity condition is satisfied whenever the feasible set of each agent is bounded. Otherwise, like in our test instance, this condition is satisfied whenever each feasible set is bounded in the L^2 sense and each \mathcal{R} is the square of the L^2 norm of the weak time derivative; see (46).*

Remark 4.4. *For the existence of a variational equilibrium, the mild Slater condition (25), involving all players' decisions, needs to hold. If one seeks a stronger characterization of the solution set of the GNEP, so that any variational equilibrium is a solution of System (30)–(31), the, we need both (27) and (29) to hold with equality. For this, a sufficient condition is that for all i , and any feasible \hat{u}_{-i} , there exists $\hat{u}_i \in \hat{U}_i^{\text{ad}}$ such that*

$$\hat{S}(\hat{u}_i, \hat{u}_{-i}) \in \text{int } K,$$

which is much more stringent.

Furthermore, if the producers have the same feasible sets and cost functionals, then a solution to the GNEP can be constructed from the solution to an optimization problem. We refer to this case as the *symmetric case*.

Proposition 4.5. *Consider the GNEP of Theorem 4.1 with a given initial state y_0 and assume that the players' problems are identical. Suppose that all the assumptions of Theorem 4.1 hold, except for the boundedness of the P^{ad} and Q^{ad} . Let $Q_{\text{opt}}^{\text{ad}} := MQ^{\text{ad}}$ and suppose that there exists*

$\bar{u} \in P^{\text{ad}} \times Q_{\text{opt}}^{\text{ad}}$ such that $S(\bar{u}) \in \text{int } K$ holds. Then, a variational equilibrium of the GNEP can be constructed from any solution of the optimization problem

$$\begin{aligned} \min_{u \in U} \quad & \int_0^T \left(\frac{M+1}{2M} \alpha(t) q^{\text{out}}(t) - \beta(t) \right) q^{\text{out}}(t) dt + g(u) \\ \text{s.t.} \quad & u \in P^{\text{ad}} \times Q_{\text{opt}}^{\text{ad}}, \quad S(u) \in K, \end{aligned} \quad (38)$$

where $g(p, q) = Mg^p(p) + Mg_i^q(M^{-1}q)$.

Proof. We use the same functional analytic setting as in the proof of Theorem 4.1 and we also perform the change of variables $\hat{p} := p - \hat{p}^0$ and $\hat{q} := q - \hat{q}^0$ with $\hat{p}^0(t) := (p_0(0), p_0(L))$ and $\hat{q}^0(t) := (q_0(0), q_0(L))$ for all $t \in [0, T]$. We also use the same functional analytic setting as in the previous proof. The space $\hat{Q}_{\text{opt}}^{\text{ad}}$ is defined as $\{\hat{q} \in \hat{Q} \mid \hat{q} + \hat{q}^0 \in Q_{\text{opt}}^{\text{ad}}\}$. We show that any (\hat{p}, \hat{q}) satisfying the first-order optimality condition of this problem can be used to construct an element $\hat{u} \in \hat{P} \times \hat{Q}$ satisfying the first-order optimality condition (34). From basic results of convex analysis, we have $\partial_{\hat{q}} \hat{g}(\hat{p}, \hat{q}) = \partial g_i^q(M^{-1}(\hat{q} + \hat{q}^0))$. Recall that $\hat{\alpha} = (1 + M^{-1})\alpha$. The revenue part $\hat{r}_{\text{opt}}: L^2(0, T) \rightarrow \mathbb{R}$ of the objective functional with the shifted variables reads

$$\hat{r}_{\text{opt}}(\hat{q}^{\text{out}}) = \int_0^T \left[\frac{1}{2} \hat{\alpha}(t) \hat{q}^{\text{out}}(t) - \beta(t) + \frac{1}{2} \hat{\alpha}(t) q_0(L) \right] [\hat{q}^{\text{out}}(t) + q_0(L)] dt.$$

It is Fréchet differentiable and its gradient is given by

$$\nabla \hat{r}_{\text{opt}}(\hat{q}^{\text{out}}) = \hat{\alpha} \hat{q}^{\text{out}} + \hat{\alpha} q_0(L) - \beta.$$

With the assumptions on the cost function and the fact that Slater's CQ holds, any solution of the optimization problem satisfies

$$\begin{aligned} 0 &= M \partial g^p(p^*) + \zeta_p + \eta^p, \\ \tilde{e} &= \begin{pmatrix} 0 \\ \alpha(1 + M^{-1})(q^{\text{out}})^* \end{pmatrix} + \partial g_i^q(M^{-1}(q^* + \hat{q}^0)) + \zeta_q + \eta^q, \end{aligned}$$

where $\zeta_p \in N_{\hat{P}^{\text{ad}}}(p^*)$, $\zeta_q \in N_{\hat{Q}_{\text{opt}}^{\text{ad}}}(q^*)$, $\mu \in N_K(\hat{S}(u^*))$, (η^p, η^q) are obtained from $\hat{S}_0^* \mu$ and \tilde{e} satisfies $\langle \tilde{e}, w_i \rangle_{\hat{Q}^*, \hat{Q}} = (\beta - \hat{\alpha} q_0(L), w_i^{\text{out}})_{L^2(0, T)}$ for any $w_i = (w_i^{\text{in}}, w_i^{\text{out}}) \in \hat{Q}$. Note that $\tilde{\zeta}_p := M^{-1} \zeta_p$ also belongs to $N_{\hat{P}^{\text{ad}}}(p^*)$. Now let $\hat{q}_i = M^{-1} q^*$ and $\hat{\mathbf{q}}$ collects all \hat{q}_i . Then, after some algebraic manipulations, we get that the following holds:

$$\begin{aligned} 0 &= \partial g^p(p^*) + \tilde{\zeta}_p + M^{-1} \eta^p, \\ \tilde{e} &= \begin{pmatrix} 0 \\ \alpha \sum_k \hat{q}_k^{\text{out}} + \alpha \hat{q}_i^{\text{out}} \end{pmatrix} + \partial g_i^q(\hat{q}_i + M^{-1} \hat{q}^0) + \zeta_q + \eta^q. \end{aligned}$$

Note that $g_i^q(\hat{q}_i + M^{-1} q_0(L)) = \hat{g}_i^q(\hat{q}_i)$ holds by the definition of the latter function. Since $\hat{S}(u^*) = \tilde{S}(p^*, \hat{\mathbf{q}})$, we have $\mu \in N_K(\tilde{S}(p^*, \hat{\mathbf{q}}))$. Then, one can see that for all i , the generalized equation (32) holds with $u_i = (p^*, \hat{q}_i)$. Therefore, the strategy profile $u = (u_1, \dots, u_M)$ is a variational equilibrium of the GNEP. \square

Existence of a solution of the optimization problem (38) is easily obtained using classic arguments. The feasible set is a non-empty, convex, and closed subset of a reflexive Banach space. In addition to the hypotheses of Proposition 4.5, assume that

$$\lim_{\|u\| \rightarrow +\infty} g(u) + \delta_{P^{\text{ad}} \times Q_{\text{opt}}^{\text{ad}}}(u) = +\infty.$$

This ensures that the level sets of the objective functional are bounded over the feasible set. The latter is also proper lower semicontinuous and convex with full domain. Then, there exists a minimizer; see, e.g., Barbu and Precupanu (2012, Theorem 2.11).

