Long-run market equilibria in coupled energy sectors: A study of uniqueness

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Abstract. We propose an equilibrium model for coupled markets of multiple energy sectors. The agents in our model are operators of sector-specific production and sector-coupling technologies, as well as price-sensitive consumers with varying demand. We analyze long-run investment in production capacity in each sector and investment in coupling capacity between sectors, as well as production decisions determined at repeated spot markets. We show that in our multi-sector model, multiplicity of equilibria may occur, even if all assumptions hold that would be sufficient for uniqueness in a single-sector model. We then contribute to the literature by deriving sufficient conditions for the uniqueness of short- and long-run market equilibrium in coupled markets of multiple energy sectors. We illustrate via simple examples that these conditions are indeed required to guarantee uniqueness in general. The uniqueness result is an important step to be able to incorporate the proposed market equilibrium problem in more complex computational multilevel equilibrium models, in which uniqueness of lower levels is a prerequisite for obtaining meaningful solutions. Our analysis also paves the way to understand and analyze more complex sector coupling models in the future.

Key words. Energy Markets, Sector Coupling, Regional Pricing, Uniqueness, Short- and Long-Run Market Equilibrium.

1. Introduction

The goal of climate neutrality in 2050 in Europe and other regions worldwide means that the entire economy must decarbonize or defossilize. On the one hand, renewable electricity will be used for the direct electrification of the heat and transport sectors as well as industry. Where this is not possible, the path is via climate-neutrally generated hydrogen and synthetic energy sources based on it. Hydrogen and renewable synthetic fuels are also needed to compensate for fluctuations of renewable energies in the electricity sector. Sector coupling will thus play an increasingly important role in both directions. In the medium run, during the time of transition, also fossil gas will play a role in the energy system. Overall, the transition towards a climate neutral world will lead to a higher degree of integration between the markets for electricity, fossil gas, and hydrogen. This integration must be taken into account in market designs for energy sectors.

For the assessment of market designs it is a common approach to develop (multilevel) equilibrium models that examine the decisions of stakeholders in energy markets and the corresponding market outcomes. In this paper, we extend approaches from the literature to coupled energy sectors and pave the way for a convincing equilibrium analysis of those more complex environments. For the case of a single energy commodity, Grimm, Schewe,

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et al. (2017) analyze a framework of peak-load pricing in electricity markets, i.e., they consider investment and operation decisions of firms when interacting in a network context with flow constraints. Krebs, Schewe, et al. (2018) extend this analysis to the case of DC power flow in electricity markets and Grübel et al. (2020) analyze uniqueness and multiplicity in a similar setting taking into account storage facilities (without a network). The extension of modeling approaches to the representation of several coupled markets for energy commodities is important to assess the impact of sector-specific market designs on investment and operation decisions in integrated energy systems. In particular, further research is needed to develop a better understanding of the impact that the interactions between different energy markets have on the uniqueness of market outcomes in order to derive necessary model assumptions. This is the main focus of our contribution. First, we provide sufficient conditions for uniqueness of equilibrium in a sector-coupled energy market model. Second, our analysis paves the way to build on the framework to analyze particular issues that arise in coupled energy markets in practice.

There are different strands of literature on the transition of energy markets with several energy commodities and a spatial market representation. Many articles focus on co-optimizing system cost for different energy commodities, which are usually referred to as energy system models. These focus exclusively on determining the cost optimal configuration and operation of an energy system that consists of several energy commodities, potentially subject to specific policy goals (e.g., climate targets or emission pricing). It is not in the nature of energy system models to take into account market design and energy pricing, however. Consequently, the optimization of system cost is not in line with the outcome of market interaction given incentives at markets for individual participants, and market equilibria in coupled energy markets are not assessed. Earlier contributions include Geidl and Andersson (2007), Martinez-Mares and Fuerte-Esquivel (2012), Clegg and Mancarella (2015b), Zhang et al. (2015) and Clegg and Mancarella (2015a), recent work is provided by, e.g., Zlotnik et al. (2017) or Felten (2020).

Another strand of literature, which is more closely related to the present work, explicitly considers the behavior of all market participants and the resulting market equilibria in coupled energy markets. In this context, compare for example Abrell and Weigt (2012) on short-run market equilibria in electricity and gas markets and Abrell and Weigt (2016) for the case of a long-run analysis including firms' investment decisions in production facilities. Similarly, Huppmann and Egging (2014) and Gil et al. (2015) provide frameworks to model the behavior of market participants in fuel markets and the electricity market subject to network constraints resulting both from electricity and fuel transport. Chen et al. (2019) build on this work and explicitly analyze strategic bidding of firms under the anticipation of nodal pricing both in the electricity and the gas market in a bilevel approach. Several other very recent contributions highlight further interesting aspects arising in the context of market interaction in coupled energy markets. Ordoudis et al. (2019) analyze different degrees of coupling day-ahead and real-time electricity and gas markets under uncertainty. And most recently, Roach and Meeus (2020) provide an iterative simulation procedure to analyze the impact of long-term gas contracts on market outcomes in coupled electricity and gas markets. All those studies provide highly valuable insights on the different aspects regarding market interaction in coupled energy markets. Whereas they determine one of the resulting market equilibria for their test instances considered, they do not analyze whether there are many different equilibria yielding potentially different policy conclusions. Moreover, they do not provide conditions when their setup indeed delivers unique market outcomes.

This paper discusses the extension of market equilibrium models that capture the behavior of individual market participants to multiple coupled markets for energy commodities. As compared to single-commodity models, the implementation of market dynamics between integrated energy markets in a joint market equilibrium model requires additional considerations. We propose a model for coupled energy markets that captures long-term investment decisions in transport and production capacity and short-term market decisions. For this kind of model, we determine sufficient conditions that ensure unique market outcomes. Our contribution is, to the best of our knowledge, the first to fill this gap for the case of coupled energy markets.

Our work lays the foundation for addressing timely research questions on energy market coupling with regard to, e.g., climate policy, regulation of networks, energy pricing and its effects on infrastructure planning, investment incentives, and market outcomes. A substantive analysis of the long-run interaction of the different agents requires techno-economic multilevel modeling of coupled energy markets similar to the sector-specific models of Grimm, Martin, et al. (2016) and Ambrosius et al. (2020) for electricity markets and Grimm, Schewe, et al. (2019) for gas markets. In those multilevel energy market models, uniqueness of the market outcome at the second level typically is key in order to enable a coherent analysis of decision makers' investments in transport capacities at the first level. In the existing literature, uniqueness for multilevel market models has only been addressed for one energy sector and additional aspects like storage and load-flow representation. Our results show that multi-sector market models, with a representation of the agents active in different markets, require additional conditions to guarantee the uniqueness of short- and long-term market equilibrium. Beyond our general analysis, we illustrate in small examples. We also discuss the current limits for ensuring uniqueness in model applications and open questions for further research.

The remainder of the paper is organized as follows. Section 2 introduces the general problem setting with timeline, model assumptions, and basic notation. Section 3 states the model formulation, Sect. 4 provides the analysis of uniqueness, and Sect. 5 concludes.

2. Setup, basic model assumptions, and notation

In liberalized energy markets, various agents with different objectives interact. If several coupled energy sectors are taken into account, some of these agents participate in multiple commodity markets. In this paper we explicitly consider network-based energy sectors such as electricity, fossil gas, and hydrogen, which have a regulated and centrally operated transmission system, or could have one in the future. The types of agents considered are sector-specific consumers and producers, sector-coupling producers, as well as sector-specific transmission companies and system operators. Our model covers their individual decisions on investment, spot market trading, and adjustments of spot market allocations necessary for technical operation of transmission systems. At the spot market stage, network constraints in energy pricing are partly considered, which requires a spatial definition of market zones and a pricing regime for transmission capacity. The private agents seek to maximize their own objectives on the spot market. While consumers decide on the level of their consumption by maximizing their consumer surpluses, producers—with the aim of maximizing profits—decide on investment in their production technologies and on the operation of established facilities. The difference between sector-specific and sector-coupling production technologies is that for sector-coupling technologies, the energy sector from which the input factors are obtained is also modeled endogenously. This is not the case for sector-specific production technologies.

In the literature that analyses semi-liberalized energy markets in multilevel equilibrium models (see, e.g., Sauma and Oren (2006), Baringo and Conejo (2012), Jenabi et al. (2013), Grimm, Martin, et al. (2016), Grimm, Schewe, et al. (2019), or Ambrosius et al. (2020)), all the above-mentioned market-driven decisions of private agents are typically captured by one level of a multilevel equilibrium model. In this paper we focus on a particular, very important aspect of this level, namely the uniqueness of the equilibrium at the "market level", for the case of multiple coupled energy markets. Only precise knowledge about the set of equilibria and the conditions under which a unique market equilibrium at the lower level obtains allows a meaningful analysis of the entire model. It is evident that uniqueness requires simplifying technical assumptions, which might seem too restrictive for many applications of interest. However, it is important to understand under which assumptions it is possible to derive coherent results and under which assumptions, due to multiplicity of equilibria, one has to be very careful in interpreting the results of more complex models. In order to understand precisely under which assumptions one can actually expect uniqueness of equilibrium, a very detailed understanding of the underlying multilevel problem is essential. For this reason we start with a description of the timeline of the overall model. We then present the basic model assumptions with regard to the economic environment. Finally, the notation is introduced.

2.1. **Setup.** Our market model for coupled energy sectors has the following multilevel structure, which is illustrated in Figure 1 for two sectors. First, network companies in the different sectors decide on network expansion, anticipating private investment and the resulting spot market outcomes, as well as subsequent adjustments of the market outcomes to ensure technical feasibility in the transmission systems. Next, the private agents observe network investment and the implied trade restrictions and decide on their investment in production capacities, both in sector-specific and sector-coupling technologies. These investment decisions depend on their expectations on spot market outcomes in all sectors. Finally, trading on the spot market takes place over several time periods, and possibly subsequent adjustments are made in the individual sectors in case of violated transport constraints. This is relevant if transmission capacity constraints are not reflected in spot market prices and therefore accounted for in the market outcome, i.e., under zonal pricing. If these adjustments are made cost-based, there is no interference with trading on the spot market since the private agents cannot realize additional profits. In this case, anticipation of decisions at the adjustment stage does not affect investment and operating decisions.

Under the assumption of perfect competition, the investment and spot market trading levels can be subsumed in one level that contains all market-driven decisions. The multilevel structure described above can thus be reduced to a techno-economic trilevel problem with the following underlying structure: network investment on the first level, private investment in production capacity as well as spot market trading on the second level, and adjustments of market outcomes on the third level.

This paper focuses on the uniqueness analysis of the second level, i.e., of investment in sector-specific and sector-coupling production capacity as well as spot market trading in all sectors considered. Uniqueness of the second level is a prerequisite for uniqueness of the overall equilibrium solution. The question of uniqueness of the first and third level is not addressed in detail in this paper. At the first level, multiplicities occur if and only if different investment decisions yield the same maximal welfare, given cost minimal adjustment decisions at the third level. While this kind of multiplicities might occur, they are much easier to handle upon applications of multilevel models to specific scenarios.

2.2. **Basic model assumptions.** Modeling private investment and spot market behavior in multiple sectors jointly results in a large market equilibrium problem. In order to keep the analysis tractable, we assume—as it is standard in many related contributions—perfect competition at all markets, i.e., upon investment and spot market decisions.

Second, we assume that each involved agent has perfect foresight, i.e., producers correctly anticipate the spot market outcomes of the sectors in which they operate when they make their investment decisions. In particular, perfect foresight implies that spot market results are not subject to uncertainty on final production and demand levels due to forecasting errors. This assumption may be relaxed by stochastic or robust optimization techniques. In this case, complementarity problems under uncertainty arise. However, this kind of equilibrium problems have only recently received increased attention in literature and are theoretically not yet well understood. In particular, the existence of a single-level counterpart as in the case of perfect foresight is not generally guaranteed. For more information on uncertain linear complementarity problems see, e.g., Krebs and Schmidt (2020) and Krebs, Müller, et al. (2019).

As third assumption, operators of storage devices are excluded as possible agents in our context since Grübel et al. (2020) already showed that uniqueness fails to hold in general as

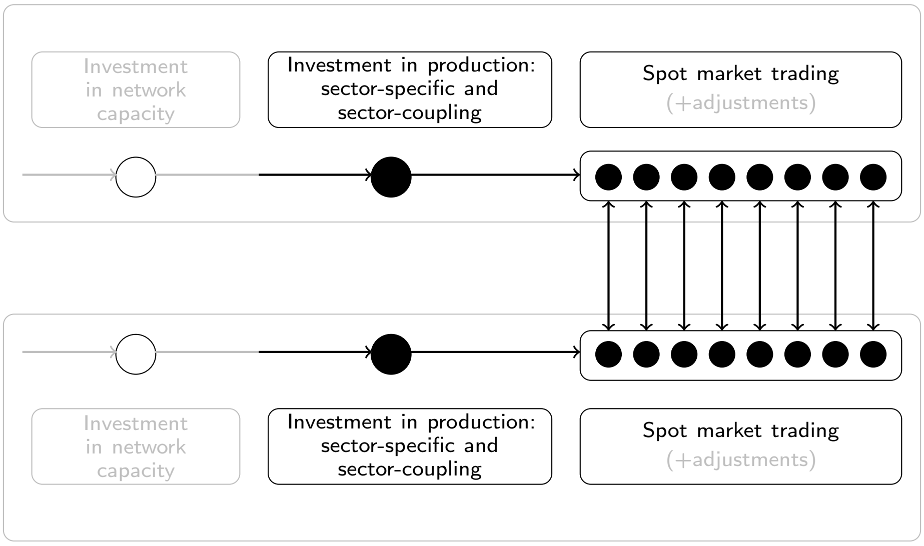

Figure 1. Multilevel structure of investment into network and production capacity and subsequent trade in and across multiple sectors—the uniqueness analysis in this paper focuses on the levels highlighted in black

soon as several storage operators are considered in a market equilibrium problem similar to the one analyzed here.

Fourth, we consider trade capacity between adjacent bidding zones at the spot market while we abstract from additional constraints on trade capacity related to potential-based network flows. For some applications, in particular in network-based energy sectors, it might be of interest to incorporate those constraints at the spot market; e.g., if nodal pricing or zonal flow-based pricing systems are modeled. In this paper, we refrain from taking potential-based trade constraints into account since, even in case of linear models, multiplicities may easily arise; see the results for the electricity market equilibrium problems in Krebs, Schewe, et al. (2018) and Krebs and Schmidt (2018) that both include as trade constraints the linear direct current (DC) lossless setup to approximate Kirchhoff's voltage law once without and once with transportation costs. Moreover, if nonconvex transportation constraints are considered in a market equilibrium problem, the equivalence between market equilibria and welfare maxima may no longer hold; see Grimm, Grübel, et al. (2019) who address the case of nonconvex gas physics.

Finally, we assume that the time horizon and the time intervals for trading the products (i.e., electricity or gas) are the same on spot markets across all sectors. If this is not the case, technologies from sectors where products are traded more frequently will face equal conditions in the coupled sector for a number of subsequent trading periods, which might obviously lead to multiplicities. This issue is left for future research.

2.3. **Notation.** In the following, we introduce the basic notation. We start with the time horizon and market structure, continue with the sector-specific demand and supply, and conclude with the sector-coupling preliminaries. For the sake of completeness, an overview of all sets, parameters, and variables used throughout the paper is presented in Tables 1, 2, and 3 in App. A.

Time horizon. The time horizon T for all considered energy markets is assumed to be equidistantly discretized, i.e., in time periods $T = \{t_1, \dots, t_{|T|}\}$ with the duration $\tau = t_i - t_{i-1}$ for $i = 2, \dots, |T|$. This time horizon typically resembles all trading periods of, e.g., one year. The related data is based either on historical or representative data, using the net present value approach for investment.

