

# Pareto Robust optimization on Euclidean vector spaces

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**ABSTRACT.** Pareto efficiency for robust linear programs was introduced by Iancu and Trichakis in [9]. We generalize their approach and theoretical results to robust optimization problems in Euclidean spaces with linear uncertainty. Additionally, we demonstrate the value of this approach in an exemplary manner in the area of robust semidefinite programming (SDP). In particular, we prove that computing a Pareto robustly optimal solution for a robust SDP is tractable and illustrate the benefit of such solutions at the example of the maximal eigenvalue problem. Furthermore, we modify the famous algorithm of Goemans and Williamson [8] in order to compute cuts for the robust MAX CUT problem that yield an improved approximation guarantee in non-worst-case scenarios.

## 1. INTRODUCTION

Pareto efficiency is a well-established concept in a variety of fields such as economy, engineering and biology, see e.g. [18] for a broad overview. In [9], Iancu and Trichakis adapted this concept to robust optimization (RO) for linear programs. In particular, they consider the robust linear program

$$\sup_{x \in \mathcal{X}} \min_{p \in \mathcal{U}} p^\top x, \quad (1)$$

where  $\mathcal{X}$  denotes a polyhedron and the uncertainty set  $\mathcal{U}$  is assumed to be a polytope. In this setting they characterize and compute so-called Pareto robustly optimal or PRO solutions. These are robustly optimal solutions  $x \in \mathcal{X}$  for which there exists no other  $\bar{x} \in \mathcal{X}$  such that

$$\begin{aligned} \forall p \in \mathcal{U}: \quad & p^\top \bar{x} \geq p^\top x, \\ \exists \bar{p} \in \mathcal{U}: \quad & \bar{p}^\top \bar{x} > \bar{p}^\top x. \end{aligned}$$

The main purpose of this article is to generalize the definition above and retrieve a characterization of PRO solutions in this setting that is similar to the one in [9]. Moreover, we show that in the case of robust semidefinite programs, computing PRO solutions is tractable.

Although the work of Iancu and Trichakis on the linear framework is rather new, it has triggered further research such as an analysis for adjustable settings, see e.g. [15] for a rolling horizon approach and [4] for a Fourier-Motzkin Elimination based approach.

In Section 2, we generalize the approach of Iancu and Trichakis to  $\mathcal{X}$  being a subset of a finite dimensional Euclidean vector space and an uncertain parameter that affects the objective linearly and is contained in a compact, convex uncertainty set  $\mathcal{U}$ . In particular, we provide a characterization of Pareto robustly optimal (PRO) solutions in this broader setting, which is our main result. This result enables us to prove the tractability of computing a PRO solution in the case of robust semidefinite programming. In Sections 3 and 4, we illustrate how to compute the robust maximal

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eigenvalue of a class of matrices and consider a variant of the SDP that is at the core of the Goemans-Williamson Algorithm [8]. The PRO solutions of the latter, are then used as an input for the algorithm and improve the computed cuts for the robust MAX-CUT problem.

**Notation.** In the remainder of this article, the feasible set  $\mathcal{X}$  and the uncertainty set  $\mathcal{U}$  are contained in finite dimensional Euclidean vector spaces  $(V, \langle \cdot, \cdot \rangle_V)$  and  $(W, \langle \cdot, \cdot \rangle_W)$ , respectively. In the present article, we will mostly choose for both spaces the space of  $n$ -dimensional real symmetric matrices  $\mathcal{S}^n$  equipped with the Frobenius inner product  $\langle \cdot, \cdot \rangle_F$ , i.e.  $(\mathcal{S}^n, \langle \cdot, \cdot \rangle_F)$ . Given a set  $S \subseteq V$ , we denote its dual cone by  $S^* = \{y \in V : \langle y, x \rangle \geq 0 \ \forall x \in S\}$  and its relative interior by  $\text{relint}(S)$ . For a real matrix  $A \in \mathbb{R}^{n \times n}$ , we denote its trace by  $\text{Tr}(A)$ . For a positive integer  $n \in \mathbb{N}$ , we use  $[n] := \{1, \dots, n\}$  to denote a set of indices and  $I_n$  to denote the  $n$ -dimensional identity matrix.

