

ROBUST STATIC AND DYNAMIC MAXIMUM FLOWS

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ABSTRACT. We study the robust maximum flow problem and the robust maximum flow over time problem where a given number of arcs Γ may fail or may be delayed. Two prominent models have been introduced for these problems: either one assigns flow to arcs fulfilling weak flow conservation in any scenario, or one assigns flow to paths where an arc failure or delay affects a whole path. We provide a unifying framework by presenting novel general models, in which we assign flow to subpaths. These models contain the known models as special cases and unify their advantages in order to obtain less conservative robust solutions.

We give a thorough analysis with respect to complexity of the general models. In particular, we show that the general models are essentially NP-hard, whereas, e.g. in the static case with $\Gamma = 1$ an optimal solution can be computed in polynomial time. Further, we answer the open question about the complexity of the dynamic path model for $\Gamma = 1$. We also compare the solution quality of the different models. In detail, we show that the general models have better robust optimal values than the known models and we prove bounds on these gaps.

1. INTRODUCTION

In this work, we study the robust maximum flow problem in a network with arc failures or delays on arcs. An optimal solution to this problem maximizes the flow reaching a sink t from a source s in the case that at most Γ arcs might fail or be delayed, following the prominent Γ -approach in robust optimization [BS03]. The robust maximum flow problem under arc failure has been subject of various publications in which different models are discussed. In [ACN01], the authors present a polynomial time algorithm for computing a solution maximizing the remaining flow when one arc fails. This model is extended to multiple arc failures by considering the s - t -path decomposition of a flow in [DC07]. In this model, an arc failure has the effect that the flow on all paths that contain the failing arc is deleted. The goal is to find a flow and its s - t -path decomposition such that the guaranteed remaining flow is maximized even if Γ arcs fail. We call this formulation path model (PM) in the following. For $\Gamma = 2$, [DC07] claimed that the problem is NP-hard. However, [DM20] pointed out a flaw in the arguments, so that the complexity of (PM) for fixed $\Gamma \geq 2$ is still open. Further, [DM20] show that the problem is NP-hard if Γ is part of the input. Another model for the robust maximum flow problem, which we call arc model (AM), is given in [BNS13]: flow is assigned to arcs and has to satisfy weak flow conservation at every node in every possible scenario, which makes this model quite restrictive. However, a great advantage of this model is that an optimal solution can be found in polynomial time for arbitrary Γ . Apart from that, [BNS13] introduce an adaptive model and also propose an approximation for (PM), which was later improved in [BNO16]. Our main contribution to this setting is the introduction of a new general model (GM), which combines the properties of the arc model (AM) and the path model (PM) and which covers

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them as special cases: instead of assigning flow to arcs or s - t -paths, we assign flow to subpaths. Therefore, (GM) extends our understanding of robust maximum flows and, in particular, on the connection between (PM) and (AM), which seemed to be unrelated. We extend most of the NP-hardness results for (PM) given in [DM20] to (GM). We also show that (GM) remains solvable in polynomial time in many special cases for which (PM) is solvable in polynomial time, for example if only one arc may fail. Additionally, we provide comparisons of the robust optimal values of the three models and present instances with specific graph structures for which (GM) indeed yields better robust optimal values, highlighting the relevance of the new model.

The flows in the former setting are independent of time and are thus called static flows. However, in many applications a time component has to be considered. In detail, every arc in the network is associated with a transit time, or travel time, the flow needs to traverse the arc. We extend our study to these dynamic flows, in the literature often also called flows over time. In the nominal dynamic maximum flow problem, the goal is to maximize the flow that reaches the sink within a given time horizon. This problem was first introduced by Ford and Fulkerson in [FF58]. For a more recent overview, we refer to e.g. [Sku09]. To the best of our knowledge, there are only two publications on the theory of robust dynamic maximum flows. In [BNS13], an adaptive robust model is introduced for the case that arcs might fail. In [Got+18], one instead assumes that at most Γ arcs are delayed. Clearly, if this delay is greater than the time horizon, this corresponds to arc failure. The model in [Got+18], which we call dynamic path model (DPM), is path-based and can be seen as a generalization of (PM) to flows over time. [Got+18] prove several results on the complexity of (DPM) and they also show that temporally repeated flows, a solution concept in the nominal case, are not optimal in the robust setting but satisfy certain approximation guarantees.

In the dynamic setting, the path model (DPM) is quite restrictive due to the strict flow conservation and capacity constraints for any possible delay and therefore yields very conservative solutions as mentioned in [Got+18]. We address this issue by introducing two new models, the dynamic arc model (DAM) and the dynamic general model (DGM). These models are analogous extensions of their static counterparts. Again, we show that (DGM) contains (DPM) and (DAM) as special cases and therefore is a relaxation of them. In terms of the computational complexity of the models, we answer an open question from [Got+18] by proving that (DPM) is already NP-hard for one delayed arc, which is in contrast to the polynomial time solvability in the static case. Further, we clarify the complexity of the new models. While (DGM) is NP-hard in all considered cases, (DAM) is solvable in polynomial time regardless of the number of delayed arcs. As in the static case, we compare the robust values of all three models and show that (DGM) yields the best robust optimal values on all instances compared to the other models. Thus, we extend the analysis of robust dynamic maximum flows to a similar basis as for the static case.

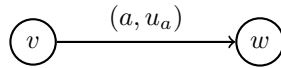
The robust maximum flow problem under arc failure can be seen as the opposite version of the network interdiction problem. While in the robust flow problem the maximum flow is sought under the knowledge that Γ arcs will be deleted, in the network interdiction problem the task is to find a subset of Γ arcs whose removal minimizes the maximum flow value in the remaining network, cf. [Wol64; Woo93], or [CZ17] for recent hardness results. Another related problem is the so called maximum multiroute flow problem, cf. [Kis96]. A solution to this problem is expressed as the conic combination of flows on arc-disjoint paths and thus the failure of an arc only destroys a certain fraction of the total flow. [BS14] obtain a $(\Gamma + 1)$ -approximation to (PM) with this concept. Establishing appropriate robust flow models and clarifying algorithmic complexities is also relevant for the study of more general tasks where a minimum-cost robust

layout of a network is to be determined, cf. [Mat13; Cac+16].

This paper is structured as follows: in the next section we state necessary definitions and notation that we will use throughout the paper. In Section 3, we consider robust static maximum flows. We present the two known robust maximum flow models, (PM) and (AM), in Section 3.1 and introduce the new model (GM). We prove several results on the complexity of (GM) in Section 3.2. In Section 3.3, we compare the solution quality of the three models. The section on robust dynamic maximum flows, Section 4, is structured similarly. In Section 4.1, we discuss (DPM) and state the new models, (DAM) and (DGM). We present complexity results for the three models in Section 4.2. In Section 4.3, we compare the solution quality of the models, followed by a brief discussion of possible approximation approaches for (DGM) in Section 4.4. Finally, in Section 5, we collect open questions and discuss further research directions.

2. DEFINITIONS AND NOTATION

We consider a directed graph $G = (V, A, s, t, u)$ with two distinct nodes in V , a source s and a sink t , and positive integral capacities $u_a \in \mathbb{N}$ for all arcs $a \in A$. In all figures, if not stated otherwise, arc labels denote the arc name and the capacity of the arc as depicted below.



The source s has no incoming arcs whereas the sink t has no outgoing arcs and additionally there is a directed s - v - t -path for every node $v \in V \setminus \{s, t\}$.

In the following, we assume that at most $\Gamma \in \mathbb{N}$ arcs may fail or be delayed. We define the resulting set of all possible scenarios by

$$\mathcal{S} := \{S \subseteq A: |S| \leq \Gamma\}$$

and its corresponding indicator set for the considered value of Γ by

$$\Lambda := \{z \in \{0, 1\}^{|A|}: \sum_{a \in A} z(a) \leq \Gamma\}.$$

We denote the set of all simple s - t -paths by \mathcal{P} and the set of all subpaths in \mathcal{P} by

$$\bar{\mathcal{P}} := \{v\text{-}w\text{-path } P: v, w \in V, \exists s\text{-}t\text{-path } P' \text{ s.t. } P \subseteq P'\}.$$

We use the standard notation to denote the set of incoming arcs at a node v by $\delta_A^-(v)$ and the set of outgoing arcs by $\delta_A^+(v)$. Analogously, we write $P \in \delta_{\bar{\mathcal{P}}}^+(v)$ or $P \in \delta_{\bar{\mathcal{P}}}^-(w)$ if the path $P \in \mathcal{P}$ starts at v or ends at w , respectively.

We denote flow by a vector x . The dimension of x depends on the currently considered model and can be the number of arcs, the number of s - t -paths or the number of subpaths in the graph. The nominal value of a flow is the amount that enters t if no arc fails, i.e. for the example of $x \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}$ this is given by

$$f(x) := \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t)} x(P).$$

The nominal maximum flow problem maximizes this value satisfying strict flow conservation at the nodes and capacity constraints at the arcs:

$$\begin{aligned} f^* &:= \max_{x \in \mathbb{R}_{\geq 0}^{|A|}} \sum_{a \in \delta_A^-(t)} x(a) \\ \text{s.t.} \quad &\sum_{a \in \delta_A^-(v)} x(a) = \sum_{a \in \delta_A^+(v)} x(a), \quad \forall v \in V \setminus \{s, t\}, \\ &x(a) \leq u_a, \quad \forall a \in A. \end{aligned}$$

We hence denote the nominal maximum flow value on a graph by f^* . Note that relaxing the strict flow conservation to weak flow conservation, meaning that the inflow in each node can be larger than the outflow, does not improve the optimal value. Further, for any flow defined on arcs, an s - t -path decomposition can be found in polynomial time. For an extensive overview on flow problems and algorithms, we refer to [AMO93].

Further, we write in the following for a positive integer $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$, and set $[n] := \emptyset$ for all $n \in \mathbb{Z} \setminus \mathbb{N}$.

3. ROBUST MAXIMUM FLOWS

In the following, we present different models for the robust maximum flow problem under arc failure, investigate their complexity and study solution quality gaps as well as the price of robustness.

3.1. Robust flow models.

Path model (PM). One of the first maximum flow models dealing with arc failures was introduced in [ACN01]. There, the goal is to maximize the flow under the assumption that one arc may fail. The model was later generalized in [DC07] to the case of $\Gamma \geq 1$. The overall flow is decomposed into flow on s - t -paths, so that the flow variable x has one entry per s - t -path. If an arc fails, the flow on each path that contains the arc is deleted. The corresponding problem, which we call path model (PM), reads as follows.

$$f_{pm}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|P|}} \sum_{P \in \mathcal{P}} x(P) - \max_{S \in \mathcal{S}} \sum_{\substack{P \in \mathcal{P}: \\ P \cap S \neq \emptyset}} x(P) \quad (\text{PM.1})$$

$$\text{s.t.} \quad \sum_{\substack{P \in \mathcal{P}: \\ a \in P}} x(P) \leq u_a, \quad \forall a \in A. \quad (\text{PM.2})$$

While for $\Gamma = 1$, this problem is solvable in polynomial time, cf. [ACN01], it is NP-hard to find an optimal solution if Γ is part of the input, cf. [DM20]. For fixed $\Gamma \geq 2$, the complexity of the model is unknown.

Arc model (AM). In the arc model introduced in [BNS13], flow is assigned to each arc so that the flow variable x has $|A|$ entries. Weak flow conservation has to hold at every node and for every possible scenario of at most Γ many arc failures. It is hence required that every node has non-negative outflow even if Γ of the incoming arcs fail. We call this property robust flow conservation in the following. The corresponding arc model (AM) is given by

$$f_{am}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|A|}} \sum_{a \in \delta_A^-(t)} x(a) - \max_{z \in \Lambda} \sum_{a \in \delta_A^+(t)} z(a)x(a) \quad (\text{AM.1})$$

$$\text{s.t.} \quad \sum_{a \in \delta_A^-(v)} (1 - z(a))x(a) - \sum_{a \in \delta_A^+(v)} x(a) \geq 0, \quad \forall v \in V \setminus \{s, t\}, z \in \Lambda, \quad (\text{AM.2})$$

$$x(a) \leq u_a, \quad \forall a \in A. \quad (\text{AM.3})$$

[BNS13] show that (AM) is equivalent to a linear program, where the number of variables and constraints is polynomial in $|V|$ and $|A|$. Therefore, an optimal solution to (AM) can be computed in polynomial time.

General model (GM). We now introduce a new general model that combines the properties of (AM) and (PM). The flow is assigned to paths in the set \mathcal{P} of all possible subpaths of s - t -paths. As in the path model, we assume that the flow on all paths that contain a failing arc is deleted. Additionally, we require weak flow conservation and hence non-negative outflow at every node in every possible scenario as in the arc model, i.e. robust flow conservation as in (AM). The general model (GM) reads as follows.

$$f_{gm}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}} \sum_{P \in \delta_{\mathcal{P}}^-(t)} x(P) - \max_{S \in \mathcal{S}} \sum_{\substack{P \in \delta_{\mathcal{P}}^-(t): \\ P \cap S \neq \emptyset}} x(P) \quad (\text{GM.1})$$

$$\text{s.t.} \quad \sum_{P \in \delta_{\mathcal{P}}^-(v)} x(P) - \sum_{\substack{P \in \delta_{\mathcal{P}}^-(v): \\ P \cap S \neq \emptyset}} x(P) \geq \sum_{P \in \delta_{\mathcal{P}}^+(v)} x(P), \quad \forall v \in V \setminus \{s, t\}, S \in \mathcal{S}, \quad (\text{GM.2})$$

$$\sum_{\substack{P \in \mathcal{P}: \\ a \in P}} x(P) \leq u_a, \quad \forall a \in A. \quad (\text{GM.3})$$

Given a feasible solution x to (GM), we denote its objective value by $f_R(x)$. We denote an optimal solution by x_{gm}^* with optimal value $f_{gm}^* = f_R(x_{gm}^*)$. We observe that any feasible solution x_{pm} to (PM) can be easily extended to a feasible solution to (GM) by setting

$$x_{gm}(P) := \begin{cases} x_{pm}(P), & P \in \mathcal{P}, \\ 0, & \text{else.} \end{cases}$$

For ease of notation, we will assume that $x_{pm} \in \mathbb{R}^{|\mathcal{P}|}$ when we refer to (PM) and $x_{pm} \in \mathbb{R}^{|\bar{\mathcal{P}}|}$ when we refer to (GM). It follows that $f_{pm}^* = f_R(x_{pm}^*)$, where x_{pm}^* denotes an optimal solution to (PM). Similarly, all feasible solutions to (AM) can be extended to feasible solutions to (GM). Again, we have $f_{am}^* = f_R(x_{am}^*)$, where x_{am}^* denotes an optimal solution to (AM). It follows that the new model (GM) in fact is a generalization of the two previous models since they are special cases in which the solutions are restricted to s - t -paths or to paths consisting of single arcs, respectively.

By adding an additional variable λ , the max-min-problem (GM) can be written as the linear program

$$f_{gm}^* = \max_{x \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|}, \lambda \in \mathbb{R}} \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t)} x(P) - \lambda \quad (1a)$$

$$\text{s.t.} \quad \sum_{\substack{P \in \delta_{\bar{\mathcal{P}}}^-(t): \\ P \cap S \neq \emptyset}} x(P) - \lambda \leq 0, \quad \forall S \in \mathcal{S}, \quad (1b)$$

$$\sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v)} x(P) - \sum_{\substack{P \in \delta_{\bar{\mathcal{P}}}^-(v): \\ P \cap S \neq \emptyset}} x(P) \geq \sum_{P \in \delta_{\bar{\mathcal{P}}}^+(v)} x(P), \quad \forall v \in V \setminus \{s, t\}, S \in \mathcal{S}, \quad (1c)$$

$$\sum_{\substack{P \in \bar{\mathcal{P}}: \\ a \in P}} x(P) \leq u_a, \quad \forall a \in A. \quad (1d)$$

For an optimal solution (x^*, λ^*) to (1), it holds that x^* is an optimal solution to (GM) and $\lambda^* = \max_{S \in \mathcal{S}} \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t): P \cap S \neq \emptyset} x^*(P)$ is the amount of flow that is deleted in the worst case. Clearly, the same rewriting can also be applied to (PM) and (AM).