5. TEST CASES: THE INFLUENCE OF LINEPACKING ON MARKET OUTCOMES

The dynamics of transport in networks as formalized in our analysis allows to capture important intertemporal aspects, which are highly relevant in real-world market interaction. An important example is given in the context of gas markets, where transport dynamics of the network are such that the network can be used as a temporal storage device. In the context of gas transport this is referred to as *linepacking*. The importance of linepacking in the operation of real-world gas transport is documented in, e.g., Koch et al. (2015, Page 212). It has been estimated that for, e.g., the Great Britain national gas transmission system, the average hourly linepack stores energy of about 3743 GW h, but that the available linepack flexibility varies widely over the year (G. Wilson and Rowley 2019). In that study, the available within-day flexibility of the transmission system ranged from 13 GW h to 427 GW h.

In what follows, based on a stylized numerical test example, we use our framework to illustrate how the ability of linepacking can influence market outcomes in gas markets. Specifically, we investigate how linepacking is used for storage by market participants and who benefits to which degree.

To this end, we consider M identical and strategically acting firms that trade natural gas, which is transported over a single pipe. The time horizon of 4 h is split into two periods of 2 h each. Purchase cost $c(t)$ and inverse demand $P(t, q(t)) = \beta(t) + \alpha(t)q(t)$ do not change within each period. With $i \in \{1, 2\}$, we use c_i , α_i , and β_i to denote their values, which are given by

$$\begin{aligned} c(t) &= \begin{cases} 388.8, & t \in [0, 2 \text{ h}), \\ 432.0, & t \in [2 \text{ h}, 4 \text{ h}]. \end{cases} \\ \alpha(t) &= \begin{cases} -16.6, & t \in [0, 2 \text{ h}), \\ -20.7, & t \in [2 \text{ h}, 4 \text{ h}], \end{cases} \\ \beta(t) &= \begin{cases} 1500.8, & t \in [0, 2 \text{ h}), \\ 2000.3, & t \in [2 \text{ h}, 4 \text{ h}], \end{cases} \end{aligned}$$

The purchase cost of gas is 10% cheaper in the first period and the inverse demand function $P(t, q(t))$ reflects higher demand in the second period. This choice of parameters thus incentivizes producers to use the pipe as a temporal storage for the gas acquired in the first period for its use in the second one to reduce the overall costs. Taking piecewise constant data allows us, moreover, to easily formulate a “static” counterpart of the considered scenario. For this counterpart, we associate a mass flow q_i and boundary pressures ($p_i^{\text{in}}, p_i^{\text{out}}$) for each period. We consider the flow to be stationary and that each triple $(q_i, p_i^{\text{in}}, p_i^{\text{out}})$ satisfies the Weymouth equation as a stationary approximation of gas flow in a pipe. Our choice of parameters ensures that no congestion occurs for the stationary flow. In the stationary case, this implies that the bounds on pressure and mass flow do not influence the market outcomes. As we will see, however, these bounds play an important role in the instationary model.

The physical parameters of the instance, inspired by Schmidt et al. (2017), are as follows: $L = 50$ km, $D = 0.5$ m, $\varphi = 0$, $k = 1 \times 10^{-4}$ m, $T_{\text{emp}} = 15$ °C, $R = 8.3145$ J mol $^{-1}$ K $^{-1}$, and $m = 0.016043$ kg mol $^{-1}$. The latter two values are those for methane, which usually dominates the composition of natural gas. The specific gas constant $R_s := R/m$ has value 518.26 J kg K $^{-1}$ and the friction coefficient is modeled using the formula by Nikuradse, i.e., $\lambda = (2 \log_{10}(\frac{D}{k}) + 1.138)^{-2}$; see Page 25 in Koch et al. (2015) for a discussion of this model.

The initial state is assumed to be stationary and is given by $q_0 = 48.289$ kg s $^{-1}$ and the pressure at the entry node being 50 bar. Remember that the pressure and mass flow admit, in a pointwise sense, upper (\bar{p}, \bar{q}) and lower (p, q) bounds in the pipe. The values for the last three bounds are fixed to

$$p = 40 \text{ bar}, \quad q = 10.9 \text{ kg s}^{-1}, \quad \bar{q} = 163 \text{ kg s}^{-1}.$$

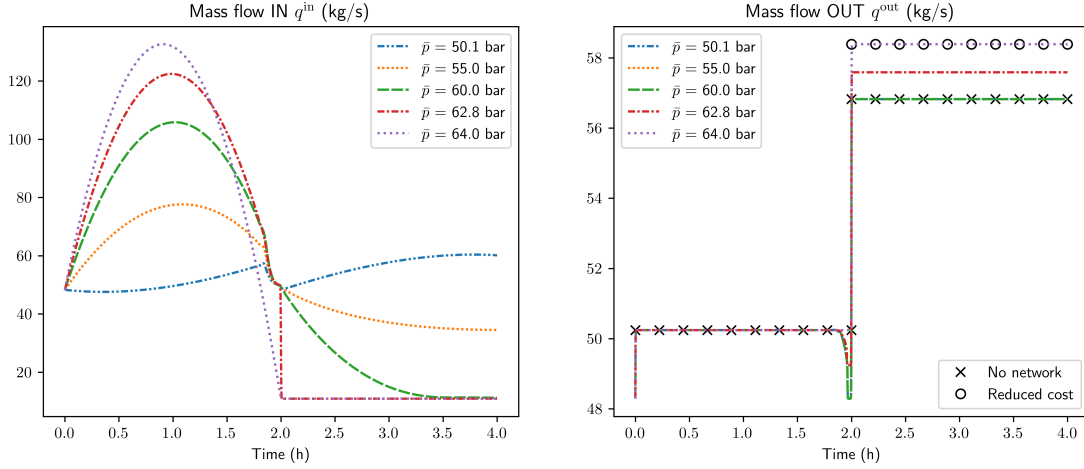


FIGURE 1. Mass flow boundary controls for different values of the upper pressure bound \bar{p} . The crosses indicate the optimal flow given by the static model. The circles indicate the optimal mass flow during the second period with the cost of gas being c_1 rather than c_2 also for the static model.

The value of \bar{p} directly influences the maximal linepack capability in our setup and, thus, we will vary it in our test cases. By using the ideal gas law $p = c_s^2 \rho$, the total amount of gas in the pipe linearly depends on the integral of the pressure profile over the pipe. Thus, pressure is the variable we monitor when studying linepack effects. In our setup, we vary \bar{p} in the range of $[50.1, 70]$ bar.

With this calibration, we solve the model described in (4). More precisely, we are in the setting of Proposition 4.5. The specific discretization of the PDE system (1) is described in Appendix B. To keep the presentation focused on the actual results, we moved the detailed problem formulations for this test case to Appendix C. There, we also present the modeling environments and solvers used for the computations.