Bidding zones. In each sector, the spot market represents the network infrastructure and the spatial distribution of demand and supply within one or multiple bidding zones and with trade products between the bidding zones. We denote the set of bidding zones per sector $s \in S$ by $Z_s = \{z_1, \dots, z_{|Z_s|}\}$. In case of a market with a single bidding zone, $|Z_s| = 1$ holds. In case of $|Z_s| > 1$, the trade capacity between adjacent bidding zones is specified and accounted for at the spot market. In our setting, the set $K_s \subseteq Z_s \times Z_s$ represents all bidding zones of the same sector $s \in S$ between which a positive trade capacity is available. Each element $k \in K_s$ is characterized by its maximum and minimum trade capacity f_k^+, f_k^-. A positive trade value $f_{t,k}$ on $k = (z, z') \in K_s$ in time period $t \in T$ implies that the respective amount of the sector commodity is traded from bidding zone z to z', while a negative trade value represents trading in the opposite direction. Finally, we follow common notation and define the set of all ingoing and outgoing trade capacities of a bidding zone $z \in Z_s$ by

$$\delta_s^{\text{in}}(z) := \{k = (z', z) \in K_s \mid z' \in Z_s\},$$
$$\delta_s^{\text{out}}(z) := \{k = (z, z') \in K_s \mid z' \in Z_s\}.$$

Sector-specific demand. The demand in each sector varies over time and is assumed to be price-elastic. For the ease of notation, it is furthermore assumed that there is one (possibly aggregated) demand function per bidding zone. The inverse demand function $P_{t,z} : [0, \infty) \to \mathbb{R}^+$ of bidding zone $z \in Z_s$ in sector $s \in S$ and in time period $t \in T$ is continuously differentiable and strictly decreasing.

Sector-specific supply. The demand is met on the one hand by production technologies that operate only within one sector and on the other hand by technologies that couple the different sectors. The difference between the two types of technologies is that the sector-specific production technologies do not rely on input from the other sectors in our model scope. All existing production facilities in bidding zone $z \in Z_s$ that are operated sector-specific form the set G_z^{ex}. In order to account for the possibility of further investment in sector-specific production technologies, we introduce the set G_z^{new} of candidate production facilities. We refer to the set of all existing and candidate sector-specific production facilities by G_z^{all}, i.e., $G_z^{\text{all}} := G_z^{\text{ex}} \cup G_z^{\text{new}}$. Each facility $g \in G_z^{\text{all}}$ is characterized by its variable costs of production c_g^{var}. Moreover, existing facilities have a given capacity y_g^{ex}, while the capacity y_g^{new} of candidate facilities is variable with associated investment costs of c_g^{inv}. Finally, we introduce the facility-specific availability $\alpha_{t,g}$. In the case of renewable energy sources, like wind and solar, $\alpha_{t,g}$ describes variability in physical availability whereas in the other cases it may be interpreted as planned availability, which includes shut-down times due to maintenance.

Sector coupling. Each sector-coupling technology transforms the commodity from one sector $s \in S$ into the commodity of another sector $s' \in S$ with $s \neq s'$. We introduce the set X^{ex} for all existing production facilities with sector-coupling technologies. An element (o, i) of this set represents the respective withdrawal point o and the respective injection point i of the facility. Moreover, we introduce the set X^{new} for all candidate sector-coupling facilities and refer to the set of all sector-coupling facilities by X^{all}, i.e., $X^{\text{all}} := X^{\text{ex}} \cup X^{\text{new}}$. Each sector-coupling facility $x \in X^{\text{all}}$ is characterized by its efficiency $\eta_x \in (0; 1)$. Note that an efficiency is not specified for the sector-specific production facilities because the respective efficiency is captured by the variable production costs.

The set O_z^{ex} represents all existing sector-coupling facilities that withdraw the commodity from sector $s \in S$ in bidding zone $z \in Z_s$, while the set O_z^{new} represents the candidate facilities. The set of all facilities is denoted by O_z^{all}, i.e., $O_z^{\text{all}} := O_z^{\text{ex}} \cup O_z^{\text{new}}$.

Analogously, the set I_z^{ex} represents all existing sector-coupling facilities that inject the commodity from sector $s \in S$ in bidding zone $z \in Z_s$. As before, we also consider the respective sets of candidate facilities I_z^{new} and all facilities I_z^{all}. All existing sector-coupling facilities have a given capacity y_i^{ex}, while the capacity y_i^{new} of candidate facilities is variable with associated investment costs of c_i^{inv}. Note that the capacity restriction needs only to

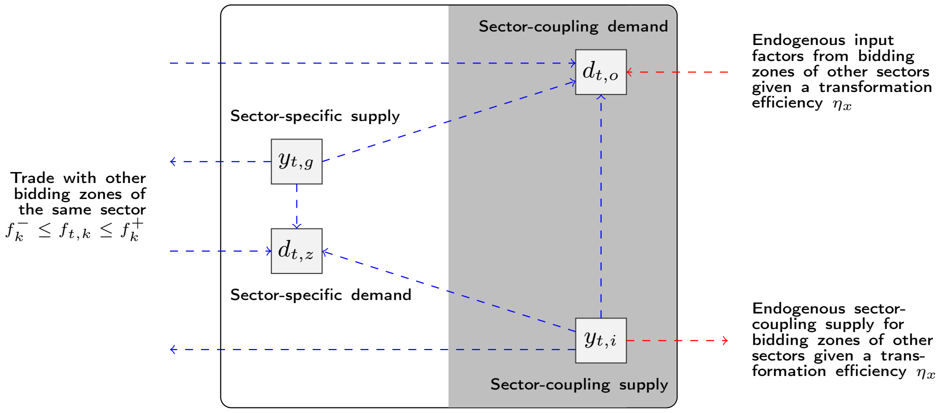

FIGURE 2. Dependencies of the smallest component of our model in one time period: One bidding zone in one sector (endogenous trade of the sector's commodity in blue and endogenous trade of sector-coupled commodities in red)

be defined for the injection point because by this, the capacity at the withdrawal point is also restricted given the efficiency of the underlying facility. The equivalent availability of sector-coupling facilities is denoted by $\alpha_{t,i}$.

Figure 2 illustrates the trading dependencies of one bidding zone in one sector, i.e., of the smallest component of our model. First of all, the sector's commodity is traded between the agents within this bidding zone. For this purpose, the sector-specific production facilities purchase their input factors at an exogenously determined price. In turn, the purchase of input factors of sector-coupling facilities is endogenously modeled. For the transformation of one commodity into another, a transfer efficiency is presumed. Hence, depending on the relation of market prices between the considered bidding zone and bidding zones of other sectors, both trade for transforming the considered commodity into others or vice versa is possible. Finally, the sector's commodity can be sold to or bought from agents of other bidding zones of the same sector. However, the trade between the considered bidding zone and an adjacent one is limited by the respective trade capacity.

3. An equilibrium model for coupled energy markets and its single-level counterpart

In this section, we proceed as follows. First, we propose a market equilibrium problem that covers the private investment in sector-specific and sector-coupling production technologies together with the subsequent spot market trading in the different coupled energy sectors. Afterward, we present a welfare maximization problem suitably chosen to form a single-level counterpart to the proposed market equilibrium problem, i.e., each welfare maximal solution corresponds to a market equilibrium and vice versa. We close with the proof of the latter relation.

To simplify notation, we denote, e.g., by y_t the vector of sector-specific and sector-coupling production given a fixed time period $t \in T$. Furthermore, when we speak, e.g., of the producer $g \in G_z^{\text{all}}$ in the following, we are actually referring to the operator of the production facility g.

3.1. The market equilibrium problem.

On the spot markets of different sectors, multiple agents interact who all pursue their own objectives. The main agents are the consumers, the operators of sector-specific production as well as sector-coupling technologies. We continue with the description of the individual optimization problems of each agent type and start with the consumers.

Consumers. Consumers maximize their gross consumer surpluses less their purchasing costs. Hence, the consumers located in bidding zone $z \in Z_s$ of sector $s \in S$ face the following optimization problem

$$\max \quad \sum_{t \in T} \left(\int_0^{d_{t,z}} P_{t,z}(\mu) \, \mathrm{d}\mu - p_{t,z} d_{t,z} \right) \tag{1a}$$

$$\text{s.t.} \quad 0 \le d_{t,z}, \quad \text{for all } t \in T, \qquad [\gamma_{t,z}] \tag{1b}$$

where $p_{t,z}$ denotes the price of the respective bidding zone in the considered time period. Due to the assumption of perfect competition, this price is—from the point of view of the individual consumers—exogenously given, i.e., effects of own or others' decisions on the price are not taken into account.

The Greek letters behind constraints denote the associated dual variables. Since the inverse demand function $P_{t,z}$ is assumed to be continuously differentiable and strictly decreasing, Problem (1) is a concave maximization problem with linear constraints. Thus, the corresponding Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for global optimality. Consequently, all solutions of the stated problem are characterized by

$$P_{t,z}(d_{t,z}) - p_{t,z} + \gamma_{t,z} = 0, \quad \text{for all } t \in T, \tag{2a}$$

$$0 \le \gamma_{t,z} \perp d_{t,z} \ge 0, \quad \text{for all } t \in T. \tag{2b}$$

Sector-specific producers. Each operator of an existing sector-specific production technology maximizes profits from trading, i.e., the revenues from trading minus the variable production costs. Thus, the producer $g \in G_z^{\text{ex}}$ located in bidding zone $z \in Z_s$ of sector $s \in S$ has the following maximization problem

$$\max \quad \sum_{t \in T} \left(p_{t,z} - c_g^{\text{var}} \right) y_{t,g} \tag{3a}$$

$$\text{s.t.} \quad 0 \le y_{t,g} \le \alpha_{t,g} y_g^{\text{ex}}, \quad \text{for all } t \in T. \qquad [\beta_{t,g}^{\pm}] \tag{3b}$$

The latter inequality implies that the production must remain within the real-time available capacities. The associated KKT conditions are necessary and sufficient for global optimality since Problem (3) is a linear maximization problem.

$$p_{t,z} - c_g^{\text{var}} + \beta_{t,g}^{-} - \beta_{t,g}^{+} = 0, \quad \text{for all } t \in T, \tag{4a}$$

$$0 \le \beta_{t,g}^{-} \perp y_{t,g} \ge 0, \quad \text{for all } t \in T, \tag{4b}$$

$$0 \le \beta_{t,g}^{+} \perp \alpha_{t,g} y_g^{\text{ex}} - y_{t,g} \ge 0, \quad \text{for all } t \in T. \tag{4c}$$

The operators of candidate sector-specific production technologies additionally consider their investment costs. Therefore, such an operator $g \in G_z^{\text{new}}$ located in bidding zone $z \in Z_s$ of sector $s \in S$ optimizes the following maximization problem

$$\max \quad \sum_{t \in T} \left(p_{t,z} - c_g^{\text{var}} \right) y_{t,g} - c_g^{\text{inv}} y_g^{\text{new}} \tag{5a}$$

$$\text{s.t.} \quad 0 \le y_{t,g} \le \alpha_{t,g} y_g^{\text{new}}, \quad \text{for all } t \in T. \qquad [\delta_{t,g}^{\pm}] \tag{5b}$$

The respective (necessary and sufficient) KKT conditions read

$$-c_g^{\text{inv}} + \sum_{t \in T} \alpha_{t,g} \delta_{t,g}^{+} = 0, \tag{6a}$$

$$p_{t,z} - c_g^{\text{var}} + \delta_{t,g}^{-} - \delta_{t,g}^{+} = 0, \quad \text{for all } t \in T, \tag{6b}$$

$$0 \le \delta_{t,g}^{-} \perp y_{t,g} \ge 0, \quad \text{for all } t \in T, \tag{6c}$$

$$0 \le \delta_{t,g}^{+} \perp \alpha_{t,g} y_g^{\text{new}} - y_{t,g} \ge 0, \quad \text{for all } t \in T. \tag{6d}$$

Note that the specified setup does not exclude the case where an operator owns more than one production facility. Since we assume perfect competition and consequently that the operators cannot influence market prices by their decisions, optimizing jointly over all owned facilities is equivalent to optimizing individually each facility (see also Theorem 3.3 below for the proof of this assertion).

Sector-coupling producers. The objective of an operator of a sector-coupling technology is to maximize profits from trading. As a consequence, the operator of the existing sector-coupling facility $x = (o, i) \in X^{\text{ex}}$ faces the following optimization problem

$$\max \quad \sum_{t \in T} (p_{t,z} y_{t,i} - p_{t,z'} d_{t,o}) \tag{7a}$$

$$\text{s.t.} \quad \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } t \in T, \quad [\zeta_{t,x}] \tag{7b}$$

$$0 \le y_{t,i} \le \alpha_{t,i} y_i^{\text{ex}}, \quad \text{for all } t \in T, \quad [\nu_{t,x}^{\pm}] \tag{7c}$$

$$0 \le d_{t,o}, \quad \text{for all } t \in T, \quad [\theta_{t,x}] \tag{7d}$$

where the injection point $i \in I_z^{\text{ex}}$ of the sector-coupling facility is located in bidding zone $z \in Z_s$ of sector $s \in S$ and the withdrawal point $o \in O_{z'}^{\text{ex}}$ is located in bidding zone $z' \in Z_{s'}$ of sector $s' \in S$ with $s \neq s'$.

Here, the KKT conditions are—due to the linearity of Problem (7)—again necessary and sufficient for global optimality

$$p_{t,z} + \nu_{t,x}^{-} - \nu_{t,x}^{+} - \zeta_{t,x} = 0, \quad \text{for all } t \in T, \tag{8a}$$

$$-p_{t,z'} + \theta_{t,x} + \eta_x \zeta_{t,x} = 0, \quad \text{for all } t \in T, \tag{8b}$$

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } t \in T, \tag{8c}$$

$$0 \le \nu_{t,x}^{-} \perp y_{t,i} \ge 0, \quad \text{for all } t \in T, \tag{8d}$$

$$0 \le \nu_{t,x}^{+} \perp \alpha_{t,i} y_i^{\text{ex}} - y_{t,i} \ge 0, \quad \text{for all } t \in T, \tag{8e}$$

$$0 \le \theta_{t,x} \perp d_{t,o} \ge 0, \quad \text{for all } t \in T. \tag{8f}$$

The operator of the candidate sector-coupling facility $x = (o, i) \in X^{\text{new}}$ additionally considers required investment costs and thus plans in accordance with the following maximization problem

$$\max \quad \sum_{t \in T} (p_{t,z} y_{t,i} - p_{t,z'} d_{t,o}) - c_i^{\text{inv}} y_i^{\text{new}} \tag{9a}$$

$$\text{s.t.} \quad \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } t \in T, \quad [\xi_{t,x}] \tag{9b}$$

$$0 \le y_{t,i} \le \alpha_{t,i} y_i^{\text{new}}, \quad \text{for all } t \in T, \quad [\tau_{t,x}^{\pm}] \tag{9c}$$

$$0 \le d_{t,o}, \quad \text{for all } t \in T. \quad [\rho_{t,x}] \tag{9d}$$

The respective (necessary and sufficient) KKT conditions read

$$-c_i^{\text{inv}} + \sum_{t \in T} \alpha_{t,i} \tau_{t,x}^{+} = 0, \tag{10a}$$

$$p_{t,z} + \tau_{t,x}^{-} - \tau_{t,x}^{+} - \xi_{t,x} = 0, \quad \text{for all } t \in T, \tag{10b}$$

$$-p_{t,z'} + \rho_{t,x} + \eta_x \xi_{t,x} = 0, \quad \text{for all } t \in T, \tag{10c}$$

$$\eta_x d_{t,o} - y_{t,i} = 0, \quad \text{for all } t \in T, \tag{10d}$$

$$0 \le \tau_{t,x}^{-} \perp y_{t,i} \ge 0, \quad \text{for all } t \in T, \tag{10e}$$

$$0 \le \tau_{t,x}^{+} \perp \alpha_{t,i} y_i^{\text{new}} - y_{t,i} \ge 0, \quad \text{for all } t \in T, \tag{10f}$$

$$0 \le \rho_{t,x} \perp d_{t,o} \ge 0, \quad \text{for all } t \in T. \tag{10g}$$

System operators. For the market allocation of the predetermined trade capacity between bidding zones an independent system operator or, alternatively, an exchange might be responsible. The objective is to exploit price differences between adjacent bidding zones to maximize profits from congestion rents. This is done by allocating as much flow as the trade capacities allow from low price bidding zones to high price bidding zones, i.e., the following optimization problem is faced

$$\max \quad \sum_{k=(z,z')\in K_s} \sum_{t\in T} (p_{t,z} - p_{t,z'})\, f_{t,k} \tag{11a}$$

$$\text{s.t.} \quad f_k^- \le f_{t,k} \le f_k^+, \quad \text{for all } k \in K_s,\ t \in T, \qquad [\kappa_{t,k}^{\pm}] \tag{11b}$$

by the responsible system operator of sector $s \in S$. The modeling mainly follows Hobbs and Helman (2004).