3.2. Complexity of (GM). We now analyze the complexity of (GM) for different choices of u and Γ . We start with the simplest case, namely $\Gamma = 1$, in which only one arc may fail. Further, we extend several hardness results from [DM20] to our setting. Thereby, we exploit the fact that (PM) is a special case of the general model. An overview of the complexity of the three robust flow models can be found in Table 1.

	Arc Model (AM)	Path Model (PM)	General Model (GM)
$\Gamma = 1$	poly time [BNS13]	poly time [ACN01]	poly time (Thm. 3.2)
fixed $\Gamma \geq 2$	poly time [BNS13]	?	?
Γ arb.	poly time [BNS13]	strongly NP-hard [DM20]	strongly NP-hard (Thm. 3.4)
Γ arb., $u \equiv 1$	poly time [BNS13]	poly time [DM20]	poly time (Prop. 3.5)
Γ arb., $u \in \{1, \infty\}$	poly time [BNS13]	NP-hard [DM20]	NP-hard (Prop. 3.6)

TABLE 1. Overview of the complexity results of the robust static flow models.

We first state an auxiliary result, which we use in the subsequent complexity proofs. The result formalizes the following observation: if the number of incoming arcs of a node $w \in V$ is not larger than Γ , then there is a scenario in which all incoming arcs of w fail. Hence, in a feasible solution to (GM), no flow can be sent on paths starting at w due to robust flow conservation (GM.2). As a consequence, all flow sent on paths ending at w can be discarded without reducing the robust flow value.

Lemma 3.1. *Let $|\delta_A^-(w)| \leq \Gamma$ for a node $w \in V \setminus \{s, t\}$ and let x be a feasible solution to (GM). Then, $x(P) = 0$ for all $P \in \delta_{\bar{\mathcal{P}}}^+(w)$. Furthermore, if $x(P) > 0$ for some $P \in \delta_{\bar{\mathcal{P}}}^-(w)$, then the solution x' defined by*

$$x'(P) := \begin{cases} 0, & P \in \delta_{\bar{\mathcal{P}}}^+(w), \\ x(P), & P \in \bar{\mathcal{P}} \setminus \delta_{\bar{\mathcal{P}}}^-(w), \end{cases}$$

is also feasible to (GM) and has the same robust flow value as x .

As mentioned above, the in general NP-hard path model is solvable in polynomial time for $\Gamma = 1$. Although the general model at first glance seems more complex considering the set of variables and constraints, we now show that (GM) lies in the same complexity class and hence is solvable in polynomial time for $\Gamma = 1$.

Theorem 3.2. *For $\Gamma = 1$, the optimal value of (GM) can be computed by solving the following linear program of polynomial size*

$$f_{gm}^* = \max_{\substack{y \in \mathbb{R}_{\geq 0}^{|V|^2|A|}, \\ \nu \in \mathbb{R}}} \sum_{a \in A} \sum_{v \in V} y(a, v, t) - \nu \quad (2a)$$

$$s.t. \quad \sum_{v \in V} y(a, v, t) - \nu \leq 0, \quad \forall a \in A, \quad (2b)$$

$$\begin{aligned} \sum_{a \in \delta_A^-(v')} \sum_{v \in V} y(a, v, v') - \sum_{v \in V} y(a', v, v') \\ \geq \sum_{a \in \delta_A^+(v')} \sum_{w \in V} y(a, v', w), \quad \forall v' \in V, a' \in A, \end{aligned} \quad (2c)$$

$$\sum_{a \in \delta_A^-(v')} y(a, v, w) = \sum_{a \in \delta_A^+(v')} y(a, v, w), \quad \forall v' \in V \setminus \{s, t\}, v, w \in V \setminus \{v'\}, \quad (2d)$$

$$\sum_{v \in V} \sum_{w \in V} y(a, v, w) \leq u_a, \quad \forall a \in A. \quad (2e)$$

Given a feasible solution (y, ν) to (2), a feasible solution to (GM) with same objective value can be computed in polynomial time.

Hence, (GM) can be solved in time polynomial in the size of the graph G if $\Gamma = 1$.

Proof. First, let (x, λ) be a feasible solution to problem (1), which is the linear programming equivalent of (GM). We now construct a feasible solution (y, ν) to (2) with same objective value. We set $\nu := \lambda$ and

$$y(a, v, w) := \sum_{\substack{P \in (\delta_P^+(v) \cap \delta_P^-(w)): \\ a \in P}} x(P) \quad \forall a \in A, v, w \in V,$$

i.e. $y(a, v, w)$ denotes the total flow on a of paths starting at v and ending at w . Thus, for any $a \in A$ we have

$$\sum_{v \in V} y(a, v, t) = \sum_{\substack{P \in \delta_P^-(t): \\ a \in P}} x(P). \quad (3)$$

Hence, (x, λ) for (1) and (y, ν) for (2) have the same objective value. Furthermore, (1b) implies (2b). Additionally, in (3) we can replace t by any node $v' \in V$ and thus robust flow conservation (1c) for x implies (2c) for y . Strict flow conservation (2d) for $y(\cdot, v, w)$ holds due to the fact that x also fulfills strict flow conservation within each path from v to w . Finally, y is non-negative since x is non-negative and y fulfills the capacity constraint (2e) due to (1d). Thus, (y, ν) is feasible for (2) with the same objective value as (x, λ) for (1).

On the other hand, let (y, ν) now be an arbitrary feasible solution to (2). Constraint (2d) ensures that $y(\cdot, v, w)$ is a flow from v to w satisfying strict flow conservation. Therefore, there exists a decomposition of this v - w -flow into flow on cycles and simple v - w -paths. Let $x \in \mathbb{R}^{|P|}$

be the union of these path decompositions, for all pairs $v, w \in V$, where we discard flow on cycles. Further, we set $\lambda := \nu$. We now show that (x, λ) is feasible for (1) with the same objective value as (y, ν) in (2). From the definition of x as path decomposition it follows that x is non-negative and fulfills the capacity constraint (1d). Furthermore, for any $a \in A$ we have $\sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t): a \in P} x(P) = \sum_{v \in V} y(a, v, t)$ and therefore $\sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t)} x(P) = \sum_{a \in \delta_A^-(t)} \sum_{v \in V} y(a, v, t)$. Thus, (x, λ) has the same objective value as (y, ν) . (1b) holds due to (2b). Finally, due to strict flow conservation of $y(\cdot, v, w)$ for every pair $v, w \in V$, for any node $v' \in V \setminus \{s, t\}$ it holds that

$$\sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v')} x(P) - \sum_{P \in \delta_{\bar{\mathcal{P}}}^+(v')} x(P) = \sum_{a \in \delta_A^-(v')} \sum_{v \in V} y(a, v, v') - \sum_{a \in \delta_A^+(v')} \sum_{w \in V} y(a, v', w).$$

Additionally, by definition of x , for any $a \in A$ we have $\sum_{v \in V} y(a, v, v') = \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v'): a \in P} x(P)$ so that (2c) implies (1c). Hence, (x, λ) is feasible for (1).

Summarizing, for every feasible solution (y, ν) to (2) there exists a feasible solution (x, λ) to (1) with same objective value and vice versa. Thus, (2) has an optimal value of f_{gm}^* . Since a path decomposition of a flow can be computed in polynomial time, we can furthermore compute an optimal solution x_{gm}^* to (GM) in polynomial time given a feasible solution (y, ν) to (2). \square

In addition to the complexity of (GM) for $\Gamma = 1$, we next show that in this case there is a nominal optimal solution which is robust optimal. A similar fact has been shown for adaptive robust flows in [BNS13].

Theorem 3.3. *For $\Gamma = 1$, there is an optimal solution x_{gm}^* to (GM) which is also nominal optimal, i.e. $f(x_{gm}^*) = f^*$.*

The proof of Theorem 3.3 is quite technical and can be found in Appendix A.1.

The following result states NP-hardness of (GM) when Γ is arbitrary.

Theorem 3.4. *(GM) is strongly NP-hard if Γ is part of the input.*

Proof. In [DM20, Theorem 2, proof], it is proven by a reduction from CLIQUE that (PM) is strongly NP-hard if Γ is part of the input. We use the same construction and extend their idea to our setting: they construct an instance $I = (G = (V, A, s, t, u), \Gamma)$ for (PM) from a given CLIQUE instance $I' = (G' = (V', A'), \Gamma')$ and show that an optimal solution to (PM) sends flow on a specific arc $(v', v'') \in A$ if and only if there exists a clique of size Γ' in G' .

Let x_{gm}^* be an arbitrary optimal solution to (GM) on the instance I . In the following, we will observe that every node $v \in V \setminus \{t\}$ has less than or equal to Γ incoming arcs. We will then apply Lemma 3.1 to construct an optimal solution x_{pm}^* to (PM) on I . First, we observe that every node $v \in V \setminus (B \cup \{s, t\})$, where $B \subset V$ is defined as in [DM20, Theorem 2, proof], has one or two incoming arcs. Second, from the construction of G and the definition of Γ as in [DM20, Theorem 2, proof], we derive for the nodes $b \in B$:

$$|\delta_A^-(b)| \leq 2 + 2|A'| \leq \Gamma.$$

Thus, from Lemma 3.1 it follows that $x_{gm}^*(P) = 0$ for all v - w -paths P in G with $v \neq s$.

Additionally, it follows that we can construct a solution x^*

$$x^*(P) := \begin{cases} x_{gm}^*(P), & P \in \mathcal{P}, \\ 0, & P \in \bar{\mathcal{P}} \setminus \mathcal{P}, \end{cases}$$

with the same robust objective value as x_{gm}^* . The solution x^* only uses s - t -paths and is therefore an optimal solution to (PM).

Summarizing, by solving (GM) on I , we can find in a time, which is polynomial in the size of the original solution, an optimal solution to (PM) by which we can decide whether there is a clique of size Γ' in G' . \square

Restrictions on the capacity u . In the following, we investigate the complexity of (GM) for several special cases in which the capacity u is restricted to certain values. Again, the results are based on analogous statements in [DM20]. We start with the special case in which all arcs have unit capacity.

Proposition 3.5. *Let $u \equiv 1$. Then, for arbitrary $\Gamma \geq 1$, an optimal solution to (GM) can be determined in polynomial time.*

Proof. The proof works analogously to the proof of Theorem 7 in [DM20]. Let C be the value of the minimal cut of a given instance. At most C units of flow can arrive at the sink in the nominal case. Due to the unit capacities, at most Γ units of flow may be deleted. If $\Gamma > C$, then no flow arrives at the sink in the robust case. If $C > \Gamma$, exactly Γ of the C flow units are deleted and the robust flow value equals $C - \Gamma$. Hence, a nominal optimal solution is also optimal for (GM). \square

Proposition 3.6. *Let Γ be part of the input and let capacities $u_a \in \{1, u_{\max}\}$ or $u_a \in \{1, \infty\}$ for all $a \in A$. Then, (GM) is NP-hard.*

The proof uses the analogous result for the problem with arbitrary capacities, following the proof for (PM) in [DM20]. For this, we exploit an idea that is similar to Lemma 3.1, and which we apply in a similar way as in the proof of Theorem 3.4. We refer to Appendix A.2.

Integral flows. So far, all flows were allowed to take fractional values. In the nominal maximum flow problem, there always exists an optimal flow with only integral values, as long as the capacities are integral, cf. [AMO93, Theorem 6.5]. Such an optimal integral flow usually can be found by standard flow algorithms, e.g. the well known Ford-Fulkerson algorithm, cf. [FF56], augments the flow by an integer value in each iteration. In the various robust models, however, there is not necessarily an optimal solution with integral values. For example, on the instance

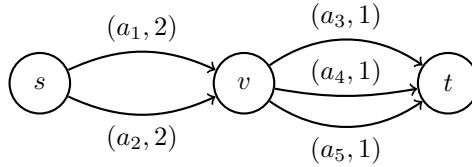


FIGURE 1. Instance with $f_{am}^* = 4/3$, $f_{pm}^* = 3/2$ and $f_{gm}^* = 2$ for $\Gamma = 1$.

in Figure 1, the unique optimal solution to (AM) sends $2/3$ units of flow on each arc from v to t , every optimal solution to (PM) uses at least two s - t -paths with fractional values, and the optimal solution to (GM) sends $1/3$ units of flow on each s - t -path as well as on each arc from v to t .

Therefore, it is of interest to study the complexity of the integral models in which we require all entries of the flows to be integral. To the best of our knowledge, there are no results on the complexity of integral (AM). For integral (PM), [ACN01] show that computing an optimal solution is possible in polynomial time if $\Gamma = 1$. Moreover, [DM20] prove that an optimal integral flow can be found in polynomial time for arbitrary Γ if the capacities are restricted to $u_a \leq 2$ for all $a \in A$. The complexity of integral (GM) remains an open question for these special cases, whereby we note that the proof techniques by [DM20] are not directly transferable to integral (GM). On the other hand, [DM20] also show that integral (PM) is NP-hard and that there is no $(3/2 - \varepsilon)$ -approximation for integral (PM) even for $\Gamma = 2$ and instances with $u_a \leq 3$ for all $a \in A$. We extend this result to integral (GM).

Theorem 3.7. *Unless $P = NP$, there is no $(3/2 - \varepsilon)$ -approximation algorithm for computing an optimal solution to integral (GM), even when restricted to instances where $\Gamma = 2$ and $u_a \leq 3$ for all $a \in A$.*

The proof extends the corresponding proof for integral (PM) from [DM20] and can be found in Appendix A.3. The proof utilizes the fact that, for an integral solution, single flow units cannot be distributed among multiple paths, which leads to a reduction from ARC-DISJOINT PATHS.

3.3. Comparison of the solution quality between the robust flow models. As discussed in Section 3.1, all feasible solutions to (AM) and (PM) are feasible for (GM). Therefore, it obviously holds that the optimal value of the general model is always greater than or equal to the optimal value of the arc model and the path model. We first show that there is no bound on how much better the optimal value of (PM) and (GM) can be compared to (AM), i.e. the extent to which the arc model is surpassed by the others is unbounded. Second, we provide a lower bound on the factor by which the arc model as well as the general model may outperform the path model. Thus, there are instances where the path model achieves a greater optimal value, but also those where the arc model leads to a greater one, so that neither of the two models can be considered to be better in general.

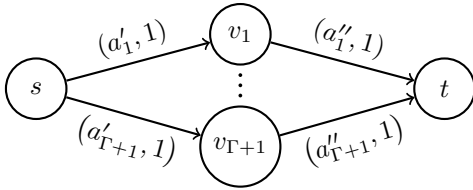
Furthermore, we conjecture that the above mentioned lower bound on the gap between (PM) and (GM) is tight. In Theorem 3.11, we prove this for the special case that G is a DAG and only one arc may fail, i.e. $\Gamma = 1$.

Proposition 3.8. *For any $\Gamma \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ there are instances such that $f_{pm}^* > \alpha f_{am}^*$ and consequently $f_{gm}^* > \alpha f_{am}^*$.*

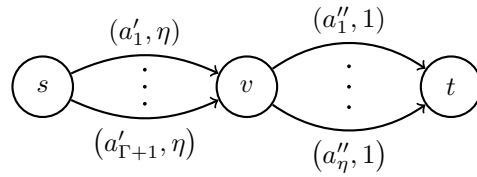
Proof. For an arbitrary $\Gamma \geq 1$, we construct an instance $I = (G, \Gamma)$ such that $f_{pm}^* = 1$ and $f_{am}^* = 0$, cf. Figure 2a. The constructed graph G consists of the nodes s , t , and v_i , $i \in [\Gamma + 1]$. For all $i \in [\Gamma + 1]$, s is connected to v_i by an arc a'_i and v_i is connected to t by an arc a''_i . All arcs have unit capacity.

