Nonconvex Equilibrium Models for Energy Markets: Exploiting Price Information to Determine the Existence of an Equilibrium

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Abstract. Motivated by examples from the energy sector, we consider market equilibrium problems (MEPs) involving players with nonconvex strategy spaces or objective functions, where the latter are assumed to be linear in market prices. We propose an algorithm that determines if an equilibrium of such an MEP exists and that computes an equilibrium in case of existence. Three key prerequisites have to be met. First, appropriate bounds on market prices have to be derived from necessary optimality conditions of some players. Second, a technical assumption is required for those prices that are not uniquely determined by the derived bounds. Third, nonconvex optimization problems have to be solved to global optimality. We test the algorithm on well-known instances from the power and gas literature that meet these three prerequisites. There, nonconvexities arise from considering the transmission system operator as an additional player besides producers and consumers who, e.g., switches lines or faces nonlinear physical laws. Our numerical results indicate that equilibria often exist, especially for the case of continuous nonconvexities in the context of gas market problems.

1. Introduction

Market equilibrium problems are an important mathematical tool to model many practically relevant applications such as energy markets for power or gas, auctions, or transport network planning. Usually, these problems consist of a number of rational players that compete for a set of goods, which they want to purchase to maximize their utility. In such situations one asks whether there exists a price for these goods so that the market clears and so that no player can improve her utility by unilaterally changing her decisions. Mathematically, rationality is typically modeled via optimization problems and a market equilibrium price then clears the market while all players choose a global optimal solution. The main economic or mathematical questions are (i) whether such an equilibrium exists, (ii) whether it is unique, and (iii) how to compute it. In this paper, we address these three topics but focus on the algorithmic aspects.

The classic results ensure existence of market equilibria under suitable convexity assumptions; see, e.g., Wald (1951), Arrow and Debreu (1954), Gale (1955), McKenzie (1959), or Debreu (1962). Unfortunately, many real-world market equilibrium problems do not satisfy these assumptions. Our study is mainly motivated by practically relevant aspects of market equilibrium problems in energy—namely gas and power market equilibrium models on networks. In these settings, the set of players includes producers and consumers that are located at the nodes of the energy network as well as the transmission system operator (TSO), who acts as an

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arbitrageur and who also controls the network itself; see, e.g., Hobbs and Helman (2004), Gabriel and Smeers (2006) in Seeger (2006), Gabriel, Conejo, et al. (2012), Grimm, Schewe, et al. (2017), or Krebs et al. (2018). The main challenge regarding these equilibrium problems arises in situations in which either the energy flow model is nonlinear or when controlling the network includes deciding on discrete switching variables. Both aspects lead to nonconvex player problems and, thus, the classic existence theory is not applicable anymore. See Scarf (1994) for the special case of integralities in a power market or Grimm, Grübel, et al. (2019) for the special case of gas market interaction on a network with nonconvex flow models.

There are different parts of the applied literature that tackle such nonconvex situations; see, e.g., Shapley and Shubik (1971), Leonard (1983), Bikhchandani, Ostroy, et al. (2002), and Bikhchandani and Ostroy (2006) for assignment problems, Bikhchandani and Mamer (1997) and Baldwin and Klemperer (2019) for general exchange economies with indivisibilities, O'Neill et al. (2005) and Guo et al. (2021) for discrete markets, Hatfield et al. (2013), Fleiner et al. (2019), and Hatfield et al. (2019) for trading networks, and Beato (1982), Brown et al. (1986), Bonnisseau and Cornet (1988), and Bonnisseau and Cornet (1990) for economies with increasing returns to scale. Very recently, Harks (2020) presented a unifying framework for many (possibly nonconvex) equilibrium problems including network tolls for transportation networks, indivisible item auctions, bilateral trade, or congestion control. This framework is based on Lagrangian duality and enables to characterize the existence of solutions to (possibly) nonconvex equilibrium problems by checking if a suitably chosen optimization problem (e.g., the overall welfare maximization problem in economic settings) has a zero duality gap.

The first main contribution of this paper is that we, based on the results of Harks (2020), derive an algorithm to decide the existence of a solution of the market equilibrium problem with convex and nonconvex player problems. If such an equilibrium exists, our algorithm computes it—otherwise, it indicates that no such equilibrium exists. However, three key prerequisites have to be met. First, we need some specific knowledge about potential candidates for equilibrium prices. More specifically, appropriate bounds on equilibrium prices have to be derived from necessary optimality conditions of some players, e.g., of those facing convex optimization problems. Second, a technical assumption is required for those prices that are not uniquely determined by the derived bounds. In particular, for all players, the part of their objective functions affected by non-unique prices has to be minimal or maximal in the global solution of the corresponding welfare optimization problem. Third and finally, the presented algorithm relies on solving nonconvex optimization problems to global optimality. This cannot be avoided in our setting that includes nonconvex player problems.

As second main contribution, we demonstrate the performance of the proposed algorithm by testing it on two practically relevant equilibrium problems in energy for which the three above-mentioned prerequisites are met: (i) a power market problem in which the TSO controls the underlying DC network by switching on or off DC power lines and (ii) a gas market problem in which the TSO's model is nonconvex due to the inherent nonlinearity of gas flow models. By doing so, we consider two very different settings of nonconvex market equilibrium problems: one in which the nonconvexity is continuous but nonlinear and one in which the nonconvexity is due to the presence of integer variables. Thus, these studies nicely illustrate the broad applicability of our methods. We present a detailed computational study for both problems and discuss the reasons why equilibria exist or why not. One main conclusion of this computational study is that in the case in which the nonconvexity is due to integrality constraints, we show that no equilibrium exists for a large

number of instances. On the other hand, we confirm the existence of an equilibrium for all solvable instances in the nonconvex but continuous case.

The remainder of the paper is organized as follows. In Section 2 we present the abstract model of the market equilibrium problem with convex and nonconvex players. Then, the existence and uniqueness of equilibria is studied in Section 3. The beginning of this section is based on the results by Harks (2020) and extends this work to obtain the algorithm that computes equilibria or proves that no such equilibria exist. Moreover, we provide a uniqueness result for the special case of players with unique best responses. In Section 4 we present the networked power and gas market equilibrium problems and derive theoretical results that are used in Section 5, where the developed algorithm is applied to these two cases. The paper closes with some concluding remarks and some further topics of future research in Section 6.

2. The Market Equilibrium Problem

We consider a special type of market equilibrium problems with a finite set of players $i \in I$. We assume perfect competition, i.e., the players act as price takers and do not anticipate the impact of their own actions on prices. Formally, for an exogenously given price vector $\pi \in \mathbb{R}^{n_\pi}$, every player $i \in I$ solves an optimization problem of the form

$$\min_{y_i} \quad f_i(y_i, \pi) := c_i(y_i) + \pi^\top h_i(y_i) \quad \text{s.t.} \quad y_i \in Y_i, \tag{1}$$

where $y_i \in \mathbb{R}^{n_i}$ are the decision variables of player i, $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{n_\pi} \to \mathbb{R}$ is the objective function consisting of $c_i : \mathbb{R}^{n_i} \to \mathbb{R}$ and $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_\pi}$. Moreover, $Y_i \subseteq \mathbb{R}^{n_i}$ is the non-empty feasible set of player i.

Solving the market equilibrium problem in this context means to find market-clearing prices π, i.e., prices to which the best responses of all players exist and satisfy predefined market-clearing conditions that depend on the variables of all players. We model these market-clearing conditions as

$$\sum_{i \in I} h_i(y_i) = 0. \tag{2}$$

Thus, we consider the following market equilibrium problem:

$$\begin{aligned} \text{optimization problems of the players:} \quad & (1) \text{ for all } i \in I, \\ \text{market-clearing conditions:} \quad & (2). \end{aligned} \tag{MEP}$$

A broad range of market equilibrium problems can be modeled this way. For instance, many economic applications, in which a large number of price-taking players purchase and sell goods at certain prices, fall into this category. In this context, $\pi^\top h_i(y_i)$ describes the part of the players' money either spent or gained by trading a specific amount of the goods. Moreover, the market-clearing conditions (2) ensure that the traded amounts are balanced, i.e., there is no excess demand or supply. Problems of this kind arise in energy market modeling; see, e.g., Hobbs and Helman (2004) or Gabriel, Conejo, et al. (2012) and the references therein. Note that, although the assumption of perfect competition might not be satisfied for all energy applications, this is a common assumption used in the energy market literature; see, e.g., Boucher and Smeers (2001), Daxhelet and Smeers (2007), Grimm, Kleinert, et al. (2019), and Grimm, Martin, et al. (2016) for power market models as well as Böttger et al. (2021) and Grimm, Schewe, et al. (2019) for gas market models. For two further examples from the field of energy markets see also Section 4, where we discuss them thoroughly. A detailed overview of applications from other fields can be found in, e.g., Harks (2020).

Although the existence of an equilibrium is not guaranteed in general if nonconvex players are part of the (MEP), it is well-known that in the case of existence, market equilibria correspond to welfare optima; cf. Part 1 of Theorem 2.3 in Harks (2020). Formally, the welfare problem is the optimization problem in which the sum of all players' objectives is minimized subject to the constraints of all players and the market-clearing conditions. This welfare optimization problem thus reads

$$\min_{y} \quad \sum_{i \in I} c_i(y_i) \quad \text{s.t.} \quad y \in Y, \quad \sum_{i \in I} h_i(y_i) = 0. \tag{WFP}$$

Here, we use $y := (y_i)_{i \in I} \in \mathbb{R}^{n_y}$ with $n_y := \sum_{i \in I} n_i$ as an abbreviation for the decision variables of all players $i \in I$. Analogously, let $Y := \bigtimes_{i \in I} Y_i$ denote the Cartesian product of the individual feasible sets.