## 2. PARETO OPTIMAL SOLUTIONS FOR LINEAR UNCERTAINTY

In contrast to (1), we consider the following more general robust optimization problem

$$\sup_{x \in \mathcal{X}} \min_{p \in \mathcal{U}} f(x, p), \quad (2)$$

where  $\mathcal{X} \subseteq V$ ,  $\mathcal{U} \subseteq W$  and  $f(\cdot, p) : \mathcal{X} \rightarrow \mathbb{R}$  is a function that is well-defined for all  $p \in \mathcal{U}$ . The parameter  $p \in \mathcal{U}$  encodes a linear uncertainty, i.e.  $f(x, \cdot) : \mathcal{U} \rightarrow \mathbb{R}$  is linear for all  $x \in \mathcal{X}$ . If  $\mathcal{X}$  is compact and  $f$  is continuous on  $\mathcal{X}$ , we replace 'sup' by 'max' in (2). We denote the set of robustly optimal solutions, i.e. the set of optimal solutions of (2), by  $\mathcal{X}^{RO}$ . In robust optimization, one usually focuses on the worst-case scenario, i.e. it suffices to find any robust solution  $x \in \mathcal{X}^{RO}$ . In contrast to this approach, we aim for a specific  $x \in \mathcal{X}^{RO}$  that also performs well under all other scenarios  $p \in \mathcal{U}$ . To this end, we use the definition of Pareto robustness from [4]:

**Definition 1.** A robustly optimal solution  $x \in \mathcal{X}^{RO}$  is called a *Pareto robustly optimal solution* (PRO) of (2) if there exists no  $\bar{x} \in \mathcal{X}$  such that

$$\forall p \in \mathcal{U}: \quad f(\bar{x}, p) \geq f(x, p), \quad (3)$$

$$\exists \bar{p} \in \mathcal{U}: \quad f(\bar{x}, \bar{p}) > f(x, \bar{p}). \quad (4)$$

In this case, we also write  $x \in \mathcal{X}^{PRO}$ . If  $x \notin \mathcal{X}^{PRO}$ , we say for an  $\bar{x}$ , which fulfills (3) and (4), that it Pareto dominates  $x$ .

It is natural to ask whether such solutions exist, if they can be characterized and whether they can be determined properly. We address the first task in an exemplary manner in Section 4. The key to characterize and determine PRO solutions is the following theorem, a generalization of Theorem 1 in [9] and our main result.

**Theorem 1.** A solution  $x^* \in \mathcal{X}^{RO}$  of (2) is PRO if and only if for an arbitrary  $\hat{p} \in \text{relint}(\mathcal{U})$  the optimization problem

$$\begin{aligned} & \sup_y f(y, \hat{p}), \\ & \text{s.t.} \quad \min_{p \in \mathcal{U}} f(y, p) - f(x^*, p) \geq 0, \\ & \quad y \in \mathcal{X} \end{aligned} \quad (5)$$

has optimal value  $f(x^*, \hat{p})$ . Otherwise, its optimal solution  $y^*$  is Pareto robustly optimal for (2).

*Proof.* First, we note that the optimal value of (5) is at least  $f(x^*, \hat{p})$ , since  $x^*$  is feasible for (5). If the optimal value is strictly greater than  $f(x^*, \hat{p})$  with optimal solution  $y^*$ , then  $y^*$  Pareto dominates  $x^*$  and hence  $x^* \notin \mathcal{X}^{PRO}$ . For the opposite direction, we assume that the optimal value equals  $f(x^*, \hat{p})$  and that  $x^*$  is not Pareto robustly optimal. Then there exists a solution  $y \in \mathcal{X}$  that Pareto dominates  $x^*$  and we obtain

$$0 < \max_{p \in \mathcal{U}} f(y, p) - f(x^*, p). \quad (6)$$

Since we optimize a linear function over a convex set  $\mathcal{U}$ , its optimal solution  $\bar{p}$  is w.l.o.g. an extreme point of  $\mathcal{U}$ . Additionally, the convexity of  $\mathcal{U}$  implies that for  $\hat{p} \in \text{relint}(\mathcal{U})$ , there exist  $p \in \mathcal{U}$  and  $\varepsilon > 0$ , such that  $\hat{p} = \varepsilon \bar{p} + (1 - \varepsilon)p$  and we conclude

$$f(y, \hat{p}) - f(x^*, \hat{p}) = \varepsilon(f(y, \bar{p}) - f(x^*, \bar{p})) + (1 - \varepsilon)(f(y, p) - f(x^*, p)) > 0,$$

where the inequality follows from the fact that  $\bar{p}$  was optimal for (6). Hence, we obtain a contradiction to the optimality of  $f(x^*, \hat{p})$ .