In all feasible solutions x_{am} to (AM) it holds that $x_{am}(a''_i) = 0$ for all $i \in [\Gamma + 1]$ due to robust flow conservation (AM.2). It follows that $f_{am}^* = 0$.

The set of s - t -paths in G is given by $\mathcal{P} = \{P_i := \{a'_i, a''_i\} \mid \forall i \in [\Gamma + 1]\}$ and the optimal solution x_{pm}^* to (PM) is given by $x_{pm}^*(P) = 1 \forall P \in \mathcal{P}$. Deleting Γ arcs results in $f_{pm}^* = (\Gamma + 1) - \Gamma = 1$, proving the first claim. Since the optimal solution to (PM) is also optimal for (GM) on this graph, the second claim follows. \square



(A) Instance with $f_{pm}^* = f_{gm}^* = 1$ and $f_{am}^* = 0$.



(B) Instance with $f_{am}^* = f_{gm}^* = \eta - \Gamma$ and $f_{pm}^* = \eta/\Gamma + 1$, i.e. $f_{am}^* \geq f_{pm}^*$ and $f_{gm}^* \geq f_{pm}^*$.

FIGURE 2. Instances in the proofs of Prop. 3.8 (A) and Prop. 3.9 (B).

Proposition 3.9. *Let $\Gamma \in \mathbb{N}$. For any $\alpha = \Gamma + 1 - 1/\beta$ with $\beta \in \mathbb{N}$ there are instances such that $f_{am}^* = \alpha f_{pm}^*$ and $f_{gm}^* = \alpha f_{pm}^*$.*

Proof. For an arbitrary $\Gamma \geq 1$ we construct an instance $I = (G, \Gamma)$ that consists of three nodes $\{s, v, t\}$. Let

$$\eta := \frac{\Gamma^2 + \Gamma}{\Gamma + 1 - \alpha} = \beta(\Gamma^2 + \Gamma) \in \mathbb{N}. \quad (4)$$

There are $\Gamma + 1$ parallel arcs a'_i , $i \in [\Gamma + 1]$, from s to v with capacity η . The node v is connected to t by η parallel arcs a''_i , $i \in [\eta]$, with unit capacity, cf. Figure 2b.

First, we show that an optimal solution x_{am}^* to (AM) on I has objective value $f_{am}^* = \eta - \Gamma$. We may assume w.l.o.g. that $x_{am}^*(a'_i) = u_{a'_i} = \eta$ for all $i \in [\Gamma + 1]$. Due to robust flow conservation (AM.2), at most η units of flow may be sent from node v to t . Since x_{am}^* is optimal, these η flow units are distributed equally along the parallel arcs from v to t , fully utilizing their capacity. Since $\eta \geq \Gamma + 1$, Γ many arcs from v to t with unit capacity are deleted in the worst case, resulting in an optimal value $f_{am}^* = \eta - \Gamma$.

Next, we show that an optimal solution x_{pm}^* to (PM) has objective value $f_{pm}^* = \eta/(\Gamma + 1)$. Obviously, x_{pm}^* saturates each of the parallel arcs from v to t and thus $\sum_{P \in \mathcal{P}} x_{pm}^*(P) = \eta$. As the optimal flow minimizes the flow that is deleted in the worst case, it is uniformly distributed on the $\Gamma + 1$ parallel arcs from s to v . Thus, $\sum_{P \in \mathcal{P}: a' \in P} x(P) = \eta/(\Gamma + 1)$ for all arcs a' from s to v . Since $\eta \geq \Gamma + 1$, Γ of the arcs connecting s and v are deleted in the worst case and the optimal value is given by $f_{pm}^* = \eta - \Gamma\eta/(\Gamma + 1) = \eta/(\Gamma + 1)$.

Combining the above yields

$$\frac{f_{am}^*}{f_{pm}^*} = \frac{\eta - \Gamma}{\eta/(\Gamma + 1)} = \frac{(\Gamma + 1)(\eta - \Gamma)}{\eta} = \Gamma + 1 - \frac{\Gamma^2 + \Gamma}{\eta} = \alpha.$$

We note that the optimal solution x_{am}^* to (AM) is also optimal to (GM) so that $f_{gm}^* = \alpha f_{pm}^*$. \square

For the special case of $\Gamma = 1$ and instances without directed cycles, i.e. directed acyclic graphs (DAGs), we prove tightness of this bound. We thereby use the following lemma.

Lemma 3.10. *Let $\Gamma = 1$ and let G be a DAG. For any feasible solution \tilde{x} to (GM) there exists a feasible solution x to (GM) with*

- (i) $\sum_{P: a \in \bar{P}} x(P) \leq \sum_{P: a \in \bar{P}} \tilde{x}(P)$ for all $a \in A$,
- (ii) $x(P) = \tilde{x}(P)$ for all $P \in \delta_{\bar{P}}^-(t)$, implying $f(x) = f(\tilde{x})$ and $f_R(x) = f_R(\tilde{x})$,

and

$$\sum_{\substack{P \in \delta_{\bar{P}}^-(v): \\ a \in P}} x(P) \leq \sum_{P \in \delta_{\bar{P}}^+(v)} x(P) \quad \forall a \in A \quad (5)$$

at every node $v \in V \setminus \{s, t\}$.

For the proof, we refer to Appendix A.4.

The following proof of tightness starts with an optimal solution to (GM) and then explicitly constructs a solution to (PM) on the same instance. We show that this construction, namely the deletion of flow on all subpaths, along with a suitable shifting of flow, does not reduce the robust flow value by more than half.

Theorem 3.11. *Let $\Gamma = 1$ and let G be a DAG. Then, $f_{gm}^* \leq 2f_{pm}^*$ and $f_{am}^* \leq 2f_{pm}^*$.*

Proof. For a feasible solution x_{gm} to (GM) and an arc a , we denote the amount of flow on paths ending at t and using arc a by

$$\mu(x_{gm}, a) := \sum_{\substack{P \in \delta_{\bar{p}}^-(t) \\ a \in P}} x_{gm}(P).$$

We then have, due to $\Gamma = 1$,

$$f_R(x_{gm}) = f(x_{gm}) - \max_{a \in A} \mu(x_{gm}, a). \quad (6)$$

Let x_{gm}^* be an optimal solution to (GM) which satisfies (5). The existence of such a solution follows from Lemma 3.10.

We construct a feasible solution x_{pm} to (PM) on G , for which we will show that $f_R(x_{pm}) \geq 1/2 f_{gm}^*$. The construction is an iterative procedure, starting with $x_0 = x_{gm}^*$. The procedure iterates over incoming paths of t and in each iteration $i = 1, 2, \dots$ of the procedure, we will construct a new flow x_i as follows: let $P_i \in \delta_{\bar{p}}^-(t)$ be a v_i - t -path with $v_i \neq s$, $x_{i-1}(P_i) > 0$ and minimum number of arcs. If no such path exists, we stop the procedure as we will discuss later. We first construct an interim solution \tilde{x}_i . For this, we shift all the flow from P_i to incoming paths of t that contain P_i as subpath, i.e. to paths in the set

$$\mathcal{P}_i := \{\bar{P} \in \delta_{\bar{p}}^-(t) : \bar{P} = P' \cup P_i \text{ with } P' \in \delta_{\bar{p}}^-(v_i)\}.$$

In detail, we set $\tilde{x}_i(P_i) := 0$ and

$$\tilde{x}_i(\bar{P}) := x_{i-1}(\bar{P}) + \frac{x_{i-1}(P')}{\sum_{P \in \delta_{\bar{p}}^-(v_i)} x_{i-1}(P)} \cdot x_{i-1}(P_i) \quad \forall \bar{P} = P' \cup P_i \in \mathcal{P}_i, \quad (7)$$

$$\tilde{x}_i(P') := x_{i-1}(P') - \frac{x_{i-1}(P')}{\sum_{P \in \delta_{\bar{p}}^-(v_i)} x_{i-1}(P)} \cdot x_{i-1}(P_i) \quad \forall P' \in \delta_{\bar{p}}^-(v_i), \quad (8)$$

$$\tilde{x}_i(P) := x_{i-1}(P) \quad \forall P \in \bar{\mathcal{P}} \setminus (\{P_i\} \cup \mathcal{P}_i \cup \delta_{\bar{p}}^-(v_i)). \quad (9)$$

By this, we reduce the flow on paths ending at v_i , namely in total by $x_{i-1}(P_i)$. The outgoing flow of v_i is also reduced by $x_{i-1}(P_i)$ on P_i . Thus, \tilde{x}_i still satisfies robust flow conservation (GM.2) at v_i and is therefore feasible for (GM) since also the capacity constraints remain satisfied.

From \tilde{x}_i , a feasible solution x_i satisfying (i), (ii) (cf. Lemma 3.10), and (5) can be constructed due to Lemma 3.10.

In some iteration K , there exists no path $P_K \in \delta_{\bar{p}}^-(t)$ with $P_K \notin \delta_{\bar{p}}^+(s)$ and $x_{K-1}(P_K) > 0$. We define $x_{pm}(P) := x_{K-1}(P)$ for all $P \in \mathcal{P}$ and $x_{pm}(P) := 0$ for all $P \in \bar{\mathcal{P}} \setminus \mathcal{P}$ and stop with the procedure. Note that x_{pm} is a feasible solution to (PM) as there is no flow on subpaths.

Furthermore, this procedure ends after finitely many iterations, i.e. $K < \infty$, as we consider every path ending at t at most once.

We now prove two further properties of x_{pm} .

First, in every iteration $i \in [K - 1]$, the inflow of t does not change, since, due to (ii),

$$\sum_{\bar{P} \in \mathcal{P}_i} x_i(\bar{P}) + x_i(P_i) = \sum_{\bar{P} \in \mathcal{P}_i} x_i(\bar{P}) = \sum_{\bar{P} \in \mathcal{P}_i} x_{i-1}(\bar{P}) + x_{i-1}(P_i),$$

which implies equality of the nominal values $f(x_{pm}) = f(x_{gm}^*)$.

Second, we investigate the amount of flow on the single arcs. Let $a' \in A$ be an arbitrary arc.

For every iteration $i \in [K - 1]$, x_{i-1} satisfies robust flow conservation (GM.2) at v_i , which in particular due to $\Gamma = 1$ implies

$$\sum_{P \in \delta_{\bar{p}}^-(v_i) : a' \notin P} x_{i-1}(P) \geq \sum_{P \in \delta_{\bar{p}}^+(v_i)} x_{i-1}(P).$$

Together with (5) we obtain

$$\frac{\sum_{P \in \delta_{\bar{p}}^-(v_i): a' \in P} x_{i-1}(P)}{\sum_{P \in \delta_{\bar{p}}^-(v_i)} x_{i-1}(P)} \leq \frac{1}{2}.$$

Therefore, by summing up (7) and (9) for all paths that end at t and contain a' , it follows that

$$\sum_{\substack{P \in \delta_{\bar{p}}^-(t): \\ a' \in P}} x_i(P) = \sum_{\substack{P \in \delta_{\bar{p}}^-(t): \\ a' \in P}} x_{i-1}(P) + \frac{\sum_{P \in \delta_{\bar{p}}^-(v_i): a' \in P} x_{i-1}(P)}{\sum_{P \in \delta_{\bar{p}}^-(v_i)} x_{i-1}(P)} x_{i-1}(P_i) \leq \sum_{\substack{P \in \delta_{\bar{p}}^-(t): \\ a' \in P}} x_{i-1}(P) + \frac{1}{2} x_{i-1}(P_i).$$

We recall that in every iteration i , we add flow on incoming paths of t of amount $x_{i-1}(P_i)$ in total. The above inequality hence says that at most half of this amount of flow uses the arc a' . Thus, for x_{pm} and the arc $a' \in A$, we have

$$\sum_{\substack{P \in \mathcal{P}: \\ a' \in P}} x_{pm}(P) \leq \mu(x_{gm}^*, a') + \sum_{\substack{P \in \mathcal{P}: \\ a' \notin P}} x_{pm}(P) \leq \max_{a \in A} \mu(x_{gm}^*, a) + \sum_{\substack{P \in \mathcal{P}: \\ a' \notin P}} x_{pm}(P).$$

Hence,

$$\sum_{\substack{P \in \mathcal{P}: \\ a' \notin P}} x_{pm}(P) \geq \sum_{\substack{P \in \mathcal{P}: \\ a' \in P}} x_{pm}(P) - \max_{a \in A} \mu(x_{gm}^*, a).$$

Putting together the above, we obtain

$$\begin{aligned} f_{gm}^* &= f(x_{gm}^*) - \max_{a \in A} \mu(x_{gm}^*, a) = f(x_{pm}) - \max_{a \in A} \mu(x_{gm}^*, a) \\ &= \sum_{\substack{P \in \mathcal{P}: \\ a' \notin P}} x_{pm}(P) + \sum_{\substack{P \in \mathcal{P}: \\ a' \in P}} x_{pm}(P) - \max_{a \in A} \mu(x_{gm}^*, a) \geq 2 \sum_{\substack{P \in \mathcal{P}: \\ a' \in P}} x_{pm}(P) - 2 \max_{a \in A} \mu(x_{gm}^*, a), \end{aligned}$$

which is equivalent to $\sum_{P \in \mathcal{P}: a' \in P} x_{pm}(P) \leq \max_{a \in A} \mu(x_{gm}^*, a) + \frac{1}{2} f_{gm}^*$.

Since $a' \in A$ has been chosen arbitrarily, we can conclude the proof of the first claim by

$$\begin{aligned} f_{pm}^* &\geq f_R(x_{pm}) = f(x_{pm}) - \max_{a \in A} \sum_{\substack{P \in \mathcal{P}: \\ a \in P}} x_{pm}(P) = f(x_{gm}^*) - \max_{a \in A} \sum_{\substack{P \in \mathcal{P}: \\ a \in P}} x_{pm}(P) \\ &\geq f(x_{gm}^*) - \max_{a \in A} \mu(x_{gm}^*, a) - \frac{1}{2} f_{gm}^* = \frac{1}{2} f_{gm}^*. \end{aligned}$$

Moreover, every optimal solution to (AM) is feasible for (GM) and thus $\frac{1}{2} f_{am}^* \leq \frac{1}{2} f_{gm}^* \leq f_{pm}^*$. \square

We recall that both, the path model and the general model, are solvable in polynomial time for $\Gamma = 1$. Thus, the former results show that (GM) yields robust optimal values up to twice as large as the ones of (PM) in this computationally tractable special case.

Note that the proof of Lemma 3.10 requires the graph to be acyclic. If this lemma can be extended to general directed graphs, Theorem 3.11 can be extended to those as well.

Due to the simple structure of the instances in Proposition 3.9 and tightness on DAGs with $\Gamma = 1$, we suppose that the lower bounds on the gap from Proposition 3.9 are tight in general. We therefore state the following conjecture.

Conjecture 3.12. For any $\Gamma \in \mathbb{N}$ the inequalities $f_{gm}^* \leq (\Gamma + 1) f_{pm}^*$ and $f_{am}^* \leq (\Gamma + 1) f_{pm}^*$ hold on every instance.

The proofs of Lemma 3.10 as well as Theorem 3.11 heavily exploit the fact that in the case of $\Gamma = 1$ it is easy to determine the arc that fails in the worst case, as it is the arc that maximizes μ . Thus, for $\Gamma > 1$, a different proof technique seems to be necessary for Conjecture 3.12.

Price of Robustness. We conclude this section by comparing the nominal value of the robust optimal solutions to the nominal optimal value f^* . We hence analyze the loss that may arise when hedging against uncertainties that do not realize which has been introduced as the *Price of Robustness (PoR)* in [BS04]. To the best of our knowledge, there are no earlier results on the PoR in the robust network flow context.

