THE TURNPIKE PROPERTY AND THE LONG-TIME BEHAVIOR OF THE HAMILTON-JACOBI EQUATION

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ABSTRACT. In this work, we analyze the consequences that the so-called turnpike property has on the long-time behavior of the value function corresponding to an optimal control problem. As a by-product, we obtain the long-time behavior of the solution to the associated Hamilton-Jacobi-Bellman equation. In order to carry out our study, we use the setting of a finite-dimensional linear-quadratic optimal control problem, for which the turnpike property is well understood. We prove that, when the time horizon T tends to infinity, the value function converges to a travelling-front like solution of the form $W(x) + cT + \lambda$. In addition, we provide a control interpretation of each of these three terms in the spirit of the turnpike theory. Finally, we compare this asymptotic decomposition with the existing results on long-time behavior for Hamilton-Jacobi equations. We stress that in our case, the Hamiltonian is not coercive in the momentum variable, a case rarely considered in the classical literature about Hamilton-Jacobi equations.

1. Introduction

1.1. Motivation.

In this work we are interested in the behavior of the value function associated to an optimal control problem when the time horizon goes to infinity. Instead of using the theory developed in the context of Hamilton-Jacobi equations, namely, the ergodic theory and the results concerning the long-time asymptotics of the solution to time-evolution Hamilton-Jacobi equations, our purpose here is to deduce the long-time behavior of the value function as a consequence of an intrinsic property that is satisfied by a large class of optimal control problems and arises when the time horizon is considered to be sufficiently large. This is the so-called *turnpike property* and establishes that the optimal strategy in a controlled system

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during a sufficiently long time interval is to quickly stabilize from the starting state to the steady optimal and do not leave this one until the time is close to the end.

In order to carry out our study, we have chosen the setting of finite-dimensional linear-quadratic optimal control problems, for which the turnpike property is well understood under stabilizability and detectability assumptions [26]. Nevertheless, we stress that our goal is not to rely on the LQ theory, but rather to use general arguments that can be applicable to a larger class of optimal control problems enjoying the aforementioned turnpike property.

As a by-product of our study, we obtain the long-time behavior of the associated Hamilton-Jacobi-Bellman equation in a case where the Hamiltonian is not strictly convex and not even coercive, a scenario much less considered in the literature. This indicates that, in some cases, this kind of assumptions on the structure of the Hamiltonian can be merely relaxed to weaker assumptions concerning the stabilizability and detectability of the optimal control problem.

Another interesting by-product of our study is the characterization of the ergodic constant with the optimal value of the stationary control problem. This constant is usually identified with the limit, as t goes to infinity, of the ratio $\frac{V(t,x)}{t}$, where V is the solution to the time-evolution Hamilton-Jacobi equation [2, 3]. We stress that our characterization of the ergodic constant is completely independent of the value function V and does not rely on the theory of Hamilton-Jacobi equations.

From the viewpoint of the optimal control theory, our analysis is interesting since it establishes the implications that the turnpike property has on the value function. In particular, it can be deduced that the presence of the turnpike property allows the design of a quasi-optimal autonomous feedback control that stabilizes the system towards the turnpike. In view of our results, the application of this autonomous feedback control can be considered nearly optimal when the time is far enough from the final time. This can be observed by explicit computations in Proposition 1.4. However, when the time approaches the end, this strategy is no longer useful since, even if we are close to the turnpike, the optimal strategy is to eventually leave it in order to minimize the final payoff.

1.2. **Mathematical setting.** Let us introduce the mathematical framework that we will use along the paper. We consider the following optimal control problem in the finite-dimensional linear-quadratic setting: for a given time horizon $T > 0$ and an initial state x in $\mathbb{R}^n$, we denote the trajectory of the system by $y(\cdot)$, which is determined by the solution to the following controlled linear ODE:

$$\begin{cases} \dot{y}(s) = A\,y(s) + B\,u(s), \quad s \in (0,T) \\ y(0) = x, \end{cases} \tag{1.1}$$

where $A \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n,m}(\mathbb{R})$, with $n, m \geq 1$, are two given matrices and u, that will be referred to as the control, can be any function in the set of admissible controls $\mathcal{U}_T := L^2(0,T;\mathbb{R}^m)$.

The optimal control problem is to minimize, over the admissible controls $u \in \mathcal{U}_T$, the cost functional

$$J_{T,x}(u) := \frac{1}{2}\int_0^T \left[\|u(s)\|^2 + \|C\, y(s) - z\|^2\right] ds + g(y(T)), \tag{1.2}$$

where $C \in \mathcal{M}_n(\mathbb{R})$ is a given matrix, $z \in \mathbb{R}^n$ is the prescribed *running target* and $g : \mathbb{R}^n \to \mathbb{R}$ is a given locally Lipschitz function bounded from below, known as the *final pay-off*. Typical final pay-off functions are for example

- quadratic functions of the form $g(y(T)) = K\|D\, y(T) - z_T\|^2$ where $K > 0$, $D \in \mathcal{M}_n(\mathbb{R})$ and $z_T \in \mathbb{R}^n$ are given. This corresponds to the fully quadratic LQ problem, and can be seen as a penalization for the final state. By letting $K \to \infty$, the problem converges to the optimal control problem with fixed final state;
- the L^1-norm of the final state, i.e. $g(y(T)) = \|y(T)\|_1 = \sum_{i=1}^n |y_i(T)|$ which has the effect of optimally sparsifying the vector $y(T)$.
- distance function to a given set of points, i.e.
$$g(y(T)) = \min\{\|y(T) - z_1\|, \dots, \|y(T) - z_N\|\}.$$

Note that considering a final pay-off in the linear-quadratic problem allows us to study the associated time-evolution Hamilton-Jacobi-Bellman equation with general initial condition. However, in the case where the final pay-off is a nonconvex function, even if it is considered to be very smooth, the gradient of the value function ceases to exist in the classical sense for T large enough (see Example 2.6), and then, the solution to the Hamilton-Jacobi equation needs to be understood in the viscosity sense [12, 13, 24].

For the linear-quadratic optimal control problem that we consider here, exponential turnpike property was established in [26] by Porretta and Zuazua. This exponential turnpike property can be stated as follows: any optimal control $u^T(\cdot)$ and its corresponding state trajectory $y^T(\cdot)$ satisfy, for all $s \in [0, T]$,

$$\|u^T(s) - \overline{u}\| + \|y^T(s) - \overline{y}\| \leq K\left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T - s\right)\right)\right], \tag{1.3}$$

where K and μ are two positive constants independent of T and $(\overline{u}, \overline{y})$ is the steady optimal control-state pair, i.e. the pair $(u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n$ minimizing the steady functional

$$J_s(u_s, y_s) := \frac{1}{2}\left(\|u_s\|^2 + \|C\, y_s - z\|^2\right), \tag{1.4}$$

over the subset of controlled steady states

$$M := \left\{(u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \mid Ay_s + Bu_s = 0\right\}. \tag{1.5}$$

We denote by V_s the optimal steady cost, that is

$$V_s := \min\left\{J_s(u_s, y_s)\ : (u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \text{ s.t. } Ay_s + Bu_s = 0,\right\} \tag{1.6}$$

In [26], it is proved that, (1.3) follows from stabilizability of (A, B) and detectability of (A, C). In fact the validity of (1.3) for any initial data x and final cost g is equivalent to the stabilizability of (A, B) and the detectability of (A, C) (Theorem A.3). In the sequel, we will sometimes refer to the steady optimal control $\overline{u}$ and its corresponding state $\overline{y}$ as the turnpike. Observe that the steady functional J_s and then also the turnpike, are independent of the final pay-off g. For self-completeness, we give the proof of the equivalence of the turnpike property and the stabilizability of (A, B) and detectability of (A, C) (Theorem A.3 in appendix A).

Our main goal here is not to improve the results in [26], but rather to make the connection between the turnpike property and the asymptotic behavior, as T tends to infinity, of the value function associated to the optimal control problem (1.1)-(1.2), defined as

$$V(x, T) := \inf_{u \in \mathcal{U}_T} J_{T,x}(u). \tag{1.7}$$

When studying the long-time behavior of the value function, one is tempted to consider the same optimal control problem (1.1)–(1.2) in an infinite time horizon. However, this approach fails in general since, in most of the cases, the running cost is not integrable in $(0, \infty)$ for any control $u \in L^2_{loc}(0, \infty; \mathbb{R}^m)$.

In view of the turnpike property (1.3), if T is sufficiently large, the running cost for the optimal control problem (1.1)-(1.2) satisfies

$$\frac{1}{2}\left[\|u^T(s)\|^2 + \|C\, y^T(s) - z\|^2\right] \sim V_s, \tag{1.8}$$

for any t far away from 0 and T, where V_s was defined in (1.4). Hence, the lack of integrability issue when considering the infinite time horizon problem can be handled by subtracting the constant V_s to the running cost. In this way, as a consequence of (1.3), one can use the optimal control to make the running cost exponentially small for any t away from 0 and T.

We therefore introduce the following infinite time horizon optimal control problem, where the dynamics are the same as (1.1) in the time interval $[0, +\infty)$ and the cost functional is given by

$$J_{\infty,x}(u) := \int_0^\infty \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s\right] ds. \tag{1.9}$$

The control is considered to be in the set of admissible controls $\mathscr{A}_x$, defined in (2.17) as the controls u for which $J_{\infty,x}(u)$ is well-defined. Observe that in this case, the set of admissible controls depends on x. In fact, as we will see in Lemma 2.4, a control is admissible if and only if the control and its associated trajectory converge sufficiently fast to $(\overline{u}, \overline{y})$ as $t \to \infty$. The existence of such controls for any $x \in \mathbb{R}^n$ follows from the stabilizability of (A, B).

For this optimal control problem, we denote by $W(x)$ the associated value function, defined as

$$W(x) := \inf_{u \in \mathscr{A}_x} J_{\infty,x}(u). \tag{1.10}$$

Note that this function is independent of the final pay-off g. To the best of our knowledge, the definition of the infinite time horizon problem in the case the target $z \neq 0$ has not been treated in the literature so far. A similar analysis is well established in case $z = 0$ using Riccati theory (see e.g. [23]).

Our definition of W is motivated by the necessity of identifying the cost of stabilization towards the turnpike in the asymptotic expansion of the value function

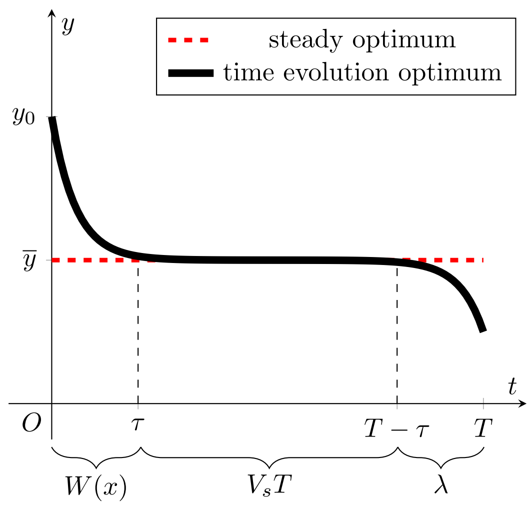

FIGURE 1. Optimal state fulfilling the turnpike property and associated asymptotic decomposition of the value function.

$V(x,T)$. This stabilizing phase (see $[0,\tau]$ in figure 1) is not visible in the classical definition of the infinite time horizon problem (see for example [15]), where the limite of the time averages only captures the transient arc where the optima are close to the turnpike.

1.3. **Main result.** Our main result states that $V(\cdot,T)-V_s T$ converges, as T goes to infinity, to the value function of the infinite time horizon optimal control problem $W(\cdot)$ plus a constant λ independent of x and T. As we shall see in Remark 1.2 below, this constant λ is related to the final arc of the optimal trajectory, when the control and the state leave the turnpike in order to minimize the final pay-off (see Figure 1).

As it well-known (see §4.1), the value function $V(x,T)$ defined in (1.7) is the unique viscosity solution to the following Cauchy problem

(1.11)
$$\begin{cases} \partial_T V + \frac{1}{2}\|B^*\nabla_x V\|^2 - Ax\cdot\nabla_x V = \frac{1}{2}\|Cx-z\|^2, & (x,T)\in\mathbb{R}^n\times(0,+\infty) \\ V(x,0)=g(x), & x\in\mathbb{R}^n \end{cases}$$

Therefore, the following result describes the long-time behavior of the solution to this problem. In the above time-evolution problem, the time-derivative is taken with respect to the time horizon T because the value function V is a function of the time horizon of the optimal control problem.

Theorem 1.1. *Assume (A,B) is stabilizable and (A,C) is detectable. Let $g:\mathbb{R}^n\to\mathbb{R}$ be a given locally Lipschitz function bounded from below and $z\in\mathbb{R}^n$. Let V and W be the value functions defined in* (1.7) *and* (1.10) *respectively. Then,* *(i) for any bounded set* $\Omega \subset \mathbb{R}^n$*, we have*

$$V(x,T) - V_s T \longrightarrow W(x) + \lambda, \quad \text{as } T \to \infty, \quad \text{uniformly in } x \in \Omega$$

where V_s *is the constant defined in* (1.6) *and the constant* λ *is given by*

$$\lambda = \lim_{T \to +\infty} V(\bar{y}, T) - V_s T,$$

where $\bar{y}$ *is the state in the pair* $(\bar{u}, \bar{y})$ *minimizing the steady functional* J_s*.*

(ii) $W \in C^1(\mathbb{R}^n)$ *and is, up to an additive constant, the unique viscosity solution bounded from below to the stationary problem*

$$V_s + \frac{1}{2}\|B^* \nabla W(x)\|^2 - A\,x \cdot \nabla W(x) = \frac{1}{2}\|C\,x - z\|^2 \qquad x \in \mathbb{R}^n. \tag{1.12}$$

In addition, (1.12) *with a different constant* $c \neq V_s$ *does not admit viscosity solutions bounded from below.*

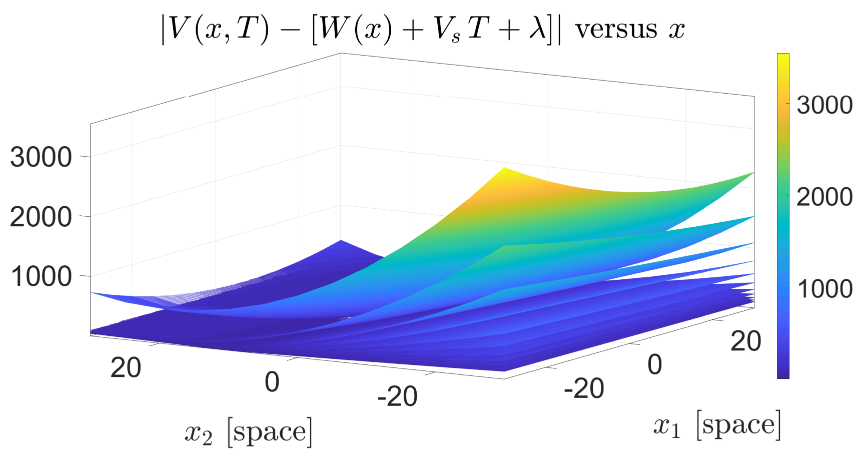

FIGURE 2. Difference between the value function and its asymptotic expansion versus the space variable x. Each layer corresponds to the difference at a certain fixed time horizon T. The layer in the top (that reaches the maximum value) corresponds to the shortest time horizon, while the layer in the bottom (that approaches zero) corresponds to a larger time horizon. As $T \to +\infty$, we observe that the difference vanishes.