If on the other hand  $x^* \notin \mathcal{X}^{PRO}$ , then there exists a solution  $y^*$  that is optimal for (5). Assume that  $y^* \notin \mathcal{X}^{PRO}$ , then there would exist  $\bar{p} \in \mathcal{U}$  and  $z \in \mathcal{X}$  with  $f(z, \bar{p}) > f(y^*, \bar{p})$  and  $f(z, p) - f(y^*, p) \geq 0 \forall p \in \mathcal{U}$ . But then,  $y^*$  would not be optimal for (5) – a contradiction.  $\square$

We observe that since the function  $f$  is linear on the convex set  $\mathcal{U}$ , if in addition  $f$  is continuously differentiable on  $\mathcal{X}$ , one could reformulate the minimization problem with its dual cone, KKT-conditions or reformulations given in [3]. This property would be beneficial to solve (5). Another way to determine a PRO solution is given by the following theorem in case one can characterize  $\mathcal{X}^{RO}$  precisely:

**Theorem 2.** *Let  $\hat{p} \in \text{relint}(\mathcal{U})$ . Then  $\text{argmax}_{x \in \mathcal{X}^{RO}} f(x, p)$  is a subset of Pareto robustly optimal solutions of (2).*

*Proof.* Assume that  $x^*$  is a maximizer for a given  $\hat{p} \in \text{relint}(\mathcal{U})$  but  $x^* \notin \mathcal{X}^{PRO}$ . Then, there exists a robustly optimal  $y \in \mathcal{X}^{RO}$  with  $f(x^*, p) \leq f(y, p)$  for all  $p \in \mathcal{U}$  and there exists a  $\bar{p} \in \mathcal{U}$  with  $f(x^*, \bar{p}) < f(y, \bar{p})$ . Similar to the previous proof,  $\hat{p} = \varepsilon \bar{p} + (1 - \varepsilon)p$  for a  $p \in \mathcal{U}$  and  $\varepsilon \in (0, 1)$ . Hence,

$$0 \geq f(y, \hat{p}) - f(x^*, \hat{p}) = \varepsilon(f(y, \bar{p}) - f(x^*, \bar{p})) + (1 - \varepsilon)(f(y, p) - f(x^*, p)) > 0,$$

where the first inequality holds since  $x^*$  was a maximizer of  $f(\cdot, \hat{p})$ .  $\square$

In contrast to the two theorems above, which aim to determine PRO solutions, the following theorem addresses the question whether there exist non-trivial PRO solutions  $x$  for (2), i.e.  $x \in \mathcal{X}^{PRO}$  and  $\mathcal{X}^{PRO} \neq \mathcal{X}^{RO}$ .

**Theorem 3.** *Let  $\hat{p} \in \text{relint}(\mathcal{U})$  and consider the optimization problem*

$$\begin{aligned} & \sup_{x, y} f(y, \hat{p}) - f(x, \hat{p}), \\ & \text{s.t. } \min_{p \in \mathcal{U}} f(y, p) - f(x, p) \geq 0, \\ & y \in \mathcal{X}, \\ & x \in \mathcal{X}^{RO}. \end{aligned} \quad (7)$$

*Then  $\mathcal{X}^{PRO} = \mathcal{X}^{RO}$  if and only if the optimal value of (7) is zero.*

*Proof.* Suppose that the objective value of (7) is strictly positive. Then, there exist feasible  $x^*, y^*$  for (7) with strictly positive objective value. We observe that

$$\min_{p \in \mathcal{U}} f(y^*, p) - f(x^*, p) \geq 0 \text{ and } f(y^*, \hat{p}) - f(x^*, \hat{p}) > 0$$

implies that  $y^*$  Pareto dominates  $x^* \in \mathcal{X}^{RO}$  and thus  $x^* \in \mathcal{X}^{RO} \setminus \mathcal{X}^{PRO}$ . For the opposite direction, we consider an arbitrary  $\bar{x} \in \mathcal{X}^{RO}$  and suppose that the optimal value of (7) is zero. This implies that  $0 \geq \sup_y f(y, \hat{p}) - f(\bar{x}, \hat{p})$  for every  $x \in \mathcal{X}^{RO}$  and hence

$$\begin{aligned} f(\bar{x}, \hat{p}) &\geq \sup_y f(y, \hat{p}), \\ \text{s.t. } \min_{p \in \mathcal{U}} f(y, p) - f(\bar{x}, p) &\geq 0, \\ y &\in \mathcal{X}. \end{aligned}$$

Moreover, equality holds since  $y = \bar{x}$  is a feasible and optimal solution and thus we can apply Theorem 1 to obtain that  $\bar{x} \in \mathcal{X}^{PRO}$  and conclude  $\mathcal{X}^{PRO} = \mathcal{X}^{RO}$ .  $\square$