We start by a result on the PoR for the arc model, for which we use the instance from the proof of Proposition 3.8, cf. Figure 2a. Note that the nominal value of all robust feasible solutions to (AM) in this instance is zero, while the nominal optimal flow has a value greater than zero. We infer that, in general, the PoR can be unbounded:

Corollary 3.13. *For any $\Gamma \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ there are instances such that $f^* > \alpha f(x_{am})$ for all robust feasible solutions x_{am} to (AM).*

In contrast, for $\Gamma = 1$ there exists an optimal solution x_{pm}^* to (PM) with $f(x_{pm}^*) = f^*$ and an optimal solution x_{gm}^* to (GM) with $f(x_{gm}^*) = f^*$, as shown in [ACN01] and Theorem 3.3, respectively. Robustness thus can be achieved without loss in the nominal flow value if $\Gamma = 1$. However, this is not the case for $\Gamma \geq 2$ anymore.

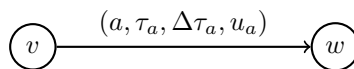
Proposition 3.14. *Let $\Gamma \geq 2$ and $\alpha \in [1, 2^{\Gamma/\Gamma+1}) \cap \mathbb{Q}$. Then, there are instances such that $f^* = \alpha f(x_{pm}^*) = \alpha f(x_{gm}^*)$, where x_{pm}^* is an optimal solution to (PM) with maximal nominal value and x_{gm}^* is an optimal solution to (GM) with maximal nominal value.*

In the proof of Proposition 3.14 in Appendix A.5, we provide such a class of instances.

Going beyond our results on the PoR for the three static models, some open questions remain, e.g., if there is an upper bound on the PoR for (PM) and (GM), or if there exist instances on which the PoR for (PM) is higher than for (GM).

4. ROBUST DYNAMIC MAXIMUM FLOWS

We now extend the concepts of robust maximum flows to a dynamic setting. In detail, we consider maximum flow problems in which a travel time is assigned to each arc. These travel times then are affected by uncertainty in the sense that delays may occur. We model such problems as follows. A graph $G = (V, A, s, t, u, \tau, \Delta\tau)$ is equipped with travel times $\tau_a \in \mathbb{N}_0$ and delays $\Delta\tau_a \in \mathbb{N}_0$ on each arc $a \in A$. Therefore, in figures throughout this section, arc labels denote the name of the arc, the travel time, the delay and the capacity as shown below.



If an amount of flow enters an arc $a = (v, w) \in A$ at a point in time, denoted by θ , then it arrives at w at time $\theta + \tau_a$. Given a time horizon $T \in \mathbb{N}$, we define the set of time steps until the time horizon by $\mathcal{T} := [T] = \{1, \dots, T\}$. We say that an amount of flow arrives at the sink within the time horizon if it reaches t at any point in time $\theta \in \mathcal{T}$. Flow solutions are again defined on arcs or paths and now also depend on time, as we have to specify when a certain amount of flow starts to travel along an arc or a path. $x(a, \theta)$ describes the inflow rate of arc a at time θ and $x(P, \theta)$ the inflow rate of path P at time θ , respectively. Furthermore, we consider uncertainties in the form that at most Γ many arcs may be delayed. The set of all possible scenarios \mathcal{S} and its corresponding indicator set Λ as well as the set of all simple s - t -paths \mathcal{P} and the set of all subpaths $\bar{\mathcal{P}}$ are defined as stated in Section 2. For simplicity, we allow flow only on simple paths. Sending flow on cycles would introduce additional complexity, as then there would be infinitely many paths. Such a problem in turn could then be modeled alternatively by allowing for waiting.

The discussion of such and other extensions of our model is beyond the scope of this work.

The nominal dynamic maximum flow problem maximizes the flow arriving at the sink t within the time horizon T while satisfying flow conservation as well as capacity constraints. We set $x(a, \theta) := 0$ for all $a \in A$, $\theta \in \mathbb{Z} \setminus \mathcal{T}$. Then, a simple discrete nominal dynamic maximum flow model using strict flow conservation is given by

$$f^* = \max_{x \in \mathbb{R}_{\geq 0}^{|A|T}} \sum_{a \in \delta_{\bar{A}}^-(t)} \sum_{\theta \in [T - \tau_a]} x(a, \theta) \quad (10a)$$

$$\text{s.t.} \quad \sum_{a \in \delta_{\bar{A}}^-(v)} x(a, \theta - \tau_a) = \sum_{a \in \delta_{\bar{A}}^+(v)} x(a, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}, \quad (10b)$$

$$x(a, \theta) \leq u_a, \quad \forall a \in A, \theta \in \mathcal{T}. \quad (10c)$$

There are several variants of this problem in the literature, for example continuous models, models using weak flow conservation and models which allow to store flow at intermediate nodes, cf. [Sku09]. In [FT98], the authors point out a close correspondence between discrete and continuous nominal dynamic flows.

4.1. Robust dynamic flow models. In the dynamic setting, flow requires a certain travel time τ to traverse an arc. If an arc a is delayed, the travel time on this arc changes from τ_a to $\tau_a + \Delta\tau_a$ for all flow sent on arc a at any point in time.

When defining flow on paths, $\tau_P = \sum_{a \in P} \tau_a$ describes the travel time of path P and $\Delta_z(P) \in \mathbb{N}_0$ the delay of path P in scenario z .

For a path P and an arc $a \in P$, we denote the point in time when flow, which enters arc a at time θ in scenario z via path P , has entered path P at its first node by $\theta_{z,a}(P)$ (cf. [Got+18]):

$$\theta_{z,a}(P) := \theta - \sum_{a' \in A: a' \prec_P a} (\tau_{a'} + z(a')\Delta\tau_{a'}),$$

where $a' \prec_P a$ means that $a', a \in P$ and a' is visited before a .

Additionally, by $\theta_z(P)$ (and $\theta_z(a)$) we denote the point in time when flow, which arrives at the end node of path P (or arc a) at time θ in scenario z , has entered path P (or arc a) at its first node:

$$\theta_z(P) := \theta - (\tau_P + \Delta_z(P)) \quad \text{and} \quad \theta_z(a) := \theta - (\tau_a + z(a)\Delta\tau_a).$$

Dynamic path model (DPM). To the best of our knowledge, there is only one model for the robust dynamic maximum flow problem with delays in the literature so far, cf. [Got+18]. This model is an extension of the path model (PM) to the dynamic case, i.e., flow is assigned to s - t -paths again. [Got+18] set up the model in a continuous framework, where the flow is then a function of the time. They show that there always exists a piecewise constant flow solution that changes its values only at integer points in time and still solves the continuous model to optimality. For this reason, we give their model, which we call dynamic path model (DPM), in a discretized version. We define $x(P, \theta) := 0$ for all $P \in \mathcal{P}$, $\theta \in \mathbb{Z} \setminus \mathcal{T}$. Then, the dynamic path model (DPM) reads

$$f_{dpm}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|\mathcal{P}|T}} \min_{z \in \Lambda} \sum_{P \in \mathcal{P}} \sum_{\theta \in [T - \tau_P - \Delta_z(P)]} x(P, \theta) \quad (\text{DPM.1})$$

$$\text{s.t.} \quad \sum_{P \in \mathcal{P}: a \in P} x(P, \theta_{z,a}(P)) \leq u_a, \quad \forall a \in A, \theta \in \mathcal{T}, z \in \Lambda. \quad (\text{DPM.2})$$

We denote an optimal solution to (DPM) by x_{dpm}^* and the optimal value by f_{dpm}^* .

An important solution concept for the nominal dynamic maximum flow problem are temporally repeated flows, cf. e.g. [Sku09]. A feasible solution is called temporally repeated if the inflow rate $x(P, \theta)$ of all paths $P \in \mathcal{P}$ is constant over all $\theta \in \mathcal{T}$. While there always exists a temporally repeated flow solving the nominal problem to optimality, there is not necessarily a temporally repeated flow that solves (DPM) to optimality, cf. [Got+18].

Dynamic arc model (DAM). We now introduce the dynamic arc model (DAM), which is the natural extension of the static arc model (AM) to the dynamic case, i.e. the flow solution has one entry for each arc and for each time step. In the static setting, robust flow conservation ensured that the inflow rate at each node is higher than the outflow rate, no matter which scenario realizes. We extend this constraint to the dynamic setting by requiring robust flow conservation for each point in time. We define $x(a, \theta) := 0$ for all $a \in A$, $\theta \in \mathbb{Z} \setminus \mathcal{T}$. Then, the dynamic arc model (DAM) is given by

$$f_{dam}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|\mathcal{A}|T}} \min_{z \in \Lambda} \sum_{a \in \delta_A^-(t)} \sum_{\theta \in [T - \tau_a - z(a)\Delta\tau_a]} x(a, \theta) \quad (\text{DAM.1})$$

$$\text{s.t.} \quad \sum_{a \in \delta_A^-(v)} x(a, \theta_z(a)) \geq \sum_{a \in \delta_A^+(v)} x(a, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}, z \in \Lambda, \quad (\text{DAM.2})$$

$$x(a, \theta) \leq u_a, \quad \forall a \in A, \theta \in \mathcal{T}. \quad (\text{DAM.3})$$

We denote an optimal solution to (DAM) by x_{dam}^* and the optimal value by f_{dam}^* .

Dynamic general model (DGM). Finally, we also extend the new general model to the dynamic setting, i.e. flow is assigned to subpaths. Analogously to the other two dynamic models, we require flow conservation and capacity constraints to hold for every possible scenario at each point in time. We define $x(P, \theta) := 0$ for all $P \in \bar{\mathcal{P}}$, $\theta \in \mathbb{Z} \setminus \mathcal{T}$. The dynamic general model (DGM) then reads

$$f_{dgm}^* := \max_{x \in \mathbb{R}_{\geq 0}^{|\bar{\mathcal{P}}|T}} \min_{z \in \Lambda} \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t)} \sum_{\theta \in [T - \tau_P - \Delta_z(P)]} x(P, \theta) \quad (\text{DGM.1})$$

$$\text{s.t.} \quad \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v)} x(P, \theta_z(P)) \geq \sum_{P \in \delta_{\bar{\mathcal{P}}}^+(v)} x(P, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}, z \in \Lambda, \quad (\text{DGM.2})$$

$$\sum_{P \in \bar{\mathcal{P}}: a \in P} x(P, \theta_{z,a}(P)) \leq u_a, \quad \forall a \in A, \theta \in \mathcal{T}, z \in \Lambda. \quad (\text{DGM.3})$$

Given a feasible solution x to (DGM), we denote its objective value by $f_R(x)$. We denote the optimal solution usually by x_{dgm}^* and the optimal value by $f_{dgm}^* = f_R(x_{dgm}^*)$.

As in the static case, we observe that (DPM) and (DAM) are special cases of (DGM) where the solutions are restricted to s - t -paths or paths consisting of single arcs, respectively. Therefore, any feasible solution to (DPM) or (DAM) can be extended such that it is feasible for (DGM) by defining the flow on the subpaths, which are not s - t -paths or single arcs, respectively, to be equal to zero. For ease of notation, we write $x_{dpm} \in \mathbb{R}^{|\mathcal{P}|}$ when talking about (DPM) and $x_{dpm} \in \mathbb{R}^{|\bar{\mathcal{P}}|}$ when talking about (DGM), which implies $f_{dpm}^* = f_R(x_{dpm}^*)$. Similarly, we proceed with feasible solutions to (DAM).

Additionally, we remark that the static robust maximum flow models are special cases of their respective dynamic counterparts. Given an instance for the static problem, we can transform it to an instance for the dynamic problem by defining $\tau \equiv 0$, $\Delta\tau \equiv 1$, and $T = 1$. The delay of any arc a increases the travel time so that flow does not arrive at the sink within the time horizon if it uses a . Therefore, the delay of an arc has the same effect as if this arc would fail. Since

the travel times are zero and the time horizon is one, an optimal solution to one of the dynamic model thus maximizes the objective value of the corresponding static model.

Continuous models. In the above dynamic flow models, we consider discretized flow solutions. The discretization has been made over the time, so that the flow that is assigned to a path and a point in time can only be assigned to integer points in time. We justify this restriction of our dynamic models to the discretized settings as follows: one can show that, even if one allows for arbitrary Lebesgue-integrable functions over time as flow solutions, the models can be solved to optimality by piecewise constant functions that change their values only at integer points in time. This has been shown for the dynamic path model in [Got+18, Proposition 1]. Here, we show the statement for the dynamic general model and the dynamic arc model. In the following, we denote by continuous dynamic model a model that allows, in contrast to the stated ones, for any Lebesgue-integrable function over time as flow solution. A formal model for such a continuous framework can be easily derived by slightly modifying the previously stated ones: the domain of x changes as described before and all sums over time are replaced by integrals over time.

Proposition 4.1. *The continuous general model and the continuous arc model can be solved to optimality by a piecewise constant function:*

- (i) *There is a piecewise constant function which changes its values only at integer points and solves the dynamic general model to optimality.*
- (ii) *There is a piecewise constant function which changes its values only at integer points and solves the dynamic arc model to optimality.*

The proof of Proposition 4.1 is similar to the proof of the analogous statement for the dynamic path model from [Got+18] and can be found in Appendix B.1.

4.2. Complexity of the dynamic models. In the following, we provide an overview of the computational complexity of the three robust dynamic maximum flow models. We start with the complexity of (DPM): [Got+18] prove that solving (DPM) is at least as hard as (PM), which implies NP-hardness for Γ being part of the input. They further investigate the complexity of computing a temporally repeated flow and provide several results. We supplement the results from [Got+18] by a complexity proof for fixed $\Gamma \geq 1$, which they have left as an open question. Subsequently, we show that (DAM) is solvable in polynomial time followed by various results concerning the computational complexity of (DGM). A summary of the complexity results can be found in Table 2.

	Arc Model (DAM)	Path Model (DPM)		General Model (DGM)
		general solutions	temporally repeated	
fixed $\Gamma \geq 1$	polynomial time (Theorem 4.4)	weakly NP-hard (Theorem 4.2)	weakly NP-hard (Corollary 4.3)	weakly NP-hard (Theorem 4.6)
Γ arb.	polynomial time (Theorem 4.4)	NP-hard [Got+18]	NP-hard [Got+18]	strongly NP-hard (Theorem 4.7)

TABLE 2. Overview of the complexity results of the robust dynamic flow models.

Complexity of (DPM).

Theorem 4.2. (DPM) is weakly NP-hard for fixed $\Gamma \geq 1$.

Proof. We first show that (DPM) is weakly NP-hard for $\Gamma = 1$ and then extend this result to fixed $\Gamma > 1$. The proof for $\Gamma = 1$ closely follows the proof by which [BNS13] show weakly NP-hardness of the adaptive maximum flow problem under arc failure and which in turn is based on

[Mel07, Theorem 1, proof]. Indeed, the delays we use in the following are sufficiently large such that they correspond to the failure of an arc.

We prove the statement by a reduction from the weakly NP-complete PARTITION problem: let $n \in \mathbb{N}$ and given a set of integers $b_i \in \mathbb{N}$, $i \in [n]$, with $\sum_{i \in [n]} b_i = 2L$ for some $L \in \mathbb{N}$, is there a subset $I \subset [n]$ such that $\sum_{i \in I} b_i = L$?

We construct an instance $(G = (V, A, s, t, u, \tau, \Delta\tau), \Gamma, T)$ of (DPM) from such a PARTITION instance as follows, cf. Figure 3. Let $\Gamma = 1$, $\bar{b} = \max_{i \in [n]} b_i$ and $T = (2n\bar{b} + 1)L + 1$. The node set is given by $V = \{s = v_1, v_2, \dots, v_n, v_{n+1} = t\}$. We connect for every $i \in [n]$ the node v_i to v_{i+1} , by two parallel arcs a_i^* and a_i' with travel times $\tau_{a_i^*} := n\bar{b}b_i$ and $\tau_{a_i'} := (n\bar{b} + 1)b_i = \tau_{a_i^*} + b_i$. We denote the set of arcs of type a_i^* by A^* and the set of arcs of type a_i' by A' , so that $A = A^* \cup A'$. We set $\Delta\tau_a := T$ and $u_a := 1$ for all $a \in A$.