Remark 1.2. Observe that, as a consequence of this result, the value function $V(x,T)$ admits the following asymptotic decomposition (figure 2):

$$V(x,T) \sim W(x) + V_s T + \lambda, \qquad \text{as} \quad T \sim \infty.$$

where $V(x,T)$ is as described in (1.7), $W(x)$ is given by (1.10), V_s is defined as in (1.6) and λ is the constant in Theorem 1.1. In view of the turnpike interpretation of the optimal strategy for the finite time horizon control problem (1.1)–(1.2), we can identify each of the three terms in the right-hand-side as follows.

1. The term $W(x)$ represents the cost of stabilizing the trajectory from the initial state x to the turnpike. Observe that the optimal strategy for the

infinite time horizon problem is to stabilize towards the turnpike and stay there forever. Indeed, since the turnpike property is fulfilled, in a large time interval, optimal strategies spend most of the time close to the turnpike. In addition, in view of (1.9), the running cost at the turnpike is zero and hence, $W(x)$ stands for the minimal cost needed to stabilize to the turnpike from the initial state.

2. The term $V_s T$ corresponds to the running cost accumulated in the intermediate arc, where the time-evolution optima are close to the steady ones.
3. The constant λ represents the cost during the final arc, when the optimal trajectory leaves the turnpike in order to minimize the final pay-off g. Observe that, although this final arc does not appear in the infinite time horizon problem, it is always present in the problem in finite time horizon, no matter how large T is considered to be, and therefore, it has to be taken into account when studying the long time behavior of the value function V. The way to single out this final arc from the rest of the trajectory is to consider the finite time horizon problem taking $\overline{y}$ as initial state, so that the cost of reaching the turnpike is 0 and then to subtract the cost during the transient arc $V_s T$ (see the definition of λ in the statement of the theorem).

It appears that for some specific initial data g, our result holds under weaker assumptions. This is the object of the next remark.

Remark 1.3. The proof of Theorem 1.1 is based on the validity of the turnpike property (1.3). In Theorem A.3 in the appendix, we prove that the turnpike property (1.3) is satisfied for any final cost g bounded from below if and only if (A, B) is stabilizable and (A, C) is detectable. However, for special final costs, the conclusion of Theorem 1.1 can be deduced from a weaker version of the turnpike property. If for instance $g \equiv 0$, it suffices that the following inequality is satisfied

$$\|u^T(s) - \overline{u}\| + \|Cy^T(s) - C\overline{y}\| \leq K\left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T - s\right)\right)\right], \tag{1.13}$$

where K and $\mu > 0$ are T-independent constants. For this particular case, the detectability of (A, C) is no longer necessary and it is sufficient to only assume the C-stabilizability, i.e. the stabilizability of observable modes (see [30] and the references therein). Actually, given a specific final cost g bounded from below, it would be interesting to obtain sharp conditions for a weaker turnpike property as (1.13) to hold.

We shall prove Theorem 1.1 using general arguments, which can be applicable to a wide variety of control problems. However, by using Riccati theory, one can prove that in fact W is real analytic (polynomial). This is the object of the following Proposition, proved in section 3.

Proposition 1.4. *Let $(\overline{u}, \overline{y})$ be the minimizer for J_s defined in* (1.4)*. Then,*

$$F(y) := -B^* \widehat{E}\left(y - \overline{y}\right) + \overline{u} \tag{1.14}$$

defines an optimal feedback law for $J_{\infty,x}$ defined in (1.9)*, meaning that, for any $x \in \mathbb{R}^n$, the unique optimal control is given by*

$$u^*(s) = -B^* \widehat{E}\left(y^*(s) - \overline{y}\right) + \overline{u}, \qquad s \in (0, T) \tag{1.15}$$

where $\widehat{E}$ is the unique symmetric positive semidefinite solution to the Algebraic Riccati Equation

$$-\widehat{E}A - A^*\widehat{E} + \widehat{E}BB^*\widehat{E} = C^*C \qquad \text{(ARE)}$$

and y^* solves the closed loop equation

$$\begin{cases} \frac{d}{ds}y^*(s) = (A + B\,F)\,y^*(s), & s \in (0,\infty) \\ y^*(0) = x. \end{cases}$$

Moreover, the value function W is given by

$$W(x) = \frac{1}{2}\,(x-\overline{y})^*\,\widehat{E}\,(x-\overline{y}) + (\overline{p}, x-\overline{y})_{\mathbb{R}^n}.$$

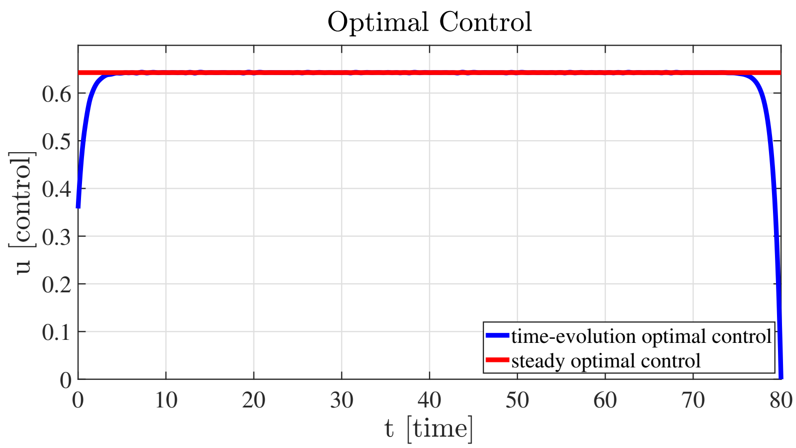

FIGURE 3. Optimal control $u(s)$ for (1.1)-(1.2) (in blue) and optimal steady control $\overline{u}$ (in red).

1.4. **Known results on long-time behavior for Hamilton-Jacobi equations.** Observe that the PDE in (1.11) is a Hamilton-Jacobi equation of the form

$$\partial_T V + H(x, \nabla_x V) = \ell(x), \quad \text{in } \mathbb{R}^n \times (0,+\infty), \tag{1.16}$$

where, in our case, the function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, known as the *Hamiltonian*, and the function $\ell : \mathbb{R}^n \to \mathbb{R}$ are given by

$$\begin{aligned} H(x,p) &= \max_u \left[-p\cdot(Ax+Bu) - \frac{1}{2}\|u\|^2 \right] = \frac{1}{2}\|B^*p\|^2 - p\cdot Ax \\ \ell(x) &= \frac{1}{2}\|Cx - z\|^2. \end{aligned} \tag{1.17}$$

The long-time behavior for equations like (1.16) has been widely studied in the literature, especially in the flat torus but also in more general settings, e.g. [5, 7, 17, 19, 20, 21, 29] and the references therein.

Here, we deal with unbounded solutions in the whole space $\mathbb{R}^n$, a scenario much less studied compared to the case in the n-dimensional tours. In the recent work

[5] it is proved, under suitable hypotheses on H, the existence of a constant $c \in \mathbb{R}$ such that

$$V(x,T) - c\,T \to \varphi(x), \qquad \text{as } T \to \infty, \tag{1.18}$$

where φ is a viscosity solution to the *stationary* Hamilton-Jacobi equation

$$c + H(x, \nabla_x \varphi) = \ell(x), \qquad \text{in } \mathbb{R}^n. \tag{1.19}$$

This is also called the *ergodic problem* [2, 3, 6] and c is known as the ergodic constant. For equations like (1.19), a solution is understood as a pair (c,φ), where c is a constant and φ is a continuous function satisfying (1.19) in the viscosity sense.

In Theorem 1.1, we have obtained long-time asymptotics of the form (1.18) for the solution to the Hamilton-Jacobi equation (1.11). Although in [5], the unbounded case (space domain $\Omega = \mathbb{R}^n$) is treated, we point out that in our setting, the Hamiltonian does not satisfy all the assumptions required in [5]. In particular, in our case, the function $H(x,p)$ defined in (1.17) is neither globally Lipschitz in the p variable nor coercive. Note that, in view of (1.17), the Hamiltonian is coercive if and only if B^* has a trivial kernel.

Concerning the solutions to the stationary equation (1.19), the ergodic constant is commonly identified in the literature as the limit of averages

$$c = \lim_{T\to+\infty} \frac{1}{T} V(x,T),$$

Here we give an alternative characterization of c as the minimum value of the steady functional J_s defined in (1.4). Observe that this characterization is based on the turnpike property and does not involve the value function V. Another interesting feature of our approach is the infinite time-horizon problem used to characterize the solution to (1.19), for which the well-posedness relies on the turnpike property. To the best of our knowledge, this particular control interpretation of (1.19) as a consequence of the turnpike property has been obtained only in [22].

The paper is structured as follows. In section 2.1, we prove a first result which is a direct consequence of the turnpike property, namely, the time-averages of the value function converge to the ergodic constant as the time horizon tends to infinity. In the subsection 2.2 we study the auxiliary infinite time horizon optimal control problem introduced above and in subsection 2.3, we give the proof of the first statement of Theorem 1.1. In Section 3, we use Riccati theory to prove the Proposition 1.4. In Section 4, we employ the existing theory based on the dynamic programming principle to verify that the value functions V and W solve the Hamilton-Jacobi-Bellman equations (1.11) and (1.12) respectively, and prove the second statement of Theorem 1.1 In Section 5, we sum up the conclusions of the paper and give a list of possible research lines. Finally, for the reader's convenience and self-consistency of the paper, we include an appendix with the proof of the turnpike property (Theorem A.3).

2. Infinite horizon problem and proof of Theorem 1.1

The proof of Theorem 1.1 relies on the use of the turnpike property (1.3), which ensures that the optimal control for the problem (1.1)–(1.2) and its corresponding state trajectory remain exponentially close to the steady optima for any t far away from 0 and T.

Let us introduce the steady version of the control problem (1.1)-(1.2). We define the subspace of controlled steady states as

$$M := \{(u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \mid 0 = Ay_s + Bu_s\}. \tag{2.1}$$

The steady optimal control problem is to minimize, over the subspace M, the cost functional

$$J_s(u_s, y_s) := \frac{1}{2}\left[\|u_s\|^2 + \|C\,y_s - z\|^2\right]. \tag{2.2}$$

We define the value of the steady optimal control problem as

$$V_s := \inf_{(u_s, y_s)\in M} J_s. \tag{2.3}$$

As we shall see in the Appendix A, the assumption (A, B) is stabilizable and (A, C) is detectable yields existence and uniqueness of a minimizer $(\overline{u}, \overline{y}) \in M$ for (2.2). In this case, we can just write $V_s = J_s(\overline{u}, \overline{y})$.

2.1. **A first consequence of the turnpike property.** We start with the following result, which is a direct consequence of the turnpike property (1.3). It ensures that the time-averages of the cost-functional $J_{T,x}(\cdot)$, evaluated in the optimal control u^T, converge to the value of the steady optimal control problem as T goes to infinity.

Proposition 2.1. *Under the assumptions of Theorem 1.1, let $V(x, T)$ be the value function defined in* (1.7) *and V_s defined as in* (2.3)*. Then, for any $x \in \mathbb{R}^n$, we have*

$$\frac{1}{T}V(x, T) \underset{T\to+\infty}{\longrightarrow} V_s. \tag{2.4}$$

In order to prove the above proposition we need to rewrite the functional $J_{T,x}$ defined in (1.2) in a different way. Roughly speaking, we need the running cost to be centered around the turnpike.

Lemma 2.2. *Let $(\overline{u}, \overline{y})$ be the steady optimal control-state pair for the functional J_s defined in* (2.2) *and $V_s := J_s(\overline{u}, \overline{y})$. Then, for any $T > 0$, $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_T$, we have*

$$\begin{aligned} J_{T,x}(u) &= T\,V_s + \frac{1}{2}\int_0^T \left[\|u(s) - \overline{u}\|^2 + \|C\,\left(y(s) - \overline{y}\right)\|^2\right] ds \\ &\quad + (\overline{p}, x - y(T))_{\mathbb{R}^n} + g(y(T)), \end{aligned}$$

where $\overline{p} \in \mathbb{R}^n$ is the optimal adjoint steady state (Lagrange multiplier) and is independent of T, x and u.