**2.1. A tractable reformulation for SDPs under linear perturbations.** We illustrate the above results by the example of semidefinite programming with uncertainties that solely affect the cost matrix. In addition, we provide a tractability result for this class of optimization problems. We consider a spectrahedron

$$\mathcal{X} = \{X \in \mathcal{S}^n : X \succeq 0, \langle A_j, X \rangle = b_j, \forall j \in [k]\},$$

and matrix interval uncertainties:

$$\mathcal{U} = \left\{ D_0 + \sum_{i=1}^N \mu_i D_i : \mu \in [\mu^-, \mu^+] \right\}$$

where  $D_0, \dots, D_N \in \mathcal{S}^n, \mu^-, \mu^+ \in \mathbb{R}^N$  are fixed parameters (cf [6]). We observe that since the Frobenius inner product  $f(X, P) = \langle P, X \rangle$  is bilinear, it encodes linearity in  $X$  and in the uncertain parameter  $P$ . Hence, it can be used as an objective function for (2). Together this defines the following case of a robust semidefinite program with interval uncertainties:

$$\begin{aligned} \sup_{X \in \mathcal{S}_{\succeq 0}^n} \min_{P \in \mathcal{U}} \langle P, X \rangle \\ \text{s.t. } \langle A_j, X \rangle = b_j, \quad \forall j \in [k]. \end{aligned} \tag{8}$$

It is worth noting that the above problem formulation differs from the more established ones in, e.g. [6] or [2] by considering uncertainties in the objective instead of uncertainties in the constraints. Although we do not investigate the exact relation between these two approaches here, we want to point out that the considered problem is a semidefinite version of the setting investigated by Beck and Ben-Tal in [1]. We recall that we aim to compute a Pareto robustly optimal solution for (8), i.e. a robustly optimal solution  $X \in \mathcal{X}^{RO}$ , such that there is no other  $\bar{X} \in \mathcal{X}^{RO}$  that satisfies

$$\begin{aligned} \forall P \in \mathcal{U} : \langle P, \bar{X} \rangle &\geq \langle P, X \rangle, \\ \exists \bar{P} \in \mathcal{U} : \langle \bar{P}, \bar{X} \rangle &> \langle \bar{P}, X \rangle. \end{aligned}$$

The following proposition shows how Theorem 1 can be used to compute Pareto robustly optimal solutions to Problem (8).

**Proposition 1.** A solution  $X \in \mathcal{X}^{RO}$  is Pareto robustly optimal for (8) if and only if the optimal value of

$$\begin{aligned} \max_Z \langle \hat{P}, Z \rangle, \\ \text{s.t. } Z \in \mathcal{U}^*, \\ X + Z \in \mathcal{X} \end{aligned} \tag{9}$$

is 0. If it is positive with optimal solution  $Z$ , then  $X + Z \in \mathcal{X}^{PRO}$ . Moreover, the above program computes a PRO solution to (8) in polynomial time.

*Proof.* Applying Theorem 1, one obtains that  $X \in \mathcal{X}^{RO}$  is Pareto robustly optimal if and only if

$$\max_Y \langle \hat{P}, Y \rangle, \quad (10a)$$

$$\text{s.t. } \min_{P \in \mathcal{U}} \langle Y - X, P \rangle \geq 0, \quad (10b)$$

$$Y \in \mathcal{X} \quad (10c)$$

has an optimal value of  $\langle \hat{P}, X \rangle$ . Let  $Z := Y - X$ . Then,  $\langle \hat{P}, Y \rangle \geq \langle \hat{P}, X \rangle$  is equivalent to  $\langle \hat{P}, Z \rangle \geq 0$  and the inequality  $\min_{P \in \mathcal{U}} \langle Y - X, P \rangle \geq 0$  is equivalent to  $Z \in \mathcal{U}^*$ , which proves the first part of the claim. In order to prove tractability for matrix interval uncertainties, we observe

$$\begin{aligned} (10b) &\Leftrightarrow 0 \leq \min_{\mu \in [\mu^-, \mu^+]} \langle Y - X, D_0 \rangle + \sum_{i=1}^N \mu_i \langle Y - X, D_i \rangle \\ &\Leftrightarrow -\langle Y - X, D_0 \rangle \leq \min_{\mu \in [\mu^-, \mu^+]} \sum_{i=1}^N \mu_i \langle Y - X, D_i \rangle \\ &\Leftrightarrow -\langle Y - X, D_0 \rangle \leq \max_{y \in \mathbb{R}_{\geq 0}^{2n}} y^\top \begin{pmatrix} -\mu^+ \\ \mu^- \end{pmatrix} : y^\top \begin{pmatrix} -I_n \\ I_n \end{pmatrix} = \begin{pmatrix} \langle Y - X, D_1 \rangle \\ \vdots \\ \langle Y - X, D_n \rangle \end{pmatrix} \end{aligned}$$

and consequently, Problem (9) can be written as an SDP which is polynomially solvable in its input length:

$$\begin{aligned} &\max_{Y, y} \langle \hat{P}, Y \rangle, \\ &\text{s.t. } y^\top \begin{pmatrix} -\mu^+ \\ \mu^- \end{pmatrix} \geq -\langle Y - X, D_0 \rangle, \\ &\quad y^\top \begin{pmatrix} -I_n \\ I_n \end{pmatrix} = \begin{pmatrix} \langle Y - X, D_1 \rangle \\ \vdots \\ \langle Y - X, D_n \rangle \end{pmatrix}, \\ &\quad Y \in \mathcal{X}, y \in \mathbb{R}_{\geq 0}^{2n}. \end{aligned}$$

□

Thus, we have proven that computing a Pareto robustly optimal solution for robust semidefinite programs (8) with interval uncertainties is tractable. In the following section we illustrate its use for a robust eigenvalue problem and the computation of max-cuts on graphs with uncertain weights.