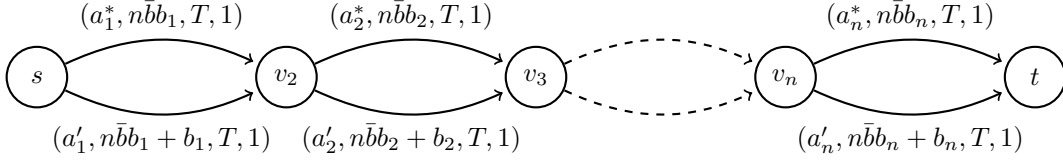


FIGURE 3. Instance constructed in the proof of Theorem 4.2. This construction has been proposed in [Mel07; BNS13].

We show that the PARTITION instance is a 'YES'-instance if and only if $f_{dpm}^* > 0$ on the constructed instance of (DPM).

First, assume that there exists a subset $I \subseteq S$ such that $\sum_{i \in I} b_i = \sum_{i \in S \setminus I} b_i = L$. We define two arc-disjoint paths P_1 and P_2 by

$$P_1 := \{a_i^* \in A^* : i \in I\} \cup \{a_i' \in A' : i \in [n] \setminus I\}, \quad P_2 := A \setminus P_1.$$

The travel times of these paths are given by

$$\tau_{P_1} = \sum_{\substack{a_i^* \in A^* \\ i \in I}} \tau_{a_i^*} + \sum_{\substack{a_i' \in A' \\ i \in [n] \setminus I}} \tau_{a_i'} = n\bar{b} \sum_{i \in I} b_i + (n\bar{b} + 1) \sum_{i \in [n] \setminus I} b_i = (2n\bar{b} + 1)L = T - 1,$$

and analogously $\tau_{P_2} = T - 1$. Let x be the solution defined by $x(P_1, \theta) = x(P_2, \theta) = 1$ for all $\theta \in \mathcal{T}$. Without any arc being delayed, two units of flow arrive at the sink within the time horizon T . By delaying any arc \tilde{a} , the travel time of the path including \tilde{a} changes into $2T - 1$, such that no flow arrives at the sink t via that path in time. However, one unit of flow arrives at t via the path not including \tilde{a} and thus $f_{dpm}^* \geq f_R(x) = 1 > 0$.

Second, assume that $f_{dpm}^* > 0$. This implies that there exist two arc-disjoint paths P_1 and P_2 with travel times at most $T - 1$, such that the flow sent on either path reaches the sink within T if the other path is delayed. We define $I := \{i \in [n] : a_i^* \in P_1\}$. Assume for contradiction that $\sum_{i \in I} b_i = L + \delta$ for some $\delta \neq 0$ and thus $\sum_{i \in S \setminus I} b_i = L - \delta$. Then, the travel times of P_1 and P_2 are given by

$$\begin{aligned} \tau_{P_1} &= (n\bar{b}) \sum_{i \in I} b_i + (n\bar{b} + 1) \sum_{i \in [n] \setminus I} b_i \\ &= (n\bar{b})(L + \delta) + (n\bar{b} + 1)(L - \delta) = (2n\bar{b} + 1)L - \delta = (T - 1) - \delta \end{aligned}$$

and similarly,

$$\tau_{P_2} = (n\bar{b}) \sum_{i \in [n] \setminus I} b_i + (n\bar{b} + 1) \sum_{i \in I} b_i = (2n\bar{b} + 1)L + \delta = (T - 1) + \delta.$$

This contradicts the existence of two arc-disjoint paths P_1 and P_2 with $\tau_{P_1}, \tau_{P_2} \leq (T-1)$. Hence, $\sum_{i \in I} b_i = \sum_{i \in [n] \setminus I} b_i = L$, which corresponds to a 'YES'-instance of the PARTITION problem. Thus, computing an optimal solution to (DPM) for $\Gamma = 1$ is weakly NP-hard, which concludes the first part of the proof.

We now extend the result to any fixed $\Gamma \geq 1$. Given $\Gamma > 1$, we add $\Gamma - 1$ parallel s - t -arcs to G . Each of these additional arcs has capacity $u \equiv 2$, travel time $\tau \equiv 0$ and delay $\Delta\tau \equiv T$, i.e. the capacity of these arcs is larger than the capacity of any arc in G and the delay of an arc again corresponds to arc failure. Let x_{dpm}^* be an optimal solution to (DPM) on this extended instance and let λ^* be the maximum amount of flow that can be deleted by delaying Γ arcs. By maximality of λ^* and x_{dpm}^* , in the worst case, all of the $\Gamma - 1$ parallel s - t -arcs and one arc out of the original arcs of G are delayed. Hence, by solving (DPM) on the extended instance with additional arcs for some fixed Γ , we solve (DPM) on G for $\Gamma = 1$ and thus solve the according PARTITION problem. It follows that solving (DPM) is weakly NP-hard for any fixed $\Gamma \geq 1$. \square

The optimal solutions to the instances in the proof of Theorem 4.2 are temporally repeated flows.

Corollary 4.3. *Computing a robust optimal temporally repeated flow to (DPM) is weakly NP-hard for any fixed $\Gamma \geq 1$.*

On the other hand, [Got+18] show that an optimal temporally repeated flow to (DPM) can be computed in polynomial time on instances satisfying the T-bounded path length property. This property on an instance requires that the travel time including the possible delay of each s - t -path is less than or equal to the time horizon T .

Complexity of (DAM). The static version, (AM), is solvable in polynomial time, as shown in [BNS13]. We prove a similar result for the dynamic case.

Theorem 4.4. *A robust optimal solution $x_{dam}^* \in \mathbb{R}_{\geq 0}^{|A|T}$ to (DAM) can be computed by solving the linear program*

$$\max_{x, \eta, \lambda, \mu, \nu} \sum_{a \in \delta_A^-(t)} \sum_{\theta \in \mathcal{T}} x(a, \theta - \tau_a) - \sum_{a \in \delta_A^-(t)} \nu(a) - \Gamma \mu \quad (12a)$$

$$s.t. \quad \sum_{a \in \delta_A^-(v)} (x(a, \theta - \tau_a) - \lambda_\theta(a)) - \Gamma \eta_{v, \theta} \geq \sum_{a \in \delta_A^+(v)} x(a, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}, \quad (12b)$$

$$\eta_{v, \theta} + \lambda_\theta(a) \geq x(a, \theta - \tau_a) - x(a, \theta - \tau_a - \Delta\tau_a), \quad \forall v \in V \setminus \{s, t\}, a \in \delta_A^-(v), \theta \in \mathcal{T}, \quad (12c)$$

$$\mu + \nu(a) \geq \sum_{i \in [\Delta\tau_a]} x(a, T - \tau_a - (i - 1)), \quad \forall a \in \delta_A^-(t), \quad (12d)$$

$$x(a, \theta) \leq u_a, \quad \forall a \in A, \theta \in \mathcal{T}, \quad (12e)$$

$$x, \eta, \lambda, \mu, \nu \geq 0, \quad (12f)$$

with $\eta \in \mathbb{R}^{T(|V|-2)}$, $\lambda \in \mathbb{R}^{T(|A| - |\delta_A^-(t)|)}$, $\mu \in \mathbb{R}$, and $\nu \in \mathbb{R}^{|\delta_A^-(t)|}$.

Hence, (DAM) can be solved in time polynomial in the size of the input (G, Γ, T) .

The proof utilizes standard techniques from robust optimization, such as dualizing the inner minimization problems, and can be found in Appendix B.2.

Complexity of (DGM). For a feasible solution x to (DPM) or (DGM), we denote the first point in time at which flow reaches t in any scenario, i.e. the point of earliest arrival, by $\theta_{ea}(x)$:

$$\theta_{ea}(x) := \min\{\theta \in \mathcal{T} : \forall z \in \Lambda \exists P_z \in \delta_{\bar{\mathcal{P}}}^-(t) : x(P_z, \theta - \tau_{P_z} - \Delta_z(P_z)) > 0\}$$

with $\theta_{ea}(x) := \infty$ if no such $\theta \in \mathcal{T}$ exists. Intuitively, it is obvious that considering (DGM) compared to (DPM) does not speed up the flow since the only difference is that we allow flow to 'seep away' at intermediate nodes in (DGM). We will use this idea as auxiliary lemma in the following and prove it formally.

Lemma 4.5. *Let x_{dgm} be a feasible solution to (DGM). Then, there exists a feasible solution x_{dpm} to (DPM) on the same instance with $\theta_{ea}(x_{dpm}) \leq \theta_{ea}(x_{dgm})$.*

Proof. We define a feasible solution x_{dpm} to (DPM) by

$$x_{dpm}(P, \theta) := c \quad \forall P \in \mathcal{P}, \theta \in \mathcal{T},$$

where $c > 0$ is the maximum value such that x_{dpm} is feasible. Such a $c > 0$ exists since all capacities u are positive.

Let x_{dgm} be a feasible solution to (DGM). Let $z \in \Lambda$ be arbitrary and $P_z \in \delta_{\bar{\mathcal{P}}}^-(t) \subseteq \bar{\mathcal{P}}$ be a path on which flow of x_{dgm} arrives at the sink at the earliest time in scenario z . If P_z is an s - t -path, it follows directly that $x_{dpm}(P_z, \theta) = c$ for all $\theta \in \mathcal{T}$ for which $x_{dgm}(P_z, \theta) > 0$. If P_z is a v - t -path for any $v \in V \setminus \{s, t\}$, there exists an s - v -path P'_z with

$$1 + \tau_{P'_z} + \Delta_z(P'_z) \leq \theta_{ea}(x_{dgm}) - \tau_{P_z} - \Delta_z(P_z),$$

as otherwise x_{dgm} would violate robust flow conservation (DGM.2) at v .

Hence, for the s - t -path $P''_z = P'_z \cup P_z$ we have

$$\theta_{ea}(x_{dgm}) - \tau_{P''_z} - \Delta_z(P''_z) = \theta_{ea}(x_{dgm}) - \tau_{P_z} - \Delta_z(P_z) - \tau_{P'_z} - \Delta_z(P'_z) \geq 1$$

with $x_{dpm}(P''_z, \theta_{ea}(x_{dgm}) - \tau_{P''_z} - \Delta_z(P''_z)) = c$. Thus, for each $z \in \Lambda$ we find an s - t -path fulfilling the property in the definition of $\theta_{ea}(x_{dpm})$ for x_{dpm} . It follows that $\theta_{ea}(x_{dpm}) \leq \theta_{ea}(x_{dgm})$. \square

We now extend the complexity result of (DPM) for fixed $\Gamma \geq 1$ to (DGM).

Theorem 4.6. (DGM) is weakly NP-hard for fixed $\Gamma \geq 1$.

Proof. Let x_{dgm}^* be an optimal solution to (DGM) on the instance from Theorem 4.2 for $\Gamma = 1$. If $f_{dgm}^* > 0$, we know that $\theta_{ea}(x_{dgm}^*) \leq T$. By Lemma 4.5, it follows that there exists a feasible solution x_{dpm} to (DPM) with $\theta_{ea}(x_{dpm}) \leq \theta_{ea}(x_{dgm}^*)$. Thus, in an optimal solution to (DPM), flow reaches the sink in time and $f_{dpm}^* > 0$. On the other hand, if $f_{dgm}^* = 0$, we directly obtain $f_{dpm}^* = 0$ since x_{dpm}^* is feasible for (DGM). Hence, by solving (DGM) on the instance from Theorem 4.2 for $\Gamma = 1$, we can solve the according partition problem.

Analogously to (DPM), we can extend the instance for $\Gamma = 1$ to any fixed $\Gamma \geq 1$. \square

The previous results show that (DPM) and (DGM) are NP-hard even for $\Gamma = 1$ although their static counterparts are solvable in polynomial time for that special case.

If Γ is part of the input, Theorem 3.4 can be extended to the dynamic setting since (GM) is a special case of (DGM).

Theorem 4.7. (DGM) is strongly NP-hard if Γ is part of the input.

Remark 4.8. Note that the graph in the proof of Theorem 4.2 is a series-parallel graph and has unit capacities on all arcs. Thus, (DPM) and (DGM) are NP-hard for fixed $\Gamma = 1$ even when restricted to instances with $u \equiv 1$. This stands in contrast to the static setting where (PM) and (GM) are solvable in polynomial time when restricted to instances with $u \equiv 1$, even if Γ is part of the input.

Integral flows. While there are no results on the complexity of integral (DAM) so far, [Got+18] prove that integral (DPM) is NP-hard and inapproximable within any factor, even for $\Gamma = 1$. For integral (DGM), we obtain NP-hardness by using the instance in the proof of Theorem 4.2.

Theorem 4.9. *For fixed $\Gamma \geq 1$, integral (DGM) is weakly NP-hard.*

Proof. Recall that in the proof of Theorem 4.2 we have shown that the optimal value of (DPM) on a constructed instance with $\Gamma = 1$ is greater than zero if and only if a given PARTITION instance is a 'YES'-instance. On this constructed instance, there is an optimal solution to (DPM) which has only integral values. Thus, the optimal value of integral (DPM) on the constructed instance is greater than zero if and only if the PARTITION instance is a 'YES'-instance. Applying the idea from Lemma 4.5, it follows directly that also the optimal value of integral (DGM) on the constructed instance is greater than zero if and only if the PARTITION instance is a 'YES'-instance. The extension to $\Gamma > 1$ works analogously as in the proof of Theorem 4.2. \square

4.3. Comparison of the solution quality between the robust dynamic flow models. In the following, we briefly compare the dynamic models to each other.

We can extend the instances in the proofs of Proposition 3.8 and Proposition 3.9 to the dynamic setting by defining $\tau \equiv 0$ and $\Delta\tau \equiv T$ with $T \in \mathbb{N}$. With analogous proofs, we obtain the following corollaries.

Corollary 4.10. *For any $\Gamma, T \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ there are instances such that $f_{dpm}^* > \alpha f_{dam}^*$ and consequently $f_{dgm}^* > \alpha f_{dam}^*$.*

Corollary 4.11. *Let $\Gamma, T \in \mathbb{N}$. Then, for any $\alpha = \Gamma + 1 - 1/\beta$ with $\beta \in \mathbb{N}$ there are instances such that $f_{dam}^* = \alpha f_{dpm}^*$ and $f_{dgm}^* = \alpha f_{dpm}^*$.*

Hence, as in the static case, neither of the two models, (DAM) or (DPM), can be considered preferably in general in terms of the robust optimal value.

Price of robustness. Next, we compare the nominal optimal flow value f^* to the nominal value of the robust optimal flow solutions for the different dynamic models, i.e. the PoR. First, since on any instance it holds that $f^* \geq f(x_{dam}^*)$, from Corollary 4.10 it follows:

Corollary 4.12. *For any $\Gamma, T \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, there are instances such that $f^* > \alpha f(x_{dam}^*)$ for all robust feasible solutions x_{dam}^* to (DAM).*

Second, we establish a lower bound on the PoR. This bound is stronger than the lower bound in the static setting, cf. Proposition 3.14.