Proof of Lemma 2.2. In view of the definition of $J_{T,x}$ in (1.2), We can compute

$$\begin{aligned} J_{T,x}(u) &= \frac{1}{2}\int_0^T \left[\|u(s)-\overline{u}+\overline{u}\|^2 + \|C\,y(s)-C\overline{y}+C\overline{y}-z\|^2\right] ds + g(y(T)) \\ &= \frac{T}{2}\left[\|\overline{u}\|^2+\|C\,\overline{y}-z\|^2\right] + \frac{1}{2}\int_0^T \left[\|u(s)-\overline{u}\|^2 + \|C\,\left(y(s)-\overline{y}\right)\|^2\right] ds \\ &\quad + \int_0^T \left[(\overline{u}, u(s)-\overline{u})_{\mathbb{R}^m} + (C\overline{y}-z,\, C\,(y(s)-\overline{y}))_{\mathbb{R}^n}\right] ds + g(y(T)) \\ &= T\,V_s + \frac{1}{2}\int_0^T \left[\|u(s)-\overline{u}\|^2 + \|C\,\left(y(s)-\overline{y}\right)\|^2\right] ds \\ &\quad + \int_0^T \left[(\overline{u}, u(s)-\overline{u})_{\mathbb{R}^m} + (C\overline{y}-z,\, C\,(y(s)-\overline{y}))_{\mathbb{R}^n}\right] ds + g(y(T)). \end{aligned} \tag{2.5}$$

We now focus on the term

$$\int_0^T (C\,\overline{y}-z,\, C\,(y(s)-\overline{y}))_{\mathbb{R}^n}\, ds. \tag{2.6}$$

We recall that the pair $(\overline{u},\overline{y})$ satisfies the steady optimality system (A.10), which reads as

$$\begin{cases} 0 = A\overline{y} - BB^*\overline{p} \\ 0 = A^*\overline{p} + C^*(C\,\overline{y}-z). \end{cases} \tag{2.7}$$

In the other hand, the pairs $(u(\cdot),y(\cdot))$ and $(\overline{u},\overline{y})$ satisfy the equation in (1.1). Hence, we have

$$\begin{cases} \frac{d}{ds}(y-\overline{y}) = A(y-\overline{y}) + B(u-\overline{u}) & s\in(0,T) \\ y(0)-\overline{y} = x-\overline{y}. \end{cases} \tag{2.8}$$

Then, using (2.7) and (2.8) and taking into account that $y(0)=x$ and $\overline{u}=-B^*\overline{p}$, we can compute the term (2.6) as follows:

$$\begin{aligned} \int_0^T (C\overline{y}-z,\, C\,(y(s)-\overline{y}))_{\mathbb{R}^n}\, ds &= \int_0^T (C^*\,(C\overline{y}-z)\,,\, y(s)-\overline{y})_{\mathbb{R}^n}\, ds \\ &= -\int_0^T (\overline{p},\, A\,(y(s)-\overline{y}))_{\mathbb{R}^n}\, ds \\ &= -\int_0^T \left(\overline{p}, \frac{d}{ds}(y-\overline{y}) - B(u-\overline{u})\right)_{\mathbb{R}^n} ds \\ &= (\overline{p}, y(0)-\overline{y})_{\mathbb{R}^n} - (\overline{p}, y(T)-\overline{y})_{\mathbb{R}^n} \\ &\quad + \int_0^T (B^*\overline{p}, u(s)-\overline{u})_{\mathbb{R}^m}\, ds \\ &= (\overline{p}, x-y(T))_{\mathbb{R}^n} - \int_0^T (\overline{u}, u(s)-\overline{u})_{\mathbb{R}^m}\, ds. \end{aligned} \tag{2.9}$$

Finally, the conclusion follows by combining (2.5) and (2.9). □

We now proceed to the prove of Proposition 2.1.

Proof of Proposition 2.1. Let u^T be the optimal control for the problem (1.1)-(1.2) and y^T its corresponding state. By Lemma 2.2, the value function V satisfies

$$\begin{aligned} V(x,T) &= \inf_{u\in\mathcal{U}_T} J_{T,x}(u) = J_{T,x}\left(u^T\right) \\ &= T\,V_s + \frac{1}{2}\int_0^T \left[\|u^T(s)-\overline{u}\|^2 + \|C\,\left(y^T(s)-\overline{y}\right)\|^2\right] ds \\ &\quad + \left(\overline{p}, x - y^T(T)\right)_{\mathbb{R}^n} + g(y^T(T)). \end{aligned}$$

Applying the turnpike property (1.3), we have the following estimate

$$\left|\frac{1}{2}\int_0^T \left[\|u^T(s)-\overline{u}\|^2 + \|C\,\left(y^T(s)-\overline{y}\right)\|^2\right] ds + \left(\overline{p}, x-y^T(T)\right)_{\mathbb{R}^n} + g(y(T))\right|$$

$$\leq K^2\int_0^T \left(e^{-\mu s} + e^{-\mu(T-s)}\right)^2 ds + 4K(\|\overline{p}\|+1),$$

where K depends on x but not on T. This yields

$$\frac{1}{T}V(x,T) \underset{T\to+\infty}{\longrightarrow} V_s.$$

□

Here, we prove the following Lipschitz estimate, uniform in T, that is also consequence of the turnpike property and will be useful in the proof of Theorem 1.1.

Lemma 2.3. *Assume (A,B) is stabilizable and (A,C) is detectable. Let V be the function defined in (1.7). Then, for any $M>0$, there exists a constant $K_M>0$ such that*

$$|V(x_2,T) - V(x_1,T)| \leq K_M\,\|x_2-x_1\|\,, \tag{2.10}$$

for all $T>0$ and all x_1 and x_2 in $\mathbb{R}^n$ satisfying $\|x_i\|\leq M$.

Proof. We prove this Lemma by using the definition of $V(T,x)$ as minimal value of $J_{T,x}$. Let $u^T_{x_1}\in\mathcal{U}_T$ be an optimal control for J_{T,x_1}. Since (A,B) is stabilizable, there exists a feedback matrix $F\in\mathcal{M}_{m\times n}(\mathbb{R})$, such that $A+BF$ generates an exponentially stable semigroup. Set the control

$$\hat{u}(s) := F\tilde{y}(s) + u^T_{x_1}(s), \qquad s\in(0,T), \tag{2.11}$$

where $\tilde{y}$ solves the closed loop equation

$$\begin{cases} \frac{d}{ds}\tilde{y}(s) = (A+B\,F)\,\tilde{y}(s), & s\in(0,\infty) \\ \tilde{y}(0) = x_2 - x_1. \end{cases}$$

We start by proving

$$\left|J_{T,x_2}(\hat{u}) - J_{T,x_1}(u^T_{x_1})\right| \leq K_M\,\|x_2-x_1\|\,, \tag{2.12}$$

where K_M is independent of $T>0$.
Step 1 **Proof of** (2.12)
Set $y^T_{x_1}$ solution to (1.1) with initial datum x_1 and control $u^T_{x_1}$ and $\hat{y}$ solution to

$$\begin{cases} \frac{d}{ds}\hat{y}(s) = A\,\hat{y}(s) + B\,\hat{u}(s), & s\in[0,T] \\ \hat{y}(0) = x_2. \end{cases} \tag{2.13}$$

By definition of F, we have for any $s \geq 0$

$$\left\|\hat{y}(s) - y^T_{x_1}(s)\right\| = \|\tilde{y}(s)\| \leq K \|x_2 - x_1\| \exp(-\mu s), \tag{2.14}$$

whence

$$\left\|\hat{u}(s) - u^T_{x_1}(s)\right\| = \|F\tilde{y}(s)\| \leq K \|x_2 - x_1\| \exp(-\mu s), \tag{2.15}$$

the constants K and $\mu > 0$ being independent of $s \geq 0$. From the above inequalities and Theorem A.3, (2.12) follows.
Step 2 **Conclusion**
For $i = 1, 2$, let $u^T_{x_i}$ be optimal controls for J_{T,x_i} and let $\hat{u}$ defined as above for $u := u^T_{x_1}$. Then, by definition of value function and (2.12)

$$\begin{aligned} V(x_2, T) - V(x_1, T) &= J_{T,x_2}(u^T_{x_2}) - J_{T,x_1}(u^T_{x_1}) \\ &\leq J_{T,x_2}(\hat{u}) - J_{T,x_1}(u^T_{x_1}) \\ &\leq K_M \|x_2 - x_1\|. \end{aligned}$$

By the arbitrariness of x_1 and x_2, we obtain the desired Lipschitz property. □

2.2. **The infinite horizon linear-quadratic problem.** Here we introduce the auxiliary infinite time horizon optimal control problem announced in the introduction, that allows us to compute the optimal cost of stabilizing the trajectory to the turnpike from the initial state. For each $x \in \mathbb{R}^n$, the dynamics are determined by the same ODE in (1.1), in this case considering the time interval $(0, \infty)$:

$$\begin{aligned} &\dot{y}(s) = A\, y(s) + B\, u(s), \quad s \in (0, \infty) \\ &y(0) = x. \end{aligned} \tag{2.16}$$

For this problem we consider the following set of admissible controls:
(2.17)

$$\mathscr{A}_x := \left\{ u \in L^2_{\text{loc}}((0, +\infty)) \ : \ \int_0^\infty \left| \frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s \right| ds < +\infty \right\},$$

where y is the solution to (2.16) and $V_s = J_s(\overline{u}, \overline{y})$ is the constant defined in (2.3). Note that the set of admissible controls is different for each x. In addition, since (A, B) is stabilizable, we deduce that it is nonempty for all x.

The problem that we consider here is to minimize, over the controls $u \in \mathscr{A}_x$, the cost functional

$$J_{\infty,x}(u) := \int_0^\infty \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s \right] ds, \tag{2.18}$$

where y is the solution to (2.16), with initial condition x and control u. The value function for this problem is then defined as

$$W(x) = \inf_{u \in \mathscr{A}_x} \int_0^\infty \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s \right] ds. \tag{2.19}$$

The following lemma follows directly from the definition of $\mathscr{A}_x$.

Lemma 2.4. *Let $(\overline{u}, \overline{y})$ be the minimizer for J_s defined in* (2.2)*. For any $x \in \mathbb{R}^n$ and any control $u \in \mathscr{A}_x$, we denote by y the solution to* (2.16) *with control u and initial datum x. Then it holds*

$$u - \overline{u} \in L^2(0, +\infty) \quad \textit{and} \quad y - \overline{y} \in L^2(0, +\infty).$$

In addition, $\{y(t)\}_{t>0}$ *is bounded in* $\mathbb{R}^n$ *and satisfies*

$$y(t) \longrightarrow \overline{y} \quad as \quad t \to +\infty.$$

The functional $J_{\infty,x}$ *can be written as*

$$J_{\infty,x}(u) = \frac{1}{2}\int_0^{\infty} \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds + (\overline{p}, x-\overline{y})_{\mathbb{R}^n} \tag{2.20}$$

and it admits a minimizer u^* *in* $\mathscr{A}_x$*.*

Proof. Step 1 **Boundedness of** $\{y(t)\}_{t>0} \subset \mathbb{R}^n$
Take any $u \in \mathscr{A}_x$ and let y be the solution to (2.16), with initial datum x and control u. By Lemma A.1 applied to $y-\overline{y}$, we have

$$\|y(s)-\overline{y}\|^2 \leq K\left[\|x-\overline{y}\|^2 + \int_0^t \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds\right],$$

whence

$$\frac{1}{2}\int_0^t \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds \geq \alpha\|y(s)-\overline{y}\|^2 - K,$$

where $\alpha = \alpha(A,C) > 0$ and $K = K(A,B,C,x,z) \geq 0$. Using the above inequality, together with Lemma 2.2, yields

$$\begin{aligned}
J_{\infty,x}(u) &= \lim_{t\to+\infty} \int_0^t \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\,y(s)-z\|^2 - V_s\right] ds \\
&= \lim_{t\to+\infty}\left[\frac{1}{2}\int_0^t \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds \right.\\
&\qquad \left. +(\overline{p}, x-y(t))_{\mathbb{R}^n}\right] \\
&\geq \limsup_{t\to+\infty}\left[\frac{1}{2}\int_0^t \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds \right.\\
&\qquad \left. -K\left(1+\|y(t)-\overline{y}\|\right)\right] \\
&\geq \limsup_{t\to+\infty}\left[\alpha\|y(t)-\overline{y}\|^2 - K\left(\|y(t)-\overline{y}\|+2\right)\right] \\
&\geq \frac{\alpha}{2}\limsup_{t\to+\infty}\|y(t)-\overline{y}\|^2 - K.
\end{aligned}$$

Now, since $u \in \mathscr{A}_x$, the functional $J_{\infty,x}(u) < +\infty$. This, together with the above estimate, implies the boundedness of $\{y(t)\}_{t>0} \subset \mathbb{R}^n$.
Step 2 **Proof of** $u-\overline{u} \in L^2(0,+\infty)$ **and** $y-\overline{y} \in L^2(0,+\infty)$
By Step 1, there exists a constant $K(u) \geq 0$, such that, for any $t > 0$,

$$\|y(t)\| \leq K(u).$$

By Lemma 2.2, one gets

$$\begin{aligned}
&\int_0^t \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\,y(s)-z\|^2 - V_s\right] ds \\
&\qquad = \frac{1}{2}\int_0^t \left[\|u(s)-\overline{u}\|^2 + \|C\ (y(s)-\overline{y})\ \|^2\right] ds + (\overline{p}, x-y(t))_{\mathbb{R}^n}
\end{aligned} \tag{2.21}$$

and using the above bound, for any $t > 0$, we have

$$\begin{aligned}(2.22)\qquad &\int_0^t \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\,y(s) - z\|^2 - V_s\right] ds \\ &\geq \frac{1}{2}\int_0^t \left[\|u(s) - \overline{u}\|^2 + \|C\,\left(y(s) - \overline{y}\right)\|^2\right] - K(u),\end{aligned}$$

whence, since $u \in \mathscr{A}_x$,

$$\begin{aligned}+\infty > J_{\infty,x}(u) &= \lim_{t\to+\infty}\int_0^t \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\,y(s) - z\|^2 - V_s\right] ds \\ &\geq \frac{1}{2}\int_0^\infty \left[\|u(s) - \overline{u}\|^2 + \|C\,\left(y(s) - \overline{y}\right)\|^2\right] ds - K(u),\end{aligned}$$

which in turn implies $u - \overline{u} \in L^2(0, +\infty)$ and $C(y - \overline{y}) \in L^2(0, +\infty)$. Now, since the pair (A, C) is detectable, adapting the techniques of the proof of Lemma A.1, we have in fact $y - \overline{y} \in L^2(0, +\infty)$.