### 3. APPLICATION I: THE ROBUST MAXIMUM EIGENVALUE PROBLEM

In the following paragraphs, we show that computing the maximal eigenvalue of a set of affine combinations of matrices fits into the setting of (2). The largest eigenvalue problem of a matrix  $C$  can be written as (see, e.g. [14]):

$$\begin{aligned} \lambda_{\max}(C) &= \max_{X \succeq 0} \langle C, X \rangle &= \min_y y \\ &\text{s.t. } \text{Tr}(X) = 1 \ (\Leftrightarrow \langle I_n, X \rangle = 1) &\text{s.t. } yI_n - C \succeq 0. \end{aligned} \quad (11)$$

An optimal matrix  $X$  for the first optimization problem corresponds to the eigenvector  $x$  with respect to the largest eigenvalue  $\lambda_{\max}$  of  $C$  by  $X = xx^\top$ . In the remainder of this section, we consider the following robust variant of (11) with

respect to a compact uncertainty set  $\mathcal{U}$ :

$$\begin{aligned} \lambda_{\max} = \max_{X \succeq 0} \min_{\mu \in [\mu^-, \mu^+]} \langle C(\mu), X \rangle \\ \text{s.t. } \text{Tr}(X) = 1, \end{aligned} \quad (12)$$

where  $C : [\mu^-, \mu^+] \rightarrow \mathcal{U}$  denotes an affine function. Note that for compact and convex uncertainty sets  $\mathcal{U}$ , Sion's minimax theorem [17] allows us to interchange the max and min operators. Thus, if the uncertainty set  $\mathcal{U}$  consists of interval uncertainties as in (8), the problem boils down to minimize the maximal eigenvalue of an affine family of symmetric matrices – a problem with a wide range of applications, e.g. in stability analysis of dynamic systems or the computation of structured singular values, see [7]. In the following example, we provide an instance with non-trivial ( $\mathcal{X}^{PRO} \neq \mathcal{X}^{RO}$ ) Pareto robustly optimal solutions for this eigenvalue problem.

**Example 1.** Let  $C(\mu) \in \mathcal{U} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} : \mu \in [0, 1] \right\}$ . Then, the matrix  $X' = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  is a robustly optimal solution for (12) since for every  $\mu \in [0, 1]$  we have:

$$\langle C(\mu), X \rangle = \langle I_2, X \rangle + \mu \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, X \right\rangle \geq \langle I_2, X \rangle = 1.$$

Thus, for every feasible  $X$ ,  $\mu = 0$  is the worst case realization of uncertainty that can occur. Consequently, every feasible solution  $X$ , such as  $X'$ , is also a robustly optimal solution. However,  $X'$  Pareto dominates every other solution  $X \in \mathcal{X}^{RO}$ , since for every  $\mu > 0$  and  $X \neq X'$ , we have

$$\langle C(\mu), X \rangle = \langle I_2, X \rangle + \mu \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, X \right\rangle < 1 + \left\langle \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, X' \right\rangle = \langle C(\mu), X' \rangle.$$

Note that the existence of more than one robustly optimal solution is non-trivial as for uncorrelated uncertainties, i.e. uncorrelated uncertainty sets for the entries of  $C$ , we often obtain a unique robustly optimal solution. In the above example, the uncertainties in the entries are linked through the matrix  $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  and thus correlated.

#### 4. APPLICATION II: ROBUST MAX CUT

The WEIGHTED MAX CUT problem is one of the fundamental combinatorial problems from Karp's list of 21 NP-complete problems [10]. Given an undirected graph  $G = (V, E)$  equipped with a weight function  $w : E \rightarrow \mathbb{R}$ , the task is to find a cut  $\delta(V') = \{e \in E : |e \cap V'| = 1\}$  defined by  $V' \subseteq V$  with maximal weight, i.e.,

$$mc(G, w) := \max_{V' \subseteq V} \sum_{e \in \delta(V')} w_e = \max_{x \in \{-1, 1\}^V} \frac{1}{4} x^\top L_w x,$$

where  $L_w$  denotes the weighted Laplacian of the graph, i.e.

$$L_w = \sum_{\{i, j\} \in E} w_{ij} E'_{ij} \text{ with } E'_{ij} = E_{ii} + E_{jj} - 2E_{ij}.$$

