

A ROBUST APPROACH FOR MODELING LIMITED OBSERVABILITY IN BILEVEL OPTIMIZATION

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ABSTRACT. Many applications of bilevel optimization contain a leader facing a follower whose reaction deviates from the one expected by the leader due to some kind of bounded rationality. We consider bilinear bilevel problems with follower's response uncertainty due to limited observability regarding the leader's decision and exploit robust optimization to model the decision making of the follower. We show that the robust counterpart of the lower level allows to tackle the problem via the lower level's KKT conditions.

1. INTRODUCTION

In bilevel optimization, some of the variables are constrained to be optimal solutions of another optimization problem, the so-called lower-level problem. The remaining variables are decided on in the so-called upper-level problem. The upper-level player (or leader) makes a decision first, anticipating the reaction of the lower-level player (or follower). The concept of this class of optimization problems dates back to the seminal publications of von Stackelberg (1932; 1954). In the last years and decades, bilevel problems have gained increasing attention due to their ability to model hierarchical decision making processes. These situations arise in various real-world applications such as in energy markets; see, e.g., Ambrosius et al. (2020), Grimm et al. (2019), and Xinmin and Ralph (2007), in transportation; see, e.g., Migdalas (1995) and Ben-Ayed et al. (1992), or in critical infrastructure defense; see, e.g., Brown et al. (2006), DeNegre (2011), Fischetti et al. (2019), Jain et al. (2010), Kiekintveld et al. (2009), Paruchuri et al. (2008), Pita, Jain, Marecki, et al. (2008), and Shieh et al. (2012). Thus, it is obvious that the capability of modeling hierarchical decision processes is important for practice. However, this ability makes bilevel problems intrinsically hard to solve. Even their easiest instantiations, namely linear bilevel problems, are strongly NP-hard; see Hansen et al. (1992).

In the classic setting of bilevel optimization, it is assumed that both players act perfectly rational. However, this assumption rarely holds in many practical applications as both players may face bounded rationality; see, e.g., Simon (1972). For instance, in Chariri (2017), it is elaborated on how decision makers are confronted with cognitive limitations preventing them from reaching a perfectly rational decision. Although Simon's theory received considerable recognition, this notion has long been abstracted from. It has been a point of controversy between economists as it has been accused of being too limited to individual psychological processes rather than that it fits the behavior of institutions and large economies; see, e.g., Dequech (2001) and Rainey (2001). For a more detailed discussion on bounded rationality, we refer to Rubinstein (1998).

Nevertheless, the consideration of bounded rationality has attained increasing attention in recent years. One possible reason for bounded rationality is data uncertainty. In the context of bilevel optimization, these types of problems have been investigated in, e.g., Haghighat (2014), Dempe et al. (2017), Ivanov (2018), Yanıkoğlu and Kuhn (2018), Burtscheidt, Claus, and Dempe (2020), and Burtscheidt and Claus (2020). Another reason

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for bounded rationality is uncertainty about the decision of the other player. Thus, it is evident that uncertainty is an important aspect of bounded rationality. In mathematical optimization, one approach to deal with uncertainties is robust optimization (see, e.g., Soyster (1973), Ben-Tal et al. (2009), and Bertsimas et al. (2010)), which we will also exploit in this paper.

In contrast to the case of uncertain data, decision uncertainty has been much less investigated in the context of bilevel optimization. Concerning this matter, we focus on the following two papers. Recently, Besançon et al. (2019) propose a robust approach to hedge against near-optimal lower-level decisions as a generalization of the “ ε -approximation” introduced in Wiesemann et al. (2013). Pita, Jain, Ordóñez, et al. (2009) consider follower’s response uncertainty due to limited observability regarding the upper-level decision using anchoring biases; see, e.g., Kelly et al. (2006).

The contribution of this paper is the following. We follow the concept of Pita, Jain, Ordóñez, et al. (2009) to consider follower’s response uncertainty due to limited observability regarding the upper-level decision. In contrast to the approach using anchoring biases, we pursue the same idea as in Besançon et al. (2019) and model bounded rationality using the toolbox of robust optimization. However, while these authors consider the effect of near-optimal decisions of the follower on the upper-level constraints, we focus on how limited observability regarding the upper-level decision affects the problem at the lower level.