Let us first present the time evolution of the physical quantities in the case of 3 firms. We start with the evolution of the mass flows at the endpoints of the pipe as shown in Figure 1. For most parts of the time horizon, the output q^{out} is constant but has rather different values in each period. Each value is very close to the one obtained by the static model, which is indicated by the crosses in the figure. Two transitions are present: (i) at the beginning from the initial state to the desired output value and (ii) at the interface of the two time periods at $t = 2$ h. The latter has a drop in the outflow, which we will discuss later. The evolution of q^{in} is distinct for each value of \bar{p} . The general trend is that when \bar{p} increases, more gas is injected during the first period and less during the second one. This leads to the situation that only the minimum amount q is injected during or towards the end of the second period. The exception to this behavior is for $\bar{p} = 50.1$ bar, where the ability to steer the system from one operational point to another one is difficult due to the small gap between the initial pressure and its upper bound.

In Figure 2, the evolution of the linepack relative to its initial value is displayed. Except for $\bar{p} = 50.1$ bar, the curves are similar with an increase in the first period, leading to a maximum around the switching time of 2 h, which is finally followed by an almost linear decrease. A higher value of \bar{p} leads to a higher linepack peak. Any value \bar{p} greater than 64 bar yields the same result for this instance. As mentioned before, the linepack is related to the pressure profile in the pipe. Figure 4 displays the values of p^{in} and p^{out} over the time horizon. For a given \bar{p} , these pressures evolve similarly except for the period just before $t = 2$ h. Except for $\bar{p} = 64$ bar, p^{in} reaches a plateau close to its upper bound. Note that this is not the case for

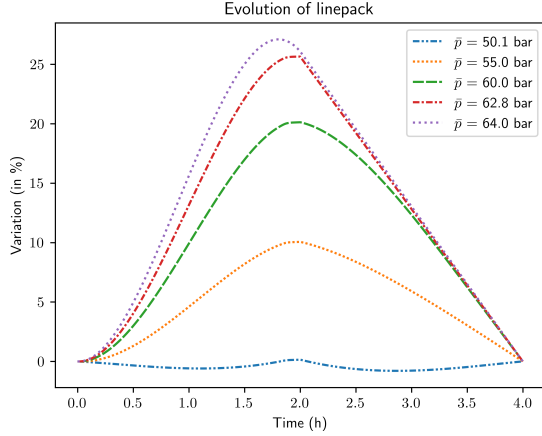


FIGURE 2. Evolution of the linepack (total mass of gas in the pipe). The instance is the same as in Figure 1.

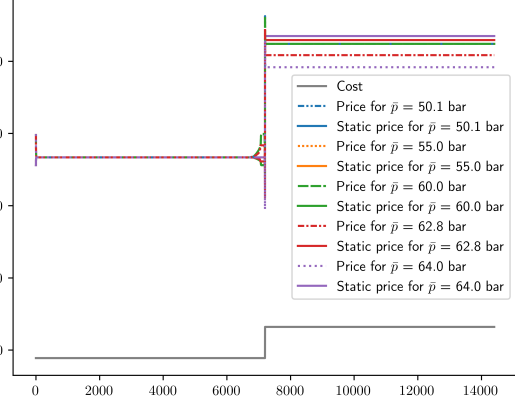


FIGURE 3. Evolution of the “static” and actual price at the output node. The instance is the same as in Figure 1.

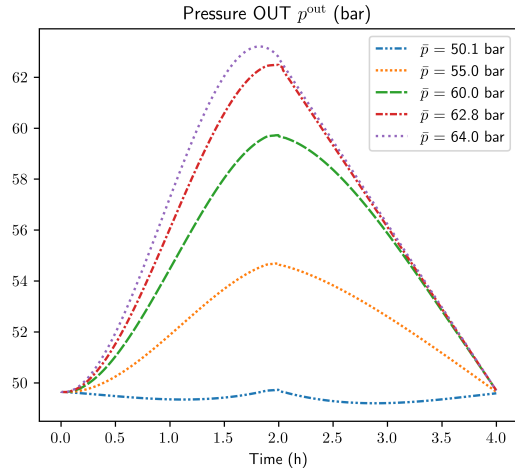
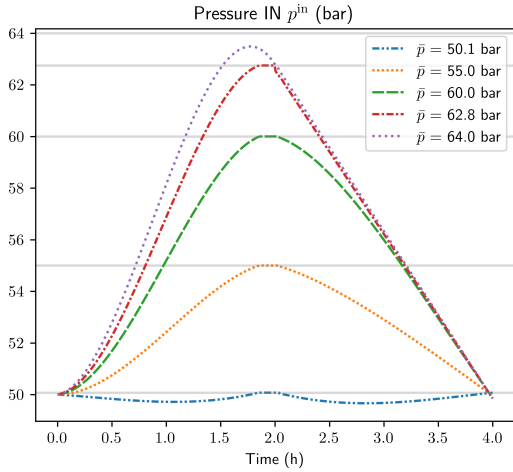


FIGURE 4. Evolution of the pressure at the endpoints. The instance is the same as in Figure 1.

p^{out} or the linepack, which is due to the drop in q^{out} for this short time interval, leading to a higher linepack. This drop disappears when the upper pressure constraint is not active; see, e.g., the plot for $\bar{p} = 64$ bar.

We now move on to the economical interpretation of these results. First, note that the cost and demand structure indeed incentivizes linepacking. The case $\bar{p} = 50.1$ bar is an exception since the physical constraints appear to be too tight—remember that $p^{\text{in}}(0) = 50$ bar—so that linepacking is not observed to a significant degree. There, the transition between the different desired output values drives the decisions. With \bar{p} large enough, the quantity of gas stored during the first period covers the amount sold during the second one with $q^{\text{in}}(t) = q$ for $t \in [2 \text{ h}, 4 \text{ h}]$. If more gas can be stored, this leads to an increased value of q^{out} . This lasts until q^{out} reaches the solution of the static problem with purchase cost c_1 rather than c_2 ; see the results for $\bar{p} = 64$ bar. Then, a further increase in the linepack ability does not yield any change.

Next, we look at the price of gas at the output node as shown in Figure 3. There, as a reference, we display the purchase cost $c(t)$ and static strategic prices given by $c(t) + \beta M^{-1} q^{\text{out}}(t)$. These would occur if firms choose standard strategic markups for a given $q^{\text{out}}(t)$. Again, the two transitions induce price variations. Furthermore, the discontinuity of P at $t = 2$ h leads to a jump in prices. For low and moderate linepack capability, i.e., $\bar{p} \in \{50.1, 55.0, 60.0\}$ bar, pricing apparently takes place based on the current purchase cost c . Firms, rather than consumers, benefit from savings due to linepacking. When the storage capacity is large, i.e., $\bar{p} \in \{62.8, 64\}$ bar, we observe a significant decrease of consumer prices. In these cases, the firms can purchase all the gas that they plan to sell minus the minimum amount induced by q . The static price prediction, as the relevant marginal purchase cost, is not only determined by the purchase cost c_2 but also takes into account c_1 . If the upper pressure bound is never active, then c_1 is the only relevant purchase cost for the entire time horizon. Market prices in hours 2–4 are then also based on c_1 only.