Even if a global solution to the welfare optimization problem (WFP) exists, this solution does not necessarily constitute a market equilibrium if nonconvexities are present in the optimization problems as given in (1). However, if these optimization problems are convex for all π, the existence of market equilibria is well-understood. In particular, there exists a market equilibrium of (MEP) if there exists a global solution to the welfare optimization problem (WFP). To see this note that, for each player i, the convexity of the objective function over the convex set Y_i yields

$$f_i(\lambda \tilde{y}_i + (1-\lambda)\hat{y}_i, \pi) \leq \lambda f_i(\tilde{y}_i, \pi) + (1-\lambda) f_i(\hat{y}_i, \pi)$$

for all $\tilde{y}_i, \hat{y}_i \in Y_i$, $\lambda \in [0,1]$, and all $\pi \in \mathbb{R}^{n_\pi}$. Inserting the definition of the objective function and rearranging then yields

$$\begin{aligned} &c_i(\lambda \tilde{y}_i + (1-\lambda)\hat{y}_i) - \lambda c_i(\tilde{y}_i) - (1-\lambda) c_i(\hat{y}_i) \\ &\leq \pi^\top (\lambda h_i(\tilde{y}_i) + (1-\lambda) h_i(\hat{y}_i) - h_i(\lambda \tilde{y}_i + (1-\lambda)\hat{y}_i)). \end{aligned} \tag{3}$$

Note that the left-hand side is independent of π, and the latter can take any value in $\mathbb{R}^{n_\pi}$. Hence, we conclude that

$$\lambda h_i(\tilde{y}_i) + (1-\lambda) h_i(\hat{y}_i) = h_i(\lambda \tilde{y}_i + (1-\lambda)\hat{y}_i). \tag{4}$$

Since the right-hand side of (3) is zero, we see that c_i has to be convex on Y_i. Let $\tilde{y}, \hat{y}$ be feasible for (WFP) and $\lambda \in [0,1]$. The point $\lambda \tilde{y} + (1-\lambda)\hat{y}$ is also feasible since the set Y is convex and the relation (4) implies that $\sum_{i \in I} h_i(\lambda \tilde{y} + (1-\lambda)\hat{y}) = 0$ holds. Hence, the optimization problem (WFP) is convex. Furthermore, suppose that Slater's condition holds for (WFP). If there exists a global solution to (WFP), then this solution together with the multipliers of the market-clearing conditions as prices is a market equilibrium. In the presence of nonconvex players, the existence of a market equilibrium to (MEP) is not guaranteed in general; see, e.g., Example 5.1 and Example 5.2. However, in many applications there is at least one player—or possibly more—whose optimization problem is not convex.

3. Existence of Equilibria

In this section we analyze how global solutions of the welfare optimization problem (WFP) can be used to either find a market equilibrium or to determine that no market equilibria exist. To this end, we build on results from Harks (2020) to observe that market equilibria of (MEP) are equivalent to primal-dual solution pairs of the corresponding welfare problem with zero duality gap and provide some practical consequences of this result. Among those are the fact that players with unique best responses to given market prices also have a unique optimal strategy over all market equilibria (if any exists). The close relation between global solutions of the welfare problem and market equilibria motivates to first compute a global solution y^* of the welfare problem and then try to find suitable market prices π^* such that (y^*, π^*) is a market equilibrium. We show that under certain technical

assumptions, which are satisfied in the applications presented in Section 4, it suffices to check one critical price to either obtain a market equilibrium or to know that none can exist. This result is then formalized in an algorithm, which we use in Section 5 to compute market equilibria of nonconvex energy market models.

If we denote the Lagrangian of the welfare optimization problem (WFP) by

$$L(y,\pi) := \sum_{i\in I} \left(c_i(y_i) + \pi^\top h_i(y_i)\right),$$

the corresponding Lagrangian dual problem is given by

$$\sup_{\pi\in\mathbb{R}^{n_\pi}} \; d(\pi), \tag{5}$$

where $d(\pi) := \inf_{y\in Y} L(y,\pi)$. Due to weak duality we always have the relation

$$\inf_{y\in Y}\left\{\sum_{i\in I} c_i(y_i)\colon y\in Y,\ \sum_{i\in I} h_i(y_i) = 0\right\} \geq \sup_{\pi\in\mathbb{R}^{n_\pi}} \; d(\pi).$$

In the case of strong duality, i.e., if the latter inequality is satisfied with equality, we refer to the welfare optimization problem as having a zero duality gap.

Definition 3.1. The welfare optimization problem has zero duality gap if there exist globally optimal solutions y^* of the primal problem (WFP) and π^* of the dual problem (5) with the same objective function value.

However, the duality gap of the welfare problem can be positive in the presence of nonconvexities. On the other hand, Part 1 of Theorem 2.3 in Harks (2020) states that a zero duality gap is equivalent to the existence of a market equilibrium. In our setting, this result reads as follows.

Theorem 3.2 (See Part 1 of Theorem 2.3 in Harks (2020))**.** *The pair (y^*,π^*) is a market equilibrium of (MEP) if and only if y^* and π^* are global solutions of the welfare optimization problem (WFP) and the corresponding dual problem (5), respectively, with zero duality gap.*

In what follows, we use this result to derive new results, which then will finally lead to an algorithm that can decide the existence of a market equilibrium and that computes such an equilibrium if it exists. Let us start with some immediate consequences from the last theorem.

Corollary 3.3.
(a) *If (y^*,π^*) is a market equilibrium of (MEP), then y^* is a global solution of the welfare problem (WFP).*
(b) *If (y^*,π^*) is a market equilibrium of (MEP), then (y,π^*) is a market equilibrium of (MEP) for all global solutions y of the welfare problem.*
(c) *If (y^*,π^*) and $(\hat{y},\hat{\pi})$ are two market equilibria of (MEP), then so are $(y^*,\hat{\pi})$ and $(\hat{y},\pi^*)$.*
(d) *If y^* is a global solution of the welfare problem (WFP) for which there exists no π such that (y^*,π) is a market equilibrium of (MEP), then the market equilibrium problem (MEP) has no solution.*

Part (a) of the corollary ensures that only global solutions y^* of the welfare problem are candidates for a market equilibrium. Part (d) states that there does not exist any market equilibrium at all, if we find a global solution y^* of the welfare problem, which is not a market equilibrium for all π. In general, neither y^* nor π^* have to be unique in a market equilibrium and Parts (b) and (c) state that we can mix and match different solutions.

In some applications, we know for a subset $S \subseteq I$ of the players that, for all possible π, their optimization problem (1) has at most one global solution. This

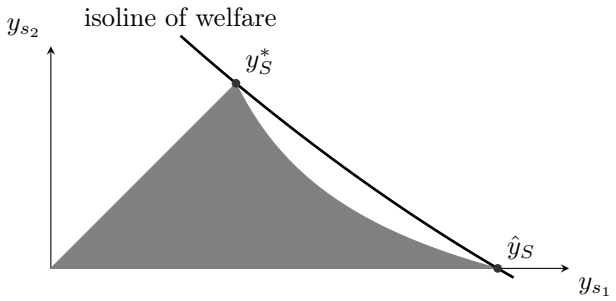

FIGURE 1. Illustration of Corollary 3.4 (based on Figures 2 and 3 of Grimm, Grübel, et al. (2019); see also Example 5.1). The nonconvexity of the feasible set of the welfare problem (WFP) (gray) leads to multiple welfare optima y_S^* and $\hat{y}_S$ that differ in the decisions of the strictly convex players. Thus, no market equilibrium to (MEP) exists.

means that these players have a unique best response if it exists at all. This is, e.g., the case for players with convex feasible sets Y_i and strictly convex objective functions $y_i \mapsto f_i(y_i, \pi)$. Those players then also have a unique optimal strategy in all market equilibria and multiplicity in global solutions of the welfare problem immediately implies that no market equilibrium exists. For a graphic illustration of this effect see Figure 1.

Corollary 3.4. *Let $S \subseteq I$ be the set of players with unique best responses for all price vectors $\pi \in \mathbb{R}^{n_\pi}$.*

(a) *If (y^*, π^*) and $(\hat{y}, \hat{\pi})$ are two market equilibria of (MEP), then $y_S^* = \hat{y}_S$.*
(b) *If y^* and $\hat{y}$ are two global solutions of the welfare problem (WFP) with $y_S^* \neq \hat{y}_S$, then the market equilibrium problem (MEP) does not have a solution.*

Proof. (a) By Corollary 3.3 (c) we know that $(\hat{y}, \pi^*)$ is also a market equilibrium and thus both y_i^* and $\hat{y}_i$ are global solutions of the optimization problem (1) of player i with price π^*. For all players $i \in S$ this solution is unique, i.e., $y_S^* = \hat{y}_S$ holds.

(b) If the market equilibrium problem would have a solution for some price vector π, then both (y^*, π) and $(\hat{y}, \pi)$ would be market equilibria, which contradicts (a). □

Note, however, that the market equilibrium problem can have more than one solution as long as the solutions differ only in the prices or in the strategies of the players in $I \setminus S$, who do not have unique best responses to given prices.

Going back to Corollary 3.3, Part (a) motivates the following approach to compute market equilibria of (MEP): First, we compute a global solution y^* of the welfare problem (WFP). Then, we find a price vector π^* such that for all players $i \in I$, the vector y_i^* is a global solution of the player's optimization problem (1) for the given price vector π^*. To determine such an equilibrium price π^*, let $\Pi(y^*) \subseteq \mathbb{R}^{n_\pi}$ be a set that includes all market equilibrium prices, i.e., it has the property

$$(y^*, \pi^*) \text{ is a market equilibrium of (MEP)} \implies \pi^* \in \Pi(y^*). \tag{6}$$

Given a welfare solution y^* and such a candidate set $\Pi(y^*)$, only prices $\pi \in \Pi(y^*)$ can be equilibrium prices. Thus, we can assume that $\Pi(y^*) \neq \emptyset$, since otherwise no market equilibrium exists.

For the forthcoming analysis, we assume that we can compute an enclosing box of $\Pi(y^*)$ that takes the form

$$\{\pi \in \mathbb{R}^{n_\pi} : \pi_k^- \leq \pi_k \leq \pi_k^+ \text{ for all } k \in \{1, \ldots, n_\pi\}\},$$

where the lower and upper bounds belong to the extended real line. We now show that it is possible in some instances to reduce the study of the existence of a market equilibrium to checking whether a global welfare optimal solution and a particular price constitute a market equilibrium.