The remainder of the paper is organized as follows. Section 2 gives a brief introduction of bilinear bilevel problems and the concept of limited observability. In Section 3, we present an illustrative example to demonstrate the importance of the proposed modeling aspect. In Section 4, the robust counterpart of the lower level is shown to be a bilinear problem as well so that we can establish an equivalent single-level reformulation by replacing the lower-level problem with its Karush–Kuhn–Tucker (KKT) conditions. We then return in Section 5 to the example to illustrate the effect of follower’s decisions under bounded rationality on the problem studied in Section 3. Further, we address the relation between limited observability regarding the upper-level decision and bilevel problems with lower-level right-hand side uncertainty in Section 6. Finally, we conclude in Section 7.

2. PROBLEM STATEMENT

In this paper, we consider the bilinear bilevel problem

$$\min_{x,y} c^\top x + d^\top y + x^\top R y \quad (1a)$$

$$\text{s.t. } Ax + By \geq a, \quad (1b)$$

$$y \in \arg \min_{y'} \{f^\top x + g^\top y' + x^\top Q y' : Cx + D y' \geq b\} \quad (1c)$$

with $x, c, f \in \mathbb{R}^n$, $y, d, g \in \mathbb{R}^m$, $R, Q \in \mathbb{R}^{n \times m}$, $A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{k \times m}$, $a \in \mathbb{R}^k$, $C \in \mathbb{R}^{\ell \times n}$, $D \in \mathbb{R}^{\ell \times m}$, and $b \in \mathbb{R}^\ell$. We refer to (1a)–(1b) as the upper level and to (1c) as the nominal lower level. Here, we consider the optimistic approach to bilevel optimization; see, e.g., Dempe (2002). This means that whenever the lower-level problem has multiple solutions y , the follower selects the most favorable one w.r.t. the leader’s objective function.

In Problem (1), we make the strong assumption that the leader and the follower act perfectly rational. In real-world applications, however, this assumption rarely holds as both players face bounded rationality; see, e.g., Simon (1972). Here, we consider follower response uncertainty due to limited observability regarding the upper-level decision as proposed, e.g., in Pita, Portway, et al. (2008). This means that the follower cannot perfectly observe the actual leader’s decision x . Nevertheless, the observed upper-level decision \bar{x} provides an insight into the leader’s scope of action. Given this knowledge, the follower’s response is based on \bar{x} , which is assumed to belong to a given uncertainty set $\mathcal{U}(x)$. This

leads to the robust bilevel problem

$$\min_{x,y} c^\top x + d^\top y + x^\top R y \quad (2a)$$

$$\text{s.t. } Ax + By \geq a, \quad (2b)$$

$$y \in \arg \min_{y'} \left\{ g^\top y' + \max_{\bar{x} \in \mathcal{U}(x)} \{ f^\top \bar{x} + \bar{x}^\top Q y' : C\bar{x} + D y' \geq b \ \forall \bar{x} \in \mathcal{U}(x) \} \right\} \quad (2c)$$

with a robustified lower-level objective function as well as a robustified feasible set in the lower-level problem. Note that due to the robustification of the lower-level's objective function, the linear term in x cannot be avoided as it is usually done in (bi)linear bilevel optimization. Throughout this paper, the uncertainty set is parameterized in an affine way by

$$\mathcal{U}(x) = \{x + P\zeta : \zeta \in \mathcal{Z} \subseteq \mathbb{R}^q\} \quad (3)$$

with $P \in \mathbb{R}^{n \times q}$ and a polyhedral perturbation set

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^q : H\zeta \geq h\}$$

for $H \in \mathbb{R}^{s \times q}$ and $h \in \mathbb{R}^s$. To ensure that the original leader's decision x is still part of the uncertainty set $\mathcal{U}(x)$ one can additionally assume that $0 \in \mathcal{Z}$ holds.