To further understand how the market participants benefit from additional linepack ability, we look at the evolution of the aggregated firms' profits, the consumer surplus, and the overall welfare over time if \bar{p} varies in $[51, 70]$ bar. In addition to the case of 3 firms, we solved the same instance in the case of perfect competition. To better see the different occurring effects, we split the overall firms profit into three components:

$$\begin{aligned}
& \int_0^{4\text{h}} (P(t, q^{\text{out}}(t)) q^{\text{out}}(t) - c(t) q^{\text{in}}(t)) dt \\
&= \int_0^{4\text{h}} (q^{\text{out}}(t) - q^{\text{in}}(t)) c(t) dt \\
& \quad + \int_0^{4\text{h}} (\alpha M^{-1} q^{\text{out}}(t)) q^{\text{out}}(t) dt \\
& \quad + \int_0^{4\text{h}} (P(t, q^{\text{out}}(t)) - (\alpha M^{-1} q^{\text{out}}(t) + c(t))) q^{\text{out}}(t) dt \\
&=: \boxed{\text{linepack savings}} + \boxed{\text{static strategic profits}} + \boxed{\text{network rent}}
\end{aligned} \tag{39}$$

The first term captures the reduction in total purchase cost thanks to linepacking that allows them to separate purchase and selling time. Next, we have profits obtained by the exercise of static market power. Last, the network rent captures the impact of the network on market prices. These are defined as the earnings obtained from the difference between the observed market price and the static price. In Figure 5, the welfare and its two components are displayed. We can observe that there are two different regimes with different solution structures: Low linepacking ability and high linepacking ability. As seen in Figure 5 (left), if the linepacking ability is low, the consumer surplus remains constant, while the welfare increases with \bar{p} . This implies that firms' profits become larger, which is confirmed in Figure 5 (right), where we see that the linepack savings are increasing. This structure of the solution stays the same for linepack abilities below a threshold of around $\bar{p} = 62$ bar for 3 firms and $\bar{p} = 67$ bar for the case of perfect competition. With \bar{p} above these values, the structure of the solution changes. In those cases additional linepack capability increases the availability of cheaper gas. As discussed previously, the market price in the second period is then based on both c_1 and c_2 . From this, the consumers benefit and their surplus increases, leading to higher welfare values. Finally, for very high linepack capability, purchase cost c_2 does no longer influence the market price at all, but it is exclusively determined based on the lower purchase cost c_1 , inducing lower market prices in the second period. This beneficial effect of linepacking, inducing lower market prices is observed for a larger range of linepack capabilities in case of strategically acting firms rather than for the perfectly competitive case. This is due to the fact that strategically behaved firms in general tend to sell less gas and smaller linepack capabilities allow to fully cover all sales by period 1 purchases. As a consequence of the drop in market prices for large linepack capabilities

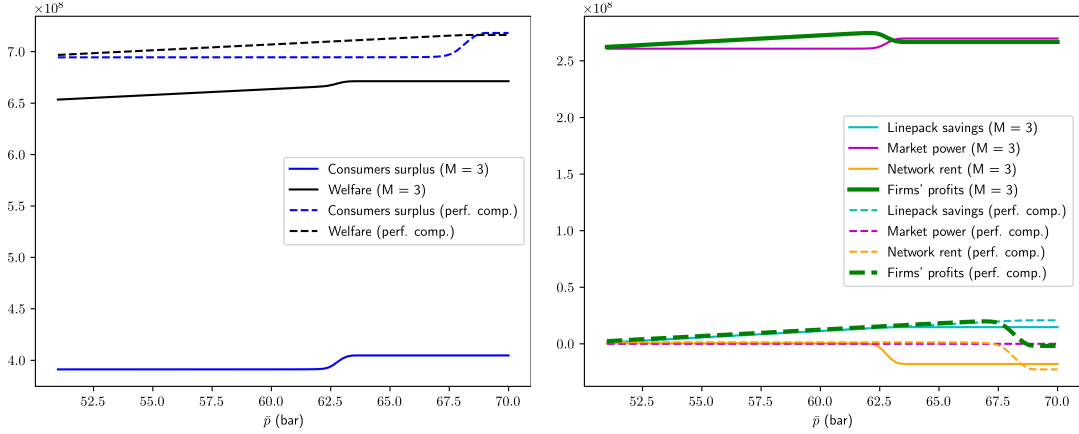


FIGURE 5. Aggregated consumer surplus and welfare (left) and firms' profits partitioned in components (right) for the test instance in the case of 3 producers and under perfect competition.

we observe a drop of firms profits in those cases, which is caused by network rents becoming negative in those cases; see Figure 5 (right). By looking at its definition in (39), it is clear that as soon as the market price drops during the second period, this quantity becomes negative. As firms are forced to purchase a minimum amount of gas for all points in time t ($q^{\text{in}}(t) \geq \underline{q}$) in the case of perfect competition, the firms' overall profit even becomes negative for large \bar{p} .

We have thus seen that our setup is perfectly suitable to analyze dynamic network transport in a market environment and thus contributes to the understanding of phenomena such as linepack, which is highly important in the context of gas markets. We are able to analyze how the ability of linepacking can influence market outcomes in gas markets and which market participant benefits to which degree. Interestingly, for low linepacking capabilities mainly firms can increase their profits due to an improved purchase cost structure. This reverts in case of high linepacking capabilities where reduced market prices induce a significant increase of consumer surplus.

6. CONCLUSION

We investigated a stylized intraday gas market under Cournot competition by modeling it as a GNEP in which the evolution equations for the gas flow are shared constraints. We showed the well-posedness of this system of PDEs with a solution regular enough to ensure the existence of Lagrangian multipliers. Then, we proved the existence of a variational equilibrium of the GNEP, which is consistent with the continuity of pressures at the nodes. When producers are identical, we identified an equivalent optimization problem in the sense that a variational equilibrium can be constructed from any solution to the latter. We illustrated the insights gained via such a PDE model compared to a static one. In particular, we highlighted how linepacking impacts the market outcomes and determined who benefits from additional linepacking ability. For this purpose, we considered a simple instance in which the time horizon is split into two periods and gas is cheaper during the first one. While the quantity of gas sold is most of the time close to the solution of the static problem, the inflow is rather different and changes with the linepacking ability. Indeed, the producers seek to reduce their overall cost by storing cheap gas as much as possible in the pipe. There, linepacking only benefits the producers and this lasts until it is possible to store enough gas to cover the sale of the second period while fulfilling the flow constraints. Then, the amount sold during that time interval increases until it reaches the value given by the solution of the static problem for the second period, but with the cost of the

first one. This indicates a decoupling between injection and withdrawal of gas from the pipe and induces an increase of the consumer surplus as well as a decrease of the firms' profit, while the overall welfare is still increasing.

These results encourage further investigation of such models. On the one hand, other models for the evolution of the gas flow can be considered; see, e.g., Domschke et al. (2021). In this case, the shared constraint consists of a semilinear system of hyperbolic PDEs. On the other hand, one can depart from the deterministic setting and investigate the market outcomes when uncertainties are present. These could originate either from the physical model, where, e.g., the friction coefficient is unknown, or from the economical data such as demand or supply.

ACKNOWLEDGMENTS

The authors thank the DFG for their support within project A05, B02, B07, B08, and B09 in CRC TRR 154. This research has been performed as part of the Energie Campus Nürnberg and is supported by funding of the Bavarian State Government.