In combinatorial optimization under uncertainty, it is common to restrict oneself to uncertainties in the objective in order to keep the structure of the underlying combinatorial problem, see [11] for a survey. In the remainder of this section, we consider interval uncertainties in the weights, i.e.,  $w_e(\mu) = w_e + g_e(\mu)$ , where  $g : \mathbb{R}^E \rightarrow \mathbb{R}$  is an affine map of the interval uncertainties  $\mu \in [\mu^-, \mu^+] \subseteq \mathbb{R}^E$  that

encodes a potential correlation. Similar to [13], we define the robust counterpart of the uncertain weighed MAX CUT problem that corresponds to  $mc(G, w)$  by

$$mc(G, w, \mathcal{U}) = \max_{x \in \{-1, 1\}^V} \min_{\mu \in [\mu^-, \mu^+]} \frac{1}{4} x^\top L_w(\mu) x, \quad (13)$$

where  $L_w(\mu) = \sum_{\{i, j\} \in E} w_{ij}(\mu) E'_{ij} = L_w + \sum_{\{i, j\} \in E} g_{ij}(\mu) E'_{ij}$  denotes the uncertain Laplacian. Moreover, since  $g$  is an affine map, the set  $\mathcal{U} = \{L_w(\mu) : \mu \in [\mu^-, \mu^+]\}$  represents an interval uncertainty as in the previous sections.

Again, we address the question whether for a given graph  $G$ , we can improve a robustly optimal solution to (13) in terms of Pareto dominance. In some instances such as  $\gamma$ -stable graphs introduced by Bilu and Linial [5], there exist solutions  $\hat{x}$  that are not only Pareto optimal but moreover ensures that there is no solution  $\bar{x} \in \mathcal{X}$  such that  $\exists \bar{p} \in \mathcal{U} : f(\bar{x}, \bar{p}) > f(\hat{x}, \bar{p})$ . Although our techniques would apply for their instances there are more efficient ways to compute these solutions. However, in general, graphs are not  $\gamma$ -stable and hence we first demonstrate the existence of two optimal solutions to an instance of robust weighted MAX-CUT problem of which one Pareto dominates the other with the following example:

**Example 2.** Consider the complete graph with three nodes equipped with the uncertain weights  $w_{12}(\mu) = w_{13}(\mu) = 4 + 2\mu$  and  $w_{23}(\mu) = 3 + \mu$  that affinely depend on  $\mu$  with  $\mu \in [-1, 1]$ . We observe that

$$8 + 4\mu = w(\delta(v_1)) \geq w(\delta(v_2)) = w(\delta(v_3)) = 7 + 3\mu,$$

where equality holds if and only if  $\mu = -1$ . Since this describes the worst case for all these three cuts, we have that every cut is a robustly optimal solution. However, the cut  $\delta(v_1)$  Pareto dominates the other cuts, since  $w(\delta(v_1)) > w(\delta(v_2)) = w(\delta(v_3))$  whenever  $\mu > -1$ .

We would like to point out that Example 2 is not a classical example since it is not a classical interval uncertainty set which is more common in combinatorial optimization under uncertainty [11]. However, in the context of Pareto optimality of binary programs with a linear objective under interval uncertainty this notion is not interesting as the following proposition illustrates.

**Proposition 2.** Consider problem (2) with  $\mathcal{X} \subseteq \{0, 1\}^n$ ,  $f(x, p) = p^\top x$  and  $\mathcal{U} := [\bar{p} - \Delta p, \bar{p}] \subseteq \mathbb{R}_+^n$  and let  $x^* \in \mathcal{X}^{RO}$ . Then,  $x^* + z$  with  $z \in \{-1, 0, 1\}^n$  Pareto dominates  $x^*$  if and only if  $z_i = -1$  implies  $\Delta p_i = 0$  for all  $i \in [n]$ , there exists at least one  $i \in [n]$  with  $z_i = 1$  and  $\Delta p_i > 0$  and  $x^* + z \in \mathcal{X}^{RO}$ .

*Proof.* Theorem 1 states that  $x^* \in \mathcal{X}^{RO}$  is Pareto robustly optimal if and only if for  $\hat{p} \in \text{relint}(\mathcal{U})$  the mixed integer program

$$\begin{aligned} & \max_z \hat{p}^\top z, \\ & \text{s.t. } z \in \mathcal{U}^*, \\ & \quad x^* + z \in \mathcal{X} \end{aligned} \quad (14)$$

has optimal value 0. We determine the dual cone:

$$\begin{aligned} z \in \mathcal{U}^* & \Leftrightarrow z^\top u \geq 0 \quad \forall u \in \mathcal{U}, \\ & \Leftrightarrow \min_{u \in [\bar{p} - \Delta p, \bar{p}]} z^\top u \geq 0, \\ & \Leftrightarrow \max_{(y, s) \in \mathbb{R}_{\geq 0}^n : y - s = z} (\bar{p} - \Delta p)^\top y - \bar{p}^\top s \geq 0, \\ & \Leftrightarrow \exists y \geq 0 : \bar{p}^\top z - \Delta p^\top y \geq 0, \quad y \geq z, \end{aligned} \quad (15)$$