Theorem 3.5. *Let y^* be a global solution of the welfare problem (WFP) and let $\Pi(y^*) \neq \emptyset$ be a set satisfying Condition (6). Assume that for all $k \in \{1, \ldots, n_\pi\}$ at least one of the following properties is satisfied:*

(a) $\pi_k^- = \pi_k^+$,
(b) $\pi_k^+ < \infty$ *and* $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$,
(c) $\pi_k^- > -\infty$ *and* $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$,
(d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$, *and* $(h_i(y_i^*))_k = (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$.

Then, there exists a market equilibrium of (MEP) if and only if $(y^*, \hat{\pi})$ is a market equilibrium, where the critical price $\hat{\pi}$ is defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

Proof. If $(y^*, \hat{\pi})$ is a market equilibrium, then obviously one exists. So let us assume that $(y^*, \hat{\pi})$ is not a market equilibrium. For every player $i \in I$ and all $y_i \in Y_i$ as well as all $\pi \in \Pi(y^*)$, the difference in the player's objective function values is

$$\begin{aligned} f_i(y_i, \pi) - f_i(y_i^*, \pi) &= c_i(y_i) - c_i(y_i^*) + \pi^\top (h_i(y_i) - h_i(y_i^*)) \\ &= c_i(y_i) - c_i(y_i^*) + \sum_{k=1}^{n_\pi} \pi_k (h_i(y_i) - h_i(y_i^*))_k. \end{aligned}$$

It immediately follows from (a)–(d) that this difference becomes maximal over $\Pi(y^*)$ for the critical price $\hat{\pi}$. Since $(y^*, \hat{\pi})$ is not a market equilibrium, there exists a player $i \in I$ and a strategy $y_i \in Y_i$ of this player such that $f_i(y_i, \hat{\pi}) < f_i(y_i^*, \hat{\pi})$ holds. For this player and this strategy, the following then holds for all $\pi \in \Pi(y^*)$:

$$f_i(y_i, \pi) - f_i(y_i^*, \pi) \leq f_i(y_i, \hat{\pi}) - f_i(y_i^*, \hat{\pi}) < 0.$$

If a market equilibrium would exist, then by Corollary 3.3 (b) there would exist an equilibrium price $\pi^* \in \Pi(y^*)$ such that (y^*, π^*) is a market equilibrium. But as we have shown above none of the equilibrium price candidates in $\Pi(y^*)$ supports an equilibrium in y^*. Consequently, no market equilibrium exists. □

Our approach relies on exploiting the interplay between the structural properties of the players' problems and the set of admissible prices $\Pi(y^*)$ for a given global solution y^* of the welfare problem. This is best understood when looking at the definition of the critical price $\hat{\pi}$. Looking at Case (b), one sees that if all players contribute in their minimum way to the market-clearing conditions, then it is sufficient to test for the upper bound as an equilibrium price candidate for this component. If, analogously, all players reach their maximum possible contribution for the market-clearing conditions, which is Case (c), then it is sufficient to check whether the lower bound for the component is an equilibrium price. Such upper

and lower bounds might implicitly result from (necessary) optimality conditions of the players.

Applications fulfilling the properties stated in Theorem 3.5 arise, e.g., in transportation networks. For applications in the context of energy markets, in which all the stated properties are satisfied, see Section 4. Since these applications are often formulated in terms of maximization problems, we provide the analogue of Theorem 3.5 for maximization problems in Appendix A.

In general, one wants to choose the candidate set $\Pi(y^*)$ as small as possible in order to satisfy the conditions for Theorem 3.5. However, one cannot choose $\Pi(y^*)$ arbitrarily small since Condition (6) needs to be satisfied, i.e., $\Pi(y^*)$ needs to include all market equilibrium prices. A straightforward approach to construct the set $\Pi(y^*)$ is to exploit necessary optimality conditions of each player's optimization problem. If (y^*, π) is a market equilibrium, then for all players $i \in I$, y_i^* is a global solution of optimization problem (1) with a price vector π and thus has to satisfy the necessary optimality conditions for (1). These optimality conditions evaluated at y_i^* provide constraints for possible market equilibrium prices π. Since we impose only very few assumptions on the optimization problems (1), different types of necessary optimality conditions might be needed for different classes of players. Fortunately, mixing varying types of necessary optimality conditions is not a problem here.

Remark 3.6. *Consider a player $i \in I$. If the feasible set Y_i is given by standard constraints, e.g., $Y_i = \{y_i \colon g_i(y_i) \leq 0\}$, if all functions c_i, h_i, and g_i are continuously differentiable, and if a constraint qualification for Y_i is satisfied at y_i^*, then the KKT conditions for (1) are necessary. Thus, only prices π, for which there exist multipliers μ_i with*

$$0 = \nabla c_i(y_i^*) + \nabla h_i(y_i^*)\pi + \nabla g_i(y_i^*)\mu_i, \quad 0 \leq \mu_i \perp g_i(y_i^*) \leq 0 \tag{7}$$

can be market equilibrium prices. If, additionally, $y_i \mapsto f_i(y_i, \pi)$ and g_i are convex functions, the KKT conditions are not only necessary but also sufficient optimality conditions for (1).

For other players, using the KKT conditions as necessary optimality conditions might not be possible, e.g., because they have discrete decision variables, their optimization problem is not differentiable, or their feasible set does not satisfy a constraint qualification. These players can either be ignored in the definition of $\Pi(y^*)$ or one can use alternative optimality conditions to generate conditions on market equilibrium prices π. For example, the textbooks by Clarke (1990), Luo et al. (1996), Mordukhovich (2018), Rockafellar (1970), and Rockafellar and Wets (1998) provide optimality conditions based on subdifferentials and variational analysis, which can be used in the presence of nondifferentiable or degenerate constraints.

In situations, in which the KKT conditions (7) of certain players are necessary and sufficient and used to obtain the candidate set $\Pi(y^*)$, it is possible to weaken the assumptions of Theorem 3.5. Note that this modification only works in the case that $\hat{\pi} \in \Pi(y^*)$ holds, which is a nontrivial assumption, because $\hat{\pi}$ is per definition a vertex of an enclosing box of $\Pi(y^*)$. However, in the application considered in Section 4, this assumption will indeed be fulfilled.

Corollary 3.7. *Let y^* be a global solution of the welfare problem (WFP). Moreover, let $C \subseteq I$ denote the subset of players, for which the KKT conditions (7) are necessary and sufficient optimality conditions and choose the candidate set $\Pi(y^*)$ such that Condition (6) as well as the KKT conditions of all players $i \in C$ are satisfied, i.e.,*

$$\Pi(y^*) \subseteq \{\pi \in \mathbb{R}^{n_\pi} \colon \text{for all } i \in C \text{ exists } \mu_i \text{ such that (7) holds}\}.$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

(a) $\pi_k^- = \pi_k^+$,
(b) $\pi_k^+ < \infty$ *and* $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I \backslash C$,
(c) $\pi_k^- > -\infty$ *and* $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I \backslash C$,
(d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ *and* $(h_i(y_i^*))_k = (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I \backslash C$.

Now, let the critical price $\hat{\pi}$ *be defined as*

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

If the critical price satisfies $\hat{\pi} \in \Pi(y^*)$, *then there exists a market equilibrium of* (MEP) *if and only if* $(y^*, \hat{\pi})$ *is a market equilibrium.*

Proof. If $(y^*, \hat{\pi})$ is a market equilibrium, then obviously one exists. So let us assume that $(y^*, \hat{\pi})$ is not a market equilibrium. Due to $\hat{\pi} \in \Pi(y^*)$, the KKT conditions of all players $i \in C$ are satisfied, i.e., y_i^* is a best response to $\hat{\pi}$ for all players $i \in C$. Thus, if $(y^*, \hat{\pi})$ is not a market equilibrium, this has to be due to one of the players $i \notin C$. For those players, one obtains a contradiction by applying the same argument as in the proof of Theorem 3.5. □

Finally, we utilize the previously derived results to formally state Algorithm 1, which terminates either with a market equilibrium or with the information that no market equilibrium exists. The presented algorithm is based on Theorem 3.5. Note that for the situation described in Corollary 3.7, the algorithm remains the same except for Line 8, where it suffices to check whether y_i^* is a best response to the price vector $\hat{\pi}$ for all players $i \in I \backslash C$.

Algorithm 1: Deciding the existence of an equilibrium of (MEP) and computing an equilibrium in case of existence

Input : Market equilibrium problem (MEP)
1 Compute a global solution y^* of the welfare optimization problem (WFP).
2 **if** (WFP) *cannot be solved* **then**
3 | **return** *"No claim regarding the existence of equilibria can be made."*
4 **else if** (WFP) *does not have a solution* **then**
5 | **return** *"No market equilibrium exists."*
6 **else**
7 | Define the critical price vector $\hat{\pi}$ as in Theorem 3.5.
8 | **if** y_i^* *is a best response to the price vector* $\hat{\pi}$ *for all players* $i \in I$ **then**
9 | | **return** $(y^*, \hat{\pi})$ *is a market equilibrium.*
10 | **else**
11 | | **return** *"No market equilibrium exists."*
12 | **end**
13 **end**

Remark 3.8. (a) *In Lines 1 and 8, we assume that it is possible to solve the potentially nonconvex problems* (WFP) *and* (1) *to global optimality.*
(b) *If the solver computing the global solution of* (WFP) *in Line 1 additionally provides dual variables such that strong duality holds, then a market equilibrium exists and the dual variables associated to the market-clearing conditions are market equilibrium prices. However, there is no guarantee*

that the solver is able to do so; see, e.g., Section 5.3 in the BARON manual by Sahinidis (2021).