The KKT conditions are again necessary and sufficient for global optimality due to the linearity of the considered optimization problem

$$p_{t,z} - p_{t,z'} + \kappa_{t,k}^- - \kappa_{t,k}^+ = 0, \quad \text{for all } k = (z,z') \in K_s,\ t \in T, \tag{12a}$$

together with

$$0 \le \kappa_{t,k}^- \perp f_{t,k} - f_k^- \ge 0, \quad \text{for all } k \in K_s,\ t \in T, \tag{12b}$$

$$0 \le \kappa_{t,k}^+ \perp f_k^+ - f_{t,k} \ge 0, \quad \text{for all } k \in K_s,\ t \in T. \tag{12c}$$

Market clearing. Finally, we add the flow balance equation for each bidding zone $z \in Z_s$ in each sector $s \in S$ and time period $t \in T$

$$d_{t,z} - \sum_{g\in G_z^{\text{all}}} y_{t,g} + \sum_{o\in O_z^{\text{all}}} d_{t,o} - \sum_{i\in I_z^{\text{all}}} y_{t,i} = \sum_{k\in\delta_s^{\text{in}}(z)} f_{t,k} - \sum_{k\in\delta_s^{\text{out}}(z)} f_{t,k}, \tag{13}$$

i.e., the inflows minus the outflows of a bidding zone must equal the sector-specific and sector-coupling demand minus the sector-specific and sector-coupling production of this bidding zone.

The game. Joining all KKT conditions (2), (4), (6), (8), (10), and (12) together with the market clearing (13) yields the following mixed complementarity problem (MCP) that models the spot market trading in different coupled energy markets:

$$\begin{aligned}
\text{Consumers}: &\quad (2), \quad \text{for all } s \in S,\ z \in Z_s,\\
\text{Sector-specific producers}&\\
\text{without investment}: &\quad (4), \quad \text{for all } s \in S,\ z \in Z_s,\ g \in G_z^{\text{ex}},\\
\text{Sector-specific producers}&\\
\text{with investment}: &\quad (6), \quad \text{for all } s \in S,\ z \in Z_s,\ g \in G_z^{\text{new}},\\
\text{Sector-coupling producers}&\\
\text{without investment}: &\quad (8), \quad \text{for all } x = (o,i) \in X^{\text{ex}},\\
\text{Sector-coupling producers}&\\
\text{with investment}: &\quad (10), \quad \text{for all } x = (o,i) \in X^{\text{new}},\\
\text{System operators}: &\quad (12), \quad \text{for all } s \in S,\\
\text{Market clearing}: &\quad (13), \quad \text{for all } s \in S,\ z \in Z_s,\ t \in T.
\end{aligned} \tag{MCP}$$

3.2. **Equivalence to a single-level counterpart.** Next, we prove that all market equilibria, i.e., all solutions of the above stated (MCP), correspond to solutions of a suitably chosen welfare maximization problem, and vice versa. This directly implies the economic efficiency of the market equilibria. The welfare maximization problem reads

$$\max \quad \sum_{s\in S} \sum_{z\in Z_s} \sum_{t\in T} \left(\int_0^{d_{t,z}} P_{t,z}(\mu)\, \mathrm{d}\mu - \sum_{g\in G_z^{\text{all}}} c_g^{\text{var}} y_{t,g} \right)$$

$$-\sum_{s\in S}\sum_{z\in Z_s}\left(\sum_{g\in G_z^{\text{new}}} c_g^{\text{inv}} y_g^{\text{new}} + \sum_{i\in I_z^{\text{new}}} c_i^{\text{inv}} y_i^{\text{new}}\right) \tag{15a}$$

$$\begin{aligned}
\text{s.t.}\quad & 0 \le d_{t,z}, \quad \text{for all } s\in S,\ z\in Z_s,\ t\in T, && [\gamma_{t,z}] && (15b)\\
& 0 \le y_{t,g} \le \alpha_{t,g} y_g^{\text{ex}}, \quad \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{ex}},\ t\in T, && [\beta_{t,g}^{\pm}] && (15c)\\
& 0 \le y_{t,g} \le \alpha_{t,g} y_g^{\text{new}}, \quad \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{new}},\ t\in T, && [\delta_{t,g}^{\pm}] && (15d)\\
& \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && [\zeta_{t,x}] && (15e)\\
& 0 \le y_{t,i} \le \alpha_{t,i} y_i^{\text{ex}}, \quad \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && [\nu_{t,x}^{\pm}] && (15f)\\
& 0 \le d_{t,o}, \quad \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && [\theta_{t,x}] && (15g)\\
& \eta_x d_{t,o} = y_{t,i}, \quad \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && [\xi_{t,x}] && (15h)\\
& 0 \le y_{t,i} \le \alpha_{t,i} y_i^{\text{new}}, \quad \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && [\tau_{t,x}^{\pm}] && (15i)\\
& 0 \le d_{t,o}, \quad \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && [\rho_{t,x}] && (15j)\\
& f_k^- \le f_{t,k} \le f_k^+, \quad \text{for all } s\in S,\ k\in K_s,\ t\in T, && [\kappa_{t,k}^{\pm}] && (15k)\\
& (13), \quad \text{for all } s\in S,\ z\in Z_s,\ t\in T, && [\lambda_{t,z}] && (15l)
\end{aligned}$$

together with its KKT conditions: first stationarity

$$\begin{aligned}
P_{t,z}(d_{t,z}) - \lambda_{t,z} + \gamma_{t,z} &= 0, && \text{for all } s\in S,\ z\in Z_s,\ t\in T, && (16a)\\
\lambda_{t,z} - c_g^{\text{var}} + \beta_{t,g}^- - \beta_{t,g}^+ &= 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{ex}},\ t\in T, && (16b)\\
-c_g^{\text{inv}} + \sum_{t\in T}\alpha_{t,g}\delta_{t,g}^+ &= 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{new}}, && (16c)\\
\lambda_{t,z} - c_g^{\text{var}} + \delta_{t,g}^- - \delta_{t,g}^+ &= 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{new}},\ t\in T, && (16d)\\
\lambda_{t,z} + \nu_{t,x}^- - \nu_{t,x}^+ - \zeta_{t,x} &= 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (16e)\\
-\lambda_{t,z'} + \theta_{t,x} + \eta_x\zeta_{t,x} &= 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (16f)\\
-c_i^{\text{inv}} + \sum_{t\in T}\alpha_{t,i}\tau_{t,x}^+ &= 0, && \text{for all } x=(o,i)\in X^{\text{new}}, && (16g)\\
\lambda_{t,z} + \tau_{t,x}^- - \tau_{t,x}^+ - \xi_{t,x} &= 0, && \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && (16h)\\
-\lambda_{t,z'} + \rho_{t,x} + \eta_x\xi_{t,x} &= 0, && \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && (16i)\\
\lambda_{t,z} - \lambda_{t,z'} + \kappa_{t,k}^- - \kappa_{t,k}^+ &= 0, && \text{for all } s\in S,\ k=(z,z')\in K_s,\ t\in T, && (16j)
\end{aligned}$$

subsequent primal feasibility

$$\begin{aligned}
\eta_x d_{t,o} - y_{t,i} &= 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (17a)\\
\eta_x d_{t,o} - y_{t,i} &= 0, && \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && (17b)
\end{aligned}$$

and finally complementarity

$$\begin{aligned}
0 \le \gamma_{t,z} \perp d_{t,z} &\ge 0, && \text{for all } s\in S,\ z\in Z_s,\ t\in T, && (18a)\\
0 \le \beta_{t,g}^- \perp y_{t,g} &\ge 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{ex}},\ t\in T, && (18b)\\
0 \le \beta_{t,g}^+ \perp \alpha_{t,g}y_g^{\text{ex}} - y_{t,g} &\ge 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{ex}},\ t\in T, && (18c)\\
0 \le \delta_{t,g}^- \perp y_{t,g} &\ge 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{new}},\ t\in T, && (18d)\\
0 \le \delta_{t,g}^+ \perp \alpha_{t,g}y_g^{\text{new}} - y_{t,g} &\ge 0, && \text{for all } s\in S,\ z\in Z_s,\ g\in G_z^{\text{new}},\ t\in T, && (18e)\\
0 \le \nu_{t,x}^- \perp y_{t,i} &\ge 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (18f)\\
0 \le \nu_{t,x}^+ \perp \alpha_{t,i}y_i^{\text{ex}} - y_{t,i} &\ge 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (18g)\\
0 \le \theta_{t,x} \perp d_{t,o} &\ge 0, && \text{for all } x=(o,i)\in X^{\text{ex}},\ t\in T, && (18h)\\
0 \le \tau_{t,x}^- \perp y_{t,i} &\ge 0, && \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && (18i)\\
0 \le \tau_{t,x}^+ \perp \alpha_{t,i}y_i^{\text{new}} - y_{t,i} &\ge 0, && \text{for all } x=(o,i)\in X^{\text{new}},\ t\in T, && (18j)
\end{aligned}$$

$$0 \le \rho_{t,x} \perp d_{t,o} \ge 0, \quad \text{for all } x = (o,i) \in X^{\text{new}},\ t \in T, \tag{18k}$$

$$0 \le \kappa^-_{t,k} \perp f_{t,k} - f^-_k \ge 0, \quad \text{for all } s \in S,\ k \in K_s,\ t \in T, \tag{18l}$$

$$0 \le \kappa^+_{t,k} \perp f^+_k - f_{t,k} \ge 0, \quad \text{for all } s \in S,\ k \in K_s,\ t \in T. \tag{18m}$$

Since all inverse demand functions are assumed to be continuously differentiable and strictly decreasing, the welfare maximization problem is a concave maximization problem with linear constraints. Consequently, its KKT conditions (16)–(18) are necessary and sufficient for global optimality. Therefore, it is sufficient to show the 1-1 correspondence of the solutions of (MCP) and the stated KKT conditions in order to prove the equivalence of market equilibria and welfare maxima. Thus, we start with the implication that each solution of the KKT conditions yields a market equilibrium.

Lemma 3.1. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, \lambda)$ be a solution of the KKT conditions* (16)*–*(18)*. Then, w corresponds to a solution of (MCP).*

Proof. Comparison of (16)–(18) to (MCP) shows that all equations coincide if $p_{t,z} := \lambda_{t,z}$ holds for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$. Hence, $(d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ is a solution of (MCP). □

Next, we continue with the implication that each market equilibrium yields a solution of the KKT conditions.

Lemma 3.2. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of (MCP). Then, w corresponds to a solution of the KKT conditions* (16)*–*(18)*.*

Proof. Again, comparison of (MCP) to (16)–(18) shows that all equations coincide if $\lambda_{t,z} := p_{t,z}$ holds for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$. Thus, $(d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, \lambda)$ is a solution of the KKT conditions (16)–(18). □

Finally, we are able to prove the equivalence of market equilibria and welfare maxima.

Theorem 3.3. *There is a 1-1 correspondence of the market equilibrium problem (MCP) and the welfare maximization problem* (15)*.*

Proof. The assertion follows from Lemmata 3.1 and 3.2. □

So, we have shown that the presented welfare maximization problem is indeed a single-level counterpart to the proposed market equilibrium problem, i.e., each welfare maximal solution corresponds to a market equilibrium and vice versa. This 1-1 correspondence is exploited in the next section during the study of existence and uniqueness of the underlying market equilibrium.

4. Existence and Uniqueness

In the following, we study existence and uniqueness of short- and long-run market equilibrium of the proposed market equilibrium problem (MCP) that models the private investment and subsequent spot market trading in different coupled energy sectors. Since we established the equivalence of (MCP) to a single-level counterpart in the previous section, it is now easy to see existence of equilibrium by applying the classical theorem of Weierstraß to the single-level counterpart. However, uniqueness is not so easily achieved. For this purpose, additional conditions have to be fulfilled. Before we address these conditions, we show in Sect. 4.1 that if multiple equilibria exist, at least two of them share some structural properties. The obtained insights are later exploited to deduce contradictions when showing uniqueness under further assumptions. In Sect. 4.2, we derive conditions that we then prove to be sufficient for guaranteeing uniqueness of equilibrium in the short-run, i.e., for the case of fixed investment decisions. Afterward, in Sect. 4.3, additional sufficient conditions for long-run uniqueness are presented. We conclude with a detailed discussion of all conditions for uniqueness in Sect. 4.4.

4.1. **Structural properties of multiple equilibria.** Grimm, Schewe, et al. (2017) have shown in the context of a single sector that if multiple equilibria exist, there also exist two equilibria that share some structural properties. By applying Lemma 1 of Grimm, Schewe, et al. (2017), we obtain the same result for the case of multiple sectors.

Lemma 4.1. *Exactly one of the following two cases occurs:*

(i) There exist a demand vector d^*, a production vector y^*, and an investment vector $(y^{\text{new}})^*$ such that the primal variables of every market equilibrium of (MCP) are of the form $(d^*, y^*, f, (y^{\text{new}})^*)$ for some flow f.

(ii) There exist two market equilibria w and $\tilde{w}$ of (MCP) with $(d, y, y^{\text{new}}) \neq (\tilde{d}, \tilde{y}, \tilde{y}^{\text{new}})$ and

$$
\begin{aligned}
\{k \in K_s : f_{t,k} = f_k^-\} &= \{k \in K_s : \tilde{f}_{t,k} = f_k^-\},\\
\{k \in K_s : f_{t,k} = f_k^+\} &= \{k \in K_s : \tilde{f}_{t,k} = f_k^+\},\\
\{g \in G_z^{\text{all}} : y_{t,g} = 0\} &= \{g \in G_z^{\text{all}} : \tilde{y}_{t,g} = 0\},\\
\{g \in G_z^{\text{ex}} : y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\} &= \{g \in G_z^{\text{ex}} : \tilde{y}_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\},\\
\{i \in I_z^{\text{all}} : y_{t,i} = 0\} &= \{i \in I_z^{\text{all}} : \tilde{y}_{t,i} = 0\},\\
\{i \in I_z^{\text{ex}} : y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\} &= \{i \in I_z^{\text{ex}} : \tilde{y}_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\},
\end{aligned}
$$

for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.[1]

Proof. This follows from Proposition 1 in Grimm, Schewe, et al. (2017). □

We establish uniqueness of short- and long-run equilibrium in Sect. 4.2 and 4.3 by showing that, under appropriate conditions, only Case (i) of Lemma 4.1 occurs. To this end, it is advantageous to introduce the concept of price clusters as in Grimm, Schewe, et al. (2017). Later, we will state conditions on the appearance of these price clusters in order to ensure long-run uniqueness. For now, we use the concept to reformulate Lemma 4.1 suitably.

Definition 4.2. Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of the market equilibrium problem (MCP). Furthermore, let $\mathcal{C}_{t,s} = \{C_1, \dots, C_{|\mathcal{C}_{t,s}|}\}$ be a partition of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$. We call $\mathcal{C}_{t,s}$ a partition into *price clusters* if prices are equal for all bidding zones in a cluster $C \in \mathcal{C}_{t,s}$. To emphasize that price clusters may depend on the considered solution, we also use the notation $\mathcal{C}_{t,s}(w)$. The adjacent bidding zones $k = (z, z') \in K_s$ are called *intercluster adjacent bidding zones* if $z \in C_i$ and $z' \in C_j$ with $i \neq j$ holds.

To simplify notation in the following, we denote, e.g., by G_C^{new} the set of all candidate sector-specific production facilities in price cluster $C \in \mathcal{C}_{t,s}$ or by $\delta_s^{\text{in}}(C)$ the set of all ingoing trade capacities of the price cluster.

In case of a uniform pricing system in sector s, $\mathcal{C}_{t,s} = \{Z_s\}$ is a partition into price clusters for all time periods. The same is true if there is never congestion between bidding zones, i.e., if inequalities (11b) are never binding over the considered time horizon. This relation is shown in the next lemma and the subsequent corollary.

Lemma 4.3. *Let adjacent bidding zones $k = (z, z') \in K_s$ of sector $s \in S$ be given. For time periods $t \in T$ with $p_{t,z} \neq p_{t,z'}$, i.e., with different prices in the adjacent bidding zones, $f_{t,k} = f_k^+$ or $f_{t,k} = f_k^-$ holds.*

Proof. For $k = (z, z') \in K_s$ and $t \in T$ with $p_{t,z} < p_{t,z'}$, $\kappa_{t,k}^+ > 0$ follows from (12a) and therefore $f_{t,k} = f_k^+$ holds by (12c). For $k = (z, z') \in K_s$ and $t \in T$ with $p_{t,z} > p_{t,z'}$, $\kappa_{t,k}^- > 0$ follows from (12a) and therefore $f_{t,k} = f_k^-$ holds by (12b). □

[1] If this case applies, it follows directly that infinitely many market equilibria exist. As shown in the previous section, the two distinct market equilibria correspond to two distinct welfare maximal solutions. All convex combinations of this two welfare maxima also form welfare maxima since the welfare maximization problem (15) is a concave maximization problem with linear constraints. However, all these welfare maxima in turn correspond to market equilibria.