Step 3 **Proof of $y(t) \longrightarrow \overline{y}$ as $t \to +\infty$.**
Now, since $y - \overline{y} \in L^2(0, +\infty)$, there exists a sequence $t_m \to +\infty$, such that

$$y(t_m) \underset{m\to+\infty}{\longrightarrow} \overline{y}.$$

By the above convergence and $u - \overline{u} \in L^2(0, +\infty)$ and $C(y - \overline{y}) \in L^2(0, +\infty)$, for any $\varepsilon > 0$, there exists $m_\varepsilon \in \mathbb{N}$ such that for every $m > m_\varepsilon$

$$\|y(t_m) - \overline{y}\| < \varepsilon \quad \text{and} \quad \int_{t_m}^{+\infty} \left[\|u(s) - \overline{u}\|^2 + \|C\,\left(y(s) - \overline{y}\right)\|^2\right] ds < \varepsilon^2.$$

Then, by Lemma A.1, for any $m > m_\varepsilon$ and for any $t > t_m$ we have

$$\|y(t) - \overline{y}\|^2 \leq K\left[\|y(t_m) - \overline{y}\|^2 + \int_{t_m}^t \left[\|u(s) - \overline{u}\|^2 + \|C\,\left(y(s) - \overline{y}\right)\|^2\right] ds\right] < 2K\varepsilon^2,$$

whence

$$y(t) \longrightarrow \overline{y}, \qquad \text{as } t \to +\infty.$$

Step 4 **Proof of** (2.20)
The representation formula (2.20) is a consequence of (2.18), (2.21), $u - \overline{u} \in L^2(0, +\infty)$ and $y - \overline{y} \in L^2(0, +\infty)$ and $y(t) \underset{t\to+\infty}{\longrightarrow} \overline{y}$. The existence of the minimizer follows from (2.20) and the Direct Method in the Calculus of Variations. □

Next we prove a local Lipschitz estimate for W that will be used in the proof of Theorem 1.1.

Lemma 2.5. *Assume (A, B) is stabilizable and (A, C) is detectable and let W be the function defined in* (2.19). *Then, for any $M > 0$, there exists a constant $K_M > 0$ such that*

$$(2.23)\qquad |W(x_2) - W(x_1)| \leq K_M \|x_2 - x_1\|,$$

for all x_1 and x_2 in $\mathbb{R}^n$ satisfying $\|x_i\| \leq M$.

Proof. The proof can be done by adapting the techniques used in the proof of Lemma 2.3. □

2.3. **Proof of Theorem 1.1 (i).** We prove now the statement *(i)* in Theorem 1.1 and the C^1 regularity of W.

Proof of Theorem 1.1 (i). Step 1 **Convergence** For any bounded set $\Omega \subset \mathbb{R}^n$, let $x \in \Omega$ be fixed and, for any $T > 0$, let $u^T(\cdot)$ and $y^T(\cdot)$ be an optimal control for problem (1.1)–(1.2) and its corresponding state trajectory. Then, as a consequence of the Dynamic Programming Principle, for any $T > 0$ we can write

$$V(x,T) = \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s) - z\|^2\right] ds + V\left(y^T\left(\frac{T}{2}\right), \frac{T}{2}\right). \tag{2.24}$$

Now, using Lemma 2.3 and that $y^T(T/2) \to \bar{y}$ as $T \to \infty$, we deduce that

$$\lim_{T\to\infty} \left| V\left(y^T\left(\frac{T}{2}\right), \frac{T}{2}\right) - V\left(\bar{y}, \frac{T}{2}\right)\right| = 0. \tag{2.25}$$

Hence, we have

$$\begin{aligned}
\lim_{T\to\infty} V\left(y^T\left(\frac{T}{2}\right), \frac{T}{2}\right) - \frac{T}{2}V_s &= \lim_{T\to\infty} \left[V\left(y^T\left(\frac{T}{2}\right), \frac{T}{2}\right) - V\left(\bar{y}, \frac{T}{2}\right) \right. \\
&\qquad \left. + V\left(\bar{y}, \frac{T}{2}\right) - \frac{T}{2}V_s \right] \\
&= \lim_{T\to\infty} V\left(\bar{y}, \frac{T}{2}\right) - \frac{T}{2}V_s =: \lambda. \qquad (2.26)
\end{aligned}$$

The existence of this limit can be justified by proving that the function

$$T \longmapsto V(\bar{y}, T) - T\,V_s$$

is decreasing and bounded from below. Indeed, observe that if u^T is an optimal control for $J_{x,T}$, then for any $T' > T$, we can use the control

$$\hat{u}(s) := \begin{cases} \bar{u} & s \in (0, T'-T) \\ u^T(s) & s \in [T'-T, T') \end{cases}$$

to prove the monotonicity. The boundedness from below can be obtained from the turnpike property.

Let us now prove that

$$\lim_{T\to+\infty} \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s) - z\|^2\right]\, ds - \frac{T}{2}V_s = W(x). \tag{2.27}$$

Let $u^* \in \mathscr{A}_x$ be the optimal control for the functional $J_{\infty,x}$ defined in (2.19) and y^* its corresponding state trajectory. For any $T > 0$, as a consequence of the dynamic programming principle in Lemma 4.3, we have

$$\begin{aligned}
W(x) &= \int_0^{\frac{T}{2}} \left[\frac{1}{2}\|u^*(s)\|^2 + \frac{1}{2}\|C\,y^*(s) - z\|^2 - V_s\right] ds + W\left(y^*\left(\frac{T}{2}\right)\right) \\
&\leq \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s) - z\|^2\right] ds - \frac{T}{2}V_s + W\left(y^T\left(\frac{T}{2}\right)\right). \qquad (2.28)
\end{aligned}$$

Now, observe that by plugging $\bar{y}$ in formula (2.20) in Lemma 2.4, one can easily see that $W(\bar{y}) = 0$. Then, using the fact that, by the turnpike property $y^T(T/2)$ converges exponentially to $\bar{y}$ and that, by Lemma 2.5, the function $W(\cdot)$ is continuous, we deduce that

$$\liminf_{T\to+\infty} \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s)-z\|^2\right]\,ds - \frac{T}{2}V_s \geq W(x). \tag{2.29}$$

Using again the dynamic programming principle, this time for the value function V, we obtain for any $T>0$:

$$\begin{aligned} V(x,T) &= \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s)-z\|^2\right] ds + V\left(y^T\left(\frac{T}{2}\right),\frac{T}{2}\right) \\ &\leq \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^*(s)\|^2 + \|C\,y^*(s)-z\|^2\right] ds + V\left(y^*\left(\frac{T}{2}\right),\frac{T}{2}\right). \end{aligned} \tag{2.30}$$

Using this time the dynamic programming principle for W (see first equality in (2.28)), we can compute

$$\frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^*(s)\|^2 + \|C\,y^*(s)-z\|^2\right] ds = W(x) + \frac{T}{2}V_s - W\left(y^*\left(\frac{T}{2}\right)\right).$$

And combining this identity with (2.30), we obtain

$$\begin{aligned} \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s)-z\|^2\right] ds - \frac{T}{2}V_s \leq W(x) - W\left(y^*\left(\frac{T}{2}\right)\right) \\ +V\left(y^*\left(\frac{T}{2}\right),\frac{T}{2}\right) - V\left(y^T\left(\frac{T}{2}\right),\frac{T}{2}\right). \end{aligned}$$

This inequality, together with $W(\bar{y})=0$, the Lipschitz continuity of V from Lemma 2.3 and the fact that, by the turnpike property and Lemma 2.4, we have that $y^T(T/2)$ and $y^*(T/2)$ converge to $\bar{y}$ as $T\to\infty$, gives

$$\limsup_{T\to+\infty} \frac{1}{2}\int_0^{\frac{T}{2}} \left[\|u^T(s)\|^2 + \|C\,y^T(s)-z\|^2\right]\,ds - \frac{T}{2}V_s \leq W(x).$$

From this inequality and (2.29), it follows (2.27).

Finally, combining (2.24), (2.26) and (2.27) we obtain

$$V(x,T) - TV_s \underset{T\to+\infty}{\longrightarrow} W(x)+\lambda. \tag{2.31}$$

Step 2 **Regularity of $W(\cdot)$.**
Since the function W is independent of the final cost g, in order to prove that W is in $C^1(\mathbb{R}^n)$, we will consider the case $g\equiv 0$.

Step 2.1 **Differential of optimal trajectories in finite time**
Set

$$\begin{aligned} \psi_T : \mathbb{R}^n &\longrightarrow C^1([0,T];\mathbb{R}^n)^2 \\ x &\longmapsto \left[y^T, p^T\right], \end{aligned} \tag{2.32}$$

where $\left[y^T, p^T\right]$ is the solution to the optimality system

$$\begin{cases} \frac{d}{ds}y^T(s) = Ay^T(s) - BB^*p^T(s) & s \in (0,T) \\ -\frac{d}{ds}p^T(s) = C^*(C\,y^T(s) - z) + A^*p^T(s) & s \in (0,T) \\ y^T(0) = x \\ p^T(T) = 0. \end{cases} \tag{2.33}$$

Whence for any $x \in \mathbb{R}^n$ and for any direction $v \in \mathbb{R}^n$ the Fréchet differential of ψ_T at x along direction v reads as

$$(D\psi_T(x), v)_{\mathbb{R}^n} = \left[\hat{y}^T, \hat{p}^T\right], \tag{2.34}$$

where

$$\begin{cases} \frac{d}{ds}\hat{y}^T(s) = A\hat{y}^T(s) - BB^*\hat{p}^T(s) & s \in (0,T) \\ -\frac{d}{ds}\hat{p}^T(s) = C^*C\hat{y}^T(s) + A^*\hat{p}^T(s) & s \in (0,T) \\ \hat{y}^T(0) = v \\ \hat{p}^T(T) = 0. \end{cases} \tag{2.35}$$

Step 2.2 **Differentials of the value function in finite time**
By Step 1 and the chain rule, for any directions $v \in \mathbb{R}^n$, we have

$$\frac{\partial}{\partial v}V(x,T) = \int_0^T \left[\left(u^T, -B^*\hat{p}^T\right)_{\mathbb{R}^m} + \left(C\,y^T - z, C\hat{y}^T\right)_{\mathbb{R}^n}\right] ds, \tag{2.36}$$

where

$$\begin{cases} \frac{d}{ds}\hat{y}^T(s) = A\hat{y}^T(s) - BB^*\hat{p}^T(s) & s \in (0,T) \\ -\frac{d}{ds}\hat{p}^T(s) = C^*C\hat{y}^T(s) + A^*\hat{p}^T(s) & s \in (0,T) \\ \hat{y}^T(0) = v \\ \hat{p}^T(T) = 0. \end{cases} \tag{2.37}$$

Step 2.3 **Bounds for the differentials of the value function, uniform in time horizon**
We start reminding that, for any $v \in \mathbb{R}^n$, (2.35) is the optimality system for the optimal control problem with state equation

$$\begin{aligned} &\tfrac{d}{ds}\hat{y}(s) = A\,\hat{y}(s) + B\,\hat{u}(s), \quad s \in (0,T) \\ &\hat{y}(0) = v, \end{aligned}$$

and cost functional

$$J_{T,v}\left(\hat{u}\right) := \frac{1}{2}\int_0^T \left[\|\hat{u}(s)\|^2 + \|C\,\hat{y}(s)\|^2\right] ds.$$

Since (A,B) is stabilizable and (A,C) is detectable, we are in position to apply Theorem A.3 both to the above problem and the original (1.1)-(1.2), with initial datum x and running target z, getting

$$\|\hat{p}^T(s)\| + \|\hat{y}^T(s)\| \le K\left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T-s\right)\right)\right], \qquad \forall s \in [0,T]. \tag{2.38}$$

and

$$\|p^T(s) - \overline{p}\| + \|y^T(s) - \overline{y}\| \le K\left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T-s\right)\right)\right], \qquad \forall s \in [0,T]. \tag{2.39}$$

for two T-independent constants K and $\mu > 0$. In the proof of Theorem A.3 one can check that K and μ can be chosen uniformly in $v \in S^1(\mathbb{R}^n)$ and $x \in \overline{B(0,R)}$, for some arbitrarily fixed $R > 0$. Then, for any $x \in \overline{B(0,R)}$ and $v \in S^1(\mathbb{R}^n)$, by (2.36), (2.39), (2.38), we can estimate

$$\begin{aligned} \left|\frac{\partial}{\partial v}V(x,T)\right| &\leq \int_0^T \left[\|u^T\| \, \|B^*\hat{p}^T\| + \|C\,y^T - z\| \, \|C\hat{y}^T\|\right] ds \\ &\leq K\int_0^T \left[\|B^*\hat{p}^T\| + \|C\hat{y}^T\|\right] ds \\ &\leq K, \end{aligned} \tag{2.40}$$

with K uniform on $T > 0$, $x \in \overline{B(0,R)}$ and $v \in S^1(\mathbb{R}^n)$.

□

Let us finish this section with an illustrative example that shows why the value function $V(x,T)$ is not in general differentiable. As we will see, for a suitable nonconvex final cost g, the global minimizer for $J_{T,x}$ with $x = 0$ and T sufficiently large is not unique. This implies in particular that $V(\cdot,T)$ is not differentiable at 0 for T sufficiently large (see for instance Theorem 7.4.17 in [10]).

Example 2.6. Let us consider the optimal control problem (1.1)–(1.2) with the pair of matrices (A,B) being controllable and C being any matrix. As final cost, we consider the function

$$g_\varepsilon(x) = \frac{1}{\varepsilon}[\|x\|^4 - \|x\|^2],$$

where $\varepsilon > 0$ will be chosen later.

Our goal is to show that if $\varepsilon > 0$ sufficiently small, the functional

$$J_{T,0}(u) := \frac{1}{2}\int_0^T \left[\|u(s)\|^2 + \|C\,y(s)\|^2\right] ds + g_\varepsilon(y(T)), \tag{2.41}$$

admits (at least) two distinguished global minimizers whenever $T > 2$.

Let us first prove that, if $\varepsilon > 0$ is sufficiently small, then for any $T > 1$, the control $u \equiv 0$ is not optimal.