(c) *To execute Line 7, Algorithm 1 relies on the existence of a procedure to compute the set $\Pi(y^*)$ for any given welfare optimal solution y^*. Please see Section 4, where we present such procedure for specific applications.*

(d) *Note that if multiple equilibria exist, Algorithm 1 computes one of those equilibria. An extension of the algorithm so that it computes all equilibria is out of reach since this corresponds to computing all globally optimal solutions of the nonconvex welfare maximization problem, which itself is an extremely challenging problem.*

(e) *Let us finally comment on that we have to solve a potentially NP-hard and nonconvex welfare optimization problem at the beginning of the algorithm. This nonconvexity is inherited from the nonconvexity of the corresponding player problems. Fortunately, there is an increasing number of NP-hard problems in the energy sector, e.g., the AC-OPF problem, for which more and more sophisticated solution techniques have been developed to solve such problems to global optimality; see, e.g., Krasko and Rebennack (2017) and the references therein.*

Theorem 3.9. *If the assumptions of Theorem 3.5 are satisfied and if the* (WFP) *can be solved to global optimality or it can be decided that it has no solution, then Algorithm 1 terminates correctly with either a market equilibrium of* (MEP) *or with the information that such an equilibrium does not exist.*

Proof. Follows from Theorem 3.5 and Corollary 3.3. □

4. Applications in Energy Market Modeling

In this section, we consider energy networks modeled as graphs $G = (V, A)$ and assume that the graph G is directed and weakly connected. The node set V can further be split in the set $V_- \subset V$ of consumer locations, the set $V_+ \subset V$ of producer locations, and the set $V_0 \subset V$ of so-called inner nodes. For the ease of presentation, we assume that these three sets are disjoint and that $V_- \cup V_+ \cup V_0 = V$ holds. A possible approach to handle nodes, where both a producer and a consumer are located, is discussed in Section 5. In the market model, there are three types of players: producers, consumers, and the transmission system operator (TSO). For the sake of simplicity, we assume that each consumer node can be identified with a single consumer and each producer node with a single producer. For publications that study similar producer and consumer models as we do see, e.g., Gabriel, Conejo, et al. (2012), Grimm, Grübel, et al. (2019), Hobbs and Helman (2004), and Krebs et al. (2018).

Consider the consumer located at node $u \in V_-$, let π_u be the market price at this node, and let $P_u(\cdot)$ be the consumer's inverse demand function. The consumer maximizes his surplus by choosing his demand d_u as a global solution of

$$\max_{d_u} \quad \int_0^{d_u} P_u(t)\,\mathrm{d}t - \pi_u d_u \quad \text{s.t.} \quad d_u \geq 0. \tag{8}$$

For the producer located at $u \in V_+$, let π_u again be the market price at this node and let $c_u(\cdot)$ denote the variable cost of production. The producer maximizes her profits by choosing her production level y_u within her production capacity $\bar{y}_u > 0$ as a global solution of

$$\max_{y_u} \quad \pi_u y_u - c_u(y_u) \quad \text{s.t.} \quad \bar{y}_u \geq y_u \geq 0. \tag{9}$$

The TSO is responsible for operating the network and his goal is to maximize congestion rents by routing as much of the commodity from low-price to high-price

areas. For all arcs $a \in A$, let q_a denote the flow along that arc and let $q := (q_a)_{a\in A}$ be the vector of all flows. Besides the flow q, the TSO can have an additional decision variable x to describe choices such as switching on or off a power line in the power network or activating a compressor station in the gas network. In general, the components of x can be continuous or integer. The optimization problem of the TSO is then given by

$$\max_{q,x} \quad \sum_{u\in V_-\cup V_+} \pi_u \left(\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a \right) - c^{\mathrm{t}}(q,x) \tag{10a}$$

$$\text{s.t.} \quad \sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a \geq 0 \quad \text{for all } u \in V_-, \tag{10b}$$

$$\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a \leq 0 \quad \text{for all } u \in V_+, \tag{10c}$$

$$\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a \geq -\bar{y}_u \quad \text{for all } u \in V_+, \tag{10d}$$

$$\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a = 0 \quad \text{for all } u \in V_0, \tag{10e}$$

$$F(q,x) \geq 0. \tag{10f}$$

Here, $c^{\mathrm{t}}(q,x)$ describes the transportation costs and $\delta^{\mathrm{in}}(u)$ ($\delta^{\mathrm{out}}(u)$) denote the sets of incoming (outgoing) arcs at node u. Finally, the mapping $F(q,x)$ summarizes the, potentially nonconvex, network-related physical and technical constraints. The first three constraints make sure that the net flow can only be positive at consumer nodes, negative at producer nodes, and that it has to be zero at inner nodes. Constraints (10d) ensure that at each supply node, the TSO does not obtain more of the respective energy carrier than the nodal capacity allows for, i.e., the TSO is informed about the capacities of all production facilities. Note that this is consistent with the assumption of perfect competition.

The model is completed by the nodal market-clearing conditions

$$\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a = d_u \quad \text{for all } u \in V_-, \tag{11a}$$

$$\sum_{a\in\delta^{\mathrm{in}}(u)} q_a - \sum_{a\in\delta^{\mathrm{out}}(u)} q_a = -y_u \quad \text{for all } u \in V_+, \tag{11b}$$

which ensure that production, consumption, and the in- and outgoing flows are balanced at every node of the network.

The complete energy market equilibrium problem is thus given by

$$\begin{aligned} \text{consumers:} &\quad (8) \text{ for all } u \in V_-, \\ \text{producers:} &\quad (9) \text{ for all } u \in V_+, \\ \text{TSO:} &\quad (10), \\ \text{market-clearing conditions:} &\quad (11). \end{aligned} \tag{MEP-E}$$

Since it has the form discussed in Sections 2 and 3, we know that market equilibria $(d^*, y^*, q^*, x^*, \pi^*)$ are related to global solutions of the welfare maximization problem

$$\begin{aligned}
\max_{d,y,q,x} \quad & \sum_{u \in V_-} \int_0^{d_u} P_u(t)\,\mathrm{d}t - \sum_{u \in V_+} c_u(y_u) - c^{\mathrm{t}}(q,x) \\
\text{s.t.} \quad & \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = 0 \quad \text{for all } u \in V_0, \\
& \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = d_u \quad \text{for all } u \in V_-, \\
& \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = -y_u \quad \text{for all } u \in V_+, \\
& F(q,x) \geq 0, \quad d \geq 0, \quad \bar{y} \geq y \geq 0.
\end{aligned} \tag{WFP-E}$$

In the remainder of this section, we show that the conditions from Corollary A.2 are satisfied under standard assumptions on the market equilibrium problem (MEP-E). Thus, we can use Algorithm 1 to decide on the existence of equilibria and to compute an equilibrium if one exists. The imposed standard assumptions read as follows.

Assumption 1. (1) *The inverse demand functions $P_u(\cdot)$ are continuous and strictly decreasing for all $u \in V_-$.*

(2) *The variable cost functions $c_u(\cdot)$ are monotonically increasing with $c_u(0) = 0$, convex, and continuously differentiable for all $u \in V_+$.*

These assumptions ensure that the producers and consumers have concave maximization problems subject to linear constraints. Consequently, any global solution of (8) and (9) is characterized by the respective KKT conditions. Exploiting this observation, we can first derive sufficient information from these KKT conditions to appropriately bound market prices and second ensure that the parts of the TSO's objective function affected by prices that are not yet uniquely determined by the obtained bounds are minimal or maximal in the global solution of the welfare problem. Hence, we can prove that Corollary A.2 is applicable.

Theorem 4.1. *Suppose Assumption 1 holds. Let (d^*, y^*, q^*, x^*) be a global solution of the welfare problem (WFP-E) and define $\hat{\pi}$ as*

$$\hat{\pi}_u := \begin{cases} P_u(d_u^*), & \text{if } u \in V_-, \\ c_u'(y_u^*), & \text{if } u \in V_+. \end{cases}$$

Then, either $(d^*, y^*, q^*, x^*, \hat{\pi})$ is a market equilibrium of (MEP-E), or there is no market equilibrium.

Proof. We use the KKT conditions of the producers and consumers to define the set $\Pi(d^*, y^*, q^*, x^*)$. For a consumer located at $u \in V_-$, these KKT conditions can be reduced to

$$0 \leq d_u^* \perp \pi_u \geq P_u(d_u^*).$$

For a producer located at $u \in V_+$, the KKT conditions read

$$\pi_u - c_u'(y_u^*) + \beta_u^- - \beta_u^+ = 0, \quad 0 \leq \beta_u^- \perp y_u^* \geq 0, \quad 0 \leq \beta_u^+ \perp \bar{y}_u - y_u^* \geq 0,$$

where β_u^- and β_u^+ denote the corresponding dual variables.

All candidates for market equilibrium prices are thus elements of the set $\Pi(d^*, y^*, q^*, x^*)$ defined by

$$\left\{ \pi \in \mathbb{R}^{|V_+ \cup V_-|} : \ \pi_u \in \begin{cases} \{P_u(d_u^*)\}, & \text{if } u \in V_-,\ d_u^* > 0, \\ [P_u(d_u^*), \infty), & \text{if } u \in V_-,\ d_u^* = 0, \\ \{c_u'(y_u^*)\}, & \text{if } u \in V_+,\ \bar{y}_u > y_u^* > 0, \\ (-\infty, c_u'(y_u^*)], & \text{if } u \in V_+,\ y_u^* = 0, \\ [c_u'(y_u^*), +\infty), & \text{if } u \in V_+,\ y_u^* = \bar{y}_u, \end{cases} \right\}.$$

Due to the simple structure of $\Pi(d^*, y^*, q^*, x^*)$, the critical price $\hat{\pi}$ can be stated explicitly and satisfies $\hat{\pi} \in \Pi(d^*, y^*, q^*, x^*)$. By definition of $\Pi(d^*, y^*, q^*, x^*)$, d^* thus is the consumers' best response to $\hat{\pi}$ and y^* is the producers' best response to $\hat{\pi}$. Consequently, it remains to show that Cases (a)–(d) of Corollary A.2 are fulfilled for the TSO.

For all production nodes $u \in V_+$ with positive but not binding production level and for all consumption nodes $u \in V_-$ with positive demand, there is exactly one candidate for an equilibrium price in $\Pi(d^*, y^*, q^*, x^*)$, namely $\hat{\pi}_u$. Hence, we are in Case (a) of Corollary A.2.

For all demand nodes $u \in V_-$ with zero demand, we have

$$\begin{aligned} (h_{\mathrm{TSO}}(q^*))_u &= \sum_{a \in \delta^{\mathrm{in}}(u)} q_a^* - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a^* \\ &= d_u^* = 0 \leq \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = (h_{\mathrm{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite minimum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (c) of Corollary A.2 are satisfied.