REFERENCES

- Adams, R. and J. Fournier (2003). *Sobolev Spaces*. 2nd ed. Academic Press.
- Barbu, V. and T. Precupanu (2012). *Convexity and Optimization in Banach Spaces*. 4th ed. Springer Monographs in Mathematics. Springer Netherlands, pp. xxi+368. DOI: [10.1007/978-94-007-2247-7](https://doi.org/10.1007/978-94-007-2247-7).
- Benchekroun, H. (2003). “Unilateral production restrictions in a dynamic duopoly.” In: *Journal of Economic Theory* 111.2, pp. 214–239. DOI: [10.1016/S0022-0531\(03\)00090-5](https://doi.org/10.1016/S0022-0531(03)00090-5).
- (2008). “Comparative dynamics in a productive asset oligopoly.” In: *Journal of Economic Theory* 138.1, pp. 237–261. DOI: [10.1016/j.jet.2006.10.012](https://doi.org/10.1016/j.jet.2006.10.012).
- Brezis, H. (2010). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York. DOI: [10.1007/978-0-387-70914-7](https://doi.org/10.1007/978-0-387-70914-7).
- Caruana, G. and L. Einav (2008). “Production targets.” In: *The RAND Journal of Economics* 39.4, pp. 990–1017. DOI: [10.1111/j.1756-2171.2008.00047.x](https://doi.org/10.1111/j.1756-2171.2008.00047.x).
- Chen, X. and Z. Wang (2014). “Differential variational inequality approach to dynamic games with shared constraints.” In: *Mathematical Programming* 146.1, pp. 379–408. DOI: [10.1007/s10107-013-0689-1](https://doi.org/10.1007/s10107-013-0689-1).
- Colombo, L. and P. Labrecciosa (2015). “On the Markovian efficiency of Bertrand and Cournot equilibria.” In: *Journal of Economic Theory* 155, pp. 332–358. DOI: [10.1016/j.jet.2014.11.007](https://doi.org/10.1016/j.jet.2014.11.007).
- (2019). “Stackelberg versus Cournot: A differential game approach.” In: *Journal of Economic Dynamics and Control* 101, pp. 239–261. DOI: [10.1016/j.jedc.2018.10.007](https://doi.org/10.1016/j.jedc.2018.10.007).
- Cournot, A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses*. Hachette Paris.
- Dautray, R. and J.-L. Lions (2000). *Mathematical Analysis and Numerical Methods for Science and Technology*. Vol. 5: *Evolution Equations I*. 6 vols. Springer Berlin Heidelberg.
- Domschke, P., B. Hiller, J. Lang, V. Mehrmann, R. Morandin, and C. Tischendorf (2021). *Gas Network Modeling: An Overview*. Tech. rep. Technische Universität Darmstadt, p. 53. URL: <https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/411>.
- Dreves, A. and J. Gwinner (2016). “Jointly Convex Generalized Nash Equilibria and Elliptic Multiobjective Optimal Control.” In: *Journal of Optimization Theory and Applications* 168.3, pp. 1065–1086. DOI: [10.1007/s10957-015-0788-7](https://doi.org/10.1007/s10957-015-0788-7).
- Dunning, I., J. Huchette, and M. Lubin (2017). “JuMP: A Modeling Language for Mathematical Optimization.” In: *SIAM Review* 59.2, pp. 295–320. DOI: [10.1137/15M1020575](https://doi.org/10.1137/15M1020575).
- Gugat, M., J. Habermann, M. Hintermüller, and O. Huber (2021). *Constrained exact boundary controllability of a semilinear model for pipeline gas flow*. Tech. rep. URL: <https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/388>.

- Gugat, M. and S. Steffensen (2018). “Dynamic boundary control games with networks of strings.” In: *ESAIM: Control, Optimisation and Calculus of Variations* 24.4, pp. 1789–1813. DOI: [10.1051/cocv/2017082](https://doi.org/10.1051/cocv/2017082).
- Gurobi Optimization, LLC (2021). *Gurobi Optimizer Reference Manual*. URL: <http://www.gurobi.com>.
- Hart, W. E., C. D. Laird, J.-P. Watson, D. L. Woodruff, G. A. Hackebeil, B. L. Nicholson, and J. D. Sirola (2017). *Pyomo—Optimization Modeling in Python*. Vol. 67. Springer Optimization and Its Applications. Springer. DOI: [10.1007/978-3-319-58821-6](https://doi.org/10.1007/978-3-319-58821-6).
- Hintermüller, M. and T. Surowiec (2013). “A PDE-constrained generalized Nash equilibrium problem with pointwise control and state constraints.” In: *Pacific Journal on Optimization* 9.2, pp. 251–273.
- Hintermüller, M., T. Surowiec, and A. Kämmler (2015). “Generalized Nash Equilibrium Problems in Banach Spaces: Theory, Nikaido–Isoda-Based Path-Following Methods, and Applications.” In: *SIAM Journal on Optimization* 25.3, pp. 1826–1856. DOI: [10.1137/14096829X](https://doi.org/10.1137/14096829X).
- Hinze, M., R. Pinnau, M. Ulbrich, and S. Ulbrich (2009). *Optimization with PDE Constraints*. Vol. 23. Mathematical Modelling: Theory and Applications. Springer Netherlands. DOI: [10.1007/978-1-4020-8839-1](https://doi.org/10.1007/978-1-4020-8839-1).
- Holmberg, P. and A. B. Philpott (2018). “On supply-function equilibria in radial transmission networks.” In: *European Journal of Operational Research* 271.3, pp. 985–1000. DOI: [10.1016/j.ejor.2018.05.070](https://doi.org/10.1016/j.ejor.2018.05.070).
- Huber, O. (2023). *PDE-Cournot-code*. Version 0.1.0. DOI: [10.5281/zenodo.7702535](https://doi.org/10.5281/zenodo.7702535). URL: <https://github.com/xhub/PDE-Cournot-code>.
- Jun, B. and X. Vives (2004). “Strategic incentives in dynamic duopoly.” In: *Journal of Economic Theory* 116.2, pp. 249–281. DOI: [10.1016/j.jet.2003.08.005](https://doi.org/10.1016/j.jet.2003.08.005).
- Kanzow, C., V. Karl, D. Steck, and D. Wachsmuth (2019). “The multiplier-penalty method for generalized Nash equilibrium problems in Banach spaces.” In: *SIAM Journal on Optimization* 29.1, pp. 767–793. DOI: [10.1137/17M114114X](https://doi.org/10.1137/17M114114X).
- Koch, T., B. Hiller, M. E. Pfetsch, and L. Schewe, eds. (2015). *Evaluating Gas Network Capacities*. Philadelphia, PA: Society for Industrial and Applied Mathematics. DOI: [10.1137/1.9781611973693](https://doi.org/10.1137/1.9781611973693).
- Lambertini, L. and A. Palestini (2014). “On the feedback solutions of differential oligopoly games with hyperbolic demand curve and capacity accumulation.” In: *European Journal of Operational Research* 236.1, pp. 272–281. DOI: [10.1016/j.ejor.2013.12.008](https://doi.org/10.1016/j.ejor.2013.12.008).
- Ledvina, A. and R. Sircar (2011). “Dynamic Bertrand Oligopoly.” In: *Applied Mathematics & Optimization* 63.1, pp. 11–44. DOI: [10.1007/s00245-010-9110-0](https://doi.org/10.1007/s00245-010-9110-0).
- Lions, P. L. (1981). “Une inégalité pour les opérateurs elliptiques du second ordre.” In: *Annali di Matematica Pura ed Applicata* 127.1, pp. 1–11. DOI: [10.1007/bf01811715](https://doi.org/10.1007/bf01811715).
- Nash, J. F. (1950). “Equilibrium points in n -person games.” In: *Proceedings of the National Academy of Sciences* 36.1, pp. 48–49. DOI: [10.1073/pnas.36.1.48](https://doi.org/10.1073/pnas.36.1.48).
- Renardy, M. and R. C. Rogers (2004). *An Introduction to Partial Differential Equations*. Vol. 13. Springer-Verlag. DOI: [10.1007/b97427](https://doi.org/10.1007/b97427).
- Schiro, D. A., J.-S. Pang, and U. V. Shanbhag (2013). “On the solution of affine generalized Nash equilibrium problems with shared constraints by Lemke’s method.” In: *Mathematical Programming* 142.1, pp. 1–46. DOI: [10.1007/s10107-012-0558-3](https://doi.org/10.1007/s10107-012-0558-3).
- Schmidt, Martin, Denis Afkman, Robert Burlacu, Jesco Humpola, Imke Joormann, Nikolaos Kanelakis, Thorsten Koch, Djamal Oucherif, Marc Pfetsch, Lars Schewe, Robert Schwarz, and Mathias Sirvent (2017). “GasLib—A Library of Gas Network Instances.” In: *Data* 2.4, p. 40. DOI: [10.3390/data2040040](https://doi.org/10.3390/data2040040).
- Schwab, C. and R. Stevenson (2009). “Space-time adaptive wavelet methods for parabolic evolution problems.” In: *Mathematics of Computation* 78.267, pp. 1293–1318. DOI: [10.1090/s0025-5718-08-02205-9](https://doi.org/10.1090/s0025-5718-08-02205-9).