Lemma 4.3 directly implies that the prices of adjacent bidding zones are the same if the trade capacities between these bidding zones are not binding.

Corollary 4.4. *Let adjacent bidding zones $k = (z, z') \in K_s$ of sector $s \in S$ be given. For time periods $t \in T$ with $f_{t,k} \neq f_k^+$ and $f_{t,k} \neq f_k^-$, i.e., with non-binding trade capacities, $p_{t,z} = p_{t,z'}$ holds.*

In particular, this guarantees the following: if all adjacent bidding zones with one binding trade capacity are disregarded, then, remaining adjacent bidding zones have the same price. Based on this insight, a partition into price clusters of the bidding zones can be constructively derived (see Theorem 2 in Grimm, Schewe, et al. (2017) for the same result in the context of one sector).

Lemma 4.5. *Let $w = (d, y, f, y^{\text{new}}, \gamma, \beta^\pm, \delta^\pm, \zeta, \nu^\pm, \theta, \xi, \tau^\pm, \rho, \kappa^\pm, p)$ be a solution of the market equilibrium problem (MCP). Consider the set $K_{t,s}^{\text{o}}$ of adjacent bidding zones with non-binding trade capacities in time period $t \in T$ and sector $s \in S$, i.e., $K_{t,s}^{\text{o}} := \{k \in K_s : f_k^- < f_{t,k} < f_k^+\}$. Furthermore, consider the graph $\mathcal{G} := (Z_s, K_{t,s}^{\text{o}})$. Let $\mathcal{C}_{t,s} = \{C_1, \dots, C_{|\mathcal{C}_{t,s}|}\}$ be the set of connected components of the graph $\mathcal{G}$. Then, $\mathcal{C}_{t,s}$ is a partition into price clusters of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$.*

Proof. The assertion follows directly from Corollary 4.4. □

We call the latter partition capacity-induced because it is induced by the binding trade capacities in the given market equilibrium.

Definition 4.6. The partition $\mathcal{C}_{t,s}$ described in Lemma 4.5 is called the *capacity-induced* partition into price clusters of the bidding zones Z_s of sector $s \in S$ in time period $t \in T$.

Now, we are able to reformulate Lemma 4.1 suitably.

Lemma 4.7. *Exactly one of the following two cases occurs:*

(i) There exist a demand vector d^*, a production vector y^*, and an investment vector $(y^{\text{new}})^*$ such that the primal variables of every market equilibrium of (MCP) are of the form $(d^*, y^*, f, (y^{\text{new}})^*)$ for some flow f.

(ii) There exist two market equilibria w and $\tilde{w}$ of (MCP) with $(d, y, y^{\text{new}}) \neq (\tilde{d}, \tilde{y}, \tilde{y}^{\text{new}})$ such that

- *the capacity-induced partitions $\mathcal{C}_{t,s}(w)$ and $\mathcal{C}_{t,s}(\tilde{w})$ into price clusters of the bidding zones Z_s are the same for all sectors and all time periods, i.e., $\mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ holds for all $s \in S$, $t \in T$,*
- *the total in- or outflow of each price cluster $C \in \mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ is unique for all $s \in S$, $t \in T$:*
$$f_C := \sum_{k \in \delta_s^{\text{in}}(C)} f_{t,k} - \sum_{k \in \delta_s^{\text{out}}(C)} f_{t,k} = \sum_{k \in \delta_s^{\text{in}}(C)} \tilde{f}_{t,k} - \sum_{k \in \delta_s^{\text{out}}(C)} \tilde{f}_{t,k} =: \tilde{f}_C,$$
- *any operator of a sector-specific or sector-coupling facility who does not produce in one market equilibrium does also not produce in the other one, i.e.,*
$$\{g \in G_z^{\text{all}} : y_{t,g} = 0\} = \{g \in G_z^{\text{all}} : \tilde{y}_{t,g} = 0\},$$
$$\{i \in I_z^{\text{all}} : y_{t,i} = 0\} = \{i \in I_z^{\text{all}} : \tilde{y}_{t,i} = 0\},$$
for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$,
- *any operator of an existing sector-specific or sector-coupling facility who does produce at maximum available capacity in one market equilibrium does also produce at maximum available capacity in the other one, i.e.,*
$$\{g \in G_z^{\text{ex}} : y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\} = \{g \in G_z^{\text{ex}} : \tilde{y}_{t,g} = \alpha_{t,g} y_g^{\text{ex}}\},$$
$$\{i \in I_z^{\text{ex}} : y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\} = \{i \in I_z^{\text{ex}} : \tilde{y}_{t,i} = \alpha_{t,i} y_i^{\text{ex}}\},$$
for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.

Proof. The lemma follows directly from Lemmata 4.1 and 4.5 with the following observation. By definition of the capacity-induced partition, $f_{t,k} = f_k^+$ or $f_{t,k} = f_k^-$ holds for all intercluster adjacent bidding zones $k \in K_s$ and $f_{t,k} \neq f_k^+$ and $f_{t,k} \neq f_k^-$ for all other adjacent bidding zones $k \in K_s$. Since binding patterns are assumed to coincide in w and $\tilde{w}$ (cf. Lemma 4.1 (ii)), $\mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ and $f_C = \tilde{f}_C$ for $C \in \mathcal{C}_{t,s}(w) = \mathcal{C}_{t,s}(\tilde{w})$ follows for all $s \in S$, $t \in T$. □

So far, we have shown that if multiple equilibria exist, there necessarily exist two different equilibria that share the structural properties listed in Lemma 4.7 (ii). Next, we show gradually that, under additional assumptions, two such market equilibria are identical in the short-run (Sect. 4.2) as well as in the long-run (Sect. 4.3), i.e., finally only Case (i) of Lemma 4.7 occurs under all additional assumptions. This implies uniqueness of equilibrium of the proposed market equilibrium problem (MCP).

4.2. **Uniqueness of equilibrium in the short-run.** For the moment, we only consider the short-run, i.e., all investment decisions have already been made and are therefore fixed. We begin our short-run uniqueness analysis with the important observation that at least all consumer demand is unique.

Theorem 4.8. *Let w and $\tilde{w}$ be two market equilibria of (MCP). Then, the sector-specific demand is unique, i.e., $d_{t,z} = \tilde{d}_{t,z}$ holds for each sector $s \in S$, bidding zone $z \in Z_s$, and time period $t \in T$.*

Proof. The assertion follows from applying Theorem 1a in Mangasarian (1988) to the single-level counterpart (15) under the assumption of continuously differentiable and strictly decreasing inverse demand functions. □

However, to obtain uniqueness of the other primal decision variables and therefore of the short-run market equilibrium, additional assumptions are necessary. The first one relates to the sector-specific demand and is similar to Assumption 5 in Grübel et al. (2020).

Assumption 1. *The sector-specific demand $d_{t,z}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is positive in all time periods $t \in T$.*

This assumption can be justified by the fact that, in real applications, there should always be consumers in a bidding zone who are willing to pay enough and especially more than the resident operators of sector-coupling facilities. The benefit of this assumption is that it enables us to prove uniqueness of market prices (cf. Lemma 3.8 in Grübel et al. (2020) for the same result in the context of a single sector).

Lemma 4.9. *Suppose Assumption 1 holds. Then, the zonal price $p_{t,z}$ is unique for all sectors $s \in S$, bidding zones $z \in Z_s$, and time periods $t \in T$.*

Proof. Since $d_{t,z} > 0$ holds for all $s \in S$, $z \in Z_s$, and $t \in T$ by Assumption 1, $\gamma_{t,z} = 0$ follows from (2b). Moreover, $P_{t,z}(d_{t,z}) = p_{t,z}$ follows from (2a). Since all sector-specific demand is unique (Theorem 4.8), uniqueness of zonal prices is given due to the assumption of continuously differentiable and strictly decreasing inverse demand functions. □

Since we assume perfect competition, all players act as price takers, i.e., view market prices as exogenously given. This means that each player does not take the effects of his own or others' decisions on the prices into account when reacting to the market prices. Hence, due to the uniqueness of market prices, multiplicity of equilibria directly implies multiplicity of primal decisions of at least one individual player, i.e., players with multiple best responses to the unique market prices exist. For this reason, we next consider each player type independently and analyze under which circumstances these types are indeed indifferent about their actions and under which circumstances this is not the case.

First, we consider the operators of sector-specific production facilities. In accordance to Lemma 3.9 in Grübel et al. (2020), we prove that the production of a sector-specific facility is unique if the zonal market price does not meet its variable costs of production.

Lemma 4.10. *Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,g}$ of producer $g \in G_z^{\text{ex}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is unique if $p_{t,z} \neq c_g^{\text{var}}$ holds.*

Proof. For $g \in G_z^{\text{ex}}$ with $c_g^{\text{var}} < p_{t,z}$, $\beta_{t,g}^+ > 0$ follows from (4a) and therefore $y_{t,g} = \alpha_{t,g} y_g^{\text{ex}}$ holds by (4c). For $g \in G_z^{\text{ex}}$ with $c_g^{\text{var}} > p_{t,z}$, $\beta_{t,g}^- > 0$ follows from (4a) and therefore $y_{t,g} = 0$ holds by (4b). □

Analogously, the following applies for candidate sector-specific production facilities.

Lemma 4.11. *Given fixed investment decisions y^{new}. Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,g}$ of producer $g \in G_z^{\text{new}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ is unique if $p_{t,z} \neq c_g^{\text{var}}$ holds.*

Proof. For $g \in G_z^{\text{new}}$ with $c_g^{\text{var}} < p_{t,z}$, $\delta_{t,g}^+ > 0$ follows from (6b) and therefore $y_{t,g} = \alpha_{t,g} y_g^{\text{new}}$ holds by (6d). For $g \in G_z^{\text{new}}$ with $c_g^{\text{var}} > p_{t,z}$, $\delta_{t,g}^- > 0$ follows from (6b) and therefore $y_{t,g} = 0$ holds by (6c). □

Next, we examine the operators of sector-coupling facilities and show that those are only indifferent about their actions if the price of the withdrawing zone z' matches the price of the injection zone z multiplied by the facility's efficiency.

Lemma 4.12. *Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,i}$ and the demand $d_{t,o}$ of the sector-coupling facility $x = (o, i) \in X^{\text{ex}}$ are unique if $p_{t,z'} \neq \eta_x p_{t,z}$ holds.*

Proof. From equations (8a) and (8b), $-p_{t,z'} + \theta_{t,x} + \eta_x(p_{t,z} + \nu_{t,x}^- - \nu_{t,x}^+) = 0$ follows for all time periods $t \in T$. For $x = (o, i) \in X^{\text{ex}}$ with $p_{t,z'} < \eta_x p_{t,z}$, $\eta_x \nu_{t,x}^+ > 0$ follows from the latter equation and therefore $y_{t,i} = \alpha_{t,i} y_i^{\text{ex}}$ holds by (8e). For $x = (o, i) \in X^{\text{ex}}$ with $p_{t,z'} > \eta_x p_{t,z}$, $\theta_{t,x} + \eta_x \nu_{t,x}^- > 0$ follows and therefore $d_{t,o} = y_{t,i} = 0$ holds by (8c), (8d), and (8f). □

Analogously, the following applies for candidate sector-coupling facilities.

Lemma 4.13. *Given fixed investment decisions y^{new}. Let w be a market equilibrium of (MCP). Then, in time period $t \in T$, the production $y_{t,i}$ and the demand $d_{t,o}$ of the sector-coupling facility $x = (o, i) \in X^{\text{new}}$ are unique if $p_{t,z'} \neq \eta_x p_{t,z}$ holds.*

Proof. From equations (10b) and (10c), $-p_{t,z'} + \rho_{t,x} + \eta_x(p_{t,z} + \tau_{t,x}^- - \tau_{t,x}^+) = 0$ follows for all time periods $t \in T$. For $x = (o, i) \in X^{\text{new}}$ with $p_{t,z'} < \eta_x p_{t,z}$, $\eta_x \tau_{t,x}^+ > 0$ follows from the latter equation and therefore $y_{t,i} = \alpha_{t,i} y_i^{\text{new}}$ holds by (10f). For $x = (o, i) \in X^{\text{new}}$ with $p_{t,z'} > \eta_x p_{t,z}$, $\rho_{t,x} + \eta_x \tau_{t,x}^- > 0$ follows and therefore $d_{t,o} = y_{t,i} = 0$ holds by (10d), (10e), and (10g). □

Before we finally prove uniqueness of equilibrium in the short-run, we formulate additional sufficient conditions required in order to carry out the proof.

Assumption 2. *Here and in what follows, we assume that the below stated properties are satisfied by the variable production costs and the efficiencies of the sector-coupling facilities.*

(a) The variable production costs are pairwise distinct in each sector, i.e., $c_g^{\text{var}} \neq c_{g'}^{\text{var}}$ for $g \neq g'$ with $g \in G_z^{\text{all}}$ and $g' \in G_{z'}^{\text{all}}$ for all $z, z' \in Z_s$ and $s \in S$.

(b) The efficiencies of all sector-coupling facilities are pairwise distinct, i.e., $\eta_x \neq \eta_{x'}$ for $x \neq x'$ with $x, x' \in X^{\text{all}}$.

(c) Given the sector-coupling facility $x = (o, i) \in X^{\text{all}}$ that produces the commodity of sector $s \in S$ from the commodity of sector $s' \in S$ with $s' \neq s$. Then, the variable production costs across the sectors are—taking into account the sector-coupling facility's efficiency—pairwise distinct, i.e., $\eta_x c_g^{\text{var}} \neq c_{g'}^{\text{var}}$ with $g \in G_z^{\text{all}}$ for all $z \in Z_s$ and with $g' \in G_{z'}^{\text{all}}$ for all $z' \in Z_{s'}$.

(d) *Moreover, the following is true*

$$c_g^{\text{var}} \prod_{i=1}^{j} \eta_{x_i} \neq c_{g'}^{\text{var}} \prod_{i=j+1}^{j'} \eta_{x_i}$$

for all $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ *with* $2 \leq j' \leq \sum_{s \in S} |Z_s|$ *and for* $g \neq g'$ *with* $g \in G_z^{\text{all}}$ *for all* $z \in Z_s$, $s \in S$, *and with* $g' \in G_{z'}^{\text{all}}$ *for all* $z' \in Z_{s'}$, $s' \in S$.

(e) *Furthermore, it holds that*

$$\prod_{i=1}^{j} \eta_{x_i} \neq \prod_{i=j+1}^{j'} \eta_{x_i}$$

for all $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ *with* $2 \leq j' \leq \sum_{s \in S} |Z_s|$.

In principle, each condition prevents facilities from producing at identical variable production costs in the same sector. For example, Assumption 2 (b) guarantees that two sector-coupling facilities with the same input and target sector have distinct variable production costs. Assumption 2 (e) ensures, e.g., that one sector-coupling facility does not produce at the same costs as sequentially producing sector-coupling facilities with the same primary input and final target sector. In Sect. 4.4, we elaborate in more detail on this topic and also show with illustrative examples that as soon as one of the above conditions is removed, then, these conditions are no longer sufficient for guaranteeing uniqueness in the short-run.

In addition, we like to observe that Assumption 2 (a) is the direct extension of the classical uniqueness assumption for one sector (see, e.g., Grimm, Schewe, et al. (2017)) to the case of multiple sectors. Moreover, it directly follows from Assumption 2 (a) and Lemmata 4.10 and 4.11 that the sector-specific production in a price cluster is not unique for at most one player. Analogously, it directly follows from Assumption 2 (b) and Lemmata 4.12 and 4.13 that the sector-coupling production and demand between two price clusters of different sectors is not unique for at most one player. With Assumption 2 (c), the subsequent case is avoided: indifference about the sector-specific production in one sector leads to indifference about the sector-coupling demand and production between two sectors that in turn triggers indifference about the sector-specific production in the other sector. However, indifference about a sector-coupling demand and production between the other and another sector may also be triggered. To avoid such paths and cycles of indifference, Assumptions 2 (d) and (e) are required. Given these assumptions, we are finally able to prove short-run uniqueness. The proof is split into two parts. First, we show that all sector-specific production is unique. In a second step, we establish uniqueness of sector-coupling demand and production.