For all supply nodes $u \in V_+$ with zero production, we have

$$\begin{aligned} (h_{\mathrm{TSO}}(q^*))_u &= \sum_{a \in \delta^{\mathrm{in}}(u)} q_a^* - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a^* \\ &= -y_u^* = 0 \geq \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = (h_{\mathrm{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite maximum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (b) of Corollary A.2 are satisfied.

For supply nodes $u \in V_+$ at full capacity, we have

$$\begin{aligned} (h_{\mathrm{TSO}}(q^*))_u &= \sum_{a \in \delta^{\mathrm{in}}(u)} q_a^* - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a^* \\ &= -y_u^* = -\bar{y}_u \leq \sum_{a \in \delta^{\mathrm{in}}(u)} q_a - \sum_{a \in \delta^{\mathrm{out}}(u)} q_a = (h_{\mathrm{TSO}}(q))_u \end{aligned}$$

for all q feasible for (10). Since, additionally, $\hat{\pi}_u$ is chosen as the finite minimum nodal price in $\Pi(d^*, y^*, q^*, x^*)$, all conditions in Case (c) of Corollary A.2 are satisfied.

Consequently, all conditions of Corollary A.2 are satisfied and the claim follows. □

In general, choosing a sufficiently tight superset of all possible equilibrium prices and computing the critical price can be difficult. However, in this application, we get the critical price $\hat{\pi}$ "for free" once a global solution (d^*, y^*, q^*, x^*) of the welfare problem (WFP-E) is known.

Additionally, since $\Pi(d^*, y^*, q^*, x^*)$ is defined via the KKT conditions of the consumers and producers, which are necessary and sufficient under Assumption 1, and since $\hat{\pi} \in \Pi(d^*, y^*, q^*, x^*)$, we immediately know that d_u^* and y_u^* are best responses of all consumers $u \in V_-$ and all producers $u \in V_+$ to the prices $\hat{\pi}$. To check if $(d^*, y^*, q^*, x^*, \hat{\pi})$ is a market equilibrium, one thus only has to verify that (q^*, x^*) is a best response of the TSO to the prices $\hat{\pi}$.

5. Computational Study

We now consider in more detail two applications in energy market modeling that fit into the framework described above. We start with a detailed description of the two applications in Section 5.1 and continue with giving information about the computational setup and our test instances in Section 5.2. Afterward, we present the numerical results for Algorithm 1; first for the gas application (Section 5.3) and then for the power application (Section 5.4). The results are discussed and insights are given on the conditions leading to non-existence of equilibria.

5.1. The Optimization Problems of the TSOs.

We examine as applications the case of a TSO operating a gas network under nonlinear stationary gas flow equations and the case of a TSO switching DC lines in a DC power network. For each application we provide an instance for which the duality gap of the welfare problem is nonzero, i.e., for which no market equilibrium exists. In order to establish non-existence of an equilibrium for the gas flow instance, we apply Corollary 3.4. For the DC line switching instance, Algorithm 1 is applied to determine that no market equilibrium exists.

We start with describing the optimization problems (10) of the respective TSOs, focusing mostly on the transportation costs $c^{\mathrm{t}}(q, x)$ and the network constraints $F(q, x)$. The optimization problems of the consumers (8) and producers (9) stay the same as in Section 4. An overview of all technical and economic parameters and variables together with the respective units used in this section can be found in Table 3 located in the Appendix B.

5.1.1. *Gas Flow.*

This application is taken from Grimm, Grübel, et al. (2019). We choose this application to study in particular the impact of continuous nonconvexities on the existence of an equilibrium. The network-related physical constraints are given by the following model of stationary gas physics:

$$\begin{aligned} p_u^2 - p_v^2 &= \Lambda_a q_a |q_a|, \quad a = (u, v) \in A, \\ p_u^- &\le p_u \le p_u^+, \quad u \in V, \\ q_a^- &\le q_a \le q_a^+, \quad a \in A. \end{aligned}$$

Here, the gas flow through the pipes is determined by the so-called Weymouth equation; see, e.g., the chapter by Fügenschuh et al. (2015) in Koch et al. (2015) for more information on this topic. This equation links the flow q_a on an arc to the pressure drop $(p_u^2 - p_v^2)$ over this arc in a nonlinear way. Finally, nodal pressure and flow bounds are imposed to, e.g., guarantee technical and contractual requirements. For more information on the general setup see Grimm, Grübel, et al. (2019).

The transportation costs are assumed to increase quadratically with the flow, i.e., we have

$$c^{\mathrm{t}}(q) = \sum_{a \in A} \alpha q_a^2.$$

Next, we present an instance in which no market equilibrium exists for the described gas application of (MEP-E). To simplify the presentation, we set the transportation costs to zero for now.

Example 5.1 (Non-existence of Equilibria)**.** *For the instance depicted in Figure 2, Grimm, Grübel, et al. (2019) show that exactly two welfare maximal solutions exist, namely*

$$d_2 = 1+\sqrt{2}, \quad d_3 = 0, \quad y_1 = 1+\sqrt{2}, \quad q_{1,2} = \sqrt{2}, \quad q_{1,3} = 1, \quad q_{2,3} = -1,$$
$$p_1 = \sqrt{2}, \quad p_2 = 0, \quad p_3 = 1,$$

and

$$\bar{d}_2 = \bar{d}_3 = 1, \quad \bar{y}_1 = 2, \quad \bar{q}_{1,2} = \bar{q}_{1,3} = 1, \quad \bar{q}_{2,3} = 0, \quad \bar{p}_1 = \sqrt{2}, \quad \bar{p}_2 = \bar{p}_3 = 1,$$

which differ in demand and generation. Hence, by Corollary 3.4, we know that there is no market equilibrium for this instance. For an illustration of the given example see Figure 1, in which the demand at node 2 is plotted on the abscissa and the demand at node 3 on the ordinate.

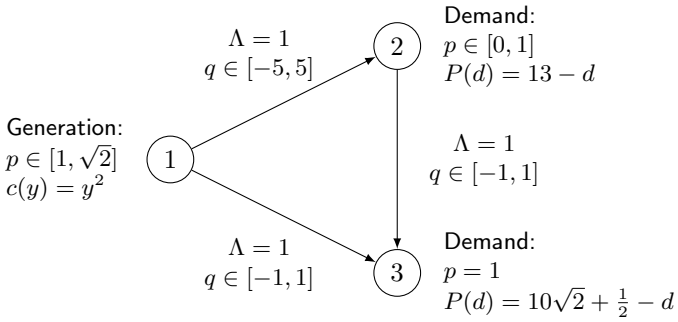

FIGURE 2. 3-node network with all physical and economic data considered in Examples 5.1 and 5.2; based on Figure 1 of Grimm, Grübel, et al. (2019).

5.1.2. *DC Line Switching.* This application addresses the problem introduced in, e.g., Fisher et al. (2008), Hedman et al. (2008), or Hedman et al. (2009) from an economic point of view. We split the set A of arcs into the set of switchable arcs A_+ and non-switchable arcs A_-. As soon as a power line $a \in A_+$ is switched off (indicated by the binary variable z_a being equal to 0), no power flow over this line is possible and no physical laws are imposed for this line. Conversely, as soon as a power line $a \in A_+$ is switched on ($z_a = 1$), the power flow over this line is bounded by its capacities and follows physical laws.

As in Fisher et al. (2008), Hedman et al. (2008), and Hedman et al. (2009), we use the lossless direct current (DC) load flow approximation to model power flow. In particular, we follow the formulation given in Section 3.7 in Zimmerman and Murillo-Sánchez (2021). In the previously presented application, we focus on the effect of continuous nonconvexities on the existence of an equilibrium. Here, we focus on integrality restrictions and therefore choose this linear power flow model. In total, the network-related physical constraints read

$$q_a^- \le q_a \le q_a^+, \quad a \in A_-, \tag{12a}$$
$$\theta_u - \theta_v - \theta_a^{\text{shift}} = X_a q_a, \quad a = (u,v) \in A_-, \tag{12b}$$
$$M_a^-(1-z_a) \le \theta_u - \theta_v - \theta_a^{\text{shift}} - X_a q_a \le M_a^+(1-z_a), \quad a = (u,v) \in A_+, \tag{12c}$$
$$q_a^- z_a \le q_a \le q_a^+ z_a, \quad a \in A_+, \tag{12d}$$
$$z_a \in \{0,1\}, \quad a \in A_+. \tag{12e}$$

First, all flows q_a are bounded from below and above by the capacities of the respective DC lines. In accordance with the DC load flow approximation, the flow on a line multiplied by its reactance X_a has to equal the nodal phase angle change $(\theta_u - \theta_v)$ for all non-switchable lines. In addition, for transformer nodes, a phase shift angle θ_a^{shift} is considered. The DC load flow approximation is only fulfilled for switchable lines if they are switched on. Finally, if a switchable line is switched off, the flow on this line must be zero.

Let us briefly comment on how we choose the big-M values M_a^- and M_a^+ in Equation (12c). On the one hand, it is crucial to find sufficiently large big-M values to obtain a correct linearization. On the other hand, the big-M values should be as tight as possible to avoid numerical problems. While such big-M values are easily obtained for (12d) by the flow bounds, computing big-M values for (12c) is harder. Due to physics, there are bounds on the differences of nodal phase angles. One can derive nodal phase angle bounds by fixing the phase angle at a reference node to zero. Since the path from the reference node to any other node leads over at most $|V| - 1$ arcs, the bounds on the nodal phase angles differences can be transferred to bounds on the nodal phase angles. These bounds could be tightened by solving an all-pairs longest-path problem. However, since this problem is known to be NP-hard, we refrain from using this approach. The big-M values finally result from the DC load flow approximation utilizing the derived phase angle bounds and the flow bounds.

As before, the transportation costs increase quadratically with the flow. In addition, a fee has to be paid for each line that is switched on. Hence, the transportation costs are given by

$$c^{\mathrm{t}}(q, z) = \sum_{a \in A} \alpha q_a^2 + \sum_{a \in A_+} \beta z_a.$$

In the following example, we provide an instance for which no market equilibrium exists for the described application of (MEP-E). For the sake of simplicity, we fix the transportation costs to zero. Transformers are also not taken into account in this example. Later in the computational study, we also provide instances without an equilibrium when transportation costs are nonzero and transformers are included.