Proposition 4.13. *Let $\Gamma \in \mathbb{N}$. For any $\alpha \in [1, \Gamma + 1) \cap \mathbb{Q}$ there exists a $T > \Gamma + 1$ such that there are instances with $f^* = \alpha f(x_{dpm}^*) = \alpha f(x_{dgm}^*)$, where x_{dpm}^* is an optimal solution to (DPM) with maximal nominal value and x_{dgm}^* is an optimal solution to (DGM) with maximal nominal value.*

Proof. For given $\Gamma \geq 1$, we initially set $T = 1$ and construct an instance $I = (G, \Gamma, T)$ as depicted in Figure 4. We define

$$\eta := \frac{(\Gamma + 1) - \alpha}{\Gamma\alpha(\Gamma + 1)}.$$

Note that $0 < \eta \leq 1/(\Gamma + 1)$. The graph G is composed of a source s , a sink t and the nodes $v_i, w_i, i \in [\Gamma]$. There are two parallel arcs, a_i^1 and a_i^2 , from v_i to w_i for all $i \in [\Gamma]$, with travel times $\tau_{a_i^1} = 1/(\Gamma + 1) - \eta$ and $\tau_{a_i^2} = 0$, and delays $\Delta\tau_{a_i^1} = 0$ and $\Delta\tau_{a_i^2} = 1/(\Gamma + 1)$. For all $i \in [\Gamma - 1]$, w_i is connected to v_{i+1} by an arc a_{i+1}^* with $\tau_{a_{i+1}^*} = \Delta\tau_{a_{i+1}^*} = 0$. In addition, each w_i is connected to v_{i+1} by an arc a_{i+1}^* , $i \in [\Gamma - 1]$. Besides, there is an arc a_1^* from s to v_1 and an arc $a_{\Gamma+1}^*$ from w_Γ to t . All of these arcs a_i^* , $i \in [\Gamma + 1]$, have travel time and delay equal to zero. All arcs have unit capacity.

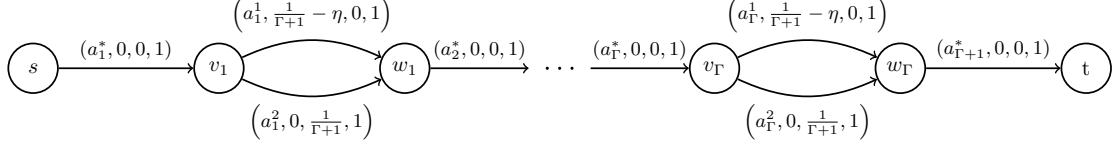


FIGURE 4. Instance with $f^* = 1$ and $f(x_{dpm}^*) = f(x_{dgm}^*) = 1/(\Gamma+1) + \Gamma\eta$ in the proof of Proposition 4.13, i.e. $f^* \geq f(x_{dpm}^*)$ and $f^* \geq f(x_{dgm}^*)$.

We define $P_1 := \{a_1^*, a_1^1, \dots, a_\Gamma^*, a_\Gamma^1, a_{\Gamma+1}^*\}$ and $P_2 := \{a_1^*, a_1^2, \dots, a_\Gamma^*, a_\Gamma^2, a_{\Gamma+1}^*\}$. As the nominal travel time of the arcs a_i^2 , $i \in [\Gamma]$ is zero, the nominal optimal solution sends one unit of flow on P_2 for all $\theta \in \mathcal{T}$ resulting in an optimal value $f^* = T = 1$. Moreover, we note that only the arcs a_i^2 , $i \in [\Gamma]$, have nonzero delay. Hence, in the worst case, all of these Γ arcs are delayed, resulting in a travel time of $\tau_{a_i^2} + \Delta\tau_{a_i^2} = 1/(\Gamma+1) > 1/(\Gamma+1) - \eta = \tau_{a_i^1} + \Delta\tau_{a_i^1}$, $i \in [\Gamma]$. Thus, the optimal solution x_{dpm}^* to (DPM) sends one unit of flow on path P_1 in each time step $\theta \in \mathcal{T}$ yielding an objective value of $f_{dpm}^* = 1 - (\Gamma/(\Gamma+1) - \Gamma\eta) = 1/(\Gamma+1) + \Gamma\eta$. As x_{dpm}^* only uses arcs with delay equal to zero, $f(x_{dpm}^*) = f_{dpm}^*$. This robust optimal solution can be shown to be unique, so that the described solution has maximal nominal value. Thus, combining the above proves the first assertion as

$$\frac{f^*}{f(x_{dpm}^*)} = \frac{1}{\frac{1}{(\Gamma+1)} + \Gamma\eta} = \frac{1}{\frac{1}{(\Gamma+1)} + \Gamma \left(\frac{(\Gamma+1) - \alpha}{\Gamma\alpha(\Gamma+1)} \right)} = \frac{1}{\frac{1}{(\Gamma+1)} + \frac{1}{\alpha} - \frac{1}{(\Gamma+1)}} = \alpha.$$

The unique optimal solution x_{dpm}^* to (DPM) is also the unique optimal solution to (DGM) yielding the second assertion.

We further note that the constructed instance contains arcs with non-integral travel times or delays. However, since α and therefore η are rational, we can w.l.o.g. scale the time horizon T and the instance to an instance with integral travel times as well as integral delays giving the same bound. \square

4.4. Approximation approaches. In the following, we present two solution concepts for (DPM) and (DGM).

Temporally repeated flows. As mentioned in Section 4.1, temporally repeated flows are a classical solution concept for the nominal dynamic maximum flow problem. We recall the results from [Got+18] which show that temporally repeated flow solutions do not solve (DPM) to optimality in general and give bounds on the corresponding approximation factor. In the following, we denote the optimal value of a temporally repeated flow by f_{tr}^* .

Theorem 4.14. [Got+18, Proposition 7, Theorem 3] *Let $\Gamma \in \mathbb{N}$ be arbitrary. We set $T := \Gamma + 1$. Then, for any $\alpha \in [1, T]$ there are instances such that $f_{dpm}^* \geq \alpha f_{tr}^*$. On the other hand, for all instances the inequality $f_{dpm}^* \leq \eta k \log(T) f_{tr}^*$ holds, where k and η are parameters depending on the given graph.*

It is clear that the lower bound in this proposition translates to (DGM). In detail, we observe that the instances used to obtain Corollary 4.10 and Corollary 4.11, both have optimal solutions to (DPM) that are temporally repeated. This leads to the following result.

Corollary 4.15. *Let $\Gamma \in \mathbb{N}$. For any $\alpha \in [1, \Gamma + 1]$ there are instances such that $f_{dgm}^* \geq \alpha f_{tr}^*$ and $f_{dam}^* \geq \alpha f_{tr}^*$. Moreover, for any $\alpha \in \mathbb{R}$ there are instances such that $f_{tr}^* \geq \alpha f_{dam}^*$.*

Hence, it does not hold in general that $f_{dam}^* > f_{tr}^*$ or $f_{tr}^* > f_{dam}^*$.

Temporally increasing flows. The previous paragraph shows that the simple concept of temporally repeated flows satisfies approximation guarantees for (DPM). However, we are interested in the question whether there is a similar approach for (DGM) also fulfilling approximation guarantees or even being optimal. We first state a natural extension of temporally repeated flows, which we call temporally increasing flows. In detail, they fulfill the condition

$$x(P, \theta) \leq x(P, \theta + 1) \quad \forall P \in \bar{\mathcal{P}}, \theta \in [T - 1].$$

We denote an optimal temporally increasing flow by x_{ti}^* and the corresponding objective value by f_{ti}^* . Clearly, all temporally repeated flows are also temporally increasing flows.

The intuitive idea for temporally increasing flows is that we choose a (sub)path decomposition and send as much flow as possible on the chosen paths satisfying the capacity and robustness constraints. However, an optimal temporally increasing flow is not necessarily an optimal solution to (DGM), i.e. we sometimes have to reroute flow within the time horizon to solve (DGM) to optimality. The following example demonstrates such a case.

Example 4.16. Let $\Gamma = 1$ and $T = 2$. We construct a graph G consisting of three nodes $\{s, v, t\}$ on which we have $f_{dgm}^* = f_{dpm}^* = f_{dam}^* = 2$ and $f_{ti}^* = 1.5$, cf. Figure 5. In G , there is one arc a_1 from s to v with $\tau_{a_1} = \Delta\tau_{a_1} = 0$. v is connected to t by one arc a_2 with $\tau_{a_2} = 0$, $\Delta\tau_{a_2} = 2$ and by one arc a_3 with $\tau_{a_3} = 1$, $\Delta\tau_{a_3} = 0$. Besides, there is one arc a_4 from s to t with $\tau_{a_4} = \Delta\tau_{a_4} = 1$. All arcs have unit capacity.

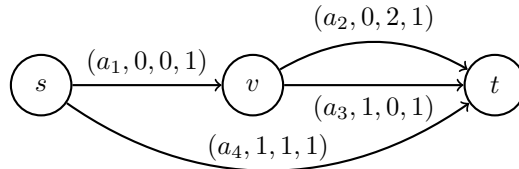


FIGURE 5. Instance with $f_{dgm}^* = f_{dam}^* = f_{dpm}^* = 2$ and $f_{ti}^* = 1.5$ in Example 4.16.

Obviously, $x_{dgm}^*(a_1, \theta) = 1$ and $x_{dgm}^*(a_4, \theta) = 1$ for all $\theta \in \mathcal{T}$. For $\theta = 1$, the flow unit arriving at v is sent via a_3 to t and arrives in any case within the time horizon $T = 2$ as $\Delta\tau_{a_3} = 0$. For $\theta = 2$, the flow unit arriving at v is sent to t via a_2 . In the worst case, a_2 or a_4 is delayed so that no flow arrives at the sink via that arc. Regardless of which of these two arcs is delayed, one unit of flow arrives at t within the time horizon T via the remaining arc, which results in an objective value of 2.

As in the general solution, $x_{ti}^*(a_1, \theta) = x_{ti}^*(a_4, \theta) = 1$ for all $\theta \in \mathcal{T}$. In each time step, the optimal temporally increasing flow solution sends 0.5 flow units via each of the arcs a_2 and a_3 . In any case, 0.5 flow units arrive at the sink via a_3 as $\Delta\tau_{a_3} = 0$. Regardless of which of the other arcs, a_2 and a_4 , is delayed, one unit of flow arrives at t within the time horizon via the remaining, non-delayed arc. This results in an optimal value of $f_{ti}^* = 1.5$.

We note that the optimal solution x_{dgm}^* to (DGM) is also feasible to (DAM), resulting in the same objective value. Combining a_1 and the v - t -arcs to s - t -paths while sending the same amount of flow as in x_{dgm}^* , the solution is feasible for (DPM) and yields the same objective value.

We remark that the example can be extended to arbitrary $\Gamma \geq 1$ and $T \geq 2$.

In contrast to the previous example, there are also instances on which the best temporally increasing flow has a greater objective value than the optimal solutions to (DAM) and (DPM), cf. the instances in Section 4.3. It remains an open question how to compute an optimal temporally increasing flow. In contrast to temporally repeated flows, for which a linear programming formulation exists if the paths in the graph are T-bounded, there is no trivial optimization model

for temporally increasing flows. Also, it is unclear whether there is an upper bound on the approximation ratio, a question which required a quite tedious proof for temporally repeated flows for (DPM), cf. [Got+18]. Finding an efficient algorithm for temporally increasing flows or other approximation approaches as well as proving approximation guarantees could be an important focus for future research.

5. CONCLUSION AND OUTLOOK

In this paper, we have introduced new models for the robust maximum flow problem as well as for the robust maximum flow over time problem. The main advantage of the proposed general models, (GM) and (DGM), is to unify the known robust flow models in order to obtain less conservative solutions. We provided a thorough analysis of the complexity of these new models and investigated the solution quality in comparison to existing models, showing e.g. that the new general models yield better robust optimal values than the known models. In particular, we also showed that (GM) remains solvable in polynomial time in the special cases for which (PM) is solvable in polynomial time, for example if only one arc may fail. These results highlight the advantages of the new general models compared to the previously known models.

We have pointed out several open questions, and highlight some of them here. We know that (PM) is NP-hard if the number of failing arcs Γ is part of the input. However, it is unknown whether the problem is already NP-hard for $\Gamma = 2$, cf. [DM20]. Thus, the complexity for fixed Γ is still an open question, for (PM) as well as for (GM). Furthermore, the complexity of the integral versions of the arc models is unclear whereas we clarified the complexity of the other integral models.

We have established several lower bounds on possible gaps and the price of robustness. It stands out that most of these lower bounds are $\Gamma + 1$. An open question is whether there exist lower bounds in the dynamic setting that depend on the time horizon T . Finally, we have proven an upper bound of two on the gap between (GM) and (PM), or (AM) and (PM) respectively, for the case of one failing arc, i.e. $\Gamma = 1$. Since the general models, (GM) and (DGM), are computationally hard to solve in general, a further investigation of those gaps and a generalization of our result to cases with $\Gamma > 1$ could lead to approximation guarantees for the general models.

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APPENDIX A. PROOFS OF SECTION 3

A.1. Proof of Theorem 3.3.

Proof. Let x be an optimal solution to (GM) with maximal nominal flow value $f(x)$ and assume for contradiction that x is not nominal optimal, i.e. $f(x) < f^*$. We split the flow into $x = g + h$, where g denotes the flow on paths ending at t , i.e. $g(P) = 0$ for all $P \notin \delta_{\bar{\mathcal{P}}}^-(t)$ and h denotes the flow on all other subpaths, i.e. $h(P) = 0$ for all $P \in \delta_{\bar{\mathcal{P}}}^-(t)$. In order to illustrate the subsequent steps of the proof, we provide a sketch in Figure 6.

Since, by assumption, x is not nominal optimal, there exists an augmenting acyclic s - t -path P in the residual graph of g , which we denote by G_g . If this augmenting path existed in G_x , it would be possible to increase the nominal flow value of x without reducing the robust flow value. This would contradict the assumption that x maximizes $f(x)$.

Therefore, there is at least one forward-arc of P in G_g that does not exist in G_x . Due to $x = g + h$, this arc thus carries flow that is part of h . Let \bar{a} be the last arc on P such that there exists a path $P' \in \bar{\mathcal{P}}$ with $h(P') > 0$ and $\bar{a} \in P'$. Let P' end at node w , i.e. $P' \in \delta_{\bar{\mathcal{P}}}^-(w)$. Furthermore, let $P'_{\leq \bar{a}}$ and $P'_{> \bar{a}}$ denote the subpaths of P' that end in the end node of \bar{a} and start at the end node of \bar{a} , respectively, i.e. $P' = P'_{\leq \bar{a}} \cup P'_{> \bar{a}}$ and $\bar{a} \in P'_{\leq \bar{a}}$, $\bar{a} \notin P'_{> \bar{a}}$. Analogously, we use this notation for P'' .

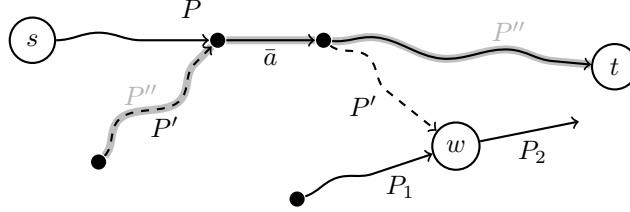


FIGURE 6. Sketch for the proof of Theorem 3.3.

We will now construct a new solution $x' = g' + h'$ that increases the nominal value without reducing the robust value compared to x , which then contradicts our assumption. The main idea of the following construction is to reduce the flow on P' by some positive amount ε while increasing flow along $P'_{> \bar{a}}$.

First, we increase flow along $P'_{> \bar{a}}$: the set of arcs that fail in the worst cases, which can be written as $\{a \in A : a \in \arg \max \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(t): a \in P} g(P)\}$, is independent of the specific path decomposition of g , as $\Gamma = 1$. Therefore, we can consider the underlying arc flow of g and execute an augmentation step on this flow. In particular, we augment the underlying arc flow of g by a small value $\varepsilon > 0$ on $P'' = P'_{\leq \bar{a}} \cup P'_{> \bar{a}}$. This increase is possible since $h(P') > 0$, implying that the arcs of $P'_{\leq \bar{a}}$ are in G_g , and since P was assumed to be in G_g . As argued above, we can now choose an arbitrary $\delta_{\bar{\mathcal{P}}}^-(t)$ -path decomposition of this new flow and obtain an interim \tilde{g} . The above modification, namely to replace g by \tilde{g} , strictly increases the nominal flow value by ε without decreasing the robust flow value.