3. EXAMPLE: BILEVEL BIMATRIX GAMES

A bimatrix game is a non-cooperative two-player simultaneous-move game. This means that two competitive players, called Player 1 and Player 2, select their strategies at the same time. Each player can choose from a finite number of possible actions, which are called pure strategies. We denote $x \in \mathbb{R}^n$ as the leader's mixed strategy if

$$\sum_{i=1}^n x_i = 1 \quad \text{and} \quad x \geq 0$$

holds. In this case, x_i represents the probability that Player 1 plays strategy $i \in [n] := \{1, \dots, n\}$. Analogously, we obtain the feasible set for the follower's mixed strategy $y \in \mathbb{R}^m$ via

$$\sum_{j=1}^m y_j = 1 \quad \text{and} \quad y \geq 0.$$

Both players attempt to minimize their objective functions $x^\top R y$ and $x^\top Q y$ with cost matrices $R, Q \in \mathbb{R}^{n \times m}$ for Player 1 and Player 2, respectively. The entries R_{ij} and Q_{ij} represent the associated costs if Player 1 chooses action $i \in [n]$ and if Player 2 selects strategy $j \in [m]$.

In what follows, we consider the sequential bimatrix game stated in Problem (4). Thus, we refer to Player 1 as the leader and to Player 2 as the follower:

$$\min_{x,y} x^\top R y \quad (4a)$$

$$\text{s.t. } \sum_{i=1}^n x_i = 1, \quad x \geq 0, \quad (4b)$$

$$y \in \arg \min_{y'} \left\{ x^\top Q y' : \sum_{j=1}^m y'_j = 1, \ y' \geq 0 \right\}. \quad (4c)$$

In Problem (4), the leader has to commit to a strategy first. Then, after observing the upper-level decision, the follower's strategy is determined. In particular, Problem (4) is a special case of Problem (1).

This type of problem has been subject to extensive research in many real-world applications in security domains such as defender-attacker scenarios and patrolling; see, e.g., Brown et al. (2006), Gatti (2008), Paruchuri et al. (2008), Pita, Portway, et al. (2008), Pita,

TABLE 1. Costs for the example in Section 3

	y_1	y_2
x_1	(2, 5)	(4, 0)
x_2	(3, 1)	(4, 2)

Jain, Marecki, et al. (2008), Kiekintveld et al. (2009), Jain et al. (2010), Shieh et al. (2012), and Yang et al. (2014) and the references therein.

Under the assumption that the follower faces limited observability regarding the upper-level decision, the reformulation of Problem (4) as a robust bilevel problem is given by

$$\begin{aligned} \min_{x,y} \quad & x^\top R y \\ \text{s.t.} \quad & \sum_{i=1}^n x_i = 1, \quad x \geq 0, \\ & y \in \arg \min_{y'} \left\{ \max_{\bar{x} \in \mathcal{U}(x)} \left\{ \bar{x}^\top Q y' : \sum_{j=1}^m y'_j = 1, y'_j \geq 0 \right\} \right\}. \end{aligned}$$

To illustrate follower's response uncertainty due to limited observability regarding the upper-level decision, we consider the example depicted in Table 1 with the respective costs for the leader and the follower. If the leader commits to a pure strategy and the follower can perfectly observe the upper-level decision, the bilevel solution is given by $x = (0, 1)$ and $y = (1, 0)$. If the leader commits to a mixed strategy of playing x_1 with probability $1/6$ and x_2 with probability $5/6$, the optimistic follower will select $y = (1, 0)$ resulting in expected costs of $17/6$ for the leader. In Section 5, we will return to this example to illustrate the case of limited observability regarding the upper-level decision. Before, we have to consider how these robust bilevel problems can be reformulated as single-level optimization problems.

4. SINGLE-LEVEL REFORMULATION

As elaborated in Bertsimas et al. (2010), the lower-level problem (2c) can be replaced with the robust formulation

$$\min_{y,\tau} \quad \tau \tag{6a}$$

$$\text{s.t.} \quad g^\top y + \max_{\bar{x} \in \mathcal{U}(x)} \{f^\top \bar{x} + \bar{x}^\top Q y\} \leq \tau, \tag{6b}$$

$$C\bar{x} + D y \geq b \quad \text{for all } \bar{x} \in \mathcal{U}(x), \tag{6c}$$

in which the uncertainties only arise in the constraints of the problem. In terms of the affine parameterization of the uncertainty set, Constraint (6b) can be stated as

$$g^\top y + f^\top x + x^\top Q y + \max_{\zeta \in \mathcal{Z}} \{(f + Q y)^\top P \zeta\} \leq \tau. \tag{7}$$