- Simon, J. (1986). “Compact sets in the space $L^p(0, T; B)$.” In: *Annali di Matematica Pura ed Applicata* 146.1, pp. 65–96. DOI: [10.1007/bf01762360](https://doi.org/10.1007/bf01762360).
- Wächter, A. and L. T. Biegler (2005). “On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming.” In: *Mathematical Programming* 106.1, pp. 25–57. DOI: [10.1007/s10107-004-0559-y](https://doi.org/10.1007/s10107-004-0559-y).
- Wei, J.-Y. and Y. Smeers (1999). “Spatial Oligopolistic Electricity Models with Cournot Generators and Regulated Transmission Prices.” In: *Operations Research* 47.1, pp. 102–112. DOI: [10.1287/opre.47.1.102](https://doi.org/10.1287/opre.47.1.102).
- Wilson, G. and P. Rowley (2019). *Flexibility in Great Britain’s gas networks: analysis of linepack and linepack flexibility using hourly data*. Briefing note. UK Energy Research Centre. URL: <https://ukerc.ac.uk/publications/linepack/>.
- Wilson, R. (2008). “Supply Function Equilibrium in a Constrained Transmission System.” In: *Operations Research* 56.2, pp. 369–382. DOI: [10.1287/opre.1070.0421](https://doi.org/10.1287/opre.1070.0421).
- Wirl, F. (2010). “Dynamic demand and noncompetitive intertemporal output adjustments.” In: *International Journal of Industrial Organization* 28.3, pp. 220–229. DOI: [10.1016/j.ijindorg.2009.08.002](https://doi.org/10.1016/j.ijindorg.2009.08.002).
- Xin, Baogui and Minghe Sun (2018). “A differential oligopoly game for optimal production planning and water savings.” In: *European Journal of Operational Research* 269.1, pp. 206–217. DOI: [10.1016/j.ejor.2017.07.016](https://doi.org/10.1016/j.ejor.2017.07.016).

APPENDIX A. FUNDAMENTAL RESULTS

In this section, we review some fundamental results from convex optimization and the theory of PDEs. For our work, we need to prove that our problems have solutions of high regularity (see Section 3) to ensure the existence of multipliers of the state constraints. For this, we need to review some results from PDE theory, starting with the Aubin–Lions lemma.

Lemma A.1 (Aubin–Lions lemma; Simon (1986, Corollary 4)). *Let X_0 , X , and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1 . For $1 \leq p, q \leq +\infty$, let*

$$W := \{u \in L^p(0, T; X_0) \mid u_t \in L^q(0, T; X_1)\}.$$

- (i) *If $p < +\infty$, then the embedding of W into $L^p(0, T; X)$ is compact.*
- (ii) *If $p = +\infty$ and $q > 1$, then the embedding of W into $C([0, T]; X)$ is compact.*

We use this lemma with $X_0 = H_0^1(\Omega)^2$, $X = C(\bar{\Omega})^2$, and $X_1 = H^{-1}(\Omega)^2$. The proof of additional regularity of the solution to the PDE system uses the Ladyzhenskaya–Sobolevski inequality for the Laplacian. For convenience, we provide a proof of this result and also state its main ingredient: the interpolation inequalities of Gagliardo–Nirenberg.

Lemma A.2 (Interpolation inequalities of Gagliardo–Nirenberg; Brezis (2010, p. 234)). *Let I be an open and bounded interval. Then, there exists a constant $C > 0$ such that for any $v \in H^2(I)$, we have*

$$\|\nabla v\|_{L^2(I)} \leq C \|v\|_{H^2(I)}^{1/2} \|v\|_{L^2(I)}^{1/2}, \quad (40)$$

and for a given $\kappa > 0$, there exists C_κ such that

$$\|\nabla v\|_{L^2(I)} \leq \|v\|_{H^1(I)} \leq \kappa \|v\|_{H^2(I)} + C_\kappa \|v\|_{L^2(I)}. \quad (41)$$

We can now state and prove the Ladyzhenskaya–Sobolevski inequality for the Laplacian.

Lemma A.3 (Ladyzhenskaya–Sobolevski inequality for Δ). *Let I be an open and bounded interval. Then, there exists a constant $C > 0$ such that for all $v \in H^2(I)$, we have*

$$(\Delta v, \Delta v)_{L^2(I)} \geq \frac{1}{2} \|v\|_{H^2(I)}^2 - C \|v\|_{L^2(I)}^2.$$

Proof. From Inequality (40) in Lemma A.2, we have

$$\|\nabla v\|_{L^2(I)}^2 \leq C' \|v\|_{H^2(I)} \|v\|_{L^2(I)}.$$

Using Cauchy's inequality, there exists $C'' > 0$ such that

$$\|\nabla v\|_{L^2(I)}^2 \leq \frac{1}{2} \|v\|_{H^2(I)}^2 + C'' \|v\|_{L^2(I)}^2$$

holds and by further using the definition of the H^2 -norm as well as by rearranging terms, we obtain

$$\begin{aligned} \|\Delta v\|_{L^2(I)}^2 &\geq \|\nabla v\|_{L^2(I)}^2 + \|v\|_{L^2(I)}^2 - 2(C'' + 1) \|v\|_{L^2(I)}^2, \\ 2\|\Delta v\|_{L^2(I)}^2 &\geq \|v\|_{H^2(I)}^2 - 2(C'' + 1) \|v\|_{L^2(I)}^2. \end{aligned}$$