Fix x_1 a minimizer of the function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined as $g(x) := \|x\|^4 - \|x\|^2$ and set

$$\tilde{u}(s) = \begin{cases} 0 & s \in (0, T-1) \\ u_1(t - T + 1) & s \in (T-1, T), \end{cases}$$

where u_1 is any control solving the controllability problem

$$\begin{aligned} &\dot{y_1}(s) = A\,y_1(s) + B\,u_1(s), \quad s \in [0,1] \\ &y_1(0) = 0, \; y_1(1) = x_1. \end{aligned} \tag{2.42}$$

Let $\tilde{y}$ be the solution to (1.1),with control $\tilde{u}$. Since $x = 0$ with control $u = 0$ is a stationary point of (1.1), by uniqueness of solution we have

$$\tilde{y}(s) = \begin{cases} 0 & s \in (0, T-1) \\ y_1(t - T + 1) & s \in (T-1, T), \end{cases}$$

Let us now evaluate the functional $J_{T,0}$ defined in (2.41) at $\tilde{u}$ and compare it with the control $u \equiv 0$. Since $\min_{\mathbb{R}^n} g(x) < 0$, we have

$$\begin{aligned} J_{T,0}(\tilde{u}) &= \frac{1}{2}\int_0^1 \left[\|u_1(s)\|^2 + \|C\,y_1(s)\|^2\right] ds + \frac{1}{\varepsilon}\left[\|x_1\|^4 - \|x_1\|^2\right] \\ &= \frac{1}{2}\int_0^1 \left[\|u_1(s)\|^2 + \|C\,y_1(s)\|^2\right] ds + \frac{1}{\varepsilon}\min_{\mathbb{R}^n} g \\ &< 0 = J_{T,x}(0), \end{aligned}$$

for a sufficiently small ε. This means that $u \equiv 0$ is not a global minimizer of (2.41).

Finally, since the final cost g_ε in (2.41) is bounded from below, by the direct method in the Calculus of variations, there exists a minimizer u^T of (2.41). Moreover, we have that $u^T \neq 0$ if $T > 1$. Now, by definition of (2.41), $J_{T,0}(-u^T) = J_{T,0}(u^T) = \min_{\mathcal{U}_T} J_{T,x}$, whence u^T and $-u^T$ are two distinguished global minimizers of (2.41).

3. Riccati theory

So far, we have proved our results by using general techniques applicable to general optimal control problems. We show now how, in the LQ setting, Riccati theory can be employed to solve explicitly the infinite time horizon minimization problem (1.10).

Our analysis is based on the following well-known Lemma, where we analyze the Algebraic Riccati Equation and the properties of the so-called hamiltonian matrix

$$\mathrm{Ham} := \begin{bmatrix} A & -BB^* \\ -C^*C & -A^*. \end{bmatrix}$$

One can realize that Ham is the associated matrix to (A.3).

Lemma 3.1. *Assume (A,B) is stabilizable and (A,C) is detectable. Then,*

(1) there exists a unique symmetric positive semidefinite solution to the Algebraic Riccati Equation

$$-\widehat{E}A - A^*\widehat{E} + \widehat{E}BB^*\widehat{E} = C^*C \qquad \text{(ARE)} \tag{3.1}$$

such that $A - BB^*\widehat{E}$ is stable, i.e. the real part of the spectrum $Re(\sigma(A - BB^*\widehat{E})) \subset (-\infty, 0)$;

(2) set

$$\Lambda := \begin{bmatrix} I_n & S \\ \widehat{E} & \widehat{E}S + I_n \end{bmatrix}, \tag{3.2}$$

where S is solution to the Lyapunov equation

$$S(A - BB^*\widehat{E})^* + (A - BB^*\widehat{E})S = BB^*.$$

Then, Λ is invertible and

$$\Lambda^{-1}\,Ham\,\Lambda = \begin{bmatrix} A - BB^*\widehat{E} & 0 \\ 0 & -(A - BB^*\widehat{E})^*. \end{bmatrix}$$

As a consequence, Ham is invertible and its spectrum does not intersect the imaginary axis.

The first part of the above Lemma is Riccati theory (see, for instance, [9, Fact 1-(a) and Fact 1-(f)] or [1]). The second part[1] is taken from [31, subsection III.B]. We are now ready to prove Proposition 1.4.

Proof of Proposition 1.4. By (2.20), the minimization of $J_{\infty,x}$ is equivalent to the minimization of

$$\begin{array}{rccl} \widehat{J}_{\infty,x}: & L^2_{\text{loc}}(0,+\infty) & \longrightarrow & \mathbb{R}\cup\{+\infty\} \\ & u & \longmapsto & \displaystyle\frac{1}{2}\int_0^\infty \left[\|u(s)-\overline{u}\|^2 + \|C\;(y(s)-\overline{y})\;\|^2\right] ds, \end{array}$$

where y is the solution to (1.1), with initial datum x and control u. By [23, Theorem 3.7 pages 237-238], there exists a unique minimizer u^* for $\widehat{J}_{\infty,x}$, given by (1.15) and

$$\inf_{L^2_{\text{loc}}(0,+\infty)} \widehat{J}_{\infty,x} = \frac{1}{2}\left(x-\overline{y}\right)^* \widehat{E}\left(x-\overline{y}\right),$$

whence, by (1.10) and (2.20),

$$W(x) = \inf_{L^2_{\text{loc}}(0,+\infty)} \widehat{J}_{\infty,x} + (\overline{p}, x-\overline{y})_{\mathbb{R}^n} = \frac{1}{2}\left(x-\overline{y}\right)^* \widehat{E}\left(x-\overline{y}\right) + (\overline{p}, x-\overline{y})_{\mathbb{R}^n},$$

as desired. □

In [22] the terms λ, V_s and $W(\cdot)$ in Theorem 1.1 have been represented by a different approach. The author introduces Riccati operators in an augmented state space, taking into account the target z as a state variable (with zero dynamics). An asymptotic behavior of such Riccati operators is provided in [22, Lemma 2] which is the key ingredient to determine the asymptotic behavior of the value function as $V(x,T) \sim W(x) + V_s T + \lambda$, when $T\to+\infty$ such that

$$V_s = \frac{1}{2}(z,\Lambda_3 z)_{\mathbb{R}^n}, \quad W(x) = \frac{1}{2}(\Lambda_1 x, x)_{\mathbb{R}^n} + (x,\Lambda_2 z)_{\mathbb{R}^n} \text{ and } \lambda = \frac{1}{2}(z,\beta z)_{\mathbb{R}^n}$$

where[2] $(\Lambda_1,\Lambda_2,\Lambda_3,\beta)\in\mathcal{S}_n^{++}(\mathbb{R})\times\mathcal{M}_n(\mathbb{R})\times\mathcal{S}_n(\mathbb{R})\times\mathcal{S}_n(\mathbb{R})$ are uniquely defined by the asymptotic behavior in [22, Lemma 2] (see also in [22] formula (39) in the proof of Theorem 1 and statement *(i)* in Corollary 2).

4. The Hamilton-Jacobi-Bellman equation

In this section we are interested in the Hamilton-Jacobi-Bellman equations associated to the optimal control problems (1.1)–(1.2) and (2.16)–(2.18) respectively. For the first one, which is a finite horizon problem, we obtain a time-evolution

[1]
$$\Lambda^{-1} = \begin{bmatrix} I_n + S\widehat{E} & -S \\ -\widehat{E} & I_n. \end{bmatrix}$$

[2] $\mathcal{S}_n(\mathbb{R})$ and $\mathcal{S}_n^{++}(\mathbb{R})$ denote respectively the subset of symmetric matrices and positive definite symmetric matrices in $\mathcal{M}_n(\mathbb{R})$.

Hamilton-Jacobi equation where the time-variable is the length of the time horizon and the initial condition is the final cost. For the second one, which is an infinite horizon problem we obtain the associated stationary equation. The arguments rely on the Dynamic Programming Principle and are standard in this kind of optimal control problems. However, since the value functions V and W are defined in a particular way that differs from the standard definitions, we include here the version of the Dynamic Programming Principle for these value functions. The rest of the proof can be concluded by adapting the standard techniques. Finally, in subsection 4.2, we conclude the proof of the statement (ii) in Theorem 1.1 about the uniqueness of the stationary solution bounded from below and the ergodic constant.

4.1. **Finite time-horizon optimal control problem.** We start with the Hamilton-Jacobi-Bellman equation associated to the optimal control problem (1.1)–(1.2). Let us recall that, for each $x \in \mathbb{R}^n$ and $T > 0$, the value function is defined as

$$V(x,T) := \inf_{u\in\mathcal{U}_T} J_{T,x}(u), \tag{4.1}$$

where the functional $J_{T,x}$ is given by

$$J_{T,x}(u) = \frac{1}{2}\int_0^T \left[\|u(s)\|^2 + \|C\,y(s) - z\|^2\right] ds + g(y(T)),$$

and for each u, the function $y : (0,T) \to \mathbb{R}^n$ is the solution to

$$\begin{cases} \dot{y}(s) = A\,y(s) + B\,u(s) & s \in (0,T) \\ y(0) = x. \end{cases}$$

This definition of the value function, which seems to be unusual in the literature, is motivated by the study of the behavior of V when the time horizon tends to infinity. An alternative way to proceed would be to consider the optimal control problem with prescribed final state and free initial state, considering the dynamics in a backward sense. However, it makes the comparison between the finite time and the infinite time horizon problems a bit awkward.

Let us justify that the value function V is the unique viscosity solution to the Hamilton-Jacobi-Bellman equation associated to this finite time horizon optimal control problem.

Theorem 4.1. *The value function V defined in* (4.1) *is the unique viscosity solution to the initial-value problem*

$$\begin{cases} \partial_T V + \frac{1}{2}\|B^*\nabla_x V\|^2 - Ax\cdot\nabla_x V = \frac{1}{2}\|Cx - z\|^2, & (x,T) \in \mathbb{R}^n \times (0,+\infty) \\ V(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

The proof can be done by adapting the methods in [14, Section 10.3] based on the Dynamic Programming Principle. Since the value function V is defined differently, for the reader's convenience, we state and prove the Dynamic Programming Principle which is satisfied by this value function. The conclusion of the theorem follows by combining the standard techniques with this version of the Dynamic Programming Principle.

The uniqueness of the viscosity solution can be deduced by using the arguments of [22, Theorem 6 page 41].

Lemma 4.2. *Set the functions*

$$L(y,u) := \|u\|^2 + \|C\,y - z\|^2 \quad \textit{and} \quad f(y,u) := A\,y + B\,u,$$

and let V *be the value function defined in* (4.1). *Then, for all* $x \in \mathbb{R}^n$, $T > 0$ *and* $0 < h < T$, *we have*

$$V(x,T) = \inf_{u\in\mathcal{U}_T} \left\{ \int_0^h L(y(s),u(s))ds + V(y(h),T-h) \right\},$$

where, for each $u \in \mathcal{U}_T$, y *is the solution to* (1.1) *with initial datum* x *and control* u.

Proof. Let us denote by $\omega(x,T)$ the right hand side in the equality. We first prove that $V(x,T) \geq \omega(x,T)$ for all $(x,T) \in \mathbb{R}^n \times (0,\infty)$.

For any control $u \in \mathcal{U}_T$ we can write

$$\begin{aligned} J_{T,x}(u) &= \int_0^h L(y(s),u(s))ds + \int_h^T L(y(s),u(s))ds + g(y(T)) \\ &= \int_0^h L(y(s),u(s))ds + \int_0^{T-h} L(\tilde{y}(s),\tilde{u}(s))ds + g(\tilde{y}(T-h)) \\ &\geq \int_0^h L(y(s),u(s))ds + V(y(h),T-h), \end{aligned}$$

where $\tilde{u}(s) = u(s+h)$ and $\tilde{y}(s) = y(s+h)$. Then, by the arbitrariness of u we obtain

$$V(T,x) = \inf_{u\in\mathcal{U}_T} J_{T,x}(u) \geq \inf_{u\in\mathcal{U}_T} \left\{ \int_0^h L(y(s),u(s))ds + V(y(h),T-h) \right\}.$$

For the reverse inequality, fix $u \in \mathcal{U}_T$ and set $z = y(h)$. For any $\varepsilon > 0$, let $u^\varepsilon \in \mathcal{U}_{T-h}$ be such that

$$V(T-h,z) \geq J_{T-h,z}(u^\varepsilon) - \varepsilon.$$

Define the control

$$u^*(s) := \begin{cases} u(s), & \text{if } \ 0 \leq s \leq h \\ u^\varepsilon(s-h), & \text{if } \ h < s < T \end{cases}$$

and let y^*, y^ε be the trajectories corresponding to u^* and u^ε respectively. Then, noticing that $y^\varepsilon(0) = y^*(h) = y(h) = z$, one has

$$\begin{aligned} V(x,T) &\leq \int_0^h L(y^*(s),u^*(s))ds + \int_h^T L(y^*(s),u^*(s))ds + g(y^*(T)) \\ &= \int_0^h L(y(s),u(s))ds + J_{T-h,z}(u^\varepsilon) \\ &\leq \int_0^h L(y(s),u(s))ds + V(T-h,z) + \varepsilon, \end{aligned}$$

and letting $\varepsilon \to 0$, we obtain

$$V(T,x) = \inf_{u\in\mathcal{U}_T} J_{T,x}(u) \le \inf_{u\in\mathcal{U}_T} \left\{ \int_0^h L(y(s),u(s))ds + V(y(h),T-h) \right\}.$$

□

4.2. **Infinite time horizon optimal control problem.** Let us recall the definition of the value function associated to the infinite-time horizon LQ problem that we stated in the introduction:

$$W(x) = \inf_{u\in\mathscr{A}_x} J_{\infty,x}(u) \tag{4.3}$$

where $\mathscr{A}_x$ defined in (2.17) is the set of admissible controls, the functional $J_{\infty,x}$ is defined as

$$J_{\infty,x}(u) = \int_0^\infty \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\,y(s) - z\|^2 - V_s \right] ds,$$

and for each $u \in \mathscr{A}_x$, the function $y : (0,\infty) \to \mathbb{R}^n$ is the solution to

$$\begin{cases} \dot{y}(s) = A\,y(s) + B\,u(s), & s \in (0,\infty) \\ y(0) = x. \end{cases} \tag{4.4}$$

Our goal here is to prove that the value function W with the constant $c = V_s$ is the unique viscosity solution[3] bounded from below to the stationary Hamilton-Jacobi-Bellman equation

$$c + \frac{1}{2}\|B^*\nabla W(x)\|^2 - \nabla W(x)^* A x = \frac{1}{2}\|Cx - z\|^2, \quad x \in \mathbb{R}^n. \tag{4.5}$$

The first step in the proof consists in showing that W satisfies a Dynamic Programming Principle (DPP for short) that we state in the next Lemma.