For each fixed $y \in \mathbb{R}^m$ in the follower's objective function, the inner optimization problem in (7) is equivalent to the solution of the linear problem

$$\max_{\zeta} \quad (f + Q y)^\top P \zeta \tag{8a}$$

$$\text{s.t.} \quad H \zeta \geq h. \tag{8b}$$

Similarly, we can replace each constraint in (6c) with the corresponding worst-case scenario

$$C_j \cdot x + \min_{\zeta \in \mathcal{Z}} \{(C_j \cdot P) \zeta\} + D_j \cdot y \geq b_j, \tag{9}$$

where C_j denotes the j th row of C for $j \in [\ell]$. Thus, (9) requires the solution of

$$\min_{\zeta} (C_j.P) \zeta \quad (10a)$$

$$\text{s.t. } H\zeta \geq h, \quad (10b)$$

for each constraint $j \in [\ell]$. We denote with $\sigma \in \mathbb{R}^s$ and $\lambda^j \in \mathbb{R}^s$, $j \in [\ell]$, the dual variables associated with Problem (8) and (10), respectively. The dual problem of (8) is given by

$$\min_{\sigma} -h^\top \sigma \quad (11a)$$

$$\text{s.t. } H^\top \sigma = -P^\top (f + Qy), \quad (11b)$$

$$\sigma \geq 0. \quad (11c)$$

Similarly, we obtain the dual formulation of (10) as

$$\max_{\lambda^j} h^\top \lambda^j \quad (12a)$$

$$\text{s.t. } H^\top \lambda^j = (C_j.P)^\top, \quad (12b)$$

$$\lambda^j \geq 0. \quad (12c)$$

By strong duality, the objective values of the respective primal and dual problems coincide for primal-dual optimal pairs. Thus, to satisfy the uncertain constraints in (6), the following inequalities must hold for all dual variables σ and λ^j

$$f^\top x + g^\top y + x^\top Qy - h^\top \sigma \leq \tau, \quad (13a)$$

$$C_j.x + D_j.y + h^\top \lambda^j \geq b_j, \quad j \in [\ell], \quad (13b)$$

as well as (11b)–(11c) and (12b)–(12c). As a result, we obtain the robust counterpart

$$\min_{y, \tau, \sigma, \lambda} \tau \quad (14a)$$

$$\text{s.t. } f^\top x + g^\top y + x^\top Qy - h^\top \sigma \leq \tau, \quad (14b)$$

$$C_j.x + D_j.y + h^\top \lambda^j \geq b_j, \quad j \in [\ell], \quad (14c)$$

$$H^\top \sigma = -P^\top (f + Qy), \quad (14d)$$

$$H^\top \lambda^j = (C_j.P)^\top, \quad j \in [\ell], \quad (14e)$$

$$\lambda^j \geq 0, \quad j \in [\ell], \quad (14f)$$

$$\sigma \geq 0, \quad (14g)$$

of the lower-level problem. In particular, (14) is a linear problem for each fixed leader's decision x . Thus, we can replace the lower level with its KKT conditions and obtain the

single-level reformulation

$$\min_{x,y,z} c^\top x + d^\top y + x^\top R y \quad (15a)$$

$$\text{s.t. } Ax + By \geq a, \quad (15b)$$

$$C_j \cdot x + D_j \cdot y + h^\top \lambda^j \geq b_j, \quad j \in [\ell], \quad (15c)$$

$$H^\top \sigma = -P^\top (f + Qy), \quad (15d)$$

$$H^\top \lambda^j = (C_j \cdot P)^\top, \quad j \in [\ell], \quad (15e)$$

$$g + Q^\top x - D^\top \alpha - Q^\top P \beta = 0, \quad (15f)$$

$$h + H\beta + \delta = 0, \quad (15g)$$

$$\alpha_j h + H\gamma^j + \varepsilon^j = 0, \quad j \in [\ell], \quad (15h)$$

$$f^\top x + g^\top y + x^\top Qy - h^\top \sigma = \tau, \quad (15i)$$

$$\alpha_j (C_j \cdot x + D_j \cdot y + h^\top \lambda^j - b_j) = 0, \quad j \in [\ell], \quad (15j)$$