Finally, we get

$$(\Delta v, \Delta v)_{L^2(I)} \geq \frac{1}{2} \|v\|_{H^2(I)}^2 - (C'' + 1) \|v\|_{L^2(I)}^2. \quad \square$$

APPENDIX B. DISCRETIZATION SCHEME

Coming back to Problem (45), as a first step towards solving such instances, we pursue a discretize-then-optimize approach. For the discretization of the system in (45b)–(45f), rather than using a generic scheme for parabolic PDEs, we retain a numerical scheme that can be related to the linearized version of the ISO2 system. Indeed, the relations in (45b)–(45c) are obtained from the semilinear Euler equations

$$\begin{aligned} p_t + \frac{c_s^2}{A} q_x &= 0, \\ q_t + A p_x &= -\frac{1}{2} \sigma A \frac{q|q|}{p} - gA \sin \varphi \frac{p}{c_s^2} \end{aligned}$$

by linearizing the right-hand side and adding a viscosity term. Using the Riemann invariants $R_{\pm} := c_s A^{-1} q \pm p$, we obtain the diagonal system

$$\begin{aligned} (R_+)_t + c_s (R_+)_x &= F(R_+, R_-), \\ (R_-)_t - c_s (R_-)_x &= F(R_+, R_-), \end{aligned} \quad (42)$$

in which the coupling between R_+ and R_- is through the right-hand side

$$F(R_+, R_-) := -\frac{\lambda c_s}{4D} (R_+ + R_-) \frac{|R_+ + R_-|}{R_+ - R_-} - g \sin \varphi \frac{R_+ - R_-}{2c}.$$

This diagonal form is useful to investigate theoretical properties of the system as well as for developing numerical schemes; see Gugat, Habermann, et al. (2021). For the relations in (45b)–(45c), the same change of variables results in

$$\begin{aligned} (R_+)_t - \varepsilon (R_+)_{xx} + c_s (R_+)_x &= F_{\text{aff}}(R_+, R_-), \\ (R_-)_t - \varepsilon (R_-)_{xx} - c_s (R_-)_x &= F_{\text{aff}}(R_+, R_-), \end{aligned} \quad (43)$$

where F_{aff} is the affine approximation of F around the operational points $\check{p} > 0$ and $\check{q} > 0$.

Let us first discretize the linearized version of the hyperbolic system (42), where F has been substituted by F_{aff} . Specifically, we integrate R_{\pm} along the characteristic lines using a triangular grid as illustrated on Figure 6. There, the solid line from node i to node k (resp. from j to k) represents the characteristic line for R_+ (resp. R_-). The characteristics passing through a given pair (\bar{x}, \bar{t}) are the images of the mappings $\xi_+(s, \bar{x}, \bar{t}) := (\bar{x} + c_s(s - \bar{t}), s)$ and $\xi_-(s, \bar{x}, \bar{t}) := (\bar{x} - c_s(s - \bar{t}), s)$, respectively. Along the characteristics, System (42) is compactly given by

$$\frac{d}{ds} (R_{\pm} \circ \xi_{\pm}) = (R_{\pm})_t \circ \xi_{\pm} + c_s (R_{\pm})_x \circ \xi_{\pm} = F(R_+, R_-) \circ \xi_{\pm}. \quad (44)$$

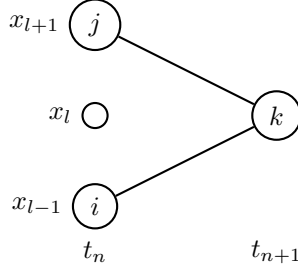


FIGURE 6. Triangular grid used for integration

Let us denote by R_{\pm}^i , R_{\pm}^j , and R_{\pm}^k the values of the Riemann invariants at the nodes i , j , and k . We have $\Delta x = x_l - x_{l-1} = x_{l+1} - x_l$, $\Delta t = t_{n+1} - t_n$, and $\Delta x = c_s \Delta t$ as well as $R_+^k = R_+(\xi_+(t_n + \Delta t, x_{l-1}, t_n))$ and $R_-^k = R_-(\xi_-(t_n + \Delta t, x_{l+1}, t_n))$. Integrating System (44) along the characteristics starting from node i (node j) for R_+ (R_-) and using the midpoint rule for approximating the right-hand side yields

$$\begin{aligned} R_+^k - R_+^i &= \frac{\Delta t}{2} [F_{\text{aff}}(R_+^i, R_-^i) + F_{\text{aff}}(R_+^k, R_-^k)], \\ R_-^k - R_-^j &= \frac{\Delta t}{2} [F_{\text{aff}}(R_+^j, R_-^j) + F_{\text{aff}}(R_+^k, R_-^k)]. \end{aligned}$$

Next, we relate this discretized system with one obtained by discretizing the viscosity regularized system. We proceed just with R_+ as the reasoning is similar with R_- . We discretize the left-hand side of the first relation in (43) using a finite difference scheme with $\Delta x = c_s \Delta t$ and $\varepsilon = c_s \Delta x$. Using a central difference formula for the second-order term, one obtains

$$\begin{aligned} & \frac{R_+^k - R_+(x_l, t_n)}{\Delta t} - c_s \Delta x \frac{R_+^j + R_+^i - 2R_+(x_l, t_n)}{(\Delta x)^2} + c_s \frac{R_+^j - R_+(x_l, t_n)}{\Delta x} \\ &= \frac{R_+^k - R_+(x_l, t_n)}{\Delta t} - \frac{R_+^j + R_+^i - 2R_+(x_l, t_n)}{\Delta t} + \frac{R_+^j - R_+(x_l, t_n)}{\Delta t} \\ &= \frac{R_+^k - R_+^i}{\Delta t}. \end{aligned}$$

With our choice of the right-hand side, the values of the Riemann invariants at (x_l, t_n) do not appear in any relation. This gives the grid its triangular shape.

APPENDIX C. OPTIMIZATION MODELS USED FOR THE TEST INSTANCE

In Section 5, all producers are identical. In this case, we can solve the optimization problem given in Proposition 4.5. Remember that the price functional in the symmetric case is given by

$$\tilde{P}(t, q(t)) = \beta(t) - \frac{M+1}{2M} \alpha(t) q(t).$$

With $u = (q^{\text{in}}, q^{\text{out}}, p^{\text{in}}, p^{\text{out}})$, the optimization problem associated with the symmetric case reads

$$\max_{u \in H^1(0, T)^4} \int_0^T \left(\tilde{P}(t, q^{\text{out}}(t)) q^{\text{out}}(t) - c(t) q^{\text{in}}(t) \right) dt - \mathcal{R}(p, q, q^{\text{in}}, q^{\text{out}}, p^{\text{in}}, p^{\text{out}}) \quad (45a)$$

$$\text{s.t. } p_t - \varepsilon p_{xx} + \frac{c_s^2}{A} q_x = 0 \quad \text{in } (0, L) \times (0, T), \quad (45b)$$

$$q_t - \varepsilon q_{xx} + A p_x = -\frac{1}{2} \sigma A \left(\frac{2\check{q}}{\check{p}} q - \frac{\check{q}^2}{\check{p}^2} p \right) \quad \text{in } (0, L) \times (0, T), \quad (45c)$$