where we apply strong duality to obtain (15). Since for all  $i \in [n]$ , there exists  $\lambda_i \in (0, 1)$ , such that  $\hat{p}_i = \bar{p}_i - \lambda_i \Delta p_i$ , problem (14) is equivalent to

$$\begin{aligned} \max_{y, z} \quad & \sum_{i \in [n]} (\bar{p}_i - \lambda_i \Delta p_i) z_i, \\ \text{s.t.} \quad & \bar{p}^T z - \Delta p^T y \geq 0, \\ & x^* + z \in \mathcal{X}, \\ & y \geq z, \\ & y \geq 0. \end{aligned} \tag{16}$$

for  $\lambda \in (0, 1)^n$ . Now, assume that  $(y^*, z^*)$  is optimal for (16) and hence  $x^* + z^* \in \mathcal{X}^{PRO}$ . Thus,  $x^* + z^*, x^* \in \mathcal{X}^{RO}$  implying

$$\min_{p \in \mathcal{U}} p^T (x^* + z^*) = \min_{p \in \mathcal{U}} p^T x^*.$$

Since  $p, x^*, x^* + z^*$  are nonnegative, the worst-case uncertainty is attained at  $\bar{p} - \Delta p$  and we obtain  $(\bar{p} - \Delta p)^T (x^* + z^*) = (\bar{p} - \Delta p)^T x^*$  implying  $\bar{p}^T z^* = \Delta p^T z^*$ . Thus, the optimal value of (16) equals

$$\sum_{i \in [n]} (1 - \lambda_i) \Delta p_i z_i^* \tag{17}$$

Since  $z^*$  is feasible for (16), we further obtain

$$\Delta p^T (z^* - y^*) \geq 0$$

and since  $y^* \geq z^*$  and  $\Delta p \geq 0$ ,

$$\Delta p^T (z^* - y^*) \leq 0,$$

finally obtaining  $\Delta p^T (z^* - y^*) = 0$ . Since  $x^* + z^* \in \mathcal{X}$  implies  $z \in \{-1, 0, 1\}^n$ , we obtain that

$$z_i^* = -1 \stackrel{y_i^* \geq 0}{\Rightarrow} z_i^* - y_i^* < 0 \Rightarrow \Delta p_i = 0.$$

Plugging this into (17) implies that

$$\sum_{i \in [n]} (1 - \lambda_i) \Delta p_i z_i^* > 0$$

is equivalent to the existence of at least one  $i \in [n]$  with  $z_i^* = 1$  and  $\Delta p_i > 0$ . Thus, we have proven our claim.  $\square$

This characterization implies that if a solution  $x \in \mathcal{X}^{RO}$  is Pareto dominated by another solution  $x' \in \mathcal{X}^{RO}$  then it contains an entry  $i$  with  $x_i = 1$  and  $\Delta p_i = 0$ . Thus, if  $\Delta p > 0$ , we have that every robustly optimal solution  $x \in \mathcal{X}^{RO}$  is also Pareto optimal, i.e.  $\mathcal{X}^{RO} = \mathcal{X}^{PRO}$ .

Since MAX CUT can be phrased as a binary program by using the cut polytope, the statement above holds true for the robust MAX CUT problem for uncorrelated uncertainties. Although the nominal MAX-CUT problem is widely considered in the literature, its robust counterpart is to the best of our knowledge not well-investigated. For the nominal case, the famous algorithm of Goemans and Williamson [8] enables us to compute a cut that satisfies an  $\alpha$ -approximation ratio with  $\alpha = 0.878...$ . Moreover, if Khot's unique games conjecture [12] holds, this is the best approximation ratio we could hope to achieve with a polynomial time algorithm. In the remainder of this section, we first derive robustly optimal cuts with the same approximation ratio and then apply our results from Section 2 to compute new cuts with improved



approximation guarantees if the worst-case uncertainty is not attained. To this end, we consider the SDP relaxation of (13):

$$\begin{aligned} sdp(G, w, \mathcal{U}) = \max_Y \min_{\mu \in [\mu^-, \mu^+]} & \left\langle \frac{1}{4} L_w(\mu), Y \right\rangle \\ \text{s.t. } & \langle I_{\{i\}}, Y \rangle = 1 \quad \forall i \in [n], \\ & Y \succeq 0. \end{aligned} \quad (18)$$

Since the inner problem in (18) is an LP that can be dualized, we can properly compute a robustly optimal solution to (18) by solving the resulting SDP. This solution can now be used to compute a cut via Goemans-Williamson's Algorithm that guarantees the same approximation ratio for the robust MAX-CUT.