Lemma 4.14. *Given fixed investment decisions* y^{new}. *Suppose that Assumptions 1 and 2 hold. Let* w *and* $\tilde{w}$ *be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-specific production is unique, i.e.,* $y_{t,g} = \tilde{y}_{t,g}$ *holds for each sector* $s \in S$, *bidding zone* $z \in Z_s$, *producer* $g \in G_z^{\text{all}}$, *and time period* $t \in T$.

Proof. In the following, we denote for sector $s \in S$ and time period $t \in T$ by $\mathcal{C}_{t,s}$ the identical capacity-induced partitions $\mathcal{C}_{t,s}(w)$ and $\mathcal{C}_{t,s}(\tilde{w})$ into price clusters of the bidding zones Z_s. Let a price cluster $C \in \mathcal{C}_{t,s}$ be given. Moreover, let $p_{t,C}$ be the price of this price cluster. If $p_{t,C} \neq c_g^{\text{var}}$ holds for all producers $g \in G_C^{\text{all}}$, all sector-specific production in this cluster is unique (follows from Lemmata 4.10 and 4.11). In turn, if $p_{t,C} = c_g^{\text{var}}$ holds for a producer $g \in G_C^{\text{all}}$, the respective sector-specific production may not be unique. It remains to show that this is not the case. Hence, for the sake of contradiction, we assume that the production of producer g is not unique, i.e., $y_{t,g} \neq \tilde{y}_{t,g}$ holds for some $t \in T$. First, we like to observe that the production is then unique for all other sector-specific facilities in this cluster (follows from Lemmata 4.10 and 4.11 under Assumption 2 (a)). Since, in addition, all sector-specific demand (Theorem 4.8) and the total in- and outflow of this cluster (Lemma 4.7 (ii)) are unique, it directly follows from the market clearing conditions (13) for this cluster that $d_{t,o}$ or $y_{t,i}$ is not unique for at least one $o \in O_C^{\text{all}}$ or $i \in I_C^{\text{all}}$. This, in turn, implies by (7b) or

(9b) that the sector-coupling demand or production in at least one price cluster $C' \in \mathcal{C}_{t,s'}$ in another sector $s' \in S$ is not unique. Again, we like to observe that the sector-coupling demand and production is then unique for all other sector-coupling facilities that operate between price clusters C and C' (follows from Lemmata 4.12 and 4.13 under Assumption 2 (b)). Moreover, Lemmata 4.12 and 4.13 establish as a prerequisite for multiplicities that

$$p_{t,C} = c_g^{\text{var}} = \eta_x p_{t,C'} \quad \text{or} \quad p_{t,C} = c_g^{\text{var}} = (1/\eta_x) p_{t,C'}$$

holds for the respective sector-coupling facility $x \in X^{\text{all}}$. By Assumption 2 (c), it directly follows that $p_{t,C'} \neq c_{g'}^{\text{var}}$ for all $g' \in G_{C'}^{\text{all}}$. Hence, all sector-specific production is unique in this price cluster. The same argumentation as above implies that $d_{t,o}$ or $y_{t,i}$ is not unique for at least another $o \in O_{C'}^{\text{all}}$ or $i \in I_{C'}^{\text{all}}$. This, in turn, implies by (7b) or (9b) that the sector-coupling demand or production in at least one price cluster of another sector is not unique. The same argumentation can be applied again and again until either the non-unique sector-coupling demand or production is met by a non-unique sector-specific production of producer $g' \neq g$, i.e., the term

$$c_g^{\text{var}} \prod_{i=1}^{j} \eta_{x_i} = c_{g'}^{\text{var}} \prod_{i=j+1}^{j'} \eta_{x_i}$$

is—in accordance with the prerequisite for multiplicities—true for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$, or, since there are at most $\sum_{s \in S} |Z_s|$ price clusters, a price cluster C'' is again reached, i.e., the term

$$p_{t,C''} \prod_{i=1}^{j} \eta_{x_i} = p_{t,C''} \prod_{i=j+1}^{j'} \eta_{x_i} \quad \Leftrightarrow \quad \prod_{i=1}^{j} \eta_{x_i} = \prod_{i=j+1}^{j'} \eta_{x_i}$$

holds—in accordance with the prerequisite for multiplicities—for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \leq j' \leq \sum_{s \in S} |Z_s|$. Both cases are forbidden by Assumptions 2 (d) and (e). Therefore, $y_{t,g} = \tilde{y}_{t,g}$ holds. □

Next, we show the short-run uniqueness of sector-coupling demand and production under Assumptions 1 and 2.

Lemma 4.15. *Given fixed investment decisions y^{new}. Suppose that Assumptions 1 and 2 hold. Let w and $\tilde{w}$ be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-coupling demand and production are unique, i.e., $d_{t,o} = \tilde{d}_{t,o}$ and $y_{t,i} = \tilde{y}_{t,i}$ hold for each sector $s \in S$, bidding zone $z \in Z_s$, withdrawing facility $o \in O_z^{\text{all}}$, injecting facility $i \in I_z^{\text{all}}$, and time period $t \in T$.*

Proof. Since all sector-specific demand and production (Theorem 4.8 and Lemma 4.14) and the total in- and outflow of each cluster (Lemma 4.7 (ii)) are unique, non-unique demand or production of one sector-coupling facility $x \in X^{\text{all}}$ in a cluster directly implies by the market clearing conditions (13) for this cluster non-unique demand or production of another sector-coupling facility $x' \in X^{\text{all}}$ in this cluster. Furthermore, since at most one sector-coupling demand and production is not unique between two price clusters of different sectors (follows from Lemmata 4.12 and 4.13 under Assumption 2 (b)), this non-uniqueness continues until, due to at most $\sum_{s \in S} |Z_s|$ price clusters, the same price cluster C is again reached, i.e., the term

$$p_{t,C} \prod_{i=1}^{j} \eta_{x_i} = p_{t,C} \prod_{i=j+1}^{j'} \eta_{x_i} \quad \Leftrightarrow \quad \prod_{i=1}^{j} \eta_{x_i} = \prod_{i=j+1}^{j'} \eta_{x_i}$$

holds for some $\{x_1, \dots, x_{j'}\} \subseteq X^{\text{all}}$ with $2 \le j' \le \sum_{s \in S} |Z_s|$. This contradicts Assumption 2 (e) and, consequently, there is uniqueness with regard to sector-coupling demand and production.[2] □

So, we finally obtain uniqueness of short-run market equilibrium in coupled markets of multiple energy sectors.

Theorem 4.16. *Suppose that Assumptions 1 and 2 hold. Then, the market equilibrium of (MCP) is unique in the short-run, i.e., in the case of fixed investment decisions.*

Proof. The assertion follows directly from Theorem 4.8 and Lemmata 4.14 and 4.15. □

4.3. **Uniqueness of equilibrium in the long-run.** We now continue our study of uniqueness with the long-run, i.e., investment decisions are no longer fixed. It is common knowledge that investment decisions are mainly driven by the generated contribution margins over time. What is special in our context is that the time periods with positive contribution margins are uniquely determined for each facility due to the uniqueness of zonal prices (Lemma 4.9).

Lemma 4.17. *Suppose that Assumption 1 holds. Let w be a market equilibrium of (MCP). Moreover, let T_g for producer $g \in G_z^{\text{new}}$ in bidding zone $z \in Z_s$ in sector $s \in S$ be the time periods in which this producer obtains positive contribution margins, i.e., $T_g := \{t \in T : \delta_{t,g}^+ > 0\}$ holds. Then, T_g is unique.*

Proof. Since the zonal price $p_{t,z}$ is unique for all time periods $t \in T$ by Lemma 4.9, the time periods with $p_{t,z} > c_g^{\text{var}}$ are also unique but $p_{t,z} > c_g^{\text{var}}$ holds in a time period t if and only if $\delta_{t,g}^+ > 0$ holds due to (6b)–(6d); see also the argumentation in the proof of Lemma 4.11. □

Analogously, the same applies for candidate sector-coupling facilities.

Lemma 4.18. *Suppose that Assumption 1 holds. Let w be a market equilibrium of (MCP). Moreover, let T_x for producer $x \in X^{\text{new}}$ be the time periods in which this producer generates positive contribution margins, i.e., $T_x := \{t \in T : \tau_{t,x}^+ > 0\}$ holds. Then, T_x is unique.*

Proof. Since the zonal price $p_{t,z}$ of the bidding zone $z \in Z_s$ of sector $s \in S$, in which the injection point of facility x is located, and the zonal price $p_{t,z'}$ of the bidding zone $z' \in Z_{s'}$ of sector $s' \in S$ with $s \neq s'$, in which the withdrawal point of facility x is located, are unique for all time periods $t \in T$ by Lemma 4.9, the time periods with $p_{t,z'} < \eta_x p_{t,z}$ are also unique but $p_{t,z'} < \eta_x p_{t,z}$ holds in a time period t if and only if $\tau_{t,x}^+ > 0$ holds due to (10b), (10c), (10e), and (10f); see also the argumentation in the proof of Lemma 4.13. □

Nevertheless, in order to achieve uniqueness of the long-run equilibrium, further assumptions are required. These are obtained by naturally extending Assumptions 3 and 4 of Grimm, Schewe, et al. (2017) to the case of multiple sectors.

Assumption 3. *Let w be a market equilibrium of (MCP). Here and in what follows, we assume that, for each sector $s \in S$, there exists a subset of time periods $\overline{T} \subseteq T$ for which the capacity-induced partitions $\mathcal{C}_{t,s}(w)$, $t \in \overline{T}$, are the same. Moreover, for each price cluster $C \in \mathcal{C}_{t,s}(w)$, there exists a bijective function $h_C : \overline{T}_C \to G_C^{\text{new}} \cup I_C^{\text{new}} \cup O_C^{\text{new}}$ with $\overline{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|\overline{T}_C|}\} \subseteq \overline{T}$ such that the following holds for a given $j \in \{1, \dots, |\overline{T}_C|\}$:*

(i) all producers $h_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, produce in time period $\bar{t}_j$ either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied in Equation (6d) or (10f), i.e., either

$$y_{\bar{t}_j, h_C(\bar{t}_{j'})} = 0 \quad \text{or} \quad y_{\bar{t}_j, h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_j, h_C(\bar{t}_{j'})} y_{h_C(\bar{t}_{j'})}^{\text{new}}$$

[2] In order to increase the understanding of the latter expression and therefore of the contradiction, we like to point out again that Lemmata 4.12 and 4.13 establish as a prerequisite for multiplicities that

$$p_{t,C} = \eta_x p_{t,C'} \quad \text{or} \quad p_{t,C} = (1/\eta_x) p_{t,C'}$$

holds for a sector-coupling facility $x \in X^{\text{all}}$ that operates between the price clusters C and C'.

with

$$\delta^+_{\bar{t}_j,h_C(\bar{t}_{j'})} > 0, \quad \text{if } h_C(\bar{t}_{j'}) \in G_C^{\text{new}}, \text{ and}$$
$$\tau^+_{\bar{t}_j,x} > 0, \quad \text{if } x = (o, h_C(\bar{t}_{j'})) \in X^{\text{new}}.$$

(ii) *all sector-coupling consumers* $h_C(\bar{t}_{j'})$ *with* $j' < j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, *demand in time period* $\bar{t}_j$ *either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied in Equation* (10f), *i.e., either*

$$d_{\bar{t}_j,h_C(\bar{t}_{j'})} = 0 \quad \text{or} \quad \eta_x d_{\bar{t}_j,h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_j,i} y_i^{\text{new}} \quad \text{with} \quad \tau^+_{\bar{t}_j,x} > 0,$$

where $x = (h_C(\bar{t}_{j'}), i) \in X^{\text{new}}$.

(iii) *if* $h_C(\bar{t}_j)$ *is a producer, then, this producer produces in time period* $\bar{t}_j$ *at maximum available capacity while strict complementarity is satisfied in Equation* (6d) *or* (10f), *i.e.,*

$$y_{\bar{t}_j,h_C(\bar{t}_j)} = \alpha_{\bar{t}_j,h_C(\bar{t}_j)} y^{\text{new}}_{h_C(\bar{t}_j)}$$

with

$$\delta^+_{\bar{t}_j,h_C(\bar{t}_j)} > 0, \quad \text{if } h_C(\bar{t}_j) \in G_C^{\text{new}}, \text{ and}$$
$$\tau^+_{\bar{t}_j,x} > 0, \quad \text{if } x = (o, h_C(\bar{t}_j)) \in X^{\text{new}}.$$

(iv) *if* $h_C(\bar{t}_j)$ *is a sector-coupling consumer, then, this consumer demands in time period* $\bar{t}_j$ *at maximum available capacity while strict complementarity is satisfied in Equation* (10f), *i.e.,*

$$\eta_x d_{\bar{t}_j,h_C(\bar{t}_j)} = \alpha_{\bar{t}_j,i} y_i^{\text{new}} \quad \text{with} \quad \tau^+_{\bar{t}_j,x} > 0,$$

where $x = (h_C(\bar{t}_j), i) \in X^{\text{new}}$.

(v) *all producers* $h_C(\bar{t}_{j'})$ *with* $j' > j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, *produce in time period* $\bar{t}_j$ *zero, i.e.,* $y_{\bar{t}_j,h_C(\bar{t}_{j'})} = 0$,

(vi) *all sector-coupling consumers* $h_C(\bar{t}_{j'})$ *with* $j' > j$, $j' \in \{1, \dots, |\overline{T}_C|\}$, *demand in time period* $\bar{t}_j$ *zero, i.e.,* $d_{\bar{t}_j,h_C(\bar{t}_{j'})} = 0$,

(vii) *all existing sector-specific producers produce in time period* $\bar{t}_j$ *either zero or at maximum available capacity, i.e.,* $y_{\bar{t}_j,g} = 0$ *or* $y_{\bar{t}_j,g} = \alpha_{\bar{t}_j,g} y_g^{\text{ex}}$ *for all* $g \in G_C^{\text{ex}}$,

(viii) *all existing sector-coupling producers produce in time period* $\bar{t}_j$ *either zero or at maximum available capacity, i.e.,* $y_{\bar{t}_j,i} = 0$ *or* $y_{\bar{t}_j,i} = \alpha_{\bar{t}_j,i} y_i^{\text{ex}}$ *for all* $i \in I_C^{\text{ex}}$, *and*

(ix) *all existing sector-coupling consumers demand in time period* $\bar{t}_j$ *either zero or at maximum available capacity, i.e.,* $d_{\bar{t}_j,o} = 0$ *or* $\eta_x d_{\bar{t}_j,o} = \alpha_{\bar{t}_j,i} y_i^{\text{ex}}$ *for all* $x = (o, i) \in X^{\text{ex}}$ *with* $o \in O_C^{\text{ex}}$.

In the following, we explain why Assumption 3 prevents multiplicities in the long-run and how likely it is that Assumption 3 is fulfilled in general. Therefore, let us assume that the long-run investment of a given facility is not unique, e.g., $y_g^{\text{new}} + \varepsilon_g = \tilde{y}_g^{\text{new}}$ holds for two market equilibria w and $\tilde{w}$ of (MCP) as described in Lemma 4.7 (ii) with $\varepsilon_g > 0$.[3] By Lemmata 4.17 and 4.18, it directly follows that also the production is not unique, i.e.,

$$\tilde{y}_{t,g} = \alpha_{t,g} \tilde{y}_g^{\text{new}} = \alpha_{t,g}(y_g^{\text{new}} + \varepsilon_g) = y_{t,g} + \alpha_{t,g}\varepsilon_g \tag{19}$$

holds for all time periods $t \in T_g$ due to $\delta^+_{t,g} > 0$, $\tilde{\delta}^+_{t,g} > 0$, and (6d). However, since the sector-specific demand is unique (Theorem 4.8) and must be served in each time period, all these changes in production have to be compensated by changes in production of other facilities of the cluster. Now, this is the point where Assumption 3 comes in. Assumption 3 (i)–(vi) involves that at least one time period exists for each candidate facility in which this facility and all other producing candidate facilities generate positive contribution margins. Moreover,

[3]The same argumentation can be applied for the sector-coupling facility $x \in X^{\text{new}}$.

in this time period, the production of all existing facilities coincides in w and $\tilde{w}$ due to Assumption 3 (vii)–(ix) and Lemma 4.7 (ii). Hence, one of the compensating facilities in this time period is a candidate facility that produces with a positive contribution margin. Thus, also the investment of this facility is not unique and resulting changes in production of this facility have to be compensated by changes in production of other facilities of the cluster. At this point, Assumption 3 again intervenes and prevents that these two facilities entirely compensate each other, i.e., the sets of time periods with positive contribution margins of facility g and the mentioned compensating candidate facility are different.[4] Furthermore, Assumption 3 imposes that there exists a time period in the union of the set differences in which all conditions of Assumption 3 are again fulfilled. Therefore, applying the same argument as above leads to multiplicities in investment of a third candidate facility. Assumption 3 allows us to apply this argument until a time period is reached in which only one candidate facility from this cluster produces while the production of all existing facilities coincides in this time period in w and $\tilde{w}$ due to Assumption 3 (vii)–(ix) and Lemma 4.7 (ii).[5] Since no compensation is possible in this time period, the initial assumption that the long-run investment of facility g is not unique must have been wrong.