$$p(0, \cdot) = p^{\text{in}}, \quad p(L, \cdot) = p^{\text{out}}, \quad \text{in } (0, T), \quad (45d)$$

$$q(0, \cdot) = q^{\text{in}}, \quad q(L, \cdot) = q^{\text{out}}, \quad \text{in } (0, T), \quad (45\text{e})$$

$$p(\cdot, 0) = p_0, \quad q(\cdot, 0) = q_0, \quad \text{in } (0, L), \quad (45\text{f})$$

$$\underline{p} \leq p \leq \bar{p}, \quad \underline{q} \leq q \leq \bar{q}, \quad \text{in } [0, L] \times [0, T], \quad (45\text{g})$$

$$p_0(0) = p^{\text{in}}(0), \quad p_0(L) = p^{\text{out}}(0), \quad q_0(0) = q^{\text{in}}(0), \quad q_0(L) = q^{\text{out}}(0), \quad (45\text{h})$$

$$\int_0^T q^{\text{out}}(t) - q^{\text{in}}(t) dt \leq 0. \quad (45\text{i})$$

Thanks to the regularity of the solution to the system in (45b)–(45f), any feasible (boundary) controls also satisfy the same bounds as in (45g) pointwise in time. As in our instance the lower and upper bounds at the nodes are identical to the ones in the pipe, we do not add explicit bounds at the nodes. The compatibility conditions between the initial state and the decision variables are given in (45h). Remember that Inequality (45i) ensures that the market participants do not get gas for free out of the initial state. The regularization functional \mathcal{R} has the structure

$$\mathcal{R}(p, q, u) = \gamma_1^p \|\nabla p^{\text{in}}\|_{L^2(0, T)^2}^2 + \gamma_1^q \|\nabla q^{\text{in}}\|_{L^2(0, T)^2}^2 + \eta_1 \|\nabla q(\cdot, T)\|_{L^2(\Omega)}^2 \quad (46)$$

with non-negative coefficients. By the regularity of the solution, we have $q \in C([0, T]; H^1(\Omega))$, which ensures that $\nabla q(\cdot, T) \in L^2(\Omega)$. The last term aims at avoiding unrealistic final states. For any sequence $u_n \in H^1(0, T)^4$ such that $\|\nabla u_n\|_{L^2(0, T)^4} \rightarrow +\infty$, we have $\lim \mathcal{R}(S(u_n), u_n) = +\infty$. As any feasible control of Problem (45) is bounded in $L^2(0, T)^4$, the level sets of the objective functional are bounded. The latter is also lower semi-continuous, proper, and subdifferentiable everywhere. Furthermore, the feasible set is non-empty, closed, and convex. Thus, there exists a solution of Problem (45). In order to discretize System (45), we use a discretization scheme that is motivated by analogous schemes for the semilinear Euler equations; for details, see Appendix B. It better captures the problem structure than a generic discretization scheme for parabolic PDE.

Problem (45) is parametric in the linearization point (\check{p}, \check{q}) . Inspired by sequential programming approaches, we seek a tuple (u^*, p^*, q^*) such that $(u^*, p^*, q^*) \in \text{SOL}(p^*, q^*)$, where SOL is the solution mapping of Problem (45). Starting with $(\bar{p}^0, \bar{q}^0) = (p_0(0), q_0)$, a sequence of solutions $(\bar{u}^k, \bar{p}^k, \bar{q}^k) \in \text{SOL}(\bar{p}^{k-1}, \bar{q}^{k-1})$ is computed. Whenever the relative difference between $(\bar{p}^{k-1}, \bar{q}^{k-1})$ and (\bar{p}^k, \bar{q}^k) falls below a given threshold, the procedure stops and a solution is returned.

The corresponding stationary model is given as follows. The time interval $[0, T]$ is partitioned into J subintervals $[t_k, t_{k+1})$. The flow q is constant along the pipe and the pressure monotonically changes according to the Weymouth equation

$$p^2(x) = p^2(0) - \sigma q |q| x;$$

see Section 2.3.1.2 in Koch et al. (2015) for a derivation of this relation. Hence, for each time subinterval, the state consists of the scalar flow q_k and the endpoint pressures p_k^{in} and p_k^{out} . With α_k , β_k , and c_k being the constant data of the inverse demand and cost functions on each subinterval $[t_k, t_{k+1})$, we obtain the problem

$$\begin{aligned} \max_{q, p^{\text{in}}, p^{\text{out}}} \quad & \sum_{k=1}^J (t_{k+1} - t_k) [(\beta_k - (M^{-1} + 1)\alpha_k q_k)q_k - c_k q_k] \\ \text{s.t.} \quad & (p_k^{\text{out}})^2 = (p_k^{\text{in}})^2 - \Lambda q_k^2, \quad 1 \leq k \leq J, \\ & \underline{p} \leq p_k^{\text{in}} \leq \bar{p}, \quad \underline{p} \leq p_k^{\text{out}} \leq \bar{p}, \quad \underline{q} \leq q_k \leq \bar{q}, \quad 1 \leq k \leq J, \end{aligned} \quad (47)$$

with $\Lambda := \sigma L$. As q_k is constant on each time subintervals, Constraint (45i) is automatically satisfied.

We use two different computational setups for the PDE-constrained and the stationary problem. The discretized version of Problem (47) is a convex QP. The latter is solved using

the barrier algorithm of Gurobi 9.5.0; see Gurobi Optimization (2021). We use JuMP as the modeling environment; see Dunning et al. (2017). The complete instance data as well as the code to perform the numerical computations and create the figures can be found at Huber (2023). The nonconvex stationary problem (47) is solved by Ipopt 3.14.1 via the Pyomo modeling environment; see Wächter and Biegler (2005) and Hart et al. (2017), respectively.

(V. Grimm) FRIEDRICH–ALEXANDER–UNIVERSITÄT ERLANGEN–NÜRNBERG, CHAIR OF ECONOMIC THEORY, LANGE GASSE 20, 90403 NÜRNBERG, GERMANY
Email address: `veronika.grimm@fau.de`

(M. Hintermüller) WEIERSTRASS INSTITUTE FOR APPLIED ANALYSIS AND STOCHASTICS, MOHRENSTRASSE 39, 10117 BERLIN, GERMANY
Email address: `michael.hintermueller@wias-berlin.de`

(O. Huber) HUMBOLDT–UNIVERSITÄT ZU BERLIN, DEPARTMENT OF MATHEMATICS, UNTER DEN LINDEN 6, 10099 BERLIN, GERMANY
Email address: `olivier.huber@hu-berlin.de`

(L. Schewe) SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, PETER GUTHRIE TAIT ROAD, EDINBURGH, EH9 3FD, UK
Email address: `lars.schewe@ed.ac.uk`

(M. Schmidt) TRIER UNIVERSITY, DEPARTMENT OF MATHEMATICS, UNIVERSITÄTSRING 15, 54296 TRIER GERMANY
Email address: `martin.schmidt@uni-trier.de`

(G. Zöttl) FRIEDRICH–ALEXANDER–UNIVERSITÄT ERLANGEN–NÜRNBERG, INDUSTRIAL ORGANIZATION AND ENERGY MARKETS, LANGE GASSE 20, 90403 NÜRNBERG, GERMANY
Email address: `gregor.zoettl@fau.de`