**Proposition 3.** Let  $w_{ij}(\mu) = w_{ji}(\mu) \geq 0$  for every  $\mu \in [\mu^-, \mu^+]$  and  $\bar{Y}$  be a robust optimal solution to (18). Then,

$$\min_{u \in [\mu^-, \mu^+]} \left\langle \frac{L_w(u)}{4}, \bar{Y} \right\rangle = sdp(G, w, \mathcal{U}) \geq mc(G, w, \mathcal{U}) \geq 0.878 \dots sdp(G, w, \mathcal{U}).$$

*Proof.* The first inequality follows by a simple relaxation argument. For the second inequality we strictly follow the arguments of Goemans and Williamson [8]:

Let  $\bar{y}_k$  denote the columns of the Cholesky decomposition of  $\bar{Y}$ . Then, we observe that  $x \in \{-1, 1\}^V$  defined by  $x_k = \text{sign}(\bar{y}_k^\top r)$  forms a cut in  $G$ . The proof of Goemans and Williamson then relies on the fact that for vectors  $r \in S^{n-1}$  drawn from the rotationally invariant probability distribution on the unit sphere and their corresponding cuts, we have that

$$\mathbb{E}(1 - x_i x_j) \geq 0.878 \dots (1 - \bar{y}_i^\top \bar{y}_j) = 0.878 \dots sdp(G, w, \mathcal{U}).$$

Finally, we conclude

$$\begin{aligned} \mathbb{E} \left( \min_{\mu \in [\mu^-, \mu^+]} \frac{1}{4} x^\top L_w(\mu) x \right) &= \mathbb{E} \left( \min_{\mu \in [\mu^-, \mu^+]} \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu) (1 - x_i x_j) \right) \\ &= \min_{\mu \in [\mu^-, \mu^+]} \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu) \mathbb{E}((1 - x_i x_j)) \\ &\geq 0.878 \dots \min_{\mu \in [\mu^-, \mu^+]} \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu) (1 - \bar{y}_i^\top \bar{y}_j) \\ &= 0.878 \dots sdp(G, w, \mathcal{U}). \end{aligned}$$

□

It is worth noting that there are already similar results for certain uncorrelated uncertainties known, see e.g. [16] for general robust combinatorial problems with linear interval uncertainties. Thus, the above result on the one hand considers more general uncertainty sets but on the other hand is restricted to the MAX-CUT problem. We observe that the quality of a cut in a graph with uncertain edge weights may not only rely on its performance in a worst case scenario but also on its performance in every other scenario  $L_w(\mu) \in \mathcal{U}$ . Hence, we show that a Pareto optimal solution  $Y^*$  to (13) outperforms any other robustly optimal solution  $\bar{Y}$  of  $sdp(G, w, \mathcal{U})$  in terms of the approximation ratio of their corresponding cuts:

**Proposition 4.** Let  $Y^*$  Pareto dominate  $\bar{Y}$  for (18) and let  $x$  and  $\bar{x}$  denote the corresponding cuts derived from  $Y^*$  and  $\bar{Y}$  respectively via the Goemans-Williamson Algorithm. Denote

$$sdp(G, w, \mu, Y) = \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu) (1 - y_i^\top y_j).$$

Then, for every  $L_w(\mu) \in \mathcal{U}$  we have

$$mc(G, w, \mu) \geq 0.878 \dots \text{sdp}(G, w, \mu, Y^*) \geq 0.878 \dots \text{sdp}(G, w, \mu, \bar{Y})$$

and there exists a  $L_w(\mu) \in \mathcal{U}$ , for which the last inequality holds strictly.

*Proof.*

$$\begin{aligned} \mathbb{E} \left( \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu)(1 - x_i x_j) \right) &= \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu) \mathbb{E}((1 - x_i x_j)) \\ &\geq 0.878 \dots \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu)(1 - (y_i^*)^\top y_j^*) \\ &\geq 0.878 \dots \frac{1}{4} \sum_{\{i,j\} \in E} w_{ij}(\mu)(1 - \bar{y}_i^\top \bar{y}_j), \end{aligned}$$

where the last inequality and its strict counterpart for at least one realization of the uncertain parameter follows from the Pareto dominance of  $Y^*$ .  $\square$

## 5. CONCLUSION

In this paper, we generalized the methods introduced in [9] to determine Pareto robustly optimal solutions for linear programs with an uncertain objective to general optimization problems whose objective function is affected linearly by the uncertainty. Moreover, we proved the tractability of these methods in the case of semidefinite programming with matrix box uncertainties and illustrated their use at the examples of the maximal eigenvalue of an affine set of matrices and the classical MAX CUT problem.

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