In what follows, we want to elaborate on how likely it is that Assumption 3 is violated in general. In the case of one sector, Assumption 3 (iii) and (iv) involve that at least one time period exists for each candidate facility in which this facility generates the lowest but still positive contribution margin. This implies that—given the merit order of all candidate facilities in a cluster C, i.e., $c_1^{\text{var}} < c_2^{\text{var}} < \ldots < c_n^{\text{var}}$ with $n := |G_C^{\text{new}}|$—there exists at least one time period $\bar{t}_1$ such that the relation $c_1^{\text{var}} < p_{\bar{t}_1,C} < c_2^{\text{var}}$ is true, at least one time period $\bar{t}_2$ such that $c_2^{\text{var}} < p_{\bar{t}_2,C} < c_3^{\text{var}}$ holds, and so on. This situation is illustrated for the case of three candidate and no existing production facility in Figure 3. In particular, the above mentioned condition implies the following: over time, there runs at least one aggregated cluster demand function through each dashed area depicted in Figure 3 (exemplary inverse demand functions are indicated). Thus, the wider the gap between two subsequent facilities in the merit order, the more likely it is that the stated assumption is satisfied. We like to observe that this result is in line with the classical peak-load pricing setting for a single sector with one bidding zone, where significant gaps between variable costs are also a prerequisite. For more information on the case of a single sector see Grimm, Schewe, et al. (2017).

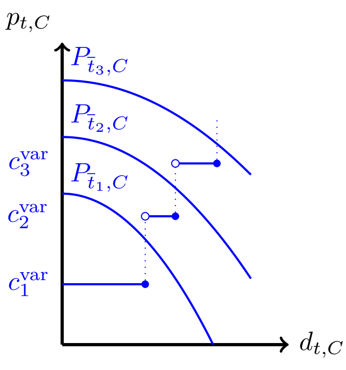

FIGURE 3. Illustration of Assumption 3 for the case of a single sector

[4]In time period $\bar{t}_1$, the only producing candidate facility with positive contribution margin is $h_C(\bar{t}_1)$ due to Assumption 3 (i)–(vi). In time period $\bar{t}_j$ with $j \in \{2, \ldots, |\overline{T}_C|\}$, the candidate facility $h_C(\bar{t}_j)$ produces with positive contribution margin due to Assumption 3 (iii) and (iv). In addition, at most the candidate facilities $h_C(\bar{t}_{j'})$ with $j' < j$, $j' \in \{1, \ldots, |\overline{T}_C|\}$, produce due to Assumption 3 (i), (ii), (v), and (vi). And in case of production, these facilities also produce with positive contribution margins. Consequently, different candidate facilities of a cluster never produce always at the same time with positive contribution margins.

[5]This is, e.g., the case in time period $\bar{t}_1$.

For the case of multiple sectors, the long-run uniqueness condition cannot be represented as easily as in the case of one sector via the merit order. Of course, it must still be true for the merit order of candidate sector-specific production facilities in each cluster that there exists at least one time period $\bar{t}_1$ such that the relation $c_1^{\mathrm{var}} < p_{\bar{t}_1,C} < c_2^{\mathrm{var}}$ is true, at least one time period $\bar{t}_2$ such that $c_2^{\mathrm{var}} < p_{\bar{t}_2,C} < c_3^{\mathrm{var}}$ holds, and so on. Hence, significant gaps between variable costs within clusters still increase the chance of Assumption 3 being valid. Nevertheless, this condition alone is not enough since also the candidate sector-coupling facilities injecting or withdrawing in this cluster have to be considered. However, it is not a priori clear where the sector-coupling facilities enter the respective merit orders. This depends heavily on the underlying demand and supply data of the coupled sectors and the resulting relation of prices between these sectors. Again, the more diverse the data, the more likely it is that Assumption 3 is satisfied.

As a first step in order to formally obtain long-run uniqueness under Assumption 3, we prove that investment decisions are unique under this assumption.

Lemma 4.19. *Suppose that Assumptions 1, 2, and 3 hold. Let w and $\tilde{w}$ be two market equilibria of (MCP) as described in Lemma 4.7 (ii). Then, the sector-specific and sector-coupling investment is unique in each bidding zone $z \in Z_s$ and sector $s \in S$, i.e., $y_g^{\mathrm{new}} = \tilde{y}_g^{\mathrm{new}}$ holds for all candidate sector-specific production facilities $g \in G_z^{\mathrm{new}}$ and $y_i^{\mathrm{new}} = \tilde{y}_i^{\mathrm{new}}$ holds for all candidate sector-coupling facilities $i \in I_z^{\mathrm{new}}$.*

Proof. In what follows, we assume that, for each sector $s \in S$, capacity-induced partitions $\mathcal{C}_{t,s}(w)$, $t \in \overline{T} \subseteq T$, as well as bijective functions $h_C : \overline{T}_C \to G_C^{\mathrm{new}} \cup I_C^{\mathrm{new}} \cup O_C^{\mathrm{new}}$ with $\overline{T}_C := \{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_{|T_C|}\} \subseteq \overline{T}$ for all price clusters $C \in \mathcal{C}_{t,s}(w)$ are given that fulfill the properties stated in Assumption 3. Now, we prove the assertion by induction over the time periods in $\overline{T}_C$. We start the induction with period $\bar{t}_1$. First of all, we know by Theorem 4.8 that

$$d_{\bar{t}_1,z} = \tilde{d}_{\bar{t}_1,z}, \qquad \text{for all } z \in C, \tag{20}$$

holds. In accordance with Assumption 3 (vii)–(ix), all existing producers produce either zero or at maximum available capacity and all existing sector-coupling consumers demand either zero or at maximum available capacity. Thus, it directly follows by Lemma 4.7 (ii) that

$$y_{\bar{t}_1,g} = \tilde{y}_{\bar{t}_1,g}, \qquad \text{for all } g \in G_C^{\mathrm{ex}}, \tag{21}$$

$$y_{\bar{t}_1,i} = \tilde{y}_{\bar{t}_1,i}, \qquad \text{for all } i \in I_C^{\mathrm{ex}}, \tag{22}$$

$$d_{\bar{t}_1,o} = \tilde{d}_{\bar{t}_1,o}, \qquad \text{for all } o \in O_C^{\mathrm{ex}}, \tag{23}$$

is true. Moreover, Assumption 3 (v) and (vi) imply that $y_{\bar{t}_1,g} = y_{\bar{t}_1,i} = d_{\bar{t}_1,o} = 0$ is valid for all $g \in G_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}$, $i \in I_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}$, and $o \in O_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}$. Hence,

$$y_{\bar{t}_1,g} = \tilde{y}_{\bar{t}_1,g} = 0, \qquad \text{for all } g \in G_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}, \tag{24}$$

$$y_{\bar{t}_1,i} = \tilde{y}_{\bar{t}_1,i} = 0, \qquad \text{for all } i \in I_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}, \tag{25}$$

$$d_{\bar{t}_1,o} = \tilde{d}_{\bar{t}_1,o} = 0, \qquad \text{for all } o \in O_C^{\mathrm{new}} \setminus \{h_C(\bar{t}_1)\}, \tag{26}$$

follows from Lemma 4.7 (ii). Furthermore, by Lemma 4.7 (ii), the capacity-induced partitions $\mathcal{C}_{\bar{t}_1,s}(w)$ and $\mathcal{C}_{\bar{t}_1,s}(\tilde{w})$ are the same for each sector $s \in S$ and, in addition, the total in- and outflow of each price cluster $C \in \mathcal{C}_{\bar{t}_1,s}(w) = \mathcal{C}_{\bar{t}_1,s}(\tilde{w})$ is unique. Consequently, by summing up the market clearing conditions (13) for all bidding zones of the price cluster C, we obtain the following relation

$$\begin{aligned} f_C &= \sum_{z \in C} d_{\bar{t}_1,z} - \sum_{g \in G_C^{\mathrm{all}}} y_{\bar{t}_1,g} + \sum_{o \in O_C^{\mathrm{all}}} d_{\bar{t}_1,o} - \sum_{i \in I_C^{\mathrm{all}}} y_{\bar{t}_1,i} \\ &= \sum_{z \in C} \tilde{d}_{\bar{t}_1,z} - \sum_{g \in G_C^{\mathrm{all}}} \tilde{y}_{\bar{t}_1,g} + \sum_{o \in O_C^{\mathrm{all}}} \tilde{d}_{\bar{t}_1,o} - \sum_{i \in I_C^{\mathrm{all}}} \tilde{y}_{\bar{t}_1,i} = \tilde{f}_C, \end{aligned}$$

which is due to (20)–(26) equivalent to

$$y_{\bar{t}_1,h_C(\bar{t}_1)} = \tilde{y}_{\bar{t}_1,h_C(\bar{t}_1)}, \quad \text{if } h_C\left(\bar{t}_1\right) \in (G_C^{\text{new}} \cup I_C^{\text{new}}), \text{ and}$$
$$d_{\bar{t}_1,h_C(\bar{t}_1)} = \tilde{d}_{\bar{t}_1,h_C(\bar{t}_1)}, \quad \text{if } h_C\left(\bar{t}_1\right) \in O_C^{\text{new}}.$$

Therefore, all production and demand decisions coincide in w and $\tilde{w}$ in time period $\bar{t}_1$.

From now on, we assume that $h_C\left(\bar{t}_1\right) \in G_C^{\text{new}}$ holds and derive the assertion of the induction start. The same argumentation can be applied to the case $h_C\left(\bar{t}_1\right) \in I_C^{\text{new}} \cup O_C^{\text{new}}$. By Assumption 3 (iii), $\delta^+_{\bar{t}_1,h_C(\bar{t}_1)} > 0$ holds. Thus, by Lemma 4.17, $\tilde{\delta}^+_{\bar{t}_1,h_C(\bar{t}_1)} > 0$ is also true. Due to (6d), we get

$$y_{\bar{t}_1,h_C(\bar{t}_1)} = \alpha_{\bar{t}_1,h_C(\bar{t}_1)} y^{\text{new}}_{h_C(\bar{t}_1)} \quad \text{and} \quad \tilde{y}_{\bar{t}_1,h_C(\bar{t}_1)} = \alpha_{\bar{t}_1,h_C(\bar{t}_1)} \tilde{y}^{\text{new}}_{h_C(\bar{t}_1)}.$$

Since we have already shown that all production decisions coincide in w and $\tilde{w}$ in time period $\bar{t}_1$, the investment in the facility $h_C\left(\bar{t}_1\right)$ also coincides by the latter two equations. Hence, we are finally finished with the start of our induction:

$$y^{\text{new}}_{h_C(\bar{t}_1)} = \tilde{y}^{\text{new}}_{h_C(\bar{t}_1)}.$$

Now, for a given time period $\bar{t}_{j+1}$ with $j \in \{1, \ldots, |\overline{T}_C|-1\}$, we assume that the investment in the facilities $h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_j\right)$ coincide in the two market equilibria w and $\tilde{w}$. It remains to show that this is also the case for the facility $h_C\left(\bar{t}_{j+1}\right)$. The same argumentation as in the start of the induction yields

$$\begin{aligned}
d_{\bar{t}_{j+1},z} &= \tilde{d}_{\bar{t}_{j+1},z}, && \text{for all } z \in C,\\
y_{\bar{t}_{j+1},g} &= \tilde{y}_{\bar{t}_{j+1},g}, && \text{for all } g \in G_C^{\text{ex}},\\
y_{\bar{t}_{j+1},i} &= \tilde{y}_{\bar{t}_{j+1},i}, && \text{for all } i \in I_C^{\text{ex}},\\
d_{\bar{t}_{j+1},o} &= \tilde{d}_{\bar{t}_{j+1},o}, && \text{for all } o \in O_C^{\text{ex}},\\
y_{\bar{t}_{j+1},g} = \tilde{y}_{\bar{t}_{j+1},g} &= 0, && \text{for all } g \in G_C^{\text{new}} \setminus \left\{h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_{j+1}\right)\right\},\\
y_{\bar{t}_{j+1},i} = \tilde{y}_{\bar{t}_{j+1},i} &= 0, && \text{for all } i \in I_C^{\text{new}} \setminus \left\{h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_{j+1}\right)\right\},\\
d_{\bar{t}_{j+1},o} = \tilde{d}_{\bar{t}_{j+1},o} &= 0, && \text{for all } o \in O_C^{\text{new}} \setminus \left\{h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_{j+1}\right)\right\}.
\end{aligned}$$

It remains to consider the production and demand decisions of the facilities $h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_{j+1}\right)$. By Assumption 3 (i) and (ii), the facilities $h_C\left(\bar{t}_1\right), \ldots, h_C\left(\bar{t}_j\right)$ produce or demand in the market equilibrium w in time period $\bar{t}_{j+1}$ either zero or at maximum available capacity, where in the latter case strict complementarity is satisfied. In accordance with Lemma 4.7 (ii), it follows for the case of zero production or demand that there is also zero production or demand in the market equilibrium $\tilde{w}$. Moreover, for the case of production or demand at maximum available capacity, strict complementarity in w implies strict complementarity in $\tilde{w}$ by Lemmata 4.17 and 4.18. Due to (6d) or (10f), we therefore obtain for $j' \in \{1, \ldots, j\}$

$$y_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} y^{\text{new}}_{h_C(\bar{t}_{j'})} \quad \text{and} \quad \tilde{y}_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} \tilde{y}^{\text{new}}_{h_C(\bar{t}_{j'})},$$

if $h_C\left(\bar{t}_{j'}\right) \in (G_C^{\text{new}} \cup I_C^{\text{new}})$, and

$$\eta_x d_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1},i} y_i^{\text{new}} \quad \text{and} \quad \eta_x \tilde{d}_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \alpha_{\bar{t}_{j+1},i} \tilde{y}_i^{\text{new}},$$

if $x = \left(h_C\left(\bar{t}_{j'}\right), i\right) \in X^{\text{new}}$ with $h_C\left(\bar{t}_{j'}\right) \in O_C^{\text{new}}$. Since, in both cases, the right-hand sides are equal by the induction hypothesis, it directly follows for all $j' \in \{1, \ldots, j\}$ that

$$y_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \tilde{y}_{\bar{t}_{j+1},h_C(\bar{t}_{j'})}, \quad \text{if } h_C\left(\bar{t}_{j'}\right) \in (G_C^{\text{new}} \cup I_C^{\text{new}}), \text{ and}$$
$$d_{\bar{t}_{j+1},h_C(\bar{t}_{j'})} = \tilde{d}_{\bar{t}_{j+1},h_C(\bar{t}_{j'})}, \quad \text{if } h_C\left(\bar{t}_{j'}\right) \in O_C^{\text{new}},$$

is true. Now, by applying the same argument as in the start of our induction, we obtain that the production or demand associated to $h_C\left(\bar{t}_{j+1}\right)$ also coincide in w and $\tilde{w}$ in time period $\bar{t}_{j+1}$. Therefore, all production and demand decisions coincide in w and $\tilde{w}$ in time period $\bar{t}_{j+1}$.

The same argumentation as in the start of our induction finally yields

$$y^{\text{new}}_{h_C(\bar{t}_{j+1})} = \tilde{y}^{\text{new}}_{h_C(\bar{t}_{j+1})}.$$

□

So, we finally obtain uniqueness of long-run market equilibrium in coupled markets of multiple energy sectors.

Theorem 4.20. *Suppose that Assumptions 1, 2, and 3 hold. Then, the market equilibrium of (MCP) is unique in the long-run.*

Proof. The assertion directly follows from Theorem 4.16 and Lemma 4.19. □

4.4. **A discussion of the sufficient conditions for uniqueness.** So far, we presented sufficient conditions that ensure uniqueness of equilibrium for coupled markets of multiple energy sectors in the short-run and in the long-run. We discuss those conditions in more detail below in order to provide further insights on when these conditions are violated and how resulting multiplicities might be resolved. In addition, we show that each condition is indeed needed for guaranteeing uniqueness in general.