$$\delta^\top \sigma = 0, \quad (15k)$$

$$(\varepsilon^j)^\top \lambda^j = 0, \quad j \in [\ell], \quad (15l)$$

$$\lambda^j, \varepsilon^j \geq 0, \quad j \in [\ell], \quad (15m)$$

$$\sigma, \alpha, \delta \geq 0, \quad (15n)$$

where z contains all primal variables used for modeling limited observability in the sense of robust optimization as well as all dual lower-level variables with $\sigma \in \mathbb{R}^s$, $\lambda^j \in \mathbb{R}^s$, $\tau \in \mathbb{R}$, $\alpha \in \mathbb{R}^\ell$, $\beta \in \mathbb{R}^q$, $\gamma^j \in \mathbb{R}^q$, $\delta \in \mathbb{R}^s$, and $\varepsilon^j \in \mathbb{R}^s$ for all $j \in [\ell]$. Note that τ models the robustified objective function value of the follower. Thus, we could—in principal—dispose of Constraint (15i) since τ is an auxiliary variable in Problem (14) that could be eliminated in (14) as well.

5. THE ILLUSTRATIVE EXAMPLE REVISITED

We now reconsider the example in Section 3 under the assumption that the follower faces limited observability regarding the upper-level decision. We focus on the case that the perceived leader's decision \bar{x} is only known to lie within a box, i.e., the uncertainty set is given by

$$\mathcal{U}(x) = \{\bar{x} = x + P\zeta : \zeta \in \mathcal{Z}\} \cap \{\bar{x} : \bar{x}_1 + \bar{x}_2 = 1\}$$

with

$$P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

and the perturbation set

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^2 : -1 \leq \zeta_i \leq 1, i = 1, 2\}.$$

In particular, this is a special case of a polyhedral uncertainty set. Before we discuss the results let us briefly comment on the interpretation of the modeling in this setting—especially w.r.t. the definition of the uncertainty set $\mathcal{U}(x)$. The first set in the definition of $\mathcal{U}(x)$ corresponds to the uncertainty modeling of the leader's decision as stated around (3). In this example, the leader commits to a mixed strategy, which is a probability distribution over a finite number of possible actions. When facing limited observability, the follower anticipates the resulting simplex structure of the leader's mixed strategy, which is captured in the second set of the intersection.

The specific example is implemented in Python 3.8 and Gurobi 9.1.0 is used to solve the single-level reformulation (15). To handle the KKT complementarity constraints we exploit special ordered sets of type 1 (SOS1) to avoid big- M reformulations that can be

TABLE 2. Leader's strategies under limited observability

$p_1 \backslash p_2$	0	0.25	0.5	0.75	1
0	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)
0.25	(0.167, 0.833)	(0.417, 0.583)	(0.417, 0.583)	(0.417, 0.583)	(0.417, 0.583)
0.5	(0.167, 0.833)	(0.417, 0.583)	(0.667, 0.333)	(0.667, 0.333)	(0.667, 0.333)
0.75	(0.167, 0.833)	(0.417, 0.583)	(0.667, 0.333)	(0.917, 0.083)	(0.917, 0.083)
1	(0.167, 0.833)	(0.417, 0.583)	(0.667, 0.333)	(0.917, 0.083)	(1, 0)

troublesome; see, e.g., Pineda and Morales (2019) and Kleinert et al. (2020). Hence, we restate Constraints (15j)–(15l) as

$$\begin{aligned}
r_j &= C_j \cdot x + D_j \cdot y + h^\top \lambda^j - b_j, & j \in [\ell], \\
\text{SOS1}(\alpha_j, r_j), & & j \in [\ell], \\
\text{SOS1}(\delta_k, \sigma_k), & & k \in [s], \\
\text{SOS1}\left(\varepsilon_k^j, \lambda_k^j\right), & & k \in [s], j \in [\ell].
\end{aligned}$$

Tables 2–3 summarize the nominal and robust leader's and follower's strategies for different uncertainty set parameterizations that model limited observability. The respective objective function values are given in Table 4, where the first element of the tuple denotes the value of the leader's objective and the second one gives the objective function value for the follower.