Set $L(x,u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|Cx - z\|^2 - V_s$. We recall that a proof of DPP in infinite time horizon, unrestricted state space and with a discount factor can be found for example in [4, Prop. III.2.5, p. 102], a proof in a standard case is in [24] and an abstract DPP can be found in [16].

Lemma 4.3. *For any $\delta > 0$ and $x \in \mathbb{R}^n$, we have*

$$W(x) = \inf_{u\in\mathscr{A}_x} \left\{ \int_0^\delta L(y(s),u(x))ds + W(y(\delta)) \right\}$$

where $y(\cdot)$ is the trajectory corresponding to the control u and initial state x.

Proof. We follow the proof in [4, Prop. III.2.5, p. 102]).
Denote by $\omega(x)$ the right-hand side in (4.3). First we show that $W(x) \ge \omega(x)$.

[3]It is in fact a classical solution since we proved in subsection 2.2 that $W \in C^1(\mathbb{R}^n)$. Here, the uniqueness is obviously understood up to an additive constant.

For any $u \in \mathscr{A}_x$, using the definition of $J_{\infty,x}$ we obtain

$$\begin{aligned} J_{\infty,x}(u) &= \int_0^\delta L(y(s),u(s))ds + \int_\delta^\infty L(y(s),u(s))ds \\ &= \int_0^\delta L(y(s),u(s))ds + \int_0^\infty L(y(s+\delta),u(s+\delta))ds \\ &= \int_0^\delta L(y(s),u(s))ds + J_{\infty,y(\delta)}(\tilde{u}), \quad \text{where } \tilde{u}(s) = u(s+\delta) \end{aligned}$$

It is easy to see that $u \in \mathscr{A}_x$ implies that $\tilde{u} \in \mathscr{A}_{y(\delta)}$. Now, taking the infimum over all controls $\tilde{u} \in \mathscr{A}_{y(\delta)}$ we get

$$\begin{aligned} J_{\infty,x}(u) &\geq \int_0^\delta L(y(s),u(s))ds + \inf_{\tilde{u}} J_{y(\delta)}(\tilde{u}) \\ &= \int_0^\delta L(y(s),u(s))ds + W(y(\delta)) \end{aligned}$$

and by the arbitrariness of $u \in \mathscr{A}_x$, it follows $W(x) \geq \omega(x)$.

In order to prove the opposite inequality, fix $u \in \mathscr{A}_x$ and set $z := y(\delta)$. For any $\varepsilon > 0$, let $u^\varepsilon \in \mathscr{A}_z$ be such that

$$W(z) \geq J_{\infty,z}(u^\varepsilon) - \varepsilon.$$

Define the control

$$u^*(s) := \begin{cases} u(s), & \text{if } s \leq \delta \\ u^\varepsilon(s-\delta), & \text{if } s > \delta \end{cases}$$

and let y^*, y^ε be the trajectories corresponding to u^* and u^ε respectively. Then, noticing that $y^\varepsilon(0) = y^*(\delta) = y(\delta) = z$, one has

$$\begin{aligned} W(x) \leq J_{\infty,x}(u^*) &= \int_0^\delta L(y^*(s),u^*(s))ds + \int_\delta^\infty L(y^*(s),u^*(s))ds \\ &= \int_0^\delta L(y^*(s),u^*(s))ds + \int_0^\infty L(y^\varepsilon(s),u^\varepsilon(s))ds \\ &= \int_0^\delta L(y^*(s),u^*(s))ds + J_{\infty,z}(u^\varepsilon) \\ &\leq \int_0^\delta L(y^*(s),u^*(s))ds + W(z) + \varepsilon \end{aligned}$$

Since u and ε are arbitrary, we get $W(x) \leq \omega(x)$ and hence the desired result. □

We now observe that we can rewrite the above Dynamic Programming Principle by taking the infimum over bounded controls.

Remark 4.4. By [10, Theorem 7.4.6], for any compact set $K \subset \mathbb{R}^n$, there exists $M = M(K) > 0$ such that for any $x \in \mathbb{R}^n$ and any $\delta \in (0,1)$, we have

$$W(x) = \inf_{u \in \mathscr{A}_M} \left\{ \int_0^\delta L(y(s),u(x))ds + W(y(\delta)) \right\},$$

where

$$\mathscr{A}_M := \left\{ u \in L^2((0,1),\mathbb{R}^m) \;:\; \|u(s)\| \leq M, \quad \text{a.e. in } (0,+\infty) \right\}.$$

We now conclude with the proof of statement *(ii)* in Theorem 1.1.

Proof Theorem 1.1 (ii). We recall that the C^1 regularity of W was proved in subsection 2.2. Let us now prove that $(V_s, W(\cdot))$ satisfies the equation (4.5). The proof relies on standard methods in optimal control.

We first show that $(V_s, W(\cdot))$ satisfies $V_s + H(x, \nabla W(x)) \leq \ell(x)$ for every $x \in \mathbb{R}^n$. For any $u_o \in \mathbb{R}^n$, let us consider a continuous admissible control $u \in \mathscr{A}_x$ such that $u(0) = u_o$. Thanks to the C^1 regularity of W proved in subsection 2.3, we can compute

$$\begin{aligned} W(y(\delta)) &= W(x) + \int_0^\delta \frac{d}{ds}\left(W(y(s)) - W(x)\right) ds \\ &= W(x) + \int_0^\delta (\nabla W(y(s)), \frac{d}{ds} y(s))_{\mathbb{R}^n} ds \\ &= W(x) + \int_0^\delta (\nabla W(y(s)), Ay(s) + Bu(s))_{\mathbb{R}^n} ds \end{aligned} \tag{4.6}$$

On the other hand, by Dynamic Programming principle, one has

$$W(x) \leq \int_0^\delta \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s\right] ds + W(y(\delta)),$$

which combined with (4.6) gives

$$0 \leq \int_0^\delta \left[\frac{1}{2}\|u(s)\|^2 + \frac{1}{2}\|C\, y(s) - z\|^2 - V_s + \langle \nabla W(y(s)), Ay(s) + Bu(s)\rangle_{\mathbb{R}^n}\right] ds.$$

Dividing the above inequality by δ and taking the limit as $\delta \to 0$, we deduce, from the arbitrariness of u_o, that

$$V_s + \max_{u_o^\varepsilon \in \mathbb{R}^m} \left\{-\frac{1}{2}\|u_o\|^2 - \nabla W(x) \cdot (Ax + Bu_o)\right\} \leq \ell(x).$$

In order to prove the converse inequality, let us assume by contradiction that for some $x_0 \in \mathbb{R}^n$ and $r > 0$, there exists $\varepsilon > 0$ such that, for any $u_o \in \mathbb{R}^m$,

$$V_s - \frac{1}{2}\|u_o\|^2 - \nabla W(x) \cdot (Ax + Bu_o) \leq \ell(x) - \varepsilon \tag{4.7}$$

for any $x \in B(x_0, r)$, where $B(x_0, r)$ is a ball of $\mathbb{R}^n$ centered at x_0 of radius $r > 0$. By remark 4.4, it suffices to consider u_o, with $\|u_o\| \leq M$, for some $M \geq 0$.

Take any $u \in L^2(0,1)$, with $\|u(t)\| \leq M$, a.e. in $(0,1)$. By continuous dependence from the data for (1.1), there exists $\overline{\delta} \in (0,1)$, such that for any $\delta \in [0, \overline{\delta})$ the state y associated to control u and initial datum x_0, verifies $y(\delta) \in B(x_0, r)$.

Now, using remark 4.4, there exists u^ε and an associated state y^ε, such that

$$\begin{aligned} W(x) &\geq \int_0^\delta \left[\frac{1}{2}\|u^\varepsilon(s)\|^2 + \frac{1}{2}\|C\,y^\varepsilon(s) - z\|^2 - V_s\right]\,ds + W(y^\varepsilon(\delta)) - \frac{\varepsilon\delta}{2} \\ &= \int_0^\delta \left[\frac{1}{2}\|u^\varepsilon(s)\|^2 + \frac{1}{2}\|C\,y^\varepsilon(s) - z\|^2 - V_s\right] ds + W(x) \\ &\quad + \int_0^\delta \langle \nabla W(y^\varepsilon(s)),\, Ay^\varepsilon(s) + Bu^\varepsilon(s)\rangle_{\mathbb{R}^n}\,ds - \frac{\varepsilon\delta}{2}, \end{aligned} \tag{4.8}$$

where in (4.8) we have employed the identity (4.6). We have then

$$\int_0^\delta \left[V_s - \frac{1}{2}\|u^\varepsilon\|^2 - \langle \nabla W(x), Ay^\varepsilon(s) + Bu^\varepsilon\rangle_{\mathbb{R}^n} - \ell(y^\varepsilon(s))\right] ds \geq -\frac{\varepsilon\delta}{2}, \tag{4.9}$$

so obtaining a contradiction (4.7). Then one recovers the second inequality and hence the desired result.

Finally, let us prove that $W(x)$ is the unique (up to an additive constant) viscosity solution to (1.12) bounded from below.

Let $c \in \mathbb{R}$, and let $W_1 \in C(\mathbb{R}^n)$ be a bounded from below continuous function satisfying the equation

$$c + H(x, \nabla W_1) = \ell(x)$$

in the viscosity sense. Here the Hamiltonian H and the function ℓ are defined as in (1.17)

Observe that the function given by

$$V_1(T, x) = c\,T + W_1(x)$$

is a viscosity solution to the problem (1.11) with initial condition $g(x) = W_1(x)$, which is bounded from below. As we have seen in subsection 4.1, the unique viscosity solution to (1.11) is given by the value function to the optimal control problem (1.1)–(1.2) with final cost $g(x) = W_1(x)$.

Since $W_1(\cdot)$ is bounded from below, we can use Theorem 1.1 to deduce that

$$\lim_{T\to+\infty} V_1(T, x) - V_s\,T = W(x) + \lambda, \qquad \text{for all } x \in \mathbb{R}^n,$$

for some $\lambda \in \mathbb{R}$ depending on the final cost $W_1(\cdot)$. Hence, using the definition of $V_1(T, x)$ we obtain

$$\lim_{T\to+\infty} W_1(x) + (c - V_s)\,T = W(x) + \lambda, \qquad \text{for all } x \in \mathbb{R}^n.$$

This implies, in one hand that $c = V_s$; and on the other hand that $W_1(x) - W(x) = \lambda$, for all $x \in \mathbb{R}^n$. □

5. Conclusions and open problems

In this manuscript, we have studied the long time behavior of the value function associated to a finite-dimensional linear-quadratic optimal control problem with any target z. To do so, we have introduced an infinite-time horizon optimal control problem and studied its value function $W(x)$. This allows us to provide an asymptotic decomposition of the value function V for the original control problem with finite time horizon and which is of the form $W(x) + V_s T + \lambda$, where V_s is the value function for the steady problem and λ is the cost of leaving the turnpike close the final time. It is well known that, by the Dynamic Programming Principle, the value function V satisfies a Hamilton-Jacobi-Bellman equation. Then, our results lead to new results in the context of long time behavior of solutions to Hamilton-Jacobi-Bellman equation.

To the best of our knowledge, our results differ from the available literature mainly for two reasons:

- we characterize the ergodic constant c both as $c = \lim_{T\to+\infty} \frac{1}{T} V(x,T)$ and as the value function for the steady problem $c = V_s$. The first characterization is well-known, whereas the second is new;
- in our case, the Hamiltonian $H(x,p)$ as defined in (1.17) is neither Lipschitz nor coercive in the p variable, as soon as B^* has a nontrivial kernel. Available results in the literature require both the Lipschitz property and the coercivity. This might indicate that the classical hypotheses in the structural properties of the Hamiltonian could be eventually replaced by other kind of assumptions concerning the stabilizability and the detectability of the associated optimal control problem.

We now present some open problems.

5.1. **Control problems governed by nonlinear state equations.** We formulate this for a special control problem. Let A be an $n \times n$ symmetric positive definite matrix and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing nonlinearity of class C^1 and with $f(0) = 0$. For a given time horizon $T > 0$, an initial state x in $\mathbb{R}^n$ and a control $u \in \mathcal{U}_T := L^2(0,T;\mathbb{R}^m)$ the corresponding trajectory $y(\cdot)$ solves

$$\begin{aligned} &y'(s) + A\,y(s) + f\left(y(s)\right) = B\,u(s), \quad \text{for } s \in [0,T] \\ &y(0) = x, \end{aligned} \tag{5.1}$$

where the control operator is given by the matrix $B \in \mathcal{M}_{n,m}(\mathbb{R})$ and the nonlinear term $f\left(y(s)\right) = \left(f\left(y_1(s)\right), \dots, f\left(y_n(s)\right)\right)$.

The optimal control problem is to minimize, over the admissible controls $u \in L^2(0,T;\mathbb{R}^m)$, the cost functional

$$J_{T,x}(u) := \frac{1}{2}\int_0^T \left[\|u(s)\|^2 + \|C\,y(s) - z\|^2\right] ds, \tag{5.2}$$

where $C \in \mathcal{M}_n(\mathbb{R})$ is a given matrix and $z \in \mathbb{R}^n$ is the prescribed running target.

The value function associated to the optimal control problem (5.1)-(5.2) is defined as

$$V(x,T) := \inf_{u\in\mathcal{U}_T} J_{T,x}(u). \tag{5.3}$$

In the same line as for the LQ problem treated in this manuscript, one can also introduce in this context the steady functional

$$J_s\left(u_s, y_s\right) := \frac{1}{2}\left[\|u_s\|^2 + \|C\,y_s - z\|^2\right], \tag{5.4}$$

to be minimized over the subset of controlled steady states

$$M := \{(u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \mid Ay_s + f(y_s) = Bu_s\}. \tag{5.5}$$

Set $V_s := \min_M J_s$. These kind of problems have been treated both in a finite dimensional framework [34] and in a PDE framework [27, 33, 32, 25]. Available results in the literature typically require smallness conditions on the running target.