Example 5.2 (Non-existence of Equilibria)**.** *The instance considered here is based on the instance considered in Example 5.1. The economic data is the same. In addition, all reactances are 1 and the line* $(2, 3)$ *is switchable. The welfare maximum computed in Step 1 of Algorithm 1 is the following:*

$$d_2 = \frac{11}{3}, \quad d_3 = 1, \quad y_1 = \frac{14}{3}, \quad q_{1,2} = \frac{11}{3}, \quad q_{1,3} = 1, \quad q_{2,3} = 0,$$
$$\theta_1 = \frac{11}{3}, \quad \theta_2 = 0, \quad \theta_3 = \frac{8}{3}, \quad z_{2,3} = 1.$$

The next step is to test whether the TSO's best response to the resulting critical price vector

$$\hat{\pi}_1 = \frac{28}{3}, \quad \hat{\pi}_2 = \frac{28}{3}, \quad \hat{\pi}_3 = 10\sqrt{2} - \frac{1}{2},$$

which is defined as in Corollary A.2, coincides with the TSO's strategy in the welfare maximum. However, these prices are not incentive-compatible for the TSO since—given these prices—the TSO's objective is to route as much as possible to node 3, neglecting node 2. This goal is achieved by the strategy

$$\bar{q}_{1,2} = 0.5, \quad \bar{q}_{1,3} = 1, \quad \bar{q}_{2,3} = 0.5, \quad \bar{\theta}_1 = 1, \quad \bar{\theta}_2 = 0.5, \quad \bar{\theta}_3 = 0, \quad \bar{z}_{2,3} = 0$$

and not by (q, θ, z)*. Thus, Algorithm 1 terminates with the indication that no market equilibrium exists.*

TABLE 1. Overview of the instances of the gas flow application

Name	$\lvert V\rvert$	$\lvert V_-\rvert$	$\lvert V_+\rvert$	$\lvert A\rvert$	# instances
Gas-134-S	134	45	3	133	60
Gas-11-H	11	3	3	10	36

5.2. **Computational Setup and Test Instances.** We implemented Algorithm 1 in Python 3.8.5 using Pyomo 5.7.3; see Hart et al. (2017). In order to solve the NLPs arising in the gas application, we use ANTIGONE 1.1; see Misener and Floudas (2014). In turn, we solve the MILPs arising in the power application with Gurobi 9.1.1; see Gurobi Optimization (2021). The computations have been carried out on the Woody compute nodes with four Xeon E3-1240 v5 CPUs running at 3.50 GHz with 32 GB of RAM; for more information on the compute cluster see Regionales Rechenzentrum Erlangen (2021). For each considered instance, we limit the runtime to 1 hour.

5.2.1. *Gas Flow Instances.* For the gas application, we adapted instances from Heitsch et al. (2021) and Schewe et al. (2020), which are both based on networks from Schmidt et al. (2017). An overview of the resulting 96 instances is presented in Table 1.

The 60 instances Gas-134-S are identical with those presented in Schewe et al. (2020) but we set fixed transportation costs to 0, choose the transportation cost factor α in $\{0.01, 0.05, 0.1\}$, and "shifts" of the intercepts of the demand functions in $\{-10, -5, 0, 5, 10\}$. We do not include fixed transportation costs here as the existence of a short-run market equilibrium is independent of considering any fixed costs. In addition, we divide all slopes of the demand functions by 10 and multiply all pressure drop coefficients by 8 to obtain network congestion for more instances.[1] This approach is chosen since Grimm, Grübel, et al. (2019) show in Theorem 4.3 that—as long as nodal pressures are not binding and further mild assumptions are fulfilled—an equilibrium exists for the considered market equilibrium problem. Therefore, to avoid too many instances with guaranteed equilibrium in our numerical study, a high percentage of instances with network congestion is desirable. We address this issue further when presenting the numerical results in Section 5.3.

The instances Gas-11-H are taken from Heitsch et al. (2021).[2] Varying the transportation cost factor α within $\{0.01, 0.05, 0.1\}$ yields 36 instances. To increase the percentage of instances with network congestion, all pressure drop coefficients in Gas-11-H are again multiplied by the factor 8. For further information on the data, see Heitsch et al. (2021).

5.2.2. *DC Line Switching Instances.* For the power application, we adapted the instances included in the Software MATPOWER 7.0; see Zimmerman and Murillo-Sánchez (2019) or Zimmerman, Murillo-Sánchez, and Thomas (2011) for details. We restrict our analysis to those instances for which the generation cost data is provided as polynomial cost functions.[3] In addition, we neglect all instances for which the reported minimum and maximum phase angle difference coincide in all nodes. The reason is that the same minimum and maximum phase angle difference imply the same lower and upper flow bound, i.e., there is a unique solution w.r.t.

[1] An increase of all pressure drop coefficients around the factor 8 corresponds, e.g., to a diameter reduction by 33 % for all pipes or the consideration of a hydrogen network instead of a natural gas network (this approximately equals the change in the specific gas constant).

[2] Since the flow values in Heitsch et al. (2021) are given as volumetric flows under normal conditions, we convert them to mass flow assuming a gas density of $0.87\,\mathrm{kg/m^3}$.

[3] For the two instances case30pwl and case_RTS_GMLC, the generation cost data is given by piecewise linear functions.

TABLE 2. Overview of the instances of the DC line switching application

Instance	$\lvert V\rvert$	$\lvert V_-\rvert$	$\lvert V_+\rvert$	$\lvert A\rvert$	$\lvert A_+\rvert$
Smallest	5	3	4	6	1
Average	56	33	15	97	10
Biggest	300	191	69	411	42

the flow in the welfare problem (WFP-E) and a matching unique solution of the TSO problem (10). Hence, for these instances, an equilibrium always exists as the TSO has no possibility to deviate from the welfare-maximal solution. After deleting the described instances, 29 instances remain. Since all instances with more than 1000 nodes and more than 1500 arcs cannot be solved to optimality within the time limit, we only report on the remaining 17 instances here. Finally, we vary the transportation cost factor α and switching costs β as follows: $\alpha \in \{0.01, 0.05, 0.1\}$ and $\beta \in \{20, 50\}$. Thus, the final test set contains 102 instances in total. An overview of this test set is given in Table 2.

The economic data is obtained in the following way. For the calibration of the demand functions, we assume an elasticity of -0.1. The respective reference price is chosen to be the mean of the suppliers' critical prices given the real power outputs y_u reported by MATPOWER, i.e., the reference price equals

$$\frac{1}{|V_+|} \sum_{u \in V_+} c'_u(y_u).$$

The respective reference quantity is the real power demand reported by MATPOWER. For generators, we utilize the reported generator cost data and generation capacities. Again, fixed costs are not considered as the existence of a short-run market equilibrium is independent of any given fixed costs. If multiple generators are located at one node, we use the average of the coefficients of the reported polynomial cost functions. We further note that there is also the possibility of a consumer and a producer being located at the same node. For these nodes, Case (a) of Corollary A.2 has to be valid in order to apply Algorithm 1. This is always the case.

Since there is not enough information available on switchable arcs, we randomly select 10 % of all arcs as switchable.[4] The reactance X_a and the transformer phase angle shift θ_a^{shift} are chosen as described in Section 3 in Zimmerman and Murillo-Sánchez (2021). The flow bounds are obtained from the minimum and maximum phase angle difference reported by MATPOWER in combination with the DC load flow approximation for all arcs $a \in A$, i.e.,

$$q_a^- = \frac{(\theta_u - \theta_v)^- - \theta_a^{\text{shift}}}{X_a} \quad \text{and} \quad q_a^+ = \frac{(\theta_u - \theta_v)^+ - \theta_a^{\text{shift}}}{X_a}$$

holds if $X_a > 0$, and

$$q_a^- = \frac{(\theta_u - \theta_v)^+ - \theta_a^{\text{shift}}}{X_a} \quad \text{and} \quad q_a^+ = \frac{(\theta_u - \theta_v)^- - \theta_a^{\text{shift}}}{X_a}$$

holds if $X_a < 0$.

Finally, the main questions that arise are the following:

(1) How often does a market equilibrium exist for the two considered applications of (MEP-E)?
(2) Under which circumstances does it become more likely that an equilibrium exists for the two considered applications of (MEP-E)?

[4]The number is rounded up to the next integer. The respective random seed to initialize the random number generator of the Python package random equals the number of arcs in the network.

We answer these questions now by applying Algorithm 1 to the presented instances.

5.3. **Numerical Results: Gas Flow.** The numerical results of Algorithm 1 applied to the described gas instances are as follows. In total, 84 instances out of the 96 instances (87.5 %) are solved within the time limit of 1 hour. For the remaining 12 instances, we cannot make any statement regarding the existence of an equilibrium within this time limit. The average runtime over all solved instances is 53.1 s and the median runtime is 18.7 s. To interpret the results, we call an instance congested if at least one of the following cases applies in the global welfare solution:

(1) the flow on an arc is strictly positive and at the arc's upper flow bound, or
(2) the flow on an arc is strictly negative and at the arc's lower flow bound, or
(3) the nodal pressure is at an upper bound and another nodal pressure is at a lower bound, while a directed flow path from this node to the other node exists.

We observe network congestion in 56 (66.7 %) solved instances. Similar arguments as used by Grimm, Grübel, et al. (2019) reveal that an equilibrium exists for the uncongested 28 solved instances. Our computational results confirm this and show that there also exists a market equilibrium in all solved and congested 56 instances.

Even though no instances (out of the 84 instances mentioned above) without equilibrium exist in our numerical study, we have seen in Example 5.1 that the non-existence of an equilibrium is possible for the considered application. We note that this example was handcrafted with very specific data to ensure that it does not have a market equilibrium. In contrast, our results suggest that non-existence of an equilibrium hardly occurs for practical instances.