First, it can be seen that limited observability regarding the upper-level decision can lead to deviations of the follower's strategy from the nominal decision. In this example, the follower moves from playing the pure strategy $y = (1, 0)$ to committing to the mixed strategy of playing y_1 with probability $1/6$ and y_2 with probability $5/6$ regardless of the extent to which the follower is limited in the observability of the leader's decision. In particular, this means that the follower's strategy shifts entirely when facing limited observability. Due to the leader's anticipation of the follower's response uncertainty, the upper-level decision can also change significantly compared to the nominal strategy, depending on the extent of the uncertainties. Under the assumption that the follower can perfectly observe the upper-level decision, the leader tends to play x_2 . However, the greater the uncertainty regarding the observability of the upper-level decision, the more the leader tends to play x_1 . In particular, the leader will commit to the pure strategy of playing x_1 for the uncertainty set parameterization with $p_1 = p_2 = 1$. Therefore, not only the follower's strategy but also the leader's strategy can shift entirely if limited observability regarding the upper-level decision is taken into account. Moreover, it can be seen that the strategies of both players are symmetric w.r.t. the parameterization of the uncertainty set, which is due to the simplex structure of the leader's feasible set. Further, it is noticeable that the leader's anticipation of follower's response uncertainty due to limited observability regarding the upper-level decision always leads to significantly increased costs for the leader. In this example, the increase in the leader's costs is up to approximately 33 % of the nominal value.

To sum up, this illustrative example shows that limited observability significantly impacts the solution of the underlying bilevel problem. Thus, this modeling aspect should not be ignored if the application problem at hand contains a lower-level player that cannot perfectly observe the leader's decision.

TABLE 3. Follower's strategies under limited observability

$p_1 \backslash p_2$	0	0.25	0.5	0.75	1
0	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
0.25	(1, 0)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)
0.5	(1, 0)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)
0.75	(1, 0)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)
1	(1, 0)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)	(0.167, 0.833)

TABLE 4. Leader's and follower's objective function values under limited observability

$p_1 \backslash p_2$	0	0.25	0.5	0.75	1
0	(2.833, 1.667)	(2.833, 1.667)	(2.833, 1.667)	(2.833, 1.667)	(2.833, 1.667)
0.25	(2.833, 1.667)	(3.764, 1.417)	(3.764, 1.417)	(3.764, 1.417)	(3.764, 1.417)
0.5	(2.833, 1.667)	(3.764, 1.417)	(3.722, 1.667)	(3.722, 1.667)	(3.722, 1.667)
0.75	(2.833, 1.667)	(3.764, 1.417)	(3.722, 1.667)	(3.681, 0.917)	(3.681, 0.917)
1	(2.833, 1.667)	(3.764, 1.417)	(3.722, 1.667)	(3.681, 0.917)	(3.667, 0.833)

6. RELATION TO BILEVEL PROBLEMS WITH UNCERTAIN LOWER-LEVEL DATA

Compared to the problem under perfect rationality, the consideration of limited observability regarding the upper-level decision yields significantly larger optimization problems in terms of the number of variables as well as the number of constraints. Therefore, it seems reasonable to strive for a more compact formulation to model limited observability. For this purpose, we now address the relation between Problem (15) and bilevel problems with uncertain lower-level data. More specifically, we consider problems with uncertainties in the right-hand sides of the lower-level's constraints, i.e., we consider

$$\begin{aligned}
& \min_{x, y} && c^\top x + d^\top y + x^\top R y \\
& \text{s.t.} && Ax + By \geq a, \\
& && y \in \arg \min_{y'} \{f^\top x + g^\top y' + x^\top Q y' : Cx + D y' \geq \bar{b} \text{ for all } \bar{b} \in \mathcal{U}(b)\}.
\end{aligned}$$

Here, b is the nominal right-hand side and $\mathcal{U}(b) = \mathcal{U}_1(b_1) \times \dots \times \mathcal{U}_\ell(b_\ell)$ holds with

$$\mathcal{U}_j(b_j) = \{b_j + (\tilde{p}^j)^\top \zeta^j : \zeta^j \in \mathcal{Z}^j\}$$

and

$$\mathcal{Z}^j = \{\zeta^j : (\tilde{H}^j)^\top \zeta^j \geq \tilde{h}^j\},$$

for all $j \in [\ell]$. For the ease of presentation, we omit the dimensions of all vectors and matrices in this section. Similar to the derivation in Section 4, we obtain the single-level

reformulation

$$\min_{x, y, \tilde{z}} \quad c^\top x + d^\top y + x^\top R y \quad (16a)$$

$$\text{s.t.} \quad Ax + By \geq a, \quad (16b)$$

$$C_j \cdot x + D_j \cdot y + \left(\tilde{h}^j\right)^\top \tilde{\lambda}^j \geq b_j, \quad j \in [\ell], \quad (16c)$$