By using the techniques developed in the above references, it is possible to get bounds on the space derivatives of the value function, which allow to apply the Ascoli-Arzelà Theorem as in the proof of Theorem 1.1. For small targets, by using the turnpike results of [25] and adapting the techniques of the present manuscript, we can deduce large time asymptotics of the value function as in Theorem 1.1. However, for large targets, to the best of our knowledge, the turnpike theory is not complete. In particular, we cannot identify the limit as we do in (2.31), because we do not have a result like

$$\frac{1}{T}V(x,T) \underset{T\to+\infty}{\longrightarrow} V_s, \tag{5.6}$$

which identifies the limit of the time-average of the value function as the value function for the steady problem. Note that, by adapting the techniques of [25, Lemma 2.1, page 12], it is possible to prove

$$\limsup_{T\to+\infty} \frac{1}{T}V(x,T) \leq V_s. \tag{5.7}$$

But we are not able to prove the converse inequality

$$\liminf_{T\to+\infty} \frac{1}{T}V(x,T) \geq V_s. \tag{5.8}$$

From a control perspective, the above inequality means that in time large there is no time-evolving strategy significantly better than the steady ones. Actually, in case the time evolving functional is restricted to time independent controls, (5.8) has been proved in [27, section 4], by Γ-convergence. However, to the best of our knowledge, the above inequality is unknown if the time-evolution functional is minimized over time dependent controls and it is an interesting open problem.

5.2. **Characterization of the ergodic constant in a more general case.** As we mentioned, one of the novelties of this manuscript is the characterization of the ergodic constant c appearing in (1.19) as V_s, which is the minimal value of the steady problem. This has been obtained as a consequence of the validity of the turnpike property for our problem. It would be interesting to generalize this characterization for more general problems, by using Hamilton-Jacobi techniques instead of turnpike theory.

5.3. **Enhance Hamilton-Jacobi literature to include the case of lack of coercivity and lack of Lipschitz property.** As we have anticipated, the function $p \longmapsto H(x,p)$ as defined in (1.17) is neither Lipschitz nor coercive, if B^* has a nontrivial kernel. This prevented us using available results in Hamilton-Jacobi literature. We have then employed turnpike theory to obtain long time behavior results in our context. In our opinion, it would be of great interest to enhance Hamilton-Jacobi techniques to deal with the case we treated in this manuscript.

Let us finally recall that for a Hamilton-Jacobi equation of the general form

$$\partial_T V + H(x, \nabla_x V) = \ell(x)$$

one can associate a control problem under some assumptions besides convexity of $p \mapsto H(x,p)$. In this case, we say that the Hamiltonian admits a *control representation* or it is of *Bellman* type. Such a result can be found in [18, Theorem 5.1] and in the more recent paper [28, Theorem 2.1] where the control problem that represents the HJ equation is explicitely specified. Roughly speaking, this comes from the fact that one should be able to construct a Lagrangian (running cost) as the Fenchel conjugate of the Hamiltonian, but since the Hamiltonian should itself be the Fenchel conjugate of the Lagrangian, then we need the Hamiltonian to be equal to its Fenchel bi-conjugate.

5.4. **The infinite dimensional case.** It is well known that the turnpike property holds as well in the infinite dimensional case (see [26, 27, 33, 32, 25]). In this setting, one can still associate to the infinite dimensional optimal control problem an analogue of the Hamilton-Jacobi-Bellman equation that captures the evolution of the value function and which is beyond the scope of our paper. Indeed, this can be handled for instance by means of the so-called *Master* equation whose characteristics are of HJB type. Such equation appears in the context of Mean Field Games and its long time behavior was studied for instance in [11].

Appendix A. Proof of the turnpike property

Turnpike theory is concerned with the study of the long time behavior of optimization problems as the time horizon T goes to ∞. For a given time-evolution optimization problem, the turnpike property is satisfied if the time-evolution problem asymptotically simplifies towards its steady version, as T tends to infinity.

Therefore, in order to study the turnpike property for our concrete problem, we need to introduce the steady version of (1.1)-(1.2). We define the subspace of controlled steady states as

$$M := \{(u_s, y_s) \in \mathbb{R}^m \times \mathbb{R}^n \mid 0 = Ay_s + Bu_s\}. \tag{A.1}$$

The steady functional is then given by

$$J_s(u_s, y_s) := \frac{1}{2}\left[\|u_s\|^2 + \|C y_s - z\|^2\right]. \tag{A.2}$$

The steady optimal control problem is to minimize, over the subspace M, the cost functional J_s.

By [10, Theorem 7.4.17], any optimal control for (1.1)-(1.2) reads as $u^T = -B^* p^T$, where (y^T, p^T) is solution to the optimality system

$$\begin{cases} \frac{d}{ds} y^T(s) = Ay^T(s) - BB^* p^T(s) & s \in (0,T) \\ -\frac{d}{ds} p^T(s) = C^*(C y^T(s) - z) + A^* p^T(s) & s \in (0,T) \\ y^T(0) = x \\ p^T(T) \in \nabla_x^+ g(y^T(T)), \end{cases} \tag{A.3}$$

where ∇_x^+ denotes the superdifferential.

One can realize that Ham is the matrix associated to (A.3) is the Hamiltonian matrix

$$\text{Ham} := \begin{bmatrix} A & -BB^* \\ -C^*C & -A^*. \end{bmatrix}$$

This matrix is studied in Lemma 3.1 (section 3). We now state a crucial Lemma.

Lemma A.1. *Assume (A, C) is detectable and take $f \in L^2(0, T; \mathbb{R}^n)$. Then, there exists a constant $K \geq 0$, independent of T and f, such that for any $T \geq 1$, for any y solution to*

$$\frac{d}{ds}y = Ay + f \qquad \textit{in } (0, T), \tag{A.4}$$

we have

$$\|y(T)\|^2 \leq K \left[\|y(0)\|^2 + \int_0^T \|C\,y\|^2 ds + \int_0^T \|f\|^2 ds \right]. \tag{A.5}$$

Proof. Step 1 **Decomposition into stable and antistable part**
Following the notation of [9], $\mathscr{L}^-(A)$ and $\mathscr{L}^{0+}(A)$ denote resp. the A-invariant subspaces of $\mathbb{R}^n$ spanned by the generalized eigenvectors of A corresponding to eigenvalues λ of A such that $\mathrm{Re}(\lambda) < 0$ and $\mathrm{Re}(\lambda) \geq 0$. By linear algebra,

$$\mathbb{R}^n = \mathscr{L}^-(A) \oplus \mathscr{L}^{0+}(A),$$

where $\oplus$ stands for the direct sum. Then, let y be a solution to (A.4). Denote by y_1 and y_2 resp. the projections of y onto $\mathscr{L}^-(A)$ and $\mathscr{L}^{0+}(A)$. Then, $y = y_1 + y_2$ and, for $i = 1, 2$,

$$\frac{d}{ds}y_i = Ay_i + f_i \qquad \text{in } (0, T),$$

where f_1 and f_2 stand for resp. the projection of f onto $\mathscr{L}^-(A)$ and $\mathscr{L}^{0+}(A)$.
Step 2 **Estimate for the stable part**
We have

$$\frac{d}{ds}y_1 = Ay_1 + f_1 \qquad \text{in } (0, T),$$

All the eigenvalues of $L_A \restriction_{\mathscr{L}^-(A)}$ are strictly negative, where we have denoted by L_A the linear operator associated to the matrix A. Then, we have, for any $s \in [0, T]$

$$\|y_1(s)\| \leq K \left[\|y_1(0)\| + \|f_1\|_{L^2(0,T)} \right] \leq K \left[\|y(0)\| + \|f\|_{L^2(0,T)} \right], \tag{A.6}$$

the constant K depending only on A.
Step 3 **Estimate for the antistable part**
By definition

$$\frac{d}{ds}y_2 = Ay_2 + f_2 \qquad \text{in } (0, T), \tag{A.7}$$

Since (A, C) is detectable, all the modes in $\mathscr{L}^{0+}(A)$ are observable (see definition of detectability in [9, at the bottom of page 232]). Then, by [26, Remark 2.1] applied to (A.7), we obtain

$$\begin{aligned}\|y_2(T)\|^2 &\leq K\left[\int_{T-1}^{T}\|C\,y_2\|^2ds+\int_{T-1}^{T}\|f_2\|^2ds\right]\\ &\leq K\left[\int_{T-1}^{T}\|C\,y_2\|^2ds+\int_{T-1}^{T}\|f\|^2ds\right]\\ &\leq K\left[\int_{T-1}^{T}\|C\,y_1\|^2ds+\int_{T-1}^{T}\|C\,y\|^2ds+\int_{T-1}^{T}\|f\|^2ds\right]\\ &\leq K\left[\|y(0)\|^2+\int_{T-1}^{T}\|C\,y\|^2ds+\int_{T-1}^{T}\|f\|^2ds\right],\end{aligned}$$

where in the last inequality we have employed (A.6). Therefore

$$\|y_2(T)\|^2\leq K\left[\|y(0)\|^2+\int_0^T\|C\,y\|^2ds+\int_0^T\|f\|^2ds\right], \tag{A.8}$$

with K depending only on (A,C).
Step 4 **Conclusion**
Putting together (A.6) and (A.8), we conclude. □

Furthermore, supposing again (A,C) detectable, we have

$$\|y_s\|^2\leq K\left[\|Ay_s\|^2+\|C\,y_s\|^2\right],\qquad \forall y_s\in\mathbb{R}^n. \tag{A.9}$$

The above inequality follows from (A.5) applied to $\tilde{y}:=ty_s$.

The steady inequality (A.9) yields strict convexity of J_s. Then, as announced, we have uniqueness of the minimizer. Furthermore, by Lemma 3.1, Ham is invertible. Hence there exists a unique solution $(\overline{y},\overline{p})$ to the linear system

$$\begin{cases}0=A\overline{y}-BB^*\overline{p}\\ 0=A^*\overline{p}+C^*(C\overline{y}-z).\end{cases} \tag{A.10}$$

Then, by computing the derivative of J_s, we have that $\overline{u}=-B^*\overline{p}$ is a stationary point for J_s. Then, by using strict convexity, $\overline{u}$ is the unique minimizer of J_s.

Let us define the notion of turnpike property.

Definition A.2. Let $A\in\mathcal{M}_{n\times n}(\mathbb{R})$, $B\in\mathcal{M}_{n\times m}(\mathbb{R})$ and $C\in\mathcal{M}_{n\times n}(\mathbb{R})$. The triplet (A,B,C) enjoys turnpike if, for any initial datum $x\in\mathbb{R}^n$, for every final cost g locally Lipschitz and bounded from below and for each target $z\in\mathbb{R}^n$, there exists $K=K(A,B,C,x_0,z,g)$ and $\mu=\mu(A,B,C)>0$, such that, for any $T\geq 1$,

$$\|u^T(s)-\overline{u}\|+\|y^T(s)-\overline{y}\|\leq K\left[\exp(-\mu s)+\exp\left(-\mu\,(T-t)\right)\right], \tag{A.11}$$

where (u^T,y^T) is any optimal pair for (1.1)-(1.2) and $(\overline{u},\overline{x})$ is a minimizer for the steady functional (1.4).

We are now in position to the following turnpike result.

Theorem A.3. *The triplet (A,B,C) enjoys turnpike if and only if (A,B) is stabilizable and (A,C) is detectable.*

An essential tool for the proof of Theorem A.3 is the following Lemma.

Lemma A.4. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $B \in \mathcal{M}_{n\times m}(\mathbb{R})$. (A, B) is stabilizable if and only if for any initial datum $x \in \mathbb{R}^n$, there exists a control $u \in L^2(0, +\infty; \mathbb{R}^m)$, such that*

$$\int_0^\infty \|y\|^2 ds < +\infty, \tag{A.12}$$

y being the solution to (1.1), with initial datum x and control u.

The above Lemma follows from [8, Remark 2.2 page 24].

Proof of Theorem A.3. Hereafter, we will assume $T \geq 1$.

We start by proving that the stabilizability of (A, B) and the detectability of (A, C) yield the validity turnpike property for (A, B, C).
Step 1 **Boundedness of $\|y^T(T) - \overline{y}\|$ uniform on $T \geq 1$**
By Lemma 2.2, for any control $u \in \mathcal{U}_T$, we have

$$\begin{aligned} J_{T,x}(u) \quad &= \quad T\, V_s + \frac{1}{2}\int_0^T \left[\|u(s) - \overline{u}\|^2 + \|C\ (y(s) - \overline{y})\,\|^2\right] ds \\ &\quad + (\overline{p}, x - y(T))_{\mathbb{R}^n} + g(y(T)), \end{aligned} \tag{A.13}$$

where $(\overline{u}, \overline{y})$ is the steady optimal control-state pair for the functional J_s defined in (2.2) and $V_s := J_s(\overline{u}, \overline{y})$.