5.4. **Numerical Results: DC Line Switching.** For the 102 instances of the DC line switching application, the average runtime of Algorithm 1 is 0.4 s and the median is 0.3 s. As already mentioned above, all 102 instances are solved to global optimality within the time limit. A market equilibrium exists for 60 out of the 102 instances. No market equilibrium exists for the MATPOWER instances case30, case30Q, case_ieee30, case39, case57, case145, and case300 for all variations of the transportation cost factor and the switching costs. To simplify notation, we use, e.g., case30-0.1-20 as an abbreviation for case30 with transportation cost factor $\alpha = 0.1$ and switching costs $\beta = 20$. Moreover, in the following, we refer by welfare solution and TSO solution to the global optimal solutions of the welfare optimization problem and of the optimization problem of the TSO.

In the following, we discuss the circumstances leading to non-existence of an equilibrium based on examples from our computational study. In particular, there does not exist an equilibrium when the welfare gains outweigh the losses induced by a network decision, while the TSO's profit gains do not. This may even result in the TSO incurring losses in the welfare solution as the corresponding gains of the producers and the consumers are greater than the losses of the TSO. Since the TSO's objective function value is bounded from below by zero (zero flows and all lines switched off), no equilibrium exists in these instances. Actually, this situation occurs in 29 out of the 42 instances without equilibrium.

All instances case30, case30Q, and case_ieee30 are based on the same network and admit negative TSO profits in the welfare solution. Non-existence of an equilibrium is caused in all instances by a single line being switched on in the welfare solution, which is switched off in the optimal solution of the TSO. In Figure 3, the underlying situation is exemplarily depicted for the instance case30-0.1-20. The welfare solution and the TSO solution differ only in the flows on the three depicted lines, of which $(10, 20)$ is the mentioned switchable line. By switching this line off, the TSO separates the consumers located at nodes 18, 19, and 20 from the rest of the network.

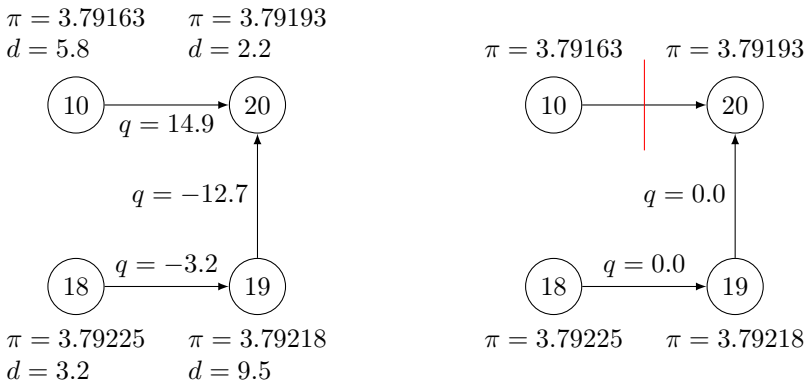

FIGURE 3. Part of the graph on which the welfare solution (left) and the best response of the TSO (right) differ for the instance case30-0.1-20

The reason for this is obvious. While the welfare gains from serving these three consumers clearly outweigh the losses induced by switching on the line $(10, 20)$ in the welfare solution, the profit gains for the TSO in case of switching this line on are by far too low to outweigh the switching costs since price differences are too low in the described subgraph. Consequently, the TSO has no incentive to switch on line $(10, 20)$.

A similar situation occurs for the instance case57. Here, non-existence of an equilibrium is caused by two lines being switched on in the welfare solution, which are switched off in the optimal solution of the TSO. In Figure 4, all differences of the welfare and the TSO solution w.r.t. the flows are presented. The switching decisions differ for the lines $(54, 55)$ and $(35, 36)$. Again, individual consumers are separated from the rest of the network in the TSO solution, namely the consumers located at node 54 and at node 35. The reason for this is the same as before. The profit gains induced by connecting these consumers to the network do not outweigh the related switching costs for the TSO.

In all instances described so far, one or multiple adjacent nodes at which consumers are located have been separated from the rest of the network in the TSO solution but not in the welfare solution. We like to note that, e.g., in the instance case39, the same is true for a node where a generator is located.

There are three main circumstances that might ensure the existence of an equilibrium in the above described situations: (i) lower switching costs, (ii) higher switching costs, or (iii) a higher transportation cost factor. If the switching costs are low enough (Case (i)), the TSO's decisions align with the welfare solution, since then the gained profits indeed outweigh the losses due to switching. If, on the other hand, switching costs are high enough (Case (ii)), the welfare solution matches the TSO's solution since then the welfare gains as well as the profit gains no longer outweigh the switching costs. Actually, in all instances of our computational study for which a market equilibrium exists, this case applies. Due to relatively high switching costs, all lines are switched off in the welfare and the TSO solution. A higher transportation cost factor (Case (iii)) leads to increased price differences within the network. Resulting profit gains for the TSO might indeed exceed possible switching costs and therefore it becomes more likely that the TSO switches on lines.

To conclude, we study the Cases (i)–(iii) by varying in more detail the switching costs and the transportation cost factor for the MATPOWER case case39. The results for varying the switching costs are depicted in Figure 5. Indeed, if the switching

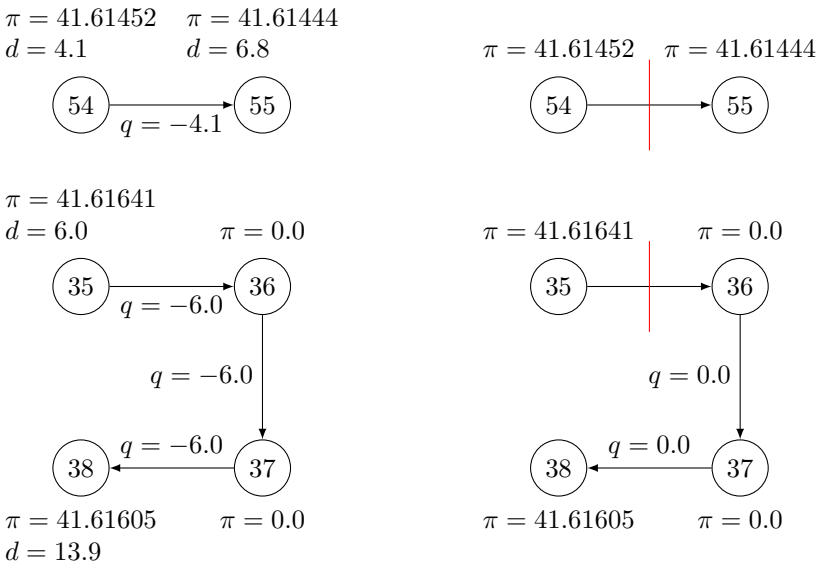

FIGURE 4. Parts of the graph on which the welfare solution (left) and the best response of the TSO (right) differ for the instance case57-0.1-20

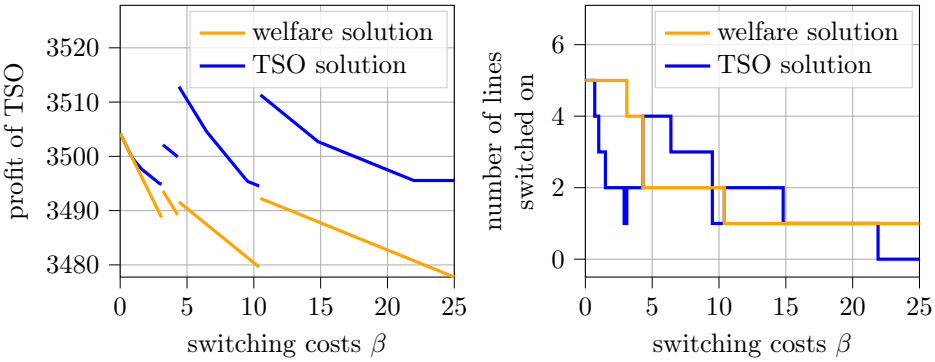

FIGURE 5. Comparison of a welfare optimum ("welfare solution") and the TSO's best response to the corresponding critical prices ("TSO solution") for varying switching costs β and constant transportation cost factor $\alpha = 0.1$. Left: Profits of the TSO. Right: Number of lines switched on in the TSO's strategies.

costs are low enough, the TSO's decisions align with the welfare solution. Since the profit gains and the welfare gains outweigh all losses due to switching, all 5 lines are switched on. Now, if the switching costs rise above 0.7, the profit gains of the TSO no longer outweigh all switching costs and the number of switched-on lines in the TSO solution decreases. This continues until the value 3.2 is reached, at which also the welfare gains no longer outweigh all losses due to switching and the number of switched-on lines in the welfare solution reduces to 4. However, this reduction does not lead to the TSO aligning again with the welfare solution. Even the reverse behavior can be observed as, e.g., when the number of switched-on lines in the welfare solution further reduces to 2. There, the TSO exploits the price differences

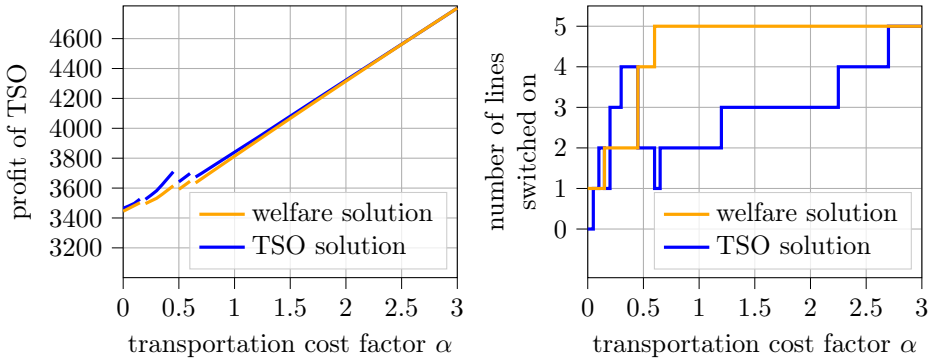

FIGURE 6. Comparison of a welfare optimum ("welfare solution") and the TSO's best response to the corresponding critical prices ("TSO solution") for constant switching costs $\beta = 20$ and varying transportation cost factor α. Left: Profits of the TSO. Right: Number of lines switched on in the TSO's strategies.

resulting from switching off the two additional lines in the welfare solution by even switching on 4 lines in total. Nevertheless, the incentive to do so decreases with further increasing switching costs. Finally, an equilibrium is obtained again when the switching costs are high enough such that neither the welfare gains nor the profit gains outweigh the losses from switching on any line. This point is reached for significantly larger switching costs of 17 524, which are omitted in the figure for the ease of readability.