$$\left(\tilde{H}^j\right)^\top \tilde{\lambda}^j = -\tilde{p}^j, \quad j \in [\ell], \quad (16d)$$

$$g + Q^\top x - D^\top \tilde{\alpha} = 0, \quad (16e)$$

$$\tilde{\alpha}_j \tilde{h}^j + \tilde{H}^j \tilde{\beta}^j + \tilde{\gamma}^j = 0, \quad j \in [\ell], \quad (16f)$$

$$\tilde{\alpha}_j \left(C_j \cdot x + D_j \cdot y + \left(\tilde{h}^j\right)^\top \tilde{\lambda}^j - b_j \right) = 0, \quad j \in [\ell], \quad (16g)$$

$$\left(\tilde{\gamma}^j\right)^\top \tilde{\lambda}^j = 0, \quad j \in [\ell], \quad (16h)$$

$$\tilde{\lambda}^j, \tilde{\gamma}^j \geq 0, \quad j \in [\ell], \quad (16i)$$

$$\tilde{\alpha} \geq 0, \quad (16j)$$

where \tilde{z} contains all primal variables used for the robustification of the uncertain right-hand sides as well as all resulting dual lower-level variables $\tilde{\lambda}^j$, $\tilde{\alpha}$, $\tilde{\beta}^j$, and $\tilde{\gamma}^j$. Before we show how Problem (15) relates to (16), we first consider an illustrative example taken and adapted from Besançon et al. (2019).

Example 1. We consider the linear bilevel problem defined by $0 \leq x, y \in \mathbb{R}$ and the data

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ -2 \end{bmatrix}, \quad a = \begin{pmatrix} -11 \\ -13 \end{pmatrix}, \quad c = 1, \quad d = -10, \quad R = 0,$$

$$C = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad b = \begin{pmatrix} 5 \\ -30 \end{pmatrix}, \quad f = 0, \quad g = 1, \quad Q = 0.$$

Further, we assume that the perceived decision of the leader as well as the uncertain lower-level data are known to lie within box constraints, i.e.,

$$\mathcal{Z} = \mathcal{Z}^j = \{\zeta \in \mathbb{R} : -1 \leq \zeta \leq 1\}, \quad j = 1, 2.$$

To model limited observability regarding the upper-level decision, we consider the uncertainty set parameterizations with $P \in \{0.5, 1\}$. The example is illustrated in Figure 1. The upper- and lower-level constraints are represented with dashed and solid lines, respectively. The optimal nominal solution and the two optimal robust solutions are illustrated with thick dots. It can be seen that limited observability regarding the leader's decision leads to a parallel shift of the follower's constraints, i.e., to lower-level right-hand side uncertainty. In particular, in this example, the corresponding lower-level right-hand side uncertainty is given by $\tilde{p}^1 = 2P$ and $\tilde{p}^2 = 5P$.

Based on the observation in Example 1, the question naturally arises on whether limited observability regarding the leader's decision can, in general, be modeled as a problem with uncertain lower-level right-hand side data. If this would be possible, it would be particularly favorable for scaling reasons, since Problem (15) can get very large. Thus, we next formally address how Problem (15) relates to Problem (16).

Theorem 2. Let (x, y, z) be a solution of Problem (15) with parameters P , h , and H modeling the uncertainty set. Furthermore, let (x, y, \tilde{z}) be a solution of Problem (16) with the uncertainty sets modeled with the parameters \tilde{p}^j , \tilde{h}^j , and \tilde{H}^j , $j \in [\ell]$. Third, suppose that the lower-level's objective function does not contain bilinear terms, i.e., $Q = 0$, and that