We introduce now a specific control, which stabilizes the system towards the optimal steady state $\overline{y}$

$$u^*(s) = -B^* \widehat{E}\,(y^*(s) - \overline{y}) + \overline{u}, \qquad s \in (0, T)$$

where $\widehat{E}$ is the unique symmetric positive semidefinite solution to the Algebraic Riccati Equation (3.1) and y^* solves the closed loop equation

$$\begin{aligned} &\tfrac{d}{ds} y^*(s) = (A + B\,F)\, y^*(s), \quad s \in (0, \infty) \\ &y^*(0) = x, \end{aligned}$$

$F(y) := -B^* \widehat{E}\,(y - \overline{y}) + \overline{u}$ being the feedback law. By Lemma 3.1, we have

$$\|u^*(s) - \overline{u}\| + \|y^*(s) - \overline{y}\| \leq K \exp(-\mu s)\,, \qquad \forall s \in [0, T], \tag{A.14}$$

where K and $\mu > 0$ are independent of the time horizon T. Hence, there exists a T-independent constant K such that

$$\begin{aligned} J_{T,x}(u^*) - T\, V_s \quad &= \quad \frac{1}{2}\int_0^T \left[\|u^*(s) - \overline{u}\|^2 + \|C\ (y^*(s) - \overline{y})\,\|^2\right] ds \\ &\quad + (\overline{p}, x - y^*(T))_{\mathbb{R}^n} + g(y^*(T)) \\ &\leq \quad K, \end{aligned}$$

whence, by definition of minimizer

$$J_{T,x}(u^T) - T\, V_s \leq J_{T,x}(u^*) - T\, V_s \leq K. \tag{A.15}$$

By Lemma A.1 applied to $y^T - \overline{y}$, we have

$$\|y^T(T) - \overline{y}\|^2 \leq K \left[\|x - \overline{y}\|^2 + \int_0^T \left[\|u^T(s) - \overline{u}\|^2 + \|C \left(y^T(s) - \overline{y} \right) \|^2 \right] ds \right],$$

whence, by (A.13), (A.15) and since g is bounded from below,

$$\begin{aligned} \|y^T(T) - \overline{y}\|^2 &\leq K \left[\|x - \overline{y}\|^2 + \int_0^T \left[\|u^T(s) - \overline{u}\|^2 + \|C \left(y^T(s) - \overline{y} \right) \|^2 \right] ds \right] \\ &\leq K \left[J_{T,x}(u^T) - \left(\overline{p}, x - y^T(T) \right)_{\mathbb{R}^n} - g(y^T(T)) + \|x - \overline{y}\|^2 + 1 \right] \\ &\leq K \left[J_{T,x}(u^T) + \|y^T(T) - \overline{y}\| + \|x - \overline{y}\|^2 + 2 \right] \\ &\leq K \left[J_{T,x}(u^*) + \|y^T(T) - \overline{y}\| + \|x - \overline{y}\|^2 + 2 \right] \\ &\leq K \left[\|x - \overline{y}\|^2 + \|y^T(T) - \overline{y}\| + 3 \right]. \end{aligned} \tag{A.16}$$

Then, there exists a T-independent constant K, such that for any $T \geq 1$

$$\left\| y^T(T) - \overline{y} \right\| \leq K. \tag{A.17}$$

Step 2 **Boundedness of $\|p^T(0) - \overline{p}\|$ uniform on $T \geq 1$**
First of all, we subtract the Optimality Systems (A.3) and (A.10), getting

$$\begin{cases} \frac{d}{ds}(y^T - \overline{y}) = A(y^T - \overline{y}) - BB^*(p^T - \overline{p}) & s \in (0,T) \\ -\frac{d}{ds}(p^T - \overline{p}) = A^*(p^T - \overline{p}) + C^*C(y^T - \overline{y}) & s \in (0,T) \\ u^T - \overline{u} = -B^*(p^T - \overline{p}) & s \in (0,T) \\ y^T(0) - \overline{y} = x - \overline{y} \\ p^T(T) - \overline{p} = q - \overline{p}, \end{cases} \tag{A.18}$$

for some $q \in \nabla_x^+ g(y(T))$. By Remark A.1 applied to $p^T(T-t) - \overline{p}$, we have

$$\begin{aligned} \|p^T(0) - \overline{p}\|^2 &\leq K \left[\|q - \overline{p}\|^2 + \int_0^T \left[\|u^T(s) - \overline{u}\|^2 + \|C \left(y^T(s) - \overline{y} \right) \|^2 \right] ds \right] \\ &\leq K \left[J_{T,x}(u^T) - \left(\overline{p}, x - y^T(T) \right)_{\mathbb{R}^n} - g(y^T(T)) + \|q - \overline{p}\|^2 + 1 \right] \\ &\leq K \left[J_{T,x}(u^T) + \|q - \overline{p}\|^2 + 2 \right] \\ &\leq K \left[J_{T,x}(u^*) + \|q - \overline{p}\|^2 + 2 \right] \\ &\leq K \left[\|q - \overline{p}\|^2 + 3 \right]. \end{aligned} \tag{A.19}$$

for a constant K independent of the time horizon $T \geq 1$.

At this point, we realize that since g is locally Lipschitz, for any compact $K \subset \mathbb{R}^n$, for any $x \in K$ and $q \in \nabla_x^+ g(x)$, we have $\|q\| \leq L$, where L is the Lipschitz constant for g in K. Therefore, by (A.17) and (A.19), there exists a T-independent constant K, such that for any $T \geq 1$

$$\left\| p^T(T) - \overline{p} \right\| \leq K. \tag{A.20}$$

At this stage, by (A.17) and (A.20), we have:

$$\|p^T(0) - \overline{p}\| + \|y^T(T) - \overline{y}\| \leq K, \tag{A.21}$$

the constant K being independent to the time horizon.
Step 3 **Conclusion**
Set

$$\begin{bmatrix} g \\ h \end{bmatrix} := \Lambda^{-1} \begin{bmatrix} y^T - \overline{y} \\ p^T - \overline{p}, \end{bmatrix}$$

where the transformation Λ has been introduced in (3.2). Then, by Lemma 3.1

$$\text{(A.22)} \qquad \begin{cases} \frac{d}{ds} g = (A - BB^* \widehat{E}) g & s \in (0,T) \\ \frac{d}{ds} h = -(A - BB^* \widehat{E})^* h & s \in (0,T). \end{cases}$$

Since the matrix $A - BB^* \widehat{E}$ is stable, there exists K and $\mu > 0$ such that

$$\| \exp(t(A - BB^* \widehat{E}))\| \leq K \exp(-\mu s).$$

Then,

$$\|g(s)\| \leq K \exp\left(-\mu s\right) \|g(0)\| \leq K \exp\left(-\mu s\right) \left[\|x - \overline{y}\| + \|p^T(0) - \overline{p}\|\right]$$

and

$$\|h(s)\| \leq K \exp\left(-\mu\left(T - s\right)\right) \|h(T)\| \leq K \exp\left(-\mu\left(T - s\right)\right) \left[\|y^T(T) - \overline{y}\| + \|q - \overline{p}\|\right].$$

Finally, by (A.21), we have

$$\|g(s)\| + \|h(s)\| \leq K \left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T - s\right)\right)\right],$$

whence

$$\|y^T(s) - \overline{y}\| + \|p^T(s) - \overline{p}\| \leq K \left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T - s\right)\right)\right].$$

We now prove that the validity turnpike property for (A, B, C) for any initial data x and final cost g entail the stabilizability of (A, B) and the detectability of (A, C).
Step 4 **Necessity of the stabilizability of** (A, B)
Suppose (A, B, C) enjoys turnpike. Then, taking target $z = 0$ and final cost $g \equiv 0$, for any initial datum $x \in \mathbb{R}^n$, we have

$$\text{(A.23)} \qquad \|u^T(s)\| + \|y^T(s)\| \leq K \left[\exp\left(-\mu s\right) + \exp\left(-\mu\left(T - t\right)\right)\right],$$

(u^T, y^T) being the optimal pair for (1.1)-(1.2), with target $z = 0$, initial datum x and final cost $g \equiv 0$, whence

$$\text{(A.24)} \qquad \int_0^T \|u^T\|^2 ds \leq K,$$

where K is independent of the time horizon T. By Banach-Alaoglu Theorem, there exists $u^\infty \in L^2((0, +\infty); \mathbb{R}^n)$, such that, up to subsequences,

$$u^T \underset{T \to +\infty}{\longrightarrow} u^\infty,$$

weakly in $L^2((0, +\infty); \mathbb{R}^m)$. We denote by y^∞ the solution to (1.1), with initial datum x and control u^∞. Arbitrarily fix $S > 0$. By definition of weak convergence, up to subsequences

$$y^T \underset{T \to +\infty}{\longrightarrow} y^\infty,$$

weakly in $L^2((0,S);\mathbb{R}^m)$. By lower-semicontinuity of the norm with respect to the weak convergence and (A.23), for any $S>0$, we have

$$\int_0^S \|y^\infty\|^2 ds \leq \liminf_{T\to+\infty} \frac{1}{2}\int_0^S \|y^T(s)\|^2 ds \leq K,$$

whence, by the arbitrariness of S,

$$\int_0^\infty \|y^\infty\|^2 ds \leq K < +\infty.$$

Then, by Lemma A.4, (A,B) is stabilizable.

Step 5 **Necessity of the detectability of** (A,C)

Suppose (A,B,C) enjoys turnpike. We introduce the set of unobservable modes

$$NO(C,A) = \bigcap_{i=0}^{n-1} \ker\left(CA^i\right).$$

To show the detectability of (A,C), we have to prove that any $x \in NO(C,A)$ is A-stable (see e.g. [35] for linear control theory). Take target $z=0$ and final cost $g \equiv 0$. Since x is not observable, for any $T>0$, the optimal control for $J_{T,x}$ is $u^T \equiv 0$ and the corresponding optimal state y^T solves

$$\begin{aligned} &y'(s) = A\,y(s), \quad \text{for } s>0 \\ &y(0) = x. \end{aligned} \tag{A.25}$$

By (A.11), we have

$$\|y^T\left(T/2\right)\| \leq K\exp(-\mu T/2),$$

for any $T>0$, whence x is A-stable. This finishes the proof. □

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*, Systems & Control: Foundations & Applications, Birkhäuser Basel, 2012.

[2] M. Arisawa, *Ergodic problem for the Hamilton-Jacobi-Bellman equation. i. existence of the ergodic attractor*, in Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 14, Elsevier, 1997, pp. 415–438.

[3] ———, *Ergodic problem for the Hamilton-Jacobi-Bellman equation. ii*, in Annales de l'Institut Henri Poincare (C) Non Linear Analysis, vol. 15, Elsevier, 1998, pp. 1–24.

[4] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Springer Science & Business Media, 2008.

[5] G. Barles, O. Ley, T.-T. Nguyen, and T. V. Phan, *Large time behavior of unbounded solutions of first-order Hamilton–Jacobi equations in rn*, Asymptotic Analysis, 112 (2019), pp. 1–22.

[6] G. Barles and J.-M. Roquejoffre, *Ergodic type problems and large time behaviour of unbounded solutions of Hamilton–Jacobi equations*, Communications in Partial Differential Equations, 31 (2006), pp. 1209–1225.

[7] G. Barles and P. E. Souganidis, *On the large time behavior of solutions of Hamilton–Jacobi equations*, SIAM Journal on Mathematical Analysis, 31 (2000), pp. 925–939.

[8] A. Bensoussan, G. Da Prato, M. Delfour, and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, Systems & Control: Foundations & Applications, Birkhäuser Boston, 2006.

[9] F. M. Callier and J. Winkin, *Convergence of the time-invariant riccati differential equation towards its strong solution for stabilizable systems*, Journal of mathematical analysis and applications, 192 (1995), pp. 230–257.

[10] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, vol. 58, Springer Science & Business Media, 2004.
[11] P. Cardaliaguet and A. Porretta, *Long time behavior of the master equation in mean field game theory*, Analysis & PDE, 12 (2019), pp. 1397–1453.
[12] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bulletin of the American mathematical society, 27 (1992), pp. 1–67.
[13] M. G. Crandall and P.-L. Lions, *Viscosity solutions of hamilton-jacobi equations*, Transactions of the American mathematical society, 277 (1983), pp. 1–42.
[14] L. C. Evans, *Partial differential equations*, vol. 19, American Mathematical Soc., 2010.
[15] W. H. Fleming and W. M. McEneaney, *Risk-sensitive control on an infinite time horizon*, SIAM Journal on Control and Optimization, 33 (1995), pp. 1881–1915.
[16] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions*, vol. 25, Springer Science & Business Media, 2006.
[17] Y. Fujita, H. Ishii, and P. Loreti, *Asymptotic solutions of Hamilton-Jacobi equations in euclidean n space*, Indiana University mathematics journal, (2006), pp. 1671–1700.
[18] H. Ishii, *On representation of solutions of hamilton-jacobi equations with convex hamiltonians*, in North-Holland Mathematics Studies, vol. 128, Elsevier, 1985, pp. 15–52.
[19] ———, *Asymptotic solutions for large time of Hamilton-Jacobi equations*, in International Congress of Mathematicians, vol. 3, 2006, pp. 213–227.
[20] ———, *Asymptotic solutions for large time of Hamilton-Jacobi equations in euclidean n space*, in Annales de l'IHP Analyse non linéaire, vol. 25, 2008, pp. 231–266.
[21] ———, *A short introduction to viscosity solutions and the large time behavior of solutions of Hamilton–Jacobi equations*, in Hamilton-Jacobi equations: approximations, numerical analysis and applications, Springer, 2013, pp. 111–249.
[22] H. Kouhkouh, *Dynamic programming interpretation of turnpike and Hamilton-Jacobi-Bellman equation*, (2018). Available online: http://bit.ly/2R7soRx.
[23] H. Kwakernaak and R. Sivan, *Linear optimal control systems*, vol. 1, Wiley-interscience New York, 1972.
[24] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, vol. 69, London Pitman, 1982.
[25] D. Pighin, *The turnpike property in semilinear control*, arXiv preprint arXiv:2004.03269, (2020).
[26] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim., 51 (2013), pp. 4242–4273.
[27] ———, *Remarks on long time versus steady state optimal control*, in Mathematical Paradigms of Climate Science, Springer, 2016, pp. 67–89.
[28] F. Rampazzo, *Faithful representations for convex hamilton–jacobi equations*, SIAM journal on control and optimization, 44 (2005), pp. 867–884.
[29] J.-M. Roquejoffre, *Convergence to steady states or periodic solutions in a class of Hamilton–Jacobi equations*, Journal de mathématiques pures et appliquées, 80 (2001), pp. 85–104.
[30] N. Sakamoto and D. Pighin, *The turnpike with lack of observability.* In preparation.
[31] N. Sakamoto and A. J. van der Schaft, *Analytical approximation methods for the stabilizing solution of the Hamilton–Jacobi equation*, IEEE Transactions on Automatic Control, 53 (2008), pp. 2335–2350.
[32] E. Trélat and C. Zhang, *Integral and measure-turnpike properties for infinite-dimensional optimal control systems*, Mathematics of Control, Signals, and Systems, 30 (2018), p. 3.
[33] E. Trélat, C. Zhang, and E. Zuazua, *Steady-state and periodic exponential turnpike property for optimal control problems in hilbert spaces*, SIAM Journal on Control and Optimization, 56 (2018), pp. 1222–1252.
[34] E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations, 258 (2015), pp. 81–114.
[35] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*, Feher/Prentice Hall Digital and, Prentice Hall, 1996.

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