Figure 6 shows the results for varying the transportation cost factor for the MATPOWER case case39. For a transportation cost factor of 0, no market equilibrium exists. The price differences are not yet high enough for the TSO to outweigh the losses due to switching. In turn, one line is switched on in the welfare solution. With increasing transportation cost factor, the number of switched-on lines in the welfare solution increases monotonically. This is because the additional transport possibilities allow for a cheaper transport despite the additional switching costs. Since an increasing transportation cost factor furthermore leads to increasing price differences, the number of switched-on lines also increases monotonically in the TSO solution for a constant number of switched-on lines in the welfare solution. At the points where additional lines are switched on in the welfare solution, the TSO responds to the corresponding critical prices by reducing the number of switched-on lines. The main reason behind this behavior is as follows. The additional switched-on lines in the welfare solution decrease the price differences. As a result, profit gains due to switching no longer outweigh the arising costs. Consequently, less lines are switched on. Finally, if the transportation cost factor is larger than 2.7, an equilibrium always exists since then all lines are switched on in the TSO and in the welfare solution.

6. CONCLUSION

In this paper we considered market equilibrium problems in which both convex as well as nonconvex player problems appear. This setting is motivated by applications from energy markets, where, e.g., nonconvexities arise in power markets due to integer decisions of certain players or in gas markets due to nonlinear flow models. In the cases studied in this paper, these nonconvexities always appear in the optimization problem of the TSO. Based on the recent results presented in Harks

(2020), we derived an algorithm that computes a solution of such nonconvex market equilibrium problems or correctly indicates the non-existence of such an equilibrium. Our computational study reveals interesting aspects. In the continuous but nonlinear and nonconvex market equilibrium problems from the gas sector, all tested instances have an equilibrium. This is different in the power application. Here, integrality restrictions lead to many instances for which no equilibrium exists.

Our results pave the way for some interesting topics of future research. First, one could try to find sufficient conditions under which an equilibrium exists, i.e., under which the resulting welfare problem has a zero duality gap. The discussed instances from the gas sector indicate that this might be possible. Second, one could consider approximate market equilibria in the nonconvex setting. A potential research question might be whether ε-relaxed optimal solutions to the players' problems give enough freedom to prove the existence of equilibria in settings in which classic equilibria fail to exist. Third, alternative pricing schemes that support an equilibrium could be tested for the DC line switching application as, e.g., the scheme of O'Neill et al. (2005) or the one of Huppmann and Siddiqui (2018). Fourth, one could investigate how the results change in a computational study in which nonlinearities and integralities are combined, as it would be the case for AC line switching models. Fifth, let us note again that our general theory does not require that a special type of player is convex. Thus, in general, it is not required that in all applications both producers and consumers have convex problems. Intuitively speaking, the prerequisite of our method is that there is enough information available to bound market prices appropriately. This information does not necessarily have to be provided by the producers and consumers as in our examples, but might also be provided by the TSO instead. This opens the door for studies of equilibrium models in which the TSO is convex but in which, e.g., the producers face some kind of mixed-integer unit commitment constraints. Sixth and finally, all the results presented in this paper rely on the assumption of perfect competition. Adding strategic interaction is, thus, a very important topic to be addressed in future research.

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Appendix A. Existence of Equilibria for Maximization Games

Instead of the minimization problem (1), each player $i \in I$ now faces the maximization problem

$$\max_{y_i} \quad f_i(y_i, \pi) := c_i(y_i) + \pi^\top h_i(y_i) \quad \text{s.t.} \quad y_i \in Y_i. \tag{13}$$

Thus, the corresponding maximization game reads

$$\begin{aligned} \text{optimization problems of the players:} \quad & (13) \text{ for all } i \in I, \\ \text{market-clearing conditions:} \quad & (2), \end{aligned} \tag{MEP-max}$$

while the corresponding welfare optimization problem now reads

$$\max_{y} \quad \sum_{i\in I} c_i(y_i) \quad \text{s.t.} \quad y \in Y, \quad \sum_{i\in I} h_i(y_i) = 0. \tag{WFP-max}$$

In this setting, Theorem 3.5 reads as follows.

Corollary A.1. *Let y^* be a global solution of the welfare problem (WFP-max) and let $\Pi(y^*) \neq \emptyset$ be a set satisfying the condition*

$$(y^*, \pi^*) \text{ is a market equilibrium of (MEP-max)} \implies \pi^* \in \Pi(y^*).$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

(a) $\pi_k^- = \pi_k^+$,
(b) $\pi_k^+ < \infty$ *and* $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$,
(c) $\pi_k^- > -\infty$ *and* $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$,
(d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ *and* $(h_i(y_i^*))_k = (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I$.

Then, there exists a market equilibrium of (MEP-max) if and only if $(y^*, \hat{\pi})$ is a market equilibrium, in which the critical price $\hat{\pi}$ is defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

Consider again a player $i \in I$ for which the feasible set Y_i is given by standard constraints, e.g., $Y_i = \{y_i \colon g_i(y_i) \leq 0\}$. If all functions c_i, h_i, and g_i are continuously differentiable, if $y_i \mapsto f_i(y_i, \pi)$ are concave functions and g_i are convex functions, and if a constraint qualification for Y_i is satisfied at y_i^*, then the KKT conditions for (13) are necessary and sufficient for optimality. As a consequence, only prices π for which there exists multipliers μ_i with

$$0 = -\nabla c_i(y_i^*) - \nabla h_i(y_i^*)\pi + \nabla g_i(y_i^*)\mu_i, \quad 0 \leq \mu_i \perp g_i(y_i^*) \leq 0 \tag{14}$$

can be market equilibrium prices. The analogue of Corollary 3.7 for maximization problems thus reads as follows.

Corollary A.2. *Let y^* be a global solution of the welfare problem (WFP-max). Moreover, let $C \subseteq I$ denote the subset of players for which the KKT conditions (14) are necessary and sufficient optimality conditions and choose the candidate set $\Pi(y^*)$ such that the condition*

$$(y^*, \pi^*) \text{ is a market equilibrium of (MEP-max)} \implies \pi^* \in \Pi(y^*)$$

as well as the KKT conditions of all players $i \in C$ are satisfied, i.e.,

$$\Pi(y^*) \subseteq \{\pi \in \mathbb{R}^{n_\pi} \colon \text{for all } i \in C \text{ exists } \mu_i \text{ such that (14) holds}\}.$$

Assume that for all $k \in \{1, \dots, n_\pi\}$ at least one of the following properties is satisfied:

(a) $\pi_k^- = \pi_k^+$,
(b) $\pi_k^+ < \infty$ *and* $(h_i(y_i^*))_k \geq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I\backslash C$,
(c) $\pi_k^- > -\infty$ *and* $(h_i(y_i^*))_k \leq (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I\backslash C$,
(d) $\pi_k^- = -\infty$, $\pi_k^+ = \infty$ *and* $(h_i(y_i^*))_k = (h_i(y_i))_k$ *for all* $y_i \in Y_i$ *and all players* $i \in I\backslash C$.

TABLE 3. Technical and economic parameters (top) and variables (bottom)

Sym.	Explanation	Unit gas	Unit power
π_u	Market price at node $u \in V_- \cup V_+$	€/(1000 Nm^3/h)	\$/MWh
a_u	Intercept of inverse demand $P_u(\cdot)$ of consumer $u \in V_-$	€/(1000 Nm^3/h)	\$/MWh
b_u	Slope of inverse demand $P_u(\cdot)$ of consumer $u \in V_-$	€/(1000Nm^3/h)^2	\$/MW^2h
$c_{u,1}$	Coefficient of linear term of variable cost $c_u(\cdot)$ of producer $u \in V_+$	€/(1000 Nm^3/h)	\$/MWh
$c_{u,2}$	Coefficient of quadratic term of variable cost $c_u(\cdot)$ of producer $u \in V_+$	€/(1000Nm^3/h)^2	\$/MW^2h
$\bar{y}_u$	Capacity of producer $u \in V_+$	1000 Nm^3/h	MW
α	Transportation cost factor	€/(1000Nm^3/h)^2	\$/MW^2h
β	Switching costs	—	\$/h
Λ_a	Pressure drop coefficient of arc $a \in A$	(bar)^2/(1000Nm^3/h)^2	—
p_u^-	Lower pressure bound at node $u \in V$	bar	—
p_u^+	Upper pressure bound at node $u \in V$	bar	—
q_a^-	Lower flow bound of arc $a \in A$	1000 Nm^3/h	MW
q_a^+	Upper flow bound of arc $a \in A$	1000 Nm^3/h	MW
X_a	Reactance of arc $a \in A$	—	p. u.
$\theta_a^{\mathrm{shift}}$	Transformer phase shift angle ($a \in A$)	—	rad
d_u	Demand of consumer $u \in V_-$	1000 Nm^3/h	MW
y_u	Production of producer $u \in V_+$	1000 Nm^3/h	MW
q_a	Flow on arc $a \in A$	1000 Nm^3/h	MW
p_u	Pressure at node $u \in V$	bar	—
θ_u	Phase angle at node $u \in V$	—	rad
z_a	Switching decision of arc $a \in A_+$	—	—

Now, let the critical price $\hat{\pi}$ be defined as

$$\hat{\pi}_k := \begin{cases} \pi_k^- = \pi_k^+, & \text{if (a) applies,} \\ \pi_k^+, & \text{if (b) applies,} \\ \pi_k^-, & \text{if (c) applies,} \\ 0, & \text{if (d) applies.} \end{cases}$$

If the critical price satisfies $\hat{\pi} \in \Pi(y^*)$, then there exists a market equilibrium of (MEP) *if and only if $(y^*, \hat{\pi})$ is a market equilibrium.*

APPENDIX B. NOTATION

All technical and economic parameters and variables used throughout the computational study in Section 5 are presented together with their respective units in Table 3. We do not use SI units here but the units that are commonly used in the literature of the respective applications, e.g., Nm^3/h denotes volumetric flow under normal conditions. This is in line with the literature from which we adapted our instances; see, e.g., Schewe et al. (2020) and Zimmerman and Murillo-Sánchez (2021).

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