Second, we reduce the flow on P' : we reduce the flow $h(P')$ by ε , i.e. we set $\tilde{h}(P') = h(P') - \varepsilon$. We distinguish the two cases that the current new flow, $\tilde{x} = \tilde{g} + \tilde{h}$, is feasible and that it is not:

- i) If $\tilde{x} = \tilde{g} + \tilde{h}$ is feasible for (GM), we set $x' = \tilde{x}$ and the construction is finished.
- ii) If $\tilde{x} = \tilde{g} + \tilde{h}$ is not feasible for (GM), we construct a new feasible solution x' as follows. The reduction of $h(P')$ to $\tilde{h}(P')$ can only lead to infeasibility in terms of robust flow conservation (GM.2) at the node w . Let P_2 be a path starting at w with $x(P_2) > 0$. Such a path exists due to robust infeasibility of \tilde{x} at w . Since x is robust feasible, there is a path P_1 , which is arc-disjoint

to P' and ends at w with $\tilde{h}(P_1) = h(P_1) > 0$. In order to ensure robust feasibility at w and all other nodes, we use the concatenation of the paths P_1 and P_2 and send a small amount of flow on this concatenation instead of the individual paths. In detail,

$$x'(P_1) = \tilde{h}(P_1) - \varepsilon, \quad x'(P_2) = \tilde{x}(P_2) - \varepsilon, \quad x'(P_1 \cup P_2) = \tilde{x}(P_1 \cup P_2) + \varepsilon. \quad (13)$$

The resulting flow x' is decomposed, as before, into flow on $\delta_{\bar{P}}^-(t)$ -paths, called g' , and the remaining flow, called h' . If $P_2 \in \delta_{\bar{P}}^-(t)$, we thus have $g'(P_1 \cup P_2) = x'(P_1 \cup P_2)$ and hence g' compared to g has been increased by ε on all arcs of P_1 . In the first construction step above, g has been increased by ε on P'' in order to obtain \tilde{g} . By construction, P_1 is arc-disjoint to P' and thus to $P''_{<\bar{a}}$. Further, since $P''_{>\bar{a}} = P_{>\bar{a}}$ does not contain any arc a with $h(a) > 0$, P_1 is arc-disjoint to P'' . Thus, g' has not been increased by more than ε on any arc compared to g . In both cases, i) and ii), we have constructed a new feasible solution x' with

$$\sum_{\substack{P \in \delta_{\bar{P}}^-(t): \\ a \in P}} g'(P) - \sum_{\substack{P \in \delta_{\bar{P}}^-(t): \\ a \in P}} g(P) \leq \varepsilon \quad \forall a \in A,$$

and $f(x') = f(x) + \varepsilon$. Thus,

$$\begin{aligned} f_R(x') &= f(x') - \max_{a \in A} \sum_{\substack{P \in \delta_{\bar{P}}^-(t): \\ a \in P}} g'(P) = f(x) + \varepsilon - \max_{a \in A} \sum_{\substack{P \in \delta_{\bar{P}}^-(t): \\ a \in P}} g'(P) \\ &\geq f(x) + \varepsilon - (\varepsilon + \max_{a \in A} \sum_{\substack{P \in \delta_{\bar{P}}^-(t): \\ a \in P}} g(P)) = f_R(x). \end{aligned}$$

Hence, x' has a strictly larger nominal flow value than x and at least the same robust flow value as x , contradicting the initial assumption on x . \square

A.2. Proof of Proposition 3.6.

Proof. We show the statement by a reduction from the static robust flow problem with arbitrary capacities. Thereby, we construct from an instance with arbitrary capacities an instance with capacities in $\{1, \infty\}$. This construction has been proposed in [DM20, Lemma 5, proof] for the path model and it remains to be proven that equivalence also holds for the general model. Let $\Gamma \geq 1$ and $G = (V, A, s, t, u)$ be an arbitrary instance for (GM). We construct a graph $G' = (V', A', s, t, u')$ from G by adding a node v_a for every $a = (v, w) \in A$ and replacing a by an arc $a' = (v, v_a)$ with infinite capacity and u_a parallel arcs $a''_i = (v_a, w)$, $i \in [u_a]$ with unit capacity (cf. Figure 7). The arc set A' thus consists of one arc a' and u_a arcs a''_i for each arc $a \in A$. We denote the set of scenarios and the set of all subpaths on G' by \mathcal{S}' and $\bar{\mathcal{P}}'$, respectively. Given a v - w -path P in G with $v, w \in V$, we denote the set of all corresponding v - w -paths in G' by $\bar{\mathcal{P}}'(P)$, meaning that every path $P' \in \bar{\mathcal{P}}'(P)$ uses for every $a \in P$ the corresponding arc a' and one of the arcs a''_i , $i \in [u_a]$.

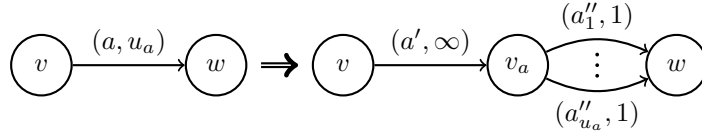


FIGURE 7. Construction in the proof of Proposition 3.6. This construction has originally been proposed in [DM20, Lemma 5, proof].

Let x'_{gm} be an optimal solution to (GM) on G' . Additionally, we define a vector $x^*_{gm} \in \mathbb{R}_{\geq 0}^{|\bar{\mathcal{P}}|}$ by

$$x^*_{gm}(P) := \sum_{P' \in \bar{\mathcal{P}}'(P)} x'_{gm}(P') \quad \forall P \in \bar{\mathcal{P}}.$$

In the following, we show that x^*_{gm} is an optimal solution to (GM) on G . First, from Lemma 3.1 it follows that $x'_{gm}(P') = 0$ for all paths starting at the intermediate nodes v_a , i.e. for all $P' \in \delta_{\bar{\mathcal{P}}'}^+(v_a)$ with $a \in A$. Additionally, we observe that for any $v \in V \setminus \{s\}$ there is at least one scenario

$$S^* \in \arg \max_{S' \in \mathcal{S}'} \sum_{\substack{P' \in \delta_{\bar{\mathcal{P}}'}^-(v): \\ P' \cap S' \neq \emptyset}} x'_{gm}(P')$$

such that S^* only contains arcs of the type a' and not arcs of the type a'_i . Hence, we can write the loss that is caused by the realization of the worst case as follows:

$$\max_{S' \in \mathcal{S}'} \sum_{\substack{P' \in \delta_{\bar{\mathcal{P}}'}^-(v): \\ P' \cap S' \neq \emptyset}} x'_{gm}(P') = \max_{S \in \mathcal{S}} \sum_{\substack{P \in \delta_{\bar{\mathcal{P}}}^-(v): \\ P \cap S \neq \emptyset}} \sum_{P' \in \bar{\mathcal{P}}'(P)} x'_{gm}(P').$$

Combining the above, it follows that for any $v \in V \setminus \{s\}$ we have

$$\begin{aligned} & \sum_{P' \in \delta_{\bar{\mathcal{P}}'}^-(v)} x'_{gm}(P') - \max_{S' \in \mathcal{S}'} \sum_{\substack{P' \in \delta_{\bar{\mathcal{P}}'}^-(v): \\ P' \cap S' \neq \emptyset}} x'_{gm}(P') \\ &= \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v)} \sum_{P' \in \bar{\mathcal{P}}'(P)} x'_{gm}(P') - \max_{S \in \mathcal{S}} \sum_{\substack{P \in \delta_{\bar{\mathcal{P}}}^-(v): \\ P \cap S \neq \emptyset}} \sum_{P' \in \bar{\mathcal{P}}'(P)} x'_{gm}(P') \\ &= \sum_{P \in \delta_{\bar{\mathcal{P}}}^-(v)} x^*_{gm}(P) - \max_{S \in \mathcal{S}} \sum_{\substack{P \in \delta_{\bar{\mathcal{P}}}^-(v): \\ P \cap S \neq \emptyset}} x^*_{gm}(P). \end{aligned}$$

Thus, x^*_{gm} is a feasible solution to (GM) on G with same optimal value as x'_{gm} on G' . Furthermore, from any feasible solution x to (GM) on G we can trivially construct a feasible solution x' to (GM) on G' with at least the same robust value by uniformly splitting the flow $x(P)$ into the paths that correspond to P . It follows that x^*_{gm} is optimal for (GM) on G . Thus, it is at least as hard to solve (GM) on a graph with capacities constrained to $u_a \in \{1, \infty\}$ as on a graph with arbitrary capacities, which is NP-hard. Note that we can substitute the arcs in G' with capacity ∞ by arcs with capacity $u_{max} = \max\{u_a : a \in A\}$. \square

A.3. Proof of Theorem 3.7.

Proof. The proof is based on [DM20, Theorem 8, proof]. They set $\Gamma = 2$ and construct a graph G for integral (PM) from a given ARC-DISJOINT PATHS instance G' and show that an optimal solution to integral (PM) has an objective value f_{pm}^* of at least 3 if and only if there exist two arc-disjoint paths in G' . Hence, it is NP-hard to distinguish instances of integral (PM) with optimal value at least 3 from those with optimal value at most 2, which implies hardness of the corresponding approximation for the problem.

Given an ARC-DISJOINT PATHS instance $G' = (V', A')$, we construct an instance $I = (G, \Gamma = 2)$ for integral (GM) analogously to [DM20, Theorem 8, proof]. The resulting graph, cf. Figure 8, consists of two parts: the inner part G' with given start and end nodes s_1, s_2, t_1, t_2 and the subgraph which we denote by H induced by the added node set $V_H = \{s, v, v', v'', w, t\}$. Note that for all nodes $v \in V_H \setminus \{t\}$, we have $\delta_{\bar{A}}^-(v) \leq 2 = \Gamma$.

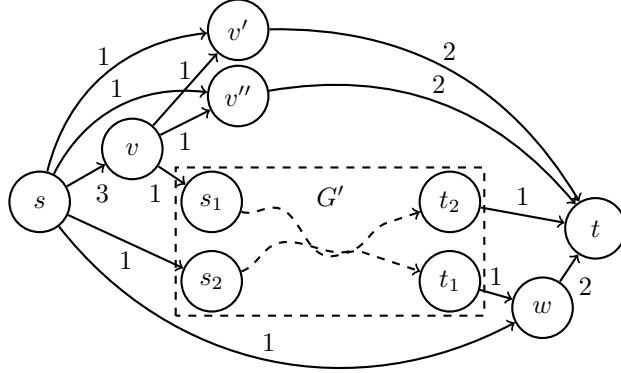


FIGURE 8. Instance in the proof of Theorem 3.7. The arc labels denote the arc capacities. This construction has originally been proposed in [DM20].

We now show that any optimal solution x_{pm}^* to integral (PM) on the constructed instance also solves integral (GM) to optimality. Let x_{gm}^* be an optimal solution to integral (GM) on the constructed instance. As $\delta_A^-(v) \leq 2$ for all $v \in V_H \setminus \{t\}$, it follows from Lemma 3.1 that all flow on subpaths in H can be deleted resulting in another optimal solution to integral (GM) with flow only on entire s - t -paths in H .

Besides the subgraph H , we consider G' and the subgraph which we denote by G'_{st} induced by the node set $V'_{st} := V' \cup \{s, v, w, t\}$. At most one unit of flow arrives at each of the nodes s_1 and s_2 in G' due to the capacities of the incoming arcs (v, s_1) and (s, s_2) . Such a flow unit cannot be distributed among different paths, as an integral solution is required. This implies that there are for each node v in G'_{st} at most two incoming paths carrying flow. As $\Gamma = 2$, it follows directly from Lemma 3.1 that all flow on s - v -paths with $v \in V'_{st} \setminus \{t\}$ can be deleted resulting in another optimal flow solution with flow only on s - t -paths. Thus, there is an optimal solution to integral (GM) which sends flow only on s - t -paths in G and is hence feasible for integral (PM).

This concludes the proof: any optimal solution to the integral path model is also optimal for the integral general model as there is a feasible solution to the integral path model that solves the integral general model to optimality and each solution to integral (PM) is feasible for integral (GM). Now, we can use the result of [DM20, Theorem 8, proof] to draw an analogous conclusion for our general model: the optimal objective value f_{gm}^* of integral (GM) is at most two if no two arc-disjoint paths exist in G' and is at least three if two arc-disjoint paths exist in G' . Deciding whether two such paths exist is NP-hard so that the claim follows. \square

A.4. Proof of Lemma 3.10.

Proof. Let \tilde{x} be a feasible solution to (GM) that does not satisfy (5) for all nodes $v \in V \setminus \{s, t\}$. Recall that we want to construct a feasible solution x satisfying

- (i) $\sum_{P: a \in \bar{P}} x(P) \leq \sum_{P: a \in \bar{P}} \tilde{x}(P)$ for all $a \in A$,
- (ii) $x(P) = \tilde{x}(P)$ for all $P \in \delta_{\bar{P}}^-(t)$, implying $f(x) = f(\tilde{x})$ and $f_R(x) = f_R(\tilde{x})$,

and

$$\sum_{\substack{P \in \delta_{\bar{P}}^-(v): \\ a \in P}} x(P) \leq \sum_{P \in \delta_{\bar{P}}^+(v)} x(P) \quad \forall a \in A \quad (5)$$

at every node $v \in V \setminus \{s, t\}$.

As G is a DAG, we can order the set of nodes $V = \{s = v_0, v_1, \dots, v_n, t = v_{n+1}\}$ such that there is no directed path in G from v_i to v_j if $i > j$. Additionally, there is a strict partial order on the arcs such that $a = (v, w) \prec a' = (v', w')$ if $w = v'$ or if there is a directed path from w to v' in G . Let $\ell \in [n+1]$ be the minimal number such that there exists a feasible solution that satisfies (i), (ii) and (5) at all nodes v_i with $i \in [n]$ and $i \geq \ell$, and we assume for contradiction that $\ell > 1$. We fix the node $v := v_{\ell-1}$ for the remainder of this proof and introduce some further notation. Let x be any feasible solution to (GM). We denote the amount of incoming and outgoing flow at v by

$$f^-(x) := \sum_{P \in \delta_{\bar{P}}^-(v)} x(P) \quad \text{and} \quad f^+(x) := \sum_{P \in \delta_{\bar{P}}^+(v)} x(P),$$

respectively. For an arc $a \in A$, we denote the set of paths that carry flow to v and use a by

$$\bar{\mathcal{P}}(x, a) := \{P \in \delta_{\bar{P}}^-(v) : a \in P, x(P) > 0\}.$$

Additionally, for every arc we denote the flow ending at v and using or not using a by

$$y(x, a) := \sum_{P \in \bar{\mathcal{P}}(x, a)} x(P) = \sum_{\substack{P \in \delta_{\bar{P}}^-(v) \\ a \in P}} x(P) \quad \text{and} \quad y^c(x, a) := \sum_{\substack{P \in \delta_{\bar{P}}^-(v) \\ a \notin P}} x(P),$$

respectively. The maximum of $y(x, a)$ over the arcs is denoted by $\bar{y}(x) := \max_{a \in A} y(x, a)$ and we define the corresponding set of maximizing arcs by

$$\bar{A}(x) := \{a \in A : y(x, a) = \bar{y}(x)\}$$

with $k(x) := |\bar{A}(x)|$. With this notation, (5) reads

$$\bar{y}(x) \leq f^+(x). \quad (14)$$

Robust flow conservation (GM.2) at v reads

$$y^c(x, a) \geq f^+(x), \quad \forall a \in A, \quad (15)$$

since $\Gamma = 1$. Furthermore, we have

$$f^-(x) = y(x, a) + y^c(x, a), \quad \forall a \in A. \quad (16)$$

Let x' be one of the solutions that satisfy (i), (ii) and (5) at the nodes v_i , $i \in [n]$ and $i \geq \ell$, and which, among them, minimizes the value

$$R(x') := \sum_{a \in A} \max\{y(x', a) - f^+(x'), 0\} > 0.$$

We first show that it holds for x' that

- (a) $f^-(x') = \bar{y}(x') + f^+(x')$ and
- (b) for all $a', a'' \in \bar{A}(x')$, there exists a path $P \in \delta_{\bar{P}}^-(v)$ with $x'(P) > 0$ and $a', a'' \in P$.