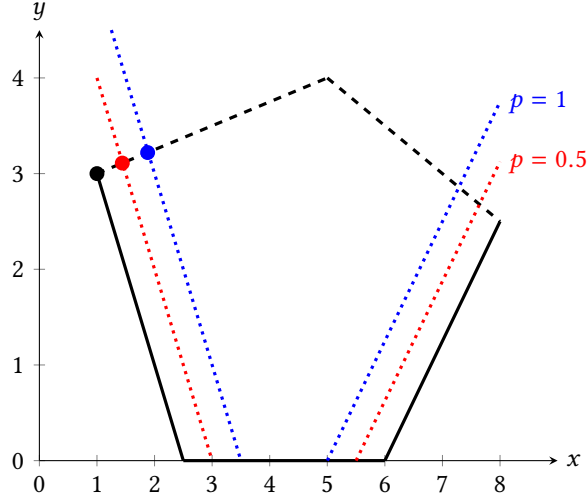


FIGURE 1. Relation between limited observability and uncertain lower-level data

$\text{rank}(D^\top) = \text{rank}([D^\top, g])$ holds. Then, the uncertainty modeling parameters satisfy

$$\begin{pmatrix} \tilde{h}^j \end{pmatrix}^\top \tilde{\lambda}^j = h^\top \lambda^j, \quad j \in [\ell], \quad (17a)$$

$$\begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \tilde{p}^j = -(\gamma^j)^\top (C_j \cdot P)^\top, \quad j \in [\ell], \quad (17b)$$

$$\begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \begin{pmatrix} \tilde{H}^j \end{pmatrix}^\top \tilde{\lambda}^j = -(\gamma^j)^\top H^\top \lambda^j, \quad j \in [\ell], \quad (17c)$$

$$\begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \begin{pmatrix} \tilde{H}^j \end{pmatrix}^\top \tilde{\lambda}^j = (\gamma^j)^\top \left(H^\top \sigma - P^\top (C_j^\top + f) \right), \quad j \in [\ell]. \quad (17d)$$

Proof. Under the imposed assumptions, we obtain $\tilde{\alpha} = \alpha$ by dual feasibility. Thus, (17a) follows immediately from KKT complementarity. For all $j \in [\ell]$, the multiplication of Constraint (16f) with $\tilde{\lambda}^j$ yields

$$\alpha_j \begin{pmatrix} \tilde{h}^j \end{pmatrix}^\top \tilde{\lambda}^j + \begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \begin{pmatrix} \tilde{H}^j \end{pmatrix}^\top \tilde{\lambda}^j + (\tilde{\gamma}^j)^\top \tilde{\lambda}^j = 0.$$

Due to (16d) and (16h), Constraint (16f) is thus equivalent to

$$\alpha_j \begin{pmatrix} \tilde{h}^j \end{pmatrix}^\top \tilde{\lambda}^j = -\begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \begin{pmatrix} \tilde{H}^j \end{pmatrix}^\top \tilde{\lambda}^j = \begin{pmatrix} \tilde{\beta}^j \end{pmatrix}^\top \tilde{p}^j, \quad j \in [\ell]. \quad (18)$$

Similarly, multiplying (15h) with λ^j and using (15e) as well as (15l) yields

$$\alpha_j h^\top \lambda^j = -(\gamma^j)^\top H^\top \lambda^j = -(\gamma^j)^\top (C_j \cdot P)^\top, \quad j \in [\ell].$$

Then, plugging in (17a) and the results in (18) yields (17b)–(17c). Finally, Equation (17d) follows immediately from (17c) as well as (15d)–(15e). \square

To sum up, there exists a connection between Problem (15) and a suitably chosen bilevel problem with lower-level right-hand side uncertainty. However, even though we impose rather strong assumptions (such as the rank condition), the established relation between the different uncertainty set parameterizations in (17) requires the knowledge of the lower-level primal and dual variables in advance, which means that the established result is only an ex-post relation. Thus, Problem (16) cannot be exploited to obtain a more compact formulation for our modeling of limited observability.

7. CONCLUSION

In this paper, we consider bilinear bilevel problems under follower's response uncertainty due to limited observability regarding the upper-level strategy. To this end, we exploit robust optimization to model decision making in the lower level under bounded rationality. An equivalent single-level reformulation is established by replacing the robust counterpart of the lower-level problem with its KKT conditions. Compared to the problem under perfect rationality, the presented modeling yields much larger optimization problems in terms of the number of variables and constraints. However, the problem remains in the same problem class as the problem without taking limited observability into account. We present an illustrative example to emphasize the importance of the proposed modeling aspect. Further, we establish an ex-post relation between the modeling of limited observability and robust bilevel problems with lower-level right-hand side uncertainty.

In this paper, polyhedral uncertainty sets have been considered. The consideration of other uncertainty set geometries might be a reasonable aspect of future work.

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