We start with the short-run and therefore assume for the moment that all investment is fixed. Multiplicities arise in the short-run mainly due to similar cost structures of different production facilities. Following Assumption 2, again five cases can be distinguished

(1) two sector-specific facilities have the same variable production costs (violates Ass. 2 (a); see Example B.1)
(2) two sector-coupling facilities with the same input and target sector have the same variable production costs since their efficiencies are not distinct (violates Ass. 2 (b); see Example B.2)
(3) a sector-coupling facility and a sector-specific facility from its target sector have the same variable production costs (violates Ass. 2 (c); see Example B.3)
(4) sequentially producing sector-coupling facilities and either a sector-specific facility from their final target sector or other sequentially producing sector-coupling facilities with the same final target sector have the same variable production costs (violates Ass. 2 (d); see Example B.4)
(5) sequentially producing sector-coupling facilities and other sequentially producing sector-coupling facilities with the same initial and final target sector have the same variable production costs (violates Ass. 2 (e); see Example B.5)

Each mentioned example illustrates how multiplicities arise if the corresponding case occurs. All examples are structured such that exactly one condition of Assumption 2 is violated, e.g., for Case (5) only Assumption 2 (e) is violated in Example B.5 while Conditions (a)–(d) of Assumption 2 hold. This demonstrates in particular that each sufficient condition identified by us is indeed needed for guaranteeing uniqueness in general. For more information on the examples see App. B.

Of course, we are aware that not all of the conditions in Assumption 2 might be fulfilled if real data is analyzed. Several sector-specific production facilities might have the same cost structure, e.g., if several power plants of the same technology are considered. One way to resolve such multiplicities is to artificially perturb same cost structures a priori by small values. Nevertheless, this can lead to undesirable boundary solutions in which, e.g., one player never produces while others with basically the same cost structure always produce. Thus, as an alternative, suitable and transparent tie-breaking rules can be implemented that select a certain equilibrium. To be able to formulate such rules, it is crucial to have a fundamental understanding of the conditions that lead to multiplicities. As explained above, identical cost structures are mainly responsible for occurring multiplicities in the short-run. Due to same variable production costs, there is no way to distinguish between two facilities. A suitable tie-breaking rule is for example to consider these facilities as a single facility for the purpose of calculating the equilibrium and then to allocate the aggregated production to the individual facilities based on their installed capacities. Consequently, facilities with the same production costs have equal capacity utilization in the resulting equilibrium.

Also in the long-run, multiplicities mainly occur due to similar cost structures of different facilities. In contrast to the short-run, the total costs of production are now the decisive factor. In particular, if all facilities are fully available and two facilities always produce at the same time with positive contribution margins and, additionally, at the same average costs, multiplicities might arise due to the possibility of shifting investment and corresponding production cost-neutrally from one facility to the other. This result is in line with the findings of Grimm, Schewe, et al. (2017) for a single sector. For the sake of completeness, we illustrate in Example C.1 of App. C how multiplicities might arise in a single sector in the long-run. The case of multiple sectors is treated in Example C.2. In both cases, it is not possible due to similar cost structures to distinguish in which facility should best be invested to satisfy the overall demand. A possible remedy to this situation is again to implement a suitable tie-breaking rule. For example, both operators could each invest the same amount. Alternatively, external costs that have not yet been taken into account might determine whether more investment is made at one location or at the other. However, if data with sufficient fluctuations in, e.g., the demand is available, such cases should actually not occur.

5. Conclusion

In this paper, we have extended the existing literature by providing a framework to analyze coupled markets of multiple energy sectors, as well as conditions for uniqueness of short- and long-run market equilibrium in such a setup. Our framework lays the foundation for addressing timely research questions on energy market coupling in the context of the transition towards a sustainable energy system. It paves the way to analyze timely questions with regard to, e.g., climate policy, regulation of network and energy pricing, and their effects on infrastructure planning, investment incentives, and market outcomes.

Our results show that multi-sector market models that account for multiple agents with different objectives require additional conditions to guarantee uniqueness of short- and long-run equilibrium compared to the single sector case. For the short-run, we derived sufficient conditions on the cost structures of production facilities that ex ante ensure uniqueness of equilibrium. Moreover, using illustrative examples, we proved that each of these conditions is indeed required to guarantee uniqueness in general. In addition, a straightforward interpretation of the circumstances that lead to multiplicities has been provided: multiplicities on the production level might arise if and only if variable production costs are the same for two distinct facilities. This understanding allows us to formulate suitable and transparent tie-breaking rules that select a certain equilibrium in case the presented conditions for short-run uniqueness are violated, as for instance when several facilities of the same technology are considered in the same bidding zone. One such possibility would be to select the equilibrium in which the facilities with similar variable production costs have equal capacity utilization.

For the long-run, we also derived sufficient conditions that guarantee uniqueness of equilibrium. However, these conditions can only be verified ex post and are fulfilled if and only if enough variability in the data is assured ex ante. It is therefore not recommendable to smooth out variability in data used to analyze specific applications. This result is consistent with the results obtained in similar contexts when sector coupling is not considered; see, e.g., Grimm, Schewe, et al. (2017). The decisive factor for multiplicities in the long-run are the total costs of production. In the case where multiplicities are caused by facilities with similar cost structures, tie-breaking rules might again be a natural choice to select a certain equilibrium, e.g., the one in which the size of these facilities is equal or the one where the facilities are ideally located given the underlying network. However, if sufficient fluctuations in, e.g., the demand are present, such multiplicities are not to be expected.

Some of the limitations in our analysis could be addressed in future research to allow for implementation of more realistic applications. First of all, one limitation of our analysis is that the frequency of spot market trade is assumed to be the same for all sectors. Of course, the case of different trade frequencies would be of interest since it allows to consider the coupling of markets that trade products on spot markets at different frequencies; in Europe, e.g., spot markets for gas define daily reference prices compared to hourly or even more frequent trade in spot markets for electricity. Second, we mentioned the issue of multiplicities in case potential-based trade constraints are considered or storage operators are modelled explicitly as agents on the spot markets of a single sector. We referred to relevant literature that analyzes uniqueness in these contexts. However, in some timely multi-sector applications, it might be absolutely necessary to model these aspects; e.g., for the production, storage, and utilization of electric fuels or for nodal pricing as benchmark in coupled energy markets. Hence, for sector-coupling applications, a more in-depth understanding of how multiplicities arise in such cases and could be resolved would be desirable.

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References

Abrell, J. and H. Weigt (2012). "Combining Energy Networks." In: *Networks and Spatial Economics* 12, pp. 377–401. DOI: 10.1007/s11067-011-9160-0.

– (2016). "Investments in a Combined Energy Network Model: Substitution between Natural Gas and Electricity?" In: *The Energy Journal* 0.4. URL: https://ideas.repec.org/a/aen/journl/ej37-4-abrell.html.

Ambrosius, M., J. Egerer, V. Grimm, and A. H. van der Weijde (2020). "Uncertain bidding zone configurations: The role of expectations for transmission and generation capacity expansion." In: *European Journal of Operational Research* 285.1, pp. 343–359. DOI: 10.1016/j.ejor.2020.01.024.

Baringo, L. and A. J. Conejo (2012). "Transmission and Wind Power Investment." In: *IEEE Transactions on Power Systems* 27.2, pp. 885–893. DOI: 10.1109/TPWRS.2011.2170441.

Chen, S., A. J. Conejo, R. Sioshansi, and Z. Wei (2019). "Equilibria in Electricity and Natural Gas Markets with Strategic Offers and Bids." In: *IEEE Transactions on Power Systems* 35.3, pp. 1956–1966. DOI: 10.1109/TPWRS.2019.2947646.

Clegg, S. and P. Mancarella (2015a). "Integrated electrical and gas network modelling for assessment of different power-and-heat options." In: *2014 Power Systems Computation Conference*. DOI: 10.1109/PSCC.2014.7038405.

– (2015b). "Integrated Modeling and Assessment of the Operational Impact of Power-to-Gas (P2G) on Electrical and Gas Transmission Networks." In: *IEEE Transactions on Sustainable Energy* 6.4, pp. 1234–1244. DOI: 10.1109/TSTE.2015.2424885.

Felten, B. (2020). "An integrated model of coupled heat and power sectors for large-scale energy system analyses." In: *Applied Energy* 266. DOI: https://doi.org/10.1016/j.apenergy.2020.114521.

Geidl, M. and G. Andersson (2007). "Optimal Power Flow of Multiple Energy Carriers." In: *IEEE Transactions on Power Systems* 22.1, pp. 145–155. DOI: 10.1109/TPWRS.2006.888988.

Gil, M., P. Dueñas, and J. Reneses (2015). "Electricity and Natural Gas Interdependency: Comparison of Two Methodologies for Coupling Large Market Models Within the European Regulatory Framework." In: *IEEE Transactions on Power Systems* 31.1, pp. 361–369. DOI: 10.1109/TPWRS.2015.2395872.

Grimm, V., J. Grübel, L. Schewe, M. Schmidt, and G. Zöttl (2019). "Nonconvex equilibrium models for gas market analysis: Failure of standard techniques and alternative modeling approaches." In: *European Journal of Operational Research* 273.3, pp. 1097–1108. DOI: 10.1016/j.ejor.2018.09.016.

Grimm, V., A. Martin, M. Schmidt, M. Weibelzahl, and G. Zöttl (2016). "Transmission and generation investment in electricity markets: The effects of market splitting and network fee regimes." In: *European Journal of Operational Research* 254.2, pp. 493–509. DOI: 10.1016/j.ejor.2016.03.044.

Grimm, V., L. Schewe, M. Schmidt, and G. Zöttl (2017). "Uniqueness of market equilibrium on a network: A peak-load pricing approach." In: *European Journal of Operational Research* 261.3, pp. 971–983. DOI: 10.1016/j.ejor.2017.03.036.

– (2019). "A multilevel model of the European entry-exit gas market." In: *Mathematical Methods of Operations Research* 89, pp. 223–255. DOI: 10.1007/s00186-018-0647-z.

Grübel, J., T. Kleinert, V. Krebs, G. Orlinskaya, L. Schewe, M. Schmidt, and J. Thürauf (2020). "On electricity market equilibria with storage: Modeling, uniqueness, and a distributed ADMM." In: *Computers & Operations Research* 114. DOI: 10.1016/j.cor.2019.104783.

Hobbs, B. F. and U. Helman (2004). "Complementarity-Based Equilibrium Modeling for Electric Power Markets." In: *Modeling Prices in Competitive Electricity Markets*. Ed. by D. Bunn. London: Wiley. Chap. 3.

Huppmann, D. and R. Egging (2014). "Market power, fuel substitution and infrastructure – A large-scale equilibrium model of global energy markets." In: *Energy* 75, pp. 483–500. DOI: 10.1016/j.energy.2014.08.004.

Jenabi, M., S. M. T. F. Ghomi, and Y. Smeers (2013). "Bi-Level Game Approaches for Coordination of Generation and Transmission Expansion Planning Within a Market Environment." In: *IEEE Transactions on Power systems* 28.3, pp. 2639–2650. DOI: 10.1109/TPWRS.2012.2236110.

Krebs, V., M. Müller, and M. Schmidt (2019). "Gamma-Robust Linear Complementarity Problems with Ellipsoidal Uncertainty Sets." In: *Optimization Online*. Preprint. URL: http://www.optimization-online.org/DB_HTML/2019/10/7434.html.

Krebs, V., L. Schewe, and M. Schmidt (2018). "Uniqueness and multiplicity of market equilibria on DC power flow networks." In: *European Journal of Operational Research* 271.1, pp. 165–178. DOI: 10.1016/j.ejor.2018.05.016.

Krebs, V. and M. Schmidt (2018). "Uniqueness of market equilibria on networks with transport costs." In: *Operations Research Perspectives* 5, pp. 169–173. DOI: 10.1016/j.orp.2018.05.002.

– (2020). "Γ-robust linear complementarity problems." In: *Optimization Methods and Software*. DOI: 10.1080/10556788.2020.1825708.

Mangasarian, O. L. (1988). "A simple characterization of solution sets of convex programs." In: *Operations Research Letters* 7.1, pp. 21–26. DOI: 10.1016/0167-6377(88)90047-8.

Martinez-Mares, A. and C. R. Fuerte-Esquivel (2012). "A Unified Gas and Power Flow Analysis in Natural Gas and Electricity Coupled Networks." In: *IEEE Transactions on Power Systems* 27.4, pp. 2156–2166. DOI: 10.1109/TPWRS.2012.2191984.

Ordoudis, C., P. Pinson, and J. M. Morales (2019). "An Integrated Market for Electricity and Natural Gas Systems with Stochastic Power Producers." In: *European Journal of Operational Research* 272.2, pp. 642–654. DOI: https://doi.org/10.1016/j.ejor.2018.06.036.

Roach, M. and L. Meeus (2020). "The welfare and price effects of sector coupling with power-to-gas." In: *Energy Economics* 86. DOI: 10.1016/j.eneco.2020.104708.

Sauma, E. E. and S. S. Oren (2006). “Proactive planning and valuation of transmission investments in restructured electricity markets.” In: *Journal of Regulatory Economics* 30, pp. 261–290. DOI: 10.1007/s11149-006-9003-y.
Zhang, X., M. Shahidehpour, A. Alabdulwahab, and A. Abusorrah (2015). “Optimal Expansion Planning of Energy Hub With Multiple Energy Infrastructures.” In: *IEEE Transactions on Smart Grid* 6.5, pp. 2302–2311. DOI: 10.1109/TSG.2015.2390640.
Zlotnik, A., L. Roald, S. Backhaus, M. Chertkov, and G. Andersson (2017). “Coordinated Scheduling for Interdependent Electric Power and Natural Gas Infrastructures.” In: *IEEE Transactions on Power Systems* 32.1, pp. 600–610. DOI: 10.1109/TPWRS.2016.2545522.

Appendix A. Notation

Here, a complete overview of all sets, parameters, and variables used throughout the paper is provided; see Tables 1, 2, and 3.

Table 1. Technical and economic sets

Symbol	Explanation	Unit
T	Set of time periods	-
S	Set of sectors	-
Z_s	Set of bidding zones in sector $s \in S$	-
K_s	Set of all adjacent bidding zones with positive trade capacity in sector $s \in S$	-
G_z^{ex}	Set of existing sector-specific production facilities in bidding zone $z \in Z_s$	-
G_z^{new}	Set of candidate sector-specific production facilities in bidding zone $z \in Z_s$	-
G_z^{all}	Set of all sector-specific production facilities in bidding zone $z \in Z_s$	-
I_z^{ex}	Set of existing sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
I_z^{new}	Set of candidate sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
I_z^{all}	Set of all sector-coupling facilities injecting in bidding zone $z \in Z_s$	-
O_z^{ex}	Set of existing sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
O_z^{new}	Set of candidate sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
O_z^{all}	Set of all sector-coupling facilities withdrawing from bidding zone $z \in Z_s$	-
X^{ex}	Set of existing sector-coupling facilities	-
X^{new}	Set of candidate sector-coupling facilities	-
X^{all}	Set of all sector-coupling facilities	-

Table 2. Technical and economic parameters (*sector-dependent unit)

Symbol	Explanation	Unit
f_k^+	Maximum trade capacity of $k \in K_s$	*
f_k^-	Minimum trade capacity of $k \in K_s$	*
$P_{t,z}(\cdot)$	Inverse demand function of bidding zone $z \in Z_s$ in time period $t \in T$	€/*
$p_{t,z}$	Price of bidding zone $z \in Z_s$ in time period $t \in T$	€/*
c_g^{var}	Variable costs of sector-specific production facility $g \in G_z^{\text{all}}$	€/*
y_g^{ex}	Capacity of existing sector-specific production facility $g \in G_z^{\text{ex}}$	*
c_g^{inv}	Investment costs of candidate sector-specific production facility $g \in G_z^{\text{new}}$	€/*
$\alpha_{t,g}$	Availability of sector-specific production facility $g \in G_z^{\text{all}}$ in time period $t \in T$	-
y_i^{ex}	Capacity of existing sector-coupling facility $i \in I_z^{\text{ex}}$	*
c_i^{inv}	Investment costs of candidate sector-coupling facility $i \in I_z^{\text{new}}$	€/*
$\alpha_{t,i}$	Availability of sector-coupling facility $i \in I_z^{\text{all}}$ in time period $t \in T$	-
η_x	Efficiency of sector-coupling facility $x \in X^{\text{all}}$	-

TABLE 3. Technical and economic variables (*sector-dependent unit)

Symbol	Explanation	Unit
$f_{t,k}$	Trade on $k \in K_s$ in time period $t \in T$	*
$d_{t,z}$	Demand in bidding zone $z \in Z_s$ in time period $t \in T$	*
$y_{t,g}$	Production of sector-specific production facility $g \in G_z^{\text{all}}$ in time period $t \in T$	*
y_g^{new}	Installed capacity of candidate sector-specific production facility $g \in G_z^{\text{new}}$	*
$d_{t,o}$	Demand of sector-coupling facility $o \in O_z^{\text{all}}$ in time period $t \in T$	*
$y_{t,i}$	Production of sector-coupling facility $i \in I_z^{\text{all}}$ in time period $t \in T$	*
y_i^{new}	Installed capacity of candidate sector-coupling facility $i \in I_z^{\text{new}}$	*

TABLE 4. Two equilibria of (MCP) for the scenarios of Example B.1 (left) and Example B.2 (right)

	d_{z_1}	y_{g_1}	y_{g_2}
1	10	10	0
2	10	5	5

	d_{z_1}	d_{z_2}	y_{g_1}	y_{g_2}	d_{x_1}	d_{x_2}	y_{x_1}	y_{x_2}
1	14	12	38	0	4	20	2	10
2	14	12	38	0	12	12	6	6

APPENDIX B. ILLUSTRATIVE EXAMPLES FOR THE SHORT-RUN

In the following, we present illustrative examples on how multiplicities arise in the short-run due to similar cost structures of different production facilities. All investment decisions are assumed to be already determined in the short-run. Hence, it is sufficient to consider only existing production facilities. Moreover, since the market equilibrium problem decouples over time for this case, it is sufficient to consider a single time period for demonstrating the underlying effects that cause multiplicities. For the ease of notation, we therefore drop the time index in the following examples. In addition, we denote by d_x and y_x the demand and production of the sector-coupling facility $x \in X^{\text{ex}}$. Furthermore, we assume without loss of generality that all facilities are fully available.