We arbitrarily choose an arc $a' \in \bar{A}(x')$. Due to the assumption that x' minimizes $R(\cdot)$, we know that modifying x' only by reducing flow on paths in $\bar{\mathcal{P}}(x', a')$ results in a flow that is infeasible in terms of robust flow conservation at v . The previous implies the following for every $P \in \bar{\mathcal{P}}(x', a')$: there exists an arc $a'' \notin P$ such that by reducing flow on P by any positive amount we would violate robust flow conservation (15) at v for the scenario in which a'' fails. Hence, in the scenario in which a'' fails robust flow conservation is fulfilled with equality, i.e. $y^c(x', a'') = f^+(x')$. Now, we have $\bar{y}(x') = y(x', a')$ since $a' \in \bar{A}(x')$, $y^c(x', a') \geq f^+(x')$ due to robust feasibility of x' , the equality (16) for a' and a'' , $y(x', a'') \leq \bar{y}(x')$ and $y^c(x', a'') = f^+(x')$. Combining this in the same order, we obtain

$$\bar{y}(x') + f^+(x') \leq y(x', a') + y^c(x', a') = f^-(x') = y(x', a'') + y^c(x', a'') \leq \bar{y}(x') + f^+(x'). \quad (17)$$

Thus, (a) holds.

Furthermore, it follows that $y(x', a'') = \bar{y}(x')$, i.e. $a'' \in \bar{A}(x')$ as well as $y^c(x', a') = f^+(x')$, i.e., robust flow conservation is satisfied with equality in the scenario in which a' fails. Additionally, since (14) is not satisfied, the above equations yield $y^c(x', a') = f^+(x') < y(x', a'') = \bar{y}(x')$. Hence, there exists at least one path that ends at v , carries flow, and contains a' as well as a'' as otherwise $y^c(x', a') \geq y(x', a'')$ would hold, which would contradict the former inequality.

Since $a' \in \bar{A}(x')$ has been chosen arbitrarily, it follows that for any $a', a'' \in \bar{A}(x')$ there exists a path $P \in \delta_{\bar{p}}^-(v)$ with $a', a'' \in P$ and $x'(P) > 0$, i.e. (b) holds.

Note that for every feasible solution x with $x(P) = x'(P)$ for all $P \notin \delta_{\bar{p}}^-(v)$ and $y(x, a) = y(x', a)$ for all $a \in A$ the properties (a) and (b) hold and additionally $\bar{A}(x) = \bar{A}(x')$.

Since G is a DAG, property (b) implies that the strict partial order on A induces a strict total order on $\bar{A}(x')$ and in particular that there exists a path $P' \in \delta_{\bar{p}}^-(v)$ such that $\bar{A}(x') \subseteq P'$ and $y(x', a) > 0$ for all $a \in P'$.

There is a path decomposition x_{pd} of $y(x', \cdot)$ with $x_{pd}(P'') > 0$ for some path $P'' \in \delta_{\bar{p}}^-(v)$ with $P' \subseteq P''$ due to the following: the vector $y(x', \cdot)$ can be extended to a nominal s - v -flow with strict flow conservation on an extended graph by introducing some artificial arcs $a_w = (s, w)$ for all $w \in V \setminus \{s, v, t\}$ and defining

$$y(x', a_w) := \sum_{a \in \delta_A^+(w)} y(x', a) - \sum_{a \in \delta_A^-(w)} y(x', a).$$

Thus, on the extended graph there exists such a path decomposition which obviously induces a path decomposition of $y(x', \cdot)$ in G .

We now define a new feasible solution x'' to (GM) by

$$x''(P) := \begin{cases} x'(P), & P \notin \delta_{\bar{p}}^-(v), \\ x_{pd}(P), & P \in \delta_{\bar{p}}^-(v). \end{cases}$$

This implies that x'' still satisfies (a) and that $x''(P'') > 0$ for the path P'' with $\bar{A}(x'') = \bar{A}(x'') \subset P''$. Hence, we can reduce the flow on P'' by some positive amount $\varepsilon > 0$ since for all $a \notin P''$ it holds that $y(x'', a) < \bar{y}(x'')$ and thus $y^c(x'', a) > f^+(x'')$ due to (a) and (16). By this reduction we obtain a new feasible solution x''' which still satisfies (i), (ii) and (5) at the nodes $v_i, i \in [n]$ and $i \geq \ell$. Additionally, due to $|\bar{A}(x')| = |\bar{A}(x'')| = k(x')$, we have

$$R(x''') \leq R(x') - k(x')\varepsilon < R(x'),$$

contradicting the assumption that x' minimizes $R(\cdot)$. This contradicts the initial assumption $\ell > 1$ so that there exists a solution x satisfying (i), (ii), and (5) at every node $v \in V \setminus \{s, t\}$. \square

A.5. Proof of Proposition 3.14.

Proof. For an arbitrary $\Gamma \geq 2$, we construct an instance $I = (G, \Gamma)$ that consists of four nodes $\{s, v_1, v_2, t\}$, cf. Figure 9. Let

$$\eta := \frac{\Gamma(\alpha - 2) + \alpha}{(\Gamma - 1)(\alpha - 2)} \in \mathbb{Q}.$$

We note that $0 < \eta \leq 1$. There are $\Gamma - 1$ parallel arcs with capacity $1 - \eta$ from s to v_1 (a'_i), from v_1 to v_2 (a''_i) and from v_2 to t (a'''_i), where $i \in [\Gamma - 1]$. Besides, there are two additional arcs, a'_Γ from s to v_1 and a''_Γ from v_2 to t with unit capacity. Furthermore, there is arc a_1 from s to v_2 and arc a_2 from v_1 to t , each with capacity equal to $\Gamma - (\Gamma - 1)\eta$.

The nominal optimal solution x_{nom}^* fully utilizes the capacity of all paths $P_i^2 = \{a'_i, a_2\}$, $i \in [\Gamma]$ and $P_i^1 = \{a_1, a'''_i\}$, $i \in [\Gamma]$, resulting in an objective value $f^* = 2(\Gamma - (\Gamma - 1)\eta)$. This is a nominal optimal flow solution, as $f^* = \sum_{a \in \delta_A^-(t)} u_a$.

Next, we describe a robust optimal solution x_{pm}^* to (PM) and prove optimality. We set $x_{pm}^*(P'_i) = 1 - \eta$ for all paths $P'_i = \{a'_i, a''_i, a'''_i\}$, $i \in [\Gamma - 1]$. Besides, let $x_{pm}^*(P_1) = x_{pm}^*(P_2) = 1$ for $P_1 = \{a_1, a'''_\Gamma\}$ and $P_2 = \{a'_\Gamma, a_2\}$. In the worst case, the flow on the arcs a_1 , a_2 and $\Gamma - 2$ of the arcs a'''_i , $i \in [\Gamma - 1]$ is deleted, resulting in an optimal value of $1 - \eta$. Since x_{pm}^* saturates all arcs except a_1 and a_2 , an increase of the flow on one of these two arcs requires a decrease on one of the paths P'_i to satisfy the capacity constraints. However, the paths containing a_1 and a_2 carry the largest amounts of flow and therefore are deleted in the original worst case scenarios and also the resulting worst case scenarios. Thus, such an increase on a_1 or a_2 would decrease the objective value and hence the above flow solution is robust optimal with maximal nominal value. The nominal value of this solution is $f(x_{pm}^*) = 2 + (\Gamma - 1)(1 - \eta)$. Combining the above proves the first assertion since

$$\frac{f^*}{f(x_{pm}^*)} = \frac{2(\Gamma - (\Gamma - 1)\eta)}{2 + (\Gamma - 1)(1 - \eta)} = \frac{2(\Gamma - \Gamma - \frac{\alpha}{\alpha - 2})}{1 + \Gamma - \Gamma - \frac{\alpha}{\alpha - 2}} = \frac{\frac{-2\alpha}{\alpha - 2}}{1 - \frac{\alpha}{\alpha - 2}} = \alpha.$$

We observe that $|\delta_A^-(v)| \leq \Gamma$ holds for all $v \in V$. Hence, it directly follows from Lemma 3.1 that the optimal solution x_{pm}^* to (PM) with maximal nominal value is also an optimal solution to (GM) maximizing the nominal value, yielding the second assertion.

We further note that the constructed instance contains arcs with non-integer capacities. However, since α and therefore η are rational, the capacities are all rational and we can w.l.o.g. scale the instance to an instance with integer capacities. \square

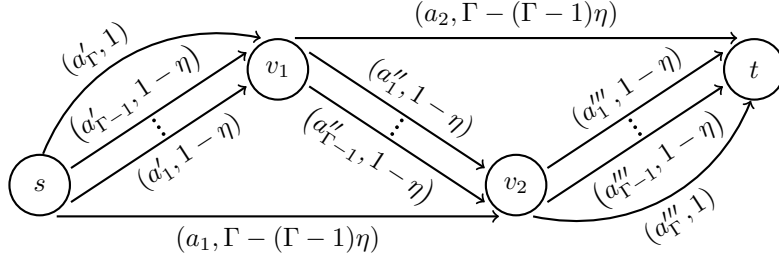


FIGURE 9. Instance with $f^* \geq f(x_{pm}^*)$ and $f^* \geq f(x_{gm}^*)$ in the proof of Prop. 3.14.

APPENDIX B. PROOFS OF SECTION 4

B.1. Proof of Proposition 4.1.

Proof. The proof works analogously to the proof of [Got+18, Proposition 1].

(i) Let \tilde{x} be an optimal solution to the continuous dynamic general model. In order to generate a piecewise constant function from the solution \tilde{x} for each path $P \in \bar{\mathcal{P}}$, the time interval $[0, T]$ is cut into unit intervals $[b - 1, b) \forall b \in \mathcal{T}$. Define $x(P, \theta)$ for all $P \in \bar{\mathcal{P}}$ as follows:

$$x(P, \theta) := \begin{cases} \int_{b-1}^b \tilde{x}(P, t) dt, & \theta \in [b - 1, b), b \in \mathcal{T} \\ 0, & \text{else.} \end{cases}$$

By definition, the objective values of \tilde{x} and x are equal and the non-negativity constraint is satisfied. We show by contradiction that x fulfills the capacity constraint. If the amount of flow on an arbitrary arc a , $\sum_{P \in \bar{\mathcal{P}}: a \in P} x(P, \theta_{z,a}(P))$, violates the capacity constraint for any point in time $\theta \in [b - 1, b)$, then it violates the capacity constraint for all $\theta \in [b - 1, b)$ due to the

integrality of τ_a and $\Delta\tau_a$. As for each path $P \in \bar{\mathcal{P}}$, the flow $x(P, \theta)$ is defined as integral of \tilde{x} in the time interval $[b-1, b)$ of length 1, there is at least one point in time, $\theta \in [b-1, b)$, such that $\tilde{x}(P, \theta)$ is greater than or equal to x . Hence, the capacity constraint would be violated by \tilde{x} as well. As a consequence, since the capacity constraint is fulfilled by \tilde{x} for all points in time, it is also satisfied by x . Analogous arguments can be employed in order to show that the piecewise constant solution x fulfills robust flow conservation.

(ii) The proof can be conducted analogously to the proof of part (i). \square

B.2. Proof of Theorem 4.4.

Proof. Every robust feasible solution x to (DAM) satisfies (DAM.2), which can be equivalently reformulated to

$$\min_{z \in \Lambda} \left\{ \sum_{a \in \delta_A^-(v)} x(a, \theta - \tau_a - z(a)\Delta\tau_a) \right\} \geq \sum_{a \in \delta_A^+(v)} x(a, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}. \quad (18)$$

For any $a \in A$, $z \in \Lambda$ and $\theta \in \mathcal{T}$, it holds that

$$x(a, \theta - \tau_a - z(a)\Delta\tau_a) = x(a, \theta - \tau_a) - z(a)(x(a, \theta - \tau_a) - x(a, \theta - \tau_a - \Delta\tau_a)).$$

Thus, constraint (18) is equivalent to

$$\sum_{a \in \delta_A^-(v)} x(a, \theta - \tau_a) - y_{v, \theta}(x) \geq \sum_{a \in \delta_A^+(v)} x(a, \theta), \quad \forall v \in V \setminus \{s, t\}, \theta \in \mathcal{T}, \quad (19)$$

with

$$\begin{aligned} y_{v, \theta}(x) &:= \max_{z_{v, \theta} \in \{0, 1\}^{|\delta_A^-(v)|}} \sum_{a \in \delta_A^-(v)} z_{v, \theta}(a) (x(a, \theta - \tau_a) - x(a, \theta - \tau_a - \Delta\tau_a)) \\ &\text{s.t.} \quad \sum_{a \in \delta_A^-(v)} z_{v, \theta}(a) \leq \Gamma. \end{aligned}$$

Since x is a fixed parameter in this maximization problem, $z_{v, \theta} \in \{0, 1\}^{|\delta_A^-(v)|}$ can be relaxed to $z_{v, \theta} \in [0, 1]^{|\delta_A^-(v)|}$ without changing the objective value $y_{v, \theta}(x)$. Hence, this relaxed problem can be dualized and we obtain

$$y_{v, \theta}(x) = \min_{\eta, \lambda} \sum_{a \in \delta_A^-(v)} \lambda_{\theta}(a) + \Gamma \eta_{v, \theta} \quad (20a)$$

$$\text{s.t.} \quad \eta_{v, \theta} + \lambda_{\theta}(a) \geq x(a, \theta - \tau_a) - x(a, \theta - \tau_a - \Delta\tau_a), \quad \forall a \in \delta_A^-(v), \quad (20b)$$

$$\lambda_{\theta}(a) \geq 0, \quad \forall a \in \delta_A^-(v), \quad (20c)$$

$$\eta_{v, \theta} \geq 0. \quad (20d)$$

Similarly, for a feasible and fixed x we can reformulate the inner optimization problem of the objective function of (DAM), i.e.

$$\min_{z \in \Lambda} \sum_{a \in \delta_A^-(t)} \sum_{\theta \in [T - \tau_a - z(a)\Delta\tau_a]} x(a, \theta) = \sum_{a \in \delta_A^-(t)} \sum_{\theta \in \mathcal{T}} x(a, \theta - \tau_a) - y_t(x) \quad (21)$$

with

$$\begin{aligned} y_t(x) &:= \max_{z \in \{0, 1\}^{|\delta_A^-(t)|}} \sum_{a \in \delta_A^-(t)} z(a) \left(\sum_{i \in [\Delta\tau_a]} x(a, T - \tau_a - (i-1)) \right) \\ &\text{s.t.} \quad \sum_{a \in \delta_A^-(t)} z(a) \leq \Gamma. \end{aligned}$$

Again, for the fixed x , the integrality of z can be relaxed to $z \in [0, 1]^{|\delta_A^-(t)|}$. Dualizing the relaxed problem gives

$$y_t(x) = \min_{\mu, \nu} \sum_{a \in \delta_A^-(t)} \nu(a) + \Gamma \mu \quad (22a)$$

$$\text{s.t. } \mu + \nu(a) \geq \sum_{i \in [\Delta \tau_a]} x(a, T - \tau_a - (i - 1)), \quad \forall a \in \delta_A^-(t), \quad (22b)$$

$$\nu(a) \geq 0, \quad \forall a \in \delta_A^-(t), \quad (22c)$$

$$\mu \geq 0. \quad (22d)$$

A substitution of (20) into (19) and (22) into the right hand side of (21) gives problem (12). This is a linear program with a polynomial number of constraints and a polynomial number of variables, which concludes the proof. \square