Example B.1 illustrates the case in which two production facilities of the same sector have the same variable production costs, i.e., Assumption 2 (a) is not satisfied. Due to the same cost structures, there is no way to distinguish between the two facilities and thus, multiple equilibria exist.

Example B.1. *Let one sector $S = \{s_1\}$ with one bidding zone $Z_{s_1} = \{z_1\}$ be given. The demand of this bidding zone is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$. Moreover, we assume that two producers $G_{z_1}^{\text{ex}} = \{g_1, g_2\}$ with the same variable production costs $c_{g_1}^{\text{var}} = c_{g_2}^{\text{var}} = 5$ and installed capacities $y_{g_1}^{\text{ex}} = 10$ and $y_{g_2}^{\text{ex}} = 8$ exist. Due to the same variable production costs, no distinction can be made between the two facilities. Therefore, it is not clear which of the producers best meets the demand. As a consequence, multiplicity of market equilibria results. Two of them are exemplarily listed in Table 4 (left).*

In addition to the case where, due to identical cost structures, no distinction can be made between sector-specific production facilities, there might also be the case where there is no way to distinguish between sector-coupling facilities. This occurs in particular if the efficiencies of sector-coupling facilities are not pairwise distinct; see Example B.2 for an illustration. The situation presented there violates Assumption 2 (b) and multiplicities arise from this violation.

Example B.2. *Let two sectors $S = \{s_1, s_2\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$ and $Z_{s_2} = \{z_2\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$ while the demand of bidding zone z_2 is determined by the inverse demand function $P_{z_2}(d_{z_2}) = 26 - 2d_{z_2}$. Moreover, we assume that one sector-specific producer produces in bidding zone z_1 and one in bidding zone z_2, i.e., $G_{z_1}^{\text{ex}} = \{g_1\}$ and*

TABLE 5. Two equilibria of (MCP) for the scenario of Example B.3

	d_{z_1}	d_{z_2}	y_{g_1}	y_{g_2}	d_{x_1}	y_{x_1}
1	14	11	18	10	4	1
2	14	11	26	8	12	3

$G^{\text{ex}}_{z_2} = \{g_2\}$. The variable production costs are $c^{\text{var}}_{g_1} = 1$ and $c^{\text{var}}_{g_2} = 4$ with installed capacities $y^{\text{ex}}_{g_1} = 40$ and $y^{\text{ex}}_{g_2} = 10$. In addition, two sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2\}$ with the same efficiency $\eta_{x_1} = \eta_{x_2} = 0.5$ exist that both withdraw from sector s_1 and inject in sector s_2. The capacities of these facilities are $y^{\text{ex}}_{x_1} = 30$ and $y^{\text{ex}}_{x_2} = 20$. Due to the same efficiency of the two sector-coupling facilities, there is no possibility to distinguish which of the operators of the two facilities best meets the demand in sector s_2. Consequently, there exist multiple market equilibria; for two of them see Table 4 (right).

Moreover, there may exist instances in which it is not possible to distinguish between a sector-specific and a sector-coupling facility. Example B.3 depicts such a case. There, multiplicity of equilibria is caused by the fact that the variable production costs across the sectors are not—taking into account the sector-coupling facility's efficiency—pairwise distinct, i.e., Assumption 2 (c) is not fulfilled.

Example B.3. *Let two sectors $S = \{s_1, s_2\}$ with the same bidding zone structure as well as demand and sector-specific production data as in Example B.2 be given. In addition, there is one sector-coupling facility x_1 with efficiency $\eta_{x_1} = 0.25$ that withdraws from sector s_1 and injects in sector s_2. The capacity of this facility is $y^{\text{ex}}_{x_1} = 10$. Due to the relation $\eta_{x_1} c^{\text{var}}_{g_2} = c^{\text{var}}_{g_1}$, there is no possibility to distinguish whether the operator of the sector-specific facility g_2 or of the sector-coupling facility x_1 best meets the demand in sector s_2. Hence, there exist multiple market equilibria. Two of them are depicted in Table 5.*

The latter situation of indistinguishable sector-specific and sector-coupling facilities can also arise across several sectors. Example B.4 presents an instance in which multiplicity of equilibria is triggered by the same variable production costs across two sectors taking into account the combined efficiency of two sequentially producing sector-coupling facilities, which contradicts Assumption 2 (d).

Example B.4. *Let three sectors $S = \{s_1, s_2, s_3\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$, $Z_{s_2} = \{z_2\}$, and $Z_{s_3} = \{z_3\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 15 - d_{z_1}$, the one of z_2 by $P_{z_2}(d_{z_2}) = 11.25 - d_{z_2}$, and the one of z_3 by $P_{z_3}(d_{z_3}) = 30.5 - 2d_{z_3}$. Moreover, we assume that one sector-specific producer produces in each bidding zone, i.e., $G^{\text{ex}}_{z_1} = \{g_1\}$, $G^{\text{ex}}_{z_2} = \{g_2\}$, and $G^{\text{ex}}_{z_3} = \{g_3\}$. The variable production costs are $c^{\text{var}}_{g_1} = 1$, $c^{\text{var}}_{g_2} = 0.75$, and $c^{\text{var}}_{g_3} = 2.5$ with installed capacities $y^{\text{ex}}_{g_1} = 50$, $y^{\text{ex}}_{g_2} = 10$, and $y^{\text{ex}}_{g_3} = 5$. In addition, two sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2\}$ with the efficiencies $\eta_{x_1} = 0.8$ and $\eta_{x_2} = 0.5$ exist. The facility x_1 withdraws from sector s_1 and injects in sector s_2 while the facility x_2 transfers the commodity of sector s_2 into the commodity of sector s_3. The capacities of these facilities are $y^{\text{ex}}_{x_1} = 40$ and $y^{\text{ex}}_{x_2} = 30$. Due to the relation $\eta_{x_1}\eta_{x_2} c^{\text{var}}_{g_3} = c^{\text{var}}_{g_1}$, there is no possibility to decide whether the supply of the sector-coupling or of the sector-specific production facility is the best to meet the demand in sector s_3. As a consequence, there exist multiple market equilibria; see Table 6 for two of them.*

Finally, the case of indistinguishable sector-coupling facilities can also appear due to sequentially producing sector-coupling facilities. In Example B.5, multiplicities occur since the efficiency of the facility that transforms commodity two into commodity three is the same as the combined efficiency of the facilities that transform commodity two into commodity one and commodity one into commodity three. This situation violates Assumption 2 (e), which states that a chain of efficiencies is not allowed to meet another chain of efficiencies.

TABLE 6. Two equilibria of (MCP) for the scenario of Example B.4

	d_{z_1}	d_{z_2}	d_{z_3}	y_{g_1}	y_{g_2}	y_{g_3}	d_{x_1}	d_{x_2}	y_{x_1}	y_{x_2}
1	14	10	14	49	10	0	35	28	28	14
2	14	10	14	39	10	4	25	20	20	10

TABLE 7. Two equilibria of (MCP) for the scenario of Example B.5

	d_{z_1}	d_{z_2}	d_{z_3}	y_{g_1}	y_{g_2}	y_{g_3}	d_{x_1}	d_{x_2}	d_{x_3}	y_{x_1}	y_{x_2}	y_{x_3}
1	7	6	23.5	0	73.5	0	67.5	47	0	54	23.5	0
2	7	6	23.5	0	73.5	0	8.75	0	58.75	7	0	23.5

Example B.5. *Let three sectors $S = \{s_1, s_2, s_3\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$, $Z_{s_2} = \{z_2\}$, and $Z_{s_3} = \{z_3\}$, be given. The demand of bidding zone z_1 is characterized by the inverse demand function $P_{z_1}(d_{z_1}) = 8.25 - d_{z_1}$, the one of z_2 by $P_{z_2}(d_{z_2}) = 7 - d_{z_2}$, and the one of z_3 by $P_{z_3}(d_{z_3}) = 26 - d_{z_3}$. Moreover, we assume that one sector-specific producer produces in each bidding zone, i.e., $G^{\text{ex}}_{z_1} = \{g_1\}$, $G^{\text{ex}}_{z_2} = \{g_2\}$, and $G^{\text{ex}}_{z_3} = \{g_3\}$. The variable production costs are $c^{\text{var}}_{g_1} = 8$, $c^{\text{var}}_{g_2} = 1$, and $c^{\text{var}}_{g_3} = 25$ with installed capacities $y^{\text{ex}}_{g_1} = 5$, $y^{\text{ex}}_{g_2} = 80$, and $y^{\text{ex}}_{g_3} = 20$. In addition, three sector-coupling facilities $X^{\text{ex}} = \{x_1, x_2, x_3\}$ with the efficiencies $\eta_{x_1} = 0.8$, $\eta_{x_2} = 0.5$, and $\eta_{x_3} = 0.4$ exist. The facility x_1 withdraws from sector s_2 and injects in sector s_1 while the facility x_2 transfers the commodity of sector s_1 into the commodity of sector s_3. Finally, the facility x_3 couples sector s_2 with sector s_3, injecting in the latter sector. The capacities of the sector-coupling facilities are $y^{\text{ex}}_{x_1} = 100$, $y^{\text{ex}}_{x_2} = 120$, and $y^{\text{ex}}_{x_3} = 140$. Due to the relation $\eta_{x_1}\eta_{x_2} = \eta_{x_3}$, there is no possibility to decide which sector-coupling producer is best to meet the demand in sector s_3. Thus, there exist multiple market equilibria. Two of them are stated in Table 7.*

APPENDIX C. ILLUSTRATIVE EXAMPLES FOR THE LONG-RUN

In the following, we introduce examples that illustrate how multiplicities arise in the long-run. Also in the long-run, multiplicities are mainly caused by similar cost structures of different production facilities. Now, the total production costs and not only the variable production costs are decisive. For all examples, we assume without loss of generality that there exist only candidate facilities and that all facilities are fully available. We do not present examples to all possible violations of the nine conditions specified in Assumption 3 but choose two exemplary cases. The first example focuses on a single sector, the second one on multiple sectors.

The single sector case is treated in Example C.1. Since, in this example, both operators of the candidate sector-specific facilities produce in all time periods and additionally at the same average costs, there is no possibility to decide which of the operators should invest best and thus, multiple equilibria exist.

Example C.1. *Given two time periods $T = \{t_1, t_2\}$. Moreover, let one sector $S = \{s_1\}$ with one bidding zone $Z_{s_1} = \{z_1\}$ be given. The inverse demand function of bidding zone z_1 in time period t_1 is determined by $P_{t_1,z_1} = 10 - d_{t_1,z_1}$, the one of time period t_2 by $P_{t_2,z_1} = 7.5 - d_{t_2,z_1}$. In addition, we assume that there are two candidate sector-specific facilities in bidding zone z_1, i.e., $G^{\text{new}}_{z_1} = \{g_1, g_2\}$. The variable production costs are $c^{\text{var}}_{g_1} = 1$ and $c^{\text{var}}_{g_2} = 3$ and the investment costs $c^{\text{inv}}_{g_1} = 6$ and $c^{\text{inv}}_{g_2} = 2$. No sector-coupling facilities exist. Due to the fact that both operators of the sector-specific facilities produce always at the same time and at the same average costs, there is no possibility to distinguish which of the producers should invest best in order to meet the demand in sector s_1. Hence, multiple market equilibria result; see Table 8 for two of them.*

TABLE 8. Two equilibria of (MCP) for the scenario of Example C.1

	d_{t_1,z_1}	d_{t_2,z_1}	y_{t_1,g_1}	y_{t_1,g_2}	y_{t_2,g_1}	y_{t_2,g_2}	$y^{\text{new}}_{g_1}$	$y^{\text{new}}_{g_2}$
1	5	4.5	2.5	2.5	2.5	2	2.5	2.5
2	5	4.5	1	4	1	3.5	1	4

TABLE 9. Two equilibria of (MCP) for the scenario of Example C.2

	d_{t_1,z_1}	d_{t_1,z_2}	d_{t_2,z_1}	d_{t_2,z_2}	y_{t_1,g_1}	y_{t_1,g_2}	y_{t_2,g_1}	y_{t_2,g_2}	$y^{\text{new}}_{g_1}$	$y^{\text{new}}_{g_2}$
1	8	5	7	4.5	8.625	4.5	7.625	4	8.625	4.5
2	8	5	7	4.5	10.5	3	9.5	2.5	10.5	3

	d_{t_1,x_1}	d_{t_2,x_1}	y_{t_1,x_1}	y_{t_2,x_1}	$y^{\text{new}}_{x_1}$
1	0.625	0.625	0.5	0.5	0.5
2	2.5	2.5	2	2	2

For multiple sectors, Example C.2 provides an instance in which multiplicities occur in the long-run. Again, multiplicities result from the fact that both operators of the sector-coupling and the sector-specific facility of sector two produce always at the same time and in addition at the same average costs.

Example C.2. *Given two time periods $T = \{t_1, t_2\}$. Furthermore, let two sectors $S = \{s_1, s_2\}$ each with one bidding zone, i.e., $Z_{s_1} = \{z_1\}$ and $Z_{s_2} = \{z_2\}$, be given. All sector-specific demand is characterized by the inverse demand functions $P_{t_1,z_1} = 10 - d_{t_1,z_1}$, $P_{t_2,z_1} = 8 - d_{t_2,z_1}$, $P_{t_1,z_2} = 10 - d_{t_1,z_2}$, and $P_{t_2,z_2} = 7.5 - d_{t_2,z_2}$. Moreover, we assume that there is one candidate sector-specific facility in each bidding zone, i.e., $G^{\text{new}}_{z_1} = \{g_1\}$ and $G^{\text{new}}_{z_2} = \{g_2\}$. The variable production costs are $c^{\text{var}}_{g_1} = 1$ and $c^{\text{var}}_{g_2} = 3$ and the investment costs are $c^{\text{inv}}_{g_1} = 1$ and $c^{\text{inv}}_{g_2} = 2$. In addition, one candidate sector-coupling facility exists that withdraws from sector s_1 and injects in sector s_2, i.e., $X^{\text{new}} = \{x_1\}$. The efficiency of this facility is $\eta_{x_1} = 0.8$ and the investment costs are $c^{\text{inv}}_{x_1} = 4.25$. Since the operators of the sector-coupling and the sector-specific facility of sector two produce always at the same time and at the same average costs, it is not clear which of these producers best meets the demand in sector s_2. Thus, there exist multiple market equilibria. Two of them are depicted in Table 9.*